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Competing on Supply Chain Quality

A Network Economics Perspective

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A Network Economics Perspective

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We dedicate this book to our families, friends, colleagues, and students.

Preface

Quality is essential to the food that we eat, the water that we drink, the air that we breathe, and every product that we consume and utilize daily, from medicines to the clothes that we wear, the homes that we live in and the places where we work, the technological devices that we depend on, and the vehicles that transport us. The quality of a product, or lack thereof, can make or break a company's reputation and affect its ultimate success or failure.

In today's network economy, supply chains weave together suppliers, manufacturers, freight service providers, and other stakeholders into intricate networks that produce, distribute, and transport the products to retailers and consumers across the globe. These networks are essential to the ultimate quality of the products. Under globalization and decentralization, the analysis of interactions among decision-makers, with a focus on quality issues, merits a new and fresh investigation and synthesis.

In this book, we explore fundamental issues concerning quality in supply chain networks theoretically, computationally, and through numerous case studies based on solved numerical examples. The book develops the fundamental methodologies for model formulation, analysis, and solution of supply chain competition problems in quality. The supply chain network topologies of the models reveal graphically the interactions among decision-makers, whose behavior can be studied and captured mathematically. In addition, this book deals with quality and information asymmetry, the imposition of minimum quality standards, R&D, outsourcing decision-making, make-or-buy decisions, supplier selection, and freight service provider selection. Both equilibrium models and many of their dynamic counterparts are presented.

The audience for this book includes researchers, practitioners, and students interested in a rigorous treatment of supply chains and quality issues from an integrated operations research and network economics perspective.

We thank Springer for permission to use parts of our previously published articles that have appeared in its journals. We also thank Wiley and Elsevier for permission to expand journal articles into book size as allowed by their copyright agreements. Full citations and acknowledgment to our published articles are provided later in this book.

We view this book as a beginning for the channeling of additional interest, research, and resources to enhance quality of products and the performance of the associated supply chains for a better world. We feel that the journey has just begun, and we welcome comments and new collaborators.

Amherst, USA
September 2015

Anna Nagurney
Dong Li

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While writing this book, we learned of the tragic death of the 1994 Nobel laureate in Economic Sciences, John F. Nash. We acknowledge the great importance of the Nash equilibrium and game theory to our work. His legacy lives on.

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Quality and Supply Chains

Chapter 1
Introduction

Abstract In this chapter, we motivate this book and demonstrate the need for the inclusion of quality in supply chain network modeling, analysis, and computations. Our perspective is that of network economics, which helps to reveal the complex interactions among supply chain decision-makers and their behaviors as well as the topology of the supply chain structures. In addition, we review definitions of quality and discuss how to quantify quality. We also highlight the foundational literature from economics, operations research and management science, and operations management on the subject of quality relevant to supply chain networks. Finally, we describe the scope of this book and the organization of its parts and chapters.

1.1 Motivation

Supply chains, as networks of economic activities, link suppliers, manufacturers, freight service providers, retailers, and consumers, along with other stakeholders, for the production and distribution of products as well as services from local to global scales. Consumers have come to expect fresh produce all year round, advanced high technology products, on demand, the latest fashionable clothing, comfortable automobiles, curative medicines, when needed, and toys that are safe for their children. Hence, the evolution of supply chains over space and time has been characterized by increasing decentralization as well as globalization.

However, as increasing numbers of firms from around the world interact and compete with one another to produce and to provide products to geographically distributed locations, supply chain networks have become far more complex than ever before. For example, Sara Lee bread, an everyday item, is made with flour from the US, vitamin supplements from China, gluten from Australia, honey from Vietnam and India, and other ingredients from Switzerland, South America, and Russia (Bailey 2007). According to Blomeyeri et al. (2012), 16 % of European fish fillet imports from China are Atlantic cod, typically, of Norwegian or Russian origin, with some product from Iceland, all processed in factories in China, demonstrating the immense length of this supply chain.

As a consequence of decentralization and globalization, firms' products may be more exposed to both domestic and international failures running the gamut from poor product quality to unfilled demand (see, e.g., Nagurney et al. 2011, 2012;

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Liu and Nagurney 2013; Yu and Nagurney 2013). Examples of recent vivid product quality failures have included adulterated infant formula (Barboza 2008), inferior pharmaceuticals (see Masoumi et al. 2012), bacteria-laden food (see, e.g., Marsden 2004), low-performing high technology products (see Goettler and Gordon 2011), defective airbags (Soble 2015) and ignition switches (Matthews and Spector 2015) as well as inferior durable goods referred to a "lemons" in the case of automobiles as noted in Akerlof's (1970) fundamental study, to name just a few. At the same time, quality has been recognized as "the single most important force leading to the economic growth of companies in international markets" (Feigenbaum 1982), and, in the long run, as the most important factor affecting a business unit's performance and competitiveness (Buzzell and Gale 1987).

High quality products make a critical contribution to a firm's long-term profitability due to the fact that consumers expect high quality products and services and are, typically, willing to pay higher prices for them. Products that are of high quality can also secure the reputation of the brand, since firms can obtain the associated certifications/labels and declarations, along with consumer support and loyalty. For example, the ISO (International Organization for Standardization) 9000 series guarantees the safety and reliability of the quality management processes of firms and, hence, the quality of their products. In addition, firms that fail to produce and deliver products of good quality may have to pay for the accompanying consequences, such as the costs of returns, replacements, the loss of customer satisfaction and loyalty, and the loss of their reputation, which can be catastrophic. Most importantly, poor quality products, whether inferior durable goods, such as automobiles, or consumables such as pharmaceuticals and food, may negatively affect the very safety and well-being of consumers, with, possibly, associated fatal consequences.

It is, hence, puzzling and paradoxical that, since firms should have sufficient incentive to produce high quality products, why do low quality products still exist?

The reality of today's supply chain networks, given their global reach from sourcing locations to points of demand, is further challenged by such issues as the growth in outsourcing and global procurement as well as the information asymmetry associated with what producers know about the quality of their products as compared to what consumers know. In addition, although much of the related literature has focused on the micro aspects of supply chain networks, considering two or three decision-makers, it is essential to be able to capture the scale of supply chain networks that occurs in practice and to evaluate and analyze a firm's performance and that of the competition in a quantifiable manner. Boeing, one of the largest manufacturers in the US, relies on hundreds of suppliers for its 737 planes (Linn 2010). Apple, the innovative high technology company, reviewed over

450 of its suppliers in 2014 (see Apple 2015). As for the demand side, the number of demand points for products can easily range from dozens to hundreds to many thousands, as in the case, of popular products, such as smartphones. Of course, demand points can also correspond to retailers. As noted in Köksalan et al. (2013), the US brewing industry, the second largest after China, includes more than 2,000 brewers (producers) and over 521,000 retailers.

The focus of this book is to provide computable supply chain network models, the associated qualitative analysis, and algorithms, that enable decision-makers to evaluate the full generality of supply chain networks with an emphasis on product quality. Towards that end, we demonstrate how to capture the objective functions that decision-makers are faced with, whether that of cost minimization or profit-maximization, etc., along with the constraints, as well as the decision-makers' interactions and the impacts on quality, product outputs, prices, and, of course, profits.

In this book, we contribute to the equilibrium and dynamic modeling and analysis of quality competition in supply chain networks in an environment of increasing economic competitiveness in order to explore such fundamental issues as: the role of information asymmetry and of product differentiation, as well as the impacts of outsourcing and of supplier selection plus freight service provider selection. Specifically, the book addresses such fundamental questions as:

1. What is the individual optimization behavior of the firms, whether, manufacturers, suppliers, contractors, or freight service providers?
2. What are the equilibrium product quality levels of competing firms and how to compute their values?
3. How do these quality levels evolve over time until the equilibrium is achieved?
4. How stable are the equilibria?
5. What are the impacts on product quality, costs, and profits, of minimum quality regulations?
6. How to capture the cost associated with R&D activities?
7. Which are the most important suppliers as well as supplier components in the supply chain network economy?
8. How to model and solve supply chain network problems with multiple suppliers, components, and manufacturing firms, and how to quantify the ensuing product quality?
9. How to capture quality in freight service provision, which is essential for the transport and delivery of product components as well as finished products?

In this book, the supply chain network structure of each model is depicted, along with the underlying economic behavior of the supply chain decision-makers. We identify the nodes and the links in the networks, the costs associated with the latter, and the link flows. We also discuss the dependence of the various functions, such as cost and demand price functions, on quality. The equilibria and the associated dynamics of production and shipments, quality, and prices, are revealed under scenarios of, respectively, information asymmetry, product differentiation, outsourcing, and under supplier selection as well as freight service provider selection.

We next present definitions of quality and the quantification of quality and associated cost. We then provide a synthesizing literature review and give the outline of the organization of this book and its chapters.

1.2 Definitions and Quantification of Quality and Cost of Quality

In order to be able to quantify quality and, hence, to measure it, it is essential to have a rigorous definition of quality. Different definitions of quality have been presented at various times by researchers in different fields. The definitions can be classified into four main categories: (1) quality is excellence, (2) quality is value, (3) quality is meeting and/or exceeding customers' expectations, and (4) quality is the conformance to a design or specification.

According to the view that quality is excellence (e.g., Tuchman 1980; Garvin 1984; Pirsig 1992), this perspective requires the investment of the best effort possible to produce the most admirable and uncompromising achievements possible. Although striving for excellence may bring significant marketing benefits for firms, one has to admit that excellence is a very abstract and subjective term, and it is very difficult to articulate precisely what excellence is, let alone explain clearly what are the standards for excellence, and how excellence can be measured, modeled, achieved, and compared in practice.

Feigenbaum (1951), Abbott (1955), and Cronin and Taylor (1992) criticized the quality-as-excellence definition and argued that the definition of quality should be value. According to them, quality is the value of a product under certain conditions, which include the actual use and the price of the product. Many attributes of quality can be included in value, such as price and durability, but quality is actually not synonymous with value (Stahl and Bounds 1991). When consumers purchase products, they consider not only their quality but also their prices, which are two separate concepts. The term *value*, hence, has the disadvantage of blending these two distinct concepts together.

The extent to which a product or service meets and/or exceeds a customer's expectations is another definition of quality (e.g., Feigenbaum 1983; Parasuraman et al. 1985; Buzzell and Gale 1987; Grönroos 1990). It is argued that customers are the only ones who judge quality, and the quality of a product should be just the perception of quality by consumers. This definition allows firms to focus on factors that consumers care about. However, it is also very subjective and, hence, very difficult to quantify and to measure. Different customers may have distinct preferences as to the attributes of a product and it is often the case that even consumers themselves may not know what their expectations are (Cameron and Whetten 1983).

Shewhart (1931), Juran (1951), Levitt (1972), Gilmore (1974), Crosby (1979), Deming (1986), and Chase and Aquilano (1992), most of whom are operations management scholars, are the major advocates of the conformance-to-specification definition of quality. They define quality as “the degree to which a specific product conforms to a design or specification,” which is how well the product is conforming to an established specification. A major advantage of this definition is that it makes it relatively straightforward to quantify quality, which is essential for firms and researchers who are eager to measure it, manage it, model it, compare it across time, and who are also making associated decisions (Shewhart 1931). Kolesar (1993) provides a survey of the classical quality literature.

At first glance, it may seem that the conformance-to-specification definition of quality focuses too much on internal quality measurement rather than on consumers’ desires at the demand markets. However, we note that consumers’ needs and desires for a product are actually governed by specific requirements or standards and these can be correctly translated to a specification by design engineers (Oliver 1981). This feature makes the conformance-to-specification definition quite general. In addition to consumers’ needs, the specification of a product can also include both international and domestic standards (Yip 1989), and, in order to gain marketing advantages in the competition with other firms, the competitors’ product specifications.

All of the above four definitions of quality are still in use today (Wankhade and Dabade 2010). As one may notice from the above, each definition has both strengths and weaknesses in criteria such as measurement, generalizability, and consumer relevance.

In this book, since the conformance-to-specification definition not only makes quality quantifiable, but also is sufficiently general to include many dimensions of quality, we define quality as “the degree to which a specific product conforms to a design or specification.” Quality, hence, may vary from a 0 % conformance level to a 100 % conformance level (see, e.g., Feigenbaum 1983; Juran and Gryna 1988; Campanella 1990; Porter and Rayner 1992; Shank and Govindarajan 1994). When the quality of a particular product is at a 0 % conformance level, the product has no quality; when the quality achieves a 100 % conformance level, the product is of perfect quality.

The reasons that we adopt this definition rather than defining quality as (a) a binary variable representing whether a product is defective or not, or (b) a value of the defect rate, are as follows. First, it is very difficult to define and identify defective goods in practice. If defective items could always be well-defined and successfully screened out before they reach consumers, quality failure incidents caused by defectives would never occur. Secondly, consumers’ demand in terms of quality is not simply based on whether a product is defective or not. What matters more to consumers is how well the product serves their needs and meets their expectations. Consumers are often willing to pay more for products that can satisfy their needs in a better way, that is, are of higher quality. Therefore, the conformance-to-specification definition is a more general expression of quality than (a) and (b). In addition, given the different characteristics of and functions that competing products can perform, quality should not only be limited to whether a product is defective or not, but must capture the degree, which can take on different values for different products.

Quality levels with lower and upper bounds can also be found in Akerlof (1970) ($q \in [0, 2]$), Leland (1979) ($q \in [0, 1]$), Chan and Leland (1982) ($q \in [q_0, q_H]$), Lederer and Rhee (1995) ($q \in [0, 1]$), Acharyya (2005) ($q \in [q_0, \bar{q}]$), and Chambers et al. (2006) ($q \in [0, q_{max}]$). Reyniers and Tapiero (1995), Tagaras and Lee (1996), Baiman et al. (2000), Hwang et al. (2006), Hsieh and Liu (2010), and Lu et al. (2012) modeled quality as probabilities, ranging from 0 to 1. The justification for quality ceilings by these authors is that, due to the laws of physics, the state of technology, and the ability of improving quality, there should be a quality ceiling. Therefore, quality levels are quantified as values between 0 and the perfect quality level in Chaps. 7, 8, and 10. Also, the sensitivity analysis for the model with information asymmetry and minimum quality standards in Chap. 3 is conducted with minimum quality standards up through 100, the perfect quality.

However, in the majority of the economics and management science literature on quality competition (see, e.g., Mussa and Rosen 1978; Gal-Or 1983; Cooper and Ross 1984; Riordan and Sappington 1987; Rogerson 1988; Ronnen 1991; Banker et al. 1998; Johnson and Myatt 2003; Xu 2009; Xie et al. 2011; Kaya 2011), quality levels are only assumed to be nonnegative, and there are no upper bounds on quality. These models assert that there is no "best" quality since there can always be a quality level that is even better than the best. Since Chaps. 3 and 5 are inspired by these papers, no upper bounds are assumed therein, although we demonstrate in Chap. 3 how they can be easily incorporated.

Based on the conformance-to-specification definition of quality, the cost of quality, consumers' sensitivity to quality, and the cost of quality disrepute (that is, the loss of reputation due to low quality), all of which are crucial elements in modeling quality competition in supply chain networks, can be quantified and measured, and the equilibrium quality level of each firm can also be determined in the competitive environment. Following the conformance-to-specification definition of quality, quality cost is defined as the "cost incurred in ensuring and assuring quality as well as the loss incurred when quality is not achieved" (ASQC 1971; BS 1990). Although, according to traditional cost accounting, quality cost may not be practically quantified in cost terms (Chiadamrong 2003), there are a variety of schemes by which quality costing can be implemented by firms, some of which have been described in Juran and Gryna (1988) and Feigenbaum (1991).

Based on the quality management literature, four categories of quality-related costs occur in the process of quality management. These are: the prevention cost, the appraisal cost, the internal failure cost, and the external failure cost. They have been developed and are widely applied in organizations (see, e.g., Crosby 1979; Harrington 1987; Juran and Gryna 1993; Rapley et al. 1999). Quality cost is usually understood as the sum of the four categories of quality-related costs, and, it is widely

Table 1.1 Categories of quality-related costs

Category	Definition	Examples	Shape of the function
Prevention costs	Investments to ensure the required quality level in the process of production	Costs in quality engineering, receiving inspection, equipment repair/maintenance, and quality training	Continuous, convex, monotonically increasing. When the quality of conformance is 0 %, this cost is zero.
Appraisal costs	Costs incurred in identifying poor quality before shipment	Incoming inspection and testing cost, in-process inspection and testing cost, final inspection and testing cost, and evaluation of stock cost	Continuous, convex, monotonically increasing. When the quality of conformance is 0 %, this cost is zero.
Internal failure costs	Failure costs incurred when defects are discovered before shipment	Scrap cost, rework cost, failure analysis cost, re-inspection and retesting cost	Continuous, convex, monotonically decreasing. When the quality of conformance is 100 %, this cost is zero.
External failure costs	Failure costs associated with defects that are found after delivery of defective goods or services	Warranty charges cost, complaint adjustment cost, returned material cost, and allowances cost	Continuous, convex, monotonically decreasing. When the quality of conformance is 100 %, this cost is zero.

believed that, the functions of the four quality-related costs are convex functions of the quality conformance level. Therefore, the cost of quality is also convex in quality (see, e.g., Feigenbaum 1983; Juran and Gryna 1988; Campanella 1990; Porter and Rayner 1992; Shank and Govindarajan 1994; Alzaman et al. 2010). Please see Table 1.1 for more details of the four quality-related costs.

Among the four quality-related costs, the external failure cost, which is the compensation cost incurred when customers are dissatisfied with the quality of the products, such as warranty charges and the complaint adjustment cost, is strongly related to consumers' satisfaction in terms of the firm's product, and, hence, can be utilized to measure the disrepute cost of the firm in addition to the cost of quality.

In addition to the cost of quality, the expenditures on R&D have also widely been recognized as a cost depending on the quality level of the firm, which is independent of production and sales (cf. Klette and Griliches 2000; Hoppe and Lehmann-Grube 2001; Symeonidis 2003). In Chap. 5, we discuss how our product cost functions can be used to include R&D cost in a supply chain network model with differentiated products.

1.3 Synthesis of the Relevant Literature

The literature review below discusses, respectively, information asymmetry in quality, quality competition, models of quality in manufacturing outsourcing, suppliers' quality, and freight services and quality. Background on supply chain network economics, but without the quality dimension, can be found in Nagurney (2006). This book also goes further by incorporating suppliers' decision-making behavior, that of contractors, and also freight service providers. For models of perishable product supply chain network competition in industries such as pharmaceuticals, food, medical nuclear, as well as fast fashion, see Nagurney et al. (2013).

1.3.1 Quality Information Asymmetry Between Firms and Consumers

One of the challenges in today's supply chains is that of information asymmetry in quality. Producers, located around the globe, are expected to have knowledge about the quality of the products that they produce but consumers may be less informed. Moreover, manufacturers who use suppliers' components, which may have been sourced from multiple locations, may not be fully aware as to the quality of these inputs to their manufacturing processes. Of course, the ultimate quality of a product from the bread that we consume to the vehicles that we drive and the planes that we fly in is a function of the quality of each of the components that make up the product.

Markets with asymmetric information have been studied by many notable economists, including Akerlof (1970), Spence (1975), and Stiglitz (1987, 2002), who shared the 2001 Nobel Prize in Economic Sciences. The seminal contribution in the area of quality information asymmetry between firms and consumers is the classic work of Akerlof (1970)'s, which has stimulated the research in this domain. Following Akerlof (1970), Leland (1979) modeled perfect competition in a market with quality information asymmetry, and argued that such markets may benefit from minimum quality standards. Smallwood and Conlisk (1979) investigated market share equilibria in a multiperiod model considering quality positively related to the probability of repeated purchases. Shapiro (1982) analyzed a monopolist's behavior in a market with imperfect information in quality, and noted that it was a reason for quality deterioration. Chan and Leland (1982) developed a model with price and quality competition among firms in which they could acquire price/quality information at a cost and the average cost functions were identical for all firms. Schwartz and Wilde (1985) considered markets where consumers were imperfectly informed about prices and quality, and provided equilibria under cases where all consumers preferred either higher quality or lower quality.

Bester (1998) studied price and quality competition between two firms and noted that imperfect information quality reduced the sellers' incentives for differentiation.

Besancenot and Vranceanu (2004) studied quality information asymmetry among firms who decided on prices and quality, and consumers who searched for the best offer in a sequential way. Armstrong and Chen (2009) presented a model in which some consumers shopped without attention to quality, and firms might cheat to exploit these consumers. Baltzer (2012) considered two firms involved in price and quality competition with specific underlying functional forms to study the impact of minimum quality standards and labels.

Moreover, product price and advertising have long been viewed as indicators of quality for consumers. Wolinsky (1983) was concerned with markets with price and quality competition in which consumers had imperfect information and concluded that price indicated quality. Cooper and Ross (1984) modeled perfect quality competition among firms to examine the degree that prices conveyed information about quality. Rogerson (1988) considered quality and price competition among identical firms and indicated that advertising was a signal of quality. Dubovik and Janssen (2012) considered a quality and price competition model with heterogeneous information on quality at the demand market, and showed that price indicated quality. Other contributors in this area are: Nelson (1974), Farrell (1980), Klein and Leffler (1981), Gerstner (1985), Milgrom and Roberts (1986), Tellis and Wernerfelt (1987), Bagwell and Riordan (1991), Linnemer (1998), Fluet and Garella (2002), and Daughety and Reinganum (2008).

In this book, we explore information asymmetry between producers and consumers in supply chain networks under imperfect, that is, oligopolistic, competition as well as under perfect competition. We also show why care must be taken by policy-makers who impose minimum quality standards in markets under such information asymmetry.

1.3.2 Competition in Quality

No firm is an island or isolated node in the Network Economy and, hence, must interact with other firms and with the consumers of its products and services. In order to be effective and profitable it is essential for a firm to compete. To scope out the competition as well as a firm's position in the network, one can utilize game theory. The noncooperative competition problem among firms, each of which acts in its own self-interest, is a classical problem in economics, with the governing equilibrium conditions constituting a Nash equilibrium (cf. Nash 1950, 1951). Well-known formalisms for oligopolistic competition include, in addition, to the Cournot (1838)-Nash framework in which firms select their optimal production quantities, the Bertrand (1883) framework, in which firms choose their product prices, as well as the von Stackelberg (1934) framework, in which decisions are made sequentially in a leader-follower type of game.

However, as argued by Abbott (1955) and Dubovik and Janssen (2012), if one focuses solely on the price or quantity competition among firms, one ignores a critical component of consumers' decision processes and the very nature of competition – that of quality. Both price/quantity and quality have to be considered as strategic variables for firms in a competitive market. In particular, as noted by Banker et al. (1998), Hotelling's (1929) paper, which considered price and quality competition between two firms and modeled quality as a location decision, has inspired the study of quality competition in economics as well as in marketing and in operations research and management science.

Pioneers in the study of quality competition assumed that firms as well as their decisions were identical. Examples are as follows. Abbott (1953) analyzed quality equilibrium in a single-fixed-price market with entry, where firms only competed in quality. Mussa and Rosen (1978) modeled a firm's decisions on the price and quality of its quality differentiated product line, and compared the associated monopoly and competitive solutions. Dixit (1979) studied quantity and quality competition by considering several cases of oligopolistic equilibria and comparing them with the social optimum. De Vany and Saving (1983) modeled quantity and capacity competition for monopolists, where quality was related to capacity captured by the waiting cost.

Furthermore, Shaked and Sutton (1982) formulated quality competition between two firms with no cost for quality improvement. Moorthy (1988) considered price and quality competition between two identical firms with heterogeneous consumers. Economides (1989) developed a model with quality and price competition between two firms with quadratic quality cost functions. Motta (1993) studied quality and price/quantity competition between two firms under cases of fixed costs and variable costs of quality. Ma and Burgess (1993) explored the role of regulation in duopoly markets where firms with costs of identical functional forms competed in both quality and price for customers. Lehmann-Grube (1997) developed a two-firm two-stage model of vertical product differentiation, where firms competed in quality in the first stage and price in the second. Johnson and Myatt (2003) presented a model of multiproduct quality competition under monopoly and duopoly cases. Acharyya (2005) modeled quality and price competition between a domestic firm and a foreign firm, and the cost of R&D was considered. Chambers et al. (2006) considered the impact of variable production costs on price and quality competition in a duopoly.

Moreover, Das and Donnenfeld (1989), Ronnen (1991), Crampes and Hollander (1995), Ecchia and Lambertini (2001), and Baliamoune-Lutz and Lutz (2010) investigated the impact of minimum quality standards on the price and quality competition between two firms. However, most models in this area, as mentioned above, are developed under duopoly settings.

Oligopoly models with quality competition that considered more than two firms have been proposed in both economics and management science. In addition to Dixit (1979), Leland (1977) considered the quality choices of a finite number of firms competing for consumers, and used the "characteristics" approach to model consumers' choices. Gal-Or (1983) developed an oligopoly model with quality heterogenous consumers, in which both prices and quality levels of the firms were determined at the equilibrium. Lederer and Rhee (1995) modeled quality competition among firms that were price-takers. Karmarkar and Pitbladdo (1997) considered the competition among several identical firms where the consumer's utility was a function related to quality. Scarpa (1998) developed a price and quality competition model with three firms to study the effects of a minimum quality standard in a vertically differentiated market. In addition, Banker et al. (1998) modeled quality competition among firms with quadratic cost functions in one demand market and investigated the impact of number of competitors on quality. Brekke et al. (2010) investigated the relationship between competition and quality via a spatial price-quality competition model.

Yu and Nagurney (2013) modeled competitive supply chain networks of fresh produce using game theory and utilized generalized networks with arc multipliers to capture perishability along the supply chain. They also noted that the framework could be adapted to handle quality deterioration over time and space (see also Nagurney et al. 2013).

> In this book, we formalize quality competition in numerous scenarios and with many different decision-makers. Our approach is general and not limited to a small number of firms and specific functional forms in terms of production costs, transportation costs, and demand or demand price functions. Moreover, the identified network structures of the supply chains allow for vivid comparisons across different scenarios and problem instances. We also utilize a range of strategic variables, whether quantity variables or price variables, as appropriate, but always include the critical strategic variable of quality in our supply chain network models in quality.

1.3.3 Quality in Manufacturing Outsourcing

Outsourcing is a strategy capable of potentially bringing large benefits to firms. Outsourcing has, nevertheless, been attributed to product quality issues in global supply chain networks. Complex products such as aircraft, for example, involve a necessary degree of outsourcing, since the manufacturer may lack expertise in some areas, such as engines and avionics (cf. Denning 2013). Boeing significantly increased the amount of outsourcing for the 787 Dreamliner airplane over its earlier planes to about 70 %, whereas for the 737 and 747 airplanes it had been at around 35–50 %. Problems with Japanese-produced lithium-ion batteries grounded several flights and resulted in widespread negative media coverage and concern for safety due to that specific airplane component (see Parker 2014).

As emphasized by Marucheck et al. (2011), although quality product problems have long been viewed as a technical problem in the domain of regulators, epidemiologists, and design engineers, there has been a growing consciousness that fresh and effective approaches to managing product quality and safety are needed. In this section, we provide a literature review of contributions to the study of quality in manufacturing outsourcing from the domain of operations management and the related fields of operations research and management science. Most of the studies, as noted earlier, focus exclusively on supply chains with a limited number of firms and contractors and without product differentiation.

The impact of outsourcing on quality, suggestions as to how to mitigate associated quality issues, and the associated decision-making problems have been studied by various scholars. Riordan and Sappington (1987) modeled the quantity and quality choices of a firm with one contractor under information asymmetry and analyzed the firm's choice of organizational mode. Sridhar and Balachandran (1997) developed a model with one firm and two sequential contractors with information asymmetry to select one of them as the inside contractor. Zhu et al. (2007) investigated the roles of different parties in quality improvement by focusing on a model between two entities. The cost of goodwill loss caused by bad quality was also considered. Kaya and Özer (2009) modeled quality in outsourcing with one firm, one contractor, and information asymmetry to determine how the firm's pricing strategy affects quality risk. Xie et al. (2011) utilized a quality standard to regulate quality in a global supply chain with one firm and one contractor under cases of vertical integration and decentralized settings.

In addition, Kaya (2011) considered a model in which the supplier makes the quality decision and another model in which the manufacturer decides on the quality with quadratic quality cost functions. Gray et al. (2011) studied the effects of location decisions on quality risk based on real data from the drug industry. Lu et al. (2012) developed a model with one firm and one contractor and argued that contract enforcement would help to mitigate the low quality led by outsourcing. Handley and Gray (2013) studied 95 contracting relationships and found that external failures had a positive effect on the contractors' perception of quality importance. Steven et al. (2014) investigated empirically how outsourcing was related to product recalls and concluded that the relationship was positive. Xiao et al. (2014) examined outsourcing decisions for two competing manufacturers who have quality improvement opportunities and differentiate their products.

In this book, we model both a single firm faced with outsourcing decisions as well as multiple competing firms considering outsourcing.

1.3.4 Suppliers' Quality

As noted earlier, the quality of a finished (final) product depends not only on the quality of the firm that produces and delivers it, but also on the quality of the components provided by the firm's suppliers (Robinson and Malhotra 2005; Foster 2008). It is actually the suppliers that determine the quality of the materials that they purchase as well as the standards of their manufacturing activities.

Therefore, there has been increasing attention paid to supply chain networks with suppliers' quality in both management science and economics. However, in the literature, most models are based on a single firm - single supplier - single component supply chain network (cf. Reyniers and Tapiero 1995; Tagaras and Lee 1996; Starbird 1997; Baiman et al. 2000; Lim 2001; Hwang et al. 2006; Zhu et al. 2007; Chao et al. 2009; Hsieh and Liu 2010; Xie et al. 2011), without the preservation/decay of quality in the assembly processes of the products, and the possible in-house component production by the firms is not considered. Given the reality of many finished product supply chains, these models may be limiting in terms of both scope and practice. For example, Ford, the second largest US car manufacturer, had 1,260 suppliers at the end of 2012 with Ford purchasing approximately 80 percent of its parts from its largest 100 suppliers (Seetharaman 2013). Apple, in 2014, conducted audits on 200 of its top suppliers in 19 countries (see Apple 2015).

Specifically, although focused, simpler models, may yield closed form analytical solutions, more general frameworks, that are computationally tractable, are also needed, given the size and complexity of real-world global supply chains.

A concise literature review of models focusing on suppliers' quality in multitiered supply chain networks is now given. Specifically, in the literature, the relationships and contracts between firms and suppliers in terms of quality and the associated decision-making problems are analyzed. Reyniers and Tapiero (1995) modeled the effect of contract parameters on the quality choice of a supplier, the inspection policy of a producer, and product quality. Tagaras and Lee (1996) studied the relationship between quality, quality cost, and the manufacturing process in a model with one vendor. Economides (1999) modeled a supplier-manufacturer problem with two components and two firms, and concluded that vertical integration could guarantee higher quality. Baiman et al. (2000) analyzed the effects of information asymmetry between one firm and one supplier on the quality that can be contracted upon. Lim (2001) studied the contract design problem between a producer and its supplier with information asymmetry of quality.

In addition, Lin et al. (2005) conducted empirical research to study the correlation between quality management and supplier selection, based on data from practicing managers. Hwang et al. (2006) examined a quality management problem in a supply chain network with one supplier and provided evidence of the increasing use of certification. Chao et al. (2009) considered two contracts with recall cost sharing between a manufacturing and a supplier to induce quality improvement. Hsieh and Liu (2010) studied the supplier's and the manufacturer's quality investment with different degrees of information revealed. Moreover, Xie et al. (2011) investigated quality and price decisions in a risk-averse supply chain with two entities under uncertain demand.

El Ouardighi and Kim (2010) formulated a dynamic game in which a supplier collaborated with two firms on design quality improvements. Pennerstorfer and Weiss (2012) studied a wine supply chain network with multiple suppliers and firms, and each firm made identical decisions on quality. Furthermore, in the models developed by Hong and Hayya (1992), Rosenthal et al. (1995), Jayaraman et al. (1999), Ghodsypour and O'Brien (2001), and Rezaei and Davoodi (2008), multiple firms and/or suppliers were involved with quality being considered as input parameters, but the decisions on quality were not provided.

In this book, we present a multitiered supply chain network model with suppliers, in which firms are faced with capacities, and wish to decide whether to produce one or more components in-house or to have suppliers produce them. We identify the most important suppliers as well as components to a firm and to the entire supply chain network, consisting of all firms. We then leverage the model to extend it to include quality competition among the suppliers as well as the firms.

1.3.5 Freight Services and Quality

Previously, we focused on quality in terms of products. It is also essential to evaluate quality associated with freight service since, via freight, components are delivered to manufacturing firms and finished products are delivered to retailers and consumers.

Products in supply chain networks may travel great distances via multiple modes of transportation, including ship, air freight, rail, and/or truck. It is well-known today that success is determined by how well the entire supply chain performs, rather than by the performance of its individual entities. Quality and price have been identified empirically as critical factors in transport mode selection for product/goods delivery (cf. Floden et al. (2010), Saxin et al. (2005), and the references therein). Although the term quality in many freight studies suffers from a somewhat vague definition (cf. Meixell and Norbis (2008) for a discussion), it, typically, encompasses factors such as on-time deliveries, reliability, frequency, and risk of damage (see also Danielis et al. 2005; Zamparini et al. 2011). Quality with respect to freight in this book corresponds to level of service as emphasized by Mancera et al. (2013). Level of service is a measure first developed for the United States Highway Capacity Manual in 1965 (see National Research Council 1965) to assess the quality of the service a user receives when utilizing a transportation infrastructure under specific conditions.

As noted by Mancera et al. (2013), quality in freight transport has been studied by few authors; nevertheless, most of them agree as to the primary features of quality of service (Gray 1982). McGinnis (1979), for example, noted that the features of freight considered to be most important by shippers were the ones associated with speed, reliability, freight rates, and loss and damage. Danielis et al. (2005), Patterson et al. (2007), and Fries (2009) have noted different priorities in shippers' behavior, depending on the type of product transported with a focus on its value and the requirements for its transport. For example, according to Murphy (2014), with the growth in cloud computing and the demand for the necessary hardware to support it, there are what are known as "white glove" handlers that operate in the United States, and which specialize in transporting high-value computer equipment and providing the associated supply chain security and coordination. Of course, perishable products also require not only timely transportation but also preservation of the quality of the product through, for example, cold chains. Products such as pharmaceuticals, in turn, may require special handling and even temperature and humidity conditions (cf. Nagurney et al. (2013) and the references therein).

Freight service providers are increasingly focused on positioning themselves as more than just a commodity business in order maintain their competitive edge, with quality of service driving logistics performance in both developed and emerging economies (Arvis et al. 2014).

In our treatment of freight service provision and quality in this book, we use two different perspectives. In the first, we assume that the transport of the freight preserves product quality (Chaps. 3, 4, 5, 6, 7, 8, and 10) and, in the second, freight service providers explicitly compete in terms of service quality (Chaps. 11 and 12). In our final chapter, Chap. 12, we formulate supply chain network competition among manufacturers and among freight service providers, with the manufacturers competing on product price and quality and the freight service providers competing on quality and price charged for the service.

1.4 Organization of the Book

This book consists of 4 parts, for a total of 12 chapters.

Part I, Quality and Supply Chains, includes this introductory chapter as well as the chapter on methodological foundations, in which we lay the groundwork for the tools used in the formulation, analysis, and solution of the supply chain network problems in quality in this book. Specifically, we overview variational inequality theory (see Nagurney 1999), projected dynamical systems theory (see Dupuis and Nagurney 1993; Nagurney and Zhang 1996) along with linkages to optimization theory and game theory and also provide a brief synopsis of multicriteria decision-making.

Part II, Information Asymmetry in Quality, consists of two chapters. Chapter 3 focuses on a supply chain network oligopoly model with and without minimum quality standards and information asymmetry and Chap. 4 presents a perfectly competitive spatial price equilibrium model, also with information asymmetry and minimum quality standards. In both these chapters the information asymmetry is between producers and consumers.

Part III, Quality in Product Differentiation and Outsourcing, turns to supply chain network models in quality in which there is product differentiation by brand. This part consists of four chapters. Chapter 5 describes a supply chain network oligopoly model with product differentiation in which there is a single freight option between firms and demand markets for transport whereas Chap. 6 describes a model with multiple freight options. Chapters 7 and 8 introduce supply chain network models with outsourcing, respectively, in a single firm case and in a competitive multiple firm setting.

Part IV, Supplier Quality and Freight Service Quality, is comprised of four chapters, with Chap. 9 constructing a multitiered supply chain network model with suppliers along with performance indicators. Chapter 10 then builds upon the model in Chap. 9 to include quality levels on the suppliers' side and the manufacturers' side.

Table 1.2 Strategic variables of supply chain network decision-makers

Modeling chapter	Manufacturing firms	Supply markets	Contractors	Suppliers	Freight service providers
Chapter 3	√ ⋆				
Chapter 4		√ ⋆			
Chapter 5	√ ⋆				
Chapter 6	√ ⋆				
Chapter 7	⋆		√ ◇		
Chapter 8	√ ⋆		√ ◇		
Chapter 9	⋆				
Chapter 10	√ ⋆			√ ◇	
Chapter 11	⋆				√ ◇
Chapter 12	√ ◇				√ ◇
	Quality - √		Quantity - ⋆		Price - ◇

The final chapters in this book, Chaps. 11 and 12, focus on the inclusion of freight service providers as explicit decision-makers in supply chains. Chapter 11 presents a supply chain network model in which freight service providers compete on quality and on price whereas Chap. 12 develops a model in which both manufacturers and freight service providers compete on quality and on price.

We have attempted to make each chapter as self-contained, as possible, and to provide references following each chapter and a Sources and Notes section for each subsequent chapter. This book is based on our publications as well as on the dissertation of Li (2015), along with new results, interpretations, and synthesis as well as standardization of notation and updated references.

For easy reference, in Table 1.2, we summarize the strategic variables (quality, quantity, and/or price) that are used in a particular modeling chapter and with the decision-makers that they are associated with.

In order to demonstrate the richness and scope of the supply chain network models presented in this book in terms of quality and competition, in Table 1.3 we summarize the functions that depend on quality in the modeling chapters. The demand functions in the model in Chap. 12 depend on both product quality and on freight service provision quality. The transportation cost functions in the model in Chap. 11 depend on freight quality service provision.

Finally, we emphasize that Chaps. 3, 4, 5, 6, 11, and 12 contain both equilibrium supply chain network models and their dynamic counterparts that provide the evolution of the decision-makers' strategic variables and describe their interactions over time until an equilibrium is achieved, under appropriate assumptions.

Table 1.3 Model functional dependencies on quality

Function	Chapters
Production cost	3, 5, 6, 8, 10, 12
Supply price	4
Demand price	3, 4, 5, 6, 10, 11
Demand	12
Transportation cost	3, 8, 10, 11, 12
Opportunity cost	4, 11
Contractor production and distribution cost	7, 8
Cost of disrepute	7, 8
Supplier production cost	10
Supplier transportation cost	10

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Chapter 2
Methodological Foundations

Abstract In this chapter, we set the groundwork for the understanding and application of the methodological tools that are utilized for the supply chain network models with quality competition in this book. We first overview the basics of variational inequality theory and the connections with optimization. We provide conditions for existence and uniqueness of solutions, along with the definitions of the essential properties. We relate the variational inequality problem to game theory since game theory models are developed throughout this book in order to formulate competition among supply chain network decision-makers. In addition, we recall the fundamentals of projected dynamical systems theory and the relationships with variational inequality theory in order to enable the description of dynamic interactions among decision-makers in supply chains. For completeness, we also provide results on stability analysis. We discuss some fundamentals of multicriteria decision-making since supply chain decision-makers may be faced with multiple criteria, even conflicting ones, that they wish to optimize. Finally, we present algorithms that are used for solving the supply chain network models with quality competition.

2.1 Introduction

Rigorous methodologies provide powerful tools for the modeling, analysis, and solution of supply chain network problems in which decision-makers interact over space and time. Such tools, which enable problem formulation, qualitative analysis, in terms of existence and uniqueness of solutions, as well as the examination of stability of solutions, along with the identification of effective and efficient computational procedures, are essential in the exploration of issues surrounding supply chains and quality and the implementation of solutions in practice.

In this chapter, we review the fundamental methodologies that are used in this book for the modeling, analysis, and solution of supply chain network problems with quality competition. Our goal is to present the foundations in an accessible way so that the reader may refer to the results, as needed, while studying the topics contained in this book.

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In Sect. 2.2, we recall the basics of variational inequality theory, which is utilized throughout this book, and also relate the variational inequality problem to optimization problems. In the real world today supply chains consist not of a single decision-maker but, rather, of several, and, in many cases, many decision-makers. Variational inequality theory provides a mechanism by which optimizing behavior of multiple, interacting decision-makers can be formalized. We provide definitions of properties that are used for qualitative analysis and also discuss existence and uniqueness results of solutions to variational inequality problems. These results are then adapted accordingly to specific models constructed in subsequent chapters of this book.

In addition, in Sect. 2.3, some of the relationships between variational inequality and game theory, specifically, noncooperative games, where the governing equilibrium concept is that of Nash (1950, 1951) equilibrium, are highlighted. Game theory is used as a conceptual framework for the competitive supply chain network models in this book to capture competition among firms in quantity and quality and competition among contractors/suppliers in price and quality as well as competition among freight service providers in price and quality.

Since decision-makers interact over time, it is also important to provide means of describing underlying competitive behavior over time. Towards that end, we review projected dynamical systems theory in Sect. 2.4, since this methodology is utilized in this book to develop dynamic counterparts of several of the equilibrium-based supply chain network models. The relationships between the variational inequality problems and projected dynamical systems are also provided, followed by qualitative properties and stability analysis of projected dynamical systems.

For completeness, concepts of multicriteria decision-making and the weighted sum method are recalled briefly in Sect. 2.5. These are here utilized in Chaps. 7 and 8 to model supply chain network problems with quality competition and outsourcing.

Finally, we review several algorithms: the Euler method and the modified projection method. The Euler method is employed to solve variational inequality problems in Chaps. 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 in this book, with the exception of Chaps. 9 and 10. The Euler method provides discrete-time realizations of the continuous-time adjustment processes associated with the projected dynamical systems models for supply chain network competition in quality developed in this book. The modified projection method is used to solve the variational inequality problems in Chaps. 9 and 10.

Additional theorems and proofs associated with finite-dimensional variational inequality theory can be found in Nagurney (1999). Further details and proofs concerning projected dynamical systems theory can be found in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996). For a spectrum of multitiered static and dynamic supply chain network models, but without quality competition, see Nagurney (2006).

2.2 Variational Inequality Theory

In this section, we provide some of the fundamentals of the theory of variational inequalities. All definitions and theorems are from Nagurney (1999) where proofs can also be found as well as additional references. Throughout this book the vectors are column vectors. We assumer that the reader is familiar with the basics of optimization theory.

Definition 2.1: Variational Inequality Problem
The finite-dimensional variational inequality problem, VI$(F, \mathscr{K})$*, is to determine a vector* $X^* \in \mathscr{K} \subset R^N$*, such that*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathscr{K}, \tag{2.1a}$$

where F *is a given continuous function from* $\mathscr{K}$ *to* R^N*,* $\mathscr{K}$ *is a given closed convex set, and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* N*-dimensional Euclidean space.*

In (2.1a), $F(X) \equiv (F_1(X), F_2(X), \ldots, F_N(X))^T$, and $X \equiv (X_1, X_2, \ldots, X_N)^T$ since $F(X)$ and X are both column vectors. Note that the variational inequality (2.1a), when expanded, term by term, is:

$$\sum_{i=1}^{N} F_i(X^*) \times (X_i - X_i^*) \geq 0, \quad \forall X \in \mathscr{K}. \tag{2.1b}$$

Recall that for two vectors $u, v \in R^N$, the inner product $\langle u, v \rangle = \|u\| \|v\| cos\theta$, where θ is the angle between the vectors u and v. Hence, the variational inequality problem (2.1a) has a geometric interpretation. In particular, it states that $F(X^*)$ is "orthogonal" to the feasible set $\mathscr{K}$ at the point X^*. In Fig. 2.1, the geometric interpretation is provided.

The variational inequality problem is a general problem that encompasses a wide spectrum of mathematical problems, including, optimization problems, complementarity problems, and fixed point problems. It has been used as a fundamental methodology for a plethora of applications, many of which are network-based, and include transportation problems (Ran and Boyce 1996;

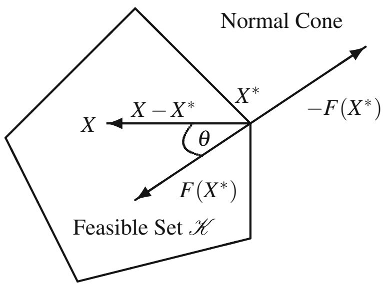


Fig. 2.1 Geometric interpretation of VI$(F, \mathscr{K})$

Nagurney 1999; Patriksson 2015), telecommunication problems (Nagurney and Dong 2002), financial networks (Nagurney and Siokos 1997), environmental problems (Dhanda et al. 1999), ecosystems (Mullon 2014), as well as supply chains (see Nagurney (2006) and Nagurney et al. (2013), and the references therein).

For example, it has been shown that optimization problems, both constrained and unconstrained, can be reformulated as variational inequality problems. The relationships between variational inequalities and optimization problems are now recalled.

Proposition 2.1
Let X^* be a solution to the optimization problem:

$$\textit{Minimize} \quad f(X) \tag{2.2}$$

subject to:

$$X \in \mathscr{K},$$

where f is continuously differentiable and $\mathscr{K}$ is closed and convex. Then X^* is a solution of the variational inequality problem:

$$\langle \nabla f(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathscr{K}, \tag{2.3}$$

where $\nabla f(X)$ is the gradient vector of f with respect to X, where $\nabla f(X) \equiv (\frac{\partial f(X)}{\partial X_1}, \ldots, \frac{\partial f(X)}{\partial X_N})^T$.

For example, in terms of supply chains, $f(X)$ might correspond to costs that the decision-maker seeks to minimize. The feasible set $\mathscr{K}$, on the other hand, would then correspond to the constraints that the decision-maker faces that are associated with his vector of decision variables X. The decision variables may be production outputs and shipment variables.

Proposition 2.2
If $f(X)$ is a convex function and X^* is a solution to $\mathrm{VI}(\nabla f, \mathscr{K})$, then X^* is a solution to the optimization problem (2.2). In the case that the feasible set $\mathscr{K} = R^N$, then the unconstrained optimization problem is also a variational inequality problem.

The definitions of positive semidefiniteness, positive definiteness, and strong positive definiteness are recalled next, followed by a theorem presenting the above relationship.

Definition 2.2
An $N \times N$ matrix $M(X)$, whose elements $m_{ij}(X)$; $i, j = 1, \ldots, N$, are functions defined on the set $\mathscr{S} \subset R^N$, is said to be positive semidefinite on $\mathscr{S}$ if

$$v^T M(X) v \geq 0, \quad \forall v \in R^N, X \in \mathscr{S}. \tag{2.4}$$

It is said to be positive definite on $\mathscr{S}$ if

$$v^T M(X) v > 0, \quad \forall v \neq 0,\ v \in R^N,\ X \in \mathscr{S}. \tag{2.5}$$

It is said to be strongly positive definite on $\mathscr{S}$ if

$$v^T M(X) v \geq \alpha \|v\|^2, \textit{ for some } \alpha > 0, \quad \forall v \in R^N,\ X \in \mathscr{S}. \tag{2.6}$$

The variational inequality problem can be reformulated as an optimization problem under certain symmetry conditions.

Theorem 2.1
Assume that $F(X)$ is continuously differentiable on $\mathscr{K}$ and that the Jacobian matrix

$$\nabla F(X) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_N} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_N}{\partial X_1} & \cdots & \frac{\partial F_N}{\partial X_N} \end{bmatrix} \tag{2.7}$$

is symmetric and positive semidefinite. Then there is a real-valued convex function $f : \mathscr{K} \longmapsto R$ satisfying

$$\nabla f(X) = F(X) \tag{2.8}$$

with X^* the solution of $\mathrm{VI}(F, \mathscr{K})$ also being the solution of the mathematical programming problem:

$$\textit{Minimize} \quad f(X)$$

subject to:

$$X \in \mathscr{K},$$

where $f(X) = \int F(X)^T dx$, and $\int$ is a line integral.

Thus, the variational inequality problem is a more general problem formulation than an optimization problem formulation, since it can also handle a function $F(X)$ with an asymmetric Jacobian (see Nagurney 1999). This feature is extremely important in applications, including supply chain network ones, since it allows one to capture asymmetric interactions among decision-makers as revealed, for example, by their production cost functions, transportation cost functions, and the demand price functions associated with their products.

Next, the qualitative properties of variational inequality problems, specifically, the conditions for existence and uniqueness of a solution, are recalled.

Theorem 2.2: Existence Under Compactness and Continuity
If $\mathscr{K}$ is a compact convex set and $F(X)$ is continuous on $\mathscr{K}$, then the variational inequality problem admits at least one solution X^*.

Theorem 2.3: Existence
If the feasible set $\mathscr{K}$ is unbounded, then $\mathrm{VI}(F, \mathscr{K})$ admits a solution if and only if there exists an $R > 0$ and a solution of $\mathrm{VI}(F, \mathscr{S})$, X_R^*, such that $\|X_R^*\| < R$, where $\mathscr{S} = \{X : \|X\| \leq R\}$.

Theorem 2.4: Existence Under Coercivity
Suppose that $F(X)$ satisfies the coercivity condition

$$\frac{\langle F(X) - F(X_0), X - X_0 \rangle}{\|X - X_0\|} \to \infty \tag{2.9}$$

as $\|X\| \to \infty$ for $X \in \mathscr{K}$ and for some $X_0 \in \mathscr{K}$. Then $\mathrm{VI}(F, \mathscr{K})$ always has a solution.

According to Theorem 2.4, the existence of a solution to a variational inequality problem can be guaranteed by the coercivity condition. Next, certain monotonicity conditions are utilized to discuss the qualitative properties of existence and uniqueness. Some basic definitions of monotonicity are reviewed first. Monotonicity plays a role in variational inequality problems similar to that of convexity in optimization problems.

Definition 2.3: Monotonicity
$F(X)$ is said to be locally monotone at X^* if there is a neighborhood $N(X^*)$ of X^* such that

$$\langle (F(X) - F(X^*)), X - X^* \rangle \geq 0, \quad \forall X \in N(X^*). \tag{2.10}$$

$F(X)$ is monotone at X^* if the above inequality holds true for all $X \in K$. $F(X)$ is said to be monotone if the above inequality holds for all $X, X^* \in \mathscr{K}$.

Definition 2.4: Strict Monotonicity
$F(X)$ is said to be locally strictly monotone at X^* if there is a neighborhood $N(X^*)$ of X^* such that

$$\langle (F(X) - F(X^*)), X - X^* \rangle > 0, \quad \forall X \in N(X^*),\ X \neq X^*. \tag{2.11}$$

$F(X)$ is strictly monotone at X^* if the above inequality holds true for all $X \in \mathscr{K}$. $F(X)$ is said to be strictly monotone if the above inequality holds for all $X, X^* \in \mathscr{K}$, $X \neq X^*$.

Definition 2.5: Strong Monotonicity
$F(X)$ is said to be locally strongly monotone at X^* if there is a neighborhood $N(X^*)$ of X^* and $\eta > 0$ such that

$$\langle (F(X) - F(X^*)), X - X^* \rangle \geq \eta \|X - X^*\|^2, \quad \forall X \in N(X^*). \tag{2.12}$$

$F(X)$ is strongly monotone at X^* if the above inequality holds true for all $X \in \mathscr{K}$. $F(X)$ is said to be strongly monotone if the above inequality holds for all $X, X^* \in \mathscr{K}$.

Definition 2.6: Lipschitz Continuity
$F : \mathscr{K} \mapsto R^N$ is locally Lipschitz continuous if for every $X \in \mathscr{K}$ there is a neighborhood $N(X)$ and a positive number $L(X) > 0$ such that

$$\|F(X') - F(X'')\| \ \leq \ L(X)\|X' - X''\|, \quad \forall X', X'' \in N(X). \tag{2.13}$$

When the above inequality holds uniformly on $\mathscr{K}$ for some constant $L > 0$, that is,

$$\|F(X') - F(X'')\| \ \leq \ L\|X' - X''\|, \quad \forall X', X'' \in \mathscr{K},$$

then F is said to be Lipschitz continuous on K.

Note that any continuously differentiable function F is locally Lipschitz.

Theorem 2.5: Uniqueness Under Strict Monotonicity
Suppose that $F(X)$ is strictly monotone on $\mathscr{K}$. Then the solution to the $\mathrm{VI}(F, \mathscr{K})$ problem is unique, if one exists.

Theorem 2.6: Existence and Uniqueness Under Strong Monotonicity
Suppose that $F(X)$ is strongly monotone on $\mathscr{K}$. Then there exists precisely one solution X^* to $\mathrm{VI}(F, \mathscr{K})$.

We now summarize Theorems 2.2, 2.5, and 2.6. Strong monotonicity of the function F guarantees both existence and uniqueness, in the case of an unbounded feasible set $\mathscr{K}$. If the feasible set $\mathscr{K}$ is compact, that is, closed and bounded, the continuity of F guarantees the existence of a solution. The strict monotonicity of F is then sufficient to guarantee its uniqueness provided its existence.

Monotonicity is closely related to positive definiteness.

Theorem 2.7
Suppose that $F(X)$ is continuously differentiable on $\mathscr{K}$ and the Jacobian matrix

$$\nabla F(X) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_N} \\ \vdots & & \vdots \\ \frac{\partial F_N}{\partial X_1} & \cdots & \frac{\partial F_N}{\partial X_N} \end{bmatrix},$$

which need not be symmetric, is positive semidefinite (positive definite). Then $F(X)$ is monotone (strictly monotone).

Proposition 2.3
Assume that $F(X)$ is continuously differentiable on $\mathscr{K}$ and that $\nabla F(X)$ is strongly positive definite. Then $F(X)$ is strongly monotone.

One obtains a stronger result in the special case where $F(X)$ is linear.

Corollary 2.1
Suppose that $F(X) = MX + b$, where M is an $N \times N$ matrix and b is a constant vector in R^N. The function F is monotone if and only if M is positive semidefinite. F is strongly monotone if and only if M is positive definite.

Proposition 2.4
Assume that $F : \mathscr{K} \mapsto R^N$ is continuously differentiable at $\bar{X}$. Then $F(X)$ is locally strictly (strongly) monotone at $\bar{X}$ if $\nabla F(\bar{X})$ is positive definite (strongly positive definite), that is,

$$v^T F(\bar{X}) v > 0, \quad \forall v \in R^N, v \neq 0,$$

$$v^T \nabla F(\bar{X}) v \geq \alpha \|v\|^2, \quad \textit{for some} \quad \alpha > 0, \quad \forall v \in R^N.$$

2.3 The Relationships Between Variational Inequalities and Game Theory

In this section, some of the relationships between variational inequalities and game theory are briefly discussed.

Nash (1950, 1951) contributed greatly to noncooperative game theory, involving multiple players, each of whom acts in his/her own interest. In particular, consider a game with m players, each player i having a strategy vector $X_i = \{X_{i1}, \ldots, X_{in}\}$ selected from a closed convex set $\mathscr{K}^i \subset R^n$. Each player i seeks to maximize his own utility function, U_i: $\mathscr{K} \to R$, where $\mathscr{K} = \mathscr{K}^1 \times \mathscr{K}^2 \times \ldots \times \mathscr{K}^m \subset R^{mn}$. The utility of player i, U_i, depends not only on his own strategy vector, X_i, but also on the strategy vectors of the other players, $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m)$. An equilibrium is achieved if no one can increase his utility by unilaterally altering the value of his strategy vector. The formal definition of the Nash equilibrium is as follows.

Definition 2.7: Nash Equilibrium
A Nash equilibrium is a strategy vector

$$X^* = (X_1^*, \ldots, X_m^*) \in \mathscr{K}, \tag{2.14}$$

such that

$$U_i(X_i^*, \hat{X}_i^*) \geq U_i(X_i, \hat{X}_i^*), \quad \forall X_i \in \mathscr{K}^i, \forall i, \tag{2.15}$$

where $\hat{X}_i^* = (X_1^*, \ldots, X_{i-1}^*, X_{i+1}^*, \ldots, X_m^*)$.

The Nash equilibrium concept is fundamental to the modeling of supply chain network competition in quality. We will present numerous supply chain network models in this book, of different degrees of complexity and numbers of decision-makers, in which the Nash equilibrium concept plays a pivotal role. Examples of appropriate strategic variables include, depending on the scenario and supply chain context, production outputs, shipment quantities, prices, as well as quality levels associated with the products and even the supply components.

It has been shown by Hartman and Stampacchia (1966) and Gabay and Moulin (1980) that given continuously differentiable and concave utility functions, U_i, $\forall i$, the Nash equilibrium problem can be formulated as a variational inequality problem defined on $\mathcal{K}$.

Theorem 2.8: Variational Inequality Formulation of Nash Equilibrium
Under the assumption that each utility function U_i is continuously differentiable and concave, X^* is a Nash equilibrium if and only if $X^* \in \mathcal{K}$ is a solution of the variational inequality

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad X \in \mathcal{K}, \tag{2.16}$$

where $F(X) \equiv (-\nabla_{X_1} U_1(X), \ldots, -\nabla_{X_m} U_m(X))^T$, *and* $\nabla_{X_i} U_i(X) = (\frac{\partial U_i(X)}{\partial X_{i1}}, \ldots, \frac{\partial U_i(X)}{\partial X_{in}})$.

The conditions for existence and uniqueness of a Nash equilibrium are now introduced. As stated in the following theorem, Rosen (1965) presented existence under the assumptions that $\mathcal{K}$ is compact and each U_i is continuously differentiable.

Theorem 2.9: Existence Under Compactness and Continuous Differentiability
Suppose that the feasible set $\mathcal{K}$ is compact and each U_i is continuously differentiable. Then existence of a Nash equilibrium is guaranteed.

Gabay and Moulin (1980), on the other hand, relaxed the assumption of the compactness of $\mathcal{K}$, and proved existence of a Nash equilibrium after imposing a coercivity condition on $F(X)$.

Theorem 2.10: Existence Under Coercivity
Suppose that $F(X)$, as given in Theorem 2.8, satisfies the coercivity condition (2.9). Then there always exists a Nash equilibrium.

Furthermore, Karamardian (1969) demonstrated existence and uniqueness of a Nash equilibrium under the strong monotonicity assumption.

Theorem 2.11: Existence and Uniqueness Under Strong Monotonicity
Assume that $F(X)$, as given in Theorem 2.8, is strongly monotone on $\mathcal{K}$. Then there exists precisely one Nash equilibrium X^*.

Additionally, based on Theorem 2.5, uniqueness of a Nash equilibrium can be guaranteed under the assumptions that $F(X)$ is strictly monotone and an equilibrium exists.

Theorem 2.12: Uniqueness Under Strict Monotonicity
Suppose that $F(X)$, as given in Theorem 2.8, is strictly monotone on $\mathcal{K}$. Then the Nash equilibrium, X^*, is unique, if it exists.

2.4 Projected Dynamical Systems

In this section, the theory of projected dynamical systems (cf. Dupuis and Nagurney 1993; Nagurney and Zhang 1996) is recalled, followed by the relationship between projected dynamical systems and variational inequality problems. Finally, some properties of the dynamic trajectories and the stability analysis of projected dynamical systems are provided. All the definitions and theorems can be found in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996).

Definition 2.8
Given $X \in \mathcal{K}$ and $v \in R^N$, define the projection of the vector v at X (with respect to $\mathcal{K}$) by

$$\Pi_{\mathcal{K}}(X, v) = \lim_{\delta \to 0} \frac{(P_{\mathcal{K}}(X + \delta v) - X)}{\delta} \tag{2.17}$$

with $P_{\mathcal{K}}$ denoting the projection map:

$$P_{\mathcal{K}}(X) = \text{argmin}_{X' \in \mathcal{K}} \|X' - X\|, \tag{2.18}$$

where $\|\cdot\| = \langle x, x \rangle$.

In Fig. 2.2, we let $y = P_{\mathcal{K}}(X)$ and we provide a graphical depiction of the projection map $P_{\mathcal{K}}(X)$.

The algorithms that we apply in this book to solve a wide range of supply chain network problems in quality are projection-type algorithms.

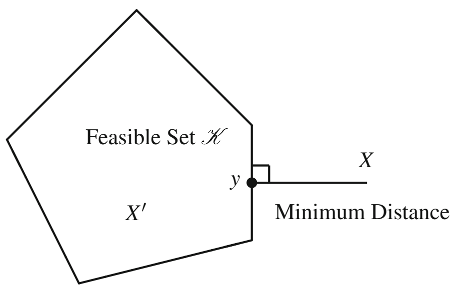


Fig. 2.2 The projection y of X on the feasible set $\mathcal{K}$

The class of ordinary differential equations that this book focuses on takes on the form:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0 \in \mathcal{K}, \tag{2.19}$$

where $\dot{X}$ denotes the rate of change of vector X, $\mathcal{K}$ is closed convex set, corresponding to the constraint set in a particular application, $F(X)$ is a vector field defined on $\mathcal{K}$, and $X(0)$ is the initial value of X. We refer to the ordinary differential equation in (2.19) as ODE$(F, \mathcal{K})$.

The classical dynamical system, in contrast to (2.19), takes the form:

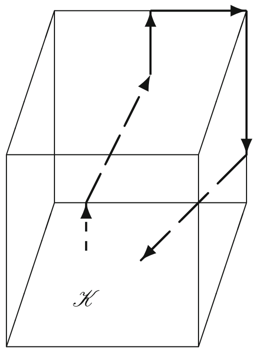
$$\dot{X} = -F(X), \quad X(0) = X_0 \in \mathcal{K}. \tag{2.20}$$

Note that in (2.20) there is no way in which to capture the constraints underlying a problem since the feasible set $\mathcal{K}$ does not appear. In the real world, supply chain network decision-makers are faced with constraints since resources are not infinite; moreover, product volumes, as well as prices, should not be negative.

Definition 2.9: The Projected Dynamical Systems
Define the projected dynamical system (referred to as PDS$(F, \mathcal{K})$*)* $X_0(t) : \mathcal{K} \times R \mapsto \mathcal{K}$ *as the family of solutions to the Initial Value Problem (*IVP*) (2.19) for all* $X_0 \in \mathcal{K}$.

The behavior of the projected dynamical system is now described. See Fig. 2.3 for a graphical depiction. Indeed, projected dynamical systems are extremely valuable in the context of supply chain networks since decision-makers may be faced with multiple constraints from nonnegativity assumptions on their production outputs and shipments to capacities associated with their manufacturing facilities to minimum quality standards imposed on their products, along with demand satisfaction at demand markets, to name just a few. A PDS guarantees that the constraints will not be violated as the system evolves over time.

Fig. 2.3 The evolution of a trajectory in $\mathcal{K}$



Specifically, if $X(t)$ lies in the interior of the feasible set $\mathcal{K}$, then the evolution of the solution is given by $F : \dot{X} = -F(X)$. However, if the vector field $-F$ drives X to the boundary of $\mathcal{K}$, that is, for some t one has $X(t) \in \partial\mathcal{K}$ and $-F(X(t))$ points "out" of $\mathcal{K}$, the right-hand side of (2.19) becomes the projection of $-F$ onto $\partial\mathcal{K}$. In this case, the solution to (2.19) then evolves along a "section" of $\partial\mathcal{K}$, that is, $\partial\mathcal{K}_i$, for some i. Later, the solution may re-enter the interior of $\mathcal{K}$, or it may enter a lower dimensional part of the boundary of $\mathcal{K}$.

Definition 2.10: A Stationary or an Equilibrium Point
The vector $X^* \in \mathcal{K}$ is a stationary point or an equilibrium point of the projected dynamical system PDS$(F, \mathcal{K})$ *if*

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)). \tag{2.21}$$

In other words, X^* is a stationary point or an equilibrium point if, once the projected dynamical system is at X^*, it will remain at X^* for all future times. Definition 2.10 demonstrates that X^* is an equilibrium point of the projected dynamical system PDS$(F, \mathcal{K})$ if the vector field F vanishes at X^*. However, it is only true when X^* is an interior point of the constraint set $\mathcal{K}$. When X^* lies on the boundary of $\mathcal{K}$, we may have $F(X^*) \neq 0$. Note that for classical dynamical systems, the necessary and sufficient condition for an equilibrium point is that the vector field vanish at that point, that is, $-F(X^*) = 0$.

We now recall the equivalence between the set of equilibria of a projected dynamical system and the set of solutions of the corresponding variational inequality problem by presenting the following theorem (see Dupuis and Nagurney 1993).

Theorem 2.13: Equivalence of Stationary (Equilibrium) Points and Solutions to the Corresponding Variational Inequality
Assume that $\mathcal{K}$ is a convex polyhedron. Then the stationary; equivalently, equilibrium points of the PDS$(F, \mathcal{K})$ *coincide with the solutions of* VI$(F, \mathcal{K})$. *Hence, for $X^* \in \mathcal{K}$ and satisfying*

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)) \tag{2.22}$$

also satisfies

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{2.23}$$

Theorem 2.13 implicitly also reveals that the variational inequality problem is a framework for formulating equilibrium problems, which we will see is, indeed, the case for the wide range of supply chain network problems in quality that we model in this book. Moreover, Theorem 2.13 implies that the PDS provides a natural underlying dynamics until the equilibrium is achieved. The dynamics, as we will demonstrate, yield rich descriptive tatonnement processes as supply chain network decision-makers interact with one another over space and time.

Before addressing the conditions for existence and uniqueness of the trajectory of a projected dynamical system, we recall the following fundamental assumption, which is implied by Lipschitz continuity (Definition 2.6).

Assumption 2.1
There exists a $B < \infty$ such that the vector field $-F : R^N \mapsto R^N$ satisfies the linear growth condition $\| F(X) \| \leq B(1+ \| X \|), X \in \mathscr{K}$, and also

$$\langle -F(X) + F(y), X - y \rangle \leq B \| X - y \|^2, \quad \forall X, y \in \mathscr{K}. \tag{2.24}$$

Theorem 2.14: Existence, Uniqueness, and Continuous Dependence
Assume Assumption 2.1. Then

(i) For any $X_0 \in \mathscr{K}$, there exists a unique solution $X_0(t)$ to the initial value problem;

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)), \quad X(0) = X_0; \tag{2.25}$$

(ii) If $X_n \to X_0$ as $n \to \infty$, then $X_n(t)$ converges to $X_0(t)$ uniformly on every compact set of $[0, \infty)$.

The second statement of Theorem 2.14 is sometimes called the continuous dependence of the solution path to ODE$(F, \mathscr{K})$ on the initial value. Therefore, the PDS$(F, \mathscr{K})$ is well-defined and inhabits $\mathscr{K}$ whenever Assumption 2.1 holds.

The stability of a system is defined as the ability of the system to maintain or restore its equilibrium when acted upon by forces tending to displace it. Since the ordinary differential equation of PDS$(F, \mathscr{K})$ (2.19) has a discontinuous right-hand side, the question of the stability of the system arises. We now recall stability concepts for projected dynamical systems at their equilibrium points, due to Zhang and Nagurney (1995).

Definition 2.11: A Stable or Unstable Equilibrium Point
An equilibrium point X^* is stable, if for any $\epsilon > 0$, there exists a $\delta > 0$, such that for all $X \in B(X^*, \delta)$ and all $t \geq 0$

$$X \cdot t \in B(X^*, \epsilon). \tag{2.26}$$

The equilibrium point X^* is unstable, if it is not stable.

$B(X, r)$ is used to denote the open ball with radius r and center X.

Definition 2.12: An Exponentially Stable Equilibrium Point
An equilibrium point X^* is exponentially stable, if there exists a $\delta > 0$ and constants $b > 0$ and $\mu > 0$ such that

$$\|X \cdot t - X^*\| \leq b\|X - X^*\| e^{-\mu t}, \quad \forall t \geq 0, \ \forall X \in B(X^*, \delta); \tag{2.27}$$

X^* is globally exponentially stable, if the above holds true for all $X^0 \in \mathscr{K}$.

Definition 2.13: Monotone Attractor
An equilibrium point X^* is a monotone attractor, if there exists a $\delta > 0$ such that for all $X \in B(X^*, \delta)$,

$$d(X,t) = \| X \cdot t - X^* \| \tag{2.28}$$

is a nonincreasing function of t; X^* is a global monotone attractor, if $d(X,t)$ is nonincreasing in t for all $X \in \mathscr{K}$.

Definition 2.14: Strictly Monotone Attractor
An equilibrium X^* is a strictly monotone attractor, if there exists a $\delta > 0$ such that for all $X \in B(X^*, \delta)$, $d(X,t)$ is monotonically decreasing to zero in t; X^* is a strictly global monotone attractor, if $d(X,t)$ is monotonically decreasing to zero in t for all $X \in \mathscr{K}$.

The stability of a projected dynamical system is actually determined by the monotonicity of the $F(X)$ in the associated variational inequality problem. Next, we recall results for local and global stability under various monotonicity conditions.

Theorem 2.15: Stability Under Monotonicity
Suppose that X^* solves $\mathrm{VI}(F, \mathscr{K})$. If $F(X)$ is locally monotone at X^*, then X^* is a monotone attractor for the $\mathrm{PDS}(F, \mathscr{K})$; if $F(X)$ is monotone, then X^* is a global monotone attractor.

Theorem 2.16: Stability Under Strict Monotonicity
Suppose that X^* solves $\mathrm{VI}(F, \mathscr{K})$. If $F(X)$ is locally strictly monotone at X^*, then X^* is a strictly monotone attractor for the $\mathrm{PDS}(F, \mathscr{K})$; if $F(X)$ is strictly monotone at X^*, then X^* is a strictly global monotone attractor.

Theorem 2.17: Stability Under Strong Monotonicity
Suppose that X^* solves $\mathrm{VI}(F, \mathscr{K})$. If $F(X)$ is locally strongly monotone at X^*, then X^* is exponentially stable for the $\mathrm{PDS}(F, \mathscr{K})$; if $F(X)$ is strongly monotone at X^*, then X^* is a globally exponentially stable.

Stability analysis is essential to the understanding of dynamic supply chain network models in quality. For example, one may wish to answer such questions as: if a supply chain network system starts near an equilibrium, will it stay at that point forever, and, given the current state of the supply chain network system, will it asymptotically approach an equilibrium?

2.5 Multicriteria Decision-Making

Multicriteria decision-making is very relevant in the context of supply chains. Decision-makers may wish to minimize costs, at the same time that they also minimize risk. In addition, they may be concerned about maximizing profits while also interested in minimizing their environmental impacts.

The goal of multicriteria decision-making (MCDM) is to evaluate a set of alternatives in terms of a number of conflicting criteria (Keeney and Raffa 1976; Cohon 1978; Triantaphyllou 2000), according to the preferences of the decision-maker (Gal et al. 1999; Jones et al. 2002). In this section, the multicriteria optimization problem and the weighted sum method are briefly reviewed.

The multicriteria optimization problem with N decision variables can be generalized as (see Marler and Arora 2004):

$$\text{Minimize} \quad \mathbf{G}(X) = [G_1(X), G_2(X), \ldots, G_k(X)]^T \tag{2.29}$$

subject to:

$$g_j(X) \leq 0, \quad j = 1, 2, \ldots, m, \tag{2.30}$$

$$h_l(X) = 0, \quad l = 1, 2, \ldots, e, \tag{2.31}$$

where k is the number of objective functions, m is the number of inequality constraints, e is the number of equality constraints, and X is the N-dimensional vector of decision variables. The feasible set $\mathscr{K}^1$ is defined as:

$$\mathscr{K}^1 \equiv \{X|(2.30)\,\text{and}\,(2.31)\,\text{are satisfied}\}. \tag{2.32}$$

The Pareto optimality of a solution to a multicriteria problem is defined by Pareto (1971), as follows.

Definition 2.15: Pareto Optimal
A point, $X^* \in \mathscr{K}^1$, is Pareto optimal iff there does not exist another point, $X^* \in \mathscr{K}^1$, such that $\mathbf{G}(X) \leq \mathbf{G}(X^*)$, and $\mathbf{G}_i(X) < \mathbf{G}_i(X^*)$ for at least one function.

The weighted sum method, which is the most common approach to multicriteria (sometimes referred to as *multiobjective*) optimization problems (see Marler and Arora 2004), is as follows. Associated with a vector of weights, denoted by w, representing the decision-maker's preferences, the multicriteria objective function (2.29) can be expressed as:

$$\Gamma = \sum_{i=1}^{k} w_i G_i(X). \tag{2.33}$$

As noted by Zadeh (1963), the optimal solution to (2.33) is Pareto optimal if all of the weights are positive.

2.6 Algorithms

In this section, we review the algorithms, the Euler method, which is based on the general iterative scheme of Dupuis and Nagurney (1993), and the modified projection method of Korpelevich (1977). These algorithms are applied in this book to compute solutions to competitive supply chain network problems in quality.

2.6.1 The Euler Method

The Euler method can be utilized to compute the solution to a variational problem (cf. (2.1a)), and can also be used for the computation of the solution to the related projected dynamic system (cf. (2.19)) (see Dupuis and Nagurney 1993; Nagurney and Zhang 1996). It is induced by the general iterative scheme developed by Dupuis and Nagurney (1993). The Euler method not only provides a discretization of the continuous-time trajectory defined by (2.19) but also yields a stationary, that is, an equilibrium, point that satisfies variational inequality (2.1a).

Specifically, at an iteration $\tau + 1$ of the Euler method (see also Nagurney and Zhang 1996) one computes:

$$X^{\tau+1} = P_{\mathscr{K}}(X^{\tau} - a_{\tau} F(X^{\tau})), \tag{2.34}$$

where F is the function in (2.1a), and $P_{\mathscr{K}}$ is the projection on the feasible set $\mathscr{K}$, defined by

$$P_{\mathscr{K}}(X) = \text{argmin}_{X' \in \mathscr{K}} \|X' - X\|. \tag{2.35}$$

We now provide the complete statement of the Euler method.

Step 0: Initialization
Set $X^0 \in \mathscr{K}$.
Let $\tau = 0$ and set the sequence $\{\alpha_\tau\}$ so that $\sum_{\tau=0}^{\infty} \alpha_\tau = \infty$, $\alpha_\tau > 0$ for all τ, and $\alpha_\tau \to 0$ as $\tau \to \infty$.
Step 1: Computation
Compute $X^{\tau+1} \in \mathscr{K}$ by solving the variational inequality subproblem:

$$\langle X^{\tau+1} + \alpha_\tau F(X^\tau) - X^\tau, X - X^{\tau+1} \rangle \geq 0, \qquad \forall X \in \mathscr{K}. \tag{2.36}$$

Step 2: Convergence Verification
If max $|X_l^{\tau+1} - X_l^\tau| \leq \epsilon$, for all l, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\tau := \tau + 1$, and go to Step 1.

The VI subproblem (2.36) is actually a quadratic programming problem. In this book, wherever appropriate, we also exploit the network structure of subproblems for computational efficiency. The Euler method has been applied to many different network problems (cf. Nagurney and Zhang 1996; Nagurney 1999) as well as supply chain network problems (see Nagurney 2006).

Before we provide conditions for convergence of the Euler method, we first state some preliminaries.

Definition 2.16
For any subset A of R^N, the ω-limit set of A is defined by:

$$\omega(A) = \{y: \ \exists X_k \in A, t_k \to \infty, \ such\ that\ X_k \cdot t_k \to y, \ as\ k \to \infty\}.$$

An assumption is recalled, followed by the convergence conditions of the Euler method in Theorem 2.18 and Corollary 2.2.

Assumption 2.2
Suppose that we fix an initial condition $X_0 \in \mathscr{K}$ and define the sequence $\{X_\tau, \tau \in T\}$ by (2.34). We assume the following conditions:

1. $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$ *as* $\tau \to \infty$.
2. $d(F_\tau(X), \overline{F}(X)) \to 0$ *uniformly on compact subsets of* $\mathscr{K}$ *as* $\tau \to \infty$.
3. *Define* ϕ_y *to be the unique solution to* $\dot{X} = \Pi_{\mathscr{K}}(X, -F(X))$ *that satisfies* $\phi_y(0) = y \in \mathscr{K}$. *The ω-limit set of* $\mathscr{K}$

$$\omega(\mathscr{K}) = \cup_{y \in \mathscr{K}} \cap_{t \geq 0} \overline{\cup_{s \geq t}\{\phi_y(s)\}}$$

   is contained in the set of stationary points of $\dot{X} = \Pi_{\mathscr{K}}(X, -F(X))$.
4. *The sequence* $\{X_\tau, \tau \in T\}$ *is bounded.*
5. *The solution to* $\dot{X} = \Pi_{\mathscr{K}}(X, -F(X))$ *are stable in the sense that given any compact set* $\mathscr{K}_1$ *there exists a compact set* $\mathscr{K}_2$ *such that* $\cup_{y \in \mathscr{K} \cap \mathscr{K}_1} \cup_{t \geq 0} \{\phi_y(t)\} \subset \mathscr{K}_2$.

Theorem 2.18
Let S denote the set of stationary points of the projected dynamical system (2.19), equivalently, the set of solutions to the variational inequality problem (2.1a). Assume Assumptions 2.1 and 2.2. Suppose $\{X_\tau, \tau \in T\}$ *is the scheme generated by (2.34). Then* $d(X_\tau, S) \to 0$ *as* $\tau \to \infty$, *where* $d(X_\tau, S) \to 0 = inf_{X \in S}\|X_\tau - X\|$.

Corollary 2.2
Assume the conditions of Theorem 2.18, and also that S consists of a finite set of points. Then $\lim_{\tau \to \infty} X_\tau$ *exists and equals a solution to the variational inequality (2.1a).*

Theorem 2.18 indicates that Assumption 2.2 is the elementary condition under which the Euler method (2.34) converges. Propositions 2.5 and 2.6 below suggest some alternative conditions that are better known in variational inequality theory as sufficient conditions for Part 3 and Part 5 of Assumption 2.2.

Proposition 2.5
If the vector field $F(X)$ is strictly monotone at some solution X^* to the variational inequality problem (2.1a), then Part 3 of Assumption 2.2 holds true.

Proposition 2.6
If the vector field $F(X)$ is monotone at some solution X^* to the variational inequality problem (2.1a), then Part 5 of Assumption 2.2 holds true.

In the subsequent chapters, as appropriate, we adapt the convergence proofs to specific supply chain network applications.

2.6.2 The Modified Projection Method

The modified projection method of Korpelevich (1977) can be utilized to solve a variational inequality problem in standard form (cf. (2.1a)). This method is guaranteed to converge if the monotonicity (cf. (2.10)) and Lipschitz continuity (cf. (2.13)) of the function F that enters the variational inequality (cf. (2.1a)) is satisfied, and a solution to the variational inequality exists.

We now recall the modified projection method, and let τ denote an iteration counter.

Step 0: Initialization
Set $X^0 \in \mathcal{K}$. Let $\tau = 1$ and let α be a scalar such that $0 < \alpha \leq \frac{1}{L}$, where L is the Lipschitz continuity constant (cf. (2.13)).

Step 1: Computation
Compute $\bar{X}^{\tau-1}$ by solving the variational inequality subproblem:

$$\langle \bar{X}^{\tau-1} + \alpha F(X^{\tau-1}) - X^{\tau-1}, X - \bar{X}^{\tau-1} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{2.37}$$

Step 2: Adaptation
Compute X^τ by solving the variational inequality subproblem:

$$\langle X^\tau + \alpha F(\bar{X}^{\tau-1}) - X^{\tau-1}, X - X^\tau \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{2.38}$$

Step 3: Convergence Verification
If $\max |X_l^\tau - X_l^{\tau-1}| \leq \epsilon$, for all l, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\tau := \tau + 1$, and go to Step 1.

Theorem 2.19: Convergence of the Modified Projection Method
If $F(X)$ is monotone and Lipschitz continuous (and a solution exists), the modified projection algorithm converges to a solution of variational inequality (2.1a).

In the following chapters, we derive the variational inequality formulations and the projected dynamical systems of the supply chain network models with quality competition with application to information asymmetry, production differentiation, outsourcing, and freight services as well as supplier selection. The computational algorithms reviewed in this chapter, which are the Euler method and the modified projection method, are also adapted accordingly.

2.7 Sources and Notes

In this chapter we laid the foundations for the methodologies that are utilized for both qualitative and quantitative analysis of supply chain network problems with quality competition in this book. This chapter is based on well-established methodological results for variational inequality theory, projected dynamical systems theory, as well as game theory and multicriteria decision-making. Hence, the primary references noted and cited in this chapter are books, with the books being by Nagurney (1999), Nagurney and Zhang (1996), Nagurney (2006), and Nagurney et al. (2013), where additional references can be found. In subsequent chapters we highlight the network structure of specific supply chains and also how this feature can be exploited both qualitatively and computationally. We adapt the theorems in this chapter to specific supply chain network problems throughout this book.

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Part II
Information Asymmetry in Quality

Chapter 3
Information Asymmetry and Minimum Quality Standards in Supply Chain Oligopolies

Abstract This chapter begins Part II of this book with Part II focusing on information asymmetry in product quality. We present a supply chain network model with information asymmetry in product quality. The competing profit-maximizing firms with multiple manufacturing plants know the quality of the product that they produce but consumers, at the demand markets, are aware only of the average quality. The framework is relevant to products ranging from certain foods to pharmaceuticals to durables such as automobiles. We propose an equilibrium model and its dynamic counterpart and show how minimum quality standards can be incorporated. Qualitative results and an algorithm are presented, along with convergence results. The numerical examples, with sensitivity analysis, provide valuable insights for firms, consumers, as well as policy-makers, who impose the minimum quality standards.

3.1 Introduction

Supply chains have transformed the ways in which goods are produced, transported, and consumed around the globe. Because of these networks, which integrate a plethora of economic activities from manufacturing through distribution, consumers today have many more product choices and options during different seasons. At the same time, given the distances that separate manufacturing plants from retailers and ultimate consumers, there may be information asymmetry as to the quality of the product. Information asymmetry in quality can occur in numerous types of products that are purchased, from food and pharmaceuticals to high technology products, such as computers and mobile phones, and even durables such as automobiles. Information asymmetry in quality is one of the major challenges faced by supply chain decision-makers, from managers to policy-makers, as well as consumers. For example, producers in different industries may be aware of their product quality whereas consumers at the demand markets may only be aware of the average quality. Such information asymmetry in quality results in products being, in effect, homogeneous at demand markets since there is no differentiation in such cases by brands or labels (see Baltzer 2012). Stiglitz (2002) said it well when he defined information asymmetry as the “fact that different people know different things.”

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When manufacturers (producers) have, at their disposal, multiple manufacturing plants, which may be located on-shore or off-shore, with the ability to monitor the quality in the latter, at times, especially challenging, information asymmetry becomes increasingly relevant and complex. Indeed, major issues and quality problems associated with distinct manufacturing plants and products ranging from food to pharmaceuticals have been the focus of increasing attention in both research (cf. Gray et al. 2011; Masoumi et al. 2012; McDonald 2013; Hogenau 2013; Yu and Nagurney 2013) as well as in practice (cf. Payne 2008; Harris 2011). For specific examples, we note several cases of major issues with quality problems associated with distinct manufacturing plants in the pharmaceutical industry. Since 2009, quality failures in several manufacturing plants of Hospira, a leading manufacturer of injectable drugs, led to several major recalls of products produced at manufacturing plants in North Carolina, California, and Costa Rica (Thomas 2012). In 2011, Ben Venue, a division of the German pharmaceutical company Boehringer Ingelheim, was forced to close one of its plants, in Bedford, Ohio, due to quality issues investigated by the Food and Drug Administration (FDA) (Lopatto 2013). Regulatory bodies, nevertheless, may not have the resources nor the jurisdiction to monitor quality across national boundaries leading to further information asymmetry from the perspective of consumers.

Akerlof (1970) utilized used automobiles, with those of inferior quality referred to as *lemons*, as a prime example in his classic work on information asymmetry in quality, which has stimulated much of the research in this domain. For example, Spence (1973, 1975) and Stiglitz (1987), all of whom shared the Nobel Prize in Economic Sciences with Akerlof, also studied markets with asymmetric information in terms of product quality. Baltzer (2012) further emphasized that firms producing the product have control over the quality but consumers may be unable to observe the level of quality as in the case not only with respect to the safety of cars but also the level of microbiological contaminants in food and even chemical residues in toys.

Given the reality of information asymmetry in today's supply chain Network Economy, it is also reasonable to explore policy interventions in the form of minimum quality standards and what the ramifications thereof might be. Leland (1979), for example, argued that markets under information asymmetry in quality may benefit from minimum quality standards. Questions, nevertheless, arise in a supply chain network context as to what are the impacts of setting minimum quality standards, which may be set regionally, nationally, or, through cross-border agreements, even internationally?

In this chapter, we utilize a network economics approach to develop both static and dynamic competitive supply chain network models with information asymmetry in quality. We consider multiple profit-maximizing firms, which are spatially separated, and may have multiple manufacturing plants at their disposal. Hence, the manufacturing plants of firms may be located in different regions of the same country or in different countries. The firms are involved in the production of a product, and compete in multiple demand markets, which are also spatially separated, in a Cournot-Nash (see Cournot 1838; Nash 1950, 1951) manner in product shipments and product quality levels. As emphasized in Chap. 1, in this chapter, and throughout this book, we define quality as “the degree to which a specific product conforms to a design or specification.”

We also reveal how minimum quality standards can be incorporated into the framework, which has wide relevance for policy-making and regulation (see, e.g., Giraud-Heraud and Soler 2006; Smith 2009). In this chapter, we consider imperfect competition in the form of a supply chain network oligopoly and note that Baltzer (2012) studied two firms involved in Bertrand (1883) competition with specific underlying functional forms. Our models, in contrast, assume Cournot-Nash competition in both quantities and quality levels, are network-based, and are not limited to two firms, among other distinctions. In Chap. 4, we explore information asymmetry in quality in perfectly competitive markets through a spatial price equilibrium model.

Firms, in this chapter, are aware of the quality of the product produced at each of their manufacturing plants, with different manufacturing plants owned by the same firm having, possibly, different quality levels. However, the quality levels perceived by consumers at the demand markets are the average quality levels of the products (see also Akerlof 1970; Leland 1979). Information asymmetry between produced and perceived quality levels and quality uncertainty are discussed in Wankhade and Dabade (2010), but no supply chain network models are constructed therein. By providing a supply chain network context for information asymmetry in quality decision-makers can conduct numerous investigations in terms of modifying the network topology, altering the production cost, transportation cost, and even demand price function structure, and assessing the resulting impacts on quality, on product flows, and on incurred prices and profits. Moreover, the impacts of minimum quality standard imposition can be evaluated to also determine which firm may benefit in terms of profits and which may lose.

The chapter is organized as follows. In Sect. 3.2, we present the static (equilibrium) supply chain network models, without and with minimum quality standards, along with their variational inequality formulations. We also demonstrate how upper bounds on manufacturing plant quality levels can be incorporated into the variational inequality formulations. In Sect. 3.3, we develop the dynamic version of the equilibrium model with minimum quality standards using projected dynamical systems theory. In Sect. 3.4, we provide qualitative properties of the equilibrium solutions and establish that the set of stationary points of our projected dynamical systems formulation coincides with the set of solutions to the corresponding variational inequality problem. In Sect. 3.5, we describe the algorithm, which yields closed form expressions in product shipments and quality levels at each iteration, and establish convergence. In Sect. 3.6, we provide numerical examples and conduct sensitivity analyses, which yield valuable insights for firms, consumers, and policy-makers. In Sect. 3.7, we summarize the results and present our conclusions. Section 3.8 contains the Sources and Notes for this chapter.

3.2 The Equilibrium Model Without and with Minimum Quality Standards

We first present the supply chain network model without minimum quality standards and then show how it can be extended to include minimum quality standards, which are useful policy instruments in practice.

In the supply chain network, depicted in Fig. 3.1, there are I firms, with a typical firm denoted by i, which compete with one another in a noncooperative Cournot-Nash manner in the production and distribution of the product. Each firm i has, at its disposal, n_i manufacturing plants. The firms determine the quantities to produce at each of their manufacturing plants and the quantities to ship to the n_R demand markets. They also control the quality level of the product at each of their manufacturing plants. Information asymmetry is present since the firms know the quality levels of the product produced at each of their manufacturing plants but the consumers are only aware of the *average* quality levels of the product at the demand markets.

The top nodes in the supply chain in Fig. 3.1 correspond to the firms, the middle nodes to the manufacturing plants, and the bottom nodes to the common demand markets. We assume that the demand at each demand market is positive; otherwise, the demand market (node) will be removed from the supply chain network.

In Fig. 3.1, the first set of links connecting the top two tiers of nodes corresponds to the process of manufacturing at each of the manufacturing plants of firm i; $i = 1, \ldots, I$. Such plants are denoted by $M_i^1, \ldots, M_i^{n_i}$, respectively, for firm i, with a typical one denoted by M_i^j; $j = 1, \ldots, n_i$. The manufacturing plants may be located

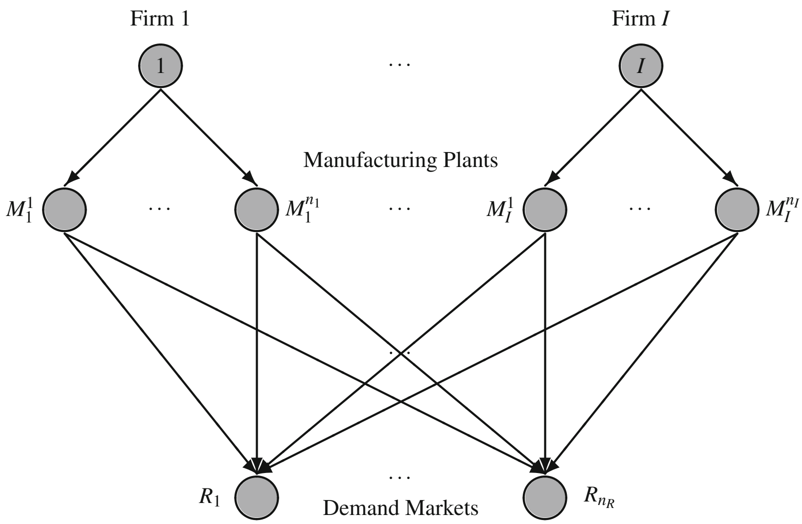

Fig. 3.1 The supply chain network topology

Table 3.1 Notation for the supply chain network models (static and dynamic) with information asymmetry

Notation	Definition
Q_{ijk}	The nonnegative amount of product produced at firm i's manufacturing plant M_i^j and shipped to demand market R_k. We group the $\{Q_{ijk}\}$ elements for all j and k into the vector $Q_i \in R_+^{n_i n_R}$ and the vectors Q_i for all i into the vector $Q \in R_+^{\sum_{i=1}^I n_i n_R}$
s_{ij}	The nonnegative production output of firm i's manufacturing plant M_i^j. We group the $\{s_i\}$ elements for all i into the vector $s \in R_+^{\sum_{i=1}^I n_i}$
q_{ij}	The nonnegative quality level of the product produced by firm i's manufacturing plant M_i^j. We group the $\{q_{ij}\}$ elements for firm i into the vector $q_j \in R_+^{n_i}$ and all the vectors q_i for all i into the vector $q \in R_+^{\sum_{i=1}^I n_i}$
d_k	The demand for the product at demand market R_k. We group the demands for all k into the vector $d \in R_+^{n_R}$
$\hat{q}_k$	The average quality level at demand market R_k as perceived by consumers. We group the average quality levels at all demand markets into the vector $\hat{q} \in R_+^{n_R}$. The average quality level at R_k, $\hat{q}_k = \frac{\sum_{i=1}^I \sum_{j=1}^{n_i} Q_{ijk} q_{ij}}{d_k}$
$f_{ij}(s, q)$	The production cost at firm i's manufacturing plant M_i^j
$\hat{c}_{ijk}(Q, q)$	The total transportation cost associated with shipping the product produced at firm i's manufacturing plant M_i^j to demand market R_k, assuming quality preservation
$\rho_k(d, \hat{q})$	The demand price at demand market R_k

not only in different regions of a country but also in different countries. The next set of links connecting the two bottom tiers of the supply chain network corresponds to the transportation links connecting the manufacturing plants with the demand markets, with a typical demand market denoted by R_k; $k = 1, \ldots, n_R$.

The variable and model function notation for the static (equilibrium) models and the dynamic version of the static one with minimum quality standards is presented in Table 3.1.

Manufacturing plants owned by a firm may have different quality levels since they may be located in different regions or countries where there may be distinct skill levels in terms of labor, as well as varying infrastructure and even technologies, plus natural resources used in production. In addition, in certain locations, there may be more or less incentive to manufacture products of higher quality due to costs, as well as levels of consumer awareness.

The output at firm i's manufacturing plant M_i^j, s_{ij}, and the demand for the product at each demand market R_k, d_k, must satisfy, respectively, the conservation of flow equations (3.1) and (3.2):

$$s_{ij} = \sum_{k=1}^{n_R} Q_{ijk}, \quad i = 1, \ldots, I; j = 1, \ldots, n_i, \tag{3.1}$$

$$d_k = \sum_{i=1}^{I} \sum_{j=1}^{n_i} Q_{ijk}, \quad k = 1, \ldots, n_R. \tag{3.2}$$

Hence, the output produced at firm i's manufacturing plant M_i^j is equal to the sum of the amounts shipped to the demand markets, and the quantity consumed at a demand market is equal to the sum of the amounts shipped by the firms to that demand market.

The product shipments must be nonnegative, that is:

$$Q_{ijk} \geq 0, \quad i = 1, \ldots, I; j = 1, \ldots, n_i; k = 1, \ldots, n_R. \tag{3.3}$$

As discussed in Chap. 1, we define and quantify quality as the quality conformance level, that is, the degree to which a specific product conforms to a design or specification (Gilmore 1974; Juran and Gryna 1988). Moreover, in the model without minimum quality standards, we have that the quality levels must be nonnegative, that is,

$$q_{ij} \geq 0, \quad i = 1, \ldots, I; j = 1, \ldots, n_i. \tag{3.4}$$

As noted in Table 3.1, the production cost f_{ij} may depend upon the entire production pattern and the entire vector of quality levels. In view of (3.1), we can define the plant manufacturing cost functions $\hat{f}_{ij}$; $i = 1, \ldots, I; j = 1, \ldots, n_i$, in shipment quantities and quality levels, such that:

$$\hat{f}_{ij} = \hat{f}_{ij}(Q, q) \equiv f_{ij}(s, q). \tag{3.5}$$

The transportation costs $\hat{c}_{ijk}$; $i = 1, \ldots, I$; $j = 1, \ldots, n_i$; $k = 1, \ldots, n_R$, (cf. Table 3.1), in turn, are such that the quality of the product is not degraded as it undergoes the shipment process and, as in the case of the production cost functions, we consider the general situation that the transportation cost functions depend both on the vector of product shipments and the quality levels since we have quality preservation during the transportation process.

The production cost functions (3.5) and the transportation functions are assumed to be convex and twice continuously differentiable.

Recall that the consumers' perception of the quality of the product, which may come from different firms, is for the *average* quality level. The demand price function at a demand market may depend, in general, on the entire demand pattern, as well as on the average quality levels at all the demand markets. Each demand price function is assumed to be monotonically decreasing in its demand but increasing in terms of the average product quality, since we assume that consumers at the demand markets are willing to pay a higher price for greater average quality. Demand functions that are functions of the prices and the average quality levels are also used by Akerlof (1970), since the producers, in the form of a supply market, are aware of their product quality levels (cf. (3.5)), while consumers at the demand markets are aware only of the average quality levels. However, Akerlof (1970) did not consider multiple manufacturing plants, transportation, and multiple demand markets. Moreover, he did not model the profit-maximizing behavior of individual, competing firms, as we do here.

In view of (3.2) and the average quality formulae in Table 3.1, we can define the demand price function $\hat{\rho}_k; k = 1, \ldots, n_R$, in quantities and quality levels of the firms, so that

$$\hat{\rho}_k = \hat{\rho}_k(Q, q) \equiv \rho_k(d, \hat{q}), \quad k = 1, \ldots, n_R. \tag{3.6}$$

We assume that the demand price functions are continuous and twice continuously differentiable.

The strategic variables of firm i are its product shipments $\{Q_i\}$ and its quality levels $\{q_i\}$. The profit/utility U_i of firm i; $i = 1, \ldots, I$, is given by:

$$U_i = \sum_{k=1}^{n_R} \rho_k(d, \hat{q}) \sum_{j=1}^{n_i} Q_{ijk} - \sum_{j=1}^{n_i} f_{ij}(s, q) - \sum_{k=1}^{n_R} \sum_{j=1}^{n_i} \hat{c}_{ijk}(Q, q), \tag{3.7}$$

which is the difference between the firm's total revenue and its total costs (production and transportation). By making use of (3.5) and (3.6), (3.7) is equivalent to

$$U_i = \sum_{k=1}^{n_R} \hat{\rho}_k(Q, q) \sum_{j=1}^{n_i} Q_{ijk} - \sum_{j=1}^{n_i} \hat{f}_{ij}(Q, q) - \sum_{k=1}^{n_R} \sum_{j=1}^{n_i} \hat{c}_{ijk}(Q, q). \tag{3.8}$$

In view of (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) and (3.8), we may express the profit as a function solely of the product shipment pattern and quality levels, that is,

$$U = U(Q, q), \tag{3.9}$$

where U is the I-dimensional vector with components: $\{U_1, \ldots, U_I\}$.

Let K^i denote the feasible set corresponding to firm i, where $K^i \equiv \{(Q_i, q_i) | Q_i \geq 0, \text{ and } q_i \geq 0\}$ and define $K \equiv \prod_{i=1}^{I} K^i$.

We consider Cournot-Nash competition, in which the I firms produce and deliver their product in a noncooperative fashion, each one trying to maximize its own profit. We seek to determine a nonnegative product shipment and quality level pattern $(Q^*, q^*) \in K$ for which the I firms will be in a state of equilibrium as defined below.

Definition 3.1: A Supply Chain Network Cournot-Nash Equilibrium with Information Asymmetry in Quality
A product shipment and quality level pattern $(Q^*, q^*) \in K$ *is said to constitute a supply chain network Cournot-Nash equilibrium with information asymmetry in quality if for each firm* i; $i = 1, \dots, I$,

$$U_i(Q_i^*, q_i^*, \hat{Q}_i^*, \hat{q}_i^*) \geq U_i(Q_i, q_i, \hat{Q}_i^*, \hat{q}_i^*), \quad \forall (Q_i, q_i) \in K^i, \tag{3.10}$$

where

$$\hat{Q}_i^* \equiv (Q_1^*, \dots, Q_{i-1}^*, Q_{i+1}^*, \dots, Q_I^*) \quad \textit{and} \quad \hat{q}_i^* \equiv (q_1^*, \dots, q_{i-1}^*, q_{i+1}^*, \dots, q_I^*).$$

According to (3.10), an equilibrium is established if no firm can unilaterally improve upon its profits by selecting an alternative vector of product shipments and quality level of its product.

3.2.1 Variational Inequality Formulations

We now present alternative variational inequality formulations of the above supply chain network Cournot-Nash equilibrium in the following theorem.

Theorem 3.1: Variational Inequality Formulations
Assume that for each firm i; $i = 1, \dots, I$, *the profit function* $U_i(Q, q)$ *is concave with respect to the variables in* Q_i *and* q_i, *and is continuous and continuously differentiable. Then the product shipment and quality pattern* $(Q^*, q^*) \in K$ *is a supply chain network Cournot-Nash equilibrium with quality information asymmetry according to Definition 3.1 if and only if it satisfies the variational inequality*

$$-\sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{k=1}^{n_R} \frac{\partial U_i(Q^*, q^*)}{\partial Q_{ijk}} \times (Q_{ijk} - Q_{ijk}^*) - \sum_{i=1}^{I}\sum_{j=1}^{n_i} \frac{\partial U_i(Q^*, q^*)}{\partial q_{ij}} \times (q_{ij} - q_{ij}^*) \geq 0,$$

$$\forall (Q, q) \in K, \tag{3.11}$$

that is,

$$\sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{k=1}^{n_R}\left[-\hat{\rho}_k(Q^*, q^*) - \sum_{l=1}^{n_R} \frac{\partial \hat{\rho}_l(Q^*, q^*)}{\partial Q_{ijk}} \sum_{h=1}^{n_i} Q_{ihl}^* + \sum_{h=1}^{n_i} \frac{\partial \hat{f}_{ih}(Q^*, q^*)}{\partial Q_{ijk}}\right.$$

$$\left. + \sum_{h=1}^{n_i}\sum_{l=1}^{n_R} \frac{\partial \hat{c}_{ihl}(Q^*, q^*)}{\partial Q_{ijk}}\right] \times (Q_{ijk} - Q_{ijk}^*)$$

$$+\sum_{i=1}^{I}\sum_{j=1}^{n_i}\left[-\sum_{k=1}^{n_R}\frac{\partial\hat{\rho}_k(Q^*,q^*)}{\partial q_{ij}}\sum_{h=1}^{n_i}Q^*_{ihk}+\sum_{h=1}^{n_i}\frac{\partial\hat{f}_{ih}(Q^*,q^*)}{\partial q_{ij}}+\sum_{h=1}^{n_i}\sum_{k=1}^{n_R}\frac{\partial\hat{c}_{ihk}(Q^*,q^*)}{\partial q_{ij}}\right]$$

$$\times (q_{ij}-q^*_{ij})\geq 0, \quad \forall (Q,q)\in K; \tag{3.12}$$

equivalently, $(d^*,s^*,Q^*,q^*)\in K^1$ *is an equilibrium demand, production, shipment, and quality level pattern if and only if it satisfies the variational inequality*

$$\sum_{k=1}^{n_R}\left[-\rho_k(d^*,\hat{q}^*)\right]\times(d_k-d^*_k)+\sum_{i=1}^{I}\sum_{j=1}^{n_i}\left[\sum_{h=1}^{n_i}\frac{\partial f_{ih}(s^*,q^*)}{\partial s_{ij}}\right]\times(s_{ij}-s^*_{ij})$$

$$+\sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{k=1}^{n_R}\left[-\sum_{l=1}^{n_R}\frac{\partial\rho_l(d^*,\hat{q}^*)}{\partial Q_{ijk}}\sum_{h=1}^{n_i}Q^*_{ihl}+\sum_{h=1}^{n_i}\sum_{l=1}^{n_R}\frac{\partial\hat{c}_{ihl}(Q^*,q^*)}{\partial Q_{ijk}}\right]\times(Q_{ijk}-Q^*_{ijk})$$

$$+\sum_{i=1}^{I}\sum_{j=1}^{n_i}\left[-\sum_{k=1}^{n_R}\frac{\partial\rho_k(Q^*,\hat{q}^*)}{\partial q_{ij}}\sum_{h=1}^{n_i}Q^*_{ihk}+\sum_{h=1}^{n_i}\frac{\partial f_{ih}(s^*,q^*)}{\partial q_{ij}}+\sum_{h=1}^{n_i}\sum_{k=1}^{n_R}\frac{\partial\hat{c}_{ihk}(Q^*,q^*)}{\partial q_{ij}}\right]$$

$$\times(q_{ij}-q^*_{ij})\geq 0, \qquad \forall(d,s,Q,q)\in K^1, \tag{3.13}$$

where $K^1\equiv\{(d,s,Q,q)\,|\,Q\geq 0,\ q\geq 0,$ *and* (3.1) *and* (3.2) *hold*$\}$.

Proof: Equation (3.12) follows directly from Gabay and Moulin (1980) and Dafermos and Nagurney (1987). For firm i's manufacturing plant M^j_i; $i=1,\ldots,I$; $j=1,\ldots,n_i$ and demand market R_k; $k=1,\ldots,n_R$, we have:

$$-\frac{\partial U_i(Q,q)}{\partial Q_{ijk}}=-\frac{\partial\left[\sum_{l=1}^{n_R}\hat{\rho}_l(Q,q)\sum_{h=1}^{n_i}Q_{ihl}-\sum_{h=1}^{n_i}\hat{f}_{ih}(Q,q)-\sum_{h=1}^{n_i}\sum_{l=1}^{n_R}\hat{c}_{ihl}(Q,q)\right]}{\partial Q_{ijk}}$$

$$=-\sum_{l=1}^{n_R}\frac{\partial\left[\hat{\rho}_l(Q,q)\sum_{h=1}^{n_i}Q_{ihl}\right]}{\partial Q_{ijk}}+\sum_{h=1}^{n_i}\frac{\partial\hat{f}_{ih}(Q,q)}{\partial Q_{ijk}}+\sum_{h=1}^{n_i}\sum_{l=1}^{n_R}\frac{\partial\hat{c}_{ihl}(Q,q)}{\partial Q_{ijk}}$$

$$=-\hat{\rho}_k(Q,q)-\sum_{l=1}^{n_R}\frac{\partial\hat{\rho}_l(Q,q)}{\partial Q_{ijk}}\sum_{h=1}^{n_i}Q_{ihl}+\sum_{h=1}^{n_i}\frac{\partial\hat{f}_{ih}(Q,q)}{\partial Q_{ijk}}+\sum_{h=1}^{n_i}\sum_{l=1}^{n_R}\frac{\partial\hat{c}_{ihl}(Q,q)}{\partial Q_{ijk}}. \tag{3.14}$$

Also, for firm i's manufacturing plant M_i^j; $i = 1, \ldots, I$; $j = 1, \ldots, n_i$, we have:

$$-\frac{\partial U_i(Q,q)}{\partial q_{ij}} = -\frac{\partial[\sum_{k=1}^{n_R} \hat{\rho}_k(Q,q) \sum_{h=1}^{n_i} Q_{ihk} - \sum_{h=1}^{n_i} \hat{f}_{ih}(Q,q) - \sum_{h=1}^{n_i} \sum_{k=1}^{n_R} \hat{c}_{ihk}(Q,q)]}{\partial q_{ij}}$$

$$= -\sum_{k=1}^{n_R} \frac{\partial \hat{\rho}_k(Q,q)}{\partial q_{ij}} \sum_{h=1}^{n_i} Q_{ihk} + \sum_{h=1}^{n_i} \frac{\partial \hat{f}_{ih}(Q,q)}{\partial q_{ij}} + \sum_{h=1}^{n_i} \sum_{k=1}^{n_R} \frac{\partial \hat{c}_{ihk}(Q,q)}{\partial q_{ij}}. \tag{3.15}$$

Thus, variational inequality (3.12) is immediate. In addition, by re-expressing the production cost functions and the demand price functions in (3.14) and (3.15) as in (3.5) and (3.6) and using the conservation of flow equations (3.1) and (3.2) and $\frac{\partial f_{ih}(s,q)}{\partial Q_{ijk}} = \frac{\partial f_{ih}(s,q)}{\partial s_{ij}} \frac{\partial s_{ij}}{\partial Q_{ijk}}$, the equivalence of variational inequalities (3.12) and (3.13) holds true. □

3.2.2 Incorporation of Minimum Quality Standards

We now present an extension of the above supply chain network model that includes minimum quality standards. The effectiveness of the imposition of minimum quality standards on quality has been studied in economics with or without information asymmetry (Leland 1979; Shapiro 1983; Besanko et al. 1988; Ronnen 1991; Lutz and Lutz 2010). Here, we integrate the supply chain network model with minimum quality standards and the one without, and present the equilibrium conditions of both through a unified variational inequality formulation.

We retain the previous notation and firm behavior and constraints but now we impose nonnegative lower bounds on the quality levels at the manufacturing plants, denoted by $\underline{q}_{ij}$; $i = 1, \ldots, I$; $j = 1, \ldots, n_i$ so that (3.4) is replaced by:

$$q_{ij} \geq \underline{q}_{ij} \quad i = 1, \ldots, I; j = 1, \ldots, n_i \tag{3.16}$$

with the understanding that, if the lower bounds are all identically equal to zero, then (3.16) collapses to (3.4) and, if the lower bounds are positive, then they represent minimum quality standards.

We define a new feasible set $K^2 \equiv \{(Q,q)|Q \geq 0, \text{and} (3.16) \text{holds}\}$. Then the following Corollary is immediate.

Corollary 3.1: Variational Inequality Formulations with Minimum Quality Standards

Assume that for each firm i; $i = 1, \ldots, I$, the profit function $U_i(Q,q)$ is concave with respect to the variables in Q_i and q_i, and is continuous and continuously differentiable. Then the product shipment and quality pattern $(Q^*, q^*) \in K^2$

is a supply chain network Cournot-Nash equilibrium with quality information asymmetry in the presence of minimum quality standards if and only if it satisfies the variational inequality

$$-\sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{k=1}^{n_R}\frac{\partial U_i(Q^*,q^*)}{\partial Q_{ijk}}\times(Q_{ijk}-Q^*_{ijk})-\sum_{i=1}^{I}\sum_{j=1}^{n_i}\frac{\partial U_i(Q^*,q^*)}{\partial q_{ij}}\times(q_{ij}-q^*_{ij})\geq 0,$$

$$\forall (Q,q)\in K^2, \tag{3.17}$$

that is,

$$\sum_{i=1}^{I}\sum_{j=1}^{n_i}\sum_{k=1}^{n_R}\Bigg[-\hat{\rho}_k(Q^*,q^*)-\sum_{l=1}^{n_R}\frac{\partial\hat{\rho}_l(Q^*,q^*)}{\partial Q_{ijk}}\sum_{h=1}^{n_i}Q^*_{ihl}+\sum_{h=1}^{n_i}\frac{\partial\hat{f}_{ih}(Q^*,q^*)}{\partial Q_{ijk}}$$

$$+\sum_{h=1}^{n_i}\sum_{l=1}^{n_R}\frac{\partial\hat{c}_{ihl}(Q^*,q^*)}{\partial Q_{ijk}}\Bigg]\times(Q_{ijk}-Q^*_{ijk})$$

$$+\sum_{i=1}^{I}\sum_{j=1}^{n_i}\Bigg[-\sum_{k=1}^{n_R}\frac{\partial\hat{\rho}_k(Q^*,q^*)}{\partial q_{ij}}\sum_{h=1}^{n_i}Q^*_{ihk}+\sum_{h=1}^{n_i}\frac{\partial\hat{f}_{ih}(Q^*,q^*)}{\partial q_{ij}}+\sum_{h=1}^{n_i}\sum_{k=1}^{n_R}\frac{\partial\hat{c}_{ihk}(Q^*,q^*)}{\partial q_{ij}}\Bigg]$$

$$\times(q_{ij}-q^*_{ij})\geq 0,\qquad \forall (Q,q)\in K^2. \tag{3.18}$$

Variational inequality (3.18) contains variational inequality (3.12) as a special case when the minimum quality standards are all zero. In fact, only the respective feasible sets K^2 and K differ. Variational inequality (3.18) plays a crucial role in the next section when we describe the underlying dynamics associated with the firms' adjustment processes in product shipments and quality levels until an equilibrium point, equivalently, a stationary point, is achieved.

We now put variational inequality (3.18) into standard form (cf. (2.1a)): determine $X^*\in\mathcal{K}$ where X is a vector in R^N, $F(X)$ is a continuous function such that $F(X):X\mapsto\mathcal{K}\subset R^N$, and

$$\langle F(X^*),X-X^*\rangle\geq 0,\quad \forall X\in\mathcal{K}, \tag{3.19}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in the N-dimensional Euclidean space, and $\mathcal{K}$ is closed and convex. We define the vector $X\equiv(Q,q)$ and the vector $F(X)\equiv(F^1(X),F^2(X))$. Also, here $N=\sum_{i=1}^{I}n_in_R+\sum_{i=1}^{I}n_i$. $F^1(X)$ consists of components $F^1_{ijk}=-\frac{\partial U_i(Q,q)}{\partial Q_{ijk}}$; $i=1,\ldots,I;j=1,\ldots,n_i;k=1,\ldots,n_R$, and $F^2(X)$ consists of components $F^2_{ij}=-\frac{\partial U_i(Q,q)}{\partial q_{ij}}$; $i=1,\ldots,I;j=1,\ldots,n_i$. In addition, we define the feasible set $\mathcal{K}\equiv K^2$. Hence, (3.18) can be put into standard form (3.19). Of course, (3.12) can also be put into standard form with $F(X)$ defined as above and with $\mathcal{K}=K$.

Remark 3.1
In certain applications, one may wish to include upper bounds on the quality that can be achieved by manufacturing plants of firms. In this case we can construct new constraints to replace (3.16), with $\overline{q}_{ij}$ denoting the upper bound of quality that is achievable (or desirable) by firm i at its manufacturing plant M_i^j; $i = 1, \ldots, I$; $j = 1, \ldots n_i$, as follows:

$$\overline{q}_{ij} \geq q_{ij} \geq \underline{q}_{ij}; \quad i = 1, \ldots I; j = 1, \ldots n_i. \tag{3.20}$$

We can define a new feasible set $K^3 \equiv \{(Q, q) | Q \geq 0$, and (3.1), (3.2), and (3.20) hold$\}$. Variational inequalities (3.17) and (3.18) will still hold with K^3 substituted for K^2 for the supply chain network game in which the firms have the same strategic vectors as before but now the quality strategies must satisfy both upper and lower bounds as in (3.20).

3.3 The Dynamic Model

The interactions of firms over time as they adjust their product shipment levels and quality levels in supply chain networks are dynamic competitive processes. We now describe the underlying dynamics for the evolution of product shipments and quality levels under information asymmetry in quality until the equilibrium satisfying variational inequality (3.18) is achieved. We identify the dynamic adjustment processes for the evolution of the firm's product shipments and quality levels. In Sect. 3.4, we provide an algorithm, which is a discrete-time version of the continuous-time adjustment processes introduced below.

Observe that, for a current vector of product shipments and quality levels at time t, $X(t) = (Q(t), q(t))$, $-F^1_{ijk}(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial Q_{ijk}}$ is the marginal utility (profit) of firm i with respect to the volume produced at its manufacturing plant j and distributed to demand market R_k, and $-F^2_{ij}(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial q_{ij}}$ is firm i's marginal utility with respect to the quality level of its manufacturing plant j. In this framework, the rate of change of the product shipment between firm i's manufacturing plant j and demand market R_k is in proportion to $-F^1_{ij}(X)$, as long as the product shipment Q_{ijk} is positive.

Namely, when $Q_{ijk} > 0$,

$$\dot{Q}_{ijk} = \frac{\partial U_i(Q, q)}{\partial Q_{ijk}}, \tag{3.21}$$

where $\dot{Q}_{ijk}$ denotes the rate of change of Q_{ijk}. However, when $Q_{ijk} = 0$, the nonnegativity condition (3.3) forces the product shipment Q_{ijk} to remain zero when $\frac{\partial U_i(Q,q)}{\partial Q_{ijk}} \leq 0$. Hence, we are only guaranteed of having possible increases of the shipment, that is, when $Q_{ijk} = 0$,

$$\dot{Q}_{ijk} = \max\{0, \frac{\partial U_i(Q,q)}{\partial Q_{ijk}}\}. \tag{3.22}$$

We can write (3.21) and (3.22) concisely as:

$$\dot{Q}_{ijk} = \begin{cases} \frac{\partial U_i(Q,q)}{\partial Q_{ijk}}, & \text{if} \quad Q_{ijk} > 0 \\ \max\{0, \frac{\partial U_i(Q,q)}{\partial Q_{ijk}}\}, & \text{if} \quad Q_{ijk} = 0. \end{cases} \tag{3.23}$$

As for the quality levels, when $q_{ij} > \underline{q}_{ij}$, then

$$\dot{q}_{ij} = \frac{\partial U_i(Q,q)}{\partial q_{ij}}, \tag{3.24}$$

where $\dot{q}_{ij}$ denotes the rate of change of q_{ij}; when $q_{ij} = \underline{q}_{ij}$,

$$\dot{q}_{ij} = \max\{\underline{q}_{ij}, \frac{\partial U_i(Q,q)}{\partial q_{ij}}\}, \tag{3.25}$$

since q_i cannot be lower than $\underline{q}_{ij}$ according to the feasible set $\mathscr{K} = K^2$.

Combining (3.24) and (3.25), we obtain:

$$\dot{q}_{ij} = \begin{cases} \frac{\partial U_i(Q,q)}{\partial q_{ij}}, & \text{if} \quad q_{ij} > \underline{q}_{ij} \\ \max\{\underline{q}_{ij}, \frac{\partial U_i(Q,q)}{\partial q_{ij}}\}, & \text{if} \quad q_{ij} = \underline{q}_{ij}. \end{cases} \tag{3.26}$$

Applying (3.23) to all firm and manufacturing plant pairs (i,j); $i = 1,\ldots,I$; $j = 1,\ldots,n_i$ and all demand markets R_k; $k = 1,\ldots,n_R$, and then applying (3.26) to all firm and manufacturing plant pairs (i,j); $i = 1,\ldots,I$; $j = 1,\ldots,n_i$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes of the product shipments and quality levels, in vector form:

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)), \tag{3.27}$$

where, since $\mathscr{K}$ is a convex polyhedron, according to Dupuis and Nagurney (1993), $\Pi_{\mathscr{K}}(X,-F(X))$ is the projection, with respect to $\mathscr{K}$, of the vector $-F(X)$ at X defined as

$$\Pi_{\mathscr{K}}(X,-F(X)) = \lim_{\delta \to 0} \frac{P_{\mathscr{K}}(X - \delta F(X)) - X}{\delta} \tag{3.28}$$

with $P_{\mathscr{K}}$ denoting the projection map:

$$P(X) = \operatorname{argmin}_{z \in \mathscr{K}} \|X - z\|, \tag{3.29}$$

and where $\|\cdot\| = \langle x, x \rangle$ and $-F(X) = \nabla U(Q,q)$, where $\nabla U(Q,q)$ is the vector of marginal utilities as described above.

We now further interpret ODE (3.27) in the context of the supply chain network model with information asymmetry in quality. First, observe that ODE (3.27) guarantees that the product shipments are always nonnegative and the quality levels never go below the minimum quality standards. In addition, ODE (3.27) reveals that the rate of change of the product shipments and the quality levels is greatest when the firm's marginal utilities are greatest. If the marginal utility of a firm with respect to its quality level is positive, then the firm will increase its quality level; if it is negative, then it will decrease the quality level, and the quality levels will also never be outside their lower bounds. A similar adjustment behavior holds for the firms in terms of their product shipments. This type of behavior is rational from an economic standpoint. Therefore, ODE (3.27) corresponds to reasonable continuous adjustment processes for the supply chain network competition model with information asymmetry in quality.

Since ODE (3.27) is nonstandard due to its discontinuous right-hand side, we further discuss the existence and uniqueness of (3.27). Dupuis and Nagurney (1993) constructed the fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by (3.27). We cite the following theorem from that paper (see also Theorem 2.13 in Chap. 2).

Theorem 3.2: Equivalence of Equilibria and Stationary Points
X^* solves the variational inequality problem (3.19) if and only if it is a stationary point of the ODE (3.27), that is,

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)). \tag{3.30}$$

This theorem demonstrates that the necessary and sufficient condition for a product shipment and quality level pattern $X^* = (Q^*, q^*)$ to be a supply chain network equilibrium with information asymmetry in quality, according to Definition 3.1, is that $X^* = (Q^*, q^*)$ is a stationary point of the adjustment processes defined by ODE (3.27), that is, X^* is the point at which $\dot{X} = 0$.

3.4 Qualitative Properties

We now investigate whether, and, under what conditions, the dynamic adjustment processes defined by (3.27) approach a Cournot-Nash equilibrium. Recall that Lipschitz continuity of $F(X)$ (cf. Chap. 2 and also Dupuis and Nagurney 1993; Nagurney and Zhang 1996) guarantees the existence of a unique solution to (3.31) below, where we have that $X^0(t)$ satisfies ODE (3.27) with initial shipment and quality level pattern (Q^0, q^0). In other words, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X^0, \tag{3.31}$$

with $X^0(0) = X^0$.

We know that, if the utility functions are twice differentiable and the Jacobian matrix of $F(X)$, denoted by $\nabla F(X)$, is positive definite, then the corresponding $F(X)$ is strictly monotone, and the solution to variational inequality (3.19) is unique, if it exists.

Assumption 3.1
Suppose that in the supply chain network model with information asymmetry in quality there exists a sufficiently large M, such that for any (i, j, k),

$$\frac{\partial U_i(Q,q)}{\partial Q_{ijk}} < 0, \tag{3.32}$$

for all shipment patterns Q with $Q_{ijk} \geq M$ *and that there exists a sufficiently large* $\bar{M}$, *such that for any* (i, j),

$$\frac{\partial U_i(Q,q)}{\partial q_{ij}} < 0, \tag{3.33}$$

for all quality level patterns q with $q_{ij} \geq \bar{M} \geq \underline{q}_{ij}$.

We now give existence and uniqueness results, the proofs of which follow from the basic theory of variational inequalities (cf. Nagurney (1999) and Chap. 2), unless explicitly stated otherwise.

Proposition 3.1: Existence
Any supply chain network problem with information asymmetry in quality that satisfies Assumption 3.1 possesses at least one equilibrium shipment and quality level pattern satisfying variational inequality (3.17) (or (3.18)).

Proof: The proof follows from Proposition 1 in Zhang and Nagurney (1995). □

Proposition 3.2: Uniqueness
Suppose that F is strictly monotone at any equilibrium point of the variational inequality problem defined in (3.19). Then it has at most one equilibrium point.

Proof: Follows from Proposition 2 in Nagurney et al. (1994).

Theorem 3.3: Existence and Uniqueness
Suppose that F is strongly monotone. Then there exists a unique solution to variational inequality (3.19); equivalently, to variational inequality (3.18).

The following theorem summarizes the stability properties of the utility gradient processes, under various monotonicity conditions on the marginal utilities.

Theorem 3.4: Stability

(i) *If* $F(X)$ *is monotone, then every supply chain network equilibrium with information asymmetry,* X^*, *provided its existence, is a global monotone attractor for the projected dynamical system. If* $F(X)$ *is locally monotone at* X^*, *then it is a monotone attractor for the projected dynamical system.*

(ii) *If $F(X)$ is strictly monotone, then there exists at most one supply chain network equilibrium with information asymmetry in quality, X^*. Furthermore, given existence, the unique equilibrium is a strictly global monotone attractor for the projected dynamical system. If $F(X)$ is locally strictly monotone at X^*, then it is a strictly monotone attractor for the projected dynamical system.*

(iii) *If $F(X)$ is strongly monotone, then the unique supply chain network equilibrium with information asymmetry in quality, which is guaranteed to exist, is also globally exponentially stable for the projected dynamical system. If $F(X)$ is locally strongly monotone at X^*, then it is exponentially stable.*

Proof: The stability assertions follow from Theorems 2.15, 2.16, and 2.17; see also Zhang and Nagurney (1995). □

3.5 The Algorithm

As mentioned in Sect. 3.3, the projected dynamical system yields continuous-time adjustment processes. For computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories, will be introduced in this section.

The algorithm that we use for the computation of the solution for supply chain network model with information asymmetry in quality is the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993), and is also presented in Chap. 2. Specifically, recall that at iteration $\tau + 1$ of the Euler method (see also Nagurney and Zhang 1996), one computes:

$$X^{\tau+1} = P_{\mathcal{K}}(X^\tau - a_\tau F(X^\tau)), \tag{3.34}$$

where $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ and F is the function that enters the variational inequality problem (3.19).

As shown in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996), for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$.

Explicit Formulae for the Euler Method Applied to the Supply Chain Network Model with Information Asymmetry in Quality

The Euler method yields, at each iteration $\tau + 1$, explicit formulae for the computation of the product shipments and quality levels. In particular, we have the following closed form expressions for the product shipments for $i = 1, \ldots, I$; $j = 1, \ldots, n_i$; $k = 1, \ldots, n_R$:

$$Q_{ijk}^{\tau+1} = \max\{0, Q_{ijk}^\tau + a_\tau(\hat{\rho}_k(Q^\tau, q^\tau) + \sum_{l=1}^{n_R} \frac{\partial \hat{\rho}_l(Q^\tau, q^\tau)}{\partial Q_{ijk}} \sum_{h=1}^{n_i} Q_{ihl}^\tau - \sum_{h=1}^{n_i} \frac{\partial \hat{f}_{ih}(Q^\tau, q^\tau)}{\partial Q_{ijk}}$$

$$- \sum_{h=1}^{n_i} \sum_{l=1}^{n_R} \frac{\partial \hat{c}_{ihl}(Q^\tau, q^\tau)}{\partial Q_{ijk}})\} \tag{3.35}$$

and the following closed form expressions for the quality levels for $i = 1, \ldots, I$; $j = 1, \ldots, n_i$:

$$q_{ij}^{\tau+1} = \max\{\underline{q}_{ij}, q_{ij}^{\tau} + a_\tau(\sum_{k=1}^{n_R} \frac{\partial \hat{\rho}_k(Q^\tau, q^\tau)}{\partial q_{ij}} \sum_{h=1}^{n_i} Q_{ihk}^{\tau} - \sum_{h=1}^{n_i} \frac{\partial \hat{f}_{ih}(Q^\tau, q^\tau)}{\partial q_{ij}}$$

$$- \sum_{h=1}^{n_i} \sum_{k=1}^{n_R} \frac{\partial \hat{c}_{ihk}(Q^\tau, q^\tau)}{\partial q_{ij}})\}. \quad (3.36)$$

Note that an iteration also has the interpretation of a time step. Therefore, (3.35) and (3.36) reveal how firms update their product shipments and quality levels in discrete time.

We now provide the convergence result. The proof follows using similar arguments as those in Theorem 5.8 in Nagurney and Zhang (1996).

Theorem 3.5: Convergence
In the supply chain network model with information asymmetry in quality, let $F(X) = -\nabla U(Q, q)$, where we group all U_i; $i = 1, \ldots, I$, into the vector $U(Q, q)$, be strictly monotone at any equilibrium shipment pattern and quality levels and assume that Assumption 3.1 is satisfied. Furthermore, assume that F is uniformly Lipschitz continuous. Then there exists a unique equilibrium product shipment and quality level pattern $(Q^*, q^*) \in \mathcal{K}^2$, and any sequence generated by the Euler method as given by (3.34) above, with explicit formulae at each iteration given by (3.35) and (3.36), where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$ converges to (Q^*, q^*).

3.6 Numerical Examples

In this section, we present numerical supply chain network examples with information asymmetry in quality, which we solve via the Euler method, as described in Sect. 3.5. We provide a spectrum of examples, accompanied by sensitivity analysis. We implemented the Euler method using Matlab on a Lenovo E46A. The convergence tolerance is 10^{-6}, so that the algorithm is deemed to have converged when the absolute value of the difference between each successive product shipment and quality level is less than or equal to 10^{-6}. The sequence $\{a_\tau\}$ is set to: $0.3\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$. We initialize the algorithm by setting the product shipments equal to 20.00 and the quality levels equal to 0.00.

Example 3.1. This example considers a monopoly case with the supply chain network topology given in Fig. 3.2. There is only one firm, firm 1, which has a single manufacturing plant M_1^1 and serves the demand market R_1. Hence, firm 1 has two strategic (decision) variables: Q_{11} and q_{11}.

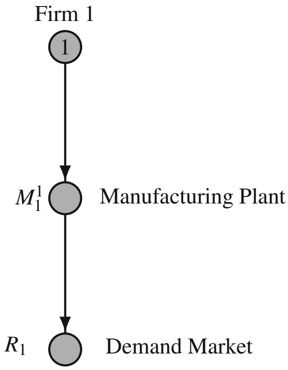

Fig. 3.2 The supply chain network topology for Example 3.1

The data are as follows.

The production cost function at manufacturing plant M_1^1 is:

$$f_{11}(s_{11}, q_{11}) = 0.8s_{11}^2 + 0.5s_{11} + 0.2s_{11}q_{11} + 0.6q_{11}^2,$$

where $s_{11} = Q_{111}$.

The total transportation cost for shipping the product from M_1^1 to the demand market R_1 is given by the function:

$$\hat{c}_{111}(Q_{111}, q_{11}) = 1.2Q_{111}^2 + Q_{111} + 0.8q_{11}^2.$$

The demand price function at demand market R_1 is:

$$\rho_1(d_1, \hat{q}_1) = -d_1 + 0.8\hat{q}_1 + 310,$$

where $d_1 = Q_{111}$, and the average quality expression is given by:

$$\hat{q}_1 = \frac{Q_{111}q_{11}}{Q_{111}} = q_{11}.$$

Note that, since this is a monopoly case, the average quality at the demand market, which represents the perceived quality by the consumers at R_1, is the same as the actual quality. Therefore, there is no information asymmetry in this example.

Also, we have that there is no positive imposed minimum quality standard, so that:

$$\underline{q}_{11} = 0.00.$$

Based on the data, the utility, that is, the profit of firm 1, takes the form:

$$\begin{aligned} U_1 =& \rho_1(d_1, \hat{q}_1) \times Q_{111} - f_{11}(s_{11}, q_{11}) - \hat{c}_{111}(Q_{111}, q_{11}) \\ =& 308.5Q_{111} - 3Q_{111}^2 + 0.6Q_{111}q_{11} - 1.4q_{11}^2. \end{aligned}$$

Theorem 3.1 implies that, if Q^*_{111} and q^*_{11} lie in the interior of the feasible set $\mathscr{K}$, that is, neither is equal to 0.00, then $F(X^*) = -\nabla U(Q^*, q^*) = 0$ must hold. Hence,

$$F(X^*) = -\nabla U(Q^*, q^*) = \begin{cases} -\frac{\partial U_1}{Q_{111}} = -308.5 + 6Q^*_{111} - 0.6q^*_{11} = 0 \\ \\ -\frac{\partial U_1}{q_{11}} = -0.6Q^*_{111} + 2.8q^*_{11} = 0. \end{cases}$$

The solution of this system of equations is:

$$Q^*_{111} = s^*_{11} = 52.54, \quad q^*_{11} = 11.26,$$

which is, indeed, in the interior of $\mathscr{K}$.

The equilibrium demand at the demand market R_1 is $d^*_1 = 52.54$. The incurred demand market price at the equilibrium is $\rho_1 = 266.46$, with the average quality level at R_1, $\hat{q}_1$, being 11.26. The profit of the firm is 8,104.69.

The Jacobian matrix of $F(X) = -\nabla U(Q, q)$, denoted by $J(Q_{111}, q_{11})$, for this problem, is:

$$J(Q_{111}, q_{11}) = \begin{pmatrix} 6.0 & -0.6 \\ -0.6 & 2.8 \end{pmatrix}.$$

Since $J(Q_{111}, q_{11})$ is strictly diagonally dominant, it is positive definite. Therefore, $F(X)$ is strongly monotone (since $F(X)$ is also linear) and the equilibrium solution is unique. The conditions for convergence of the algorithm are satisfied (cf. Theorem 3.5). Moreover, according to Theorem 3.4, the equilibrium solution X^* to this example is globally exponentially stable.

Example 3.2

Example 3.2 is based on Example 3.1, except that there is one more firm, firm 2, entering the market, and its manufacturing plant is located in a different region than firm 1's plant. The supply chain network topology is given in Fig. 3.3. Firm 2 has a single manufacturing plant M^1_2 and serves the same demand market R_1 as does firm 1.

In addition to the data in Example 3.1, the new data are as below:

The production cost function at the new manufacturing plant M^1_2 is:

$$f_{21}(s_{21}, q_{21}) = s^2_{21} + 0.8s_{21} + 0.28s_{21}q_{21} + 0.7q^2_{21},$$

where $s_{21} = Q_{211}$.

Due to congestion in transportation, the total transportation cost functions from the plants to the demand market R_1 become:

$$\hat{c}_{111}(Q_{111}, Q_{211}, q_{11}) = 1.2Q^2_{111} + Q_{111} + 0.25Q_{211} + 0.8q^2_{11},$$

$$\hat{c}_{211}(Q_{211}, Q_{111}, q_{21}) = Q^2_{211} + Q_{211} + 0.35Q_{111} + q^2_{21}.$$

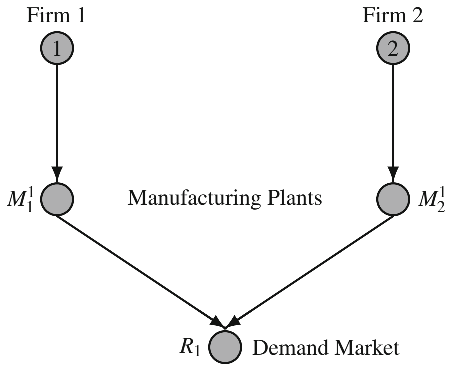

Fig. 3.3 The supply chain network topology for Example 3.2

The average quality expression at R_1 now becomes:

$$\hat{q}_1 = \frac{Q_{111}q_{11} + Q_{211}q_{21}}{Q_{111} + Q_{211}},$$

and the demand is now $d_1 = Q_{111} + Q_{211}$.

In this example, the average quality level at R_1, which is the perceived quality, is different from the actual quality levels q_{11} and q_{21}. Thus, there is now information asymmetry associated with knowledge of the product quality of consumers versus the firms.

Also, there is no positive imposed minimum quality standard on firm 2's plant; hence:

$$\underline{q}_{21} = 0.00.$$

The Euler method converges in 91 iterations and yields the following equilibrium solution. The equilibrium product outputs and shipments are:

$$s_{11}^* = Q_{111}^* = 44.32, \quad s_{21}^* = Q_{211}^* = 44.17,$$

with the equilibrium demand at the demand market being, hence, $d_1^* = 88.49$.

The equilibrium quality levels are:

$$q_{11}^* = 3.18, \quad q_{21}^* = 1.55,$$

with the average quality level at R_1, $\hat{q}_1 = 2.36$.

The incurred demand market price at the equilibrium is:

$$\rho_1 = 223.40.$$

The profits of the firms are, respectively, 5,852.88 and 5,847.45.

Since, in Example 3.2, there is an additional firm that produces and delivers to R_1 as compared to Example 3.1, the total product shipment, which is the demand at R_1, increases. Due to competition, the equilibrium shipment quantity and profit of firm 1 decrease, as compared to the corresponding values in Example 3.1. In addition, as mentioned, there is no information asymmetry in Example 3.1. Because of the quality information asymmetry in Example 3.2, consumers cannot identify the producer of the product and the quality associated with their product. As a result, firms tend to cheat on quality. The quality of firm 1 and the average quality decrease by 71.76 % and by 79.04 %, respectively. The price of the product increases at R_1.

The Jacobian of $F(X) = -\nabla U(Q, q)$, denoted by $J(Q_{111}, Q_{211}, q_{11}, q_{21})$, evaluated at the equilibrium point $X^* = (Q^*_{111}, Q^*_{211}, q^*_{11}, q^*_{21})$ is:

$$J(Q^*_{111}, Q^*_{211}, q^*_{11}, q^*_{21}) = \begin{pmatrix} 5.99 & 1.01 & -0.40 & -0.20 \\ 0.99 & 6.01 & -0.20 & -0.32 \\ -0.40 & 0.20 & 2.80 & 0.00 \\ 0.20 & -0.32 & 0.00 & 3.40 \end{pmatrix}.$$

The eigenvalues of $\frac{1}{2}(J + J^T)$ are: 2.75, 3.36, 5.07, and 7.03, and are all positive. Thus, the matrix is positive definite, and $F(X)$ is locally strictly monotone at X^*. Therefore, the equilibrium solution X^* to this example is a strictly monotone attractor.

A graphical depiction of the iterates for Example 3.2, consisting of the product shipments and the quality levels, is provided, respectively, in Figs. 3.4 and 3.5. Due to the fact that Q^*_{111} and Q^*_{211} are close, the associated trajectories appear to overlap in Fig. 3.4.

Sensitivity Analysis

We conducted sensitivity analysis by varying $\underline{q}_{11}$ and $\underline{q}_{21}$ beginning with their values set at 0 and increasing them to reflect the imposition of minimum quality standards

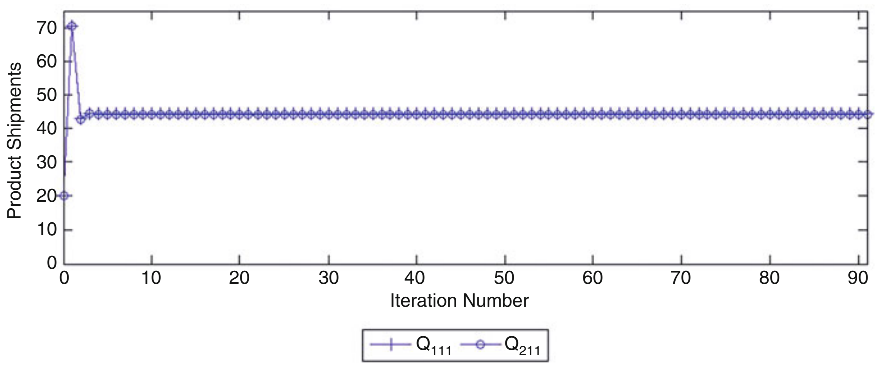

Fig. 3.4 Product shipment trajectories for Example 3.2

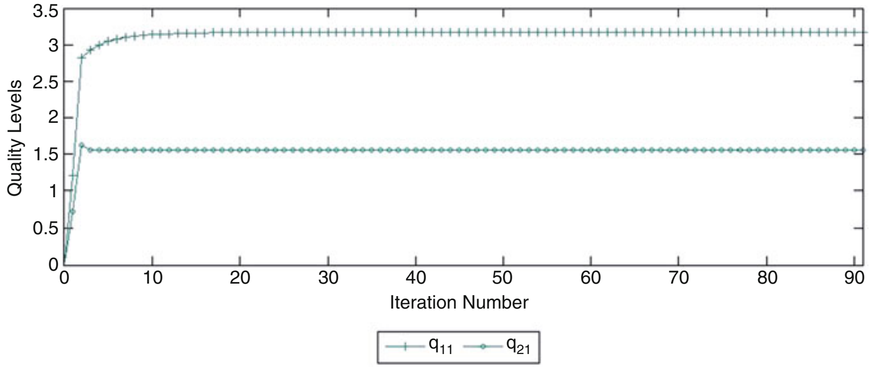

Fig. 3.5 Quality level trajectories for Example 3.2

set to: 20, 40, 60, 80, and, then, 100, with 100 representing "perfect" quality. We display the results of this sensitivity analysis in Figs. 3.6 and 3.7.

A very high minimum quality standard imposed on a firm may lead to a negative profit if the firm still stays in the market. Therefore, when this happens, the firm will leave the market because of the negative profit, and produce nothing for the market. Hence, the actual profit of the firm should be 0 in this case, and there exists no quality level associated with the product of the firm.

As the imposed minimum quality standard of a firm increases, its equilibrium quality level increases (cf. Fig. 3.6c, d), which results in increasing production and transportation costs for the firm.

Therefore, in order to alleviate increasing costs, its equilibrium shipment quantity decreases as does its profit (cf. Fig. 3.7b, c). However, due to competition, its competitor's product shipment increases or at least remains the same (cf. Fig. 3.6a, b).

Moreover, since consumers at the demand market do not differentiate between the products from different firms, and there is information asymmetry in quality between the firms (sellers) and the consumers (buyers) at the demand market, the average quality level at the demand market, as well as the price, which is determined by the quality levels of both firms, is for both firms' products. Firms prefer a higher average quality, since, at the same demand level, a higher average quality results in a higher price of the product. However, once a firm increases its own quality level, of course, the average quality level and, hence, the price increases, but its total cost will also increase due to the higher quality. Furthermore, the price increase is not only for the firm's own product, but also for its competitor's product. If a firm increases its own quality, both the firm and its competitor would get the benefits of the price increase, but only the firm itself would pay for the quality improvement.

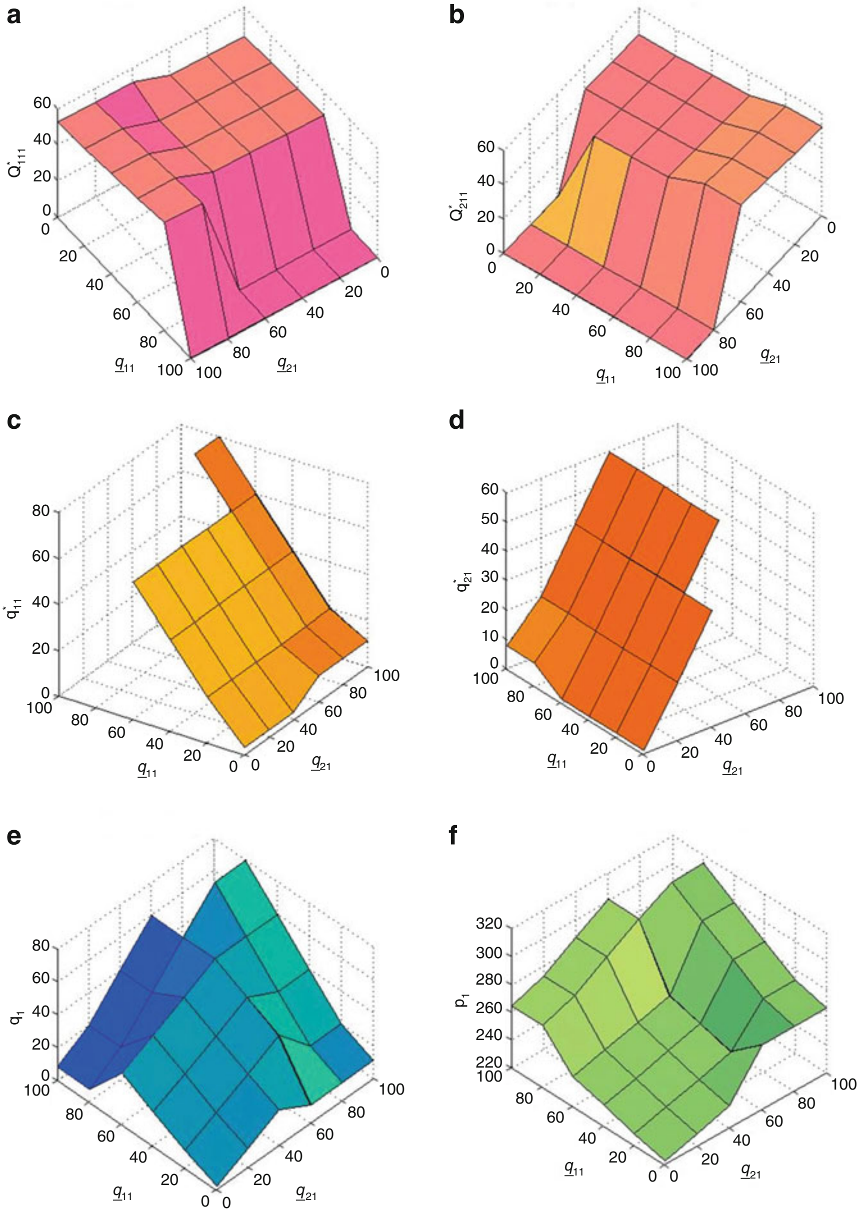

Fig. 3.6 Equilibrium product shipments, equilibrium quality levels, average quality at the demand market, and price at the demand market as the minimum quality standards vary in Example 3.2. (**a**) Equilibrium product shipment of firm 1. (**b**) Equilibrium product shipment of firm 2. (**c**) Equilibrium quality level of firm 1. (**d**) Equilibrium quality level of firm 2. (**e**) Average quality at the demand market. (**f**) Price at the demand market

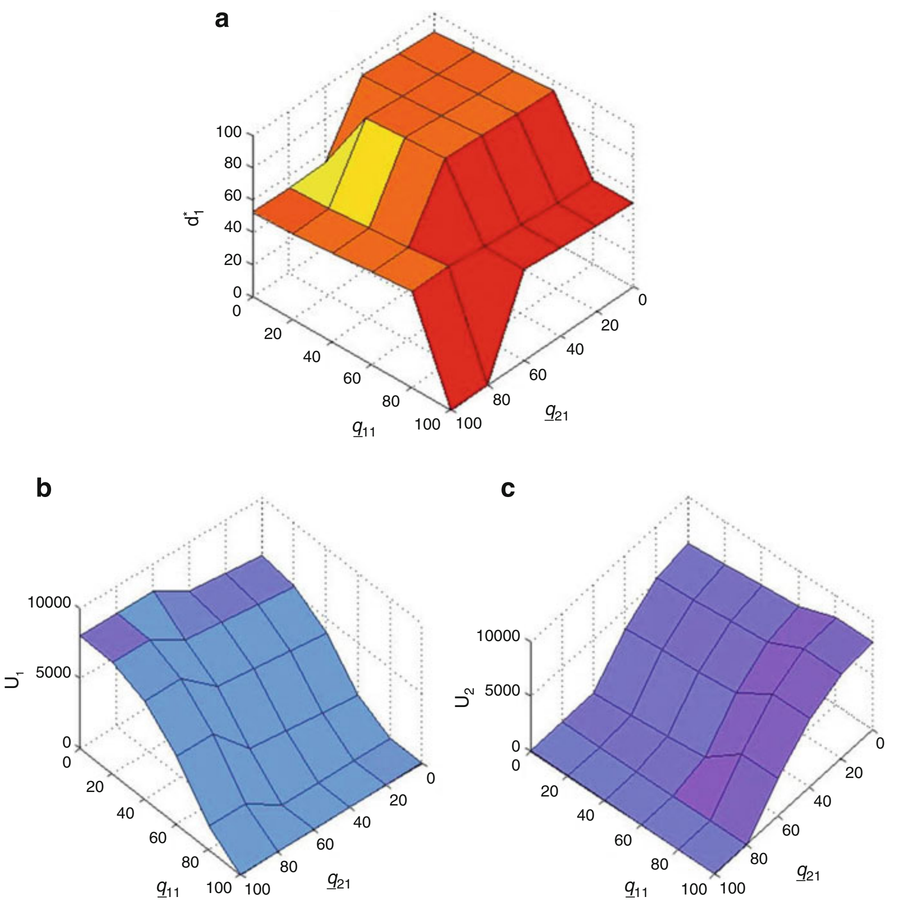

Fig. 3.7 Demand at R_1 and the profits of the firms as the minimum quality standards vary in Example 3.2. (**a**) Demand at the demand market. (**b**) Profit of firm 1. (**c**) Profit of firm 2

Thus, a firm prefers a "free ride," that is, it prefers that the other firm improve its product quality and, hence, the price, rather than have it increase its own quality. Consequently, a firm may not be willing to increase its quality levels, while the other firm is, unless it is beneficial both cost-wise and profit-wise. This explains why, as the minimum quality standard of one firm increases, its competitor's quality level increases slightly or remains the same (cf. Fig. 3.6c, d).

When there is an enforced higher minimum quality standard imposed on a firm's manufacturing facility, the firm is forced to achieve a higher quality level, which may bring its own profit down but raise the competitor's profit (cf. Fig. 3.7b, c), even though the latter firm may actually face a lower minimum quality standard. When the minimum quality standard of a firm increases to a very high value, but that of its competitor is low, the former firm will not be able to afford the high associated cost with decreasing profit, and, hence, it will produce no product for the demand market and will be forced to leave the market. As shown in Fig. 3.7c, when $\underline{q}_{21}$ is 60 and $\underline{q}_{11}$ is 0, because of the big difference between them, firm 2 will leave the market, and, hence, produce nothing and gain a profit of 0 in the market. When $\underline{q}_{21}$ is 80 and greater, firm 2 will still not be able to afford the high quality and still be out of the market. For firm 1, as in Fig. 3.7b, when $\underline{q}_{11}$ is 80 and greater, firm 1 will leave the market. However, when $\underline{q}_{11}$ is 80 but $\underline{q}_{21}$ is 80 or greater, since firm 2 will leave the market, firm 1 will be able to stay in the market and make a profit. When $\underline{q}_{11}$ and $\underline{q}_{21}$ are both 100, both firms will be out of the market.

The above results and discussion indicate the same result, but in a much more general supply chain network context, as found in Ronnen (1991), who, in speaking about minimum quality standards, on page 492, noted that: "low-quality sellers can be better off ... and high-quality sellers are worse off." Also the computational results support the statement on page 490 in Akerlof (1970) that "good cars may be driven out of the market by lemons." Moreover, our results also show that the lower the competitor's quality level, the more harmful the competitor is to the firm with the high minimum quality standard, as shown in Fig. 3.7b, c.

The implications of the sensitivity analysis for policy-makers are clear – the imposition of a one-sided quality standard can have a negative impact on the firm in one's region (or country). Moreover, policy-makers, who are concerned about the products at particular demand markets, should prevent firms located in regions with very low minimum quality standards from entering the market; otherwise, they may not only bring the average quality level at the demand market(s) down and hurt the consumers, but such products may also harm the profits of the other firms with much higher quality levels and even drive them out of the market.

Therefore, it would be beneficial and fair for both firms and consumers if the policy-makers in the same or in different regions or even countries would impose the same or at least similar minimum quality standards on plants serving the same demand market(s). In addition, the minimum quality standards should be such that they will not negatively impact either the high quality firms' survival or the consumers at the demand market(s).

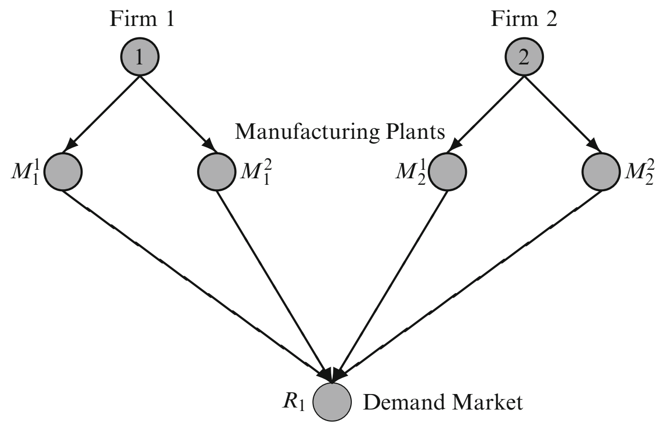

Fig. 3.8 The supply chain network topology for Examples 3.3 and 3.4

Example 3.3

Example 3.3 is constructed from Example 3.2. In Example 3.3, there is an additional manufacturing plant available for each of the two firms, and the new plant for each firm has the same associated data as its original one. This would represent a scenario in which each firm builds an identical plant in proximity to its original one. Thus, the forms of the production cost functions associated with the new plants M_1^2 and M_2^2 and the total transportation cost functions associated with the new links to R_1 are the same as those for their counterparts in Example 3.2 (but depend on the corresponding variables). This example has the topology given in Fig. 3.8.

The production cost functions at the new manufacturing plants M_1^2 and M_2^2 are:

$$f_{12}(s_{12}, q_{12}) = 0.8s_{12}^2 + 0.5s_{12} + 0.2s_{12}q_{12} + 0.6q_{12}^2,$$

$$f_{22}(s_{22}, q_{22}) = s_{22}^2 + 0.8s_{22} + 0.28s_{22}q_{22} + 0.7q_{22}^2.$$

Now $s_{11} = Q_{111}$, $s_{12} = Q_{121}$, $s_{21} = Q_{211}$, and $s_{22} = Q_{221}$.

The total transportation cost functions on the new transportation links are:

$$\hat{c}_{121}(Q_{121}, Q_{221}, q_{12}) = 1.2Q_{121}^2 + Q_{121} + 0.25Q_{221} + 0.8q_{12}^2,$$

$$\hat{c}_{221}(Q_{221}, Q_{121}, q_{22}) = Q_{221}^2 + Q_{221} + 0.35Q_{121} + q_{22}^2.$$

Now, $d_1 = Q_{111} + Q_{121} + Q_{211} + Q_{221}$, and $\hat{q}_1$ is:

$$\hat{q}_1 = \frac{Q_{111}q_{11} + Q_{211}q_{21} + Q_{121}q_{12} + Q_{221}q_{22}}{Q_{111} + Q_{211} + Q_{121} + Q_{211}}.$$

Also, at the new manufacturing plants, as in the original ones:

$$\underline{q}_{12} = \underline{q}_{22} = 0.00.$$

The Euler method converges in 80 iterations to the following equilibrium solution:

$$s_{11}^* = Q_{111}^* = 30.98, \quad s_{12}^* = Q_{121}^* = 30.98,$$

$$s_{21}^* = Q_{211}^* = 30.88, \quad s_{22}^* = Q_{221}^* = 30.88,$$

$$q_{11}^* = 2.22, \quad q_{12}^* = 2.22, \quad q_{21}^* = 1.08, \quad q_{22}^* = 1.08,$$

$$d_1^* = 123.72, \quad \hat{q}_1 = 1.65, \quad \rho_1 = 187.60.$$

Note that the average quality level has dropped from its value of 2.36 in Example 3.2. The profits of the firms are, respectively, 7,635.27 and 7,616.31.

We now discuss the results. Since, for each firm, its new manufacturing plant and the original one are assumed to be identical, the equilibrium product shipments and the quality levels associated with the two plants are identical for each firm.

The availability of an additional manufacturing plant for each firm leads to the following results. First, the total cost of manufacturing and transporting the same amount of products is now less than in Example 3.2 for each firm, which can be verified by substituting $Q_{111} + Q_{121}$ for s_{11} (i.e., Q_{111}) and $Q_{211} + Q_{221}$ for s_{21} (i.e., Q_{211}) in the production and transportation cost functions in Example 3.2, and comparing the total cost of each firm in Example 3.2 with that in Example 3.3. Hence, although the product shipments produced by the same manufacturing plant decrease in comparison to the associated values in Example 3.2, the total amount supplied by each firm increases, as does the total demand. The strategy of building an identical plant at the same location as the original one appears to be cost-wise and profitable for the firms; however, at the expense of a decrease in the average quality level at the demand market, as reflected in the results. Policy-makers may wish to take note of this.

The Jacobian matrix of $F(X) = -\nabla U(Q, q)$ evaluated at X^* for this example, is

$$J(Q_{111}^*, Q_{121}^*, Q_{211}^*, Q_{221}^*, q_{11}^*, q_{12}^*, q_{21}^*, q_{22}^*)$$

$$= \begin{pmatrix} 6.00 & 2.00 & 1.00 & 1.00 & -0.30 & -0.10 & -0.10 & -0.10 \\ 2.00 & 6.00 & 1.00 & 1.00 & -0.10 & -0.30 & -0.10 & -0.10 \\ 1.00 & 1.00 & 6.00 & 2.00 & -0.10 & -0.10 & -0.22 & -0.10 \\ 1.00 & 1.00 & 2.00 & 6.00 & -0.10 & -0.10 & -0.10 & -0.22 \\ -0.30 & -0.10 & 0.10 & 0.10 & 2.80 & 0.00 & 0.00 & 0.00 \\ -0.10 & -0.30 & 0.10 & 0.10 & 0.00 & 2.80 & 0.00 & 0.00 \\ 0.10 & 0.10 & -0.22 & -0.10 & 0.00 & 0.00 & 3.40 & 0.00 \\ 0.10 & 0.10 & -0.10 & -0.22 & 0.00 & 0.00 & 0.00 & 3.40 \end{pmatrix}.$$

The Jacobian matrix is strictly diagonally dominant at X^*. Hence, the equilibrium solution X^* is a strictly monotone attractor.

Example 3.4
Example 3.4 is constructed from Example 3.3, but now the new plant for firm 1, M_1^2, is relocated to a region where the production cost is much lower but the total transportation cost to the demand market R_1 is higher, in comparison to the data in Example 3.3. In addition, the second plant of firm 2, M_2^2, is also relocated, resulting in both a higher production cost and a higher transportation cost to R_1. Thus, the new manufacturing plants for each firm now have different associated cost functions as given below.

The production cost functions of the new plants, M_1^2 and M_2^2, are:

$$f_{12}(s_{12}, q_{12}) = 0.3s_{12}^2 + 0.1s_{12} + 0.2s_{12}q_{12} + 0.4q_{12}^2,$$

$$f_{22}(s_{22}, q_{22}) = 1.2s_{22}^2 + 0.8s_{22} + 0.3s_{22}q_{22} + 0.7q_{22}^2.$$

The total transportation cost functions on the new transportation links are now:

$$\hat{c}_{121}(Q_{121}, Q_{221}, q_{12}) = 1.8Q_{121}^2 + Q_{121} + 0.25Q_{221} + 0.8q_{12}^2,$$

$$\hat{c}_{221}(Q_{221}, Q_{121}, q_{22}) = 1.5Q_{221}^2 + Q_{221} + 0.35Q_{121} + q_{22}^2.$$

The Euler method converges in 122 iterations, yielding the equilibrium solution:

$$s_{11}^* = Q_{111}^* = 31.89, \quad s_{12}^* = Q_{121}^* = 30.48,$$

$$s_{21}^* = Q_{211}^* = 32.97, \quad s_{22}^* = Q_{221}^* = 24.41,$$

$$q_{11}^* = 2.47, \quad q_{12}^* = 2.75, \quad q_{21}^* = 1.00, \quad q_{22}^* = 0.60,$$

$$d_1^* = 119.76, \quad \hat{q}_1 = 1.76, \quad \rho_1 = 191.64.$$

The profits of the firms are, respectively, 7,822.39 and 7,072.89,

Although the production cost of firm 1's plant M_1^2 is lower than that of the original plant, M_1^1, because of the high transportation cost to the demand market, the quantity produced at and shipped from M_1^2 decreases slightly, in comparison to the value in Example 3.3. Moreover, because of the higher manufacturing cost at firm 2's plant, M_2^2, the total supply of the product from firm 2 decreases. The other results are: the demand at demand market R_1 decreases and the average quality increases slightly.

The eigenvalues of the symmetric part of the Jacobian matrix of $F(X)$ with $F(X) = -\nabla U(Q, q)$ at equilibrium are all positive. Thus, the equilibrium solution for Example 3.4 has the same stability property as the solution to Examples 3.2 and 3.3.

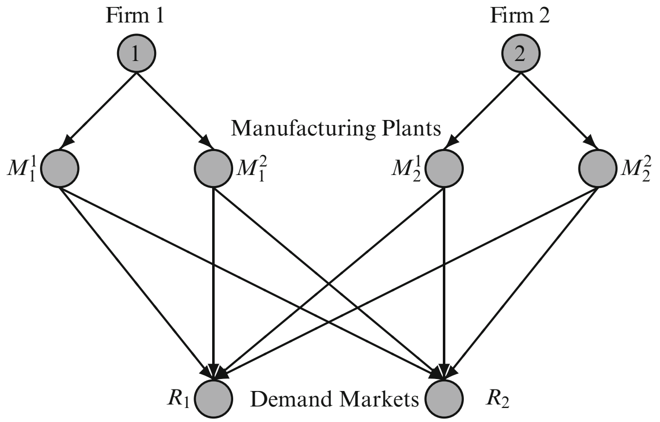

Fig. 3.9 The supply chain network topology for Example 3.5

Example 3.5
Example 3.5 considers the following scenario. Please refer to Fig. 3.9 for the supply chain network topology for this example.

Now, firms 1 and 2 also deliver to demand market R_2, which is located closer to both firms' manufacturing plants than the original demand market R_1. The total transportation cost functions for transporting the product to R_2 for both firms, respectively, are:

$$\hat{c}_{112}(Q_{112}, Q_{212}, q_{11}) = 0.8Q_{112}^2 + Q_{112} + 0.2Q_{212} + 0.05q_{11}^2,$$

$$\hat{c}_{122}(Q_{122}, Q_{222}, q_{12}) = 0.75Q_{122}^2 + Q_{122} + 0.25Q_{222} + 0.03q_{12}^2,$$

$$\hat{c}_{212}(Q_{212}, Q_{112}, q_{21}) = 0.6Q_{212}^2 + Q_{212} + 0.3Q_{112} + 0.02q_{21}^2,$$

$$\hat{c}_{222}(Q_{222}, Q_{122}, q_{22}) = 0.5Q_{222}^2 + 0.8Q_{222} + 0.25Q_{122} + 0.05q_{22}^2.$$

The production cost functions at the manufacturing plants have the same functional forms as in Example 3.3, but now $s_{11} = Q_{111} + Q_{112}$, $s_{12} = Q_{121} + Q_{122}$, $s_{21} = Q_{211} + Q_{212}$, and $s_{22} = Q_{221} + Q_{222}$.

In addition, consumers are now willing to pay more for the product, and consumers at the new demand market R_2 are more sensitive to the quality of the product than consumers at the original demand market R_1. The demand price functions for both the demand markets are now, respectively:

$$\rho_1(d, \hat{q}) = -d_1 + 0.8\hat{q}_1 + 2000, \quad \rho_2(d, \hat{q}) = -d_2 + 0.9\hat{q}_2 + 2000,$$

where $d_1 = Q_{111} + Q_{211} + Q_{121} + Q_{221}$, $d_2 = Q_{112} + Q_{122} + Q_{212} + Q_{222}$,

$$\hat{q}_1 = \frac{Q_{111}q_{11} + Q_{211}q_{21} + Q_{121}q_{12} + Q_{221}q_{22}}{Q_{111} + Q_{211} + Q_{121} + Q_{211}},$$

and

$$\hat{q}_2 = \frac{Q_{112}q_{11} + Q_{212}q_{21} + Q_{122}q_{12} + Q_{222}q_{22}}{Q_{112} + Q_{212} + Q_{122} + Q_{222}}.$$

The Euler method converges in 822 iterations, and the equilibrium solution is as below.

$$Q^*_{111} = 184.81, \quad Q^*_{121} = 188.26, \quad Q^*_{211} = 180.72, \quad Q^*_{221} = 114.90,$$

$$Q^*_{112} = 145.88, \quad Q^*_{122} = 312.76, \quad Q^*_{212} = 161.37, \quad Q^*_{222} = 177.04,$$

$$q^*_{11} = 28.72, \quad q^*_{12} = 50.27, \quad q^*_{21} = 7.45, \quad q^*_{22} = 5.31,$$

$$s^*_{11} = 330.69, \quad s^*_{12} = 501.03, \quad s^*_{21} = 342.09, \quad s^*_{22} = 291.94,$$

$$d^*_1 = 668.96, \quad d^*_2 = 797.05,$$

$$\hat{q}_1 = 25.02, \quad \hat{q}_2 = 27.67, \quad \rho_1 = 1,351.32, \quad \rho_2 = 1,227.85.$$

The profits of the firms are, respectively, 696,264.45 and 509,336.09.

Due to the addition of R_2, which has associated lower transportation costs, each firm ships more product to demand market R_2 than to R_1. Since now consumers are willing to pay much more for the product, firms produce more than before, and the total demand increases significantly.

In addition, firm 1 is the one with larger market shares, and is able to achieve higher profit by attaining higher quality levels. Thus, as the total demand increases significantly, the quality levels of firm 1 increase tremendously. However, since it is not cost-wise for firm 2 to do so, due to its higher costs and lower market shares, firm 2 prefers a "free ride" from firm 1 with its quality levels increasing a little. The average quality levels, nevertheless, increase substantially anyway, which leads to the increase in the prices and both firms' profits.

Since the eigenvalues of the symmetric part of the Jacobian matrix at equilibrium are all positive, the equilibrium solution to Example 3.5 is a strictly monotone attractor.

Sensitivity Analysis

We now explore the impact of the firms' proximity to R_2. We multiply the coefficient of the second Q_{ijk} term, that is, the linear one, in each of the transportation cost functions $\hat{c}_{ijk}$ by a positive factor β, but retain the other transportation cost functions as in Example 3.5. We vary β from 0 to 20, 40, 60, 80, 100, and 120. The results are reported in Fig. 3.10.

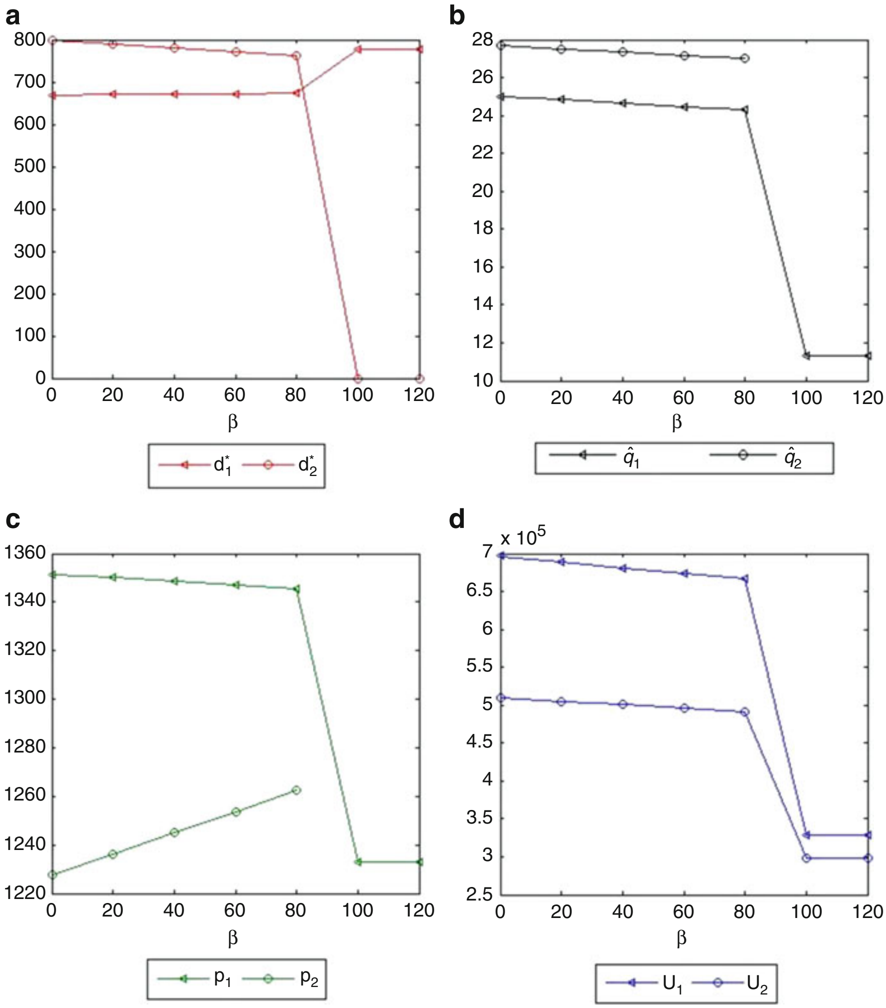

Fig. 3.10 The equilibrium demands, average quality levels, prices at the demand markets, and the profits of the firms as β varies in Example 3.5. (**a**) Equilibrium demands at the demand markets. (**b**) Average quality levels at the demand markets. (**c**) Prices at the demand markets. (**d**) Profit of firms

As β increases, that is, as R_2 is located farther, the transportation costs to R_2 increase. In order to decrease their total costs and increase their profits, firms ship less of the product to R_2 while their shipments to R_1 increase, as shown in Fig. 3.10a. In addition, at the same time, firms cannot maintain the same quality as the total costs of both firms increase, so the average quality levels at both demand markets decrease, as indicated in Fig. 3.10b. Due to the changes in the demand and the average quality levels, the price at R_1 decreases, but that at R_2 increases, and the profits of both firms decrease, as in Fig. 3.10c, d. When $\beta = 100$, firms will no longer ship to demand market R_2, since it is too distant to make a profit, and, hence, demand market R_2 will be removed from the supply chain network.

The numerical examples in this section, along with the sensitivity analysis results, reveal the type of questions that can be explored and addressed through computations. Moreover, the analyses demonstrate the impacts of minimum quality standards even "across borders" as well as the importance of the location of manufacturing plants vis á vis the demand markets. The insights gained from the numerical examples are useful to firms, to consumers at demand markets, as well as to policy-makers.

3.7 Summary and Conclusions

In this chapter, we developed a rigorous framework for the modeling, analysis, and computation of solutions to oligopolistic supply chain network problems in static and dynamic settings in which there is information asymmetry in quality. We also demonstrated how our framework can capture the inclusion of policy interventions in the form of minimum quality standards. Importantly, through the sensitivity analysis results, we show that policy-makers should be careful in imposing minimum quality standards unilaterally since certain firms may suffer profit-wise.

This research adds to the literature on information asymmetry with imperfect competition, which has only recently attracted attention, and which has focused primarily on analytical results for stylized problems. This chapter also contributes to the literature on supply chains with quality competition and reveals the spectrum of insights that can be obtained through computations, supported by theoretical analysis. Finally, it contributes to the integration of economics with operations research and the management sciences.

Specifically, the novelty of the framework described in this chapter consists of the following: (1) Static (equilibrium) and dynamic versions of supply chain network competition in an oligopolistic manner (cf. Tirole 1988) are captured under information asymmetry in quality with and without minimum quality standards using, respectively, variational inequality theory and projected dynamical systems theory. (2) Firms have, as their strategies, the product shipment amounts produced at their manufacturing plants and the product quality levels. The firms know the quality of the products produced at their plants but consumers at the demand markets are only aware of the average quality since the consumers cannot distinguish among the producers. (3) Quality is associated not only with the manufacturing plants but also tracked through the transportation process, which is assumed to preserve (at the appropriate cost) the product quality. as it is transported from the manufacturing plants to the demand markets. (4) No specific functional forms are imposed on the production cost, transportation cost, and demand price functions and we do not limit the supply chain to only one or two manufacturers, manufacturing plants, or demand markets. (5) Theoretical results, in the form of existence and uniqueness results, and stability analysis, and an effective and efficient algorithmic scheme are provided with convergence results. We also provide solutions to numerical supply chain network examples, followed by sensitivity analyses, in order to demonstrate the generality and usefulness of the models for firms, for consumers, and for policy-makers.

3.8 Sources and Notes

This chapter is based on the paper by Nagurney and Li (2014) but with entirely new numerical examples provided in Sect. 3.6, which further support the managerial and policy-making insights obtained in that paper with respect to impacts of the setting of minimum quality standards. In addition, we demonstrate that not only lower bounds in the form of minimum quality standards can be imposed on the quality levels and incorporated into a variational inequality formulation but that upper bounds can be, as well. For example, an upper bound can correspond to perfect quality, if achievable, or assume a lower value, if perfect quality is not realizable.

Aspects of information asymmetry in the supply chain literature have addressed the value, effects, and/or incentives of information and information sharing (see Corbett et al. 2004; Mishra et al. 2009; Thomas et al. 2009; Ren et al. 2010; Esmaeili and Zeephongsekul 2010). Chen (2003) provides a thorough review of the early literature on information sharing to that date.

As noted in Nagurney and Li (2014), there is also a significant literature on information asymmetry that focuses on supply chain contracting problems with an early review of the literature by Cachon (2003). Hasija et al. (2008) study contracts for a call center outsourcing problem with information asymmetry in worker productivity. Xu et al. (2010) investigate a contract setting problem of a manufacturer who has a single prime supplier and a single urgent supplier. Lee and Yang (2013) examine supply chain contracting problems involving a retailer and two suppliers. Examples of recent quantity discount contracting models with information asymmetry are given in Burnetas et al. (2007), and Zhou (2007). All of the above models, except where noted, are based on two entity supply chain "networks," and the asymmetric information considered is primarily in terms of demand and cost.

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Chapter 4
Information Asymmetry in Perfectly Competitive Spatial Price Equilibrium Problems

Abstract In this chapter, we continue our investigations into information asymmetry and product quality. We now focus on perfectly competitive markets and present a spatial price equilibrium model with information asymmetry in quality in both static and dynamic versions. Producers at the supply markets know the quality of their products, whereas consumers, at the demand markets, are aware only of the average quality of the products that are shipped to their demand markets. Minimum quality standards are included in order to be able to evaluate the impacts of this policy instrument. We provide qualitative results, in the form of existence, uniqueness, and stability analysis. An algorithm is proposed, accompanied by a convergence proof. The algorithm is applied to compute solutions to a spectrum of spatial price equilibrium numerical examples in order to explore the impacts of information asymmetry under different scenarios. The numerical examples reveal that, as the number of supply markets increases, the "anonymizing" effect leads to a decrease in the average quality. In contrast, as the number of demand markets increases, the pressure to improve quality increases, and the average quality increases. Finally, the results reveal that, after minimum quality standards are imposed, the average quality at the demand markets increases and the prices also increase.

4.1 Introduction

Consumers in the Network Economy have come to expect fresh produce in all seasons, fuel and energy to power their vehicles, homes, and equipment, upon demand, pharmaceutical products when needed, and clothing and high technology, when desired. At the same time, despite the immense distances that may be involved from production locations to ultimate consumption locations, quality is what consumers seek in food that they eat, clothes that they wear, high tech products that they use and depend on, in addition to numerous products such as cars that they drive and even planes that they fly in, plus, of course, medicines, when needed. Indeed, as we are emphasizing in this book, quality is emerging as an important characteristic in numerous products, ranging from food (see, e.g., Marsden 2004; Trienekens and Zuurbier 2008) to pharmaceuticals (see Masoumi et al. 2012 and Bennett and Yin 2013) to durable manufactured products such as

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automobiles (see Shank and Govindarajan 1994) to high tech products, including microprocessors (see Goettler and Gordon 2011), and even services associated with the Internet (cf. Kruse 2010 and Nagurney et al. 2013).

As emphasized in Chap. 3, given the great distances that may separate supply markets from demand markets, there may exist information asymmetry when it comes to the quality of certain products, especially those that are considered to be, more or less, homogeneous and are not differentiated by their brands. Recent shortcomings in product quality, which have even resulted in illnesses as well as in deaths, have drawn increasing attention to information asymmetry. Notable examples of serious quality product shortcomings around the world have ranged from the adulteration of milk and infant formula in China (see Yang et al. 2009) to the heparin adulteration, also in China (cf. Cox 2009), which led to a pharmaceutical identity crisis, to substandard medicines in developing countries (see Bate and Boateng 2007 and Gaudiano et al. 2007) to food-borne illnesses in the US and in Europe (see, respectively, Jaslow (2013) and European Food Safety Authority and European Centre for Disease Prevention and Control (2013)).

As also noted in Chap. 3, where the focus was on imperfect competition in the form of oligopolistic supply chain networks, markets with asymmetric information in terms of product quality have been studied by many notable economists, including the 2001 Nobel laureates Akerlof (1970), Spence (1973, 1975), and Stiglitz (1987). Leland (1979) further argued that such markets may benefit from minimum quality standards. However, information asymmetry in a spatial context has been less explored research-wise.

In this chapter, we turn to perfectly competitive markets and present a spatial price equilibrium model with information asymmetry in quality in that the producers at the supply markets are aware of their product quality whereas consumers at the demand markets are only aware of the average quality of the products. Examples of products that can be modeled this way include: fresh produce, oil and fuel, rice and grains, wood, generic medicines, and other non-branded products and commodities, in which there are many producers.

Our framework builds on spatial price equilibrium modeling dating to the classical work of Samuelson (1952) and Takayama and Judge (1964, 1971), but uses a variational inequality approach to include the critical quality dimension and an expanded set of equilibrium conditions followed by projected dynamical systems theory to describe the dynamic adjustment processes and the associated stability analysis. The framework is applicable to agricultural industries (cf. Thompson 1989), such as eggs (cf. Judge 1956), potatoes (Howard 1984), beef (Sohn 1970), cereal grains (Ruijs et al. 2001), soybeans (Barraza De La Cruz et al. 2010), and dairy (Bishop et al. 1994). Moreover, spatial price equilibrium models have been applied to the forestry sector (Hieu and Harrison 2011). Such perfectly competitive models are also relevant to the mineral ore and energy industries (see Hwang et al. 1994; Labys and Yang 1991, and Labys 1999), in particular, to the coal (Newcomb and Fan 1980), aluminum (Newcomb et al. 1990), and natural gas (Irwin and Yang 1996) sectors. Spatial price equilibrium models have also been developed for predator-prey networks and food webs (see Nagurey and Nagurney 2011).

As we did for the oligopolistic supply chain network model in Chap. 3, here we also incorporate minimum quality standards, which are valuable policy tools to protect consumers. Minimum quality standards are especially relevant to industries in which spatial price equilibrium models are applicable, notably, agricultural products (including food), as well as energy and mineral ores. Indeed, as noted in Metzger (1988), page 1: *minimum quality standards have been applied to such diverse items as medical drugs, auto safety and fuel efficiency features, electrical appliances, foods, clothing, and cosmetics.*

This chapter is organized as follows. In Sect. 4.2, we develop the spatial price equilibrium model with information asymmetry in quality and then extend it to include minimum quality standards at the supply markets. We present a unified variational inequality for the integration of the two models. The projected dynamical systems model is, subsequently, constructed. It captures the dynamic adjustment processes for the evolution of the product shipments and quality levels over time until the equilibrium point, equivalently, stationary point, is achieved. In Sect. 4.3, we present qualitative properties of the variational inequality formulation of the integrated model in product shipments and quality levels, in the form of existence and uniqueness results. We then provide stability analysis results for the projected dynamical system. Notably, the set of stationary points of the latter corresponds to the set of solutions to the former.

In Sect. 4.4, we present the algorithm, accompanied by convergence results. The algorithm yields closed form expressions, at each iteration, for the product shipments and the quality levels. The algorithm tracks the dynamic trajectories and approximates them until an equilibrium is achieved. In Sect. 4.5, we illustrate the model and the computational scheme through several numerical examples in order to also gain insights into the impacts of information asymmetry in quality in a spatial context. In Sect. 4.6, we summarize our results and present our conclusions. We conclude this chapter with Sources and Notes in Sect. 4.7.

4.2 Spatial Price Equilibrium with Asymmetric Information in Quality

We first develop the spatial price equilibrium model with asymmetric information in quality and derive the variational inequality formulation. We then demonstrate how the model can be modified to include policies in the form of minimum quality standards and present a unified version integrating both the models. We also describe the underlying dynamics associated with the product shipments and the quality levels and present the projected dynamical systems model whose set of stationary points corresponds to the set of solutions of the variational inequality problem governing the integrated model.

Figure 4.1 depicts the underlying network structure of the spatial price equilibrium problem. We focus, here, on the bipartite problem, for clarity and definiteness.

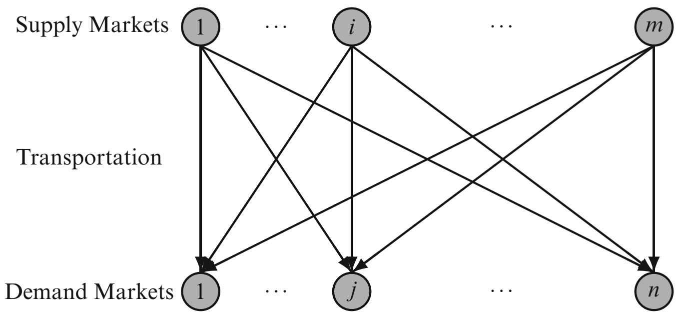

Fig. 4.1 The bipartite network structure of the spatial price equilibrium problem

Table 4.1 Notation for the spatial price equilibrium models (static and dynamic) with information asymmetry

Notation	Definition
Q_{ij}	The nonnegative shipment of the product from supply market i to demand market j. We group the $\{Q_{ij}\}$ elements for all i and j into the vector $Q \in R^{mn}_{+}$
s_i	The nonnegative product output (supply) produced at supply market i. We group the $\{s_i\}$ elements for all i into the vector $s \in R^m_+$
q_i	The quality level, or, simply, the quality, of product i, which is produced at supply market i. We group the $\{q_i\}$ elements for all i into the vector $q \in R^m_+$
d_j	The demand for the product at demand market j. We group the demands for all j into the vector $d \in R^n_+$
$\hat{q}_j$	The average quality level at demand market j as perceived by consumers. We group the average quality levels at all demand markets j into the vector $\hat{q} \in R^n_+$. The average quality level at j, $\hat{q}_j = \frac{\sum_{i=1}^m q_i Q_{ij}}{\sum_{i=1}^m Q_{ij}}$
$\pi_i(s, q_i)$	The supply price of the product at supply market i
$OC_i(q_i)$	The opportunity cost associated with the product produced at supply market i
$c_{ij}(Q)$	The unit transportation cost associated with shipping the product produced at supply market i to demand market j, assuming quality preservation
$\rho_j(d, \hat{q})$	The demand price at demand market j

As depicted in Fig. 4.1, there are m supply markets and n demand markets that are, generally, spatially separated. The product that is produced and that is consumed is homogeneous in that the consumers at the demand markets do not differentiate by point of origin. Associated with the supply markets are supply price functions and with the demand markets – demand price functions. The product is produced at the supply markets and then transported to the demand markets where it is consumed. Associated with each link joining a pair of supply and demand markets is a unit transportation cost. Table 4.1 contains the notation for the models in this chapter.

4.2.1 The Equilibrium Model

We first develop the equilibrium model. The following conservation of flow equations must hold:

$$s_i = \sum_{j=1}^{n} Q_{ij}, \quad i = 1, \ldots, m; \tag{4.1}$$

$$d_j = \sum_{i=1}^{m} Q_{ij}, \quad j = 1, \ldots, n; \tag{4.2}$$

$$Q_{ij} \geq 0, \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{4.3}$$

and

$$q_i \geq 0, \quad i = 1, \ldots, m. \tag{4.4}$$

According to (4.1), the supply of the product at each supply market is equal to the sum of the amounts of the product shipped to all the demand markets, and, according to (4.2), the quantity of the product consumed at a demand market is equal to the sum of the amounts of the product shipped from all the supply markets to that demand market. Both the shipment quantities and the quality levels must be nonnegative, as in (4.3) and (4.4), respectively. We define the feasible set $\mathscr{K}^1 \equiv \{(s, d, Q, q) | \text{ (4.1), (4.2), (4.3) and (4.4) holds}\}$.

The supply price function for each supply market i is assumed to be monotonically increasing in the s_i and q_i variables. Each demand price function is, typically, monotonically decreasing in demand at its demand market but increasing in the average quality. Akerlof (1970) used (but in an aspatial context) demand functions that were a function of the price and the average quality. The information asymmetry in our models is similar to that in Akerlof (1970) and Leland (1979), in that the producers at the supply markets are aware of the quality of their product, as expressed by the supply price functions, but consumers at the demand markets are aware only of the average quality, as expressed by the demand price functions. Our model captures the crucial spatial dimension, which is especially relevant under globalization. The unit transportation cost between each pair of supply and demand markets can include, as relevant, any unit transaction cost. Examples of transaction costs may be tariffs and taxes. We assume that the transportation cost functions are monotonically increasing.

Average quality, determined via the expression given in Table 4.1, is a valid statistic for quality since in our perfectly competitive framework there are many producers/sellers in each supply market and they are price-takers. Quality, hence, is not directly observable in our model by the buyers/consumers at the demand markets but consumers can estimate the quality of the product by the average quality of the product in the demand markets (cf. Akerlof 1970; Stiglitz 1987; Metzger

1988; Barbieri 2013; Nagurney and Li 2014). Such an assumption was also made in Chap. 3 in the case of imperfectly competitive supply chain networks. Price information also implies a certain level of quality. Moreover, average quality can be conveyed among consumers through word of mouth, their own consumption experiences, advertising, etc.

We assume that there is a positive demand at each demand market; otherwise, we remove that demand node from the bipartite network in Fig. 4.1. Hence, the denominator in the average quality expression for each demand market is positive for each demand market (cf. Table 4.1).

It is also assumed that an increase in consumers' perception of average quality increases the benefits of the product to them. Therefore, consumers are willing to pay more for a product with a higher average quality, and, hence, at the demand markets, prices increase as the average quality increases. Kaya and Özer (2009) and Kaya (2011) used demand functions in price and quality variables, which were increasing functions of quality, and of the form: $q = a - bp + e + \varepsilon$, where q is the demand and e is the quality level. Xie et al. (2011), in turn, utilized a demand price function of the form $D = a + \alpha x - \beta p$, which extended the function used by Banker et al. (1998). Anderson and Palma (2001) captured the utility of each consumer u expressed as $u = q - p + \varepsilon$ in their research on asymmetric oligopolies.

We do not limit ourselves, however, to linear demand price functions. Moreover, we allow the demand price of a product to depend not only on its demand but also on those of the other products: the same holds for the dependence of the prices on the product average quality levels. According to an interview with the Nobel laureate Michael Spence (see Shah 2012), with informational asymmetry significant quality differences between the supply and demand sides are manifested. Moreover, *product differentiation disappears, prices will reflect the average quality rather than differential quality*, as revealed through our demand price functions.

The supply price, demand price, unit transportation cost, and opportunity cost functions are assumed to be continuous. Opportunity costs are also used in imperfectly competitive supply chain network models in Chaps. 7, 8, and 10. Leland (1979), building on the work of Akerlof (1970) in quality, utilized opportunity cost functions and related them to the supply price functions at the markets. This we also do in our extension of the supply price equilibrium conditions to include quality, under asymmetric information.

We are now ready to state the spatial price equilibrium conditions, which are a generalization of the well-known spatial price equilibrium conditions of Samuelson (1952) and Takayama and Judge (1971) (see also Nagurney 1999), to include quality.

Definition 4.1: Spatial Price Equilibrium Conditions with Information Asymmetry in Quality
We say that a supply, product shipment, demand, and quality pattern $(s^*, Q^*, d^*, q^*) \in \mathscr{K}^1$ *is a spatial equilibrium with information asymmetry in quality if it satisfies the following conditions: for each pair of supply and demand markets* (i, j)*;* $i = 1, \ldots, m; j = 1, \ldots, n$*:*

$$\pi_i(s^*, q_i^*) + c_{ij}(Q^*) \begin{cases} = \rho_j(d^*, \hat{q}^*), & \text{if } Q_{ij}^* > 0, \\ \geq \rho_j(d^*, \hat{q}^*), & \text{if } Q_{ij}^* = 0, \end{cases} \tag{4.5}$$

and for each supply market i; $i = 1, \ldots, m$*:*

$$OC_i(q_i^*) \begin{cases} = \pi_i(s^*, q_i^*), & \text{if } q_i^* > 0, \\ \geq \pi_i(s^*, q_i^*), & \text{if } q_i^* = 0. \end{cases} \tag{4.6}$$

According to (4.5), there is a positive quantity of the product shipped from a supply market to a demand market, in equilibrium, if the supply price at the originating supply market plus the unit transportation cost is equal to the demand price at the demand market. If the supply price plus the unit transportation cost exceeds that demand price, then there will be no trade of the product between the pair of supply and demand markets for that product. According to (4.6), the equilibrium quality of the product produced at a supply market is positive if the opportunity cost at the supply market is equal to the supply price. If the opportunity cost at the supply market exceeds the supply price at that market, then the equilibrium quality at that supply market will be zero.

We now establish the variational inequality formulation of the above spatial price equilibrium conditions.

Theorem 4.1: Variational Inequality Formulation of Spatial Price Equilibrium with Information Asymmetry in Quality
A supply, product shipment, demand, and quality pattern $(s^*, Q^*, d^*, q^*) \in \mathscr{K}^1$ *is a spatial price equilibrium with information asymmetry in quality according to Definition 4.1 if and only if it satisfies the variational inequality problem:*

$$\sum_{i=1}^{m} \pi_i(s^*, q_i^*) \times (s_i - s_i^*) + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}(Q^*) \times (Q_{ij} - Q_{ij}^*) - \sum_{j=1}^{n} \rho_j(d^*, \hat{q}^*) \times (d_j - d_j^*)$$
$$+ \sum_{i=1}^{m} (OC_i(q_i^*) - \pi_i(s^*, q_i^*)) \times (q_i - q_i^*) \geq 0, \quad \forall (s, Q, d, q) \in \mathscr{K}^1. \tag{4.7}$$

Proof: We first establish necessity, that is, if $(s^*, Q^*, d^*, q^*) \in \mathscr{K}^1$ satisfies the spatial price equilibrium conditions according to Definition 4.1, then it also satisfies variational inequality (4.7).

Note that, for a fixed pair of supply and demand markets (i, j), (4.5) implies that

$$(\pi_i(s^*, q_i^*) + c_{ij}(Q^*) - \rho_j(d^*, \hat{q}^*)) \times (Q_{ij} - Q_{ij}^*) \geq 0, \quad \forall Q_{ij} \geq 0. \tag{4.8}$$

Indeed, since, if $Q_{ij}^* > 0$, we know, from the equilibrium conditions, that the expression to the left of the multiplication sign in (4.8) will be identically zero, so (4.8) holds true; also, if $Q_{ij}^* = 0$, then the expression preceding and following the multiplication sign in (4.8) will be nonnegative and, hence, the product is also nonnegative and (4.8) holds true for this case, as well. Summing (4.8) over all supply markets i and over all demand markets j, we obtain:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\pi_i(s^*, q_i^*) + c_{ij}(Q^*) - \rho_j(d^*, \hat{q}^*)) \times (Q_{ij} - Q_{ij}^*) \geq 0, \quad \forall Q \in R_+^{mn}. \tag{4.9}$$

Rewriting (4.9) as:

$$\sum_{i=1}^{m}\pi_i(s^*, q_i^*) \times \sum_{j=1}^{n}(Q_{ij} - Q_{ij}^*) + \sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}(Q^*) \times (Q_{ij} - Q_{ij}^*)$$
$$-\sum_{j=1}^{n}\rho_j(d^*, \hat{q}^*) \times (\sum_{i=1}^{m}Q_{ij} - \sum_{i=1}^{m}Q_{ij}^*) \geq 0, \tag{4.10}$$

and then simplifying (4.10) by using the supply and demand conservation of flow equations (4.1) and (4.2) yields:

$$\sum_{i=1}^{m}\pi_i(s^*, q_i^*) \times (s_i - s_i^*) + \sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}(Q^*) \times (Q_{ij} - Q_{ij}^*) - \sum_{j=1}^{n}\rho_j(d^*, \hat{q}^*) \times (d_j - d_j^*) \geq 0. \tag{4.11}$$

Analogously, it follows that if q^* satisfies (4.6), then:

$$(OC_i(q_i^*) - \pi_i(s^*, q_i^*)) \times (q_i - q_i^*) \geq 0, \quad \forall q_i \geq 0. \tag{4.12}$$

Summing (4.12) over all i, we obtain

$$\sum_{i=1}^{m}(OC_i(q_i^*) - \pi_i(s^*, q_i^*)) \times (q_i - q_i^*) \geq 0, \quad \forall q \in R_+^m. \tag{4.13}$$

Combining now (4.11) and (4.13), yields variational inequality (4.7).

We now turn to establishing sufficiency, that is, if $(s^*, Q^*, d^*, q^*) \in \mathcal{K}^1$ satisfies variational inequality (4.7) then it also satisfies the spatial price equilibrium conditions (4.5) and (4.6).

We first expand variational inequality (4.7), with the use of the conservation of flow equations (4.1) and (4.2), to obtain:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\pi_i(s^*, q_i^*) + c_{ij}(Q^*) - \rho_j(d^*, \hat{q}^*)) \times (Q_{ij} - Q_{ij}^*) \geq 0, \quad \forall Q \in R_+^{mn}. \tag{4.14}$$

Let $q_i = q_i^*$, $\forall i$, and $Q_{ij} = Q_{ij}^*$, $\forall (i,j) \neq (k,l)$, and substitute into (4.14). The resultant is:

$$(\pi_k(s^*, q_k^*) - c_{kl}(Q^*) - \rho_l(d^*, \hat{q}^*)) \times (Q_{kl} - Q_{kl}^*) \geq 0, \quad \forall Q_{kl} \geq 0. \tag{4.15}$$

But (4.15) implies that, if $Q_{kl}^* = 0$, then $(\pi_k(s^*, q_k^*) - c_{kl}(Q^*) - \rho_l(d^*, \hat{q}^*)) \geq 0$, and, if $Q_{kl}^* > 0$, then, for (4.15) to hold, $(\pi_k(s^*, q_k^*) - c_{kl}(Q^*) - \rho_l(d^*, \hat{q}^*)) = 0$. But since these results hold for any pair (k, l), we can conclude that the equilibrium conditions (4.5) are satisfied by the product shipment pattern satisfying (4.7).

Similarly, we now let $Q_{ij} = Q_{ij}^*$, $\forall (i,j)$, and $q_i = q_i^*$, $\forall i \neq k$. Substitution into (4.7) yields:

$$(OC_k(q_k^*) - \pi_k(s^*, q_k^*)) \times (q_k - q_k^*) \geq 0, \quad \forall q_k \geq 0. \tag{4.16}$$

According to (4.16), if $q_k^* = 0$, then $(OC_k(q_k^*) - \pi_k(s^*, q_k^*)) \geq 0$, since $q_k \geq 0$, and, if $q_k^* > 0$, then $(OC_k(q_k^*) - \pi_k(s^*, q_k^*)) = 0$, since $(q_k - q_k^*)$ can be positive, negative, or zero. Since these results hold for any supply market k, we know that $q^* \in R_+^m$ satisfying variational inequality (4.7) also satisfies equilibrium conditions (4.6). The proof is complete. □

We now demonstrate how policy interventions in the form of minimum quality standards will change the above equilibrium conditions. Specifically, assume that a regulator (or regulators, since the supply markets may be in different regions or even nations) imposes minimum quality standards at each supply market, denoted by $\underline{q}_i$, with $\underline{q}_i$ positive for $i = 1, \ldots, m$, so that, now, instead of (4.4) being satisfied, we must have that

$$q_i \geq \underline{q}_i; \quad i = 1, \ldots, m. \tag{4.17}$$

We define a new feasible set $\mathcal{K}^2 \equiv \{(s, Q, d, q) |$ (4.1), (4.2), (4.3) and (4.17) hold$\}$.

Definition 4.2: Spatial Price Equilibrium with Information Asymmetry in Quality and Minimum Quality Standards
In the case of minimum positive quality standards, then $(s^*, Q^*, d^*, q^*) \in \mathcal{K}^2$ is a spatial price equilibrium if (4.5) holds and (4.6) is modified to: for each supply market i; $i = 1, \ldots, m$:

$$OC_i(q_i^*) \begin{cases} = \pi_i(s^*, q_i^*), \; \textit{if} \; q_i^* > \underline{q}_i, \\ \geq \pi_i(s^*, q_i^*), \; \textit{if} \; q_i^* = \underline{q}_i. \end{cases} \tag{4.18}$$

The variational inequality formulation corresponding to an equilibrium satisfying Definition 4.2 has the same structure as variational inequality (4.7) but the feasible set is now $\mathcal{K}^2$ rather than $\mathcal{K}^1$. The proof follows using similar arguments as those used in the proof of Theorem 4.1. Hence, we have the following

Theorem 4.2: Variational Inequality Formulation of Spatial Price Equilibrium with Information Asymmetry in Quality and Minimum Quality Standards
A supply, product shipment, demand, and quality pattern $(s^*, Q^*, d^*, q^*) \in \mathscr{K}^2$ *is a spatial price equilibrium with information asymmetry in quality and minimum quality standards according to Definition 4.2 if and only if it satisfies the variational inequality problem:*

$$\sum_{i=1}^{m} \pi_i(s^*, q_i^*) \times (s_i - s_i^*) + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}(Q^*) \times (Q_{ij} - Q_{ij}^*) - \sum_{j=1}^{n} \rho_j(d^*, \hat{q}^*) \times (d_j - d_j^*)$$
$$+ \sum_{i=1}^{m}(OC_i(q_i^*) - \pi_i(s^*, q_i^*)) \times (q_i - q_i^*) \geq 0, \quad \forall (s, Q, d, q) \in \mathscr{K}^2. \tag{4.19}$$

We now provide an alternative variational inequality to that of (4.7) and of (4.19) in which the variables are product shipment and quality levels only. Such a formulation will allow for a more transparent identification of the evolution of the product shipments and quality levels over time via a projected dynamical system (PDS). In addition, we will utilize such a PDS to construct a computational procedure, which will provide us with a discretization of the continuous-time adjustment processes provided by the PDS.

Specifically, we first define supply price functions and demand price functions, denoted, respectively, by $\hat{\pi}_i(Q, q_i)$ for $i = 1, \ldots, m$, and by $\hat{\rho}_j(Q, q)$ for $j = 1, \ldots, n$, that are functions of product shipments and quality levels exclusively. This can be done because of constraints (4.1) and (4.2). Hence, we have:

$$\hat{\pi}_i = \hat{\pi}_i(Q, q_i) \equiv \pi_i(s, q_i), \quad i = 1, \ldots, m \tag{4.20}$$

and

$$\hat{\rho}_j = \hat{\rho}_j(Q, q) \equiv \rho_j(d, \hat{q}), \quad j = 1, \ldots, n. \tag{4.21}$$

Also, we can unify the equilibrium conditions (4.6) and (4.18) by redefining $\underline{q}_i$; $i = 1, \ldots, m$, as taking on nonnegative values so that, if $\underline{q}_i$ is equal to zero, then there is no minimum standard at supply market i, and, if $\underline{q}_i$ is positive, then there is. Thus, we now have

$$q_i \geq \underline{q}_i, \quad i = 1, \ldots, m. \tag{4.22}$$

Of course, a minimum quality standard may also be decided upon by the producers at a supply market and may not need to be imposed by a regulatory authority.

We define the feasible set $\mathscr{K}^3 \equiv \{(Q, q) | Q \in R_+^{mn}$ and (4.22) holds$\}$.

The following equilibrium conditions now integrate those in Definitions 4.1 and 4.2.

Definition 4.3: Integrated Spatial Price Equilibrium with Information Asymmetry in Quality
We say that a product shipment and quality pattern $(Q^*, q^*) \in \mathcal{K}^3$ is an integrated spatial equilibrium with information asymmetry in quality if it satisfies the following conditions: for each pair of supply and demand markets (i,j); $i = 1, \ldots, m$; $j = 1, \ldots, n$:

$$\hat{\pi}_i(Q^*, q_i^*) + c_{ij}(Q^*) \begin{cases} = \hat{\rho}_j(Q^*, q^*), & \text{if } Q_{ij}^* > 0, \\ \geq \hat{\rho}_j(Q^*, q^*), & \text{if } Q_{ij}^* = 0, \end{cases} \tag{4.23}$$

and for each supply market i; $i = 1, \ldots, m$:

$$OC_i(q_i^*) \begin{cases} = \hat{\pi}_i(Q^*, q_i^*), & \text{if } q_i^* > \underline{q}_i, \\ \geq \hat{\pi}_i(Q^*, q_i^*), & \text{if } q_i^* = \underline{q}_i. \end{cases} \tag{4.24}$$

The following Theorem is immediate.

Theorem 4.3: Variational Inequality Formulation of Integrated Spatial Price Equilibrium with Information Asymmetry
A product shipment and quality level pattern $(Q^*, q^*) \in \mathcal{K}^3$ is an integrated spatial price equilibrium with information asymmetry in quality according to Definition 4.3 if and only if it satisfies the variational inequality problem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{\pi}_i(Q^*, q_i^*) + c_{ij}(Q^*) - \hat{\rho}_j(Q^*, q^*)) \times (Q_{ij} - Q_{ij}^*)$$

$$+ \sum_{i=1}^{m}(OC_i(q_i^*) - \hat{\pi}_i(Q^*, q_i^*)) \times (q_i - q_i^*) \geq 0, \quad \forall (Q, q) \in \mathcal{K}^3. \tag{4.25}$$

We now put variational inequality (4.25) into standard form (2.1a) as in Chap. 2: determine $X^* \in \mathcal{K}$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{4.26}$$

where $\mathcal{K}$ is the feasible set, which must be closed and convex. The vector X is an N-dimensional vector, as is $F(X)$, with $F(X)$ being continuous and given, and maps X from $\mathcal{K}$ into R^N. $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space. We define the vector $X \equiv (Q, q)$ and the vector $F(X) \equiv (F^1(X), F^2(X))$ with $F^1(X)$ consisting of components $F_{ij}^1(X) = \hat{\pi}_i(Q, q_i) + c_{ij}(Q) - \hat{\rho}_j(Q, q)$, $i = 1, \ldots, m$; $j = 1, \ldots, n$ and $F^2(X)$ consisting of components: $F_i^2(X) = OC_i(q_i) - \hat{\pi}_i(Q, q_i)$; $i = 1, \ldots, m$. Here $N = mn + m$. Also, we define the feasible set $\mathcal{K} \equiv \mathcal{K}^3$. Then, variational inequality (4.25) can be placed into standard form (4.26).

4.2.2 The Dynamic Model

We now propose a dynamic adjustment process for the evolution of the product shipments and product quality levels under information asymmetry. Observe that, for a current product shipment and quality level pattern at time t, $X(t) = (Q(t), q(t))$, $-F^1_{ij}(X(t)) = \hat{\rho}_j(Q(t), q(t)) - c_{ij}(Q(t)) - \hat{\pi}_i(Q(t), q_i(t))$ is the excess price between demand market j and supply market i and $-F^2_i(X(t)) = \hat{\pi}_i(Q(t), q_i(t)) - OC_i(q_i(t))$ is the difference between the supply price and the opportunity cost at supply market i. In our framework, the rate of change of the product shipment between a supply and demand market pair (i, j) is in proportion to $-F^1_{ij}(X)$, as long as the product shipment Q_{ij} is positive, that is, when $Q_{ij} > 0$,

$$\dot{Q}_{ij} = \hat{\rho}_j(Q, q) - c_{ij}(Q) - \hat{\pi}_i(Q, q_i), \tag{4.27}$$

where $\dot{Q}_{ij}$ denotes the rate of change of Q_{ij}. However, when $Q_{ij} = 0$, the nonnegativity condition (4.3) forces the product shipment Q_{ij} to remain zero when $\hat{\rho}_j(Q, q) - c_{ij}(Q) - \hat{\pi}_i(Q, q_i) \leq 0$. Hence, in this case, we are only guaranteed of having possible increases in the shipment. Namely, when $Q_{ij} = 0$,

$$\dot{Q}_{ij} = \max\{0, \hat{\rho}_j(Q, q) - c_{ij}(Q) - \hat{\pi}_i(Q, q_i)\}. \tag{4.28}$$

We can write (4.27) and (4.28) compactly as:

$$\dot{Q}_{ij} = \begin{cases} \hat{\rho}_j(Q, q) - c_{ij}(Q) - \hat{\pi}_i(Q, q_i), & \text{if} \quad Q_{ij} > 0 \\ \max\{0, \hat{\rho}_j(Q, q) - c_{ij}(Q) - \hat{\pi}_i(Q, q_i)\}, & \text{if} \quad Q_{ij} = 0. \end{cases} \tag{4.29}$$

As for the quality levels, when $q_i > \underline{q}_i$, then

$$\dot{q}_i = \hat{\pi}_i(Q, q_i) - OC_i(q_i), \tag{4.30}$$

where $\dot{q}_i$ denotes the rate of change of q_i; otherwise:

$$\dot{q}_i = \max\{\underline{q}_i, \hat{\pi}_i(Q(t), q_i(t)) - OC_i(Q_i(t))\}, \tag{4.31}$$

since q_i cannot be lower than $\underline{q}_i$ according to (4.22) (and the feasible set $\mathscr{K}^3$).

Combining (4.30) and (4.31), we obtain:

$$\dot{q}_i = \begin{cases} \hat{\pi}_i(Q(t), q_i(t)) - OC_i(Q_i(t)), & \text{if} \quad q_i > \underline{q}_i \\ \max\{\underline{q}_i, \hat{\pi}_i(Q(t), q_i(t)) - OC_i(q_i(t))\}, & \text{if} \quad q_i = \underline{q}_i. \end{cases} \tag{4.32}$$

Applying (4.29) to all supply and demand market pairs (i, j); $i = 1, \ldots, m$; $j = 1, \ldots, n$, and applying (4.32) to all supply markets i; $i = 1, \ldots, m$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes of the product shipments and quality levels, in vector form:

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)). \tag{4.33}$$

Recall that, as noted in Chaps. 2 and 3, $\mathscr{K}$ is a convex polyhedron, $\Pi_{\mathscr{K}}(X, -F(X))$ is the projection, with respect to $\mathscr{K}$, of the vector $-F(X)$ at X defined as

$$\Pi_{\mathscr{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathscr{K}}(X - \delta F(X)) - X}{\delta} \tag{4.34}$$

with $P_{\mathscr{K}}$ denoting the projection map:

$$P_{\mathscr{K}}(X) = \text{argmin}_{z \in \mathscr{K}} \|X - z\|, \tag{4.35}$$

and where $\| \cdot \| = \langle x, x \rangle$.

We now interpret the ODE (4.33) in the context of the integrated spatial model with information asymmetry in quality. First, note that ODE (4.33) ensures that the production shipments are always nonnegative and the quality levels never go below the imposed lower bounds (which are never negative by assumption). ODE (4.33), however, retains the interpretation that if X at time t lies in the interior of $\mathscr{K}$, then the rate at which X changes is greatest when the vector field $-F(X)$ is greatest. Moreover, when the vector field $-F(X)$ pushes X to the boundary of the feasible set $\mathscr{K}$, then the projection $\Pi_{\mathscr{K}}$ ensures that X stays within $\mathscr{K}$.

As emphasized in Chaps. 2 and 3, Dupuis and Nagurney (1993) developed the fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by (4.33). We cite the following theorem from that paper (see also Theorem 2.13).

Theorem 4.4: Equivalence of Equilibria and Stationary Points
X^* solves the variational inequality problem (4.26), equivalently, (4.25), if and only if it is a stationary point of the ODE (4.33), that is,

$$\dot{X} = 0 = \Pi_{\mathscr{K}}(X^*, -F(X^*)). \tag{4.36}$$

This theorem demonstrates that the necessary and sufficient condition for a product shipment and quality level pattern $X^* = (Q^*, q^*)$ to be an integrated spatial price equilibrium with information asymmetry in quality, according to Definition 4.3, is that $X^* = (Q^*, q^*)$ is a stationary point of the adjustment process defined by ODE (4.33), that is, X^* is the point at which $\dot{X} = 0$. We refer to (4.33) as PDS $(F, \mathscr{K})$.

4.3 Qualitative Properties

We now investigate whether and under what conditions the adjustment process defined by ODE (4.33) approaches a spatial price equilibrium with information asymmetry in quality. Lipschitz continuity of $F(X)$ guarantees the existence of a unique solution to (4.37), where we have that $X^0(t)$ satisfies ODE (4.33) with initial shipment and quality level pattern (Q^0, q^0). In other words, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X^0, \tag{4.37}$$

with $X^0(0) = X^0$.

We first establish existence and uniqueness results for the solution of variational inequality (4.26), equivalently, (4.25). We then provide some stability analysis results.

We know, from the standard theory of variational inequalities (see Chap. 2), that if the Jacobian of $F(X)$, denoted by $\nabla F(X)$, is positive definite, then $F(X)$ is strictly monotone, and the solution to variational inequality (4.26) is unique, if it exists.

For the model, the Jacobian matrix $\nabla F(X)$, which is $N \times N$, can be partitioned as:

$$\nabla F(X) = \begin{pmatrix} \nabla F^{11}(X) & \nabla F^{12}(X) \\ \nabla F^{21}(X) & \nabla F^{22}(X) \end{pmatrix}, \tag{4.38}$$

where

$$\nabla F^{11}(X) \equiv \begin{pmatrix} \frac{\partial(\hat{\pi}_1(Q,q_1)+c_{11}(Q)-\hat{\rho}_1(Q,q))}{\partial Q_{11}} & \cdots & \frac{\partial(\hat{\pi}_1(Q,q_1)+c_{11}(Q)-\hat{\rho}_1(Q,q))}{\partial Q_{mn}} \\ \vdots & & \vdots \\ \frac{\partial(\hat{\pi}_m(Q,q_m)+c_{mn}(Q)-\hat{\rho}_n(Q,q))}{\partial Q_{11}} & \cdots & \frac{\partial(\hat{\pi}_m(Q,q)+c_{mn}(Q)-\hat{\rho}_n(Q,q))}{\partial Q_{mn}} \end{pmatrix},$$

$$\nabla F^{12}(X) \equiv \begin{pmatrix} \frac{\partial(\hat{\pi}_1(Q,q_1)+c_{11}(Q)-\hat{\rho}_1(Q,q))}{\partial q_1} & \cdots & \frac{\partial(\hat{\pi}_1(Q,q_1)+c_{11}(Q)-\hat{\rho}_1(Q,q))}{\partial q_m} \\ \vdots & & \vdots \\ \frac{\partial(\hat{\pi}_m(Q,q_m)+c_{mn}(Q)-\hat{\rho}_n(Q,q))}{\partial q_1} & \cdots & \frac{\partial(\hat{\pi}_m(Q,q_m)+c_{mn}(Q)-\hat{\rho}_n(Q,q))}{\partial q_m} \end{pmatrix},$$

$$\nabla F^{21}(X) \equiv \begin{pmatrix} \frac{\partial(-\hat{\pi}_1(Q,q_1))}{\partial Q_{11}} & \cdots & \frac{\partial(-\hat{\pi}_1(Q,q_1))}{\partial Q_{mn}} \\ \vdots & & \vdots \\ \frac{\partial(-\hat{\pi}_m(Q,q_m))}{\partial Q_{11}} & \cdots & \frac{\partial(-\hat{\pi}_m(Q,q_m))}{\partial Q_{mn}} \end{pmatrix},$$

$$\nabla F^{22}(X) \equiv \begin{pmatrix} \frac{\partial(OC_1(q_1)-\hat{\pi}_1(Q,q_1))}{\partial q_1} & \cdots & \frac{\partial(OC_1(q_1)-\hat{\pi}_1(Q,q_1))}{\partial q_m} \\ \vdots & & \vdots \\ \frac{\partial(OC_m(q_m)-\hat{\pi}_m(Q,q_m))}{\partial q_1} & \cdots & \frac{\partial(OC_m(q_m)-\hat{\pi}_m(Q,q_m))}{\partial q_m} \end{pmatrix}.$$

In constructing $\nabla F^{21}(X)$ we have made some algebraic simplifications by observing that the opportunity costs do not depend on the product shipments.

Assumption 4.1
Suppose that for our integrated spatial price equilibrium model there exists a sufficiently large B, such that for any supply and demand market pair (i, j):

$$F^1_{ij}(X) = \hat{\pi}_i(Q, q_i) + c_{ij}(Q) - \hat{\rho}_j(Q, q) > 0, \tag{4.39}$$

for all shipment patterns Q with $Q_{ij} \geq B$ and that there exists a sufficiently large $\bar{B}$, such that for any supply market i:

$$F^2_i = OC_i(q_i) - \hat{\pi}_i(Q, q_i) > 0, \tag{4.40}$$

for all quality level patterns q with $q_i \geq \bar{B} \geq \underline{q}_i$.

We now provide an existence result, whose proof can be established using similar arguments as the proof of Proposition 6.1 in Nagurney and Zhang (1996a) for the spatial price equilibrium problem without any quality variables.

Proposition 4.1: Existence
Any integrated spatial price equilibrium problem with information asymmetry in quality, as described in Sect. 4.2, that satisfies Assumption 4.1 possesses at least one equilibrium shipment and quality level pattern.

We now present the uniqueness result, the proof of which follows from Proposition 2 in Nagurney et al. (1994).

Proposition 4.2: Uniqueness
Suppose that F is strictly monotone at any solution (equilibrium point) of the variational inequality problem defined in (4.26). Then the variational inequality problem has at most one equilibrium point.

Of course, if $F(X)$ is strongly monotone, a property that would hold if $\nabla F(X)$ were strongly positive definite over $\mathcal{K}$, both existence and uniqueness of a solution X^* to (4.26) (equivalently, to (4.25)) would be guaranteed.

The following Theorem is a natural extension/adaptation and integration of Theorems 3.5, 3.6, and 3.7 in Nagurney (1999) to the more general spatial price equilibrium model with information asymmetry in quality (see also Nagurney and Zhang 1996b; Zhang and Nagurney 1995, and Theorems 2.15, 2.16, and 2.17). Definitions of global attractors and global exponential stability can be found in Chap. 2.

Theorem 4.5: Stability

(i). *If $F(X)$ is monotone, then every spatial price equilibrium with information asymmetry in quality, X^*, provided its existence, is a global monotone attractor for the the $PDS(F, \mathcal{K})$. If $F(X)$ is locally monotone at X^*, then it is a monotone attractor for the $PDS(F, \mathcal{K})$.*

(ii). *If $F(X)$ is strictly monotone, then there exists at most one spatial price equilibrium with information asymmetry in quality, X^*. Furthermore, provided* *existence, the unique spatial price equilibrium is a strictly global monotone attractor for the PDS$(F, \mathcal{K})$. If $F(X)$ is locally strictly monotone at X^*, then it is a strictly monotone attractor for the PDS$(F, \mathcal{K})$.*

(iii). *If $F(X)$ is strongly monotone, then there exists a unique spatial price equilibrium with information asymmetry in quality, which is globally exponentially stable for the PDS$(F, \mathcal{K})$. If $F(X)$ is locally strongly monotone at X^*, then X^* is exponentially stable.*

4.4 The Algorithm

The projected dynamical system (4.33) yields a continuous-time adjustment process in product shipments and in quality. However, for computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories is needed. We use the Euler method (cf. (2.34)), discussed in Chap. 2, which was also used in Chap. 3.

Explicit Formulae for the Euler Method Applied to the Integrated Spatial Price Equilibrium Model with Information Asymmetry in Quality

The elegance of this procedure for the computation of solutions to our spatial price equilibrium model can be seen in the following explicit formulae for iteration $\tau + 1$. In particular, we have the following closed form expression for the product shipments $i = 1, \dots, m; j = 1, \dots, n$:

$$Q_{ij}^{\tau+1} = \max\{0, Q_{ij}^{\tau} + a_\tau(\hat{\rho}_j(Q^\tau, q^\tau) - c_{ij}(Q^\tau) - \hat{\pi}_i(Q^\tau, q_i^\tau))\}, \quad (4.41)$$

and the following closed form expression for all the quality levels $i = 1, \dots, m$:

$$q_i^{\tau+1} = \max\{\underline{q}_i, q_i^\tau + a_\tau(\hat{\pi}_i(Q^\tau, q_i^\tau) - OC_i(q_i^\tau))\}. \quad (4.42)$$

Expressions (4.41) and (4.42) can also be interpreted as discrete-time adjustment processes.

We now provide the convergence result. The proof is direct from Theorem 6.10 in Nagurney and Zhang (1996a).

Theorem 4.6: Convergence

In the spatial price equilibrium problem with information asymmetry in quality let $F(X)$ be strictly monotone at any equilibrium pattern and assume that Assumption 4.1 is satisfied. Also, assume that F is Lipschitz continuous (cf. (2.13)). Then there exists a unique equilibrium product shipment and quality level pattern $(Q^*, q^*) \in \mathcal{K}$ and any sequence generated by the Euler method as given by (4.41) and (4.42), where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$ converges to (Q^*, q^*).

In the next Section, we apply the Euler method to compute solutions to numerical spatial price equilibrium problems with information asymmetry in quality.

4.5 Numerical Examples

We implemented the Euler method, as described in Sect. 4.4, in FORTRAN, using a Linux system at the University of Massachusetts Amherst. The convergence criterion is $\epsilon = 10^{-6}$; that is, the Euler method is considered to have converged if, at a given iteration, the absolute value of the difference of each product shipment and each quality level differs from its respective value at the preceding iteration by no more than ϵ. The sequence $\{a_\tau\}$ is: $0.1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \ldots)$. We initialize the algorithm by setting each product shipment, $Q_{ij} = 1.00$, $\forall (i,j)$, and by setting the quality level of the product at each supply market, $q_i = 0.00$, $\forall i$.

4.5.1 Examples in Which the Number of Supply Markets Is Increased

In the first set of examples we explore the impact of adding a supply market. Such an experiment is relevant since the number of supply markets may increase in practice due, for example, to increasing demand, the opening up of new markets, the elimination of trade barriers, new competitive entrants, etc.

The network topologies for Examples 4.1, 4.2, and 4.3 are depicted in Fig. 4.2. The input data and the computed equilibrium solutions for these examples are reported, respectively, in Tables 4.2 and 4.3. The results for the average quality levels, demands, and demand prices are summarized in Table 4.4 and in Fig. 4.3.

Example 4.1
The first example has the network topology given in Fig. 4.2a, that is, there are two supply markets and a single demand market. In this example, the second supply market is located further from the demand market than the first supply market.

The input data are provided in Table 4.2. The Euler method converges in 254 iterations to the equilibrium solution given in Table 4.3, which shows that the second supply market produces (and ships) more of the product to the demand market than the first supply market does but at a lower quality.

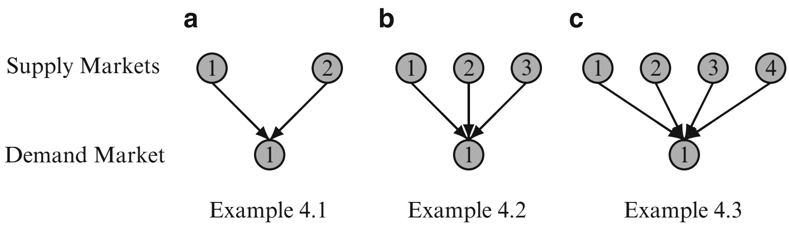

Fig. 4.2 The network topologies for Examples 4.1, 4.2, and 4.3

Table 4.2 Input data for Examples 4.1, 4.2, and 4.3

Example	m	n	$\hat{\pi}(Q,q)$	$c(Q)$	$\hat{\rho}_1(Q,q)$	$OC(q)$
4.1	2	1	$\hat{\pi}_1(Q,q_1) = 5Q_{11} + q_1 + 5,$ $\hat{\pi}_2(Q,q_2) = 2Q_{21} + q_2 + 10$	$c_{11}(Q) = Q_{11} + 15,$ $c_{21}(Q) = 2Q_{21} + 20$	$\hat{\rho}_1(Q,q) = -2(Q_{11} + Q_{21}) + (\frac{q_1Q_{11}+q_2Q_{21}}{Q_{11}+Q_{21}}) + 100$	$OC_1(q_1) = 5q_1,$ $OC_2(q_2) = 10q_2$
4.2	3	1	$\hat{\pi}_1(Q,q_1),$ $\hat{\pi}_2(Q,q_2),$ $\hat{\pi}_3(Q,q_3) = 2Q_{31} + 0.5q_3 + 5$	$c_{11}(Q),$ $c_{21}(Q),$ $c_{31}(Q) = Q_{31} + 20$	$\hat{\rho}_1(Q,q) = -2(Q_{11} + Q_{21} + Q_{31})$ $+(\frac{q_1Q_{11}+q_2Q_{21}+q_3Q_{31}}{Q_{11}+Q_{21}+Q_{31}}) + 100$	 $OC_1(q_1),$ $OC_2(q_2),$ $OC_3(q_3) = 5q_3$
4.3	4	1	$\hat{\pi}_1(Q,q_1),$ $\hat{\pi}_2(Q,q_2),$ $\hat{\pi}_3(Q,q_3),$ $\hat{\pi}_4(Q,q_4) = Q_{41} + 0.7q_4 + 5$	$c_{11}(Q),$ $c_{21}(Q),$ $c_{31}(Q),$ $c_{41}(Q) = 2Q_{41} + 10$	$\hat{\rho}_1(Q,q) = -2(Q_{11} + Q_{21} + Q_{31} + Q_{41})$ $+(\frac{q_1Q_{11}+q_2Q_{21}+q_3Q_{31}+q_4Q_{41}}{Q_{11}+Q_{21}+Q_{31}+Q_{41}}) + 100$	 $OC_1(q_1),$ $OC_2(q_2),$ $OC_3(q_3),$ $OC_4(q_4) = 10q_4$

Table 4.3 Equilibrium solutions for Examples 4.1, 4.2, and 4.3

	Example 4.1	Example 4.2	Example 4.3
Q^*	$Q_{11}^* = 7.06$, $Q_{21}^* = 9.79$	$Q_{11}^* = 5.32$, $Q_{21}^* = 6.80$, $Q_{31}^* = 10.64$	$Q_{11}^* = 3.90$, $Q_{21}^* = 4.37$, $Q_{31}^* = 7.44$, $Q_{41}^* = 11.11$
q^*	$q_1^* = 10.08$, $q_2^* = 3.29$	$q_1^* = 7.90$, $q_2^* = 2.62$, $q_3^* = 5.84$	$q_1^* = 6.13$, $q_2^* = 2.08$, $q_3^* = 4.42$, $q_4^* = 1.73$

Table 4.4 Summary of the results for the average quality levels, demands, and demand prices for Examples 4.1, 4.2, and 4.3

	Example 4.1	Example 4.2	Example 4.3
m	2	3	4
n	1	1	1
$\hat{q}_1$	6.13	5.36	3.17
d_1	16.85	22.76	26.82
ρ_1	72.44	59.83	49.54

For completeness, we now demonstrate how some of the qualitative results in Sect. 4.3 can be applied to this example. Specifically, we determine $\nabla F(X)$ according to (4.38) and evaluate it at the equilibrium solution $X^* = (Q_{11}^*, Q_{21}^*, q_1^*, q_2^*)$.

We obtain:

$$\nabla F(X^*) = \begin{pmatrix} 7.77 & 2.17 & 0.58 & -0.58 \\ 1.77 & 6.17 & -0.42 & 0.42 \\ -5.00 & 0.00 & 4.00 & 0.00 \\ 0.00 & -2.00 & 0.00 & 9.00 \end{pmatrix}.$$

We note that the eigenvalues of $\frac{1}{2}(\nabla F(X^*) + \nabla F(X^*)^T)$ are: 2.8333, 5.2333, 8.7444, and 10.1290 and since these are all positive, according to Theorem 4.5, we know that the computed X^* is a strictly monotone attractor.

Example 4.2
In the second example, the data are as in Example 4.1, except that we add a new supply market 3 as depicted in Fig. 4.2b. The new supply market is located closer to the demand market than supply market 2, but further than supply market 1.

The input data for this example are given in Table 4.2. The Euler method converges in 305 iterations to the equilibrium solution shown in Table 4.3. We note that both supply markets 1 and 2 now ship less of the product to the demand market than they did in Example 4.1, and they each lowered the quality of the product that they produce.

Example 4.3
The final example in this set, Example 4.3, has the topology given in Fig. 4.2c. There are four supply markets and a single demand market. The example has the same data as Example 4.2 except for the new data associated with supply market 4, which are provided in Table 4.2.

The Euler method converges in 337 iterations to the equilibrium solution given in Table 4.3. We observe that the first three supply markets have reduced both the amounts that they ship to the demand market as well as the quality level of their product in comparison to their corresponding values in Example 4.2.

We now provide the results for the average quality levels, demands, and demand prices for this set of examples in Table 4.4 and Fig. 4.3. From these numerical results, we observe that, as the number of supply markets increases, the average quality at the demand market decreases, the demand increases, and the price decreases. Akerlof (1970) observed that, as the price falls, normally the quality would also fall.

One can say that there is an "anonymizing" effect here in that with more producers the consumers at the demand markets are less likely to know from which supply market the product comes. Moreover, since there is no loss in reputation of producers at specific supply markets on the supply-side, the producers at the supply markets have no incentive to maintain or to increase their product quality.

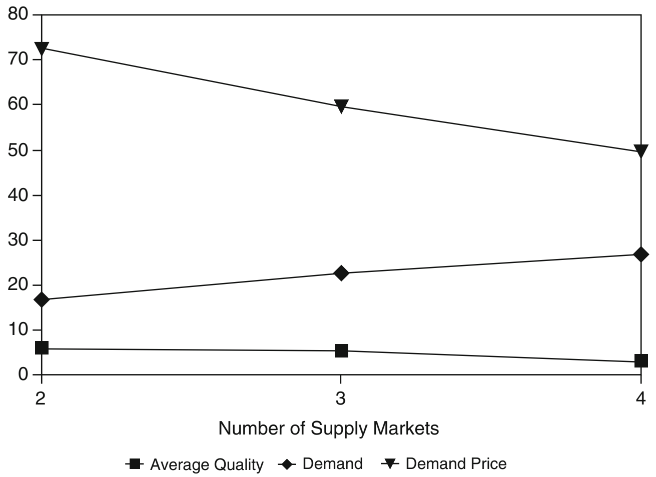

Fig. 4.3 Impact of additional supply markets on average quality, demand, and demand price

These examples clearly exhibit the effects of information asymmetry and, as Akerlof (1970) noted on page 488: *There are many markets in which buyers use some market statistic to judge the quality of prospective purchases. In this case there is incentive for sellers to market poor quality merchandise, since the returns for good quality accrue mainly to the entire group whose statistic is affected rather than to the individual seller.* In our model, as in Akerlof's, the statistic is the average quality.

4.5.2 Examples in Which the Number of Demand Markets Is Increased

In the second set of examples we investigate the impact of the addition of demand markets. Such an experiment is valuable since new demand markets may open up and it is worthwhile to examine the impact on prices and the average quality on both existing and the new demand markets. New demand markets may arise because of demanding consumers, additional marketing, the opening up of new markets, reduction of trade restrictions, etc.

The network topologies for Examples 4.4, 4.5, and 4.6 are depicted in Fig. 4.4, and the input data and the computed equilibrium solutions for these examples are reported, respectively, in Tables 4.5 and 4.6. The results for the average quality levels, demands, and demand prices are summarized in Table 4.7 and Fig. 4.5.

Example 4.4
The network topology for Example 4.4 is given in Fig. 4.4a. This example consists of two supply markets and two demand markets. The input data are as in Example 4.1, except for the added data associated with the new demand market 2, which are provided in Table 4.5.

The Euler method requires 342 iterations for convergence to the equilibrium solution shown in Table 4.6. Note that supply market 1 only serves demand market 1 since $Q^*_{12} = 0.00$.

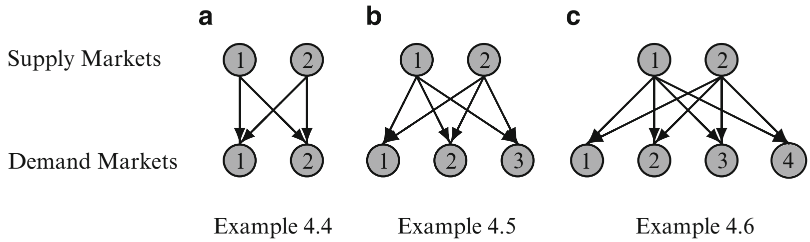

Fig. 4.4 The network topologies for Examples 4.4, 4.5, and 4.6

Table 4.5 Input data for Examples 4.4, 4.5, and 4.6

Example	m	n	$\hat{\pi}(Q,q)$	$c(Q)$	$\hat{\rho}(Q,q)$	$OC(q)$
4.4	2	2	$\hat{\pi}_1(Q,q_1)=5(Q_{11}+Q_{12})+q_1+5$, $\hat{\pi}_2(Q,q_2)=2(Q_{21}+Q_{22})+q_2+10$	$c_{11}(Q)$, $c_{21}(Q)$, $c_{12}(Q)=Q_{12}+10$, $c_{22}(Q)=2Q_{22}+10$	$\hat{\rho}_1(Q,q)=-2(Q_{11}+Q_{21})+(\frac{q_1Q_{11}+q_2Q_{21}}{Q_{11}+Q_{21}})+100$, $\hat{\rho}_2(Q,q)=-(Q_{12}+Q_{22})+(\frac{q_1Q_{12}+q_2Q_{22}}{Q_{12}+Q_{22}})+50$	$OC_1(q_1)$, $OC_2(q_2)$
4.5	2	3	$\hat{\pi}_1(Q,q_1)=5(Q_{11}+Q_{12}+Q_{13})+q_1+5$, $\hat{\pi}_2(Q,q_2)=2(Q_{21}+Q_{22}+Q_{23})+q_2+10$	$c_{11}(Q)$, $c_{21}(Q)$, $c_{12}(Q)$, $c_{22}(Q)$, $c_{13}(Q)=2Q_{13}+10$, $c_{23}(Q)=Q_{23}+5$	$\hat{\rho}_1(Q,q)=-2(Q_{11}+Q_{21})+(\frac{q_1Q_{11}+q_2Q_{21}}{Q_{11}+Q_{21}})+100$, $\hat{\rho}_2(Q,q)$, $\hat{\rho}_3(Q,q)=-(Q_{13}+Q_{23})+(\frac{q_1Q_{13}+q_2Q_{23}}{Q_{13}+Q_{23}})+50$	$OC_1(q_1)$, $OC_2(q_2)$
4.6	2	4	$\hat{\pi}_1(Q,q_1)=5(Q_{11}+Q_{12}+Q_{13}+Q_{14})+q_1+5$, $\hat{\pi}_2(Q,q_2)=2(Q_{21}+Q_{22}+Q_{23}+Q_{24})+q_2+10$	$c_{11}(Q)$, $c_{21}(Q)$, $c_{12}(Q)$, $c_{22}(Q)$, $c_{13}(Q)$, $c_{23}(Q)$, $c_{14}(Q)=Q_{14}+5$, $c_{24}(Q)=2Q_{24}+5$	$\hat{\rho}_1(Q,q)=-2(Q_{11}+Q_{21})+(\frac{q_1Q_{11}+q_2Q_{21}}{Q_{11}+Q_{21}})+100$, $\hat{\rho}_2(Q,q)$, $\hat{\rho}_3(Q,q)$, $\hat{\rho}_4(Q,q)=-(Q_{14}+Q_{24})+0.5(\frac{q_1Q_{14}+q_2Q_{24}}{Q_{14}+Q_{24}})+60$	$OC_1(q_1)$, $OC_2(q_2)$

Table 4.6 Equilibrium solutions for Examples 4.4, 4.5, and 4.6

	Example 4.4	Example 4.5	Example 4.6
Q^*	$Q^*_{11} = 7.30$, $Q^*_{12} = 0.00$, $Q^*_{21} = 8.93$, $Q^*_{22} = 2.43$	$Q^*_{11} = 7.60$, $Q^*_{12} = 0.00$, $Q^*_{13} = 0.00$, $Q^*_{21} = 7.89$, $Q^*_{22} = 1.16$, $Q^*_{23} = 4.23$	$Q^*_{11} = 7.11$, $Q^*_{12} = 0.00$, $Q^*_{13} = 0.00$, $Q^*_{14} = 0.94$, $Q^*_{21} = 7.43$, $Q^*_{22} = 0.18$, $Q^*_{23} = 0.18$, $Q^*_{24} = 4.34$
q^*	$q^*_1 = 10.38$, $q^*_2 = 3.64$	$q^*_1 = 10.75$, $q^*_2 = 4.06$	$q^*_1 = 11.32$, $q^*_2 = 4.38$

Table 4.7 Summary of the results for the average quality levels, demands, and demand prices for Examples 4.1, 4.4, 4.5, and 4.6

	Example 4.1	Example 4.4	Example 4.5	Example 4.6
m	2	2	2	2
n	1	2	3	4
$\hat{q}$	$\hat{q}_1 = 6.13$	$\hat{q}_1 = 6.67$, $\hat{q}_2 = 3.64$	$\hat{q}_1 = 7.35$, $\hat{q}_2 = 4.06$, $\hat{q}_3 = 4.06$	$\hat{q}_1 = 7.77$, $\hat{q}_2 = 4.38$, $\hat{q}_3 = 4.38$, $\hat{q}_4 = 5.62$
d	$d_1 = 16.85$	$d_1 = 16.23$, $d_2 = 2.43$	$d_1 = 15.49$, $d_2 = 1.16$, $d_3 = 4.23$	$d_1 = 14.54$, $d_2 = 0.18$, $d_3 = 2.77$, $d_4 = 5.29$
ρ	$\rho_1 = 72.44$	$\rho_1 = 74.21$, $\rho_2 = 51.21$	$\rho_1 = 76.37$, $\rho_2 = 52.91$, $\rho_3 = 49.83$	$\rho_1 = 78.70$, $\rho_2 = 54.20$, $\rho_3 = 51.61$, $\rho_4 = 57.53$

Example 4.5
Example 4.5 is constructed from Example 4.4, but it includes a new demand market 3 as depicted in Fig. 4.4b. New data are added as reported in Table 4.5.

The Euler method requires 408 iterations for convergence to the solution in Table 4.6. Supply market 1 continues to supply only demand market 1. Also, interestingly, the quality levels of the product produced at supply market 1 and at supply market 2 have increased in comparison to the respective values for Example 4.4.

Example 4.6
Example 4.6 is constructed from Example 4.5 and has the same data except for the additional data associated with the new demand market 4. The network topology for this example is depicted in Fig. 4.4c. The input data are provided in Table 4.5.

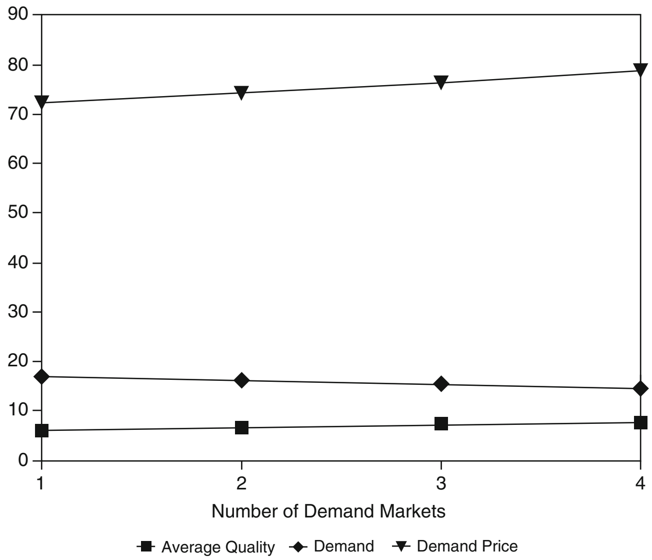

Fig. 4.5 Impact of additional demand markets on average quality, demand, and demand price at demand market 1

The Euler method required 1,081 iterations for convergence to the equilibrium solution reported in Table 4.6.

We now provide the results for the average quality levels, demands, and demand prices for the examples in this set. Please refer to Table 4.7. In Fig. 4.5 the results for the first demand market are depicted. The respective results for the other demand markets follow a similar trend.

Our numerical examples indicate that, as the number of demand markets increases, the average quality at the demand markets increases. Also, as the number of demand markets increases, the demand prices at the demand markets increase, whereas the demands decrease slightly. A greater number of demand markets provides greater economic pressure on the producers at the supply markets to improve quality.

4.5.3 Examples in Which Minimum Quality Standards Are Imposed

In the last set of numerical examples, we impose minimum quality standards. Specifically, for each of the preceding examples, we set the minimum quality standard at all supply markets equal to the maximum equilibrium quality level at the supply markets computed for the corresponding example without imposed minimum quality standards. The results for these examples are reported in Table 4.8. The results for the average quality levels, demands, and demand prices for all 12 examples are summarized in Table 4.9 in order to show the impact of the minimum quality standards.

Example 4.7
This example is constructed from Example 4.1, except that now we have that:

$$\underline{q}_1 = \underline{q}_2 = 10.80.$$

The Euler method converges in 218 iterations to the equilibrium solution given in Table 4.8.

Table 4.8 Results for Examples 4.7 through 4.12

	Example 4.7	Example 4.8	Example 4.9	Example 4.10	Example 4.11	Example 4.12
Q^*	$Q^*_{11} = 7.66$, $Q^*_{21} = 9.17$	$Q^*_{11} = 5.66$, $Q^*_{21} = 6.10$, $Q^*_{31} = 11.11$	$Q^*_{11} = 4.24$, $Q^*_{21} = 3.97$, $Q^*_{31} = 7.98$, $Q^*_{41} = 10.90$	$Q^*_{11} = 7.91$, $Q^*_{12} = 0.00$, $Q^*_{21} = 8.18$, $Q^*_{22} = 2.73$	$Q^*_{11} = 8.20$, $Q^*_{12} = 0.00$, $Q^*_{13} = 0.00$, $Q^*_{21} = 7.03$, $Q^*_{22} = 1.37$, $Q^*_{23} = 4.55$	$Q^*_{11} = 7.65$, $Q^*_{12} = 0.00$, $Q^*_{13} = 0.00$, $Q^*_{14} = 1.01$, $Q^*_{21} = 6.72$, $Q^*_{22} = 0.59$, $Q^*_{23} = 3.39$, $Q^*_{24} = 3.40$
q^*	$q^*_1 = 10.82$, $q^*_2 = 10.80$	$q^*_1 = 8.32$, $q^*_2 = 7.90$, $q^*_3 = 7.90$	$q^*_1 = 6.55$, $q^*_2 = 6.13$, $q^*_3 = 6.13$, $q^*_4 = 6.13$	$q^*_1 = 11.13$, $q^*_2 = 10.38$	$q^*_1 = 11.50$, $q^*_2 = 10.75$	$q^*_1 = 12.07$, $q^*_2 = 11.32$
$\hat{q}$	$\hat{q}_1 = 10.81$	$\hat{q}_1 = 8.00$	$\hat{q}_1 = 6.20$	$\hat{q}_1 = 10.75$, $\hat{q}_2 = 10.38$	$\hat{q}_1 = 11.15$, $\hat{q}_2 = 10.75$, $\hat{q}_3 = 10.75$	$\hat{q}_1 = 11.72$, $\hat{q}_2 = 11.32$, $\hat{q}_3 = 11.32$, $\hat{q}_4 = 11.49$
d	$d_1 = 16.83$	$d_1 = 22.86$	$d_1 = 27.10$	$d_1 = 16.09$, $d_2 = 2.73$	$d_1 = 15.23$, $d_2 = 1.37$, $d_3 = 4.55$	$d_1 = 14.37$, $d_2 = 0.59$, $d_3 = 3.39$, $d_4 = 4.41$
ρ	$\rho_1 = 76.76$	$\rho_1 = 62.28$	$\rho_1 = 52.00$	$\rho_1 = 78.57$, $\rho_2 = 57.65$	$\rho_1 = 80.70$, $\rho_2 = 59.38$, $\rho_3 = 56.20$	$\rho_1 = 82.98$, $\rho_2 = 60.73$, $\rho_3 = 57.93$, $\rho_4 = 61.34$

Table 4.9 Summary of the results for the average quality levels, demands, and demand prices for Examples 4.1 through 4.12

Example	m	n	$\hat{q}$	d	ρ
4.1	2	1	6.13	16.85	72.44
4.7	2	1	10.81	16.83	76.76
4.2	3	1	5.36	22.76	59.83
4.8	3	1	8.00	22.86	62.28
4.3	4	1	3.17	26.82	49.54
4.9	4	1	6.20	27.10	52.00
4.4	2	2	6.67, 3.64	16.23, 2.43	74.21, 51.21
4.10	2	2	10.75, 10.38	16.09, 2.73	78.57, 57.65
4.5	2	3	7.35, 4.06, 4.06	15.49, 1.16, 4.23	76.37, 52.91, 49.83
4.11	2	3	11.15, 10.75, 10.75	15.23, 1.37, 4.55	80.70, 59.38, 56.20
4.6	2	4	7.77, 4.38, 4.38, 5.62	14.54, 0.18, 2.77, 5.29	78.70, 54.20, 51.61, 57.53
4.12	2	4	11.72, 11.32, 11.32, 11.49	14.37, 0.59, 3.39, 4.41	82.98, 60.73, 57.93, 61.34

An interesting question is whether the imposition of minimum quality standards has affected the stability results as obtained for Example 4.1. We determine $\nabla F(X)$ according to (4.38) for Example 4.7 (which recall was constructed from Example 4.1) and evaluate it at the equilibrium solution $X^* = (Q^*_{11}, Q^*_{21}, q^*_1, q^*_2)$, which yields:

$$\nabla F(X^*) = \begin{pmatrix} 8.00 & 2.00 & 0.54 & -0.54 \\ 2.00 & 6.00 & -0.45 & 0.45 \\ -5.00 & 0.00 & 4.00 & 0.00 \\ 0.00 & -2.00 & 0.00 & 9.00 \end{pmatrix}.$$

The eigenvalues of $\frac{1}{2}(\nabla F(X^*)+\nabla F(X^*)^T)$ are: 2.86, 5.11, 8.81, and 10.22. Since these eigenvalues are all positive, the same type of stability holds for this example with minimum quality standards, as without.

Example 4.8
This example is constructed from Example 4.2, except that now we impose the minimum quality standard (which was the highest achieved equilibrium quality level in Example 4.2):

$$\underline{q}_1 = \underline{q}_2 = 7.90.$$

The Euler method converges in 302 iterations and yields the equilibrium solution provided in Table 4.8. Interestingly, the quality level of the product at the first supply market exceeds the imposed minimum quality standard.

Example 4.9
Example 4.9 is constructed from Example 4.3 with the imposition of the following minimum quality standard (which was the highest computed equilibrium quality level in Example 4.3):

$$\underline{q}_1 = \underline{q}_2 = 6.13.$$

The Euler method converges in 332 iterations to the equilibrium solution in Table 4.8.

Example 4.10
Example 4.10 has the same data as Example 4.4 except for the imposition of the following minimum quality standards:

$$\underline{q}_1 = \underline{q}_2 = 10.38.$$

The Euler method requires 202 iterations for convergence to the equilibrium solution shown in Table 4.8.

Example 4.11
Example 4.11 is built from Example 4.5 and, hence, has the same data as Example 4.5 except for the inclusion of the following minimum quality standards:

$$\underline{q}_1 = \underline{q}_2 = 10.75.$$

The Euler method requires 482 iterations for convergence to the equilibrium solution in Table 4.8.

Example 4.12
In our final example, we proceeded as in the preceding five examples. We constructed Example 4.12 from Example 4.6, using its data with the inclusion of minimum quality standards, which were set as follows:

$$\underline{q}_1 = 11.32, \quad \underline{q}_2 = 11.32.$$

The value 11.32 was selected since it is the highest equilibrium quality obtained at a supply market for Example 4.6.

The Euler method requires 950 iterations for convergence to the equilibrium solution.

As revealed in Tables 4.3 and 4.6, when no minimum quality standards are imposed, supply market 1 is always the one that is able to produce at the highest quality level. However, in Examples 4.7 through 4.12, with minimum quality standards set to the highest quality levels of supply market 1, the other supply markets are forced to produce higher quality at higher costs. Under this circumstance, in order to break-even, some products, which were originally produced by supply markets 2, 3, and/or 4, are given up to supply market 1. With more products to produce, supply market 1 produces at even higher quality in the cases with minimum quality standards (cf. Table 4.8) than in the cases without (cf. Tables 4.3 and 4.6).

In Table 4.9, a summary of the results for all 12 examples is provided in order to demonstrate the impacts of minimum quality standards. The examples are grouped without and with the corresponding minimum quality standards. As depicted in Table 4.9, after imposing minimum quality standards, the average quality at the demand markets increases, and the prices also increase.

4.6 Summary and Conclusions

In this chapter, we focused on perfect competition, rather than imperfect competition, which was the theme in Chap. 3. We described both static and dynamic models of spatial competition in the case of asymmetric information in terms of product quality. The models are significant extensions to classical spatial price equilibrium models to include the possibility, which is not infrequent in practice, that producers at supply markets may know more about the quality of their product than consumers know. Such a framework may apply to products as varied as certain food products, oil and gas, and even medicines that are produced far from points of consumption.

Variational inequality theory was used for the development of the equilibrium model and projected dynamical systems theory for the dynamic version. Qualitative results in terms of conditions for existence and uniqueness of equilibria as well as stability analysis for the solutions of the associated projected dynamical system were also given.

The proposed algorithm was accompanied by convergence results, and then applied to solve numerical examples. We found that as the number of supply markets increases, the average quality at the demand market decreases, the demand increases, and the price decreases. In contrast, as the number of demand markets increases, the average quality at the demand markets increases, the demand decreases, and the demand price at the demand markets increases. In addition, we showed how minimum quality standards can be incorporated into the model(s) and provided numerical examples that illustrated the impacts of such policy regulations on the flows, prices, and average quality. Notably, after the imposition of minimum quality standards, the average quality at the demand markets increases and the prices also increase.

The novelty of the contributions in this chapter is as follows. 1. We introduce the important spatial dimension using a network construct in the case of perfect competition and information asymmetry in quality. 2. We construct rigorous static (equilibrium) and dynamic models, without and with minimum quality standards, an important policy instrument. 3. We provide a qualitative analysis of the equilibrium product shipments and average quality level pattern, in particular, existence and uniqueness results, as well as stability analysis results. 4. We propose an effective computational procedure, along with conditions for convergence. Our model can handle nonlinear functions and as many supply markets and demand markets as mandated by the specific application. 5. Our numerical examples illustrate the impact of the imposition of quality standards, as well as the impact of an increasing number of supply markets or demand markets on the equilibrium product shipments, the average quality levels at the demand markets, and on the incurred prices.

4.7 Sources and Notes

This chapter is based on the paper by Nagurney et al. (2014). Spatial price equilibrium models have many features of supply chains, since both have supply markets and demand markets and transportation costs between them, and can be viewed as precursors to supply chains. In this chapter, we focus on supply chains with a bipartite structure in order to clearly extract the impacts of information asymmetry. Spatial price equilibrium models on general networks can be found in Dafermos and Nagurney (1984); see also Nagurney (1999) and the references therein. For a variety of static and dynamic supply chain network models, but without quality variables as decision variables, see Nagurney (2006). Projected dynamical systems theory was used by Nagurney et al. (1995a,b) and Nagurney and Zhang (1996b) to formulate, analyze, and solve dynamic spatial price equilibrium problems but without quality elements or information asymmetry. For a survey of spatial price equilibrium models see Labys and Yang (1997) and for a related survey with a focus on spatial economic location, see Kilkenny and Thisse (1999).

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Part III
Quality in Product Differentiation and Outsourcing

Chapter 5
Supply Chain Network Oligopolies with Product Differentiation

Abstract This chapter is the first chapter of Part III of this book, which concentrates on quality in product differentiation and outsourcing settings. In Chap. 3, we noted that quality information asymmetry under product homogeneity could lead to a quality "free ride," which might jeopardize certain firms' profits and product quality. Therefore, in order to prevent such harmful results, it is important for firms to differentiate their products from that of their competitors, so that consumers will be able to identify the products of different firms and their quality. In this chapter, we present a supply chain network model with quality competition in differentiated products, where the product of each firm is substitutable but differentiated by a brand or label. We first present the equilibrium model and derive alternative variational inequality formulations. We then construct the projected dynamical systems model, which provides a continuous-time evolution of the firms' product shipments and product quality levels. The stability analysis results are also presented, and a discrete-time version of the continuous-time adjustment process is constructed, which yields an algorithm with closed form expressions at each iteration. The algorithm is then utilized to compute solutions to several numerical examples. We also include sensitivity analysis results for minimum quality standards. The framework developed in this chapter can serve as the foundation for the modeling and analysis of competition among firms in industries ranging from food to pharmaceuticals to durable goods and high tech products, as well as Internet services, where quality and product differentiation are seminal.

5.1 Introduction

Oligopolies are a fundamental industrial organization market structure of numerous industries ranging from airplane manufacturers, as well as airlines, to wireless service providers, certain food manufacturers and retailers, as well as utilities and energy companies, and even specific banks. In classical oligopoly problems, as in Chap. 3, the product that is produced is assumed to be homogeneous and, hence, consumers at the demand markets do not distinguish among the firms that produce the product.

Increasingly, however, in the case of many products in imperfect markets such as oligopolies, consumers may consider the products to be differentiated according

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to the producer and, hence, oligopolies with product differentiation have been gaining in attention, dating to the work of Hotelling (1929) (see also Dixit and Stiglitz (1977), D'Aspremont et al. (1979), Economides (1989), Anderson and Palma (2001), Johnson and Myatt (2003), and the references therein).

Indeed, since quality is emerging as an important feature of numerous products, Holcombe (2009) argues that firms, in reality, do not differentiate their products to make them different, or to give consumers more variety but, rather, to make them better so that consumers purchase the firm's product. Moreover, although the differentiated product may even cost more to produce, it may result in higher profits since consumers may be drawn to such products. Hence, quality is implicit in product differentiation. Holcombe (2009) also notes that, according to Schumpeter (1943, page 82), "The essential point to grasp is that in dealing with capitalism we are dealing with an evolutionary process." Hence, new dynamic models with quality considered in product differentiation are imperative.

Moreover, one cannot ignore that research and development (R&D) activities play a significant role in the improvement of technology, and, hence, quality (Bernstein and Nadiri 1991; Motta 1993; Cohen and Klepper 1996; Acharyya 2005). In the process of R&D, for example, firms may gain competitive advantages from increased specialization of scientific and technological knowledge, skills and resources, and the state of knowledge of a firm may, typically, be reflected in the quality of its products (Lilien and Yoon 1990; Aoki 1991; Berndt et al. 1995; Shankar et al. 1998). Because of the influence of R&D on quality improvement, value adding, and profit enhancement, firms may invest in R&D activities, which is referred to as the cost of quality improvement. Eli Lilly, one of the world's largest pharmaceutical companies, invested billions of dollars in profits back into its R&D (Steiner et al. 2007). The 18th biggest R&D spender, Apple, invested 2.6 billion US dollars in 2011 in R&D and maintained a net profit of 13 billion dollars every 3 months (Krantz 2012).

Therefore, in this chapter, we present supply chain network oligopoly models with differentiated products and quality levels, which capture the cost associated with R&D activities. The framework for our model is that of Cournot (1838)-Nash (1950, 1951) competition in which the firms compete by determining their optimal product shipments as well as the quality levels of their particular products. We present both the static model, in an equilibrium context, which we formulate as a finite-dimensional variational inequality problem, and the dynamics, using projected dynamical systems theory.

In our modeling framework, we explicitly consider the spatial component and include transportation costs associated with shipping the products to the demand markets via a supply chain network. The relevance of transportation costs was also recognized by Hotelling (1929). Our theoretical and computational framework can be applied to supply chain network oligopolies with product differentiation and quality levels in many different settings. Recent contributions to dynamic oligopolies can be found in the book by Bischi et al. (2009) and the paper by Matsumoto and Szidarovsky (2011). Our approach, in contrast, integrates oligopolies in a supply chain network setting along with differentiated products and quality levels. Firms select both the product shipments and the quality levels of their specific products subject to the costs associated with production, quality, and transportation, as well as the demand price functions at the demand markets.

This chapter is organized as follows. In Sect. 5.2, we first present the static version of the supply chain network oligopoly model with product differentiation, and establish alternative variational inequality formulations of the governing Nash-Cournot equilibrium conditions. We then present its dynamic counterpart. In Sect. 5.3, we present stability analysis results and illustrate the concepts with several numerical examples. In Sect. 5.4, we propose the discrete-time adjustment process, which provides an approximation to the continuous-time trajectories of the firm's product shipments and quality levels over time. We then apply the algorithm to compute solutions to several numerical examples of supply chain network oligopoly problems with differentiated products in Sect. 5.5, and demonstrate how the algorithm can be used to track the trajectories and to compute the equilibria. We also conduct sensitivity analysis on minimum quality standards. We summarize our results and present our conclusions in Sect. 5.6. The Sources and Notes for this chapter are provided in Sect. 5.7.

5.2 The Supply Chain Network Oligopoly Models with Product Differentiation

In this section, we first develop the equilibrium supply chain network oligopoly model with product differentiation and quality competition and derive the variational inequality formulation. We then describe the underlying dynamics associated with the firms' production outputs as well as quality levels and present the projected dynamical systems model.

Please refer to Fig. 5.1 for the underlying network structure of the supply chain network oligopoly with product differentiation. As depicted in Fig. 5.1, the I firms and n_R demand markets are generally spatially separated. There is a distinct (but substitutable) product produced by each of the firms and is consumed at the demand markets. The I firms compete with one another in a noncooperative manner in the production and distribution of their products, and select both their product shipments and the quality levels of their products. A typical firm is denoted by i.

In the supply chain network in Fig. 5.1, a top-tiered node corresponds to a firm i; $i = 1, \ldots, I$, a second-tiered node to firm i's manufacturing plant i'; $i' = 1', \ldots, I'$, and the nodes at the bottom to the n_R common demand markets. The link connecting each firm i and its manufacturing plant i' represents the manufacturing process at manufacturing plant i'. The link emanating from manufacturing plant i' to demand market R_k; $k = 1, \ldots, n_R$, corresponds to the transportation activities from firm i's plant to that demand market. The notation for this chapter is provided in Table 5.1. All vectors here are assumed to be column vectors.

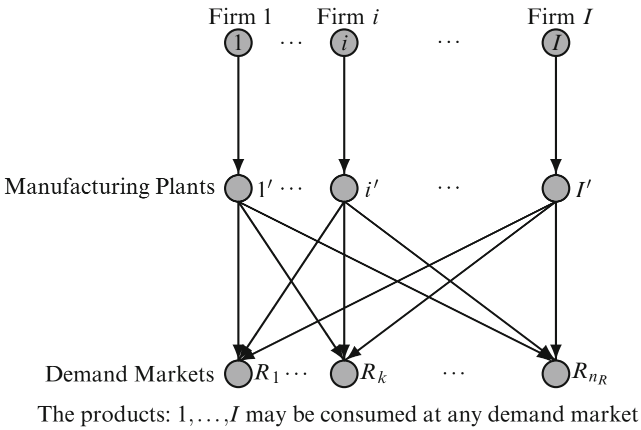

Fig. 5.1 The supply chain network topology of the oligopoly problem with product differentiation

Table 5.1 Notation for the supply chain network oligopoly models (static and dynamic) with product differentiation

Notation	Definition
Q_{ik}	The nonnegative shipment of firm i's product to demand market R_k. The $\{Q_{ik}\}$ elements for all i and k are grouped into the vector $Q \in R^{In_R}_{+}$
s_i	The nonnegative product output produced by firm i. We group the production outputs for all i into the vector $s \in R^I_+$
q_i	The quality level, or, simply, the quality, of product i, which is produced by firm i. The quality levels of all firms are grouped into the vector $q \in R^I_+$
d_{ik}	The demand for the product of firm i at demand market R_k. We group the demands for all i and k into the vector $d \in R^{In_R}_{+}$
$\hat{f}_i(s, q_i)$	The production cost at firm i's manufacturing plant i'
$\hat{c}_{ik}(Q_{ik})$	The total transportation cost associated with shipping firm i's product, produced at manufacturing plant i', to demand market R_k
$\rho_{ik}(d, q)$	The demand price function for firm i's product at demand market R_k

5.2.1 The Equilibrium Model

We now develop the equilibrium supply chain network oligopoly model with product differentiation and quality competition. The following conservation of flow equations must hold:

$$s_i = \sum_{k=1}^{n_R} Q_{ik}, \quad i = 1, \ldots, I; \tag{5.1}$$

$$d_{ik} = Q_{ik}, \quad i = 1, \ldots, I; k = 1, \ldots, n_R, \tag{5.2}$$

$$Q_{ik} \geq 0, \quad i = 1, \ldots, I; k = 1, \ldots, n_R, \tag{5.3}$$

and

$$q_i \geq 0, \quad i = 1, \ldots, I. \tag{5.4}$$

According to (5.1), the quantity of the product produced by each firm is equal to the sum of the amounts shipped to all the demand markets, and, according to (5.2), the quantity of a firm's product consumed at a demand market is equal to the amount shipped from the firm to that demand market. Both the shipment volumes and the quality levels must be nonnegative.

Although the products produced by the firms are differentiated by brands, they are still substitutes. Common factors may be utilized in the process of manufacturing. Hence, we allow for the general situation where the production cost of a firm i may depend upon the entire production pattern and on its own quality level, as shown in Table 5.1.

It is assumed here that the production cost functions also capture the quality cost, since, as a special case, they can take on the form

$$\hat{f}_i(s, q_i) = f_i(s, q_i) + g_i(q_i), \quad i = 1, \ldots, I, \tag{5.5}$$

where the first term depends on both production outputs and quality and the second term only depends on the quality. Interestingly, the second term in (5.5) can also be interpreted as the R&D cost of the firm, which is assumed to depend on the quality level of its products.

In addition, in contrast to the total transportation cost functions in Chap. 3, we assume that transportation activities will not affect product quality in this chapter. Hence, transportation cost functions are general functions of only the vector of product shipments.

The production cost functions and the total transportation cost functions are convex, continuous, and twice continuously differentiable.

Unlike in Chaps. 3 and 4, the demand price functions we consider in this chapter are under product differentiation. They also depend, in general, not only on the entire consumption pattern, but also on all the levels of quality of all the products. The demand price functions are, typically, assumed to be continuous, twice continuously differentiable, and monotonically decreasing in demand at their respective demand markets but increasing in terms of product quality.

The strategic variables of firm i are its product shipments $\{Q_i\}$ where $Q_i = (Q_{i1}, \ldots, Q_{in_R})$ and its quality level q_i. The profit or utility U_i of firm i; $i = 1, \ldots, I$, is given by the expression

$$U_i = \sum_{k=1}^{n_R} \rho_{ik} d_{ik} - \hat{f}_i - \sum_{k=1}^{n_R} \hat{c}_{ik}, \tag{5.6}$$

which is the difference between its total revenue and its total cost.

In view of (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6), one may write the profit as a function solely of the shipment pattern and quality levels, that is,

$$U = U(Q, q), \tag{5.7}$$

where U is the I-dimensional vector with components: $\{U_1, \ldots, U_I\}$.

Let K^i denote the feasible set corresponding to firm i, where $K^i \equiv \{(Q_i, q_i) | Q_i \geq 0, \text{ and } q_i \geq 0\}$ and define $K \equiv \prod_{i=1}^{I} K^i$.

We consider the market mechanism in which the I firms supply their products in a noncooperative fashion, each one trying to maximize its own profit. They compete in Cournot (1838)-Nash (1950, 1951) fashion, and seek to determine a nonnegative product shipment and quality level pattern (Q^*, q^*) for which the I firms will be in a state of equilibrium as defined below.

Definition 5.1: A Supply Chain Network Cournot-Nash Equilibrium with Product Differentiation
A product shipment and quality level pattern $(Q^*, q^*) \in K$ is said to constitute a supply chain network Cournot-Nash equilibrium if for each firm i; $i = 1, \ldots, I$,

$$U_i(Q_i^*, q_i^*, \hat{Q_i^*}, \hat{q}_i^*) \geq U_i(Q_i, q_i, \hat{Q_i^*}, \hat{q}_i^*), \quad \forall (Q_i, q_i) \in K^i, \tag{5.8}$$

where

$$\hat{Q_i^*} \equiv (Q^*{}_1, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_I^*); \quad \textit{and} \quad \hat{q_i^*} \equiv (q^*{}_1, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_I^*).$$

According to (5.8), an equilibrium is established if no firm can unilaterally improve upon its profits by selecting an alternative vector of product shipments and quality level of its product.

We now present alternative variational inequality formulations of the above supply chain network Cournot-Nash equilibrium with product differentiation in the following theorem.

Theorem 5.1: Variational Inequality Formulations of the Supply Chain Network Cournot-Nash Equilibrium with Product Differentiation
Assume that, for each firm i, the profit function $U_i(Q, q)$ is concave with respect to the variables $\{Q_{i1}, \ldots, Q_{in_R}\}$, and q_i, and is continuous and continuously differentiable. Then $(Q^*, q^*) \in K$ is a supply chain network Cournot-Nash equilibrium according to Definition 5.1 if and only if it satisfies the variational inequality

$$-\sum_{i=1}^{I}\sum_{k=1}^{n_R} \frac{\partial U_i(Q^*, q^*)}{\partial Q_{ik}} \times (Q_{ik} - Q_{ik}^*) - \sum_{i=1}^{I} \frac{\partial U_i(Q^*, q^*)}{\partial q_i} \times (q_i - q_i^*) \geq 0,$$

$$\forall (Q, q) \in K, \tag{5.9}$$

or, equivalently, $(s^*, Q^*, d^*, q^*) \in K^1$ *is an equilibrium production, shipment, consumption, and quality level pattern if and only if it satisfies the variational inequality*

$$\sum_{i=1}^{I} \frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial s_i} \times (s_i - s_i^*) + \sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[\frac{\partial \hat{c}_{ik}(Q_{ik}^*)}{\partial Q_{ik}} - \sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*, q^*)}{\partial d_{ik}} \times d_{il}^*\right] \times (Q_{ik} - Q_{ik}^*)$$
$$- \sum_{i=1}^{I}\sum_{k=1}^{n_R} \rho_{ik}(d^*, q^*) \times (d_{ik} - d_{ik}^*)$$
$$+ \sum_{i=1}^{I}\left[\frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial q_i} - \sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*, q^*)}{\partial q_i} \times d_{il}^*\right] \times (q_i - q_i^*) \geq 0, \quad \forall (s, Q, d, q) \in K^1, \tag{5.10}$$

where $K^1 \equiv \{(s, Q, d, q) \,|\, Q \geq 0,\ q \geq 0,$ *and* (5.1) *and* (5.2) *hold*$\}$.

Proof: Equation (5.9) follows directly from Gabay and Moulin (1980) and Dafermos and Nagurney (1987).

In order to obtain variational inequality (5.10) from variational inequality (5.9), it is noted that:

$$-\frac{\partial U_i(Q^*, q^*)}{\partial Q_{ik}} = \left[\frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial s_i} + \frac{\partial \hat{c}_{ik}(Q_{ik}^*)}{\partial Q_{ik}} - \rho_{ik}(d^*, q^*) - \sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*, q^*)}{\partial d_{ik}} \times d_{il}^*\right];$$
$$i = 1, \ldots, I;\ k = 1, \ldots, n_R, \tag{5.11}$$

and

$$-\frac{\partial U_i(Q^*, q^*)}{\partial q_i} = \left[\frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial q_i} - \sum_{k=1}^{n_R}\frac{\partial \rho_{il}(d^*, q^*)}{\partial q_i} \times d_{il}^*\right]; \quad i = 1, \ldots, I. \tag{5.12}$$

Multiplying the right-most expression in (5.11) by $(Q_{ik} - Q_{ik}^*)$ and summing the resultant over all i and all k; similarly, multiplying the right-most expression in (5.12) by $(q_i - q_i^*)$ and summing the resultant over all i yields, respectively:

$$\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[\frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial s_i} + \frac{\partial \hat{c}_{ik}(Q_{ik}^*)}{\partial Q_{ik}} - \rho_{ik}(d^*, q^*) - \sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*, q^*)}{\partial d_{ik}} \times d_{il}^*\right] \times (Q_{ik} - Q_{ik}^*) \tag{5.13}$$

and

$$\sum_{i=1}^{I}\left[\frac{\partial \hat{f}_i(s^*, q_i^*)}{\partial q_i} - \sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*, q^*)}{\partial q_i} \times d_{il}^*\right] \times (q_i - q_i^*). \tag{5.14}$$

Finally, summing (5.13) and (5.14) and then using constraints (5.1) and (5.2), yields variational inequality (5.10). □

We now put the above supply chain network equilibrium problem with product differentiation and quality levels into standard variational inequality form, as in (2.1a), that is, determine $X^* \in \mathscr{K} \subset R^N$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathscr{K}, \tag{5.15}$$

where F is a given continuous function from $\mathscr{K}$ to R^N and $\mathscr{K}$ is a closed and convex set.

We define the $(In_R + I)$-dimensional vector $X \equiv (Q, q)$ and the $(In_R + I)$-dimensional vector $F(X) \equiv (F^1(X), F^2(X))$ with the (i,k)-th component, F^1_{ik}, of $F^1(X)$ given by

$$F^1_{ik}(X) \equiv -\frac{\partial U_i(Q,q)}{\partial Q_{ik}}, \tag{5.16}$$

the i-th component, F^2_i, of $F^2(X)$ given by

$$F^2_i(X) \equiv -\frac{\partial U_i(Q,q)}{\partial q_i}, \tag{5.17}$$

and with the feasible set $\mathscr{K} \equiv K$. Then, clearly, variational inequality (5.9) can be put into standard form (2.1a).

In a similar manner, one can establish that variational inequality (5.10) can also be put into standard variational inequality form (2.1a).

Remark 5.1
Lower bounds and upper bounds on quality (cf. Chap. 1) can also be included into the model with lower bounds representing associated minimum quality standards (cf. Chap. 3) and upper bounds capturing the perfect quality conformance level (cf. Chap. 1). In this case, a new set of constraints can be constructed to replace (5.4), that is:

$$\overline{q}_i \geq q_i \geq \underline{q}_i; \quad i = 1, \ldots I, \tag{5.18}$$

where $\overline{q}_i$ denotes the upper bound of quality that is achievable by firm i, and $\underline{q}_i$ denotes the minimum quality standard faced by firm i.

We define $K^2 \equiv \{(Q,q) | Q \geq 0 \text{ and } (5.18) \text{ holds}\}$ and $K^3 \equiv \{(Q,q) | Q \geq 0, \text{ and } (5.1), (5.2), \text{ and } (5.18) \text{ hold}\}$. Variational inequalities (5.9) and (5.10) will still hold with K^2 substituted for K and K^3 substituted for K^1. In this case, firms in the supply chain network will face the same decision-making problems as before, but the quality decision of each firm will have to satisfy constraint (5.18).

5.2.2 The Dynamic Model

We now propose a dynamic adjustment process for the evolution of the firms' product shipments and product quality levels. Observe that, for a current product shipment and quality level pattern at time t, $X(t) = (Q(t), q(t))$, $-F^1_{ik}(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial Q_{ik}}$, given by (5.16), is the marginal utility (profit) of firm i with respect to its product shipment to demand market R_k. Similarly, $-F^2_i(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial q_i}$, given by (5.17), is the firm's marginal utility (profit) with respect to its quality level. In this framework, the rate of change of the product shipment between a firm and demand market pair (i, k) is in proportion to $-F^1_{ik}(X)$, as long as the product shipment Q_{ik} is positive. Namely, when $Q_{ik} > 0$,

$$\dot{Q}_{ik} = \frac{\partial U_i(Q, q)}{\partial Q_{ik}}, \tag{5.19}$$

where $\dot{Q}_{ik}$ denotes the rate of change of Q_{ik}. However, when $Q_{ik} = 0$, the nonnegativity condition (5.3) forces the product shipment Q_{ik} to remain zero when $\frac{\partial U_i(Q,q)}{\partial Q_{ik}} \leq 0$. Hence, in this case, One is only guaranteed of having possible increases of the shipment. Namely, when $Q_{ik} = 0$,

$$\dot{Q}_{ik} = \max\{0, \frac{\partial U_i(Q, q)}{\partial Q_{ik}}\}. \tag{5.20}$$

One may write (5.19) and (5.20) concisely as:

$$\dot{Q}_{ik} = \begin{cases} \frac{\partial U_i(Q,q)}{\partial Q_{ik}}, & \text{if} \quad Q_{ik} > 0 \\ \max\{0, \frac{\partial U_i(Q,q)}{\partial Q_{ik}}\}, & \text{if} \quad Q_{ik} = 0. \end{cases} \tag{5.21}$$

As for the quality levels, when $q_i > 0$, then

$$\dot{q}_i = \frac{\partial U_i(Q, q)}{\partial q_i}, \tag{5.22}$$

where $\dot{q}_i$ denotes the rate of change of q_i; otherwise:

$$\dot{q}_i = \max\{0, \frac{\partial U_i(Q, q)}{\partial q_i}\}, \tag{5.23}$$

since q_i must be nonnegative.

Combining (5.22) and (5.23), one may write:

$$\dot{q}_i = \begin{cases} \frac{\partial U_i(Q,q)}{\partial q_i}, & \text{if} \quad q_i > 0 \\ \max\{0, \frac{\partial U_i(Q,q)}{\partial q_i}\}, & \text{if} \quad q_i = 0. \end{cases} \tag{5.24}$$

Applying (5.21) to all firm and demand market pairs (i,k); $i = 1,\ldots,I$; $k = 1,\ldots,n_R$, and applying (5.24) to all firms i; $i = 1,\ldots,I$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes of the product shipments and quality levels, in vector form, as:

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)), \tag{5.25}$$

where, $F(X) = -\nabla U(Q,q)$, where $\nabla U(Q,q)$ is the vector of marginal utilities with components given by (5.16) and (5.17).

We now interpret the ODE (5.24) in the context of the supply chain network model with product differentiation and quality competition. First, note that ODE (5.25) ensures that the production shipments and quality levels are always nonnegative. Indeed, if one were to consider, instead, the ordinary differential equation: $\dot{X} = -F(X)$, or, equivalently, $\dot{X} = \nabla U(X)$, such an ODE would not ensure that $X(t) \geq 0$, for all $t \geq 0$, unless additional restrictive assumptions were to be imposed. ODE (5.25), however, retains the interpretation that if X at time t lies in the interior of $\mathscr{K}$, then the rate at which X changes is greatest when the vector field $-F(X)$ is greatest. Moreover, when the vector field $-F(X)$ pushes X to the boundary of the feasible set $\mathscr{K}$, then the projection $\Pi_{\mathscr{K}}$ ensures that X stays within $\mathscr{K}$. Hence, the product shipments and quality levels are always nonnegative.

Although the use of the projection on the right-hand side of ODE (5.25) guarantees that the product shipments and the quality levels are always nonnegative, it also raises the question of existence of a solution to ODE (5.25), since this ODE is nonstandard due to its discontinuous right-hand side. Dupuis and Nagurney (1993) developed the fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by (5.25).

Furthermore, the necessary and sufficient condition for a product shipment and quality level pattern $X^* = (Q^*, q^*)$ to be a supply chain network Cournot-Nash equilibrium (cf. Definition 5.1) is that $X^* = (Q^*, q^*)$ is a stationary point of the adjustment process defined by ODE (5.25), that is, X^* is the point at which $\dot{X} = 0$, as described by the following theorem from Dupuis and Nagurney (1993) (Theorem 2.13).

Theorem 5.2: Equivalence of Equilibria and Stationary Points
X^* solves the variational inequality problem (5.10), equivalently, (5.9), if and only if it is a stationary point of the ODE (5.25), that is,

$$\dot{X} = 0 = \Pi_{\mathscr{K}}(X^*, -F(X^*)). \tag{5.26}$$

5.3 Qualitative Properties

We now present qualitative properties for the supply chain network Cournot-Nash equilibrium with product differentiation, under the above utility gradient process.

Recall that Lipschitz continuity of $F(X)$ (cf. Chap. 2) guarantees the existence of a unique solution to (5.25), where $X^0(t)$ satisfies ODE (5.25) with initial shipment and quality level pattern (Q^0, q^0). In other words, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)), \quad X(0) = X^0, \tag{5.27}$$

with $X^0(0) = X^0$.

We know that, in the context of the supply chain network problem with product differentiation and quality competition, where $F(X)$ is the vector of negative marginal utilities as in (5.16) – (5.17), if the utility functions are twice differentiable and the Jacobian of the negative marginal utility functions (or, equivalently, the negative of the Hessian matrix of the utility functions) is positive definite, then the corresponding $F(X)$ is strictly monotone. The solution to variational inequality (5.9) is then unique, if it exists.

Assumption 5.1
Suppose that in the supply chain network model there exists a sufficiently large M, such that for any (i, k),

$$\frac{\partial U_i(Q,q)}{\partial Q_{ik}} < 0, \tag{5.28}$$

for all shipment patterns Q with $Q_{ik} \geq M$ and that there exists a sufficiently large $\bar{M}$, such that for any i,

$$\frac{\partial U_i(Q,q)}{\partial q_i} < 0, \tag{5.29}$$

for all quality level patterns q with $q_i \geq \bar{M}$.

We now provide existence and uniqueness results.

Proposition 5.1: Existence
Any supply chain network problem, as described above, that satisfies Assumption 5.1 possesses at least one equilibrium shipment and quality level pattern satisfying variational inequality (5.9) (or (5.10)).

Proof: The proof follows from Proposition 1 in Zhang and Nagurney (1995). □

Proposition 5.2: Uniqueness
Suppose that F is strictly monotone at any equilibrium point of the variational inequality problem defined in (5.15). Then it has at most one equilibrium point.

Proof: The proof follows from Proposition 2 in Nagurney et al. (1994). □

Theorem 5.3: Existence and Uniqueness
Suppose that F is strongly monotone. Then there exists a unique solution to variational inequality (5.15); equivalently, to variational inequality (5.9).

The following theorem presents the stability properties of the projected dynamical system described in (5.25); see Theorems 2.15, 2.16, and 2.17.

Theorem 5.4: Stability

(i). *If $F(X)$ is monotone, then every supply chain network Cournot-Nash equilibrium, X^*, provided its existence, is a global monotone attractor for the projected dynamical system. If $F(X)$ is locally monotone at X^*, then it is a monotone attractor for the projected dynamical system.*

(ii). *If $F(X)$ is strictly monotone, then there exists at most one supply chain network Cournot-Nash equilibrium, X^*. Furthermore, provided existence, the unique spatial Cournot-Nash equilibrium is a strictly global monotone attractor for the projected dynamical system. If $F(X)$ is locally strictly monotone at X^*, then it is a strictly monotone attractor for the projected dynamical system.*

(iii). *If $F(X)$ is strongly monotone, then there exists a unique supply chain network Cournot-Nash equilibrium, X^*, which is globally exponentially stable for the projected dynamical system. If $F(X)$ is locally strongly monotone at X^*, then it is exponentially stable.*

We now present two examples in order to illustrate some of the above concepts and results.

Example 5.1
Consider a supply chain network oligopoly problem consisting of two firms and a single demand market, R_1, as depicted in Fig. 5.2.

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 s_2 + q_2^2 + 37,$$

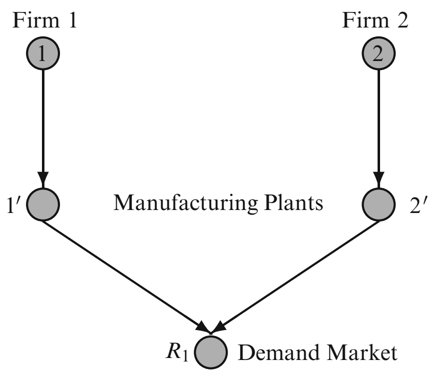

Fig. 5.2 The supply chain network topology for Example 5.1

the total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10,$$

and the demand price functions are:

$$\rho_{11}(d, q) = -d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2 + 100,$$
$$\rho_{21}(d, q) = -0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2 + 100.$$

The Jacobian matrix of $F(X) = -\nabla U(Q, q)$, denoted by $J(Q_{11}, Q_{21}, q_1, q_2)$, is

$$J(Q_{11}, Q_{21}, q_1, q_2) = \begin{pmatrix} 6.0 & 1.4 & -0.3 & -0.5 \\ 2.6 & 21.0 & -0.1 & -0.5 \\ -0.3 & 0.0 & 4.0 & 0.0 \\ 0.0 & -0.5 & 0.0 & 2.0 \end{pmatrix}.$$

This Jacobian matrix is positive definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, $F(X)$, is strongly monotone (since $F(X)$ is linear). Thus, both the existence and uniqueness of the solution to variational inequality (5.9) with respect to this example are guaranteed (Theorem 5.3). Moreover, the equilibrium solution, which is: $Q_{11}^* = 16.08$, $Q_{21}^* = 2.79$, $q_1^* = 1.21$, and $q_2^* = 0.70$, is globally exponentially stable, according to Theorem 5.4.

Example 5.2
Another example with two firms and two demand markets, R_1 and R_2, as depicted in Fig. 5.3, is presented.

The production cost functions of the two firms are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 s_2 + q_2^2 + 37,$$

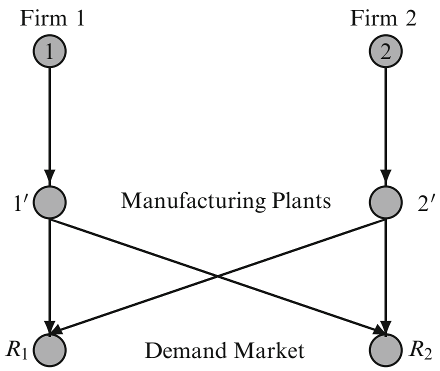

Fig. 5.3 The supply chain network topology for Example 5.2

the total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{12}(Q_{12}) = 5Q_{12}^2 + 7,$$
$$\hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10, \quad \hat{c}_{22}(Q_{22}) = 2Q_{22}^2 + 5,$$

and the demand price functions are:

$$\rho_{11}(d,q) = -d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2 + 100,$$
$$\rho_{12}(d,q) = -2d_{12} - d_{22} + 0.4q_1 + 0.2q_2 + 100,$$
$$\rho_{21}(d,q) = -0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2 + 100,$$
$$\rho_{22}(d,q) = -0.7d_{12} - 1.7d_{22} + 0.01q_1 + 0.6q_2 + 100.$$

The Jacobian of $F(X) = -\nabla U(Q,q)$, $J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, q_1, q_2)$, is

$$J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, q_1, q_2) = \begin{pmatrix} 6.0 & 2.0 & 1.4 & 1.0 & -0.3 & -0.05 \\ 2.0 & 16.0 & 1.0 & 2.0 & -0.4 & -0.2 \\ 2.6 & 2.0 & 21.0 & 4.0 & -0.1 & -0.5 \\ 2.0 & 2.7 & 4.0 & 7.4 & -0.01 & -0.6 \\ -0.3 & -0.4 & 0.0 & 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & -0.5 & -0.6 & 0.0 & 2.0 \end{pmatrix}.$$

This Jacobian matrix is also positive definite. Since F(X) is also linear, $F(X)$ is strongly monotone, and both the existence and the uniqueness of the equilibrium solution to this example are guaranteed. Moreover, the equilibrium solution (stationary point) is: $Q_{11}^* = 14.27$, $Q_{12}^* = 3.81$, $Q_{21}^* = 1.76$, $Q_{22}^* = 4.85$, $q_1^* = 1.45$, $q_2^* = 1.89$, and it is globally exponentially stable.

The stationary points of both Examples 5.1 and 5.2 are computed using the Euler method. In the next section, we present the induced closed form expressions at each iteration of the Euler method, along with the convergence result.

5.4 The Algorithm

The projected dynamical system (5.25) yields continuous-time adjustment processes. However, for computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories, is needed.

In this chapter, we use the Euler method (cf. Chap. 2) as the algorithm for the computation of the solution to the supply chain network oligopoly model with product differentiation. The Euler method is also used in Chaps. 3 and 4.

Explicit Formulae for the Euler Method Applied to the Supply Chain Network Oligopoly Model with Product Differentiation

The explicit formulae yielded by the Euler method at iteration $\tau+1$ are as follows. In particular, the closed form expression for the product shipments, $i = 1,\dots,I; k = 1,\dots,n_R$, is:

$$Q_{ik}^{\tau+1} = \max\{0, Q_{ik}^{\tau} + a_\tau(\rho_{ik}(d^\tau, q^\tau) + \sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^\tau, q^\tau)}{\partial d_{ik}} d_{il}^\tau - \frac{\partial \hat{f}_i(s^\tau, q_i^\tau)}{\partial s_i} - \frac{\partial \hat{c}_{ik}(Q_{ik}^\tau)}{\partial Q_{ik}})\}, \tag{5.30}$$

and the closed form expression for the quality levels, $i = 1,\dots,I$, is:

$$q_i^{\tau+1} = \max\{0, q_i^\tau + a_\tau(\sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^\tau, q^\tau)}{\partial q_i} d_{il}^\tau - \frac{\partial \hat{f}_i(s^\tau, q_i^\tau)}{\partial q_i})\} \tag{5.31}$$

with the demands being updated according to:

$$d_{ik}^{\tau+1} = Q_{ik}^{\tau+1}; \quad i = 1,\dots,I; k = 1,\dots,n_R, \tag{5.32}$$

and the supplies being updated according to:

$$s_i^{\tau+1} = \sum_{k=1}^{n_R} Q_{ik}^{\tau+1}, \quad i = 1,\dots,I. \tag{5.33}$$

We now provide the convergence result. The proof follows similar arguments as that for Theorem 3.5.

Theorem 5.5: Convergence
In the supply chain network oligopoly problem with product differentiation and quality levels let $F(X) = -\nabla U(Q,q)$ be strictly monotone at any equilibrium pattern and assume that Assumption 5.1 is satisfied. Also, assume that F is uniformly Lipschitz continuous. Then there exists a unique equilibrium product shipment and quality level pattern $(Q^*, q^*) \in K$ and any sequence generated by the Euler method as given by (5.30)–(5.31) with updates of the demands and supplies via (5.32) and (5.33), where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$ converges to (Q^*, q^*).

5.5 Numerical Examples

We implemented the Euler method (cf. (5.30), (5.31), (5.32) and (5.33)), using Matlab on a Lenovo E46A. The convergence criterion is $\epsilon = 10^{-6}$; that is, the Euler method is considered to have converged if, at a given iteration, the absolute value of the difference between each successive product shipment and quality level is no more than ϵ.

The sequence $\{a_\tau\}$ is: $0.1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \ldots)$. We initialize the algorithm by setting each product shipment $Q_{ik} = 2.5$, $\forall i, k$, and by setting the quality level of each firm $q_i = 0.00$, $\forall i$.

In Sect. 5.3, we discussed stability analysis and presented results for two numerical examples. We now provide additional results for these two examples.

Example 5.1 Revisited

The Euler method requires 39 iterations for convergence to the equilibrium pattern for Example 5.1. A graphical depictions of the iterates, consisting of the product shipments and the quality levels, is given, respectively, in Figs. 5.4 and 5.5. The utility/profit of firm 1 is 723.89 and that of firm 2 is 34.44.

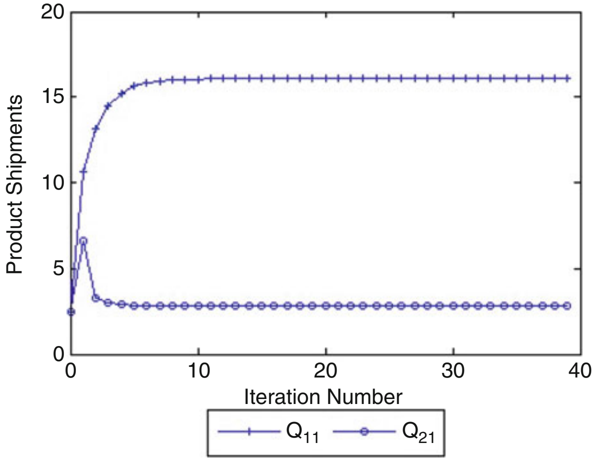

Fig. 5.4 Product shipment trajectories for Example 5.1

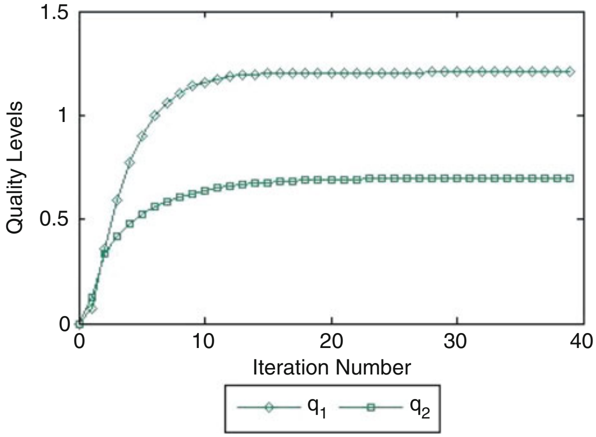

Fig. 5.5 Quality level trajectories for Example 5.1

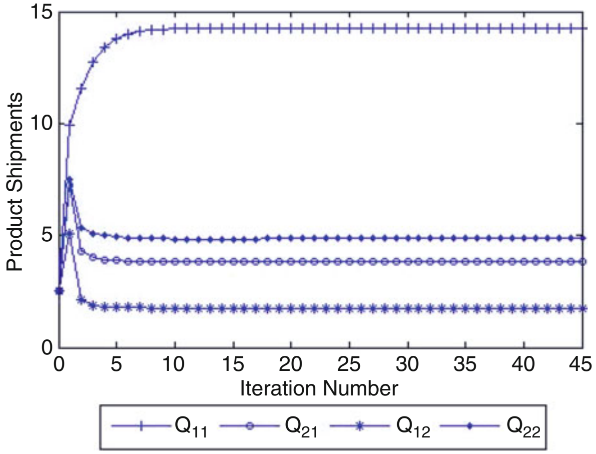

Fig. 5.6 Product shipment trajectories for Example 5.2

One can see from these figures, that, as predicted by the stability analysis results, the convergence is exponentially fast.

As shown in Fig. 5.5, the equilibrium quality level of firm 1 is 42.15 % higher than that of firm 2, which happens because customers are more quality-sensitive to firm 1's product, as revealed by the demand price functions.

Example 5.2 Revisited

For Example 5.2, the Euler method requires 45 iterations for convergence. A graphical depiction of the product shipment and quality level iterates is given in Figs. 5.6 and 5.7. One can see from these figures that the convergence to the equilibrium solution is exponentially fast. The profit of firm 1 is 775.19, whereas that of firm 2 is 145.20.

In the next example, there is another firm, firm 3, entering the market, and its quality cost is much higher than those of firms 1 and 2.

Example 5.3

Example 5.3 consists of three firms and two demand markets, R_1 and R_2, as depicted in Fig. 5.8.

This example is built from Example 5.2 with the production cost functions of the original two firms expanded and the original demand price functions as well. The complete data for this example are given below.

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 s_2 + s_1 s_3 + 2q_1^2 + 39,$$

$$\hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 s_2 + 2s_3 s_2 + q_2^2 + 37,$$

$$\hat{f}_3(s, q_3) = s_3^2 + s_1 s_3 + s_3 s_2 + 8q_3^2 + 60.$$

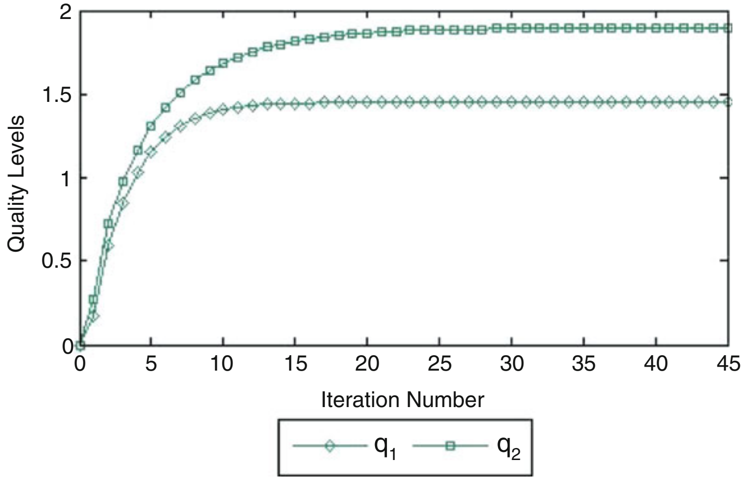

Fig. 5.7 Quality level trajectories for Example 5.2

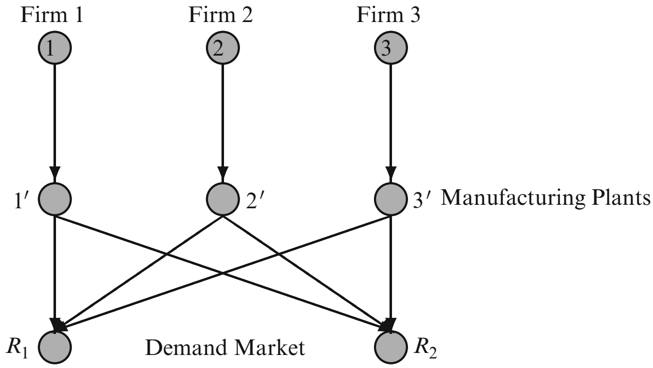

Fig. 5.8 The supply chain network topology for Example 5.3

The total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{12}(Q_{12}) = 5Q_{12}^2 + 7,$$
$$\hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10, \quad \hat{c}_{22}(Q_{22}) = 2Q_{22}^2 + 5,$$
$$\hat{c}_{31}(Q_{31}) = 2Q_{31}^2 + 9, \quad \hat{c}_{32}(Q_{32}) = 3Q_{32}^2 + 8,$$

and the demand price functions are:

$$\rho_{11}(d,q) = -d_{11} - 0.4d_{21} - 0.1d_{31} + 0.3q_1 + 0.05q_2 + 0.05q_3 + 100,$$
$$\rho_{12}(d,q) = -2d_{12} - d_{22} - 0.1d_{32} + 0.4q_1 + 0.2q_2 + 0.2q_3 + 100,$$

$$\rho_{21}(d,q) = -0.6d_{11} - 1.5d_{21} - 0.1d_{31} + 0.1q_1 + 0.5q_2 + 0.1q_3 + 100,$$

$$\rho_{22}(d,q) = -0.7d_{12} - 1.7d_{22} - 0.1d_{32} + 0.01q_1 + 0.6q_2 + 0.01q_3 + 100,$$

$$\rho_{31}(d,q) = -0.2d_{11} - 0.4d_{21} - 1.8d_{31} + 0.2q_1 + 0.2q_2 + 0.7q_3 + 100,$$

$$\rho_{32}(d,q) = -0.1d_{12} - 0.3d_{22} - 2d_{32} + 0.2q_1 + 0.1q_2 + 0.4q_3 + 100.$$

The Jacobian of $F(X) = -\nabla U(Q,q)$, $J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$, is

$$J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, q_1, q_2) = \begin{pmatrix} 6.0 & 2.0 & 1.4 & 1.0 & -0.3 & -0.05 \\ 2.0 & 16.0 & 1.0 & 2.0 & -0.4 & -0.2 \\ 2.6 & 2.0 & 21.0 & 4.0 & -0.1 & -0.5 \\ 2.0 & 2.7 & 4.0 & 7.4 & -0.01 & -0.6 \\ -0.3 & -0.4 & 0.0 & 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & -0.5 & -0.6 & 0.0 & 2.0 \end{pmatrix}.$$

This Jacobian matrix is positive definite. Thus, $F(X)$ is strongly monotone (F(X) is linear), and both the existence and uniqueness of the solution to this example are guaranteed.

The Euler method converges to the equilibrium solution: $Q_{11}^* = 12.63$, $Q_{12}^* = 3.45$, $Q_{21}^* = 1.09$, $Q_{22}^* = 3.21$, $Q_{31}^* = 6.94$, $Q_{32}^* = 5.42$, $q_1^* = 1.29$, $q_2^* = 1.23$, $q_3^* = 0.44$, in 42 iterations. The profits of the firms are: $U_1 = 601.67$, $U_2 = 31.48$, and $U_3 = 403.97$. Graphical depictions of the product shipment and the quality level iterates are given, respectively, in Figs. 5.9 and 5.10.

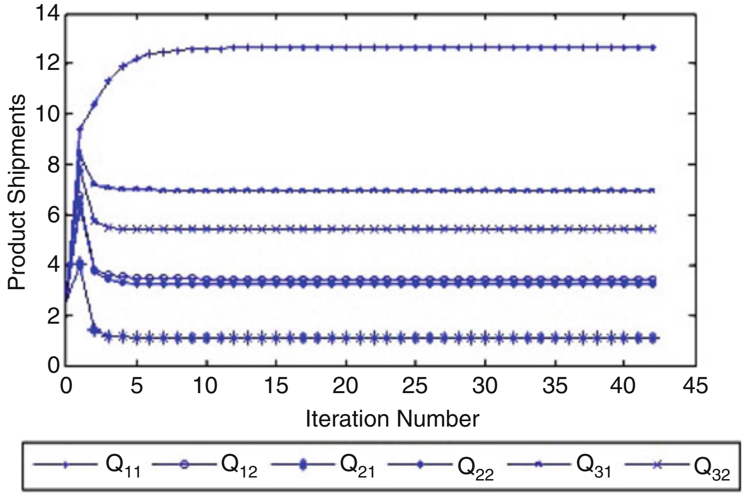

Fig. 5.9 Product shipment trajectories for Example 5.3

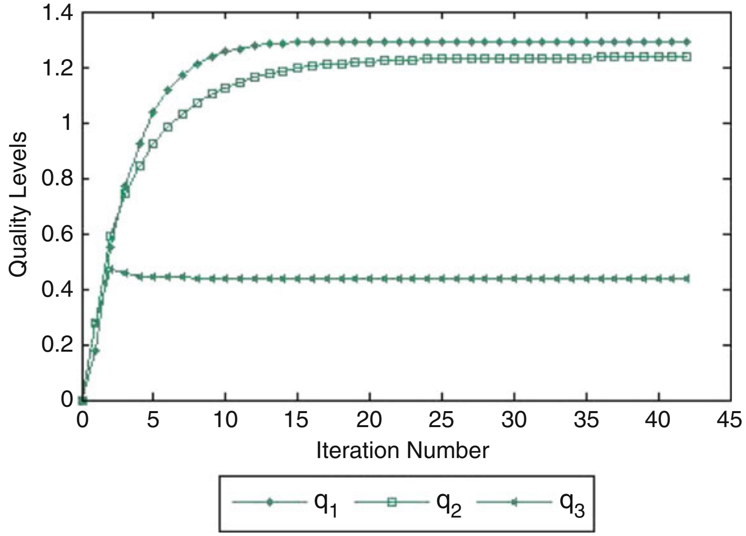

Fig. 5.10 Quality level trajectories for Example 5.3

The properties of the Jacobian matrix are verified above in order to also evaluate the stability of the utility gradient process as well as to check whether conditions for convergence of the algorithm are satisfied. One should realize, however, that the algorithm does not require strong monotonicity of minus the gradient of the utility functions for convergence (cf. Theorem 5.5). Moreover, if the algorithm converges, it converges to a stationary point of the projected dynamical systems; equivalently, to a solution of the variational inequality problem governing the Cournot-Nash equilibrium conditions for the supply chain network model.

In addition, with these examples, we illustrate the types of problems with not unrealistic features and underlying functions that can be theoretically effectively analyzed as to their qualitative properties and also their solutions computed.

Example 5.4
Example 5.4 is constructed from Example 5.3 to consider the following scenario. The consumers at demand market R_2 have become more sensitive as to the quality of the products. To reflect this, the new demand price functions associated with demand market R_2 are now:

$$\rho_{12}(d,q) = -2d_{12} - d_{22} - 0.1d_{32} + 0.49q_1 + 0.2q_2 + 0.2q_3 + 100,$$
$$\rho_{22}(d,q) = -0.7d_{12} - 1.7d_{22} - 0.1d_{32} + 0.01q_1 + 0.87q_2 + 0.01q_3 + 100,$$

and

$$\rho_{32}(d,q) = -0.1d_{12} - 0.3d_{22} - 2d_{32} + 0.2q_1 + 0.1q_2 + 1.2q_3 + 100.$$

The Jacobian of $F(X)$ is now:

$$J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3) =$$
$$\begin{pmatrix} 6.0 & 2.0 & 1.4 & 1.0 & 1.1 & 1.0 & -0.3 & -0.05 & -0.05 \\ 2.0 & 16.0 & 1.0 & 2.0 & 1.0 & 1.1 & -0.49 & -0.2 & -0.2 \\ 2.6 & 2.0 & 21.0 & 4.0 & 2.1 & 2.0 & -0.1 & -0.5 & -0.5 \\ 2.0 & 2.7 & 4.0 & 7.4 & 2.0 & 2.1 & -0.01 & -0.87 & -0.01 \\ 1.2 & 1.0 & 1.4 & 1.0 & 9.6 & 2.0 & -0.2 & -0.2 & -0.7 \\ 1.0 & 1.1 & 1.0 & 1.3 & 2.0 & 12.0 & -0.2 & -0.1 & -1.2 \\ -0.3 & -0.49 & 0.0 & 0.0 & 0.0 & 0.0 & 4.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.5 & -0.87 & 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.7 & -1.2 & 0.0 & 0.0 & 16.0 \end{pmatrix}.$$

This Jacobian matrix is also positive definite. Therefore, for this example, the existence and the uniqueness of the equilibrium product shipment and quality level pattern are guaranteed, since $F(X)$ is also linear. Moreover, the equilibrium solutions for both Examples 5.3 and 5.4 are globally exponentially stable.

The computed equilibrium solution is now: $Q_{11}^* = 13.41$, $Q_{12}^* = 3.63$, $Q_{21}^* = 1.41$, $Q_{22}^* = 4.08$, $Q_{31}^* = 3.55$, $Q_{32}^* = 2.86$, $q_1^* = 1.45$, $q_2^* = 2.12$, $q_3^* = 0.37$. The profits of the firms are now: $U_1 = 682.44$, $U_2 = 82.10$, and $U_3 = 93.19$.

The Euler method requires 47 iterations for convergence. Please refer to Figs. 5.11 and 5.12 to view the iterates of the product shipments and the quality levels generated by the Euler method. Due to the fact that, in each iteration, the values of Q_{12} and Q_{22} are very close, the trajectories of these two almost overlap in Fig. 5.11.

In this example, as consumers become more sensitive to the quality of the substitutable product, the equilibrium quality levels of the three firms change, with those of firm 1 and firm 2 increasing, relative to their values in Example 5.3. Since it costs much more for firm 3 to achieve higher quality levels than for firm 1 and firm 2, the profit of firm 3 decreases by 76.9 %, while the profits of firms 1 and 2 increase by 13.4 % and 160.8 %, respectively. Hence, the pressure on the consumers' side through the demand price functions in quality can result not only in higher quality but also in higher profits for those firms that have lower quality costs.

Example 5.5

In this example, we take a possible relationship between output and quality improvement into consideration, such that, as output increases, it may be more costly to improve product quality, due to limited resources. The data are as in Example 5.4 except for the production cost functions, which are now:

$$\hat{f}_1(s, q_1) = 2s_1^2 + 0.005s_1q_1 + 2q_1^2 + 30, \quad \hat{f}_2(s, q_2) = 4s_2^2 + 0.005s_2q_2 + q_2^2 + 30,$$
$$\hat{f}_3(s, q_3) = 4s_3^2 + 0.005s_3q_3 + 8q_3^2 + 50.$$

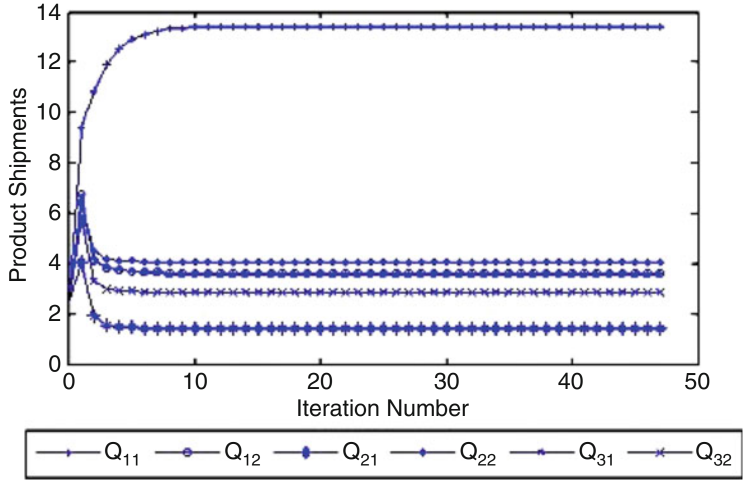

Fig. 5.11 Product shipment trajectories for Example 5.4

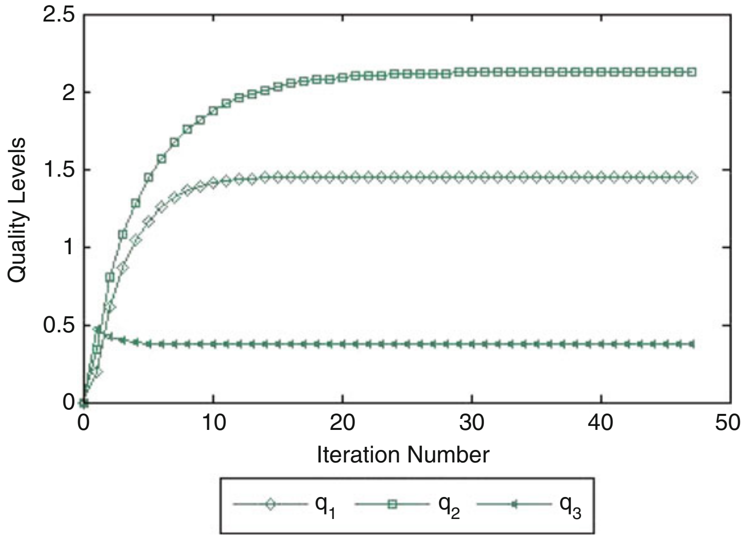

Fig. 5.12 Quality level trajectories for Example 5.4

The Jacobian of $F(X)$, denoted by $J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$, is

$$J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$$
$$= \begin{pmatrix} 8.0 & 4.0 & 0.4 & 0.0 & 0.1 & 0.0 & -0.295 & -0.05 & -0.05 \\ 4.0 & 18.0 & 0.0 & 1.0 & 0.0 & 0.1 & -0.395 & -0.2 & -0.2 \\ 0.6 & 0.0 & 25.0 & 8.0 & 0.1 & 0.0 & -0.1 & -0.495 & -0.1 \\ 0.0 & 0.7 & 8.0. & 15.4 & 0.0 & 0.1 & -0.01 & -0.595 & -0.01 \\ 0.2 & 0.0 & 0.4 & 0.0 & 9.6 & 2.0 & -0.2 & -0.2 & -0.695 \\ 0.0 & 0.1 & 0.0 & 0.3 & 2.0 & 12.0 & -0.2 & -0.1 & -0.395 \\ -0.295 & -0.395 & 0.0 & 0.0 & 0.0 & 0.0 & 4.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.495 & -0.595 & 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.695 & -0.395 & 0.0 & 0.0 & 16.0 \end{pmatrix}.$$

This Jacobian matrix is positive definite, and both the existence and the uniqueness of the equilibrium solution to this example are guaranteed.

The Euler method converges to the equilibrium solution: $Q_{11}^* = 10.97$, $Q_{12}^* = 2.86$, $Q_{21}^* = 2.04$, $Q_{22}^* = 5.44$, $Q_{31}^* = 4.49$, $Q_{32}^* = 3.51$, $q_1^* = 1.15$, $q_2^* = 2.86$, $q_3^* = 0.46$ in 49 iterations. The profits of the firms are: $U_1 = 631.47$, $U_2 = 315.75$, and $U_3 = 325.63$. Graphical depictions of the product shipment and the quality level iterates are given in Figs. 5.13 and 5.14.

Compared to Example 5.4, firm 1 is the one with more cost in improving quality than the other firms when output increases, so the product shipments, quality levels, and profit of firm 1 decrease, but those of the other firms increase.

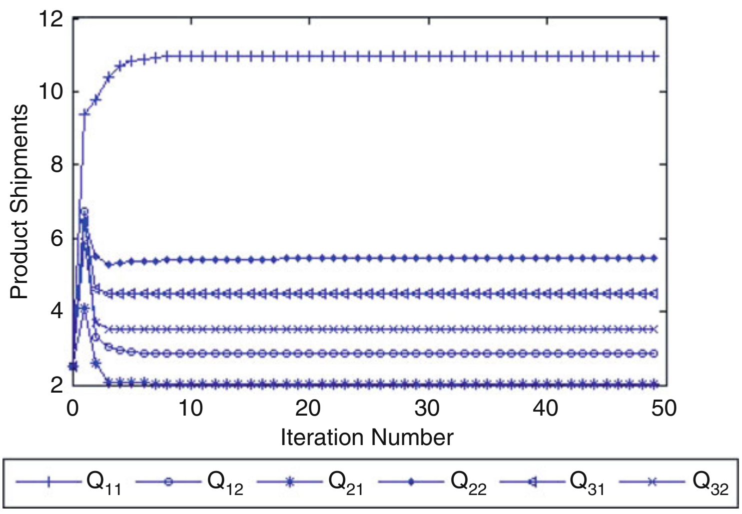

Fig. 5.13 Product shipment trajectories for Example 5.5

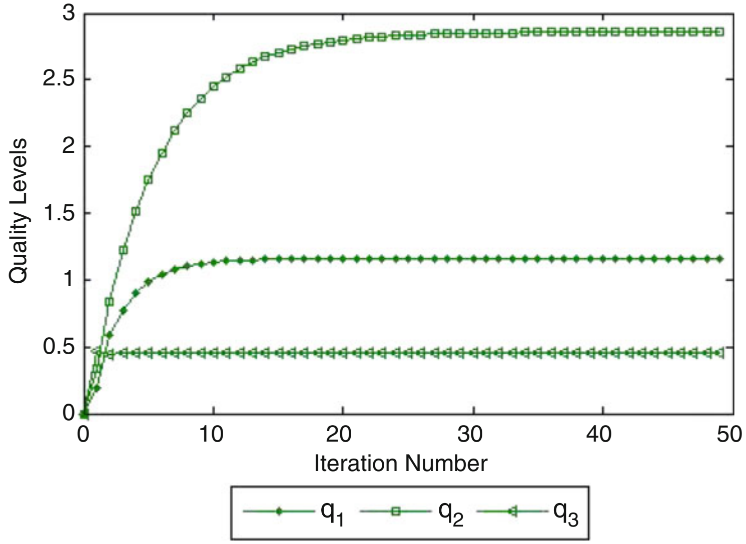

Fig. 5.14 Quality level trajectories for Example 5.5

Sensitivity Analysis for Example 5.1

For comparison purposes with the sensitivity analysis for Example 3.2, we now explore the impacts of quality lower bounds for Example 5.1. For the sensitivity analysis computations, we used the Euler method (5.30), (5.32), and (5.33), with (5.31) modified in that $\underline{q}_i$ replaces the 0.

We conduct a sensitivity analysis by varying quality lower bounds $\underline{q}_1$ and $\underline{q}_2$. Their initial values are set at 0, and they are increased to 5, 10, 15, 20, 25, and 30 to reflect the imposition of minimum quality standards. The results of this sensitivity analysis are shown in Figs. 5.15 and 5.16.

Similar to the results of the sensitivity analysis for Example 3.2, a very high minimum quality standard imposed on a firm may force it to leave the market and produce nothing for the market (cf. Figs. 5.15a, b and 5.16).

> In contrast to Chap. 3, in this chapter, consumers do differentiate the products and the quality levels of different firms, so there is no quality information asymmetry as in Chap. 3. Due to the fact that the products are substitutes and firms are competing, the price of one firm's product is still determined by both firms' quality standards, but, in contrast to Example 3.2, it is mainly affected by its own minimum quality standard when the firm is still in the market (Fig. 5.15e, f). Hence, the impact of a quality "free ride" is no longer as critical as it was in Example 3.2.

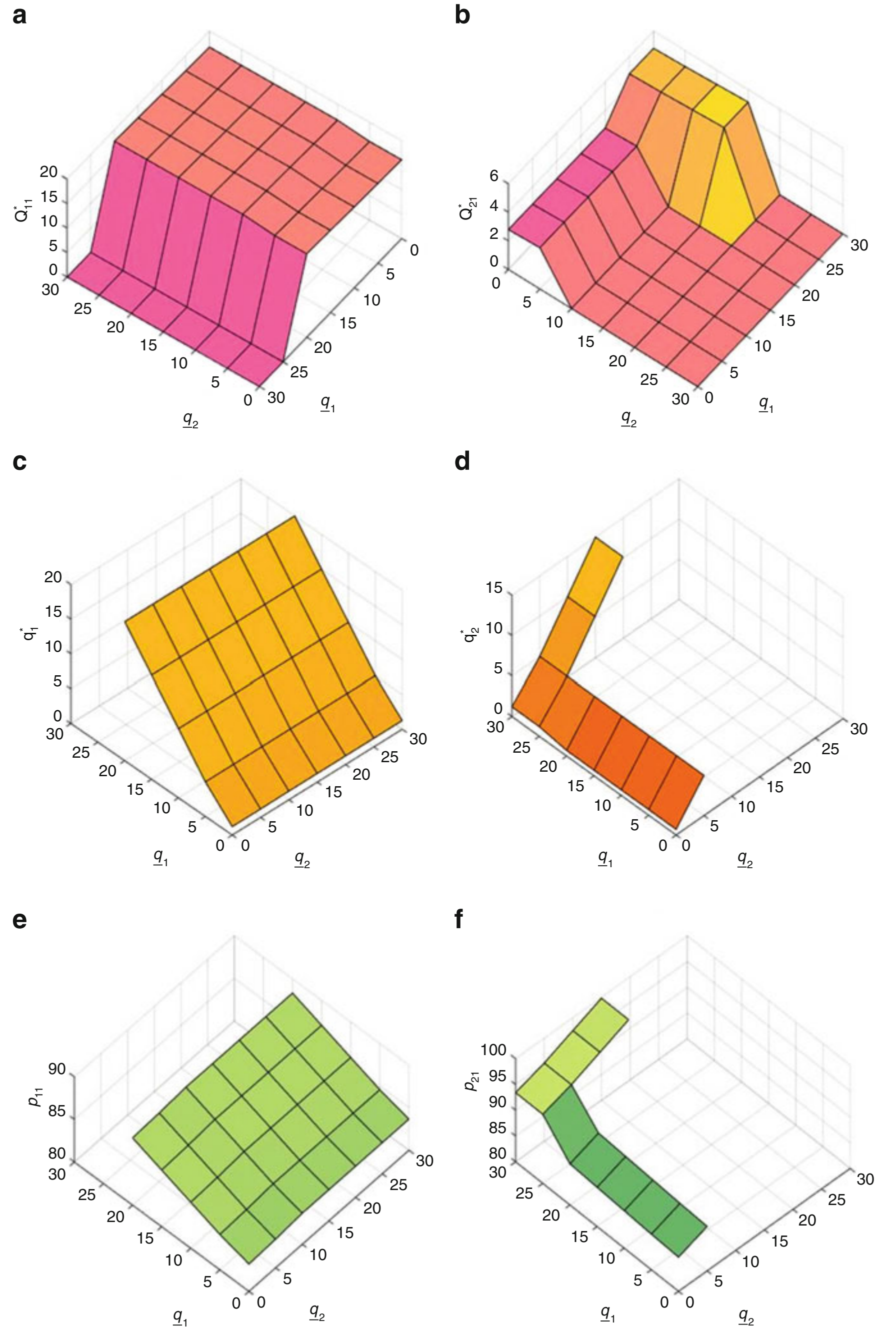

Fig. 5.15 Equilibrium product shipments, equilibrium quality levels, and prices as the minimum quality standards vary in Example 5.1. (**a**) Equilibrium product shipment (demand) of firm 1. (**b**) Equilibrium product shipment (demand) of firm 2. (**c**) Equilibrium quality level of firm 1. (**d**) Equilibrium quality level of firm 2. (**e**) Price of firm 1's product. (**f**) Price of firm 2's product

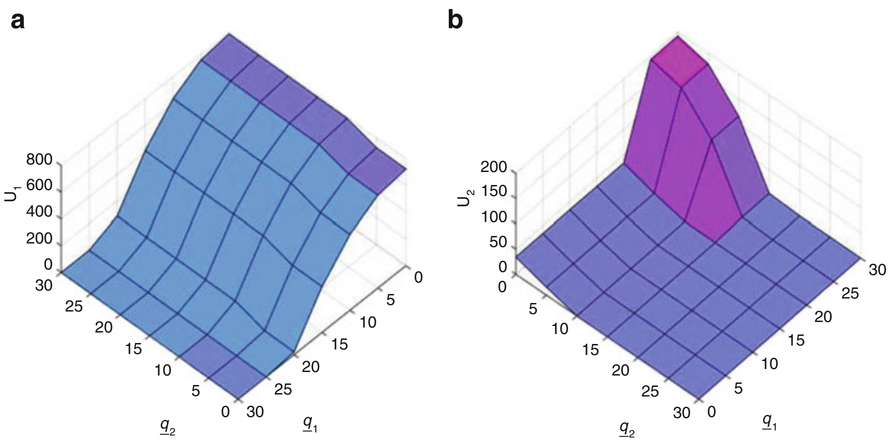

Fig. 5.16 Profits of the firms as the minimum quality standards vary in Example 5.1. (**a**) Profit of firm 1. (**b**) Profit of firm 2

As the imposed minimum quality standard of a firm increases, its equilibrium quality level increases (cf. Fig. 5.15c, d), which results in an increasing production cost of the firm. In this example, since the impact of "free ride" is no longer as dominant as in Example 3.2, before a firm leaves the market, as its minimum quality standard increases, its own equilibrium product shipment increases in order to improve revenue, its competitor's equilibrium quality level and product shipment remain the same, and its competitor's profit decreases only slightly due to competition (cf. Figs. 5.15a–d and 5.16).

From the results of this sensitivity analysis and those for Example 3.2, we can draw the conclusion that the product differentiation strategy is, indeed, able to significantly decrease the harmful impact of a quality free ride and the quality information asymmetry caused by product homogeneity in oligopoly competition. Firms and policy makers may want to take notice of this conclusion and act accordingly based on specific circumstances.

5.6 Summary and Conclusions

In this chapter, we developed a supply chain network oligopoly model in both static and dynamic realizations. The model handles product differentiation and includes transportation costs, R&D costs, as well as demand price functions that capture both demand for the substitutable products and their quality levels. The model is a Cournot-Nash model in which the strategic variables associated with each firm are its product shipments as well as the quality level of each firm's product. The consumers, in turn, signal their preferences for the differentiated products through the demand price functions for each product associated with the demand markets, which are spatially separated.

We derived the governing equilibrium conditions and provided alternative variational inequality formulations. We then proposed a continuous-time adjustment process for the dynamics of product shipments and quality levels. The qualitative properties of existence and uniqueness of the dynamic trajectories and the conditions for stability analysis are also provided.

We also described an algorithm, which yields closed form expressions for the product shipment and quality levels at each iteration and which provides a discrete-time discretization of the continuous-time product shipment and quality level trajectories. A convergence result is provided. Through several numerical examples, we illustrated the model and theoretical results, in order to demonstrate how the contributions in this chapter could be applied in practice. A sensitivity analysis on the impacts of minimum quality standards under product differentiation was conducted.

Both the static and the dynamic versions of our supply chain network oligopoly network model with product differentiation and quality levels contribute to the literature in the following ways: 1. The models are not limited to a preset number of firms (such as two, in the case of duopoly) or to specific functional forms (linear demand functions, for example). 2. Because of the generality of the production cost functions, which also include quality, R&D costs are also captured. 3. Both qualitative results, including stability analysis results, as well as an effective, and easy to implement, computational procedure are provided, along with numerical examples.

> Given that supply chain network oligopolies with differentiated products as well as quality issues are relevant to many industries, ranging from food to high tech, fashion, and even the Internet, the results in this chapter are relevant to many application domains.

5.7 Sources and Notes

This chapter is based on the paper by Nagurney and Li (2014). Here we standardized the notation and expanded the supply chain networks so that the economic activity of production is link-based, rather than node-based. We also included a remark on upper and lower bounds on quality and a sensitivity analysis on quality lower bounds to study the impacts of minimum quality standards on product shipments, quality levels, prices, and profits under product differentiation. We then compared these results with those under quality information asymmetry in Chap. 3.

In Nagurney et al. (1994), a dynamic network oligopoly model was developed using projected dynamical systems theory (cf. Dupuis and Nagurney 1993; Zhang and Nagurney 1995; Nagurney and Zhang 1996; Nagurney 1999). It extended the spatial oligopoly model of Dafermos and Nagurney (1987) to the dynamic domain. In this chapter, the static and dynamic network models that we construct generalize the former models to include both product differentiation and quality competition.

In addition, quality has also been studied in competitive perishable product supply chains in pharmaceuticals (Masoumi et al. 2012) and food (Yu and Nagurney 2013) using generalized networks, game theory, and variational inequality theory in which arc multipliers are utilized to capture decay of product, that is, product loss, over space and time on links. For other applications, ranging from perishable fresh produce to medical nuclear supply chains, see Nagurney et al. (2013).

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Chapter 6
Supply Chain Network Competition with Multiple Freight Options

Abstract This chapter extends the results in Chap. 5 to include multiple freight options for the manufacturers to ship their products from their manufacturing plants to consumers at the demand markets. We first develop a static supply chain network model of Cournot-Nash competition with product differentiation, multiple freight options, and quality competition. Each manufacturing firm seeks to maximize its own profit by determining its product shipments and product quality. We utilize variational inequality theory for the formulation of the governing Cournot-Nash equilibrium. We then construct the projected dynamical systems model, which provides a continuous-time evolution of the product shipments of the firms and the product quality levels, and whose set of stationary points coincides with the set of solutions to the variational inequality problem. We establish stability analysis results using a monotonicity approach and construct a discrete-time version of the continuous-time adjustment processes, which yields an algorithm, with closed form expressions at each iteration. The algorithm is then utilized to compute the solutions to several numerical examples. A sensitivity analysis on changes in the demand price functions is also conducted.

6.1 Introduction

In Chap. 5, we developed a competitive supply chain network model with product differentiation and quality competition in which there was a single freight option between the manufacturing plant of each firm and each of their demand markets, which could differ by mode and/or carrier. Hence, a firm, in effect, would have known a priori its most appropriate means of transportation from the plant to each demand market but would still need to determine the volume of the product shipment to each demand market (as well as the quality of its product). In this chapter, we develop a static model and its dynamic counterpart that allow each competing firm in the supply chain network to choose among the freight options that are available. Hence, multiple freight options may be used between a manufacturing plant and a demand market if the volume of shipments calls for this under the optimizing behavior of the firms in competition. Such a framework allows each firm to determine its optimal freight service provision for the shipment of its product. Having the capability to explore different freight options provides greater flexibility

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for a firm in the case of a freight option reaching its capacity. In addition, it can increase a firm's responsiveness to consumers and its agility when demand increases for its products. Groothede et al. (2005) describe the Sony case, which used a combination of long distance container transport by sea combined with air transport. The demand that can be predicted well in advance uses the sea mode, and the excess demand uses the air mode. As also noted by these authors, the possibilities of transportation in parallel using alternative modes, may include the combination of truck transport and rail transport or the combination of truck transport and short sea transport. Bookbinder and Prentice (2013) refers to such parallel freight transport as the *new intermodal*.

Also, as emphasized in Qiang et al. (2009), alternative transportation modes can be used in the case of the failure of a transportation mode. Indeed, many authors have emphasized that redundancy needs to be considered in supply chains in order to prevent supply chain disruptions. Wilson (2007) found that the existence of transportation alternatives significantly improved supply chain performance in the case of transportation disruptions.

For example, in terms of real world applications, and as noted by Nagurney et al. (2013b), a surge in demand for apparel and related products based on the top-grossing Disney animated film Frozen, led to shipments from manufacturing plants in China increasingly being airlifted in addition to using maritime transport (see Palmeri 2014). Also, biomass supply chains may utilize multiple freight modes for shipment, including truck, rail, as well as pipelines and ships, with, of course, distinct associated costs (see Bonilla and Whittaker 2009; Floden 2015).

Tavasszy and de Jong (2014) emphasized that freight models have been around since the 1960s, and have evolved essentially in parallel with passenger transport models, which are much more highly developed and have been widely applied in practice (Boyce and Williams 2015). The edited volume by Ben-Aliva et al. (2013) stated the need to develop new freight transport models on local, regional, and global levels. Wigan and Southworth (2005) noted that freight systems are increasingly complex placing high demands on freight system modeling.

In terms of supply chain network equilibria, Nagurney et al. (2002) developed a dynamic multilevel model with logistical, financial, and informational flows. Nagurney and Nagurney (2010) proposed a multicriteria supply chain network design model with environmental concerns, which allowed for the possibility of multiple freight and production possibilities. Nagurney and Yu (2011) focused on fashion supply chain management with cost and time minimization that also allowed for multiple freight options (see also Nagurney and Yu 2012). Nagurney et al. (2015), in turn, focused on disaster relief and presented an integrated supply chain network model with alternative freight modes for delivery of disaster relief products, where time was also of utmost concern. The book by Nagurney et al. (2013b) presented a spectrum of supply chain network models for perishable products in which generalized networks are used in both optimization and game theoretical constructs to formulate and solve supply chains ranging from medical nuclear products to fresh produce to pharmaceuticals. The above supply chain network models, however, did not consider product quality as a strategic variable of a firm.

The chapter is organized as follows. In Sect. 6.2, we develop both the supply chain network equilibrium model with product differentiation, multiple freight options, and quality competition and its dynamic counterpart. In Sect. 6.3, we provide some qualitative results including stability analysis results. In Sect. 6.4, we present the algorithm, which yields closed form expressions for the product shipment volumes and the quality levels, at each iteration. We then apply the algorithm to several numerical examples and conduct a sensitivity analysis on the demand price functions in Sect. 6.5 to gain insights into the network economics and the evolutionary process. We summarize and present our conclusions in Sect. 6.6. The chapter ends with the Sources and Notes Sect. 6.7

6.2 The Supply Chain Network Model with Multiple Freight Options

In this section, we develop both the static and the dynamic supply chain network models with differentiated products, multiple freight options, and quality competition. In the supply chain network economy under study, as depicted in Fig. 6.1 there are I firms, with a typical firm denoted by i, o freight service options, with a typical one denoted by j, and n_R demand markets. A typical demand market is denoted by R_k. While we consider different quality levels among the firms' products, we do not explicitly handle the quality level among freight service options (but allow for differing costs). Hence, the models in this chapter extend those in Chap. 5 to include multiple freight options between the manufacturing plants of the firms and the consumers at the demand markets.

The demand for a product is reflected in the demand price function at a demand market. We allow for consumers to differentiate among the products provided by the firms in terms of the product quality. It is assumed that the firms compete under the Cournot (1938)-Nash (1950, 1951) equilibrium concept of noncooperative behavior, as was also done in Chap. 5, and that they select both their production quantities (and shipment levels) as well as the quality levels of their products. The consumers, in turn, signal their preferences for the products through the demand price functions associated with the demand markets. The demand price functions are, in general, functions of the demands for the products at all the demand markets as well as their quality levels.

We first develop the equilibrium model and derive alternative variational inequality formulations. We then describe the underlying dynamics associated with the product shipments as well as the quality levels and present the projected dynamical systems model whose set of stationary points corresponds to the set of solutions of the variational inequality problem.

There is a distinct (but substitutable) product produced by each of the I firms, represented by the top-tiered nodes in Fig. 6.1. These nodes are joined by links, respectively, to the manufacturing plant node i' of each firm i; $i = 1, \ldots, I$. The links

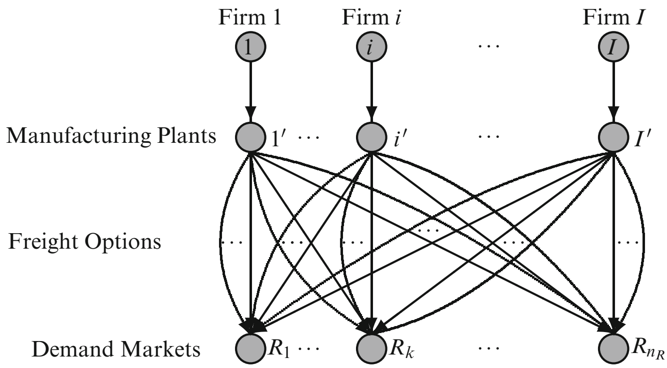


Fig. 6.1 The supply chain network topology with multiple freight options

joining the manufacturing plant nodes correspond to the freight option links with transportation of the products taking place on them. The manufacturing plant nodes are joined by such links to the demand market, bottom-tiered, nodes in Fig. 6.1. A freight option here corresponds to a specific shipment mode and carrier choice that the firm may have at its disposal and can include, for example, if feasible, an express shipment by a specific freight carrier, next day delivery, or even a combination of freight service providers engaged in intermodal transportation. Each link joining a manufacturing plant node in Fig. 6.1 represents a specific option that the firm wishes to evaluate in terms of freight service provision cost. The transportation cost can be a generalized cost that can include a weighting of time and even emissions, provided that the firm or firms are concerned about such costs. Note that here capacity is endogenously captured in the transportation cost functions, which can be nonlinear.

Of course, if a firm has its own freight transport capabilities then a link would correspond to such an option. The solution of the model provides information as to which freight options should be used by each firm and at which level in terms of the volume of product shipment to each specific demand market.

The notation for the models is given in Table 6.1. All vectors here are assumed to be column vectors, as was done in Chaps. 2 through 5.

The following conservation of flow equations must hold:

$$s_i = \sum_{j=1}^{o} \sum_{k=1}^{n_R} Q_{ijk}, \quad i = 1, \ldots, I, \tag{6.1}$$

$$d_{ik} = \sum_{j=1}^{o} Q_{ijk}, \quad i = 1, \ldots, I; k = 1, \ldots, n_R, \tag{6.2}$$

$$Q_{ijk} \geq 0, \quad i = 1, \ldots, I; j = 1, \ldots, o; k = 1, \ldots, n_R, \tag{6.3}$$

Table 6.1 Notation for the supply chain network models (static and dynamic) with product differentiation and multiple freight options

Notation	Definition
Q_{ijk}	The nonnegative shipment of firm i's product to demand market R_k via freight option j. We group the $\{Q_{ijk}\}$ elements for all j and k into the vector $Q_i \in R^{on_R}_+$ and the vectors Q_i for all i into the vector $Q \in R^{Ion_R}_+$
s_i	The nonnegative product output produced by firm i. We group the production outputs for all i into the vector $s \in R^I_+$
q_i	The quality level, or, simply, the quality, of product i, which is produced by firm i. The quality levels of all firms are grouped into the vector $q \in R^I_+$
d_{ik}	The demand for the product of firm i at demand market R_k. We group the demands for all i and k into the vector $d \in R^{In_R}_+$
$\hat{f}_i(s, q_i)$	The production cost at firm i's manufacturing plant i'
$\hat{c}_{ijk}(Q_{ijk})$	The total cost of transporting the volume Q_{ijk} of the product from firm i to demand market k via freight option j
$\hat{c}_{ik}(Q_{ik})$	The total transportation cost associated with shipping firm i's product, produced at manufacturing plant i', to demand market R_k
$\rho_{ik}(d, q)$	The demand price function for firm i's product at demand market R_k

and, since the quality levels must also be nonnegative, we must have that

$$q_i \geq 0, \quad i = 1, \ldots, I. \tag{6.4}$$

Hence, according to (6.1), the quantity of the product produced by each manufacturing firm at its manufacturing plant is equal to the sum of the amounts of the product transported to all the demand markets, and the quantity of a firm's product consumed at a demand market, according to (6.2), is equal to the amount transported from the manufacturing plant to the demand market via all the freight options. Both the product shipments and the quality levels must be nonnegative.

From Table 6.1 we see that $\hat{c}_{ik}$ denotes the total transportation cost associated with transporting firm i's product to demand market R_k, which is given by the function:

$$\hat{c}_{ik} = \sum_{j=1}^{o} \hat{c}_{ijk}(Q_{ijk}), \quad i = 1, \ldots, I; k = 1, \ldots, n_R. \tag{6.5}$$

In our model, it is the manufacturer that pays for the transportation of the product. The production cost functions, the transportation cost functions, and the demand price functions are assumed to be continuous and twice continuously differentiable.

The profit or utility U_i of firm i; $i = 1, \ldots, I$, is given by the expression

$$U_i = \sum_{k=1}^{n_R} \rho_{ik} d_{ik} - \hat{f}_i - \sum_{k=1}^{n_R} \hat{c}_{ik}, \tag{6.6}$$

which is the difference between its total revenue and its total cost.

In view of (6.1), (6.2), (6.3), (6.4), (6.5) and (6.6), one may write the profit as a function solely of the product shipments and quality levels, that is,

$$U = U(Q, q), \tag{6.7}$$

where U is the I-dimensional vector with components: $\{U_1, \ldots, U_I\}$.

Let K^i denote the feasible set corresponding to firm i, where $K^i \equiv \{(Q_i, q_i) | Q_i \geq 0, \text{ and } q_i \geq 0\}$ and define $K \equiv \prod_{i=1}^{I} K^i$.

We consider the competitive oligopolistic market mechanism, as was also done for the supply chain network models with differentiated products in Chap. 5, in which the I firms supply their products in a noncooperative manner, each one trying to maximize its own profit. We seek to determine a nonnegative product shipment pattern (from which the product outputs via (6.1) can then be determined) and quality level pattern (Q^*, q^*) for which the I firms will be in a state of equilibrium as defined below. In particular, as emphasized in Chap. 2, and also in a similar vein in Chap. 5, Nash (1950, 1951) generalized Cournot's (1938) concept of an equilibrium among several players, in what has been come to be called a noncooperative game.

Definition 6.1: A Supply Chain Network Cournot-Nash Equilibrium with Product Differentiation, Multiple Freight Options, and Quality Levels
A product shipment and quality level pattern $(Q^*, q^*) \in K$ is said to constitute a Cournot-Nash equilibrium if for each firm i,

$$U_i(Q_i^*, q_i^*, \hat{Q}_i^*, \hat{q}_i^*) \geq U_i(Q_i, q_i, \hat{Q}_i^*, \hat{q}_i^*), \quad \forall (Q_i, q_i) \in K^i, \tag{6.8}$$

where

$$\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_I^*); \text{ and } \hat{q}_i^* \equiv (q_1^*, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_I^*). \tag{6.9}$$

According to (6.8), an equilibrium is established if no manufacturing firm can unilaterally improve upon its profits by selecting an alternative vector of product shipments and quality level of its products. Observe that the vector of strategic product shipment variables for each firm is of higher dimension than in the models in Chap. 6 since firms now explicitly select the size of shipment associated with each freight option.

6.2.1 Alternative Variational Inequality Formulations

We now present alternative variational inequality formulations of the above Cournot-Nash equilibrium with product differentiation in the following theorem.

Theorem 6.1: Variational Inequality Formulation of the Supply Chin Network Cournot-Nash Equilibrium with Product Differentiation and Multiple Freight Options

Assume that for each firm i the profit function $U_i(Q, q)$ is concave with respect to the variables $\{Q_{i11}, \ldots, Q_{ion_R}\}$, and q_i, and is continuous and continuously differentiable. Then $(Q^*, q^*) \in K$ is a Cournot-Nash equilibrium according to Definition 6.1 if and only if it satisfies the variational inequality

$$-\sum_{i=1}^{I}\sum_{j=1}^{o}\sum_{k=1}^{n_R}\frac{\partial U_i(Q^*,q^*)}{\partial Q_{ijk}}\times(Q_{ijk}-Q^*_{ijk})-\sum_{i=1}^{I}\frac{\partial U_i(Q^*,q^*)}{\partial q_i}\times(q_i-q^*_i)\geq 0,$$

$$\forall (Q,q)\in K, \tag{6.10}$$

or, equivalently, $(s^*, d^*, Q^*, q^*) \in \mathcal{K}^1$ is an equilibrium production, transport, consumption, and quality level pattern if and only if it satisfies the variational inequality

$$\sum_{i=1}^{I}\frac{\partial \hat{f}_i(s^*,q^*_i)}{\partial s_i}\times(s_i-s^*_i)-\sum_{i=1}^{I}\sum_{k=1}^{n_R}\rho_{ik}(d^*,q^*)\times(d_{ik}-d^*_{ik})$$

$$+\sum_{i=1}^{I}\sum_{j=1}^{o}\sum_{k=1}^{n_R}\left[\frac{\partial \hat{c}_{ijk}(Q^*_{ijk})}{\partial Q_{ijk}}-\sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*,q^*)}{\partial d_{ik}}\times d^*_{il}\right]\times(Q_{ijk}-Q^*_{ijk})$$

$$+\sum_{i=1}^{I}\left[\frac{\partial \hat{f}_i(s^*,q^*_i)}{\partial q_i}-\sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*,q^*)}{\partial q_i}\times d^*_{il}\right]\times(q_i-q^*_i)\geq 0,\ \ \forall (s,d,Q,q)\in\mathcal{K}^1, \tag{6.11}$$

where $\mathcal{K}^1 \equiv \{(s, d, Q, q) \,|\, Q \geq 0,\ q \geq 0, \text{and } (6.1) \text{ and } (6.2) \text{ hold}\}$.

Proof: Equation (6.10) follows directly from Gabay and Moulin (1980) and Dafermos and Nagurney (1987).

In order to obtain (6.11) from (6.10), we note that, in light of (6.1) and (6.2): for each i, j, and k,

$$-\frac{\partial U_i(Q^*,q^*)}{\partial Q_{ijk}}$$

$$=\left[\frac{\partial \hat{f}_i(s^*,q^*_i)}{\partial Q_{ijk}}+\frac{\partial \hat{c}_{ijk}(Q^*_{ijk})}{\partial Q_{ijk}}-\rho_{ik}(d^*,q^*)\times\frac{\partial d_{ik}}{\partial Q_{ijk}}-\sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*,q^*)}{\partial Q_{ijk}}\times d^*_{il}\right]$$

$$=\left[\frac{\partial \hat{f}_i(s^*,q^*_i)}{\partial s_i}\frac{\partial s_i}{\partial Q_{ijk}}+\frac{\partial \hat{c}_{ijk}(Q^*_{ijk})}{\partial Q_{ijk}}-\rho_{ik}(d^*,q^*)-\sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*,q^*)}{\partial d_{ik}}\frac{\partial d_{ik}}{\partial Q_{ijk}}\times d^*_{il}\right]$$

$$=\left[\frac{\partial \hat{f}_i(s^*,q^*_i)}{\partial s_i}+\frac{\partial \hat{c}_{ijk}(Q^*_{ijk})}{\partial Q_{ijk}}-\rho_{ik}(d^*,q^*)-\sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*,q^*)}{\partial d_{ik}}\times d^*_{il}\right], \tag{6.12}$$

and for each i,

$$-\frac{\partial U_i(Q^*,q^*)}{\partial q_i} = \left[\frac{\partial \hat{f}_i(s^*,q_i^*)}{\partial q_i} - \sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^*,q^*)}{\partial q_i} \times d_{il}^*\right]. \tag{6.13}$$

Multiplying the right-most expression in (6.12) by $(Q_{ijk} - Q_{ijk}^*)$ and summing the resultant over all i, j, and k; similarly, multiplying the right-most expression in (6.13) by $(q_i - q_i^*)$ and summing the resultant over all i yields, respectively:

$$\sum_{i=1}^{I}\sum_{j=1}^{o}\sum_{k=1}^{n_R}\left[\frac{\partial \hat{f}_i(s^*,q^*)}{\partial s_i} + \frac{\partial \hat{c}_{ijk}(Q_{ijk}^*)}{\partial Q_{ijk}} - \rho_{ik}(d^*,q^*) - \sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*,q^*)}{\partial d_{ik}} \times d_{il}^*\right] \times (Q_{ijk} - Q_{ijk}^*) \tag{6.14}$$

and

$$\sum_{i=1}^{I}\left[\frac{\partial \hat{f}_i(s^*,q_i^*)}{\partial q_i} - \sum_{l=1}^{n_R}\frac{\partial \rho_{il}(d^*,q^*)}{\partial q_i} \times d_{il}^*\right] \times (q_i - q_i^*). \tag{6.15}$$

Finally, summing (6.14) and (6.15) and then using constraints (6.1) and (6.2), yields variational inequality (6.11). □

We now put the above variational inequalities into standard variational inequality form (see (2.1a)), that is, determine $X^* \in \mathcal{K} \subset R^N$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{6.16}$$

where F is a given continuous function from $\mathcal{K}$ to R^N, and $\mathcal{K}$ is a closed and convex set.

We define the $(Ion_R + I)$-dimensional vector $X \equiv (Q, q)$ and the $(Ion_R + I)$-dimensional vector $F(X) \equiv (F^1(X), F^2(X))$ with the (i, j, k)-th component, F_{ijk}^1, of $F^1(X)$ given by

$$F_{ijk}^1(X) \equiv -\frac{\partial U_i(Q,q)}{\partial Q_{ijk}}, \tag{6.17}$$

the i-th component, F_i^2, of $F^2(X)$ given by

$$F_i^2(X) \equiv -\frac{\partial U_i(Q,q)}{\partial q_i}, \tag{6.18}$$

and with the feasible set $\mathcal{K} \equiv K$. Then, clearly, variational inequality (6.10) can be put into standard form (6.16).

In a similar manner, one can establish that variational inequality (6.11) can also be put into standard variational inequality form (6.16).

6.2.2 *The Dynamic Model*

We now propose dynamic adjustment processes for the evolution of the shipment volumes and the quality levels. Observe that, for a current product shipment and quality level pattern at time t, $X(t) = (Q(t), q(t))$, $-F^1_{ijk}(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial Q_{ijk}}$ given by (6.17), is the marginal utility (profit) of firm i with respect to its transport of the product to demand market j via k. Similarly, $-F^2_i(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial q_i}$, given by (6.18), is for i's marginal utility (profit) with respect to its quality level. In this framework, the rate of change of the product flow between a firm's manufacturing plant and demand market pair using k, (i, j, k), is in proportion to $-F^1_{ijk}(X)$, as long as the product shipment volume Q_{ijk} is positive. Namely, when $Q_{ijk} > 0$,

$$\dot{Q}_{ijk} = \frac{\partial U_i(Q,q)}{\partial Q_{ijk}}, \tag{6.19}$$

where $\dot{Q}_{ijk}$ denotes the rate of change of Q_{ijk}. However, when $Q_{ijk} = 0$, the nonnegativity condition (6.3) forces the product shipment volume Q_{ijk} to remain zero when $\frac{\partial U_i(Q,q)}{\partial Q_{ijk}} \leq 0$. Hence, in this case, we are only guaranteed of having possible increases of the shipment volume. Namely, when $Q_{ijk} = 0$,

$$\dot{Q}_{ijk} = \max\left\{0, \frac{\partial U_i(Q,q)}{\partial Q_{ijk}}\right\}. \tag{6.20}$$

We may write (6.19) and (6.20) concisely as:

$$\dot{Q}_{ijk} = \begin{cases} \frac{\partial U_i(Q,q)}{\partial Q_{ijk}}, & \text{if} \quad Q_{ijk} > 0 \\ \max\{0, \frac{\partial U_i(Q,q)}{\partial Q_{ijk}}\}, & \text{if} \quad Q_{ijk} = 0. \end{cases} \tag{6.21}$$

As for the quality levels, when $q_i > 0$, then

$$\dot{q}_i = \frac{\partial U_i(Q,q)}{\partial q_i}, \tag{6.22}$$

where $\dot{q}_i$ denotes the rate of change of q_i; otherwise:

$$\dot{q}_i = \max\left\{0, \frac{\partial U_i(Q,q)}{\partial q_i}\right\}, \tag{6.23}$$

since q_i must be nonnegative.

Combining (6.22) and (6.23), we may write:

$$\dot{q}_i = \begin{cases} \frac{\partial U_i(Q,q)}{\partial q_i}, & \text{if} \quad q_i > 0 \\ \max\{0, \frac{\partial U_i(Q,q)}{\partial q_i}\}, & \text{if} \quad q_i = 0. \end{cases} \tag{6.24}$$

Applying (6.21) to all firm and demand market pairs (i,k); $i = 1,\ldots,I$; $k = 1,\ldots,n_R$, and all freight options $j = 1,\ldots,o$, and applying (6.24) to all firms i; $i = 1,\ldots,I$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes associated with the firm product shipment volumes and quality levels, in vector form, as:

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)), \tag{6.25}$$

where, since $\mathscr{K}$ is a convex polyhedron, according to Dupuis and Nagurney (1993) (see also Chaps. 2, 3, and 4), $\Pi_{\mathscr{K}}(X, -F(X))$ is the projection, with respect to $\mathscr{K}$, of the vector $-F(X)$ at X defined as

$$\Pi_{\mathscr{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathscr{K}}(X - \delta F(X)) - X}{\delta} \tag{6.26}$$

with $P_{\mathscr{K}}$ denoting the projection map:

$$P(X) = \text{argmin}_{z \in \mathscr{K}} \|X - z\|, \tag{6.27}$$

and where recall that $\|\cdot\| = \langle x, x \rangle$. Hence, $F(X) = -\nabla U(Q, q)$, where $\nabla U(Q, q)$ is the vector of marginal utilities with components given by (6.17) and (6.18).

We now interpret the ODE (6.25) in the context of the supply chain network model with product differentiation, multiple freight options, and quality competition. We do so, for completeness, and easy reference. First, note that ODE (6.25) ensures that the product shipments and quality levels are always nonnegative. Indeed, if one were to consider, instead, the ordinary differential equation: $\dot{X} = -F(X)$, or, equivalently, $\dot{X} = \nabla U(X)$, such an ODE would not ensure that $X(t) \geq 0$, for all $t \geq 0$, unless additional restrictive assumptions were to be imposed. ODE (6.25), however, retains the interpretation that if X at time t lies in the interior of $\mathscr{K}$, then the rate at which X changes is greatest when the vector field $-F(X)$ is greatest. Also, when the vector field $-F(X)$ pushes X to the boundary of the feasible set $\mathscr{K}$, then the projection $\Pi_{\mathscr{K}}$ ensures that X stays within $\mathscr{K}$. Hence, the shipment volumes and quality levels are always nonnegative.

Recall now the definition of $F(X)$ for the supply chain network model, in which case the dynamical system (6.25) states that the rate of change of the product shipments and quality levels is greatest when the firms' marginal utilities (profits) are greatest. If the marginal utilities with respect to these product shipments are positive, then the firms will increase their product shipments; if they are negative, then they will decrease them. The same adjustment behavior holds for the product quality levels. The ODE (6.25) is a reasonable continuous adjustment process for the supply chain network problem with product differentiation and multiple freight options.

As noted in Chap. 2, Dupuis and Nagurney (1993) developed the fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by (6.26). We cite the following theorem from that paper (see also Chaps. 2 and 5).

Theorem 6.2: Equivalence of Equilibria and Stationary Points
X^* solves the variational inequality problem (6.16) if and only if it is a stationary point of the ODE (6.25), that is,

$$\dot{X} = 0 = \Pi_{\mathscr{K}}(X^*, -F(X^*)). \tag{6.28}$$

This theorem demonstrates that the necessary and sufficient condition for a product shipment and quality level pattern $X^* = (Q^*, q^*)$ to be a Cournot-Nash equilibrium, according to Definition 6.1, is that $X^* = (Q^*, q^*)$ is a stationary point of the adjustment process defined by ODE (6.25), that is, X^* is the point at which $\dot{X} = 0$.

Consider now the competitive supply chain network economic system consisting of the manufacturing firms, who, in order to maximize their utilities, adjust their product flow and quality level patterns by instantly responding to the marginal utilities, according to (6.25). The following questions naturally arise and are of interest. Does the utility gradient process defined by (6.25), approach a Cournot-Nash equilibrium, and how does it approach an equilibrium in term of the convergence rate? Also, for a given Cournot-Nash equilibrium, do all the disequilibrium product shipment and quality level patterns that are close to this equilibrium always stay near by? Motivated by these questions, we now present the stability analysis of Cournot-Nash equilibrium, under the above utility gradient process.

The stability of Cournot-Nash equilibrium has been well-studied in the history of oligopoly theory. Among others, Arrow and Hurwicz (1977) investigated the asymptotical stability of Cournot-Nash equilibrium (see also Vives 1999). In that paper, in place of the projection operator, $\Pi_{\mathscr{K}}$, a discontinuous matrix function, γ, was used to multiply the utility gradient on the right-hand side of the ODE (but in a much simpler model than developed here), to ensure that the adjustment process would evolve within the nonnegative orthant. Okuguchi and Szidarovszky (1990) also studied the asymptotical stability of the utility gradient process at the Cournot-Nash equilibrium, under the assumptions of linear price functions and quadratic cost functions, and with no quality levels as strategic variables or with the spatial dimension.

6.3 Stability Under Monotonicity

We now turn to the questions raised in the previous section, that is, whether and under what conditions do the adjustment processes defined by ODE (6.25) approaches a Cournot-Nash equilibrium? We first note that Lipschitz continuity of $F(X)$ (cf. Chap. 2) guarantees the existence of a unique solution to (6.29) below, where we have that $X^0(t)$ satisfies ODE (6.25) with initial product shipment and quality level pattern (Q^0, q^0). In other words, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)), \quad X(0) = X^0, \tag{6.29}$$

with $X^0(0) = X^0$.

For the definitions of stability and monotonicity, the stability properties of the gradient process under various monotonicity conditions, and the associated proofs, please refer to Chap. 2 and, for the single freight option model between a firm and demand market, to Chap. 5.

We now turn to establishing existence and uniqueness results of the equilibrium pattern by utilizing the theory of variational inequalities.

In the context of the supply chain network problem, where $F(X)$ is the vector of negative marginal utilities as in (6.17)–(6.18), we point out that if the utility functions are twice differentiable and the Jacobian of the negative marginal utility functions (or, equivalently, the negative of the Hessian matrix of the utility functions) is positive definite, then the corresponding $F(X)$ is strictly monotone.

In a practical supply chain network model, it is reasonable to expect that the utility of any firm i, $U_i(Q,q)$, would decrease whenever its output has become sufficiently large, that is, when U_i is differentiable, $\frac{\partial U_i(Q,q)}{\partial Q_{ijk}}$ is negative for sufficiently large Q_{ijk}, because $q_i \geq Q_{ijk}$, for all j; the same holds for sufficiently large q_i. Hence, the following assumption is not unreasonable:

Assumption 6.1
Suppose that in the supply chain network model there exists a sufficiently large M, such that for any (i,j,k),

$$\frac{\partial U_i(Q,q)}{\partial Q_{ijk}} < 0, \tag{6.30}$$

for all product shipment patterns Q with $Q_{ijk} \geq M$ and that there exists a sufficiently large $\bar{M}$, such that for any i,

$$\frac{\partial U_i(Q,q)}{\partial q_i} < 0, \tag{6.31}$$

for all quality level patterns q with $q_i \geq \bar{M}$.

We now give an existence result.

Proposition 6.1: Existence
Any supply chain network problem, as described above, that satisfies Assumption 6.1 possesses at least one equilibrium product shipment and quality level pattern.

Proof: The proof follows from Proposition 1 in Zhang and Nagurney (1995). □

We now present the uniqueness result, the proof of which follows from the basic theory of variational inequalities (cf. Chap. 2).

Proposition 6.2: Uniqueness
Suppose that F is strictly monotone at any equilibrium point of the variational inequality problem defined in (6.16). Then it has at most one equilibrium point.

Theorem 6.3: Existence and Uniqueness
Suppose that F is strongly monotone. Then there exists a unique solution to variational inequality (6.16); equivalently, to variational inequality (6.10) (and (6.11)).

The following theorem presents the stability properties of the projected dynamical system described in (6.25) (cf. Theorems 2.15, 2.16, and 2.17).

Theorem 6.4: Stability

(i). *If $F(X)$ is monotone, then every supply chain network Cournot-Nash equilibrium, X^*, provided its existence, is a global monotone attractor for the projected dynamical system. If $F(X)$ is locally monotone at X^*, then it is a monotone attractor for the projected dynamical system.*
(ii). *If $F(X)$ is strictly monotone, then there exists at most one supply chain network Cournot-Nash equilibrium, X^*. Furthermore, provided existence, the unique spatial Cournot-Nash equilibrium is a strictly global monotone attractor for the projected dynamical system. If $F(X)$ is locally strictly monotone at X^*, then it is a strictly monotone attractor for the projected dynamical system.*
(iii). *If $F(X)$ is strongly monotone, then there exists a unique supply chain network Cournot-Nash equilibrium, X^*, which is globally exponentially stable for the projected dynamical system. If $F(X)$ is locally strongly monotone at X^*, then it is exponentially stable.*

We now present two examples in order to illustrate some of the above concepts and results.

Example 6.1
Consider a supply chain network problem consisting of two firms, each with two freight options, and a single demand market R_1, as depicted in Fig. 6.2.

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 + s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 + s_2 + q_2^2 + 37,$$

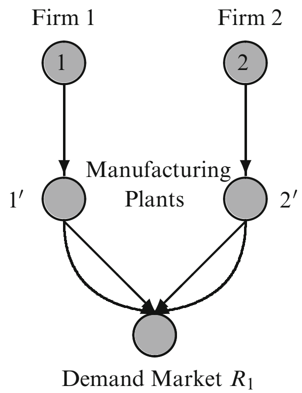


Fig. 6.2 The supply chain network topology for Example 6.1

the total transportation cost functions are:

$$\hat{c}_{111}(Q_{111}) = 0.5Q_{111}^2 + 0.4Q_{111}, \quad \hat{c}_{121}(Q_{121}) = 0.7Q_{121}^2 + 0.5Q_{121},$$
$$\hat{c}_{211}(Q_{211}) = 0.6Q_{211}^2 + 0.4Q_{211}, \quad \hat{c}_{221}(Q_{221}) = 0.4Q_{221}^2 + 0.2Q_{221},$$

and the demand price functions are:

$$\rho_{11}(d,q) = -d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2 + 70,$$
$$\rho_{21}(d,q) = -0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2 + 70.$$

The Jacobian matrix of $-\nabla U(Q,q)$, denoted by $J(Q_{111}, Q_{121}, Q_{211}, Q_{221}, q_1, q_2)$, is

$$J(Q_{111}, Q_{121}, Q_{211}, Q_{221}, q_1, q_2) = \begin{pmatrix} 5.0 & 4.0 & 0.4 & 0.4 & -0.3 & -0.05 \\ 4.0 & 5.4 & 0.4 & 0.4 & -0.3 & -0.05 \\ 0.6 & 0.6 & 8.2 & 7.0 & -0.1 & -0.5 \\ 0.6 & 0.6 & 7.0 & 7.8 & -0.1 & -0.5 \\ -0.3 & -0.3 & 0.0 & 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & -0.5 & -0.5 & 0.0 & 2.0 \end{pmatrix}.$$

This Jacobian matrix is positive definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, $-\nabla U(Q,q)$ is strongly monotone. Thus, both the existence and the uniqueness of the solution to variational inequality (6.11) with respect to this example are guaranteed. Moreover, the equilibrium solution, which is: $Q_{111}^* = 8.40$, $Q_{121}^* = 5.93$, $Q_{211}^* = 3.18$, $Q_{221}^* = 5.01$, $q_1^* = 1.08$, and $q_2^* = 2.05$ is globally exponentially stable.

Example 6.2
We now present Example 6.2 with the supply chain network topology depicted in Fig. 6.3. The supply chain consists of two competing firms, each with two Freight options, and two demand markets R_1 and R_2.

The production cost functions are:

$$\hat{f}_1(s,q_1) = s_1^2 + s_1 + s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s,q_2) = 2s_2^2 + 2s_1 + s_2 + q_2^2 + 37,$$

the total transportation cost functions are:

$$\hat{c}_{111}(Q_{111}) = 0.5Q_{111}^2 + 0.4Q_{111}, \quad \hat{c}_{121}(Q_{121}) = 0.7Q_{121}^2 + 0.5Q_{121}$$
$$\hat{c}_{211}(Q_{211}) = 0.6Q_{211}^2 + 0.4Q_{211}, \quad \hat{c}_{221}(Q_{221}) = 0.4Q_{221}^2 + 0.2Q_{221},$$
$$\hat{c}_{112}(Q_{112}) = 0.3Q_{112}^2 + 0.1Q_{112}, \quad \hat{c}_{122}(Q_{122}) = 0.5Q_{122}^2 + 0.3Q_{122},$$
$$\hat{c}_{212}(Q_{212}) = 0.4Q_{212}^2 + 0.3Q_{212}, \quad \hat{c}_{222}(Q_{222}) = 0.4Q_{222}^2 + 0.2Q_{222},$$

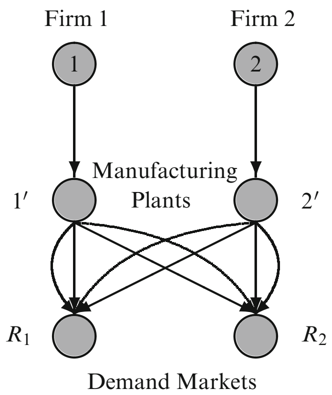


Fig. 6.3 The supply chain network topology for Example 6.2

and the demand price functions are:

$$\rho_{11}(d,q) = -d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2 + 70,$$
$$\rho_{12}(d,q) = -2d_{12} - d_{22} + 0.4q_1 + 0.2q_2 + 70,$$
$$\rho_{21}(d,q) = -0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2 + 70,$$
$$\rho_{22}(d,q) = -0.7d_{12} - 1.7d_{22} + 0.01q_1 + 0.6q_2 + 70.$$

The Jacobian of $-\nabla U(Q,q)$, denoted by $J(Q_{111}, Q_{121}, Q_{112}, Q_{122}, Q_{211}, Q_{221}, Q_{212}, Q_{222}, q_1, q_2)$, is

$$J(Q_{111}, Q_{121}, Q_{112}, Q_{122}, Q_{211}, Q_{221}, Q_{212}, Q_{222}, q_1, q_2)$$
$$= \begin{pmatrix} 5.0 & 4.0 & 2.0 & 2.0 & 0.4 & 0.4 & 0.0 & 0.0 & -0.3 & -0.05 \\ 4.0 & 5.4 & 2.0 & 2.0 & 0.4 & 0.4 & 0.0 & 0.0 & -0.3 & -0.05 \\ 2.0 & 2.0 & 6.6 & 6.0 & 0.0 & 0.0 & 1.0 & 1.0 & -0.4 & -0.2 \\ 2.0 & 2.0 & 6.0 & 7.0 & 0.0 & 0.0 & 1.0 & 1.0 & -0.4 & -0.2 \\ 0.6 & 0.6 & 0.0 & 0.0 & 8.2 & 7.0 & 4.0 & 4.0 & -0.1 & -0.5 \\ 0.6 & 0.6 & 0.0 & 0.0 & 7.0 & 7.8 & 4.0 & 4.0 & -0.1 & -0.5 \\ 0.0 & 0.0 & 0.7 & 0.7 & 4.0 & 4.0 & 8.2 & 7.4 & -0.01 & -0.6 \\ 0.0 & 0.0 & 0.7 & 0.7 & 4.0 & 4.0 & 7.4 & 8.2 & -0.01 & -0.6 \\ -0.3 & -0.3 & -0.4 & -0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & -0.5 & -0.6 & -0.6 & 0.0 & 2.0 \end{pmatrix}.$$

Clearly, this Jacobian matrix is also positive definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, $-\nabla U(Q,q)$ is strongly monotone. Thus, both the existence and uniqueness of the solution to variational inequality (6.11) with respect to this example are also guaranteed. Moreover, the equilibrium solution (stationary point) is: $Q^*_{111} = 6.97$, $Q^*_{121} = 4.91$, $Q^*_{112} = 2.40$, $Q^*_{122} = 3.85$, $Q^*_{211} = 3.58$, $Q^*_{221} = 1.95$, $Q^*_{212} = 2.77$, $Q^*_{222} = 2.89$, $q^*_1 = 1.52$, $q^*_2 = 3.08$ and it is globally exponentially stable.

The stationary points of both Examples 6.1 and 6.2 were computed using the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993). In the next section, we present the induced closed form expressions at each iteration, along with convergence results.

6.4 The Algorithm

As mentioned in Sect. 6.1, the projected dynamical system yields continuous-time adjustment processes. However, for computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories is needed.

We now recall the Euler method (see also Chaps. 2 and 5). Specifically, iteration $\tau + 1$ of the Euler method is given by:

$$X^{\tau+1} = P_{\mathscr{K}}(X^{\tau} - a_{\tau} F(X^{\tau})), \tag{6.32}$$

where $P_{\mathscr{K}}$ is the projection on the feasible set $\mathscr{K}$ and F is the function that enters the variational inequality problem (6.16).

As shown in Chap. 2, for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_{\tau}\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_{\tau} = \infty$, $a_{\tau} > 0$, $a_{\tau} \to 0$, as $\tau \to \infty$.

Explicit Formulae for the Euler Method Applied to the Supply Chain Network Model with Product Differentiation, Multiple Freight Options, and Quality Competition

The elegance of this procedure for the computation of solutions to the supply chain network model with product differentiation, multiple freight options, and quality level competition can be seen in the following explicit formulae for iteration $\tau + 1$. In particular, the closed form expression for the product shipments $i = 1, \ldots, I; j = 1, \ldots, o; k = 1, \ldots, n_R$ is:

$$Q_{ijk}^{\tau+1} = \max\left\{0, Q_{ijk}^{\tau} + a_{\tau}\left(\rho_{ik}(d^{\tau}, q^{\tau}, p) + \sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^{\tau}, q^{\tau})}{\partial d_{ik}} d_{il}^{\tau} - \frac{\partial \hat{f}_i(s^{\tau}, q_i^{\tau})}{\partial s_i} - \frac{\partial \hat{c}_{ijk}(Q_{ijk}^{\tau})}{\partial Q_{ijk}}\right)\right\}, \tag{6.33}$$

and the closed form expression for all the quality levels $i = 1, \ldots, I$ is:

$$q_i^{\tau+1} = \max\left\{0, q_i^{\tau} + a_{\tau}\left(\sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^{\tau}, q^{\tau}, p)}{\partial q_i} d_{il}^{\tau} - \frac{\partial \hat{f}_i(s^{\tau}, q_i^{\tau})}{\partial q_i}\right)\right\} \tag{6.34}$$

with the demands being updated according to:

$$d_{ik}^{\tau+1} = \sum_{k=1}^{n_R} Q_{ijk}^{\tau+1}; \quad i = 1,\ldots,I; k = 1,\ldots,n_R, \tag{6.35}$$

and the supplies being updated according to:

$$s_i^{\tau+1} = \sum_{j=1}^{o}\sum_{k=1}^{n_R} Q_{ijk}^{\tau+1}, \quad i = 1,\ldots,I. \tag{6.36}$$

We now provide the convergence result. The proof is direct from Theorem 5.8 in Nagurney and Zhang (1996).

Theorem 6.5: Convergence
In the supply chain network model with product differentiation, multiple freight options, and quality levels let $F(X) = -\nabla U(Q,q)$ *be strictly monotone at any equilibrium pattern and assume that Assumption 6.1 is satisfied. Also, assume that* F *is uniformly Lipschitz continuous. Then there exists a unique equilibrium service volume and quality level pattern* $(Q^*, q^*) \in K$ *and any sequence generated by the Euler method as given by (6.33) and (6.34) above, with updates of the demands and supplies via (6.35) and (6.36), respectively, where* $\{a_\tau\}$ *satisfies* $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, *as* $\tau \to \infty$ *converges to* (Q^*, q^*).

In the next section, we apply the Euler method to compute solutions to numerical supply chain network problems with multiple freight options and quality competition.

6.5 Numerical Examples

We implemented the Euler method, as described in Sect. 6.4, using Matlab on a Lenovo Z580 computer. The convergence criterion is $\epsilon = 10^{-6}$; that is, the Euler method is considered to have converged if, at a given iteration, the absolute value of the difference of each product shipment and each quality level differs from its respective value at the preceding iteration by no more than ϵ.

The sequence $\{a_\tau\}$ is: $0.1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \ldots)$. We initialize the algorithm by setting each product shipment $Q_{ijk} = 2.5$, $\forall i, j, k$, and by setting the quality level of each firm $q_i = 0.00$, $\forall i$.

Example 6.1 Revisited
In Sect. 6.3, we discussed stability analysis and presented results for two numerical examples. We now provide additional results for these examples.

The Euler method requires 72 iterations for convergence to the equilibrium pattern for Example 6.1 described in Sect. 6.3. A graphical depiction of the iterates, consisting of the product shipments and the quality levels is given in Fig. 6.4. The utility/profit of firm 1 is 567.35 and that of firm 2 is 216.94.

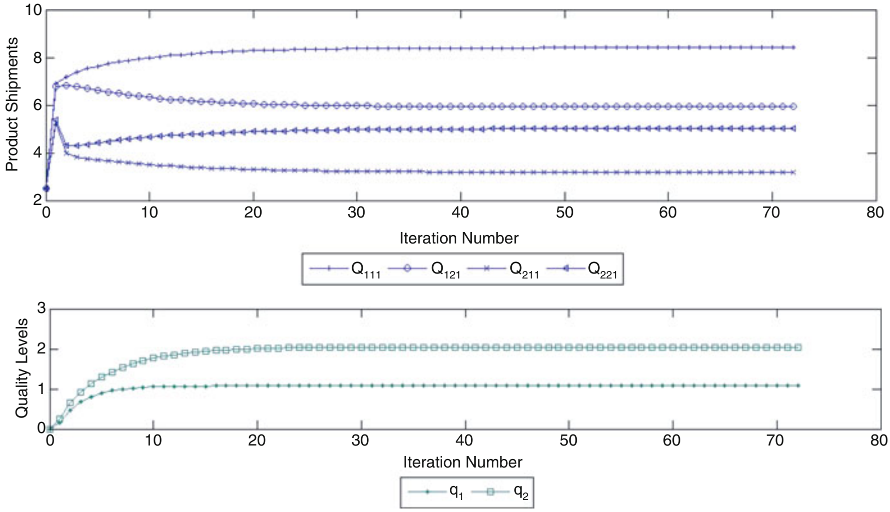


Fig. 6.4 Product shipment and quality level trajectories for Example 6.1

One can see from Fig. 6.4, that, as was predicted by the stability analysis results, the convergence was exponentially fast. Moreover, we know that the equilibrium solution is globally exponentially stable.

Example 6.2 Revisited
For Example 6.2 described in Sect. 6.3, in which there are two firms, two freight options for each firm, and two demand markets, the Euler method requires 84 iterations for convergence. A graphical depiction of the product shipment and quality level iterates is given in Fig. 6.5. One can see from the figure that, also, as predicted by the theory, the convergence to the equilibrium solution (stationary point) is exponentially fast and the gradient process is also globally exponentially stable for this example. The profit of firm 1 is 547.60, whereas that of firm 2 is 292.79.

The trajectories in Figs. 6.4 and 6.5 provide a discrete-time evolution of the product shipments and quality levels of the firms as they respond to the feedback from the consumers as to the demands for the products and the quality levels from the preceding iteration (time period).

We investigate the properties of the Jacobian matrix above in order to also evaluate the stability of the utility gradient process as well as to check whether conditions for convergence of the algorithm are satisfied. One should realize, however, that the algorithm does not require strong monotonicity of minus the gradient of the utility functions for convergence. Moreover, if the algorithm converges, it converges to a stationary point of the projected dynamical system; equivalently, to a solution of the variational inequality problem governing the Nash-Cournot equilibrium conditions for the supply chain network model.

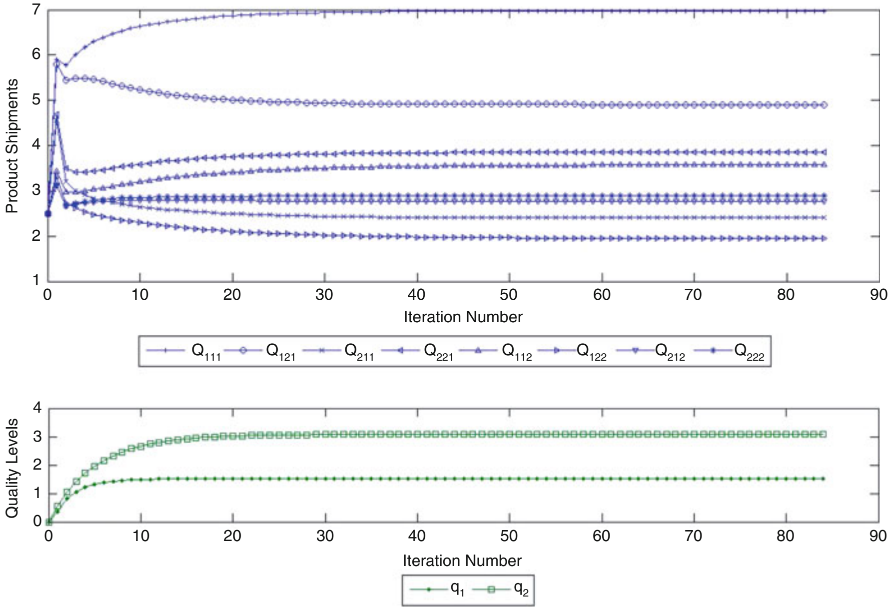
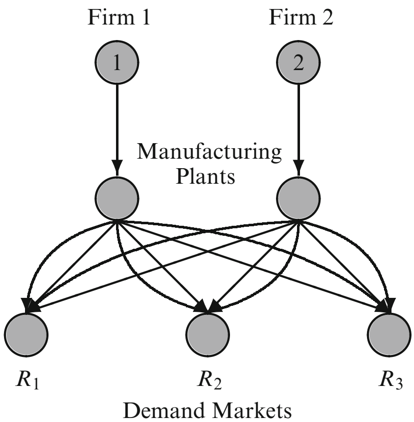


Fig. 6.5 Product shipment and quality level trajectories for Example 6.2

Fig. 6.6 The supply chain network topology for Example 6.3



Example 6.3

Example 6.3 consists of two firms and three demand markets, as depicted in Fig. 6.6.

This example is built from Example 6.2 with the production cost functions of the original two demand markets expanded and the original demand price functions as well. The new demand market, R_3, is farther than demand markets R_1 and R_2. We also add new data for the new firm. The complete data for this example are given below.

The production cost functions are:

$$\hat{f}_1(s,q_1) = s_1^2 + s_1 + s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s,q_2) = 2s_2^2 + 2s_1 + s_2 + q_2^2 + 37.$$

The total transportation cost functions are:

$$\hat{c}_{111}(Q_{111}) = 0.5Q_{111}^2 + 0.4Q_{111}, \quad \hat{c}_{121}(Q_{121}) = 0.7Q_{121}^2 + 0.5Q_{121}$$
$$\hat{c}_{211}(Q_{211}) = 0.6Q_{211}^2 + 0.4Q_{211}, \quad \hat{c}_{221}(Q_{221}) = 0.4Q_{221}^2 + 0.2Q_{221},$$
$$\hat{c}_{112}(Q_{112}) = 0.3Q_{112}^2 + 0.1Q_{112}, \quad \hat{c}_{122}(Q_{122}) = 0.5Q_{122}^2 + 0.3Q_{122},$$
$$\hat{c}_{212}(Q_{212}) = 0.4Q_{212}^2 + 0.3Q_{212}, \quad \hat{c}_{222}(Q_{222}) = 0.4Q_{222}^2 + 0.2Q_{222},$$
$$\hat{c}_{113}(Q_{113}) = Q_{113}^2 + 0.5Q_{113}, \quad \hat{c}_{123}(Q_{123}) = Q_{123}^2 + 0.6Q_{123},$$
$$\hat{c}_{213}(Q_{213}) = 0.8Q_{213}^2 + 0.5Q_{213}, \quad \hat{c}_{223}(Q_{223}) = Q_{223}^2 + 0.7Q_{223},$$

and the demand price functions are:

$$\rho_{11}(d,q) = -d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2 + 70,$$
$$\rho_{12}(d,q) = -2d_{12} - d_{22} + .4q_1 + 0.2q_2 + 70,$$
$$\rho_{13}(d,q) = -1.7d_{13} - 0.7d_{23} + 0.5q_1 + 0.1q_2 + 70,$$
$$\rho_{21}(d,q) = -0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2 + 70,$$
$$\rho_{22}(d,q) = -0.7d_{12} - 1.7d_{22} + 0.01q_1 + 0.6q_2 + 70,$$
$$\rho_{23}(d,q) = -0.9d_{13} - 2d_{23} + 0.2q_1 + 0.7q_2 + 70,$$

The Jacobian of $-\nabla U(Q,q)$, denoted by $J(Q_{111}, Q_{121}, Q_{112}, Q_{122}, Q_{113}, Q_{123}, Q_{211}, Q_{221}, Q_{212}, Q_{222}, Q_{213}, Q_{223}, q_1, q_2)$=$[J_1|J_2]$, where

$$J_1 = \begin{pmatrix} 5.0 & 4.0 & 2.0 & 2.0 & 2.0 & 2.0 & 0.4 \\ 4.0 & 5.4 & 2.0 & 2.0 & 2.0 & 2.0 & 0.4 \\ 2.0 & 2.0 & 6.6 & 6.0 & 2.0 & 2.0 & 0.0 \\ 2.0 & 2.0 & 6.0 & 7.0 & 2.0 & 2.0 & 0.0 \\ 2.0 & 2.0 & 2.0 & 2.0 & 7.4 & 5.4 & 0.0 \\ 2.0 & 2.0 & 2.0 & 2.0 & 5.4 & 7.4 & 0.0 \\ 0.6 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 8.2 \\ 0.6 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 7.0 \\ 0.0 & 0.0 & 0.7 & 0.7 & 0.0 & 0.0 & 4.0 \\ 0.0 & 0.0 & 0.7 & 0.7 & 0.0 & 0.0 & 4.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.9 & 0.9 & 4.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.9 & 0.9 & 4.0 \\ -0.3 & -0.3 & -0.4 & -0.4 & -0.5 & -0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -0.5 \end{pmatrix}.$$

$$J_2 = \begin{pmatrix} 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.05 \\ 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 & -0.05 \\ 0.0 & 1.0 & 1.0 & 0.0 & 0.0 & -0.4 & -0.2 \\ 0.0 & 1.0 & 1.0 & 0.0 & 0.0 & -0.4 & -0.2 \\ 0.0 & 0.0 & 0.0 & 0.7 & 0.7 & -0.5 & -0.1 \\ 0.0 & 0.0 & 0.0 & 0.7 & 0.7 & -0.5 & -0.1 \\ 7.0 & 4.0 & 4.0 & 4.0 & 4.0 & -0.1 & -0.5 \\ 7.8 & 4.0 & 4.0 & 4.0 & 4.0 & -0.1 & -0.5 \\ 4.0 & 8.2 & 7.4 & 4.0 & 4.0 & -0.01 & -0.6 \\ 4.0 & 7.4 & 8.2 & 4.0 & 4.0 & -0.01 & -0.6 \\ 4.0 & 4.0 & 4.0 & 9.6 & 8.0 & -0.2 & -0.7 \\ 4.0 & 4.0 & 4.0 & 8.0 & 10.0 & -0.2 & -0.7 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 4.0 & 0.0 \\ -0.5 & -0.6 & -0.6 & -0.7 & -0.7 & 0.0 & 2 \end{pmatrix}.$$

The above Jacobian matrix J is positive definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, $-\nabla U(Q, q)$ is strongly monotone. Thus, both the existence and uniqueness of the solution to variational inequality (6.11) with respect to this example are guaranteed.

The Euler method converges to the equilibrium solution: $Q^*_{111} = 5.80$, $Q^*_{121} = 4.07$, $Q^*_{112} = 3.50$, $Q^*_{122} = 1.90$, $Q^*_{113} = 2.91$, $Q^*_{123} = 2.86$,$Q^*_{211} = 1.65$, $Q^*_{221} = 2.73$, $Q^*_{212} = 2.25$, $Q^*_{222} = 2.38$, $Q^*_{213} = 1.94$, $Q^*_{223} = 1.45$, $q^*_1 = 2.00$, $q^*_2 = 3.67$, in 84 iterations. The profits of the firms are: $U_1 = 655.28$ and $U_2 = 324.18$. Graphical depictions of the product shipment and the quality level iterates are given, respectively, in Fig. 6.7.

With the above examples, we illustrate the types of supply chain network problems with not unrealistic features and underlying functions that can be theoretically effectively analyzed as to their qualitative properties and also their solutions computed.

Sensitivity Analysis for Example 6.3

After obtaining the equilibrium solution to Example 6.3, we are interested in the following question: How do changes in the fixed demand price function term for all the firm and demand market pairs, which is now equal to 70 and which we refer to as p, influence the equilibrium solution and the profits? We conduct a sensitivity analysis for this parameter based on the data given in Example 6.3, and attain the results reported in Table 6.2 and Fig. 6.8.

As indicated in Table 6.2 and Fig. 6.8, the product shipments, quality levels, and the profits are positively related to the demand price parameter. The reason is the following. In Example 6.3, as the demand prices become higher, consumers are willing to pay more for purchasing one unit product at a certain quality level. As a result, firms will produce more in order to maximize their profits, and there is more incentive for them to improve quality. Therefore, the quality levels and the profits of the two firms also increase.

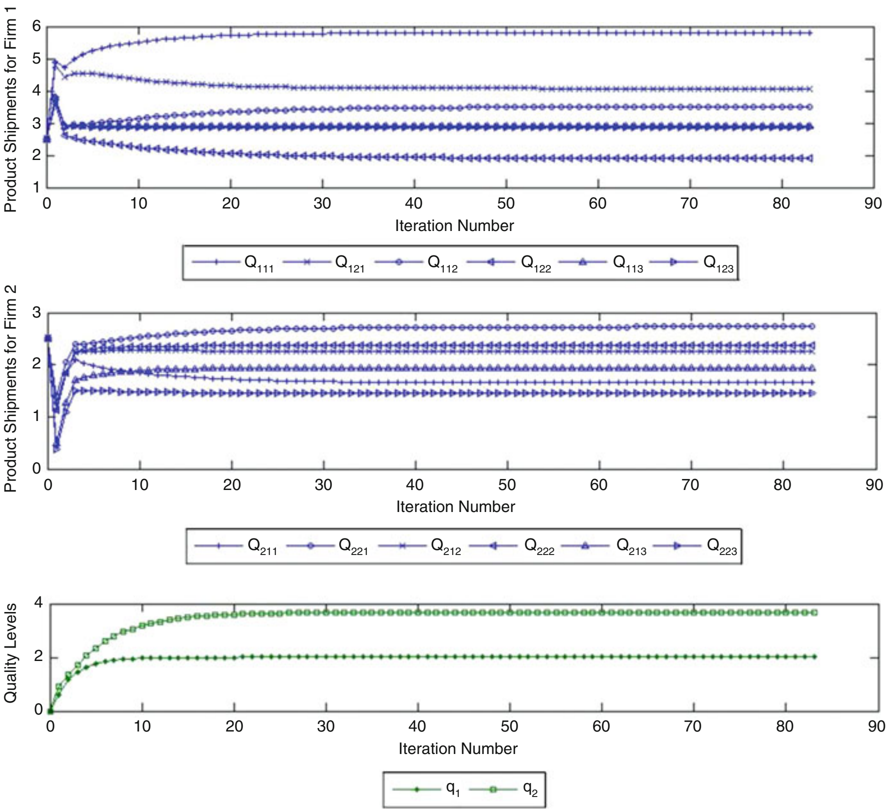


Fig. 6.7 Product shipment and quality level trajectories for Example 6.3

6.6 Summary and Conclusions

In this chapter, we developed static and dynamic supply chain network models with product differentiation, multiple freight options, and quality competition. The models generalize the respective ones presented in Chap. 5 to include multiple freight options. Being able to evaluate multiple freight options and the optimal and equilibrium solutions provides firms with valuable information and enhances their flexibility and agility in the case of demand variability as well as capacity changes in freight options or supply chain disruptions in transport. Here we implicitly include capacity in the transportation cost functions, which can be generalized, and nonlinear. We derived the governing equilibrium conditions and provided alternative variational inequality formulations. We also proposed continuous-time adjustment processes for the evolution of the product shipment and the quality levels of the products of the firms. We described an algorithm, which yields closed

Table 6.2 Computed equilibrium product shipments, quality levels, and profits as p Increases

p	30	40	50	60	70	80	90
Q_{111}^*	2.44	3.28	4.12	4.96	5.80	6.64	7.48
Q_{121}^*	1.67	2.27	2.87	3.47	4.07	4.67	5.27
Q_{112}^*	1.55	2.04	2.53	3.01	3.50	3.99	4.48
Q_{122}^*	0.73	1.02	1.32	1.61	1.90	2.20	2.49
Q_{113}^*	1.22	1.64	2.06	2.48	2.91	3.33	3.75
Q_{123}^*	1.17	1.59	2.01	2.43	2.86	3.28	3.70
Q_{211}^*	0.64	0.89	1.14	1.40	1.65	1.90	2.16
Q_{221}^*	1.20	1.58	1.97	2.35	2.73	3.11	3.49
Q_{212}^*	0.91	1.24	1.58	1.92	2.25	2.59	2.92
Q_{222}^*	1.03	1.37	1.71	2.04	2.38	2.71	3.05
Q_{213}^*	0.83	1.11	1.38	1.66	1.94	2.22	2.50
Q_{223}^*	0.56	0.79	1.01	1.23	1.45	1.68	1.90
q_1^*	0.83	1.13	1.42	1.71	2.00	2.29	2.59
q_2^*	1.53	2.07	2.60	3.14	3.67	4.21	4.74
U_1	86.08	183.89	311.36	468.49	655.28	871.72	1117.81
U_2	21.94	71.94	139.20	223.28	324.18	441.90	576.43

form expressions for the product shipments and quality levels at each iteration, and applied it to solve numerical examples. We also conducted a sensitivity analysis on the demand price functions.

6.7 Sources and Notes

This chapter is based on the paper by Nagurney et al. (2013a). That paper, however, focused on a service-oriented Internet in which the service providers played the role of the manufacturing firms in this chapter, and the freight mode options corresponded to different network providers for Internet transport provision. The goal was to provide more choices for consumers as argued for new Internet architectures by Wolf et al. (2012).

In this chapter, we expanded the network topologies, since we focus on supply chain network economic activities (and associated costs) on links. We also provide an entirely new motivation for including multiple freight options in supply chain networks with product differentiation and quality competition.

In Chaps. 11 and 12, we include freight service providers as another tier of decision-makers in supply chain networks in which freight service providers also compete and do so in terms of quality and prices charged.

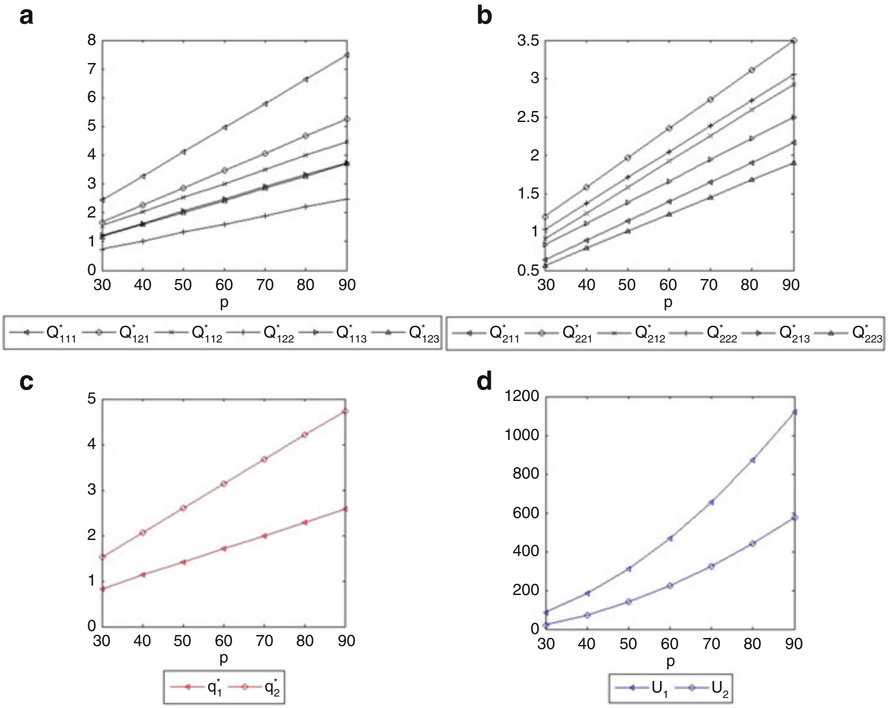


Fig. 6.8 Sensitivity analysis for demand price parameter p for Example 6.3. (**a**) Equilibrium product shipments of firm 1. (**b**) Equilibrium product shipments of firm 2. (**c**) Equilibrium quality levels. (**d**) Profits

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Chapter 7
Outsourcing Under Price and Quality Competition: Single Firm Case

Abstract In this chapter, we add another set of decision-makers to a supply chain network – that of the contractors. We present a supply chain network equilibrium model with outsourcing under price and quality competition. We consider a firm that is engaged in determining the optimal product flows associated with its supply chain network activities in the form of manufacturing and distribution. In addition to multimarket demand satisfaction, the firm seeks to minimize its cost, with the associated function also capturing the firm's weighted disrepute cost caused by possible quality issues associated with the contractors. Simultaneously, the contractors, who compete with one another in a noncooperative manner in prices a la Bertrand, and in quality, seek to secure manufacturing and distribution of the product from the firm. This game theory model allows for the determination of the optimal product flows associated with the supply chain in-house and outsourcing network activities and provides the firm with its optimal make-or-buy decisions and the optimal contractor-selections. We state the governing equilibrium conditions and derive the equivalent variational inequality formulation. We propose an algorithm and apply it to compute solutions to numerical examples to illustrate the generality and applicability of the framework.

7.1 Introduction

The reality of today's supply chain networks, given their global reach from sourcing locations to points of demand, is further challenged by such issues as the growth in outsourcing and in global procurement. Outsourcing is defined as the behavior of moving some of a firm's responsibilities and/or internal processes, such as product design or manufacturing, to a third party company (Chase et al. 2004). Outsourcing of manufacturing/production has long been noted in operations and supply chain management in such industries as computer engineering and manufacturing, financial analysis, fast fashion apparel, and pharmaceuticals (cf. Austin et al. 2003; Nagurney and Yu 2011; Hayes et al. 2005).

One of the main arguments for the outsourcing of production, as well as distribution, is cost reduction (Insinga and Werle 2000; Cecere 2005; Jiang et al. 2007). Outsourcing, as a supply chain strategy, may also increase operational

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efficiency and agility (Klopack 2000; John 2006), enhance a firm's competitiveness (cf. Narasimhan and Das 1999), and even yield benefits from supportive government policies (Zhou 2007).

In the pharmaceutical industry, for example, in 2010, up to 40 % of the drugs that Americans consumed were imported, and more than 80 % of the active ingredients for drugs sold in the United States were outsourced (Ensinger 2010), with the market for outsourced pharmaceutical manufacturing expanding at the rate of 10–12 % annually in the US (Olson and Wu 2011). In the fashion industry, according to the ApparelStats Report released by the American Apparel and Footwear Association, 97.7 % of the apparel sold in the United States in 2011 was produced outside the US (AAFA 2012). In addition, in the electronics industry, in the fourth quarter of 2012, 100 % of the 26.9 million iPhones sold by Apple were designed in California, but assembled in China (Apple 2012; Rawson 2012).

However, parallel to the dynamism of and growth in outsourcing, the nation's growing reliance on sometimes uninspected contractors has raised public and governmental awareness and concern, with outsourcing firms being faced with quality-related risks (cf. Dong et al. 2005; Helm 2006; Steven et al. 2014). In 2003, the suspension of the license of Pan Pharmaceuticals, the world's fifth largest contract manufacturer of health supplements, due to quality failure, caused costly consequences in terms of product recalls and credibility losses (Allen 2003). In 2008, fake heparin made by a Chinese manufacturer not only led to recalls of drugs in over ten European countries (Payne 2008), but also resulted in the deaths of 81 Americans (Harris 2011). Furthermore, in 2009, more than 400 peanut butter products were recalled after 8 people died and more than 500 people in 43 states, half of them children, were sickened by salmonella poisoning, the source of which was a peanut butter plant in Georgia (Harris 2009).

Therefore, with the increasing volume of outsourcing, it is imperative for firms to be prepared to adopt best practices aimed at safeguarding the quality of their supply chain networks and their reputations. Outsourcing makes supply chain networks more complex and, hence, more vulnerable to quality risks (cf. Bozarth et al. 2009). In outsourcing, since contract manufacturers are not of the same brand names as the original firms, they may have fewer incentives to be concerned with quality (Amaral et al. 2006), which may lead them to expend less effort to ensure high quality. Consequently, quality should be incorporated into the make-or-buy as well as the contractor-selection decisions of firms.

In this chapter, we develop a supply chain network model utilizing a game theory approach which takes into account the quality concerns in the context of global outsourcing. This model captures the behaviors of the firm and its potential contractors with consideration of the transactions between them and the quality of the outsourced product. The objective of each contractor is to maximize its profit. The firm seeks to minimize its total cost, which includes its weighted disrepute cost, which is influenced by the quality of the product produced by its contractors and the amount of product that is outsourced. The contractors compete with one another by determining the prices that they charge the firm for manufacturing and delivering the product to the demand markets and the quality levels in order to maximize their profits.

In Chaps. 3 through 6 of this book, the supply chain network models focused on two parties, firms (or supply markets) and demand markets. In this chapter and Chap. 8, in contrast, quality competition in supply chain networks with outsourcing is formulated and analyzed with another party added to the supply chain networks, that of the contractors of the firms. Moreover, in this chapter and Chap. 8, the demands are no longer elastic, but, rather, fixed. Products with inelastic demand include life-saving pharmaceuticals and infant formula, for example. In both of these chapters on outsourcing we incorporate upper bounds on the quality of the product. Also, for the first time in this book, we focus not only on Cournot (1838) competition in quantity variables but also on Bertrand (1883) competition in prices, for the contractors.

This chapter is organized as follows. In Sect. 7.2, we describe the decision-making behavior of the firm and that of the competing contractors. We then develop the game theory model, state the equilibrium conditions, and derive the equivalent variational inequality formulation. We assume that the demand for the product is known at the various demand markets since the firm can be expected to have good in-house forecasting abilities. Hence, we focus on cost minimization associated with the firm but profit maximization for the contractors who compete on prices and quality.

In Sect. 7.3, we describe the algorithm, which yields closed form expressions, at each iteration, for the contractor prices and the quality levels, with the product flows being solved, at each iteration, exactly using an equilibration algorithm. We illustrate the concepts through small examples, and include sensitivity analysis results. We then apply the algorithm to demonstrate the modeling and computational framework on larger examples. We explore the case of a disruption in the supply chain network and discuss two cases focused on opportunity costs. We summarize our results and give our conclusions in Sect. 7.4. Section 7.5 contains the Sources and Notes for this chapter.

7.2 The Supply Chain Network Model with Outsourcing and Price and Quality Competition

In this section, we develop the supply chain network model with outsourcing and with price and quality competition among the contractors. We assume that the firm is involved in the processes of in-house manufacturing and distribution of a product, and may also contract its manufacturing and distribution activities to contractors, who may be located overseas. We seek to determine the optimal product flows of the firm to its demand markets, along with the prices the contractors charge the firm for production and distribution, and the quality levels of their products.

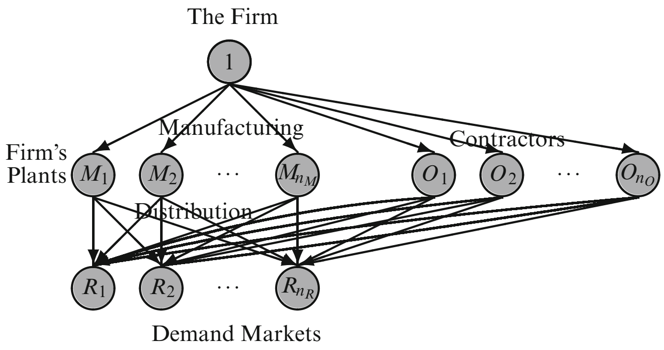


Fig. 7.1 The supply chain network topology with outsourcing

The supply chain network topology is depicted in Fig. 7.1. In the supply chain network, there are n_M manufacturing facilities or plants that the firm owns and n_R demand markets. Some of the links from the top-tiered node 1, representing the firm, are connected to its manufacturing facility/plant nodes, which are denoted, respectively, by: $M_1, \ldots, M_{n_M}$ and these, in turn, are connected to the demand nodes: $R_1, \ldots, R_{n_R}$.

As also depicted in Fig. 7.1, we capture the outsourcing of the product in terms of its production and delivery. There are n_O contractors available to the firm. The firm may potentially contract to any of these contractors who then also distribute the outsourced product that they manufacture to the n_R demand markets. The first set of outsourcing links directly links the top-most node 1 to the n_O contractor nodes, $O_1, \ldots, O_{n_O}$, which correspond to their respective manufacturing activities. The next set of outsourcing links emanates from the contractor nodes to the demand markets and reflect the delivery of the outsourced product to the demand markets.

The top set of links in the supply chain network in Fig. 7.1 consists of the manufacturing links, whether in-house or outsourced (contracted), whereas the next set of links consists of the distribution links. For simplicity, we let $n = n_M + n_O$ denote the number of manufacturing plants, whether in-house or belonging to the contractors. The notation for the model is given in Table 7.1. As in the previous chapters, vectors are assumed to be column vectors and the optimal/equilibrium solution is denoted by "*".

7.2.1 *The Behavior of the Firm*

We assume that in-house activities ensure a 100 % perfect quality conformance level. The quality conformance level of contractor j is denoted by q_j, which varies from a 0 % quality conformance level to a 100 % quality conformance level, such that

$$0 \leq q_j \leq q^U, \quad j = 1, \ldots, n_O, \tag{7.1}$$

Table 7.1 Notation for the game theoretic supply chain network model with outsourcing

Notation	Definition
Q_{jk}	The nonnegative amount of product produced at manufacturing plant j and delivered to demand market R_k. We group the $\{Q_{jk}\}$ elements into the vector $Q \in R_+^{nn_R}$
d_k	The demand for the product at demand market R_k, assumed known and fixed
q_j	The nonnegative quality level of the pharmaceutical product produced by contractor j. We group the $\{q_j\}$ elements into the vector $q \in R_+^{n_O}$
π_{jk}	The price charged by contractor j for producing and delivering a unit of the product to R_k. We group the $\{\pi_{jk}\}$ elements for contractor j into the vector $\pi_j \in R_+^{n_R}$ and then group all such vectors for all the contractors into the vector $\pi \in R_+^{n_O n_R}$
$\hat{f}_j(\sum_{k=1}^{n_R} Q_{jk})$	The total production cost at manufacturing plant j; $j = 1, \ldots, n_M$ owned by the firm
$q\prime$	The average quality level
$tc_j(\sum_{k=1}^{n_R} Q_{n_M+j,k})$	The total transaction cost associated with the firm transacting with contractor j; $j = 1, \ldots, n_O$
$\hat{c}_{jk}(Q_{jk})$	The total transportation cost associated with delivering the product manufactured at j to R_k; $j = 1, \ldots, n_M$; $k = 1, \ldots, n_R$
$\widehat{sc}_{jk}(Q, q)$	The total cost of contractor j; $j = 1, \ldots, n_O$, to produce and distribute the product to demand market R_k; $k = 1, \ldots, n_R$
$\widehat{qc}_j(q)$	Quality cost faced by contractor j; $j = 1, \ldots, n_O$
$oc_{jk}(\pi)$	The opportunity cost associated with pricing the product by contractor j; $j = 1, \ldots, n_O$ and delivering it to k; $k = 1, \ldots, n_R$
$dc(q\prime)$	The cost of disrepute, which corresponds to the external failure quality cost

where q^U is the value representing the perfect quality conformance level achieved by the firm in its in-house manufacturing.

The quality level associated with the product of the firm is, hence, an average quality level that is determined by the quality levels decided upon by the contractors and the outsourced product amounts. Thus, the average quality level for the firm's product, both in-house and outsourced, can be expressed as

$$q\prime = \frac{\sum_{j=n_M+1}^{n} \sum_{k=1}^{n_R} Q_{jk} q_{j-n_M} + (\sum_{j=1}^{n_M} \sum_{k=1}^{n_R} Q_{jk}) q^U}{\sum_{k=1}^{n_R} d_k}. \tag{7.2}$$

Equation (7.2) is a variant of the average quality measures developed in Chaps. 3 and 4 which assess the average quality level of homogeneous products from multiple firms with information asymmetry in quality, but without outsourcing.

The firm selects the product flows Q, whereas the contractors, who compete with one another, select their respective quality level q_j and price vector π_j for contractor $j = 1, \ldots, n_O$.

The objective of the firm is to maximize its utility (cf. (7.3)), represented by minus its total costs that include the production costs, the transportation costs, the payments to the contractors, the total transaction costs, along with the weighted cost of disrepute (loss of reputation), with the nonnegative term ω denoting the weight that the firm imposes on the disrepute cost function. The firm's utility function is denoted by U_0 and, hence, the firm seeks to

$$\text{Maximize}_Q \quad U_0(Q, q^*, \pi^*) = -\sum_{j=1}^{n_M} \hat{f}_j(\sum_{k=1}^{n_R} Q_{jk}) - \sum_{j=1}^{n_M}\sum_{k=1}^{n_R} \hat{c}_{jk}(Q_{jk})$$

$$-\sum_{j=1}^{n_O}\sum_{k=1}^{n_R} \pi^*_{jk} Q_{n_M+j,k} - \sum_{j=1}^{n_O} tc_j(\sum_{k=1}^{n_R} Q_{n_M+j,k}) - \omega dc(q\prime) \tag{7.3}$$

subject to:

$$\sum_{j=1}^{n} Q_{jk} = d_k, \quad k = 1, \ldots, n_R, \tag{7.4}$$

$$Q_{jk} \geq 0, \quad j = 1, \ldots, n; k = 1, \ldots, n_R, \tag{7.5}$$

with $q\prime$ in (7.3) as in (7.2).

Note that (7.3) is equivalent to minimizing the total costs. Also, according to (7.4) the demand at each demand market must be satisfied. This is important since the firm is dealing with products that are essential, such as pharmaceuticals, for example. We assume that all the cost functions in (7.3) are continuous, twice continuously differentiable, and convex. We define the feasible set K^0 as follows: $K^0 \equiv \{Q|Q \in R_+^{nn_R}$ with (7.4) satisfied$\}$. K^0 is closed and convex. The following theorem is immediate (see Chap. 2).

Theorem 7.1: Variational Inequality Formulation of Firm's Optimization Problem

The optimality conditions for the firm, faced with (7.3) and subject to (7.4) and (7.5), with $q\prime$ as in (7.2) embedded into $dc(q\prime)$, and under the above imposed assumptions, coincide with the solution of the following variational inequality: determine $Q^* \in K^0$

$$-\sum_{h=1}^{n}\sum_{l=1}^{n_R} \frac{\partial U_0(Q^*, q^*, \pi^*)}{\partial Q_{hl}} \times (Q_{hl} - Q^*_{hl}) \geq 0, \quad \forall Q \in K^0, \tag{7.6}$$

with notice that for $h = 1, \ldots, n_M$; $l = 1, \ldots, n_R$:

$$-\frac{\partial U_0(Q^*, q^*, \pi^*)}{\partial Q_{hl}} = \left[\frac{\partial \hat{f}_h(\sum_{k=1}^{n_R} Q^*_{hk})}{\partial Q_{hl}} + \frac{\partial \hat{c}_{hl}(Q^*_{hl})}{\partial Q_{hl}} + \omega \frac{\partial dc(q\prime^*)}{\partial Q_{hl}}\right]$$

$$= \left[\frac{\partial \hat{f}_h(\sum_{k=1}^{n_R} Q^*_{hk})}{\partial Q_{hl}} + \frac{\partial \hat{c}_{hl}(Q^*_{hl})}{\partial Q_{hl}} + \omega \frac{\partial dc(q\prime^*)}{\partial q\prime} \frac{q^U}{\sum_{k=1}^{n_R} d_k}\right],$$

and for $h = n_M + 1, \ldots, n; l = 1, \ldots, n_R$*:*

$$-\frac{\partial U_0(Q^*, q^*, \pi^*)}{\partial Q_{hl}} = \left[\pi^*_{h-n_M,l} + \frac{\partial tc_{h-n_M}(\sum_{k=1}^{n_R} Q^*_{hk})}{\partial Q_{hl}} + \omega \frac{\partial dc(q'^*)}{\partial Q_{hl}}\right]$$
$$= \left[\pi^*_{h-n_M,l} + \frac{\partial tc_{h-n_M}(\sum_{k=1}^{n_R} Q^*_{hk})}{\partial Q_{hl}} + \omega \frac{\partial dc(q'^*)}{\partial q'} \frac{q^*_h}{\sum_{k=1}^{n_R} d_k}\right].$$

7.2.2 The Behavior of the Contractors and Their Optimality Conditions

The objective of the contractors is profit maximization. Their revenues are obtained from the purchasing activities of the firm, while their costs are the costs of production and distribution, the quality cost, and the opportunity cost. Opportunity cost is defined as "the loss of potential gain from other alternatives when one alternative is chosen" (New Oxford American Dictionary 2010). In this model, the contractors' opportunity costs are functions of the prices charged, since, if the values are too low, they may not recover all of their costs, whereas if they are too high, then the firm may select another contractor. In our model, these are the only costs that depend on the prices that the contractors charge the firm and, hence, there is no double counting. We note that the concept of opportunity cost (cf. Mankiw 2011) is very relevant to both economics and operations research. It has been emphasized in firm competition by Grabowski and Vernon (1990), Palmer and Raftery (1999), and Cockburn (2004). Gan and Litvinov (2003) also constructed opportunity cost functions that are functions of prices as we consider here (see Table 7.1) but in an energy application.

Interestingly, Leland (1979), inspired by the work of the Nobel laureate Akerlof (1970) on quality, noted that drugs (pharmaceuticals) must satisfy federal safety standards and, in his model, introduced opportunity costs that are functions of quality levels.

General opportunity cost functions include both explicit and implicit costs (Mankiw 2011) with the explicit opportunity costs requiring monetary payment, and including possible anticipated regulatory costs, wage expenses, and the opportunity cost of capital (see Porteus 1986), etc. Implicit opportunity costs are those that do not require payment, but to the decision-maker, still need to be monetized, for the purposes of decision-making, and can include the time and effort put in (see Payne et al. 1996), and the profit that the decision-maker could have earned, if he had made other choices (Sandoval-Chavez and Beruvides 1998).

As presented in Table 7.1, $\widehat{sc}_{jk}(Q, q)$, which is contractor j's cost function associated with producing and delivering the firm's product to demand market R_k, only captures the cost of production and delivery. It depends on both the quantities and the quality levels. However, $\widehat{qc}_j(q)$ is the cost associated with quality management, and reflects the "cost incurred in ensuring and assuring quality as well as the loss incurred when quality is not achieved," and is over and above the cost of production and delivery activities. The $\widehat{qc}_j(q)$ is a convex function in the quality levels. Thus, $\widehat{sc}_{jk}(Q, q)$ and $\widehat{qc}_j(q)$ are two entirely different costs, and they do not overlap.

Each contractor has, as its strategic variables, its quality level, and the prices that it charges the firm for production and distribution to the demand markets. We denote the utility of each contractor j by U_j, with $j = 1, \ldots, n_O$, and note that it represents the profit. Hence, each contractor $j; j = 1, \ldots, n_O$ seeks to:

$$\text{Maximize}_{q_j,\pi_j} \quad U_j(Q^*, q, \pi) = \sum_{k=1}^{n_R} \pi_{jk} Q^*_{n_M+j,k} - \sum_{k=1}^{n_R} \widehat{sc}_{jk}(Q^*, q) - \widehat{qc}_j(q) - \sum_{k=1}^{n_R} oc_{jk}(\pi) \tag{7.7}$$

subject to:

$$\pi_{jk} \geq 0, \quad k = 1, \ldots, n_R, \tag{7.8}$$

and (7.1) for each j.

We assume that the cost functions in each contractor's utility function are continuous, twice continuously differentiable, and convex. Moreover, we assume that the contractors compete in a noncooperative in the sense of Nash (1950, 1951), with each one trying to maximize its own profits.

We define the feasible sets

$$K^j \equiv \{(q_j, \pi_j) | \pi_j \text{ satisfies } (7.8) \text{ and } q_j \text{ satisfies } (7.1) \text{ for } j\}; \quad j = 1, \ldots, n_O.$$

We also define the feasible set $\mathcal{K}^1 \equiv \prod_{j=1}^{n_O} K^j$ and $\mathcal{K} \equiv K^0 \times \mathcal{K}^1$. We observe that all the above-defined feasible sets are closed and convex.

Definition 7.1: A Nash-Bertrand Equilibrium with Price and Quality Competition

A quality level and price pattern $(q^*, \pi^*) \in \mathcal{K}^1$ is said to constitute a Bertrand-Nash equilibrium if for each contractor j; $j = 1, \ldots, n_O$

$$U_j(Q^*, q_j^*, \widehat{q_j^*}, \pi_j^*, \widehat{\pi_j^*}) \geq U_j(Q^*, q_j, \widehat{q_j^*}, \pi_j, \widehat{\pi_j^*}), \quad \forall (q_j, \pi_j) \in K^j, \tag{7.9}$$

where

$$\widehat{q_j^*} \equiv (q_1^*, \ldots, q_{j-1}^*, q_{j+1}^*, \ldots, q_{n_O}^*), \tag{7.10}$$

$$\widehat{\pi_j^*} \equiv (\pi_1^*, \ldots, \pi_{j-1}^*, \pi_{j+1}^*, \ldots, \pi_{n_O}^*). \tag{7.11}$$

According to (7.9), a Nash-Bertrand equilibrium is established if no contractor can unilaterally improve upon its profits by selecting an alternative vector of quality levels and prices charged to the pharmaceutical firm.

Next, we present the variational inequality formulation of the Bertrand-Nash equilibrium according to Definition 7.1 (see Bertrand 1883; Nash 1950, 1951; Gabay and Moulin 1980; Nagurney 2006).

Theorem 7.2: Variational Inequality Formulation of the Contractors' Problems
Assume that, for each contractor j; $j = 1, \ldots, n_O$, the profit function $U_j(Q, q, \pi)$ is concave with respect to the variables $\{\pi_{j1}, \ldots, \pi_{jn_R}\}$ and q_j, and is continuous and continuously differentiable. Then $(q^*, \pi^*) \in \mathcal{K}^1$ is a Bertrand-Nash equilibrium according to Definition 7.1 if and only if it satisfies the variational inequality:

$$-\sum_{j=1}^{n_O} \frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial q_j} \times (q_j - q_j^*) - \sum_{j=1}^{n_O} \sum_{k=1}^{n_R} \frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial \pi_{jk}} \times (\pi_{jk} - \pi_{jk}^*) \geq 0,$$
$$\forall (q, \pi) \in \mathcal{K}^1. \tag{7.12}$$

with notice that: for $j = 1, \ldots, n_O$:

$$-\frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial q_j} = \sum_{k=1}^{n_R} \frac{\partial \widehat{sc}_{jk}(Q^*, q^*)}{\partial q_j} + \frac{\partial \widehat{qc}_j(q^*)}{\partial q_j}, \tag{7.13}$$

and for $j = 1, \ldots, n_O$; $k = 1, \ldots, n_R$:

$$-\frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial \pi_{jk}} = \sum_{r=1}^{n_R} \frac{\partial oc_{jr}(\pi^*)}{\partial \pi_{jk}} - Q^*_{n_M+j,k}. \tag{7.14}$$

7.2.3 The Equilibrium Conditions for the Supply Chain Network with Outsourcing and with Price and Quality Competition

In equilibrium, the optimality conditions for all contractors and the optimality conditions for the firm must hold simultaneously, according to the definition below.

Definition 7.2: Supply Chain Network Equilibrium with Outsourcing and with Price and Quality Competition
The equilibrium state of the supply chain network with outsourcing is one where both variational inequalities (7.6) and (7.12) hold simultaneously.

The following theorem is then immediate.

Theorem 7.3: Variational Inequality Formulation of the Supply Chain Network Equilibrium with Outsourcing and Price and Quality Competition
The equilibrium conditions governing the supply chain network model with outsourcing are equivalent to the solution of the variational inequality problem: determine $(Q^*, q^*, \pi^*) \in \mathscr{K}$*, such that:*

$$-\sum_{h=1}^{n}\sum_{l=1}^{n_R} \frac{\partial U_0(Q^*, q^*, \pi^*)}{\partial Q_{hl}} \times (Q_{hl} - Q^*_{hl}) - \sum_{j=1}^{n_O} \frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial q_j} \times (q_j - q^*_j)$$

$$-\sum_{j=1}^{n_O}\sum_{k=1}^{n_R} \frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial \pi_{jk}} \times (\pi_{jk} - \pi^*_{jk}) \geq 0, \quad \forall (Q, q, \pi) \in \mathscr{K}. \tag{7.15}$$

We now put variational inequality (7.15) into standard form (2.1a): determine $X^* \in \mathscr{K}$ where X is a vector in R^N, $F(X)$ is a continuous function such that $F(X) : X \mapsto \mathscr{K} \subset R^N$, and

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathscr{K}, \tag{7.16}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the N-dimensional Euclidean space, and $\mathscr{K}$ is closed and convex. We define the vector $X \equiv (Q, q, \pi)$. Also, here $N = nn_R + n_O + n_O n_R$. Note that (7.16) may be rewritten as:

$$\sum_{i=1}^{N} F_i(X^*) \times (X_i - X_i^*) \geq 0, \quad \forall X \in \mathscr{K}. \tag{7.17}$$

The components of F are as follows. The first nn_R components of F are given by: $-\frac{\partial U_0(Q,q,\pi)}{\partial Q_{hl}}$ for $h = 1, \ldots, n$; $l = 1, \ldots, n_R$; the next n_O components of F are given by: $-\frac{\partial U_j(Q,q,\pi)}{\partial q_j}$ for $j = 1, \ldots, n_O$, and the subsequent $n_O n_R$ components of F are given by: $-\frac{\partial U_j(Q,q,\pi)}{\partial \pi_{jk}}$ with $j = 1, \ldots, n_O$; $k = 1, \ldots, n_R$. Hence, (7.15) can be put into standard form (7.16).

The following theorem is immediate from the classical theory of variational inequalities (see Chap. 2).

Theorem 7.4: Existence and Uniqueness
Suppose that F *is strongly monotone. Then there exists a unique solution to variational inequality (7.16); equivalently, to variational inequality (7.15).*

7.3 The Algorithm and Numerical Examples

We now discuss the form that the Euler method (see Chap. 2) takes for the solution of the supply chain network equilibrium model with outsourcing and price and quality competition governed by variational inequality (7.16); equivalently, (7.15).

Note that, at each iteration $\tau + 1$, of the Euler method, $X^{\tau+1}$ is actually the solution to the strictly convex quadratic programming problem given by:

$$X^{\tau+1} = \text{Minimize}_{X\in\mathscr{K}} \quad \frac{1}{2}\langle X, X\rangle - \langle X^{\tau} - a_{\tau}F(X^{\tau}), X\rangle. \tag{7.18}$$

As for solving (7.18), in order to obtain the values of the product flows at each iteration, we can apply the exact equilibration algorithm, originated by Dafermos and Sparrow (1969), and used for the solution of many different applications of networks with special structure (cf. Nagurney 1999; Nagurney and Zhang 1996). See also Nagurney and Zhang (1997) for an application to fixed demand traffic network equilibrium problems.

Furthermore, in light of the nice structure of the underlying feasible set $\mathscr{K}$, we can obtain the values for the quality variables explicitly according to the following closed form expression for contractor j; $j = 1, \ldots, n_O$:

$$q_j^{\tau+1} = \min\{q^U, \max\{0, q_j^{\tau} + a_{\tau}(-\sum_{k=1}^{n_R} \frac{\partial \widehat{sc}_{jk}(Q^{\tau}, q^{\tau})}{\partial q_j} - \frac{\partial \widehat{qc}_j(q^{\tau})}{\partial q_j})\}\}. \tag{7.19}$$

Also, we have the following explicit formula for the contractor prices: for $j = 1, \ldots, n_O$; $k = 1, \ldots, n_R$:

$$\pi_{jk}^{\tau+1} = \max\{0, \pi_{jk}^{\tau} + a_{\tau}(-\sum_{r=1}^{n_R} \frac{\partial oc_{jr}(\pi^{\tau})}{\partial \pi_{jk}} + Q_{n_M+j,k}^{\tau})\}. \tag{7.20}$$

We now provide the convergence result. The proof is direct from Theorem 6.10 in Nagurney and Zhang (1996).

Theorem 7.5: Convergence
In the supply chain network model with outsourcing, let $F(X) = -\nabla U(Q, q, \pi)$ be strongly monotone. Also, assume that F is uniformly Lipschitz continuous. Then there exists a unique equilibrium product flow, quality level, and price pattern $(Q^*, q^*, \pi^*) \in \mathscr{K}$ and any sequence generated by the Euler method as given by (2.34), where $\{a_{\tau}\}$ satisfies $\sum_{\tau=0}^{\infty} a_{\tau} = \infty$, $a_{\tau} > 0$, $a_{\tau} \to 0$, as $\tau \to \infty$ converges to (Q^*, q^*, π^*).

Note that convergence also holds if $F(X)$ is strictly monotone (cf. Theorem 8.6 in Nagurney and Zhang 1996) provided that the price iterates are bounded. We know that the product flow iterates as well as the quality level iterates will be bounded due to the constraints. Clearly, in practice, contractors cannot charge unbounded

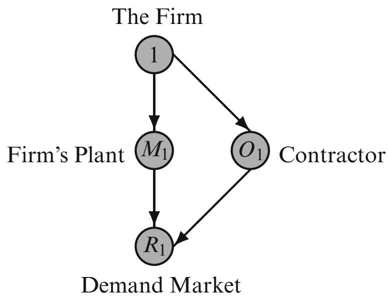


Fig. 7.2 The supply chain network for an illustrative numerical example

prices for production and delivery. Hence, we can also expect the existence of a solution, given the continuity of the functions that make up $F(X)$, under less restrictive conditions that of strong monotonicity.

7.3.1 *An Illustrative Example, a Variant, and Sensitivity Analysis*

We now provide a small example to clarify ideas, along with a variant, and also conduct a sensitivity analysis exercise. The supply chain network consists of the firm, a single contractor, and a single demand market R_1, as depicted in Fig. 7.2.

The data are as follows. The firm's production cost function is:

$$\hat{f}_1(Q_{11}) = Q_{11}^2 + Q_{11}$$

and its total transportation cost function is:

$$\hat{c}_{11}(Q_{11}) = 0.5Q_{11}^2 + Q_{11}.$$

The firm's transaction cost function associated with the contractor is given by:

$$tc_1(Q_{21}) = 0.05Q_{21}^2 + Q_{21}.$$

The demand for the product at demand market R_1 is 1,000, q^U is 100, and the weight ω is 1.

The contractor's total cost of production and distribution function is:

$$\hat{sc}_{11}(Q_{21}, q_1) = Q_{21}q_1.$$

Its total quality cost function is given by:

$$\hat{qc}_1(q_1) = 10(q_1 - 100)^2.$$

The contractor's opportunity cost function is:

$$oc_{11}(\pi_{11}) = 0.5(\pi_{11} - 10)^2.$$

The firm's cost of disrepute function is:

$$dc(q\prime) = 100 - q\prime$$

where $q\prime$ (cf. (7.1)) is given by: $\frac{Q_{21}q_1 + Q_{11}100}{1{,}000}$.

We set the convergence tolerance to 10^{-3} so that the Euler method is deemed to have converged when the absolute value of the difference between each product flow, each quality level, and each price is less than or equal to 10^{-3}. The Euler method is initialized with $Q^0_{11} = Q^0_{21} = 500.00$, $q^0_1 = 1.00$, and $\pi^0_{11} = 0.00$. The sequence $\{a_\tau\}$ is set to: $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$.

The Euler method converges in 87 iterations and yields the following product flow, quality level, and price pattern:

$$Q^*_{11} = 270.50, \quad Q^*_{21} = 729.50, \quad q^*_1 = 63.52, \quad \pi^*_{11} = 739.50.$$

The total cost incurred by the firm is 677,128.65 with the contractor earning a profit of 213,786.67. The value of $q\prime$ is 73.39.

The Jacobian matrix of $F(X) = -\nabla U(Q, q, \pi)$, for this example, denoted by $J(Q_{11}, Q_{21}, q_1, \pi_{11})$, is

$$J(Q_{11}, Q_{21}, q_1, \pi_{11}) = \begin{pmatrix} 3.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.1 & -.001 & 1.0 \\ 0.0 & 1.0 & 20.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 1.0 \end{pmatrix}.$$

This Jacobian matrix is positive definite, and, hence, $-\nabla U(Q, q, \pi)$ is strongly monotone (since F(X) is linear). Thus, both the existence and the uniqueness of the solution to variational inequality (7.15) with respect to this example are guaranteed. Moreover, the equilibrium solution, reported above, is globally exponentially stable (see Chap. 2).

We then construct a variant of this example. The transportation cost function is reduced by a factor of 10 so that it is now:

$$\hat{c}_{11}(Q_{11}) = 0.05Q^2_{11} + 0.1Q_{11}$$

with the remainder of the data as in the original example above. Such a change, for example, captures the situation of the firm moving its production facility closer to the demand market.

The Euler method again requires 87 iterations for convergence and yields the equilibrium solution:

$$Q_{11}^* = 346.86, \quad Q_{21}^* = 653.14, \quad q_1^* = 67.34, \quad \pi_{11}^* = 663.15.$$

The firm's total costs are now 581,840.07 and the contractor's profit is 165,230.62. The value of $q\prime$ is now 78.67. The average quality increased, with the quantity of the product produced by the firm having increased. Also, the price charged by the contractor decreased but the quality level of its product increased.

For both these examples, the underlying constraints are satisfied, consisting of the demand constraint, the nonnegativity constraints, as well as the upper bound on the contractor's quality level. In addition, the variational inequality for this problem is satisfied.

It is easy to verify that the Jacobian of F for the variant is positive definite with the only change in the Jacobian matrix above being that the 3.0 is replaced by 2.1.

We then proceeded to conduct a sensitivity analysis exercise. We returned to the original example and increased the demand for the product at R_1 in increments of 1,000. The results of the computations are reported in Figs. 7.3, 7.4, and 7.5 for the equilibrium product flows, the quality levels, and the average quality $q\prime$, and, finally, the equilibrium prices.

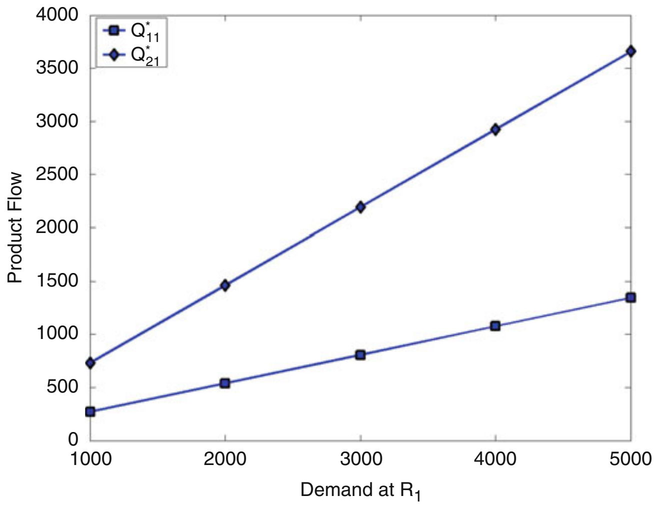


Fig. 7.3 Equilibrium product flows as the demand increases for the illustrative example

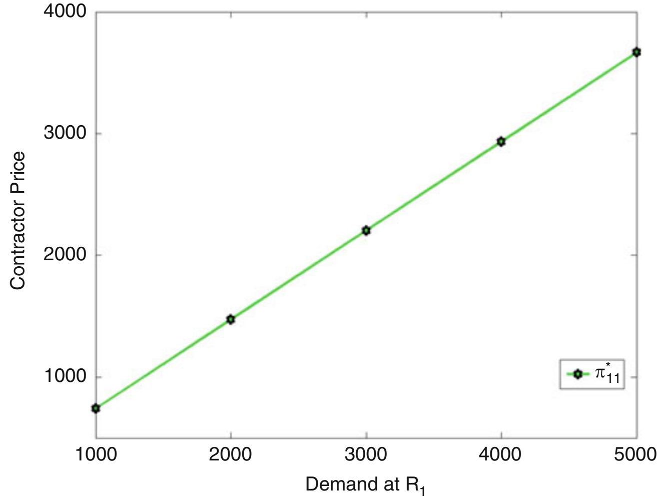


Fig. 7.4 Equilibrium contractor prices as the demand increases for the illustrative example

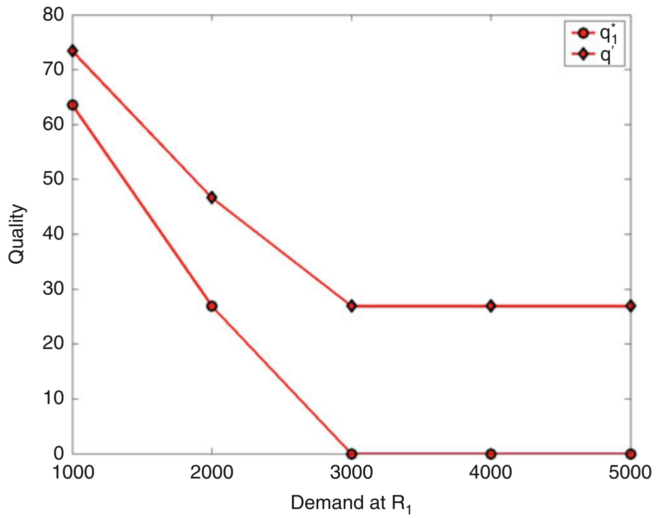


Fig. 7.5 Equilibrium contractor quality level and the average quality as the demand increases for the illustrative example

It is interesting to observe that, when the demand increases past a certain point, the contractor's equilibrium quality level decreases to zero and stays at that level. Such unexpected insights may be obtained through a modeling and computational framework that we have developed. Of course, the results in this subsection are based on constructed examples. One may conduct other sensitivity analysis exercises and also utilize different underlying cost functions in order to tailor the general framework to specific firms' needs and situations, in specific, relevant industries.

7.3.2 Additional Numerical Examples

In this section, we apply the Euler method to compute solutions to examples that are larger than those in the preceding section. We report all of the input data as well as the output. The Euler method is initialized as in the illustrative example, except that the initial product flows are equally distributed among the available options for each demand market. We use the same convergence tolerance as previously.

Example 7.1 and Sensitivity Analysis
Example 7.1 consists of the topology given in Fig. 7.6. There are two manufacturing plants owned by the firm and two possible contractors. The firm must satisfy the demands for its product at the two demand markets. The demand for the product at demand market R_1 is 1,000 and it is 500 at demand market R_2. q^U is 100 and the weight ω is 1.

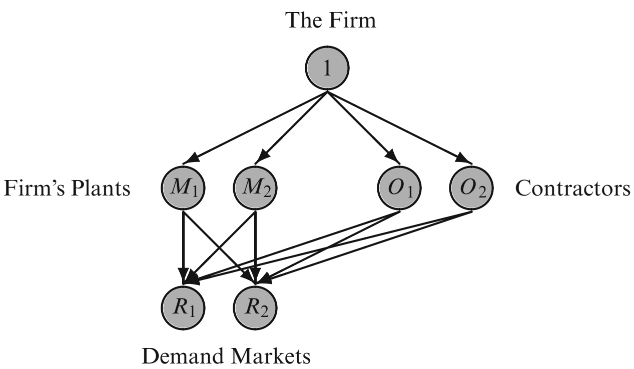


Fig. 7.6 The supply chain network topology for Example 7.1

The production cost functions at the plants are:

$$\hat{f}_1(\sum_{k=1}^{2} Q_{1k}) = (Q_{11} + Q_{12})^2 + 2(Q_{11} + Q_{12}),$$

$$\hat{f}_2(\sum_{k=1}^{2} Q_{2k}) = 1.5(Q_{21} + Q_{22})^2 + 2(Q_{21} + Q_{22}).$$

The total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = 1.5Q_{11}^2 + 10Q_{11}, \quad \hat{c}_{12}(Q_{12}) = 1Q_{12}^2 + 25Q_{12},$$
$$\hat{c}_{21}(Q_{21}) = 1Q_{21}^2 + 5Q_{21}, \quad \hat{c}_{22}(Q_{22}) = 2.5Q_{22}^2 + 40Q_{22}.$$

The transaction cost functions are:

$$tc_1(Q_{31} + Q_{32}) = 0.5(Q_{31} + Q_{32})^2 + 0.1(Q_{31} + Q_{32}),$$
$$tc_2(Q_{41} + Q_{42}) = 0.25(Q_{41} + Q_{42})^2 + 0.2(Q_{41} + Q_{42}).$$

The contractors' total cost of production and distribution functions are:

$$\hat{sc}_{11}(Q_{31}, q_1) = Q_{31}q_1, \quad \hat{sc}_{12}(Q_{32}, q_1) = Q_{32}q_1,$$
$$\hat{sc}_{21}(Q_{41}, q_2) = 2Q_{41}q_2, \quad \hat{sc}_{22}(Q_{42}, q_2) = 2Q_{42}q_2.$$

Their total quality cost functions are given by:

$$\hat{qc}_1(q_1) = 5(q_1 - 100)^2, \quad \hat{qc}_2(q_2) = 10(q_1 - 100)^2.$$

The contractors' opportunity cost functions are:

$$oc_{11}(\pi_{11}) = 0.5(\pi_{11} - 10)^2, \quad oc_{12}(\pi_{12}) = (\pi_{12} - 10)^2,$$
$$oc_{21}(\pi_{21}) = (\pi_{21} - 5)^2, \quad oc_{22}(\pi_{22}) = 0.5(\pi_{22} - 20)^2.$$

The firm's cost of disrepute function is:

$$dc(q\prime) = 100 - q\prime$$

where $q\prime$ (cf. (7.1)) is given by: $\frac{Q_{31}q_1+Q_{32}q_1+Q_{41}q_2+Q_{42}q_2+Q_{11}100+Q_{12}100+Q_{21}100+Q_{22}100}{1{,}500}$.

The Euler method converges in 153 iterations and yields the following equilibrium solution. The computed product flows are:

$$Q_{11}^* = 95.77, \quad Q_{12}^* = 85.51, \quad Q_{21}^* = 118.82, \quad Q_{22}^* = 20.27,$$
$$Q_{31}^* = 213.59, \quad Q_{32}^* = 224.59, \quad Q_{41}^* = 571.83, \quad Q_{42}^* = 169.63.$$

The computed quality levels of the contractors are:

$$q_1^* = 56.18, \quad q_2^* = 25.85,$$

and the computed prices are:

$$\pi_{11}^* = 223.57, \quad \pi_{12}^* = 122.30, \quad \pi_{21}^* = 290.92, \quad \pi_{22}^* = 189.61.$$

The total cost of the firm is 610,643.26 and the profits of the contractors are: 5,733.83 and 9,294.44.

The value of q' is 50.55.

In order to investigate the stability of the computed equilibrium for Example 7.1 (and a similar analysis holds for the subsequent examples), we constructed the Jacobian matrix as follows. The Jacobian matrix of $F(X) = -\nabla U(Q, q, \pi)$, for this example, denoted by $J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, Q_{41}, Q_{42}, q_1, q_2, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22})$, is

$$J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, Q_{41}, Q_{42}, q_1, q_2, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22})$$

$$= \begin{pmatrix}
5&2&0&0&0&0&0&0&0&0&0&0&0&0\\
2&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&5&3&0&0&0&0&0&0&0&0&0&0\\
0&0&3&8&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&-6.67\times10^{-4}&0&1&0&0&0\\
0&0&0&0&1&1&0&0&-6.67\times10^{-4}&0&0&1&0&0\\
0&0&0&0&0&0&0.5&0.5&0&-6.67\times10^{-4}&0&0&1&0\\
0&0&0&0&0&0&0.5&0.5&0&-6.67\times10^{-4}&0&0&0&1\\
0&0&0&0&0&0&0&0&10&0&0&0&0&0\\
0&0&0&0&0&0&2&2&0&20&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&1
\end{pmatrix}.$$

This Jacobian matrix is positive semidefinite. Hence, since $F(X)$ is monotone, every supply chain network equilibrium, as defined in Definition 7.2, is a global monotone attractor for the utility gradient process.

Interestingly, in this example, the firm pays relatively higher prices for the product with a lower quality level from contractor O_2. This happens because the firm's demand is fixed in each demand market, and, therefore, there is no pressure for quality improvement from the demand side (as would be the case if the demands were elastic). As reflected in the transaction costs with contractors O_1 and O_2, the firm is willing to pay higher prices to O_2, despite a lower quality level.

We then conducted sensitivity analysis. In particular, we investigate the effect of increases in the demands at both demand markets R_1 and R_2. The results for the new equilibrium product flows are depicted in Fig. 7.7, with the results

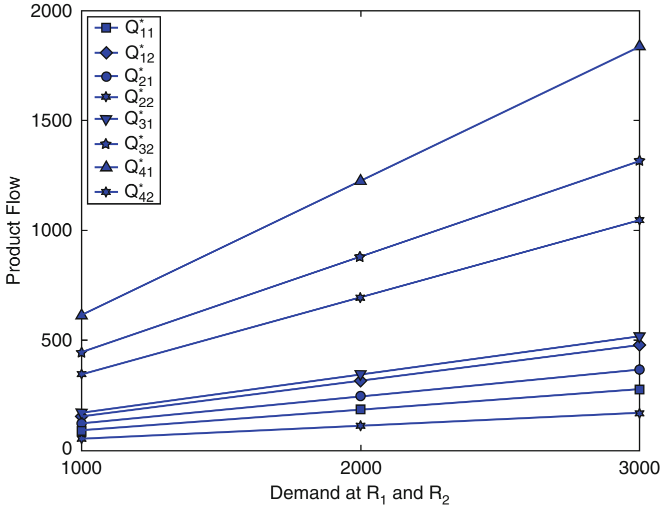


Fig. 7.7 Equilibrium product flows as the demand increases for Example 7.1

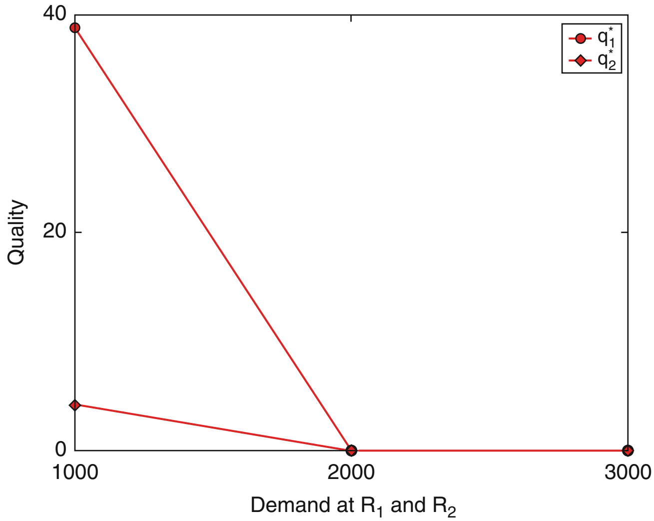


Fig. 7.8 Equilibrium quality levels as the demand increases for Example 7.1

for the equilibrium quality levels and the equilibrium prices in Figs. 7.8 and 7.9, respectively. The contractors consistently provide the majority of the product to the demand markets.

Observe from Fig. 7.8 that, as the demand increases, the quality levels for both contractors drops to zero.

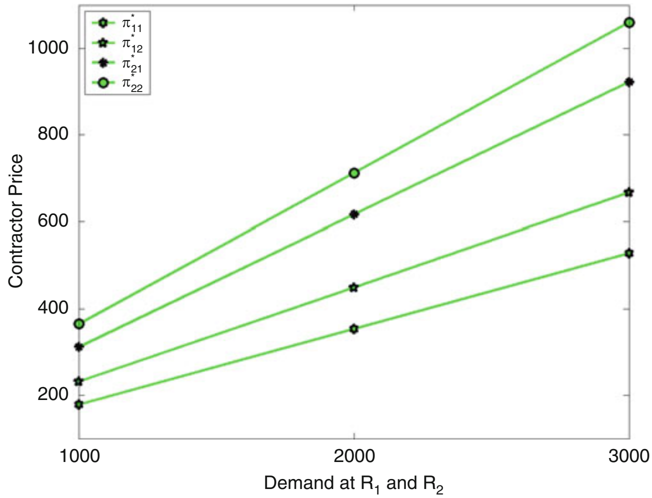
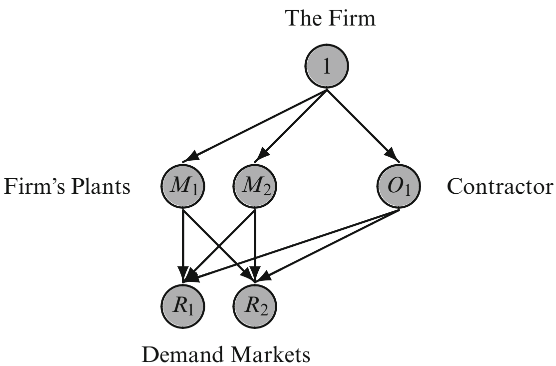


Fig. 7.9 Equilibrium contractor prices as the demand increases for Example 7.1

Fig. 7.10 The supply chain network topology for Example 7.2



Example 7.2
We then considered a disruption to the original supply chain in Example 7.1. No business is immune from supply chain disruptions and, as noted by Purtell (2010), pharmaceuticals are especially vulnerable since they are high-value, highly regulated products. Moreover, pharmaceutical disruptions may not only increase costs but may also create health hazards and expose the pharmaceutical companies to damage to their brands and reputations (see also Nagurney et al. 2011; Masoumi et al. 2012; Qiang and Nagurney 2012).

Specifically, we considered the following disruption. The data are as in Example 7.1 but contractor O_2 is not able to provide any production and distribution services. This could arise due to a natural disaster, adulteration in its production process, and/or an inability to procure an ingredient. Hence, the topology of the disrupted supply chain network is as in Fig. 7.10.

The Euler method converges in 73 iterations to the following new equilibrium solution. The computed product flows are:

$$Q_{11}^* = 218.06, \quad Q_{12}^* = 141.79, \quad Q_{21}^* = 260.20, \quad Q_{22}^* = 25.96,$$
$$Q_{31}^* = 521.74, \quad Q_{32}^* = 332.25.$$

The computed quality level of the remaining contractor is:

$$q_1^* = 14.60,$$

and the computed prices are:

$$\pi_{11}^* = 531.74, \quad \pi_{12}^* = 176.12.$$

With the decrease in competition among the contractors, since there is now only one, rather than two, as in Example 7.1, the quality level of contractor O_1 drops, but the prices that it charges increases. The new average quality level is $q' = 51.38$. The total cost of the firm is now 1,123,226.62 whereas the profit of the first contractor is now 123,460.67, a sizable increase relative to that in the case of competition as in Example 7.1.

Example 7.3

The data for Example 7.3 are the same as for Example 7.1, except that the opportunity cost functions oc_{11}, oc_{12}, oc_{21}, oc_{22} are all equal to 0.00.

The Euler method converges in 76 iterations to the following equilibrium solution. The computed product flows are:

$$Q_{11}^* = 451.17, \quad Q_{12}^* = 396.50, \quad Q_{21}^* = 548.73, \quad Q_{22}^* = 103.39,$$
$$Q_{31}^* = 0.00, \quad Q_{32}^* = 0.00, \quad Q_{41}^* = 0.00, \quad Q_{42}^* = 0.00.$$

The computed quality levels of the contractors are:

$$q_1^* = 100.00, \quad q_2^* = 100.00,$$

and the computed prices are:

$$\pi_{11}^* = 3,060.70, \quad \pi_{12}^* = 2,515.08, \quad \pi_{21}^* = 3,060.56, \quad \pi_{22}^* = 2,515.16.$$

The total cost of the firm is 2,171,693.16 and the profits of the contractors are 0.00 and 0.00. The value of q' is 100.00.

Because the opportunity costs are all zero, in order to improve the total profit, the contractors will charge the firm very high prices. Thus, the original firm would rather produce by itself than outsource to the contractors.

One can see, from this example, that, in addition to the total revenue term, each contractor must consider an outsourcing price related term, such as the opportunity cost, in its objective function. Without considering such a function, the outsourcing quantities will all be zero (cf. (7.14)), and, hence, a contractor would not secure any contracts from the firm.

Example 7.4
The data for Example 7.4 are the same as for Example 7.1, except for the opportunity cost functions and the demand. The demand in R_1 is now 700 and the demand in R_2 is 100.

The contractors' opportunity cost functions now become:

$$oc_{11}(\pi_{11}) = 0.5(\pi_{11} - 2)^2 - 15{,}265.29, \quad oc_{12}(\pi_{12}) = (\pi_{12} - 1)^2 - 513.93,$$
$$oc_{21}(\pi_{21}) = (\pi_{21} - 1)^2 - 35{,}751.25, \quad oc_{22}(\pi_{22}) = 0.5(\pi_{22} - 2)^2 - 613.20.$$

The Euler method converges in 14 iterations to the following equilibrium solution. The computed product flows are:

$$Q_{11}^* = 69.12, \quad Q_{12}^* = 19.65, \quad Q_{21}^* = 77.99, \quad Q_{22}^* = 0.00,$$
$$Q_{31}^* = 174.72, \quad Q_{32}^* = 45.34, \quad Q_{41}^* = 378.17, \quad Q_{42}^* = 35.00.$$

The computed quality levels of the contractors are:

$$q_1^* = 77.99, \quad q_2^* = 58.68,$$

and the computed prices are:

$$\pi_{11}^* = 176.73, \quad \pi_{12}^* = 23.67, \quad \pi_{21}^* = 190.08, \quad \pi_{22}^* = 37.02.$$

The total cost of the firm is 204,701.28 and the profits of the contractors are: 12,366.75 and 7,614.84. The incurred opportunity costs at the equilibrium prices are all zero. Thus, in this example, the equilibrium prices that the contractors charge the firm are such that they are able to adequately recover their costs, and secure contracts. The value of $q\prime$ is 72.61.

7.4 Summary and Conclusions

In this chapter, we developed a supply chain network game theory model to capture contractor selection, based on the competition among the contractors in the prices that they charge as well as the quality levels of the products that they produce. We introduced a disrepute cost associated with the average quality at the demand markets. We assumed that the firm is cost-minimizing whereas the contractors are profit-maximizing.

We utilized variational inequality theory for the formulation of the governing Nash-Bertrand equilibrium conditions and proposed an algorithm. We illustrated the methodological framework through a series of numerical examples for which we reported the complete input and output data for transparency purposes. Our numerical studies included sensitivity analysis results as well as a disruption to the supply chain network in that a contractor is no longer available for production and distribution. We also discussed the scenario in which the opportunity costs on the contractors' side are identically equal to zero and the scenario in which the opportunity costs at the equilibrium are all zero.

This chapter is a contribution to the literature on outsourcing with a focus on quality with an emphasis on an industry where the quality of a product is paramount (such as pharmaceuticals, for example). It also is an interesting application of game theory and associated methodologies. The ideas in this chapter may be adapted, with appropriate modifications, to other relevant industries.

The game theory supply chain network model developed in this chapter is based on the following assumptions: 1. The firm may contract the manufacturing and the delivery tasks to the contractors. 2. According to regulations (such as, for example, as described in FDA (2002), U.S. Department of Health and Human Services, CDER, FDA, and CBER (2009), and the European Commission Enterprise and Industry Directorate-General (2010)) and the literature, before signing the contract, the firm should have reviewed and evaluated the contractors' ability to perform the outsourcing tasks. Therefore, the production/distribution costs and the quality cost information of the contractors are assumed to be known by the firm. 3. In addition to paying the contractors, the firm also pays the transaction cost and one category of the quality-related costs, the external failure costs. The transaction cost is the "cost of making each contract" (cf. Coase (1937) and also Aubert et al. (1996)), which includes the costs of evaluating suppliers, negotiation costs, the monitoring and the enforcement of the contract in order to ensure the quality (Picot et al. (1996); Franceschini et al. 2003; Heshmati 2003; Liu and Nagurney 2013). External failure costs are the compensation costs incurred when customers are unsatisfied with the quality of the products. The objective of the firm is to minimize the total operational costs and transaction costs, along with the weighted individual disrepute. 4. For the in-house supply chain activities, it is assumed that the firm can ensure a 100 % perfect quality conformance level (see Schneiderman 1986; Kaya 2011). 5. The contractors, who produce for the same firm, compete a la Bertrand (cf. Bertrand 1883) by determining their optimal prices and quality levels in order to maximize their profits.

7.5 Sources and Notes

This chapter is based on the paper by Nagurney et al. (2013) but the model in this chapter extends the model in that paper in that the opportunity cost of each supplier is now a function of the vector of prices charged by all suppliers, rather than only of the specific supplier's price. Due to competition among suppliers in pricing their products, the opportunity cost of a supplier, which is its "loss of potential gain" in pricing, may also be affected by the prices charged by the other suppliers. Moreover, the outsourcing model presented in the Nagurney et al. (2013) paper was done in a pharmaceutical industry setting context. In this chapter, we consider an industry-independent setting. Furthermore, we note that only a small portion of the supply chain literature (cf. Sect. 1.3.3) directly addresses and models the risk of quality and safety issues associated with outsourcing, as we do in this chapter.

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Chapter 8
Outsourcing Under Price and Quality Competition: Multiple Firms

Abstract In this chapter, we extend the results of Chap. 7 and develop a supply chain network game theory model with product differentiation, possible outsourcing of production and distribution, and quality and price competition among the contractors. The original firms compete with one another in terms of in-house quality levels and product flows, whereas the contractors, aiming at maximizing their own profits, engage in competition for the outsourced production and distribution in terms of prices that they charge and their quality levels. The solution of the model provides each original firm with its optimal in-house quality level as well as its optimal in-house and outsourced production and shipment quantities that minimize the total cost and the weighted cost of disrepute, associated with lower quality levels and the impact on a firm's reputation. The governing equilibrium conditions of the model are formulated as a variational inequality problem. An algorithm is proposed and numerical supply chain network examples are provided to illustrate how such a supply chain network game theory model can be applied in practice. The model is relevant to products ranging from pharmaceuticals to fast fashion to high technology products.

8.1 Introduction

Since the mid-1990s, outsourcing of production has exerted a huge impact on manufacturing industries as wide-ranging as pharmaceuticals, fast fashion, and high technology. As the volume of outsourcing has increased, the supply chain networks weaving the original manufacturers and the contractors are becoming increasingly complex. Firms may no longer outsource exclusively to specific contractors, and there may be contractors engaging with multiple firms, who are actually competitors. For example, Apple relies heavily on processors for its iPhones and iPads produced by its competitor, Samsung (Whittaker 2015). This is notable since about 60 % of Apple's revenue is due to iPhone sales. Moreover, Apple, Compaq, Dell, Gateway, Lenovo, and Hewlett-Packard are consumers of Quanta Computer Incorporated, a Taiwan-based Chinese manufacturer of notebook computers (Landler 2002), and Foxconn, another Taiwan-based manufacturer, is currently producing tablet computers for Apple, Google, Android, and Amazon (Nystedt 2010; Topolsky 2010). As another illustration, the US head office of Volvo

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has been outsourcing the production of components to companies such as Minda HUF, Visteon, Arvin Meritor, and Rico Auto. These major sources, in turn, obtain almost 100 % of their components, ranging from the engine parts, the suspension and braking parts, to the electric parts, from Indian contractors (Klum 2007).

However, parallel to the dynamism of and growth in outsourcing, issues of quality have gradually emerged. In 2007, the international toy giant Mattel recalled 19 million outsourced toy cars because of lead paint and small, poorly designed magnets, which could harm children, if ingested (Story and Barboza 2007). In 2010, the suicide scandal at Foxconn, which revealed the poor working conditions in its contract manufacturing plants, led to negative impacts on the reputation of multiple original electronic product manufacturers (McEntegart 2010).

In addition, the apparel industry, recently, has had many significant quality failures associated with the outsourcing of production, notably, in Bangladesh. A fire at an unauthorized sub-contracted garment factory producing for Walmart, Sears, Disney, and other apparel corporations in Bangladesh took at least 112 lives in 2012, which was the deadliest in the history of Bangladesh (Alam 2012). Only 5 months later, a similar illegally constructed commercial building producing clothing for major European and American brands collapsed. 1,127 people were killed and 2,500 injured, which makes this the deadliest accidental structural failure in modern human history (BBC 2013; Alam and Hossain 2013).

Clearly, cost reduction due to outsourcing, including offshore outsourcing, is an extremely important issue for original manufacturing firms. However, as discussed in Chap. 7, with the increasing volume of outsourcing and the growing complexity of the supply chain networks associated with outsourcing, it is essential for firms to be able to rigorously assess not only the possible benefits due to outsourcing but also the potential costs associated with disrepute (loss of reputation) resulting from the potential quality degradation due to outsourcing.

This chapter, in contrast to the literature on outsourcing reviewed in Chap. 7, focuses on the supply chain network of multiple competing original firms and the contractors. For each contractor, the number of contracted original firms is not predetermined. Some contractors may end up contracting with one or more original firms, while others may contract to none. At the same time, each original firm can also outsource the production of all of its products, along with their delivery to the demand markets, or outsource some to any number of contractors, or choose not to outsource. This supply chain network model provides each original firm with the equilibrium in-house quality level and the equilibrium make-or-buy and contractor-selection policy, with the demand for its product being satisfied in multiple demand markets. We assume that the original firms compete in the sense of Nash (1950, 1951)-Cournot (1838). Each firm aims at determining its equilibrium in-house quality level, in-house production quantities and shipments, and outsourcing quantities, which satisfy demand requirements, so as to minimize its total cost and the weighted cost of disrepute. The contractors, in turn, are competing a la Nash (1950, 1951)-Bertrand (1883) in determining their optimal quality and price levels in order to maximize their individual total profits, as they did in the single firm model of Chap. 7.

In the supply chain network model in this chapter, the products from different original firms are considered to be differentiated by their brands, as was also the case in the supply chain network models in Chaps. 5 and 6 (see also, e.g., Nagurney et al. 2013; Yu and Nagurney 2013). When consumers observe a brand of a product, they consider the quality, function, and reputation of that particular brand name and the product. With outsourcing, chances are that the product was manufactured by a completely different company than the brand indicates, but the level of quality and the reputation associated with the outsourced product still remain with the "branded" original firm. If a product is recalled for a faulty part and that part was outsourced, the original firm is the one that carries the burden of correcting its damaged reputation.

Chapter 8, which extends the model in Chap. 7, investigates further the supply chain network problem with outsourcing and quality competition. A supply chain network game theory model with multiple original firms competing with one another is developed, and the products of the distinct original firms are differentiated by their brands. Moreover, the in-house quality levels are no longer assumed to be perfect, as was the case in Chap. 7, but, rather, are strategic variables of the firms, since in-house quality failures may also occur (cf. Beamish and Bapuji 2008).

In contrast to Chap. 7, we here also associate quality with respect to the distribution activities of the products to the demand markets (cf. Chap. 3). We accomplish this by having the transportation (distribution) functions depend explicitly on both flows and on product quality levels, with the assumption that the product will be delivered at the same quality level that it was produced at. As noted in Chap. 1, quality, as well as price, have been identified empirically as critical factors in transport mode selection for product delivery (cf. Floden et al. 2010; Saxin et al. 2005, and the references therein).

This chapter is organized as follows. In Sect. 8.2, we describe the decision-making behavior of the original firms, who compete in quantity and quality, and that of the contractors, who compete in price and quality. Then, we construct the supply chain network game theory model with product differentiation, the possible outsourcing of production and distribution, and quality and price competition. We obtain the governing equilibrium conditions, and derive the equivalent variational inequality formulation. For consistency, we define and quantify the quality levels, the quality costs, and the disrepute cost in a manner similar to that in Chap. 7.

In Sect. 8.3, we describe the algorithm, the Euler method, which yields closed form expressions, at each iteration, for the prices and the quality levels, with the product flows being solved exactly by an equilibration algorithm, as we did in Chap. 7 for the single firm supply chain network model with outsourcing. The Euler method, in the context of our supply chain network game theory model, can be interpreted as a discrete-time adjustment process until the equilibrium state is achieved. Convergence results are also provided. It is applied in Sect. 8.4 to compute solutions to numerical examples, along with sensitivity analysis, in order to demonstrate the generality and the applicability of the proposed framework. In Sect. 8.5, we summarize our results and present our conclusions. Sources and Notes are provided in Sect. 8.6.

8.2 The Supply Chain Network Model with Product Differentiation and Outsourcing

In this section, we develop the supply chain network game theory model with product differentiation, possible outsourcing, price and quality competition among the contractors, and quantity and quality competition among the original firms. We consider a finite number of I original firms, with a typical firm denoted by i, who compete noncooperatively. The products of the I firms are not homogeneous but, rather, are differentiated by brands. Firm i; $i = 1, \ldots, I$, is involved in the processes of in-house manufacturing and distribution of its brand name product, and may subcontract its manufacturing and distribution activities to contractors who may be located overseas. We seek to determine the optimal product flows from each firm to its demand markets, along with the prices the contractors charge the firms, and the quality levels of the in-house manufactured products and the outsourced products.

The supply chain network topology of the I firms depicted is given in Fig. 8.1. Each firm i; $i = 1, \ldots, I$, is considering in-house and outsourcing manufacturing facilities and serves the same n_R demand markets. A link from each top-tiered node i, representing original firm i, is connected to its in-house manufacturing facility node M^i. The in-house distribution activities of firm i, in turn, are represented by links connecting M^i to the demand nodes: $R_1, \ldots, R_{n_R}$.

In this model, we capture the possible outsourcing of the products from the I firms in terms of their production and delivery. As depicted in Fig. 8.1, there are n_O contractors available to each of the I firms. Each firm may potentially contract to any of these contractors who then produce and distribute the product to the same n_R demand markets. In Fig. 8.1, hence, there are additional links from each top-most node i; $i = 1, \ldots, I$, to the n_O contractor nodes, $O_1, \ldots, O_{n_O}$, each of which corresponds to the transaction activity of firm i with contractor j. The next set of links, which emanates from the contractor nodes to the demand markets, reflects the production and delivery of the outsourced products to the n_R demand markets.

As shown in Fig. 8.1, the outsourced flows of different firms are represented by links and in the processes of transaction and outsourcing of manufacturing and distribution, the outsourced products are still differentiated by brands. The mathematical notation, which is given in Table 8.1, explicitly handles such options.

The top set of links in Fig. 8.1 consists of the manufacturing links, whether in-house or outsourcing, whereas the next set of links consists of the associated

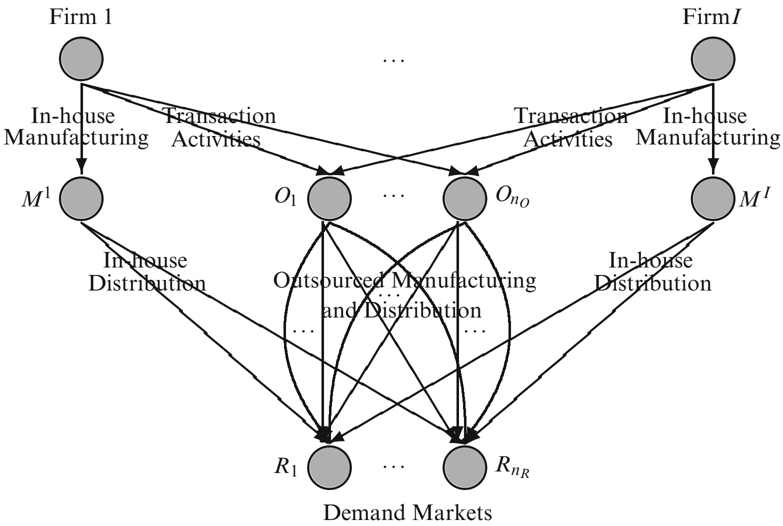

Fig. 8.1 The supply chain network topology with product differentiation and outsourcing

distribution links. For simplicity, we let $n = 1 + n_O$, where n_O is the number of potential contractors, denote the number of manufacturing plants, whether in-house or belonging to the contractors.

The notation for the model is given in Table 8.1. As in preceding chapters, the vectors are assumed to be column vectors and the optimal/equilibrium solution is denoted by "*". Inn_R, In_R, $In_R n_O$, In_O, In, and nn_R are equivalent to $I \times n \times n_R$, $I \times n_R$, $I \times n_R \times n_O$, $I \times n_O$, $I \times n$, and $n \times n_R$, respectively.

8.2.1 The Behavior of the Original Firms and Their Optimality Conditions

Recall that the quality level of firm i's product produced in-house is denoted by q_i, where $i = 1, \ldots, I$, and the quality level of firm i's product produced by contractor j is denoted by q_{ij}, where $j = 1, \ldots, n_O$. Both vary from a 0 % quality conformance level to a 100 % quality conformance level, so that, respectively,

$$0 \leq q_{ij} \leq q^U, \quad i = 1, \ldots, I; j = 1, \ldots, n_O, \tag{8.1}$$

$$0 \leq q_i \leq q^U, \quad i = 1, \ldots, I, \tag{8.2}$$

where q^U is the value representing perfect quality level associated with the 100 % quality conformance level.

Table 8.1 Notation for the supply chain network game theory model with outsourcing

Notation	Definition
Q_{ijk}	The nonnegative amount of firm i's product produced at manufacturing plant j, whether in-house or contracted, and delivered to demand market R_k, where $j = 1, \ldots, n$. For firm i, we group its own $\{Q_{ijk}\}$ elements into the vector Q_i, and group all such vectors for all original firms into the vector Q, where $Q \in R_+^{Inn_R}$. All in-house quantities are grouped into the vector $Q^1 \in R_+^{In_R}$, with all outsourcing quantities into the vector $Q^2 \in R_+^{In_R n_O}$
d_{ik}	The demand for firm i's product at demand market R_k; $k = 1, \ldots, n_R$
q_i	The nonnegative quality level of firm i's product produced in-house. We group the $\{q_i\}$ elements into the vector $q^1 \in R_+^I$
q_{ij}	The nonnegative quality level of firm i's product produced by contractor j; $j = 1, \ldots, n_O$. We group all the $\{q_{ij}\}$ elements for firm i's product into the vector $q_i^2 \in R_+^{n_O}$. For each contractor j, we group its own $\{q_{ij}\}$ elements into the vector q_j, and then group all such vectors for all contractors into the vector $q^2 \in R_+^{In_O}$
q	We group all the q_i and q_{ij} into the vector $q \in R_+^{In}$
π_{ijk}	The price charged by contractor j; $j = 1, \ldots, n_O$, for producing and delivering a unit of firm i's product to demand market R_k. We group the $\{\pi_{ijk}\}$ elements for contractor j into the vector $\pi_j \in R_+^{In_R}$, and group all such vectors for all the contractors into the vector $\pi \in R_+^{In_R n_O}$
$\hat{f}_i(Q^1, q^1)$	The total in-house production cost of firm i; $i = 1, \ldots, I$
$qc_i(q^1)$	The total quality cost of firm i; $i = 1, \ldots, I$
$tc_{ij}(\sum_{k=1}^{n_R} Q_{i,1+j,k})$	The total transaction cost associated with firm i transacting with contractor j; $j = 1, \ldots, n_O$. The detailed definition will be given later
$\hat{c}_{ik}(Q^1, q^1)$	The total transportation cost associated with delivering firm i's product manufactured in-house to demand market R_k; $k = 1, \ldots, n_R$
$\widehat{sc}_{ijk}(Q^2, q^2)$	The total cost of contractor j; $j = 1, \ldots, n_O$, to produce and distribute the product of firm i to demand market R_k; $k = 1, \ldots, n_R$
$\widehat{qc}_j(q^2)$	The total quality cost faced by contractor j; $j = 1, \ldots, n_O$
$oc_{ijk}(\pi)$	The opportunity cost associated with pricing the product of firm i at π_{ijk}, and delivering to demand market R_k, by contractor j; $j = 1, \ldots, n_O$
$q_i'(Q_i, q_i, q_i^2)$	The average quality level of firm i's product (cf. (8.3))
$dc_i(q_i'(Q_i, q_i, q_i^2))$	The cost of disrepute (loss of reputations) of firm i

The average quality level of firm i's product is, hence, an average quality level determined by the in-house quality level, the in-house product flows, the quality levels of the contractors, and the outsourced product flows. Thus, the average quality level for firm i's product, both in-house and outsourced, can be expressed as

$$q_i'(Q_i, q_i, q_i^2) = \sum_{k=1}^{n_R} \frac{\sum_{j=2}^{n} Q_{ijk} q_{i,j-1} + \sum_{k=1}^{n_R} Q_{i1k} q_i}{\sum_{k=1}^{n_R} d_{ik}}, \quad i = 1, \ldots, I. \tag{8.3}$$

Following the results in Chap. 7, we assume that the disrepute cost of firm i, $dc_i(q_i'(Q_i, q_i, q_i^2))$, is a monotonically decreasing function of the average quality level. Here, however, we no longer assume as was done in Chap. 7, which considered only a single original firm, that the original firms produce their branded products with perfect quality. Hence, the average quality level in (8.3) is a generalization of (7.2).

Each original firm i selects the product flows Q_i and the in-house quality level q_i, whereas each contractor j, who competes for contracts in quality and price, selects its outsourcing quality level vector q_j and outsourcing price vector π_j.

Firm i's utility function is denoted by U_i^1, where $i = 1, \ldots, I$. The objective of original firm i is to maximize its utility (cf. (8.4)) represented by minus its total costs that include the production cost, the quality cost, the transportation costs, the payments to the contractors, the transaction costs, along with the weighted cost of disrepute, with the nonnegative term ω_i denoting the weight that firm i imposes on the disrepute cost function. As discussed in Chap. 7, the production and the transportation costs and the quality cost are entirely different costs, and they do not overlap.

Hence, firm i seeks to

$$\text{Maximize}_{Q_i, q_i} \quad U_i^1 = -\hat{f}_i(Q^1, q^1) - qc_i(q^1) - \sum_{k=1}^{n_R} \hat{c}_{ik}(Q^1, q^1) - \sum_{j=1}^{n_O} \sum_{k=1}^{n_R} \pi_{ijk}^* Q_{i,1+j,k}$$
$$- \sum_{j=1}^{n_O} tc_{ij}\left(\sum_{k=1}^{n_R} Q_{i,1+j,k}\right) - \omega_i dc_i(q_i'(Q_i, q_i, q_i^{2^*})) \tag{8.4}$$

subject to:

$$\sum_{j=1}^{n} Q_{ijk} = d_{ik}, \quad i = 1, \ldots, I; k = 1, \ldots, n_R, \tag{8.5}$$

$$Q_{ijk} \geq 0, \quad i = 1, \ldots, I; j = 1, \ldots, n; k = 1, \ldots, n_R, \tag{8.6}$$

and (8.2).

According to (8.4), the prices and the contractors' quality levels are evaluated at their equilibrium values.

We assume that all the cost functions in (8.4) are continuous, continuously differentiable, and convex. The original firms compete in a noncooperative in the sense of Nash (1950, 1951) with each one trying to maximize its own utility. The strategic variables for each original firm i are all the in-house and the outsourcing flows produced and shipped by firm i and its in-house quality level.

We define the feasible set K_i as $K_i \equiv \{(Q_i, q_i) | Q_i \in R_+^{nn_R}$ with (8.5) satisfied and q_i satisfying (8.2)$\}$. All K_i; $i = 1, \ldots, I$, are closed and convex. We also define the feasible set $\mathscr{K}^1 \equiv \Pi_{i=1}^I K_i$.

Definition 8.1: Supply Chain Network Cournot-Nash Equilibrium with Product Differentiation and Outsourcing of Production and Distribution
An in-house and outsourced product flow and in-house quality level pattern $(Q^*, q^{1^*}) \in \mathscr{K}^1$ *is said to constitute a Cournot-Nash equilibrium if for each firm i;* $i = 1, \ldots, I$,

$$U_i^1(Q_i^*, \hat{Q}_i^*, q_i^*, \hat{q}_i^*, q_i^{2^*}, \pi_i^*) \geq U_i^1(Q_i, \hat{Q}_i^*, q_i, \hat{q}_i^*, q_i^{2^*}, \pi_i^*), \quad \forall (Q_i, q_i) \in K_i, \tag{8.7}$$

where

$$\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_I^*),$$
$$\hat{q}_i^* \equiv (q_1^*, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_I^*).$$

According to (8.7), a Cournot-Nash equilibrium is established if no firm can unilaterally improve upon its utility by selecting an alternative vector of in-house or outsourced product flows and quality level.

We now present the variational inequality formulation of the Cournot-Nash equilibrium with product differentiation and outsourcing according to Definition 8.1 (see Cournot 1838; Nash 1950, 1951; Gabay and Moulin 1980; Liu and Nagurney 2009; Cruz et al. 2006).

Theorem 8.1: Variational Inequality Formulation of Firms' Problems
Assume that, for each firm i; i $= 1, \ldots, I$, *the utility function* $U_i^1(Q, q^1, q_i^{2^*}, \pi_i^*)$ *is concave with respect to its variables* Q_i *and* q_i, *and is continuous and continuously differentiable. Then* $(Q^*, q^{1^*}) \in \mathscr{K}^1$ *is a Counot-Nash equilibrium according to Definition 8.1 if and only if it satisfies the variational inequality:*

$$-\sum_{i=1}^{I}\sum_{h=1}^{n}\sum_{m=1}^{n_R} \frac{\partial U_i^1(Q^*, q^{1^*}, q_i^{2^*}, \pi_i^*)}{\partial Q_{ihm}} \times (Q_{ihm} - Q_{ihm}^*)$$
$$-\sum_{i=1}^{I} \frac{\partial U_i^1(Q^*, q^{1^*}, q_i^{2^*}, \pi_i^*)}{\partial q_i} \times (q_i - q_i^*) \geq 0, \quad \forall (Q, q^1) \in \mathscr{K}^1, \tag{8.8}$$

with notice that: for $h = 1$; $i = 1, \ldots, I$; $m = 1, \ldots, n_R$:

$$-\frac{\partial U_i^1(Q^*, q^{1^*}, q_i^{2^*}, \pi_i^*)}{\partial Q_{ihm}}$$
$$= \left[\frac{\partial \hat{f}_i(Q^{1^*}, q^{1^*})}{\partial Q_{ihm}} + \sum_{k=1}^{n_R} \frac{\partial \hat{c}_{ik}(Q^{1^*}, q^{1^*})}{\partial Q_{ihm}} + \omega_i \frac{\partial dc_i(q_i'^*)}{\partial q_i'} \frac{\partial q_i'(Q_i^*, q_i^*, q_i^{2^*})}{\partial Q_{ihm}} \right],$$

for $h = 2, \ldots, n$; $i = 1, \ldots, I$; $m = 1, \ldots, n_R$:

$$-\frac{\partial U_i^1(Q^*,q^{1^*},q_i^{2^*},\pi_i^*)}{\partial Q_{ihm}}$$

$$=\left[\pi^*_{i,h-1,m}+\frac{\partial tc_{i,h-1}(\sum_{k=1}^{n_R}Q^*_{ihk})}{\partial Q_{ihm}}+\omega_i\frac{\partial dc_i(q_i'^*)}{\partial q_i'}\frac{\partial q_i'(Q_i^*,q_i^*,q_i^{2^*})}{\partial Q_{ihm}}\right],$$

for $i=1,\ldots,I$*:*

$$-\frac{\partial U_i^1(Q^*,q^{1^*},q_i^{2^*},\pi_i^*)}{\partial q_i}$$

$$=\left[\frac{\partial\hat{f}_i(Q^{1^*},q^{1^*})}{\partial q_i}+\frac{\partial qc_i(q^{1^*})}{\partial q_i}+\sum_{k=1}^{n_R}\frac{\partial\hat{c}_{ik}(Q^{1^*},q^{1^*})}{\partial q_i}+\omega_i\frac{\partial dc_i(q_i'^*)}{\partial q_i'}\right.$$
$$\left.\frac{\partial q_i'(Q_i^*,q_i^*,q_i^{2^*})}{\partial q_i}\right].$$

8.2.2 The Behavior of the Contractors and Their Optimality Conditions

The objective of contractor $j; j=1,\ldots,n_O$, is profit maximization. The contractors' revenues are obtained from the purchasing activities of the original firms, while their costs are the costs of production and distribution, the total quality costs, and the opportunity costs. In Chap. 7, we also utilized opportunity costs on the contractors' side. The contractors' opportunity costs are, again, functions of the prices that they charge the firms, as in Table 8.1. As discussed in Chap. 7, if the prices charged are too low, they may not recover all their costs, whereas if they are too high, the firms may select another contractor.

Similar to that in Chap. 7, in this model, each contractor has, as its strategic variables, its quality levels for producing and distributing the original firms' products, and the prices that charges the firms. We denote the utility of each contractor j by U_j^2, with $j=1,\ldots,n_O$, and note that it represents the profit. Hence, each contractor $j; j=1,\ldots,n_O$, seeks to:

$$\text{Maximize}_{q_j,\pi_j}\quad U_j^2=\sum_{k=1}^{n_R}\sum_{i=1}^{I}\pi_{ijk}Q^*_{i,1+j,k}-\sum_{k=1}^{n_R}\sum_{i=1}^{I}\widehat{sc}_{ijk}(Q^{2^*},q^2)-\widehat{qc}_j(q^2)$$
$$-\sum_{k=1}^{n_R}\sum_{i=1}^{I}oc_{ijk}(\pi)\tag{8.9}$$

subject to:

$$\pi_{ijk}\geq 0,\quad j=1,\ldots,n_O;k=1,\ldots,n_R,\tag{8.10}$$

and (8.1) for each j. The outsourced product flows are at their equilibrium values.

According to (8.9), the original firms' outputs are evaluated at the equilibrium, since the contractors do not control these variables, and, hence, must respond to these outputs.

We assume that the cost functions in each contractor's utility function are continuous, continuously differentiable, and convex. The contractors compete in a noncooperative in the sense of Nash (1950, 1951), with each one trying to maximize its own profits.

We define the feasible sets $K_j \equiv \{(q_j, \pi_j)|q_j$ satisfies (8.1) and π_j satisfies (8.10) for $j\}$, $\mathcal{K}^2 \equiv \Pi_{j=1}^{n_O} K_j$, and $\mathcal{K} \equiv \mathcal{K}^1 \times \mathcal{K}^2$. All the above-defined feasible sets are convex.

Definition 8.2: A Bertrand-Nash Equilibrium with Price and Quality Competition

A quality level and price pattern $(q^{2^*}, \pi^*) \in \mathcal{K}^2$ is said to constitute a Bertrand-Nash equilibrium if for each contractor j; $j = 1, \ldots, n_O$,

$$U_j^2(Q^{2^*}, q_j^*, \hat{q}_j^*, \pi_j^*, \hat{\pi}_j^*) \geq U_j^2(Q^{2^*}, q_j, \hat{q}_j^*, \pi_j, \hat{\pi}_j^*), \quad \forall (q_j, \pi_j) \in K_j, \tag{8.11}$$

where

$$\hat{q}_j^* \equiv (q_1^*, \ldots, q_{j-1}^*, q_{j+1}^*, \ldots, q_{n_O}^*),$$
$$\hat{\pi}_j^* \equiv (\pi_1^*, \ldots, \pi_{j-1}^*, \pi_{j+1}^*, \ldots, \pi_{n_O}^*).$$

According to (8.11), a Bertrand-Nash equilibrium is established if no contractor can unilaterally improve upon its profits by selecting an alternative vector of quality levels or prices charged to the original firms.

Next, we present the variational inequality formulation of the Bertrand-Nash equilibrium according to Definition 8.2 (see, Bertrand 1883; Nash 1950, 1951; Gabay and Moulin 1980; Nagurney 2006) in the following theorem.

Theorem 8.2: Variational Inequality Formulation of the Contractors' Problems

Assume that, for each contractor j; $j = 1, \ldots, n_O$, the profit function $U_j^2(Q^{2^*}, q^2, \pi)$ is concave with respect to the variables π_j and q_j, and is continuous and continuously differentiable. Then $(q^{2^*}, \pi^*) \in \mathcal{K}^2$ is a Bertrand-Nash equilibrium according to Definition 8.2 if and only if it satisfies the variational inequality:

$$-\sum_{l=1}^{I}\sum_{j=1}^{n_O} \frac{\partial U_j^2(Q^{2^*}, q^{2^*}, \pi^*)}{\partial q_{lj}} \times (q_{lj} - q_{lj}^*)$$
$$-\sum_{l=1}^{I}\sum_{j=1}^{n_O}\sum_{k=1}^{n_R} \frac{\partial U_j^2(Q^{2^*}, q^{2^*}, \pi^*)}{\partial \pi_{ljk}} \times (\pi_{ljk} - \pi_{ljk}^*) \geq 0, \quad \forall (q^2, \pi) \in \mathcal{K}^2. \tag{8.12}$$

with notice that: for $j = 1, \ldots, n_O$*;* $l = 1, \ldots, I$*:*

$$-\frac{\partial U_j^2(Q^{2^*}, q^{2^*}, \pi^*)}{\partial q_{lj}} = \sum_{i=1}^{I}\sum_{k=1}^{n_R} \frac{\partial \hat{s}c_{ijk}(Q^{2^*}, q^{2^*})}{\partial q_{lj}} + \frac{\partial \widehat{qc}_j(q^{2^*})}{\partial q_{lj}},$$

and for $j = 1, \ldots, n_O; l = 1, \ldots, I; k = 1, \ldots, n_R$*:*

$$-\frac{\partial U_j^2(Q^{2^*}, q^{2^*}, \pi^*)}{\partial \pi_{ljk}} = \sum_{i=1}^{I}\sum_{r=1}^{n_R} \frac{\partial oc_{ijr}(\pi^*)}{\partial \pi_{ljk}} - Q^*_{l,1+j,k}.$$

8.2.3 The Equilibrium Conditions for the Supply Chain Network with Product Differentiation, Outsourcing of Production and Distribution, and Quality Competition

In equilibrium, the optimality conditions for all contractors and the optimality conditions for all the original firms must hold simultaneously, according to the definition below.

Definition 8.3: Supply Chain Network Equilibrium with Product Differentiation, Outsourcing of Production and Distribution, and Quality and Price Competition
The equilibrium state of the supply chain network with product differentiation, outsourcing of production and distribution, and quality and price competition is one where both variational inequalities (8.8) and (8.12) hold simultaneously.

Theorem 8.3: Variational Inequality Formulation of the Supply Chain Network Equilibrium with Product Differentiation, Outsourcing of Production and Distribution, and Quality and Price Competition
The equilibrium conditions governing the supply chain network model with product differentiation, outsourcing of production and distribution, and quality competition are equivalent to the solution of the variational inequality problem: determine $(Q^*, q^{1^*}, q^{2^*}, \pi^*) \in \mathcal{K}$*, such that:*

$$-\sum_{i=1}^{I}\sum_{h=1}^{n}\sum_{m=1}^{n_R} \frac{\partial U_i^1(Q^*, q^{1^*}, q_i^{2^*}, \pi_i^*)}{\partial Q_{ihm}} \times (Q_{ihm} - Q^*_{ihm})$$

$$-\sum_{i=1}^{I} \frac{\partial U_i^1(Q^*, q^{1^*}, q_i^{2^*}, \pi_i^*)}{\partial q_i} \times (q_i - q_i^*)$$

$$-\sum_{l=1}^{I}\sum_{j=1}^{n_O}\frac{\partial U_j^2(Q^{2^*},q^{2^*},\pi^*)}{\partial q_{lj}}\times(q_{lj}-q_{lj}^*)$$

$$-\sum_{l=1}^{I}\sum_{j=1}^{n_O}\sum_{k=1}^{n_R}\frac{\partial U_j^2(Q^{2^*},q^{2*},\pi^*)}{\partial \pi_{ljk}}\times(\pi_{ljk}-\pi_{ljk}^*)\geq 0,\quad \forall(Q,q^1,q^2,\pi)\in\mathcal{K}. \tag{8.13}$$

Proof: Summation of variational inequalities (8.8) and (8.12) yields variational inequality (8.13). A solution to variational inequality (8.13) satisfies the sum of (8.8) and (8.12) and, hence, is an equilibrium according to Definition 8.3. □

Variational inequality (8.13) can be put into standard form (2.1a): determine $X^*\in\mathcal{K}$ such that:

$$\langle F(X^*),X-X^*\rangle\geq 0,\quad \forall X\in\mathcal{K}, \tag{8.14}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in N-dimensional Euclidean space, where $N=Inn_R+I+In_O+In_On_R$. Indeed, if we define the vectors $X\equiv(Q,q^1,q^2,\pi)$ and $F(X)\equiv(F_1(X),F_2(X),F_3(X),F_4(X))$, such that:

$$F_1(X)=\left[\frac{\partial U_i^1(Q,q^1,q_i^2,\pi_i)}{\partial Q_{ihm}};\ h=1,\ldots,n;i=1,\ldots,I;m=1,\ldots,n_R\right],$$

$$F_2(X)=\left[\frac{\partial U_i^1(Q,q^1,q_i^2,\pi_i)}{\partial q_i};\ i=1,\ldots,I\right],$$

$$F_3(X)=\left[\frac{\partial U_j^2(Q^2,q^2,\pi)}{\partial q_{lj}};\ l=1,\ldots,I;j=1,\ldots,n_O\right],$$

$$F_4(X)=\left[\frac{\partial U_j^2(Q^2,q^2,\pi)}{\partial \pi_{ljk}};\ l=1,\ldots,I;j=1,\ldots,n_O;k=1,\ldots,n_R\right], \tag{8.15}$$

and $\mathcal{K}\equiv\mathcal{K}$ then (8.13) can be re-expressed as (8.14).

8.3 The Algorithm and Numerical Examples

The algorithm that we employed for the computation of the solution for the supply chain network game theory model is the Euler method (cf. (2.34)), which is induced by the general iterative scheme of Dupuis and Nagurney (1993), and which we have also utilized in Chaps. 3 through 7. Specifically, recall that at iteration $\tau+1$ of the Euler method (see also Nagurney and Zhang 1996), one computes:

$$X^{\tau+1} = P_{\mathscr{K}}(X^{\tau} - a_{\tau}F(X^{\tau})), \tag{8.16}$$

where $P_{\mathscr{K}}$ is the projection on the feasible set $\mathscr{K}$ and F is the function that enters the variational inequality problem (8.14).

At each iteration τ, $X^{\tau+1}$ in (8.16) is actually the solution to the following strictly convex quadratic programming problem:

$$X^{\tau+1} = \text{Minimize}_{X\in\mathscr{K}} \quad \frac{1}{2}\langle X, X\rangle - \langle X^{\tau} - a_{\tau}F(X^{\tau}), X\rangle. \tag{8.17}$$

As for solving (8.17), we employ a similar strategy to that used for the solution of the single firm model with outsourcing in Chap. 7. In order to obtain the values of the product flows at each iteration τ, we apply the exact equilibration algorithm (Dafermos and Sparrow 1969), which has been applied to many different applications of networks with special structure (cf. Nagurney 1999; Nagurney and Zhang 1996).

Furthermore, we can determine the values for the in-house and the outsourced quality variables explicitly, at an iteration, according to the following closed form expressions:

for each original firm i; $i = 1, \ldots, I$:

$$q_i^{\tau+1} = \min\{q^U, \max\{0, q_i^{\tau} - a_{\tau}(\frac{\partial \hat{f}_i(Q^{1^{\tau}}, q^{1^{\tau}})}{\partial q_i} + \frac{\partial qc_i(q^{1^{\tau}})}{\partial q_i} + \sum_{k=1}^{n_R} \frac{\partial \hat{c}_{ik}(Q^{1^{\tau}}, q^{1^{\tau}})}{\partial q_i}$$

$$+\omega_i \frac{\partial dc_i(q_i'^{\tau})}{\partial q_i'} \frac{\partial q_i'(Q_i^{\tau}, q_i^{\tau}, q_i^{2^{\tau}})}{\partial q_i})\}\}; \tag{8.18}$$

and for the contractor and firm pairs: $l = 1, \ldots, I; j = 1, \ldots, n_O$:

$$q_{lj}^{\tau+1} = \min\{q^U, \max\{0, q_{lj}^{\tau} - a_{\tau}(\sum_{i=1}^{I}\sum_{k=1}^{n_R} \frac{\partial \widehat{sc}_{ijk}(Q^{2^{\tau}}, q^{2^{\tau}})}{\partial q_{lj}} + \frac{\partial \widehat{qc}_j(q^{2^{\tau}})}{\partial q_{lj}})\}\}. \tag{8.19}$$

Also, we have the following explicit formulae for the outsourced product prices: for $l = 1, \ldots, I; j = 1, \ldots, n_O; k = 1, \ldots, n_R$:

$$\pi_{ljk}^{\tau+1} = \max\{0, \pi_{ljk}^{\tau} - a_{\tau}(\sum_{i=1}^{I}\sum_{r=1}^{n_R} \frac{\partial oc_{ijr}(\pi^{\tau})}{\partial \pi_{ljk}} - Q_{l,1+j,k}^{\tau})\}. \tag{8.20}$$

We now provide the convergence result. The proof follows using similar arguments as those in Theorem 6.10 in Nagurney and Zhang (1996).

Theorem 8.4: Convergence
In the supply chain network game theory model with product differentiation, outsourcing of production and distribution, and quality competition, let $F(X) = -\nabla U(Q, q^1, q^2, \pi)$*, where we group all* U_i^1*;* $i = 1, \ldots, I$*, and* U_j^2*;* $j = 1, \ldots, n_O$*,*

into the vector $U(Q, q^1, q^2, \pi)$, be strongly monotone. Also, assume that F is uniformly Lipschitz continuous. Then there exists a unique equilibrium product flow, quality level, and price pattern $(Q^*, q^{1^*}, q^{2^*}, \pi^*) \in \mathcal{K}$, and any sequence generated by the Euler method as given by (8.16) above, where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$ converges to $(Q^*, q^{1^*}, q^{2^*}, \pi^*)$.

Note that convergence also holds if $F(X)$ is strictly monotone (cf. Theorem 8.6 in Nagurney and Zhang 1996) provided that the price iterates are bounded, and, in practice, contractors cannot charge unbounded prices for production and delivery. Hence, similar to that in Chap. 7, we can also expect the existence of a solution, given the continuity of the functions that make up $F(X)$, under less restrictive conditions that of strong monotonicity.

The Euler method, as outlined above for our model, can be interpreted as a discrete-time adjustment process in which each iteration reflects a time step. The original firms determine, at each time step, their optimal production (and shipment) outputs and quality levels, whereas the contractors, at each time step (iteration), compute their optimal quality levels and the prices that they charge. The process evolves over time until the equilibrium product flows, quality levels, and contractor prices are achieved, at which point no one has any incentive to switch their strategies.

8.4 Numerical Examples

In this section, we present numerical supply chain network examples for which we apply the Euler method, as outlined in Sect. 8.3, to compute the equilibrium solutions. We present a spectrum of examples, accompanied by sensitivity analysis.

The supply chain network topology of the numerical examples is given in Fig. 8.2. There are two original firms, both of which are located in North America. Their products are substitutes but are differentiated by brands in the two demand markets, R_1 and R_2. Demand market R_1 is in North America, whereas demand market R_2 is in Asia.

Each original firm has one in-house manufacturing plant and two potential contractors. Contractor 1 and contractor 2 are located in North America and Asia, respectively. Each firm must satisfy the demands for its product at the two demand markets. The demands for firm 1's product at R_1 and at R_2 are 50 and 100, respectively. The demands for firm 2's product at R_1 and at R_2 are 75 and 150.

For the computation of solutions to the numerical examples, we implemented the Euler method, as discussed in Sect. 8.3, using Matlab on a Lenovo E46A. The convergence tolerance is 10^{-6}, so that the algorithm is deemed to have converged when the absolute value of the difference between each successive product flow, quality level, and price is less than or equal to 10^{-6}. The sequence $\{a_\tau\}$ is set to:

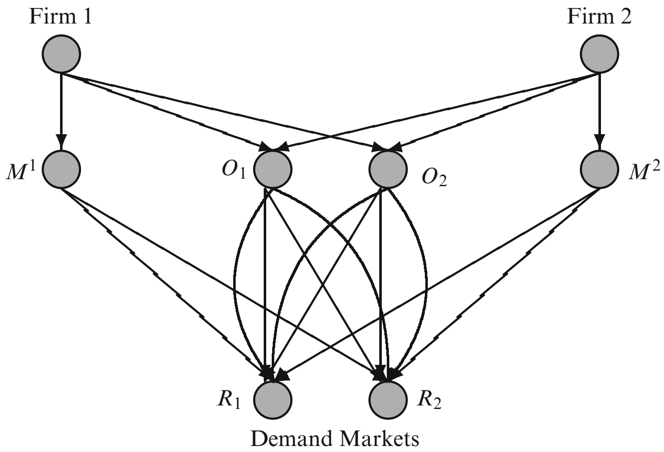

Fig. 8.2 The supply chain network topology for the numerical examples

$\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$. We initialize the algorithm by equally distributing the product flows among the paths joining the firm top-node to the demand market, by setting the quality levels equal to 1.00 and the prices equal to 0.00.

Example 8.1
The data are as follows.

The production cost functions at the in-house manufacturing plants are:

$$\hat{f}_1(Q^1, q^1) = (Q_{111} + Q_{112})^2 + 1.5(Q_{111} + Q_{112}) + 2(Q_{211} + Q_{212}) + 0.2q_1(Q_{111} + Q_{112}),$$

$$\hat{f}_2(Q^1, q^1) = 2(Q_{211} + Q_{212})^2 + 0.5(Q_{211} + Q_{212}) + (Q_{111} + Q_{112}) + 0.1q_2(Q_{211} + Q_{212}).$$

The total transportation cost functions for the in-house manufactured products are:

$$\hat{c}_{11}(Q_{111}) = Q_{111}^2 + 5Q_{111}, \quad \hat{c}_{12}(Q_{112}) = 2.5Q_{112}^2 + 10Q_{112},$$

$$\hat{c}_{21}(Q_{211}) = 0.5Q_{211}^2 + 3Q_{211}, \quad \hat{c}_{22}(Q_{212}) = 2Q_{212}^2 + 5Q_{212}.$$

The in-house total quality cost functions for the two original firms are given by:

$$qc_1(q_1) = (q_1 - 80)^2 + 10, \quad qc_2(q_2) = (q_2 - 85)^2 + 20.$$

The transaction cost functions are:

$$tc_{11}(Q_{121}+Q_{122}) = 0.5(Q_{121}+Q_{122})^2 + 2(Q_{121}+Q_{122}) + 100,$$
$$tc_{12}(Q_{131}+Q_{132}) = 0.7(Q_{131}+Q_{132})^2 + 0.5(Q_{131}+Q_{132}) + 150,$$
$$tc_{21}(Q_{221}+Q_{222}) = 0.5(Q_{221}+Q_{222})^2 + 3(Q_{221}+Q_{222}) + 75,$$
$$tc_{22}(Q_{221}+Q_{222}) = 0.75(Q_{231}+Q_{232})^2 + 0.5(Q_{231}+Q_{232}) + 100.$$

The contractors' total cost functions of production and distribution are:

$$\widehat{sc}_{111}(Q_{121},q_{11}) = 0.5Q_{121}q_{11}, \quad \widehat{sc}_{112}(Q_{122},q_{11}) = 0.5Q_{122}q_{11},$$
$$\widehat{sc}_{121}(Q_{131},q_{12}) = 0.5Q_{131}q_{12}, \quad \widehat{sc}_{122}(Q_{132},q_{12}) = 0.5Q_{132}q_{12},$$
$$\widehat{sc}_{211}(Q_{221},q_{21}) = 0.3Q_{221}q_{21}, \quad \widehat{sc}_{212}(Q_{222},q_{21}) = 0.3Q_{222}q_{21},$$
$$\widehat{sc}_{221}(Q_{231},q_{22}) = 0.25Q_{231}q_{22}, \quad \widehat{sc}_{222}(Q_{232},q_{22}) = 0.25Q_{232}q_{22}.$$

The total quality cost functions of the contractors are:

$$\widehat{qc}_1(q_{11},q_{21}) = (q_{11}-75)^2 + (q_{21}-75)^2 + 15,$$
$$\widehat{qc}_2(q_{12},q_{22}) = 1.5(q_{12}-75)^2 + 1.5(q_{22}-75)^2 + 20.$$

The contractors' opportunity cost functions are:

$$oc_{111}(\pi_{111}) = (\pi_{111}-10)^2, \quad oc_{121}(\pi_{121}) = 0.5(\pi_{121}-5)^2,$$
$$oc_{112}(\pi_{112}) = 0.5(\pi_{112}-5)^2, \quad oc_{122}(\pi_{122}) = (\pi_{122}-15)^2,$$
$$oc_{211}(\pi_{211}) = 2(\pi_{211}-20)^2, \quad oc_{221}(\pi_{221}) = 0.5(\pi_{221}-5)^2,$$
$$oc_{212}(\pi_{212}) = 0.5(\pi_{212}-5)^2, \quad oc_{222}(\pi_{222}) = (\pi_{222}-15)^2.$$

The original firms' disrepute cost functions are:

$$dc_1(q_1') = 100 - q_1', \quad dc_2(q_2') = 100 - q_2',$$

where

$$q_1' = \frac{Q_{121}q_{11} + Q_{131}q_{12} + Q_{111}q_1 + Q_{122}q_{11} + Q_{132}q_{12} + Q_{112}q_1}{d_{11}+d_{12}},$$

and

$$q_2' = \frac{Q_{221}q_{21} + Q_{231}q_{22} + Q_{211}q_2 + Q_{222}q_{21} + Q_{232}q_{22} + Q_{212}q_2}{d_{21}+d_{22}}.$$

The weights are ω_1 and ω_2 are 1. The value representing perfect quality, q^U, is 100.

The Euler method converges in 255 iterations and yields the following equilibrium solution.

The computed product flows are:

$$Q^*_{111} = 13.64, \quad Q^*_{121} = 26.87, \quad Q^*_{131} = 9.49, \quad Q^*_{112} = 9.34, \quad Q^*_{122} = 42.85,$$
$$Q^*_{132} = 47.81, \quad Q^*_{211} = 16.54, \quad Q^*_{221} = 47.31, \quad Q^*_{231} = 11.16, \quad Q^*_{212} = 12.65,$$
$$Q^*_{222} = 62.90, \quad Q^*_{232} = 74.45.$$

The computed quality levels of the original firms and the contractors are:

$$q^*_1 = 77.78, \quad q^*_2 = 83.61,$$
$$q^*_{11} = 57.57, \quad q^*_{12} = 65.45, \quad q^*_{21} = 58.47, \quad q^*_{22} = 67.87.$$

The equilibrium prices are:

$$\pi^*_{111} = 23.44, \quad \pi^*_{112} = 47.85, \quad \pi^*_{121} = 14.49, \quad \pi^*_{122} = 38.91,$$
$$\pi^*_{211} = 31.83, \quad \pi^*_{212} = 67.90, \quad \pi^*_{221} = 16.16, \quad \pi^*_{222} = 52.23.$$

Notice that, although the North American contractor produces at a lower quality and at a higher price at equilibrium, it produces and distributes more than the offshore contractor to R_1, who is located in North America. This happens for two reasons. First, because of the fixed demands, no pressure for quality improvement is imposed from the demand side. Secondly, as reflected in the transaction costs with the North America contractor, firms are willing to outsource more to this contractor. The Asian contractor, who produces at higher quality levels and at lower prices at equilibrium, produces and distributes more to R_2. This happens because the contractor who charges lower prices and produces at higher quality levels is highly preferable in the demand market, R_2, with the larger demand for which the original firms need to outsource more.

The total costs of the original firms' are, respectively, 11,419.90 and 24,573.94, with their incurred disrepute costs being 36.32 and 34.69. The profits of the contractors are 567.84 and 440.92. The values of q'_1 and q'_2 are, respectively, 63.68 and 65.31.

We conduct sensitivity analysis by varying the weights that the firms impose on their disrepute costs, ω, the vector of ω_i; $i = 1, 2$, with $\omega = (0, 0), (1{,}000, 1{,}000), (2{,}000, 2{,}000), (3{,}000, 3{,}000), (4{,}000, 4{,}000), (5{,}000, 5{,}000)$.

We display the equilibrium product flows and the equilibrium quality levels, both the in-house and the outsourced ones, and the average quality levels, in Fig. 8.3, with the equilibrium prices changed by each contractor, the disrepute cost, and the total cost of each original firm displayed in Fig. 8.4.

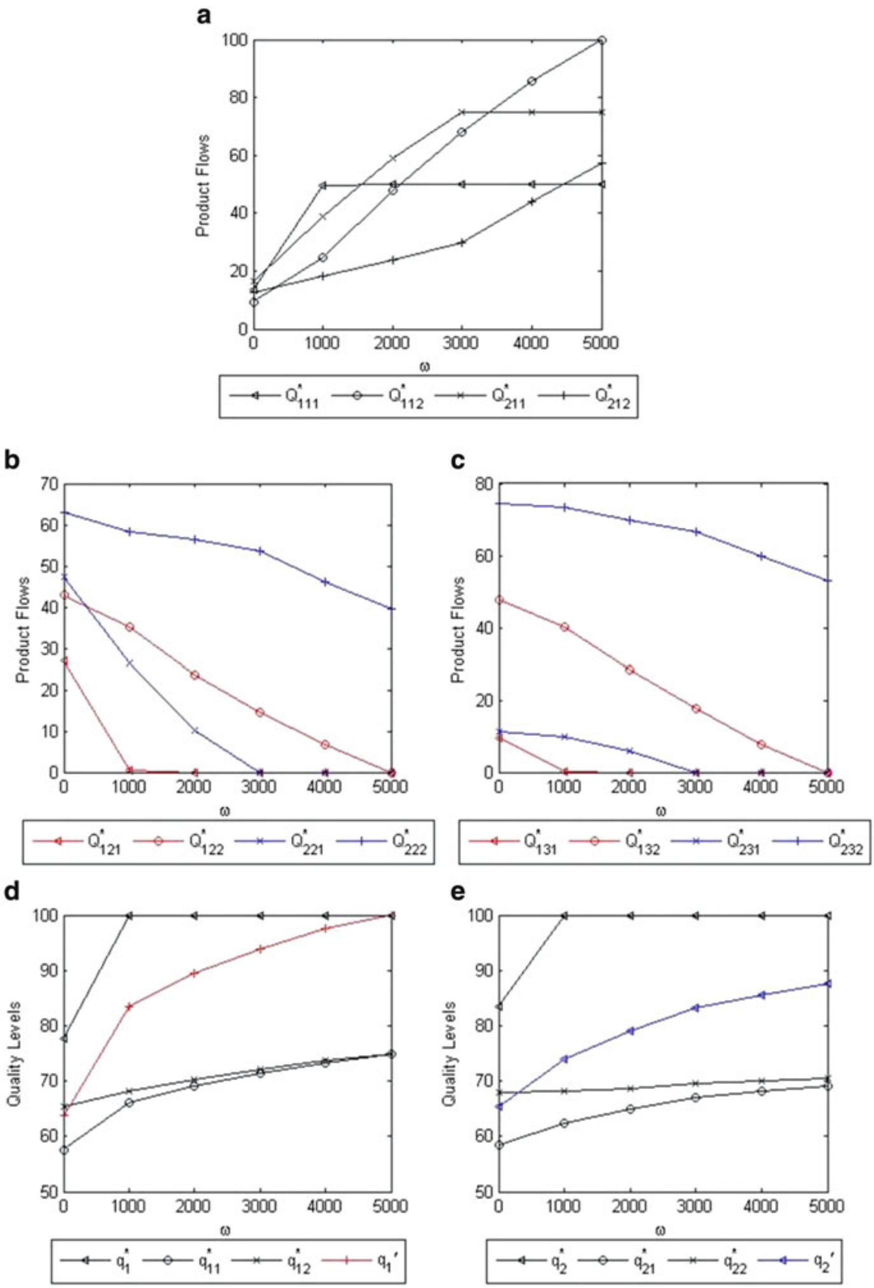

Fig. 8.3 Equilibrium product flows and quality levels as ω increases for Example 8.1. (**a**) Equilibrium in-house product flows. (**b**) Equilibrium product flows via contractor 1. (**c**) Equilibrium product flows via contractor 2. (**d**) Equilibrium and average quality levels of firm 1. (**e**) Equilibrium and average quality levels of firm 2

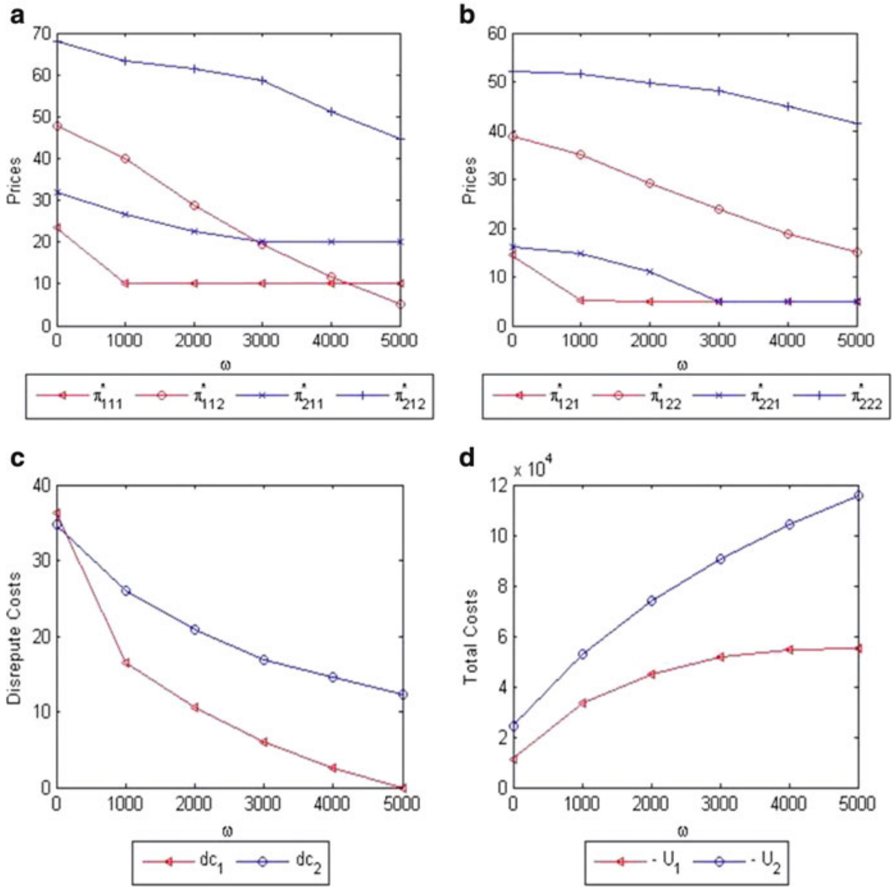

Fig. 8.4 Equilibrium prices, disrepute costs, and total costs of the firms as ω increases for Example 8.1. (**a**) Equilibrium prices changed by contractor 1. (**b**) Equilibrium prices changed by contractor 2. (**c**) Disrepute costs. (**d**) Total costs

As the weights of the disrepute costs increase, there is more pressure for firms to improve quality. Thus, all the quality levels increase. In addition, because the in-house activities are more capable of guaranteeing higher quality, the outputs of both firms are shifted in-house as the weights increase. As a result, the outsourcing prices decrease (see (8.12)). Moreover, as shown in Fig. 8.4, as expected, the values of the incurred disrepute costs decrease as ω increases, but the total costs of the original firms increase.

Example 8.2
In Example 8.2, both firms consider quality levels as variables affecting their in-house transportation costs. Recall, as mentioned in Sect. 8.1, we assume here that the costs reflect that the transportation activity will deliver the products at the same quality levels as they were produced at. The transportation cost functions of the original firms, hence, now depend on in-house quality levels as follows:

$$\hat{c}_{11}(Q_{111}, q_1) = Q_{111}^2 + 1.5Q_{111}q_1, \quad \hat{c}_{12}(Q_{112}, q_1) = 2.5Q_{112}^2 + 2Q_{112}q_1,$$
$$\hat{c}_{21}(Q_{211}, q_2) = 0.5Q_{211}^2 + 3Q_{211}q_2, \quad \hat{c}_{22}(Q_{212}, q_2) = 2Q_{212}^2 + 2Q_{212}q_2.$$

The remaining data are identical to those in Example 8.1.

The Euler method converges in 298 iterations and yields the following equilibrium solution.

The computed product flows are:

$$Q_{111}^* = 0.00, \quad Q_{121}^* = 36.42, \quad Q_{131}^* = 13.58, \quad Q_{112}^* = 0.00, \quad Q_{122}^* = 46.42,$$
$$Q_{132}^* = 53.58, \quad Q_{211}^* = 0.00, \quad Q_{221}^* = 60.13, \quad Q_{231}^* = 14.87, \quad Q_{212}^* = 3.83,$$
$$Q_{222}^* = 65.50, \quad Q_{232}^* = 80.68.$$

The computed quality levels of the original firms and the contractors are:

$$q_1^* = 80, \quad q_2^* = 80.90, \quad q_{11}^* = 54.29, \quad q_{12}^* = 63.81, \quad q_{21}^* = 56.16, \quad q_{22}^* = 67.04.$$

The equilibrium prices are:

$$\pi_{111}^* = 28.21, \quad \pi_{112}^* = 51.42, \quad \pi_{121}^* = 18.58, \quad \pi_{122}^* = 41.79,$$
$$\pi_{211}^* = 35.03, \quad \pi_{212}^* = 70.50, \quad \pi_{221}^* = 19.87, \quad \pi_{222}^* = 55.34.$$

The total costs of the original firms are, respectively, 13,002.64 and 27,607.44, with incurred disrepute costs of 41.45 and 38.80. The profits of the contractors are, respectively, 967.96 and 656.78. The average quality levels of the original firms, q_1' and q_2', are 58.55 and 61.20.

We conduct sensitivity analysis by varying the weights associated with the disrepute costs, ω, for $\omega = (0, 0)$, (1,000, 1,000), (2,000, 2,000), (3,000, 3,000), (4,000, 4,000), (5,000, 5,000). We display the results of this sensitivity analysis in Figs. 8.5 and 8.6.

We now discuss the results of the sensitivity analysis for Example 8.2. As shown in Figs. 8.5 and 8.6, as the weights of the disrepute costs increase, the trends of the variables and costs in Figs. 8.5 and 8.6 are the same as those in Example 8.1, except the trends for Q_{231}^*, q_{22}^*, and π_{221}^*. As ω increases from 0 to 2,000, Q_{231}^* increases. However, as ω increases further, Q_{231}^* decreases. The trends of q_{22}^* and π_{221}^* then change accordingly. The reason is as follows. Because now in-house transportation

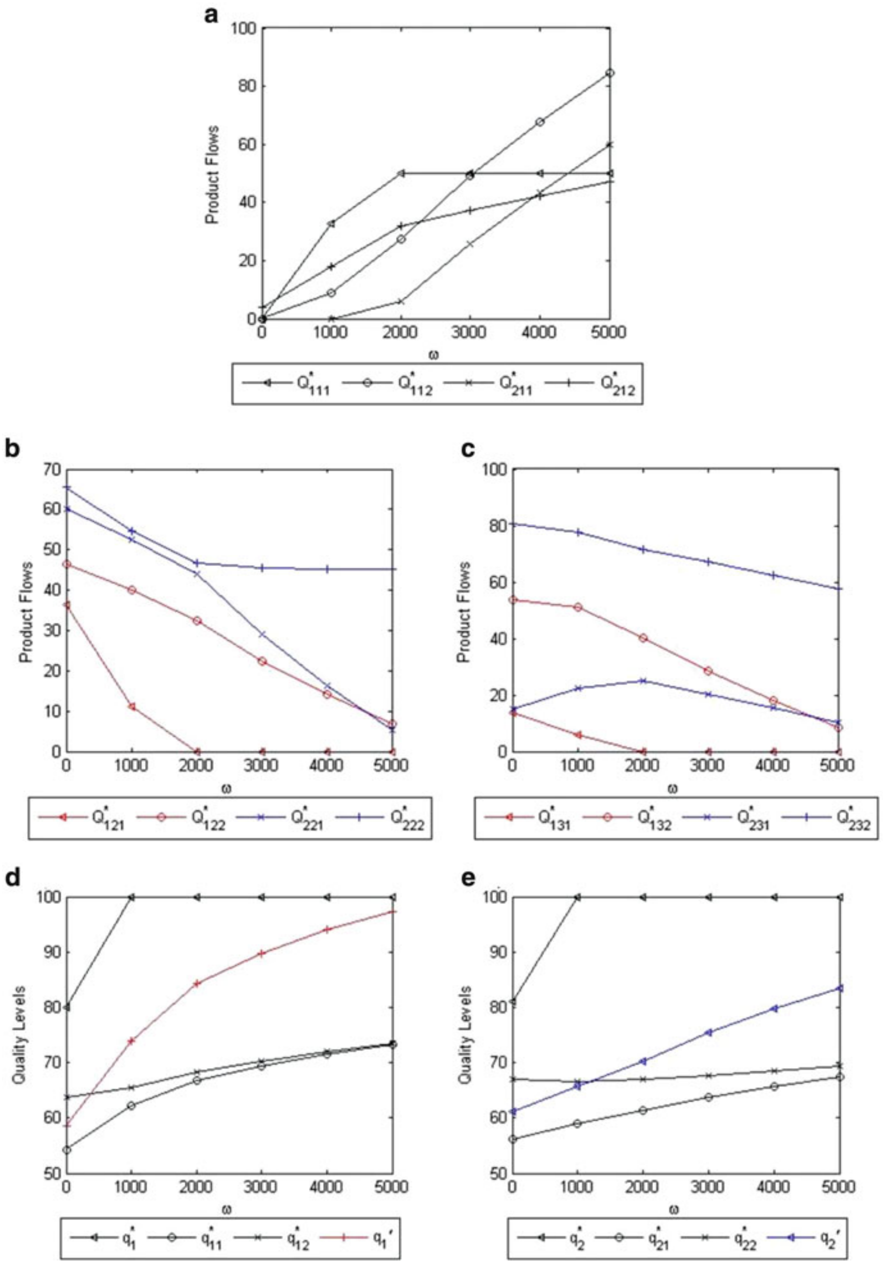

Fig. 8.5 Equilibrium product flows and quality levels as ω increases for Example 8.2. (**a**) Equilibrium in-house product flows. (**b**) Equilibrium product flows via contractor 1. (**c**) Equilibrium product flows via contractor 2. (**d**) Equilibrium and average quality levels of firm 1's product. (**e**) Equilibrium and average quality levels of firm 2's product

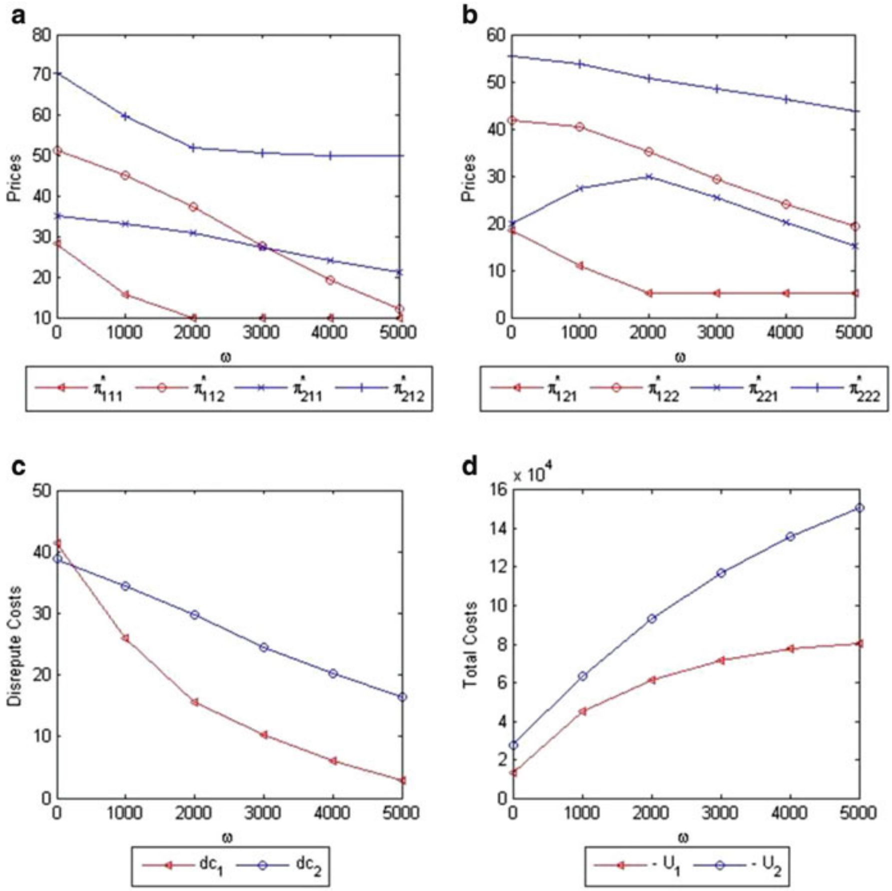

Fig. 8.6 Equilibrium prices, disrepute costs, and total costs of the firms as ω increases for Example 8.2. (**a**) Equilibrium prices changed by contractor 1. (**b**) Equilibrium prices changed by contractor 2. (**c**) Disrepute costs. (**d**) Total costs

costs more than before, in order to satisfy the fixed demand d_{21}, firm 2 tends to shift more production and distribution to the contractor with a good quality level, when ω is small. This is why Q^*_{231} increases as ω increases from 0 to Nevertheless, as ω increases further, firm 2 is under greater pressure to improve quality. Therefore, firm 2 then shifts more production in-house, and, as a result, Q^*_{231} decreases.

For firm 1, whose cost functions are completely distinct from those of firm 2, it is always more cost-wise for it to improve quality by shifting product flows in-house. Thus, the changing trends of all the variables and costs of firm 1 in Figs. 8.5 and 8.6 are either monotonically increasing or decreasing.

Further Comparison of Examples 8.1 and 8.2

We now compare Examples 8.1 and 8.2 with ω in both examples equal to 0. Please refer to the preceding figures. This case is interesting and illuminating since it represents the scenario that neither firm 1 nor firm 2 cares about its possible reputation loss due to its product/brand having a lower quality. After the incorporation of quality levels into both firms' in-house transportation costs, it now costs more for both firms to transport the same amounts of their products manufactured in-house and to maintain the same in-house quality levels as in Example 8.1, as reflected by the in-house transportation functions in Examples 8.1 and 8.2. Hence, in Example 8.2, the equilibrium in-house product flows are lower as compared to those in Example 8.1, and, in order to satisfy the demands, the equilibrium outsourced flows of both firms are higher. The corresponding results are: the contractors charge the firms more; the outsourcing quality levels of the contractors are lower; the contractors' total profits increase, and the firms' total costs are higher than those in Example 8.1.

Example 8.3

In Example 8.3, we consider the scenario that the in-house transportation from the two firms to each demand market gets much more congested than before and each firm's in-house quantities also affect the other firm's in-house transportation costs. The total in-house transportation cost functions of the two firms now become:

$$\hat{c}_{11}(Q_{111}, Q_{211}, q_1) = Q_{111}^2 + 1.5Q_{111}q_1 + 7Q_{211},$$

$$\hat{c}_{12}(Q_{112}, Q_{212}.q_1) = 2.5Q_{112}^2 + 2Q_{112}q_1 + 10Q_{212},$$

$$\hat{c}_{21}(Q_{211}, Q_{111}, q_2) = 0.5Q_{211}^2 + 3Q_{211}q_2 + 8Q_{111},$$

$$\hat{c}_{22}(Q_{212}, Q_{112}, q_2) = 2Q_{212}^2 + 2Q_{212}q_2 + 10Q_{112}.$$

The remaining data are identical to those in Example 8.1.

The total costs of firm 1 and firm 2 associated with different ω values are displayed in Tables 8.2 and 8.3, respectively. Note that, in Table 8.2, the total cost of firm 1 increases monotonically, whether ω_1 or ω_2 increases. The same result is inferred from Table 8.3 for firm 2.

The reason is the following. As discussed for Examples 8.1 and 8.2, when $\omega_i; i = 1, 2$, increases, the in-house production quantities of firm i increase. Now, in Example 8.3, because of the in-house production costs (as in Example 8.1) and the new in-house transportation costs, the increase of firm i's in-house quantities would also increase the other firm's total cost.

According to the results in Tables 8.2 and 8.3, strategically, if a firm has to increase the weight of its own disrepute cost, it is more cost-wise to increase it before the other firm does. If the firm increases its weight at the same time as, or after the other firm does, it would incur more cost under the same disrepute cost weight.

Table 8.2 Total costs of firm 1 with different sets of ω_1 and ω_2

ω	$\omega_1 = 0$	$\omega_1 = 1{,}000$	$\omega_1 = 2{,}000$	$\omega_1 = 3{,}000$	$\omega_1 = 4{,}000$	$\omega_1 = 5{,}000$
$\omega_2 = 0$	12,999.09	45,135.09	61,322.22	71,463.36	77,437.89	80,462.63
$\omega_2 = 1{,}000$	13,218.71	45,348.05	61,535.18	71,676.32	77,650.85	80,675.60
$\omega_2 = 2{,}000$	13,425.67	45,571.40	61,758.53	71,899.67	77,874.20	80,898.94
$\omega_2 = 3{,}000$	13,666.29	45,812.52	61,999.65	72,140.79	78,115.32	81,114.01
$\omega_2 = 4{,}000$	14,091.85	46,034.08	62,221.20	72,362.34	78,336.88	81,361.62
$\omega_2 = 5{,}000$	14,091.85	46,239.00	62,426.12	72,567.26	78,541.80	81,566.54

Table 8.3 Total costs of firm 2 with different sets of ω_1 and ω_2

ω	$\omega_1 = 0$	$\omega_1 = 1{,}000$	$\omega_1 = 2{,}000$	$\omega_1 = 3{,}000$	$\omega_1 = 4{,}000$	$\omega_1 = 5{,}000$
$\omega_2 = 0$	27,585.65	28,203.96	28,561.15	28,798.10	29,005.24	29,187.92
$\omega_2 = 1{,}000$	62,896.33	63,626.00	63,983.19	64,220.14	64,427.28	64,609.96
$\omega_2 = 2{,}000$	92,753.88	93,312.11	93,669.30	93,906.25	94,113.39	94,296.07
$\omega_2 = 3{,}000$	116,378.40	116,981.94	117,339.13	117,576.08	117,783.22	117,965.90
$\omega_2 = 4{,}000$	135,237.43	135,872.91	136,230.10	136,467.05	136,674.19	136,856.87
$\omega_2 = 5{,}000$	150,231.01	150,886.51	151,243.69	151,480.65	151,687.79	151,870.47

8.5 Summary and Conclusions

In this chapter, we constructed a supply chain network game theory model with product differentiation, outsourcing of production and distribution, and price and quality competition, that extends the single firm model with outsourcing of Chap. 7. The original firms compete with one another in in-house quality levels and in-house and outsourced production (and shipment) flows in order to minimize their total costs and the weighted disrepute costs. The contractors, in turn, compete in their quality levels and the prices that they charge the original firms for manufacturing and distributing the products to the demand markets. This model provides the optimal make-or-buy as well as contractor selection decisions for each original firm.

Similar to Chap. 7, we modeled the impact of quality on in-house and outsourced production and transportation and on the reputation of each firm through the quantification of the quality levels, quality cost, and the disrepute cost, with the production and the transportation cost functions depending on both quantities and quality levels. The product quality levels and quality costs were defined and quantified based on concepts and ideas in classic quality management literature, as we have also done in the preceding model chapters. The disrepute cost, which captures the impact of quality on a firm's reputation, was formulated as a function of the average quality level of the firm.

Variational inequality theory was employed in the formulations of the equilibrium conditions of the original firms, the contractors, and the supply chain network game theory model with product differentiation, possible outsourcing of production and distribution, and quality and price competition. The algorithm adopted is the

Euler method, which provides a discrete-time adjustment process and tracks the evolution of the in-house and outsourced production (and shipment) flows, the in-house and the outsourced quality levels, and the prices over time. It also yields closed form explicit formulae at each iteration with nice features for computation for all variables except for the production/shipment ones, which are computed via an exact equilibration algorithm.

In order to demonstrate the generality of the model and the computational scheme, we then provided solutions to a series of numerical examples, accompanied by sensitivity analysis.

8.6 Sources and Notes

This chapter is based on the paper by Nagurney and Li (2015).

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Part IV
Supplier Quality and Freight Service Quality

Chapter 9
The General Multitiered Supply Chain Network Model with Performance Indicators

Abstract This chapter begins Part IV of this book, which is on supplier quality and freight service quality. In order to set the stage for quality competition with suppliers in Chap. 10, in this chapter we present a multitiered competitive supply chain network game theory model, which includes the supplier tier. The firms are differentiated by brands and can produce their own components, as reflected by their capacities, and/or obtain components from one or more suppliers, who also are capacitated. The firms compete in a Cournot-Nash fashion, whereas the suppliers compete a la Bertrand since firms are sensitive to prices. All decision-makers seek to maximize their profits with consumers reflecting their preferences through the demand price functions associated with the demand markets for the firms' products. We construct supply chain network performance measures for the full supply chain and the individual firm levels that assess the efficiency of the supply chain or firm, respectively. They allow for the identification and ranking of the importance of suppliers as well as the components of suppliers with respect to the full supply chain or individual firm. The framework is illustrated through a series of numerical supply chain network examples.

9.1 Introduction

In Part III of this book, we presented a series of models with product differentiation and also outsourcing options. In this and the next chapter we introduce suppliers explicitly into multitiered supply chain network game theory models. Suppliers are critical in providing essential components and resources for finished goods in today's globalized supply chain networks. The number of components comprising a finished product may be small or immense as in aircraft manufacturing and other complex high-tech products. Even in the case of simpler products, such as bread, ingredients may travel across the globe as inputs into production processes. Suppliers are also decision-makers and they compete with one another to provide components to downstream manufacturing firms. When suppliers are faced with disruptions, whether due to man-made activities or errors, natural disasters, unforeseen events, or even terrorist attacks, the ramifications and effects may propagate through a supply chain or multiple supply chains. Hence, capturing supplier behavior is

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essential in modeling the full scope of supply chain network competition and in identifying the importance of a supplier and the components that it provides to the firms.

There are many vivid examples of supplier failures, due to natural disasters, and associated supply chain disruptions. A famous example is the Royal Philips Electronics cell phone chip manufacturing plant fire caused by a lightning strike on March 17, 2000, with subsequent water and smoke damage that adversely affected Ericsson. Ericsson, unlike Nokia, did not have a backup, and suffered a second quarter operating loss in 2000 of $200 million in its mobile phone division (cf. Mukherjee 2008). The Fukushima triple disaster on March 11, 2011 in Japan resulted in shortages of memory chips, automotive sensors, silicon wafers, and even certain colors of automotive paints, because of the affected suppliers (see Lee and Pierson 2011). The worst floods in 50 years that followed in October 2011 in Thailand impacted both Apple and Toyota supply chains, since Thailand is the world's largest producer of computer hard disk drives and also a big automotive manufacturing hub (Yang 2011). However, not all supplier shortcomings need be due to disasters. Boeing, facing challenges with its 787 Dreamliner supply chain design and numerous delays, ended up having to buy two suppliers for $2.4 billion because the units were underperforming in the chain (Tang et al. 2009).

In this chapter, we develop a multitiered competitive supply chain network game theory model, which includes the supplier tier. The firms are differentiated by brands, as were the models in Chaps. 5, 6, and 8. The firms can produce their own components, up to their capacities, and/or obtain components from one or more suppliers, who also are capacitated. The firms compete in a Cournot (1838)-Nash (1950, 1951) fashion, whereas the suppliers compete a la Bertrand (1883) since firms are sensitive to prices. We also assumed Bertrand competition among contractors in our supply chain network models in Chaps. 7 and 8. All decision-makers seek to maximize their profits with consumers reflecting their preferences through the demand price functions associated with the demand markets for the firms' products.

> The framework adds to the growing literature on supply chain disruptions (cf. Chopra and Sodhi 2004; Wu and Blackhurst 2009; Nagurney and Qiang 2009; Qiang and Nagurney 2012; Chen et al. 2015) by providing metrics that allow individual firms, industry overseers or regulators, and/or government policy-makers to identify the importance of suppliers and the components that they produce for various product supply chains.

This chapter is organized as follows. In Sect. 9.2, we present the supply chain network model, describe the behavior of the firms and the suppliers, identify the governing equilibrium conditions, and provide the variational inequality formulation. In Sect. 9.3, we propose the supply chain network performance measures at the full supply chain and individual firm levels, and define the supplier and supplier component importance indicators. In Sect. 9.4, we describe an algorithm, which is then applied in Sect. 9.5 to compute solutions to numerical supply chain network examples to illustrate the model and methodology and how the performance measures and the supplier and component importance indicators can be applied in practice. We summarize our results and present our conclusions in Sect. 9.6. Section 9.7 contains the Sources and Notes.

9.2 The Multitiered Supply Chain Network Game Theory Model with Suppliers

In this section, we develop a multitiered supply chain network game theory model with suppliers and firms that procure components from the suppliers for their products, which are differentiated by brand. We consider a supply chain network consisting of I firms, with a typical firm denoted by i, n_S suppliers, with a typical supplier denoted by j, and a total of n_R demand markets, with a typical demand market denoted by k.

The firms compete noncooperatively, and each firm corresponds to an individual brand representing the product that it produces. We assume that product i, which is the product produced by firm i, requires n_{l^i} different components, and the total number of different components required by the I products is n_l. We allow for the situation that each supplier may be able to produce a variety of components for each firm.

The I firms are involved in the processes of assembling the products using the components needed, transporting the products to the demand markets, and, possibly, producing one or more of the components of the products. The suppliers, in turn, are involved in the processes of producing and delivering the components of the products to the firms. Both in-house and contracted component production activities are captured in the model. The capacity of production is also considered.

The supply chain network topology G of the problem is depicted in Fig. 9.1, where G consists of the set of nodes N and the set of links L, so that $G = [N, L]$. Firm i's supply chain network topology; $i = 1, \ldots, I$, is denoted by G_i. G_i consists of the sets of nodes and links that represent the economic activities associated with firm i and its suppliers. In Fig. 9.1, the first two sets of links from the top nodes are links corresponding to distinct supplier components. The links from the top-tiered nodes $j; j = 1, \ldots, n_S$, representing the suppliers, are connected to the associated manufacturing nodes, denoted by nodes $1, \ldots, n_l$. These links represent the manufacturing activities of the suppliers. The next set of links that emanate from $1, \ldots, n_l$ to the firms, denoted by nodes $1, \ldots, I$, reflects the transportation of the components to the associated firms. In addition, the links that connect nodes $1^i, \ldots, n^i_{l^i}$, which are firm i's component manufacturing nodes, and firm i are the manufacturing links of firm i for producing its components.

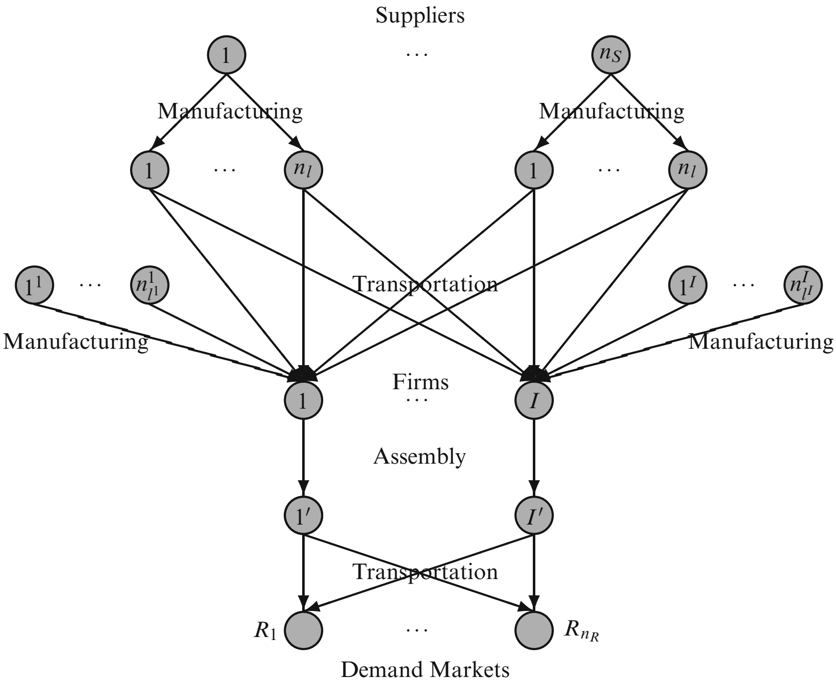

Fig. 9.1 The multitiered supply chain network topology

The rest of the links in Fig. 9.1 are links corresponding to the finished products. The link connecting firm i and node i', which is the assembly node of firm i, represents the activity of assembling firm i's product using the components needed, which may be produced by firm i, the suppliers, or both. Finally, the links joining nodes $1', \ldots, I'$ with demand market nodes $R_1, \ldots, R_{n_R}$ correspond to the transportation of the products to the demand markets.

We wish to determine the optimal component production quantities, both by the firms and by the suppliers, the optimal product shipments from the firms to the demand markets, and the prices that the suppliers charge the firms for producing and delivering their components. The firms compete noncooperatively under the Cournot-Nash equilibrium concept in product shipments and component production quantities, while the suppliers compete in Bertrand fashion in the prices that they charge the firms.

The notation for the model is given in Table 9.1. The vectors are assumed to be column vectors, as in the previous chapters, with the equilibrium solutions denoted by "*".

Table 9.1 Notation for the multitiered supply chain network game theory model with suppliers

Notation	Definition
Q^S_{jil}	The nonnegative amount of firm i's component l produced by supplier j; $j = 1, \ldots, n_S$; $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$. For firm i, we group its $\{Q^S_{jil}\}$ elements into the vector $Q^S_i \in R^{n_S n_{l^i}}_+$. All the $\{Q^S_{jil}\}$ elements are grouped into the vector $Q^S \in R^{n_S \sum_{i=1}^I n_{l^i}}_+$
CAP^S_{jil}	The capacity of supplier j for producing firm i's component l
Q^F_{il}	The nonnegative amount of firm i's component l produced by firm i itself. For firm i, we group its $\{Q^F_{il}\}$ elements into the vector $Q^F_i \in R^{n_{l^i}}_+$, and group all such vectors into the vector $Q^F \in R^{\sum_{i=1}^I n_{l^i}}_+$
CAP^F_{il}	The capacity of firm i for producing its component l
Q_{ik}	The nonnegative shipment of firm i's product from firm i to demand market R_k; $k = 1, \ldots, n_R$. For firm i, we group its $\{Q_{ik}\}$ elements into the vector $Q_i \in R^{n_R}_+$, and group all such vectors into the vector $Q \in R^{In_R}_+$
π_{jil}	The price charged by supplier j for producing one unit of firm i's component l. For supplier j, we group its $\{\pi_{jil}\}$ elements into the vector $\pi_j \in R^{\sum_{i=1}^I n_{l^i}}_+$, firm i's $\{\pi_{jil}\}$ elements into the vector $\pi_i \in R^{n_S n_{l^i}}_+$, and group all such vectors into the vector $\pi \in R^{n_S \sum_{i=1}^I n_{l^i}}_+$
d_{ik}	The demand for firm i's product at demand market R_k. We group all $\{d_{ik}\}$ elements into the vector $d \in R^{In_R}_+$
θ_{il}	The amount of component l needed by firm i to produce one unit product i
$\hat{f}^S_{jl}(Q^S)$	Supplier j's production cost for producing component l; $l = 1, \ldots, n_l$; $j = 1, \ldots, n_S$
$\hat{c}^S_{jil}(Q^S)$	Supplier j's transportation cost for shipping firm i's component l; $j = 1, \ldots, n_S$; $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$
$oc_j(\pi)$	Supplier j's opportunity cost; $j = 1, \ldots, n_S$
$\hat{f}_i(Q)$	Firm i's cost for assembling its product using the components needed; $i = 1, \ldots, I$
$\hat{f}^F_{il}(Q^F)$	Firm i's production cost for producing its component l; $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$
$\hat{c}^F_{ik}(Q)$	Firm i's transportation cost for shipping its product to demand market R_k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$
$tc_{ijl}(Q^S)$	The transaction cost paid by firm i for transacting with supplier j for its component l; $i = 1, \ldots, I$; $j = 1, \ldots, n_S$; $l = 1, \ldots, n_{l^i}$
$\rho_{ik}(d)$	The demand price for firm i's product at demand market k; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$

9.2.1 The Behavior of the Firms and Their Optimality Conditions

Given the prices π^* of the components charged by the suppliers, the objective of firm i; $i = 1, \ldots, I$, is to maximize its utility/profit U_i^F, which is the difference between its total revenue and its total cost. The total cost includes the assembly cost, the production cost, the transportation costs, the payments to the suppliers, and the transaction costs. As noted in Table 9.1, the assembly cost functions, the production cost functions, the transportation cost functions, and the demand price functions are general functions in vectors of quantities, which capture the competition among firms for resources.

Hence, firm i seeks to

$$\text{Maximize}_{Q_i, Q_i^F, Q_i^S} \quad U_i^F = \sum_{k=1}^{n_R} \rho_{ik}(d) d_{ik} - \hat{f}_i(Q) - \sum_{l=1}^{n_{l^i}} \hat{f}_{il}^F(Q^F) - \sum_{k=1}^{n_R} \hat{c}_{ik}^F(Q)$$

$$- \sum_{j=1}^{n_S} \sum_{l=1}^{n_{l^i}} \pi_{jil}^* Q_{jil}^S - \sum_{j=1}^{n_S} \sum_{l=1}^{n_{l^i}} tc_{ijl}(Q^S) \tag{9.1a}$$

subject to:

$$Q_{ik} = d_{ik}, \quad i = 1, \ldots, I; k = 1, \ldots, n_R, \tag{9.2}$$

$$\sum_{k=1}^{n_R} Q_{ik} \theta_{il} \leq \sum_{j=1}^{n_S} Q_{jil}^S + Q_{il}^F, \quad i = 1, \ldots, I; l = 1, \ldots, n_{l^i}, \tag{9.3}$$

$$Q_{ik} \geq 0, \quad i = 1, \ldots, I; k = 1, \ldots, n_R, \tag{9.4}$$

$$CAP_{jil}^S \geq Q_{jil}^S \geq 0, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}, \tag{9.5}$$

$$CAP_{il}^F \geq Q_{il}^F \geq 0, \quad i = 1, \ldots, I; l = 1, \ldots, n_{l^i}. \tag{9.6}$$

We assume that all the cost functions and the demand price functions in (9.1a) are continuous and continuously differentiable. The cost functions are convex and each demand price function is monotonically decreasing in demand at its demand market. According to constraint (9.2), the product shipment from a firm to a demand market should be equal to the quantity of the firm's product consumed at that demand market. Constraint (9.3) captures the material requirements in the assembly process. Constraint (9.4) is the nonnegativity constraint for product shipments. Constraints (9.5) and (9.6) indicate that the component production quantities should be nonnegative and limited by the associated capacities, which can capture the abilities of producing. If a supplier or a firm is not capable of producing a certain component, the associated capacity would be 0.

In light of (9.2), we can define the demand price function $\hat{\rho}_{ik}$ in product shipments of the firms, so that $\hat{\rho}_{ik}(Q) \equiv \rho_{ik}(d); i = 1, \ldots, I, k = 1, \ldots, n_R$. Therefore, (9.1a) is equivalent to:

$$\text{Maximize}_{Q_i, Q_i^F, Q_i^S} \quad U_i^F = \sum_{k=1}^{n_R} \hat{\rho}_{ik}(Q)Q_{ik} - \hat{f}_i(Q) - \sum_{l=1}^{n_{l^i}} \hat{f}_{il}^F(Q^F) - \sum_{k=1}^{n_R} \hat{c}_{ik}^F(Q)$$

$$- \sum_{j=1}^{n_S} \sum_{l=1}^{n_{l^i}} \pi_{jil}^* Q_{jil}^S - \sum_{j=1}^{n_S} \sum_{l=1}^{n_{l^i}} tc_{ijl}(Q^S). \tag{9.1b}$$

The firms compete in the sense of Nash (1950, 1951). The strategic variables for each firm i are the product shipments to the demand markets, the in-house component production quantities, and the contracted component production quantities produced by the suppliers.

We define the feasible set $\overline{K}_i^F$ as $\overline{K}_i^F \equiv \{(Q_i, Q_i^F, Q_i^S)|$ (9.3), (9.4), (9.5) and (9.6) are satisfied$\}$. All $\overline{K}_i^F; i = 1, \ldots, I$, are closed and convex. We also define the feasible set $\overline{\mathcal{K}}^F \equiv \Pi_{i=1}^I \overline{K}_i^F$.

Definition 9.1: A Cournot-Nash Equilibrium
A product shipment, in-house component production, and contracted component production pattern $(Q^*, Q^{F^*}, Q^{S^*}) \in \overline{\mathcal{K}}^F$ is said to constitute a Cournot-Nash equilibrium if for each firm i; $i = 1, \ldots, I$,

$$U_i^F(Q_i^*, \hat{Q}_i^*, Q_i^{F^*}, \hat{Q}_i^{F^*}, Q_i^{S^*}, \hat{Q}_i^{S^*}, \pi_i^*) \geq U_i^F(Q_i, \hat{Q}_i^*, Q_i^F, \hat{Q}_i^{F^*}, Q_i^S, \hat{Q}_i^{S^*}, \pi_i^*),$$
$$\forall (Q_i, Q_i^F, Q_i^S) \in \overline{K}_i^F, \tag{9.7}$$

where

$$\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_I^*),$$
$$\hat{Q}_i^{F^*} \equiv (Q_1^{F^*}, \ldots, Q_{i-1}^{F^*}, Q_{i+1}^{F^*}, \ldots, Q_I^{F^*}),$$
$$\hat{Q}_i^{S^*} \equiv (Q_1^{S^*}, \ldots, Q_{i-1}^{S^*}, Q_{i+1}^{S^*}, \ldots, Q_I^{S^*}).$$

According to (9.7), a Cournot-Nash equilibrium is established if no firm can unilaterally improve upon its profit by selecting an alternative vector of product shipments, in-house component production quantities, and contracted component production quantities produced by the suppliers.

We now derive the variational inequality formulation of the Cournot-Nash equilibrium (see Cournot 1838; Nash 1950, 1951; Gabay and Moulin 1980) in the following theorem.

Theorem 9.1: Variational Inequality Formulation of Firms' Problems
Assume that, for each firm i; $i = 1, \ldots, I$, the utility function $U_i^F(Q, Q^F, Q^S, \pi_i^*)$ is concave with respect to its variables in Q_i, Q_i^F, and Q_i^S, and is continuous and continuously differentiable. Then $(Q^*, Q^{F^*}, Q^{S^*}) \in \overline{\mathscr{K}}^F$ is a Counot-Nash equilibrium according to Definition 9.1 if and only if it satisfies the variational inequality:

$$-\sum_{i=1}^{I}\sum_{k=1}^{n_R}\frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, \pi_i^*)}{\partial Q_{ik}} \times (Q_{ik} - Q_{ik}^*)$$
$$-\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, \pi_i^*)}{\partial Q_{il}^F} \times (Q_{il}^F - Q_{il}^{F^*})$$
$$-\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, \pi_i^*)}{\partial Q_{jil}^S} \times (Q_{jil}^S - Q_{jil}^{S^*}) \geq 0, \quad \forall (Q, Q^F, Q^S) \in \overline{\mathscr{K}}^F, \tag{9.8}$$

with notice that: for $i = 1, \ldots, I$; $k = 1, \ldots, n_R$:

$$-\frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, \pi_i^*)}{\partial Q_{ik}}$$
$$= \left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ik}} + \sum_{h=1}^{n_R}\frac{\partial \hat{c}_{ih}^F(Q^*)}{\partial Q_{ik}} - \sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*)}{\partial Q_{ik}}Q_{ih}^* - \hat{\rho}_{ik}(Q^*)\right],$$

for $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$-\frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, \pi_i^*)}{\partial Q_{il}^F} = \left[\sum_{m=1}^{n_{l^i}}\frac{\partial \hat{f}_{im}^F(Q^{F^*})}{\partial Q_{il}^F}\right],$$

for $j = 1, \ldots, n_S$; $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$-\frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, \pi_i^*)}{\partial Q_{jil}^S} = \left[\pi_{jil}^* + \sum_{g=1}^{n_S}\sum_{m=1}^{n_{l^i}}\frac{\partial tc_{igm}(Q^{S^*})}{\partial Q_{jil}^S}\right],$$

or, equivalently, $(Q^*, Q^{F^*}, Q^{S^*}, \lambda^*) \in \mathscr{K}^F$ is a vector of the equilibrium product shipment, in-house component production, contracted component production pattern, and Lagrange multipliers if and only if it satisfies the variational inequality

$$\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ik}}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}^F_{ih}(Q^*)}{\partial Q_{ik}}-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*)}{\partial Q_{ik}}Q^*_{ih}-\hat{\rho}_{ik}(Q^*)+\sum_{l=1}^{n_{l^i}}\lambda^*_{il}\theta_{il}\right]$$
$$\times(Q_{ik}-Q^*_{ik})+\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\sum_{m=1}^{n_{l^i}}\frac{\partial \hat{f}^F_{im}(Q^{F^*})}{\partial Q^F_{il}}-\lambda^*_{il}\right]\times(Q^F_{il}-Q^{F^*}_{il})$$
$$+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\pi^*_{jil}+\sum_{g=1}^{n_S}\sum_{m=1}^{n_{l^i}}\frac{\partial tc_{igm}(Q^{S^*})}{\partial Q^S_{jil}}-\lambda^*_{il}\right]\times(Q^S_{jil}-Q^{S^*}_{jil})$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\sum_{j=1}^{n_S}Q^{S^*}_{jil}+Q^{F^*}_{il}-\sum_{k=1}^{n_R}Q^*_{ik}\theta_{il}\right]\times(\lambda_{il}-\lambda^*_{il})\geq 0,\quad \forall(Q,Q^F,Q^S,\lambda)\in\mathscr{K}^F,\tag{9.9}$$

where $\mathscr{K}^F \equiv \Pi_{i=1}^{I}K_i^F$ *and* $K_i^F \equiv \{(Q_i,Q_i^F,Q_i^S,\lambda_i)|\lambda_i \geq 0$ *with (9.4), (9.5) and (9.6) satisfied*$\}$. λ_i *is the* n_{l^i}*-dimensional vector with component* l *being the element* λ_{il} *corresponding to the Lagrange multiplier associated with the* (i,l)*-th constraint (9.3). Both the above-defined feasible sets are convex.*

Proof: For a given firm i, under the imposed assumptions, (9.8) holds if and only if (see Bertsekas and Tsitsiklis 1989, page 287) the following holds:

$$\sum_{k=1}^{n_R}\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ik}}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}^F_{ih}(Q^*)}{\partial Q_{ik}}-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*)}{\partial Q_{ik}}Q^*_{ih}-\hat{\rho}_{ik}(Q^*)+\sum_{l=1}^{n_{l^i}}\lambda^*_{il}\theta_{il}\right]$$
$$\times(Q_{ik}-Q^*_{ik})+\sum_{l=1}^{n_{l^i}}\left[\sum_{m=1}^{n_{l^i}}\frac{\partial \hat{f}^F_{im}(Q^{F^*})}{\partial Q^F_{il}}\right]\times(Q^F_{il}-Q^{F^*}_{il})$$
$$+\sum_{j=1}^{n_S}\sum_{l=1}^{n_{l^i}}\left[\pi^*_{jil}+\sum_{g=1}^{n_S}\sum_{m=1}^{n_{l^i}}\frac{\partial tc_{igm}(Q^{S^*})}{\partial Q^S_{jil}}-\lambda^*_{il}\right]\times(Q^S_{jil}-Q^{S^*}_{jil})$$
$$+\sum_{l=1}^{n_{l^i}}\left[\sum_{j=1}^{n_S}Q^{S^*}_{jil}+Q^{F^*}_{il}-\sum_{k=1}^{n_R}Q^*_{ik}\theta_{il}\right]\times(\lambda_{il}-\lambda^*_{il})\geq 0,\quad \forall(Q_i,Q_i^F,Q_i^S,\lambda_i)\in K_i^F.\tag{9.10}$$

Variational inequality (9.10) holds for each firm i; $i = 1,\ldots,I$, and, hence, the summation of (9.10) yields variational inequality (9.9). □

9.2.2 The Behavior of the Suppliers and Their Optimality Conditions

The suppliers' opportunity costs are functions of the prices that they charge the firms for producing and delivering the components, as in Table 9.1. The suppliers may not be able to recover their costs if the prices that they charge are too low. If the prices are too high, the suppliers may lose the contracts. Here, we capture the opportunity cost of a supplier with a general function that depends on the vector of prices, since the opportunity cost of a supplier may also be affected by the prices charged by the other suppliers.

Given the Q^{S^*} determined by the firms, the objective of supplier $j; j = 1, \ldots, n_S$, is to maximize its total profit, denoted by U_j^S. Its revenue is obtained from the payments of the firms, while its costs are the costs of production and delivery, and the opportunity cost. The strategic variables of a supplier are the prices that it charges the firms.

The decision-making problem for supplier j is the following:

$$\text{Maximize}_{\pi_j} \quad U_j^S = \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \pi_{jil} Q_{jil}^{S^*} - \sum_{l=1}^{n_l} \hat{f}_{jl}^S(Q^{S^*}) - \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \hat{c}_{jil}^S(Q^{S^*}) - oc_j(\pi) \tag{9.11}$$

subject to:

$$\pi_{jil} \geq 0, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}. \tag{9.12}$$

We assume that the cost functions of each supplier are continuous, continuously differentiable, and convex.

The suppliers also compete in a noncooperative in the sense of Nash (1950, 1951), with each one trying to maximize its own profit. We define the feasible sets $K_j^S \equiv \{\pi_j | \pi_j \in R_+^{\sum_{i=1}^{I} n_{l^i}}\}$, $\mathscr{K}^S \equiv \Pi_{j=1}^{n_S} K_j^S$, and $\overline{\mathscr{K}} \equiv \overline{\mathscr{K}}^F \times \mathscr{K}^S$. All the above-defined feasible sets are convex.

Definition 9.2: A Bertrand-Nash Equilibrium
A price pattern $\pi^* \in \mathscr{K}^S$ is said to constitute a Bertrand-Nash equilibrium if for each supplier $j; j = 1, \ldots, n_S$,

$$U_j^S(Q^{S^*}, \pi_j^*, \hat{\pi}_j^*) \geq U_j^S(Q^{S^*}, \pi_j, \hat{\pi}_j^*), \quad \forall \pi_j \in K_j^S, \tag{9.13}$$

where

$$\hat{\pi}_j^* \equiv (\pi_1^*, \ldots, \pi_{j-1}^*, \pi_{j+1}^*, \ldots, \pi_{n_S}^*).$$

According to (9.13), a Bertrand-Nash equilibrium is established if no supplier can unilaterally improve upon its profit by selecting an alternative vector of prices charged to the firms.

The variational inequality formulation of the Bertrand-Nash equilibrium according to Definition 9.2 (see Bertrand 1883; Nash 1950, 1951; Gabay and Moulin 1980; Nagurney 2006) is given in the following theorem.

Theorem 9.2: Variational Inequality Formulation of the Suppliers' Problems
Assume that, for each supplier j; $j = 1, \ldots, n_S$, the profit function $U_j^S(Q^{S^*}, \pi)$ is concave with respect to the variables in π_j, and is continuous and continuously differentiable. Then $\pi^* \in \mathcal{K}^S$ is a Bertrand-Nash equilibrium according to Definition 9.2 if and only if it satisfies the variational inequality:

$$-\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_j^S(Q^{S^*}, \pi^*)}{\partial \pi_{jil}} \times (\pi_{jil} - \pi_{jil}^*) \geq 0,$$

$$\forall \pi \in \mathcal{K}^S, \tag{9.14}$$

with notice that: for $j = 1, \ldots, n_S$; $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$-\frac{\partial U_j^S(Q^{S^*}, \pi^*)}{\partial \pi_{jil}} = \frac{\partial oc_j(\pi^*)}{\partial \pi_{jil}} - Q_{jil}^{S^*}.$$

9.2.3 The Equilibrium Conditions for the Multitiered Supply Chain Network with Suppliers

In equilibrium, the optimality conditions for all firms and the optimality conditions for all suppliers must hold simultaneously, according to the definition below.

Definition 9.3: Multitiered Supply Chain Network Equilibrium with Suppliers
The equilibrium state of the multitiered supply chain network with suppliers is one where both variational inequalities (9.8) (or (9.9)) and (9.14) hold simultaneously.

Theorem 9.3: Variational Inequality Formulation of the Multitiered Supply Chain Network Equilibrium with Suppliers
The equilibrium conditions governing the multitiered supply chain network model with suppliers are equivalent to the solution of the variational inequality problem: determine $(Q^*, Q^{F^*}, Q^{S^*}, \pi^*) \in \overline{\mathcal{K}}$, such that:

$$-\sum_{i=1}^{I}\sum_{k=1}^{n_R} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, \pi_i^*)}{\partial Q_{ik}} \times (Q_{ik} - Q_{ik}^*) - \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, \pi_i^*)}{\partial Q_{il}^F}$$
$$\times (Q_{il}^F - Q_{il}^{F^*}) - \sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, \pi_i^*)}{\partial Q_{jil}^S} \times (Q_{jil}^S - Q_{jil}^{S^*})$$

$$-\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\frac{\partial U_j^S(Q^{S^*},\pi^*)}{\partial \pi_{jil}}\times(\pi_{jil}-\pi_{jil}^*)\geq 0, \quad \forall (Q,Q^F,Q^S,\pi)\in\overline{\mathscr{K}}, \tag{9.15}$$

or, equivalently: determine $(Q^*, Q^{F^*}, Q^{S^*}, \lambda^*, \pi^*) \in \mathscr{K}$*, such that:*

$$\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ik}}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}_{ih}^F(Q^*)}{\partial Q_{ik}}-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*)}{\partial Q_{ik}}Q_{ih}^*-\hat{\rho}_{ik}(Q^*)+\sum_{l=1}^{n_{l^i}}\lambda_{il}^*\theta_{il}\right]$$

$$\times(Q_{ik}-Q_{ik}^*)+\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\sum_{m=1}^{n_{l^i}}\frac{\partial \hat{f}_{im}^F(Q^{F^*})}{\partial Q_{il}^F}-\lambda_{il}^*\right]\times(Q_{il}^F-Q_{il}^{F^*})$$

$$+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\pi_{jil}^*+\sum_{g=1}^{n_S}\sum_{m=1}^{n_{l^i}}\frac{\partial tc_{igm}(Q^{S^*})}{\partial Q_{jil}^S}-\lambda_{il}^*\right]\times(Q_{jil}^S-Q_{jil}^{S^*})$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\sum_{j=1}^{n_S}Q_{jil}^{S^*}+Q_{il}^{F^*}-\sum_{k=1}^{n_R}Q_{ik}^*\theta_{il}\right]\times(\lambda_{il}-\lambda_{il}^*)$$

$$+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\frac{\partial oc_j(\pi^*)}{\partial \pi_{jil}}-Q_{jil}^{S^*}\right]\times(\pi_{jil}-\pi_{jil}^*)\geq 0, \quad \forall (Q,Q^F,Q^S,\lambda,\pi)\in\mathscr{K}, \tag{9.16}$$

where $\mathscr{K} \equiv \mathscr{K}^F \times \mathscr{K}^S$.

Proof: Summation of variational inequalities (9.8) (or (9.9)) and (9.14) yields variational inequality (9.15) (or (9.16)). A solution to variational inequality (9.15) (or (9.16)) satisfies the sum of (9.8) (or (9.9)) and (9.14) and, hence, is an equilibrium according to Definition (9.3). □

We now put variational inequality (9.16) into standard form (2.1a), as we have been doing in all of the modeling chapters: determine $X^* \in \mathscr{K}$ where X is a vector in R^N, $F(X)$ is a continuous function such that $F(X) : X \mapsto \mathscr{K} \subset R^N$, and

$$\langle F(X^*), X - X^*\rangle \geq 0, \quad \forall X \in \mathscr{K}, \tag{9.17}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in the N-dimensional Euclidean space, $N = In_R + 2n_S\sum_{i=1}^{I} n_{l^i} + 2\sum_{i=1}^{I} n_{l^i}$, and $\mathscr{K}$ is closed and convex. We define the vector $X \equiv (Q, Q^F, Q^S, \lambda, \pi)$ and the vector $F(X) \equiv (F^1(X), F^2(X), F^3(X), F^4(X), F^5(X))$, such that:

$$F^1(X)=\left[\frac{\partial \hat{f}_i(Q)}{\partial Q_{ik}}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}_{ih}^F(Q)}{\partial Q_{ik}}-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q)}{\partial Q_{ik}}Q_{ih}-\hat{\rho}_{ik}(Q)+\sum_{l=1}^{n_{l^i}}\lambda_{il}\theta_{il};\right.$$

$$i = 1, \ldots, I; k = 1, \ldots, n_R \Bigg], \tag{9.18a}$$

$$F^2(X) = \left[\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^F)}{\partial Q_{il}^F} - \lambda_{il}; i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right], \tag{9.18b}$$

$$F^3(X) = \left[\pi_{jil} + \sum_{g=1}^{n_S} \sum_{m=1}^{n_{l^i}} \frac{\partial tc_{igm}(Q^S)}{\partial Q_{jil}^S} - \lambda_{il}; j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right], \tag{9.18c}$$

$$F^4(X) = \left[\sum_{j=1}^{n_S} Q_{jil}^S + Q_{il}^F - \sum_{k=1}^{n_R} Q_{ik}\theta_{il}; i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right], \tag{9.18d}$$

$$F^5(X) = \left[\frac{\partial oc_j(\pi)}{\partial \pi_{jil}} - Q_{jil}^S; j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right]. \tag{9.18e}$$

Similarly, we also put variational inequality (9.15) into standard form (9.1a): determine $Y^* \in \overline{\mathcal{K}}$ where Y is a vector in R^M, $G(Y)$ is a continuous function such that $G(Y) : Y \mapsto \overline{\mathcal{K}} \subset R^M$, and

$$\langle G(Y^*), Y - Y^* \rangle \geq 0, \quad \forall Y \in \overline{\mathcal{K}}, \tag{9.19}$$

where $M = In_R + \sum_{i=1}^{I} n_{l^i} + 2n_S \sum_{i=1}^{I} n_{l^i}$, and $\overline{\mathcal{K}}$ is closed and convex. We define $Y \equiv (Q, Q^F, Q^S, \pi)$, $G(Y) \equiv (-\frac{\partial U_i^F}{\partial Q_{ik}}, -\frac{\partial U_i^F}{\partial Q_{il}^F}, -\frac{\partial U_i^F}{\partial Q_{jil}^S}, -\frac{\partial U_j^S}{\partial \pi_{jil}}); j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}$. Hence, (9.15) can be put into standard form (9.19).

The equilibrium solution $(Q^*, Q^{F^*}, Q^{S^*}, \pi^*)$ to (9.19) and the $(Q^*, Q^{F^*}, Q^{S^*}, \pi^*)$ in the equilibrium solution to (9.17) are equivalent for this multitiered supply chain network problem with suppliers. In addition to $(Q^*, Q^{F^*}, Q^{S^*}, \pi^*)$, the equilibrium solution to (9.17) also contains the equilibrium Lagrange multipliers (λ^*).

9.3 Qualitative Properties

We now present some qualitative properties of the solution to variational inequalities (9.17) and (9.19), equivalently, (9.16) and (9.15). In particular, we provide the existence result and the uniqueness result.

In a supply chain network with suppliers, it is reasonable to expect that the price charged by each supplier j for producing one unit of firm i's component l, π_{jil}, is bounded by a sufficiently large value, since, in practice, each supplier cannot charge unbounded prices to the firms. Therefore, the following assumption is not unreasonable:

Assumption 9.1
Suppose that in our supply chain network model with suppliers there exists a sufficiently large Π, such that,

$$\pi_{jil} \le \Pi, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}. \tag{9.20}$$

With this assumption, we have the following existence result.

Theorem 9.4: Existence
With Assumption 9.1 satisfied, there exists at least one solution to variational inequalities (9.17) and (9.19); equivalently, (9.16) and (9.15).

Proof: We first prove that there exists at least one solution to variational inequality (9.19) (cf. (9.15)). Due to constraint (9.3), the product quantities Q_{ik}; $i = 1, \ldots, I; k = 1, \ldots, n_S$ are bounded, since the components quantities are nonnegative and capacitated (cf. (9.5) and (9.6)). Therefore, with Assumption 9.1, the feasible set of variational inequality (9.19) is bounded. Since the cost functions and the demand price functions are continuously differentiable, and the feasible set is convex and compact, the existence of a solution to (9.19) is then guaranteed. Since (9.19) and (9.17) (cf. (9.16)) are equivalent (see Theorem 3 in Nagurney and Dhanda (2000)), the existence of (9.17) is guaranteed. □

Theorem 9.5: Uniqueness
If Assumption 9.1 is satisfied, the equilibrium product shipment, in-house component production, contracted component production, and suppliers' price pattern $(Q^*, Q^{F^*}, Q^{S^*}, \pi^*)$ in variational inequality (9.19), equivalently, in (9.17), is unique under the following conditions:

(i) one of the two families of convex functions $\hat{f}_i(Q)$; $i = 1, \ldots, I$, and $\hat{c}^F_{ik}(Q)$; $k = 1 \ldots . n_R$, is strictly convex in Q_{ik};
(ii) the $\hat{f}^F_{il}(Q^F)$; $i = 1, \ldots, I, l = 1, \ldots . n_{l^i}$, are strictly convex in Q^F_{il};
(iii) the $tc_{ijl}(Q^S)$; $j = 1, \ldots, n_S$, $i = 1, \ldots, I, l = 1, \ldots . n_{l^i}$, are strictly convex in Q^S_{jil};
(iv) the $oc_j(\pi)$; $j = 1, \ldots, n_S$, are strictly convex in π_{jil};
(v) the $\rho_{ik}(d)$; $i = 1, \ldots, I, k = 1, \ldots . n_R$, are strictly monotonically decreasing of d_{ik}.

Proof: Assume the above conditions. Then the negative utility functions, $-U^F_i$ and $-U^S_j$; $\forall i = 1, \ldots, I, j = 1, \ldots, n_S$, are strictly convex in associated variables (cf. (9.1b), (9.11), and Theorems 9.1 and 9.2). Therefore,

$$\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[\left(-\frac{\partial U^F_i(Q', Q^{F'}, Q^{S'}, \pi')}{\partial Q_{ik}}\right) - \left(-\frac{\partial U^F_i(Q'', Q^{F''}, Q^{S''}, \pi'')}{\partial Q_{ik}}\right)\right] \times (Q'_{ik} - Q''_{ik})$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\left(-\frac{\partial U^F_i(Q', Q^{F'}, Q^{S'}, \pi')}{\partial Q^F_{il}}\right) - \left(-\frac{\partial U^F_i(Q'', Q^{F''}, Q^{S''}, \pi'')}{\partial Q^F_{il}}\right)\right] \times (Q^{F'}_{il} - Q^{F''}_{il})$$

$$+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\left(-\frac{\partial U_i^F(Q',Q^{F'},Q^{S'},\pi')}{\partial Q_{jil}^S}\right)-\left(-\frac{\partial U_i^F(Q'',Q^{F''},Q^{S''},\pi'')}{\partial Q_{jil}^S}\right)\right]\times(Q_{jil}^{S'}-Q_{jil}^{S''})$$

$$+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\left(-\frac{\partial U_j^S(Q^{S'},\pi')}{\partial \pi_{jil}}\right)-\left(-\frac{\partial U_j^S(Q^{S''},\pi'')}{\partial \pi_{jil}}\right)\right]\times(\pi'_{jil}-\pi''_{jil})>0,$$

$$\forall (Q',Q^{F'},Q^{S'},\pi'),(Q'',Q^{F''},Q^{S''},\pi'')\in\overline{\mathscr{K}},\quad (Q',Q^{F'},Q^{S'},\pi')\neq(Q'',Q^{F''},Q^{S''},\pi''), \tag{9.21}$$

that is,

$$\langle G(Y')-G(Y''),Y'-Y''\rangle>0,\quad \forall Y',Y''\in\overline{\mathscr{K}},Y'\neq Y'', \tag{9.22}$$

where $Y'=(Q',Q^{F'},Q^{S'},\pi')$, $Y''=(Q'',Q^{F''},Q^{S''},\pi'')$. Equation (9.22) proves that $G(Y)$ is strictly monotone. Under the existence (Theorem 9.4) and the strict monotonicity, the proof of uniqueness follows the standard variational inequality theory. □

Theorem 9.6: Lipschitz Continuity
The function that enters the variational inequality problem (9.17) is Lipschitz continuous, that is,

$$\| F(X')-F(X'') \|\leq L \| X'-X'' \|,\quad \forall X',X''\in\mathscr{K},\ \textit{where}\ L>0. \tag{9.23}$$

Proof: Since we have assumed that all the cost functions have bounded second-order partial derivatives, and the demand price functions have bounded first-order and second-order partial derivatives, the result is direct by applying a mid-value theorem from calculus to the $F(X)$ that enters variational inequality (9.17). □

9.4 Supply Chain Network Performance Measures

We now present the supply chain network performance measure for the whole competitive supply chain network G and that for the supply chain network of each individual firm i, G_i; $i=1,\ldots,I$, under competition. Such measures capture the efficiency of the supply chains in that the higher the demand to price ratios normalized over associated firm and demand market pairs, the higher the efficiency. Hence, a supply chain network is deemed to perform better if it can satisfy higher demands, on the average, relative to the product prices.

Definition 9.4: The Supply Chain Network Performance Measure for the Entire Competitive Supply Chain Network *G*
The supply chain network performance/efficiency measure, $\mathscr{E}(G)$, for a given competitive supply chain network topology G and the equilibrium demand vector d^*, is defined as follows:

$$\mathscr{E}=\mathscr{E}(G)=\frac{\sum_{i=1}^{I}\sum_{k=1}^{n_R}\frac{d_{ik}^*}{\rho_{ik}(d^*)}}{I\times n_R}, \tag{9.24}$$

where recall that I is the number of firms and n_R *is the number of demand markets in the competitive supply chain network, and* d^*_{ik} *and* $\rho_{ik}(d^*)$ *denote the equilibrium demand and the equilibrium price, respectively, associated with firm i and demand market k.*

Definition 9.5: The Supply Chain Network Performance Measure for an Individual Firm under Competition
The supply chain network performance/efficiency measure, $\mathcal{E}_i(G_i)$*, for the supply chain network topology of a given firm i,* G_i*, under competition and the equilibrium demand vector* d^**, is defined as:*

$$\mathcal{E}_i = \mathcal{E}_i(G_i) = \frac{\sum_{k=1}^{n_R} \frac{d^*_{ik}}{\rho_{ik}(d^*)}}{n_R}, \quad i = 1, \ldots, I. \tag{9.25}$$

9.4.1 The Importance of Supply Chain Network Suppliers and Their Components

With our supply chain network performance/efficiency measures, we are ready to investigate the importance of suppliers and their components, which correspond to nodes in our supply chain, for the entire competitive supply chain network and for each individual firm under competition. The importance is determined by studying the impact of the suppliers and the components on the supply chain efficiency through their removal.

We define the importance of a supplier for the entire competitive supply chain network as follows:

Definition 9.6: Importance of a Supplier for the Entire Competitive Supply Chain Network *G*
The importance of a supplier j, corresponding to a supplier node $j \in G$*,* $I(j)$*, for the entire competitive supply chain network, is measured by the relative supply chain network efficiency drop after j is removed from the entire supply chain:*

$$I(j) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G) - \mathcal{E}(G-j)}{\mathcal{E}(G)}, \quad j = 1, \ldots, n_S, \tag{9.26}$$

where $G - j$ *is the resulting supply chain after supplier j is removed from the competitive supply chain network G.*

The upper bound of the importance of a supplier is 1. The higher the value, the more important a supplier is to the supply chain.

We also can construct using an adaptation of (9.26) a robustness-type measure for the entire competitive supply chain by evaluating how the supply chain is impacted if *all* the suppliers are eliminated due to a major disruption. One may recall the triple disaster in Fukushima, Japan in March 2011 as an illustration of such an event. Specifically, we let:

$$I(\sum_{j=1}^{n_S} j) = \frac{\triangle \mathscr{E}}{\mathscr{E}} = \frac{\mathscr{E}(G) - \mathscr{E}(G - \sum_{j=1}^{n_S} j)}{\mathscr{E}(G)} \quad (9.27)$$

measure how the entire supply chain responds if all of its suppliers are unavailable.

The importance of a supplier for the supply chain network of an individual firm under competition is defined as follows:

Definition 9.7: Importance of a Supplier for the Supply Chain Network of an Individual Firm under Competition
The importance of a supplier j, corresponding to a supplier node $j \in G_i$, $I_i(j)$, for the supply chain network of a given firm i under competition, is measured by the relative supply chain network efficiency drop after j is removed from G_i:

$$I_i(j) = \frac{\triangle \mathscr{E}_i}{\mathscr{E}_i} = \frac{\mathscr{E}_i(G_i) - \mathscr{E}_i(G_i - j)}{\mathscr{E}_i(G_i)}, \quad i = 1, \ldots, I; j = 1, \ldots, n_S. \quad (9.28)$$

The corresponding robustness measure for the supply chain of firm i if all the suppliers are eliminated is:

$$I_i(\sum_{j=1}^{n_S} j) = \frac{\triangle \mathscr{E}_i}{\mathscr{E}_i} = \frac{\mathscr{E}_i(G_i) - \mathscr{E}_i(G_i - \sum_{j=1}^{n_S} j)}{\mathscr{E}_i(G_i)}, \quad i = 1, \ldots, I. \quad (9.29)$$

In addition, we define the importance of a supplier's component for the entire competitive supply chain network as follows:

Definition 9.8: Importance of a Supplier's Component for the Entire Competitive Supply Chain Network G
The importance of a supplier j's component l_j; $l_j = 1_j, \ldots, n_{lj}$, corresponding to j's component node $l_j \in G$, $I(l_j)$, for the entire competitive supply chain network, is measured by the relative supply chain network efficiency drop after l_j is removed from G:

$$I(l_j) = \frac{\triangle \mathscr{E}}{\mathscr{E}} = \frac{\mathscr{E}(G) - \mathscr{E}(G - l_j)}{\mathscr{E}(G)}, \quad j = 1, \ldots, n_S; l_j = 1_j, \ldots, n_{lj}. \quad (9.30)$$

where $G - l_j$ is the resulting supply chain after supplier j's component l_j is removed from the entire competitive supply chain network.

The corresponding robustness measure for the entire competitive supply chain network if all suppliers' component l_j; $l_j = 1_j, \ldots, n_{lj}$, are eliminated is:

$$I(\sum_{j=1}^{n_S} l_j) = \frac{\triangle \mathscr{E}}{\mathscr{E}} = \frac{\mathscr{E}(G) - \mathscr{E}(G - \sum_{j=1}^{n_S} l_j)}{\mathscr{E}(G)}, \quad l_j = 1_j, \ldots, n_{lj}. \quad (9.31)$$

The importance of a supplier's component for the supply chain network of an individual firm is defined as:

Definition 9.9: Importance of a Supplier's Component for the Supply Chain Network of an Individual Firm under Competition
The importance of supplier j's component l_j; $l_j = 1_j, \ldots, n_{lj}$, corresponding to a component node $l_j \in G_i$, $I_i(l_j)$, for the supply chain network of a given firm i under competition, is measured by the relative supply chain network efficiency drop after l_j is removed from G_i:

$$I_i(l_j) = \frac{\triangle \mathscr{E}_i}{\mathscr{E}_i} = \frac{\mathscr{E}_i(G_i) - \mathscr{E}_i(G_i - l_j)}{\mathscr{E}_i(G_i)}, \quad i = 1, \ldots, I; j = 1, \ldots, n_S; l_j = 1_j, \ldots, n_{lj}. \tag{9.32}$$

The corresponding robustness measure for the supply chain network of firm i if all suppliers' component l_j, $l_j = 1_j, \ldots, n_{lj}$, are eliminated is:

$$I_i(\sum_{j=1}^{n_S} l_j) = \frac{\triangle \mathscr{E}_i}{\mathscr{E}_i} = \frac{\mathscr{E}_i(G_i) - \mathscr{E}_i(G_i - \sum_{j=1}^{n_S} l_j)}{\mathscr{E}_i(G_i)}, \quad i = 1, \ldots, I; l_j = 1_j, \ldots, n_{lj}. \tag{9.33}$$

Note that, in removing a supplier node, we also remove all the links emanating from the node and the subsequent component nodes and links. Similarly, in removing a component node of a supplier, we remove from the supply chain network topology that node and the links that emanate to and from the node.

9.5 The Algorithm

We employ the modified projection method for the computation of the solution for the multitiered supply chain network game theory model with suppliers (cf. Sect. 2.6.2). The realization of the modified projection method for our model results in the following explicit formulae.

Explicit Formulae for the Modified Projection Method Applied to the Multitiered Supply Chain Network Game Theory Model with Suppliers
At each iteration τ of the modified projection method, variational inequality (2.37) yields explicit formulae for the computation of the product shipment, in-house component production, and contracted component production pattern, the Lagrange multipliers, and the prices charged by the suppliers, as the following:

for the product shipments: for $i = 1, \ldots, I$; $k = 1, \ldots, n_R$:

$$\bar{Q}_{ik}^{\tau-1} = \max\{0, Q_{ik}^{\tau-1} + a(-\frac{\partial \hat{f}_i(Q^{\tau-1})}{\partial Q_{ik}} - \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^{\tau-1})}{\partial Q_{ik}} + \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^{\tau-1})}{\partial Q_{ik}} Q_{ih}^{\tau-1}$$
$$+ \hat{\rho}_{ik}(Q^{\tau-1}) - \sum_{l=1}^{n_{l^i}} \lambda_{il}^{\tau-1} \theta_{il})\}; \tag{9.34a}$$

for the in-house component production pattern: for $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$\bar{Q}_{il}^{F^{\tau-1}} = \min\{CAP_{il}^F, \max\{0, Q_{il}^{F^{\tau-1}} + a(-\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^{F^{\tau-1}})}{\partial Q_{il}^F} + \lambda_{il}^{\tau-1})\}\}; \tag{9.34b}$$

for the contracted component production pattern: for $j = 1, \ldots, n_S$; $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$\bar{Q}_{jil}^{S^{\tau-1}} = \min\{CAP_{jil}^S, \max\{0, Q_{jil}^{S^{\tau-1}} + a(-\pi_{jil}^{\tau-1} - \sum_{g=1}^{n_S}\sum_{m=1}^{n_{l^i}} \frac{\partial tc_{igm}(Q^{S^{\tau-1}})}{\partial Q_{jil}^S} + \lambda_{il}^{\tau-1})\}\}, \tag{9.34c}$$

and for the Lagrange multipliers: for $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$\bar{\lambda}_{il}^{\tau-1} = \max\{0, \lambda_{il}^{\tau-1} + a(-\sum_{j=1}^{n_S} Q_{jil}^{S^{\tau-1}} - Q_{il}^{F^{\tau-1}} + \sum_{k=1}^{n_R} Q_{ik}^{\tau-1}\theta_{il})\}. \tag{9.34d}$$

Also, the following closed form expressions are for the prices charged by the suppliers: for $j = 1, \ldots, n_S$; $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$\bar{\pi}_{jil}^{\tau-1} = \max\{0, \pi_{jil}^{\tau-1} + a(-\frac{\partial oc_j(\pi^{\tau-1})}{\partial \pi_{jil}} + Q_{jil}^{S^{\tau-1}})\}. \tag{9.34e}$$

In addition, at each iteration τ of the modified projection method, the explicit formulae yielded from variational inequality (2.38) are:

$$Q_{ik}^{\tau} = \max\{0, Q_{ik}^{\tau-1} + a(-\frac{\partial \hat{f}_i(\bar{Q}^{\tau-1})}{\partial Q_{ik}} - \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(\bar{Q}^{\tau-1})}{\partial Q_{ik}} + \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(\bar{Q}^{\tau-1})}{\partial Q_{ik}} \bar{Q}_{ih}^{\tau-1}$$

$$+ \hat{\rho}_{ik}(\bar{Q}^{\tau-1}) - \sum_{l=1}^{n_{l^i}} \bar{\lambda}_{il}^{\tau-1}\theta_{il})\}; i = 1, \ldots, I; k = 1, \ldots, n_R; \tag{9.35a}$$

$$Q_{il}^{F^{\tau}} = \min\{CAP_{il}^F, \max\{0, Q_{il}^{F^{\tau-1}} + a(-\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(\bar{Q}^{F^{\tau-1}})}{\partial Q_{il}^F} + \bar{\lambda}_{il}^{\tau-1})\}\};$$

$$i = 1, \ldots, I; l = 1, \ldots, n_{l^i}; \tag{9.35b}$$

$$Q_{jil}^{S^{\tau}} = \min\{CAP_{jil}^S, \max\{0, Q_{jil}^{S^{\tau-1}} + a(-\bar{\pi}_{jil}^{\tau-1} - \sum_{g=1}^{n_S}\sum_{m=1}^{n_{l^i}} \frac{\partial tc_{igm}(\bar{Q}^{S^{\tau-1}})}{\partial Q_{jil}^S} + \bar{\lambda}_{il}^{\tau-1})\}\};$$

$$j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}; \tag{9.35c}$$

$$\lambda_{il}^{\tau} = \max\{0, \lambda_{il}^{\tau-1} + a(-\sum_{j=1}^{n_S} \bar{Q}_{jil}^{S^{\tau-1}} - \bar{Q}_{il}^{F^{\tau-1}} + \sum_{k=1}^{n_R} \bar{Q}_{ik}^{\tau-1}\theta_{il})\};$$

$$i = 1, \ldots, I; l = 1, \ldots, n_{l^i}; \tag{9.35d}$$

$$\pi_{jil}^{\tau-1} = \max\{0, \pi_{jil}^{\tau-1} + a(-\frac{\partial oc_j(\bar{\pi}^{\tau-1})}{\partial \pi_{jil}} + \bar{Q}_{jil}^{S^{\tau-1}})\};$$

$$j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}. \tag{9.35e}$$

Theorem 9.7: Convergence
If Assumption 9.1 is satisfied, and the function $F(X)$ is monotone and Lipschitz continuous, then the modified projection method described above converges to the solution of variational inequality (9.17).

Proof: According to Theorem 2.19, the modified projection method converges to the solution of the variational inequality problem of the form (9.17), provided that the function F that enters the variational inequality is monotone (cf. Definition 2.3) and Lipschitz continuous (cf. (9.23)) and that a solution exists. Equation (9.18d) is monotone in each λ_{il}, and since all cost functions are convex and continuously differentiable and each demand price function is monotonically decreasing in its demand, the functions (9.18a), (9.18b), (9.18c) and (9.18e) are monotone in associated variables. Existence of a solution follows from Theorem 9.4 and Lipschitz continuity follows from Theorem 9.6. □

As we mentioned, the monotonicity of $F(X)$ is necessary for the convergence of the modified projection method. Nevertheless, as discussed in Chaps. 3, 4, 5, 6, 7 and 8, for the convergence of the Euler method, at least strict monotonicity of $F(X)$ is needed. This is not hard to be achieved for the product shipment, in-house component production, and contracted component production pattern, and the prices charged by the suppliers, since it is reasonable for the cost functions to be strictly monotone and the demand price functions to be strictly decreasing in demands. However, the strict monotonicity of the function (9.18d) is difficult to guarantee. Therefore, the modified projection method is used in this chapter in order to ensure convergence.

9.6 Numerical Examples

In this section, we present numerical supply chain network examples with suppliers, which we solve via the modified projection method, as described in the preceding section. We implemented the modified projection method using Matlab on a Lenovo Z580. The convergence tolerance is 10^{-6}, so that the algorithm is deemed to have converged when the absolute value of the difference between each successive quantities, prices, and Lagrange multipliers is less than or equal to 10^{-6}. a is set to 0.05. We initialize the algorithm by setting the product and component quantities equal to 50.00 and the prices and the Lagrange multipliers equal to 0.00.

Example 9.1

The supply chain network topology of Example 9.1 is given in Fig. 9.2. There are two firms serving demand markets R_1 and R_2. The firms procure the components of their products from supplier 1. They also have the option of producing the components needed by themselves.

The product of firm 1 requires two components, which are 1^1 and 2^1. Two units of component 1^1 and 3 units of component 2^1 are needed for producing one unit of firm 1's product. The product of firm 2 requires two components, 1^2 and 2^2. To produce one unit of firm 2's product, 2 units of component 1^2 and 2 units of component 2^2 are needed. Therefore,

$$\theta_{11} = 2, \quad \theta_{12} = 3, \quad \theta_{21} = 2, \quad \theta_{22} = 2.$$

Components 1^1 and 1^2 are the same component, which corresponds to node 1 in the second tier in Fig. 9.2 below. Components 2^1 and 2^2 correspond to nodes 2 and 3, respectively.

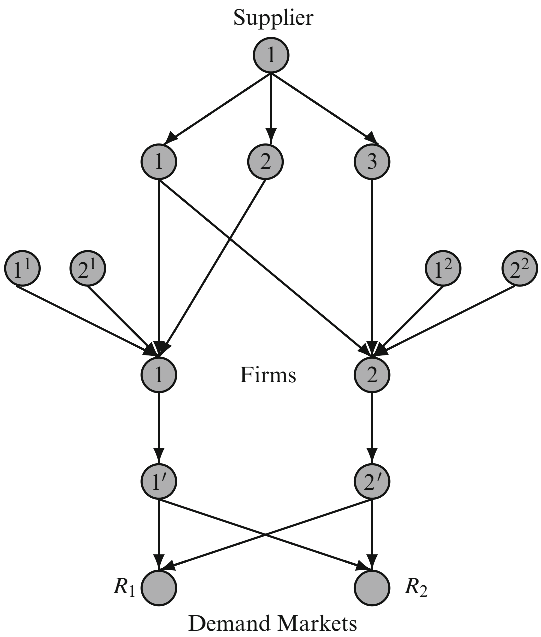

Fig. 9.2 The supply chain network topology for Example 9.1

The data are as follows.

The capacities of the suppliers are:

$$CAP^S_{111} = 80, \quad CAP^S_{112} = 90, \quad CAP^S_{121} = 80, \quad CAP^S_{122} = 50,$$

Thus, supplier 1 is capable of producing components 1^1, 2^1, 1^2, and 2^2 for the firms.

The firms are not capable of producing components 1^1 or 1^2, so their capacities are:

$$CAP^F_{11} = 0, \quad CAP^F_{12} = 20, \quad CAP^F_{21} = 0, \quad CAP^F_{22} = 30.$$

The supplier's production costs are:

$$\hat{f}^S_{11}(Q^S_{111}, Q^S_{121}) = 2(Q^S_{111} + Q^S_{121}), \quad \hat{f}^S_{12}(Q^S_{112}) = 3Q^S_{112}, \quad f^S_{13}(Q^S_{122}) = Q^S_{122}.$$

The supplier's transportation costs are:

$$\hat{c}^S_{111}(Q^S_{111}, Q^S_{112}) = 0.75Q^S_{111} + 0.1Q^S_{112}, \quad \hat{c}^S_{112}(Q^S_{112}, Q^S_{111}) = 0.1Q^S_{112} + 0.05Q^S_{111},$$
$$\hat{c}^S_{121}(Q^S_{121}, Q^S_{122}) = Q^S_{121} + 0.2Q^S_{122}, \quad \hat{c}^S_{122}(Q^S_{122}, Q^S_{121}) = 0.6Q^S_{122} + 0.25Q^S_{121}.$$

The opportunity cost of the supplier is:

$$oc_1(\pi_{111}, \pi_{112}, \pi_{121}, \pi_{122})$$
$$= 0.5(\pi_{111} - 10)^2 + (\pi_{112} - 5)^2 + 0.5(\pi_{121} - 10)^2 + 0.75(\pi_{122} - 7)^2.$$

The firms' assembly costs are:

$$\hat{f}_1(Q_{11}, Q_{12}, Q_{21}, Q_{22})$$
$$= 2(Q_{11} + Q_{12})^2 + 2(Q_{11} + Q_{12}) + (Q_{11} + Q_{12})(Q_{21} + Q_{22}),$$
$$\hat{f}_2(Q_{11}, Q_{12}, Q_{21}, Q_{22})$$
$$= 1.5(Q_{21} + Q_{22})^2 + 2(Q_{21} + Q_{22}) + (Q_{11} + Q_{12})(Q_{21} + Q_{22}).$$

The firms' production costs for producing their components are:

$$\hat{f}^F_{11}(Q^F_{11}, Q^F_{21}) = 3Q^{F^2}_{11} + Q^F_{11} + 0.5Q^F_{11}Q^F_{21}, \quad \hat{f}^F_{12}(Q^F_{12}) = 2Q^{F^2}_{12} + 1.5Q^F_{12},$$
$$\hat{f}^F_{21}(Q^F_{11}, Q^F_{21}) = 3Q^{F^2}_{21} + 2Q^F_{21} + 0.75Q^F_{11}Q^F_{21}, \quad \hat{f}^F_{22}(Q^F_{22}) = 1.5Q^{F^2}_{22} + Q^F_{22}.$$

The firms' transportation costs for shipping their products to the demand markets are:

$$\hat{c}^F_{11}(Q_{11}, Q_{21}) = Q^2_{11} + Q_{11} + 0.5Q_{11}Q_{21},$$

$$\hat{c}^F_{12}(Q_{12}, Q_{22}) = 2Q^2_{12} + Q_{12} + 0.5Q_{12}Q_{22},$$
$$\hat{c}^F_{21}(Q_{21}, Q_{11}) = 1.5Q^2_{21} + Q_{21} + 0.25Q_{11}Q_{21},$$
$$\hat{c}^F_{22}(Q_{12}, Q_{22}) = Q^2_{22} + 0.5Q_{22} + 0.25Q_{12}Q_{22}.$$

The transaction costs of the firms are:

$$tc_{111}(Q^S_{111}) = 0.5Q^{S^2}_{111} + 0.25Q^S_{111}, \quad tc_{112}(Q^S_{112}) = 0.25Q^{S^2}_{112} + 0.3Q^S_{112},$$
$$tc_{211}(Q^S_{121}) = 0.3Q^{S^2}_{121} + 0.2Q^S_{121}, \quad tc_{212}(Q^S_{122}) = 0.2Q^{S^2}_{122} + 0.1Q^S_{122}.$$

The demand price functions are:

$$\rho_{11}(d_{11}, d_{21}) = -1.5d_{11} - d_{21} + 500, \quad \rho_{12}(d_{12}, d_{22}) = -2d_{12} - d_{22} + 450,$$
$$\rho_{21}(d_{11}, d_{21}) = -2d_{21} - 0.5d_{11} + 500, \quad \rho_{22}(d_{12}, d_{22}) = -2d_{22} - d_{12} + 400.$$

The modified projection method converges in 479 iterations. The equilibrium product shipments are:

$$Q^*_{11} = 13.39, \quad Q^*_{12} = 4.51, \quad Q^*_{21} = 18.62, \quad Q^*_{22} = 5.87.$$

The equilibrium demands are:

$$d^*_{11} = 13.39, \quad d^*_{12} = 4.51, \quad d^*_{21} = 18.62, \quad d^*_{22} = 5.87$$

with the induced demand prices being

$$\rho_{11} = 461.30, \quad \rho_{12} = 435.11, \quad \rho_{21} = 456.07, \quad \rho_{22} = 383.75.$$

The equilibrium in-house component production pattern is:

$$Q^{F^*}_{11} = 0.00, \quad Q^{F^*}_{12} = 11.50, \quad Q^{F^*}_{21} = 0.00, \quad Q^{F^*}_{22} = 14.35.$$

The equilibrium contracted component production pattern is:

$$Q^{S^*}_{111} = 35.78, \quad Q^{S^*}_{112} = 42.18, \quad Q^{S^*}_{121} = 48.99, \quad Q^{S^*}_{122} = 34.64.$$

The equilibrium Lagrange multipliers are:

$$\lambda^*_{11} = 81.82, \quad \lambda^*_{12} = 47.48, \quad \lambda^*_{21} = 88.58 \quad \lambda^*_{22} = 44.05.$$

The equilibrium prices charged by the supplier are:

$$\pi^*_{11} = 45.78, \quad \pi^*_{12} = 26.09, \quad \pi^*_{21} = 58.99, \quad \pi^*_{22} = 30.09.$$

The profits of the firms are, respectively, 2,518.77 and 3,485.51. The profit of the supplier is 3,529.19.

We now apply the supply chain network performance measures and the supplier and component importance indicators presented in Sect. 9.3 to this example.

The supply chain network performance measure $\mathscr{E}(G)$ for the entire competitive supply chain network (cf. (9.24)) for Example 9.1 is:

$$\mathscr{E}(G) = \frac{\frac{d_{11}}{\rho_{11}} + \frac{d_{12}}{\rho_{12}} + \frac{d_{21}}{\rho_{21}} + \frac{d_{22}}{\rho_{22}}}{I \times n_R}$$
$$= \frac{\frac{13.39}{461.30} + \frac{4.51}{435.11} + \frac{18.62}{456.07} + \frac{5.87}{383.75}}{2 \times 2}$$
$$= 0.0239.$$

The supply chain network performance measure for the supply chain network topology of firm 1 (cf. (9.25)) is then given by:

$$\mathscr{E}_1(G_1) = \frac{\frac{d_{11}}{\rho_{11}} + \frac{d_{12}}{\rho_{12}}}{n_R}$$
$$= \frac{\frac{13.39}{461.30} + \frac{4.51}{435.11}}{2}$$
$$= 0.0197,$$

and that of firm 2 is:

$$\mathscr{E}_2(G_2) = \frac{\frac{d_{21}}{\rho_{21}} + \frac{d_{22}}{\rho_{22}}}{n_R}$$
$$= \frac{\frac{18.62}{456.07} + \frac{5.87}{383.75}}{2}$$
$$= 0.0281.$$

Note that, in this example, only supplier 1 is able to produce components 1^1 and 1^2, which is the first component of supplier 1 (i.e., node 1 in the second tier in Fig. 9.2), and neither of the firms can produce these components. Without supplier 1, no products of the firms can be assembled or delivered to the demand markets. Therefore,

$$\mathscr{E}(G-1) = 0, \quad \mathscr{E}_1(G_1-1) = 0, \quad \mathscr{E}_2(G_2-1) = 0.$$

According to (9.26) and (9.28),

$$I(1) = 1, \quad I_1(1) = 1, \quad I_2(1) = 1.$$

Hence, the importance of supplier 1 for the entire competitive supply chain network, for the supply chain of firm 1, and that for the supply chain of firm 2 is 1. Without supplier 1, the supply chain network in Fig. 9.2 will collapse.

In addition, the supply chain network performance for the supply chain without supplier 1's component 1 (i.e., node 1 in the second tier in Fig. 9.2) is:

$$\mathscr{E}(G - 1_1) = 0, \quad \mathscr{E}_1(G_1 - 1_1) = 0, \quad \mathscr{E}_2(G_2 - 1_1) = 0$$

and the importance of supplier 1's component 1 is:

$$I(1_1) = 1, \quad I_1(1_1) = 1, \quad I_2(1_1) = 1.$$

Therefore, supplier 1's component 1 is the most important component compared to its components 2 (i.e., node 2 in the second tier in Fig. 9.2) and 3 (i.e., node 3 in the second tier in Fig. 9.2).

Now suppose that supplier 1's component 2 is removed from Fig. 9.2. The modified projection method converges in 795 iterations to the equilibrium solution in Table 9.2. The profits of the firms are now 1,519.08 and 3,755.89. The profit of the supplier is 2,458.92.

The associated supply chain network performance measure values are now:

$$\mathscr{E}(G - 2_1) = 0.0181, \quad \mathscr{E}_1(G_1 - 2_1) = 0.0071, \quad \mathscr{E}_2(G_2 - 2_1) = 0.0292.$$

After supplier 1's component 3 is removed (cf. Fig. 9.2), the modified projection method converges in 1,020 iterations to the equilibrium solution in Table 9.3.

The profits of the firms are 2,724.82 and 3,043.42, and the profit of the supplier is 2,177.26.

The associated supply chain network performance measure values are now:

$$\mathscr{E}(G - 3_1) = 0.0183, \quad \mathscr{E}_1(G_1 - 3_1) = 0.0203, \quad \mathscr{E}_2(G_2 - 3_1) = 0.0163.$$

We summarize the supply chain network performance measure values in Table 9.4. The importance of supplier 1's components 1, 2, and 3 (cf. (9.30))

Table 9.2 Equilibrium solution and incurred demand prices after the removal of supplier 1's component 2

Q^*	$Q^*_{11} = 6.49$	$Q^*_{121} = 0.17$	$Q^*_{21} = 19.08$	$Q^*_{22} = 6.46$
Q^{F*}	$Q^{F*}_{11} = 0.00$	$Q^{F*}_{12} = 20.00$	$Q^{F*}_{21} = 0.00$	$Q^{F*}_{22} = 14.90$
Q^{S*}	$Q^{S*}_{111} = 13.33$	$Q^{S*}_{121} = 51.08$	$Q^{S*}_{122} = 36.18$	
λ^*	$\lambda^*_{11} = 36.92$	$\lambda^*_{12} = 103.29$	$\lambda^*_{21} = 91.93$	$\lambda^*_{22} = 45.70$
π^*	$\pi^*_{111} = 23.33$	$\pi^*_{121} = 61.08$	$\pi^*_{122} = 31.12$	
d^*	$d^*_{11} = 6.49$	$d^*_{12} = 0.17$	$d^*_{21} = 19.08$	$d^*_{22} = 6.46$
ρ	$\rho_{11} = 471.18$	$\rho_{12} = 443.19$	$\rho_{21} = 458.59$	$\rho_{22} = 386.91$

Table 9.3 Equilibrium solution and incurred demand prices after the removal of supplier 1's component 3

Q^*	$Q^*_{11} = 13.75$	$Q^*_{121} = 4.88$	$Q^*_{21} = 14.25$	$Q^*_{22} = 0.75$
Q^{F^*}	$Q^{F^*}_{11} = 0.00$	$Q^{F^*}_{12} = 11.94$	$Q^{F^*}_{21} = 0.00$	$Q^{F^*}_{22} = 30.00$
Q^{S^*}	$Q^{S^*}_{111} = 37.26$	$Q^{S^*}_{112} = 43.96$	$Q^{S^*}_{121} = 30.00$	
λ^*	$\lambda^*_{11} = 84.78$	$\lambda^*_{12} = 49.26$	$\lambda^*_{21} = 58.20$	$\lambda^*_{22} = 103.44$
π^*	$\pi^*_{111} = 47.26$	$\pi^*_{112} = 26.98$	$\pi^*_{121} = 40.00$	
d^*	$d^*_{11} = 13.75$	$d^*_{12} = 4.88$	$d^*_{21} = 14.25$	$d^*_{22} = 0.75$
ρ	$\rho_{11} = 465.12$	$\rho_{12} = 439.50$	$\rho_{21} = 464.62$	$\rho_{22} = 393.63$

Table 9.4 Supply chain network performance measure values for Example 9.1

	$\mathscr{E}(G)$	$\mathscr{E}(G-1)$	$\mathscr{E}(G-1_1)$	$\mathscr{E}(G-2_1)$	$\mathscr{E}(G-3_1)$
Entire supply chain	0.0239	0	0	0.0181	0.0183
	$\mathscr{E}_i(G_i)$	$\mathscr{E}_i(G_i-1)$	$\mathscr{E}_i(G_i-1_1)$	$\mathscr{E}_i(G_i-2_1)$	$\mathscr{E}_i(G_i-3_1)$
Firm 1's supply chain	0.0197	0	0	0.0071	0.0203
Firm 2's supply chain	0.0281	0	0	0.0292	0.0163

and (9.32)) and their rankings, not only for the entire supply chain network but also for each firm's supply chain, are reported in Table 9.5.

Because supplier 1's component 2 is produced exclusively for firm 1, it is more important for firm 1 than supplier 1's component 3, but not as important as component 1. After removing it from the supply chain, firm 1's profit decreases, but firm 1's competitor, firm 2's profit, increases because of competition. The supply chain performance of firm 2's supply chain also increases after the removal. In addition, component 2 is most important for firm 1 than for firm 2 and for the entire supply chain network.

For a similar reason, since supplier 1's component 3 is made exclusively for firm 2, it is more important than supplier 1's component 2 for firm 2. After dropping component 3 from the supply chain, firm 2's profit decreases and its competitor's, firm 1's, profit increases. The supply chain performance of firm 1's supply chain also increases. Component 3 is most important for firm 2 than for firm 1 and for the entire supply chain.

Example 9.2

Example 9.2 is the same as Example 9.1 except that supplier 1 is no longer the only entity that can produce components 1^1 and 1^2. Both firms recover their capacities for producing components 1^1 and 1^2 and, hence, they are raised from 0 to 20. The capacities of the firms are now:

$$CAP^F_{11} = 20, \quad CAP^F_{12} = 20, \quad CAP^F_{21} = 20, \quad CAP^F_{22} = 30.$$

Table 9.5 Importance and rankings of supplier 1's components 1, 2, and 3 for Example 9.1

	Importance for the entire supply chain	Ranking	Importance for firm 1's supply chain	Ranking	Importance for firm 2's supply chain	Ranking
Supplier 1	1		1		1	
Component 1	1	1	1	1	1	1
Component 2	0.2412	2	0.6401	2	−0.0387	3
Component 3	0.2331	3	−0.0329	3	0.4197	2

	Importance for the entire supply chain	Importance for firm 1's supply chain	Importance for firm 2's supply chain
Supplier 1	1	1	1
Ranking	1	1	1
Component 1	1	1	1
Ranking	1	1	1
Component 2	0.2412	0.6401	−0.0387
Ranking	2	1	3
Component 3	0.2331	−0.0329	0.4197
Ranking	2	3	1

Table 9.6 Equilibrium solution and incurred demand prices for Example 9.2

Q^*	$Q^*_{11} = 14.43$	$Q^*_{121} = 5.13$	$Q^*_{21} = 19.60$	$Q^*_{22} = 7.02$
Q^{F^*}	$Q^{F^*}_{11} = 10.23$	$Q^{F^*}_{12} = 12.50$	$Q^{F^*}_{21} = 11.28$	$Q^{F^*}_{22} = 15.47$
Q^{S^*}	$Q^{S^*}_{111} = 28.89$	$Q^{S^*}_{112} = 46.19$	$Q^{S^*}_{121} = 41.97$	$Q^{S^*}_{122} = 37.78$
λ^*	$\lambda^*_{11} = 68.04$	$\lambda^*_{12} = 51.49$	$\lambda^*_{21} = 77.35$	$\lambda^*_{22} = 47.40$
π^*	$\pi^*_{111} = 38.89$	$\pi^*_{112} = 28.10$	$\pi^*_{121} = 51.97$	$\pi^*_{122} = 32.19$
d^*	$d^*_{11} = 14.43$	$d^*_{12} = 5.13$	$d^*_{21} = 19.60$	$d^*_{22} = 7.02$
ρ	$\rho_{11} = 458.75$	$\rho_{12} = 432.72$	$\rho_{21} = 453.58$	$\rho_{22} = 380.83$

Table 9.7 Supply chain network performance measure values for Example 9.2

	$\mathcal{E}(G)$	$\mathcal{E}(G-1)$	$\mathcal{E}(G-1_1)$	$\mathcal{E}(G-2_1)$	$\mathcal{E}(G-3_1)$
Entire supply chain	0.0262	0.0086	0.0105	0.0197	0.0195
	$\mathcal{E}_i(G_i)$	$\mathcal{E}_i(G_i-1)$	$\mathcal{E}_i(G_i-1_1)$	$\mathcal{E}_i(G_i-2_1)$	$\mathcal{E}_i(G_i-3_1)$
Firm 1's supply chain	0.0217	0.0067	0.0106	0.0071	0.0226
Firm 2's supply chain	0.0308	0.0105	0.0105	0.0324	0.0163

The modified projection method converges in 408 iterations. The equilibrium solution is presented in Table 9.6.

The profits of the firms are now 2,968.88 and 4,110.89. The profit of the supplier is now 3,078.45. With recovered capacities, the profits of the firms increase, but that of the supplier decreases, compared to the corresponding values in Example 9.1. If there are costs for capacity increment for each firm and if the costs are less than the associated profit increment it is profitable for firms to recover their capacities and to produce more components in-house. If not, purchasing from the supplier would be a wise choice. In Example 9.2, the demand prices decrease due to more demand.

The supply chain network performance measure values and the importance of supplier 1's components 1, 2, and 3 and their rankings are reported as in Tables 9.7 and 9.8.

With firms' recovered capacities for producing components 1^1 and 1^2, supplier 1's component 1 is still the most important component for the entire supply chain network and for firm 2, compared to the other components. However, for firm 1's supply chain, component 2 is now the most important component.

In addition, supplier 1 is now most important for firm 1. Therefore, in the case of a disruption on the supplier's side, firm 1's supply chain will be affected the most. Moreover, components 1 and 3 are most important for firm 2, and component 2 is most important for firm 1.

Table 9.8 Importance and rankings of supplier 1 and its components 1, 2, and 3 for Example 9.2

	Importance for the entire supply chain	Ranking	Importance for firm 1's supply chain	Ranking	Importance for firm 2's supply chain	Ranking
Supplier 1	0.6721		0.6897		0.6598	
Component 1	0.5984	1	0.5121	2	0.6590	1
Component 2	0.2476	3	0.6721	1	−0.0505	3
Component 3	0.2586	2	−0.0438	3	0.4710	2

	Importance for the entire supply chain	Importance for firm 1's supply chain	Importance for firm 2's supply chain
Supplier 1	0.6721	0.6897	0.6598
Ranking	2	1	3
Component 1	0.5984	0.5121	0.6590
Ranking	2	3	1
Component 2	0.2476	0.6721	−0.0505
Ranking	2	1	3
Component 3	0.2586	−0.0438	0.4710
Ranking	2	3	1

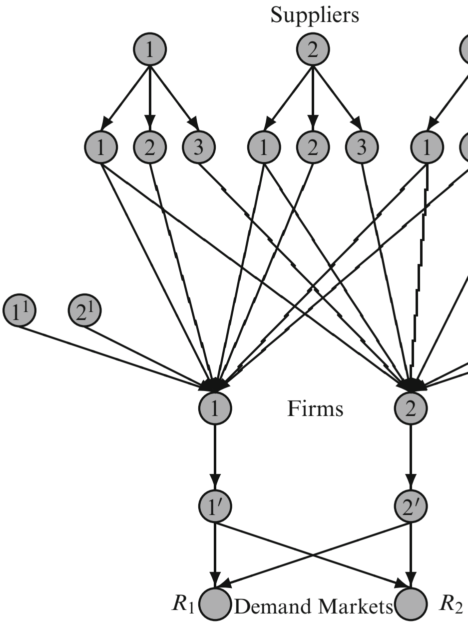

Fig. 9.3 The supply chain network topology for Example 9.3

Example 9.3
Example 9.3 is the same as Example 9.2, except that two more suppliers are now available to the firms in addition to supplier 1. The supply chain network topology of Example 9.3 is given in Fig. 9.3.

The data associated with suppliers 2 and 3 are following.
The capacities of suppliers 2 and 3 are:

$$CAP^S_{211} = 60, \quad CAP^S_{212} = 70, \quad CAP^S_{221} = 50, \quad CAP^S_{222} = 60,$$
$$CAP^S_{311} = 50, \quad CAP^S_{312} = 80, \quad CAP^S_{321} = 80, \quad CAP^S_{322} = 60.$$

Suppliers 2 and 3 are capable of providing components 1^1, 2^1, 1^2, and 2^2 for the firms.

The production costs of the suppliers are:

$$\hat{f}^S_{21}(Q^S_{211}, Q^S_{221}) = Q^S_{211} + Q^S_{221}, \quad \hat{f}^S_{22}(Q^S_{212}) = 3Q^S_{212}, \quad \hat{f}^S_{23}(Q^S_{222}) = 2Q^S_{222},$$
$$\hat{f}^S_{31}(Q^S_{311}, Q^S_{321}) = 10(Q^S_{311} + Q^S_{321}), \quad \hat{f}^S_{32}(Q^S_{312}) = Q^S_{312}, \quad \hat{f}^S_{33}(Q^S_{322}) = 2.5Q^S_{322}.$$

The transportation costs are:

$$\hat{c}^S_{211}(Q^S_{211}, Q^S_{212}) = 0.5Q^S_{211} + 0.2Q^S_{212}, \quad \hat{c}^S_{212}(Q^S_{212}, Q^S_{211}) = 0.3Q^S_{212} + 0.1Q^S_{211},$$

$$\hat{c}^S_{221}(Q^S_{221}, Q^S_{222}) = 0.8Q^S_{221} + 0.2Q^S_{222}, \quad \hat{c}^S_{222}(Q^S_{222}, Q^S_{221}) = 0.75Q^S_{222} + 0.1Q^S_{221},$$
$$\hat{c}^S_{311}(Q^S_{311}, Q^S_{312}) = 0.4Q^S_{311} + 0.05Q^S_{312}, \quad \hat{c}^S_{312}(Q^S_{312}, Q^S_{311}) = 0.4Q^S_{312} + 0.2Q^S_{311},$$
$$\hat{c}^S_{321}(Q^S_{321}, Q^S_{322}) = 0.7Q^S_{321} + 0.1Q^S_{322}, \quad \hat{c}^S_{322}(Q^S_{322}, Q^S_{321}) = 0.6Q^S_{322} + 0.1Q^S_{321}.$$

The opportunity costs are:

$$oc_2(\pi_{211}, \pi_{212}, \pi_{221}, \pi_{222}) =$$
$$(\pi_{211} - 6)^2 + 0.75(\pi_{212} - 5)^2 + 0.3(\pi_{221} - 8)^2 + 0.5(\pi_{222} - 4)^2,$$
$$oc_3(\pi_{311}, \pi_{312}, \pi_{321}, \pi_{322})$$
$$= 0.5(\pi_{311} - 5)^2 + 1.5(\pi_{312} - 5)^2 + 0.5(\pi_{321} - 3)^2 + 0.5(\pi_{322} - 4)^2.$$

The transaction costs of the firms now become:

$$tc_{121}(Q^S_{211}) = 0.5Q^{S^2}_{211} + Q^S_{211}, \quad tc_{122}(Q^S_{212}) = 0.25Q^{S^2}_{212} + 0.3Q^S_{212},$$
$$tc_{221}(Q^S_{221}) = Q^{S^2}_{221} + 0.1Q^S_{221}, \quad tc_{222}(Q^S_{222}) = Q^{S^2}_{222} + 0.5Q^S_{222},$$
$$tc_{131}(Q^S_{311}) = 0.2Q^{S^2}_{311} + 0.3Q^S_{311}, \quad tc_{132}(Q^S_{312}) = 0.5Q^{S^2}_{312} + 0.2Q^S_{312},$$
$$tc_{231}(Q^S_{321}) = 0.1Q^{S^2}_{321} + 0.1Q^S_{321}, \quad tc_{232}(Q^S_{322}) = 0.5Q^{S^2}_{322} + 0.1Q^S_{322}.$$

The rest of the data for firms 1 and 2 and the demand price functions are the same as in Example 9.2.

The modified projection method converges in 421 iterations and achieves the equilibrium solution shown in Table 9.9.

The profits of the firms are now 4,968.67 and 5,758.13. The profits of the suppliers are 1,375.22, 725.17, and 837.44, respectively. With more competition

Table 9.9 Equilibrium solution and incurred demand prices for Example 9.3

Q^*	$Q^*_{11} = 21.82$	$Q^*_{12} = 9.61$	$Q^*_{21} = 24.23$	$Q^*_{22} = 12.41$
Q^{F*}	$Q^{F*}_{11} = 5.57$	$Q^{F*}_{12} = 9.11$	$Q^{F*}_{21} = 6.48$	$Q^{F*}_{22} = 12.94$
Q^{S*}	$Q^{S*}_{111} = 13.71$	$Q^{S*}_{112} = 32.64$	$Q^{S*}_{121} = 21.77$	$Q^{S*}_{122} = 30.68$
	$Q^{S*}_{211} = 20.45$	$Q^{S*}_{212} = 27.98$	$Q^{S*}_{221} = 10.07$	$Q^{S*}_{222} = 11.78$
	$Q^{S*}_{311} = 23.13$	$Q^{S*}_{312} = 24.56$	$Q^{S*}_{321} = 34.94$	$Q^{S*}_{322} = 17.86$
λ^*	$\lambda^*_{11} = 37.68$	$\lambda^*_{12} = 37.94$	$\lambda^*_{21} = 45.03$	$\lambda^*_{22} = 39.83$
π^*	$\pi^*_{111} = 23.71$	$\pi^*_{112} = 21.32$	$\pi^*_{121} = 31.77$	$\pi^*_{122} = 27.45$
	$\pi^*_{211} = 16.23$	$\pi^*_{212} = 23.65$	$\pi^*_{221} = 24.79$	$\pi^*_{222} = 15.78$
	$\pi^*_{311} = 28.13$	$\pi^*_{312} = 13.19$	$\pi^*_{321} = 37.94$	$\pi^*_{322} = 21.86$
d^*	$d^*_{11} = 21.82$	$d^*_{12} = 9.61$	$d^*_{21} = 24.23$	$d^*_{22} = 12.41$
ρ	$\rho_{11} = 443.04$	$\rho_{12} = 418.38$	$\rho_{21} = 440.64$	$\rho_{22} = 365.58$

Table 9.10 Supply chain network performance measure values for Example 9.3

	$\mathscr{E}(G)$	$\mathscr{E}(G-1)$	$\mathscr{E}(G-2)$	$\mathscr{E}(G-3)$	$\mathscr{E}(G-\sum_{j=1}^{ns} j)$
Entire supply chain	0.0403	0.0334	0.0361	0.0332	0.0086
	$\mathscr{E}_i(G_i)$	$\mathscr{E}_i(G_i-1)$	$\mathscr{E}_i(G_i-2)$	$\mathscr{E}_i(G_i-3)$	$\mathscr{E}_i(G_i-\sum_{j=1}^{ns} j)$
Firm 1's supply chain	0.0361	0.0309	0.0303	0.0309	0.0067
Firm 2's supply chain	0.0445	0.0358	0.0419	0.0355	0.0105

on the supplier's side, the prices of supplier 1 decrease and its profit also decreases, compared to the values in Example 9.2. However, the profits of the firms increase. In addition, with more products made, the prices at the demand markets decrease.

The supply chain network performance measure values and the importance of the suppliers are reported in Tables 9.10 and 9.11.

As shown in Table 9.11, supplier 2 is the most important supplier for firm 1's supply chain. Supplier 3 is the most important supplier for firm 2 and the entire supply chain network, compared to the other suppliers. In addition, suppliers 1 and 3 are most important for firm 2. Supplier 2 is most important for firm 1's supply chain.

The group of suppliers consisting of suppliers 1, 2, and 3 is most important for firm 1. If a major disaster occurs and all the suppliers are unavailable to the firms, firm 1's supply chain will be affected the most.

9.7 Summary and Conclusions

Supply chains provide the critical infrastructure for the production and distribution of products around the globe. In the case of many products from simple ones to high tech ones, components that comprise the product are produced by suppliers and then assembled by firms. Hence, the behavior of both suppliers and firms needs to be captured in order to be able to assess supply chain network performance and vulnerabilities.

In this chapter, we propose a new multitiered model consisting of competing firms, who can procure components for their products, which are represented by brands, from suppliers or can make them, as appropriate, in-house. The firms compete in terms of quantities whereas the suppliers compete in terms of prices charged for the components. The optimizing behavior of the decision-makers is captured and a unified variational inequality constructed, whose solution yields the equilibrium quantities of the components, produced in-house and/or contracted for, the prices charged by the suppliers, as well as the Lagrange multipliers associated with the capacities. Qualitative properties of the solution are also discussed.

Table 9.11 Importance and rankings of suppliers for Example 9.3

	Importance for the entire supply chain	Ranking	Importance for firm 1's supply chain	Ranking	Importance for firm 2's supply chain	Ranking
Supplier 1	0.1717	2	0.1443	2	0.1939	2
Supplier 2	0.1035	3	0.1612	1	0.0566	3
Supplier 3	0.1760	1	0.1438	3	0.2021	1
All suppliers	0.7864		0.8139		0.7641	

	Importance for the entire supply chain	Importance for firm 1's supply chain	Importance for firm 2's supply chain
Supplier 1	0.1717	0.1443	0.1939
Ranking	2	3	1
Supplier 2	0.1035	0.1612	0.0566
Ranking	2	1	3
Supplier 3	0.1760	0.1438	0.2021
Ranking	2	3	1
All suppliers	0.7864	0.8139	0.7641
Ranking	2	1	3

The model is then used as the setting for the introduction of supply chain network performance measures for the entire supply chain network economy consisting of all the firms as well as for that of an individual firm. Importance indicators are constructed that allow for the ranking of suppliers for the entire supply chain or that of an individual firm, as well as for the supplier components. This rigorous methodology can be used for planning purposes and for investment purposes. Moreover, it can be utilized as a tool for regulators since information about both individual firms as well as the entire supply chain network is revealed.

The model and the performance measures are illustrated through a series of examples, the solutions of which are computed using a proposed algorithm with nice features for implementation.

The originality of contributions in this chapter are the following: 1. We develop a general multitiered competitive supply chain network equilibrium model with suppliers and firms that includes capacities and constraints to capture the production activities. 2. We construct supply chain network performance measures on the full supply chain and on the individual firm levels that assess the efficiency of the supply chain or firm, respectively, and also allow for the identification and ranking of the importance of suppliers and the components of suppliers with respect to the full supply chain or individual firm.

The supply chain network performance measure is inspired by our work on network performance assessment in a variety of network systems ranging from transportation to the Internet (see Nagurney and Qiang 2008, 2009 and the references therein) as well as in supply chains (cf. Qiang et al. 2009; Cruz 2011; Qiang and Nagurney 2012) but with the addition of the supplier tier, which is the focus here. 3. The performance measure and importance indicators are used to assess the relative importance of components as well as suppliers to both a firm's supply chain network and that of the entire supply chain network economy for that particular application. 4. The framework provides tools for regulators and government policy-makers to identify the importance of both suppliers and the components that they produce to various industries.

9.8 Sources and Notes

This chapter is based on the paper by Li and Nagurney (2015). Here we have updated the notation for consistency with the other modeling chapters in this book. In addition, we utilize the modified projection method in this chapter for the solution of the numerical supply chain network examples, whereas in Li and Nagurney (2015) we applied the Euler method.

Although there has been extensive research on multitiered supply chain network equilibrium problems, beginning with the work of Nagurney et al. (2002); see, e.g., Nagurney et al. (2005), Cruz et al. (2006), Nagurney (2006), Cruz (2008), Qiang et al. (2013), Liu and Nagurney (2013), Qiang (2015), and Toyasaki et al.

(2014), there has been less work done on integrating suppliers and their behavior into general multitiered supply chain network equilibrium frameworks. Also, since there has been a dearth of general supply chain network models with suppliers, the identification of which suppliers are important in a supply chain has received less research attention, although it is a very important issue in practice (see Glendon and Bird 2013). Some examples, nevertheless, include the work of Liu and Nagurney (2009), who developed an integrated supply chain network equilibrium model with fuel suppliers focusing on the electric power industry in New England, and that of Liu and Cruz (2012), who modeled supply chains with credit trade and financial risk. However, those papers did not identify which suppliers or the components that they produce are the most important from a supply chain network efficiency perspective.

As noted in Qiang et al. (2009), most supply disruption studies have focused on a local point of view, in the form of a single-supplier problem (see, e.g., Gupta 1996; Parlar 1997) or a two-supplier problem (see, e.g., Parlar and Perry 1996). Very few research papers have examined supply chain risk management with multiple decision-makers (cf. Tomlin 2006). We believe that it is imperative to formulate and solve supply chains from a system-wide holistic perspective and to include both supplier and firm decision-makers in the supply chain network tiers. Indeed, such an approach has also been argued by Wu et al. (2006).

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Chapter 10
The General Multitiered Supply Chain Model of Quality Competition with Suppliers

Abstract In this chapter, we extend the model in Chap. 9 to include quality competition among suppliers. We develop a general multitiered supply chain network equilibrium model consisting of competing suppliers and competing firms who purchase components for the assembly of their final branded products and, if capacity permits, and it enhances profits, produce their own components. The competitive behavior of each tier of decision-makers is described along with their strategic variables, which include quality of the components and, in the case of the firms, the quality of the assembly process itself. The governing equilibrium conditions of the supply chain network are formulated as a variational inequality and qualitative properties are presented. The algorithm, accompanied with convergence results, is then applied to numerical supply chain network examples, along with sensitivity analysis in which the impacts of capacity disruptions and complete supplier elimination are investigated.

10.1 Introduction

In recent years, there have been numerous examples of finished product failures due to the poor quality of a suppliers' components. For example, as noted in Chap. 8, the toy manufacturer, Mattel, in 2007, recalled 19 million toy cars because of suppliers' lead paint and poorly designed magnets, which could harm children if ingested (Story and Barboza 2007). In 2013, four Japanese car-makers, along with BMW, recalled 3.6 million vehicles because the airbags supplied by Takata Corp., the world's second-largest supplier of airbags, were at risk of rupturing and injuring passengers (Kubota and Klayman 2013). The recalls are still ongoing and have expanded to other companies as well (Tabuchi and Jensen 2014). Most recently, the defective ignition switches in General Motors (GM) vehicles, which were produced by Delphi Automotive in Mexico, have been linked to 13 deaths, due to the fact that the switches could suddenly shut off engines with no warning (Stout et al. 2014; Bomey 2014). In addition, serious quality shortcomings and failures associated with suppliers have also occurred in finished products such as aircraft (Drew 2014), pharmaceuticals (Rao 2014), and also food (Strom 2013; McDonald 2014). In 2009,

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over 400 peanut butter products were recalled after 8 people died and more than 500 people, half of them children, were sickened by salmonella poisoning, the source of which was a peanut butter plant in Georgia (Harris 2009).

Clearly, the quality of a finished/final product depends not only on the quality of the firm that produces and delivers it, but also on the quality of the components provided by the firm's suppliers (Robinson and Malhotra 2005; Foster 2008). It is the suppliers that determine the quality of the materials that they supply as well as the standards of their manufacturing activities. However, suppliers may have less reason to be concerned with quality (cf. Chaps. 7 and 8 and Amaral et al. 2006). In Mattel's case, some of the suppliers were careless, others flouted rules, and others simply avoided obeying the rules (Tang 2008). With non-conforming components, it may be challenging and very difficult for firms to produce high quality finished products even if they utilize the most superior production and transportation delivery techniques.

Furthermore, since suppliers may be located both on-shore and off-shore, supply chain networks of firms may be more vulnerable to disruptions around the globe than ever before. Photos of Honda automobiles under 15 feet of water were some of the most appalling images of the impacts of the Thailand floods of 2011. Asian manufacturing plants affected by the catastrophe were unable to supply components for cars, electronics, and many other products (Kageyama 2011). In the same year, the triple disaster in Fukushima affected far more than the manufacturing industry in Japan. A General Motors plant in Louisiana had to shut down due to a shortage of Japanese-made components after the disaster took place (Lohr 2011). Under such disruptions, suppliers may not even be capable of performing their production tasks, let alone guaranteeing the quality of the components.

In addition, the number of suppliers that a firm may be dealing with can be vast. For example, according to Seetharaman (2013), Ford, the second largest US car manufacturer, had 1,260 suppliers at the end of 2012 with Ford purchasing approximately 80 % of its parts from its largest 100 suppliers. Due to increased demand, many of the suppliers were running "flat out" with the consequence that there were quality issues.

In this chapter, we formulate the supply chain network problem with multiple competing firms and their potential suppliers. The proposed game theory model captures the relationships between firms and suppliers in supply chain networks with quality competition. Along with the general multitiered supply chain network model, we also provide a computational procedure so as not to limit the number of suppliers and competing firms.

Specifically, the potential suppliers may either provide distinct components to the firms or provide the same component, in which case they compete noncooperatively with one another in terms of quality and prices. The firms, in turn, are responsible for assembling the products under their brand names using the components needed and transporting the products to multiple demand markets. They also have the option of producing their own components, if necessary. The firms compete in product quantities, the quality preservation levels of their assembly processes, the contracted component quantities produced by the suppliers, and in in-house component quantities and quality levels. Each of the firms aims to maximize profits. The quality of an end product is determined by the qualities and quality levels of its components, produced both by the firms and the suppliers, the importance of the quality of each component to that of the end product, and the quality preservation level of its assembly process. Consumers at the demand markets respond to both the prices and the quality of the end products.

As in our previous chapters, we define quality as "the degree to which a specific product conforms to a design or specification," which is how well the product is conforming to an established specification (Shewhart 1931; Juran 1951; Levitt 1972; Gilmore 1974; Crosby 1979; Deming 1986). As noted in Chap. 1, this definition makes quality relatively straightforward to quantify, which is essential for firms and researchers who are eager to measure it, manage it, model it, compare it across time, and to also make associated decisions (Shewhart 1931).

Since we are dealing with supply chain networks in which finished products are assembled from multiple components, we also need to characterize and quantify the quality of the finished product. Hence, we provide a formula to quantify quality of the finished product based on the quality of the individual components. We assume in the model that each component's quality ranges from a lower bound of 0 to an imposed upper bound, which, depending upon the application, can represent perfect quality, if it is achievable by the manufacturer/producer.

This chapter is organized as follows. In Sect. 10.2, we develop the multitiered supply chain network model with competing suppliers and competing firms. We describe their strategic variables and their competitive behavior and derive the variational inequality formulations for each tier followed by a unified variational inequality. In Sect. 10.3, we present qualitative properties of the equilibrium pattern, in particular, existence and uniqueness results. In Sect. 10.4, we outline the algorithm, along with conditions for convergence, which is then applied in Sect. 10.5 to compute solutions to numerical supply chain network examples accompanied by sensitivity analysis. We also discuss the results in order to provide managerial insights. We summarize our results and present our conclusions in Sect. 10.6. The last section of this chapter, Sect. 10.7, is the Sources and Notes section.

10.2 The Multitiered Supply Chain Model with Suppliers and Quality Competition

In this section, we develop a multitiered supply chain network game theory model with suppliers and firms that procure components from the suppliers for their products, which are differentiated by brand. We consider a supply chain network consisting of I firms, with a typical firm denoted by i, n_S suppliers, with a typical supplier denoted by j, and a total of n_R demand markets, with a typical demand market denoted by R_k.

The firms compete noncooperatively. Each firm corresponds to an individual brand representing the product that it produces. We assume that product i, which is the product produced by firm i, requires n_{l^i} different components. The total number of different components required by the I products is n_l. We allow for the situation that each supplier may be able to produce a variety of components for each firm.

The I firms are involved in the processes of assembling the products using the components needed, transporting the products to the demand markets, and, possibly, producing the components of the products. The suppliers, in turn, are involved in the processes of producing and delivering the components of the products to the firms. Both in-house and contracted component production activities are captured in the model. Firms' and suppliers' production capacities/abilities are also considered.

The network topology of the supply chain problem is as depicted in Fig. 9.1 since the model in this chapter is an extension of the model in Chap. 9 to include quality competition with the additional notation and generalized functions to capture quality. Recall that the first two sets of links from the top nodes are links corresponding to distinct supplier components. The links from the top-tiered nodes $j; j = 1, \ldots, n_S$, representing the suppliers, are connected to the associated manufacturing nodes, denoted by nodes $1, \ldots, n_l$. These links represent the manufacturing activities of the suppliers. The next set of links that emanates from $1, \ldots, n_l$ to the firms, denoted by nodes $1, \ldots, I$, reflects the transportation of the components to the associated firms. In addition, the links that connect nodes $1^i, \ldots, n^i_{l^i}$, which are firm i's component manufacturing nodes, and firm i are the manufacturing links of firm i for producing its components.

The rest of the links in Fig. 9.1 are links corresponding to the finished products. The link connecting firm i and node i', which is the assembly node of firm i, represents the activity of assembling firm i's product using the components needed, which may be produced by firm i, the suppliers, or both. Finally, the links joining nodes $1', \ldots, I'$ with demand market nodes $R_1, \ldots, R_{n_R}$ correspond to the transportation of the products to the demand markets.

In this chapter, we seek to determine the optimal component production quantities and quality levels, both by the firms and by the suppliers, the optimal product shipments from the firms to the demand markets, the optimal quality preservation levels of the assembly processes of the firms, and the prices that the suppliers charge the firms for producing and delivering the components. The firms compete noncooperatively under the Cournot-Nash equilibrium concept in product shipments, in-house and contracted component production quantities, in-house component quality levels, and the quality preservation levels of the assembly processes. The suppliers, in turn, compete in Bertrand fashion in the prices that they charge the firms and the quality levels of the components produced by them. We assume that there is no information asymmetry between the firms and the suppliers.

The notation for the variables and parameters in the model is given in Table 10.1. The functions in the model are given in Table 10.2. As in the preceding Chaps. 2 through 9, the vectors are assumed to be column vectors and the optimal/equilibrium solution is denoted by "*".

Table 10.1 Notation for the variables and parameters in the multitiered supply chain network game theory model with suppliers and quality

Notation	Definition
Q^S_{jil}	The nonnegative amount of firm i's component l produced by supplier j; $j = 1,\ldots,n_S$; $i = 1,\ldots,I$; $l = 1,\ldots,n_{l^i}$. For firm i, we group its $\{Q^S_{jil}\}$ elements into the vector $Q^S_i \in R^{n_S n_{l^i}}_+$. All the $\{Q^S_{jil}\}$ elements are grouped into the vector $Q^S \in R^{n_S \sum_{i=1}^{I} n_{l^i}}_+$
q^S_{jil}	The quality of firm i's component l produced by supplier j. For supplier j, we group its $\{q^S_{jil}\}$ elements into the vector $q^S_j \in R^{\sum_{i=1}^{I} n_{l^i}}_+$, and group all such vectors into the vector $q^S \in R^{n_S \sum_{i=1}^{I} n_{l^i}}_+$
CAP^S_{jil}	The capacity of supplier j for producing firm i's component l
π_{jil}	The price charged by supplier j for producing one unit of firm i's component l. For supplier j, we group its $\{\pi_{jil}\}$ elements into the vector $\pi_j \in R^{\sum_{i=1}^{I} n_{l^i}}_+$, and group all such vectors into the vector $\pi \in R^{n_S \sum_{i=1}^{I} n_{l^i}}_+$
Q^F_{il}	The nonnegative amount of firm i's component l produced by firm i itself. For firm i, we group its $\{Q^F_{il}\}$ elements into the vector $Q^F_i \in R^{n_{l^i}}_+$, and group all such vectors into the vector $Q^F \in R^{\sum_{i=1}^{I} n_{l^i}}_+$
q^F_{il}	The quality of firm i's component l produced by firm i itself. For firm i, we group its $\{q^F_{il}\}$ elements into the vector $q^F_i \in R^{n_{l^i}}_+$, and group all such vectors into the vector $q^F \in R^{\sum_{i=1}^{I} n_{l^i}}_+$
CAP^F_{il}	The capacity of firm i for producing its component l
q_{il}	The average quality of firm i's component l, produced both by the firm and by the suppliers
Q_{ik}	The nonnegative shipment of firm i's product from firm i to demand market R_k; $k = 1,\ldots,n_R$. For firm i, we group its $\{Q_{ik}\}$ elements into the vector $Q_i \in R^{n_R}_+$, and group all such vectors into the vector $Q \in R^{In_R}_+$
α^F_i	The quality preservation level of the assembly process of firm i. We group all $\{\alpha^F_i\}$ elements into the vector $\alpha^F \in R^I_+$
q_i	The quality associated with firm i's product. We group all $\{q_i\}$ elements into the vector $q \in R^I_+$
d_{ik}	The demand for firm i's product at demand market R_k. We group all $\{d_{ik}\}$ elements into the vector $d \in R^{In_R}_+$
θ_{il}	The amount of component l needed by firm i to produce one unit product i
ω_{il}	The ratio of the importance of the quality of firm i's component l in one unit product i to the quality associated with one unit product i (i.e., q_i)

Observe, from Table 10.2, that the production cost functions of the firms and of the suppliers now also depend on quality explicitly as do the transportation cost functions and the demand price functions. In the model in Chap. 9, there was no competition in quality and, hence, there were no quality variables.

Table 10.2 Functions for the multitiered supply chain network game theory model with suppliers and quality

Function	Definition
$\hat{f}^F_{il}(Q^F, q^F)$	Firm i's production cost for producing its component l; $i = 1,\ldots,I$; $l = 1,\ldots,n_l$
$\hat{f}_i(Q, \alpha^F)$	Firm i's cost for assembling its product using the components needed; $i = 1,\ldots,I$
$c^F_{ik}(Q, q)$	Firm i's transportation cost for shipping its product to demand market R_k; $i = 1,\ldots,I$; $k = 1,\ldots,n_R$
$tc_{ijl}(Q^S)$	The transaction cost paid by firm i for transacting with supplier j; $i = 1,\ldots,I$; $j = 1,\ldots,n_S$, for its component l
$\hat{f}^S_{jl}(Q^S, q^S)$	Supplier j's production cost for producing component l; $j = 1,\ldots,n_S$; $l = 1,\ldots,n_l$
$\hat{c}^S_{jil}(Q^S, q^S)$	Supplier j's transportation cost for shipping firm i's component l; $j = 1,\ldots,n_S$; $i = 1,\ldots,I$; $l = 1,\ldots,n_l$
$oc_{jil}(\pi)$	The opportunity cost of supplier j associated with pricing firm i's component l at π_{jil} for producing and transporting it; $j = 1,\ldots,n_S$; $i = 1,\ldots,I$; $l = 1,\ldots,n_l$
$\rho_{ik}(d, q)$	The demand price for firm i's product at demand market R_k; $i = 1,\ldots,I$; $k = 1,\ldots,n_R$

The following conservation of flow equations must hold:

$$Q_{ik} = d_{ik}, \quad i = 1,\ldots,I; k = 1,\ldots,n_R. \tag{10.1}$$

Hence, the quantity of a firm's brand-name product consumed at a demand market is equal to the amount shipped from the firm to that demand market. In addition, the shipment volumes must be nonnegative, that is:

$$Q_{ik} \geq 0, \quad i = 1,\ldots,I; k = 1,\ldots,n_R. \tag{10.2}$$

We quantify the quality levels of the components as values between 0 and the perfect quality, that is:

$$\bar{q}_{il} \geq q^S_{jil} \geq 0, \quad j = 1,\ldots,n_S; i = 1,\ldots,I; l = 1,\ldots,n_{l^i}, \tag{10.3}$$

$$\bar{q}_{il} \geq q^F_{il} \geq 0, \quad i = 1,\ldots,I; l = 1,\ldots,n_{l^i}. \tag{10.4}$$

The parameter $\bar{q}_{il}$ is the value representing the perfect quality level, according to the conformance specifications, associated with firm i's component l; $i = 1,\ldots,I$; $l = 1,\ldots,n_{l^i}$.

The average quality level of product i's component l is determined by all the quantities and quality levels of that component, produced both by firm i and by the suppliers, that is:

$$q_{il} = \frac{q^F_{il} Q^F_{il} + \sum_{j=1}^{n_S} Q^S_{jil} q^S_{jil}}{Q^F_{il} + \sum_{j=1}^{n_S} Q^S_{jil}}, \quad i = 1, \dots, I; l = 1, \dots, n_{l^i}. \tag{10.5}$$

In Chao et al. (2009), the quality failure rate of a finished product was modeled as a weighted summation of those of its components. However, Economides (1999) considered that the quality of the composite good is the minimum quality of the quality levels of its components. Combining the above approaches, Pennerstorfer and Weiss (2012) presented three forms of quality aggregation that the quality of the final product can be modeled as the weighted summation, the minimum, or the maximum of the quality of suppliers.

We all know that a product is composed of or is a mixture of different components/materials with distinct functions and they synergize in contributing to the performance of the final product. This synergy mechanism should be captured and considered in the quality expression of the final product. In addition, different components are of different importance to the quality of the final product. For example, for an automobile, the quality of the spare tire is not as critical as that of the ignition switch, the engine, or the airbags, so it is far from the truth to consider that the quality of a car can be represented by the quality of the spare tire only. Therefore, the weighted summation expression is the most general among the three expressions of product quality in Pennerstorfer and Weiss (2012). Moreover, one should not ignore the fact that the quality of a product is also affected by its assembly processes in which the components are fitted together in order to achieve specific functions of the product.

Thus, in this chapter, we model the quality of a finished product i as a function determined by the average quality levels of its components, the importance of the quality of the components to the quality of the product, and the quality preservation level of the assembly process of firm i. It is expressed as:

$$q_i = \alpha^F_i \Big(\sum_{l=1}^{n_{l^i}} \omega_{il} q_{il}\Big), \quad i = 1, \dots, I; l = 1. \tag{10.6}$$

Note that α^F_i captures the percentage of the quality preservation of product i in the assembly process of the firm and lies between 0 and 1, that is:

$$0 \le \alpha^F_i \le 1, \quad i = 1, \dots, I. \tag{10.7}$$

The decay of quality can be captured by the factor $1 - \alpha^F_i$. In Nagurney and Masoumi (2012), Masoumi et al. (2012), Nagurney and Nagurney (2012), Yu and Nagurney (2013), and in Nagurney et al. (2013), arc multipliers that are similar to α^F_i are used to model the perishability of particular products, such as

(continued)

pharmaceuticals, human blood, medical nuclear products, and fresh produce, in terms of the percentages of flows that reach the successor nodes in supply chain networks.

We also assume that the values of the importance of the quality levels of all components of product i sum up to 1, that is:

$$\sum_{l=1}^{n_{li}} \omega_{il} = 1, \quad i = 1, \ldots, I. \tag{10.8}$$

In view of (10.1), (10.5), and (10.6), we can redefine the transportation cost functions of the firms $c^F_{ik}(Q, q)$ and the demand price functions $\rho_{ik}(d, q)$; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$, as functions of quantities and quality levels of the components, both by the firms and by the suppliers, the quantities of the products, and the quality preservation levels of the assembly processes, as follows:

$$\hat{c}^F_{ik} = \hat{c}^F_{ik}(Q, Q^F, Q^S, q^F, q^S, \alpha^F) = c^F_{ik}(Q, q), \quad i = 1, \ldots, I; k = 1, \ldots, n_R, \tag{10.9}$$

$$\hat{\rho}_{ik} = \hat{\rho}_{ik}(Q, Q^F, Q^S, q^F, q^S, \alpha^F) = \rho_{ik}(d, q), \quad i = 1, \ldots, I; k = 1, \ldots, n_R. \tag{10.10}$$

The generality of the expressions in (10.9) and (10.10) allows for modeling and application flexibility. The demand price functions (10.10) are, typically, assumed to be monotonically decreasing in demand at its demand market but increasing in terms of product quality levels.

As noted in Table 10.2, the assembly cost functions, the production cost functions, the transportation cost functions, the transaction cost functions, and the demand price functions are general functions in vectors of quantities and/or quality levels, since one supplier's or one firm's decisions will affect their competitors' costs and, hence, their decisions as well. In this way, the interactions among firms and those among suppliers in their competition for resources and technologies are captured.

Furthermore, the cost functions measure not only the monetary costs in the corresponding processes, but also other important factors, such as the time spent in conforming the processes and the costs of ensuring and assuring quality in these processes (e.g., scrap costs, screening costs, rework costs, and the investments for quality engineering and training). The compensation costs incurred when customers are dissatisfied with the quality of the products, such as warranty charges and the complaint adjustment cost, are also included, which can be utilized to measure the disrepute costs of the firms. The costs related to quality are all convex functions of quality conformance levels (see, e.g., Feigenbaum 1983; Juran and Gryna 1988;

Campanella 1990; Porter and Rayner 1992; Shank and Govindarajan 1994). In addition, since, as argued by Bender et al. (1985), one of the most important factors that must be considered in selecting suppliers is their cost, we assume that the cost functions of the suppliers are known by the firms.

In addition, in this chapter, we assume that transportation activities affect quality in terms of quality preservation and, thus, quality does not deteriorate during transportation but, as mentioned earlier, it may deteriorate in the assembly processes.

This model is also capable of handling the case of outsourcing by setting each n_{l^i}; $i = 1, \ldots, I$, to 1. In such a case, the contractors do the outsourced jobs of producing products and transporting them to the firms. The firms do the packaging and labeling for their products and may also produce in-house. In addition, when the number of firms and the number of the suppliers are one, this model is able to capture the case of a single firm – single supplier supply chain where the firm procures from one exclusive supplier, without competition, as in the models in related literature (cf. Sect. 10.1 and Example 10.1).

10.2.1 The Behavior of the Firms and Their Optimality Conditions

Given the prices π_i^* of the components that the suppliers charge firm i and the quality q^{S^*} of the components produced by the suppliers, the objective of firm i; $i = 1, \ldots, I$, is to maximize its utility/profit U_i^F. It is the difference between its total revenue and its total cost. The total cost includes the assembly cost, the production costs, the transportation costs, the transaction costs, and the payments to the suppliers.

Hence, firm i seeks to

$$\text{Maximize}_{Q_i, Q_i^F, Q_i^S, q_i^F, \alpha_i^F} \quad U_i^F = \sum_{k=1}^{n_R} \hat{\rho}_{ik}(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F) d_{ik} - \hat{f}_i(Q, \alpha^F)$$

$$- \sum_{l=1}^{n_{l^i}} \hat{f}_{il}^F(Q^F, q^F) - \sum_{k=1}^{n_R} \hat{c}_{ik}^F(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F) - \sum_{j=1}^{n_S} \sum_{l=1}^{n_{l^i}} tc_{ijl}(Q^S) - \sum_{j=1}^{n_S} \sum_{l=1}^{n_{l^i}} \pi_{jil}^* Q_{jil}^S \tag{10.11}$$

subject to:

$$\sum_{k=1}^{n_R} Q_{ik}\theta_{il} \leq \sum_{j=1}^{n_S} Q_{jil}^S + Q_{il}^F, \quad i = 1, \ldots, I; l = 1, \ldots, n_{l^i}, \tag{10.12}$$

$$CAP_{jil}^S \geq Q_{jil}^S \geq 0, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}, \tag{10.13}$$

$$CAP_{il}^F \geq Q_{il}^F \geq 0, \quad i = 1, \ldots, I; l = 1, \ldots, n_{l^i}, \tag{10.14}$$

and (10.1), (10.2), (10.4), and (10.7).

We assume that all the cost functions and demand price functions in (10.11) are continuous and twice continuously differentiable. The cost functions are convex in quantities and/or quality levels and have bounded second-order partial derivatives. The demand price functions have bounded first-order and second-order partial derivatives. Constraint (10.12) captures the material requirements in the assembly process. Constraints (10.13) and (10.14) indicate that the component production quantities should be nonnegative and limited by the associated capacities, which can capture the abilities of producing. If a supplier or a firm is not capable of producing a certain component, the associated capacity should be 0.00.

The firms compete in the sense of Nash (1950, 1951). The strategic variables for each firm i are the product shipments to the demand markets, the in-house component production quantities, the contracted component production quantities, which are produced by the suppliers, the quality levels of the in-house produced components, and the quality preservation level of its assembly process.

We define the feasible set $\overline{K}_i^F$ as

$$\overline{K}_i^F \equiv \{(Q_i, Q_i^F, Q_i^S, q_i^F, \alpha_i^F) | (10.1), (10.2), (10.4), (10.7), (10.12), (10.13) \text{ and } (10.14) \text{ hold}\}.$$

All $\overline{K}_i^F$; $i = 1, \ldots, I$, are closed and convex. We also define the feasible set $\overline{\mathscr{K}}^F \equiv \Pi_{i=1}^I \overline{K}_i^F$.

Definition 10.1: A Cournot-Nash Equilibrium
A product shipment, in-house component production, contracted component production, in-house component quality, and assembly quality preservation pattern $(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}) \in \overline{\mathscr{K}}^F$ *is said to constitute a Cournot-Nash equilibrium if for each firm i;* $i = 1, \ldots, I$,

$$U_i^1(Q_i^*, \hat{Q}_i^*, Q_i^{F^*}, \hat{Q}_i^{F^*}, Q_i^{S^*}, \hat{Q}_i^{S^*}, q_i^{F^*}, \hat{q}_i^{F^*}, \alpha_i^{F^*}, \hat{\alpha}_i^{F^*}, \pi_i^*, q^{S^*}) \geq$$
$$U_i^1(Q_i, \hat{Q}_i^*, Q_i^F, \hat{Q}_i^{F^*}, Q_i^S, \hat{Q}_i^{S^*}, q_i^F, \hat{q}_i^{F^*}, \alpha_i^F, \hat{\alpha}_i^{F^*}, \pi_i^*, q^{S^*}),$$
$$\forall (Q_i, Q_i^F, Q_i^S, q_i^F, \alpha_i^F) \in \overline{K}_i^F, \tag{10.15}$$

where

$$\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_I^*),$$
$$\hat{Q}_i^{F^*} \equiv (Q_1^{F^*}, \ldots, Q_{i-1}^{F^*}, Q_{i+1}^{F^*}, \ldots, Q_I^{F^*}),$$
$$\hat{Q}_i^{S^*} \equiv (Q_1^{S^*}, \ldots, Q_{i-1}^{S^*}, Q_{i+1}^{S^*}, \ldots, Q_I^{S^*}),$$
$$\hat{q}_i^{F^*} \equiv (q_1^{F^*}, \ldots, q_{i-1}^{F^*}, q_{i+1}^{F^*}, \ldots, q_I^{F^*}),$$

and

$$\hat{\alpha}_i^{F^*} \equiv (\alpha_1^{F^*}, \ldots, \alpha_{i-1}^{F^*}, \alpha_{i+1}^{F^*}, \ldots, \alpha_I^{F^*}).$$

According to (10.15), a Cournot-Nash equilibrium is established if no firm can unilaterally improve upon its profit by selecting an alternative vector of product shipments, in-house component production quantities, contracted component production quantities, in-house component quality levels, and the quality preservation level of its assembly process.

We now derive the variational inequality formulation of the Cournot-Nash equilibrium as done in a basic setting in Chap. 2 (see also Cournot 1838; Nash 1950, 1951; Gabay and Moulin 1980) in the following theorem.

Theorem 10.1: Variational Inequality Formulations of the Firms' Problems *Assume that, for each firm i; $i = 1, \ldots, I$, the utility function $U_i^F(Q, Q^F, Q^S, q^F, \alpha^F, \pi_i^*, q^{S^*})$ is concave with respect to its variables in Q_i, Q_i^F, Q_i^S, q_i^F, and α_i^F, and is continuous and continuously differentiable. Then $(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}) \in \overline{\mathcal{K}}^F$ is a Counot-Nash equilibrium according to Definition 10.1 if and only if it satisfies the variational inequality:*

$$-\sum_{i=1}^{I}\sum_{k=1}^{n_R} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial Q_{ik}} \times (Q_{ik} - Q_{ik}^*)$$
$$-\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial Q_{il}^F} \times (Q_{il}^F - Q_{il}^{F^*})$$
$$-\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial Q_{jil}^S} \times (Q_{jil}^S - Q_{jil}^{S^*})$$
$$-\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial q_{il}^F} \times (q_{il}^F - q_{il}^{F^*})$$
$$-\sum_{i=1}^{I} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial \alpha_i^F} \times (\alpha_i^F - \alpha_i^{F^*}) \geq 0,$$
$$\forall (Q, Q^F, Q^S, q^F, \alpha^F) \in \overline{\mathcal{K}}^F, \tag{10.16}$$

with notice that: for $i = 1, \ldots, I$; $k = 1, \ldots, n_R$:

$$-\frac{\partial U_i^F}{\partial Q_{ik}} = \left[\frac{\partial \hat{f}_i(Q, \alpha^F)}{\partial Q_{ik}} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial Q_{ik}} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial Q_{ik}} \times d_{ih} \right.$$
$$\left. - \hat{\rho}_{ik}(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)\right],$$

for $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$-\frac{\partial U_i^F}{\partial Q_{il}^F} = \left[\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^F, q^F)}{\partial Q_{il}^F} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial Q_{il}^F} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial Q_{il}^F} \times d_{ih} \right],$$

for $j = 1, \ldots, n_S$; $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$-\frac{\partial U_i^F}{\partial Q_{jil}^S} = \left[\pi_{jil}^* + \sum_{h=1}^{n_S} \sum_{m=1}^{n_{l^i}} \frac{\partial tc_{ihm}(Q^S)}{\partial Q_{jil}^S} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial Q_{jil}^S} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial Q_{jil}^S} \times d_{ih} \right],$$

for $i = 1, \ldots, I$; $l = 1, \ldots, n_{l^i}$:

$$-\frac{\partial U_i^F}{\partial q_{il}^F} = \left[\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^F, q^F)}{\partial q_{il}^F} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial q_{il}^F} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial q_{il}^F} \times d_{ih} \right],$$

for $i = 1, \ldots, I$:

$$-\frac{\partial U_i^F}{\partial \alpha_i^F} = \left[\frac{\partial \hat{f}_i(Q^F, \alpha^F)}{\partial \alpha_i^F} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial \alpha_i^F} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^{S^*}, \alpha^F)}{\partial \alpha_i^F} \times d_{ih} \right],$$

or, equivalently, in view of (10.1) and (10.12), $(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \lambda^*) \in \mathcal{K}^F$ *is a vector of the equilibrium product shipment, in-house component production, contracted component production, in-house component quality, and assembly quality preservation pattern and Lagrange multipliers if and only if it satisfies the variational inequality*

$$\sum_{i=1}^{I} \sum_{k=1}^{n_R} \left[\frac{\partial \hat{f}_i(Q^*, \alpha^{F^*})}{\partial Q_{ik}} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, q^{S^*}, \alpha^{F^*})}{\partial Q_{ik}} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, q^{S^*}, \alpha^{F^*})}{\partial Q_{ik}} \times Q_{ih}^* \right.$$
$$\left. - \hat{\rho}_{ik}(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, q^{S^*}, \alpha^{F^*}) + \sum_{l=1}^{n_{l^i}} \lambda_{il}^* \theta_{il} \right] \times (Q_{ik} - Q_{ik}^*)$$
$$+ \sum_{i=1}^{I} \sum_{l=1}^{n_{l^i}} \left[\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^{F^*}, q^{F^*})}{\partial Q_{il}^F} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, q^{S^*}, \alpha^{F^*})}{\partial Q_{il}^F} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, q^{S^*}, \alpha^{F^*})}{\partial Q_{il}^F} \times Q_{ih}^* \right.$$
$$\left. - \lambda_{il}^* \right] \times (Q_{il}^F - Q_{il}^{F^*})$$
$$+ \sum_{j=1}^{n_S} \sum_{i=1}^{I} \sum_{l=1}^{n_{l^i}} \left[\pi_{jil}^* + \sum_{h=1}^{n_S} \sum_{m=1}^{n_{l^i}} \frac{\partial tc_{ihm}(Q^{S^*})}{\partial Q_{jil}^S} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, q^{S^*}, \alpha^{F^*})}{\partial Q_{jil}^S} \right.$$
$$\left. - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, q^{S^*}, \alpha^{F^*})}{\partial Q_{jil}^S} \times Q_{ih}^* - \lambda_{il}^* \right] \times (Q_{jil}^S - Q_{jil}^{S^*})$$
$$+ \sum_{i=1}^{I} \sum_{l=1}^{n_{l^i}} \left[\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^{F^*}, q^{F^*})}{\partial q_{il}^F} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, q^{S^*}, \alpha^{F^*})}{\partial q_{il}^F} \right.$$
$$\left. - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, q^{S^*}, \alpha^{F^*})}{\partial q_{il}^F} \times Q_{ih}^* \right] \times (q_{il}^F - q_{il}^{F^*})$$

$$+\sum_{i=1}^{I}\left[\frac{\partial \hat{f}_i(Q^{F^*},\alpha^{F^*})}{\partial \alpha_i^F}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}_{ih}^F(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial \alpha_i^F}\right.$$
$$\left.-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial \alpha_i^F}\times Q_{ih}^*\right]\times(\alpha_i^F-\alpha_i^{F^*})$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{n_{li}}\left[\sum_{j=1}^{n_S}Q_{jil}^{S^*}+Q_{il}^{F^*}-\sum_{k=1}^{n_R}Q_{ik}^*\theta_{il}\right]\times(\lambda_{il}-\lambda_{il}^*)\geq 0,\quad \forall(Q,Q^F,Q^S,q^F,\alpha^F,\lambda)\in\mathscr{K}^F, \tag{10.17}$$

where $\mathscr{K}^F\equiv\Pi_{i=1}^{I}K_i^F$ *and* $K_i^F\equiv\{(Q_i,Q_i^F,Q_i^S,q_i^F,\alpha_i^F,\lambda_i)|\lambda_i\geq 0\}$ *with (10.2), (10.4), (10.7), (10.13), and (10.14) satisfied. The vector* λ_i *is the* n_{li}*-dimensional vector with component* l *being the element* λ_{il} *corresponding to the Lagrange multiplier associated with the* (i,l)*-th constraint (10.12). Both the above-defined feasible sets are convex.*

Proof: For a given firm i, under the imposed assumptions, (10.16) holds if and only if (see Bertsekas and Tsitsiklis (1989), page 287) the following holds:

$$\sum_{k=1}^{n_R}\left[\frac{\partial \hat{f}_i(Q^*,\alpha^{F^*})}{\partial Q_{ik}}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}_{ih}^F(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q_{ik}}\right.$$
$$\left.-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q_{ik}}\times Q_{ih}^*-\hat{\rho}_{ik}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})+\sum_{l=1}^{n_{li}}\lambda_{il}^*\theta_{il}\right]\times(Q_{ik}-Q_{ik}^*)$$
$$+\sum_{l=1}^{n_{li}}\left[\sum_{m=1}^{n_{li}}\frac{\partial \hat{f}_{im}^F(Q^{F^*},q^{F^*})}{\partial Q_{il}^F}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}_{ih}^F(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q_{il}^F}\right.$$
$$\left.-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q_{il}^F}\times Q_{ih}^*-\lambda_{il}^*\right]\times(Q_{il}^F-Q_{il}^{F^*})$$
$$+\sum_{j=1}^{n_S}\sum_{l=1}^{n_{li}}\left[\pi_{jil}^*+\sum_{h=1}^{n_S}\sum_{m=1}^{n_{li}}\frac{\partial tc_{ihm}(Q^{S^*})}{\partial Q_{jil}^S}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}_{ih}^F(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q_{jil}^S}\right.$$
$$\left.-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q_{jil}^S}\times Q_{ih}^*-\lambda_{il}^*\right]\times(Q_{jil}^S-Q_{jil}^{S^*})$$
$$+\sum_{l=1}^{n_{li}}\left[\sum_{m=1}^{n_{li}}\frac{\partial \hat{f}_{im}^F(Q^{F^*},q^{F^*})}{\partial q_{il}^F}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}_{ih}^F(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial q_{il}^F}\right.$$
$$\left.-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial q_{il}^F}\times Q_{ih}^*\right]\times(q_{il}^F-q_{il}^{F^*})$$
$$+\left[\frac{\partial \hat{f}_i(Q^{F^*},\alpha^{F^*})}{\partial \alpha_i^F}+\sum_{h=1}^{n_R}\frac{\partial \hat{c}_{ih}^F(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial \alpha_i^F}\right.$$
$$\left.-\sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial \alpha_i^F}\times Q_{ih}^*\right]\times(\alpha_i^F-\alpha_i^{F^*})$$
$$+\sum_{l=1}^{n_{li}}\left[\sum_{j=1}^{n_S}Q_{jil}^{S^*}+Q_{il}^{F^*}-\sum_{k=1}^{n_R}Q_{ik}^*\theta_{il}\right]\times(\lambda_{il}-\lambda_{il}^*)\geq 0,\quad \forall(Q_i,Q_i^F,Q_i^S,q_i^F,\alpha_i^F,\lambda_i)\in K_i^F. \tag{10.18}$$

Variational inequality (10.18) holds for each firm i; $i = 1, \ldots, I$, and, hence, the summation of (10.18) yields variational inequality (10.17). □

For additional background on the variational inequality problem, please refer to the book by Nagurney (1999).

10.2.2 The Behavior of the Suppliers and Their Optimality Conditions

Opportunity costs of the suppliers are considered in this model as they were in Chap. 9.

Given the Q^{S^*} determined by the firms, the objective of supplier j; $j = 1, \ldots, n_S$, is to maximize its total profit, denoted by U_j^S. Its revenue is obtained from the payments of the firms, while its costs are the costs of production and delivery and the opportunity costs. The strategic variables of a supplier are the prices that it charges the firms and the quality levels of the components that it produces.

The decision-making problem for supplier j is as the following:

$$\text{Maximize}_{\pi_j, q_j^S} \quad U_j^S = \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \pi_{jil} Q_{jil}^{S*} - \sum_{l=1}^{n_l} \hat{f}_{jl}^S(Q^{S^*}, q^S) - \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \hat{c}_{jil}^S(Q^{S^*}, q^S)$$
$$- \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} oc_{jil}(\pi) \tag{10.19}$$

subject to:

$$\pi_{jil} \geq 0, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}, \tag{10.20}$$

and (10.3).

We assume that the cost functions of each supplier are continuous, twice continuously differentiable, and convex, and have bounded second-order partial derivatives.

The suppliers compete in a noncooperative in the sense of Nash (1950, 1951), with each one trying to maximize its own profit. We define the feasible sets $K_j^S \equiv \{(\pi_j, q_j^S) | \pi_j \in R_+^{\sum_{i=1}^{I} n_{l^i}}$ and q_j^S satisfies (10.3) for $j\}$, $\mathcal{K}^S \equiv \Pi_{j=1}^{n_S} K_j^S$, and $\overline{\mathcal{K}} \equiv \overline{\mathcal{K}}^F \times \mathcal{K}^S$. All the above-defined feasible sets are convex.

Definition 10.2: A Bertrand-Nash Equilibrium
A price and contracted component quality pattern $(\pi^*, q^{S^*}) \in \mathcal{K}^S$ *is said to constitute a Bertrand-Nash equilibrium if for each supplier* j*;* $j = 1, \ldots, n_S$*,*

$$U_j^S(Q^{S^*}, \pi_j^*, \hat{\pi}_j^*, q_j^{S^*}, \hat{q}_j^{S^*}) \geq U_j^S(Q^{S^*}, \pi_j, \hat{\pi}_j^*, q_j^S, \hat{q}_j^{S^*}), \quad \forall (\pi_j, q_j^S) \in K_j^S, \tag{10.21}$$

where

$$\hat{\pi}_j^* \equiv (\pi_1^*, \ldots, \pi_{j-1}^*, \pi_{j+1}^*, \ldots, \pi_{n_S}^*)$$

and

$$\hat{q}_j^{S^*} \equiv (q_1^{S^*}, \ldots, q_{j-1}^{S^*}, q_{j+1}^{S^*}, \ldots, q_{n_S}^{S^*}).$$

According to (10.21), a Bertrand-Nash equilibrium is established if no supplier can unilaterally improve upon its profit by selecting an alternative vector of prices that it charges the firms and the quality levels of the components that it produces.

The variational inequality formulation of the Bertrand-Nash equilibrium according to Definition 10.2 (see Bertrand 1883; Nash 1950, 1951; Gabay and Moulin 1980; Nagurney 2006) is given in the following theorem.

Theorem 10.2: Variational Inequality Formulations of the Suppliers' Problems
Assume that, for each supplier j; $j = 1, \ldots, n_S$, the profit function $U_j^S(Q^{S^*}, \pi, q^S)$ is concave with respect to the variables in π_j and q_j^S, and is continuous and continuously differentiable. Then $(\pi^*, q^{S^*}) \in \mathcal{K}^S$ is a Bertrand-Nash equilibrium according to Definition 10.2 if and only if it satisfies the variational inequality:

$$-\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_j^S(Q^{S^*}, \pi^*, q^{S^*})}{\partial \pi_{jil}} \times (\pi_{jil} - \pi_{jil}^*)$$
$$-\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_j^S(Q^{S^*}, \pi^*, q^{S^*})}{\partial q_{jil}^S} \times (q_{jil}^S - q_{jil}^{S^*}) \geq 0,$$
$$\forall (\pi, q^S) \in \mathcal{K}^S, \tag{10.22}$$

with notice that: for $j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}$:

$$-\frac{\partial U_j^S}{\partial \pi_{jil}} = \sum_{g=1}^{I}\sum_{m=1}^{n_{l^i}} \frac{\partial oc_{jgm}(\pi)}{\partial \pi_{jil}} - Q_{jil}^{S^*},$$

for $j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}$:

$$-\frac{\partial U_j^S}{\partial q_{jil}^S} = \sum_{m=1}^{n_l} \frac{\partial \hat{f}_{jm}^S(Q^{S^*}, q^S)}{\partial q_{jil}^S} + \sum_{g=1}^{I}\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{c}_{jgm}^S(Q^{S^*}, q^S)}{\partial q_{jil}^S}.$$

10.2.3 The Equilibrium Conditions for the Multitiered Supply Chain Network with Suppliers and Quality Competition

In equilibrium, the optimality conditions for all firms and the optimality conditions for all suppliers must hold simultaneously, according to the definition below.

Definition 10.3: Multitiered Supply Chain Network Equilibrium with Suppliers and Quality Competition
The equilibrium state of the multitiered supply chain network with suppliers is one where both variational inequalities (10.16), or, equivalently, (10.17), and (10.22) hold simultaneously.

Theorem 10.3: Unified Variational Inequality
The equilibrium conditions governing the multitiered supply chain network model with suppliers and quality competition are equivalent to the solution of the variational inequality problem: determine $(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi^*, q^{S^*}) \in \overline{\mathscr{K}}$, *such that:*

$$
\begin{aligned}
&-\sum_{i=1}^{I}\sum_{k=1}^{n_R} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial Q_{ik}} \times (Q_{ik} - Q_{ik}^*) \\
&-\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial Q_{il}^F} \times (Q_{il}^F - Q_{il}^{F^*}) \\
&-\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial Q_{jil}^S} \times (Q_{jil}^S - Q_{jil}^{S^*}) \\
&-\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial q_{il}^F} \times (q_{il}^F - q_{il}^{F^*}) \\
&-\sum_{i=1}^{I} \frac{\partial U_i^F(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi_i^*, q^{S^*})}{\partial \alpha_i^F} \times (\alpha_i^F - \alpha_i^{F^*}) \\
&-\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_j^S(Q^{S^*}, \pi^*, q^{S^*})}{\partial \pi_{jil}} \times (\pi_{jil} - \pi_{jil}^*) \\
&-\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}} \frac{\partial U_j^S(Q^{S^*}, \pi^*, q^{S^*})}{\partial q_{jil}^S} \times (q_{jil}^S - q_{jil}^{S^*}) \geq 0, \quad \forall (Q, Q^S, Q^F, q^F, \alpha^F, \pi, q^S) \in \overline{\mathscr{K}},
\end{aligned}
\tag{10.23}
$$

or, equivalently: determine $(Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \lambda^*, \pi^*, q^{S^*}) \in \mathcal{K}$*, such that:*

$$
\begin{aligned}
&\sum_{i=1}^{I}\sum_{k=1}^{n_R}\Bigg[\frac{\partial \hat{f}_i(Q^*,\alpha^{F^*})}{\partial Q_{ik}} + \sum_{h=1}^{n_R}\frac{\partial \hat{c}^F_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q_{ik}} \\
&- \sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q_{ik}} \times Q^*_{ih} - \hat{\rho}_{ik}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*}) + \sum_{l=1}^{n_{l^i}}\lambda^*_{il}\theta_{il}\Bigg] \times (Q_{ik}-Q^*_{ik}) \\
&+ \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\Bigg[\sum_{m=1}^{n_{l^i}}\frac{\partial \hat{f}^F_{im}(Q^{F^*},q^{F^*})}{\partial Q^F_{il}} + \sum_{h=1}^{n_R}\frac{\partial \hat{c}^F_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q^F_{il}} \\
&- \sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q^F_{il}} \times Q^*_{ih} - \lambda^*_{il}\Bigg] \times (Q^F_{il}-Q^{F^*}_{il}) \\
&+ \sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\Bigg[\pi^*_{jil} + \sum_{h=1}^{n_S}\sum_{m=1}^{n_{l^i}}\frac{\partial tc_{ihm}(Q^{S^*})}{\partial Q^S_{jil}} + \sum_{h=1}^{n_R}\frac{\partial \hat{c}^F_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q^S_{jil}} \\
&- \sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial Q^S_{jil}} \times Q^*_{ih} - \lambda^*_{il}\Bigg] \times (Q^S_{jil}-Q^{S^*}_{jil}) \\
&+ \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\Bigg[\sum_{m=1}^{n_{l^i}}\frac{\partial \hat{f}^F_{im}(Q^{F^*},q^{F^*})}{\partial q^F_{il}} + \sum_{h=1}^{n_R}\frac{\partial \hat{c}^F_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial q^F_{il}} \\
&- \sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial q^F_{il}} \times Q^*_{ih}\Bigg] \times (q^F_{il}-q^{F^*}_{il}) \\
&+ \sum_{i=1}^{I}\Bigg[\frac{\partial \hat{f}_i(Q^{F^*},\alpha^{F^*})}{\partial \alpha^F_i} + \sum_{h=1}^{n_R}\frac{\partial \hat{c}^F_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial \alpha^F_i} \\
&- \sum_{h=1}^{n_R}\frac{\partial \hat{\rho}_{ih}(Q^*,Q^{F^*},Q^{S^*},q^{F^*},q^{S^*},\alpha^{F^*})}{\partial \alpha^F_i} \times Q^*_{ih}\Bigg] \times (\alpha^F_i-\alpha^{F^*}_i) \\
&+ \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\Bigg[\sum_{j=1}^{n_S}Q^{S^*}_{jil} + Q^{F^*}_{il} - \sum_{k=1}^{n_R}Q^*_{ik}\theta_{il}\Bigg] \times (\lambda_{il}-\lambda^*_{il}) + \sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\Bigg[\sum_{g=1}^{I}\sum_{m=1}^{n_{l^i}}\frac{\partial oc_{jgm}(\pi^*)}{\partial \pi_{jil}} - Q^{S^*}_{jil}\Bigg] \times (\pi_{jil}-\pi^*_{jil}) \\
&+ \sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\Bigg[\sum_{m=1}^{n_l}\frac{\partial \hat{f}^S_{jm}(Q^{S^*},q^{S^*})}{\partial q^S_{jil}} + \sum_{g=1}^{I}\sum_{m=1}^{n_{l^i}}\frac{\partial \hat{c}^S_{jgm}(Q^{S^*},q^{S^*})}{\partial q^S_{jil}}\Bigg] \times (q^S_{jil}-q^{S^*}_{jil}) \geq 0, \\
&\forall (Q, Q^F, Q^S, q^F, \alpha^F, \lambda, \pi, q^S) \in \mathcal{K},
\end{aligned}
\tag{10.24}
$$

where $\mathcal{K} \equiv \mathcal{K}^F \times \mathcal{K}^S$.

Proof: Summation of variational inequalities (10.16) (or (10.17)) and (10.22) yields variational inequality (10.23) (or (10.24)). A solution to variational inequality (10.23) (or (10.24)) satisfies the sum of (10.16) (or (10.17)) and (10.22) and, hence, is an equilibrium according to Definition 10.3. □

We now put variational inequality (10.24) into standard form (cf. (2.1a)): determine $X^* \in \mathcal{K}$ where X is a vector in R^N, $F(X)$ is a continuous function such that $F(X): X \mapsto \mathcal{K} \subset R^N$, and

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{10.25}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the N-dimensional Euclidean space, $N = In_R + 3\sum_{i=1}^{I} n_{l^i} + 3n_S \sum_{i=1}^{I} n_{l^i} + I$, and $\mathscr{K}$ is closed and convex. We define the vector $X \equiv (Q, Q^F, Q^S, q^F, \alpha^F, \lambda, \pi, q^S)$ and the vector

$$F(X) \equiv (F^1(X), F^2(X), F^3(X), F^4(X), F^5(X), F^6(X), F^7(X), F^8(X)),$$

such that:

$$F^1(X) = \left[\frac{\partial \hat{f}_i(Q, \alpha^F)}{\partial Q_{ik}} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q_{ik}} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q_{ik}} \right.$$
$$\left. \times Q_{ih} - \hat{\rho}_{ik}(Q, Q^F, Q^S, q^F, q^S, \alpha^F) + \sum_{l=1}^{n_{l^i}} \lambda_{il}\theta_{il}; i = 1, \ldots, I; k = 1, \ldots, n_R \right],$$

$$F^2(X) = \left[\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^F, q^F)}{\partial Q_{il}^F} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q_{il}^F} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q_{il}^F} \times Q_{ih} - \lambda_{il}; \right.$$
$$\left. i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right],$$

$$F^3(X) = \left[\pi_{jil} + \sum_{h=1}^{n_S} \sum_{m=1}^{n_{l^i}} \frac{\partial tc_{ihm}(Q^S)}{\partial Q_{jil}^S} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q_{jil}^S} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q_{jil}^S} \times Q_{ih} - \lambda_{il}; \right.$$
$$\left. j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right],$$

$$F^4(X) = \left[\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^F, q^F)}{\partial q_{il}^F} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial q_{il}^F} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial q_{il}^F} \times Q_{ih}; \right.$$
$$\left. i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right],$$

$$F^5(X) = \left[\frac{\partial \hat{f}_i(Q^F, \alpha^F)}{\partial \alpha_i^F} + \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial \alpha_i^F} - \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial \alpha_i^F} \times Q_{ih}; i = 1, \ldots, I \right],$$

$$F^6(X) = \left[\sum_{j=1}^{n_S} Q_{jil}^S + Q_{il}^F - \sum_{k=1}^{n_R} Q_{ik}\theta_{il}; i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right],$$

$$F^7(X) = \left[\sum_{g=1}^{I} \sum_{m=1}^{n_{l^i}} \frac{\partial oc_{jgm}(\pi)}{\partial \pi_{jil}} - Q_{jil}^S; j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right],$$

$$F^8(X) = \left[\sum_{m=1}^{n_l} \frac{\partial \hat{f}_{jm}^S(Q^S, q^S)}{\partial q_{jil}^S} + \sum_{g=1}^{I} \sum_{m=1}^{n_{l^i}} \frac{\partial \hat{c}_{jgm}^S(Q^S, q^S)}{\partial q_{jil}^S}; j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i} \right]. \tag{10.26}$$

Hence, (10.24) can be put into standard form (10.25).

Similarly, we also put variational inequality (10.23) into standard form: determine $Y^* \in \overline{\mathscr{K}}$ where Y is a vector in R^M, $G(Y)$ is a continuous function such that $G(Y) : Y \mapsto \overline{\mathscr{K}} \subset R^M$, and

$$\langle G(Y^*), Y - Y^* \rangle \geq 0, \quad \forall Y \in \overline{\mathscr{K}}, \tag{10.27}$$

where $M = In_R + 2\sum_{i=1}^{I} n_{l^i} + 3n_S \sum_{i=1}^{I} n_{l^i} + I$, and $\overline{\mathscr{K}}$ is closed and convex. We define $Y \equiv (Q, Q^F, Q^S, q^F, \alpha^F, \pi, q^S)$ and

$$G(Y) \equiv \left(-\frac{\partial U_i^F}{\partial Q_{ik}}, -\frac{\partial U_i^F}{\partial Q_{il}^F}, -\frac{\partial U_i^F}{\partial Q_{jil}^S}, -\frac{\partial U_i^F}{\partial q_{il}^F}, -\frac{\partial U_i^F}{\partial \alpha_i^F}, -\frac{\partial U_j^S}{\partial \pi_{jil}}, -\frac{\partial U_j^S}{\partial q_{jil}^S} \right);$$

$$j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}.$$

Hence, (10.23) can be put into standard form (10.27).

10.3 Qualitative Properties

In this section, we present some qualitative properties of the solution to variational inequality (10.25), equivalently, (10.24). In particular, we provide the existence result and the uniqueness result. We also investigate the equivalence in the properties of the function F given by (10.26) that enters variational inequality (10.25) and the function G that enters variational inequality (10.27).

In a multitiered supply chain network with suppliers, it is reasonable to expect that the price charged by each supplier j for producing one unit of firm i's component l, π_{jil}, is bounded by a sufficiently large value, since, in practice, each supplier cannot charge unbounded prices to the firms. Therefore, the following assumption is not unreasonable:

Assumption 10.1
Suppose that in our multitiered supply chain network model with suppliers and quality competition, there exist a sufficiently large Π, such that,

$$\pi_{jil} \leq \Pi, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}. \tag{10.28}$$

With this assumption, we have the following existence result.

Theorem 10.4: Existence
With Assumption 10.1 satisfied, there exists at least one solution to variational inequality (10.25), equivalently, (10.24).

Proof: We first prove that there exists at least one solution to variational inequality (10.23). Note that the quality levels q_{jil}^S and q_{il}^F and the quality preservation levels α_i are bounded due to constraints (10.3), (10.4), and (10.7). Due to constraint (10.12), the product quantities Q_{ik}; $i = 1, \ldots, I; k = 1, \ldots, n_S$ are also bounded, since the components quantities are nonnegative and capacitated (cf. (10.13) and (10.14)). Therefore, with Assumption 10.1, the feasible set of variational inequality (10.23) is bounded. Since the cost functions and the demand price functions are continuously differentiable, and the feasible set is convex and compact, the existence of a solution to (10.23) is then guaranteed. The rest of the proof is an analog to Theorem 10.3 in Nagurney and Dhanda (2000). □

Theorem 10.5: Equivalence in Monotonicity
If the $G(Y)$ that enters variational inequality (10.27) is monotone, that is,

$$\langle G(Y') - G(Y''), Y' - Y'' \rangle \geq 0, \quad \forall Y', Y'' \in \overline{\mathscr{K}}, \tag{10.29}$$

the $F(X)$ in variational inequality (10.25) is also monotone,

$$\langle F(X') - F(X''), X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathscr{K}. \tag{10.30}$$

Proof: Let

$$Y' = (Q', Q^{S'}, Q^{F'}, q^{F'}, \alpha^{F'}, \pi', q^{S'}), Y'' = (Q'', Q^{S''}, Q^{F''}, q^{F''}, \alpha^{F''}, \pi'', q^{S''}),$$
$$X' = (Q', Q^{S'}, Q^{F'}, q^{F'}, \alpha^{F'}, \lambda', \pi', q^{S'}),$$

and

$$X'' = (Q'', Q^{S''}, Q^{F''}, q^{F''}, \alpha^{F''}, \lambda'', \pi'', q^{S''}).$$

Then the left-hand-side of (10.29) is

$$\begin{aligned}
&\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial Q_{ik}}) - (-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial Q_{ik}})\right] \times (Q'_{ik} - Q''_{ik})\\
&+\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial Q^F_{il}}) - (-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial Q^F_{il}})\right] \times (Q^{F'}_{il} - Q^{F''}_{il})\\
&+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial Q^S_{jil}}) - (-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial Q^S_{jil}})\right] \times (Q^{S'}_{jil} - Q^{S''}_{jil})\\
&+\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial q^F_{il}}) - (-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial q^F_{il}})\right] \times (q^{F'}_{il} - q^{F''}_{il})\\
&+\sum_{i=1}^{I}\left[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial \alpha^F_{i}}) - (-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial \alpha^F_{i}})\right] \times (\alpha^{F'}_{i} - \alpha^{F''}_{i})\\
&+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[(-\frac{\partial U_j^S(Q^{S'},\pi',q^{S'})}{\partial \pi_{jil}}) - (-\frac{\partial U_j^S(Q^{S''},\pi'',q^{S''})}{\partial \pi_{jil}})\right] \times (\pi'_{jil} - \pi''_{jil})\\
&+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[(-\frac{\partial U_j^S(Q^{S'},\pi',q^{S'})}{\partial q^S_{jil}}) - (-\frac{\partial U_j^S(Q^{S''},\pi'',q'')}{\partial q^S_{jil}})\right] \times (q^{S'}_{jil} - q^{S''}_{jil})\\
&+\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[\sum_{l=1}^{n_{l^i}}\lambda'_{il}\theta_{il} - \sum_{l=1}^{n_{l^i}}\lambda'_{il}\theta_{il}\right] \times (Q'_{ik} - Q''_{ik}) + \sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[\sum_{l=1}^{n_{l^i}}\lambda''_{il}\theta_{il} - \sum_{l=1}^{n_{l^i}}\lambda''_{il}\theta_{il}\right] \times (Q'_{ik} - Q''_{ik})\\
&+\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\lambda'_{il} - \lambda'_{il}\right] \times (Q^{F'}_{il} - Q^{F''}_{il}) + \sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\lambda''_{il} - \lambda''_{il}\right] \times (Q^{F'}_{il} - Q^{F''}_{il})\\
&+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\lambda'_{il} - \lambda'_{il}\right] \times (Q^{S'}_{jil} - Q^{S''}_{jil}) + \sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{l^i}}\left[\lambda''_{il} - \lambda''_{il}\right] \times (Q^{S'}_{jil} - Q^{S''}_{jil}).
\end{aligned} \tag{10.31}$$

After combining terms, (10.31) reduces to

$$\sum_{i=1}^{I}\sum_{k=1}^{n_R}\Big[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial Q_{ik}}+\sum_{l=1}^{n_{f^i}}\lambda'_{il}\theta_{il})-(-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial Q_{ik}}+\sum_{l=1}^{n_{f^i}}\lambda''_{il}\theta_{il})\Big]$$
$$\times(Q'_{ik}-Q''_{ik})$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{n_{f^i}}\Big[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial Q^F_{il}}-\lambda'_{il})-(-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial Q^F_{il}}-\lambda''_{il})\Big]$$
$$\times(Q^{F'}_{il}-Q^{F''}_{il})$$
$$+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{f^i}}\Big[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial Q^S_{jil}}-\lambda'_{il})-(-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial Q^S_{jil}}-\lambda''_{il})\Big]$$
$$\times(Q^{S'}_{jil}-Q^{S''}_{jil})$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{n_{f^i}}\Big[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial q^F_{il}})-(-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial q^F_{il}})\Big]\times(q^{F'}_{il}-q^{F''}_{il})$$
$$+\sum_{i=1}^{I}\Big[(-\frac{\partial U_i^F(Q',Q^{S'},Q^{F'},q^{F'},\alpha^{F'},\pi',q^{S'})}{\partial \alpha^F_i})-(-\frac{\partial U_i^F(Q'',Q^{S''},Q^{F''},q^{F''},\alpha^{F''},\pi'',q^{S''})}{\partial \alpha^F_i})\Big]\times(\alpha^{F'}_i-\alpha^{F''}_i)$$
$$+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{f^i}}\Big[(-\frac{\partial U_j^S(Q^{S'},\pi',q^{S'})}{\partial \pi_{jil}})-(-\frac{\partial U_j^S(Q^{S''},\pi'',q^{S''})}{\partial \pi_{jil}})\Big]\times(\pi'_{jil}-\pi''_{jil})$$
$$+\sum_{j=1}^{n_S}\sum_{i=1}^{I}\sum_{l=1}^{n_{f^i}}\Big[(-\frac{\partial U_j^S(Q^{S'},\pi',q^{S'})}{\partial q^S_{jil}})-(-\frac{\partial U_j^S(Q^{S''},\pi'',q'')}{\partial q^S_{jil}})\Big]\times(q^{S'}_{jil}-q^{S''}_{jil})$$
$$+\sum_{i=1}^{I}\sum_{l=1}^{n_{f^i}}\Big[(\sum_{j=1}^{n_S}Q^{S'}_{jil}+Q^{F'}_{il}-\sum_{k=1}^{n_R}Q'_{ik}\theta_{il})-(\sum_{j=1}^{n_S}Q^{S''}_{jil}+Q^{F''}_{il}-\sum_{k=1}^{n_R}Q''_{ik}\theta_{il})\Big]\times(\lambda'_{il}-\lambda''_{il}).$$
$$(10.32)$$

As shown above, (10.32) is derived from the left-hand-side of (10.29). Therefore, if (10.32) is greater than or equal to 0, the $G(Y)$ that enters variational inequality (10.27) is monotone (cf. (10.29)). Since (10.32) is also the left-hand-side of (10.30), the $F(X)$ that enters variational inequality (10.25) is also monotone under this condition. □

Theorem 10.6: Equivalence in Strict Monotonicity
If the $G(Y)$ that enters variational inequality (10.27) is strictly monotone, that is,

$$\langle G(Y')-G(Y''),Y'-Y''\rangle>0,\quad \forall Y',Y''\in\overline{\mathscr{K}},Y'\neq Y'', \tag{10.33}$$

the $F(X)$ in variational inequality (10.25) is also strictly monotone,

$$\langle F(X')-F(X''),X'-X''\rangle>0,\quad \forall X',X''\in\mathscr{K},X'\neq X''. \tag{10.34}$$

Proof: The proof is similar to that for Theorem 10.5.

Theorem 10.7: Uniqueness
Assume that the strict monotonicity condition (10.34) is satisfied. Then, if variational inequality (10.25) admits a solution, $(Q^*,Q^{F*},Q^{S*},q^{F*},\alpha^{F*},\lambda^*,\pi^*,q^{S*})$, that is the only solution.

Proof: Under the strict monotonicity assumption given by (10.34), the proof follows the standard variational inequality theory. □

Theorem 10.8: Lipschitz Continuity
The function that enters the variational inequality problem (10.25) is Lipschitz continuous, that is,

$$\| F(X') - F(X'') \| \leq L \| X' - X'' \|, \quad \forall X', X'' \in \mathscr{K}, \text{ where } L > 0. \tag{10.35}$$

Proof: Since we have assumed that all the cost functions have bounded second-order partial derivatives, and the demand price functions have bounded first-order and second-order partial derivatives, the result is direct by applying a mid-value theorem from calculus to the $F(X)$ that enters variational inequality (10.25). □

10.4 The Algorithm

We employ the modified projection method (see also Chap. 2) for the computation of the solution for the multitiered supply chain network game theory model with suppliers and quality competition. It has been effectively used in large-scale supply chain network equilibrium problems (cf. Liu and Nagurney 2009). Recall that the statement of the modified projection method is as follows, where τ denotes an iteration counter:

10.4.1 The Modified Projection Method

Step 0: Initialization

Start with $X^0 \in \mathscr{K}$ (cf. (10.25)). Set $\tau = 1$ and select a, such that $0 < a \leq \frac{1}{L}$, where L is the Lipschitz continuity constant (cf. (10.35)) for $F(X)$ (cf. (10.25))

Step 1: Construction and Computation

Compute $\overline{X}^{\tau-1}$ by solving the variational inequality subproblem:

$$\langle \overline{X}^{\tau-1} + (aF(X^{\tau-1}) - X^{\tau-1}), X - \overline{X}^{\tau-1} \rangle \geq 0, \quad \forall X \in \mathscr{K}. \tag{10.36}$$

In particular, the explicit formulae for the solution to variational inequality (10.36) are as the following:

$$\bar{Q}_{ik}^{\tau-1} = \max\{0, Q_{ik}^{\tau-1} + a(-\frac{\partial \hat{f}_i(Q^{\tau-1}, \alpha^{F^{\tau-1}})}{\partial Q_{ik}} - \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial Q_{ik}}$$
$$+ \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial Q_{ik}} \times Q_{ih}^{\tau-1}$$

$$+\hat{\rho}_{ik}(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}}) - \sum_{l=1}^{n_{l^i}} \lambda_{il}^{\tau-1}\theta_{il})\}; i = 1,\ldots,I; k = 1,\ldots,n_R, \quad (10.37a)$$

$$\begin{aligned}\bar{Q}_{il}^{F^{\tau-1}} &= \min\{CAP_{il}^F, \max\{0, Q_{il}^{F^{\tau-1}} + a(-\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^{F^{\tau-1}}, q^{F^{\tau-1}})}{\partial Q_{il}^F} \\ &- \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial Q_{il}^F} \\ &+ \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial Q_{il}^F} \times Q_{ih}^{\tau-1} + \lambda_{il}^{\tau-1})\}\}; i = 1,\ldots,I; l = 1,\ldots,n_{l^i},\end{aligned} \quad (10.37b)$$

$$\begin{aligned}\bar{Q}_{jil}^{S^{\tau-1}} &= \min\{CAP_{jil}^S, \max\{0, Q_{jil}^{S^{\tau-1}} + a(-\pi_{jil}^{\tau-1} - \sum_{h=1}^{n_S}\sum_{m=1}^{n_{l^i}} \frac{\partial tc_{ihm}(Q^{S^{\tau-1}})}{\partial Q_{jil}^S} \\ &- \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial Q_{jil}^S} \\ &+ \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial Q_{jil}^S} \times Q_{ih}^{\tau-1} + \lambda_{il}^{\tau-1})\}\}; j = 1,\ldots,n_S; i = 1,\ldots,I; l = 1,\ldots,n_{l^i},\end{aligned} \quad (10.37c)$$

$$\begin{aligned}\bar{q}_{il}^{F^{\tau-1}} &= \min\{\bar{q}_{il}, \max\{0, q_{il}^{F^{\tau-1}} + a(-\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F(Q^{F^{\tau-1}}, q^{F^{\tau-1}})}{\partial q_{il}^F} \\ &- \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial q_{il}^F} \\ &+ \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial q_{il}^F} \times Q_{ih}^{\tau-1})\}\}; i = 1,\ldots,I; l = 1,\ldots,n_{l^i},\end{aligned} \quad (10.37d)$$

$$\begin{aligned}\bar{\alpha}_i^{F^{\tau-1}} &= \min\{1, \max\{0, \alpha_i^{F^{\tau-1}} + a(-\frac{\partial \hat{f}_i(Q^{F^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial \alpha_i^F} \\ &- \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial \alpha_i^F} \\ &+ \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}(Q^{\tau-1}, Q^{F^{\tau-1}}, Q^{S^{\tau-1}}, q^{F^{\tau-1}}, q^{S^{\tau-1}}, \alpha^{F^{\tau-1}})}{\partial \alpha_i^F} \times Q_{ih}^{\tau-1}\}; i = 1,\ldots,I,\end{aligned} \quad (10.37e)$$

$$\bar{\lambda}_{il}^{\tau-1} = \max\{0, \lambda_{il}^{\tau-1} + a(-\sum_{j=1}^{n_S} Q_{jil}^{S^{\tau-1}} - Q_{il}^{F^{\tau-1}} + \sum_{k=1}^{n_R} Q_{ik}^{\tau-1}\theta_{il})\}; i = 1,\ldots,I; l = 1,\ldots,n_{l^i}, \quad (10.37f)$$

$$\bar{\pi}_{jil}^{\tau-1} = \max\{0, \pi_{jil}^{\tau-1} + a(-\sum_{g=1}^{I}\sum_{m=1}^{n_{l^i}} \frac{\partial oc_{jgm}(\pi^{\tau-1})}{\partial \pi_{jil}} + Q_{jil}^{S^{\tau-1}})\}; j = 1,\ldots,n_S; i = 1,\ldots,I; l = 1,\ldots,n_{l^i}, \quad (10.37g)$$

$$\begin{aligned}\bar{q}_{jil}^{S^{\tau-1}} &= \min\{\bar{q}_{il}, \max\{0, q_{jil}^{S^{\tau-1}} + a(-\sum_{m=1}^{n_l} \frac{\partial \hat{f}_{jm}^S(Q^{S^{\tau-1}}, q^{S^{\tau-1}})}{\partial q_{jil}^S} - \sum_{g=1}^{I}\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{c}_{jgm}^S(Q^{S^{\tau-1}}, q^{S^{\tau-1}})}{\partial q_{jil}^S})\}\}; \\ &j = 1,\ldots,n_S; i = 1,\ldots,I; l = 1,\ldots,n_{l^i}.\end{aligned} \quad (10.37h)$$

Step 2: Adaptation

Compute X^τ by solving the variational inequality subproblem:

$$\langle X^\tau + (aF(\overline{X}^{\tau-1}) - X^{\tau-1}), X - X^\tau \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{10.38}$$

The explicit formulae for the solution to variational inequality (10.38) are as the following:

$$Q_{ik}^\tau = \max\Big\{0, Q_{ik}^{\tau-1} + a\Big(-\frac{\partial \hat{f}_i\big(\bar{Q}^{\tau-1},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial Q_{ik}} - \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial Q_{ik}}$$
$$+ \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial Q_{ik}} \times \bar{Q}_{ih}^{\tau-1}$$
$$+\hat{\rho}_{ik}\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big) - \sum_{l=1}^{n_{l^i}} \bar{\lambda}_{il}^{\tau-1}\theta_{il}\Big)\Big\}; i = 1,\ldots,I; k = 1,\ldots,n_R, \tag{10.39a}$$

$$Q_{il}^{F^\tau} = \min\Big\{CAP_{il}^F, \max\Big\{0, Q_{il}^{F^{\tau-1}} + a\Big(-\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F\big(\bar{Q}^{F^{\tau-1}},\bar{q}^{F^{\tau-1}}\big)}{\partial Q_{il}^F}$$
$$- \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial Q_{il}^F}$$
$$+ \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial Q_{il}^F} \times \bar{Q}_{ih}^{\tau-1} + \bar{\lambda}_{il}^{\tau-1}\Big)\Big\}\Big\}; i = 1,\ldots,I; l = 1,\ldots,n_{l^i}, \tag{10.39b}$$

$$Q_{jil}^{S^\tau} = \min\Big\{CAP_{jil}^S, \max\Big\{0, Q_{jil}^{S^{\tau-1}} + a\Big(-\bar{\pi}_{jil}^{\tau-1} - \sum_{h=1}^{n_S}\sum_{m=1}^{n_{l^i}} \frac{\partial tc_{ihm}\big(\bar{Q}^{S^{\tau-1}}\big)}{\partial Q_{jil}^S}$$
$$- \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial Q_{jil}^S}$$
$$+ \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial Q_{jil}^S} \times \bar{Q}_{ih}^{\tau-1}$$
$$+\bar{\lambda}_{il}^{\tau-1}\Big)\Big\}\Big\}; j = 1,\ldots,n_S; i = 1,\ldots,I; l = 1,\ldots,n_{l^i}, \tag{10.39c}$$

$$q_{il}^{F^\tau} = \min\Big\{\bar{q}_{il}, \max\Big\{0, q_{il}^{F^{\tau-1}} + a\Big(-\sum_{m=1}^{n_{l^i}} \frac{\partial \hat{f}_{im}^F\big(\bar{Q}^{F^{\tau-1}},\bar{q}^{F^{\tau-1}}\big)}{\partial q_{il}^F}$$
$$- \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial q_{il}^F}$$
$$+ \sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial q_{il}^F} \times \bar{Q}_{ih}^{\tau-1}\Big)\Big\}\Big\}; i = 1,\ldots,I; l = 1,\ldots,n_{l^i}, \tag{10.39d}$$

$$\alpha_i^{F^\tau} = \min\Big\{1, \max\Big\{0, \alpha_i^{F^{\tau-1}} + a\Big(-\frac{\partial \hat{f}_i\big(\bar{Q}^{F^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial \alpha_i^F}$$
$$- \sum_{h=1}^{n_R} \frac{\partial \hat{c}_{ih}^F\big(\bar{Q}^{\tau-1},\bar{Q}^{F^{\tau-1}},\bar{Q}^{S^{\tau-1}},\bar{q}^{F^{\tau-1}},\bar{q}^{S^{\tau-1}},\bar{\alpha}^{F^{\tau-1}}\big)}{\partial \alpha_i^F}$$

$$+\sum_{h=1}^{n_R} \frac{\partial \hat{\rho}_{ih}\left(\bar{Q}^{\tau-1}, \bar{Q}^{F^{\tau-1}}, \bar{Q}^{S^{\tau-1}}, \bar{q}^{F^{\tau-1}}, \bar{q}^{S^{\tau-1}}, \bar{\alpha}^{F^{\tau-1}}\right)}{\partial \alpha_i^F} \times \bar{Q}_{ih}^{\tau-1}; i = 1, \ldots, I, \tag{10.39e}$$

$$\lambda_{il}^{\tau} = \max\left\{0, \lambda_{il}^{\tau-1} + a\left(-\sum_{j=1}^{n_S} \bar{Q}_{jil}^{S^{\tau-1}} - \bar{Q}_{il}^{F^{\tau-1}} + \sum_{k=1}^{n_R} \bar{Q}_{ik}^{\tau-1}\theta_{il}\right)\right\}; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}, \tag{10.39f}$$

$$\pi_{jil}^{\tau} = \max\left\{0, \pi_{jil}^{\tau-1} + a\left(-\sum_{g=1}^{I} \sum_{m=1}^{n_{l^i}} \frac{\partial oc_{jgm}\left(\bar{\pi}^{\tau-1}\right)}{\partial \pi_{jil}} + \bar{Q}_{jil}^{S^{\tau-1}}\right)\right\}; j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}, \tag{10.39g}$$

$$q_{jil}^{S^{\tau}} = \min\left\{\bar{q}_{il}, \max\left\{0, q_{jil}^{S^{\tau-1}} + a\left(-\sum_{m=1}^{n_l} \frac{\partial \hat{f}_{jm}^S\left(\bar{Q}^{S^{\tau-1}}, \bar{q}^{S^{\tau-1}}\right)}{\partial q_{jil}^S} - \sum_{g=1}^{I} \sum_{m=1}^{n_{l^i}} \frac{\partial \hat{c}_{jgm}^S\left(\bar{Q}^{S^{\tau-1}}, \bar{q}^{S^{\tau-1}}\right)}{\partial q_{jil}^S}\right)\right\}\right\};$$
$$j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{l^i}. \tag{10.39h}$$

Step 3: Convergence Verification

If $| X^{\tau} - X^{\tau-1} | \leq \epsilon$, with $\epsilon > 0$, a prespecified tolerance, then stop; else set $\tau := \tau + 1$, and go to step 1.

Theorem 10.9: Convergence
If Assumption 10.1 is satisfied, and the function $F(X)$ is monotone and Lipschitz continuous, then the modified projection method described above converges to the solution of variational inequality (10.25).

Proof: According to Korpelevich (1977) and Nagurney (1999), the modified projection method converges to the solution of the variational inequality problem of the form (10.25), provided that the function F that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 10.4, monotonicity follows from Theorem 10.5, and Lipschitz continuity, in turn, follows from Theorem 10.8. □

With strict monotonicity being a stronger condition than monotonicity, the algorithm will converge given the existence of a solution (cf. Assumption 10.1 and Theorem 10.4) and the strict monotonicity (cf. Theorem 10.6) and Lipschitz continuity (cf. Theorem 10.8) of the $F(X)$ that enters variational inequality (10.25).

10.5 Numerical Examples and Sensitivity Analysis

In this section, we apply the modified projection method, as described in Sect. 10.4, to several numerical examples accompanied by extensive sensitivity analysis. The modified projected method was implemented in Matlab on a Lenovo Z580. We set $a = 0.003$ in the algorithm with the convergence tolerance $\epsilon = 10^{-4}$. The product and component quantities are initialized to 30.00 and the prices, quality levels, quality preservation levels, and the Lagrange multipliers to 0.00.

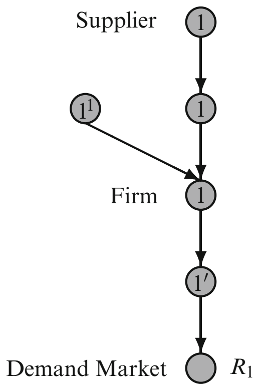

Fig. 10.1 The supply chain network topology for Example 10.1

Example 10.1

Consider the supply chain network topology given in Fig. 10.1 in which firm 1 serves demand market R_1 and procures the components of its product from supplier 1. The firm also has the option of producing the components that it needs. The product of firm 1 requires only one component 1^1. 2 units of 1^1 are needed for producing one unit of firm 1's product. Thus,

$$\theta_{11} = 2.$$

Component 1^1 corresponds to node 1 in the second tier and node 1^1 in the third tier in Fig. 10.1 below.

The data are as follows.

The capacity of the supplier is:

$$CAP^S_{111} = 120.$$

The firm's capacity for producing its component is:

$$CAP^F_{11} = 80.$$

The value that represents the perfect component quality is:

$$\bar{q}_{11} = 75.$$

The supplier's production cost is:

$$\hat{f}^S_{11}(Q^S_{111}, q^S_{111}) = 5Q^S_{111} + 0.8(q^S_{111} - 62.5)^2.$$

The supplier's transportation cost is:

$$\hat{c}^S_{111}(Q^S_{111}, q^S_{111}) = 0.5Q^S_{111} + 0.2(q^S_{111} - 125)^2 + 0.3Q^S_{111}q^S_{111},$$

and its opportunity cost is:

$$oc_{111}(\pi_{111}) = 0.7(\pi_{111} - 100)^2.$$

The firm's assembly cost is:

$$\hat{f}_1(Q_{11}, \alpha^F_1) = 0.75Q^2_{11} + 200\alpha^{F2}_1 + 200\alpha^F_1 + 25Q_{11}\alpha^F_1.$$

The firm's production cost for producing its component is:

$$\hat{f}^F_{11}(Q^F_{11}, q^F_{11}) = 2.5Q^{F2}_{11} + 0.5(q^F_{11} - 60)^2 + 0.1Q^F_{11}q^F_{11},$$

and its transaction cost is:

$$tc_{111}(Q^S_{111}) = 0.5Q^{S2}_{111} + Q^S_{111} + 100.$$

The firm's transportation cost for shipping its product to the demand market is:

$$c^F_{11}(Q_{11}, q_1) = 0.5Q^2_{11} + 0.02q^2_1 + 0.1Q_{11}q_1,$$

and the demand price function at demand market R_1 is:

$$\rho_{11}(d_{11}, q_1) = -d_{11} + 0.7q_1 + 1{,}000,$$

where $q_1 = \alpha^F_1\omega_{11}\frac{Q^F_{11}q^F_{11} + Q^S_{111}q^S_{111}}{Q^F_{11} + Q^S_{111}}$ and $\omega_{11} = 1$.

The equilibrium solution that we obtain using the modified projection method is:

$$Q^*_{11} = 89.26, \quad Q^{F*}_{11} = 60.16, \quad Q^{S*}_{111} = 118.38, \quad q^{F*}_{11} = 71.17,$$
$$q^{S*}_{111} = 57.25, \quad \pi^*_{11} = 184.53, \quad \alpha^{F*}_1 = 1.00, \quad \lambda^*_{11} = 305.25.$$

with the induced demand, demand price, and product quality being

$$d_{11} = 89.26, \quad \rho_{11} = 954.10, \quad q_1 = 61.94.$$

The profit of the firm is 33,331.69 and the profit of the supplier is 13,218.67.

For this example, the eigenvalues of the symmetric part of the Jacobian matrix of $G(Y)$ (cf. (10.27)) are 0.0016, 0.0101, 0.0140, 0.0169, 0.0439, 0.0503, 5.5468, which are all positive. Therefore, $\nabla G(Y^*)$ is positive definite, and $G(Y^*)$ is locally strictly monotone at Y^*.

10.5.1 Sensitivity Analysis

In Example 10.1, the capacities of the firm and the supplier do not constrain the production of the components, since, at the equilibrium, the component quantities are lower than the associated capacities. However, in some cases, due to disruptions to capacities, such as disasters and strikes, firms and suppliers may not always be able to operate under desired capacities. In this sensitivity analysis, we investigate the impacts of the capacities that constrain the production of the components on the quantities, prices, quality levels, and the profits of the firm and the supplier.

First, we maintain the capacity of the firm at 80, and vary the capacity of the supplier from 0 to 20, 40, 60, 80, 100, and 120. The results of equilibrium quantities, quality levels, prices, and profits are shown in Figs. 10.2 and 10.3.

As indicated in Fig. 10.2.b, when the capacity of the supplier is 0, the firm has to produce the components for its product by itself, at full capacity, which is 80. This production pressure limits the firm's ability to produce with high quality, which causes a low in-house component quality (cf. Fig. 10.2d). Based on the data in this example, purchasing components from the supplier is always cheaper than producing them in-house. Therefore, as the capacity of the supplier increases, the firm buys more components from the supplier and tends to be more dependent on the supplier in component production. Thus, the contracted component quantity increases (cf. Fig. 10.2a), and the in-house component quantity decreases (cf. Fig. 10.2b).

In addition, with more components provided by the supplier, the firm is now able to assemble more products for profit maximization, which leads to an increase in demand (cf. Fig. 10.2e) and in profit (cf. Fig. 10.3f).

Since there is no competition on the supplier's side, as the firm becomes more dependent on the supplier, it charges more to the firm to maximize its profit (cf. Fig. 10.3d). For the same reason, the supplier's incentive to improve quality decreases, which leads to a reduction in contracted quality (cf. Fig. 10.2c). After the capacity of the supplier achieves a certain value (e.g., 100), as the capacity of the supplier increases, the contracted quantity and price keep increasing.

This results in an extremely high payment to the supplier and a large transaction cost, and, hence, a decline in the profit of the firm (cf. Fig. 10.3f). The profit of the supplier always increases as its capacity expands (cf. Fig. 10.3e).

Moreover, when the supplier's capacity is 20, the in-house component quality achieves a higher value than before (cf. Fig. 10.3d), because the firm is able to pay for quality improvement for more profit at this point. However, it decreases ever after, since, given the high payment to the supplier and the high transaction cost, the firm is unable to produce a higher quality anymore. This also explains the trend of the product quality (cf. Fig. 10.3f) and that of the demand price (cf. Fig. 10.3c). The highest product quality and the highest demand price are achieved when the supplier's capacity is 20, after which they decrease.

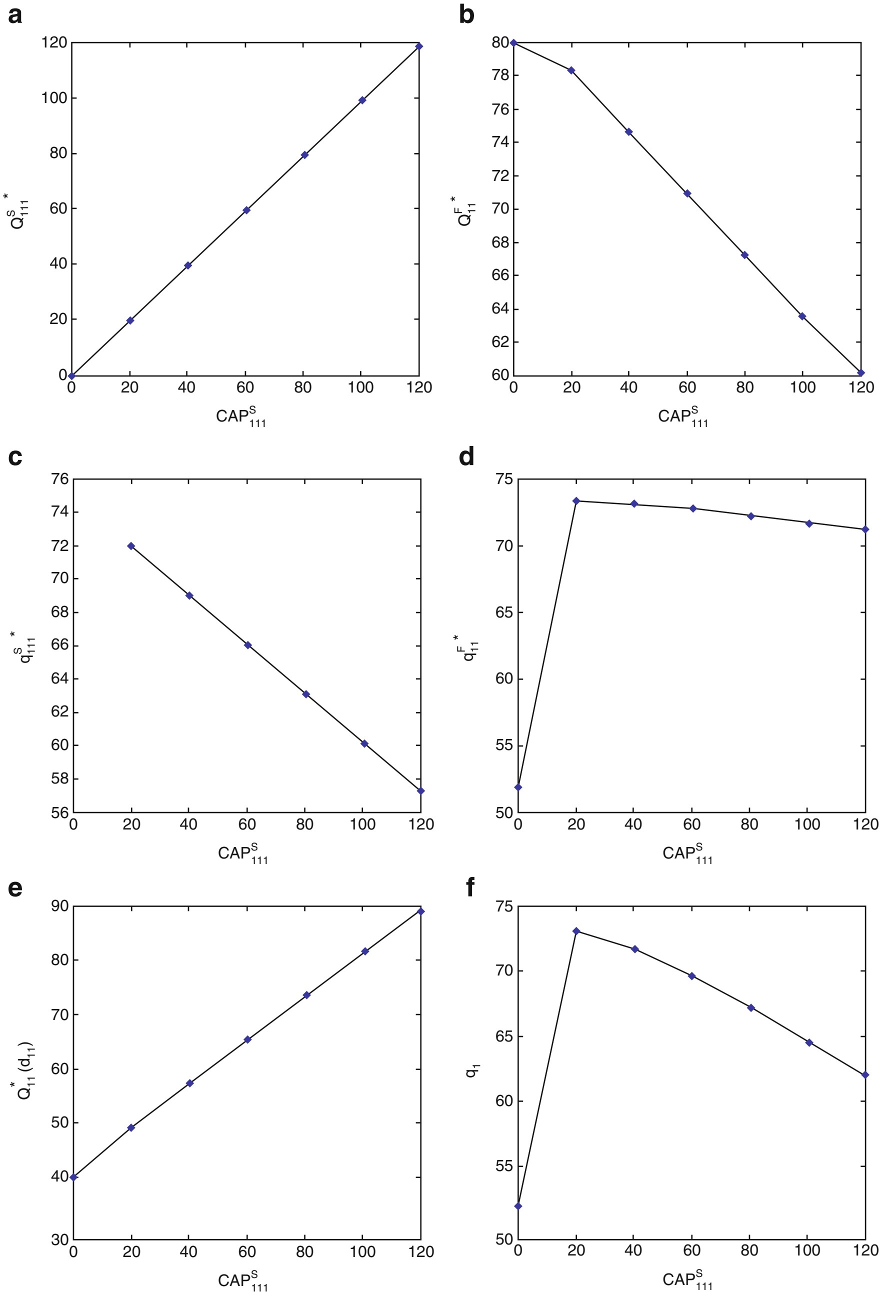

Fig. 10.2 Equilibrium component quantities, equilibrium component quality levels, equilibrium product quantity (demand), and product quality as the capacity of the supplier varies. (**a**) Equilibrium contracted component quantity. (**b**) Equilibrium in-house component quantity. (**c**) Equilibrium contracted component quality. (**d**) Equilibrium in-house component quality. (**e**) Equilibrium product quantity (demand). (**f**) Product quality

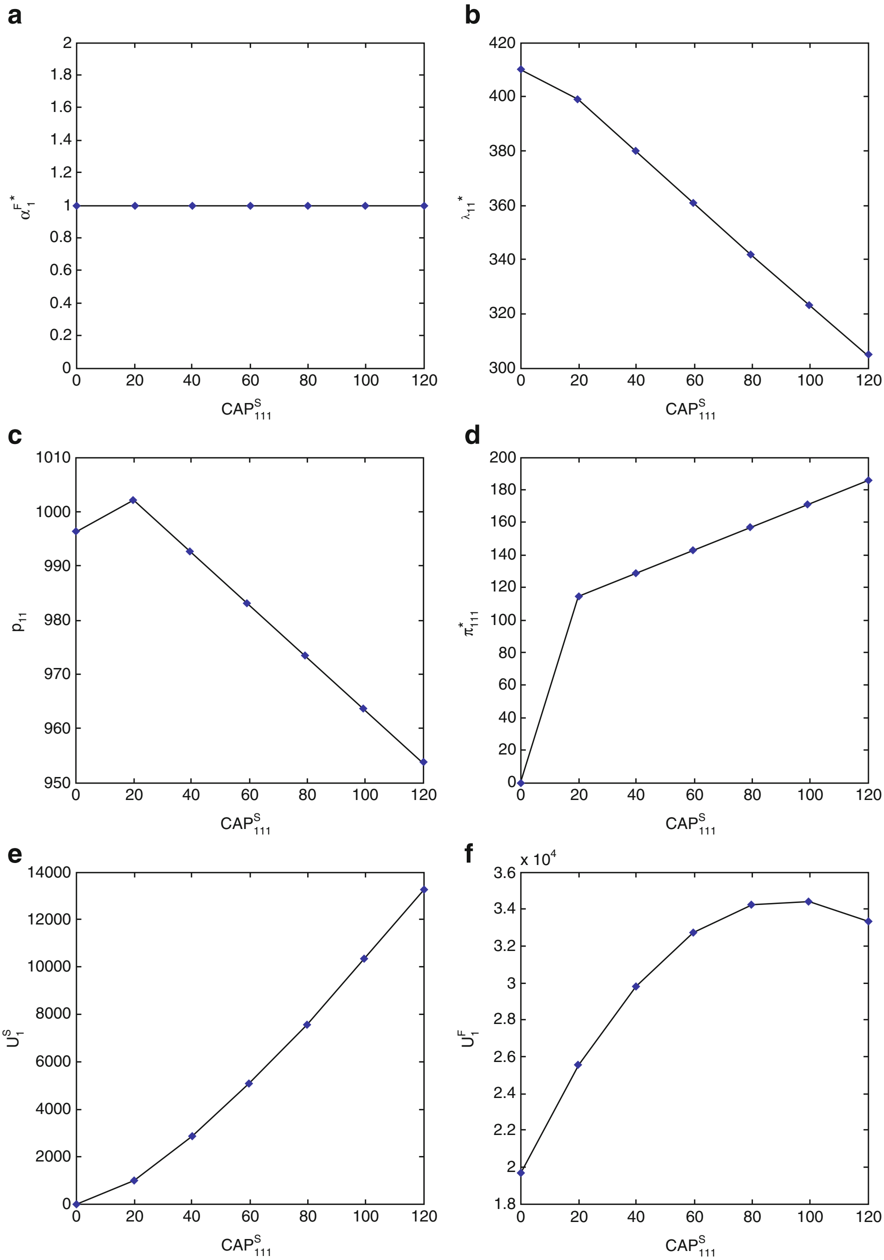

Fig. 10.3 Equilibrium quality preservation level, equilibrium lagrange multiplier, demand price, equilibrium contracted price, the supplier's profit, and the firm's profit as the capacity of the supplier varies. (**a**) Equilibrium quality preservation level. (**b**) Equilibrium Lagrange multiplier. (**c**) Demand price. (**d**) Equilibrium contracted price. (**e**) Profit of the supplier. (**f**) Profit of the firm

Therefore, in the case of this example, the supplier would want to prevent disruptions to its own capacity in order to maintain a good profit. However, such disruptions may be beneficial for the firm's profit and the quality of the product at the demand market. Hence, it may be wise for the firm to contract with competing suppliers who have capacities that are not so high as to harm the profit of the firm.

As already noted, when the capacity of the supplier is 0, the quantity of the in-house produced component is bounded by the capacity of the firm, which is 80. This happens because the firm can actually produce more to improve its profit with higher capacity. When the capacity of the firm is 80.78 or higher, the in-house component production does not have to operate at full capacity.

We then maintain the capacity of the supplier at 120, and vary the capacity of the firm from 0 to 20, 40, 60, and 80. The results are reported in Figs. 10.4 and 10.5.

Most of the trends in Figs. 10.4 and 10.5 follow a similar logic as that for Figs. 10.2 and 10.3. However, as revealed in Fig. 10.5e, f, as the capacity of the firm increases, the profit of the supplier decreases, but that of the firm increases. With higher capacity, the firm is capable of producing more to satisfy the greater demand by itself, which weakens its dependence on the supplier and leads to a decline in the supplier's profit. Therefore, disruptions to the firm's capacity would benefit the profit of the supplier but jeopardize the profit of the firm and the quality of the product at the demand market. Thus, the supplier would want to produce for firms who have low capacities and are, hence, more dependent on suppliers in component production.

As shown in Fig. 10.4a, when the capacity of the firm is 0, 20, and 40, the quantity of contracted component production is bounded by the capacity of the supplier. Actually, when the capacity of the supplier is no less than 141.71, 133.99, and 126.20, respectively, the supplier does not need to operate at full capacity.

10.5.2 Investing in Capacity Changing

The sensitivity analysis sheds light on the investments in capacity changing for the supplier and for the firm. If the investment is higher than the associated profit improvement, it is not wise for the supplier or the firm to invest in themselves' or each other's capacity changing. Tables 10.3 and 10.4 show the maximum acceptable investments for capacity changing for this sensitivity analysis. The first number in each cell is the maximum acceptable investment for the supplier and the second is that for the firm. In the italicized cells, the two numbers are with different signs.

In Tables 10.3 and 10.4, for the cells in which both numbers are negative, it is not wise for the firm or the supplier to change the capacities, because their profits would decrease with the associated capacity change. For the italicized cells that are with two opposite-sign numbers, the one with the negative number should prevent the other from investing in the associated capacity change or it should ask the other for a compensation which will prevent its profit from being compromised. This situation may occur only in four cases when the supplier's capacity varies (cf. Table 10.3).

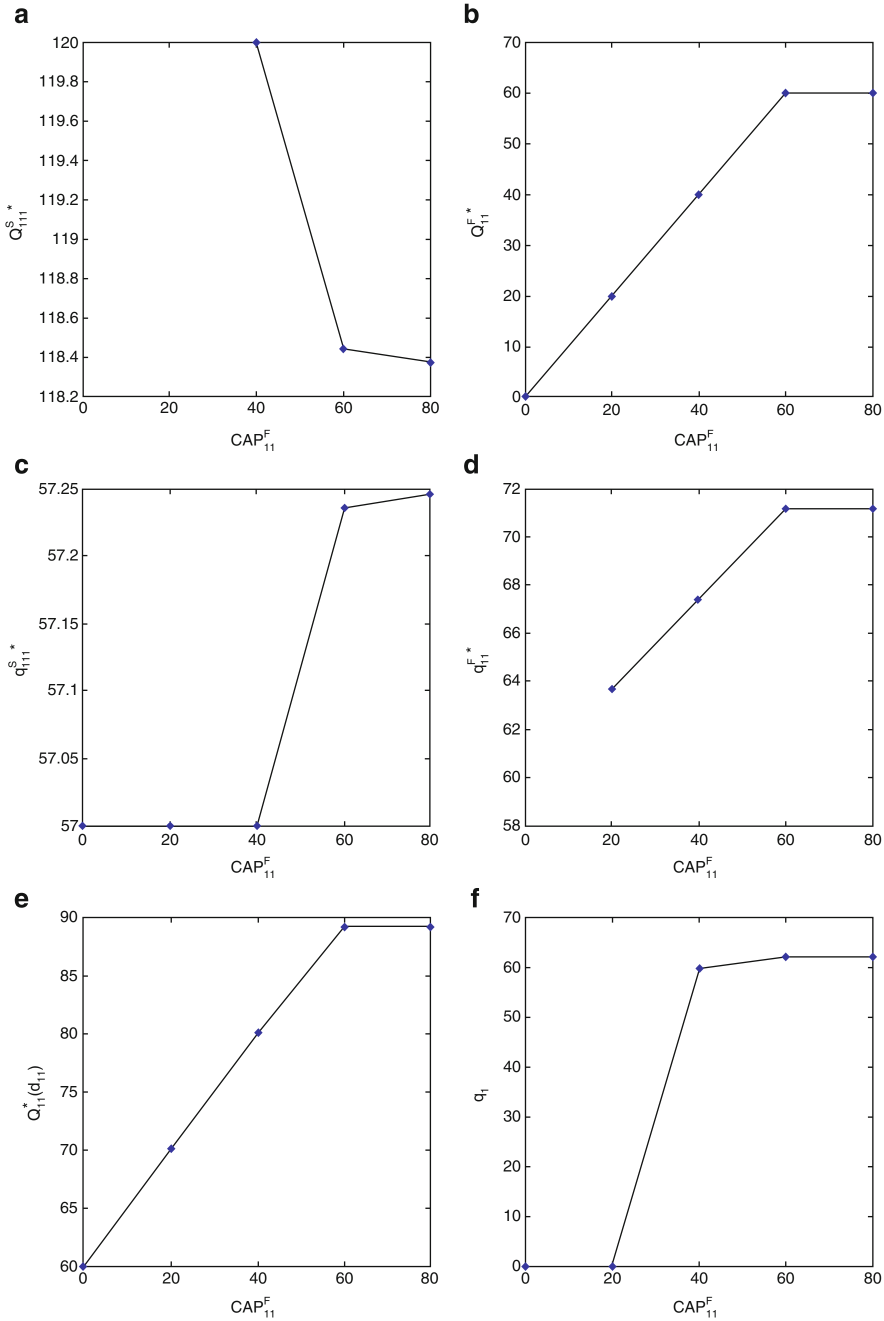

Fig. 10.4 Equilibrium component quantities, equilibrium component quality levels, equilibrium product quantity (demand), and product quality as the capacity of the firm varies. (**a**) Equilibrium contracted component quantity. (**b**) Equilibrium in-house component quantity. (**c**) Equilibrium contracted component quality. (**d**) Equilibrium in-house component quality. (**e**) Equilibrium product quantity (demand). (**f**) Product quality

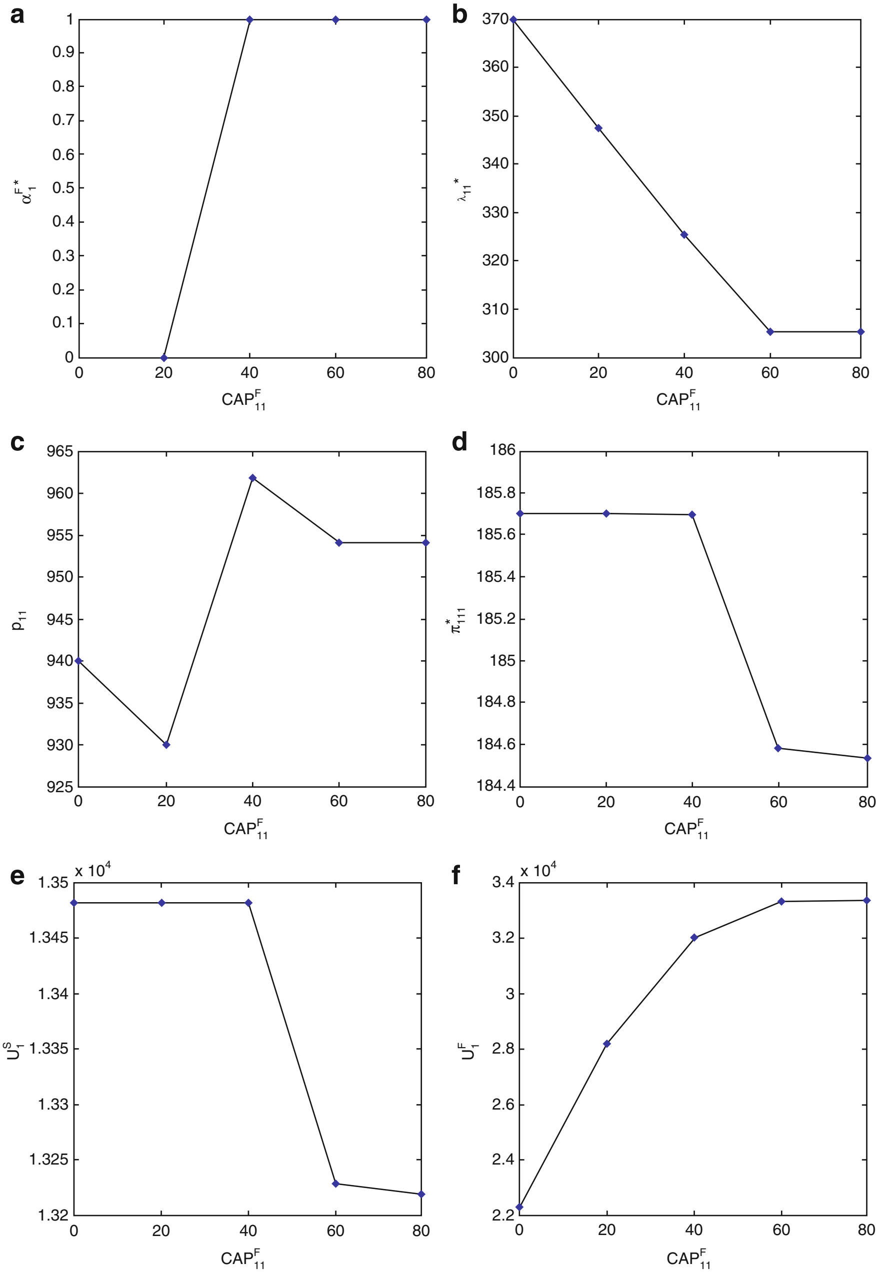

Fig. 10.5 Equilibrium quality preservation level, equilibrium lagrange multiplier, demand price, equilibrium contracted price, the supplier's profit, and the firm's profit as the capacity of the firm varies. (**a**) Equilibrium quality preservation level. (**b**) Equilibrium Lagrange multiplier. (**c**) Demand price. (**d**) Equilibrium contracted price. (**e**) Profit of the supplier. (**f**) Profit of the firm

However, in Table 10.3, it happens very often when the firm's capacity varies, which is consistent with the results in the above sensitivity analysis. For the numbers that are 0, the associated profits will not be affected by the corresponding capacity changes.

In addition, if there is a capacity changing offer that costs more than the summation of the two numbers in the associated cell, it is not worthwhile for the supplier or the firm to accept the offer, since more profit cannot be obtained by doing so. If the offer costs less, the two parties should consider investing in the associated capacity change, and, if possible, negotiate on the separation of the payment between themselves.

Example 10.2 In Example 10.2, there are two firms competing with each other with differentiated but substitutable products in demand market R_1. The firms can procure the components for producing their products from suppliers 1 and 2 who also compete noncooperatively; they can also produce the components themselves.

Two components are required by the product of firm 1, components 1^1 and 2^1. 1 unit of 1^1 and 2 units of 2^1 are required for producing 1 unit of firm 1's product. In order to produce 1 unit firm 2's product, 2 units of 1^2 and 1 unit of 1^2 are needed. Therefore,

$$\theta_{11} = 1, \quad \theta_{12} = 2, \quad \theta_{21} = 2, \quad \theta_{22} = 1.$$

The ratio of the importance of the quality of the components to the quality of one unit product is:

$$\omega_{11} = 0.2, \quad \omega_{12} = 0.8, \quad \omega_{21} = 0.4, \quad \omega_{22} = 0.6.$$

The network topology of Example 10.2 is as in Fig. 10.6. Components 1^1 and 2^1 are the same component, which correspond to nodes 1's in the second tier of the figure. Components 2^1 and 2^2 are the same component, and they correspond to nodes 2's in the second tier.

The other data are as follows:

The capacities of the suppliers are:

$$CAP^S_{111} = 80, \quad CAP^S_{112} = 100, \quad CAP^S_{121} = 100, \quad CAP^S_{122} = 60,$$
$$CAP^S_{211} = 60, \quad CAP^S_{212} = 100, \quad CAP^S_{221} = 100, \quad CAP^S_{222} = 50.$$

The firms' capacities for in-house component production are:

$$CAP^F_{11} = 30, \quad CAP^F_{12} = 30, \quad CAP^F_{21} = 30, \quad CAP^F_{22} = 30.$$

The values representing the perfect component quality are:

$$\bar{q}_{11} = 60, \quad \bar{q}_{12} = 75, \quad \bar{q}_{21} = 60, \quad \bar{q}_{22} = 75.$$

Table 10.3 Maximum acceptable investments (×10^3) for capacity changing when the capacity of the firm maintains 80 but that of the supplier varies

From \ To	$CAP^S_{111}=0$	20	40	60	80	100	120
$CAP^S_{111}=0$	–	0.97, 5.89	2.86, 10.17	5.08, 13.09	7.57, 14.62	10.37, 14.77	13.22, 13.69
20	−0.97, −5.89	–	1.90, 4.28	4.09, 7.20	6.60, 8.73	9.40, 8.88	12.25, 7.80
40	−2.86, −10.17	−1.90, −4.28	–	2.20, 2.92	4.70, 4.45	7.51, 4.60	10.36, 3.52
60	−5.06, −13.09	−4.09, −7.20	−2.20, −2.92	–	2.50, 1.53	5.31, 1.68	8.16, 0.60
80	−7.57, −14.62	−6.60, −8.73	−4.70, −4.45	−2.50, −1.53	–	2.81, 0.15	*5.66, −0.93*
100	−10.37, −14.77	−9.40, −8.88	−7.51, −4.60	−5.31, −1.68	−2.81, −0.15	–	*2.85, −1.08*
120	−13.22, −13.69	−12.25, −7.80	−10.36, −3.52	−8.16, −0.60	*−5.65, 0.93*	*−2.85, 1.08*	–

Table 10.4 Maximum acceptable investments ($\times 10^3$) for capacity changing when the capacity of the supplier maintains 120 but that of the firm varies

From \ To	$CAP^F_{11} = 0$	20	40	60	80
$CAP^F_{11} = 0$	–	0.00, 5.94	0.00, 9.77	*−0.25, 11.10*	*−0.26, 11.10*
20	0.00, −5.94	–	0.00, 3.83	*−0.25, 5.16*	*−0.26, 5.16*
40	0.00, −9.77	0.00, −3.83	–	*−0.25, 1.33*	*−0.26, 1.33*
60	*0.25, −11.10*	*0.25, −5.16*	*0.25, −1.33*	–	*−0.01, 0.004*
80	*0.26, −11.10*	*0.26, −5.16*	*0.26, −1.33*	*0.01, −0.004*	–

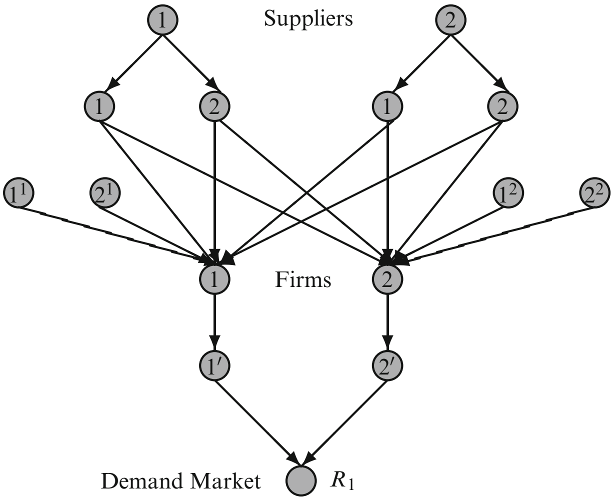

Fig. 10.6 The supply chain network topology for Example 10.2

The suppliers' production costs are:

$$\hat{f}^S_{11}(Q^S_{111}, Q^S_{121}, q^S_{111}, q^S_{121}, q^S_{211}, q^S_{221})$$
$$= 0.4(Q^S_{111} + Q^S_{121}) + 1.5(q^S_{111} - 50)^2 + 1.5(q^S_{121} - 50)^2 + q^S_{211} + q^S_{221},$$
$$\hat{f}^S_{12}(Q^S_{112}, Q^S_{122}, q^S_{112}, q^S_{122}, q^S_{212}, q^S_{222})$$
$$= 0.4(Q^S_{112} + Q^S_{122}) + 2(q^S_{112} - 45)^2 + 2(q^S_{122} - 45)^2 + q^S_{212} + q^S_{222},$$
$$\hat{f}^S_{21}(Q^S_{211}, Q^S_{221}, q^S_{211}, q^S_{221}, q^S_{111}, q^S_{121})$$
$$= Q^S_{211} + Q^S_{221} + 2(q^S_{211} - 31.25)^2 + 2(q^S_{221} - 31.25)^2 + q^S_{111} + q^S_{121},$$
$$\hat{f}^S_{12}(Q^S_{212}, Q^S_{222}, q^S_{212}, q^S_{222}, q^S_{112}, q^S_{122})$$
$$= Q^S_{212} + Q^S_{222} + (q^S_{212} - 85)^2 + (q^S_{222} - 85)^2 + q^S_{112} + q^S_{122}.$$

Their transportation costs are:

$$\hat{c}^S_{111}(Q^S_{111}, q^S_{111}) = 0.2Q^S_{111} + 1.2(q^S_{111} - 41.67)^2,$$
$$\hat{c}^S_{112}(Q^S_{112}, q^S_{112}) = 0.1Q^S_{112} + 1.2(q^S_{112} - 37.5)^2,$$
$$\hat{c}^S_{121}(Q^S_{121}, q^S_{121}) = 0.2Q^S_{121} + 1.4(q^S_{121} - 39.29)^2,$$
$$\hat{c}^S_{122}(Q^S_{122}, q^S_{122}) = 0.1Q^S_{122} + 1.1(q^S_{122} - 36.36)^2,$$
$$\hat{c}^S_{211}(Q^S_{211}, q^S_{211}) = 0.3Q^S_{211} + 1.3(q^S_{211} - 30.77)^2,$$
$$\hat{c}^S_{212}(Q^S_{212}, q^S_{212}) = 0.4Q^S_{212} + 1.7(q^S_{212} - 32.35)^2,$$
$$\hat{c}^S_{221}(Q^S_{221}, q^S_{221}) = 0.2Q^S_{221} + 1.3(q^S_{221} - 30.77)^2,$$
$$\hat{c}^S_{222}(Q^S_{222}, q^S_{222}) = 0.1Q^S_{222} + 1.5(q^S_{222} - 30)^2.$$

The opportunity costs of the suppliers are:

$$oc_{111}(\pi_{111}, \pi_{211}) = 5(\pi_{111} - 80)^2 + 0.5\pi_{211},$$
$$oc_{112}(\pi_{112}, \pi_{212}) = 9(\pi_{112} - 80)^2 + \pi_{212},$$
$$oc_{121}(\pi_{121}, \pi_{221}) = 5(\pi_{121} - 100)^2 + \pi_{221},$$
$$oc_{122}(\pi_{122}, \pi_{222}) = 7.5(\pi_{122} - 50)^2 + 0.1\pi_{222},$$
$$oc_{211}(\pi_{211}, \pi_{111}) = 5(\pi_{211} - 50)^2 + 2\pi_{111},$$
$$oc_{212}(\pi_{212}, \pi_{112}) = 8(\pi_{212} - 70)^2 + 0.5\pi_{112},$$
$$oc_{221}(\pi_{221}, \pi_{121}) = 9(\pi_{221} - 60)^2 + \pi_{121},$$
$$oc_{222}(\pi_{222}, \pi_{122}) = 8(\pi_{222} - 60)^2 + 0.5\pi_{122}.$$

The firms' assembly costs are:

$$\hat{f}_1(Q_{11}, \alpha^F_1) = 3Q^2_{11} + 0.5Q_{11}\alpha^F_1 + 100\alpha^{F^2}_1 + 50\alpha^F_1,$$
$$\hat{f}_2(Q_{21}, \alpha^F_2) = 2.75Q^2_{21} + 0.6Q_{21}\alpha^F_2 + 100\alpha^{F^2}_2 + 50\alpha^F_2.$$

Their production costs for producing components are:

$$\hat{f}^F_{11}(Q^F_{11}, q^F_{11}) = Q^{F^2}_{11} + 0.0001Q^F_{11}q^F_{11} + 1.1(q^F_{11} - 36.36)^2,$$
$$\hat{f}^F_{12}(Q^F_{12}, q^F_{12}) = 1.25Q^{F^2}_{12} + 0.0001Q^F_{12}q^F_{12} + 1.2(q^F_{12} - 41.67)^2,$$
$$\hat{f}^F_{21}(Q^F_{21}, q^F_{21}) = Q^{F^2}_{21} + 0.0001Q^F_{21}q^F_{21} + 1.5(q^F_{21} - 33.33)^2,$$
$$\hat{f}^F_{22}(Q^F_{22}, q^F_{22}) = 0.75Q^{F^2}_{22} + 0.0001Q^F_{22}q^F_{22} + 1.25(q^F_{22} - 36)^2,$$

The transaction costs are:

$$tc_{111}(Q^S_{111}) = 0.5Q^{S^2}_{111} + Q^S_{111} + 100, \quad tc_{112}(Q^S_{112}) = 0.5Q^{S^2}_{112} + 0.5Q^S_{112} + 150,$$

$$tc_{121}(Q^S_{211}) = 0.75Q^{S^2}_{211} + 0.75Q^S_{211} + 150, \quad tc_{122}(Q^S_{212}) = Q^{S^2}_{212} + Q^S_{212} + 100,$$

$$tc_{211}(Q^S_{121}) = 0.75Q^{S^2}_{121} + Q^S_{121} + 150, \quad tc_{212}(Q^S_{122}) = 0.5Q^{S^2}_{122} + 0.75Q^S_{122} + 100,$$

$$tc_{221}(Q^S_{221}) = 0.8Q^{S^2}_{221} + 0.25Q^S_{221} + 100, \quad tc_{222}(Q^S_{222}) = 0.5Q^{S^2}_{222} + Q^S_{222} + 175,$$

The firms' transportation costs are:

$$c^F_{11}(Q_{11}, q_1) = 3Q^2_{11} + 0.3Q_{11}q_1 + 0.25q_1,$$

$$c^F_{21}(Q_{21}, q_2) = 3Q^2_{21} + 0.3Q_{21}q_2 + 0.1q_2,$$

and the demand price functions are:

$$\rho_{11}(d_{11}, d_{21}, q_1, q_2) = -3d_{11} - 1.3d_{21} + q_1 + 0.74q_2 + 2{,}200,$$

$$\rho_{21}(d_{21}, d_{11}, q_2, q_1) = -3.5d_{21} - 1.4d_{11} + 1.1q_2 + 0.9q_1 + 1{,}800,$$

where $q_1 = \alpha^F_1(\omega_{11}\frac{Q^F_{11}q^F_{11}+Q^S_{111}q^S_{111}+Q^S_{211}q^S_{211}}{Q^F_{11}+Q^S_{111}+Q^S_{211}} + \omega_{12}\frac{Q^F_{12}q^F_{12}+Q^S_{112}q^S_{112}+Q^S_{212}q^S_{212}}{Q^F_{12}+Q^S_{112}+Q^S_{212}})$ and $q_2 = \alpha^F_2(\omega_{21}\frac{Q^F_{21}q^F_{21}+Q^S_{121}q^S_{121}+Q^S_{221}q^S_{221}}{Q^F_{21}+Q^S_{121}+Q^S_{221}} + \omega_{22}\frac{Q^F_{22}q^F_{22}+Q^S_{122}q^S_{122}+Q^S_{222}q^S_{222}}{Q^F_{22}+Q^S_{122}+Q^S_{222}})$.

The modified projection method converges to the following equilibrium solution:

$$Q^*_{11} = 93.56, \quad Q^*_{21} = 71.34,$$

$$Q^{F^*}_{11} = 30.00, \quad Q^{F^*}_{12} = 30.00, \quad Q^{F^*}_{21} = 30.00, \quad Q^{F^*}_{22} = 30.00,$$

$$Q^{S^*}_{111} = 27.37, \quad Q^{S^*}_{112} = 100.00, \quad Q^{S^*}_{121} = 45.44, \quad Q^{S^*}_{122} = 23.35,$$

$$Q^{S^*}_{211} = 36.19, \quad Q^{S^*}_{212} = 57.12, \quad Q^{S^*}_{221} = 67.24, \quad Q^{S^*}_{222} = 17.99,$$

$$q^{F^*}_{11} = 38.26, \quad q^{F^*}_{12} = 45.15, \quad q^{F^*}_{21} = 34.93, \quad q^{F^*}_{22} = 41.71,$$

$$q^{S^*}_{111} = 46.30, \quad q^{S^*}_{112} = 42.19, \quad q^{S^*}_{121} = 44.83, \quad q^{S^*}_{122} = 41.94,$$

$$q^{S^*}_{211} = 31.06, \quad q^{S^*}_{212} = 51.85, \quad q^{S^*}_{221} = 31.06, \quad q^{S^*}_{222} = 52.00,$$

$$\pi^*_{111} = 82.74, \quad \pi^*_{112} = 85.56, \quad \pi^*_{121} = 104.54, \quad \pi^*_{122} = 51.56,$$

$$\pi^*_{211} = 53.62, \quad \pi^*_{212} = 73.57, \quad \pi^*_{221} = 63.74, \quad \pi^*_{222} = 61.12,$$

$$\alpha^{F^*}_1 = 1.00, \quad \alpha^{F^*}_2 = 1.00,$$

$$\lambda^*_{11} = 109.83, \quad \lambda^*_{12} = 187.06, \quad \lambda^*_{21} = 172.34, \quad \lambda^*_{22} = 76.58,$$

and the induced demands, demand prices, and product quality levels are:

$$d_{11} = 93.56, \quad d_{21} = 71.34, \quad \rho_{11} = 1,901.07, \quad \rho_{21} = 1,504.22,$$
$$q_1 = 44.06, \quad q_2 = 41.13.$$

The firms' profits are 94,610.69 and 57,787.69, respectively, and those of the suppliers are 15,671.13 and 6,923.20.

The eigenvalues of the symmetric part of the Jacobian matrix of $G(Y)$ (cf. (27)) are 0.0089, 0.0098, 0.0100, 0.0102, 0.0107, 0.0135, 0.0147, 0.0151, 0.0158, 0.0164, 0.0198, 0.0198, 0.0201, 0.0224, 0.0254, 0.0298, 0.0409, 0.0492, 0.0540, 0.0564, 0.0578, 0.0605, 0.0650, 0.0660, 0.1000, 0.1000, 0.1000, 0.1063, 0.1500, 0.1600, 0.1600, 0.1600, 0.1800, 0.1800, 2.0280, 2.1399, which are all positive. Thus, $G(Y^*)$ is locally strictly monotone at Y^*.

10.5.3 Supplier Disruption Analysis and the Values of Suppliers

As mentioned in Sect. 10.1, the manufacturing plants of suppliers may be located in different geographical locations around the globe, which increases the vulnerability of the supply chain networks of the firms to the disruptions that happen to the suppliers, such as those caused by natural disasters. In this analysis, we model and analyze the impacts of the disruptions to suppliers 1 and 2 on the profits of the firms and the demands, prices, and quality levels of the products.

We also evaluate the values of the two suppliers and which one of them is more important to the firms. With the values of the suppliers and the importance level of them to the firms, the firms can make more specific and targeted efforts in their supplier management strategies and in the contingency plans in handling the disruptions to their suppliers.

First, we present the following disruption. The data are as in Example 10.2, except that supplier 1 is no longer available for the firms to contract with or to produce or transport the components needed. The supply chain network topology with this disruption is presented in Fig. 10.7.

The equilibrium solution achieved by the modified projection method is:

$$Q_{11}^* = 65.00, \quad Q_{21}^* = 65.00,$$
$$Q_{11}^{F^*} = 30.00, \quad Q_{12}^{F^*} = 30.00, \quad Q_{21}^{F^*} = 30.00, \quad Q_{22}^{F^*} = 30.00,$$
$$Q_{111}^{S^*} = 0.00, \quad Q_{112}^{S^*} = 0.00, \quad Q_{121}^{S^*} = 0.00, \quad Q_{122}^{S^*} = 0.00,$$
$$Q_{211}^{S^*} = 35.00, \quad Q_{212}^{S^*} = 100.00, \quad Q_{221}^{S^*} = 100.00, \quad Q_{222}^{S^*} = 35.00,$$
$$q_{11}^{F^*} = 38.26, \quad q_{12}^{F^*} = 45.16, \quad q_{21}^{F^*} = 34.93, \quad q_{22}^{F^*} = 41.75,$$

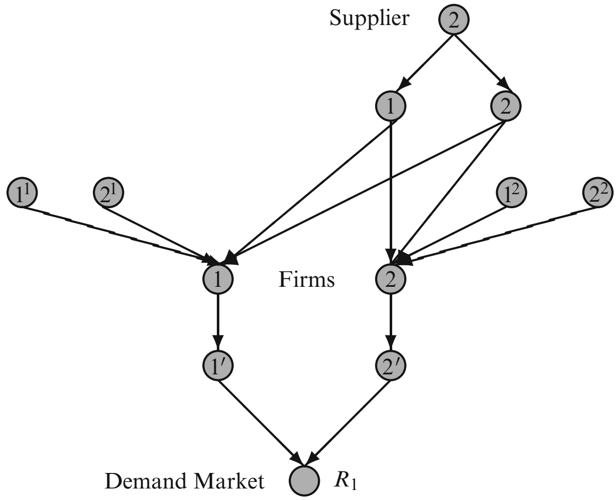

Fig. 10.7 The supply chain network topology with disruption to supplier 1

$$q^{S*}_{211} = 31.06, \quad q^{S*}_{212} = 51.85, \quad q^{S*}_{221} = 31.06, \quad q^{S*}_{222} = 52.00,$$
$$\pi^*_{211} = 53.50, \quad \pi^*_{212} = 76.25, \quad \pi^*_{221} = 65.56, \quad \pi^*_{222} = 62.19,$$
$$\alpha^{F^*}_1 = 1.00, \quad \alpha^{F^*}_2 = 1.00,$$
$$\lambda^*_{11} = 107.53, \quad \lambda^*_{12} = 448.93, \quad \lambda^*_{21} = 242.02, \quad \lambda^*_{22} = 95.98,$$

and the induced demands, demand prices, and product quality levels are:

$$d_{11} = 65.00, \quad d_{21} = 65.00 \quad \rho_{11} = 1{,}998.07, \quad \rho_{21} = 1,569.17,$$
$$q_1 = 47.12, \quad q_2 = 41.14.$$

The firms' profits are 80,574.83 and 57,406.47, respectively, and Supplier 2's profit is 13,635.49.

Without supplier 1 and with the firms' limited in-house production capacities, there is no competition on the suppliers' side and the firms have to depend more on the supplier in component production. Therefore, as shown by the results, 3 out of the 4 contracted component quantities produced by supplier 2 increase. Supplier 2 charges the firms more than before and its profit improves. Without supplier 1, the firms are not able to provide as many products previously and, hence, the demands at the demand market decrease. The quality of the products of firms 1 and 2 increases and the prices at the demand market increase.

Under this disruption, the profit of firm 1 decreases by 14.84 % and that of firm 2 decreases by 0.66 %. Therefore, from this perspective, Supplier 1 is more important to firm 1 than to firm 2. The value of supplier 1 to firm 1 is 14,035.86, and that to firm 2 is 381.22, which are measured by the associated profit declines.

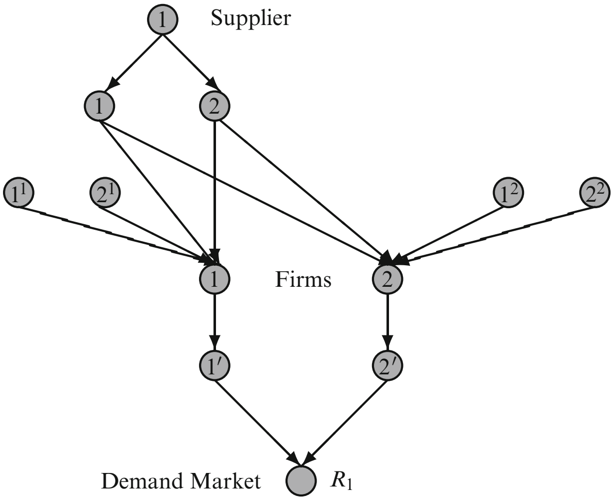

Fig. 10.8 The supply chain network topology with disruption to supplier 2

We then present the disruption in which supplier 2 is no longer available to the firms. The other data are the same as in Example 10.2. The network topology is as in Fig. 10.8.

The modified projection method converges to the following equilibrium solution:

$$Q^*_{11} = 65.00, \quad Q^*_{21} = 63.79,$$

$$Q^{F*}_{11} = 30.00, \quad Q^{F*}_{12} = 30.00, \quad Q^{F*}_{21} = 30.00, \quad Q^{F*}_{22} = 30.00,$$

$$Q^{S*}_{111} = 35.00, \quad Q^{S*}_{112} = 100.00, \quad Q^{S*}_{121} = 97.58, \quad Q^{S*}_{122} = 33.79,$$

$$Q^{S*}_{211} = 0.00, \quad Q^{S*}_{212} = 0.00, \quad Q^{S*}_{221} = 0.00, \quad Q^{S*}_{222} = 0.00,$$

$$q^{F*}_{11} = 38.26, \quad q^{F*}_{12} = 45.16, \quad q^{F*}_{21} = 34.93, \quad q^{F*}_{22} = 41.75,$$

$$q^{S*}_{111} = 46.30, \quad q^{S*}_{112} = 42.19, \quad q^{S*}_{121} = 44.83, \quad q^{S*}_{122} = 41.94,$$

$$\pi^*_{111} = 83.50, \quad \pi^*_{112} = 85.56, \quad \pi^*_{121} = 109.76, \quad \pi^*_{122} = 52.25,$$

$$\alpha^{F*}_1 = 1.00, \quad \alpha^{F*}_2 = 1.00,$$

$$\lambda^*_{11} = 119.17, \quad \lambda^*_{12} = 442.79, \quad \lambda^*_{21} = 256.75, \quad \lambda^*_{22} = 86.75.$$

The induced demands, demand prices, and product quality levels are:

$$d_{11} = 65.00, \quad d_{21} = 63.79, \quad \rho_{11} = 1,996.05, \quad \rho_{21} = 1,570.59,$$

$$q_1 = 42.82, \quad q_2 = 42.11.$$

The firms' profits are 83,895.42 and 53,610.96, respectively, and Supplier 1's profit is 22,729.18.

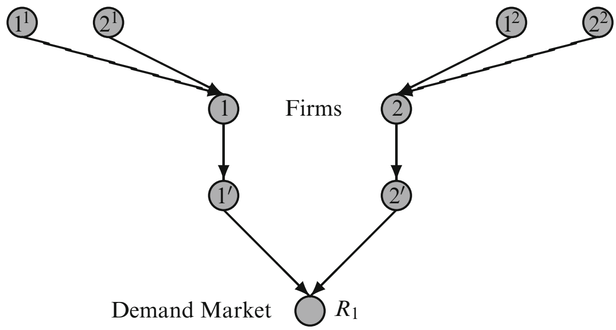

Fig. 10.9 The supply chain network topology with disruption to suppliers 1 and 2

The impacts of the disruption to supplier 2 follow similar logic as those brought about by the disruption to supplier 1. The contracted component quantities by supplier 1 increase and its profit increases. The demands at the demand market decrease. Firm 1's product quality decreases and that of firm 2 increases. The prices at the demand market increase.

Without supplier 2, firm 1's profit declines by 11.33 % and that of firm 2 is reduced by 7.23 %. Thus, supplier 2 is more important to firm 1 than to firm 2 under this disruption. The value of supplier 2 to firm 1 is 10,715.27 and that to firm 2 is 4,176.73.

In addition, according to the above results, supplier 1 is more important than supplier 2 to firm 1, whereas supplier 2 is more important to firm 2.

For completeness, the disruption in which neither supplier is available to the firms is also considered. The other data are the same as in Example 10.2. The network topology is depicted in Fig. 10.9.

The equilibrium solution obtained using the modified projection method is:

$$Q^*_{11} = 15.00, \quad Q^*_{21} = 15.00,$$

$$Q^{F*}_{11} = 15.00, \quad Q^{F*}_{12} = 30.00, \quad Q^{F*}_{21} = 30.00, \quad Q^{F*}_{22} = 30.00,$$

$$Q^{S*}_{111} = 0.00, \quad Q^{S*}_{112} = 0.00, \quad Q^{S*}_{121} = 0.00, \quad Q^{S*}_{122} = 0.00,$$

$$Q^{S*}_{211} = 0.00, \quad Q^{S*}_{212} = 0.00, \quad Q^{S*}_{221} = 0.00, \quad Q^{S*}_{222} = 0.00,$$

$$q^{F*}_{11} = 37.29, \quad q^{F*}_{12} = 45.08, \quad q^{F*}_{21} = 35.71., \quad q^{F*}_{22} = 37.90,$$

$$\alpha^{F*}_1 = 1.00, \quad \alpha^{F*}_2 = 1.00,$$

$$\lambda^*_{11} = 30.46, \quad \lambda^*_{12} = 967.28, \quad \lambda^*_{21} = 772.88, \quad \lambda^*_{22} = 22.63.$$

The induced demands, demand prices, and the product quality levels are:

$$d_{11} = 15.00, \quad d_{21} = 15.00, \quad \rho_{11} = 2{,}206.42, \quad \rho_{21} = 1{,}806.40,$$
$$q_1 = 43.52, \quad q_2 = 37.02.$$

The firms' profits are 30,016.91 and 24,391.32, respectively.

Compared to Example 10.2, without the suppliers, the demands at the demand market decrease, the firms' product quality levels decrease, and the prices at the demand market increase. Firm 1's profit deceases by 68.27 %. Firm 2's profit is reduced by 57.79 %. The value of the suppliers to firm 1 is 64,593.78 and that to firm 2 is 33,396.37.

10.6 Summary and Conclusions

In this chapter, we developed a general multitiered supply chain network equilibrium model with a focus on quality in which suppliers compete to produce components that are utilized by competing firms as they assemble final products that are differentiated by brands. The firms can also produce components in-house, depending on their capacities. We modeled the competitive behavior of the two tiers of decision-makers as they identify their optimal strategies in terms of quantity and quality with the assembling firms also identifying their assembly quality preservation levels. The suppliers charge the firms prices for the components that they supply.

The novelty of our framework lies in its generality and its computability. Rather than focus, as some of the literature does, on one supplier-one manufacturer studies, here we do not limit the number of components needed for the finished product, the number of suppliers, the number of firms, nor the number of demand markets. Moreover, we provide a framework for tracking the quality of the product from the component level, through the assembly process into the final product, and ultimate distribution to the demand markets.

We derived the unified variational inequality formulation of the governing equilibrium conditions, provided qualitative properties of the equilibrium solution pattern, in terms of existence and uniqueness results, and proposed an algorithm along with conditions for convergence. Our framework is illustrated with numerical examples, accompanied by sensitivity analysis that explores such critical issues as the impacts of capacity disruptions and the potential investments in capacity enhancements. We also conducted sensitivity analysis to reveal the impacts of specific supplier unavailability on the profits of the firms and on the quality of the finished products. With knowledge of the value of the suppliers to the firms, the firms can make more specific, targeted efforts in their supplier management strategies and in their contingency plans in the case of supplier disruptions.

The main contributions of this chapter to the existing literature are: 1. We formulate the supply chain network problem with multiple nonidentical competing firms and their potential suppliers who also compete in quality. 2. The model is general and not limited to a small number of firms, suppliers, or components or limited to specific functional forms in terms of costs or demand price functions. 3. The solution of the proposed game theory model provides equilibrium decisions on the in-house and contracted component production and quality levels, component prices, product quantities, and the quality preservation/decay levels of the assembly processes simultaneously. Decisions on the prices and quality levels of the final products are determined through information provided via the demand price functions and the quality aggregation functions. 4. Based on this model, the value of each supplier to each firm can be identified, as illustrated in the analysis in Sect. 10.5. This information is crucial in facilitating strategy design and development in supplier management especially in response to supplier disruptions. 5. Along with the general multitiered supply chain network model, we also provide a general computational procedure with explicit formulae at each iteration. 6. The qualitative properties of the solution to the proposed model, in terms of existence and uniqueness, and the convergence criteria of the computational procedure are presented.

10.7 Sources and Notes

In this chapter, we described a comprehensive framework for supply chain network competition in quality with the inclusion of suppliers. It can be applied to many different industries. This chapter is based on the paper by Li and Nagurney (2015). Here we have standardized the notation.

The model in this chapter, as others in this book, aims at providing the final equilibrium decisions for the supply chain network decision-makers, which, in this chapter, are the firms and the suppliers. Since there is no information asymmetry among firms and suppliers and estimations can be made, we assume that all firms and suppliers make their decisions simultaneously. Moreover, each firm and each supplier make quality and quantity/price decisions at the same stage. Analogous assumptions hold for other models in this book. In Hotelling (1929), Shaked and Sutton (1982), Motta (1993), Aoki and Prusa (1997), Lehmann-Grube (1997), and Banker et al. (1998), firms first decided on product quality, and, at the second stage, product quantity/price was determined. Nevertheless, in Leland (1977), Dixit (1979), Gal-Or (1983), Porteus (1986), Cheng (1991), Lederer and Rhee (1995), Starbird (1997), Zhu et al. (2007), Xu (2009), Shi et al. (2013), and El Ouardighi (2013), and in Pennerstorfer and Weiss (2012)'s model for the wine industry, decision-makers determined quality and quantity/price in one stage. Brekke et al. (2010) modeled both one-stage and two-stage scenarios. The above two-stage models reflect the presumption that quantity and price decisions entail more flexibility than firms' quality positioning. However, this is not always the case.

For example, for critical needs products and products with a steady production rate and demand, such as vaccines, medicines, food, and important agricultural products, the quantity and price decisions can be as flexible (or not) as that for quality.

Moreover, as we have seen in the equilibrium models in Chaps. 3, 4, 5, and 6, associated with each variational inequality problem is a projected dynamical system, which provides for tatonnement processes describing the evolution of the decision variables (whether quantity and/or price, as well as quality) over time and the interactions of the supply chain decision-makers as they adjust the values of their strategic variables until an equilibrium is achieved (under suitable assumptions). Such a projected dynamical system may also be constructed for the supply chain network problem in this chapter using the toolset outlined in this book in Chap. 2.

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Chapter 11
The Supply Chain Network Model with Freight Service Provider Competition

Abstract With this chapter we turn to the inclusion of the behavior of freight service providers engaged in competition in supply chain networks. The manufacturing firms are profit-maximizing and provide substitutable (but not identical) products and compete in quantities in a Cournot-Nash manner. The freight service providers, which transport the products to the consumers at the demand markets, are also profit-maximizers, but compete in prices in Bertrand fashion and on quality. The consumers respond to the composition of product and freight service provision through the demand price functions, which are both quantity and quality dependent. We derive the governing equilibrium conditions of the integrated supply chain network game theory model and show that it satisfies a variational inequality problem. We then describe the underlying dynamics and provide some qualitative properties, including stability analysis. The proposed algorithmic scheme tracks, in discrete-time, the dynamic evolution of the product shipments, the quality levels, and the prices until an approximation of a stationary point (within the desired convergence tolerance) is achieved. Numerical examples demonstrate the modeling and computational framework.

11.1 Introduction

In this chapter, we turn to the modeling of the behavior of another set of decision-makers in supply chain networks – that of freight service providers. Freight service providers are essential in the delivery of products to demand markets. Specifically, in this chapter, we capture the quality of freight service provision under competition. Firms nowadays may have multiple freight service providers to choose from and their reputations depend on the delivery of products that are not only timely but that are not damaged. As noted in Chap. 1 of this book, quality and price have been identified empirically as critical factors in transport mode selection for product/goods delivery (cf. Floden et al. (2010); Saxin et al. (2005), and the references therein).

In this chapter, we focus on the development of a supply chain network game theory model, in both equilibrium and dynamic settings, that captures competition among manufacturing firms (producers) and among freight service providers. The former competition is assumed to be that of Cournot-Nash since the firms compete

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in quantities (cf. Cournot 1838), whereas the latter is that of Bertrand (1883) competition, since the freight service providers compete in prices, along with quality levels. In our framework, we do not restrict the number of firms, nor the number of freight service providers, nor demand markets. Moreover, we allow for product differentiation. Our model also enables the tracking not only of the volume of products provided but also the evolution of the quality levels as well as the prices that the freight service providers charge. The methodology that we utilize for the integrated supply chain network economic model under Cournot-Nash-Bertrand equilibrium is variational inequality theory (cf. Chap. 2). For its dynamic counterpart, we use projected dynamical systems theory (see Chap. 2 and also Nagurney and Zhang 1996 and Nagurney 2006).

A notable feature of our modeling approach is that it allows for *composition*, in that consumers at demand markets have associated demand price functions that reflect how much they are willing to pay for the product and the freight service provision combination, as a function of product shipments and quality levels. Such an idea is motivated, in part, by the desire to provide consumers with more choices (see also the work of Wolf et al. (2012) in another application context). Consequently, our framework can be used as the foundation for the further disaggregation of decision-making and the inclusion of additional topological constructs in the supply chain network topology as given in Fig. 11.1. One can, for example, expand the links joining the freight service provider nodes with the demand markets into paths, which may reflect the transport of the products at a more detailed level of expanded sequences of links. Such paths, for example, may correspond to actual roadways, air links, waterways associated with maritime shipping, or a combination thereof to reflect intermodal transport.

This chapter is organized as follows. In Sect. 11.2, we develop the supply chain network game theory model by explicitly describing the firms' and the freight (transport) service providers' competitive behavior and their interactions with the consumers at the demand markets and with one another. We demonstrate that the governing integrated Cournot-Nash-Bertrand equilibrium conditions are equivalent to the solution of a variational inequality problem. In Sect. 11.3, we then identify the underlying dynamics associated with the time evolution of the product shipments, the quality levels, and the prices charged by the freight service providers. We show that the dynamics correspond to a projected dynamical system, whose set of stationary points coincides with the set of solutions to the variational inequality problem governing the Cournot-Nash-Bertrand model in Sect. 11.2. We also provide some qualitative properties, including stability analysis.

In Sect. 11.4, we propose an algorithm, which yields a time-discretization of the continuous-time adjustment processes in product shipments, quality levels, and

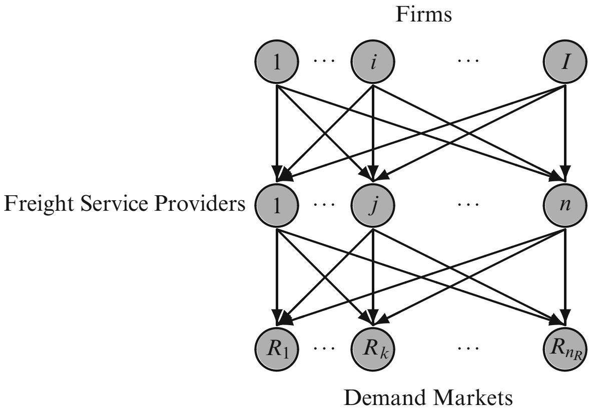

Fig. 11.1 The supply chain network topology with competing freight service providers

prices until an approximation of the stationary point (within the desired convergence tolerance) is achieved. We also give convergence results. The algorithm is then applied to compute solutions to several numerical examples, in Sect. 11.5, in order to illustrate the modeling and computational framework. We summarize our results and present our conclusions in Sect. 11.6. Sources and Notes in Sect. 11.7 conclude this chapter.

11.2 The Cournot-Nash-Bertrand Game Theory Model with Price and Quality Competition

In this section, we develop the game theory supply chain network model in which we capture the behavior of both the manufacturing firms and the freight service providers. We assume that there are I firms, with a typical firm denoted by i, n freight service providers, which provide transport of the products to the demand markets, with a typical one denoted by j, and n_R demand markets associated with the consumers of the products and freight service provision. A typical demand market is denoted by R_k. The firms offer differentiated, but substitutable, products.

It is assumed that the firms compete under the Cournot-Nash equilibrium concept of noncooperative behavior and select their product quantities. The freight service providers, in turn, compete with prices a la Bertrand and with quality levels. The consumers, in turn, signal their preferences for the products and freight service provision via the demand price functions associated with the demand markets. The demand price functions are, in general, functions of the product quantities at all the demand markets as well as the quality levels of freight service provision, since the focus here is on *composition* and having choices.

The notation for the supply chain network game theory model is given in Table 11.1. All vectors here are assumed to be column vectors, as throughout this book. An optimal/equilibrium solution is denoted by a "*".

Table 11.1 Notation for the supply chain Cournot-Nash-Bertrand model

Notation	Definition
Q_{ijk}	The nonnegative amount of product produced by firm i and transported to demand market R_k via j. We group the $\{Q_{ijk}\}$ elements for all j and k into the vector $Q_i \in R^{nn_R}_+$ and then we group all the vectors Q_i for all i into the vector $Q \in R^{Inn_R}_+$
s_i	The nonnegative production output produced by firm i. We group the $\{s_i\}$ elements into the vector $s \in R^I_+$
q_{ijk}	The nonnegative quality level of freight service provider j transporting the product of i to demand market R_k. We group the q_{ijk} for all i and k into the vector $q_j \in R^{In_R}_+$ and all the vectors q_j for all j into the vector $q \in R^{Inn_R}_+$
π_{ijk}	The price charged by freight service provider j for transporting a unit of product produced by i via j to demand market R_k. We group the π_{ijk} for all i and R_k into the vector $\pi_j \in R^I_+$ and then we group all the vectors π_j for all j into the vector $\pi \in R^{Inn_R}_+$
$f_i(s)$	The total production cost of firm i
$\hat{\rho}_{ijk}(Q,q)$	The demand price at demand market R_k associated with product of firm i transported via j
$\hat{c}_{ijk}(Q,q)$	The total transportation cost associated with delivering i's product via j to R_k
$oc_{ijk}(\pi_{ijk})$	The opportunity cost associated with pricing by freight service provider j in transporting from i to R_k

In Sect. 11.2.1 we present the behavior of the firms, along with the Nash-Cournot definition and formulation. In Sect. 11.2.2, we then describe the analogues for the freight service providers, but under Bertrand competition. In Sect. 11.2.3, we present the integrated Cournot-Nash-Bertrand equilibrium and derive the variational inequality formulation of the governing equilibrium conditions.

11.2.1 The Behavior of the Firms and Their Optimality Conditions

The firms seek to maximize their individual profits, where the profit function for firm i; $i = 1, \ldots, I$ is given by the expression:

$$\sum_{j=1}^{n}\sum_{k=1}^{n_R}\hat{\rho}_{ijk}(Q,q^*)Q_{ijk} - f_i(s) - \sum_{j=1}^{n}\sum_{k=n_R}^{o}\pi^*_{ijk}Q_{ijk} \tag{11.1}$$

subject to the constraints:

$$s_i = \sum_{j=1}^{n}\sum_{k=1}^{n_R}Q_{ijk}, \quad i = 1, \ldots, I, \tag{11.2}$$

$$Q_{ijk} \geq 0, \quad j = 1, \ldots, n; k = 1, \ldots, n_R. \tag{11.3}$$

The first term in (11.1) is the revenue for firm i, the second term is its production cost, and the third term in (11.1) is the total payout to the freight service providers for delivering the product to the consumers at the demand markets. Note that firm i controls its vector of product shipments Q_i, whereas, as we show in the next subsection, the freight service providers control the prices charged for the transport of the product from the firms to the demand markets as well as the quality levels of such transport. Hence, we have q^* and π^* in (11.1).

According to constraint (11.2), the quantity of the product produced by each firm is equal to the sum of the amounts of the product transported to all the demand markets via all the freight service providers. Constraint (11.3) guarantees that the product shipments are nonnegative.

In view of constraint (11.2), we can define the production cost functions $\hat{f}_i(Q)$; $i = 1, \ldots, I$, as follows:

$$\hat{f}_i(Q) \equiv f_i(s). \tag{11.4}$$

We assume that the production cost and the demand price functions are continuous and twice continuously differentiable. We also assume that the production cost functions are convex and that the demand price functions are monotonically decreasing in the firm's product volume at the specific demand market but increasing in the quality of freight service provision.

Therefore, the profit maximization problem for firm i; $i = 1, \ldots, I$, with its profit expression denoted by U_i^1, which also represents its utility function, with the superscript 1 reflecting the first (top) tier of decision-makers in Fig. 11.1, can be reexpressed as:

$$\text{Maximize } U_i^1(Q, q^*, \pi^*) = \sum_{j=1}^{n} \sum_{k=1}^{n_R} \hat{\rho}_{ijk}(Q, q^*) Q_{ijk} - \hat{f}_i(Q) - \sum_{j=1}^{n} \sum_{k=1}^{n_R} \pi^*_{ijk} Q_{ijk} \tag{11.5}$$

subject to: $Q_i \in K^{1i}$, where $K^{1i} \equiv \{Q_i | Q_i \geq 0\}$. We also define $K^1 \equiv \prod_{i=1}^{I} K^{1i}$.

Note that the consumers pay the firms according to the incurred prices $\{\hat{\rho}_{ijk}\}$, whereas the firms pay the freight service providers according to the equilibrium prices $\{\pi^*_{ijk}\}$ with the former evaluated at the equilibrium product shipments and quality levels (as described further below). The consumers at the demand markets reflect their preferences for the combination of product and freight service provision through the demand price functions.

We assume that the firms compete according to Cournot-Nash. Indeed, note that the production cost functions (11.4) capture competition for resources since the production cost of a particular firm depends not only on its product shipments (and, hence, outputs), but also on those of the other firms. Also, the demand price functions (see Table 11.1) reveal that consumers at a demand market care not only about the quality level associated with their specific product/freight service provision combination but also on that of the other combinations, as well as the product volumes.

In view of (11.1), (11.2), (11.3), (11.4) and (11.5), we may write the profit functions of the firms as functions of the product shipment, freight quality level, and price pattern, that is,

$$U^1 = U^1(Q, q^*, \pi^*), \tag{11.6}$$

where U^1 is the I-dimensional vector with components: $\{U_1^1, \ldots, U_I^1\}$.

We consider the oligopolistic market mechanism, in which the I firms supply their products in a noncooperative fashion, each one trying to maximize its own profit. We seek to determine a nonnegative product shipment pattern Q^* for which the I firms will be in a state of equilibrium as defined below. In particular, as noted in the preceding modeling chapters, Nash (1950, 1951) generalized Cournot's concept of an equilibrium among several players, in what has been come to be called a noncooperative game.

Definition 11.1: Cournot-Nash Equilibrium with Product Differentiation and Freight Service Provision Choices
A product shipment pattern $Q^* \in K^1$ is said to constitute a Cournot-Nash equilibrium if for each firm i; $i = 1, \ldots, I$:

$$U_i^1(Q_i^*, \hat{Q_i^*}, q^*, \pi^*) \geq U_i^1(Q_i, \hat{Q_i^*}, q^*, \pi^*), \quad \forall Q_i \in K^{1i}, \tag{11.7}$$

where

$$\hat{Q_i^*} \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_I^*). \tag{11.8}$$

According to (11.7), a Cournot-Nash equilibrium is established if no firm can unilaterally improve upon its profits by selecting an alternative vector of product shipments. Alternative variational inequality formulations of the above equilibrium are given below.

Theorem 11.1: Variational Inequality Formulations of Cournot-Nash Equilibrium
Assume that for each firm i the profit function $U_i^1(Q, q, \pi)$ is concave with respect to the variables in $\{Q_i\}$ and is continuous and continuously differentiable. Then, $Q^* \in K^1$ is a Cournot-Nash equilibrium according to Definition 11.1 if and only if it satisfies the variational inequality

$$-\sum_{i=1}^{I}\sum_{j=1}^{n}\sum_{k=1}^{n_R} \frac{\partial U_i^1(Q^*, q^*, \pi^*)}{\partial Q_{ijk}} \times (Q_{ijk} - Q_{ijk}^*) \geq 0, \quad \forall Q \in K^1, \tag{11.9}$$

or, equivalently, $Q^* \in K^1$ is a Cournot-Nash equilibrium product shipment pattern if and only if it satisfies the variational inequality

$$\sum_{i=1}^{I}\sum_{j=1}^{n}\sum_{k=1}^{n_R}\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ijk}}+\pi_{ijk}^*-\hat{\rho}_{ijk}(Q^*,q^*)-\sum_{h=1}^{n}\sum_{l=1}^{n_R}\frac{\partial \hat{\rho}_{ihl}(Q^*,q^*)}{\partial Q_{ijk}}\times Q_{ihl}^*\right]$$
$$\times(Q_{ijk}-Q_{ijk}^*)\geq 0, \quad \forall Q\in K^1. \tag{11.10}$$

Proof: Equation (11.9) follows directly from Gabay and Moulin (1980) and Dafermos and Nagurney (1987).

In order to obtain (11.10) from (11.9), we note that $\forall i, j, k$:

$$-\frac{\partial U_i^1(Q^*,q^*,\pi^*)}{\partial Q_{ijk}}=\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ijk}}+\pi_{ijk}^*-\hat{\rho}_{ijk}(Q^*,q^*)-\sum_{h=1}^{n}\sum_{l=1}^{n_R}\frac{\partial \hat{\rho}_{ihl}(Q^*,q^*)}{\partial Q_{ijk}}\times Q_{ihl}^*\right]. \tag{11.11}$$

Multiplying the expression in (11.11) by $(Q_{ijk}-Q_{ijk}^*)$ and summing the resultant over all i, j, and k yields (11.10). □

11.2.2 The Behavior of the Freight Service Providers and Their Optimality Conditions

The freight service providers also seek to maximize their individual profits. They have as their strategic variables the prices that they charge for the transport of the products and the quality levels of the transport.

We denote the profit function associated with freight service provider j by U_j^2 since this is the second tier of decision-makers (cf. Fig. 11.1). The optimization problem faced by freight service provider j; $j = 1, \ldots, n$ is given by

$$\text{Maximize } U_j^2(Q^*,q,\pi)=\sum_{i=1}^{I}\sum_{k=1}^{n_R}\pi_{ijk}Q_{ijk}^*-\sum_{i=1}^{m}\sum_{k=1}^{o}(\hat{c}_{ijk}(Q^*,q)+oc_{ijk}(\pi_{ijk})) \tag{11.12}$$

subject to:

$$\pi_{ijk}\geq 0, \quad i=1,\ldots,I; k=1,\ldots,n_R, \tag{11.13}$$

$$q_{ijk}\geq 0, \quad i=1,\ldots,I; k=1,\ldots,n_R. \tag{11.14}$$

The first term in (11.12), after the equal sign, is the revenue, whereas the second term is the total transportation cost. Note that the total transportation cost is quite general and captures also competition in quality as well as possible congestion associated with transport to the demand markets. The third term in (11.12) is the opportunity cost, which captures that, if the price charged by the freight service provider is too high then there is an associated business cost; similarly, if it is too low, since then other costs may not be adequately covered. The opportunity cost may include, for example, possible anticipated regulatory costs, loss of potential revenue if the price charged is too high, etc. We, hence, assume that the costs in (11.12) are convex, continuous, and twice continuously differentiable. Observe that the freight service providers have, as their strategic variables, the prices and quality levels of the transport provided. They do not directly control the volume of products that they transport and, therefore, the use of Q^* in (11.12).

We group the freight service provider utility functions, as given in (11.12), into the vector U^2:

$$U^2 = U^2(Q^*, q, \pi). \tag{11.15}$$

Let K^{2j} denote the feasible set corresponding to freight service provider j, such that $K^{2j} \equiv \{(q_j, \pi_j) | q_j \geq 0, \pi_j \geq 0\}$ and define $K^2 \equiv \prod_{j=1}^{n} K^{2j}$.

We now define the Bertrand equilibrium that captures the freight service providers' behavior.

Definition 11.2: Bertrand Equilibrium in Transport Prices and Quality
A quality level pattern and transport price pattern $(q^*, \pi^*) \in K^2$ is said to constitute a Bertrand equilibrium if for each freight service provider j; $j = 1, \ldots, n$:

$$U_j^2(Q^*, q_j^*, \hat{q_j^*}, \pi_j^*, \hat{\pi_j^*}) \geq U_j^2(Q^*, q_j, \hat{q_j^*}, \pi_j, \hat{\pi_j^*}), \quad \forall (q_j, \pi_j) \in K^{2j}, \tag{11.16}$$

where

$$\hat{q_j^*} \equiv (q_1^*, \ldots, q_{j-1}^*, q_{j+1}^*, \ldots, q_n^*), \tag{11.17}$$

$$\hat{\pi_j^*} \equiv (\pi_1^*, \ldots, \pi_{j-1}^*, \pi_{j+1}^*, \ldots, \pi_n^*). \tag{11.18}$$

According to (11.16), a Bertrand equilibrium is established if no freight service provider can unilaterally improve upon its profits by selecting an alternative vector of quality levels and transport prices. Alternative variational inequality formulations of the above equilibrium are as follows.

Theorem 11.2: Variational Inequality Formulations of Bertrand Equilibrium
Assume that for each freight service provider j the profit function $U_j^2(Q, q, \pi)$ is concave with respect to the variables in $\{q_j\}$ and in $\{\pi_j\}$ and is continuous and continuously differentiable. Then, $(q^*, \pi^*) \in K^2$ is a Bertrand equilibrium according to Definition 11.2 if and only if it satisfies the variational inequality

$$-\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R} \frac{\partial U_j^2(Q^*, q^*, \pi^*)}{\partial q_{ijk}} \times (q_{ijk} - q_{ijk}^*)$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R} \frac{\partial U_j^2(Q^*, q^*, \pi^*)}{\partial \pi_{ijk}} \times (\pi_{ijk} - \pi_{ijk}^*) \geq 0, \quad \forall (q, \pi) \in K^2, \tag{11.19}$$

or, equivalently, $(q^*, \pi^*) \in K^2$ *is a Bertrand price and quality level equilibrium pattern if and only if it satisfies the variational inequality*

$$\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[\sum_{h=1}^{I}\sum_{l=1}^{n_R}\frac{\partial \hat{c}_{hjl}(Q^*, q^*)}{\partial q_{ijk}}\right] \times (q_{ijk} - q^*_{ijk})$$

$$+\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[-Q^*_{ijk} + \frac{\partial oc_{ijk}(\pi^*_{ijk})}{\partial \pi_{ijk}}\right] \times (\pi_{ijk} - \pi^*_{ijk}) \geq 0, \qquad \forall (q, \pi) \in K^2. \tag{11.20}$$

Proof: Similar to the proof of Theorem 11.1.

11.2.3 The Integrated Cournot-Nash-Bertrand Equilibrium Conditions and Variational Inequality Formulations

We are now ready to present the Cournot-Nash-Bertrand equilibrium conditions. We let $K^3 \equiv K^1 \times K^2$ denote the feasible set for the integrated model. We assume the same assumptions on the functions as in Sects. 11.2.1 and 11.2.2.

Definition 11.3: Cournot-Nash-Bertrand Equilibrium in Product Differentiation, Freight Service Prices, and Quality
A product shipment, quality level, and freight service price pattern $(Q^*, q^*, \pi^*) \in K^3$ *is a Cournot-Nash-Bertrand equilibrium if it satisfies (11.7) and (11.16) simultaneously.*

Given Definition 11.3, Theorem 11.3 below is immediate.

Theorem 11.3: Variational Inequality Formulations of Cournot-Nash-Bertrand Equilibrium
Under the same assumptions as given in Theorems 11.1 and 11.2, $(Q^*, q^*, \pi^*) \in K^3$ *is a Cournot-Nash-Bertrand equilibrium according to Definition 11.3 if and only if it satisfies the variational inequality:*

$$-\sum_{i=1}^{I}\sum_{j=1}^{n}\sum_{k=1}^{n_R}\frac{\partial U_i^1(Q^*, q^*, \pi^*)}{\partial Q_{ijk}} \times (Q_{ijk} - Q^*_{ijk})$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\frac{\partial U_j^2(Q^*, q^*, \pi^*)}{\partial q_{ijk}} \times (q_{ijk} - q^*_{ijk})$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\frac{\partial U_j^2(Q^*,q^*,\pi^*)}{\partial \pi_{ijk}}\times(\pi_{ijk}-\pi_{ijk}^*)\geq 0, \quad \forall (Q,q,\pi)\in K^3, \tag{11.21}$$

or, equivalently, the variational inequality problem:

$$\sum_{i=1}^{I}\sum_{j=1}^{n}\sum_{k=1}^{n_R}\left[\frac{\partial \hat{f}_i(Q^*)}{\partial Q_{ijk}}+\pi_{ijk}^*-\hat{\rho}_{ijk}(Q^*,q^*)-\sum_{h=1}^{n}\sum_{l=1}^{o}\frac{\partial \hat{\rho}_{ihl}(Q^*,q^*)}{\partial Q_{ijk}}\times Q_{ihl}^*\right]$$
$$\times(Q_{ijk}-Q_{ijk}^*)+\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[\sum_{h=1}^{I}\sum_{l=1}^{n_R}\frac{\partial \hat{c}_{hjl}(Q^*,q^*)}{\partial q_{ijk}}\right]\times(q_{ijk}-q_{ijk}^*)$$
$$+\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\left[-Q_{ijk}^*+\frac{\partial oc_{ijk}(\pi_{ijk}^*)}{\partial \pi_{ijk}}\right]\times(\pi_{ijk}-\pi_{ijk}^*)\geq 0, \quad \forall (Q,q,\pi)\in K^3. \tag{11.22}$$

Indeed, note that if we set $(q,\pi) = (q^*,\pi^*)$ and substitute into variational inequality (11.22), we obtain (11.10). Similarly, if we let $Q = Q^*$ and substitute into (11.22) we obtain (11.20). Hence, the solution of (11.22) also provides us with the solutions to (11.10) and (11.22).

We now put variational inequality (11.22) into standard form (cf. (2.1a)): determine $X^* \in \mathcal{K}$ where X is a vector in R^N, $F(X)$ is a continuous function such that $F(X): X \mapsto \mathcal{K} \subset R^N$, and

$$\langle F(X^*), X-X^*\rangle \geq 0, \quad \forall X\in\mathcal{K}, \tag{11.23}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in the N-dimensional Euclidean space, and $\mathcal{K}$ is closed and convex. We define the vector $X \equiv (Q,q,\pi)$ and $\mathcal{K} \equiv K^3$. Also, here $N = 3Inn_R$. The components of F are then given by: for $i = 1,\ldots,I$; $j = 1,\ldots,n$; $k = 1,\ldots,n_R$:

$$F_{ijk}^1(X)=\frac{\partial \hat{f}_i(Q)}{\partial Q_{ijk}}+\pi_{ijk}-\hat{\rho}_{ijk}(Q,q)-\sum_{h=1}^{n}\sum_{l=1}^{o}\frac{\partial \hat{\rho}_{ihl}(Q,q)}{\partial Q_{ijk}}\times Q_{ihl}, \tag{11.24}$$

$$F_{ijk}^2(X)=\sum_{h=1}^{m}\sum_{l=1}^{o}\frac{\partial \hat{c}_{hjl}(Q,q)}{\partial q_{ijk}}, \tag{11.25}$$

$$F_{ijk}^3(X)=-Q_{ijk}+\frac{\partial oc_{ijk}(\pi_{ijk})}{\partial \pi_{ijk}}. \tag{11.26}$$

Hence, (11.22) can be put into standard form (11.23).

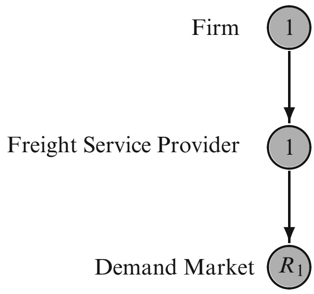

Fig. 11.2 The supply chain network topology for an illustrative example

An Illustrative Example and a Variant

We now present a simple example for illustrative purposes. Please refer to Fig. 11.2. The problem consists of a single firm, a single freight service provider, and a single demand market.

The functions are as follows. The production cost function is:

$$\hat{f}_1(Q_{111}) = Q_{111}^2 + Q_{111}.$$

The demand price function is:

$$\hat{\rho}_{111}(Q_{111}, q_{111}) = -2Q_{111} + q_{111} + 78,$$

and the total transportation cost function is:

$$\hat{c}_{111}(Q_{111}, q_{111}) = (q_{111} - 1)^2,$$

with an opportunity cost of:

$$oc_{111}(\pi_{111}) = \pi_{111}^2.$$

Hence, according to (11.24):

$$\begin{aligned} F_{111}^1(X) &= 2Q_{111} + 1 + \pi_{111} + 2Q_{111} - q_{111} - 78 + 2Q_{111} \\ &= 6Q_{111} + \pi_{111} - q_{111} - 77, \end{aligned}$$

whereas, according to (11.25):

$$F_{111}^2(X) = 2q_{111} - 2$$

and, according to (11.26):

$$F_{111}^3(X) = -Q_{111} + 2\pi_{111}.$$

We will assume that the Cournot-Nash-Bertrand equilibrium solution $X^* = (Q^*, q^*, \pi^*)$ in this example lies in the interior, so we can then explicitly solve for

X^* in (11.22), with notice that $\mathscr{K}$ is the nonnegative orthant R^3_+ as follows. Hence, we can set $F^1_{111}(X^*) = 0$, $F^2_{111}(X^*) = 0$, and $F^3_{111}(X^*) = 0$. Since $F^2_{111}(X^*) = 0$, this means that

$$2q^*_{111} - 2 = 0$$

so that $q^*_{111} = 1$. Also, since $F^3_{111}(X^*) = 0$. we know that

$$Q^*_{111} = 2\pi^*_{111}.$$

Noting that $F^1_{111}(X^*) = 0$, simplifies to

$$6Q^*_{111} + \pi^*_{111} - q^*_{111} - 77 = 0,$$

which, with the above substitutions, yields:

$$13\pi^*_{111} = 78$$

or

$$\pi^*_{111} = 6.$$

Hence, $X^* = (12, 1, 6)$ and the profit of firm 1, $U^1_1 = 432$, and that of freight service provider 1, $U^2_1 = 36$.

We now construct a variant of the above example. All the data remain the same except that we change the transportation cost function of the freight service provider, which is now:

$$\hat{c}_{111}(Q_{111}, q_{111}) = (q_{111} - 1)^2 + Q_{111}q_{111}$$

so that the product shipment volume explicitly appears now in the last term.

The new equilibrium solution is now: $X^* = (11.82, 0, 5.91)$. Note that the quality level has dropped to zero. The profit $U^1_1(X^*)$ for the firm is 421.14, whereas that for the freight service provider, $U^2_1(X^*)$, is 33.10. Hence, both the firm and the freight service provider now have lower profits than in the original example.

Another Illustrative Example

We now present another example. There are two firms, a single freight service provider, and a single demand market, as depicted in Fig. 11.3.

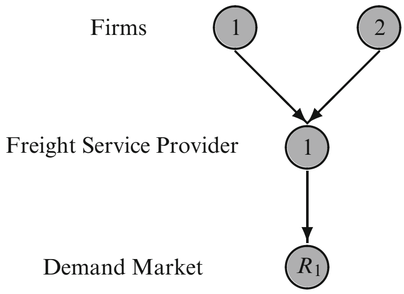

Fig. 11.3 The supply chain network topology for another illustrative example

The data are as follows.

The production cost functions are:

$$\hat{f}_1(Q) = Q_{111}^2 + Q_{111}, \quad \hat{f}_2(Q) = 2Q_{211}^2 + Q_{211}.$$

The demand price functions are:

$$\hat{\rho}_{111}(Q, q) = -Q_{111} - 0.5Q_{211} + 0.5q_{111} + 100,$$
$$\hat{\rho}_{211}(Q, q) = -Q_{211} - 0.5Q_{111} + 0.5q_{211} + 200.$$

The transportation cost functions are:

$$\hat{c}_{111}(Q, q) = 0.5(q_{111} - 20)^2, \quad \hat{c}_{211}(Q, q) = 0.5(q_{211} - 10)^2,$$

with the opportunity cost functions being:

$$oc_{111}(\pi_{111}) = \pi_{111}^2, \quad oc_{211}(\pi_{211}) = \pi_{211}^2.$$

Using (11.24) through (11.26), we construct the following:

$$F_{111}^1(X) = 2Q_{111} + 1 + \pi_{111} + Q_{111} + 0.5Q_{211} - 0.5q_{111} - 100 + Q_{111},$$
$$F_{211}^1(X) = 4Q_{211} + 1 + \pi_{211} + Q_{211} + 0.5Q_{111} - 0.5q_{211} - 200 + Q_{211},$$
$$F_{111}^2(X) = q_{111} - 20, \quad F_{211}^2(X) = q_{211} - 10,$$
$$F_{111}^3(X) = -Q_{111} + 2\pi_{111}, \quad F_{211}^3(X) = -Q_{211} + 2\pi_{211}.$$

Solving, as for the first example in section "An Illustrative Example and a Variant", we obtain:

$$Q_{111}^* = 21.00, \quad Q_{211}^* = 30.00,$$
$$q_{111}^* = 20.00, \quad q_{211}^* = 10.00,$$
$$\pi_{111}^* = 10.50, \quad \pi_{211}^* = 15.00.$$

The profits for the firms are: $U_1^1(X^*) = 875.00$ and $U_2^1(X^*) = 2{,}660.00$, whereas the profit for the freight service provider, $U_1^2(X^*) = 331.00$.

11.3 The Underlying Dynamics and Stability Analysis

We now describe the underlying dynamics until the equilibrium satisfying variational inequality (11.22) is achieved. Specifically, we propose a dynamic adjustment process for the evolution of the firms' shipments, and that of the freight service providers' quality levels and transport prices. The consumers provide feedback through the demand price functions. Observe that, for a current shipment, quality level, and price pattern at time t, $X(t) = (Q(t), q(t), \pi(t))$, $-F_{ijk}^1(X(t)) = \frac{\partial U_i^1(Q(t),q(t),\pi(t))}{\partial Q_{ijk}}$, given by minus the expression in (11.24), is the marginal utility (profit) of firm i with respect to its product shipment to demand market R_k via freight service provider j. Similarly, $-F_{ijk}^2(X(t)) = \frac{\partial U_j^2(Q(t),q(t),\pi(t))}{\partial q_{ijk}}$, given by minus the value in (11.25), is the freight service provider j's marginal utility (profit) with respect to its quality level associated with transporting the product from i to R_k. Finally, $-F_{ijk}^3(X(t)) = \frac{\partial U_j^2(Q(t),q(t),\pi(t))}{\partial \pi_{ijk}}$, given by minus the value in (11.26), is the freight service provider j's marginal utility (profit) with respect to its price charged for transporting the product from i to R_k. Below we provide the continuous-time adjustment processes and the corresponding projected dynamical system.

In this framework, the rate of change of the product shipment between firm i and demand market R_k via freight service provider j is in proportion to $-F_{ijk}^1(X)$, as long as the product shipment Q_{ijk} is positive. Namely, when $Q_{ijk} > 0$,

$$\dot{Q}_{ijk} = \frac{\partial U_i^1(Q, q, \pi)}{\partial Q_{ijk}}, \tag{11.27}$$

where $\dot{Q}_{ijk}$ denotes the rate of change of Q_{ijk}. However, when $Q_{ijk} = 0$, the nonnegativity condition (11.3) forces the Q_{ijk} to remain zero when $\frac{\partial U_i^1(Q,q,\pi)}{\partial Q_{ijk}} \leq 0$. Hence, in this case, we are only guaranteed of having possible increases of the product shipment. Namely, when $Q_{ijk} = 0$,

$$\dot{Q}_{ijk} = \max\{0, \frac{\partial U_i^1(Q, q, \pi)}{\partial Q_{ijk}}\}. \tag{11.28}$$

We may write (11.27) and (11.28) concisely as:

$$\dot{Q}_{ijk} = \begin{cases} \frac{\partial U_i^1(Q,q,\pi)}{\partial Q_{ijk}}, & \text{if} \quad Q_{ijk} > 0 \\ \max\{0, \frac{\partial U_i^1(Q,q,\pi)}{\partial Q_{ijk}}\}, & \text{if} \quad Q_{ijk} = 0. \end{cases} \tag{11.29}$$

Using similar arguments as above, we may write:

$$\dot{q}_{ijk} = \begin{cases} \frac{\partial U_j^2(Q,q,\pi)}{\partial q_{ijk}}, & \text{if} \quad q_{ijk} > 0 \\ \max\{0, \frac{\partial U_j^2(Q,q,k)}{\partial q_{ijk}}\}, & \text{if} \quad q_{ijk} = 0, \end{cases} \tag{11.30}$$

and

$$\dot{\pi}_{ijk} = \begin{cases} \frac{\partial U_j^2(Q,q,\pi)}{\partial \pi_{ijk}}, & \text{if} \quad \pi_{ijk} > 0 \\ \max\{0, \frac{\partial U_j^2(Q,q,\pi)}{\partial \pi_{ijk}}\}, & \text{if} \quad \pi_{ijk} = 0. \end{cases} \tag{11.31}$$

Applying (11.29), (11.30), and (11.31) to all $i = 1, \ldots, I$; $j = 1, \ldots, n$, and $k = 1, \ldots, n_R$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes of the product shipments, quality levels, and freight service provision prices, in vector form, as:

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)), \tag{11.32}$$

where, since $\mathscr{K}$ is a convex polyhedron, according to Dupuis and Nagurney (1993), and as emphasized in Chap. 2, $\Pi_{\mathscr{K}}(X, -F(X))$ is the projection, with respect to $\mathscr{K}$, of the vector $-F(X)$ at X defined as

$$\Pi_{\mathscr{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathscr{K}}(X - \delta F(X)) - X}{\delta} \tag{11.33}$$

with $P_{\mathscr{K}}$ denoting the projection map:

$$P(X) = \text{argmin}_{z \in \mathscr{K}} \|X - z\|, \tag{11.34}$$

and where $\| \cdot \| = \langle x, x \rangle$. Hence, $F(X) = -\nabla U(Q, q, \pi)$, where $\nabla U(Q, q, \pi)$ is the vector of marginal utilities (profits) with components given by (11.24), (11.25), and (11.26).

We now interpret the ODE (11.32) in the context of the Cournot-Nash-Bertrand model with price and quality competition among the freight service providers. First, note that ODE (11.32) ensures that the product shipments, freight service quality levels, and prices are always nonnegative. Indeed, if one were to consider, instead, the ordinary differential equation: $\dot{X} = -F(X)$, or, equivalently, $\dot{X} = \nabla U(X)$, such an ODE would not ensure that $X(t) \geq 0$, for all $t \geq 0$, unless additional restrictive assumptions were to be imposed. Moreover, ODE (11.32) retains the interpretation that if X at time t lies in the interior of $\mathscr{K}$, then the rate at which X changes is greatest when the vector field $-F(X)$ is greatest. In addition, when the vector field $-F(X)$ pushes X to the boundary of the feasible set $\mathscr{K}$, then the projection $\Pi_{\mathscr{K}}$ ensures that X stays within $\mathscr{K}$. Hence, the product shipments, quality levels, and prices are always nonnegative.

Recall now the definition of $F(X)$ (see (11.24), (11.25) and (11.26)) for the integrated model, in which case the projected dynamical system (11.32) states that the rate of change of the product shipments, freight service quality levels, and prices is greatest when the firms' and freight service providers' marginal utilities (profits) are greatest. If the marginal utilities with respect to the product shipments are positive, then the firms will increase their shipments; if they are negative, then they will decrease them. A similar adjustment behavior holds for the freight service providers in terms of their transport quality levels and prices. This type of behavior is rational from an economic standpoint. Therefore, ODE (11.32) is a reasonable continuous adjustment process for the Cournot-Nash-Bertrand supply chain network model.

Although the use of the projection on the right-hand side of ODE (11.32) guarantees that the underlying variables are always nonnegative, it also raises the question of existence of a solution to ODE (11.32), since this ODE is nonstandard due to its discontinuous right-hand side. As noted in Chap. 2, Dupuis and Nagurney (1993) developed the fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by (11.32). We cite the following theorem from that paper. See also the book by Nagurney and Zhang (1996).

Theorem 11.4: Equivalence of Equilibria and Stationary Points
X^* solves the variational inequality problem (11.22) if and only if it is a stationary point of the ODE (11.32), that is,

$$\dot{X} = 0 = \Pi_{\mathscr{K}}(X^*, -F(X^*)). \tag{11.35}$$

This theorem demonstrates that the necessary and sufficient condition for a pattern $X^* = (Q^*, q^*, \pi^*)$ to be a Cournot-Nash-Bertrand equilibrium, according to Definition 11.3, is that $X^* = (Q^*, q^*, \pi^*)$ is a stationary point of the adjustment process defined by ODE (11.32), that is, X^* is the point at which $\dot{X} = 0$.

Consider now the competitive system consisting of the firms and the freight service providers, who, in order to maximize their profits, adjust, respectively, their product shipments, their freight service quality levels, and prices by instantly responding to the marginal profits, according to (11.32). The following questions naturally arise and are of interest. Does the profit (utility) gradient process defined by (11.32), approach a Cournot-Nash-Bertrand equilibrium, and how does it approach an equilibrium in term of the convergence rate? Also, for a given Cournot-Nash-Bertrand equilibrium, do all the disequilibrium product shipment, quality level, and price patterns that are close to this equilibrium always stay nearby? Motivated by these questions, we now present the stability analysis of Cournot-Nash-Bertrand equilibrium, under the above gradient process.

11.4 Stability Under Monotonicity

We now turn to the questions raised above, that is, whether and under what conditions does the adjustment process defined by ODE (11.32) approaches a Cournot-Nash-Bertrand equilibrium? We first recall that Lipschitz continuity of

$F(X)$ (cf. Dupuis and Nagurney 1993; Nagurney and Zhang 1996) guarantees the existence of a unique solution to (11.36) below, where we have that $X^0(t)$ satisfies ODE (11.32) with product shipment, quality level, and price pattern (Q^0, q^0, π^0). In other words, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X^0, \tag{11.36}$$

with $X^0(0) = X^0$. For convenience, we will sometimes write $X^0 \cdot t$ for $X^0(t)$.

We adapt the definitions of stability provided in Chap. 2 for this adjustment process (see also Zhang and Nagurney 1995; Nagurney and Zhang 1996). As therein, we use $B(X, r)$ to denote the open ball with radius r and center X.

We now adapt some fundamental definitions from Chap. 2, for completeness, and recall some basic qualitative results.

Definition 11.4: A Stable or Unstable Equilibrium Point
An equilibrium product shipment, quality level, and price pattern X^* is stable, if for any $\epsilon > 0$, there exists a $\delta > 0$, such that for all initial $X \in B(X^*, \delta)$ and all $t \geq 0$

$$X(t) \in B(X^*, \epsilon). \tag{11.37}$$

The equilibrium point X^* is unstable, if it is not stable.

Definition 11.5: An Asymptotically Stable Equilibrium Point
An equilibrium product shipment, quality level, and price pattern X^* is asymptotically stable, if it is stable and there exists a $\delta > 0$ such that for all initial service volumes, quality levels, and prices $X \in B(X^*, \delta)$

$$\lim_{t \to \infty} X(t) \longrightarrow X^*. \tag{11.38}$$

Definition 11.6: A Globally Exponentially Stable Equilibrium Point
An equilibrium product shipment, quality level, and price pattern X^* is globally exponentially stable, if there exist constants $b > 0$ and $\mu > 0$ such that

$$\|X^0(t) - X^*\| \leq b\|X^0 - X^*\|e^{-\mu t}, \quad \forall t \geq 0, \ \forall X^0 \in \mathcal{K}. \tag{11.39}$$

Definition 11.7: A Global Monotone Attractor
An equilibrium product shipment, quality level, and price pattern X^* is a global monotone attractor, if the Euclidean distance $\|X(t) - X^*\|$ is nonincreasing in t for all $X \in \mathcal{K}$.

Definition 11.8: A Strictly Global Monotone Attractor
An equilibrium X^* is a strictly global monotone attractor, if $\|X(t) - X^*\|$ is monotonically decreasing to zero in t for all $X \in \mathcal{K}$.

We now investigate the stability of the adjustment process under various monotonicity conditions.

Recall (cf. Chap. 2) that $F(X)$ is *monotone* if

$$\langle F(X) - F(X^*), X - X^* \rangle \geq 0, \quad \forall X, X^* \in \mathcal{K}. \tag{11.40}$$

$F(X)$ is *strictly monotone* if

$$\langle F(X) - F(X^*), X - X^* \rangle > 0, \quad \forall X, X^* \in \mathcal{K}, X \neq X^*. \tag{11.41}$$

$F(X)$ is *strongly monotone*, if there is an $\eta > 0$, such that

$$\langle F(X) - F(X^*), X - X^* \rangle \geq \eta \|X - X^*\|^2, \quad \forall X, X^* \in \mathcal{K}. \tag{11.42}$$

The monotonicity of a function F is closely related to the positive definiteness of its Jacobian ∇F (cf. Nagurney 1999). Specifically, if ∇F is positive semidefinite, then F is monotone; if ∇F is positive definite, then F is strictly monotone; and, if ∇F is strongly positive definite, in the sense that the symmetric part of ∇F, $(\nabla F^T + \nabla F)/2$, has only positive eigenvalues, then F is strongly monotone.

In the context of the Cournot-Nash-Bertrand, where $F(X)$ is the vector of negative marginal utilities as in (11.24), (11.25) and (11.26), we point out that if the utility functions are twice continuously differentiable and the Jacobian of the negative marginal utility functions (or, equivalently, the negative of the Hessian matrix of the utility functions) for the integrated model is positive definite, then the corresponding $F(X)$ is strictly monotone.

We now present an existence and uniqueness result, the proof of which follows from the basic theory of variational inequalities.

Theorem 11.5: Existence and Uniqueness
Suppose that F is strongly monotone. Then there exists a unique solution to variational inequality (11.22); equivalently, to variational inequality (11.23).

We summarize in the following theorem the stability properties of the utility gradient process, under various monotonicity conditions on the marginal utilities.

Theorem 11.6: Stability

(i). *If $F(X)$ is monotone, then every Cournot-Nash-Bertrand equilibrium, as defined in Definition 11.3, provided its existence, is a global monotone attractor for the utility gradient process.*

(ii). *If $F(X)$ is strictly monotone, then there exists at most one Cournot-Nash-Bertrand equilibrium. Furthermore, given existence, the unique Cournot-Nash-Bertrand equilibrium is a strictly global monotone attractor for the utility gradient process.*

(iii). *If $F(X)$ is strongly monotone, then the unique Cournot-Nash-Bertrand equilibrium, which is guaranteed to exist, is also globally exponentially stable for the utility gradient process.*

Proof: The stability assertions follow from Theorems 3.5, 3.6, and 3.7 in Nagurney and Zhang (1996), respectively. The uniqueness in (ii) is a classical variational inequality result, whereas existence and uniqueness as in (iii) follows from Theorem 11.5. □

11.4.1 Examples

We now return to the examples in sections "An Illustrative Example and a Variant" and "Another Illustrative Example" in order to illustrate some of the above concepts and results.

We begin with the first example in section "An Illustrative Example and a Variant". The Jacobian matrix of $F(X) = -\nabla U(Q, q, \pi)$, for this example, denoted by $J(Q_{111}, q_{111}, \pi_{111})$, is

$$J(Q_{111}, q_{111}, \pi_{111}) = \begin{pmatrix} 6.0 & -1.0 & 1.0 \\ 0.0 & 2.0 & 0.0 \\ -1.0 & 0.0 & 2.0 \end{pmatrix}.$$

This Jacobian matrix is positive definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, $-\nabla U(Q, q, \pi)$ is strongly monotone. Thus, both the existence and the uniqueness of the solution to variational inequality (11.22) with respect to this example are guaranteed. Moreover, the equilibrium solution, which is: $Q^*_{111} = 12$, $q^*_{111} = 1$, and $\pi^*_{111} = 6$ is globally exponentially stable.

The variant of this example, in turn, as described in section "An Illustrative Example and a Variant", has the Jacobian matrix of its $F(X) = -\nabla U(Q, q, \pi)$ given by:

$$J(Q_{111}, q_{111}, \pi_{111}) = \begin{pmatrix} 6.0 & -1.0 & 1.0 \\ 1.0 & 2.0 & 0.0 \\ -1.0 & 0.0 & 2.0 \end{pmatrix}.$$

We note that this Jacobian matrix is also positive definite, since it is also strictly diagonally dominant and, hence, the same conclusions as above hold for its equilibrium solution $X^* = (11.82, 0, 5.91)$ and the associated gradient process.

We now turn to the numerical example in section "Another Illustrative Example". The Jacobian matrix of its $F(X) = -\nabla U(Q, q, \pi)$, denoted by $J(Q_{111}, Q_{211}, q_{111}, q_{211}, \pi_{111}, \pi_{211})$, is

$$J(Q_{111}, Q_{211}, q_{111}, q_{211}, \pi_{111}, \pi_{211}) = \begin{pmatrix} 4.0 & 0.50 & -0.50 & 0.0 & 1.0 & 0.0 \\ 0.50 & 6.0 & 0.0 & -0.50 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{pmatrix}.$$

Given the positive definiteness of this Jacobian matrix, similar conclusions to those obtained from the two preceding examples follow in terms of existence and uniqueness of the equilibrium $(21, 30, 20, 10, 10.5, 15)$ as well as its stability.

11.5 The Algorithm

As mentioned in Sect. 11.3, the projected dynamical system yields continuous-time adjustment processes. However, for computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories is needed.

We now recall the Euler method, described in Chap. 2, which is induced by the general iterative scheme of Dupuis and Nagurney (1993). Specifically, iteration $\tau+1$ of the Euler method (see also Nagurney and Zhang 1996) is given by:

$$X^{\tau+1} = P_{\mathscr{K}}(X^{\tau} - a_{\tau} F(X^{\tau})), \tag{11.43}$$

where $P_{\mathscr{K}}$ is the projection on the feasible set $\mathscr{K}$ and F is the function that enters the variational inequality problem (11.23).

As shown in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996), for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_{\tau}\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_{\tau} = \infty$, $a_{\tau} > 0$, $a_{\tau} \to 0$, as $\tau \to \infty$. Specific conditions for convergence of this scheme as well as various applications to the solutions of other game theory models can be found in Nagurney et al. (1994, 1995, 2002), Cruz (2008), Nagurney (2010), and Nagurney and Li (2014).

Explicit Formulae for the Euler Method Applied to the Cournot-Nash-Bertrand Game Theory Model

The elegance of this procedure for the computation of solutions to our model (in both the dynamic and static, that is, equilibrium, versions) can be seen in the following explicit formulae for $\tau + 1$. In particular, we have the following closed form expression for the product shipments for $i = 1, \ldots, I$; $j = 1, \ldots, n$; $k = 1, \ldots, n_R$:

$$Q_{ijk}^{\tau+1}$$

$$= \max\{0, Q_{ijk}^{\tau} + a_\tau(\hat{\rho}_{ijk}(Q^\tau, q^\tau) + \sum_{h=1}^{n}\sum_{l=1}^{o} \frac{\partial \hat{\rho}_{ihl}(Q^\tau, q^\tau)}{\partial Q_{ijk}} \times Q_{ihl}^{\tau} - \pi_{ijk}^{\tau} - \frac{\partial \hat{f}_i(Q^\tau)}{\partial Q_{ijk}})\}, \tag{11.44}$$

and the following closed form expression for all the quality levels for $i = 1, \ldots, I$; $j = 1, \ldots, n$; $k = 1, \ldots, n_R$:

$$q_{ijk}^{\tau+1} = \max\{0, q_{ijk}^{\tau} + a_\tau(-\sum_{h=1}^{m}\sum_{l=1}^{o} \frac{\partial \hat{c}_{hjl}(Q^\tau, q^\tau)}{\partial q_{ijk}})\} \tag{11.45}$$

with the explicit formulae for the freight service provision prices being: for $i = 1, \ldots, I; j = 1, \ldots, n; k = 1, \ldots, n_R$:

$$\pi_{ijk}^{\tau+1} = \max\{0, \pi_{ijk}^{\tau} + a_\tau(Q_{ijk}^{\tau} - \frac{\partial oc_{ijk}(\pi_{ijk})}{\partial \pi_{ijk}})\}. \tag{11.46}$$

We now provide the convergence result. The proof is direct from Theorem 5.8 in Nagurney and Zhang (1996).

Theorem 11.7: Convergence
In the Cournot-Nash-Bertrand model for the supply chain network model with competition in prices and quality among freight service providers, let $F(X) = -\nabla U(Q, q, \pi)$ *be strongly monotone. Also, assume that* F *is uniformly Lipschitz continuous. Then there exists a unique equilibrium product shipment, quality level, and price pattern* $(Q^*, q^*, \pi^*) \in \mathcal{K}$ *and any sequence generated by the Euler method as given by (11.43) above, where* $\{a_\tau\}$ *satisfies* $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, *as* $\tau \to \infty$ *converges to* (Q^*, q^*, π^*).

In the next section, we apply the Euler method to compute solutions to several numerical problems.

11.6 Larger Numerical Examples

In this section, we apply the Euler method, described in the preceding section, to compute the Cournot-Nash-Bertrand equilibrium for several supply chain network examples. We set the sequence $\{a_\tau\}$=$1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$. The convergence criterion is that the absolute value of the difference of the iterates at two successive iterations is less than or equal to 10^{-4}. All the variables (product shipments, quality levels, and freight service provider prices) are initialized to 0.00.

In the examples (cf. Fig. 11.4) there are three firms, two freight service providers, and two demand markets. We implemented the algorithm in FORTRAN and used a LINUX system at the University of Massachusetts Amherst for the computations. All the examples in this section satisfy the conditions for convergence as given

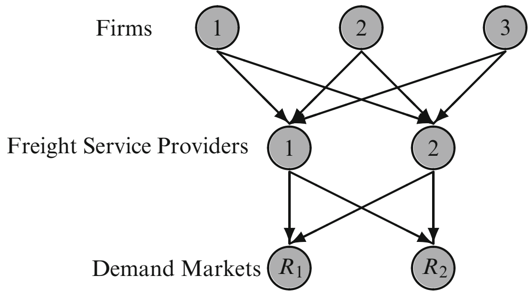

Fig. 11.4 The supply chain network topology for larger numerical examples

in Theorem 11.7. The computed equilibria for the numerical examples below, which are guaranteed to exist, are unique, since the respective Jacobians of their $-\nabla U(Q, q, \pi)$ are positive definite, and, hence, the function F that enters the variational inequality (11.23) for each of these numerical examples is strongly monotone. Moreover, these Fs are also uniformly Lipschitz continuous since the utility functions have bounded second order partial derivatives.

11.6.1 Baseline Example 11.1

The data for the first numerical example in this section, from which we then construct subsequent variants, are as follows.

The production cost functions are:

$$\hat{f}_1(Q) = 2(Q_{111} + Q_{112} + Q_{121} + Q_{122})^2 + (Q_{111} + Q_{112} + Q_{121} + Q_{122}),$$

$$\hat{f}_2(Q) = (Q_{211} + Q_{212} + Q_{221} + Q_{222})^2 + (Q_{211} + Q_{212} + Q_{221} + Q_{222}),$$

$$\hat{f}_3(Q) = 3(Q_{311} + Q_{312} + Q_{321} + Q_{322})^2 + (Q_{311} + Q_{312} + Q_{321} + Q_{322}).$$

The demand price functions are:

$$\hat{\rho}_{111}(Q, q) = -Q_{111} - 0.5Q_{112} + q_{111} + 100,$$

$$\hat{\rho}_{112}(Q, q) = -2Q_{112} - 1Q_{111} + q_{112} + 200,$$

$$\hat{\rho}_{121}(Q, q) = -2Q_{121} - 0.5Q_{111} + 0.5q_{121} + 100,$$

$$\hat{\rho}_{122}(Q, q) = -3Q_{122} - Q_{112} + 0.5q_{122} + 150,$$

$$\hat{\rho}_{211}(Q, q) = -1Q_{211} - 0.5Q_{212} + 0.3q_{211} + 100,$$

$$\hat{\rho}_{212}(Q, q) = -3Q_{212} + 0.8q_{212} + 200,$$

$$\hat{\rho}_{221}(Q, q) = -2Q_{221} - 1Q_{222} + q_{221} + 140,$$

$$\hat{\rho}_{222}(Q, q) = -3Q_{222} - Q_{121} + q_{221} + 300,$$

$$\hat{\rho}_{311}(Q,q) = -4Q_{311} + 0.5q_{311} + 230,$$
$$\hat{\rho}_{312}(Q,q) = -2Q_{312} - Q_{321} + 0.3q_{312} + 150,$$
$$\hat{\rho}_{321}(Q,q) = -3Q_{321} - Q_{311} + 0.2q_{321} + 200,$$
$$\hat{\rho}_{322}(Q,q) = -4Q_{322} + 0.7q_{322} + 300.$$

The transportation cost functions are:

$$\hat{c}_{111}(Q,q) = q_{111}^2 - 0.5q_{111}, \quad \hat{c}_{112}(Q,q) = 0.5q_{112}^2 - q_{112},$$
$$\hat{c}_{121}(Q,q) = 0.1q_{121}^2 - q_{121}, \quad \hat{c}_{122}(Q,q) = q_{122}^2,$$
$$\hat{c}_{211}(Q,q) = 0.1q_{211}^2 - q_{211}, \quad \hat{c}_{212}(Q,q) = q_{212}^2 - 0.5q_{212},$$
$$\hat{c}_{221}(Q,q) = 2q_{221}^2, \quad \hat{c}_{222}(Q,q) = 0.5q_{222}^2 - q_{222},$$
$$\hat{c}_{311}(Q,q) = q_{311}^2 - q_{311}, \quad \hat{c}_{312}(Q,q) = 0.5q_{312}^2 - q_{312},$$
$$\hat{c}_{321}(Q,q) = q_{321}^2 - q_{321}, \quad \hat{c}_{322}(Q,q) = 2q_{322}^2 - 2q_{322}.$$

The opportunity cost functions are:

$$oc_{111}(\pi_{111}) = 2\pi_{111}^2, \quad oc_{112}(\pi_{112}) = 2\pi_{112}^2,$$
$$oc_{121}(\pi_{121}) = \pi_{121}^2, \quad oc_{122}(\pi_{122}) = 0.5\pi_{122}^2,$$
$$oc_{211}(\pi_{211}) = \pi_{211}^2, \quad oc_{212}(\pi_{212}) = 0.5\pi_{212}^2,$$
$$oc_{221}(\pi_{221}) = 2\pi_{221}^2, \quad oc_{222}(\pi_{222}) = 1.5\pi_{222}^2,$$
$$oc_{311}(\pi_{311}) = \pi_{311}^2, \quad oc_{312}(\pi_{312}) = 2.5\pi_{312}^2,$$
$$oc_{321}(\pi_{321}) = 1.5\pi_{321}^2, \quad oc_{322}(\pi_{322}) = \pi_{322}^2.$$

The Euler method converges in 432 iterations and yielded the approximation to the equilibrium solution reported in Table 11.2.

The profit of firm 1 is: 2,402.31, that of firm 2: 6,086.77, and of firm 3: 3,549.49. The profit of freight service provider 1 is: 184.04 and that of freight service provider 2: 241.54.

It is interesting to see that demand market R_1 obtains no product from firm 1 since Q_{111}^* and Q_{121}^* are equal to 0.00 and only obtains product from firms 2 and 3. Demand market R_2, however, obtains products from all three firms. Freight service provider 1 handles positive shipments of products from all firms as does freight service provider 2. It is also interesting to see that two of the quality levels are equal to zero.

Noting that $Q_{111}^* = 0.00$ we then constructed Variant 1 as described below.

Table 11.2 Equilibrium solution for the baseline Example 11.1

Firm i	Freight service provider j	Demand market k	Q^*_{ijk}	q^*_{ijk}	π^*_{ijk}
1	1	1	0.00	0.25	0.00
1	1	2	22.67	1.00	5.67
1	2	1	0.00	5.00	0.00
1	2	2	3.24	0.00	3.24
2	1	1	0.00	5.00	0.00
2	1	2	14.53	0.25	14.53
2	2	1	2.24	0.00	0.56
2	2	2	31.97	1.00	10.66
3	1	1	7.55	0.50	3.77
3	1	2	0.00	1.00	0.00
3	2	1	4.18	0.50	1.39
3	2	2	15.80	0.50	7.90

11.6.2 Example 11.2: Variant 1 of Example 11.1

In this example, we explore the effects of a change in the price function $\hat{\rho}_{111}$ since recall that, in Example 11.1, $Q^*_{111} = 0.00$. Such a change in a price function could occur, for example, through enhanced marketing. Specifically, we seek to determine the change in the equilibrium pattern if the consumers at demand market R_1 are willing to pay more for the product of the firm 1 and freight service provider 1 combination. The new demand price function is:

$$\hat{\rho}_{111}(Q, q) = -Q_{111} - 0.5Q_{112} + q_{111} + 200,$$

with the remainder of the data as in Example 11.1. The new computed solution is reported in Table 11.3. The algorithm converges in 431 iterations.

The profit of firm 1 is: 3,168.18. The profits of the other two firms remain as in Example 11.1. The profit of freight service provider 1 is: 209.85 and that of freight service provider 2: 236.35. Hence, both firm 1 and freight service provider 1 have higher profits than in Example 11.1 and the product shipment Q^*_{111} increases from 0.00 to 25.40. There is a reduction in product shipment Q^*_{112} and in Q^*_{122}.

11.6.3 Example 11.3: Variant 2 of Example 11.1

In the next example, we return to Example 11.1 and modify all of the transportation cost functions to include an additional term: $Q_{ijk}q_{ijk}$ to reflect that cost can depend on both congestion level and on the quality of transport. The solution obtained via the Euler method for this example is given in Table 11.4. The Euler method requires 705 iterations for convergence.

Table 11.3 Equilibrium solution for Example 11.2: variant 1 of Example 11.1

Firm i	Freight service provider j	Demand market k	Q^*_{ijk}	q^*_{ijk}	π^*_{ijk}
1	1	1	25.40	0.25	6.35
1	1	2	8.67	1.00	2.17
1	2	1	0.00	4.45	0.00
1	2	2	0.37	0.00	0.37
2	1	1	0.00	4.45	0.00
2	1	2	14.52	0.25	14.53
2	2	1	2.24	0.00	0.56
2	2	2	31.97	1.00	10.66
3	1	1	7.55	0.50	3.77
3	1	2	0.00	1.00	0.00
3	2	1	4.18	0.50	1.39
3	2	2	15.80	0.50	7.90

Table 11.4 Equilibrium solution for Example 11.3: variant 2 of Example 11.1

Firm i	Freight service provider j	Demand market k	Q^*_{ijk}	q^*_{ijk}	π^*_{ijk}
1	1	1	0.00	0.25	0.00
1	1	2	22.52	0.00	5.63
1	2	1	0.00	4.98	0.00
1	2	2	3.31	0.00	3.31
2	1	1	0.00	4.99	0.00
2	1	2	14.52	0.00	14.52
2	2	1	2.31	0.00	0.58
2	2	2	31.84	0.00	10.61
3	1	1	7.53	0.00	3.77
3	1	2	0.00	1.00	0.00
3	2	1	4.19	0.00	1.40
3	2	2	15.77	0.00	7.89

The profit of firm 1 is: 2,380.87. The profit of firm 2 is: 6,053.76 and that of firm 3 is: 3,541.93. The profit of freight service provider 1 is now: 181.89 and that of freight service provider 2: 237.21.

Observe that, in this, as in the previous two examples, if $Q^*_{ijk} = 0.00$, then the price $\pi^*_{ijk} = 0.00$, which is reasonable. It is interesting to note that, in this example, the inclusion of an additional term $Q_{ijk}q_{ijk}$ to each transportation cost function c_{ijk}, with the remainder of the data as in Example 11.1, results in a decrease in the quality levels in 8 out of the 12 computed equilibrium variable values, with the other quality values remaining unchanged. A quality level of zero implies that no extra features are provided. Having an effective modeling and computational framework allows one to explore the effects of changes in the underlying functions on the equilibrium pattern to gain insights that may not be apparent from smaller scale, analytical solutions.

Table 11.5 Equilibrium solution for Example 11.4: variant 3 of Example 11.1

Service provider i	Network provider j	Demand market k	Q^*_{ijk}	q^*_{ijk}	π^*_{ijk}
1	1	1	2.44	0.00	0.61
1	1	2	27.48	0.00	6.87
1	2	1	4.00	0.00	2.00
1	2	2	5.96	0.00	5.96
2	1	1	0.00	4.98	0.00
2	1	2	14.59	0.00	14.59
2	2	1	2.55	0.00	0.64
2	2	2	31.28	0.00	10.43
3	1	1	7.54	0.00	3.77
3	1	2	0.00	1.00	0.00
3	2	1	4.19	0.00	1.40
3	2	2	15.77	0.00	7.89

11.6.4 Example 11.4: Variant 3 of Example 11.1

In the next example, we use the same data as in Example 11.3 but we reduce the production cost function for firm 1 to see the effects on its product shipments (observe that in Table 11.4, two of its equilibrium product shipments are 0.00).

The new production cost function is:

$$\hat{f}_1(Q) = (Q_{111} + Q_{112} + Q_{121} + Q_{122})^2 + (Q_{111} + Q_{112} + Q_{121} + Q_{122}),$$

The algorithm converges in 638 iterations and yielded the equilibrium pattern reported in Table 11.5.

The profit of firm 1 increases and is now: 3,245.84. The profit of firm 2 is reduced to 5,933.49. The profit of firm 3 remains unchanged and is 3,541.93. The profit of freight service provider 1 increases to 214.54 and that of freight service provider 2 also increases to 247.33.

Observe that firm 1, by lowering his production costs, now has all positive equilibrium product shipments.

11.7 Summary and Conclusions

In this chapter, we developed a game theory model for a competitive supply chain network model in which freight service providers compete in prices and quality levels associated with their service provision. The motivation for the research stems, in part, from a need to understand the underlying network economics of supply chains with more choices as well as to demonstrate the integration of complex competitive behaviors on multitiered networks. We developed both static and dynamic versions of the Cournot-Nash-Bertrand game theory model in which the firms offer differentiated, but substitutable, products and the freight service providers transport the products to consumers at the demand markets. Consumers respond to the composition of firms and freight service provision choices and to the quality levels and product shipment volumes, through the prices. The firms compete in a Cournot-Nash manner, whereas the freight service providers compete a la Bertrand in prices charged for the transport of the products, as well as with the quality levels associated with the transport.

We derived the governing equilibrium conditions of the integrated supply chain network game theory model and showed that it satisfies a variational inequality problem. We then described the underlying dynamics, using the theory of projected dynamical systems, and also presented stability analysis and other qualitative results. An algorithm was presented, along with convergence results, which provides a discrete-time version of the continuous-time adjustment processes for the service volumes, quality levels, and prices. We demonstrated the generality of the modeling and computational framework with several numerical examples.

11.8 Sources and Notes

This chapter is inspired by the paper by Nagurney and Wolf (2014). In that paper, the authors developed a game theory model for a service-oriented Internet in which service providers compete with one another and can provide content, which may be viewed as products, which are then transported to users at demand markets via competing transport network providers, which play a similar role to the freight service providers in our framework in this chapter. Indeed, we believe that Future Internet Architectures (FIAs) will be similar in concept and design to many of today's complex, multitiered supply chain networks. In this chapter, we have modified the notation from that in Nagurney and Wolf (2014) and provided a motivation and context in the framework of supply chain networks with competing freight service providers.

Ballot et al. (2014), in turn, argue for a physical Internet for logistics networks in order to achieve more efficient and sustainable logistics. Hence, there are clear synergies in concept, possible network representations, and accompanying stakeholder behaviors, between the two application domains of the Internet (especially FIA) and supply chain networks.

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Chapter 12
Supply Chain Network Competition in Prices and Quality

Abstract In this chapter, we develop static and dynamic competitive supply chain network models with multiple manufacturers and freight service providers. The manufacturers compete with one another in terms of price and quality of the product manufactured, whereas the freight service providers compete on price and quality of the transportation service that they provide for multiple modes. In contrast to the models in preceding chapters in which either demand price functions or fixed demands were utilized, here we use direct demand functions. In addition, in this chapter, we consider both product quality as well as freight service quality in consumer decision-making. Manufacturers and freight service providers maximize their profits while considering the consequences of the competitors' prices and quality levels. Bounds on prices and quality levels are included that have relevant policy-related implications. The governing equilibrium conditions of the static model are formulated as a variational inequality problem. The underlying dynamics are then described, with the stationary point corresponding to the variational inequality solution. An algorithm, which provides a discrete-time adjustment process and tracks the evolution of the quality levels and prices over time is proposed, and convergence results given. Numerical examples illustrate how such a supply chain network framework, which is relevant to products ranging from high value to low value ones, can be applied in practice.

12.1 Introduction

In our global Network Economy, success is determined by how well the entire supply chain performs, rather than by the performance of individual entities, whether manufacturing firms or freight service providers, since multiple decision-makers are involved in chaining the resources into products and the ultimate delivery to consumers. Hence, examining the interactions among manufacturing firms as well as those of freight service providers as they compete to provide products and transport services, respectively, to consumers at demand markets is highly relevant.

The growth of intercontinental multi-channel distribution, containerization, and direct to business and direct to customer shipping has led to fierce competition among freight service providers who are subjected to pricing pressures and increased expectations to handle more complex services (Crainic et al. 2013;

A. Nagurney, D. Li, *Competing on Supply Chain Quality*, Springer Series in Supply Chain Management 2, DOI 10.1007/978-3-319-25451-7_12

Hakim 2014; DHL 2014) . In order to maintain their competitive edge, freight service providers are increasingly concentrating on positioning themselves as more than just a commodity business. Freight service providers may offer flexibility to meet customer needs of safety, and/or traceability and, furthermore, differentiate themselves from the rest of the competition, thereby migrating towards being more value-oriented than cost-oriented (Bowman 2014; Glave et al. 2014). The quality of freight service provision is driving logistics performance in both developed and emerging economies (Arvis et al. 2014).

Increasingly, challenging customer demands are also putting the transport system under pressure. The online retailer Amazon.com recently submitted a patent (United States Patent 2013) for anticipatory shipping and speculative shipping, meaning that, based on advanced forecasts of customer behavior (previous purchases, behavior during homepage visits, demographics, etc.) the company actually ships the product before the customer orders it! The product is transported towards a region where a purchase is expected and is redirected during transport when the order is placed, thus, allowing almost instant deliveries (Bensinger 2014). Transport owners that cannot offer the desired level of quality are forced to leave the market, as was the case when the intermodal company CargoNet withdrew from the Swedish rail market, claiming unreliable infrastructure as one of the main reasons (Floden and Woxenius 2013).

In this chapter, as in Chap. 11, we focus on the development of supply chain network game theory models in both equilibrium and dynamic settings with the inclusion of the behavior of freight service providers. We consider a supply chain network with multiple manufacturers and multiple freight service providers handling freight transportation. The decision-makers at each echelon compete in prices, in contrast to the static and dynamic models in Chap. 11 and those in earlier modeling chapters, in which the firms competed in quantities. Quality of the product is traced along the supply chain with consumers differentiating among the products offered by the manufacturers. Also, quality of freight service provision is accounted for in the model and the providers compete on both price and quality. In addition, in this chapter, in contrast to other modeling chapters in this book, we utilize direct demand functions at the demand markets, rather than demand price functions or fixed demands for the products. At times, it may be easier to estimate demand functions directly. Moreover, in our framework in this chapter, consumers respond directly to the product and freight service choice composition.

Our framework is inspired, in part, by the work of Nagurney et al. (2013) and Saberi et al. (2014). The latter proposed network economic game theory models of service-oriented Internet architectures with price and quality competition occurring between content and network providers. Here, we go further in that we allow for multiple modes of transportation and each freight service provider can have a distinct number of mode options. In addition, we consider a mode in a general way in that it can correspond to intermodal transportation. The former studied a network economic game theory model of a service-oriented Internet with choices and quality competition. For additional background on freight transportation modeling, we refer the reader to the books by Tavasszy and de Jong (2013), Ben-Akiva et al. (2013), and Bookbinder (2013), and the references therein.

The static and dynamic models in this chapter also build on the work of Nagurney et al. (2002), which introduced supply chain network equilibrium models but here the competition is in price and quality and not in quantities. See, also, the dynamic multilevel financial/informational/logistical framework of Nagurney et al. (2002), the supernetwork model with freight carriers in Yamada et al. (2011), and the maritime chain model with carriers, ports and shippers of Talley and Ng (2013). For a plethora of supply chain network equilibrium models, along with the underlying dynamics, but without the distinctive quality aspect, see the book by Nagurney (2006). However, none of the above multitiered competitive supply chain network equilibrium models with freight service provider behavior captured quality either in transportation or in production.

Our framework is not in the context of supply chain network design; for an extensive review of the overall supply chain network design literature, see Farahani et al. (2014). For supply chain network models, including design ones, for perishable products, see Nagurney et al. (2013).

The structure of the chapter is as follows. Section 12.2 presents the multitiered supply chain network game theory model with manufacturers and freight service providers. We describe the firms' behavior that accounts for the prices and quality levels of the products at the demand markets. In parallel, we model the freight service providers' behavior that deals with the prices and quality levels of their services for various modes. The freight service providers compete in terms of price and quality that differ by mode. A variational inequality formulation is derived, which unifies the firms' and freight service providers' behaviors. An existence result for a solution to the unified variational inequality formulation is also given. A projected dynamical systems model is, subsequently, constructed in Sect. 12.3 to capture the underlying dynamics of the competitive behavior. In Sect. 12.4, we present an algorithm for solving the proposed variational inequality formulation, accompanied by convergence results. At each iteration, the algorithm yields closed form expressions for the prices and qualities of the firms and freight service providers. It also serves as a time-discretization of the continuous-time adjustment processes in prices and quality levels. Section 12.5 illustrates the model and the computational algorithm through several numerical examples in order to gain managerial insights. In Sect. 12.6, we summarize our results and present our conclusions. The chapter concludes with Sources and Notes in Sect. 12.7.

12.2 The Supply Chain Network Model with Price and Quality Competition

In the supply chain network there are I manufacturing firms involved in the production of differentiated, substitutable products that are transported by n freight service providers or carriers to n_R demand markets. We denote a typical manufacturing firm by i; $i = 1, \dots, I$, a typical freight service provider by j; $j = 1, \dots, n$, and a typical

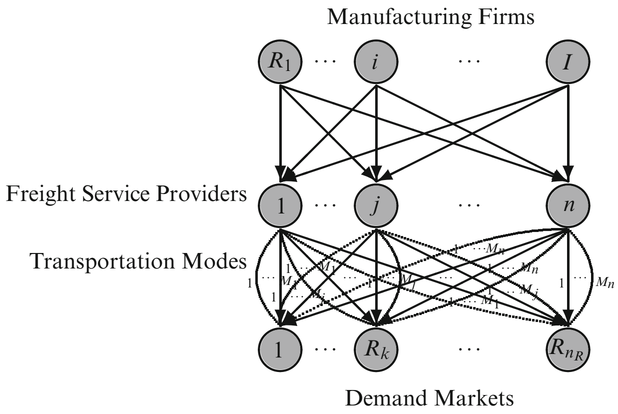

Fig. 12.1 The supply chain network topology of the game theory model with price and quality competition

demand market by R_k; $k = 1, \ldots, n_R$. Each freight service provider j; $j = 1, \ldots, n$, has M_j possible modes of transport/shipment, associated with which is also a distinct quality. The modes of shipment may include rail, air, truck, sea, or even bicycles for last mile deliveries, etc.

For the sake of modeling flexibility and generality, a *mode* in our framework may represent a composition of modes as in the case of intermodal transportation. The freight service providers are responsible for picking up the products at the manufacturers and delivering them to consumers at the demand markets. The demand markets may also correspond to retail outlets, depending on the application. Each freight service provider may have a different number of mode options based on vehicle ownership and access, contracts, prior relationships, geographical issues, etc. The supply chain network representation of our game theory model is depicted in Fig. 12.1. The manufacturing firms compete with one another as do the freight service providers.

The notation for the model is given in Table 12.1.

12.2.1 The Firms' Behavior

Firm i manufactures a product of quality q_i at price ρ_i. As in the previous chapters in this book, we define and quantify product quality as the quality conformance level, that is, the degree to which a specific product conforms to a design or specification.

Table 12.1 Notation for the supply chain model with price and quality competition

Notation	Definition
q_i	The nonnegative quality level of firm i's product. We group the quality levels of all firms' products into the vector $q_F \in R^I_+$
q^m_{ijk}	The nonnegative quality level of freight service provider j transporting the product of i to demand market R_k via mode m. We group the q_{ijk} for all i, k, and m into the vector $q_j \in R^{In_R M_j}_+$ and all the vectors q_j for all j into the vector $q_C \in R^{In n_R \sum_{j=1}^n M_j}_+$
ρ_i	The nonnegative price of firm i's product. We group the prices of all firms' products into the vector $p_F \in R^I_+$
ρ^m_{ijk}	The nonnegative price charged by freight service provider j for transporting a unit of product produced by i via mode m of j to demand market R_k. We group the ρ^m_{ijk} for all i, and for all k and m, into the vector $\rho_j \in R^{In_R M_j}_+$ and then we group all the vectors ρ_j for all j into the vector $\rho_C \in R^{In n_R \sum_{j=1}^n M_j}_+$
$s_i(\rho_F, q_F, \rho_C, q_C)$	The nonnegative production output produced by firm i. We group the $\{s_i\}$ elements into the vector $s_F \in R^I_+$
$\hat{f}_i(s_F, q_F)$	The total production cost of firm i
$d^m_{ijk}(\rho_F, q_F, \rho_C, q_C)$	The demand for firm i's product transported by j via mode m to demand market R_K. We group the demands into the $In n_R \sum_{j=1}^n M_j$-dimensional vector $d(\rho_F, q_F, \rho_C, q_C)$

Quality with respect to freight in our model, as noted in Chap. 1, corresponds to level of service as emphasized by Mancera et al. (2013).

The supply of firm i's product, s_i, is equal to the demand, that is,

$$s_i(\rho_F, q_F, \rho_C, q_C) = \sum_{k=1}^{n_R} \sum_{j=1}^{n} \sum_{m=1}^{M_j} d^m_{ijk}(\rho_F, q_F, \rho_C, q_C), \quad i = 1, \ldots, I, \tag{12.1}$$

since we expect the markets to clear.

The generality of the demand functions, given in Table 12.1, allows for the modeling of competition on the demand side for the products and freight service provision. We expect that the demand d^m_{ijk} will increase (decrease) as the price (quality) of firm i's product at its own demand market or the shipment price (quality) of freight service provider j decreases.

Note that the production cost of firm i, $\hat{f}_i$, depends, in general, upon the entire production (supply) pattern, as well as on the product quality levels, that is:

$$\hat{f}_i = \hat{f}_i\big(s_F(\rho_F, q_F, \rho_C, q_C), q_F\big), \quad i = 1, \ldots, I. \tag{12.2}$$

The generality of the production cost functions allows us to capture competition for resources in manufacturing, whether natural, human, and/or capital.

The utility of firm i, U_i; $i = 1, \ldots, I$, represents its profit, and is the difference between the firm's revenue and the production cost:

$$U_i(\rho_F, q_F, \rho_C, q_C) = \rho_i \Big[\sum_{k=1}^{n_R} \sum_{j=1}^{n} \sum_{m=1}^{M_j} d_{ijk}^m(\rho_F, q_F, \rho_C, q_C) \Big] - \hat{f}_i\big(s_F(\rho_F, q_F, \rho_C, q_C), q_F\big). \tag{12.3}$$

Each firm i is faced with a nonnegative lower bound $\underline{q}_i$ on the quality of his product as well as an upper bound $\bar{q}_i$, so that

$$\underline{q}_i \leq q_i \leq \bar{q}_i, \quad i = 1, \ldots, I. \tag{12.4}$$

Typically, $\bar{q}_i = 100$ corresponds to perfect quality conformance as discussed in Chap. 1. If that is not achievable by a firm, then the upper bound would be set to a lower value. Also, a positive lower bound $\underline{q}_i$ corresponds to a minimum quality standard as discussed in Chaps. 1, 3, and 4.

In addition, each firm i is faced with an upper bound on the price that it charges for its product, that is,

$$0 \leq \rho_i \leq \bar{\rho}_i, \quad i = 1, \ldots, I. \tag{12.5}$$

The price that firm i charges and its quality level correspond to its strategic variables in the competitive game.

Let K_i^1 denote the feasible set corresponding to firm i, where $K_i^1 \equiv \{(\rho_{F_i}, q_{F_i}) \mid$ (12.4) and (12.5) hold$\}$. We define: $K^1 \equiv \prod_{i=1}^{I} K_i^1$. We assume that all the above functions are continuous and twice continuously differentiable.

The manufacturers compete in a noncooperative manner which we formalize in Sect. 12.2.3.

12.2.2 The Freight Service Providers' Behavior

Recall that freight service provider j transports a product from firm i to demand market R_k via mode m at a quality level q_{ijk}^m at a unit price of ρ_{ijk}^m. Its quality levels and prices charged are its strategic variables.

The transportation cost between firm i and demand market R_k via mode m of freight service provider j is denoted by $\hat{c}_{ijk}^m$. We assume that:

$$\hat{c}_{ijk}^m = \hat{c}_{ijk}^m\big(d(p_F, q_F, p_C, q_C), q_C\big),$$

$$i = 1, \ldots, I; j = 1, \ldots, n; k = 1, \ldots, n_R; m = 1, \ldots, M_j, \tag{12.6}$$

that is, the transportation cost may depend, in general, on the vector of demands and the vector of quality levels of the freight service providers. In the transportation costs we also include handling costs associated with, for example, loading and unloading and, perhaps, also, storage of the products over a period of time.

The utility or profit function of freight service provider j, U_j, is the difference between its revenue and transportation costs:

$$U_j(\rho_F, q_F, \rho_C, q_C) = \sum_{i=1}^{I} \sum_{k=1}^{n_R} \sum_{m=1}^{M_j} \rho_{ijk}^m d_{ijk}^m(\rho_F, q_F, \rho_C, q_C)$$
$$- \sum_{i=1}^{I} \sum_{k=1}^{n_R} \sum_{m=1}^{M_j} \hat{c}_{ijk}^m \big(d(\rho_F, q_F, \rho_C, q_C), q_C \big). \tag{12.7}$$

Each freight service provider j; $j = 1, \ldots, n$, is faced with a lower and upper bound on the quality of transport shipment, $\underline{q}_{ijk}^m$, $\bar{q}_{ijk}^m$, respectively, and an upper bound for price, $\bar{\rho}_{ijk}^m$, between i and R_k so that

$$\underline{q}_{ijk}^m \leq q_{ijk}^m \leq \bar{q}_{ijk}^m, \quad i = 1, \ldots, I; k = 1, \ldots, n_R; m = 1, \ldots, M_j, \tag{12.8}$$

$$0 \leq \rho_{ijk}^m \leq \bar{\rho}_{ijk}^m, \quad i = 1, \ldots, I; k = 1, \ldots, n_R; m = 1, \ldots, M_j. \tag{12.9}$$

The freight service provider lower bounds are assumed to be nonnegative as in the case of product quality with a positive value corresponding to a minimum quality standard.

Let K_j^2 denote the feasible set corresponding to j, where $K_j^2 \equiv \{(\rho_j, q_j) \mid$ (12.8) and (12.9) hold$\}$. We then define $K^2 \equiv \prod_{j=1}^{n} K_j^2$. We assume that all the above functions associated with the freight service providers are continuous and twice continuously differentiable.

The freight service providers also compete in a noncooperative manner, as per below.

12.2.3 The Bertrand-Nash Equilibrium Conditions and Variational Inequality Formulation

We now present the Bertrand (1883), Nash (1950, 1951) equilibrium definition that captures the decision-makers' competitive behavior in our model.

Definition 12.1: Bertrand-Nash Equilibrium in Prices and Quality Levels
A price and quality level pattern $(\rho_F^*, q_F^*, \rho_C^*, q_C^*) \in K^3 \equiv \prod_{i=1}^{I} K_i^1 \times \prod_{j=1}^{n} K_j^2$, *is said to constitute a Nash equilibrium if for each firm i;* $i = 1, \ldots, I$*:*

$$U_i(\rho_i^*, \hat{\rho_i^*}, q_i^*, \hat{q_i^*}, \rho_C^*, q_C^*) \geq U_i(\rho_i, \hat{\rho_i^*}, q_i, \hat{q_i^*}, \rho_C^*, q_C^*), \quad \forall (\rho_i, q_i) \in K_i^1, \tag{12.10}$$

where

$$\hat{\rho_i^*} \equiv (\rho_1^*, \ldots, \rho_{i-1}^*, \rho_{i+1}^*, \ldots, \rho_I^*) \text{ and } \hat{q_i^*} \equiv (q_1^*, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_I^*), \tag{12.11}$$

and if for each freight service provider j; $j = 1, \ldots, n$:

$$U_j(\rho_F^*, q_F^*, \rho_j^*, \hat{\rho_j^*}, q_j^*, \hat{q_j^*}) \geq U_j(\rho_F^*, q_F^*, \rho_j, \hat{\rho_j^*}, q_j, \hat{q_j^*}), \quad \forall (\rho_j, q_j) \in K_j^2, \tag{12.12}$$

where

$$\hat{\rho_j^*} \equiv (\rho_1^*, \ldots, \rho_{j-1}^*, \rho_{j+1}^*, \ldots, \rho_n^*) \text{ and } \hat{q_j^*} \equiv (q_1^*, \ldots, q_{j-1}^*, q_{j+1}^*, \ldots, q_n^*). \tag{12.13}$$

According to (12.10) and (12.12), a Bertrand-Nash equilibrium is established if no decision-maker, whether a manufacturing firm or freight service provider, can unilaterally improve upon its profits by selecting an alternative vector of prices and quality levels.

We assume that the above utility functions are concave. Under our previously imposed assumptions on the production cost, transportation cost, and demand functions, we know that the utility functions are continuous and continuously differentiable. We now derive the variational inequality formulation of the governing equilibrium conditions.

Theorem 12.1: Variational Inequality Formulations of Bertrand-Nash Equilibrium in Prices and Quality
Assume that the manufacturing firms' and freight service providers' utility functions are concave, continuous, and continuously differentiable. Then $(\rho_F^*, q_F^*, \rho_C^*, q_C^*) \in \mathcal{K}^3$ is a Bertrand-Nash equilibrium according to Definition 12.1 if and only if it satisfies the variational inequality:

$$-\sum_{i=1}^{I} \frac{\partial U_i(\rho_F^*, q_F^*, \rho_C^*, q_C^*)}{\partial \rho_i} \times (\rho_i - \rho_i^*) - \sum_{i=1}^{I} \frac{\partial U_i(\rho_F^*, q_F^*, \rho_C^*, q_C^*)}{\partial q_i} \times (q_i - q_i^*)$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j} \frac{\partial U_j(\rho_F^*, q_F^*, \rho_C^*, q_C^*)}{\partial \rho_{ijk}^m} \times (\rho_{ijk}^m - \rho_{ijk}^{m*})$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j} \frac{\partial U_j(\rho_F^*, q_F^*, \rho_C^*, q_C^*)}{\partial q_{ijk}^m} \times (q_{ijk}^m - q_{ijk}^{m*}) \geq 0, \quad \forall (\rho_F, q_F, \rho_C, q_C) \in \mathcal{K}^3, \tag{12.14}$$

or, equivalently,

$$\sum_{i=1}^{I}\left[\sum_{l=1}^{I}\frac{\partial \hat{f}_i\big(s_F(\rho_F^*,q_F^*,\rho_C^*,q_C^*),q_F^*\big)}{\partial s_l}\times\frac{\partial s_l(\rho_F^*,q_F^*,\rho_C^*,q_C^*)}{\partial \rho_i}\right.$$
$$\left.-\sum_{j=1}^{n}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j} d_{ijk}^m(\rho_F^*,q_F^*,\rho_C^*,q_C^*)-\rho_i^*\sum_{j=1}^{n}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j}\frac{\partial d_{ijk}^m(\rho_F^*,q_F^*,\rho_C^*,q_C^*)}{\partial \rho_i}\right]\times(\rho_i-\rho_i^*)$$
$$+\sum_{i=1}^{I}\left[\sum_{l=1}^{I}\frac{\partial \hat{f}_i\big(s_F(p_F^*,q_F^*,p_C^*,q_C^*),q_F^*\big)}{\partial s_l}\times\frac{\partial s_l(\rho_F^*,q_F^*,\rho_C^*,q_C^*)}{\partial q_i}\right.$$
$$\left.+\frac{\partial \hat{f}_i\big(s_F^*,q_F^*\big)}{\partial q_i}-\rho_i^*\sum_{j=1}^{n}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j}\frac{\partial d_{ijk}^m(\rho_F^*,q_F^*,\rho_C^*,q_C^*)}{\partial q_i}\right]\times(q_i-q_i^*)$$
$$+\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j}\left[\sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j}\left[\sum_{r=1}^{I}\sum_{v=1}^{n}\sum_{w=1}^{n_R}\sum_{z=1}^{M_v}\frac{\partial \hat{c}_{ljs}^t(d(\rho_F^*,q_F^*,\rho_C^*,q_C^*),q_C^*)}{\partial d_{rvw}^z}\right.\right.$$
$$\left.\times\frac{\partial d_{rvw}^z(\rho_F^*,q_F^*,\rho_C^*,q_C^*)}{\partial \rho_{ijk}^m}\right]$$
$$\left.-d_{ijk}^m(\rho_F^*,q_F^*,\rho_C^*,q_C^*)-\sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j}\frac{\partial d_{ljs}^t(\rho_F^*,q_F^*,\rho_C^*,q_C^*)}{\partial \rho_{ijk}^m}\times\rho_{ljs}^{t*}\right]\times(\rho_{ijk}^m-\rho_{ijk}^{m*})$$
$$+\sum_{j=1}^{n}\sum_{i=1}^{I}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j}\left[\sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j}\left[\sum_{r=1}^{I}\sum_{v=1}^{n}\sum_{w=1}^{n_R}\sum_{z=1}^{M_v}\frac{\partial \hat{c}_{ljs}^t(d(\rho_F^*,q_F^*,\rho_C^*,q_C^*),q_C^*)}{\partial d_{rvw}^z}\right.\right.$$
$$\left.\times\frac{\partial d_{rvw}^z(\rho_F^*,q_F^*,\rho_C^*,q_C^*)}{\partial q_{ijk}^m}\right]$$
$$\left.+\sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j}\frac{\partial \hat{c}_{ljs}^t(d(\rho_F^*,q_F^*,\rho_C^*,q_C^*),q_C^*)}{\partial q_{ijk}^m}-\sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j}\frac{\partial d_{ljs}^t(\rho_F^*,q_F^*,\rho_C^*,q_C^*)}{\partial q_{ijk}^m}\times p_{ljs}^{t*}\right]$$
$$\times(q_{ijk}^m-q_{ijk}^{m*})\geq 0,\quad \forall(\rho_F,q_F,\rho_C,q_C)\in\mathscr{K}^3, \tag{12.15}$$

where $s_F^*\equiv s_F(\rho_F^*,q_F^*,\rho_C^*,q_C^*)$ *and* $d^*\equiv d(\rho_F^*,q_F^*,\rho_C^*,q_C^*)$.

Proof: The feasible set $\mathscr{K}^3$, underlying both variational inequalities (12.14) and (12.15), is convex since it consists of the box-type constraints (12.4), (12.5), and (12.8), (12.9). Equation (12.14) then follows from Gabay and Moulin (1980) and Dafermos and Nagurney (1987). In order to obtain (12.15) from (12.14), for each i we have:

$$-\frac{\partial U_i}{\partial \rho_i} = \sum_{l=1}^{I} \frac{\partial \hat{f}_i}{\partial s_l} \times \frac{\partial s_l}{\partial \rho_i} - \sum_{j=1}^{n}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j} d_{ijk}^m - \rho_i \sum_{j=1}^{n}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j} \frac{\partial d_{ijk}^m}{\partial \rho_i}, \tag{12.16}$$

$$-\frac{\partial U_i}{\partial q_i} = \sum_{l=1}^{I} \frac{\partial \hat{f}_i}{\partial s_l} \times \frac{\partial s_l}{\partial q_i} + \frac{\partial \hat{f}_i}{\partial q_i} - \rho_i \sum_{j=1}^{n}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j} \frac{\partial d_{ijk}^m}{\partial q_i}, \tag{12.17}$$

and, for each i, j, k and m, we have:

$$-\frac{\partial U_j}{\partial \rho_{ijk}^m} = \sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j}\left[\sum_{r=1}^{I}\sum_{v=1}^{n}\sum_{w=1}^{n_R}\sum_{z=1}^{M_v} \frac{\partial \hat{c}_{ljs}^t}{\partial d_{rvw}^z} \times \frac{\partial d_{rvw}^z}{\partial \rho_{ijk}^m}\right]$$
$$-d_{ijk}^m - \sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j} \frac{\partial d_{ljs}^t}{\partial \rho_{ijk}^m} \times \rho_{ljs}^t, \tag{12.18}$$

$$-\frac{\partial U_j}{\partial q_{ijk}^m} = \sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j}\left[\sum_{r=1}^{I}\sum_{v=1}^{n}\sum_{w=1}^{n_R}\sum_{z=1}^{M_v} \frac{\partial \hat{c}_{ljs}^t}{\partial d_{rvw}^z} \times \frac{\partial d_{rvw}^z}{\partial q_{ijk}^m}\right]$$
$$+\sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j} \frac{\partial \hat{c}_{ljs}^t}{\partial q_{ijk}^m} - \sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j} \frac{\partial d_{ljs}^t}{\partial q_{ijk}^m} \times \rho_{ljs}^t. \tag{12.19}$$

Substituting expressions (12.16), (12.17), (12.18) and (12.19) into (12.14) yields variational inequality (12.15). □

We now put the above Bertrand-Nash equilibrium problem into standard variational inequality form (cf. (2.1a)) that is: determine $X^* \in \mathcal{K}$ where X is a vector in R^N, $F(X)$ is a continuous function such that $F(X) : X \mapsto \mathcal{K} \subset R^N$, and

$$\langle F(X^*), X - X^* \rangle \geq 0, \qquad \forall X \in \mathcal{K}, \tag{12.20}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space. We set $\mathcal{K} \equiv \mathcal{K}^3$, which is a closed and convex set, and $N = 2I + 2(Inn_R \sum_{j=1}^{n} M_j)$. We define the vector $X \equiv (\rho_F, q_F, \rho_C, q_C)$ and $F(X) \equiv (F_{\rho_F}, F_{q_F}, F_{\rho_C}, F_{q_C})$ with the i-th component of F_{p_F} and F_{q_F} given, respectively, by:

$$F_{\rho_i} = -\frac{\partial U_i}{\partial \rho_i}, \tag{12.21}$$

$$F_{q_i} = -\frac{\partial U_i}{\partial q_i}, \tag{12.22}$$

and the (i,j,k,m)-th component of F_{ρ_C} and F_{q_C}, respectively, given by:

$$F_{\rho_{ijk}^m} = -\frac{\partial U_j}{\partial \rho_{ijk}^m}, \tag{12.23}$$

$$F_{q_{ijk}^m} = -\frac{\partial U_j}{\partial q_{ijk}^m}. \tag{12.24}$$

Then, clearly, variational inequality (12.15) can be put into standard form (12.20).

Theorem 12.2: Existence of a Solution
A solution to variational inequality (12.14), equivalently, (12.15), exists.

Proof: The feasible set $\mathcal{K}^3$ is convex and compact since it consists of box-type constraints (12.4), (12.5), and (12.8), (12.9), which are bounded below and above, resulting in bounded prices and quality levels for both manufacturers and freight service providers. Existence of a solution to variational inequality (12.14), equivalently, variational inequality (12.15), is, thus, guaranteed since the feasible set $\mathcal{K}$ is compact and the function $F(X)$ (cf. (12.20)) in our model is continuous, under the assumptions made on the underlying functions. □

12.3 The Dynamics

We now propose dynamic adjustment processes for the evolution of the firms' product prices and quality levels and those of the freight service providers (carriers). Each manufacturing firm adjusts the prices and quality of its products in a direction that maximizes its utility while maintaining the price and quality bounds. Also, each freight service provider adjusts its prices and quality levels in order to maximize its utility while keeping the prices and quality levels within their minimum and maximum levels. This kind of behavior, as we show below, yields a projected dynamical system. In addition, it follows that the stationary point of the projected dynamical system coincides with the solution of the variational inequality governing the Bertrand-Nash equilibrium of the supply chain network model introduced in Sect. 12.2. Hence, the adjustment processes provide a reasonable economic and behavioral description of the underlying competitive interactions.

For a current price and quality level pattern at time t, $X(t) = \big(\rho_F(t), q_F(t), \rho_C(t), q_C(t)\big)$, $-F_{\rho_i}(X(t)) = \frac{\partial U_i\left(\rho_F(t), q_F(t), \rho_C(t), q_C(t)\right)}{\partial \rho_i}$, given by (12.21), is the marginal utility (profit) of firm i with respect to the price that it charges for its product, $-F_{q_i}(X(t)) = \frac{\partial U_i\left(\rho_F(t), q_F(t), \rho_C(t), q_C(t)\right)}{\partial q_i}$, defined in (12.22), is the marginal utility of firm i with respect to the quality of its product.

$-F_{\rho_{ijk}^m}(X(t)) = \frac{\partial U_j\left(\rho_F(t), q_F(t), \rho_C(t), q_C(t)\right)}{\partial \rho_{ijk}^m}$, given by (12.23), and $-F_{q_{ijk}^m}(X(t)) = \frac{\partial U_j\left(\rho_F(t), q_F(t), \rho_C(t), q_C(t)\right)}{\partial q_{ijk}^m}$, defined in (12.24), are, respectively, the marginal utility of freight service provider j with respect to price and with respect to quality of shipment, from manufacturing firm i to demand market R_k by mode m. In this framework, the rate of change of the price that firm i charges is in proportion to $-F_{\rho_i}(X)$, as long as the price ρ_i is positive and less than $\bar{\rho}_i$. Namely, when $0 < \rho_i < \bar{\rho}_i$, then

$$\dot{\rho}_i = \frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_i}, \tag{12.25}$$

where $\dot{\rho}_i$ denotes the rate of change of ρ_i. However, when $\frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_i} \leq 0$ or $\frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_i} \geq \bar{\rho}_i$, constraint (12.5) forces the price to remain zero or equal to $\bar{\rho}_i$, hence

$$\dot{\rho}_i = \max\left\{0, \min\left\{\frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_i}, \bar{\rho}_i\right\}\right\}. \tag{12.26}$$

We may write (12.25) and (12.26) concisely as:

$$\dot{\rho}_i = \begin{cases} \frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_i}, & \text{if} \quad 0 < \rho_i < \bar{\rho}_i \\ \max\{0, \min\{\frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_i}, \bar{\rho}_i\}\}, & \text{if} \quad \rho_i = 0 \text{ or } \rho_i = \bar{\rho}_i. \end{cases} \tag{12.27}$$

The rate of change of the product quality of firm i, in turn, is in proportion to $-F_{q_i}(X)$, if $\underline{q}_i < q_i < \bar{q}_i$, so that

$$\dot{q}_i = \frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial q_i}, \tag{12.28}$$

where $\dot{q}_i$ denotes the rate of change of q_i. However, when $\frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial q_i} \leq \underline{q}_i$ or $\frac{\partial U_{F_i}(p_F, q_F, p_C, q_C)}{\partial q_i} \geq \bar{q}_i$, constraint (12.4) forces the quality level to remain at least $\underline{q}_i$ or at most $\bar{q}_i$, respectively. Therefore,

$$\dot{q}_i = \max\left\{\underline{q}_i, \min\left\{\frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial q_i}, \bar{q}_i\right\}\right\}. \tag{12.29}$$

Combining (12.28) and (12.29), we may write:

$$\dot{q}_i = \begin{cases} \frac{\partial U_{F_i}(\rho_F, q_F, \rho_C, q_C)}{\partial q_i}, & \text{if} \quad \underline{q}_i < q_i < \bar{q}_i \\ \max\{\underline{q}_i, \min\{\frac{\partial U_i(\rho_F, q_F, \rho_C, q_C)}{\partial q_i}, \bar{q}_i\}\}, & \text{if} \quad q_i = \underline{q}_i \text{ or } q_i = \bar{q}_i. \end{cases} \tag{12.30}$$

The rate of change of price ρ_{ijk}^m, in turn, that freight service provider j charges demand market R_k to transport the product from firm i via mode m, is in proportion to $-F_{\rho_{ijk}^m}$, as long as $0 < \rho_{ijk}^m < \bar{\rho}_{ijk}^m$, so that

$$\dot{\rho}_{ijk}^m = \frac{\partial U_j(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_{ijk}^m}, \tag{12.31}$$

where $\dot{\rho}_{ijk}^m$ is the rate of change of ρ_{ijk}^m. Otherwise, constraint (12.9) forces the price to be zero or at most equal to $\bar{\rho}_{ijk}^m$. Thus,

$$\dot{\rho}_{ijk}^m = \max\left\{0, \min\left\{\frac{\partial U_j(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_{ijk}^m}, \bar{\rho}_{ijk}^m\right\}\right\}. \tag{12.32}$$

We can write (12.31) and (12.32) compactly as:

$$\dot{\rho}_{ijk}^m = \begin{cases} \frac{\partial U_j(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_{ijk}^m}, & \text{if} \quad 0 < \rho_{ijk}^m < \bar{\rho}_{ijk}^m \\ \max\left\{0, \min\left\{\frac{\partial U_j(\rho_F, q_F, \rho_C, q_C)}{\partial \rho_{ijk}^m}, \bar{\rho}_{ijk}^m\right\}\right\}, & \text{if} \quad \rho_{ijk}^m = 0 \text{ or } \rho_{ijk}^m = \bar{\rho}_{ijk}^m. \end{cases} \tag{12.33}$$

Finally, the rate of change of q_{ijk}^m, which is given by $\dot{q}_{ijk}^m$, is in proportion to $-F_{q_{ijk}^m}$, while the quality of mode m of freight service provider j for transporting the product from firm i to demand market R_k, q_{ijk}^m, is more than its lower bound and less than its upper bound. In other words, when $\underline{q}_{ijk}^m < q_{ijk}^m < \bar{q}_{ijk}^m$, we have

$$\dot{q}_{ijk}^m = \frac{\partial U_j(\rho_F, q_F, \rho_C, q_C)}{\partial q_{ijk}^m}, \tag{12.34}$$

otherwise:

$$\dot{q}_{ijk}^m = \max\left\{\underline{q}_{ijk}^m, \min\left\{\frac{\partial U_j(\rho_F, q_F, \rho_C, q_C)}{\partial q_{ijk}^m}, \bar{q}_{ijk}^m\right\}\right\}. \tag{12.35}$$

Combining (12.34) and (12.35), the quality level q_{ijk}^m evolves according to

$$\dot{q}_{ijk}^m = \begin{cases} \frac{\partial U_j(\rho_F, q_F, \rho_C, q_C)}{\partial q_{ijk}^m}, & \text{if} \quad \underline{q}_{ijk}^m < q_{ijk}^m < \bar{q}_{ijk}^m \\ \max\left\{\underline{q}_{ijk}^m, \min\left\{\frac{\partial U_j(\rho_F, q_F, \rho_C, q_C)}{\partial q_{ijk}^m}, \bar{q}_{ijk}^m\right\}\right\}, & \text{if} \quad q_{ijk}^m = \underline{q}_{ijk}^m \text{ or } q_{ijk}^m = \bar{q}_{ijk}^m. \end{cases} \tag{12.36}$$

Applying (12.27) and (12.30) to all manufacturing firms i; $i = 1, \ldots, I$, and applying (12.33) and (12.36) to all modes $m = 1, \ldots, M_j$ of freight service providers j; $j = 1, \ldots, n$ used in transporting the product from firm i; $i = 1, \ldots, I$ to all demand markets R_k; $k = 1, \ldots, n$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes of the prices and quality levels of firms and freight service providers, in vector form:

$$\dot{X} = \Pi_{\mathscr{K}}(X, -F(X)), \quad X(0) = X^0. \tag{12.37}$$

Recall that $\Pi_{\mathscr{K}}$ is the projection operator of $-F(X)$ onto $\mathscr{K}$ and X^0 is the initial point $(\rho_F^0, q_F^0, \rho_C^0, q_C^0)$ corresponding to the initial price and quality levels of the manufacturing firms and freight service providers.

The dynamical system (12.37) is a projected dynamical system. Please refer to Chap. 2 for additional theoretical results, which can be adapted to the specific dynamic model above.

12.4 The Algorithm

The feasible set underlying variational inequality (12.15) consists of box-type constraints, a feature that we exploit for computational purposes. Specifically, PDS (12.37) yields continuous-time adjustment processes in prices and quality levels of firms and freight service providers. However, for computational purposes, a discrete-time algorithm, which can serve as an approximation to the continuous-time trajectories is needed.

We can apply the Euler method (see (2.34)), which, at each iteration, yields the following closed form expressions for our model (12.30).

Explicit Formulae for the Euler Method Applied to the Multitiered Supply Chain Network Problem

In particular, we have the following closed form expressions for the firms' product prices $\rho_i; i = 1, \ldots, I$ and their product quality levels $q_i; i = 1, \ldots, I$, respectively:

$$\rho_i^{\tau+1} = \max\Bigg\{0\,, \min\Bigg\{\bar{\rho}_i\,, \rho_i^\tau + a_\tau\Bigg[\sum_{j=1}^{n}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j} d_{ijk}^m(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)$$
$$+\rho_i^\tau \sum_{j=1}^{n}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j} \frac{\partial d_{ijk}^m(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)}{\partial \rho_i}$$
$$-\sum_{l=1}^{I} \frac{\partial \hat{f}_i(s_F(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau), q_F^\tau)}{\partial s_l} \times \frac{\partial s_l(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)}{\partial \rho_i}\Bigg]\Bigg\}\Bigg\}, \tag{12.38}$$

$$q_i^{\tau+1} = \max\left\{\underline{q}_i\,, \min\left\{\bar{q}_i\,, q_i^\tau + a_\tau\left[\rho_i^\tau \sum_{j=1}^{n}\sum_{k=1}^{n_R}\sum_{m=1}^{M_j} \frac{\partial d_{ijk}^m(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)}{\partial q_i}\right.\right.\right.$$

$$\left.\left.\left. - \sum_{l=1}^{I} \frac{\partial \hat{f}_i(s_F(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau), q_F^\tau)}{\partial s_l} \times \frac{\partial s_l(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)}{\partial q_i} - \frac{\partial \hat{f}_i(s_F^\tau, q_F^\tau)}{\partial q_i}\right]\right\}\right\}. \tag{12.39}$$

Also, we have the following closed form expressions for the freight service provider prices, ρ_{ijk}^m, and the quality levels, q_{ijk}^m: $i = 1, \ldots, I$; $j = 1, \ldots, n$; $k = 1, \ldots, n_R$; $m = 1, \ldots, M_j$, respectively:

$$\rho_{ijk}^{m(\tau+1)} = \max\left\{0\,, \min\left\{\bar{\rho}_{ijk}^m\,, \rho_{ijk}^{m\tau} + a_\tau\left[d_{ijk}^m(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)\right.\right.\right.$$

$$+ \sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j} \frac{\partial d_{ljs}^t(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)}{\partial \rho_{ijk}^m} \times \rho_{ljs}^{t\tau}$$

$$- \sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j}\left(\sum_{r=1}^{I}\sum_{v=1}^{n}\sum_{w=1}^{n_R}\sum_{z=1}^{M_v} \frac{\partial \hat{c}_{ljs}^t(d(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau), q_C^\tau)}{\partial d_{rvw}^z}\right.$$

$$\left.\left.\left.\left. \times \frac{\partial d_{rvw}^z(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)}{\partial \rho_{ijk}^m}\right)\right]\right\}\right\}, \tag{12.40}$$

$$q_{ijk}^{m(\tau+1)} = \max\left\{\underline{q}_{ijk}^m\,, \min\left\{\bar{q}_{ijk}^m\,, q_{ijk}^{m\tau} + a_\tau\left[\sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j} \frac{\partial d_{ljs}^t(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)}{\partial q_{ijk}^m} \times \rho_{ljs}^{t\tau}\right.\right.\right.$$

$$- \sum_{l=1}^{I}\sum_{s=1}^{n}\sum_{t=1}^{M_j}\left(\sum_{r=1}^{I}\sum_{v=1}^{n}\sum_{w=1}^{n_R}\sum_{z=1}^{M_v} \frac{\partial \hat{c}_{ljs}^t(d(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau), q_C^\tau)}{\partial d_{rvw}^z} \times \frac{\partial d_{rvw}^z(\rho_F^\tau, q_F^\tau, \rho_C^\tau, q_C^\tau)}{\partial q_{ijk}^m}\right)$$

$$\left.\left.\left. - \sum_{l=1}^{I}\sum_{s=1}^{n_R}\sum_{t=1}^{M_j} \frac{\partial \hat{c}_{ljs}^t(d^\tau, q_C^\tau)}{\partial q_{ijk}^m}\right]\right\}\right\}. \tag{12.41}$$

Note that all the functions to the left of the equal signs in (12.38), (12.39), (12.40) and (12.41) are evaluated at their respective variables computed at the τ-th iteration.

Also, the below convergence result is immediate following Nagurney and Zhang (1996) since the feasible set $\mathcal{K}$ is compact.

Theorem 12.3: Convergence

In our multitiered supply chain network game theory model, assume that $F(X) = -\nabla U(\rho_F, q_F, \rho_C, q_C)$ is strictly monotone. Also, assume that F is uniformly

Lipschitz continuous. Then, there exists a unique equilibrium price and quality pattern $(\rho_F^*, q_F^*, \rho_C^*, q_C^*) \in \mathcal{K}$ *and any sequence generated by the Euler method as given by (12.38), (12.39), (12.40) and (12.41), where* $\{a_\tau\}$ *satisfies* $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, *as* $\tau \to \infty$ *converges to* $(\rho_F^*, q_F^*, \rho_C^*, q_C^*)$.

12.5 Numerical Examples

In this section, we present numerical examples illustrating the multitiered supply chain network game theory framework developed in Sects. 12.2 and 12.3. The equilibrium solutions of the model are computed via the Euler method as outlined in Sect. 12.4. Specifically, we present a spectrum of examples with various combinations of manufacturing firms, freight service providers, and modes. The supply chain network topology for each numerical example is described before the data and solution are presented.

The computations via the Euler method are carried out using Matlab. The algorithm was implemented on a VAIO S Series laptop with an Intel Core i7 processor and 12 GB RAM. The convergence tolerance is 10^{-6}, so that the algorithm is deemed to have converged when the absolute value of the difference between each successive price and quality level is less than or equal to 10^{-6}. The sequence $\{\alpha_\tau\}$ is set to: $0.1\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$. We initialize the algorithm by setting the prices and quality levels at their lower bounds. The ranges in which the prices and quality levels vary are noted for each example.

The first two examples are simple examples, for exposition purposes and clarity. The subsequent examples, along with their variants, reveal various aspects of the underlying competition. For the first two examples, we also provide the trajectories of the evolution of the prices and quality.

> Our framework can be applied to both high value and low value products with appropriate modifications in the underlying functions. For example, valuable goods would require greater quality in freight service provision, but at a higher associated cost; also, their production/manufacturing costs, given the components, we would also expect to be higher.

Example 12.1

In the first example, we have a single manufacturing firm, 1, a single freight service provider, 1, with one mode of transport, and a single demand market, R_1, as depicted in the supply chain network in Fig. 12.2.

The demand function for demand market R_1 is:

$$d_{111}^1 = -1.62p_{111}^1 + 1.6q_{111}^1 - 1.45p_1 + 1.78q_1 + 43.$$

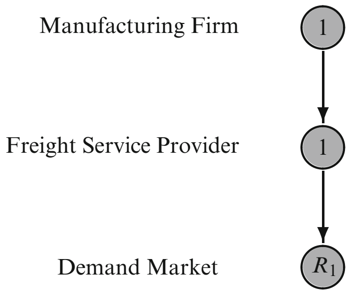

Fig. 12.2 The supply chain network topology for Example 12.1

The supply of firm 1 is:

$$s_1 = d^1_{111}.$$

The production cost of firm 1 is:

$$\hat{f}_1 = 1.55(s_1 + 1.15q_1^2).$$

The utility/profit expression of firm 1 is:

$$U_1 = \rho_1 s_1 - \hat{f}_1.$$

The quality and price of the firm are bounded as below:

$$0 \leq \rho_1 \leq 80, \qquad 10 \leq q_1 \leq 100.$$

The transportation cost of freight service provider 1 is:

$$\hat{c}^1_{111} = 0.5d^1_{111} + (q^1_{111})^2.$$

The utility/profit expression of freight service provider 1 is:

$$U_1 = \rho^1_{111} d^1_{111} - \hat{c}^1_{111},$$

with the following limitations on its price and quality:

$$0 \leq \rho^1_{111} \leq 70, \quad 9 \leq q^1_{111} \leq 100.$$

The Jacobian of $-\nabla U(\rho^1_{111}, \rho_1, q^1_{111}, q_1)$, denoted by $J(\rho^1_{111}, \rho_1, q^1_{111}, q_1)$, is

$$J = \begin{pmatrix} 3.24 & 1.45 & -1.60 & -1.78 \\ 1.62 & 2.90 & -1.60 & -1.78 \\ -1.60 & 0.00 & 2.00 & 0.00 \\ 0.00 & -1.78 & 0.00 & 3.57 \end{pmatrix}.$$

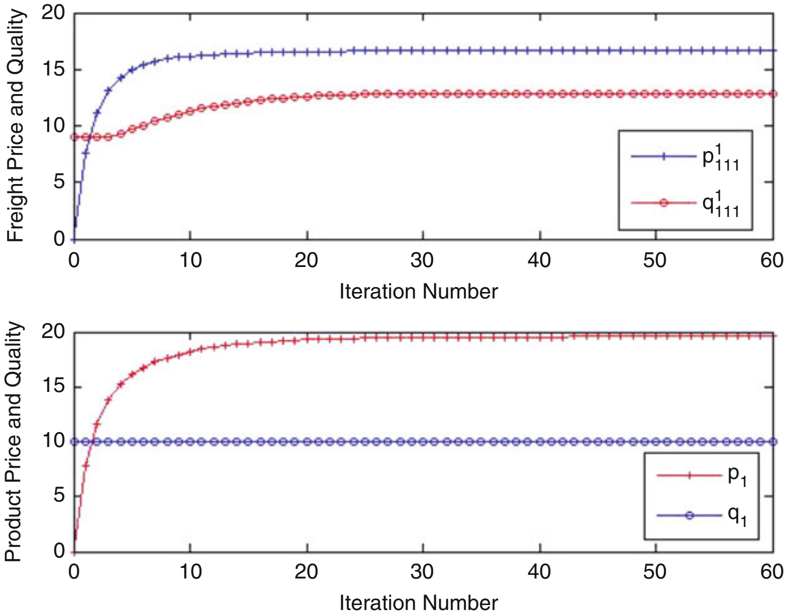

Fig. 12.3 Prices and quality level iterates for the product and freight service for Example 12.1

The eigenvalues of the symmetric part of J, $(J + J^T)/2$, are all positive and they are: 0.79, 1.14, 3.28, and 6.47. The equilibrium result, after 60 iterations, is:

$$\rho_{111}^{1*} = 16.63, \qquad \rho_1^* = 19.57, \qquad q_{111}^{1*} = 12.90, \qquad q_1^* = 10.00.$$

The iterates displayed in Fig. 12.3 provide a discrete-time evolution of the prices and quality levels of the manufacturer and the freight service provider as they respond through the time periods to the demands for the product and service. We observe that the prices move much above the quality levels and reach significantly higher values than their initial ones, while the quality levels do not gain as much. This can be attributed to a lack of competition and enough scope at the demand market for gaining revenues. The manufacturer and freight service provider would try to extract the maximum price out of the market while offering a low quality product and services.

Indeed, in the absence of competition, the manufacturing firm and the freight service provider produce and transport at low quality levels. This explains the low equilibrium values of q_1^* and q_{111}^{1*}. The profit of firm 1 is 292.60 and that of freight service provider 1 is 254.95. Also, the demand d_{111}^1 at equilibrium is 26.13. The demand function is such that more weight is given to the quality of the product than of the freight service provision and the price of the service provider than the product price. Since there is no competition, the manufacturing firm ends up with a higher profit by selling a low quality product, while the service provider gains but not as much as the manufacturer.

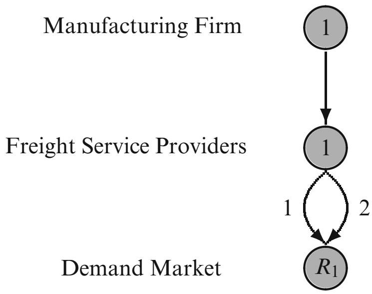

Fig. 12.4 The supply chain network topology for Example 12.2

Example 12.2
In Example 12.2, we extend Example 12.1 by adding another mode of shipment for freight service provider 1. The supply chain network topology is now as depicted in Fig. 12.4.

The demand functions are:

$$d^1_{111} = -1.62\rho^1_{111} + 1.6q^1_{111} - 1.45\rho_1 + 1.78q_1 + 0.03\rho^2_{111} - 0.2q^2_{111} + 43,$$
$$d^2_{111} = -1.75\rho^2_{111} + 1.21q^2_{111} - 1.45\rho_1 + 1.78q_1 + 0.03\rho^1_{111} - 0.2q^1_{111} + 52.$$

The contribution of quality of the product is higher in the demand functions than its price. Also, the contribution of price of the freight service provider is higher in the demand functions than the quality it offers. Here, the freight service providers are striving to position themselves as a value added service.

The supply of manufacturing firm 1 is changed to:

$$s_1 = d^1_{111} + d^2_{111}$$

since there are two modes of shipment available now.

The production cost function of firm 1 is the same as Example 12.1. The transportation costs of the freight service provider 1 for modes 1 and 2 are:

$$\hat{c}^1_{111} = 0.5d^1_{111} + (q^1_{111})^2,$$
$$\hat{c}^2_{111} = 0.45d^2_{111} + 0.54(q^2_{111})^2 + 0.0035d^2_{111}q^2_{111}.$$

Note that mode 1's cost remains as in Example 12.1.

The utility/profit expression of freight service provider 1 is:

$$U_1 = \rho^1_{111}d^1_{111} + \rho^2_{111}d^2_{111} - \hat{c}^1_{111} - \hat{c}^2_{111},$$

with the constraints on the price and quality of shipment kept for the first mode as in Example 12.1 and for the added second mode as below:

$$0 \leq \rho^2_{111} \leq 70, \quad 9 \leq q^2_{111} \leq 100.$$

The symmetric part of J, $(J+J^T)/2$, has positive eigenvalues, which guarantees the strict monotonicity of $F(X)$. The equilibrium solution, after 166 iterations, is:

$$\rho_{111}^{1*}=21.68, \qquad \rho_{111}^{2*}=24.16, \qquad \rho_1^*=27.18,$$
$$q_{111}^{1*}=14.58, \qquad q_{111}^{2*}=22.43, \qquad q_1^*=25.59.$$

The trajectories in Fig. 12.5 provide a discrete-time evolution of the prices and quality levels of the manufacturer and freight service provider. Such trajectories provide valuable information as the decision-makers interact over space and time adjusting their strategic variables so as to maximize their respective profits.

As compared to Fig. 12.3, the quality levels, and, therefore, the prices, of both manufacturer and freight service provider increase. This is a consequence of the competing modes. We observe that the quality of mode 2 is much higher than that of mode 1. Hence, the freight service provider quotes a higher price for mode 2. At the manufacturer's level, we continue to obtain a higher price in comparison to the quality level. However, we see the difference between the prices and quality levels to be much less than Fig. 12.3 (the trajectories move along more closely in Fig. 12.5 than in Fig. 12.3 for the manufacturer).

At equilibrium, the profit of manufacturing firm 1 is 737.29 and that of freight service provider 1 is 1,190.05. The amount shipped via mode 1, d_{111}^1, is 33.59 and that shipped via mode 2, d_{111}^2, is 40.73. Interestingly, even though the price offered by freight service provider 1 for mode 2 is slightly higher, the quality level of mode 2 is much better than that of mode 1, which increases the demand satisfied by mode 2 as compared to mode 1. Also, the fixed component of the demand function, d_{111}^2 is higher than that of d_{111}^1. This also contributes to the higher demand shipped by mode 2 to demand market R_1.

The differences in the profits of the manufacturer (737.29) and the service provider (1,190.05) are explained mainly by the production costs and transportation costs, respectively. It is reasonable to assume that the production costs of a manufacturing firm would be higher than the transportation costs incurred by a freight service provider. This difference gets captured in the (comparatively) higher coefficients of the production cost function.

Example 12.3 and Variant

In Example 12.3 and its variant, we extend Example 12.2 by including another freight service provider with one mode of shipment as illustrated in Fig. 12.6.

The demand functions are:

$$d_{111}^1=-1.62\rho_{111}^1+1.6q_{111}^1-1.45\rho_1+1.78q_1$$
$$+0.03\rho_{111}^2-0.2q_{111}^2+0.04\rho_{121}^1-0.1q_{121}^1+43,$$

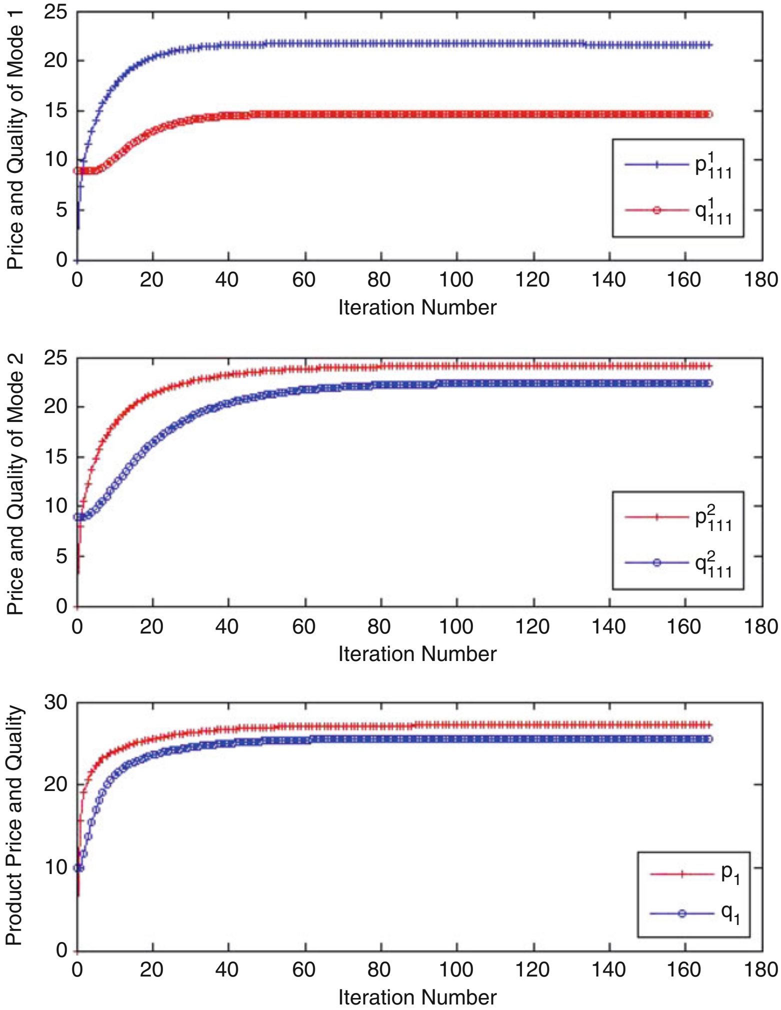

Fig. 12.5 Prices and quality levels for products and modes 1 and 2 of Example 12.2

$$d^2_{111} = -1.75\rho^2_{111} + 1.21q^2_{111} - 1.45\rho_1 + 1.78q_1$$
$$+0.03\rho^1_{111} - 0.2q^1_{111} + 0.04\rho^1_{121} - 0.1q^1_{121} + 52,$$
$$d^1_{121} = -1.79\rho^1_{121} + 1.41q^1_{121} - 1.45\rho_1 + 1.78q_1$$
$$+0.03\rho^1_{111} - 0.2q^1_{111} + 0.04\rho^2_{111} - 0.1q^2_{111} + 47.$$

The supply of firm 1 is:

$$s_1 = d^1_{111} + d^2_{111} + d^1_{121}.$$

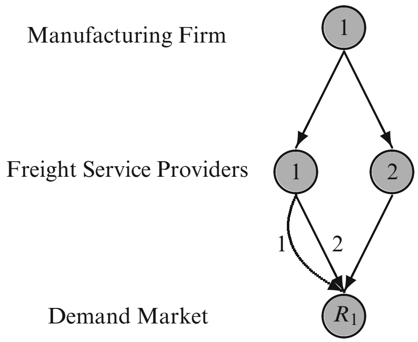

Fig. 12.6 The supply chain network topology for Example 12.3 and variant

The production cost of 1 is the same as in Example 12.2. Therefore, the utility function of 1 has not changed. The transportation costs of freight service provider 1 are:

$$\hat{c}^1_{111} = 0.5d^1_{111} + (q^1_{111})^2 + 0.045d^1_{121},$$
$$\hat{c}^2_{111} = 0.45d^2_{111} + 0.54(q^2_{111})^2 + 0.005d^2_{111}q^2_{111},$$

and that of freight service provider 2 is:

$$\hat{c}^1_{121} = 0.64d^1_{121} + 0.76(q^1_{121})^2.$$

The utility/profit function of firm 1 and its price and quality constraints have not changed. The utility/profit expression of freight service provider 2 is:

$$U_2 = \rho^1_{121}d^1_{121} - \hat{c}^1_{121}.$$

The maximum and minimum levels of price and quality of 2 are:

$$0 \leq \rho^1_{121} \leq 65, \qquad 12 \leq q^1_{121} \leq 100.$$

The Jacobian of $F(X)$ for this example is also positive definite. The new equilibrium solution, computed after 218 iterations, is:

$$\rho^{1*}_{111} = 45.69, \qquad \rho^{2*}_{111} = 45.32, \qquad \rho^{1*}_{121} = 44.82, \qquad \rho^*_1 = 53.91,$$
$$q^{1*}_{111} = 31.69, \qquad q^{2*}_{111} = 41.32, \qquad q^{1*}_{121} = 41.24, \qquad q^*_1 = 78.43.$$

In addition to the competition between modes captured in Example 12.2, in Example 12.3, we capture the competition among freight service providers. This adds pragmatism and generality. The assumption regarding the demand functions being more inclined towards the quality of the product manufactured and the prices of the freight service providers remains valid in this instance as well. This supposition induced by the assumed coefficients of the demand and cost functions gets clearly reflected in the equilibrium solution ($\rho_1^* = 53.91; q_1^* = 78.43$).

At equilibrium, the profit of manufacturing firm 1 is 961.39 and those of freight service providers 1 and 2 are: 4,753.06 and 2,208.92, respectively. Demand market R_1 receives amounts of 71.88 and 76.81 via modes 1 and 2 from freight service provider 1, and 79.07 from freight service provider 2. The inclusion of an additional freight service provider helps to increase the total demand as compared to Example 12.2. The increasing demand provides an incentive for manufacturing firm 1 to increase its quality level and, consequently, its price. This surge in demand also has a positive effect on the utilities (profits) of the manufacturing firm and both freight service providers. Higher demand gets satisfied by freight service provider 2 since its price is lower and the quality level is at par with the quality provided by freight service provider 1 for both modes. Clearly, mode 1 of freight service provider 1 carries the lowest amount of the total demand due to the higher price and lower quality combination he offers.

Variant of Example 12.3

We consider a variant of Example 12.3 wherein the demand function is more sensitive to the price of the product manufactured and the quality offered by the service providers. Keeping the other data consistent, the demand functions are, hence, modified to the following:

$$d_{111}^1 = -1.44\rho_{111}^1 + 1.53q_{111}^1 - 1.82\rho_1 + 1.21q_1$$
$$+0.03\rho_{111}^2 - 0.2q_{111}^2 + 0.04\rho_{121}^1 - 0.1q_{121}^1 + 43,$$
$$d_{111}^2 = -1.49\rho_{111}^2 + 1.65q_{111}^2 - 1.82\rho_1 + 1.21q_1$$
$$+0.03\rho_{111}^1 - 0.2q_{111}^1 + 0.04\rho_{121}^1 - 0.1q_{121}^1 + 52,$$
$$d_{121}^1 = -1.57\rho_{121}^1 + 1.64q_{121}^1 - 1.82\rho_1 + 1.21q_1$$
$$+0.03\rho_{111}^1 - 0.2q_{111}^1 + 0.04\rho_{111}^2 - 0.1q_{111}^2 + 47.$$

The equilibrium solution, computed after 553 iterations, is:

$$\rho_{111}^{1*} = 8.71, \qquad \rho_{111}^{2*} = 63.17, \qquad \rho_{121}^{1*} = 16.22, \qquad \rho_1^* = 24.80,$$
$$q_{111}^{1*} = 9.00, \qquad q_{111}^{2*} = 93.15, \qquad q_{121}^{1*} = 16.92, \qquad q_1^* = 23.67.$$

The quality levels offered by the freight service providers take on higher values than their prices as opposed to a vice versa situation in the case of Example 12.3. At equilibrium, the profit of manufacturing firm 1 is 1,952.19 and the profits of freight service providers 1 and 2 are: 1,073.86 and 164.99, respectively. The transportation costs increase to ensure high quality transportation. Thus, the utility of the manufacturing firm is higher than the utilities of both freight service

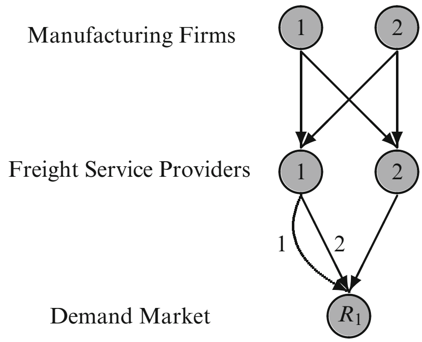

Fig. 12.7 The supply chain network topology for Example 12.4 and variant

providers. This can be explained by the fact that, apart from the price and quality level of the second mode of freight service provider 1, the prices and quality levels of the other mode and the other freight service provider take on much smaller values than in the equilibrium solution of the previous assumption. Since the emphasis is given to the quality of the freight service provider in the demand functions, the low quality levels result in lower demand. Demand market R_1 receives amounts of 9.96 and 92.51 via modes 1 and 2 of freight service provider 1, and 24.46 via freight service provider 2. The low demand further reduces the profits.

Example 12.4 and Variant

Example 12.4 and its variant extend the previous numerical examples through the addition of another manufacturing firm, as shown in Fig. 12.7. These manufacturers offer substitutable products to the demand markets.

The demand functions for manufacturing firm 1 are:

$$\begin{aligned} d^1_{111} &= -1.62\rho^1_{111} + 1.6q^1_{111} - 1.45\rho_1 + 1.78q_1 + 0.08\rho_2 - 0.04q_2 \\ &\quad +0.03\rho^2_{111} - 0.2q^2_{111} + 0.04\rho^1_{121} - 0.1q^1_{121} + 43, \\ d^2_{111} &= -1.75\rho^2_{111} + 1.21q^2_{111} - 1.45\rho_1 + 1.78q_1 + 0.08\rho_2 - 0.04q_2 \\ &\quad +0.03\rho^1_{111} - 0.2q^1_{111} + 0.04\rho^1_{121} - 0.1q^1_{121} + 52, \\ d^1_{121} &= -1.79\rho^1_{121} + 1.41q^1_{121} - 1.45\rho_1 + 1.78q_1 + 0.08\rho_2 - 0.04q_2 \\ &\quad +0.03\rho^1_{111} - 0.2q^1_{111} + 0.04\rho^2_{111} - 0.1q^2_{111} + 47, \end{aligned}$$

and that of manufacturing firm 2 are:

$$\begin{aligned} d^1_{211} &= -1.57\rho^1_{211} + 1.26q^1_{211} - 1.65\rho_2 + 1.98q_2 + 0.08\rho_1 - 0.04q_1 \\ &\quad +0.04\rho^2_{211} - 0.1q^2_{211} + 0.02\rho^1_{221} - 0.12q^1_{221} + 51, \\ d^2_{211} &= -1.63\rho^2_{211} + 1.21q^2_{211} - 1.65\rho_2 + 1.98q_2 + 0.08\rho_1 - 0.04q_1 \\ &\quad +0.04\rho^1_{211} - 0.1q^1_{211} + 0.02\rho^1_{221} - 0.12q^1_{221} + 44, \end{aligned}$$

$$d_{221}^1 = -1.46\rho_{221}^1 + 1.41q_{221}^1 - 1.65\rho_2 + 1.98q_2 + 0.08\rho_1 - 0.04q_1$$
$$+0.04\rho_{211}^1 - 0.1q_{211}^1 + 0.02\rho_{211}^2 - 0.12q_{211}^2 + 56.$$

The supply of manufacturing firm 1 is similar to that in Example 12.3 and that of manufacturing firm 2 is:

$$s_2 = d_{211}^1 + d_{211}^2 + d_{221}^1.$$

The production cost functions of firms 1 and 2 are:

$$\hat{f}_1 = 1.55s_1 + 1.88q_1^2 + 0.02s_2 + 0.06q_2,$$
$$\hat{f}_2 = 1.47s_2 + 1.94q_2^2 + 0.041s_1 + 0.032q_1.$$

Manufacturing firm 1 has the same utility/profit function and price and quality bounds as in Example 12.3. The utility/profit expression of manufacturing firm 2 is:

$$U_2 = \rho_2 s_2 - \hat{f}_2,$$

and the price and quality of his product are constrained in the following manner:

$$0 \le \rho_2 \le 95, \qquad 8 \le q_2 \le 100.$$

The transportation cost functions of freight service provider 1 are changed to:

$$\hat{c}_{111}^1 = 0.5d_{111}^1 + (q_{111}^1)^2 + 0.0045d_{121}^1 + 0.0045d_{221}^1 + 0.0045d_{211}^1,$$
$$\hat{c}_{111}^2 = 0.45d_{111}^2 + 0.54(q_{111}^2)^2 + 0.0011d_{211}^2,$$
$$\hat{c}_{211}^1 = 0.68d_{211}^1 + 0.79(q_{211}^1)^2 + 0.002d_{211}^1 + 0.002d_{221}^1,$$
$$\hat{c}_{211}^2 = 0.57d_{211}^2 + 0.74(q_{211}^2)^2 + 0.005d_{111}^2,$$

and the functions of freight service provider 2 are changed to:

$$\hat{c}_{121}^1 = 0.64d_{121}^1 + 0.76(q_{121}^1)^2 + 0.0015d_{221}^1,$$
$$\hat{c}_{221}^1 = 0.59d_{221}^1 + 0.80(q_{221}^1)^2 + 0.01d_{121}^1 + 0.01d_{111}^1 + 0.01d_{211}^1.$$

The utility of freight service provider 1 is:

$$U_1 = \rho_{111}^1 d_{111}^1 + \rho_{111}^2 d_{111}^2 + \rho_{211}^1 d_{211}^1 + \rho_{211}^2 d_{211}^2 - \hat{c}_{111}^1 - \hat{c}_{111}^2 - \hat{c}_{211}^1 - \hat{c}_{211}^2,$$

and that of freight service provider 2 is:

$$U_2 = \rho_{121}^1 d_{121}^1 + \rho_{221}^1 d_{221}^1 - \hat{c}_{121}^1 - \hat{c}_{221}^1.$$

The lower and upper bounds of the prices for freight service providers are now:

$$0 \leq \rho_{i1k}^{M_1} \leq 90, \qquad \forall i, k, M_1, \text{ for } M_1 = 2,$$
$$0 \leq \rho_{i2k}^{M_2} \leq 85, \qquad \forall i, k, M_2, \text{ for } M_2 = 1.$$

The equilibrium solution, computed after 231 iterations, is:

$$\rho_{111}^{1*} = 40.20, \quad \rho_{111}^{2*} = 40.72, \quad \rho_{121}^{1*} = 39.79, \quad \rho_1^* = 48.08,$$
$$\rho_{211}^{1*} = 51.17, \quad \rho_{211}^{2*} = 42.88, \quad \rho_{221}^{1*} = 69.18, \quad \rho_2^* = 50.89,$$
$$q_{111}^{1*} = 27.73, \quad q_{111}^{2*} = 37.76, \quad q_{121}^{1*} = 36.53, \quad q_1^* = 66.25,$$
$$q_{211}^{1*} = 37.64, \quad q_{211}^{2*} = 29.42, \quad q_{221}^{1*} = 63.97. \quad q_2^* = 75.65.$$

In this example, we consider competition at the manufacturers' level, the freight service providers' level, and between modes of a particular service provider. This, further, increases the generality, as well as the complexity, of the problem when compared with Example 12.3. The assumption regarding the demand functions being more inclined towards the quality of the product manufactured and the prices of the service providers remains valid in this instance as well. The equilibrium solution ($\rho_1^* = 48.08$; $q_1^* = 66.25$; $\rho_2^* = 50.89$; $q_2^* = 75.65$) supports this assumption.

The utilities of manufacturing firms 1 and 2 are 1,179.39 and 976.85, respectively. Moreover, the utilities of freight service providers 1 and 2 are: 8,743.66 and 5,340.84, respectively. The demand market receives an amount of 132.37 of the product manufactured by 1 from freight service provider 1 and an amount of 70.05 from freight service provider 2. Firm 2 sends 144.51 units via 1 and 100.14 units by 2.

Due to the added competition at the manufacturers' level, the quality and price of the product manufactured at firm 1 have declined as compared to Example 12.3. This is expected since to attain more market share, the prices would be lowered, which would result in a lowering of quality levels. The profit of firm 1 is higher than that of firm 2. A product with reduced prices and quality levels would require cheaper prices (and, hence, quality) of the transporters. Hence, prices and quality levels of freight service provider 1 carrying products from firm 1 have also been reduced. It is interesting to note that even though the price and quality level of freight service provider 2 transporting the product manufactured by firm 2 are the highest of all ($\rho_{221}^{1*}; q_{221}^{1*}$), more demand for the product of firm 2 is satisfied by freight service provider 2 (100.14) than that of firm 1 (70.05). The prices and quality levels of freight service provider 2 transporting goods of manufacturer 1 are at par with that of freight service provider 1.

Variant of Example 12.4

We now construct a variant of Example 12.4 in which the demand is more sensitive to the price of the product manufactured and the quality offered by the freight service providers. Keeping the other data consistent, the demand functions are, hence, modified to the following:

$$d^1_{111} = -1.44\rho^1_{111} + 1.53q^1_{111} - 1.82\rho_1 + 1.21q_1 + 0.08p\rho_2 - 0.04q_2$$
$$+0.03\rho^2_{111} - 0.2q^2_{111} + 0.04\rho^1_{121} - 0.1q^1_{121} + 43,$$
$$d^2_{111} = -1.49\rho^2_{111} + 1.65q^2_{111} - 1.82\rho_1 + 1.21q_1 + 0.08\rho_2 - 0.04q_2$$
$$+0.03\rho^1_{111} - 0.2q^1_{111} + 0.04\rho^1_{121} - 0.1q^1_{121} + 52,$$
$$d^1_{121} = -1.57\rho^1_{121} + 1.64q^1_{121} - 1.82\rho_1 + 1.21q_1 + 0.08\rho_2 - 0.04q_2$$
$$+0.03\rho^1_{111} - 0.2q^1_{111} + 0.04\rho^2_{111} - 0.1q^2_{111} + 47,$$
$$d^1_{211} = -1.39\rho^1_{211} + 1.66q^1_{211} - 1.88\rho_2 + 1.25q_2 + 0.08\rho_1 - 0.04q_1$$
$$+0.04\rho^2_{211} - 0.1q^2_{211} + 0.02\rho^1_{221} - 0.12q^1_{221} + 51,$$
$$d^2_{211} = -1.42\rho^2_{211} + 1.58q^2_{211} - 1.88\rho_2 + 1.25q_2 + 0.08\rho_1 - 0.04q_1$$
$$+0.04\rho^1_{211} - 0.1q^1_{211} + 0.02\rho^1_{221} - 0.12q^1_{221} + 44,$$
$$d^1_{221} = -1.40\rho^1_{221} + 1.63q^1_{221} - 1.88\rho_2 + 1.25q_2 + 0.08\rho_1 - 0.04q_1$$
$$+0.04\rho^1_{211} - 0.1q^1_{211} + 0.02\rho^2_{211} - 0.12q^2_{211} + 56.$$

The equilibrium solution, computed after 568 iterations, is:

$$\rho^{1*}_{111} = 8.30, \quad \rho^{2*}_{111} = 64.70, \quad \rho^{1*}_{121} = 15.54, \quad \rho^*_1 = 25.02,$$
$$\rho^{1*}_{211} = 28.70, \quad \rho^{2*}_{211} = 18.47, \quad \rho^{1*}_{221} = 36.15, \quad \rho^*_2 = 21.38,$$
$$q^{1*}_{111} = 9.00, \quad q^{2*}_{111} = 96.71, \quad q^{1*}_{121} = 16.16, \quad q^*_1 = 22.71,$$
$$q^{1*}_{211} = 28.34, \quad q^{2*}_{211} = 17.19, \quad q^{1*}_{221} = 38.55. \quad q^*_2 = 19.24.$$

At equilibrium, the profits of manufacturing firms 1 and 2 are: 2,037.45 and 1,511.87, respectively, and those of freight service providers 1 and 2 are: 1,729.44 and 737.02. It is important to note that, based on the previous equilibrium solution, the profits of the freight service providers are higher than those of the manufacturers. However, based on the variant's solution, the profits of the freight service providers (focus on quality) are lower than the profits of the manufacturers (focus on price). This is directly connected to the transportation costs which increase in order to ensure high quality transportation. Demand market R_1 receives 104.81 units of firm 1's product from freight service provider 1 and 23.37 units from freight service provider 2. Also, the demand market receives 62.52 units of firm 2's product via freight service provider 1 and 49.79 via freight service provider 2.

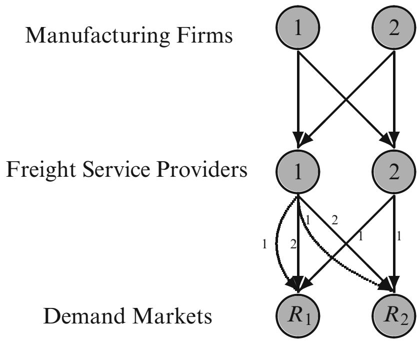

Fig. 12.8 The supply chain network topology for Example 12.5 and variant

Example 12.5 and Variant
In this example and its variant, we extend the previous ones by adding another demand market, demand market R_2, to the supply chain network; see Fig. 12.8. The manufacturers and freight service providers compete to serve two demand markets now.

The demand functions at demand market R_2 for manufacturing firm 1 are:

$$\begin{aligned}
d^1_{112} &= -1.63\rho^1_{112} + 1.55q^1_{112} - 1.48\rho_1 + 1.74q_1 + 0.06p_2 - 0.05q_2 \\
&\quad +0.05\rho^2_{112} - 0.23q^2_{112} + 0.02\rho^1_{122} - 0.13q^1_{122} + 50, \\
d^2_{112} &= -1.78\rho^2_{112} + 1.21q^2_{112} - 1.48\rho_1 + 1.74q_1 + 0.06\rho_2 - 0.05q_2 \\
&\quad +0.05\rho^1_{112} - 0.23q^1_{112} + 0.02\rho^1_{122} - 0.13q^1_{122} + 39, \\
d^1_{122} &= -1.66\rho^1_{122} + 1.41q^1_{122} - 1.48\rho_1 + 1.74q_1 + 0.06\rho_2 - 0.05q_2 \\
&\quad +0.05\rho^1_{112} - 0.23q^1_{112} + 0.02\rho^2_{112} - 0.13q^2_{112} + 42,
\end{aligned}$$

and for manufacturing firm 2:

$$\begin{aligned}
d^1_{212} &= -1.49\rho^1_{212} + 1.34q^1_{212} - 1.61\rho_2 + 1.86q_2 + 0.06\rho_1 - 0.05q_1 \\
&\quad +0.05\rho^2_{212} - 0.09q^2_{212} + 0.03\rho^1_{222} - 0.08q^1_{222} + 38, \\
d^2_{212} &= -1.57\rho^2_{212} + 1.26q^2_{212} - 1.61\rho_2 + 1.86q_2 + 0.06\rho_1 - 0.05q_1 \\
&\quad +0.05\rho^1_{212} - 0.09q^1_{212} + 0.03\rho^1_{222} - 0.08q^1_{222} + 43, \\
d^1_{222} &= -1.53\rho^1_{222} + 1.31q^1_{222} - 1.61\rho_2 + 1.86q_2 + 0.06\rho_1 - 0.05q_1 \\
&\quad +0.05\rho^1_{212} - 0.09q^1_{212} + 0.03\rho^2_{212} - 0.08q^2_{212} + 58.
\end{aligned}$$

The supply functions for both manufacturers are changed in the following manner:

$$s_1 = d^1_{111} + d^2_{111} + d^1_{121} + d^1_{112} + d^2_{112} + d^1_{122},$$
$$s_2 = d^1_{211} + d^2_{211} + d^1_{221} + d^1_{212} + d^2_{212} + d^1_{222}.$$

There is no change to the utility functions of the manufacturing firms. However, the transportation functions of freight service provider 1 have been changed to:

$$\hat{c}^1_{111} = 0.5d^1_{111} + (q^1_{111})^2 + 0.0045d^1_{121} + 0.0045d^1_{221} + 0.0045d^1_{211} + 0.0045d^1_{112},$$
$$\hat{c}^2_{111} = 0.45d^2_{111} + 0.54(q^2_{111})^2 + 0.0011d^2_{211} + 0.0011d^2_{212},$$
$$\hat{c}^1_{211} = 0.68d^1_{211} + 0.79(q^1_{211})^2 + 0.002d^1_{111} + 0.002d^1_{121} + 0.002d^1_{212},$$
$$\hat{c}^2_{211} = 0.57d^2_{211} + 0.74(q^2_{211})^2 + 0.005d^2_{111} + 0.005d^2_{212},$$
$$\hat{c}^1_{112} = 0.61d^1_{112} + 0.7(q^1_{112})^2 + 0.0037d^1_{111} + 0.0037d^1_{122} + 0.0037d^1_{212},$$
$$\hat{c}^2_{112} = 0.52d^2_{112} + 0.58(q^2_{112})^2 + 0.0024d^2_{212},$$
$$\hat{c}^1_{212} = 0.49d^1_{212} + 0.59(q^1_{212})^2 + 0.0017d^1_{112} + 0.0017d^1_{122},$$
$$\hat{c}^2_{212} = 0.43d^2_{212} + 0.55(q^2_{212})^2 + 0.0023d^2_{112},$$

and that of freight service provider 2 to:

$$\hat{c}^1_{121} = 0.64d^1_{121} + 0.76(q^1_{121})^2 + 0.0015d^1_{221},$$
$$\hat{c}^1_{221} = 0.59d^1_{221} + 0.80(q^1_{221})^2 + 0.014d^1_{121} + 0.014d^1_{111} + 0.014d^1_{211},$$
$$\hat{c}^1_{122} = 0.67d^1_{122} + 0.73(q^1_{122})^2 + 0.0031d^1_{222} + 0.0031d^1_{212},$$
$$\hat{c}^1_{222} = 0.45d^1_{222} + 0.58(q^1_{222})^2 + 0.012d^1_{122} + 0.012d^1_{112} + 0.012d^1_{212}.$$

With the same constraints on the prices and quality levels, the profit expressions of freight service providers become:

$$U_1 = \rho^1_{111}d^1_{111} + \rho^2_{111}d^2_{111} + \rho^1_{211}d^1_{211} + \rho^2_{211}d^2_{211} + \rho^1_{112}d^1_{112}$$
$$+\rho^2_{112}d^2_{112} + \rho^1_{212}d^1_{212} + \rho^2_{212}d^2_{212}$$
$$-\hat{c}^1_{111} - \hat{c}^2_{111} - \hat{c}^1_{211} - \hat{c}^2_{211} - \hat{c}^1_{112} - \hat{c}^2_{112} - \hat{c}^1_{212} - \hat{c}^2_{212},$$
$$U_2 = \rho^1_{121}d^1_{121} + \rho^1_{221}d^1_{221} + \rho^1_{122}d^1_{122} + \rho^1_{222}d^1_{222} - \hat{c}^1_{121} - \hat{c}^1_{221} - \hat{c}^1_{122} - \hat{c}^1_{222}.$$

The equilibrium solution, after 254 iterations, is:

$$\rho^{1*}_{111} = 56.79, \quad rho^{2*}_{111} = 55.45, \quad \rho^{1*}_{112} = 72.96, \quad \rho^{2*}_{112} = 36.93,$$
$$\rho^{1*}_{121} = 55.19, \quad \rho^{1*}_{122} = 53.55, \quad \rho^{1*}_{211} = 62.77, \quad \rho^{2*}_{211} = 53.28,$$

$$\rho_{212}^{1*} = 72.94, \quad \rho_{212}^{2*} = 65.91, \quad \rho_{221}^{1*} = 76.15, \quad \rho_{222}^{1*} = 83.73,$$

$$\rho_1^* = 63.76, \quad \rho_2^* = 64.90, \quad q_1^* = 100.00, \quad q_2^* = 100.00,$$

$$q_{111}^{1*} = 39.53, \quad q_{111}^{2*} = 51.20, \quad q_{112}^{1*} = 74.61, \quad q_{112}^{2*} = 23.54,$$

$$q_{121}^{1*} = 50.93, \quad q_{122}^{1*} = 51.05, \quad q_{211}^{1*} = 46.25, \quad q_{211}^{2*} = 36.72,$$

$$q_{212}^{1*} = 76.89, \quad q_{212}^{2*} = 69.56, \quad q_{221}^{1*} = 61.18, \quad q_{222}^{1*} = 94.70.$$

In this example, we consider competition at the manufacturers' level, the freight service providers' level, and between modes of a particular service provider, wherein all these players are competing to satisfy the demands at two different demand markets. This makes the problem quite complex. The assumption regarding the demand functions being more sensitive to the quality of the product manufactured and the prices of the service providers remains valid in this example as well. The equilibrium solution ($\rho_1^* = 63.76; q_1^* = 100.00; \rho_2^* = 64.90; q_2^* = 100.00$) supports this assumption. The price and quality levels have gone up as compared to Example 4 since there are two demand markets to be satisfied now as opposed to one.

The utilities/profits of manufacturers 1 and 2 have increased to 15,244.22 and 19,922.55, respectively. Also, the freight service providers 1 and 2 are now witnessing higher utilities/profits of 29,256.82 and 16,905.45, respectively. Since more demand from multiple demand markets has increased the prices and quality levels of products, the utilities have increased. Freight service provider 1 transports an amount of 279.46 to demand market R_1 and an amount of 381.13 to demand market R_2. Also, freight service provider 2 carries an amount of 207.96 to demand market R_1 and 215.20 to demand market R_2.

Since there is enough demand for products of both manufacturers 1 and 2, the prices of the products are high and the quality levels are at their upper bounds of 100. This happens since the emphasis is on quality rather than price for manufacturers. Hence, the prices and quality levels of the two freight service providers also go up as compared to Example 12.4.

Variant of Example 12.5

Once again, we consider a variant wherein the demand function is more sensitive to the price of the product manufactured and the quality offered by the freight service providers. Keeping the other data consistent, the demand functions are, hence, modified to the following:

$$d_{112}^1 = -1.37\rho_{112}^1 + 1.67q_{112}^1 - 1.91\rho_1 + 1.33q_1 + 0.06\rho_2 - 0.05q_2$$
$$+0.05\rho_{112}^2 - 0.23q_{112}^2 + 0.02\rho_{122}^1 - 0.13q_{122}^1 + 50,$$

$$d_{112}^2 = -1.41\rho_{112}^2 + 1.65q_{112}^2 - 1.91\rho_1 + 1.33q_1 + 0.06\rho_2 - 0.05q_2$$
$$+0.05\rho_{112}^1 - 0.23q_{112}^1 + 0.02\rho_{122}^1 - 0.13q_{122}^1 + 39,$$

$$d^1_{122} = -1.35\rho^1_{122} + 1.70q^1_{122} - 1.91\rho_1 + 1.33q_1 + 0.06\rho_2 - 0.05q_2$$
$$+0.05\rho^1_{112} - 0.23q^1_{112} + 0.02\rho^2_{112} - 0.13q^2_{112} + 42,$$
$$d^1_{212} = -1.33\rho^1_{212} + 1.59q^1_{212} - 1.87\rho_2 + 1.29q_2 + 0.06\rho_1 - 0.05q_1$$
$$+0.05\rho^2_{212} - 0.09q^2_{212} + 0.03\rho^1_{222} - 0.08q^1_{222} + 38,$$
$$d^2_{212} = -1.36\rho^2_{212} + 1.67q^2_{212} - 1.87\rho_2 + 1.29q_2 + 0.06\rho_1 - 0.05q_1$$
$$+0.05\rho^1_{212} - 0.09q^1_{212} + 0.03\rho^1_{222} - 0.08q^1_{222} + 43,$$
$$d^1_{222} = -1.42\rho^1_{222} + 1.68q^1_{222} - 1.87\rho_2 + 1.29q_2 + 0.06\rho_1 - 0.05q_1$$
$$+0.05\rho^1_{212} - 0.09q^1_{212} + 0.03\rho^2_{212} - 0.08q^2_{212} + 58.$$

The equilibrium solution, after 769 iterations, is:

$$\rho^{1*}_{111} = 22.05, \quad \rho^{2*}_{111} = 80.01, \quad \rho^{1*}_{112} = 44.02, \quad \rho^{2*}_{112} = 77.79,$$
$$\rho^{1*}_{121} = 46.56, \quad \rho^{1*}_{122} = 71.98, \quad \rho^{1*}_{211} = 62.01, \quad \rho 211^{2*} = 47.77,$$
$$\rho^{1*}_{212} = 82.80, \quad \rho^{2*}_{212} = 85.62, \quad \rho^{1*}_{221} = 64.72, \quad \rho^{1*}_{222} = 85.00,$$
$$\rho^*_1 = 43.78, \quad \rho^*_2 = 52.86, \quad q^*_1 = 85.79, \quad q^*_2 = 100.00,$$
$$q^{1*}_{111} = 9.00, \quad q^{2*}_{111} = 100.00, \quad q^{1*}_{112} = 39.34, \quad q^{2*}_{112} = 100.00,$$
$$q^{1*}_{121} = 49.85, \quad q^{1*}_{122} = 82.99, \quad q^{1*}_{211} = 61.55, \quad q^{2*}_{211} = 46.18,$$
$$q^{1*}_{212} = 100.00, \quad q^{2*}_{212} = 100.00, \quad q^{1*}_{221} = 65.62, \quad q^{1*}_{222} = 100.00.$$

The profits of firms 1 and 2 are 6,333.31 and 10,285.25, respectively. The profits of freight service providers 1 and 2 are 18,654.58 and 10,277.76, respectively. As expected, the profits are higher than those in Example 12.3 onwards. This particular variant registers the highest. Since the focus of the freight service providers is on quality, there are multiple cases wherein the quality levels of the providers are at their upper bounds. The demand markets have grown which lets the manufacturers and freight service providers increase their prices and quality levels. Higher quality levels, however, ensure that the transportation costs go up which, in turn, reduces the profits of the freight service providers.

12.6 Summary and Conclusions

In this chapter, we developed a game theory supply chain network model in both static and dynamic versions with multiple manufacturers and freight service providers competing on price and quality. This multi-faceted inclusion of competition in the model assesses the quality conformance level of the product and the level of service of freight service providers along with the prices at which the products and the transportation services are offered. The model handles multiple modes of transportation for delivery of shipments. The utility/profit of each manufacturer (or freight service provider) depends on the prices and on the quality levels offered by its competitors as well as those of the others.

Variational inequality theory was employed in the formulation of the equilibrium governing the manufacturers' and freight service providers' behaviors with respect to price and quality followed by the rigorous description of the underlying dynamic interactions until a stationary point; equivalently, an equilibrium is achieved. The dynamics were shown to satisfy a projected dynamical system. The computational procedure utilized was the Euler method. The discrete-time algorithm, also serving as an approximation to the continuous-time trajectories, yields equilibrium price and quality patterns for the manufacturers and the freight service providers.

In order to illustrate the generality of the framework and the computational scheme, we provided solutions to a series of numerical examples, beginning with smaller scale examples. In the larger examples, a scenario and its variant were explored while computing and analyzing the solutions for various combinations of manufacturing firms, freight service providers, and modes of transportation. The competition within echelons of the different examples altered the price and quality levels, and, hence, the profits, of the entities. We considered a scenario wherein the demand functions were more sensitive to the quality of the product manufactured and the price charged by the freight service providers. The variant took a contrasting position, whereby the demand markets were giving more importance to the price of the product manufactured and the quality levels offered by the freight service providers. These contradictory situations brought about interesting comparisons between the profits of the manufacturers and the freight service providers, and how they changed when the emphasis on price and quality levels changed.

There are many aspects to our proposed framework that are worthy of further discussion and investigation. For instance, additional tiers of supply chain decision-makers could be included. The quality levels might be explicitly modeled for the freight service providers in terms of time-conformance of delivery, reliability of the service, emission standards (to compare the environmental viability of various modes), the quality of in-house transportation infrastructure, and so on. It is interesting to note from the results of this chapter that in order to capture a higher market share, manufacturers or freight service providers might try to quote a lower price and offer a lower quality level (leading to a lower cost). However, a lower quality product/service might not be able to sustain the market share.

Our contributions to the existing literature are: (1) We model explicit competition among manufacturing firms and freight service providers (carriers) in terms of prices and quality of the products that the firms offer and the prices and quality of the freight services provided. This multi-faceted inclusion of competition from price and quality dimensions leads to results that not just quantify quality at the product and service ends, but also helps to assess the trade-offs between quality and costs at each echelon of the supply chain that ultimately influences the demand. A model that considers oligopolistic competition among manufacturers and freight service providers under price and quality with multiple modes of transportation and non-separable, nonlinear, and asymmetric demand and cost functions is constructed for the first time with this framework. (2) The analysis for freight service providers contains price and quality evaluations for multiple modes of transportation. The transportation costs, consequently, differ by mode, leading to a pertinent evaluation of quality vs. costs for the freight service providers and the modes of transportation that they offer to the customers. In our frame of reference, modes can also imply intermodal transportation of products. (3) We handle heterogeneity in the freight service providers' cost functions and in the consumers' demands and do not limit ourselves to specific functional forms. The utility/profit of each manufacturing firm considers price and quality for not just its own products, but that of other manufacturing firms as well. Similarly, the utility/profit of each freight service provider includes the implications of other providers' prices and quality for various modes in addition to its own. Also, we impose bounds on the prices and quality levels with positive minimum quality levels corresponding to minimum quality standards, relevant for policy-making. (4) We provide qualitative properties of the equilibrium price and quality pattern and also present the underlying dynamics associated with the evolution of the prices and quality levels over time until the equilibrium is achieved. (5) The theoretical framework is supported by a rigorous algorithm that is well-suited for implementation. (6) The computational scheme is applied to a spectrum of numerical examples in order to illustrate the generality of the framework. Specifically, we provide complete input and output data for five examples and three variants, for a total of eight examples.

Our work fills the gap in the existing literature by capturing quality in transportation as well as production in multitiered competitive supply chain networks, along with prices as strategic variables. It provides a critical foundation for future research in this area.

12.7 Sources and Notes

This chapter is based on the paper by Nagurney et al. (2015). In this chapter, we added a table, standardized the notation, and also added new references. Additional references can be found in Nagurney et al. (2015).

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Glossary of Notation

This is a glossary of symbols used in this book. Other symbols are defined in the book as needed. A vector is assumed to be a column vector, unless noted otherwise.

$\in$	an element of
$\subset$	subset of
$\subseteq$	subset of or equal to
$\cup, \cap$	union, intersection
$\forall$	for all
$\exists$	there exists
R	the real line
R^N	Euclidean N-dimensional space
R^N_+	Euclidean N-dimensional space on the nonnegative orthant
$:$	such that; also $\mid$
$\equiv$	is equivalent to
$\mapsto$	maps to
$\rightarrow$	tends to
$\circ$	composition
$\|x\| = (\sum_{i=1}^N x_i^2)^{\frac{1}{2}}$	length of $x \in R^N$ with components $(x_1, x_2, \ldots, x_N)$
x^T	transpose of a vector x
$\langle x, x \rangle$	inner product of vector x in R^N where $\langle x, x \rangle = x_1^2 + \ldots + x_N^2$
$x^T \cdot x$	also denotes the inner product of x
$\lvert y \rvert$	absolute value of y
$[a, b]$; (a, b)	a closed interval; an open interval in R
∇f	gradient of $f : R^N \mapsto R$

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∇F	the $N \times N$ Jacobian of a mapping $F : R^N \mapsto R^N$
$\frac{\partial f}{\partial x}$	partial derivative of f with respect to x
$H(X)$	Hessian of a twice differentiable function $f : R^N \rightarrow R$
	$H = \begin{pmatrix} \frac{\partial^2 f}{\partial X_1 \partial X_1} & \frac{\partial^2 f}{\partial X_1 \partial X_2} & \cdots & \frac{\partial^2 f}{\partial X_1 \partial X_N} \\ \frac{\partial^2 f}{\partial X_2 \partial X_1} & \frac{\partial^2 f}{\partial X_2 \partial X_2} & \cdots & \frac{\partial^2 f}{\partial X_2 \partial X_N} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial X_N \partial X_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial X_N \partial X_N} \end{pmatrix}$
$\text{argmin}_{x \in \mathscr{K}} f(x)$	the set of $x \in \mathscr{K}$ attaining the minimum of $f(x)$
A^T	transpose of the matrix A
lim	limit
$\int$	integral
barS	the closure of S, where S is a subset of the Euclidean space
S^0	the interior of S, where S is a subset of the Euclidean space
∂S	the boundary of S, where S is a subset of the Euclidean space
$X_0 \cdot t$	the dynamic system at time t that passes through X_0 at time 0; also $X_0(t)$

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