Vibrations of Shells and Plates

Third Edition, Revised and Expanded

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MARCEL DEKKER, INC. NEW YORK

Vibrations of Shells and Plates

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This edition published in the Taylor & Francis e-Library, 2005.

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Library of Congress Cataloging-in-Publication Data
A catalog record for this book is available from the Library of Congress.

ISBN 0-203-02630-6 Master e-book ISBN

ISBN: 0-8247-5629-0 (Print Edition)

Headquarters
Marcel Dekker, Inc., 270 Madison Avenue, New York, NY 10016, U.S.A.
tel: 212-696-9000; fax: 212-685-4540

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To my grandchildren: Amber, Nicholas, Ashleigh, April, Jackson,
Broderic, Thomas, Carter, Kathleen and the yet unborn

Preface to the Third Edition

The third edition of *Vibrations of Shells and Plates* contains a significant amount of new material, in part fundamental type, and in part it consists of important application examples. Several of the added topics were suggested by readers of the earlier editions.

In Chapter 2, on deep shell equations, Section 2.12 describes how to obtain radii of curvature for any shell geometry analytically if they cannot easily be determined by inspection. To Section 3.5, Other Geometries, an example of a parabolic cylindrical shell has been added. To Chapter 4, on nonshell structures, Section 4.5 was added to show that Love's equations can also be reduced to the special case of a circular cylindrical tube that oscillates in torsional motion. This equation is further reduced to the classical torsion shaft. The reduction is not obvious because transverse shear deformation is assumed to be small in the standard Love's theory. It is therefore illustrative from an educational viewpoint that this reduction is possible without resorting to the material of Chapter 12, where transverse shear deformation is considered.

A significant amount of new material has been added to Chapter 5, on natural frequencies and modes. Section 5.14 describes the in-plane vibration of rectangular plates and 5.15 discusses a case of in-plane vibration of circular plates, because of the importance of this type of vibration to piezoelectric crystals and spur gears, for example. The new Section 5.16 describes the closed-form solution of the natural frequencies and modes of a circular cylindrical shell segment, which supplements Section 5.5, which examines the closed cylindrical shell. Finally, a relatively substantial Section 5.17 has been added on natural frequency and mode solutions by power series because of the importance of this approach to solving differential vibration equations where the solutions cannot be expressed in terms of trigonometric, hyperbolic, Bessel, Legendre, or other functions. This approach is usually not discussed in typical standard textbooks on vibration, despite its potential usefulness and historical importance.

Three more cases of technical significance were added to Chapter 6, on simplified shell equations. Section 6.16 was added to present the case of a closed-form solution of a special type of toroidal shell, which is limited in its application but useful from a theoretical viewpoint. While the barrel-shaped shell is discussed in Section 6.13 using a Donnell-Mushtari-Vlasov simplification, a more exact solution is now also given in the new Section 6.17, where the importance of avoiding shells of zero Gaussian curvature, if higher natural frequencies are desired, is now clearly illustrated. Finally, an example of a doubly curved plate solution based on the Donnell-Mushtari-Vlasov theory is now given in Section 6.18. Again, all new cases are significant from an applications viewpoint and should be helpful to researchers and practicing engineers.

While the first and second editions identify strain energy expressions in a general way, and can be worked out for any case, the new edition includes explicit strain energy equations for a variety of standard cases, for the purpose of quick reference. These expressions, now given in the new Section 7.7, are typically used in energy methods of vibration analysis, notably in the Rayleigh-Ritz method.

In forced vibrations, an initial value example has been added as Section 8.17 that shows the response of a plate to an initial displacement that is equal to static sag due to the weight of the plate. The concept of modal mass, stiffness, damping, and forcing is now introduced in Section 8.18. Explicitly introduced in Section 8.19 is the response of shells to periodic forcing. The general solution for shells is illustrated by the special case of a plate in Section 8.20. Finally, in Section 8.21, the phenomenon of beating is discussed by way of an example.

In Section 9.9, plate examples illustrating the application of the dynamic Green's function have been added. Also solved by way of the dynamic Green's function is the case of a ring that is impacted by a point mass.

The response of a ring on an elastic foundation to a harmonic point moment excitation is solved in Section 10.6. Following this, for the first time a moment loading dynamic Green's function is formulated for shells and plates in general in Section 10.7 and illustrated by an example.

Added to the subchapter on complex receptance is a description of how to express such complex receptances in terms of magnitudes and phase angles. The new Section 13.12 shows how one can subtract systems from each other, which is more subtle than reversing additions by changing plus signs to minus signs in the receptance expressions. The receptance treatment of three or more systems connected by one displacement each was added in Section 13.13 and illustrated on hand of connected plates in Section 13.14. Also in this chapter is the solution of a continuous plate on two interior knife edges by way of a receptance formulation of three plates connected by moments.

While Chapter 14 in the first edition pointed out that the complex modulus model of hysteretic damping is valid for harmonic forcing, Section 14.4 now describes how it is also used for steady-state periodic response calculations.

Added as Section 15.9 is the analysis of shells composed of homogeneous and isotropic lamina (so-called sandwich shells), because of their technical importance, and examples are presented in Section 15.10.

Also, because of their general technical importance as a class of cases, the equations or motion of shells of revolution that spin about their axes are now derived explicitly in Section 16.7 and a reduced example, the spinning disk, is discussed in Section 16.8.

A significant amount of important new material has been added to the chapter on elastic foundations. The force transmission into the base of the elastic foundation is analyzed in Section 18.6, and a special illustration taken from simplified tire analysis—namely, the vertical force transmission through a rigid wheel that supports by way of an elastic foundation an elastic ring—was added in Section 18.7. This case has implications beyond the tire application, however. The general response of shells on elastic foundations on base excitations is now presented in Section 18.8 and plate examples are given in Section 18.9. As stimulated by tire applications, Section 18.10 shows how natural frequencies and modes of a ring on an elastic foundation in point ground contact may be obtained from the natural frequencies and rings not in ground contact. This leads indirectly to the results of Section 18.11 in which the ground contact motion creates a harmonic point excitation. Important resonance effects are discussed.

In closing, the goals of the first and second editions are preserved by the additions made in this third edition, namely: (1) to present the foundation of the theory of vibration of shells and other structures, (2) to present analytical solutions that illustrate the behavior of vibrating shells and other structures and to give important general information to designers of such structures, (3) to present basic information needed for the development of finite element and finite difference programs (see also Chapter 21), and (4) to allow such programs to be checked out against some of the exact results collected in this book.

The remarks in the Preface to the Second Edition on how to use this book in teaching are still valid. The book contains too much material to be covered in a standard three-credit course. Chapters 2 to 8 should be treated in depth, comprising a major part of the 45 lectures per semester that are typically available. Then, approximately six chapters can typically be added, with the topics to be treated a function of the interests of the graduate students and/or lecturer. The new material that has been added to the third edition contributes to the range of choices, and certain new examples will enrich the fundamental lectures. The book may, of course, also be used in a mode of self-study.

I am indebted to Prof. J. Kim of the University of Cincinnati and his students, who scrutinized the second edition carefully and gave me a list of (fortunately minor) transcription, typing, and typesetting corrections (which were corrected in the second printing without thanking them in print). I am also grateful for contributions by D. T. Soedel, F. P. Soedel, and S. M. Soedel and by various students of my graduate course ME 664, "Vibrations of Continuous Systems," who either checked many of the new additions by way of independent assignments, made suggestions, or found a small remnant of typesetting errors. In my 2003 class, they were: S. Basak, N. Bilal, R. Deng, M. R. Duncan, J. C. Huang, R. J. Hundhausen, J. W. Kim, U. J. Kim, A. A. Kulkarni, A. Kumar, T. Puri, L. B. Sharos, T. S. Slack, M. C. Strus, D. N. Vanderlugt, A. Vyas, F. X. Wang, C. L. Yang, and K. H. Yum. Also contributing, from the 2001 class, were: H. V. Chowdhari, R. S. Grinnip III, Y. J. Kim, Y. Pu, B. H. Song, S. J. Thorpe, and M. R. Tiller. Earlier classes contributed also, directly or indirectly, and I regret that the names of these individuals have not been recorded by me. Finally, general assistance in preparing lecture notes and generating from them this third edition was provided by D. K. Cackley, M. F. Schaaf-Soedel, and A. S. Greiber-Soedel.

Werner Soedel

Preface to the Second Edition

The second edition of *Vibration of Shells and Plates* contains some revisions and a significant amount of new material. The new material reflects the latest developments in this field and meets the need of graduate students and practicing engineers to become acquainted with additional topics such as traveling modes in rotating shells, thermal effects, and fluid loading.

Love's theory remains the fundamental theory for deep shell equations (Chapter 2) since it can be shown that all the other linear thin shell theories (Flügge's, Novozhilov's, etc.) are based on relatively minor—in a practical sense, most likely unimportant—extensions. This edition includes a new section on other deep shell theories and another section shows that the derived equations are also valid for shells of nonuniform thickness, except where bending and membrane stiffnesses become functions of the surface coordinates. Because Hamilton's principle is used for derivations throughout the book, a discussion of it and a simple example are now included in Chapter 2.

While there are obviously a very large number of potential shell geometries, two more have been added to the chapter on equations of motion for commonly occurring geometries (Chapter 3). Torodial shells occur in engineering as aircraft and automobile tires, space station designs, and segments of such shells from impellers of pumps and fluid couplings. The equations for a cylindrical shell of noncircular cross section have been added in order to have one example of a shell that is not a shell of revolution, and also because it occurs quite commonly in pressure vessels.

In Chapter 5, where natural frequencies and modes are discussed, a formal separation of space and time variables has been added based on the observations that the common way of arguing that the motion in time is obviously harmonic at a natural frequency is not necessarily accepted by new students of shell behavior. For optimal learning, frequent detailed explanations are provided. Also included now is a section on how simultaneous partial differential equations of the type treated here can be uncoupled.

Because inextensional approximation is particularly useful for rings, a section on this topic has been added to Chapter 6 on simplified equations.

Chapter 7 covers approximate solution techniques. The discussion of the Galerkin technique, which in the first edition had been condensed to a point where clarity suffered, is now significantly expanded in Chapter 7.

Again based on teaching experiences, it has become desirable to discuss more extensively the use of the Dirac delta function when describing point forces in space and impulses in time. Also added to Chapter 8 is a discussion of the necessary two orthogonal sets of natural modes for shells of revolution, described by two different phase angles. Finally, two relatively detailed examples for a circular cylindrical shell have been included, one dealing with a harmonic response, the other with an initial value problem.

Chapter 9 has been expanded to include the harmonic Green's function as an introduction to transfer function techniques such as the receptance method.

A significant amount of new material can be found in Chapter 13, on combinations of structures, because of a strong interest in modal synthesis by industry. Sections added show the forced response of combined structures—how to treat systems joined by springs (important from a vibration isolation point of view) and how to approach displacement excitations—and discuss receptances that are complex numbers. The section on dynamic absorbers is now expanded to include the forced behavior.

As additional examples of composite structures, two examples on the vibration of net or textile sheets have been added to Chapter 15.

Because Coriolis effects in spinning shells of revolution create the phenomenon of traveling modes, Chapter 16, on rotating structures, has been added to develop the theory and give several illustrative examples of significance. The subject is introduced by way of spinning strings and beams, and the rotating ring is discussed extensively because of its many practical applications. Also given is an example of a rotating circular cylindrical shell.

Heating can influence or excite vibrations; thus the new Chapter 17 extends the basic theory to include thermal effects.

At times, one encounters shells or plates that are supported by an elastic medium. Often, the elastic medium can be modeled as an elastic foundation consisting of linear springs, as presented in Chapter 18. This chapter introduces similitude arguments and is therefore also an introduction to Chapter 19.

In Chapter 19, because of the importance of scaling to practical engineers who often study small models of structures, and because of its importance to rules of design, specialized similitudes for various structural elements are presented. Exact and approximate scaling relationships are derived. Also, the proper way of nondimensionalizing results is discussed.

Shell and plate structures often contain or are in contact with liquids or gases. The equations of motion of liquids or gases are derived by reduction from the equations of motion of three-dimensional elastic solids, and the necessary boundary conditions are discussed. One section gives an introduction to noise radiation from a shell by way of an example. The study of engineering acoustics is closely related to the vibration of shell structures, and Chapter 20 is a natural lead-in to this tropic. Also discussed by way of an example is the topic of the interaction of structures with incompressible liquids having free surfaces.

A new Chapter 21, on discretizing approaches, discusses finite difference and finite element techniques for obtaining natural modes and frequencies and also the forced response from the resulting matrix equations. Also included is an example of a finite element for shells of revolution.

At this point it is appropriate to suggest how the second edition is best used for teaching. The prerequisites remain: an introductory vibration course and some knowledge of boundary value problem mathematics. Also, it is still true that Chapters 2 through 8 should be treated in depth. My usual way of operating, considering a full semester of 45 lectures, is to accomplish this in approximately half of the available time. Then I select approximately six chapters of additional material from the remaining 13 chapters, with the topics changing from year to year (depending to some extent on the interest areas of the students). I treat these in relative depth and then allow myself three lectures at the end of the semester to survey the rest of the chapters.

Paradoxically, the material presented in this edition has also been used by me several times in two- and three-day courses for practicing engineers in industry, without requiring an appreciable amount of mathematics. In this case I use the book to outline the mathematical developments but dwell extensively on the physical principles and on the practical implications of the results. I have found that this is very useful to engineers who work mainly with ready-made finite element codes, work purely experimentally, or are designers of shell structures, and even to engineering managers who need an overview of the subject. Those who have the proper background, and are so inclined, seem able to use the book later in a program of self-study.

Several persons need to be mentioned for their direct or indirect help on this second edition. They are, in no particular order, S. M. Soedel, F. P. Soedel, D. T. Soedel, J. Alfred, R. Zadoks, Y. Chang, L. E. Kung, J. Blinka, D. Allaei, J. S. Kim, J. Kim, H. W. Kim, S. Saigal, S. C. Huang, J. L. Lin, M. P. Hsu, R. M. V. Pidaparti, D. S. Stutts, D. Huang, D. C. Conrad, H. J. Kim, S. H. Kim, Z. Liu and G. P. Adams. I apologize if I have forgotten someone. I would also like to express my appreciation to former students in my graduate course who were able to detect nagging small errors (all of them, I hope) that occurred in the writing and proofreading stages of the original notes used in my lectures and on which the new material in this second edition is based. I also thank those whose persistent questions helped me determine how the material should be organized and presented.

Just as the first edition did, the second edition attempts to provide information that is useful to the practicing engineer without losing sight of the fact that the primary purpose is graduate education. Its usefulness as a reference book has also been enhanced.

Werner Soedel

Preface to the First Edition

This book attempts to give engineering graduate students and practicing engineers an introduction to the vibration behavior of shells and plates. It is also hoped that it will prove to be a useful reference to the vibration specialist. It fills a need in the present literature on this subject, since it is the current practice to either discuss shell vibrations in a few chapters at the end of texts on shell statics that may be well written but are too limited in the selection of material, or to ignore shells entirely in favor of plates and membranes, as in some of the better known vibration books. There are a few excellent monographs on very specialized topics, for instance, on natural frequencies and modes of cylindrical and conical shells. But a unified presentation of shell and plate vibration, both free and forced, and with complicating effects as they are encountered in engineering practice, is still missing. This collection attempts to fill the gap.

The state of the art modern engineering demands that engineers have a good knowledge of the vibration behavior of structures beyond the usual beam and rod vibration examples. Vibrating shell and plate structures are not only encountered by the civil, aeronautical, and astronautical engineer, but also by the mechanical, nuclear, chemical, and industrial engineer. Parts or devices such as engine liners, compressor shells, tanks, heat exchangers, life support ducts, boilers, automotive tires, vehicle bodies, valve read plates, and saw disks, are all composed of structural elements that cannot be approximated as vibrating beams. Shells especially exhibit certain effects that are not present in beams or even plates and cannot be interpreted by engineers who are only familiar with beam-type vibration theory. Therefore, this book stresses the understanding of basic phenomena in shell and plate vibrations and it is hoped that the material covered will be useful in explaining experimental measurements or the results of the ever-increasing number of finite element programs. While it is the goal of every engineering manager that these programs will eventually be used as black boxes, with input provided and output obtained by relatively untrained technicians, reality shows that the interpretation of results of these programs requires a good background in finite element theory and, in the case of shell and plate vibrations, in vibration theory of greater depth and breadth than usually provided in standard texts.

It is hoped that the book will be of interest to both the stress analyst whose task it is to prevent failure and to the acoustician whose task it is to control noise. The treatment is fairly complete as far as the needs of the stress analysts go. For acousticians, this collection stresses those applications in which boundary conditions cannot be ignored.

The note collection begins with a historical discussion of vibration analysis and culminates in the development of Love's equations of shells. These equations are derived in Chapter 2 in curvilinear coordinates. Curvilinear coordinates are used throughout as much as possible, because of the loss of generality that occurs when specific geometries are singled out. For instance, the effect of the second curvature cannot be recovered from a specialized treatment of cylindrical shells. Chapter 3 shows the derivation by reduction of the equations of some standard shell geometries that have a tendency to occur in standard engineering practice, like the circular cylindrical shell, the spherical shell, the conical shell, and so on. In Chapter 4 the equations of motion of plates, arches, rings, beams, and rods are obtained. Beams and rings are sometimes used as supplementary examples in order to tie in the knowledge of beams that the reader may have with the approaches and results of shell and plate analysis.

Chapter 5 discusses natural frequencies and modes. It starts with the transversely vibrating beam, followed by the ring and plate. Finally, the exact solution of the simply supported circular cylindrical shell is derived. The examples are chosen in such a way that the essential behavior of these structures is unfolded with the help of each previous example; the intent is not to exhaust the number of possible analytical solutions. For instance, in order to explain why there are three natural frequencies for any mode number combination of the cylindrical shell, the previously given case of the vibrating ring is used to illustrate modes in which either transverse or circumferential motions dominate.

In the same chapter, the important property of orthogonality of natural modes is derived and discussed. It is pointed out that when two or more different modes occur at the same natural frequency, a superposition mode may be created that may not be orthogonal, yet is measured by the experimenter as the governing mode shape. Ways of dealing with this phenomenon are also pointed out.

For some important applications, it is possible to simplify the equations of motion. Rayleigh's simplification, in which either the bending stiffness or the membrane stiffness is ignored, is presented. However, the main thrust of Chapter 6 is the derivation and use of the Donnell-Mushtari-Vlasov equations.

While the emphasis of Chapter 5 was on so-called exact solutions (series solutions are considered exact solutions), Chapter 7 presents some of the more common approximate techniques to obtain solutions for geometrical shapes and boundary condition combinations that do not lend themselves to exact analytical treatment. First, the variational techniques known as the Rayleigh-Ritz technique and Galerkin's method and variational method are presented. Next, the purely mathematical technique of finite differences is outlined, with examples. The finite element method follows. Southwell's and Dunkerley's principles conclude the chapter.

The forced behavior of shells and plates is presented in Chapters 8, 9, and 10. In Chapter 8, the model analysis approach is used to arrive at the general solution for distributed dynamic loads in transverse and two orthogonal in-plane directions. The Dirac delta function is then used to obtain the solutions for point and line loads. Chapter 9 discusses the dynamic Green's functions approach and applies it to traveling load problems. An interesting resonance condition that occurs when a load travels along the great circles of closed shells of revolution is shown. Chapter 10 extends the types of possible loading to the technically significant set of dynamic moment loading, and illustrates it by investigating the action of a rotation point moment as it may occur when rotating unbalanced machinery is acting on a shell structure.

The influence of large initial stress fields on the response of shells and plates is discussed in Chapter 11. First, Love's equations are extended to take this effort into account. It is then demonstrated that the equations of motion of pure membranes and strings are a subset of these extended equations. The effect of initial stress fields on the natural frequencies of structures is then illustrated by examples.

In the original derivation of Love's equations, transverse shear strains, and therefore shear deflections, were neglected. This becomes less and less permissible as the average distance between node lines associated with the highest frequency of interest approaches the thickness of the structure. In Chapter 12, the shear deformations are included in the shell equations. It is shown that these equations reduce in the case of a rectangular plate and the case of a uniform beam to equations that are well known in the vibration literature. Sample cases are solved to illustrate the effect shear deformation has on natural frequencies.

Rarely are practical engineering structures simple geometric shapes. In most cases the shapes are so complicated that finite element or difference methods have to be used for accurate numerical results. However, there is a category of cases in which the engineering structures can be interpreted as being assembled of two or more classic shapes or parts. In Chapter 13, the method of receptance is presented and used to obtain, for instance, very general design rules for stiffening panels by ring- or beam-type stiffeners. It is also shown that the receptance method gives elegant and easily interpretable results for cases in which springs or masses are added to the basic structure.

The formulation and use of equivalent viscous damping was advocated in the forced vibration chapters. For steady-state harmonic response problems a complex modulus is often used. In Chapter 14, this type of structural damping, also called hystereses damping, is presented and tied in with the viscous damping formulation.

Because of the increasing importance of composite material structures, the equations of motion of laminated shells are presented and discussed in Chapter 15, along with some simple examples.

This book evolved over a period of almost ten years from lecture notes on the vibration of shells and plates. To present the subject in a unified fashion made it necessary to do some original work in areas where the available literature did not provide complete information. Some of it was done with the help of graduate students attending my lectures, for instance, R. G. Jacquot, U. R. Kristiansen, J. D. Wilken, M. Dhar, U. Bolleter, and D. P. Powder. Especially talented in detecting errors were M. G. Prasad, F. D. Wilken, M. Dhar, S. Azimi, and D. P. Egolf. Realizing that I have probably forgotten some significant contributions, I would like to single out in addition O. B. Dale, J. A. Adams, D. D. Reynolds, M. Moaveni, R. Shashaani, R. Singh, J. R. Friley, J. DeEskinazi, F. Laville, E. T. Buehlmann, N. Kaemmer, C. Hunckler, and J. Thompson, and extend my appreciation to all my former students.

I would also like to thank my colleagues on the Purdue University faculty for their direct or indirect advice.

If this book is used for an advanced course in structural vibrations of about forty-five lectures, it is recommended that Chapters 2 through 8 be treated in depth. If there is time remaining, highlights of the other chapters can be presented. Recommend prerequisites are a first course in mechanical vibrations and knowledge of boundary value problem mathematics.

Werner Soedel

Contents

1

Historical Development of Vibration Analysis of Continuous Structural Elements

Vibration analysis has its beginnings with Galilei (1564–1642), who solved by geometrical means the dependence of the natural frequency of a simple pendulum on the pendulum length (Galilei, 1939). He proceeded to make experimental observations on the vibration behavior of strings and plates, but could not offer any analytical treatment. He was partially anticipated in his observations of strings by his contemporary Mersenne (1588–1648), a French priest. Mersenne (1635) recognized that the frequency of vibration is inversely proportional to the length of the string and directly proportional to the square root of the cross-sectional area. This line of approach found its culmination in Sauveur (1653–1716), who coined the terminology "nodes" for zero-displacement points on a string vibrating at its natural frequency and also actually calculated an approximate value for the fundamental frequency as a function of the measured sag at its center, similar to the way the natural frequency of a single-degree-of-freedom spring–mass system can be calculated from its static deflection (Sauveur, 1701).

The foundation for a more precise treatment of the vibration of continuous systems was laid by Robert Hooke (1635–1703) when he established the basic law of elasticity, by Newton (1642–1727) when he established that force was equal to mass times acceleration, and by Leibnitz (1646–1716) when he established differential calculus. An approach similar to differential calculus called *fluxions* was developed by Newton independently at the same time. In 1713 the English mathematician Taylor (1685–1731) actually used the fluxion approach, together with Newton's second law applied to an element of the continuous string, to calculate the true value of the first natural frequency of a string (Taylor, 1713). The approach was based on an assumed first mode shape. This is where work in vibration analysis stagnated in England, since the fluxion method and especially its notation proved to be too clumsy to allow anything but the attack of simple problems. Because of the controversy between followers of Newton and Leibnitz as to the origin of differential calculus, patriotic Englishmen refused to use anything but fluxions and left the fruitful use of the Leibnitz notation and approach to investigators on the continent. There the mathematics of differential calculus prospered and paved the way for Le Rond d'Alembert (1717–1783), who derived in 1747 the partial differential equation which today is referred to as the wave equation and who found the wave travel solution (Le Rond d'Alembert, 1747). He was ably assisted in this by Bernoulli (1700–1782) and Euler (1707–1783), both German-speaking Swiss and friends, but did not give them due credit. It is still a controversial subject to decide who did actually what, especially since the participants were not too bashful to insult each other and claim credit right and left. However, it seems fairly clear that the principle of superposition of modes was first noted in 1747 by Bernoulli (1755) and proven by Euler (1753). These two must, therefore, be credited as being the fathers of the modal expansion technique or of eigenvalue expansion in general. The technique did not find immediate general acceptance. Fourier (1768–1830) used it to solve certain problems in the theory of heat (Fourier, 1822). The resulting Fourier series can be viewed as a special case of the use of orthogonal functions and might as well carry the name of Bernoulli. However, it is almost a rule in the history of science that people who are credited with an achievement do not completely deserve it. Progress moves in small steps and it is often the person who publishes at the right developmental step and at the right time who gets the public acclaim.

The longitudinal vibration of rods was investigated experimentally by Chladni (1787) and Biot (1816). However, not until 1824 do we find the published analytical equation and solutions, done by Navier. This is interesting since the analogous problem of the longitudinal vibration of air columns was already done in 1727 by Euler (1727).

The equation for the transverse vibration of flexible thin beams was derived by Bernoulli (1735), and the first solutions for simply supported ends, clamped ends, and free ends were found by Euler (1744).

The first torsional vibration solution, but not in a continuous sense, was given by Coulomb (1784). But not until 1827 do we find an attempt to derive the continuous torsional equation (Cauchy, 1827). This was done by Cauchy (1789–1857) in an approximate fashion. Poisson (1781–1840) is generally credited with having derived the one-dimensional torsional wave equation in 1827 (Poisson, 1829). The credit for deriving the complete torsional wave equation and giving some rigorous results belongs to Saint-Venant (1797–1886), who published on this subject (de Saint-Venant, 1849).

In the field of membrane vibrations, Euler (1766a) published equations for a rectangular membrane that were incorrect for the general case but reduce to the correct equation for the uniform tension case. It is interesting to note that the first membrane vibration case investigated analytically was not that dealing with the circular membrane, even though the latter, in the form of a drumhead, would have been the more obvious shape. The reason is that Euler was able to picture the rectangular membrane as a superposition of a number of crossing strings. In 1828, Poisson read a paper to the French Academy of Science on the special case of uniform tension. Poisson (1829) showed the circular membrane equation and solved it for the special case of axisymmetric vibration. One year later, Pagani (1829) furnished a nonaxisymmetric solution. Lamé (1795–1870) published lectures that gave a summary of the work on rectangular and circular membranes and contained an investigation of triangular membranes (Lamé, 1852).

Work on plate vibration analysis went on in parallel. Influenced by Euler's success in deriving the membrane equation by considering the superposition of strings, James Bernoulli, a nephew of Daniel Bernoulli, attempted to derive the plate equation by considering the superposition of beams. The resulting equation was wrong. In his 1788 presentation to the St. Petersburg Academy, Bernoulli (1789) acknowledged that he was stimulated in his attempt by the German experimentalist Chladni (1787), who demonstrated the beautiful node lines of vibrating plates at the courts of Europe. A presentation by Chladni before emperor Napoleon, who was a trained military engineer and very interested in technology and science, caused the latter to transfer money to the French Academy of Sciences for a prize to the person who could best explain the vibration behavior of plates. The prize was won, after several attempts, by a woman, Germaine (1776–1831), in 1815. Germaine (1821) gave an almost correct form of the plate equation. The bending stiffness and density constants were not defined. Neither were the boundary conditions stated correctly. These errors are the reason that her name is not associated today with the equation, despite the brilliance or her approach. Contributing to this was

Todhunter (1886), who compiled a fine history of the theory of elasticity, published posthumously, in which he is unreasonably critical of her work, demanding a standard of perfection that he does not apply to the works of the Bernoullis, Euler, Lagrange, and others, where he is quite willing to accept partial results. Also, Lagrange (1736–1813) entered into the act by correcting errors that Germaine made when competing for the prize in 1811. Thus, indeed, we find the equation first stated in its modern form by Lagrange (1811) in response to Germaine's submittal of her first competition paper.

What is even more interesting is that, Germaine (1821) published a very simplified equation for the vibration of a cylindrical shell. Unfortunately, again it contained mistakes. This equation can be reduced to the current rectangular plate equation, but when it is reduced to the ring equation, a sign error is passed on. But for the sign difference in one of its terms, the ring equation is identical to one given by Euler (1766b).

The correct bending stiffness was first identified by Poisson (1829). Consistent boundary conditions were not developed until 1850 by Kirchhoff (1824–1887), who also gave the correct solution for a circular plate example (Kirchhoff, 1850).

The problem of shell vibrations was first attacked by Germaine before 1821, as already pointed out. She assumed that the in-plane deflection of the neutral surface of a cylindrical shell was negligible. Her result contained errors. Aron (1874) derived a set of five equations, which he shows to reduce to the plate equation when curvatures are set to zero. The equations are complicated because of his reluctance to employ simplifications. They are in curvilinear coordinate form and apply in general. The simplifications that are logical extensions of the beam and plate equations for both transverse and in-plane motion were introduced by Love (1863–1940) in 1888 (Love, 1888). Between Aron and Love, Lord Rayleigh (1842–1919) proposed various simplifications that viewed the shell neutral surface as either extensional or inextensional (Lord Rayleigh, 1882). His simplified solutions are special cases of Love's general theory. Love's equations brought the basic development of the theory of vibration of continuous structures which have a thickness that is much less than any length or surface dimensions to a satisfying end. Subsequent development, concerned with higher-order or complicating effects, is discussed in this book when appropriate.

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2

Deep Shell Equations

The term *deep* is used to distinguish the set of equations used in this chapter from the "shallow" shell equations discussed later. The equations are based on the assumptions that the shells are thin with respect to their radii of curvature and that deflections are reasonably small. On these two basic assumptions secondary assumptions rest. They are discussed as the development warrants it.

The basic theoretical approach is due to Love (1888), who published the equations in their essential form toward the end of the 19th century. Essentially, he extended work on shell vibrations by Rayleigh, who divided shells into two classes: one where the middle surface does not stretch and bending effects are the only important ones, and one where only the stretching of the middle surface is important and the bending stiffness can be neglected (Rayleigh, 1945). Love allowed the coexistence of these two classes. He used the principle of virtual work to derive his equations, following Kirchhoff (1850), who had used it when deriving the plate equation. The derivation given here uses Hamilton's principle, following Reissner's derivation (Reissner, 1941; Kraus, 1967).

2.1. SHELL COORDINATES AND INFINITESIMAL DISTANCES IN SHELL LAYERS

We assume that thin, isotropic, and homogeneous shells of constant thickness have neutral surfaces, just as beams in transverse deflection have neutral fibers. That this is true will become evident later. Stresses in such a neutral surface can be of the membrane type but cannot be bending stresses. Locations on the neutral surface, placed into a three-dimensional Cartesian coordinate system, can also be defined by two-dimensional curvilinear surface coordinates α_1 and α_2. The location of point P on the neutral surface (Fig. 1) in Cartesian coordinates is related to the location of the point in surface coordinates by

$$x_1 = f_1(\alpha_1, \alpha_2), \quad x_2 = f_2(\alpha_1, \alpha_2), \quad x_3 = f_3(\alpha_1, \alpha_2) \tag{2.1.1}$$

The location of P on the neutral surface can also be expressed by a vector: for example,

$$\bar{r}(\alpha_1, \alpha_2) = f_1(\alpha_1, \alpha_2)\bar{e}_1 + f_2(\alpha_1, \alpha_2)\bar{e}_2 + f_3(\alpha_1, \alpha_2)\bar{e}_3 \tag{2.1.2}$$

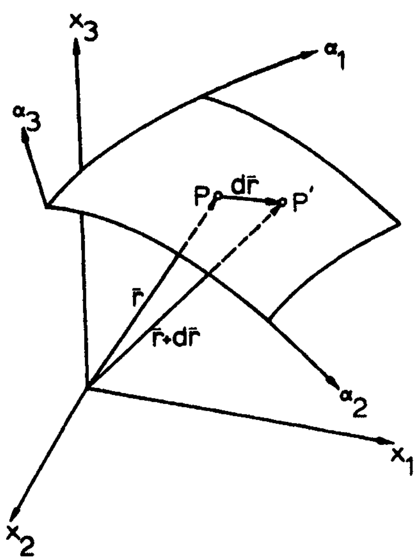

Fig. 1 Reference surface.

Now let us define the infinitesimal distance between points P and P' on the neutral surface. The differential change $\mathrm{d}\bar{r}$ of the vector $\bar{r}$ as we move from P to P' is

$$\mathrm{d}\bar{r} = \frac{\partial \bar{r}}{\partial \alpha_1}\mathrm{d}\alpha_1 + \frac{\partial \bar{r}}{\partial \alpha_2}\mathrm{d}\alpha_2 \tag{2.1.3}$$

The magnitude $\mathrm{d}s$ of $\mathrm{d}\bar{r}$ is obtained by

$$(\mathrm{d}s)^2 = \mathrm{d}\bar{r}\cdot\mathrm{d}\bar{r} \tag{2.1.4}$$

or

$$(\mathrm{d}s)^2 = \frac{\partial \bar{r}}{\partial \alpha_1}\cdot\frac{\partial \bar{r}}{\partial \alpha_1}(\mathrm{d}\alpha_1)^2 + \frac{\partial \bar{r}}{\partial \alpha_2}\cdot\frac{\partial \bar{r}}{\partial \alpha_2}(\mathrm{d}\alpha_2)^2 + 2\frac{\partial \bar{r}}{\partial \alpha_1}\cdot\frac{\partial \bar{r}}{\partial \alpha_2}\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{2.1.5}$$

In the following, we limit ourselves to orthogonal curvilinear coordinates which coincide with the lines of principal curvature of the neutral surface.

The third term in Eq. (2.1.5) thus becomes

$$2\frac{\partial \bar{r}}{\partial \alpha_1}\cdot\frac{\partial \bar{r}}{\partial \alpha_2}\mathrm{d}\alpha_1\mathrm{d}\alpha_2 = 2\left|\frac{\partial \bar{r}}{\partial \alpha_1}\right|\left|\frac{\partial \bar{r}}{\partial \alpha_2}\right|\cos\frac{\pi}{2}\mathrm{d}\alpha_1\mathrm{d}\alpha_2 = 0 \tag{2.1.6}$$

When we define

$$\frac{\partial \bar{r}}{\partial \alpha_1}\cdot\frac{\partial \bar{r}}{\partial \alpha_1} = \left|\frac{\partial \bar{r}}{\partial \alpha_1}\right|^2 = A_1^2$$

$$\frac{\partial \bar{r}}{\partial \alpha_2}\cdot\frac{\partial \bar{r}}{\partial \alpha_2} = \left|\frac{\partial \bar{r}}{\partial \alpha_2}\right|^2 = A_2^2 \tag{2.1.7}$$

Equation (2.1.5) becomes

$$(\mathrm{d}s)^2 = A_1^2(\mathrm{d}\alpha_1)^2 + A_2^2(\mathrm{d}\alpha_2)^2 \tag{2.1.8}$$

This equation is called the *fundamental form* and A_1 and A_2 are the *fundamental form parameters* or *Lamé parameters*.

As an example, let us look at the circular cylindrical shell shown in Fig. 2. The lines of principal curvature (for each shell surface point there exists a maximum and a minimum radius of curvature, whose directions are at an angle of $\pi/2$) are in this case parallel to the axis of revolution, where the radius of curvature $R_x = \infty$ or the curvature $1/R_x = 0$, and along circles, where the radius of curvature $R_\theta = a$ or the curvature $1/R_\theta = 1/a$. We then proceed to obtain the fundamental form parameters from definition (2.1.7). The curvilinear coordinates are

$$\alpha_1 = x, \quad \alpha_2 = \theta \tag{2.1.9}$$

and Eq. (2.1.2) becomes

$$\bar{r} = x\bar{e}_1 + a\cos\theta\,\bar{e}_2 + a\sin\theta\,\bar{e}_3 \tag{2.1.10}$$

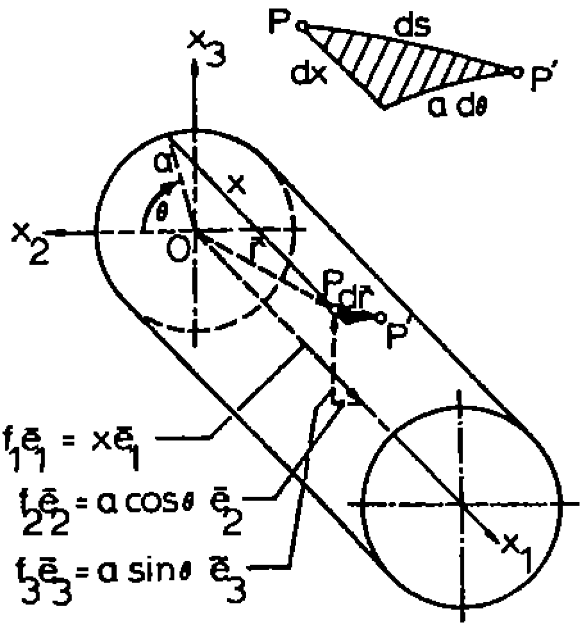

FIG. 2 Obtaining the Lamé parameters for a circular cylindrical shell.

Thus

$$\frac{\partial\bar{r}}{\partial\alpha_1} = \frac{\partial\bar{r}}{\partial x} = \bar{e}_1 \tag{2.1.11}$$

or

$$\left|\frac{\partial\bar{r}}{\partial\alpha_1}\right| = A_1 = 1 \tag{2.1.12}$$

and

$$\frac{\partial\bar{r}}{\partial\alpha_2} = \frac{\partial\bar{r}}{\partial\theta} = -a\sin\theta\bar{e}_2 + a\cos\theta\bar{e}_3 \tag{2.1.13}$$

or

$$\left|\frac{\partial\bar{r}}{\partial\theta}\right| = A_2 = a\sqrt{\sin^2\theta + \cos^2\theta} = a \tag{2.1.14}$$

The fundamental form is therefore

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + a^2(\mathrm{d}\theta)^2 \tag{2.1.15}$$

Recognizing that the fundamental form can be interpreted as defining the hypotenuse ds of a right triangle whose sides are infinitesimal distances

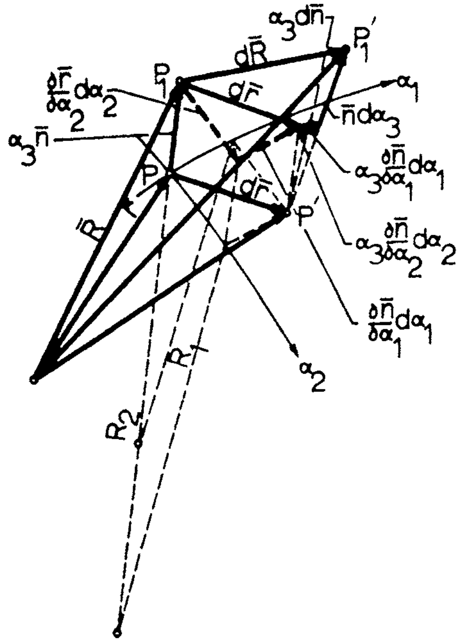

Fig. 3 Distance between two points removed from the reference surface.

along the surface coordinates of the shell, we may obtain A_1 and A_2 in a simpler fashion by expressing ds directly using inspection:

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + a^2(\mathrm{d}\theta)^2$$

By comparison with Eq. (2.1.8), we obtain $A_1 = 1$ and $A_2 = a$.

For the general case, let us now define the infinitesimal distance between a point P_1 that is normal to P and a point P_1' which is normal to P' (see Fig. 3). P_1 is located at a distance α_3 from the neutral surface (α_3 is defined to be along a normal straight line to the neutral surface). P_1' is located at a distance $\alpha_3 + \mathrm{d}\alpha_3$ from the neutral surface. We may therefore express the location of P_1 as

$$\overline{R}(\alpha_1, \alpha_2, \alpha_3) = \bar{r}(\alpha_1, \alpha_2) + \alpha_3 \bar{n}(\alpha_1, \alpha_2) \tag{2.1.16}$$

where $\bar{n}$ is a unit vector normal to the neutral surface. The differential change d$\overline{R}$, as we move from P_1 to P_1', is

$$\mathrm{d}\overline{R} = \mathrm{d}\bar{r} + \alpha_3 \mathrm{d}\bar{n} + \bar{n}\mathrm{d}\alpha_3 \tag{2.1.17}$$

where

$$\mathrm{d}\bar{n} = \frac{\partial \bar{n}}{\partial \alpha_1}\mathrm{d}\alpha_1 + \frac{\partial \bar{n}}{\partial \alpha_2}\mathrm{d}\alpha_2 \tag{2.1.18}$$

The magnitude ds of d$\overline{R}$ is obtained by

$$(\mathrm{d}s)^2 = \mathrm{d}\overline{R} \cdot \mathrm{d}\overline{R} \tag{2.1.19}$$

or

$$\begin{aligned}(\mathrm{d}s)^2 &= \mathrm{d}\bar{r} \cdot \mathrm{d}\bar{r} + \alpha_3^2 \mathrm{d}\bar{n} \cdot \mathrm{d}\bar{n} + \bar{n} \cdot \bar{n} (\mathrm{d}\alpha_3)^2 + 2\alpha_3 \mathrm{d}\bar{r} \cdot \mathrm{d}\bar{n} \\ &\quad + 2\mathrm{d}\alpha_3 \mathrm{d}\bar{r} \cdot \bar{n} + 2\alpha_3 \mathrm{d}\alpha_3 \mathrm{d}\bar{n} \cdot \bar{n} = \mathrm{d}\bar{r} \cdot \mathrm{d}\bar{r} \\ &\quad + \alpha_3^2 \mathrm{d}\bar{n} \cdot \mathrm{d}\bar{n} + (\mathrm{d}\alpha_3)^2 + 2\alpha_3 \mathrm{d}\bar{r} \cdot \mathrm{d}\bar{n}\end{aligned} \tag{2.1.20}$$

We have already seen that

$$\mathrm{d}\bar{r} \cdot \mathrm{d}\bar{r} = A_1^2 (\mathrm{d}\alpha_1)^2 + A_2^2 (\mathrm{d}\alpha_2)^2 \tag{2.1.21}$$

Next,

$$\alpha_3^2 \mathrm{d}\bar{n} \cdot \mathrm{d}\bar{n} = \alpha_3^2 \left[\frac{\partial \bar{n}}{\partial \alpha_1} \cdot \frac{\partial \bar{n}}{\partial \alpha_1} (\mathrm{d}\alpha_1)^2 + \frac{\partial \bar{n}}{\partial \alpha_2} \cdot \frac{\partial \bar{n}}{\partial \alpha_2} (\mathrm{d}\alpha_2)^2 + 2 \frac{\partial \bar{n}}{\partial \alpha_1} \cdot \frac{\partial \bar{n}}{\partial \alpha_2} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \right] \tag{2.1.22}$$

The third term of this expression is 0 because of orthogonality (see also Fig. 3). The second term may be written

$$\alpha_3^2 \frac{\partial \bar{n}}{\partial \alpha_2} \cdot \frac{\partial \bar{n}}{\partial \alpha_2} (\mathrm{d}\alpha_2)^2 = \left| \alpha_3 \frac{\partial \bar{n}}{\partial \alpha_2} \right|^2 (\mathrm{d}\alpha_2)^2 \tag{2.1.23}$$

From Fig. 3 we recognize the following relationship to the radius of curvature R_2:

$$\frac{|\partial \bar{r} / \partial \alpha_2|}{R_2} = \frac{|\alpha_3 (\partial \bar{n} / \partial \alpha_2)|}{\alpha_3} \tag{2.1.24}$$

Since

$$\left| \frac{\partial \bar{r}}{\partial \alpha_2} \right| = A_2 \tag{2.1.25}$$

we get

$$\left| \alpha_3 \frac{\partial \bar{n}}{\partial \alpha_2} \right| = \frac{\alpha_3 A_2}{R_2} \tag{2.1.26}$$

and therefore

$$\alpha_3^2 \frac{\partial \bar{n}}{\partial \alpha_2} \cdot \frac{\partial \bar{n}}{\partial \alpha_2} (\mathrm{d}\alpha_2)^2 = \alpha_3^2 \frac{A_2^2}{R_2^2} (\mathrm{d}\alpha_2)^2 \tag{2.1.27}$$

Similarly, the first term becomes

$$\alpha_3^2 \frac{\partial \bar{n}}{\partial \alpha_1} \cdot \frac{\partial \bar{n}}{\partial \alpha_1} (\mathrm{d}\alpha_1)^2 = \alpha_3^2 \frac{A_1^2}{R_1^2} (\mathrm{d}\alpha_1)^2 \tag{2.1.28}$$

and expression (2.1.22) becomes

$$\alpha_3^2 \mathrm{d}\bar{n}\cdot\mathrm{d}\bar{n} = \alpha_3^2\left[\frac{A_1^2}{R_1^2}(\mathrm{d}\alpha_1)^2 + \frac{A_2^2}{R_2^2}(\mathrm{d}\alpha_2)^2\right] \tag{2.1.29}$$

Finally, the last expression of Eq. (2.1.20) becomes

$$2\alpha_3 \mathrm{d}\bar{r}\cdot\mathrm{d}\bar{n} = 2\alpha_3\left[\frac{\partial\bar{r}}{\partial\alpha_1}\cdot\frac{\partial\bar{n}}{\partial\alpha_1}(\mathrm{d}\alpha_1)^2 + \frac{\partial\bar{r}}{\partial\alpha_2}\cdot\frac{\partial\bar{n}}{\partial\alpha_2}(\mathrm{d}\alpha_2)^2 + \frac{\partial\bar{r}}{\partial\alpha_1}\cdot\frac{\partial\bar{n}}{\partial\alpha_2}\mathrm{d}\alpha_1\mathrm{d}\alpha_2 + \frac{\partial\bar{r}}{\partial\alpha_2}\cdot\frac{\partial\bar{n}}{\partial\alpha_1}\mathrm{d}\alpha_1\mathrm{d}\alpha_2\right] \tag{2.1.30}$$

The last two terms are 0 because of orthogonality. The first term may be written

$$\frac{\partial\bar{r}}{\partial\alpha_1}\cdot\frac{\partial\bar{n}}{\partial\alpha_1}(\mathrm{d}\alpha_1)^2 = \left|\frac{\partial\bar{r}}{\partial\alpha_1}\right|\left|\frac{\partial\bar{n}}{\partial\alpha_1}\right|(\mathrm{d}\alpha_1)^2 = \frac{A_1^2}{R_1}(\mathrm{d}\alpha_1)^2 \tag{2.1.31}$$

Similarly,

$$\frac{\partial\bar{r}}{\partial\alpha_2}\cdot\frac{\partial\bar{n}}{\partial\alpha_2}(\mathrm{d}\alpha_2)^2 = \frac{A_2^2}{R_2}(\mathrm{d}\alpha_2)^2 \tag{2.1.32}$$

Expression (2.1.30) therefore becomes

$$2\alpha_3 \mathrm{d}\bar{r}\cdot\mathrm{d}\bar{n} = 2\alpha_3\left[\frac{A_1^2}{R_1}(\mathrm{d}\alpha_1)^2 + \frac{A_2^2}{R_2}(\mathrm{d}\alpha_2)^2\right] \tag{2.1.33}$$

Substituting expressions (2.1.33), (2.1.29), and (2.1.21) in Eq. (2.1.20) gives

$$(\mathrm{d}s)^2 = A_1^2\left(1+\frac{\alpha_3}{R_1}\right)^2(\mathrm{d}\alpha_1)^2 + A_2^2\left(1+\frac{\alpha_3}{R_2}\right)^2(\mathrm{d}\alpha_2)^2 + (\mathrm{d}\alpha_3)^2 \tag{2.1.34}$$

2.2. STRESS–STRAIN RELATIONSHIPS

Having chosen the mutually perpendicular lines of principal curvature as coordinates, plus the normal to the neutral surface as the third coordinate, we have three mutually perpendicular planes of strain and three shear strains. Assuming that Hooke's law applies, we have for a three-dimensional element

$$\varepsilon_{11} = \frac{1}{E}[\sigma_{11} - \mu(\sigma_{22}+\sigma_{33})] \tag{2.2.1}$$

$$\varepsilon_{22} = \frac{1}{E}[\sigma_{22} - \mu(\sigma_{11}+\sigma_{33})] \tag{2.2.2}$$

$$\varepsilon_{33} = \frac{1}{E}[\sigma_{33} - \mu(\sigma_{11} + \sigma_{22})] \tag{2.2.3}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{G} \tag{2.2.4}$$

$$\varepsilon_{13} = \frac{\sigma_{13}}{G} \tag{2.2.5}$$

$$\varepsilon_{23} = \frac{\sigma_{23}}{G} \tag{2.2.6}$$

where σ_{11}, σ_{22}, and σ_{33} are normal stresses and σ_{12}, σ_{13}, and σ_{23} are shear stresses as shown in Fig. 4. Note that

$$\sigma_{12} = \sigma_{21}, \quad \sigma_{13} = \sigma_{31}, \quad \sigma_{23} = \sigma_{32} \tag{2.2.7}$$

We will later assume that transverse shear deflections can be neglected. This implies that

$$\varepsilon_{13} = 0, \quad \varepsilon_{23} = 0 \tag{2.2.8}$$

However, we will not neglect the integrated effect of the transverse shear stresses σ_{13} and σ_{23}. This is discussed later.

The normal stress σ_{33}, which acts in the normal direction to the neutral surface, will be neglected:

$$\sigma_{33} = 0 \tag{2.2.9}$$

This is based on the argument that on an unloaded outer shell surface it is 0, or if a load acts on the shell, it is equivalent in magnitude to the external

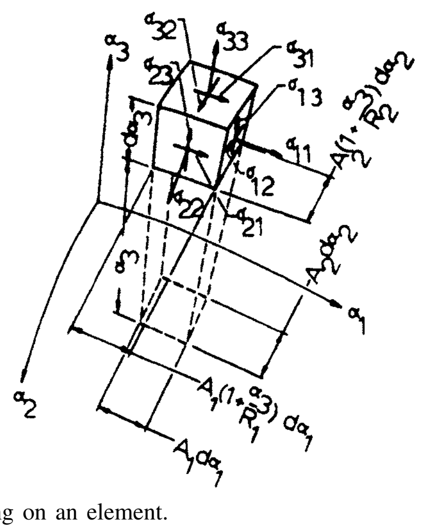

Fig. 4 Stresses acting on an element.

load on the shell, which is a relatively small value in most cases. Only in the close vicinity of a concentrated load do we reach magnitudes that would make the consideration of σ_{33} worthwhile. Our equation system therefore reduces to

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \mu\sigma_{22}) \tag{2.2.10}$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \mu\sigma_{11}) \tag{2.2.11}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{G} \tag{2.2.12}$$

and

$$\varepsilon_{33} = -\frac{\mu}{E}(\sigma_{11} + \sigma_{22}) \tag{2.2.13}$$

Only the first three relationships will be of importance in the following. Equation (2.2.13) can later be used to calculate the constriction of the shell thickness during vibration, which is of some interest to acousticians since it is an additional noise generating mechanism, along with transverse deflection.

2.3. STRAIN–DISPLACEMENT RELATIONSHIPS

We have seen that the infinitesimal distance between two points P_1 and P_1' of an undeflected shell is given by Eq. (2.1.34). Defining, for the purpose of a short notation,

$$A_1^2\left(1 + \frac{\alpha_3}{R_1}\right)^2 = g_{11}(\alpha_1, \alpha_2, \alpha_3) \tag{2.3.1}$$

$$A_2^2\left(1 + \frac{\alpha_3}{R_2}\right)^2 = g_{22}(\alpha_1, \alpha_2, \alpha_3) \tag{2.3.2}$$

$$1 = g_{33}(\alpha_1, \alpha_2, \alpha_3) \tag{2.3.3}$$

We may write Eq. (2.1.34) as

$$(\mathrm{d}s)^2 = \sum_{i=1}^{3} g_{ii}(\alpha_1, \alpha_2, \alpha_3)(\mathrm{d}\alpha_i)^2 \tag{2.3.4}$$

If point P_1, originally located at $(\alpha_1, \alpha_2, \alpha_3)$, is deflected in the α_1 direction by U_1, in the α_2 direction by U_2, and in the α_3 (normal) direction by U_3, it will be located at $(\alpha_1 + \xi_1, \alpha_2 + \xi_2, \alpha_3 + \xi_3)$. Deflections U_1 and coordinate changes ξ_i are related by

$$U_i = \sqrt{g_{ii}(\alpha_1, \alpha_2, \alpha_3)}\,\xi_i \tag{2.3.5}$$

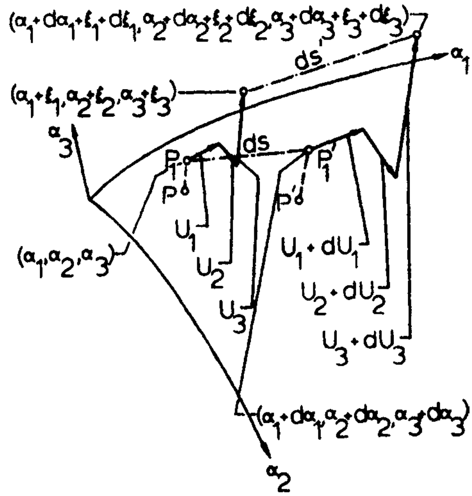

Fig. 5 Two infinitesimally close points before and after deflection.

Point P_1', originally at $(\alpha_1+\mathrm{d}\alpha_1, \alpha_2+\mathrm{d}\alpha_2, \alpha_3+\mathrm{d}\alpha_3)$, will be located at $(\alpha_1+\mathrm{d}\alpha_1+\xi_1+\mathrm{d}\xi_1, \alpha_2+\mathrm{d}\alpha_2+\xi_2+\mathrm{d}\xi_2, \alpha_3+\mathrm{d}\alpha_3+\xi_3+\mathrm{d}\xi_3)$ after deflection (Fig. 5). The distance $\mathrm{d}s'$ between P_1 and P_1' in the deflected state will therefore be

$$(\mathrm{d}s')^2 = \sum_{i=1}^{3} g_{ii}(\alpha_1+\xi_1, \alpha_2+\xi_2, \alpha_3+\xi_3)(\mathrm{d}\alpha_i+\mathrm{d}\xi_i)^2 \tag{2.3.6}$$

Since $g_{ii}(\alpha_1, \alpha_2, \alpha_3)$ varies in a continuous fashion as α_1, α_2 and α_3 change, we may utilize as an approximation the first few terms of a Taylor series expansion of $g_{ii}(\alpha_1+\xi_1, \alpha_2+\xi_2, \alpha_3+\xi_3)$ about the point $(\alpha_1, \alpha_2, \alpha_3)$:

$$g_{ii}(\alpha_1+\xi_1, \alpha_2+\xi_2, \alpha_3+\xi_3) = g_{ii}(\alpha_1, \alpha_2, \alpha_3) + \sum_{j=1}^{3} \frac{\partial g_{ii}(\alpha_1, \alpha_2, \alpha_3)}{\partial \alpha_j}\xi_j + \cdots \tag{2.3.7}$$

For the special case of an arch that deflects only in the plane of its curvature, the Taylor series expansion is illustrated in Fig. 6. In this example, $g_{22}(\alpha_1, \alpha_2, \alpha_3)=0$, $g_{33}(\alpha_1, \alpha_2, \alpha_3)=0$, and $g_{11}(\alpha_1, \alpha_2, \alpha_3)=g_{11}(\alpha_1)$. Equation (2.3.7) becomes

$$g_{11}(\alpha_1+\xi_1) = g_{11}(\alpha_1) + \frac{\partial g_{11}(\alpha_1)}{\partial \alpha_1}\xi_1 \tag{2.3.8}$$

Continuing with the general case, we may write

$$(\mathrm{d}\alpha_i+\mathrm{d}\xi_i)^2 = (\mathrm{d}\alpha_i)^2 + 2\mathrm{d}\alpha_i\mathrm{d}\xi_1 + (\mathrm{d}\xi_i)^2 \tag{2.3.9}$$

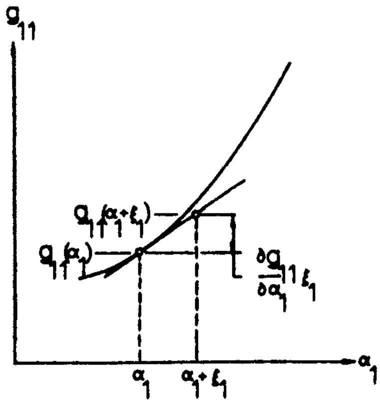

FIG. 6 Illustration of the Taylor Series expansion.

In the order of approximation consistent with linear theory, $(d\xi_i)^2$ can be neglected. Thus

$$(d\alpha_i + d\xi_i)^2 = (d\alpha_i)^2 + 2d\alpha_i d\xi_i \tag{2.3.10}$$

The differential $d\xi_i$ is

$$d\xi_i = \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j \tag{2.3.11}$$

Therefore,

$$(d\alpha_i + d\xi_i)^2 = (d\alpha_i)^2 + 2d\alpha_i \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j \tag{2.3.12}$$

Substituting Eqs. (2.3.12) and (2.3.7) in Eq. (2.3.6) gives

$$(ds')^2 = \sum_{i=1}^{3} \left[g_{ii}(\alpha_1, \alpha_2, \alpha_3) + \sum_{j=1}^{3} \frac{\partial g_{ii}(\alpha_1, \alpha_2, \alpha_3)}{\partial \alpha_j} \xi_j \right] \times \left[(d\alpha_i)^2 + 2d\alpha_i \sum_{j=1}^{3} \frac{\partial \xi_j}{\partial \alpha_j} d\alpha_j \right] \tag{2.3.13}$$

Expanding this equation and writing

$$g_{ii}(\alpha_1, \alpha_2, \alpha_3) = g_{ii} \tag{2.3.14}$$

gives

$$(\mathrm{d}s')^2 = \sum_{i=1}^{3}\left[\left(g_{ii}+\sum_{j=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_j}\xi j\right)(\mathrm{d}\alpha_j)^2+2\mathrm{d}\alpha_i g_{ii}\sum_{j=1}^{3}\frac{\partial \xi_j}{\partial \alpha_j}\mathrm{d}\alpha_j + 2\mathrm{d}\alpha_i\sum_{j=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_j}\xi_j\sum_{j=1}^{3}\frac{\partial \xi_j}{\partial \alpha_j}\mathrm{d}\alpha_j\right] \tag{2.3.15}$$

The last term is negligible except for cases where high initial stresses exist in the shell. We have, therefore, replacing j by k in the first term,

$$(\mathrm{d}s')^2 = \sum_{i=1}^{3}\left(g_{ii}+\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi k\right)(\mathrm{d}\alpha_i)^2 + \sum_{i=1}^{3}\sum_{j=1}^{3}g_{ii}\frac{\partial \xi_j}{\partial \alpha_j}\mathrm{d}\alpha_j\mathrm{d}\alpha_i+\sum_{i=1}^{3}\sum_{j=1}^{3}g_{ii}\frac{\partial \xi_j}{\partial \alpha_j}\mathrm{d}\alpha_j\mathrm{d}\alpha_i \tag{2.3.16}$$

Utilizing the Kronecker delta notation

$$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases} \tag{2.3.17}$$

we may write the first term of Eq. (2.3.16) as

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\left(g_{ii}+\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k\right)\delta_{ij}\mathrm{d}\alpha_i\mathrm{d}\alpha_j \tag{2.3.18}$$

The last two terms of Eq. (2.3.16) we may write in symmetric fashion by noting that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}g_{ii}\frac{\partial \xi_i}{\partial \alpha_j}\mathrm{d}\alpha_j\mathrm{d}\alpha_i=\sum_{i=1}^{3}\sum_{j=1}^{3}g_{jj}\frac{\partial \xi_j}{\partial \alpha_i}\mathrm{d}\alpha_i\mathrm{d}\alpha_j \tag{2.3.19}$$

Thus

$$(\mathrm{d}s')^2=\sum_{i=1}^{3}\sum_{j=1}^{3}\left[\left(g_{ii}+\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k\right)\delta_{ij}+g_{ii}\frac{\partial \xi_i}{\partial \alpha_j}+g_{jj}\frac{\partial \xi_j}{\partial \alpha_i}\right]\mathrm{d}\alpha_i\mathrm{d}\alpha_j \tag{2.3.20}$$

Denoting

$$G_{ij}=\left(g_{ii}+\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k\right)\delta_{ij}+g_{ii}\frac{\partial \xi_i}{\partial \alpha_j}+g_{jj}\frac{\partial \xi_j}{\partial \alpha_i} \tag{2.3.21}$$

gives

$$(\mathrm{d}s')^2=\sum_{i=1}^{3}\sum_{j=1}^{3}G_{ij}\mathrm{d}\alpha_i\mathrm{d}\alpha_j \tag{2.3.22}$$

Note that

$$G_{ij} = G_{ji} \tag{2.3.23}$$

Equation (2.3.22) defines the distance between two points P and P' after deflection, where point P was originally located at $(\alpha_1, \alpha_2, \alpha_3)$ and point P' at $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2, \alpha_3 + d\alpha_3)$. For example, if P' was originally located at $(\alpha_1 + d\alpha_1, \alpha_2, \alpha_3)$, that is, $d\alpha_2 = 0$ and $d\alpha_3 = 0$,

$$(ds)^2 = g_{11}(d\alpha_1)^2 = (ds)^2_{11} \tag{2.3.24}$$

$$(ds')^2 = G_{11}(d\alpha_1)^2 = (ds')^2_{11} \tag{2.3.25}$$

If point P' was originally located at $(\alpha_1, \alpha_2 + d\alpha_2, \alpha_3)$, that is, $d\alpha_1 = 0$ and $d\alpha_3 = 0$,

$$(ds)^2 = g_{22}(d\alpha_2)^2 = (ds)^2_{22} \tag{2.3.26}$$

$$(ds')^2 = G_{22}(d\alpha_2)^2 = (ds')^2_{22} \tag{2.3.27}$$

Now let us investigate the case shown in Fig. 7, where P was originally located at $(\alpha_1 + d\alpha_1, \alpha_2, \alpha 3)$ and P' was originally located at $(\alpha_1, \alpha_2 + d\alpha_2, \alpha_3)$. This is equivalent to saying that P was originally located at $(\alpha_1, \alpha_2, \alpha_3)$ and P' at $(\alpha_1 - d\alpha_1, \alpha 2 + d\alpha_2, \alpha_3)$. We then get

$$(ds)^2 = g_{11}(d\alpha_1)^2 + g_{22}(d\alpha_2)^2 = (ds)^2_{12} \tag{2.3.28}$$

$$(ds')^2 = G_{11}(d\alpha_1)^2 + G_{22}(d\alpha_2)^2 - 2G_{12}\, d\alpha_1\, d\alpha_2 = (ds')^2_{12} \tag{2.3.29}$$

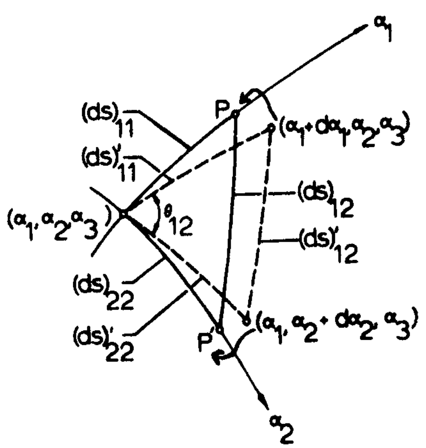

FIG. 7 Shear deformation in the plane of the reference surface.

In general,

$$(\mathrm{d}s)_{ii}^2 = g_{ii}(\mathrm{d}\alpha_i)^2 \tag{2.3.30}$$

$$(\mathrm{d}s')_{ii}^2 = G_{ii}(\mathrm{d}\alpha_i)^2 \tag{2.3.31}$$

and

$$(\mathrm{d}s)_{ij}^2 = g_{ii}(\mathrm{d}\alpha_i)^2 + g_{jj}(\mathrm{d}\alpha_j)^2 \tag{2.3.32}$$

$$(\mathrm{d}s')_{ij}^2 = G_{ii}(\mathrm{d}\alpha_i)^2 + G_{jj}(\mathrm{d}\alpha_j)^2 - 2G_{ij}\,\mathrm{d}\alpha_i\,\mathrm{d}\alpha_j \tag{2.3.33}$$

We are ready now to formulate strains. The normal strains are

$$\varepsilon_{ii} = \frac{(\mathrm{d}s')_{ii} - (\mathrm{d}s)_{ii}}{(\mathrm{d}s)_{ii}} = \sqrt{\frac{G_{ii}}{g_{ii}}} - 1 = \sqrt{1 + \frac{G_{ii} - g_{ii}}{g_{ii}}} - 1 \tag{2.3.34}$$

Noting that since

$$\frac{G_{ii} - g_{ii}}{g_{ii}} \ll 1 \tag{2.3.35}$$

we have the expansion

$$\sqrt{1 + \frac{G_{ii} - g_{ii}}{g_{ii}}} = 1 + \frac{1}{2}\frac{G_{ii} - g_{ii}}{g_{ii}} - \cdots \tag{2.3.36}$$

Thus

$$\varepsilon_{ii} = \frac{1}{2}\frac{G_{ii} - g_{ii}}{g_{ii}} \tag{2.3.37}$$

Shear strains $\varepsilon_{ij} (i \neq j)$ are defined as the angular change of an infinitesimal element:

$$\varepsilon_{ij} = \frac{\pi}{2} - \theta_{ij} \tag{2.3.38}$$

θ_{ij} for $i = 1$ and $j = 2$ is shown in Fig. 7. Utilizing the cosine law, we may compute this angle

$$(\mathrm{d}s')_{ij}^2 = (\mathrm{d}s')_{ii}^2 + (\mathrm{d}s')_{jj}^2 - 2(\mathrm{d}s')_{ii}(\mathrm{d}s')_{jj}\cos\theta_{ij} \tag{2.3.39}$$

Substituting Eqs. (2.3.31) and (2.3.33) and solving for $\cos\theta_{ij}$ gives

$$\cos\theta_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \tag{2.3.40}$$

Substituting Eq. (2.3.38) results in

$$\cos\left(\frac{\pi}{2} - \varepsilon_{ij}\right) = \sin\varepsilon_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \tag{2.3.41}$$

and since for reasonable shear strain magnitudes

$$\sin\varepsilon_{ij} \cong \varepsilon_{ij} \tag{2.3.42}$$

and

$$\frac{G_{ij}}{\sqrt{G_{ii}G_{jj}}} \cong \frac{G_{ij}}{\sqrt{g_{ii}g_{jj}}} \tag{2.3.43}$$

we may express the shear strain as

$$\varepsilon_{ij} = \frac{G_{ij}}{\sqrt{g_{ii}g_{jj}}} \tag{2.3.44}$$

Substituting Eqs. (2.3.21), (2.3.5), and (2.3.1) to (2.3.3) in Eq. (2.3.37) gives, for instance for $i=1$,

$$\begin{aligned}\varepsilon_{11} &= \frac{1}{2A_1^2(1+\alpha_3/R_1)^2}\left\{\frac{\partial[A_1^2(1+\alpha_3/R_1)^2]}{\partial\alpha_1}\frac{U_1}{A_1(1+\alpha_3/R_1)}\right.\\ &\quad\left.+\frac{\partial[A_1^2(1+\alpha_3/R_1)^2]}{\partial\alpha_2}\frac{U_2}{A_2(1+\alpha_3/R_2)}+\frac{\partial[A_1^2(1+\alpha_3/R_1)^2]}{\partial\alpha_3}U_3\right\}\\ &\quad+\frac{\partial}{\partial\alpha_1}\frac{U_1}{A_1(1+\alpha_3/R_1)}\\ &= \frac{1}{A_1(1+\alpha_3/R_1)}\left\{\frac{\partial[A_1(1+\alpha_3/R_1)]}{\partial\alpha_1}\frac{U_1}{A_1(1+\alpha_3/R_1)}\right.\\ &\quad\left.+\frac{\partial[A_1(1+\alpha_3/R_1)]}{\partial\alpha_2}\frac{U_2}{A_2(1+\alpha_3/R_2)}+\frac{A_1}{R_1}U_3\right\}+\frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial U_1}{\partial\alpha_1}\\ &\quad-\frac{\partial[A_1(1+\alpha_3/R_1)]}{\partial\alpha_1}\frac{U_1}{A_1^2(1+\alpha_3/R_1)^2}\end{aligned} \tag{2.3.45}$$

Next, we will utilize the equalities

$$\frac{\partial[A_1(1+\alpha_3/R_1)]}{\partial\alpha_2} = \left(1+\frac{\alpha_3}{R_2}\right)\frac{\partial A_1}{\partial\alpha_2} \tag{2.3.46}$$

and

$$\frac{\partial[A_2(1+\alpha_3/R_2)]}{\partial\alpha_1} = \left(1+\frac{\alpha_3}{R_1}\right)\frac{\partial A_2}{\partial\alpha_1} \tag{2.3.47}$$

These relations are named after Codazzi (see Ref. Kraus, 1967)

Substituting them in Eq. (2.3.45), we get

$$\varepsilon_{11} = \frac{1}{A_1(1+\alpha_3/R_1)}\left(\frac{\partial U_1}{\partial\alpha_1}+\frac{U_2}{A_2}\frac{\partial A_1}{\partial\alpha_2}+U_3\frac{A_1}{R_1}\right) \tag{2.3.48}$$

Similarly,

$$\varepsilon_{22} = \frac{1}{A_2(1+\alpha_3/R_2)}\left(\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1}\frac{\partial A_2}{\partial \alpha_1} + U_3\frac{A_2}{R_2}\right) \tag{2.3.49}$$

$$\varepsilon_{33} = \frac{\partial U_3}{\partial \alpha_3} \tag{2.3.50}$$

Substituting Eqs. (2.3.21), (2.3.5), and (2.3.1) to (2.3.3) in Eq. (2.3.44) gives, for instance for $i=1$, $j=2$,

$$\begin{aligned}\varepsilon_{12} = {}& \frac{A_1(1+\alpha_3/R_1)}{A_2(1+\alpha_3/R_2)}\frac{\partial}{\partial \alpha_2}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right) \\ &+ \frac{A_2(1+\alpha_3/R_2)}{A_1(1+\alpha_3/R_1)}\frac{\partial}{\partial \alpha_1}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)\end{aligned} \tag{2.3.51}$$

Similarly,

$$\varepsilon_{13} = A_1\left(1+\frac{\alpha_3}{R_1}\right)\frac{\partial}{\partial \alpha_3}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right) + \frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial U_3}{\partial \alpha_1} \tag{2.3.52}$$

$$\varepsilon_{23} = A_2\left(1+\frac{\alpha_3}{R_2}\right)\frac{\partial}{\partial \alpha_3}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right) + \frac{1}{A_2(1+\alpha_3/R_2)}\frac{\partial U_3}{\partial \alpha_2} \tag{2.3.53}$$

2.4. LOVE SIMPLIFICATIONS

If the shell is thin, we may assume that the displacements in the α_1 and α_2 directions vary linearly through the shell thickness, whereas displacements in the α_3 direction are independent of α_3:

$$U_1(\alpha_1,\alpha_2,\alpha_3) = u_1(\alpha_1,\alpha_2)+\alpha_3\beta_1(\alpha_1,\alpha_2) \tag{2.4.1}$$

$$U_2(\alpha_1,\alpha_2,\alpha_3) = u_2(\alpha_1,\alpha_2)+\alpha_3\beta_2(\alpha_1,\alpha_2) \tag{2.4.2}$$

$$U_3(\alpha_1,\alpha_2,\alpha_3) = u_3(\alpha_1,\alpha_2) \tag{2.4.3}$$

where β_1 and β_2 represents angles. If we assume that we may neglect shear deflection, which implies that the normal shear strains are 0,

$$\varepsilon_{13} = 0 \tag{2.4.4}$$

$$\varepsilon_{23} = 0 \tag{2.4.5}$$

we obtain, for example, a definition of β_1 from Eq. (2.3.52),

$$0 = A_1\left(1+\frac{\alpha_3}{R_1}\right)\frac{\partial}{\partial\alpha_3}\left(\frac{u_1+\alpha_3\beta_1}{A_1(1+\alpha_3/R_1)}\right)+\frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial u_3}{\partial\alpha_1}$$

$$= \beta_1 - \frac{u_1}{R_1} + \frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1} \tag{2.4.6}$$

or

$$\beta_1 = \frac{u_1}{R_1} - \frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1} \tag{2.4.7}$$

Similarly, we get

$$\beta_2 = \frac{u_2}{R_2} - \frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2} \tag{2.4.8}$$

Substituting Eqs. (2.4.1)–(2.4.3) into Eqs. (2.3.48)–(2.3.51), and recognizing that

$$\frac{\alpha_3}{R_1} \ll 1, \quad \frac{\alpha_3}{R_2} \ll 1 \tag{2.4.9}$$

we get

$$\varepsilon_{11} = \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}(u_1+\alpha_3\beta_1) + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}(u_2+\alpha_3\beta_2) + \frac{u_3}{R_1} \tag{2.4.10}$$

$$\varepsilon_{22} = \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}(u_2+\alpha_3\beta_2) + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}(u_1+\alpha_3\beta_1) + \frac{u_3}{R_2} \tag{2.4.11}$$

$$\varepsilon_{33} = 0 \tag{2.4.12}$$

$$\varepsilon_{13} = 0 \tag{2.4.13}$$

$$\varepsilon_{23} = 0 \tag{2.4.14}$$

$$\varepsilon_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2+\alpha_3\beta_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_1+\alpha_3\beta_1}{A_1}\right) \tag{2.4.15}$$

It is convenient to express Eqs. (2.4.10)–(2.4.15) in a form where membrane strains (independent of α_3) and bending strains (proportional to α_3) are separated:

$$\varepsilon_{11} = \varepsilon_{11}^0 + \alpha_3 k_{11} \tag{2.4.16}$$

$$\varepsilon_{22} = \varepsilon_{22}^0 + \alpha_3 k_{22} \tag{2.4.17}$$

$$\varepsilon_{12} = \varepsilon_{12}^0 + \alpha_3 k_{12} \tag{2.4.18}$$

where the membrane strains are

$$\varepsilon_{11}^0 = \frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} + \frac{u_3}{R_1} \tag{2.4.19}$$

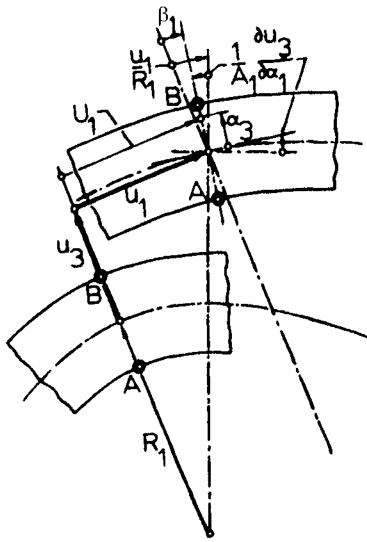

Fig. 8 Illustration of the Love assumption.

$$\varepsilon_{22}^0 = \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{u_3}{R_2} \quad (2.4.20)$$

$$\varepsilon_{12}^0 = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right) \quad (2.4.21)$$

and where the change-in-curvature terms (bending strains) are

$$k_{11} = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} \quad (2.4.22)$$

$$k_{22} = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} \quad (2.4.23)$$

$$k_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\beta_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\beta_1}{A_1}\right) \quad (2.4.24)$$

The displacement relations of Eqs. (2.4.1) and (2.4.7) are illustrated in Fig. 8.

2.5. MEMBRANE FORCES AND BENDING MOMENTS

In the following, we integrate all stresses acting on a shell element whose dimensions are infinitesimal in the α_1 and α_2 directions and equal to the

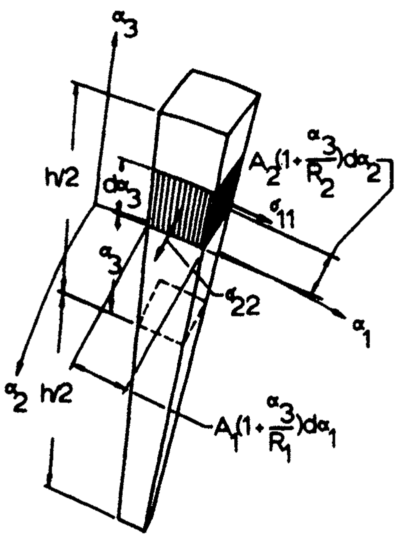

FIG. 9 An element cut from a shell that is of infinitesimal crossectional dimensions, but extends through the entire thickness of the shell.

shell thickness in the normal direction. Solving Eqs. (2.2.10)–(2.2.12) for stresses yields

$$\sigma_{11} = \frac{E}{1-\mu^2}(\varepsilon_{11}+\mu\varepsilon_{22}) \tag{2.5.1}$$

$$\sigma_{22} = \frac{E}{1-\mu^2}(\varepsilon_{22}+\mu\varepsilon_{11}) \tag{2.5.2}$$

$$\sigma_{12} = \varepsilon_{12}G \tag{2.5.3}$$

Substituting Eqs. (2.4.16)–(2.4.18) gives

$$\sigma_{11} = \frac{E}{1-\mu^2}[\varepsilon_{11}^0+\mu\varepsilon_{22}^0+\alpha_3(k_{11}+\mu k_{22})] \tag{2.5.4}$$

$$\sigma_{22} = \frac{E}{1-\mu^2}[\varepsilon_{22}^0+\mu\varepsilon_{11}^0+\alpha_3(k_{22}+\mu k_{11})] \tag{2.5.5}$$

$$\sigma_{12} = G(\varepsilon_{12}^0+\alpha_3 k_{12}) \tag{2.5.6}$$

For instance, referring to Fig. 9, the force in the α_1 direction acting on a strip of the element face of height $d\alpha_3$ and width $A_2(1+\alpha_3/R_2)d\alpha_2$ is

$$\sigma_{11}A_2\left(1+\frac{\alpha 3}{R_2}\right)d\alpha_2 d\alpha_3$$

Thus the total force acting on the element in the α_1 direction is

$$\int_{\alpha_3=-h/2}^{\alpha_3=h/2} \sigma_{11} A_2 \left(1+\frac{\alpha_3}{R_2}\right) d\alpha_2 d\alpha_3$$

and the force per unit length of neutral surface $A_2 d\alpha_2$ is

$$N_{11} = \int_{-h/2}^{h/2} \sigma_{11} \left(1+\frac{\alpha_3}{R_2}\right) d\alpha_3 \tag{2.5.7}$$

Neglecting the second term in parentheses, we obtain

$$N_{11} = \int_{-h/2}^{h/2} \sigma_{11} d\alpha_3 \tag{2.5.8}$$

Substituting Eq. (2.5.4) results in

$$N_{11} = K(\varepsilon_{11}^0 + \mu\varepsilon_{22}^0) \tag{2.5.9}$$

where

$$K = \frac{Eh}{1-\mu^2} \tag{2.5.10}$$

K is called the membrane stiffness. Similarly, integrating σ_{22} on the α_2 face of the element with the shear stresses $\sigma_{12} = \sigma_{21}$ gives

$$N_{22} = K(\varepsilon_{22}^0 + \mu\varepsilon_{11}^0) \tag{2.5.11}$$

$$N_{12} = N_{21} = \frac{K(1-\mu)}{2} \varepsilon_{12}^0 \tag{2.5.12}$$

To obtain bending moments, we first express the bending moment about the neutral surface due to the element strip $A_2(1+\alpha_3/R_2)d\alpha_2 d\alpha_3$:

$$\sigma_{11}\alpha_3 A_2 \left(1+\frac{\alpha_3}{R_2}\right) d\alpha_2 d\alpha_3$$

Thus the total bending moment acting on the element in the α_1 direction is

$$\int_{\alpha_3=-h/2}^{\alpha_3=h/2} \sigma_{11}\alpha_3 A_2 \left(1+\frac{\alpha_3}{R_2}\right) d\alpha_2 d\alpha_3$$

and the bending moment per unit length of neutral surface is

$$M_{11} = \int_{-h/2}^{h/2} \sigma_{11}\alpha_3 \left(1+\frac{\alpha_3}{R_2}\right) d\alpha_3 \tag{2.5.13}$$

Neglecting the second term in parentheses, we have

$$M_{11} = \int_{-h/2}^{h/2} \sigma_{11} \alpha_3 \, d\alpha_3 \tag{2.5.14}$$

Substituting Eq. (2.5.4) results in

$$M_{11} = D(k_{11} + \mu k_{22}) \tag{2.5.15}$$

where

$$D = \frac{Eh^3}{12(1-\mu^2)} \tag{2.5.16}$$

D is called the bending stiffness. Similarly, integrating σ_{22} and $\sigma_{12} = \sigma_{21}$ gives

$$M_{22} = D(k_{22} + \mu k_{11}) \tag{2.5.17}$$

$$M_{12} = M_{21} = \frac{D(1-\mu)}{2} k_{12} \tag{2.5.18}$$

While we have assumed that strains ε_{13} and ε_{23} due to transverse shear stresses σ_{13} and σ_{23} are negligible, we will by no means neglect the transverse shear forces in the following:

$$Q_{13} = \int_{-h/2}^{h/2} \sigma_{13} \, d\alpha_3 \tag{2.5.19}$$

and

$$Q_{23} = \int_{-h/2}^{h/2} \sigma_{23} \, d\alpha_3 \tag{2.5.20}$$

These forces will be defined by the resulting equations themselves.

Finally, if we solve Eqs. (2.5.9), (2.5.11), (2.5.12), (2.5.15), (2.5.17), and (2.5.18) for the strains, we may write Eqs. (2.5.4)–(2.5.6) as

$$\sigma_{11} = \frac{N_{11}}{h} + \frac{12M_{11}}{h^3} \alpha_3 \tag{2.5.21}$$

$$\sigma_{22} = \frac{N_{22}}{h} + \frac{12M_{22}}{h^3} \alpha_3 \tag{2.5.22}$$

$$\sigma_{12} = \frac{N_{12}}{h} + \frac{12M_{12}}{h^3} \alpha_3 \tag{2.5.23}$$

It was assumed in this section that the reference surface is halfway between the inner and outer surfaces of the shell. If this is not the case, see Sec. 2.11.

2.6. ENERGY EXPRESSIONS

The strain energy stored in one infinitesimal element that is acted on by stresses σ_{ij} is

$$dU=\frac{1}{2}(\sigma_{11}\varepsilon_{11}+\sigma_{22}\varepsilon_{22}+\sigma_{12}\varepsilon_{12}+\sigma_{13}\varepsilon_{13}+\sigma_{23}\varepsilon_{23}+\sigma_{33}\varepsilon_{33})\,dV \quad (2.6.1)$$

The last term is neglected in line with assumption (2.4.3). We do, however, have to keep the transverse shear terms, even though we have previously assumed ε_{13} and ε_{23} to be negligible, to obtain expressions for β_1 and β_2. The infinitesimal volume is given by

$$dV=A_1A_2\left(1+\frac{\alpha_3}{R_1}\right)\left(1+\frac{\alpha_3}{R_2}\right)d\alpha_1\,d\alpha_2\,d\alpha_3 \quad (2.6.2)$$

Integrating Eq. (2.6.1) over the volume of the shell gives

$$U=\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}F\,dV \quad (2.6.3)$$

where

$$F=\frac{1}{2}(\sigma_{11}\varepsilon_{11}+\sigma_{22}\varepsilon_{22}+\sigma_{12}\varepsilon_{12}+\sigma_{13}\varepsilon_{13}+\sigma_{23}\varepsilon_{23}) \quad (2.6.4)$$

The Kinetic energy of one infinitesimal element is given by

$$dK=\frac{1}{2}\rho(\dot{U}_1^2+\dot{U}_2^2+\dot{U}_3^2)\,dV \quad (2.6.5)$$

The dot indicates a time derivative.

Substituting Eqs. (2.4.1)–(2.4.3) and considering all the elements of the shell gives

$$K=\frac{\rho}{2}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\Big[\dot{u}_1^2+\dot{u}_2^2+\dot{u}_3^2+\alpha_3^2(\dot{\beta}_1^2+\dot{\beta}_2^2) +2\alpha_3(\dot{u}_1\dot{\beta}_1+\dot{u}_2\dot{\beta}_2)\Big]A_1A_2\left(1+\frac{\alpha_3}{R_1}\right)\left(1+\frac{\alpha_3}{R_2}\right)d\alpha_1\,d\alpha_2\,d\alpha_3 \quad (2.6.6)$$

Neglecting the α_3/R_1 and α_3/R_2 terms, we integrate over the thickness of the shell ($\alpha_3=-h/2$ to $\alpha_3=h/2$). This gives

$$K=\frac{\rho h}{2}\int_{\alpha_1}\int_{\alpha_2}\left[\dot{u}_1^2+\dot{u}_2^2+\dot{u}_3^2+\frac{h^2}{12}(\dot{\beta}_1^2+\dot{\beta}_2^2)\right]A_1A_2\,d\alpha_1\,d\alpha_2 \quad (2.6.7)$$

The variation of energy put into the shell by possible applied boundary force resultants and moment resultants is, along typical

$\alpha_2 =$ constant and $\alpha_1 =$ constant lines,

$$\delta E_B = \int_{\alpha_1} (\delta u_2 N^*_{22} + \delta u_1 N^*_{21} + \delta u_3 Q^*_{23} + \delta\beta_2 M^*_{22} + \delta\beta_1 M^*_{21}) A_1 \, d\alpha_1 + \int_{\alpha_2} (\delta u_1 N^*_{11} + \delta u_2 N^*_{12} + \delta u_3 Q^*_{13} + \delta\beta_1 M^*_{11} + \delta\beta_2 M^*_{12}) A_2 \, d\alpha_2 \tag{2.6.8}$$

Taking, for example, the α_2 = constant edge, N^*_{22} is the boundary force normal to the boundary in the tangent plane to the neutral surface. The units are newtons per meter. Q^*_{23} is a shear force acting on the boundary normal to the shell surface, and N^*_{21} is a shear force acting along the boundary in the tangent plane. M^*_{22} is a moment in the α_2 direction, and M^*_{21} is a twisting moment in the α_1 direction. Figure 10 illustrates this.

The variation of energy introduced into the shell by distributed load components in the α_1, α_2, and α_3 directions, namely q_1, q_2, and $q_3 (\mathrm{N/m^2})$, is

$$\delta E_L = \int_{\alpha_1}\int_{\alpha_2} (q_1 \delta u_1 + q_2 \delta u_2 + q_3 \delta u_3) A_1 A_2 \, d\alpha_1 \, d\alpha_2 \tag{2.6.9}$$

All loads are assumed to act on the neutral surface of the shell. The components are shown in Fig. 11.

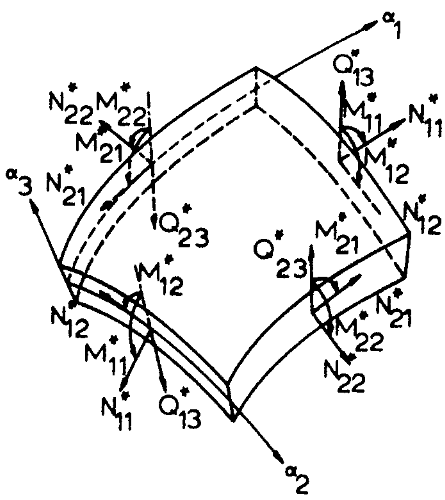

FIG. 10 Boundary force and moment resultants.

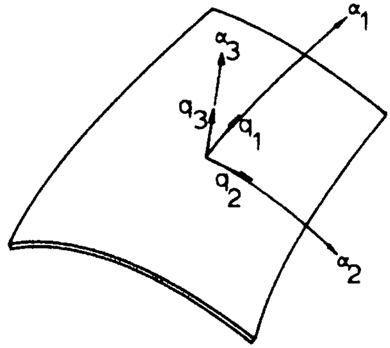

Fig. 11 Distributed loading on the reference surface.

2.7. LOVE'S EQUATIONS BY WAY OF HAMILTON'S PRINCIPLE

Hamilton's principle is given as [note the discussion in Sec. 2.9 and that Eq. (2.9.13) is multiplied here by −1 for convenience]

$$\delta\int_{t_0}^{t_1}(U-K-W_{\mathrm{ln}})\,\mathrm{d}t=0 \tag{2.7.1}$$

where W_{ln} is the total input energy. In our case

$$W_{\mathrm{ln}}=E_B+E_L \tag{2.7.2}$$

The times t_1 and t_0 are arbitrary, except that at $t=t_1$ and $t=t_0$, all variations are 0. The symbol δ is the variational symbol and is treated mathematically like a differential symbol. Variational displacements are arbitrary.

Substituting Eq. (2.7.2) for Eq. (2.7.1) and taking the variational operator inside the integral, we obtain

$$\int_{t_0}^{t_1}(\delta U-\delta E_B-\delta E_L-\delta K)\,\mathrm{d}t=0 \tag{2.7.3}$$

Let us examine these variations one by one. First, from Eq. (2.6.7),

$$\int_{t_0}^{t_1}\delta K\,\mathrm{d}t = \rho h\int_{t_0}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\Big[\dot{u}_1\delta\dot{u}_1+\dot{u}_2\delta\dot{u}_2+\dot{u}_3\delta\dot{u}_3 + \frac{h^2}{12}(\dot{\beta}_1\delta\dot{\beta}_1+\dot{\beta}_2\delta\dot{\beta}_2)\Big]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}t \tag{2.7.4}$$

Integrating by parts, for instance, the first term becomes

$$\int_{t_0}^{t_1} \dot{u}_1 \delta \dot{u}_1 \, \mathrm{d}t = [\dot{u}_1 \delta \dot{u}_1]_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{u}_1 \delta u_1 \, \mathrm{d}t \tag{2.7.5}$$

Since the virtual displacement is 0 at $t = t_0$ and $t = t_1$, we are left with

$$\int_{t_0}^{t_1} \dot{u}_1 \delta \dot{u}_1 \, \mathrm{d}t = -\int_{t_0}^{t_1} \ddot{u}_1 \delta u_1 \, \mathrm{d}t \tag{2.7.6}$$

Proceeding similarly with the other terms in the integral, we get

$$\int_{t_0}^{t_1} \delta K \, \mathrm{d}t = -\rho h \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \left[\ddot{u}_1 \delta u_1 + \ddot{u}_2 \delta u_2 + \ddot{u}_3 \delta u_3 + \frac{h^2}{12} (\ddot{\beta}_1 \delta \beta_1 + \ddot{\beta}_2 \delta \beta_2) \right] A_1 A_2 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}t \tag{2.7.7}$$

As in the classical Bernoulli–Euler beam theory, we neglect the influence of rotatory inertia, which we recognize as the terms involving $\ddot{\beta}_1$ and $\ddot{\beta}_2$. It will be shown later that rotatory inertia has to be considered only for very short wavelengths of vibration, and even then shear deformation is a more important effect.

$$\int_{t_0}^{t_1} \delta K \, \mathrm{d}t = -\rho h \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} (\ddot{u}_1 \delta u_1 + \ddot{u}_2 \delta u_2 + \ddot{u}_3 \delta u_3) A_1 A_2 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}t \tag{2.7.8}$$

Next, let us evaluate the variation of the energy due to the load. From Eq. (2.6.9),

$$\int_{t_0}^{t_1} \delta E_L \, \mathrm{d}t = \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} (q_1 \delta u_1 + q_2 \delta u_2 + q_3 \delta u_3) A_1 A_2 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}t \tag{2.7.9}$$

The integral of the variation of boundary energy is, using Eq. (2.6.8),

$$\int_{t_0}^{t_1} \delta E_B \, \mathrm{d}t = \int_{t_0}^{t_1} \int_{\alpha_1} (N_{22}^* \delta u_2 + N_{21}^* \delta u_1 + Q_{23}^* \delta u_3 + M_{22}^* \delta \beta_2 + M_{21}^* \delta \beta_1) A_1 \, \mathrm{d}\alpha_1 \, \mathrm{d}t + \int_{t_0}^{t_1} \int_{\alpha_2} (N_{11}^* \delta u_1 + N_{12}^* \delta u_2 + Q_{13}^* \delta u_3 + M_{11}^* \delta \beta_1 + M_{12}^* \delta \beta_2) A_2 \, \mathrm{d}\alpha_2 \, \mathrm{d}t \tag{2.7.10}$$

It is more complicated to evaluate the integral of the variation in strain energy. Starting with Eq. (2.6.3), we have

$$\int_{t_0}^{t_1} \delta U \, \mathrm{d}t = \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} \delta F \, \mathrm{d}V \, \mathrm{d}t \tag{2.7.11}$$

where

$$\delta F = \frac{\partial F}{\partial \varepsilon_{11}} \delta \varepsilon_{11} + \frac{\partial F}{\partial \varepsilon_{22}} \delta \varepsilon_{22} + \frac{\partial F}{\partial \varepsilon_{12}} \delta \varepsilon_{12} + \frac{\partial F}{\partial \varepsilon_{13}} \delta \varepsilon_{13} + \frac{\partial F}{\partial \varepsilon_{23}} \delta \varepsilon_{23} \tag{2.7.12}$$

Examining the first term of this equation, we see that

$$\frac{\partial F}{\partial \varepsilon_{11}} \delta \varepsilon_{11} = \frac{1}{2} \left(\frac{\partial \sigma_{11}}{\partial \varepsilon_{11}} \varepsilon_{11} + \sigma_{11} + \varepsilon_{22} \frac{\partial \sigma_{22}}{\partial \varepsilon_{11}} \right) \delta \varepsilon_{11} \tag{2.7.13}$$

Substituting Eqs. (2.5.1) and (2.5.2) gives

$$\frac{\partial F}{\partial \varepsilon_{11}} \delta \varepsilon_{11} = \sigma_{11} \delta \varepsilon_{11} \tag{2.7.14}$$

Thus we can show that

$$\int_{t_0}^{t_1} \delta U \, \mathrm{d}t = \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} (\sigma_{11} \delta \varepsilon_{11} + \sigma_{22} \delta \varepsilon_{22} + \sigma_{12} \delta \varepsilon_{12} + \sigma_{13} \delta \varepsilon_{13} + \sigma_{23} \delta \varepsilon_{23}) A_1 A_2 \left(1 + \frac{\alpha_3}{R_1}\right) \left(1 + \frac{\alpha_3}{R_2}\right) \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3 \, \mathrm{d}t \tag{2.7.15}$$

We neglect the α_3/R_1 and α_3/R_2 terms as small. Substituting Eqs. (2.4.10)–(2.4.15) allows us to express the strain variations in terms of displacement variations. Integration with respect to α_3 allows us to introduce force resultants and moment resultants. Integration by parts will put the integral into a manageable form. Let us illustrate all this on the first term of Eq. (2.7.15):

$$\int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} \sigma_{11} \delta \varepsilon_{11} A_1 A_2 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3 \, \mathrm{d}t$$

$$= \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} \left[\sigma_{11} \left(A_2 \frac{\partial(\delta u_1)}{\partial \alpha_1} + \delta u_2 \frac{\partial A_1}{\partial \alpha_2} + \frac{A_1 A_2}{R_1} \delta u_3 \right) + \alpha_3 \sigma_{11} \left(A_2 \frac{\partial(\delta \beta_1)}{\partial \alpha_1} + \delta \beta_2 \frac{\partial A_1}{\partial \alpha_2} \right) \right] \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3 \, \mathrm{d}t$$

$$= \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \left[N_{11} \left(A_2 \frac{\partial(\delta u_1)}{\partial \alpha_1} + \delta u_2 \frac{\partial A_1}{\partial \alpha_2} + \frac{A_1 A_2}{R_1} \delta u_3 \right) \right.$$

$$\left. + M_{11} \left(A_2 \frac{\partial(\delta \beta_1)}{\partial \alpha_1} + \delta \beta_2 \frac{\partial A_1}{\partial \alpha_2} \right) \right] \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}t \qquad (2.7.16)$$

Now we illustrate the integration by parts on the first term of Eq. (2.7.16):

$$\int_{\alpha_2} \int_{\alpha_1} N_{11} A_2 \frac{\partial(\delta u_1)}{\partial \alpha_1} \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2$$

$$= \int_{\alpha_2} N_{11} A_2 \delta u_1 \, \mathrm{d}\alpha_2 - \int_{\alpha_1} \int_{\alpha_2} \frac{\partial(N_{11} A_2)}{\partial \alpha_1} \delta u_1 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \qquad (2.7.17)$$

Proceeding with all terms of Eq. (2.7.15) in this fashion, we get

$$\int_{t_0}^{t_1} \delta U \, \mathrm{d}t = \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \left[\left(-\frac{\partial(N_{11} A_2)}{\partial \alpha_1} - \frac{\partial(N_{21} A_1)}{\partial \alpha_2} - N_{12} \frac{\partial A_1}{\partial \alpha_2} \right. \right.$$

$$\left. + N_{22} \frac{\partial A_2}{\partial \alpha_1} - Q_{13} \frac{A_1 A_2}{R_1} \right) \delta u_1 + \left(-\frac{\partial(N_{12} A_2)}{\partial \alpha_1} \right.$$

$$\left. - \frac{\partial(N_{22} A_1)}{\partial \alpha_2} + N_{11} \frac{\partial A_1}{\partial \alpha_2} - N_{21} \frac{\partial A_2}{\partial \alpha_1} - Q_{23} \frac{A_1 A_2}{R_2} \right) \delta u_2$$

$$+ \left(N_{11} \frac{A_1 A_2}{R_1} + N_{22} \frac{A_1 A_2}{R_2} - \frac{\partial(Q_{13} A_2)}{\partial \alpha_1} - \frac{\partial(Q_{23} A_1)}{\partial \alpha_2} \right) \delta u_3$$

$$+ \left(-\frac{\partial(M_{21} A_1)}{\partial \alpha_2} - M_{12} \frac{\partial A_1}{\partial \alpha_2} + M_{22} \frac{\partial A_2}{\partial \alpha_1} - \frac{\partial(M_{11} A_2)}{\partial \alpha_1} \right.$$

$$\left. + Q_{13} A_1 A_2 \right) \delta \beta_1 + \left(-\frac{\partial(M_{12} A_2)}{\partial \alpha_1} - \frac{\partial(M_{22} A_1)}{\partial \alpha_2} \right.$$

$$\left. \left. - M_{21} \frac{\partial A_2}{\partial \alpha_1} + M_{11} \frac{\partial A_1}{\partial \alpha_2} + Q_{23} A_1 A_2 \right) \delta \beta_2 \right] \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}t$$

$$+ \int_{t_0}^{t_1} \int_{\alpha_2} (N_{11} \delta u_1 + M_{11} \delta \beta_1 + N_{12} \delta u_2 + M_{12} \delta \beta_2 + Q_{13} \delta u_3) A_2 \, \mathrm{d}\alpha_2 \, \mathrm{d}t$$

$$+ \int_{t_0}^{t_1} \int_{\alpha_1} (N_{22} \delta u_2 + M_{22} \delta \beta_2 + N_{21} \delta u_1 + M_{21} \delta \beta_1$$

$$+ Q_{23} \delta u_3) A_1 \, \mathrm{d}\alpha_1 \, \mathrm{d}t \qquad (2.7.18)$$

We are now ready to substitute Eqs. (2.7.18), (2.7.10), (2.7.9), and (2.7.8) in Eq. (2.7.3). This gives

$$\int_{t_0}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\left\{\left[\frac{\partial(N_{11}A_2)}{\partial\alpha_1}+\frac{\partial(N_{21}A_1)}{\partial\alpha_2}+N_{12}\frac{\partial A_1}{\partial\alpha_2}-N_{22}\frac{\partial A_2}{\partial\alpha_1}\right.\right.$$

$$\left.+Q_{13}\frac{A_1A_2}{R_1}+(q_1-\rho h\ddot{u}_1)A_1A_2\right]\delta u_1$$

$$+\left[\frac{\partial(N_{12}A_2)}{\partial\alpha_1}+\frac{\partial(N_{22}A_1)}{\partial\alpha_2}+N_{21}\frac{\partial A_2}{\partial\alpha_1}-N_{11}\frac{\partial A_1}{\partial\alpha_2}+Q_{23}\frac{A_1A_2}{R_2}\right.$$

$$\left.+(q_2-\rho h\ddot{u}_2)A_1A_2\right]\delta u_2+\left[\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}+\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}\right.$$

$$\left.-\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)A_1A_2+(q_3-\rho h\ddot{u}_3)A_1A_2\right]\delta u_3$$

$$+\left(\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(M_{21}A_1)}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_1}-M_{22}\frac{\partial A_2}{\partial\alpha_1}-Q_{13}A_1A_2\right)\delta\beta_1$$

$$\left.+\left(\frac{\partial(M_{12}A_2)}{\partial\alpha_1}+\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}-Q_{23}A_1A_2\right)\delta\beta_2\right\}$$

$$\times\mathbf{d}\alpha_1\mathbf{d}\alpha_2\mathbf{d}t+\int_{t_0}^{t_1}\int_{\alpha_1}[(N_{22}^*-N_{22})\delta u_2+(N_{21}^*-N_{21})\delta u_1+(Q_{23}^*-Q_{23})\delta u_3$$

$$+(M_{22}^*-M_{22})\delta\beta_2+(M_{21}^*-M_{21})\delta\beta_1]A_1\mathbf{d}\alpha_1\mathbf{d}t$$

$$+\int_{t_0}^{t_1}\int_{\alpha_2}[(N_{11}^*-N_{11})\delta u_1+(N_{12}^*-N_{12})\delta u_2+(Q_{13}^*-Q_{13})\delta u_3$$

$$+(M_{11}^*-M_{11})\delta\beta_1+(M_{12}^*-M_{12})\delta\beta_2]A_2\mathbf{d}\alpha_2\mathbf{d}t=0 \qquad \text{(2.7.19)}$$

The equation can be satisfied only if each of the triple and double integral parts is 0 individually. Moreover, since the variational displacements are arbitrary, each integral equation can be satisfied only if the coefficients of the variational displacement are 0. Thus the coefficients of the triple integral set to zero give the following five equations:

$$-\frac{\partial(N_{11}A_2)}{\partial\alpha_1}-\frac{\partial(N_{21}A_1)}{\partial\alpha_2}-N_{12}\frac{\partial A_1}{\partial\alpha_2}+N_{22}\frac{\partial A_2}{\partial\alpha_1}$$

$$-A_1A_2\frac{Q_{13}}{R_1}+A_1A_2\rho h\ddot{u}_1=A_1A_2q_1 \qquad \text{(2.7.20)}$$

$$-\frac{\partial(N_{12}A_2)}{\partial\alpha_1}-\frac{\partial(N_{22}A_1)}{\partial\alpha_2}-N_{21}\frac{\partial A_2}{\partial\alpha_1}+N_{11}\frac{\partial A_1}{\partial\alpha_2}$$
$$-A_1A_2\frac{Q_{23}}{R_2}+A_1A_2\rho h\ddot{u}_2=A_1A_2q_2 \tag{2.7.21}$$

$$-\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}+A_1A_2\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)$$
$$+A_1A_2\rho h\ddot{u}_3=A_1A_2q_3 \tag{2.7.22}$$

where Q_{13} and Q_{23} are defined by

$$\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(M_{21}A_1)}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}-Q_{13}A_1A_2=0 \tag{2.7.23}$$

$$\frac{\partial(M_{12}A_2)}{\partial\alpha_1}+\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}-Q_{23}A_1A_2=0 \tag{2.7.24}$$

These five equations are known as *Love's equations*. They define the motion (or static deflection, for all it matters) due to any type of pressure load. Shear deflection and rotatory inertia are not included.

2.8. BOUNDARY CONDITIONS

Examining Love's equations, the stress–strain and strain–displacement relations, we see that the equations are eighth-order partial differential equations in space. This means that we can accommodate at most four boundary conditions on each edge.

However, when set to 0, the two line integrals of Eq. (2.7.19) are satisfied only if the five coefficients in each are 0 or if the virtual displacements are 0. This would define, as necessary, five boundary conditions. A similar problem was encountered by Kirchhoff (1850) in the 19th century, when he investigated the boundary conditions of a plate. It appeared as if three conditions at each edge were needed, but the fourth order equation would allow only two. Kirchhoff combined the three conditions into two by noting that the twisting moment and shear boundary conditions were related.

Following Kirchhoff's lead, the two line integrals are rewritten utilizing the definitions of Eqs. (2.4.7) and (2.4.8). For instance, for the first line integral equation, we get

$$\int_{t_0}^{t_1}\int_{\alpha_1}\Bigg\{(N_{22}^*-N_{22})\delta u_2+(N_{21}^*-N_{21})\delta u_1+(Q_{23}^*-Q_{23})\delta u_3+(M_{22}^*-M_{22})\delta\beta_2$$
$$+(M_{21}^*-M_{21})\left[\frac{\delta u_1}{R_1}-\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}(\delta u_3)\right]\Bigg\}A_1\,\mathrm{d}\alpha_1\,\mathrm{d}t=0 \tag{2.8.1}$$

Before collecting coefficients of δu_3, we have to perform an integration by parts:

$$\int_{t_0}^{t_1}\int_{\alpha_1}(M_{21}^*-M_{21})\frac{\partial}{\partial\alpha_1}(\delta u_3)\,\mathrm{d}\alpha_1\,\mathrm{d}t$$

$$=\left|\int_{t_0}^{t_1}(M_{21}^*-M_{21})\delta u_3\,\mathrm{d}t\right|_{\alpha_1}-\int_{t_0}^{t_1}\int_{\alpha_1}\frac{\partial}{\partial\alpha_1}(M_{21}^*-M_{21})\,\mathrm{d}\alpha_1\,\delta u_3\,\mathrm{d}t \tag{2.8.2}$$

Since $M_{21}=M_{21}^*$ along the entire edge, the term in parentheses is 0. Thus substituting Eq. (2.8.2) in Eq. (2.8.1) and collecting coefficients of virtual displacements yields

$$\int_{t_0}^{t_1}\int_{\alpha_1}\left\{(N_{22}^*-N_{22})\delta u_2+\left[\left(N_{21}^*+\frac{M_{21}^*}{R_1}\right)-\left(N_{21}+\frac{M_{21}}{R_1}\right)\right]\delta u_1\right.$$
$$\left.+(M_{22}^*-M_{22})\delta\beta_2+\left[\left(Q_{23}^*+\frac{1}{A_1}\frac{\partial M_{21}^*}{\partial\alpha_1}\right)-\left(Q_{23}+\frac{1}{A_1}\frac{\partial M_{21}}{\partial\alpha_1}\right)\right]\delta u_3\right\}$$
$$\times A_1\,\mathrm{d}\alpha_1\,\mathrm{d}t=0 \tag{2.8.3}$$

Similarly, for the second line integral, we get

$$\int_{t_0}^{t_1}\int_{\alpha_1}\left\{(N_{11}^*-N_{11})\delta u_1+\left[\left(N_{12}^*+\frac{M_{12}^*}{R_2}\right)-\left(N_{12}+\frac{M_{12}}{R_2}\right)\right]\delta u_2\right.$$
$$\left.+(M_{11}^*-M_{11})\delta\beta_1+\left[\left(Q_{13}^*+\frac{1}{A_2}\frac{\partial M_{12}^*}{\partial\alpha_2}\right)-\left(Q_{13}+\frac{1}{A_2}\frac{\partial M_{12}}{\partial\alpha_2}\right)\right]\delta u_3\right\}$$
$$\times A_2\,\mathrm{d}\alpha_2\,\mathrm{d}t=0 \tag{2.8.4}$$

These equations are satisfied if either the virtual displacements vanish or the coefficients of the virtual displacements vanish. Defining, in memory of Kirchhoff, the Kirchhoff effective shear stress resultants of the first kind

$$V_{13}=Q_{13}+\frac{1}{A_2}\frac{\partial M_{12}}{\partial\alpha_2} \tag{2.8.5}$$

and

$$V_{23}=Q_{23}+\frac{1}{A_1}\frac{\partial M_{21}}{\partial\alpha_1} \tag{2.8.6}$$

and as the Kirchhoff effective shear stress resultants of the second kind

$$T_{12}=N_{12}+\frac{M_{12}}{R_2} \tag{2.8.7}$$

and

$$T_{21} = N_{21} + \frac{M_{21}}{R_1} \tag{2.8.8}$$

we may write the integrals as

$$\int_{t_0}^{t_1} \int_{\alpha_1} [(N_{22}^* - N_{22})\delta u_2 + (T_{21}^* - T_{21})\delta u_1 + (M_{22}^* - M_{22})\delta\beta_2$$
$$+ (V_{23}^* - V_{23})\delta u_3] A_1 \, d\alpha_1 \, dt = 0 \tag{2.8.9}$$

and

$$\int_{t_0}^{t_1} \int_{\alpha_2} [(N_{11}^* - N_{11})\delta u_1 + (T_{12}^* - T_{12})\delta u_2 + (M_{11}^* - M_{11})\delta\beta_1$$
$$+ (V_{13}^* - V_{13})\delta u_3] A_2 \, d\alpha_2 \, dt = 0 \tag{2.8.10}$$

Now we may argue that each of these integrals can be satisfied only if the coefficients of the virtual displacements, the virtual displacements, or one of the two for each term are 0. Since virtual displacements are 0 only when the boundary displacements are prescribed, this translates into the following possible boundary conditions for an α_1 = constant edge [Eq. (2.8.10)]:

$$N_{11} = N_{11}^* \quad \text{or} \quad u_1 = u_1^* \tag{2.8.11}$$

$$M_{11} = M_{11}^* \quad \text{or} \quad \beta_1 = \beta_1^* \tag{2.8.12}$$

$$V_{13} = V_{13}^* \quad \text{or} \quad u_3 = u_3^* \tag{2.8.13}$$

$$T_{12} = T_{12}^* \quad \text{or} \quad u_2 = u_2^* \tag{2.8.14}$$

This states the intuitively obvious fact that we have to prescribe at a boundary either forces (moments) or displacements (angular displacements). However, four conditions have to be identified per edge. In a later chapter, we see that under certain simplifying assumptions, we may reduce this number even further. Similarly, examining Eq. (2.8.9), which describes an α_2 = constant edge, the four boundary conditions have to be

$$N_{22} = N_{22}^* \quad \text{or} \quad u_2 = u_2^* \tag{2.8.15}$$

$$M_{22} = M_{22}^* \quad \text{or} \quad \beta_2 = \beta_2^* \tag{2.8.16}$$

$$V_{23} = V_{23}^* \quad \text{or} \quad u_3 = u_3^* \tag{2.8.17}$$

$$T_{21} = T_{21}^* \quad \text{or} \quad u_1 = u_1^* \tag{2.8.18}$$

We can therefore state in general that if n denotes the subscript that defines the normal direction to the edge and if t denotes the subscript that defines the tangential direction to the edge, the necessary boundary conditions are

$$N_{nn} = N_{nn}^* \quad \text{or} \quad u_n = u_n^* \tag{2.8.19}$$

$$M_{nn} = M_{nn}^* \quad \text{or} \quad \beta_n = \beta_n^* \tag{2.8.20}$$

$$V_{n3} = V_{n3}^* \quad \text{or} \quad u_3 = u_3^* \tag{2.8.21}$$

$$T_{nt} = T_{nt}^* \quad \text{or} \quad \dot{u}_t = u_t^* \tag{2.8.22}$$

Let us consider a few examples. First there is the case where the edge is completely free. This means that no forces or moments act on this edge:

$$N_{nn} = 0 \tag{2.8.23}$$

$$M_{nn} = 0 \tag{2.8.24}$$

$$V_{n3} = 0 \tag{2.8.25}$$

$$T_{nt} = 0 \tag{2.8.26}$$

The other extreme is the case where the edge is completely prevented from deflecting by being clamped:

$$u_n = 0 \tag{2.8.27}$$

$$u_t = 0 \tag{2.8.28}$$

$$u_3 = 0 \tag{2.8.29}$$

$$\beta_n = 0 \tag{2.8.30}$$

If the edge is supported on knife edges such that it is free to rotate in normal direction but is prevented from having any transverse deflection, clearly the two conditions

$$u_3 = 0 \tag{2.8.31}$$

$$M_{nn} = 0 \tag{2.8.32}$$

apply. If the knife edges are such that the shell is free to slide between them, the other two conditions are

$$N_{nn} = 0 \tag{2.8.33}$$

$$T_{nt} = 0 \tag{2.8.34}$$

If the shell is somehow prevented from sliding, the conditions

$$u_n = 0 \tag{2.8.35}$$

$$u_t = 0 \tag{2.8.36}$$

should be used.

2.9. HAMILTON'S PRINCIPLE

Hamilton's principle is a minimization principle that seems to apply to all of mechanics and most classical physics. It is the end of a development that started in the second century B.C. with Hero of Alexandria, who stated that light always takes the shortest path. This indeed governs reflections, but the minimum principle that includes refraction was not found until Fermat in 1657 postulated that light travels from point to point in the shortest time. On theological grounds, Maupertius in 1747 asserted that dynamical motion takes place with minimum action, where action is defined as the product of distance and momentum, or energy and time. Lagrange formulated the mathematical foundation of this principle in 1760. In 1828, Gauss formulated the principle of least constraint, which was extended later by Hertz when formulating the principle of least curvature. Finally, in 1834, Hamilton announced his general principle, which included all the others. He postulated that while there are usually several possible paths along which a dynamic system may move from one point to another in space and time, the path actually followed is the one that minimizes the time integral of the difference between the kinetic and potential energies. In terms of the calculus of variations, developed primarily by Euler and Bernoulli in the 18th century, it is usually stated is

$$\delta\int_{t_1}^{t_2}(K-U+W_{nc})\,\mathrm{d}t=0, \qquad \delta\bar{r}_i=0, \quad t=t_1,t_2 \tag{2.9.1}$$

where $\delta\bar{r}_i$ are the variations of displacements (virtual displacements), T the kinetic energy, U the strain energy, W_{nc} any additional energy input to the system, and δ is the variation, operationally equivalent to a total differential. In general, Hamilton's principle can be viewed as an axiom, replacing the axiom of Newton's second law for dynamic problems. With other words, we either accept Newton's second law as an axiom and derive Hamilton's principle from it for dynamic problems, or we accept Hamilton's principle as an axiom and derive Newton's second law from it. In the following, let us derive Hamilton's principle from the axiom of Newton's second law, utilizing D'Alembert's principle for the restricted case of interest here, namely, the motion of masses under constraints. Let the virtual displacements $\delta x_1,\delta y_1,\delta z_1,\ldots,\delta x_n,\delta y_n,\delta z_n$, be infinitesimal, arbitrary changes in the displacement coordinates of a system. They must be compatible with the constraints of the system. If each mass particle $i=1,\ldots,n$, is acted on by forces with the resultant $\overline{F}_i$, it must be that

$$\sum_{i=1}^{n}(\overline{F}_i-\dot{\bar{p}}_i)\cdot\delta\bar{r}_i=0 \tag{2.9.2}$$

where $\dot{\bar{p}}_i$ is the rate of change of the linear momentum vector $\bar{p}_i$ and $\delta\bar{r}_i = \delta x_i\bar{i} + \delta y_i\bar{j} + \delta z_i\bar{k}$. Since

$$\dot{\bar{p}}_i = m_i\ddot{\bar{r}}_i \tag{2.9.3}$$

Equation (2.9.2) may be written as

$$\sum_{i=1}^{n} m_i\ddot{\bar{r}}_i\delta\bar{r}_i = \delta W \tag{2.9.4}$$

where

$$\delta W = \sum_{i=1}^{n} \overline{F}_i \cdot \delta\bar{r}_i \tag{2.9.5}$$

and represents the virtual work due to the applied forces alone. Using the mathematical identity

$$\ddot{\bar{r}}_i \cdot \delta\bar{r}_i = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\bar{r}}_i \cdot \delta\bar{r}_i) - \delta\left(\frac{1}{2}\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right) \tag{2.9.6}$$

gives, after multiplying it by m_i and summing over all particles,

$$\sum_{i=1}^{n} m_i\ddot{\bar{r}}_i \cdot \delta\bar{r}_i = \sum_{i=1}^{n} m_i\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\bar{r}}_i \cdot \delta\bar{r}_i) - \delta\sum_{i=1}^{n}\frac{1}{2}m_i\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i \tag{2.9.7}$$

Recognizing that the kinetic energy is

$$K = \sum_{i=1}^{n}\frac{1}{2}m_i\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i \tag{2.9.8}$$

and that work done by the applied forces is equal to the input work minus what is stored in terms of strain energy,

$$W = W_{\mathrm{ln}} - U \tag{2.9.9}$$

we obtain, utilizing Eq. (2.9.4),

$$\delta(K - U + W_{\mathrm{ln}}) = \sum_{i=1}^{n} m_i\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\bar{r}}_i \cdot \delta\bar{r}_i) \tag{2.9.10}$$

Integrating between two points in time, t_1 and t_2, where the virtual displacements or variations are 0, we obtain

$$\delta\int_{t_1}^{t_2}(K - U + W_{\mathrm{ln}})\mathrm{d}t = \sum_{i=1}^{n}\int_{t_1}^{t_2} m_i\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\bar{r}}_i \cdot \delta\bar{r}_i)\mathrm{d}t = \sum_{i=1}^{n} m_i\dot{\bar{r}}_i \cdot \delta\bar{r}_i|_{t_1}^{t_2} \tag{2.9.11}$$

If we select $\delta\bar{r}_1$ such that

$$\delta\bar{r}_i = 0, \quad t = t_1, t_2 \tag{2.9.12}$$

the final result is

$$\delta\int_{t_1}^{t_2}(K-U+W_{\text{ln}})\,\mathrm{d}t=0 \tag{2.9.13}$$

Equation (2.9.12) is part of Hamilton's principle.

2.9.1. Example: Longitudinally Vibrating Rod

For a rod, it can be shown that the strain energy and the kinetic energy as a function of longitudinal displacement u_x are

$$U=\frac{1}{2}\int_0^L EA\left(\frac{\partial u_x}{\partial x}\right)^2\mathrm{d}x \tag{2.9.14}$$

and

$$K=\frac{1}{2}\int_0^L \rho A\left(\frac{\partial u_x}{\partial t}\right)^2\mathrm{d}x \tag{2.9.15}$$

where

$E=$ Young's modulus ($\mathrm{N/m^2}$), $A=$ cross-sectional area ($\mathrm{m^2}$),

$\rho=$ mass density ($\mathrm{Ns^2/m^4}$),

$t=$ time (s), $x=$ coordinate (m), and

$L=$ length (m).

We assume that $W_{\text{ln}}=0$.

Applying Hamilton's principle gives

$$\delta\int_{t_1}^{t_2}\left[\frac{1}{2}\int_0^L \rho A\left(\frac{\partial u_x}{\partial t}\right)^2\mathrm{d}x-\frac{1}{2}\int_0^L EA\left(\frac{\partial u_x}{\partial x}\right)^2\mathrm{d}x\right]\mathrm{d}t=0 \tag{2.9.16}$$

or

$$\int_{t_1}^{t_2}\int_0^L \rho A\frac{\partial u_x}{\partial t}\delta\left(\frac{\partial u_x}{\partial t}\right)\mathrm{d}x\,\mathrm{d}t-\int_{t_1}^{t_2}\int_0^L EA\frac{\partial u_x}{\partial x}\delta\left(\frac{\partial u_x}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t=0 \tag{2.9.17}$$

Examining the first double integral, we my integrate by parts:

$$\int_{t_1}^{t_2}\rho A\frac{\partial u_x}{\partial t}\delta\left(\frac{\partial u_x}{\partial t}\right)\mathrm{d}t=\int_{t_1}^{t_2}\rho A\frac{\partial u_x}{\partial t}\frac{\partial}{\partial t}(\delta u_x)\,\mathrm{d}t \tag{2.9.18}$$

$$=\left[\rho A\frac{\partial u_x}{\partial t}\delta u_x\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\rho A\frac{\partial^2 u_x}{\partial t^2}\delta u_x\,\mathrm{d}t \tag{2.9.19}$$

Evaluating the bracketed expression at the limits t_1 and t_2 gives 0 because by definition $\delta u = 0$ at t_1 and t_2. This leaves the integral in a form where the integrant is the product of δu and a coefficient.

With the same objective in mind, we examine the second double integral of Eq. (2.9.17) and integrate by parts:

$$\int_0^L EA\frac{\partial u_x}{\partial x}\delta\left(\frac{\partial u_x}{\partial x}\right)\mathrm{d}x = \int_0^L EA\frac{\partial u_x}{\partial x}\frac{\partial}{\partial x}(\delta u_x)\mathrm{d}x$$
$$= \left[EA\frac{\partial u_x}{\partial x}\delta u_x\right]_0^L - \int_0^L \frac{\partial}{\partial x}\left(EA\frac{\partial u_x}{\partial x}\right)\delta u_x \mathrm{d}x \tag{2.9.20}$$

Evaluating the bracketed quantity, it is 0 if at $x = 0$ or $x = L$

$$EA\frac{\partial u_x}{\partial x} = 0 \quad \text{or} \quad \sigma_{xx}A = 0 \tag{2.9.21}$$

$$u_x = 0 \tag{2.9.22}$$

The first expression describes a free boundary and the second expression a clamped boundary. Equation (2.9.17) therefore becomes

$$\int_{t_1}^{t_2}\int_0^L \left[\frac{\partial}{\partial x}\left(EA\frac{\partial u_x}{\partial x}\right) - \rho A\frac{\partial^2 u_x}{\partial t^2}\right]\delta u_x \mathrm{d}x\,\mathrm{d}t = 0 \tag{2.9.23}$$

Since δu_x is arbitrary, this equation can be satisfied only if the coefficient of δu_x is 0. Therefore,

$$\frac{\partial}{\partial x}\left(EA\frac{\partial u_x}{\partial x}\right) - \rho A\frac{\partial^2 u_x}{\partial t^2} = 0 \tag{2.9.24}$$

is the equation of motion.

We could have been more general if we had allowed for the possibility of boundary forces F_0 and F_L . Hamilton's principle becomes

$$\delta\int_{t_1}^{t_2}(K - U + E_B)\mathrm{d}t = 0 \tag{2.9.25}$$

where $W_{\mathrm{In}} = E_B$ is the boundary energy input. In our example, the variation δE_B is

$$\delta E_B = F_0\delta u_x(x=0) + F_L\delta u_x(x=L) \tag{2.9.26}$$

In this case, we substitute Eqs. (2.9.19), (2.9.20), and (2.9.26) in Eq. (2.9.25) and obtain

$$\int_{t_1}^{t_2}\left\{\int_0^L\left[\frac{\partial}{\partial x}\left(EA\frac{\partial u_x}{\partial x}\right)-\rho A\frac{\partial^2 u_x}{\partial t^2}\right]\delta u_x\,\mathrm{d}x\right.$$
$$+\left[F_0+EA\frac{\partial u_x}{\partial x}(x=0)\right]\delta u_x(x=0)$$
$$\left.+\left[F_L-EA\frac{\partial u_x}{\partial x}(x=L)\right]\delta u_x(x=L)\right\}\mathrm{d}t=0 \tag{2.9.27}$$

Because δu_x is arbitrary and independent, the equation of motion of Eq. (2.9.24) follows. In addition, it must be that at $x=0$,

$$EA\frac{\partial u_x}{\partial x}=-F_0 \quad \text{or} \quad u_x=u_0 \tag{2.9.28}$$

and at $x=L$,

$$EA\frac{\partial u_x}{\partial x}=F_L \quad \text{or} \quad u_x=u_L \tag{2.9.29}$$

Equations (2.9.28) and (2.9.29) indicate that permissible boundary conditions for this problem are either to specify the force at a boundary or to specify a boundary displacement. These specified forces can, for example, be 0 (free boundary), they can be equal to $F_0=ku_0, F_L=-ku_L$ (linear grounded springs of rate k at each boundary), or the displacements can be specified; for example, they can be 0 (clamped boundary).

2.10. OTHER DEEP SHELL THEORIES

After Love's basic development, several investigators tried to improve on his theory. If we restrict ourselves to linear theories for thin shells, I confess to some prejudice in preferring, based on my experience, Love's theory to all others. The Timoshenko–Mindlin shear deformation approach for beams and plates, as extended to shells in Ref. Soedel (1982) for moderately thick shells, *thick* being a relative term as related to the characteristic "wave-length" of mode shapes, is discussed in a later chapter. The approximate differences between the various theories are discussed below [Leissa (1973)].

2.10.1. Strain–Displacement Considerations

In the theories of Novozhilov (1964); Flügge (1934); Byrne (1944), and Goldenveizer (1961), the simplification of Eq. (2.4.9) that $\alpha_3/R_1 \ll 1$ and $\alpha_3/R_2 \ll 1$ is not made. The strain–displacement relationships become

$$\varepsilon_{11} = \frac{1}{1+\alpha_3/R_1}(\varepsilon_{11}^0+\alpha_3 k_{11}) \quad (2.10.1)$$

$$\varepsilon_{22} = \frac{1}{1+\alpha_3/R_2}(\varepsilon_{22}^0+\alpha_3 k_{22}) \quad (2.10.2)$$

$$\varepsilon_{12} = \frac{1}{(1+\alpha_3/R_1)(1+\alpha_3/R_2)} \times\left[\left(1-\frac{\alpha_3^2}{R_1R_2}\right)\varepsilon_{12}^0+\alpha_3\left(1+\frac{\alpha_3}{2R_1}+\frac{\alpha_3}{2R_2}\right)k_{12}\right] \quad (2.10.3)$$

where $\varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{12}^0, k_{11}$ and k_{22} are the same as for the Love theory, but k_{12} is

$$k_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\beta_1}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\beta_2}{A_2}\right)+\frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u_1}{\partial\alpha_2}-\frac{u_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right) +\frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial\alpha_1}-\frac{u_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right) \quad (2.10.4)$$

The theories of Reissner (1941) and Naghdi and Berry (1964) are essentially equal to Love's original theory. Love originally expressed k_{12} as in Eq. (2.10.4). It was recognized by these investigators that the assumptions $\alpha_3/R_1 \ll 1$ and $\alpha_3/R_2 \ll 1$, consequentially applied, lead to

$$k_{12}=\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\beta_1}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\beta_2}{A_2}\right) \quad (2.10.5)$$

since the last two terms of Eq. (2.10.4), when substituted in Eq. (2.4.18), lead to terms of the α_3/R_1 and α_3/R_2 type.

2.10.2. Force and Moment Resultant Considerations

Love's theory (1888) and theories of Reissner (1941); Naghdi and Berry (1964); Sanders (1959) utilize the relationships of Eqs. (2.5.9)–(2.5.12) and Eqs. (2.5.15)–(2.5.18). In the theories by Flügge (1934), Byrne (1944), and Lur'ye (1940), quotients of the type $1/(1+\alpha_3/R_i)$, $i=1,2$, are replaced by a truncated geometric series. This gives

$$N_{11} = K\left[\varepsilon_{11}^0+\mu\varepsilon_{22}^0-\frac{h^2}{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(k_{11}-\frac{\varepsilon_{11}^0}{R_1}\right)\right] \quad (2.10.6)$$

$$N_{22} = K\left[\varepsilon_{11}^0 + \mu\varepsilon_{22}^0 - \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(k_{22} - \frac{\varepsilon_{22}^0}{R_2}\right)\right] \tag{2.10.7}$$

$$N_{12} = \frac{K(1-\mu)}{2}\left[\varepsilon_{12}^0 - \frac{h^2}{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{k_{12}}{2} - \frac{\varepsilon_{12}^0}{R_1}\right)\right] \tag{2.10.8}$$

$$N_{21} = \frac{K(1-\mu)}{2}\left[\varepsilon_{12}^0 - \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\frac{k_{12}}{2} - \frac{\varepsilon_{12}^0}{R_2}\right)\right] \tag{2.10.9}$$

$$M_{11} = D\left[k_{11} + \mu k_{22} - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\varepsilon_{11}^0\right] \tag{2.10.10}$$

$$M_{22} = D\left[k_{22} + \mu k_{11} - \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\varepsilon_{22}^0\right] \tag{2.10.11}$$

$$M_{12} = \frac{D(1-\mu)}{2}\left(k_{12} - \frac{\varepsilon_{12}^0}{R_1}\right) \tag{2.10.12}$$

$$M_{21} = \frac{D(1-\mu)}{2}\left(k_{12} - \frac{\varepsilon_{12}^0}{R_2}\right) \tag{2.10.13}$$

A theory by Vlasov (1964) uses similar idea, except that he also expanded the quotients $1/(1+\alpha_3/R_i)$ in terms of truncated geometric series for strain–displacement relationships. Basically, he obtained the same results as above, except for some difference in Eqs. (2.10.8), (2.10.9), (2.10.12), and (2.10.13):

$$N_{12} = \frac{K(1-\mu)}{2}\left[\varepsilon_{12}^0 - \frac{h^2}{24}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)k_{12}\right] \tag{2.10.14}$$

$$N_{21} = \frac{K(1-\mu)}{2}\left[\varepsilon_{12}^0 - \frac{h^2}{24}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)k_{12}\right] \tag{2.10.15}$$

$$M_{12} = \frac{D(1-\mu)}{2}\left(k_{12} + \frac{\varepsilon_{12}^0}{R_2}\right) \tag{2.10.16}$$

$$M_{21} = \frac{D(1-\mu)}{2}\left(k_{12} + \frac{\varepsilon_{12}^0}{R_1}\right) \tag{2.10.17}$$

Utilizing strain energy expressions and arguing about the permissibility of neglecting terms from the energy viewpoint, again employing truncated geometric expansions of quotients of the $1/(1+\alpha_3/R_i)$ type, Goldenveizer (1961), and Novozhilov (1964) obtain

$$N_{11} = K(\varepsilon_{11}^0 + \mu\varepsilon_{22}^0) \tag{2.10.18}$$

$$N_{22} = K(\varepsilon_{22}^0 + \mu\varepsilon_{11}^0) \tag{2.10.19}$$

$$N_{12} = \frac{K(1-\mu)}{2}\left(\varepsilon_{12}^0 + \frac{h^2}{12R_2}k_{12}\right) \tag{2.10.20}$$

$$N_{21} = \frac{K(1-\mu)}{2}\left(\varepsilon_{12}^0 + \frac{h^2}{12R_1}k_{12}\right) \tag{2.10.21}$$

$$M_{11} = D(k_{11}+\mu k_{22}) \tag{2.10.22}$$

$$M_{22} = D(k_{22}+\mu k_{11}) \tag{2.10.23}$$

$$M_{12} = M_{21} = \frac{D(1-\mu)}{2}k_{12} \tag{2.10.24}$$

In conclusion, the thin shell theories discussed here are basically all of the Love type. The basic differences are how the approximation is handled that α_3/R_1 and α_3/R_2 are small.

2.11. SHELLS OF NONUNIFORM THICKNESS

Equation (2.7.20)–(2.7.24) are also valid for shells of nonuniform thickness if the nonuniform thickness $h(\alpha_1,\alpha_2)$ is always halved by the reference surface. In this case, the membrane and bending stiffnesses become simply functions of location and we replace Eqs. (2.5.10) and (2.5.16) by

$$D = D(\alpha_1,\alpha_2) = \frac{Eh^3(\alpha_1,\alpha_2)}{12(1-\mu^2)} \tag{2.11.1}$$

$$K = K(\alpha_1,\alpha_2) = \frac{Eh(\alpha_1,\alpha_2)}{1-\mu^2} \tag{2.11.2}$$

Membrane force and moment resultants are now

$$M_{11} = D(\alpha_1,\alpha_2)(k_{11}+\mu k_{22}) \tag{2.11.3}$$

$$M_{22} = D(\alpha_1,\alpha_2)(k_{22}+\mu k_{11}) \tag{2.11.4}$$

$$M_{12} = \frac{D(\alpha_1,\alpha_2)(1-\mu)}{2}k_{12} \tag{2.11.5}$$

$$N_{11} = K(\alpha_1,\alpha_2)(\varepsilon_{11}^0+\mu\varepsilon_{22}^0) \tag{2.11.6}$$

$$N_{22} = K(\alpha_1,\alpha_2)(\varepsilon_{22}^0+\mu\varepsilon_{11}^0) \tag{2.11.7}$$

$$N_{12} = \frac{K(\alpha_1,\alpha_2)(1-\mu)}{2}\varepsilon_{12}^0 \tag{2.11.8}$$

and can be substituted into Eqs. (2.7.20)–(2.7.24) and boundary conditions (2.8.19)–(2.8.22).

If the nonuniform thickness is not symmetric with respect to the reference surface, the reference surface must either be redefined to satisfy the symmetry condition (which is often not feasible), or the development of Sec. 2.5 will have to be extended to the more general case where force and moment resultants are obtained by an unsymmetrical integration. For example, Eq. (2.5.8) would become, after substitution of Eq. (2.5.4),

$$N_{11} = \frac{E}{1-\mu^2} \int_{\alpha_3=-h_1(\alpha_1,\alpha_2)}^{\alpha_3=h_2(\alpha_1,\alpha_2)} [\varepsilon_{11}^0 + \mu\varepsilon_{22}^0 + \alpha_3(k_{11}+\mu k_{22})]\,\mathrm{d}\alpha_3 \qquad (2.11.9)$$

where

$$h_1(\alpha_1,\alpha_2)+h_2(\alpha_1,\alpha_2)=h(\alpha_1,\alpha_2) \qquad (2.11.10)$$

This results in

$$N_{11} = \frac{E(h_1+h_2)}{1-\mu^2}(\varepsilon_{11}^0+\mu\varepsilon_{22}^0) + \frac{E(h_2^2-h_1^2)}{2(1-\mu^2)}(k_{11}+\mu k_{22}) \qquad (2.11.11)$$

The other relationships are derived similarly:

$$N_{22} = \frac{E(h_2+h_1)}{1-\mu^2}(\varepsilon_{22}^0+\mu\varepsilon_{11}^0) + \frac{E(h_2^2-h_1^2)}{2(1-\mu^2)}(k_{22}+\mu k_{11}) \qquad (2.11.12)$$

$$N_{12} = \frac{E(h_1+h_2)}{2(1+\mu)}\varepsilon_{12}^0 + \frac{E(h_2^2-h_1^2)}{4(1+\mu)}k_{12} \qquad (2.11.13)$$

$$M_{11} = \frac{E(h_2^2-h_1^2)}{2(1-\mu^2)}(\varepsilon_{11}^0+\mu\varepsilon_{22}^0) + \frac{E(h_2^3+h_1^3)}{3(1-\mu^2)}(k_{11}+\mu k_{22}) \qquad (2.11.14)$$

$$M_{22} = \frac{E(h_2^2-h_1^2)}{2(1-\mu^2)}(\varepsilon_{22}^0+\mu\varepsilon_{11}^0) + \frac{E(h_2^3+h_1^3)}{3(1-\mu^2)}(k_{22}+\mu k_{11}) \qquad (2.11.15)$$

$$M_{12} = \frac{E(h_2^2-h_1^2)}{4(1+\mu)}\varepsilon_{12}^0 + \frac{E(h_2^3+h_1^3)}{6(1+\mu)}k_{12} \qquad (2.11.16)$$

Therefore, if the reference surface is not halfway between the inner and outer surfaces of the shell, there will be coupling between membrane force resultants and bending strains, or between bending moment resultants and membrane strains. However, Eqs. (2.7.20)–(2.7.24) will continue to be valid as long as the new definitions are used.

2.12. RADII OF CURVATURE

The radii of curvature can either be obtained by inspection or from identities (2.1.31) and (2.1.32), which give

$$R_1 = \frac{A_1^2}{\frac{\partial \bar{r}}{\partial \alpha_1} \cdot \frac{\partial \bar{n}}{\partial \alpha_1}} \qquad (2.12.1)$$

$$R_2 = \frac{A_2^2}{\frac{\partial \bar{r}}{\partial \alpha_2} \cdot \frac{\partial \bar{n}}{\partial \alpha_2}} \qquad (2.12.2)$$

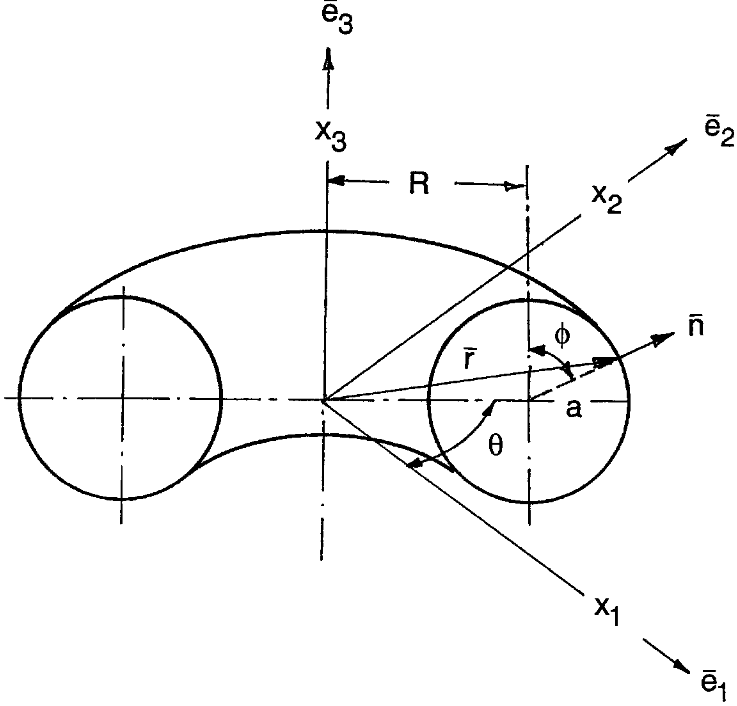

Fig. 12 The toroidal shell.

For example, for the circular cylindrical shell of Fig. 2 $\alpha_1 = x$ and $\alpha_2 = \theta$, and we obtain

$$\bar{n} = \sin\theta \bar{e}_3 + \cos\theta \bar{e}_2 \tag{2.12.3}$$

Thus,

$$\frac{\partial \bar{n}}{\partial \theta} = \cos\theta \bar{e}_3 - \sin\theta \bar{e}_2 \tag{2.12.4}$$

$$\frac{\partial \bar{n}}{\partial x} = 0 \tag{2.12.5}$$

We obtain

$$\frac{\partial \bar{r}}{\partial \theta} \cdot \frac{\partial \bar{n}}{\partial \theta} = (-a\sin\theta \bar{e}_2 + a\cos\theta \bar{e}_2) \cdot (\cos\theta \bar{e}_3 - a\sin\theta \bar{e}_2) \tag{2.12.6}$$

Since $A_2 = a$, Eq. (2.12.2) gives

$$R_\theta = a \tag{2.12.7}$$

Next, since

$$\frac{\partial \bar{r}}{\partial x} \cdot \frac{\partial \bar{n}}{\partial x} = 0 \tag{2.12.8}$$

and $A_1 = 1$, Eq. (2.12.1) gives

$$R_x = \infty \tag{2.12.9}$$

Let us take as second example the toroidal shell. We have

$$\bar{r} = (R + a\sin\phi)\cos\theta\bar{e}_1 + (R + a\sin\phi)\sin\theta\bar{e}_2 + a\cos\phi\bar{e}_3 \tag{2.12.10}$$

Thus

$$\frac{\partial \bar{r}}{\partial \theta} = -(R + a\sin\phi)\sin\theta\bar{e}_1 + (R + a\sin\phi)\cos\theta\bar{e}_2 \tag{2.12.11}$$

and

$$\left|\frac{\partial \bar{r}}{\partial \theta}\right| = \sqrt{\frac{\partial \bar{r}}{\partial \theta} \cdot \frac{\partial \bar{r}}{\partial \theta}} = (R + a\sin\phi)\sqrt{\sin^2\theta + \cos^2\theta} = A_1 = A_\theta \tag{2.12.12}$$

or

$$A_\theta = R + a\sin\phi \tag{2.12.13}$$

Next,

$$\frac{\partial \bar{r}}{\partial \theta} = a\cos\phi\cos\theta\bar{e}_1 + a\cos\phi\sin\theta\bar{e}_2 - a\sin\phi\bar{e}_3 \tag{2.12.14}$$

$$\left|\frac{\partial \bar{r}}{\partial \phi}\right| = \sqrt{\frac{\partial \bar{r}}{\partial \phi} \cdot \frac{\partial \bar{r}}{\partial \phi}} = \sqrt{a^2\cos^2\phi(\cos^2\theta + \sin^2\theta) + a^2\sin^2\phi} = a \tag{2.12.15}$$

Thus,

$$A_\phi = a \tag{2.12.16}$$

Next, we formulate $\bar{n}$:

$$\bar{n} = \sin\phi\cos\theta\bar{e}_1 + \sin\phi\sin\theta\bar{e}_2 + \cos\phi\bar{e}_3 \tag{2.12.17}$$

Thus,

$$\frac{\partial \bar{n}}{\partial \theta} = -\sin\phi\sin\theta\bar{e}_1 + \sin\phi\cos\theta\bar{e}_2 \tag{2.12.18}$$

and Eq. (2.12.1) gives

$$R_\theta = \frac{(R + a\sin\phi)^2}{(R + a\sin\phi)(\sin\phi\sin^2\theta + \sin\phi\cos^2\theta)} = \frac{R}{\sin\phi} + a \tag{2.12.19}$$

Next, since

$$\frac{\partial \bar{n}}{\partial \phi} = \cos\phi\cos\theta\,\bar{e}_1 + \cos\phi\sin\theta\,\bar{e}_2 - \sin\phi\,\bar{e}_3 \tag{2.12.20}$$

Equation (2.12.2) gives

$$R_\phi = \frac{a^2}{a\cos^2\phi\cos^2\theta + a\cos^2\phi\sin^2\theta + a\sin^2\phi} = a \tag{2.12.21}$$

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3

Equations of Motion for Commonly Occurring Geometries

In the following we derive the general shell-of-revolution equations by reduction from the general Love equations. The shell-of-revolution equations are then further reduced to specific cases, such as the conical shell and the circular cylindrical shell. Note that one can obtain the specific cases directly, without going through the general shell-of-revolution case, by direct substitution into Love's equations of the proper values for α_1, α_2, A_1, A_2, R_1, and R_2. For literature that uses reduction, see Kraus (1967), Nowacki (1963), Vlasov (1964), Novozhilov (1965) and Kilchevskiy (1965). For literature where equations for specific geometries are derived directly, see Flūgge (1932), Timoshenko and Woinowsky-Krieger (1959), and Donnell (1976).

3.1. SHELLS OF REVOLUTION

Consider a shell whose neutral surface is a surface of revolution. For such a shell, the lines of principal curvature are its meridians and its parallel circles, as shown in Fig. 1. Thus

$$\alpha_1 = \phi \tag{3.1.1}$$

$$\alpha_2 = \theta \tag{3.1.2}$$

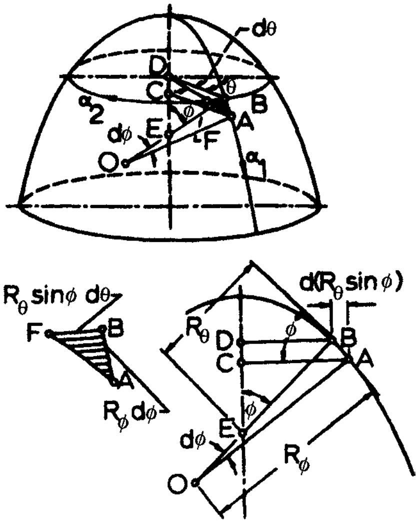

FIG. 1 Obtaining the Lamé parameters for a shell of revolution.

$$R_1 = R_\phi \tag{3.1.3}$$

$$R_2 = R_\theta \tag{3.1.4}$$

The infinitesimal distances $\overline{\mathrm{BA}}$ and $\overline{\mathrm{BF}}$ are

$$\overline{\mathrm{BA}} = R_\phi \mathrm{d}\phi \tag{3.1.5}$$

$$\overline{\mathrm{BF}} = R_\theta \sin\phi \mathrm{d}\theta \tag{3.1.6}$$

The fundamental form is therefore

$$(\mathrm{d}s)^2 = R_\phi^2(\mathrm{d}\phi)^2 + R_\theta^2 \sin^2\phi(\mathrm{d}\theta)^2 \tag{3.1.7}$$

and therefore

$$A_1 = R_\phi \tag{3.1.8}$$

$$A_2 = R_\theta \sin\phi \tag{3.1.9}$$

Inserting Eqs. (3.1.1)–(3.1.4) and Eqs. (3.1.8) and (3.1.9) in Eqs. (2.7.20) to (2.7.24) gives, with subscripts 1 and 2 changed to ϕ and θ, and with $R_\phi \cos\phi \mathrm{d}\phi = \mathrm{d}(R_\theta \sin\phi)$ from Fig. 1,

$$\frac{\partial}{\partial\phi}(N_{\phi\phi}R_\theta \sin\phi) + R_\phi \frac{\partial N_{\theta\phi}}{\partial\theta} - N_{\theta\theta}R_\phi \cos\phi + R_\phi R_\theta \sin\phi\left(\frac{Q_{\phi 3}}{R_\phi} + q_\phi\right)$$
$$= R_\phi R_\theta \sin\phi \rho h \frac{\partial^2 u_\phi}{\partial t^2} \tag{3.1.10}$$

$$\frac{\partial}{\partial\phi}(N_{\phi\theta}R_\theta\sin\phi)+R_\phi\frac{\partial N_{\theta\theta}}{\partial\theta}+N_{\theta\phi}R_\phi\cos\phi$$
$$+R_\phi R_\theta\sin\phi\left(\frac{Q_{\theta 3}}{R_\theta}+q_\theta\right)=R_\phi R_\theta\sin\phi\rho h\frac{\partial^2 u_\theta}{\partial t^2} \quad (3.1.11)$$

$$\frac{\partial}{\partial\phi}(Q_{\phi 3}R_\theta\sin\phi)+R_\phi\frac{\partial Q_{\theta 3}}{\partial\theta}-\left(\frac{N_{\phi\phi}}{R_\phi}+\frac{N_{\theta\theta}}{R_\theta}\right)R_\phi R_\theta\sin\phi$$
$$+q_3R_\phi R_\theta\sin\phi=R_\phi R_\theta\sin\phi\rho h\frac{\partial^2 u_3}{\partial t^2} \quad (3.1.12)$$

where

$$Q_{\phi 3}=\frac{1}{R_\phi R_\theta\sin\phi}\left[\frac{\partial}{\partial\phi}(M_{\phi\phi}R_\theta\sin\phi)+R_\phi\frac{\partial M_{\theta\phi}}{\partial\theta}-M_{\theta\theta}R_\phi\cos\phi\right] \quad (3.1.13)$$

$$Q_{\theta 3}=\frac{1}{R_\phi R_\theta\sin\phi}\left[\frac{\partial}{\partial\phi}(M_{\phi\theta}R_\theta\sin\phi)+R_\phi\frac{M_{\theta\theta}}{\partial\theta}+M_{\theta\phi}R_\phi\cos\phi\right] \quad (3.1.14)$$

The strain–displacement relations (2.4.19)–(2.4.24) become

$$\varepsilon^0_{\phi\phi}=\frac{1}{R_\phi}\left(\frac{\partial u_\phi}{\partial\phi}+u_3\right) \quad (3.1.15)$$

$$\varepsilon^0_{\theta\theta}=\frac{1}{R_\theta\sin\phi}\left(\frac{\partial u_\theta}{\partial\theta}+u_\phi\cos\phi+u_3\sin\phi\right) \quad (3.1.16)$$

$$\varepsilon^0_{\phi\theta}=\frac{R_\theta}{R_\phi}\sin\phi\frac{\partial}{\partial\phi}\left(\frac{u_\theta}{R_\theta\sin\phi}\right)+\frac{1}{R_\theta\sin\phi}\frac{\partial u_\phi}{\partial\theta} \quad (3.1.17)$$

$$k_{\phi\phi}=\frac{1}{R_\phi}\frac{\partial\beta_\phi}{\partial\phi} \quad (3.1.18)$$

$$k_{\theta\theta}=\frac{1}{R_\theta\sin\phi}\left(\frac{\partial\beta_\theta}{\partial\theta}+\beta_\phi\cos\phi\right) \quad (3.1.19)$$

$$k_{\phi\theta}=\frac{R_\theta}{R_\phi}\sin\phi\frac{\partial}{\partial\phi}\left(\frac{\beta_\theta}{R_\theta\sin\phi}\right)+\frac{1}{R_\theta\sin\phi}\frac{\partial\beta_\phi}{\partial\theta} \quad (3.1.20)$$

and β_ϕ and β_θ are, from Eqs. (2.4.7) and (2.4.8),

$$\beta_\phi=\frac{1}{R_\phi}\left(u_\phi-\frac{\partial u_3}{\partial\phi}\right) \quad (3.1.21)$$

$$\beta_\theta=\frac{1}{R_\theta\sin\phi}\left(u_\theta\sin\phi-\frac{\partial u_3}{\partial\theta}\right) \quad (3.1.22)$$

The relations between the force and moment resultants and the strains are given by Eqs. (2.5.9), (2.5.11), (2.5.12), (2.5.15), (2.5.17), and (2.5.18):

$$N_{\phi\phi} = K(\varepsilon^0_{\phi\phi} + \mu\varepsilon^0_{\theta\theta}) \tag{3.1.23}$$

$$N_{\theta\theta} = K(\varepsilon^0_{\theta\theta} + \mu\varepsilon^0_{\phi\phi}) \tag{3.1.24}$$

$$N_{\phi\theta} = N_{\theta\phi} = \frac{K(1-\mu)}{2}\varepsilon^0_{\phi\theta} \tag{3.1.25}$$

$$M_{\phi\phi} = D(k_{\phi\phi} + \mu k_{\theta\theta}) \tag{3.1.26}$$

$$M_{\theta\theta} = D(k_{\theta\theta} + \mu k_{\phi\phi}) \tag{3.1.27}$$

$$M_{\phi\theta} = M_{\theta\phi} = \frac{D(1-\mu)}{2}k_{\phi\theta} \tag{3.1.28}$$

3.2. CIRCULAR CONICAL SHELL

For the case of a circular conical shell, as shown in Fig. 2, we see that

$$\frac{1}{R_\phi} = 0 \tag{3.2.1}$$

$$R_\theta = x\tan\alpha \tag{3.2.2}$$

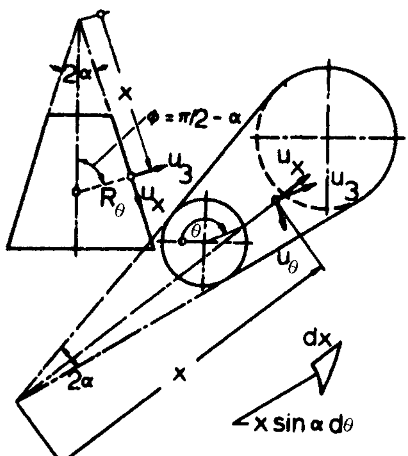

Fig. 2 Coordinate definitions for a conical shell.

and since

$$\phi = \frac{\pi}{2} - \alpha \tag{3.2.3}$$

we get

$$\sin\phi = \cos\alpha \tag{3.2.4}$$

$$\cos\phi = \sin\alpha \tag{3.2.5}$$

Furthermore, the fundamental form is now

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + x^2 \sin^2\alpha (\mathrm{d}\theta)^2 \tag{3.2.6}$$

Comparing this to Eq. (3.1.7), we obtain

$$R_\phi \mathrm{d}\phi = \mathrm{d}x \tag{3.2.7}$$

and

$$R_\theta \sin\phi = x\sin\alpha \tag{3.2.8}$$

The subscript ϕ is replaced by x and Eqs. (3.1.10)–(3.1.14) read

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{x\sin\alpha}\frac{\partial N_{\theta x}}{\partial\theta} + \frac{1}{x}(N_{xx} - N_{\theta\theta}) + q_x = \rho h \frac{\partial^2 u_x}{\partial t^2} \tag{3.2.9}$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{2}{x}N_{\theta x} + \frac{1}{x\sin\alpha}\frac{\partial N_{\theta\theta}}{\partial\theta} + \frac{1}{x\tan\alpha}Q_{\theta 3} + q_\theta = \rho h \frac{\partial^2 u_\theta}{\partial t^2} \tag{3.2.10}$$

$$\frac{\partial Q_{x3}}{\partial x} + \frac{1}{x}Q_{x3} + \frac{1}{x\sin\alpha}\frac{\partial Q_{\theta 3}}{\partial\theta} - \frac{1}{x\tan\alpha}N_{\theta\theta} + q_3 = \rho h \frac{\partial^2 u_3}{\partial t^2} \tag{3.2.11}$$

where

$$Q_{x3} = \frac{\partial M_{xx}}{\partial x} + \frac{M_{xx}}{x} + \frac{1}{x\sin\alpha}\frac{\partial M_{\theta x}}{\partial\theta} - \frac{M_{\theta\theta}}{x} \tag{3.2.12}$$

$$Q_{\theta 3} = \frac{\partial M_{x\theta}}{\partial x} + \frac{2}{x}M_{\theta x} + \frac{1}{x\sin\alpha}\frac{\partial M_{\theta\theta}}{\partial\theta} \tag{3.2.13}$$

The strain–displacement relations become

$$\varepsilon_{xx}^0 = \frac{\partial u_x}{\partial x} \tag{3.2.14}$$

$$\varepsilon_{\theta\theta}^0 = \frac{1}{x\sin\alpha}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{x}u_x + \frac{1}{x\tan\alpha}u_3 \tag{3.2.15}$$

$$\varepsilon_{x\theta}^0 = \frac{\partial u_\theta}{\partial x} - \frac{1}{x}u_\theta + \frac{1}{x\sin\alpha}\frac{\partial u_x}{\partial\theta} \tag{3.2.16}$$

$$k_{xx} = \frac{\partial\beta_x}{\partial x} \tag{3.2.17}$$

$$k_{\theta\theta} = \frac{1}{x\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta} + \frac{1}{x}\beta_x \tag{3.2.18}$$

$$k_{x\theta} = x\frac{\partial}{\partial x}\left(\frac{\beta_\theta}{x}\right) + \frac{1}{x\sin\alpha}\frac{\partial \beta_x}{\partial \theta} \tag{3.2.19}$$

and β_1 and β_2 become

$$\beta_x = -\frac{\partial u_3}{\partial x} \tag{3.2.20}$$

$$\beta_\theta = \frac{1}{x\tan\alpha}u_\theta - \frac{1}{x\sin\alpha}\frac{\partial u_3}{\partial \theta} \tag{3.2.21}$$

Note that as $\alpha \to 0$, a circular cylindrical shell results, with

$$x\tan\alpha \to a, \quad \sin\alpha \to 0, \quad \cos\alpha \to 1$$

As $\alpha \to \pi/2$, we approach the circular plate equations.

3.3. CIRCULAR CYLINDRICAL SHELL

An important subcase of the circular conical shell is the circular cylindrical shell (Fig. 3), which has the fundamental form

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + a^2(\mathrm{d}\theta)^2 \tag{3.3.1}$$

Letting α approach zero with $x\sin\alpha$ and $x\tan\alpha$ approaching a and $1/x$ approaching zero, from Eqs. (3.2.9)–(3.2.13) we have

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{a}\frac{\partial N_{\theta x}}{\partial \theta} + q_x = \rho h\frac{\partial^2 u_x}{\partial t^2} \tag{3.3.2}$$

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{a}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{Q_{\theta 3}}{a} + q_\theta = \rho h\frac{\partial^2 u_\theta}{\partial t^2} \tag{3.3.3}$$

$$\frac{\partial Q_{x3}}{\partial x} + \frac{1}{a}\frac{\partial Q_{\theta 3}}{\partial \theta} - \frac{N_{\theta\theta}}{a} + q_3 = \rho h\frac{\partial^2 u_3}{\partial t^2} \tag{3.3.4}$$

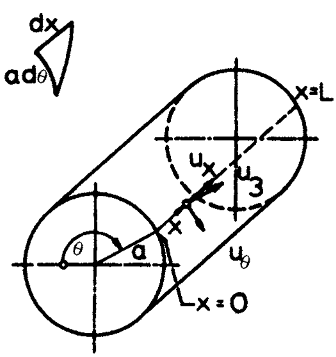

Fig. 3 Coordinate definitions for a circular cylindrical shell.

where

$$Q_{x3} = \frac{\partial M_{xx}}{\partial x} + \frac{1}{a}\frac{\partial M_{\theta x}}{\partial \theta} \tag{3.3.5}$$

$$Q_{\theta 3} = \frac{\partial M_{x\theta}}{\partial x} + \frac{1}{a}\frac{\partial M_{\theta\theta}}{\partial \theta} \tag{3.3.6}$$

The strain–displacement relations become

$$\varepsilon_{xx}^0 = \frac{\partial u_x}{\partial x} \tag{3.3.7}$$

$$\varepsilon_{\theta\theta}^0 = \frac{1}{a}\frac{\partial u_\theta}{\partial \theta} + \frac{u_3}{a} \tag{3.3.8}$$

$$\varepsilon_{x\theta}^0 = \frac{\partial u_\theta}{\partial x} + \frac{1}{a}\frac{\partial u_x}{\partial \theta} \tag{3.3.9}$$

$$k_{xx} = \frac{\partial \beta_x}{\partial x} \tag{3.3.10}$$

$$k_{\theta\theta} = \frac{1}{a}\frac{\partial \beta_\theta}{\partial \theta} \tag{3.3.11}$$

$$k_{x\theta} = \frac{\partial \beta_\theta}{\partial x} + \frac{1}{a}\frac{\partial \beta_x}{\partial \theta} \tag{3.3.12}$$

and β_1 and β_2 become

$$\beta_x = -\frac{\partial u_3}{\partial x} \tag{3.3.13}$$

$$\beta_\theta = \frac{u_\theta}{a} - \frac{1}{a}\frac{\partial u_3}{\partial \theta} \tag{3.3.14}$$

3.4. SPHERICAL SHELL

In this particular case, from Fig. 4,

$$R_\phi = a \tag{3.4.1}$$

$$R_\theta = a \tag{3.4.2}$$

and the fundamental form becomes

$$(\mathrm{d}s)^2 = a^2(\mathrm{d}\phi)^2 + a^2\sin^2\phi(\mathrm{d}\theta)^2 \tag{3.4.3}$$

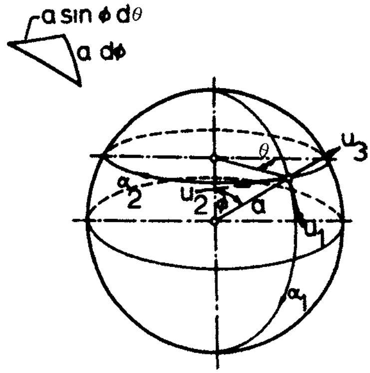

Fig. 4 Coordinate definitions for a spherical shell.

From Eqs. (3.1.10)–(3.1.14) we get

$$\frac{\partial}{\partial\phi}(N_{\phi\phi}\sin\phi)+\frac{\partial N_{\theta\phi}}{\partial\theta}-N_{\theta\theta}\cos\phi+Q_{\phi 3}\sin\phi+aq_{\phi}\sin\phi$$
$$=a\sin\phi\rho h\frac{\partial^2 u_\phi}{\partial t^2} \tag{3.4.4}$$

$$\frac{\partial}{\partial\phi}(N_{\phi\theta}\sin\phi)+\frac{\partial N_{\theta\theta}}{\partial\theta}+N_{\theta\phi}\cos\phi+Q_{\theta 3}\sin\phi+aq_{\theta}\sin\phi$$
$$=a\sin\phi\rho h\frac{\partial^2 u_\theta}{\partial t^2} \tag{3.4.5}$$

$$\frac{\partial}{\partial\phi}(Q_{\phi 3}\sin\phi)+\frac{\partial Q_{\theta 3}}{\partial\theta}-(N_{\phi\phi}+N_{\theta\theta})\sin\phi+aq_3\sin\phi$$
$$=a\sin\phi\rho h\frac{\partial^2 u_3}{\partial t^2} \tag{3.4.6}$$

and

$$Q_{\phi 3}=\frac{1}{a\sin\phi}\left[\frac{\partial}{\partial\phi}(M_{\phi\phi}\sin\phi)+\frac{\partial M_{\theta\phi}}{\partial\theta}-M_{\theta\theta}\cos\phi\right] \tag{3.4.7}$$

$$Q_{\theta 3}=\frac{1}{a\sin\phi}\left[\frac{\partial}{\partial\phi}(M_{\phi\theta}\sin\phi)+\frac{\partial M_{\theta\theta}}{\partial\theta}+M_{\theta\phi}\cos\phi\right] \tag{3.4.8}$$

The strain–displacement relations become

$$\varepsilon^0_{\phi\phi}=\frac{1}{a}\left(\frac{\partial u_\phi}{\partial\phi}+u_3\right) \tag{3.4.9}$$

$$\varepsilon^0_{\theta\theta}=\frac{1}{a\sin\phi}\left(\frac{\partial u_\theta}{\partial\theta}+u_\phi\cos\phi+u_3\sin\phi\right) \tag{3.4.10}$$

$$\varepsilon^0_{\phi\theta} = \frac{1}{a}\left(\frac{\partial u_\theta}{\partial \theta} - u_\theta \cot\phi + \frac{1}{\sin\phi}\frac{\partial u_\phi}{\partial \theta}\right) \tag{3.4.11}$$

$$k_{\phi\phi} = \frac{1}{a}\frac{\partial \beta_\phi}{\partial \phi} \tag{3.4.12}$$

$$k_{\theta\theta} = \frac{1}{a}\left(\frac{1}{\sin\phi}\frac{\partial \beta_\theta}{\partial \theta} + \beta_\phi \cot\phi\right) \tag{3.4.13}$$

$$k_{\phi\theta} = \frac{1}{a}\left(\frac{\partial \beta_\theta}{\partial \phi} - \beta_\theta \cot\phi + \frac{1}{\sin\phi}\frac{\partial \beta_\phi}{\partial \theta}\right) \tag{3.4.14}$$

and where

$$\beta_\phi = \frac{1}{a}\left(u_\phi - \frac{\partial u_3}{\partial \phi}\right) \tag{3.4.15}$$

$$\beta_\theta = \frac{1}{a}\left(u_\theta - \frac{1}{\sin\phi}\frac{\partial u_3}{\partial \theta}\right) \tag{3.4.16}$$

3.5. OTHER GEOMETRIES

3.5.1. Toroidal Shell of Circular Cross-Section

Toroidal shells of noncircular cross-section are described by Eqs. (3.1.10)–(3.1.22), because toroidal shells are shells of revolution. For the special case of a circular cross-section, we may simplify the equations. Using $\alpha_1 = \theta$ and $\alpha_2 = \phi$, as shown in Fig. 5, results in the fundamental form

$$(\mathrm{d}s)^2 = (R + a\sin\phi)^2(\mathrm{d}\theta)^2 + a^2(\mathrm{d}\phi)^2 \tag{3.5.1}$$

Therefore,

$$A_1 = A_\theta = R + a\sin\phi, \quad A_2 = A_\phi = a \tag{3.5.2}$$

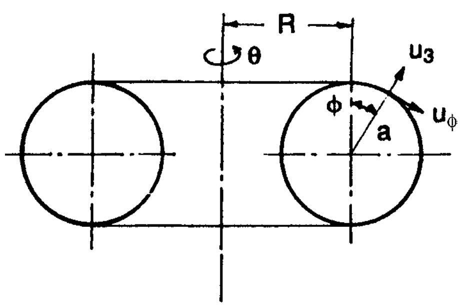

Fig. 5 Coordinate definitions for a toroidal shell.

The radii of curvature are

$$R_1 = R_\theta = \frac{R}{\sin\phi} + a \tag{3.5.3}$$

$$R_2 = R_\phi = a \tag{3.5.4}$$

These values may now be substituted into Eqs. (2.7.20)–(2.7.24) and the associated strain–displacement relationships (2.4.19)–(2.4.24). Or, we may utilize Eqs. (3.1.10)–(3.1.22) directly, in which case we need to substitute only Eqs. (3.5.3) and (3.5.4).

3.5.2. Cylindrical Shells of Noncircular Cross-Section

For cylindrical shells of noncircular cross-section, the set of orthogonal coordinates consists of straight axial lines ($\alpha_1 = x$) and lines normal to them ($\alpha_2 = s$), as shown in Fig. 6. We establish the origin of s at a convenient point and define distances from it. Designating the fundamental form diagonal as ds', we have

$$(\mathrm{d}s')^2 = (\mathrm{d}x)^2 + (\mathrm{d}s)^2 \tag{3.5.5}$$

and therefore

$$A_1 = A_x = 1, \quad A_2 = A_s = 1 \tag{3.5.6}$$

The radii of curvature are

$$R_1 = R_x = \infty, \quad R_2 = R_s \tag{3.5.7}$$

where R_s is defined by the geometry of the noncircular cross-section.

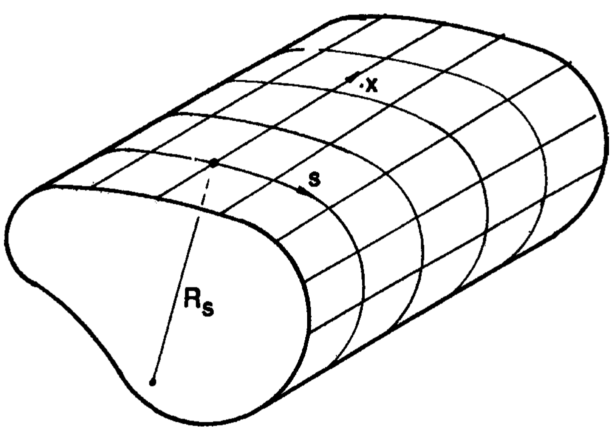

FIG. 6 Coordinate definitions for a cylindrical shell of noncircular cross-section.

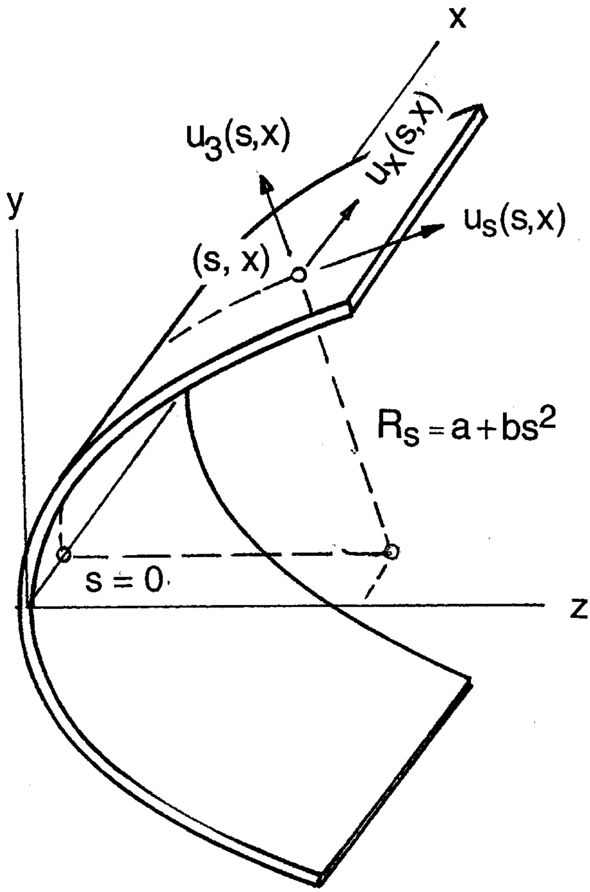

FIG. 7 Case where the base of a cylindrical shell is defined by the circumferential radius of curvature.

For example, let us consider a parabolic cylindrical shell as in Fig. 7. The surface coordinate s is measured from the apex of the parabola, which lies on the x-axis. The radius of curvature, and thus the parabola, is defined by

$$R_s = a + bs^2 \tag{3.5.8}$$

where a is the radius at the apex of the parabola and b is a constant that has to be fitted to the design of interest. Therefore, $\alpha_1 = x$, $\alpha_2 = s$, $A_1 = 1$,

$A_2=1$, $R_1=\infty$ and $R_2=R_s=a+bs^2$. Substituting this into Eqs. (2.7.20)–(2.7.24) gives

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{sx}}{\partial s}+q_x=\rho h\frac{\partial^2 u_x}{\partial t^2} \tag{3.5.9}$$

$$\frac{\partial N_{xs}}{\partial x}+\frac{\partial N_{ss}}{\partial s}+\frac{Q_{s3}}{a+bs^2}+q_s=\rho h\frac{\partial^2 u_s}{\partial t^2} \tag{3.5.10}$$

$$\frac{\partial Q_{x3}}{\partial x}+\frac{\partial Q_{s3}}{\partial s}-\frac{N_{ss}}{a+bs^2}+q_3=\rho h\frac{\partial^2 u_3}{\partial t^2} \tag{3.5.11}$$

$$Q_{x3}=\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{sx}}{\partial s} \tag{3.5.12}$$

$$Q_{s3}=\frac{\partial M_{xs}}{\partial x}+\frac{\partial M_{ss}}{\partial s} \tag{3.5.13}$$

Equations (2.4.19)–(2.4.24) become

$$\varepsilon^0_{xx}=\frac{\partial u_x}{\partial x} \tag{3.5.14}$$

$$\varepsilon^0_{ss}=\frac{\partial u_s}{\partial s}+\frac{u_3}{a+bs^2} \tag{3.5.15}$$

$$\varepsilon^0_{xs}=\frac{\partial u_s}{\partial x}+\frac{\partial u_x}{\partial s} \tag{3.5.16}$$

$$k_{xx}=\frac{\partial \beta_x}{\partial x} \tag{3.5.17}$$

$$k_{ss}=\frac{\partial \beta_s}{\partial s} \tag{3.5.18}$$

$$k_{xs}=\frac{\partial \beta_s}{\partial s}+\frac{\partial \beta_x}{\partial x} \tag{3.5.19}$$

where, from Eqs. (2.4.7) and (2.4.8),

$$\beta_x=-\frac{\partial u_3}{\partial x} \tag{3.5.20}$$

$$\beta_s=\frac{u_s}{a+bs^2}-\frac{\partial u_3}{\partial s} \tag{3.5.21}$$

Note that if we set $b=0$, and $s=a\theta$, Eqs. (3.5.9)–(3.5.21) reduce to Eqs. (3.3.2)–(3.3.14), the equations for a circular, cylindrical shell, as one would expect from the definition (3.5.3) of R_s.

3.5.3. Remarks on Curvature

The definition of what a positive radius of curvature is (Chapter 2) must be retained. This means that after the direction of positive transverse displacement has been selected, an observer viewing the geometry of the undeflected shell in direction of the positive transverse displacement will perceive the shell along a given coordinate line as concave if it has a positive radius of curvature at this point. A good illustration is the example above of the toroidal shell of circular cross-section. As described by Eq. (3.5.3), R_θ will be positive for $\phi = 0$ to π and negative for $\phi = \pi$ to 2π. This agrees with the definition, because an observer stationed at the center of the circle of radius a in Fig. 5 will perceive the shell in the direction of the θ coordinate as concave between $\phi = 0$ and π and as convex between $\phi = \pi$ and 2π. (This is not to be confused with the curvature in the ϕ coordinate direction, which will always appear concave to the observer, as Eq. (3.5.4) indicates.) When trying to visualize this, it helps to imagine spheres of radius R_θ, as described by Eq. (3.5.3), touching the toroidal shell at the θ lines of interest.

It will be shown in later chapters that the natural frequencies of shells are highly sensitive to curvature. Therefore, in the opinion of this author, the frequently reported practical problems with finite element programs giving unsatisfactory predictions are often related to an inaccurate determination of the radii of curvature. This is especially true if the radii of curvature are obtained by numerical differentiation of the reference surface if the references surface is given by discrete points. Small errors, magnified by the differentiation, will in effect create corrugated surfaces. Thus spline fits should be used whenever possible.

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4

Nonshell Structures

In the following discussion, we treat rings and beams as special cases of arches. The arch equation is derived by reduction from Love's equations for shells. Also by direct reduction from Love's equation, we obtain the plate equation. In literature, reduction is usually not used for these relatively simple structures, and each special case is derived from basic principles (Timoshenko, 1955; Biezeno and Grammel, 1954; Thomson, 1972; Meirovitch, 1972)

4.1. ARCH

The arch is a curved beam where all curvature is in one plane only, as shown in Fig. 1. Vibratory motion is assumed to occur only in that plane. Designating s as the coordinate along the neutral axis of the arch and y as the coordinate perpendicular to the neutral axis, the fundamental form becomes

$$(ds')^2 = (ds)^2 + (dy)^2 \tag{4.1.1}$$

where (ds') is the fundamental form diagonal to avoid confusion with (ds).

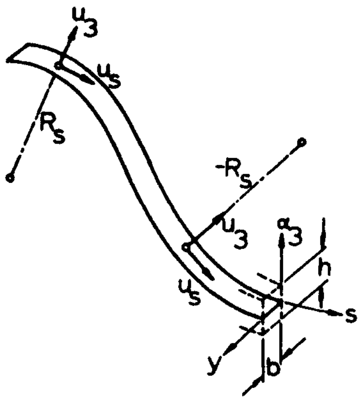

Fig. 1 Coordinate definitions for an arch.

Thus $A_1 = 1, A_2 = 1, d\alpha_1 = ds$, and $d\alpha_2 = dy$. Furthermore,

$$R_1 = R_s \tag{4.1.2}$$

$$\frac{1}{R_2} = \frac{1}{R_y} = 0 \tag{4.1.3}$$

and

$$\frac{\partial(\cdot)}{\partial\alpha_2} = \frac{\partial(\cdot)}{\partial y} = 0 \tag{4.1.4}$$

Also, stresses on the arch in the y direction can be assumed to be 0. From the fact that there are no deflections in the y direction, shear stresses are 0

$$N_{yy} = 0, \quad M_{yy} = 0$$

$$N_{ys} = N_{sy} = 0, \quad M_{ys} = M_{sy} = 0 \tag{4.1.5}$$

Love's equations become

$$\frac{\partial N_{ss}}{\partial s} + \frac{Q_{s3}}{R_s} + q_s = \rho h \frac{\partial^2 u_s}{\partial t^2} \tag{4.1.6}$$

$$\frac{\partial Q_{s3}}{\partial s} - \frac{N_{ss}}{R_s} + q_3 = \rho h \frac{\partial^2 u_3}{\partial t^2} \tag{4.1.7}$$

where

$$Q_{s3} = \frac{\partial M_{ss}}{\partial s} \tag{4.1.8}$$

The strain–displacement equations become

$$e_{ss}^0 = \frac{\partial u_s}{\partial s} + \frac{u_3}{R_s} \tag{4.1.9}$$

$$k_{ss} = \frac{\partial \beta_s}{\partial s} \tag{4.1.10}$$

where

$$\beta_s = \frac{u_s}{R_s} - \frac{\partial u_3}{\partial s} \tag{4.1.11}$$

While in an approximate sense we could proceed using Eqs. (2.5.9) and (2.5.15), it is more appropriate to start with the basic fact that $\sigma_{yy} = 0$, and thus

$$\varepsilon_{ss} = \frac{1}{E}(\sigma_{ss} - \mu\sigma_{yy}) = \frac{\sigma_{ss}}{E} \tag{4.1.12}$$

Also utilizing Eq. (2.5.21), we obtain

$$\varepsilon_{ss} = \frac{\sigma_{ss}}{E} = \frac{N_{ss}}{Eh} + \frac{12}{Eh^3} M_{ss}\alpha_3 \tag{4.1.13}$$

Therefore,

$$\varepsilon^0_{ss} = \frac{N_{ss}}{Eh}, \quad N_{ss} = Eh\varepsilon^0_{ss} \tag{4.1.14}$$

and

$$k_{ss} = \frac{12}{Eh^3} M_{ss}, \quad M_{ss} = \frac{Eh^3}{12} k_{ss} \tag{4.1.15}$$

The membrane stiffness $K = Eh/(1-\mu^2)$ reduces, therefore, to Eh, and the bending stiffness $D = Eh^3/[12(1-\mu^2)]$ reduces to $Eh^3/12$. The reason is that by isolating an arch strip from the adjoining material of a cylindrical shell, we have removed the restraining effect caused by Poisson's ratio.

Proceeding with the definition of N_{ss} and M_{ss}, we obtain

$$N_{ss} = Eh\left(\frac{\partial u_s}{\partial s} + \frac{u_3}{R_s}\right) \tag{4.1.16}$$

$$M_{ss} = \frac{Eh^3}{12}\frac{\partial}{\partial s}\left(\frac{u_s}{R_s} - \frac{\partial u_3}{\partial s}\right) \tag{4.1.17}$$

Inserting this in Eqs. (4.1.6)–(4.1.8) and multiplying through by the width b gives

$$\frac{EI}{R_s}\left[\frac{\partial^2}{\partial s^2}\left(\frac{u_s}{R_s}\right) - \frac{\partial^3 u_3}{\partial s^3}\right] + EA\left[\frac{\partial^2 u_s}{\partial s^2} + \frac{\partial}{\partial s}\left(\frac{u_3}{R_s}\right)\right] + q'_s = \rho A \frac{\partial^2 u_s}{\partial t^2} \tag{4.1.18}$$

$$EI\left[\frac{\partial^3}{\partial s^3}\left(\frac{u_s}{R_s}\right) - \frac{\partial^4 u_3}{\partial s^4}\right] - \frac{EA}{R_s}\left(\frac{\partial u_s}{\partial s} + \frac{u_3}{R_s}\right) + q'_3 = \rho A \frac{\partial^2 u_3}{\partial t^2} \tag{4.1.19}$$

where, strictly speaking, the area moment $I = bh^3/12$ and the cross-sectional area $A = hb$ apply to a rectangular cross-section. At least, this is the cross-section we obtain when we isolate an arch strip from a cylindrical shell of the same shape as the arch. It can be shown, however, that any cross-sectional shape where the shear center coincides with the area center is described by these two equations as long as the appropriate I and A values are introduced. The values $q'_s = q_s b$ and $q'_3 = q_3 b$, by the way, are now forces per unit length (N/m).

Instead of four boundary conditions, we now need only three at each edge. Examining Eqs. (2.8.11)–(2.8.14), we obtain the following necessary boundary conditions that are to be specified at each end of the arch:

$$N_{ss} = N^*_{ss} \text{ or } u_s = u^*_s \tag{4.1.20}$$

$$M_{ss} = M^*_{ss} \text{ or } \beta_s = \beta^*_s \tag{4.1.21}$$

$$Q_{s3} = Q^*_{s3} \text{ or } u_3 = u^*_3 \tag{4.1.22}$$

4.2. BEAM AND ROD

For a thin straight beam

$$\frac{1}{R_s} = 0 \tag{4.2.1}$$

Therefore, Eqs. (4.1.18) and (4.1.19) reduce to

$$EA\frac{\partial^2 u_x}{\partial x^2} + q'_x = \rho A\frac{\partial^2 u_x}{\partial t^2} \tag{4.2.2}$$

$$-EI\frac{\partial^4 u_3}{\partial x^4} + q'_3 = \rho A\frac{\partial^2 u_3}{\partial t^2} \tag{4.2.3}$$

Note that Eqs. (4.2.2) and (4.2.3) are independent of each other. Equation (4.2.2) describes longitudinal vibrations along the axis (commonly called *rod vibrations*), and Eq. (4.2.3) describes vibrations transverse to the beam neutral axis. For the rod vibration of Eq. (4.2.2), we have to specify one boundary condition at each end of the rod:

$$N_{ss} = N^*_{ss} \text{ or } u_3 = u^*_s \tag{4.2.4}$$

For the transverse vibration equation of the beam, we must specify two boundary conditions at each end:

$$M_{ss} = M^*_{ss} \text{ or } \frac{\partial u_3}{\partial s} = \left(\frac{\partial u_3}{\partial s}\right)^* \tag{4.2.5}$$

$$Q_{s3} = Q^*_{s3} \text{ or } u_3 = u^*_3 \tag{4.2.6}$$

More specifically, the boundary conditions become for the longitudinal case

$$EA\frac{\partial u_x}{\partial x} = N^*_x \text{ or } u_x = u^*_x \tag{4.2.7}$$

where $N_x^* = bN_{xx}^*$(N). For the transverse vibration case, we have

$$-EI\frac{\partial^2 u_3}{\partial x^2} = M_x^* \text{ or } \frac{\partial u_3}{\partial x} = \left(\frac{\partial u_3}{\partial x}\right)^* \tag{4.2.8}$$

$$-EI\frac{\partial^3 u_3}{\partial x^3} = Q_x^* \text{ or } u_3 = u_3^* \tag{4.2.9}$$

where $M_x^* = bM_{xx}^*$ (N m) and $Q_x^* = bQ_{x3}^*$(N).

4.3. CIRCULAR RING

For a circular ring, the radius of arch curvature is constant (Fig. 2),

$$R_s = a \tag{4.3.1}$$

and the coordinate s is commonly expressed as

$$s = a\theta \tag{4.3.2}$$

Therefore, Eqs. (4.1.18) and (4.1.19) reduce to

$$\frac{EI}{a^4}\left(\frac{\partial^2 u_\theta}{\partial \theta^2} - \frac{\partial^3 u_3}{\partial \theta^3}\right) + \frac{EA}{a^2}\left(\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial u_3}{\partial \theta}\right) + q'_\theta = \rho A\frac{\partial^2 u_\theta}{\partial t^2} \tag{4.3.3}$$

$$\frac{EI}{a^4}\left(\frac{\partial^3 u_\theta}{\partial \theta^3} - \frac{\partial^4 u_3}{\partial \theta^4}\right) - \frac{EA}{a^2}\left(\frac{\partial u_\theta}{\partial \theta} + u_3\right) + q'_3 = \rho A\frac{\partial^2 u_3}{\partial t^2} \tag{4.3.4}$$

If there is no circumferential forcing and if the circumferential inertia term can be assumed to be negligible, it is possible to eliminate u_θ and obtain Prescott's equation (1924):

$$\frac{EI}{a^4}\left(\frac{\partial^4 u_3}{\partial \theta^4} + 2\frac{\partial^2 u_3}{\partial \theta^2} + u_3\right) + \rho A\frac{\partial^2 u_3}{\partial t^2} = q_3 \tag{4.3.5}$$

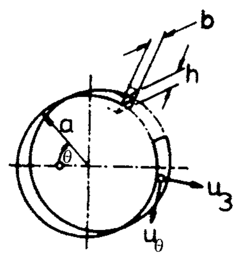

Fig. 2 Circular ring.

4.4. PLATE

A plate is a shell of zero curvatures. Thus

$$\frac{1}{R_1}=0 \tag{4.4.1}$$

$$\frac{1}{R_2}=0 \tag{4.4.2}$$

Love's equations become, after substituting Eqs. (4.4.1) and (4.4.2),

$$-\frac{\partial(N_{11}A_2)}{\partial\alpha_1}-\frac{\partial(N_{21}A_1)}{\partial\alpha_2}-N_{12}\frac{\partial A_1}{\partial\alpha_2}+N_{22}\frac{\partial A_2}{\partial\alpha_1}+A_1A_2\rho h\ddot{u}_1 =A_1A_2q_l \tag{4.4.3}$$

$$-\frac{\partial(N_{12}A_2)}{\partial\alpha_1}-\frac{(\partial N_{22}A_1)}{\partial\alpha_2}-N_{21}\frac{\partial A_2}{\partial\alpha_1}+N_{11}\frac{\partial A_1}{\partial\alpha_2}+A_1A_2\rho h\ddot{u}_2 =A_1A_2q_2 \tag{4.4.4}$$

and uncoupled from these two equations,

$$-\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}+A_1A_2\rho h\ddot{u}_3=A_1A_2q_3 \tag{4.4.5}$$

where

$$Q_{13}=\frac{1}{A_1A_2}\left[\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(M_{21}A_1)}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}\right] \tag{4.4.6}$$

$$Q_{23}=\frac{1}{A_1A_2}\left[\frac{\partial(M_{12}A_2)}{\partial\alpha_1}+\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}\right] \tag{4.4.7}$$

Equation (4.4.5) describes the transverse vibrations of plates. Equations (4.4.3) and (4.4.4) describe the in-plane oscillations. For small amplitudes of vibration, these oscillations are independent from each other.

The strain-displacement relationships of Eqs. (2.4.19)–(2.4.21) become

$$\varepsilon_{11}^0 = \frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1}+\frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} \tag{4.4.8}$$

$$\varepsilon_{22}^0 = \frac{1}{A_2}\frac{\partial u_2}{\partial\alpha_2}+\frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} \tag{4.4.9}$$

$$\varepsilon_{12}^0 = \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2}{A_2}\right)+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_1}{A_1}\right) \tag{4.4.10}$$

Since

$$\beta_1 = -\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1} \tag{4.4.11}$$

$$\beta_2 = -\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2} \tag{4.4.12}$$

we get for Eqs. (2.4.22)–(2.4.24),

$$k_{11} = -\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}\right) - \frac{1}{A_1 A_2^2}\frac{\partial u_3}{\partial \alpha_2}\frac{\partial A_1}{\partial \alpha_2} \tag{4.4.13}$$

$$k_{22} = -\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}\right) - \frac{1}{A_1^2 A_2}\frac{\partial u_3}{\partial \alpha_1}\frac{\partial A_2}{\partial \alpha_1} \tag{4.4.14}$$

$$k_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_2^2}\frac{\partial u_3}{\partial \alpha_2}\right) - \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_1^2}\frac{\partial u_3}{\partial \alpha_1}\right) \tag{4.4.15}$$

Inserting Eqs. (4.4.13)–(4.4.15) in Eqs. (2.5.15), (2.5.17) and (2.5.18) yields

$$\begin{aligned} M_{11} &= -D\left\{\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}\right) + \frac{1}{A_1 A_2^2}\frac{\partial u_3}{\partial \alpha_2}\frac{\partial A_1}{\partial \alpha_2}\right. \\ &\quad \left. + \mu\left[\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}\right) + \frac{1}{A_1^2 A_2}\frac{\partial u_3}{\partial \alpha_1}\frac{\partial A_2}{\partial \alpha_1}\right]\right\} \end{aligned} \tag{4.4.16}$$

$$\begin{aligned} M_{22} &= -D\left\{\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}\right) + \frac{1}{A_1^2 A_2}\frac{\partial u_3}{\partial \alpha_1}\frac{\partial A_2}{\partial \alpha_1}\right. \\ &\quad \left. + \mu\left[\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}\right) + \frac{1}{A_1 A_2^2}\frac{\partial u_3}{\partial \alpha_2}\frac{\partial A_1}{\partial \alpha_2}\right]\right\} \end{aligned} \tag{4.4.17}$$

$$\begin{aligned} M_{12} &= M_{21} = -\frac{D(1-\mu)}{2} \\ &\quad \times \left[\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_2^2}\frac{\partial u_3}{\partial \alpha_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_1^2}\frac{\partial u_3}{\partial \alpha_1}\right)\right] \end{aligned} \tag{4.4.18}$$

Inserting Eqs. (4.4.16)–(4.4.18) in Eqs. (4.4.6) and (4.4.7) and then the resulting expressions in Eq. (4.4.5) gives

$$D\nabla^4 u_3 + \rho h \ddot{u}_3 = q_3 \tag{4.4.19}$$

where

$$\nabla^4(\cdot) = \nabla^2\nabla^2(\cdot) \tag{4.4.20}$$

$$\nabla^2(\cdot) = \frac{1}{A_1 A_2}\left[\frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{A_1}\frac{\partial(\cdot)}{\partial \alpha_1}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{A_2}\frac{\partial(\cdot)}{\partial \alpha_2}\right)\right] \tag{4.4.21}$$

The operator ∇^2 is the Laplacian operator. Since it is expressed in curvilinear coordinates, it is now very easy to express it in the coordinate system of our choice. For instance, for Cartesian coordinates, $A_1 = 1, \mathrm{d}\alpha_1 = \mathrm{d}x, A_2 = 1, \mathrm{d}\alpha_2 = \mathrm{d}y$. Thus

$$\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2} \tag{4.4.22}$$

and therefore

$$\nabla^4(\cdot) = \frac{\partial^4(\cdot)}{\partial x^4} + 2\frac{\partial^4(\cdot)}{\partial x^2\,\partial y^2} + \frac{\partial^4(\cdot)}{\partial y^4} \tag{4.4.23}$$

Thus Eq. (4.4.19) becomes

$$D\left(\frac{\partial^4 u_3}{\partial x^4} + 2\frac{\partial^4 u_3}{\partial x^2\,\partial y^2} + \frac{\partial^4 u_3}{\partial y^4}\right) + \rho h \ddot{u}_3 = q_3 \tag{4.4.24}$$

For circular plates it is of advantage to employ polar coordinates. In this cases, $A_1 = 1, \mathrm{d}\alpha_1 = \mathrm{d}r, A_2 = r, \mathrm{d}\alpha_2 = \mathrm{d}\theta$, and

$$\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial r^2} + \frac{1}{r}\frac{\partial(\cdot)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\cdot)}{\partial \theta^2} \tag{4.4.25}$$

Elliptical plates are defined by elliptical coordinates: $A_1 = A_2 = (a^2 - b^2)(\sin^2 v + \sinh^2 u), \mathrm{d}\alpha_1 = \mathrm{d}u, \mathrm{d}\alpha_2 = \mathrm{d}v$, where a and b are the major half-axes of the ellipse. The Laplacian operator becomes

$$\nabla^2(\cdot) = \frac{1}{[(a^2 - b^2)(\sin^2 v + \sinh^2 u)]^2}\left(\frac{\partial^2(\cdot)}{\partial u^2} + \frac{\partial^2(\cdot)}{\partial v^2}\right) \tag{4.4.26}$$

In general, there are two boundary conditions that are required on each edge. They are, by reduction from Eqs. (2.8.20) and (2.8.21),

$$M_{nn} = M_{nn}^* \text{ or } \beta_n = \beta_n^* \tag{4.4.27}$$

$$V_{n3} = V_{n3}^* \text{ or } u_3 = u_3^* \tag{4.4.28}$$

For example, for Cartesian coordinates these boundary conditions are, at an $x =$ constant edge,

$$-D\left(\frac{\partial^2 u_3}{\partial x^2} + \mu\frac{\partial^2 u_3}{\partial y^2}\right) = M_{xx}^* \quad \text{or} \quad \frac{\partial u_3}{\partial x} = \left(\frac{\partial u_3}{\partial x}\right)^* \tag{4.4.29}$$

$$-D\left[\frac{\partial^3 u_3}{\partial x^3} + (2-\mu)\frac{\partial^3 u_3}{\partial y^2\,\partial x}\right] = V_{x3}^* \quad \text{or} \quad u_3 = u_3^* \tag{4.4.30}$$

4.5. TORSIONAL VIBRATION OF CIRCULAR CYLINDRICAL SHELL AND REDUCTION TO A TORSION BAR

Assuming that a circular cylindrical shell vibrates in torsion only, such that the cross-section is not elastically deformed, we set (see Fig. 3)

$$u_3 = 0 \tag{4.5.1}$$

$$u_\theta = a\gamma \tag{4.5.2}$$

$$u_x = 0 \tag{4.5.3}$$

where γ is the torsional deflection angle

$$\gamma = \gamma(x, t) \tag{4.5.4}$$

It also follows that

$$\frac{\partial(\cdot)}{\partial\theta} = 0 \tag{4.5.5}$$

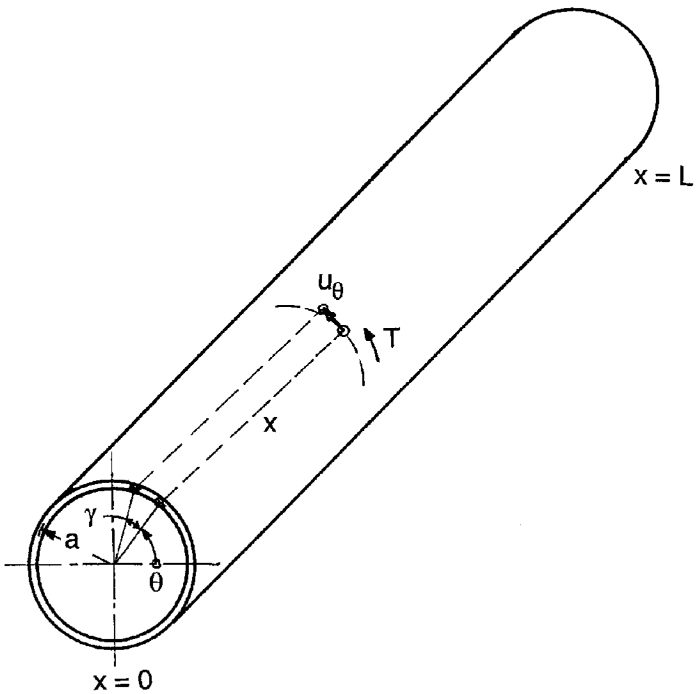

Fig. 3 Circular cylindrical shell vibrating in torsion.

From Eq. (3.3.3), we obtain

$$\frac{\partial N_{x\theta}}{\partial x} + \frac{Q_{\theta 3}}{a} + q_\theta = \rho h \ddot{u}_\theta \tag{4.5.6}$$

where from Eq. (3.3.6),

$$Q_{\theta 3} = \frac{\partial M_{x\theta}}{\partial x} \tag{4.5.7}$$

Further, $\varepsilon^0_{xx} = \varepsilon^0_{\theta\theta} = 0$, and

$$\varepsilon^0_{x\theta} = \frac{\partial u_\theta}{\partial x} \tag{4.5.8}$$

Also, $\beta_x = 0$, and

$$\beta_\theta = \frac{u_\theta}{a} \tag{4.5.9}$$

Therefore, $k_{xx} = k_{\theta\theta} = 0$, and

$$k_{x\theta} = \frac{1}{a}\frac{\partial u_\theta}{\partial x} \tag{4.5.10}$$

This gives

$$M_{x\theta} = \frac{D(1-\mu)}{2a}\frac{\partial u_\theta}{\partial x} \tag{4.5.11}$$

$$N_{x\theta} = \frac{K(1-\mu)}{2}\frac{\partial u_\theta}{\partial x} \tag{4.5.12}$$

$$Q_{\theta 3} = \frac{D(1-\mu)}{2a}\frac{\partial^2 u_\theta}{\partial x^2} \tag{4.5.13}$$

Thus, the equation of motion becomes

$$\frac{(1-\mu)}{2}\left(K + \frac{D}{2a^2}\right)\frac{\partial^2 u_\theta}{\partial x^2} + q_\theta = \rho h \ddot{u}_\theta \tag{4.5.14}$$

or

$$-\frac{Eh}{2(1+\mu)}\left(1 + \frac{h^2}{12a^2}\right)\frac{\partial^2 u_\theta}{\partial x^2} + \rho h \ddot{u}_\theta = q_\theta \tag{4.5.15}$$

Since the shear modulus is $G = E/2(1+\mu)$, utilizing Eq. (4.5.2) gives

$$-G\left(1 + \frac{h^2}{12a^2}\right)\frac{\partial^2 \gamma}{\partial x^2} + \rho\ddot{\gamma} = \frac{q_\theta}{ah} \tag{4.5.16}$$

Since we may assume that $h^2/12a^2 \ll 1$, and introducing the shear speed of sound

$$c = \sqrt{\frac{G}{\rho}} \tag{4.5.17}$$

we obtain

$$\ddot{\gamma} - c^2 \frac{\partial^2 \gamma}{\partial x^2} = \frac{q_\theta}{ah\rho} \tag{4.5.18}$$

Since torque per unit length, T, is related to q_θ by

$$T = 2\pi a^2 q_\theta \tag{4.5.19}$$

and the polar area moment of a thin circular cylindrical shell is

$$J = 2\pi a^3 h, \tag{4.5.20}$$

we may write

$$\frac{q_\theta}{ah\rho} = \frac{T}{J\rho} \tag{4.5.21}$$

or

$$\ddot{\gamma} - c^2 \frac{\partial^2 \gamma}{\partial x^2} = \frac{T}{J\rho} \tag{4.5.22}$$

This is the equation of motion of the circular cylindrical shell in torsion and it is also the equation of motion of uniform torsion bars for cross-sections of any kind if we take a leap of faith by arguing that any polar area moment can be substituted (but strictly speaking, we have proved this equation only for the thin walled circular tube or pipe). This reduction has proven that in the limit Love's equations are consistent with torsion bars (just as it was shown earlier that they are consistent with the plate, beam, and rod vibration equations). Finally, it should be noted that it is impossible to obtain Eq. (4.5.22) by cutting mathematically a strip from a rectangular plate since in Love's equation transverse shear deformations are neglected. The reason that the reduction shown in this section works is that the in-plane shear deformation is included in the Love theory.

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5

Natural Frequencies and Modes

Not only is knowledge of natural frequencies and modes important from a design viewpoint (to avoid resonance conditions, for instance), but it is also the foundation for forced response calculations. In the following, first some generalities are outlined and then specific examples are given.

5.1. GENERAL APPROACH

Love's equations can be written, after substitution of the strain–displacement relations, as

$$L_1\{u_1, u_2, u_3\} + q_1 = \rho h \frac{\partial^2 u_1}{\partial t^2} \tag{5.1.1}$$

$$L_2\{u_1, u_2, u_3\} + q_2 = \rho h \frac{\partial^2 u_2}{\partial t^2} \tag{5.1.2}$$

$$L_3\{u_1, u_2, u_3\} + q_3 = \rho h \frac{\partial^2 u_3}{\partial t^2} \tag{5.1.3}$$

or in short,

$$L_i\{u_1, u_2, u_3\} + q_i = \rho h \frac{\partial^2 u_i}{\partial t^2} \tag{5.1.4}$$

Setting $q_i = 0$ $(i = 1, 2, 3)$ and recognizing that at a natural frequency every point in the elastic system moves harmonically, we may assume that (see also Sec. 5.12)

$$u_1(\alpha_1, \alpha_2, t) = U_1(\alpha_1, \alpha_2)e^{j\omega t} \tag{5.1.5}$$

$$u_2(\alpha_1, \alpha_2, t) = U_2(\alpha_1, \alpha_2)e^{j\omega t} \tag{5.1.6}$$

$$u_3(\alpha_1, \alpha_2, t) = U_3(\alpha_1, \alpha_2)e^{j\omega t} \tag{5.1.7}$$

or, in short,

$$u_i(\alpha_1, \alpha_2, t) = U_i(\alpha_1, \alpha_2)e^{j\omega t} \tag{5.1.8}$$

All three of the $U_i(\alpha_1, \alpha_2)$ functions together constitute a natural mode. Substituting (5.1.8) in (5.1.4) gives, with $q_i = 0$,

$$L_i\{U_1, U_2, U_3\} + \rho h \omega^2 U_i = 0 \tag{5.1.9}$$

Boundary conditions can in general be written

$$B_k\{u_1, u_2, u_3\} = 0 \tag{5.1.10}$$

where $K = 1, 2, \ldots, N$ and where N is the total number of boundary conditions. In the general case of a four-sided shell segment, we have $N = 16$. For a beam, $N = 4$. For a rectangular plate, $N = 8$.

After the substitution of (5.1.8), Eq. (5.1.10) becomes

$$B_k\{U_1, U_2, U_3\} = 0 \tag{5.1.11}$$

The next step is to try separation of variables on Eqs. (5.1.9) and (5.1.11):

$$U_i(\alpha_1, \alpha_2) = R_i(\alpha_1)S_i(\alpha_2) \tag{5.1.12}$$

If this is possible, a set of ordinary differential equations results. Solutions of these equations have N unknown coefficients. Substitution of these solutions into the separated boundary conditions will give a homogeneous set of N equations. The determinant of these equations will furnish the characteristic equation. The roots of this equation will give the natural frequencies.

Often, it is not possible to obtain a general solution that is valid for all boundary condition combinations, but solutions for certain boundary conditions can be guessed. Obviously, if the guess satisfies the equations of motion and the particular set of boundary conditions, it is a valid solution even though we cannot be sure that it is the complete solution. However, experimental evidence usually takes care of this objection.

5.2. TRANSVERSELY VIBRATING BEAMS

The equation of motion is

$$EI\frac{\partial^4 u_3}{\partial x^4}+\rho'\frac{\partial^2 u_3}{\partial t^2}=0 \tag{5.2.1}$$

where I is the area moment and ρ' is the mass per unit length, having multiplied Eq. (4.2.3) by the width of the beam. Substituting

$$u_3(x,t)=U_3(x)\mathrm{e}^{j\omega t} \tag{5.2.2}$$

gives

$$\frac{\mathrm{d}^4 U_3}{\mathrm{d}x^4}-\lambda^4 U_3=0 \tag{5.2.3}$$

where

$$\lambda^4=\frac{\omega^2\rho'}{EI} \tag{5.2.4}$$

We approach the solution utilizing the Laplace transform. We get

$$(s^4-\lambda^4)U_3(s)-s^3U_3(0)-s^2\frac{\mathrm{d}U_3}{\mathrm{d}x}(0)-s\frac{\mathrm{d}^2U_3}{\mathrm{d}x^2}(0)-\frac{\mathrm{d}^3U_3}{\mathrm{d}x^3}(0)=0 \tag{5.2.5}$$

Thus

$$U_3(s)=\frac{1}{s^4-\lambda^4}\left[s^3U_3(0)+s^2\frac{\mathrm{d}U_3}{\mathrm{d}x}(0)+s\frac{\mathrm{d}^2U_3}{\mathrm{d}x^2}(0)+\frac{\mathrm{d}^3U_3}{\mathrm{d}x^3}(0)\right] \tag{5.2.6}$$

Taking the inverse transformation yields

$$\begin{aligned}U_3(x)\ &=U_3(0)A(\lambda x)+\frac{1}{\lambda}\frac{\mathrm{d}U_3}{\mathrm{d}x}(0)B(\lambda x)+\frac{1}{\lambda^2}\frac{\mathrm{d}^2U_3}{\mathrm{d}x^2}(0)C(\lambda x)\\&\quad+\frac{1}{\lambda^3}\frac{\mathrm{d}^3U_3}{\mathrm{d}x^3}(0)D(\lambda x)\end{aligned} \tag{5.2.7}$$

where

$$A(\lambda x)=\tfrac{1}{2}(\cosh\lambda x+\cos\lambda x) \tag{5.2.8}$$

$$B(\lambda x)=\tfrac{1}{2}(\sinh\lambda x+\sin\lambda x) \tag{5.2.9}$$

$$C(\lambda x)=\tfrac{1}{2}(\cosh\lambda x-\cos\lambda x) \tag{5.2.10}$$

$$D(\lambda x)=\tfrac{1}{2}(\sinh\lambda x-\sin\lambda x) \tag{5.2.11}$$

Note that $A(0)=1$, $B(0)=0$, $C(0)=0$, and $D(0)=0$. Since we will need, for application to specific boundary conditions, the derivatives of Eq. (5.2.7), they are given in the following:

$$\begin{aligned}\frac{\mathrm{d}U_3}{\mathrm{d}x}(x)\ &=\lambda U_3(0)D(\lambda x)+\frac{\mathrm{d}U_3}{\mathrm{d}x}(0)A(\lambda x)+\frac{1}{\lambda}\frac{\mathrm{d}^2U_3}{\mathrm{d}x^2}(0)B(\lambda x)\\&\quad+\frac{1}{\lambda^2}\frac{\mathrm{d}^3U_3}{\mathrm{d}x^3}(0)C(\lambda x)\end{aligned} \tag{5.2.12}$$

$$\frac{d^2U_3}{dx^2}(x) = \lambda^2 U_3(0)C(\lambda x) + \lambda\frac{dU_3}{dx}(0)D(\lambda x) + \frac{d^2U_3}{dx^2}(0)A(\lambda x) + \frac{1}{\lambda}\frac{d^3U_3}{dx^3}(0)B(\lambda x) \quad (5.2.13)$$

$$\frac{d^3U_3}{dx^3}(x) = \lambda^3 U_3(0)B(\lambda x) + \lambda^2\frac{dU_3}{dx}(0)C(\lambda x) + \lambda\frac{d^2U_3}{dx^2}(0)D(\lambda x) + \frac{d^3U_3}{dx^3}(0)A(\lambda x) \quad (5.2.14)$$

Let us treat the clamped-free beam as an example. From Eqs. (4.2.5) and (4.2.6), we formulate the boundary conditions for the clamped end at $x=0$ as

$$u_3(x=0,t)=0 \quad (5.2.15)$$

$$\frac{\partial u_3}{\partial x}(x=0,t)=0 \quad (5.2.16)$$

and at the free end $(x=L)$ as

$$M_{xx}(x=L,t)=0 \quad (5.2.17)$$

$$Q_{x3}(x=L,t)=0 \quad (5.2.18)$$

Substituting the strain–displacement relations and substituting Eq. (5.2.2) gives

$$U_3(x=0)=0 \quad (5.2.19)$$

$$\frac{dU_3}{dx}(x=0)=0 \quad (5.2.20)$$

$$\frac{d^2U_3}{dx^2}(x=L)=0 \quad (5.2.21)$$

$$\frac{d^3U_3}{dx^3}(x=L)=0 \quad (5.2.22)$$

Substituting Eqs. (5.2.7) and (5.2.12) to (5.2.14) in these conditions gives

$$0=\frac{d^2U_3}{dx^2}(0)A(\lambda L)+\frac{1}{\lambda}\frac{d^3U_3}{dx^3}(0)B(\lambda L) \quad (5.2.23)$$

$$0=\lambda\frac{d^2U_3}{dx^2}(0)D(\lambda L)+\frac{d^3U_3}{dx^3}(0)A(\lambda L) \quad (5.2.24)$$

or

$$\begin{bmatrix} A(\lambda L) & \frac{1}{\lambda}B(\lambda L) \\ \lambda D(\lambda L) & A(\lambda L) \end{bmatrix} \begin{Bmatrix} \dfrac{d^2U_3}{dx^2}(0) \\ \dfrac{d^3U_3}{dx^3}(0) \end{Bmatrix} = 0 \quad (5.2.25)$$

Since

$$\left\{ \begin{matrix} \dfrac{d^2U_3}{dx^2}(0) \\ \dfrac{d^3U_3}{dx^3}(0) \end{matrix} \right\} \neq 0 \tag{5.2.26}$$

it must be that

$$\begin{vmatrix} A(\lambda L) & \frac{1}{\lambda}B(\lambda L) \\ \lambda D(\lambda L) & A(\lambda L) \end{vmatrix} = 0 \tag{5.2.27}$$

or

$$A^2(\lambda L) - D(\lambda L)B(\lambda L) = 0 \tag{5.2.28}$$

Substituting Eqs. (5.2.8) to (5.2.11) gives

$$\cosh \lambda L \cos \lambda L + 1 = 0 \tag{5.2.29}$$

The roots of this equation are

$$\lambda_1 L = 1.875, \quad \lambda_2 L = 4.694, \quad \lambda_3 L = 7.855, \quad \lambda_4 L = 10.996, \quad \lambda_5 L = 14.137, \ldots \text{ etc.} \tag{5.2.30}$$

From Eq. (5.2.4)

$$\omega_m = \frac{(\lambda_m L)^2}{L^2}\sqrt{\frac{EI}{\rho'}} \tag{5.2.31}$$

The natural mode is given by Eq. (5.2.7):

$$U_{3m}(x) = \frac{1}{\lambda_m^2}\frac{d^2U_{3m}}{dx^2}(0)\left[C(\lambda_m x) + \frac{1}{\lambda_m}\frac{\dfrac{d^3U_{3m}}{dx^3}(0)}{\dfrac{d^2U_{3m}}{dx^2}(0)} D(\lambda_m x) \right] \tag{5.2.32}$$

From Eq. (5.2.25), we get

$$\frac{\dfrac{d^3U_{3m}}{dx^3}(0)}{\dfrac{d^2U_{3m}}{dx^2}(0)} = -\frac{\lambda_m D(\lambda_m L)}{A(\lambda_m L)} = -\frac{\lambda_m A(\lambda_m L)}{B(\lambda_m L)} \tag{5.2.33}$$

Thus

$$U_{3m}(x) = \frac{1}{\lambda_m^2}\frac{d^2U_{3m}}{dx^2}(0)\left[C(\lambda_m x) - \frac{A(\lambda_m L)}{B(\lambda_m L)} D(\lambda_m x) \right] \tag{5.2.34}$$

The mode shape is determined by the bracketed quantity. The magnitude of the coefficient

$$\frac{1}{\lambda_m^2}\frac{d^2U_{3m}}{dx^2}(0) \tag{5.2.35}$$

is arbitrary as far as the mode shape is concerned and is a function of the excitation.

As a final example, let us look at the simply supported beam. The boundary conditions are

$$u_3(x=0)=0 \tag{5.2.36}$$

$$u_3(x=L)=0 \tag{5.2.37}$$

$$M_{xx}(x=0)=0 \tag{5.2.38}$$

$$M_{xx}(x=L)=0 \tag{5.2.39}$$

Substituting the strain–displacement relations and substituting Eq. (5.2.2) in Eqs. (5.2.36) to (5.2.39) gives

$$U_3(x=0)=0 \tag{5.2.40}$$

$$U_3(x=L)=0 \tag{5.2.41}$$

$$\frac{\mathrm{d}^2U_3}{\mathrm{d}x^2}(x=0)=0 \tag{5.2.42}$$

$$\frac{\mathrm{d}^2U_3}{\mathrm{d}x^2}(x=L)=0 \tag{5.2.43}$$

Substituting Eqs. (5.2.7) and (5.2.13) in these relations gives

$$0=\frac{1}{\lambda}\frac{\mathrm{d}U_3}{\mathrm{d}x}(0)B(\lambda L)+\frac{1}{\lambda^3}\frac{\mathrm{d}^3U_3}{\mathrm{d}x^3}(0)D(\lambda L) \tag{5.2.44}$$

$$0=\lambda\frac{\mathrm{d}U_3}{\mathrm{d}x}(0)D(\lambda L)+\frac{1}{\lambda}\frac{\mathrm{d}^3U_3}{\mathrm{d}x^3}(0)B(\lambda L) \tag{5.2.45}$$

or

$$\begin{bmatrix} \frac{1}{\lambda}B(\lambda L) & \frac{1}{\lambda^3}D(\lambda L) \\ \lambda D(\lambda L) & \frac{1}{\lambda}B(\lambda L) \end{bmatrix} \begin{Bmatrix} \frac{\mathrm{d}U_3}{\mathrm{d}x}(0) \\ \frac{\mathrm{d}^3U_3}{\mathrm{d}x^3}(0) \end{Bmatrix} = 0 \tag{5.2.46}$$

and therefore, following the same argument as before,

$$\frac{1}{\lambda_2}[B^2(\lambda L)-D^2(\lambda L)]=0 \tag{5.2.47}$$

or

$$\sinh\lambda L\sin\lambda L=0 \tag{5.2.48}$$

Since

$$\lambda\neq 0 \tag{5.2.49}$$

this equation reduces to

$$\sin \lambda L = 0 \tag{5.2.50}$$

or

$$\lambda_m L = m\pi \quad (m = 1, 2, \ldots) \tag{5.2.51}$$

and

$$\omega_m = \frac{m^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho'}} \tag{5.2.52}$$

The natural mode is, from Eq. (5.2.7),

$$U_{3m}(x) = \frac{1}{\lambda_m}\frac{dU_{3m}}{dx}(0)\left[B(\lambda_m x) + \frac{1}{\lambda_m^2}\frac{\frac{d^3U_{3m}}{dx^3}(0)}{\frac{dU_{3m}}{dx}(0)}D(\lambda_m x)\right] \tag{5.2.53}$$

From Eq. (5.2.46),

$$\frac{\frac{d^3U_{3m}}{dx^3}(0)}{\frac{dU_{3m}}{dx}(0)} = -\lambda_m^2\frac{B(\lambda_m L)}{D(\lambda_m L)} = -\lambda_m^2\frac{D(\lambda_m L)}{B(\lambda_m L)} = -\lambda_m^2\frac{\sinh m\pi}{\sinh m\pi} = -\lambda_m^2 \tag{5.2.54}$$

and Eq. (4.2.53) becomes

$$U_{3m}(x) = \frac{1}{\lambda_m}\frac{dU_{3m}}{dx}(0)\sin\lambda_m x \tag{5.2.55}$$

The modes consist of sine waves, as shown in Fig. 1. Note for later reference that we could have guessed the modes for this particular case and by substituting the guess in Eq. (5.2.3) could have found the natural frequencies.

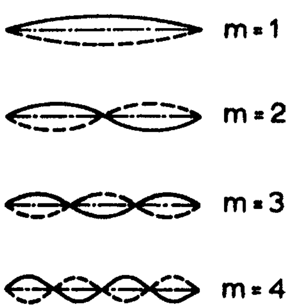

Fig. 1 Natural modes of a simply supported beam.

5.3. CIRCULAR RING

Equations governing the vibrations of a circular ring in its plane of curvature are given in Sec. 4.3. For no load, substituting

$$u_\theta(\theta,t)=U_{\theta n}(\theta)e^{j\omega_n t} \quad (5.3.1)$$

$$u_3(\theta,t)=U_{3n}(\theta)e^{j\omega_n t} \quad (5.3.2)$$

gives

$$\frac{D}{a^4}\left(\frac{d^2U_{\theta n}}{d\theta^2}-\frac{d^3U_{3n}}{d\theta^3}\right)+\frac{K}{a^2}\left(\frac{d^2U_{\theta n}}{d\theta^2}+\frac{dU_{3n}}{d\theta}\right)+\rho h\omega_n^2U_{\theta n}=0 \quad (5.3.3)$$

$$\frac{D}{a^4}\left(\frac{d^3U_{\theta n}}{d\theta^3}-\frac{d^4U_{3n}}{d\theta^4}\right)-\frac{K}{a^2}\left(\frac{dU_{\theta n}}{d\theta}+U_{3n}\right)+\rho h\omega_n^2U_{3n}=0 \quad (5.3.4)$$

It is possible to approach the solution using the Laplace transformation in a manner similar to that in Sec. 5.2. However, in certain cases, it is possible to take a shortcut. The approach in these cases is an inspired guess. Let us take, for example, the free-floating closed ring:

$$U_{3n}(\theta)=A_n\cos n(\theta-\phi) \quad (5.3.5)$$

$$U_{\theta n}(\theta)=B_n\sin n(\theta-\phi) \quad (5.3.6)$$

(See Fig. 2 for a physical interpretation of the following assumption.) We assume that ϕ is an arbitrary phase angle that must be included since the ring does not show a preference for the orientation of its modes; rather, the orientation is determined later by the distribution of the external forces. In Fig. 2 we have sketched the $n=2$ mode, setting $\phi=0$. Note that physical intuition confirms Eqs. (5.3.5) and (5.3.6) since obviously point A will move along the x axis to point A′ $[U_{32}(0)=A_2; U_{\theta 2}(0)=0]$, point B will move along the y axis to point B′ $[U_{32}(\pi/2)=-A_2; U_{\theta 2}(\pi/2)=0]$, and point C will not move in the normal direction but rather will move in

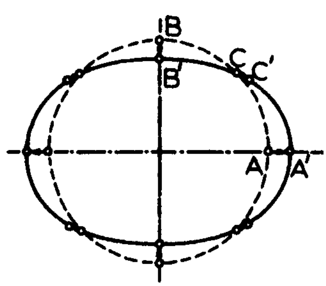

FIG. 2 Explanation of the phasing between transverse motion and circumferential motion using the example of a ring.

the circumferential direction $[U_{32}(\pi/4) = -\theta; U_{\theta 2}(\pi/4) = B_2]$. Substituting Eqs. (5.3.5) and (5.3.6) in Eqs. (5.3.3) and (5.3.4) gives

$$\begin{bmatrix} \rho h \omega_n^2 - \dfrac{n^4 D}{a^4} - \dfrac{K}{a^2} & -\dfrac{n^3 D}{a^4} - \dfrac{nK}{a^2} \\ -\dfrac{n^3 D}{a^4} - \dfrac{nK}{a^2} & \rho h \omega_n^2 - \dfrac{n^2 D}{a^4} - \dfrac{n^2 K}{a^2} \end{bmatrix} \begin{Bmatrix} A_n \\ B_n \end{Bmatrix} = 0 \tag{5.3.7}$$

Since, in general,

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix} \neq 0 \tag{5.3.8}$$

it must be that the determinant is 0. Thus

$$\omega_n^4 - K_1 \omega_n^2 + K_2 = 0 \tag{5.3.9}$$

where

$$K_1 = \frac{n^2+1}{a^2 \rho h}\left(\frac{n^2 D}{a^2} + K\right) \tag{5.3.10}$$

$$K_2 = \frac{n^2(n^2-1)^2}{a^6 (\rho h)^2} DK \tag{5.3.11}$$

so

$$\omega_n^2 = \frac{K_1}{2}\left(1 \pm \sqrt{1 - 4\frac{K_2}{K_1^2}}\right) \tag{5.3.12}$$

Therefore, for each value of n, we encounter a frequency

$$\omega_{n1}^2 = \frac{K_1}{2}\left(1 - \sqrt{1 - 4\frac{K_2}{K_1^2}}\right) \tag{5.3.13}$$

and a frequency

$$\omega_{n2}^2 = \frac{K_1}{2}\left(1 + \sqrt{1 - 4\frac{K_2}{K_1^2}}\right) \tag{5.3.14}$$

As it turns out, for typical rings,

$$\omega_{n2} \gg \omega_{n1} \tag{5.3.15}$$

From Eqs. (5.3.7), we get

$$\frac{A_{ni}}{B_{ni}} = \frac{(n/a^2)[(n^2/a^2)D + K]}{\rho h \omega_{ni}^2 - (1/a^2)[(n^4 D/a^2) + K]} = \frac{\rho h \omega_{ni}^2 - (n^2/a^2)[(D/a^2) + K]}{(n/a^2)[(n^2 D/a^2) + K]} \tag{5.3.16}$$

where $i=1,2$. To gain an intuitive feeling of this ratio, let us look at lower n numbers, where

$$\frac{n^2 D}{a^2} \ll K \tag{5.3.17}$$

We get

$$\rho h \omega_{n1}^2 \ll \frac{K}{a^2} \tag{5.3.18}$$

$$\omega_{n2}^2 \cong K_1 \cong \frac{n^2+1}{a^2 \rho h} K \tag{5.3.19}$$

Thus

$$\frac{A_{n1}}{B_{n1}} \cong -n \tag{5.3.20}$$

and

$$\frac{A_{n2}}{B_{n2}} \cong \frac{1}{n} \tag{5.3.21}$$

The conclusion is that at ω_{n1} frequencies, transverse deflections dominate: The ring is essentially vibrating in bending, analogous to the transverse bending vibration of a beam. At the ω_{n2} frequencies, circumferential deflections dominate, analogous to the longitudinal vibrations of a beam.

Now let us look at some specific values of n. At $n=0$,

$$K_1 = \frac{K}{a^2 \rho h} \tag{5.3.22}$$

$$K_2 = 0 \tag{5.3.23}$$

and therefore

$$\omega_{01}^2 = 0 \tag{5.3.24}$$

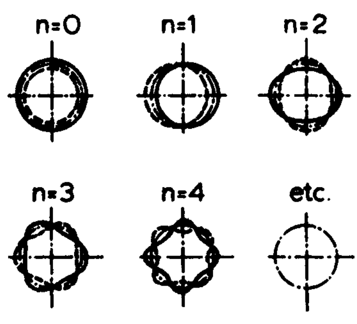

Fig. 3 Natural modes of a closed ring.

$$\omega_{02}^2 = K_1 = \frac{K}{a^2 \rho h} \tag{5.3.25}$$

and

$$\frac{A_{01}}{B_{01}} = 0, \quad \frac{B_{02}}{A_{02}} = 0 \tag{5.3.26}$$

The mode shape is shown in Fig. 3 and is sometimes called the *breathing mode* of the ring. At $n = 1$, we get

$$K_1 = \frac{2}{a^2 \rho h}\left(\frac{D}{a^2} + K\right) \tag{5.3.27}$$

$$K_2 = 0 \tag{5.3.28}$$

and therefore

$$\omega_{11}^2 = 0 \tag{5.3.29}$$

Thus a bending vibration still does not exist; we have to think of the ring as simply being displaced in a rigid-body motion as shown in Fig. 3. However, a ω_{n2} frequency does occur and the mode is one compression and one tension region around the ring.

Starting with $n = 2$, two nonzero sets of natural frequencies and modes exist. Frequencies as a function of n value are plotted in Fig. 4 ($E = 20.6 \times 10^4 \text{N/mm}^2$, $\rho = 7.85 \times 10^{-9} \text{Ns}^2/\text{mm}^4$, $\mu = 0.3$, $h = 2\text{mm}$, $a = 100\text{mm}$).

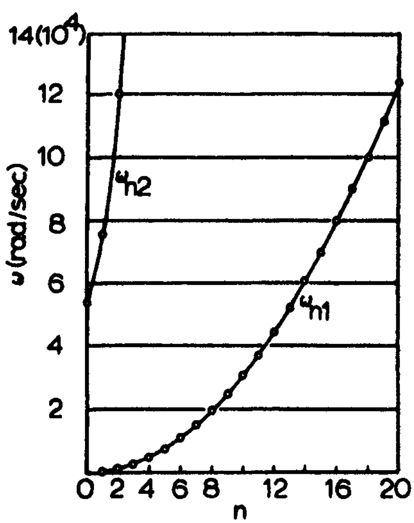

Fig. 4 Natural frequencies of a ring.

5.4. RECTANGULAR PLATES THAT ARE SIMPLY SUPPORTED ALONG TWO OPPOSING EDGES

Let us suppose that the simply supported edges occur always along the $x=0$ and $x=a$ edges as shown in Fig. 5. The boundary conditions on these edges are therefore

$$u_3(0,y,t)=0 \tag{5.4.1}$$

$$u_3(a,y,t)=0 \tag{5.4.2}$$

$$M_{xx}(0,y,t)=0 \tag{5.4.3}$$

$$M_{xx}(a,y,t)=0 \tag{5.4.4}$$

The equation of motion is, from Eq. (4.4.24),

$$D\left(\frac{\partial^4 u_3}{\partial x^4}+2\frac{\partial^4 u_3}{\partial x^2\partial y^2}+\frac{\partial^4 u_3}{\partial y^4}\right)+\rho h\frac{\partial^2 u_3}{\partial t^2}=0 \tag{5.4.5}$$

Substituting

$$u_3(x,y,t)=U_3(x,y)e^{j\omega t} \tag{5.4.6}$$

gives

$$D\left(\frac{\partial^4 U_3}{\partial x^4}+2\frac{\partial^4 U_3}{\partial x^2\partial y^2}+\frac{\partial^4 U_3}{\partial y^4}\right)-\rho h\omega^2 U_3=0 \tag{5.4.7}$$

Substituting the strain–displacement relations and Eq. (5.4.6) in Eqs. (5.4.1)–(5.4.4) gives

$$U_3(0,y)=0 \tag{5.4.8}$$

$$U_3(a,y)=0 \tag{5.4.9}$$

$$\frac{\partial^2 U_3}{\partial x^2}(0,y)=0 \tag{5.4.10}$$

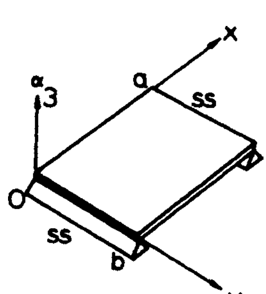

FIG. 5 Rectangular plate simply supported along two opposing edges.

$$\frac{\partial^2 U_3}{\partial x^2}(a,y)=0 \tag{5.4.11}$$

Variables become separated and the boundary conditions (5.4.8–5.4.11) are satisfied if we assume as solution of the form

$$U_3(x,y)=Y(y)\sin\frac{m\pi x}{a} \tag{5.4.12}$$

Substituting Eq. (5.4.12) in Eq. (5.4.7) gives

$$\frac{d^4Y}{dy^4}-2\left(\frac{m\pi}{a}\right)^2\frac{d^2Y}{dy^2}+\left[\left(\frac{m\pi}{a}\right)^4-\frac{\rho h}{D}\omega^2\right]Y=0 \tag{5.4.13}$$

The solution to this ordinary fourth-order differential equation must satisfy four boundary conditions. Substituting

$$Y(y)=\sum_{i=1}^{4}C_i e^{\lambda_i(y/b)} \tag{5.4.14}$$

gives

$$\left(\frac{\lambda_i}{b}\right)^4-2\left(\frac{m\pi}{a}\right)^2\left(\frac{\lambda_i}{b}\right)^2+\left[\left(\frac{m\pi}{a}\right)^4-\frac{\rho h}{D}\omega^2\right]=0 \tag{5.4.15}$$

or

$$\lambda_i=\pm\frac{b}{a}m\pi\sqrt{1\pm K} \tag{5.4.16}$$

where

$$K=\frac{\omega}{(m^2\pi^2/a^2)\sqrt{D/\rho h}} \tag{5.4.17}$$

Since the frequency of a simple supported beam of length a is

$$\omega_b=\frac{m^2\pi^2}{a^2}\sqrt{\frac{D}{\rho h}} \tag{5.4.18}$$

we get

$$K=\frac{\omega}{\omega_b} \tag{5.4.19}$$

Thus

$$K>1 \tag{5.4.20}$$

and thus

$$\lambda_1=+\rho_1,\quad \lambda_2=-\rho_1,\quad \lambda_3=+j\rho_2,\quad \lambda_4=-j\rho_2 \tag{5.4.21}$$

where

$$\rho_1=\frac{b}{a}m\pi\sqrt{K+1},\quad \rho_2=\frac{b}{a}m\pi\sqrt{K-1} \tag{5.4.22}$$

and solution (5.4.14) reads

$$Y(y)=C_1 e^{p_1(y/b)}+C_2 e^{-p_1(y/b)}+C_3 e^{jp_2(y/b)}+C_4 e^{-jp_2(y/b)} \tag{5.4.23}$$

Now let

$$C_1=\frac{A+B}{2}, \quad C_2=\frac{A-B}{2}, \quad C_3=\frac{C+D}{2j}, \quad C_4=\frac{C-D}{2j} \tag{5.4.24}$$

and we get

$$\begin{aligned} Y(y) = {} & A\frac{e^{p_1(y/b)}+e^{-p_1(y/b)}}{2}+B\frac{e^{p_1(y/b)}-e^{-p_1(y/b)}}{2} \\ & +C\frac{e^{jp_2(y/b)}+e^{-jp_2(y/b)}}{2j}+D\frac{e^{jp_2(y/b)}-e^{-jp_2(y/b)}}{2j} \end{aligned} \tag{5.4.25}$$

or

$$Y(y)=A\cosh\rho_1\frac{y}{b}+B\sinh\rho_1\frac{y}{b}+C\cos\rho_2\frac{y}{b}+D\sin\rho_2\frac{y}{b} \tag{5.4.26}$$

where we have set $C'=C/j$ and then dropped the prime. Let us now consider a few examples.

5.4.1. Two Other Edges Clamped

When the two other edges of the rectangular plate are clamped, the additional four boundary conditions are

$$u_3(x,0,t)=0 \tag{5.4.27}$$

$$u_3(x,b,t)=0 \tag{5.4.28}$$

$$\frac{\partial u_3}{\partial y}(x,0,t)=0 \tag{5.4.29}$$

$$\frac{\partial u_3}{\partial y}(x,b,t)=0 \tag{5.4.30}$$

and substituting Eqs. (5.4.6) and (5.4.12) yields

$$Y(0)=0 \tag{5.4.31}$$

$$Y(b)=0 \tag{5.4.32}$$

$$\frac{dY}{dy}(0)=0 \tag{5.4.33}$$

$$\frac{dY}{dy}(b)=0 \tag{5.4.34}$$

Substituting Eq. (5.4.26) in Eqs. (5.4.31)–(5.4.34) gives

$$0=A+C \tag{5.4.35}$$

$$0 = A\cosh\rho_1 + B\sinh\rho_1 + C\cos\rho_2 + D\sin\rho_2 \tag{5.4.36}$$

$$0 = B\frac{\rho_1}{b} + D\frac{\rho_2}{b} \tag{5.4.37}$$

$$0 = A\frac{\rho_1}{b}\sinh\rho_1 + B\frac{\rho_1}{b}\cosh\rho_1 - C\frac{\rho_2}{b}\sin\rho_2 + D\frac{\rho_2}{b}\cos\rho_2 \tag{5.4.38}$$

or

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \cosh\rho_1 & \sinh\rho_1 & \cos\rho_2 & \sin\rho_2 \\ 0 & \rho_1 & 0 & \rho_2 \\ \rho_1\sinh\rho_1 & \rho_1\cosh\rho_1 & -\rho_2\sin\rho_2 & \rho_2\cos\rho_2 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0 \tag{5.4.39}$$

This equation is satisfied if the determinant is 0. Expanding the determinant, we obtain

$$2\rho_1\rho_2(\cosh\rho_1\cos\rho_2 - 1) + (\rho_2^2 - \rho_1^2)\sinh\rho_1\sin\rho_2 = 0 \tag{5.4.40}$$

Solving this equation for its roots $K_n\,(n = 1, 2, \ldots)$ and substituting them into Eq. (5.4.17) gives

$$\omega_{mn}a^2\sqrt{\frac{\rho h}{D}} = m^2\pi^2 K_n \tag{5.4.41}$$

For example, for a square plate where $a/b = 1.0$, for $m^2\pi^2 K_n$, we get the values

	n		
m	1	2	3
1	28.9	69.2	129.1
2	54.8	94.6	154.8
3	102.2	140.2	199.9

The mode shape is obtained by letting

$$\rho_{1n} = \frac{b}{a}m\pi\sqrt{K_n + 1} \tag{5.4.42}$$

$$\rho_{2n} = \frac{b}{a}m\pi\sqrt{K_n - 1} \tag{5.4.43}$$

and by solving for three of the four coefficients of Eq. (5.4.39) in terms of the fourth. Thus

$$\begin{bmatrix} 0 & 1 & 0 \\ \sinh\rho_{1n} & \cos\rho_{2n} & \sin\rho_{2n} \\ \rho_{1n} & 0 & \rho_{2n} \end{bmatrix} \begin{Bmatrix} B \\ C \\ D \end{Bmatrix} = -A \begin{Bmatrix} 1 \\ \cosh\rho_{1n} \\ 0 \end{Bmatrix} \tag{5.4.44}$$

Thus

$$\frac{B}{A} = -\frac{\begin{vmatrix} 1 & 1 & 0 \\ \cosh\rho_{1n} & \cos\rho_{2n} & \sin\rho_{2n} \\ 0 & 0 & \rho_{2n} \end{vmatrix}}{D'} \tag{5.4.45}$$

$$\frac{C}{A} = -\frac{\begin{vmatrix} 0 & 1 & 0 \\ \sinh\rho_{1n} & \cos\rho_{1n} & \sin\rho_{2n} \\ \rho_{1n} & 0 & \rho_{2n} \end{vmatrix}}{D'} \tag{5.4.46}$$

$$\frac{D}{A} = -\frac{\begin{vmatrix} 0 & 1 & 1 \\ \sinh\rho_{1n} & \cos\rho_{2n} & \cosh\rho_{1n} \\ \rho_{1n} & 0 & 0 \end{vmatrix}}{D'} \tag{5.4.47}$$

$$D' = \begin{vmatrix} 0 & 1 & 0 \\ \sinh\rho_{1n} & \cos\rho_{2n} & \sin\rho_{2n} \\ \rho_{1n} & 0 & \rho_{2n} \end{vmatrix} = \rho_{1n}\sin\rho_{2n} - \rho_{2n}\sinh\rho_{1n} \tag{5.4.48}$$

or

$$\frac{B}{A} = -\frac{\rho_{2n}(\cos\rho_{2n} - \cosh\rho_{1n})}{\rho_{1n}\sin\rho_{2n} - \rho_{2n}\sinh\rho_{1n}} \tag{5.4.49}$$

$$\frac{C}{A} = -\frac{\rho_{1n}\sin\rho_{2n} - \rho_{2n}\sinh\rho_{1n}}{\rho_{1n}\sin\rho_{2n} - \rho_{2n}\sinh\rho_{1n}} = -1 \tag{5.4.50}$$

$$\frac{D}{A} = -\frac{\rho_{1n}(\cosh\rho_{1n} - \cos\rho_{2n})}{\rho_{1n}\sin\rho_{2n} - \rho_{2n}\sinh\rho_{1n}} = -\frac{\rho_{1n}}{\rho_{2n}}\frac{B}{A} \tag{5.4.51}$$

and from Eqs. (5.4.26) and (5.4.12) we have

$$U_{3mn}(x,y) = A\left[\left(\cosh\rho_{1n}\frac{y}{b} - \cos\rho_{2n}\frac{y}{b}\right) - \frac{\rho_{2n}(\cos\rho_{2n} - \cosh\rho_{1n})}{\rho_{1n}\sin\rho_{2n} - \rho_{2n}\sinh\rho_{1n}} \times \left(\sinh\rho_{1n}\frac{y}{b} - \frac{\rho_{1n}}{\rho_{2n}}\sin\rho_{2n}\frac{y}{b}\right)\right]\sin\frac{m\pi x}{a} \tag{5.4.52}$$

Lines where

$$U_{3mn}(x,y) = 0 \tag{5.4.53}$$

are node lines and can be obtained by searching for the x, y points that satisfy Eq. (5.4.53). Node lines for a square plate are shown in Fig. 6.

The Levy-type solution schema presented here has been extended to other combinations of boundary conditions for the rectangular plate by a superposition approach. For this technique Gorman (1982) should be consulted.

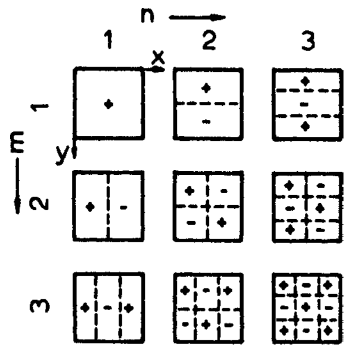

Fig. 6 Node lines of a simply supported plate.

5.4.2. Two Other Edges Simply Supported

When the two other edges are also simply supported, the additional four boundary conditions are

$$u_3(x,0,t)=0 \tag{5.4.54}$$

$$u_3(x,b,t)=0 \tag{5.4.55}$$

$$M_{yy}(x,0,t)=0 \tag{5.4.56}$$

$$M_{yy}(x,b,t)=0 \tag{5.4.57}$$

Substituting the strain–displacement relations and Eqs. (5.4.6) and (5.4.12) results in

$$Y(0)=0 \tag{5.4.58}$$

$$Y(b)=0 \tag{5.4.59}$$

$$\frac{d^2Y}{dy^2}(0)=0 \tag{5.4.60}$$

$$\frac{d^2Y}{dy^2}(b)=0 \tag{5.4.61}$$

Substituting Eq. (5.4.26) gives

$$0=A+C \tag{5.4.62}$$

$$0=A\cosh\rho_1+B\sinh\rho_1+C\cos\rho_2+D\sin\rho_2 \tag{5.4.63}$$

$$0=A\rho_1^2-C\rho_2^2 \tag{5.4.64}$$

$$0=A\rho_1^2\cosh\rho_1+B\rho_1^2\sinh\rho_1-C\rho_2^2\cos\rho_2-D\rho_2^2\sin\rho_2 \tag{5.4.65}$$

or

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \cosh\rho_1 & \sinh\rho_1 & \cos\rho_2 & \sin\rho_2 \\ \rho_1^2 & 0 & -\rho_2^2 & 0 \\ \rho_1^2\cosh\rho_1 & \rho_1^2\sinh\rho_1 & -\rho_2^2\cos\rho_2 & -\rho_2^2\sin\rho_2 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0 \tag{5.4.66}$$

This equation is satisfied if the determinant is 0. Expanding the determinant gives

$$(\rho_1^2+\rho_2^2)^2 \sinh\rho_1 \sin\rho_2 = 0 \tag{5.4.67}$$

Since neither $(\rho_1^2+\rho_2^2)^2$ nor $\sinh\rho_1$ are 0 for nontrivial solutions,

$$\sin\rho_2 = 0 \tag{5.4.68}$$

or

$$\rho_2 = n\pi \quad (n=1,2,\ldots) \tag{5.4.69}$$

or

$$K = \left(\frac{n}{m}\right)^2 \left(\frac{a}{b}\right)^2 + 1 \tag{5.4.70}$$

and therefore

$$\omega_{mn} = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \sqrt{\frac{D}{\rho h}} \tag{5.4.71}$$

For example, for a square plate where $a/b = 1.0$, for $\omega_{mn} a^2 \sqrt{\rho h/D}$ we get the values

	n		
m	1	2	3
1	19.72	49.30	98.60
2	49.30	78.88	128.17
3	98.60	128.17	177.47

To obtain the natural modes, we solve for three of the four coefficient of Eq. (5.4.66):

$$\begin{bmatrix} 1 & 0 & 1 \\ \cosh\rho_1 & \sinh\rho_1 & \cos\rho_2 \\ \rho_1^2 & 0 & -\rho_2^2 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = -D \begin{Bmatrix} 0 \\ \sin\rho_2 \\ 0 \end{Bmatrix} \tag{5.4.72}$$

This gives

$$\frac{A}{D} = 0 \tag{5.4.73}$$

$$\frac{B}{D} = 0 \tag{5.4.74}$$

$$\frac{C}{D}=0 \tag{5.4.75}$$

and we get

$$Y(y)=D\sin\frac{n\pi y}{b} \tag{5.4.76}$$

or

$$U_{3mn}(x,y)=D\sin\frac{n\pi y}{b}\sin\frac{m\pi x}{a} \tag{5.4.77}$$

This is an example of a result that could have been guessed.

5.5. Circular Cylindrical Shell Simply Supported

A circular cylindrical shell simply supported is shown in Fig. 7. It is assumed that the boundaries are such that

$$u_3(0,\theta,t)=0 \tag{5.5.1}$$

$$u_\theta(0,\theta,t)=0 \tag{5.5.2}$$

$$M_{xx}(0,\theta,t)=0 \tag{5.5.3}$$

$$N_{xx}(0,\theta,t)=0 \tag{5.5.4}$$

and

$$u_3(L,\theta,t)=0 \tag{5.5.5}$$

$$u_\theta(L,\theta,t)=0 \tag{5.5.6}$$

$$M_{xx}(L,\theta,t)=0 \tag{5.5.7}$$

$$N_{xx}(L,\theta,t)=0 \tag{5.5.8}$$

the equations of motion are, from Eqs. (3.3.2) to (3.3.4),

$$\frac{\partial N_{xx}}{\partial x}+\frac{1}{a}\frac{\partial N_{x\theta}}{\partial \theta}-\rho h\frac{\partial^2 u_x}{\partial t^2}=0 \tag{5.5.9}$$

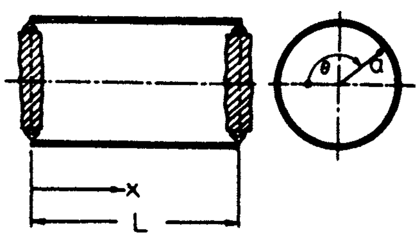

Fig. 7 Simply supported, closed circular cylindrical plate.

$$\frac{\partial N_{x\theta}}{\partial x}+\frac{1}{a}\frac{\partial N_{\theta\theta}}{\partial\theta}+\frac{Q_{\theta 3}}{a}-\rho h\frac{\partial^2 u_\theta}{\partial t^2}=0 \tag{5.5.10}$$

$$\frac{\partial Q_{x3}}{\partial x}+\frac{1}{a}\frac{\partial Q_{\theta 3}}{\partial\theta}-\frac{N_{\theta\theta}}{a}-\rho h\frac{\partial^2 u_3}{\partial t^2}=0 \tag{5.5.11}$$

with all terms defined by Eqs. (3.3.5) to (3.3.14). At a natural frequency

$$u_x(x,\theta,t)=U_x(x,\theta)e^{j\omega t} \tag{5.5.12}$$

$$u_\theta(x,\theta,t)=U_\theta(x,\theta)e^{j\omega t} \tag{5.5.13}$$

$$u_3(x,\theta,t)=U_3(x,\theta)e^{j\omega t} \tag{5.5.14}$$

This gives

$$\frac{\partial N'_{xx}}{\partial x}+\frac{1}{a}\frac{\partial N'_{x\theta}}{\partial\theta}+\rho h\omega^2 U_x=0 \tag{5.5.15}$$

$$\frac{\partial N'_{x\theta}}{\partial x}+\frac{1}{a}\frac{\partial N'_{\theta\theta}}{\partial\theta}+\frac{1}{a}Q'_{\theta 3}+\rho h\omega^2 U_\theta=0 \tag{5.5.16}$$

$$\frac{\partial Q'_{x3}}{\partial x}+\frac{1}{a}\frac{\partial Q'_{\theta 3}}{\partial\theta}-\frac{N'_{\theta\theta}}{a}+\rho h\omega^2 U_3=0 \tag{5.5.17}$$

where

$$Q'_{x3}=\frac{\partial M'_{xx}}{\partial x}+\frac{1}{a}\frac{\partial M'_{x\theta}}{\partial\theta} \tag{5.5.18}$$

$$Q'_{\theta 3}=\frac{\partial M'_{x\theta}}{\partial x}+\frac{1}{a}\frac{\partial M'_{\theta\theta}}{\partial\theta} \tag{5.5.19}$$

$$N'_{xx}=K(\varepsilon'^0_{xx}+\mu\varepsilon'^0_{\theta\theta}) \tag{5.5.20}$$

$$N'_{\theta\theta}=K(\varepsilon'^0_{\theta\theta}+\mu\varepsilon'^0_{xx}) \tag{5.5.21}$$

$$N'_{x\theta}=\frac{K(1-\mu)}{2}\varepsilon'^0_{x\theta} \tag{5.5.22}$$

$$M'_{xx}=D(k'_{xx}+\mu k'_{\theta\theta}) \tag{5.5.23}$$

$$M'_{\theta\theta}=D(k'_{\theta\theta}+\mu k'_{xx}) \tag{5.5.24}$$

$$M'_{x\theta}=\frac{D(1-\mu)}{2}k'_{x\theta} \tag{5.5.25}$$

$$\varepsilon'^0_{xx}=\frac{\partial U_x}{\partial x} \tag{5.5.26}$$

$$\varepsilon'^0_{\theta\theta}=\frac{1}{a}\frac{\partial U_\theta}{\partial\theta}+\frac{U_3}{a} \tag{5.5.27}$$

$$\varepsilon'^0_{x\theta}=\frac{\partial U_\theta}{\partial x}+\frac{1}{a}\frac{\partial U_x}{\partial\theta} \tag{5.5.28}$$

$$k'_{xx} = \frac{\partial \beta'_x}{\partial x} \tag{5.5.29}$$

$$k'_{\theta\theta} = \frac{1}{a}\frac{\partial \beta'_\theta}{\partial \theta} \tag{5.5.30}$$

$$k'_{x\theta} = \frac{\partial \beta'_\theta}{\partial x} + \frac{1}{a}\frac{\partial \beta'_x}{\partial \theta} \tag{5.5.31}$$

$$\beta'_x = -\frac{\partial U_3}{\partial x} \tag{5.5.32}$$

$$\beta'_\theta = \frac{U_\theta}{a} - \frac{1}{a}\frac{\partial U_3}{\partial \theta} \tag{5.5.33}$$

The boundary conditions become

$$U_3(0,\theta) = 0 \tag{5.5.34}$$

$$U_\theta(0,\theta) = 0 \tag{5.5.35}$$

$$M'_{xx}(0,\theta) = 0 \tag{5.5.36}$$

$$N'_{xx}(0,\theta) = 0 \tag{5.5.37}$$

$$U_3(L,\theta) = 0 \tag{5.5.38}$$

$$U_\theta(L,\theta) = 0 \tag{5.5.39}$$

$$M'_{xx}(L,\theta) = 0 \tag{5.5.40}$$

$$N'_{xx}(L,\theta) = 0 \tag{5.5.41}$$

Based on our experience with the ring and the simply supported beam, we assume the following solution:

$$U_x(x,\theta) = A\cos\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.42}$$

$$U_\theta(x,\theta) = B\sin\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{5.5.43}$$

$$U_3(x,\theta) = C\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.44}$$

While the assumptions for $U_\theta(x,\theta)$ and $U_3(x.\theta)$ are fairly obvious, the assumption for $U_x(x,\theta)$ needs some explanation. First, the term $\cos n(\theta-\phi)$ was chosen since it is to expected that a longitudinal node line will not experience deflections in the x direction. Next, the term $\cos(m\pi x/L)$ is based on the boundary condition requirement that

$$N'_{xx}(0,\theta) = 0 \tag{5.5.45}$$

$$N'_{xx}(L,\theta) = 0 \tag{5.5.46}$$

Substituting Eqs. (5.5.42)–(5.5.44) in all boundary conditions shows that all are satisfied.

Next, we substitute these assumed solutions into Eqs. (5.5.15)–(5.5.33), starting with Eq. (5.5.33) and working backward. We get

$$\beta'_\theta = \frac{1}{a}(B+nC)\sin\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{5.5.47}$$

$$\beta'_x = -\frac{m\pi}{L}C\cos\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.48}$$

$$k'_{x\theta} = \frac{m\pi}{La}(B+2nC)\cos\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{5.5.49}$$

$$k'_{\theta\theta} = \frac{n}{a^2}(B+nC)\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.50}$$

$$k'_{xx} = \left(\frac{m\pi}{L}\right)^2 C\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.51}$$

$$\varepsilon'^{0}_{x\theta} = \left(\frac{m\pi}{L}B - \frac{n}{a}A\right)\cos\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{5.5.52}$$

$$\varepsilon'^{0}_{\theta\theta} = \frac{1}{a}(Bn+C)\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.53}$$

$$\varepsilon'^{0}_{xx} = -A\frac{m\pi}{L}\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.54}$$

$$M'_{x\theta} = \frac{D(1-\mu)}{2}\frac{m\pi}{La}(B+2nC)\cos\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{5.5.55}$$

$$M'_{\theta\theta} = D\left\{\frac{n}{a^2}B + \left[\frac{n^2}{a^2} + \mu\left(\frac{m\pi}{L}\right)^2\right]C\right\}\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.56}$$

$$M'_{xx} = D\left\{\frac{\mu n}{a^2}B + \left[\mu\frac{n^2}{a^2} + \left(\frac{m\pi}{L}\right)^2\right]C\right\}\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.57}$$

$$N'_{x\theta} = \frac{K(1-\mu)}{2}\left(\frac{m\pi}{L}B - \frac{n}{a}A\right)\cos\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{5.5.58}$$

$$N'_{\theta\theta} = K\left(\frac{n}{a}B + \frac{1}{a}C - \mu\frac{m\pi}{L}A\right)\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.59}$$

$$N'_{xx} = K\left(\frac{\mu n}{a}B + \frac{\mu}{a}C - \frac{m\pi}{L}A\right)\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.60}$$

$$Q'_{x3} = D\frac{m\pi}{L}\left\{\frac{n}{a^2}\frac{1+\mu}{2}B + \left[\left(\frac{n}{a}\right)^2 + \left(\frac{m\pi}{L}\right)^2\right]C\right\}\cos\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{5.5.61}$$

$$Q'_{\theta 3} = -\frac{D}{a}\left\{\left[\frac{1-\mu}{2}\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]B + n\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]C\right\} \times\sin\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{5.5.62}$$

Thus Eqs. (5.5.15)–(5.5.17) become

$$\left\{\rho h\omega^2 - K\left[\left(\frac{m\pi}{L}\right)^2 + \frac{1-\mu}{2}\left(\frac{n}{a}\right)^2\right]\right\}A + \left(K\frac{1+\mu}{2}\frac{n}{a}\frac{m\pi}{L}\right)B + \left(K\frac{\mu}{a}\frac{m\pi}{L}\right)C = 0 \quad (5.5.63)$$

$$\left(K\frac{1+\mu}{2}\frac{m\pi}{L}\frac{n}{a}\right)A + \left\{\rho h\omega^2 - \left(K+\frac{D}{a^2}\right)\left[\frac{1-\mu}{2}\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]\right\}B + \left\{-\frac{K}{a}\frac{n}{a} - \frac{D}{a}\frac{n}{a}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]\right\}C = 0 \quad (5.5.64)$$

$$\left(\frac{\mu K}{a}\frac{m\pi}{L}\right)A + \left\{-\frac{K}{a}\frac{n}{a} - \frac{D}{a}\frac{n}{a}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]\right\}B + \left\{\rho h\omega^2 - D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2 - \frac{K}{a^2}\right\}C = 0 \quad (5.5.65)$$

or

$$\begin{bmatrix} \rho h\omega^2 - k_{11} & k_{12} & k_{13} \\ k_{21} & \rho h\omega^2 - k_{22} & k_{23} \\ k_{31} & k_{32} & \rho h\omega^2 - k_{33} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0 \quad (5.5.66)$$

where

$$k_{11} = K\left[\left(\frac{m\pi}{L}\right)^2 + \frac{1-\mu}{2}\left(\frac{n}{a}\right)^2\right] \quad (5.5.67)$$

$$k_{12} = k_{21} = K\frac{1+\mu}{2}\frac{m\pi}{L}\frac{n}{a} \quad (5.5.68)$$

$$k_{13} = k_{31} = \frac{\mu K}{a}\frac{m\pi}{L} \quad (5.5.69)$$

$$k_{22} = \left(K+\frac{D}{a^2}\right)\left[\frac{1-\mu}{2}\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right] \quad (5.5.70)$$

$$k_{23} = k_{32} = -\frac{K}{a}\frac{n}{a} - \frac{D}{a}\frac{n}{a}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right] \quad (5.5.71)$$

$$k_{33} = D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2 + \frac{K}{a^2} \quad (5.5.72)$$

For a nontrivial solution, the determinant of Eq. (5.5.66) has to be 0. Expanding the determinant gives

$$\omega^6 + a_1\omega^4 + a_2\omega^2 + a_3 = 0 \quad (5.5.73)$$

where

$$a_1 = -\frac{1}{\rho h}(k_{11}+k_{22}+k_{33}) \tag{5.5.74}$$

$$a_2 = \frac{1}{(\rho h)^2}(k_{11}k_{33}+k_{22}k_{33}+k_{11}k_{22}-k_{23}^2-k_{12}^2-k_{13}^2) \tag{5.5.75}$$

$$a_3 = \frac{1}{(\rho h)^3}(k_{11}k_{23}^2+k_{22}k_{13}^2+k_{33}k_{12}^2+2k_{12}k_{23}k_{13}-k_{11}k_{22}k_{33}) \tag{5.5.76}$$

The solutions of this equation are

$$\omega_{1mm}^2 = -\frac{2}{3}\sqrt{a_1^2-3a_2}\cos\frac{\alpha}{3}-\frac{a_1}{3} \tag{5.5.77}$$

$$\omega_{2mn}^2 = -\frac{2}{3}\sqrt{a_1^2-3a_2}\cos\frac{\alpha+2\pi}{3}-\frac{a_1}{3} \tag{5.5.78}$$

$$\omega_{3mn}^2 = -\frac{2}{3}\sqrt{a_1^2-3a_2}\cos\frac{\alpha+4\pi}{3}-\frac{a_1}{3} \tag{5.5.79}$$

where

$$\alpha=\cos^{-1}\frac{27a_3+2a_1^3-9a_1a_2}{2\sqrt{(a_1^2-3a_2)^3}} \tag{5.5.80}$$

For every m,n combination, we thus have three frequencies. The lowest is associated with the mode where the transverse component dominates, while the other two are usually higher by an order of magnitude and are associated with the mode where the displacements in the tangent plane dominate. For every m,n combination, we therefore have three different combinations of A, B, and C. Solving A and B in terms of C, we have

$$\begin{bmatrix} \rho h\omega_{imn}^2-k_{11} & k_{12} \\ k_{21} & \rho h\omega_{imn}^2-k_{22} \end{bmatrix}\begin{Bmatrix} A_i \\ B_i \end{Bmatrix}=-C_i\begin{Bmatrix} k_{13} \\ k_{23} \end{Bmatrix} \tag{5.5.81}$$

where $i=1,2,3$. Thus

$$\frac{A_i}{C_i} = -\frac{\begin{vmatrix} k_{13} & k_{12} \\ k_{23} & \rho h\omega_{imn}^2-k_{22} \end{vmatrix}}{D} \tag{5.5.82}$$

$$\frac{B_i}{C_i} = -\frac{\begin{vmatrix} \rho h\omega_{imn}^2-k_{11} & k_{13} \\ k_{21} & k_{23} \end{vmatrix}}{D} \tag{5.5.83}$$

where

$$D=\begin{vmatrix} \rho h\omega_{imn}^2-k_{11} & k_{12} \\ k_{21} & \rho h\omega_{imn}^2-k_{22} \end{vmatrix} \tag{5.5.84}$$

or

$$\frac{A_i}{C_i}=-\frac{k_{13}(\rho h\omega_{imn}^2-k_{22})-k_{12}k_{23}}{(\rho h\omega_{imn}^2-k_{11})(\rho h\omega_{imn}^2-k_{22})-k_{12}^2} \tag{5.5.85}$$

$$\frac{B_i}{C_i}=-\frac{k_{23}(\rho h\omega_{imn}^2-k_{11})-k_{21}k_{13}}{(\rho h\omega_{imn}^2-k_{11})(\rho h\omega_{imn}^2-k_{22})-k_{12}^2} \tag{5.5.86}$$

Thus, in summary, the three natural modes that are associated with the three natural frequencies ω_{imn} at each m, n combination are

$$\begin{Bmatrix} U_x \\ U_\theta \\ U_3 \end{Bmatrix}_i = C_i \begin{Bmatrix} \frac{A_i}{C_i}\cos\frac{m\pi x}{L}\cos n(\theta-\phi) \\ \frac{B_i}{C_i}\sin\frac{m\pi x}{L}\sin n(\theta-\phi) \\ \sin\frac{m\pi x}{L}\cos n(\theta-\phi) \end{Bmatrix} \tag{5.5.87}$$

where the C_i are arbitrary constants.

Let us assume that we have a steel shell ($E=20.6\times10^4$ N/mm^2, $\rho=7.85\times10^{-9}$ N s^2/mm^4, $\mu=0.3$) of thickness $h=2$ mm, radius $a=100$ mm, and length $L=200$ mm. The three frequencies ω_{imn} and the ratios A_i/C_i and B_i/C_i are plotted in Figs. 8–12.

For further examples of shell solutions, see Leissa (1973) and Flügge (1957).

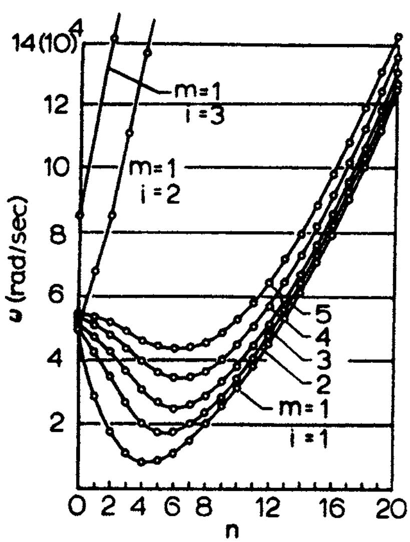

FIG. 8 Natural frequencies of a simply supported cylindrical shell.

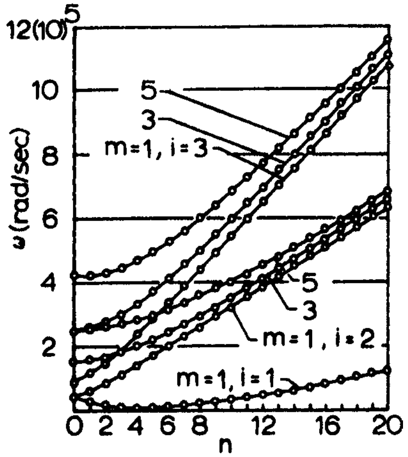

Fig. 9 Same as Fig. 8, but at a different frequency scale.

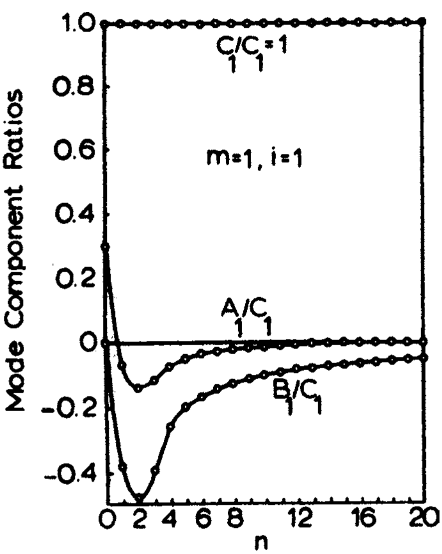

Fig. 10 Natural mode component amplitude ratios for $m=1, i=1$.

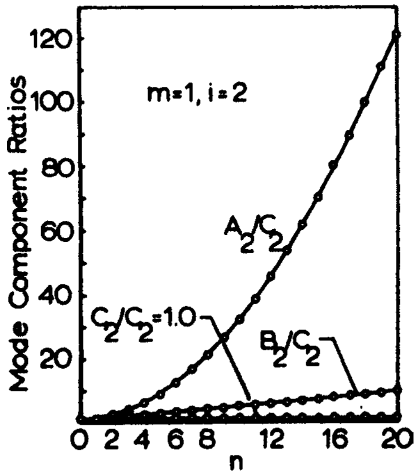

Fig. 11 Natural mode component amplitude ratios for $m=1, i=2$.

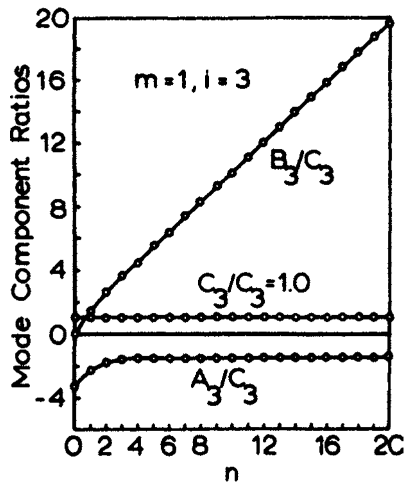

Fig. 12 Natural mode component amplitude ratios for $m=1, i=3$.

5.6. CIRCULAR PLATES VIBRATING TRANSVERSELY

Another category for which exact solutions are available (if series solutions can be termed *exact*) is that of circular plates. Circular plates are common structural elements in engineering. The first circular plate solution is due to Kirchhoff (1850).

The equation of motion for free vibration is

$$D\nabla^4 u_3 + \rho h \frac{\partial^2 u_3}{\partial t^2} = 0 \tag{5.6.1}$$

where

$$\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial r^2} + \frac{1}{r}\frac{\partial(\cdot)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\cdot)}{\partial \theta^2} \tag{5.6.2}$$

At a natural frequency

$$u_3(r,\theta,t) = U_3(r,\theta)\mathrm{e}^{j\omega t} \tag{5.6.3}$$

If we substitute this into Eq. (5.6.1), we obtain

$$D\nabla^4 U_3 - \rho h \omega^2 U_3 = 0 \tag{5.6.4}$$

Let

$$\lambda^4 = \frac{\rho h \omega^2}{D} \tag{5.6.5}$$

Equation (5.6.4) can then be written

$$(\nabla^2 + \lambda^2)(\nabla^2 - \lambda^2)U_3 = 0 \tag{5.6.6}$$

This equation is satisfied by every solution of

$$(\nabla^2 \pm \lambda^2)U_3 = 0 \tag{5.6.7}$$

It is possible to separate variables by substituting

$$U_3(r,\theta) = R(r)\ominus(\theta) \tag{5.6.8}$$

This gives

$$r^2\left[\left(\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}\right)\frac{1}{R} \pm \lambda^2\right] = -\frac{1}{\ominus}\frac{\mathrm{d}^2\ominus}{\mathrm{d}\theta^2} \tag{5.6.9}$$

This equation can be satisfied only if each expression is equal to the same constant k^2. This allows us to write

$$\frac{\mathrm{d}^2\ominus}{\mathrm{d}\theta^2} + k^2\ominus = 0 \tag{5.6.10}$$

and

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \left(\pm\lambda^2 - \frac{k^2}{r^2}\right)R = 0 \tag{5.6.11}$$

The solution of Eq. (5.6.10) is

$$\ominus = A' \cos k\theta + B \sin k\theta \tag{5.6.12}$$

or

$$\ominus = A \cos k(\theta - \phi) \tag{5.6.13}$$

where ϕ is a constant. In general, k can be a fractional number. But for plates that are closed in θ direction, $\ominus$ must be a function of period 2π. In this case. k becomes an integer,

$$k = n = 0, 1, 2, 3, \ldots \tag{5.6.14}$$

Let us now introduce a new variable

$$\xi = \begin{cases} \lambda r & \text{for} + \lambda^2 \\ \mathrm{j}\lambda r & \text{for} - \lambda^2 \end{cases} \tag{5.6.15}$$

Equation (5.6.11) becomes

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}R}{\mathrm{d}\xi} + \left(1 - \frac{k^2}{\xi^2}\right) R = 0 \tag{5.6.16}$$

This is Bessel's equation of fractional order. The solutions are in the form of series. They are classified in terms of Bessel functions. For $\xi = \lambda r$, the solution is in terms of Bessel functions of the first and second kind, $J_k(\lambda r)$ and $Y_k(\lambda r)$. For $\xi = \mathrm{j}\lambda r$, the solution is in terms of modified Bessel functions of the first and second kind, $I_k(\lambda r)$ and $K_k(\lambda r)$.

For the special category of circular plates that are closed in the θ direction so that $k = n$, the solution R is therefore

$$R = CJ_n(\lambda r) + DI_n(\lambda r) + EY_n(\lambda r) + FK_n(\lambda r) \tag{5.6.17}$$

Both $Y_n(\lambda r)$ and $K_n(\lambda r)$ are singular at $\lambda r = 0$. Thus for a plate with no central hole, we set $E = F = 0$. Typical plots of the Bessel functions are shown in Figs. 13 and 14. The general solution was first given by Kirchhoff (1850). Numerous examples are collected in Leissa (1969).

5.7. EXAMPLE: PLATE CLAMPED AT BOUNDARY

If a circular plate has no central hole,

$$E = F = 0 \tag{5.7.1}$$

The boundary conditions are, at the boundary radius $r = a$,

$$u_3(a, \theta, t) = 0 \tag{5.7.2}$$

$$\frac{\partial u_3}{\partial r}(a, \theta, t) = 0 \tag{5.7.3}$$

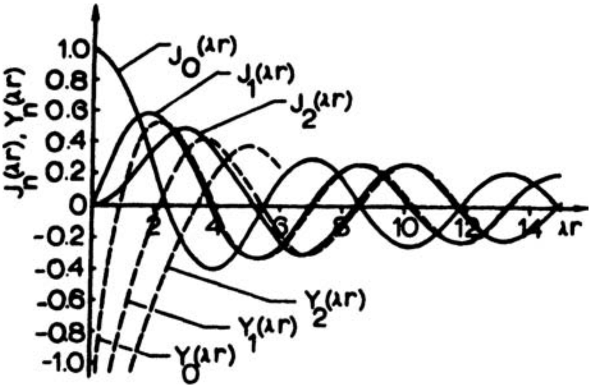

Fig. 13 Illustration of Bessel functions of the first and second kind.

This translates into

$$R(a) = 0 \tag{5.7.4}$$

$$\frac{dR}{dr}(a) = 0 \tag{5.7.5}$$

Substituting Eq. (5.6.17) in these conditions gives

$$\begin{bmatrix} J_n(\lambda a) & I_n(\lambda a) \\ \frac{dJ_n}{dr}(\lambda a) & \frac{dI_n}{dr}(\lambda a) \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix} = 0 \tag{5.7.6}$$

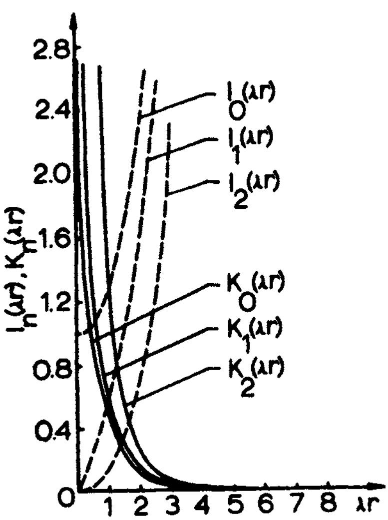

Fig. 14 Illustration of modified Bessel functions of the first and second kind.

This equation is satisfied in a meaningful way only if the determinant is 0. This gives the frequency equation

$$J_n(\lambda a)\frac{dI_n}{dr}(\lambda a)-\frac{dJ_n}{dr}(\lambda a)I_n(\lambda a)=0 \tag{5.7.7}$$

Searching this equation for its roots λa, labeled successively $m=0,1,2,\ldots$ for each $n=0,1,2,\ldots$, gives the natural frequencies. Values of the roots λa are collected in Table 1. The natural frequencies are related to these roots by

$$\omega_{mn}=\frac{(\lambda a)^2_{mn}}{a^2}\sqrt{\frac{D}{\rho h}} \tag{5.7.8}$$

Equation (5.7.7) can be simplified by using the identities

$$a\frac{dJ_n}{dr}(\lambda a)=nJ_n(\lambda a)-\lambda aJ_{n+1}(\lambda a) \tag{5.7.9}$$

$$a\frac{dI_n}{dr}(\lambda a)=nI_n(\lambda a)+\lambda aI_{n+1}(\lambda a) \tag{5.7.10}$$

Equation (5.7.7) is then replaced by

$$J_n(\lambda a)I_{n+1}(\lambda a)+I_n(\lambda a)J_{n+1}(\lambda a)=0 \tag{5.7.11}$$

To find the mode shapes, we formulate from Eq. (5.7.6):

$$\frac{D}{C}=-\frac{J_n(\lambda a)}{I_n(\lambda a)} \tag{5.7.12}$$

This then gives the mode-shape expression

$$U_3(r,\theta)=A\left[J_n(\lambda r)-\frac{J_n(\lambda a)}{I_n(\lambda a)}I_n(\lambda r)\right]\cos n(\theta-\phi) \tag{5.7.13}$$

Setting this expression equal to 0 defines the node lines. It turns out that there will be concentric circles and diametral lines. The number of concentric circles will be m and the number of diametral lines will be n. Examples are shown in Fig. 15. The values of the ratio of the nodal circles are shown in Table 2, in terms of the ratios r/a.

Table 1 Values for $(\lambda a)_{mn}$

	n			
m	0	1	2	3
0	3.196	4.611	5.906	7.143
1	6.306	7.799	9.197	10.537
2	9.440	10.958	12.402	13.795
3	12.577	14.108	15.579	17.005

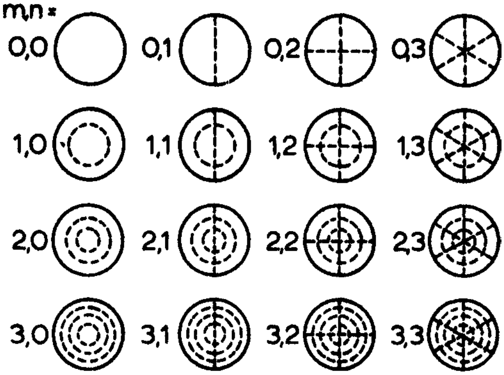

FIG. 15 Node lines for a clamped circular plate.

5.8. ORTHOGONALITY PROPERTY OF NATURAL MODES

Natural modes have the same property that is utilized in Fourier series formulations where sine and cosine functions are used. This property is *orthogonality.*

Let us start with Hamilton's principle,

$$\delta\int_{t_0}^{t_1}(\Pi-K)\,dt=0 \tag{5.8.1}$$

Because natural modes satisfy all boundary conditions, the energy put into a shell by boundary force resultants and moment resultants expressed by

TABLE 2 Nodal Radii in Terms of r/a

	n			
m	0	1	2	3
0	1.00	1.00	1.00	1.00
1	1.00	1.00	1.00	1.00
	0.38	0.49	0.56	0.61
2	1.00	1.00	1.00	1.00
	0.58	0.64	0.68	0.71
	0.26	0.35	0.41	0.46
3	1.00	1.00	1.00	1.00
	0.69	0.72	0.75	0.77
	0.44	0.50	0.54	0.57
	0.19	0.27	0.33	0.38

Eq. (2.6.8) is 0

$$E_B = 0 \tag{5.8.2}$$

Because in the eigenvalue problem forcing is not considered,

$$E_L = 0 \tag{5.8.3}$$

Therefore, as the shell vibrates in a natural mode, the potential energy is equal to the strain energy

$$\Pi = U \tag{5.8.4}$$

Hamilton's principle becomes, for this case,

$$\int_{t_0}^{t_1} \delta U \,\mathrm{d}t - \int_{t_0}^{t_1} \delta K \,\mathrm{d}t = 0 \tag{5.8.5}$$

where, from Eq. (2.7.15),

$$\begin{aligned}\int_{t_0}^{t_1} \delta U \,\mathrm{d}t &= \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} (\sigma_{11}\delta\epsilon_{11} + \sigma_{22}\delta\epsilon_{22} + \sigma_{12}\delta\epsilon_{12} \\ &\quad + \sigma_{13}\delta\epsilon_{13} + \sigma_{23}\delta\epsilon_{23}) A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2 \,\mathrm{d}\alpha_3 \,\mathrm{d}t\end{aligned} \tag{5.8.6}$$

and where, from Eq. (2.7.8),

$$\int_{t_0}^{t_1} \delta K \,\mathrm{d}t = -\rho h \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} (\ddot{u}_1 \delta u_1 + \ddot{u}_2 \delta U_2 + \ddot{u}_3 \delta U_3) A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2 \,\mathrm{d}t \tag{5.8.7}$$

The displacements when the shell is vibrating with mode k are

$$u_i(\alpha_1, \alpha_2, t) = U_{ik}(\alpha_1, \alpha_2) \mathrm{e}^{j\omega_k t} \tag{5.8.8}$$

We substitute this in Eqs. (5.8.6) and (5.8.7). Since the virtual displacements have to satisfy the boundary conditions also, but are in any other respect arbitrary, let us select mode p to represent the virtual displacement

$$\delta u_i(\alpha_1, \alpha_2, t) = U_{ip} \mathrm{e}^{j\omega_p t} \tag{5.8.9}$$

Substituting this also gives

$$\begin{aligned}&\int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} (\sigma_{11}^{(k)} \delta\epsilon_{11}^{(p)} + \sigma_{22}^{(k)} \delta\epsilon_{22}^{(p)} + \sigma_{12}^{(k)} \delta\epsilon_{12}^{(p)} + \sigma_{13}^{(k)} \delta\epsilon_{13}^{(p)} + \sigma_{23}^{(k)} \delta\epsilon_{23}^{(p)}) \\ &\quad \times A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2 \,\mathrm{d}\alpha_3 \\ &\quad = \omega_k^2 \rho h \int_{\alpha_1} \int_{\alpha_2} (U_{1k} U_{1p} + U_{2k} U_{2p} + U_{3k} U_{3p}) A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2\end{aligned} \tag{5.8.10}$$

We note that the time integrals have canceled out. The superscripts on the stresses and variations of strain signify the modes with which they are associated.

Let us now go through the same procedure, except that we assign the mode p to the deflection description

$$u_i(\alpha_1, \alpha_2, t) = U_{ip}(\alpha_1 \alpha_2)\mathrm{e}^{j\omega_p t} \tag{5.8.11}$$

and the mode k to the virtual displacements

$$\delta u_i = U_{ik}\mathrm{e}^{j\omega_k t} \tag{5.8.12}$$

We then obtain

$$\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3} (\sigma_{11}^{(p)}\delta\varepsilon_{11}^{(k)} + \sigma_{22}^{(p)}\delta\epsilon_{22}^{(k)} + \sigma_{12}^{(p)}\delta\varepsilon_{12}^{(k)} + \sigma_{13}^{(p)}\delta\varepsilon_{13}^{(k)} + \sigma_{23}^{(p)}\delta\varepsilon_{23}^{(k)}) \times A_1 A_2 \mathbf{d}\alpha_1 \mathbf{d}\alpha_2 \mathbf{d}\alpha_3 = \omega_p^2 ph \int_{a_1}\int_{a_2} (U_{1p}U_{1k} + U_{2p}U_{2k} + U_{3p}U_{3k}) \times A_1 A_2 \mathbf{d}\alpha_1 \mathbf{d}\alpha_2 \tag{5.8.13}$$

Let us now examine the first two terms of the left space integral of Eq. (5.8.10) Since

$$\varepsilon_{11}^{(p)} = \frac{1}{E}(\sigma_{11}^{(p)} - \mu\sigma_{22}^{(p)}), \quad \varepsilon_{22}^{(p)} = \frac{1}{E}(\sigma_{22}^{(p)} - \mu\sigma_{11}^{(p)}) \tag{5.8.14}$$

we obtain

$$\sigma_{11}^{(k)}\delta\varepsilon_{11}^{(p)} + \sigma_{22}^{(k)}\delta\varepsilon_{22}^{(p)} = \frac{1}{E}\delta(\sigma_{11}^{(k)}\sigma_{11}^{(p)} + \sigma_{22}^{(k)}\sigma_{22}^{(p)} - \mu\sigma_{11}^{(k)}\sigma_{22}^{(p)} - \mu\sigma_{22}^{(k)}\sigma_{11}^{(p)}) \tag{5.8.15}$$

Similarly, if we examine the first two terms of Eq. (5.8.13), we obtain

$$\sigma_{11}^{(p)}\delta\epsilon_{11}^{(k)} + \sigma_{22}^{(p)}\delta\epsilon_{22}^{(k)} = \frac{1}{E}\delta(\sigma_{11}^{(p)}\sigma_{11}^{(k)} + \sigma_{22}^{(p)}\sigma_{22}^{(k)} - \mu\sigma_{11}^{(p)}\sigma_{22}^{(k)} - \mu\sigma_{22}^{(p)}\sigma_{11}^{(k)}) \tag{5.8.16}$$

Therefore, subtracting Eq. (5.8.16) from Eq. (5.8.15) gives

$$\sigma_{11}^{(k)}\delta\epsilon_{11}^{(p)} - \sigma_{11}^{(p)}\delta\epsilon_{11}^{(k)} + \sigma_{22}^{(k)}\delta\epsilon_{22}^{(p)} - \sigma_{22}^{(p)}\delta\epsilon_{22}^{(k)} = 0 \tag{5.8.17}$$

Since the shear terms subtract out to 0 also, we may subtract Eq. (5.8.13) from Eq. (5.8.10) and obtain

$$\rho h(\omega_k^2 - \omega_\rho^2)\int_{a_1}\int_{a_2} (U_{1k}U_{1p} + U_{2k}U_{2p} + U_{3k}U_{3p})A_1 A_2 \mathbf{d}\alpha_1 \mathbf{d}\alpha_2 = 0 \tag{5.8.18}$$

This equation is satisfied whenever $p = k$ since

$$\omega_k^2 - \omega_k^2 = 0 \tag{5.8.19}$$

In this case, the integral has a numerical value which we designate as N_k:

$$N_k = \int_{a_1}\int_{a_2} (U_{1k}^2 + U_{2k}^2 + U_{3k}^2)A_1 A_2 \mathbf{d}\alpha_1 \mathbf{d}\alpha_2 \tag{5.8.20}$$

Whenever $k \neq p$, the only way that Eq. (5.8.18) can be satisfied is when

$$\int_{a_1}\int_{a_1}(U_{1k}U_{1p}+U_{2k}U_{2p}+U_{3k}U_{3p})A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2=0 \qquad (5.8.21)$$

We may summarize this by using the Kronecker delta symbol,

$$\int_{a_1}\int_{a_2}(U_{1k}U_{1p}+U_{2k}U_{2p}+U_{3k}U_{3p})A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2=\delta_{pk}N_k \qquad (5.8.22)$$

where

$$\delta_{pk}=\begin{cases}1, & p=k\\ 0, & p\neq k\end{cases} \qquad (5.8.23)$$

It is important to recognize the generality of this relationship. Any two modes of any system of uniform thickness, when multiplied with each other in the prescribed way, will integrate out to 0. This fact can, for instance, be used to check the accuracy of experimentally determined modes. But most important, it allows us to express a general solution of the forced equation in terms of an infinite series of modes. For shells and plates of non-uniform thickness and non-homogeneous mass density, ph has to be taken inside the integral (5.8.18). This will modify Eqs. (5.8.20) –(5.8.22).

5.9. SUPERPOSITION MODES

A standard procedure in structural vibrations is to determine natural frequencies and mode shapes experimentally. To find the natural frequencies and modes, the structural system is excited by a shaker, a magnetic driver, a periodic airblast, and so on. One natural frequency after the other is identified and the characteristic shape of vibration (mode shape) at each of these natural frequencies is recorded. As long as the natural frequencies are spaced apart as in beam or rod applications, no experimental difficulty is encountered. But in plate and shell structures, it is possible that two or more entirely different mode shapes occur at the same frequency. These mode shapes superimpose in a ratio that is dependent on the location of the exciter. An infinite variety of shapes can thus be created. Experimenters, however, do not usually become aware of this if they go about their task with the standard procedures, which do not necessarily require them to move their excitation location, and will possibly record only a single mode shape at such a superposition frequency where they should have measured a complete set. Proof that this actually happens can be found in the experimental literature. There is extensive published material on experimental mode shapes that form incomplete sets.

The fact that modal superposition can occur in membrane, plate and shell structures has been known for a long time. One of the earliest published discussions of it dates back to Byerly (1959) in 1893. There it is shown for the examples of a rectangular membrane how the superposition of two modes can produce an infinite variety of Chladni figures at the same natural frequency. However, what has to be shown also is the procedure to extract from superposition modes information that is useful for further study of the system (Soedel and Dhar, 1978).

The classical superposition of modes occurs when two distinct modes are associated with the same natural frequency. Let us look, as an example, at a simply supported square plate. This is a case that has a particularly large number of superposition mode possibilities. The natural frequencies are given by

$$\omega_{mn} = \left(\frac{\pi}{a}\right)^2 (m^2+n^2)\sqrt{\frac{D}{\rho h}} \tag{5.9.1}$$

where m and n can be any combination of integers $(m, n = 1, 2, 3, \ldots)$ The mode shapes are given by

$$U_{3mn} = A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \tag{5.9.2}$$

It is clear that superposition modes occur for those combinations of m and n for which m^2+n^2 is the same. This is shown in Table 3. We see that any combination $(m, n) = (i, j)$ and (j, i) has the identical natural frequency. In one case, namely $(m, n) = (1, 7), (7, 1)$ and $(5, 5)$, we find that three distinct mode shapes are associated with the same natural frequency. Such triple occurrences happen more often as we increase the values of (m, n). In general, as m^2+n^2 becomes large, the number of modes that have the identical natural frequency also becomes large.

Let us consider as an example the mode $(m, n) = (1, 2)$. Thus

$$U_{312} = A_{12} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{a} \tag{5.9.3}$$

Table 3 Numerical Values of m^2+n^2

	m						
n	1	2	3	4	5	6	7
1	2	5	10	17	26	37	50
2	5	8	13	20	29	40	53
3	10	13	18	25	34	45	58
4	17	20	25	32	41	52	65
5	26	29	34	41	50	61	74
6	37	40	45	52	61	72	85
7	50	53	58	65	74	85	98

The associated natural frequency is

$$\omega_{12} = 5\left(\frac{\pi}{a}\right)^2 \sqrt{\frac{D}{\rho h}} \tag{5.9.4}$$

At the same natural frequency, the mode $(m, n) = (2, 1)$ exists, and

$$U_{321} = A_{21} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{a} \tag{5.9.5}$$

The two modes are distinctly different, as shown in Fig. 16, where the Chladni figures (node lines) are given. Node lines define locations of zero transverse displacement. Let us now suppose that the experimenter does not know in advance what mode pattern to expect. If he happens to excite the plate with a harmonically varying point force at the node line of the $(m, n) = (1, 2)$ mode, he will find the $(m, n) = (2, 1)$ mode excited. If he does not move his excitation point, he will never be aware of the existence of the (1,2) mode. The reason for this is, of course, that any transverse plate mode has an infinite transverse impedance along its node lines. This can easily be shown theoretically or verified by experiment.

If the experimenter happens to locate his excitation force at any place of the plate that is not a node line of the (1,2) and (2,1) mode, he will excite both modes. A superposition mode will be generated of the form

$$\psi = U_{312} + hU_{321} \tag{5.9.6}$$

where h is a number that depends on the exciter location.

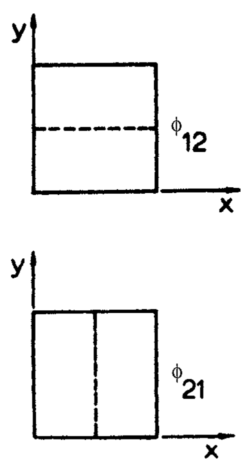

Fig. 16 Two different natural modes that have the same natural frequency.

Let us suppose that his exciter location is along the line $y=x$ but not at $y=x=0, a/2, a$. In this case, he will excite each of the two fundamental modes equally strongly:

$$A_{12}=A_{21} \tag{5.9.7}$$

This gives $h=1$ or

$$\psi_1=A_{12}\left(\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}+\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right) \tag{5.9.8}$$

Let us find the configuration of the resulting new node line. From the requirement that $\psi_1=0$ along a node line, we find the new node line equation to be

$$y=a-x \tag{5.9.9}$$

This is illustrated in Fig. 17 where the new node line is shown. This mode shape is a superposition mode, and by itself, it is not able to replace the two fundamental modes in information content. At least a second mode shape has to be found. For instance, if we locate the exciter such that

$$A_{21}=2A_{12} \tag{5.9.10}$$

we get as the superposition mode ($h=2$)

$$\psi_2=A_{12}\left(\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}+2\sin\frac{2\pi x}{x}\sin\frac{\pi y}{a}\right) \tag{5.9.11}$$

and the resulting node line is a wavy line, as shown in Fig. 18. By moving the excitation location around, a theoretically infinite number of super position mode shapes can be found.

Note that the two superposition modes that were generated are not necessarily orthogonal to each other:

$$\int_A\int \psi_1\psi_2\,\mathrm{d}A\neq 0 \tag{5.9.12}$$

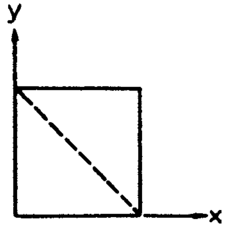

FIG. 17 A natural mode formed by superposition from the two modes of Fig. 16.

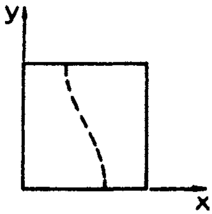

FIG. 18 A different superposition mode formed by the same two modes of Fig. 16.

To prove this in general, let

$$\psi_1 = U_{3(1)} + a_1 U_{3(2)} \tag{5.9.13}$$

$$\psi_2 = U_{3(1)} + a_2 U_{3(2)} \tag{5.9.14}$$

where $U_{3(1)}$ and $U_{3(2)}$ are two basic modes that are orthogonal and have the same natural frequency. Formulation of the integral gives

$$\int_A\int (U_{3(1)}^2 + a_1 a_2 U_{3(2)}^2 + (a_1 + a_2) U_{3(1)} U_{3(2)})\,\mathrm{d}A \neq 0 \tag{5.9.15}$$

Since, by definition, the two basic modes $U_{3(1)}$ and $U_{3(2)}$ are orthogonal, the third term disappears, but the other terms will not be in general 0. Thus it is necessary to go through an orthogonalization process to make the information useful for forced vibration prediction. The experimenter may argue that this is not his affair as long as he produces two superposition modes for two basic modes. This is certainly true, but how does the experimenter know that there were only two basic modes and not three or more? Only by performing the orthogonalization process before the experimental setup is removed can he be certain that he has measured enough superposition modes to give all the information that is needed. This is discussed next.

5.10. ORTHOGONAL MODES FROM NONORTHOGONAL SUPERPOSITION MODES

In this section, the Schmidt orthogonalization (Hadley, 1961) procedure for vectors is adapted to the superposition mode problem [Soedel and Dhar (1978)]. Let us assume that we have found two superposition modes, ψ_1 and ψ_2, of the many possible ones. These two modes are in general not orthogonal. To make them useful experimental information, one has to go through an orthogonalization process. We select one of the modes as a base mode. Let us choose ψ_1. To obtain a second mode, ψ_2', that is orthogonal to ψ_1, we subtract from ψ_2 a scalar multiple of ψ_1 (Fig. 19):

$$\psi_2' = \psi_2 - a_1 \psi_1 \tag{5.10.1}$$

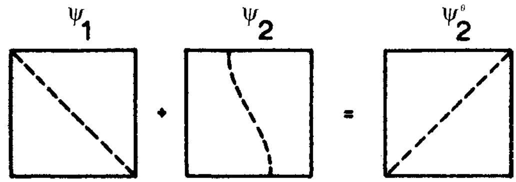

FIG. 19 Illustration of the generation of a natural mode from nonorthogonal superposition modes that is orthogonal to one of the superposition modes.

The requirement is that

$$\int_A\int \psi_2'\psi_1 \mathrm{d}A = 0 \tag{5.10.2}$$

Multiplying Eq. (5.10.1) by ψ_1, and integrating, we obtain

$$\int_A\int \psi_2'\psi_1 \mathrm{d}A = \int_A\int \psi_2\psi_1 \mathrm{d}A - a_1\int_A\int (\psi_1)^2 \mathrm{d}A \tag{5.10.3}$$

This gives

$$a_1 = \frac{\int_A\int \psi_2\psi_1 \mathrm{d}A}{\int_A\int (\psi_1)^2 \mathrm{d}A} \tag{5.10.4}$$

In the case of our example of a square plate, let us use

$$\psi_1 = A_{12}\left(\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + \sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right) \tag{5.10.5}$$

$$\psi_2 = A_{12}\left(\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + 2\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right) \tag{5.10.6}$$

These modes are not orthogonal. They are shown in Figs. 17 and 18. Let us now use ψ_1 as the base mode and construct a mode orthogonal to it. Since

$$\int_A\int (\psi_1)^2 \mathrm{d}A = \int_0^a\int_0^a \left(\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + \sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right)^2 \mathrm{d}x\mathrm{d}y$$
$$= \frac{a^2}{2} \tag{5.10.7}$$

$$\int_A\int \psi_1\psi_2 \mathrm{d}A = \int_0^a\int_0^a \left(\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + \sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right)$$
$$\times\left(\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + 2\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right)\mathrm{d}x\mathrm{d}y = \frac{3a^2}{4} \tag{5.10.8}$$

we get $a_1 = \frac{3}{2}$ and

$$\psi_2' = A_{12}'\left(\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} - \sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right) \tag{5.10.9}$$

where A_{12}' is again an arbitrary constant. This mode is now indeed orthogonal to ψ_1, as a check will easily reveal. The mode, in terms of its node line, is sketched in Fig. 19.

If there are three or more basic modes that superimpose, we proceed in a similar manner, except that we now have to measure n superposition modes $\psi_1, \psi_2, \ldots, \psi_n$, where n is the number of base modes that are superimposed. We choose one of these as the base mode, let us say ψ_1. The procedure is then, as before, except that we have to go through the orthogonalization process $n-1$ times. Let us illustrate this for the case of superposition of three modes ψ_1, ψ_2, and ψ_3. We choose ψ_1 as the base mode. Next, we obtain a mode ψ_2' that is orthogonal to ψ_1, utilizing information from ψ_2

$$\psi_2' = \psi_2 - a_{12}\psi_1 \tag{5.10.10}$$

As before, we obtain from the requirement that

$$\int_A\int \psi_2'\psi_1 \, \mathrm{d}A = 0 \tag{5.10.11}$$

the value for a_{12}:

$$a_{12} = \frac{\int_A\int \psi_1\psi_2 \, \mathrm{d}A}{\int_A\int (\psi_1)^2 \, \mathrm{d}A} \tag{5.10.12}$$

Next, we obtain the mode ψ_3' that is orthogonal to both ψ_2 and ψ_2', utilizing ψ_2 and ψ_3 information

$$\psi_3' = \psi_3 - a_{13}\psi_1 - a_{23}\psi_2' \tag{5.10.13}$$

This time we have two requirements, namely that ψ_3' be orthogonal to both ψ_1 and ψ_2'

$$\int_A\int \psi_3'\psi_1 \, \mathrm{d}A = 0 \tag{5.10.14}$$

$$\int_A\int \psi_3'\psi_2' \, \mathrm{d}A = 0 \tag{5.10.15}$$

Multiplying Eq. (5.10.13) by ψ_1 and integrating, then repeating the process by multiplying by ψ_2' and integrating, we have

$$a_{13} = \frac{\int_A\int \psi_3\psi_1 \, \mathrm{d}A}{\int_A\int \psi_1^2 \, \mathrm{d}A} \tag{5.10.16}$$

$$a_{23} = \frac{\int_A\int \psi_3\psi_2' \, \mathrm{d}A}{\int_a\int \psi_2'^2 \, \mathrm{d}A} \tag{5.10.17}$$

Note that all integrations have to be performed numerically since experimental mode data are almost never available in functional form but rather in the form of numerical arrays.

In general, if there is a superposition of n modes, the rth orthogonal mode is

$$\psi_1' = \psi_r - \sum_{i=1}^{r-1} a_{ir}\psi_i' \quad (r=2,3,\ldots,n) \tag{5.10.18}$$

where

$$\psi_1' = \psi_1 \tag{5.10.19}$$

$$a_{ir} = \frac{\int_A \int \psi_r \psi_i \,\mathrm{d}A}{\int_A \int \psi_i^2 \,\mathrm{d}A} \tag{5.10.20}$$

Returning to the square plate as an example, we get at an excitation frequency that is equivalent to $m^2+n^2=50$ a superposition of three modes: $(m,n)=(1,7),(7,1),(5,5)$. Let us assume that we have found by experiment the following three superposition modes:

$$\psi_1 = \sin\frac{\pi x}{a}\sin\frac{7\pi y}{a} + \sin\frac{7\pi x}{a}\sin\frac{\pi y}{a} + \sin\frac{5\pi x}{a}\sin\frac{5\pi y}{a} \tag{5.10.21}$$

$$\psi_2 = 2\sin\frac{\pi x}{a}\sin\frac{7\pi y}{a} + \sin\frac{7\pi x}{a}\sin\frac{\pi y}{a} + \frac{3}{2}\sin\frac{5\pi x}{a}\sin\frac{5\pi y}{a} \tag{5.10.22}$$

$$\psi_3 = \frac{1}{2}\sin\frac{\pi x}{a} + \sin\frac{7\pi y}{a} + \frac{3}{2}\sin\frac{7\pi x}{a}\sin\frac{\pi y}{a} + 2\sin\frac{5\pi x}{a}\sin\frac{5\pi y}{a} \tag{5.10.23}$$

We obtain $a_{12}=\frac{3}{2}$ and therefore

$$\psi_2' = \sin\frac{\pi x}{a}\sin\frac{7\pi y}{a} - \sin\frac{7\pi x}{a}\sin\frac{\pi y}{a} \tag{5.10.24}$$

Next, we obtain $a_{13}=\frac{4}{3}, a_{23}=-1$, and

$$\psi_3' = -\sin\frac{\pi x}{a}\sin\frac{7\pi y}{a} - \sin\frac{7\pi x}{a}\sin\frac{\pi y}{a} + 2\sin\frac{5\pi x}{a}\sin\frac{5\pi y}{a} \tag{5.10.25}$$

When we check for orthogonality, we find indeed that we now have

$$\int_0^a \int_0^a \psi_1 \psi_2' \,\mathrm{d}x\,\mathrm{d}y = 0 \tag{5.10.26}$$

$$\int_0^a \int_0^a \psi_1 \psi_3' \,\mathrm{d}x\,\mathrm{d}y = 0 \tag{5.10.27}$$

$$\int_0^a \int_0^a \psi_2' \psi_3' \,\mathrm{d}x\,\mathrm{d}y = 0 \tag{5.10.28}$$

This example also illustrates the point that it is not necessary for an orthogonalized superposition mode to be composed of all basic modes. In our case, ψ'_2 is composed of only two instead of all three. The same applies for the measured modes also, since it is always possible that the exciter is located at the node line of one of the basic modes. This will not affect the method. After all, there is a minute probability that the experimenter becomes lucky and chooses his exciter location such that he excites modes that are already orthogonal and, even more improbable, that they are equal to the mathematically generated basic modes.

We have now seen how we can construct orthogonal modes from the superposition information. Let us now ask some very practical questions. First, how does the experimenter know that he has a superposition effect? Answer: when the experimenter produces an apparently different mode shape at the same natural frequency as he moves the exciter to a different location. Second, how does the experimenter know how many basic modes are contributing to the superposition? Answer: he does not. What one has to do is to go through the orthogonalization procedure, assuming the worst, namely many contributing modes. To this end one should obtain three or four superposition modes. One is to select the base mode ψ_1 and then take one of the measured superposition modes, ψ_2, and generate a mode ψ'_2 that is orthogonal to ψ_1. Next, take the third measured superposition mode, ψ_3, and try to generate ψ'_3. If there were only two basic modes, ψ'_3 will come out to be 0, subject to the limits of experimental error. If there were more than two basic modes contributing to the superposition, ψ'_3 will turn out to be an orthogonal mode and the experimenter should try to generate a ψ'_4, and so on.

The procedure will suggest how many superposition modes have to be measured. Deciding if one has obtained a bonafide orthogonal mode or if resulting shapes are due only to experimental error may turn out to be tricky in some cases, but with some experience it should usually be possible to tell the difference because the quasimode resulting from experimental error should have a rather random distribution of numerous node lines. Also, any contributions from nonsuperposition modes because of coupling effects (which will be discussed next) should be recognizable. As long as one remembers that mathematical procedures alone cannot replace good judgment, reasonably good experimental results should be possible.

5.11. DISTORTION OF EXPERIMENTAL MODES BECAUSE OF DAMPING

Damping is always present, but usually in such small amounts that experimental modes approximate undamped natural modes quite well inside usual limits of experimental accuracy. One exception is the phenomenon of noncrossing node lines. This will be discussed using the example of a plate.

When a plate is excited harmonically, it will respond with a vibration that consists of a superposition of all of its natural modes. This will be discussed in a later chapter. Each mode participates with different intensity. Mathematically, this can be expressed as

$$u_3(\alpha_1,\alpha_2,t)=q_1(t)U_{31}(\alpha_1,\alpha_2)+q_2(t)U_{32}(\alpha_1,\alpha_2)+\cdots \tag{5.11.1}$$

where q_i is the modal participation factor and u_3 is the transverse deflection. If the system damping is negligible and if the excitation frequency is equal to the jth natural frequency, all ratios $q_1/q_j, q_2.q_j,\ldots,$ approach 0 except for q_j/q_j, which, of course, approaches unity. This means that at a natural frequency ω_j,

$$u_3=q_jU_{3j} \tag{5.11.2}$$

This is the reason why we are able experimentally to isolate one mode after the other.

Things become different if there is damping, either of a structural or air resistance nature. If we express the damping in terms of an equivalent viscous damping coefficient, the modal participation factors are of the form

$$q_i=\frac{F_i}{\omega_i^2\sqrt{[1-(\omega/\omega_i)^2]^2+4\xi_i^2(\omega/\omega_i)^2}} \tag{5.11.3}$$

where F_i, parameter dependent on mode shape, excitation force location, and distribution; ξ_i, damping factor; ω, excitation frequency; ω_i ith natural frequency.

We can now easily see that the presence of damping will tend to make the ratios q_i/q_j approach a small amount of ε_i instead of 0. It turns out that in the case of plates, the largest ε_i will be most likely ε_1 since ω_1 is smaller than $\omega_2,\omega_3,\ldots$ Thus we will often see an experimental mode shape that approaches a superposition of the resonant mode plus a small amount of the first mode:

$$\psi_j=U_{3j}+\varepsilon_1U_{31} \tag{5.11.4}$$

where ψ_j, superposition mode at the jth natural frequency; U_{3j}, jth natural mode; U_{31}, first natural mode.

Typically, the presence of this type of superposition will tend to prevent crossing of node lines. For a square plate with the $(m,n)=(2,2)$ mode for which no superposition modes of the first kind exist, a small amount of damping will produce a superposition mode

$$\psi_{32}=U_{322}+\varepsilon_{11}U_{311} \tag{5.11.5}$$

where

$$U_{322} = \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a} \tag{5.11.6}$$

$$U_{311} = \sin\frac{\pi x}{a}\sin\frac{\pi y}{a} \tag{5.11.7}$$

Superposition modes for various values of ε_{11} are shown in Fig. 20. The larger the damping effect, the larger the distance between the noncrossing lines. It is also possible that ε_{11} is a negative number. This depends on the location of the exciter. However, all that the negative sign does is to exchange the quadrants where the node line does not cross over.

This noncrossing behavior is also well known in shell structures. A typical case is shown in Fig. 21. However, due to the special frequency characteristic of shells, several damping coupled modes may participate to produce the experimental mode shape.

Let us now ask the question: What should the experimenter do when he encounters mode shapes that look like damping superpositions? On a first-order approximation level, it is recommended to him that he should simply indicate in his result report that the true node lines most likely do cross and offer as a choice corrected experimental data using his intuition. On a higher level of experimental fidelity, he should assume that the damping couples primarily the fundamental mode and use this idea to reconstruct the true mode. Let the superposition mode be

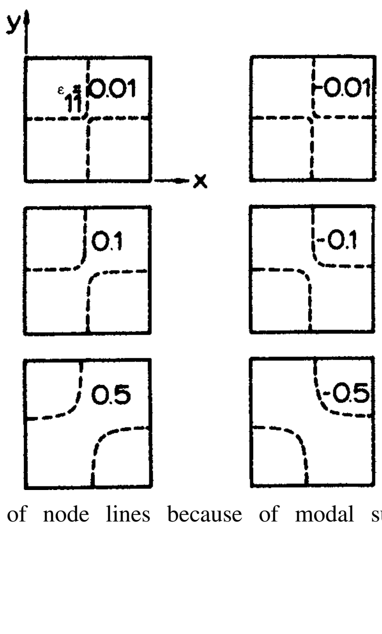

Fig. 20 Noncrossing of node lines because of modal superposition caused by damping.

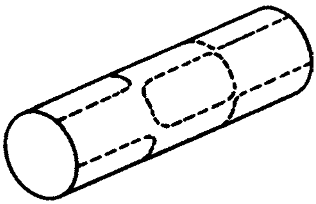

FIG. 21 A cylindrical shell example.

$$\psi_j = U_{3j} + \varepsilon_1 U_{31} \tag{5.11.8}$$

He does not know U_{3j} and ε_1, but he has measured U_{31} and, or course, ψ_j. Let us multiply the equation by U_{31} and integrate over the plate area. Utilizing the fact that

$$\int_A\int U_{3j}U_{31}\,\mathrm{d}A = 0 \quad (j \neq 1) \tag{5.11.9}$$

gives

$$\varepsilon_1 = \frac{\int_A\int \psi_j U_{31}\,\mathrm{d}A}{\int_A\int U_{31}^2\,\mathrm{d}A} \tag{5.11.10}$$

Thus the true mode shape is

$$U_{3j} = \psi_j - \varepsilon_1 U_{31} \tag{5.11.11}$$

Note again that all integration will have to be done numerically because mode shape data will not be available as a function but as an array.

5.12. SEPARATING TIME FORMALLY

Natural frequencies and modes are obtained from Eqs. (5.1.1) to (5.1.3), with $q_i = 0$:

$$L_1\{u_1, u_2, u_3\} = \rho h \frac{\partial^2 u_1}{\partial t^2} \tag{5.12.1}$$

$$L_2\{u_1, u_2, u_3\} = \rho h \frac{\partial^2 u_2}{\partial t^2} \tag{5.12.2}$$

$$L_3\{u_1, u_2, u_3\} = \rho h \frac{\partial^2 u_3}{\partial t^2} \tag{5.12.3}$$

To separate time, we try

$$u_1(\alpha_1, \alpha_2, t) = U_1(\alpha_1, \alpha_2)T(t) \tag{5.12.4}$$

$$u_2(\alpha_1, \alpha_2, t) = U_2(\alpha_1, \alpha_2)T(t) \tag{5.12.5}$$

$$u_3(\alpha_1, \alpha_2, t) = U_3(\alpha_1, \alpha_2)T(t) \tag{5.12.6}$$

Substituting Eqs. (5.12.4)–(5.12.6) into Eqs. (5.12.1)–(5.12.3) gives

$$TL_1\{U_1, U_2, U_3\} = \rho h U_1 \frac{d^2T}{dt^2} \tag{5.12.7}$$

$$TL_2\{U_1, U_2, U_3\} = \rho h U_2 \frac{d^2T}{dt^2} \tag{5.12.8}$$

$$TL_3\{U_1, U_2, U_3\} = \rho h U_3 \frac{d^2T}{dt^2} \tag{5.12.9}$$

Dividing the first equation by $\rho h U_1 T$, the second by $\rho h U_2 T$, and the third by $\rho h U_3 T$ gives

$$\frac{L_1\{U_1, U_2, U_3\}}{\rho h U_1} = \frac{1}{T}\frac{d^2T}{dt^2} = -\omega^2 \tag{5.12.10}$$

$$\frac{L_2\{U_1, U_2, U_3\}}{\rho h U_2} = \frac{1}{T}\frac{d^2T}{dt^2} = -\omega^2 \tag{5.12.11}$$

$$\frac{L_3\{U_1, U_2, U_3\}}{\rho h U_3} = \frac{1}{T}\frac{d^2T}{dt^2} = -\omega^2 \tag{5.12.12}$$

Because the left sides of the equation are functions, of space and the right sides are functions of time, each side of each equation must be equal to the same common constant. Common, because the right sides of all three equations are the same. By foresight, we may name this constant $-\omega^2$. We obtain, therefore the following equations:

$$L_1\{U_1, U_2, U_3\} + \rho h \omega^2 U_1 = 0 \tag{5.12.13}$$

$$L_2\{U_1, U_2, U_3\} + \rho h \omega^2 U_2 = 0 \tag{5.12.14}$$

$$L_3\{U_1, U_2, U_3\} + \rho h \omega^2 U_3 = 0 \tag{5.12.15}$$

and

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \tag{5.12.16}$$

This last equation has solutions of the following forms:

$$T = A \sin \omega t + B \cos \omega t \tag{5.12.17}$$

or

$$T = C \sin(\omega t - \phi) \tag{5.12.18}$$

or

$$T = De^{j\omega t} \tag{5.12.19}$$

The meaning of the constant ω is therefore that it is a frequency in radians per second at which the system wants to vibrate, namely, one of the natural frequencies in Sec. 5.1.

5.13. UNCOUPLING OF EQUATIONS OF MOTION

5.13.1. Ring

It is often possible to uncouple simultaneous equations of motion (Lang, 1962). For example, starting with the ring equations (4.3.3) and (4.3.4), and multiplying through by the ring width so that $D \cong EI, K = EA$, where $I = bh^3/12$ and $A = bh$, we obtain for zero forcing

$$L_1 u_\theta - L_2 u_3 = 0 \tag{5.13.1}$$

$$L_2 u_\theta - L_3 u_3 = 0 \tag{5.13.2}$$

where

$$L_1 = (1+p)\frac{\partial^2}{\partial\theta^2} - \frac{1}{\omega_0^2}\frac{\partial^2}{\partial t^2} \tag{5.13.3}$$

$$L_2 = p\frac{\partial^3}{\partial\theta^3} - \frac{\partial}{\partial\theta} \tag{5.13.4}$$

$$L_3 = 1 + p\frac{\partial^4}{\partial\theta^4} + \frac{1}{\omega_0^2}\frac{\partial^2}{\partial t^2} \tag{5.13.5}$$

and where

$$\omega_0^2 = \frac{E}{\rho a^2} \tag{5.13.6}$$

$$p = \frac{I}{Aa^2} \tag{5.13.7}$$

These equations are identical to those of Lang (1962) if we take into account that the transverse deflection is defined in opposite direction.

Multiplying Eq. (5.13.1) by L_2 and Eq. (5.13.2) by L_1 and subtracting one from the other results in

$$(L_1 L_3 - L_2^2) u_3 = 0 \tag{5.13.8}$$

Multiplying Eq. (5.13.1) by L_3 and Eq. (5.13.2) by L_2 gives, after subtraction,

$$(L_1 L_3 - L_2^2) u_\theta = 0 \tag{5.13.9}$$

Expanding the operator gives

$$L_1 L_3 - L_2^2 = p\left(\frac{\partial^6}{\partial\theta^6} + 2\frac{\partial^4}{\partial\theta^4} + \frac{\partial^2}{\partial\theta^2} - \frac{1}{\omega_0^2}\frac{\partial^6}{\partial\theta^4\partial t^2} + \frac{1+p}{p\omega_0^2}\frac{\partial^4}{\partial\theta^2\partial t^2} - \frac{1}{p\omega_0^2}\frac{\partial^2}{\partial t^2} - \frac{1}{p\omega_0^4}\frac{\partial^4}{\partial t^4}\right) \tag{5.13.10}$$

We may now solve either Eq. (5.13.8) or Eq. (5.13.9). To solve Eq. (5.13.8) for eigenvalues, we set

$$u_3(\theta,t)=U_3(\theta)e^{j\omega t} \tag{5.13.11}$$

Equation (5.13.8) becomes

$$\frac{\partial^6 U_3}{\partial\theta^6}+\left(2+\frac{\omega^2}{\omega_0^2}\right)\frac{\partial^4 U_3}{\partial\theta^4}+\left(1-\frac{\omega^2}{\omega_0^2}-\frac{\omega^2}{p\omega_0^2}\right)\frac{\partial^2 U_3}{\partial\theta^2}+\frac{\omega^2}{p\omega_0^2}\left(1-\frac{\omega^2}{\omega_0^2}\right)U_3=0 \tag{5.13.12}$$

Substituting

$$u_\theta(\theta,t)=U_\theta e^{j\omega t} \tag{5.13.13}$$

in Eq. (5.13.9) gives a similar expression, with U_3 replaced by U_θ.

Equation (5.13.13) may be solved in general. For the special case of a closed circular ring, we set by inspection

$$U_3(\theta)=A_n\cos(n\theta-\phi) \tag{5.13.14}$$

where $n=0,1,2,\ldots$ and A_n and ϕ are arbitrary constants, except that orthogonality requirements for forced solutions suggest that $\phi=0,\pi/2$. Equation (5.13.13) becomes

$$\omega^4-\omega^2[\omega_0^2(1+n^2)(1+pn^2)]+n^2(n^2-1)^2p\omega_0^4=0 \tag{5.13.15}$$

or

$$\omega_{\pi 1,2}^2=\frac{\omega_0^2(1+n^2)(1+pn^2)}{2}\left[1\pm\sqrt{1-\left(\frac{n^2-1}{n^2+1}\right)^2\frac{4n^2p}{(1+pn^2)^2}}\right] \tag{5.13.16}$$

As expected, this result can be shown to be identical to Eq. (5.3.12). The general solution of Eq. (5.13.12) is approached by setting

$$U_3(\theta)=\sum_{i=1}^{6}A_{ni}e^{\lambda_{ni}\theta} \tag{5.13.17}$$

5.13.2. General Uncoupling of the Equations of Motion

In general, all shell equations may be written in operator form. If the variables are u_1, u_2, and u_3,

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}=0 \tag{5.13.18}$$

Operating on the first row with L_{21}, L_{31}, the second row with $L_{11}L_{31}$, and the third row with $L_{11}L_{21}$ gives

$$\begin{bmatrix} L_{21}L_{31}L_{11} & L_{21}L_{31}L_{12} & L_{21}L_{31}L_{13} \\ L_{11}L_{31}L_{21} & L_{11}L_{31}L_{22} & L_{11}L_{31}L_{23} \\ L_{11}L_{21}L_{31} & L_{11}L_{21}L_{32} & L_{11}L_{21}L_{32} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0 \tag{5.13.19}$$

Subtracting the second row from the first and the third row from the first eliminates u_i and gives

$$\begin{bmatrix} L_{21}L_{31}L_{12} - L_{11}L_{31}L_{22} & L_{21}L_{31}L_{13} - L_{11}L_{31}L_{23} \\ L_{21}L_{31}L_{12} - L_{11}L_{21}L_{32} & L_{21}L_{31}L_{13} - L_{11}L_{21}L_{33} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 0 \tag{5.13.20}$$

One may now operate on the first row with $L_{21}L_{31}L_{11} - L_{11}L_{21}L_{31}$ and on the second row with $L_{21}L_{31}L_{12} - L_{11}L_{31}L_{22}$ and subtract the rows from each other, eliminating u_2

$$\begin{aligned} &[(L_{21}L_{31}L_{12} - L_{11}L_{21}L_{32})(L_{21}L_{31}L_{13} - L_{11}L_{31}L_{23}) \\ &\quad -(L_{21}L_{31}L_{12} - L_{11}L_{31}L_{22})(L_{21}L_{31}L_{13} - L_{11}L_{21}L_{33})]u_3 = 0 \end{aligned} \tag{5.13.21}$$

For shell theories where $L_{ij} = L_{ji}$, which some investigators feel is a requirement for a good theory, Eq. (5.13.21) reduces to

$$L_{12}L_{13}[(L_{12}L_{13} - L_{11}L_{23})^2 - (L_{12}^2 - L_{11}L_{22})(L_{13}^2 - L_{11}L_{33})]u_3 = 0 \tag{5.13.22}$$

5.14. IN-PLANE VIBRATIONS OF RECTANGULAR PLATE

For $\alpha_1 = x, \alpha_2 = y, A_1 = 1 \text{and} A_2 = 1$, Eqs.(4.4.3) and (4.4.4) become

$$-\frac{\partial Nxx}{\partial x} - \frac{\partial Nxy}{\partial y} + \rho h \ddot{u}_x = q_x \tag{5.14.1}$$

$$-\frac{\partial Nxy}{\partial x} - \frac{\partial N_{yy}}{\partial y} + \rho h \ddot{u}_y = q_y \tag{5.14.2}$$

Equations (4.4.8)–(4.4.10) become

$$\varepsilon_{xx}^0 = \frac{\partial u_x}{\partial x} \tag{5.14.3}$$

$$\varepsilon_{yy}^0 = \frac{\partial u_y}{\partial y} \tag{5.14.4}$$

$$\varepsilon_{xy}^0 = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \tag{5.14.5}$$

Substituting Eqs. (5.14.3)–(5.14.5) into Eqs. (2.5.9), (2.5.11) and (2.5.12)

$$N_{xx} = K(\varepsilon^0_{xx} + \mu\varepsilon^0_{yy}) = K\left(\frac{\partial u_x}{\partial x} + \mu\frac{\partial u_y}{\partial y}\right) \tag{5.14.6}$$

$$N_{yy} = K(\varepsilon^0_{yy} + \mu\varepsilon^0_{xx}) = K\left(\frac{\partial u_y}{\partial y} + \mu\frac{\partial u_x}{\partial x}\right) \tag{5.14.7}$$

$$N_{xy} = K\frac{(1-\mu)}{2}\varepsilon^0_{xy} = K\frac{(1-\mu)}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \tag{5.14.8}$$

Finally, Eqs. (5.14.1) and (5.14.2) become

$$-K\left[\frac{\partial^2 u_x}{\partial x^2} + \frac{(1-\mu)}{2}\frac{\partial^2 u_x}{\partial y^2} + \frac{(1+\mu)}{2}\frac{\partial^2 u_y}{\partial x \partial y}\right] + \rho h \ddot{u}_x = q_x \tag{5.14.9}$$

$$-K\left[\frac{\partial^2 u_y}{\partial y^2} + \frac{(1-\mu)}{2}\frac{\partial^2 u_y}{\partial x^2} + \frac{(1+\mu)}{2}\frac{\partial^2 u_x}{\partial x \partial y}\right] + \rho h \ddot{u}_y = q_y \tag{5.14.10}$$

These are the general equations of in-plane motion of plates in Cartesian coordinates. See also Bardell, Langley and Dunsdon (1996).

To obtain the natural frequencies and modes, we set $q_x = 0$ and $q_y = 0$, and eliminate time by substituting the fact that at natural frequencies the free vibration is harmonic:

$$u_x(x, y, t) = U_x(x, y)e^{j\omega t} \tag{5.14.11}$$

$$u_y(x, y, t) = U_y(x, y)e^{j\omega t} \tag{5.14.12}$$

Equations (5.14.9) and (5.14.10) become

$$\frac{\partial^2 U_x}{\partial x^2} + \left(\frac{1-\mu}{2}\right)\frac{\partial^2 U_x}{\partial y^2} + \left(\frac{1+\mu}{2}\right)\frac{\partial^2 U_y}{\partial x \partial y} + \frac{\rho h}{K}\omega^2 U_x = 0 \tag{5.14.13}$$

$$\frac{\partial^2 U_y}{\partial y^2} + \left(\frac{1-\mu}{2}\right)\frac{\partial^2 U_y}{\partial x^2} + \left(\frac{1+\mu}{2}\right)\frac{\partial^2 U_x}{\partial x \partial y} + \frac{\rho h}{K}\omega^2 U_y = 0 \tag{5.14.14}$$

To obtain a general solution of these equations is not a realistic option. However, it is possible to obtain analytical, closed form solutions for certain sets of boundary conditions, for example, for the case of a rectangular plate whose edges are composed of rectangular teeth that fit into boundary receptacles in such a way that deflections normal to the edges are possible, but deflections tangential to the edges are not. This case occurs in engineering praxis only very infrequently. As a matter of fact, it may be even difficult to reproduce in an experimental setting. But the case does serve as an example that illustrates in-plane vibration behavior and it can also serve as a test case for checking finite element programs.

The boundary conditions for this case that the solutions must satisfy are:

$$u_x(x,0,t) = U_x(x,0)=0 \tag{5.14.15}$$

$$u_x(x,b,t) = U_x(x,b)=0 \tag{5.14.16}$$

$$u_y(0,y,t) = U_y(0,y)=0 \tag{5.14.17}$$

$$u_y(a,y,t) = U_y(a,y)=0 \tag{5.14.18}$$

$$N_{xx}(0,y,t) = \left(\frac{\partial U_x}{\partial x}+\mu\frac{\partial U_y}{\partial y}\right)(0,y)=0 \tag{5.14.19}$$

$$N_{xx}(a,y,t) = \left(\frac{\partial U_x}{\partial x}+\mu\frac{\partial U_y}{\partial y}\right)(a,y)=0 \tag{5.14.20}$$

$$N_{yy}(x,0,t) = \left(\frac{\partial U_y}{\partial y}+\mu\frac{\partial U_x}{\partial x}\right)(x,0)=0 \tag{5.14.21}$$

$$N_{yy}(x,b,t) = \left(\frac{\partial U_y}{\partial y}+\mu\frac{\partial U_x}{\partial x}\right)(x,b)=0 \tag{5.14.22}$$

The natural mode components can be obtained by inspection:

$$U_x(x,y) = A\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{5.14.23}$$

$$U_y(x,y) = B\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{5.14.24}$$

Equations (5.14.23) and (5.14.24) satisfy all boundary conditions. Substituting them into Eqs. (5.14.13) and (5.14.14) gives

$$\begin{vmatrix} k_{11}-\frac{\omega^2\rho h}{K} & k_{12} \\ k_{21} & k_{22}-\frac{\omega^2\rho h}{K} \end{vmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.14.25}$$

where

$$k_{11}=\left(\frac{m\pi}{a}\right)^2+\frac{(1-\mu)}{2}\left(\frac{n\pi}{b}\right)^2 \tag{5.14.26}$$

$$k_{12}=k_{21}=\frac{(1+\mu)}{2}\left(\frac{m\pi}{a}\cdot\frac{n\pi}{b}\right) \tag{5.14.27}$$

$$k_{22}=\left(\frac{n\pi}{b}\right)^2+\frac{(1-\mu)}{2}\left(\frac{m\pi}{a}\right)^2 \tag{5.14.28}$$

This proves that Eqs. (5.14.23) and (5.14.24) are a valid set of solutions. If they were not, Eqs. (5.14.13) and (5.14.14) could not be converted to a set of algebraic equations. The drawback of the inspection approach is, however, that completeness is not assured; it has to be proven experimentally.

To satisfy Eq. (5.14.25) in a nontrivial way, the determinant of its matrix has to be 0. Expanding the determinant gives

$$\left(k_{11}-\frac{\omega^2\rho h}{K}\right)\left(k_{22}-\frac{\omega^2\rho h}{K}\right)-k_{12}^2=0 \tag{5.14.29}$$

or

$$\omega^4-C_1\omega^2+C_2=0 \tag{5.14.30}$$

where

$$C_1=\frac{K(k_{11}+k_{22})}{\rho h} \tag{5.14.31}$$

$$C_2=\frac{K^2(k_{11}k_{22}-k_{12}^2)}{(\rho h)^2} \tag{5.14.32}$$

Solving Eq. (5.14.30), we obtain for every (m,n) combination two natural frequencies, given by (C_1 and C_2 are functions of (m,n))

$$\omega_{mn1}=\sqrt{\frac{C_1}{2}-\frac{1}{2}\sqrt{C_1^2-4C_2}} \tag{5.14.33}$$

$$\omega_{mn2}=\sqrt{\frac{C_1}{2}+\frac{1}{2}\sqrt{C_1^2-4C_2}} \tag{5.14.34}$$

The natural mode component amplitudes are obtained by substituting in turn Eqs. (5.14.33) and (5.14.34) into Eq. (5.14.25).

This gives for ω_{mni}, where $i=1,2$,

$$A_{mni}\left(k_{11}-\frac{\omega_{mni}^2\rho h}{K}\right)+B_{mni}k_{12}=0 \tag{5.14.35}$$

where the mode component amplitudes A and B of Eqs. (5.14.20) and (5.14.21) have now the subscript (mni) to signify that the ratio of these amplitudes is different depending if $i=1$ or 2, and that this ratio is, of course, also a function of (m,n). We obtain

$$\frac{B_{mni}}{A_{mni}}=\frac{\omega_{mni}^2\rho h/K-k_{11}}{k_{12}} \tag{5.14.36}$$

Substituting this into the mode Eq. (5.14.23) and (5.14.24) gives the natural modes

$$U_{xmni}=A_{mni}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{5.14.37}$$

$$U_{ymni}=A_{mni}\left(\frac{\omega_{mni}^2\rho h/K-k_{11}}{k_{12}}\right)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{5.14.38}$$

where A_{mni} is an arbitrary constant and can be set to unity.

5.15. IN-PLANE VIBRATION OF CIRCULAR PLATES

Using polar coordinates $\alpha_1 = r, \alpha_2 = \theta$ with $A_1 = 1, A_2 = r$, Eqs. (4.4.3) and (4.4.4) become, in general,

$$\frac{-\partial(N_{rr}r)}{\partial r} - \frac{\partial(N_{\theta r})}{\partial\theta} + N_{\theta\theta} + r\rho h\ddot{u}_r = 0 \tag{5.15.1}$$

$$\frac{-\partial(N_{r\theta}r)}{\partial r} - \frac{\partial(N_{\theta\theta})}{\partial\theta} - N_{\theta r} + r\rho h\ddot{u}_\theta = 0 \tag{5.15.2}$$

The membrane strain expressions of Eqs. (4.4.8)–(4.4.10) become

$$\varepsilon_{rr}^0 = \frac{\partial u_r}{\partial r} \tag{5.15.3}$$

$$\varepsilon_{\theta\theta}^0 = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} \tag{5.15.4}$$

$$\varepsilon_{r\theta}^0 = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} \tag{5.15.5}$$

Equations (5.15.1)–(5.15.5) define the general in-plane, free vibration of circular plates.

If we confine ourselves in the following to the axisymmetric vibrations of circular plates, we set $u_\theta = 0$ and $\partial(\cdot)/\partial\theta = 0$. This results in

$$N_{r\theta} = N_{\theta r} = 0, \quad \ddot{u}_\theta = 0, \quad \partial N_{\theta\theta}/\partial\theta = 0 \tag{5.15.6}$$

Equation (5.15.2) disappears and Eq. (5.15.1) becomes

$$-\frac{\partial(N_{rr}r)}{\partial r} + N_{\theta\theta} + r\rho h\ddot{u}_r = 0 \tag{5.15.7}$$

where

$$N_{rr} = K\left(\frac{\partial u_r}{\partial r} + \mu\frac{u_r}{r}\right) \tag{5.15.8}$$

$$N_{\theta\theta} = K\left(\frac{u_r}{r} + \mu\frac{\partial u_r}{\partial r}\right) \tag{5.15.9}$$

Combining all equations gives the equation of motion in radial displacement form:

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}u_r - \frac{\rho h}{K}\ddot{u}_r = 0 \tag{5.15.10}$$

To solve this equation for the natural frequencies and modes, we separate time by setting

$$u_r(r,t) = U_r(r)\mathrm{e}^{j\omega t} \tag{5.15.11}$$

and obtain

$$\frac{d^2U_r}{dr^2}+\frac{1}{r}\frac{dU_r}{dr}-\frac{1}{r^2}U_r+\frac{\rho h\omega^2}{K}U_r=0 \tag{5.15.12}$$

Substituting a new variable

$$\xi=\lambda r \tag{5.15.13}$$

where

$$\lambda^2=\frac{\rho h\omega^2}{K} \tag{5.15.14}$$

and rearranging gives

$$\frac{d^2U_r}{d\xi^2}+\frac{1}{\xi}\frac{dU_r}{d\xi}+\left(1-\frac{1}{\xi^2}\right)U_r=0 \tag{5.15.15}$$

This is Bessel's equation of first order and has as solution

$$U_r=AJ_1(\xi)+BY_1(\xi) \tag{5.15.16}$$

or

$$U_r=AJ_1(\lambda r)+BY_1(\lambda r) \tag{5.15.17}$$

In the following, let us consider first the example of a circular plate without annulus that is constraint at the boundary so that

$$u_r(a,t)=0 \tag{5.15.18}$$

or

$$U_r(a)=0 \tag{5.15.19}$$

where $r=a$ is the radius of the plate at the boundary. The other boundary condition is the fact that the plate has no central annular boundary and deflection at $r=0$ has to be finite. This eliminates the $Y_1(\lambda r)$ function:

$$B=0 \tag{5.15.20}$$

Substituting Eq. (5.15.17) into Eq. (5.15.19) and applying Eq. (5.15.20) gives

$$J_1(\lambda a)=0 \tag{5.15.21}$$

The (λa) values that satisfy this equation are listed in Table 4 for the first four roots, numbered $m=0,1,2,3$. The physical meaning of m is that it represents the number of interior node circles.

The associated natural frequencies are obtained from Eq. (5.15.14):

$$\omega_m=\frac{(\lambda a)_m}{a}\sqrt{\frac{K}{\rho h}}=\frac{(\lambda a)_m}{a}\sqrt{\frac{E}{(1-\mu^2)\rho}} \tag{5.15.22}$$

Table 4 Values for $(\lambda a)_m$ for Fixed Edge

m	$(\lambda a)_m$
0	2.404
1	3.832
2	5.135
3	6.379

We note again as for the rectangular plate, that thickness changes of the plate will not, within the basic assumptions, change the natural frequencies associated with the in-plane portion of the plate. This result can be generalized for all plates; see Sec. (19.3).

The natural modes are, from Eq. (5.15.17),

$$U_{rm}(r) = A_m J_1(\lambda_m r) \tag{5.15.23}$$

where $\lambda_m = (\lambda a)_m / a$ and A_m is an arbitrary constant.

If the circular plate is again without central hole, but is free to move radially at $r = a$, the boundary condition (5.25.18) is replaced by

$$N_{rr}(a, t) = 0 \tag{5.15.24}$$

or

$$\left(\frac{\partial u_r}{\partial r} + \mu \frac{u_r}{r}\right)(a, t) = 0 \tag{5.15.25}$$

Applying Eq. (5.15.11), this condition becomes in terms of the mode U_r,

$$\frac{\mathrm{d}U_r}{\mathrm{d}r}(a) + \frac{\mu}{a} U_r(a) = 0 \tag{5.15.26}$$

Substituting the general solution (5.15.17) into this equation gives

$$\frac{\mathrm{d}J_1}{\mathrm{d}r}(\lambda a) + \frac{\mu}{a} J_1(\lambda a) = 0 \tag{5.15.27}$$

Since, from identity (5.7.9), we obtain

$$a \frac{\mathrm{d}J_1}{\mathrm{d}r}(\lambda a) = J_1(\lambda a) - (\lambda a) J_2(\lambda a) \tag{5.15.28}$$

Equation (5.15.27) becomes

$$(1 + \mu) J_1(\lambda a) - (\lambda a) J_2(\lambda a) = 0 \tag{5.15.29}$$

The (λa) that satisfy this equation are listed in Table 5 for the first four roots, labeled $m = 0, 1, 2, 3$. The physical meaning of m is again that it represents the number of interior nodal circles.

The associated natural frequencies are given by Eq. (5.15.22) with the $(\lambda a)_m$ values given by Table 5. The natural modes are again given by Eq. (5.15.23).

TABLE 5 Values for $(\lambda a)_m$ for Free Edge

m	$(\lambda a)_m$
0	2.049
1	5.389
2	8.572
3	11.732

5.16. DEEP CIRCULAR CYLINDRICAL PANEL SIMPLY SUPPORTED AT ALL EDGES

In this case, the circular clyindrical shell has the same boundary conditions at $x=0$ and $x=L$ as Eqs. (5.5.1)–(5.5.8), but is open in the θ direction; see Fig. 22. Therefore, the continuity conditions of Sec. 5.5 in θ direction, which expressed themselves in a solution selection of sine and cosine functions in θ-direction, do not any longer apply. They are now replaced by boundary conditions at $\theta=0$ and $\theta=\gamma$, for which an exact, closed form solution can be found. They are

$$u_3(x,0,t)=0 \tag{5.16.1}$$

$$u_x(x,0,t)=0 \tag{5.16.2}$$

$$M_{\theta\theta}(x,0,t)=0 \tag{5.16.3}$$

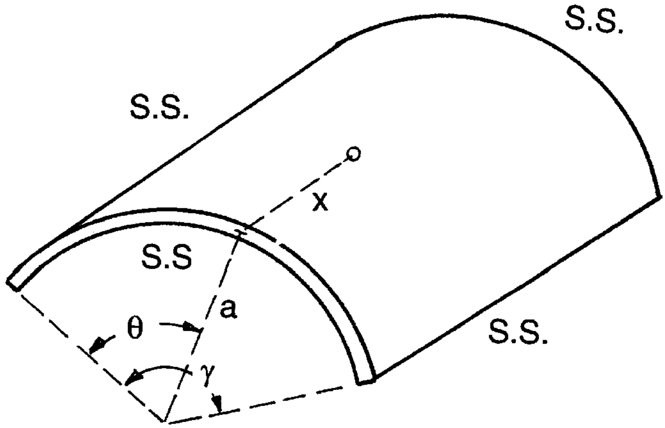

FIG. 22 Simply supported, circular cylindrical shell segment.

$$N_{\theta\theta}(x,0,t)=0 \tag{5.16.4}$$

$$u_3(x,\gamma,t)=0 \tag{5.16.5}$$

$$u_x(x,\gamma,t)=0 \tag{5.16.6}$$

$$M_{\theta\theta}(x,\gamma,t)=0 \tag{5.16.7}$$

$$N_{\theta\theta}(x,\gamma,t)=0 \tag{5.16.8}$$

The governing Eqs. (5.5.9)–(5.5.11) apply, and the solution process follows Eqs. (5.5.12)–(5.5.33).

With time separated by Eqs. (5.5.12)–(5.5.14), the boundary conditions of Eqs. (5.16.1)–(5.16.8) become

$$U_3(x,0)=0 \tag{5.16.9}$$

$$U_x(x,0)=0 \tag{5.16.10}$$

$$M'_{\theta\theta}(x,0)=0 \tag{5.16.11}$$

$$N'_{\theta\theta}(x,0)=0 \tag{5.16.12}$$

$$U_3(x,\gamma)=0 \tag{5.16.13}$$

$$U_x(x,\gamma)=0 \tag{5.16.14}$$

$$M'_{\theta\theta}(x,\gamma)=0 \tag{5.16.15}$$

$$N'_{\theta\theta}(x,\gamma)=0 \tag{5.16.16}$$

By inspection, the following solution will satisfy Eqs. (5.5.15)–(5.5.17), and all boundary conditions (5.5.34)–(5.5.41) and (5.16.9)–(5.16.16):

$$U_x(x,\theta)=A\cos\frac{m\pi x}{L}\sin\frac{n\pi\theta}{\gamma} \tag{5.16.17}$$

$$U_\theta(x,\theta)=B\sin\frac{m\pi x}{L}\cos\frac{n\pi\theta}{\gamma} \tag{5.16.18}$$

$$U_3(x,\theta)=C\sin\frac{m\pi x}{L}\sin\frac{n\pi\theta}{\gamma} \tag{5.16.19}$$

Substituting the equations into Eqs. (5.5.15–(5.5.17) gives again an algebraic matrix equation of the form of Eq. (5.5.66), namely

$$\begin{bmatrix} \rho h\omega^2-k_{11} & k_{12} & k_{13} \\ k_{21} & \rho h\omega^2-k_{22} & k_{23} \\ k_{31} & k_{32} & \rho h\omega^2-k_{33} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix}=0 \tag{5.16.20}$$

where

$$k_{11} = K\left[\left(\frac{m\pi}{L}\right)^2+\frac{(1-\mu)}{2}\left(\frac{n\pi}{a\gamma}\right)^2\right] \tag{5.16.21}$$

$$k_{12} = -K\left[\left(\frac{n\pi}{a\gamma}\right)\left(\frac{m\pi}{L}\right)\frac{(1+\mu)}{2}\right] = k_{21} \tag{5.16.22}$$

$$k_{13} = \frac{\mu K}{a}\left(\frac{m\pi}{L}\right) \tag{5.16.23}$$

$$k_{22} = \left(K + \frac{D}{a^2}\right)\left(\frac{(1-\mu)}{2}\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{a\gamma}\right)^2\right) \tag{5.16.24}$$

$$k_{23} = \left(\frac{K}{a}\right)\left(\frac{n\pi}{a\gamma}\right) + \left(\frac{D}{a}\right)\left(\frac{n\pi}{a\gamma}\right)\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{a\gamma}\right)^2\right] = k_{32} \tag{5.16.25}$$

$$k_{31} = \left(\frac{\mu K}{a}\right)\left(\frac{m\pi}{L}\right) = k_{13} \tag{5.16.26}$$

$$k_{33} = \left(D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{a\gamma}\right)^2\right]^2 + \frac{K}{a^2}\right) \tag{5.16.27}$$

Setting the determinant to 0 and expanding it gives Eqs. (5.5.73)–(5.5.80) and then the natural frequencies. Again, for every (m, n) combination there will be three natural frequencies.

The natural modes are given by Eqs. (5.5.85)–(5.5.87), as before, but with the modified k_{ij} values of Eqs. (5.16.21)–(5.16.27).

5.17. NATURAL MODE SOLUTIONS BY POWER SERIES

From a historical perspective, power series solutions to eigenvalue problems of strings, beams, membranes, plates, rings, and shells were early choices of approach. For example, Bessel solved the equation named after him, Eq. (5.6.16), by the power series method. Recurring series expressions he defined as functions which were later named Bessel functions. All common functions that appear in this text were originally obtained by power series approaches, such as Legendre functions, a power series solution to Legendre's equation (6.2.20), hyperbolic functions, and sin and cos functions. Even the basic approach of using the $e^{\lambda x}$ solution assumption for solving ordinary differential equations in space is indirectly a power series approach and $e^{\lambda x}$ is defined in terms of a power series.

Even today, the power series method, while rarely used in its pure form, remains attractive. Since the power series approach is not typically taught in introductory vibration courses, it is illustrated in the following by examples in a rather slow and pedestrian fashion, for ease of understanding.

5.17.1. Vibrating Rod

As a first example, we consider a vibrating rod. The equation of motion is

$$-EA\frac{\partial^2 u_x}{\partial x^2}+\rho A\frac{\partial^2 u_x}{\partial t^2}=0 \tag{5.17.1}$$

At a natural frequency,

$$u_x(x,t)=U_x(x)\mathrm{e}^{j\omega t} \tag{5.17.2}$$

We obtain upon substitution

$$\frac{\mathrm{d}^2 U_x}{\mathrm{d}x^2}+\lambda^2 U_x=0 \tag{5.17.3}$$

where

$$\lambda^2=\frac{\rho\omega^2}{E} \tag{5.17.4}$$

We introduce the power series

$$U_x(x)=c_0+c_1x+c_2x^2+c_3x^3+\cdots=\sum_{i=0}^{\infty}c_ix^i \tag{5.17.5}$$

and its derivatives

$$\frac{\mathrm{d}U_x}{\mathrm{d}x}=\sum_{i=0}^{\infty}ic_ix^{i-1} \tag{5.17.6}$$

$$\frac{\mathrm{d}^2U_x}{\mathrm{d}x^2}=\sum_{i=0}^{\infty}i(i-1)c_ix^{i-2} \tag{5.17.7}$$

Note that we may write the second derivative in a more convenient for by replacing i by $i+2$

$$\frac{\mathrm{d}^2U_x}{\mathrm{d}x^2}=\sum_{i=0}^{\infty}(i+2)(i+1)c_{i+2}x^i \tag{5.17.8}$$

Substituting these expressions into the differential equation, we obtain

$$\sum_{i=0}^{\infty}\left[(i+2)(i+1)c_{i+2}+\lambda^2c_i\right]x^i=0 \tag{5.17.9}$$

This equation can only be satisfied if the coefficients of the x^i are 0:

$$(i+2)(i+1)c_{i+2}+\lambda^2c_i=0 \quad (i=0,1,2,\ldots) \tag{5.17.10}$$

This gives

$$c_{i+2} = -\frac{\lambda^2 c_i}{(i+2)(i+1)} \tag{5.17.11}$$

or

$$i=0, \quad c_2 = -\frac{\lambda^2 c_0}{1\cdot 2} = -\frac{\lambda^2 c_0}{2!} \tag{5.17.12}$$

$$i=1, \quad c_3 = -\frac{\lambda^2 c_1}{1\cdot 2\cdot 3} = -\frac{\lambda^2 c_1}{3!} \tag{5.17.13}$$

$$i=2, \quad c_4 = -\frac{\lambda^2 c_2}{3\cdot 4} = \frac{\lambda^4 c_0}{1\cdot 2\cdot 3\cdot 4} = \frac{\lambda^4 c_0}{4!} \tag{5.17.14}$$

$$i=3, \quad c_5 = -\frac{\lambda^2 c_3}{4\cdot 5} = \frac{\lambda^4 c_1}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{\lambda^4 c_1}{5!} \tag{5.17.15}$$

$$i=4, \quad c_6 = -\frac{\lambda^2 c_4}{5\cdot 6} = \frac{-\lambda^6 c_0}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} = -\frac{\lambda^6 c_0}{6!} \tag{5.17.16}$$

etc.

Thus

$$U_x(x) = c_0\left[1 - \frac{(\lambda x)^2}{2!} + \frac{(\lambda x)^4}{4!} - \frac{(\lambda x)^6}{6!} + \cdots\right] + \frac{c_1}{\lambda}\left[(\lambda x) - \frac{(\lambda x)^3}{3!} + \frac{(\lambda x)^5}{5!} - \cdots\right] \tag{5.17.17}$$

We may now evaluate the solution for a particular boundary condition, or we can replace the two series by their functional names, namely $\sin(\lambda x)$ for the first series in brackets and $\cos(\lambda x)$ for the second series in brackets:

$$U_x(x) = A\sin(\lambda x) + B\cos(\lambda x) \tag{5.17.18}$$

where

$$A = \frac{c_1}{\lambda} \tag{5.17.19}$$

$$B = c_0 \tag{5.17.20}$$

and proceed in this form.

5.17.2. Transversely Vibrating Beam

Next, let us find the general natural mode expression for the transversely vibrating beam. The equation of motion is

$$EI\frac{\partial^4 u_3}{\partial x^4} + \rho A\frac{\partial^2 u_3}{\partial t^2} = 0 \tag{5.17.21}$$

Substituting, for the vibration at a natural frequency,

$$u_3(x,t) = U_3(x)\mathrm{e}^{j\omega t} \tag{5.17.22}$$

gives

$$\frac{\mathrm{d}^4 U_3}{\mathrm{d}x^4} - \lambda^4 U_3 = 0 \tag{5.17.23}$$

where

$$\lambda^4 = \frac{\rho A \omega^2}{EI} \tag{5.17.24}$$

As before, we use the series

$$U_x(x) = \sum_{i=0}^{\infty} c_i x^i \tag{5.17.25}$$

Its derivatives are

$$\frac{\mathrm{d}U_x}{\mathrm{d}x} = \sum_{i=0}^{\infty} i c_i x^{i-1} \tag{5.17.26}$$

$$\frac{\mathrm{d}^2 U_x}{\mathrm{d}x^2} = \sum_{i=0}^{\infty} i(i-1)c_i x^{i-2} = \sum_{i=0}^{\infty} (i+2)(i+1)c_{i+2} x^i \tag{5.17.27}$$

$$\frac{\mathrm{d}^3 U_x}{\mathrm{d}x^3} = \sum_{i=0}^{\infty} i(i-1)(i-2)c_i x^{i-3}$$

$$= \sum_{i=0}^{\infty} (i+3)(i+2)(i+1)c_{i+3} x^i \tag{5.17.28}$$

$$\frac{\mathrm{d}^4 U_x}{\mathrm{d}x^4} = \sum_{i=0}^{\infty} i(i-1)(i-2)(i-3)c_i x^{i-4}$$

$$= \sum_{i=0}^{\infty} (i+4)(i+3)(i+2)(i+1)c_{i+4} x^i \tag{5.17.29}$$

Substituting this into the differential equation gives

$$\sum_{i=0}^{\infty} \left[(i+4)(i+3)(i+2)(i+1)c_{i+4} - \lambda^4 c_i\right] x^i = 0 \tag{5.17.30}$$

Again, this equation can only be satisfied if all coefficients of the x^i are 0. This gives

$$c_{i+4} = \frac{\lambda^4 c_i}{(i+1)(i+2)(i+3)(i+4)} \tag{5.17.31}$$

or

$$i=0, \quad c_4 = \frac{\lambda^4 c_0}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{\lambda^4 c_0}{4!} \tag{5.17.32}$$

$$i=1, \quad c_5=\frac{\lambda^4 c_1}{2\cdot 3\cdot 4\cdot 5}=\frac{\lambda^4 c_1}{5!} \tag{5.17.33}$$

$$i=2, \quad c_6=\frac{\lambda^4 c_2}{3\cdot 4\cdot 5\cdot 6} \tag{5.17.34}$$

$$i=3, \quad c_7=\frac{\lambda^4 c_3}{4\cdot 5\cdot 6\cdot 7} \tag{5.17.35}$$

$$i=4, \quad c_8=\frac{\lambda^4 c_4}{5\cdot 6\cdot 7\cdot 8}=\frac{\lambda^8 c_0}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}=\frac{\lambda^8 c_0}{8!} \tag{5.17.36}$$

$$i=5, \quad c_9=\frac{\lambda^4 c_5}{6\cdot 7\cdot 8\cdot 9}=\frac{\lambda^8 c_1}{9!} \tag{5.17.37}$$

$$i=6, \quad c_{10}=\frac{\lambda^4 c_6}{7\cdot 8\cdot 9\cdot 10}=\frac{\lambda^8 c_2 1\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}=\frac{2\lambda^8 c_2}{10!} \tag{5.17.38}$$

$$i=7, \quad c_{11}=\frac{\lambda^4 c_7}{8\cdot 9\cdot 10\cdot 11}=\frac{\lambda^8 c_3 1\cdot 2\cdot 3}{11!} \tag{5.17.39}$$

etc.

Thus, the series solution becomes

$$\begin{aligned} U_3(x) = c_0&\left[1+\frac{(\lambda x)^4}{4!}+\frac{(\lambda x)^8}{8!}+\frac{(\lambda x)^{12}}{12!}+\cdots\right] \\ &+\frac{c_1}{\lambda}\left[(\lambda x)+\frac{(\lambda x)^5}{5!}+\frac{(\lambda x)^9}{9!}+\cdots\right] \\ &+2\frac{c_2}{\lambda^2}\left[\frac{(\lambda x)^2}{2!}+\frac{(\lambda x)^6}{6!}+\frac{(\lambda x)^{10}}{10!}+\cdots\right] \\ &+6\frac{c_3}{\lambda^3}\left[\frac{(\lambda x)^3}{3!}+\frac{(\lambda x)^7}{7!}+\frac{(\lambda x)^{11}}{11!}+\cdots\right] \end{aligned} \tag{5.17.40}$$

While this is a perfectly fine solution, we recognize that we can bring it into the more conventional form by utilizing the power series definitions of $\sin(\lambda x), \cos(\lambda x), \sinh(\lambda x)$, and $\cosh(\lambda x)$:

$$\frac{1}{2}(\cosh\lambda x+\cos\lambda x)=1+\frac{(\lambda x)^4}{4!}+\frac{(\lambda x)^8}{8!}+\cdots \tag{5.17.41}$$

$$\frac{1}{2}(\sinh\lambda x+\sin\lambda x)=(\lambda x)+\frac{(\lambda x)^5}{5!}+\frac{(\lambda x)^9}{9!}+\cdots \tag{5.17.42}$$

$$\frac{1}{2}(\cosh\lambda x-\cos\lambda x)=\frac{(\lambda x)^2}{2!}+\frac{(\lambda x)^6}{6!}+\frac{(\lambda x)^{10}}{10!}+\cdots \tag{5.17.43}$$

$$\frac{1}{2}(\sinh\lambda x-\sin\lambda x)=\frac{(\lambda x)^3}{3!}+\frac{(\lambda x)^7}{7!}+\frac{(\lambda x)^{11}}{11!}+\cdots \tag{5.17.44}$$

This result can be brought into the form

$$U_3(x)=A\sin\lambda x+B\cos\lambda x+C\sinh\lambda x+D\cosh\lambda x \tag{5.17.45}$$

Another way of solving this problem is to note that we may write the equation of motion as

$$\left(\frac{\mathrm{d}^2}{\mathrm{dx}^2}+\lambda^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{dx}^2}-\lambda^2\right)U_3=0 \tag{5.17.46}$$

Solutions of

$$\left(\frac{\mathrm{d}^2}{\mathrm{dx}^2}\pm\lambda^2\right)U_3=0 \tag{5.17.47}$$

are solutions of the total problem. Thus, we get

$$\sum_{i=0}^{\infty}\left[(i+2)(i+1)c_{i+2}\pm\lambda^2 c_i\right]x^i=0 \tag{5.17.48}$$

or

$$c_{i+2}=\frac{\pm\lambda^2 c_i}{(i+2)(i+1)} \tag{5.17.49}$$

Taking the (+) sign first, or

$$c_{i+2}^{+}=\frac{\lambda^2 c_i^{+}}{(i+2)(i+1)} \tag{5.17.50}$$

we get

$$c_2^{+}=\frac{\lambda^2 c_0^{+}}{2!} \tag{5.17.51}$$

$$c_3^{+}=\frac{\lambda^2 c_1^{+}}{3!} \tag{5.17.52}$$

$$c_4^{+}=\frac{\lambda^4 c_0^{+}}{4!} \tag{5.17.53}$$

$$c_5^{+}=\frac{\lambda^4 c_1^{+}}{4!} \tag{5.17.54}$$

etc.

For the (−) sign, we obtain

$$c_2^- = -\frac{\lambda^2 c_0^-}{2!} \tag{5.17.55}$$

$$c_3^- = -\frac{\lambda^2 c_1^-}{3!} \tag{5.17.56}$$

$$c_4^- = \frac{\lambda^2 c_0^-}{4!} \tag{5.17.57}$$

$$c_5^- = \frac{\lambda^4 c_1^-}{5!} \tag{5.17.58}$$

$$c_6^- = -\frac{\lambda^6 c_0^-}{6!} \tag{5.17.59}$$

etc.

Thus, we get

$$\begin{aligned} U_3 &= c_0^+\left[1+\frac{(\lambda x)^2}{2!}+\frac{(\lambda x)^4}{4!}+\cdots\right] \\ &+\frac{c_1^+}{\lambda}\left[(\lambda x)+\frac{(\lambda x)^3}{3!}+\frac{(\lambda x)^5}{5!}+\cdots\right] \\ &+c_0^-\left[1-\frac{(\lambda x)^2}{2!}+\frac{(\lambda x)^4}{4!}-\frac{(\lambda x)^6}{6!}+\cdots\right] \\ &+\frac{c_1^-}{\lambda}\left[(\lambda x)-\frac{(\lambda x)^3}{3!}+\frac{(\lambda x)^5}{5!}-\cdots\right] \end{aligned} \tag{5.17.60}$$

Letting $A = c_1^-/\lambda, B = c_0^-, C = c_1^+/\lambda, D = c_0^+$, we may write this as

$$U_3(x) = A\sin\lambda x + B\cos\lambda x + C\sinh\lambda x + D\cosh\lambda x \tag{5.17.61}$$

This solution is identical to the previous solution.

5.17.3. Vibrating Ring Described by Prescott's Equation

As a next example, let us obtain the solution to Prescott's ring equation (4.3.5). Prescott's equation is not particularly recommended, but it serves as a vehicle for illustrating the power series approved on a less familiar equation:

$$\frac{EI}{a^4}\left(\frac{\partial^4 u_3}{\partial\theta^4}+2\frac{\partial^2 u_3}{\partial\theta^2}+u_3\right)+\rho A\frac{\partial^2 u_3}{\partial t^2}=0 \tag{5.17.62}$$

At a natural frequency,

$$u_3(\theta, t) = U_3(\theta)e^{j\omega t} \tag{5.17.63}$$

This gives, upon substitution,

$$\frac{d^4 U_3}{d\theta^4} + 2\frac{d^2 U_3}{d\theta^2} + \left(1 - \lambda^4\right) U_3 = 0 \tag{5.17.64}$$

where

$$\lambda^4 = \rho A \frac{\omega^2 a^4}{EI} \tag{5.17.65}$$

We may write the equation also as

$$\left(\frac{d^2}{d\theta^2} + 1 + \lambda^2\right)\left(\frac{d^2}{d\theta^2} + 1 - \lambda^2\right) U_3 = 0 \tag{5.17.66}$$

Thus, we have to obtain solutions to

$$\frac{d^2 U_3}{d\theta^2} + \left(1 \pm \lambda^2\right) U_3 = 0 \tag{5.17.67}$$

Inserting the series

$$U_3(\theta) = \sum_{i=0}^{\infty} c_i \theta^i \tag{5.17.68}$$

we obtain

$$\sum_{i=0}^{\infty} \left[(i+2)(i+1)c_{i+2} + (1 \pm \lambda^2)c_i\right]\theta^i = 0 \tag{5.17.69}$$

or

$$c_{i+2} = \frac{(1 \pm \lambda^2)c_i}{(i+1)(i+2)} \tag{5.17.70}$$

Taking the (+) sign first, we obtain

$$c_2^+ = \frac{(1+\lambda^2)c_0^+}{2!} \tag{5.17.71}$$

$$c_3^+ = \frac{(1+\lambda^2)c_1^+}{3!} \tag{5.17.72}$$

$$c_4^+ = \frac{(1+\lambda^2)^2 c_0^+}{4!} \tag{5.17.73}$$

$$c_5^+ = \frac{(1+\lambda^2)^2 c_1^+}{5!} \tag{5.17.74}$$

$$c_6^+ = \frac{(1+\lambda^2)^4 c_0^+}{6!} \tag{5.17.75}$$

etc.

For the (−) sign, we obtain

$$c_2^- = \frac{(1-\lambda^2)c_0^-}{2!} \tag{5.17.76}$$

$$c_3^- = \frac{(1-\lambda^2)c_1^-}{3!} \tag{5.17.77}$$

$$c_4^- = \frac{(1-\lambda^2)^2 c_0^-}{4!} \tag{5.17.78}$$

$$c_5^- = \frac{(1-\lambda^2)^2 c_1^-}{5!} \tag{5.17.79}$$

$$c_6^- = \frac{(1-\lambda^2)^4 c_0^-}{6!} \tag{5.17.80}$$

etc.

Thus, we obtain

$$\begin{aligned} U_3(\theta) = c_0^+&\left[1+\frac{(1+\lambda^2)\theta^2}{2!}+\frac{(1+\lambda^2)^2\theta^4}{4!}+\frac{(1+\lambda^2)^3\theta^6}{6!}+\ldots\right] \\ &+c_1^+\left[x+\frac{(1+\lambda^2)\theta^3}{3!}+\frac{(1+\lambda^2)^2\theta^5}{5!}+\frac{(1+\lambda^2)^3\theta^7}{7!}+\ldots\right] \\ &+c_0^-\left[1-\frac{(\lambda^2-1)\theta^2}{2!}+\frac{(\lambda^2-1)^2\theta^4}{4!}-\frac{(\lambda^2-1)^3\theta^6}{6!}+\ldots\right] \\ &+c_1^-\left[\theta-\frac{(\lambda^2-1)\theta^3}{3!}+\frac{(\lambda^2-1)^2\theta^5}{5!}-\frac{(\lambda^2-1)^3\theta^7}{7!}+\ldots\right] \end{aligned} \tag{5.17.81}$$

Since $\lambda^2 > 1$, we let

$$\lambda^2+1=\xi^2 \tag{5.17.82}$$

$$\lambda^2-1=\eta^2 \tag{5.17.83}$$

Then,

$$\begin{aligned} U_3(\theta) &= c_0^+\left[1+\frac{(\xi\theta)^2}{2!}+\frac{(\xi\theta)^4}{4!}+\frac{(\xi\theta)^6}{6!}+\ldots\right] \\ &+\frac{c_1^+}{\xi}\left[(\xi\theta)+\frac{(\xi\theta)^3}{3!}+\frac{(\xi\theta)^5}{5!}+\frac{(\xi\theta)^7}{7!}+\ldots\right] \\ &+c_0^-\left[1-\frac{(\eta\theta)^2}{2!}+\frac{(\eta\theta)^4}{4!}-\frac{(\eta\theta)^6}{6!}+\ldots\right] \\ &+\frac{c_1}{\eta}\left[(\eta\theta)-\frac{(\eta\theta)^3}{3!}+\frac{(\eta\theta)^5}{5!}-\frac{(\eta\theta)^7}{7!}+\ldots\right] \end{aligned} \tag{5.17.84}$$

Again, while this is a perfectly fine solution (the four arbitrary constants are obtained from the four boundary conditions that Prescott's equation admits), it is instructive to bring it into a form that employs more familiar functions.

If we let $A = c_1^-/\eta, B = c_0^-, C = c_1^+/\xi, D = c_0^+$, we may write this as

$$U_3(\theta) = A \sin \eta\theta + B \cos \eta\theta + C \sinh \xi\theta + D \cosh \xi\theta \tag{5.17.85}$$

Note that in the foregoing examples, it was always possible to identify the solution in terms of sin, cos, sinh, and cosh functions. This is not necessarily always the case. In such an event, the series solutions may have to be tested for convergence. Certain recurring series expressions could be given a name, and properties of their functions could be derived.

Examples of other equations for which the power series approach may yield useful results are Eq. (6.15.10) for the inextensional ring, the more exact Eq. (5.13.12) for the ring, and the Donnell–Mushtari–Vlasov Eq. (6.9.4) for the cylindrical shell.

5.17.4. ON REGULARITIES CONCERNING NODE LINES

Because of recurring needs to sketch expected node lines, attempts have been made, from time to time, to investigate if there is any physical significance to node lines, say for plates, beyond that they are lines of zero transverse deflection. It seems that no useful, general physical significance has been found so far which could aid the estimation of natural mode shapes. The obvious speculation that node lines perhaps divide free vibration kinetic energy into equal parts is not true. Neither do lines of inflection (lines where the curvature of deflection is 0) divide strain energy equally. Some regularities were found for special cases such as beams, and plates that are simply supported along two opposing edges (Soedel and Soedel, 1989).

For example, for such a beam or Section 5.4 plate with a clamped and a free edge, the distances from node points to the free edge are equal to the distances of inflection points to the clamped edge. Or, when plotting the accumulated, nondimensional kinetic and strain energies against distance for beams in general, the former plots average to lines of unity slope while the latter average to lines of slope $\lambda_n^4 = \rho A L^4 \omega_n^2/(EI)$, relating these plots to Rayleigh's principle. But this is neither here nor there as far as universal significance is concerned.

What is left is that when sketching expected node lines, or when assessing possible errors in the numerical prediction or measurements of natural modes of plates, it is of some use to keep in mind that node lines divide regions of positive and negative deflections. For example, this is why closed circular plates cannot have node lines that are odd numbers of radial lines. It is not possible to have a node line arrangement of 1, 3, 5,… radial lines, The requirement is 2,4,6,… radial lines (or in terms of nodal diameters, it is required that the node lines be represented by 1, 2, 3, … diametral lines). As another example, experimentally obtained, noncrossing diametral lines of superposition modes, as discussed in Sec. 5.9, can easily be checked if they are valid by dividing the areas formed by the dividing node lines into positive and negative regions. Because of resolution problems when using experimental techniques, one encounters from time to time, in engineering practice, experimental mode plots that violate this principle of positive and negative division across node lines; an application of the described simple checking procedure would have pointed out immediately the error.

For shells, things are more complicated because of the coupling of transverse and in-plane motion. One still speaks of node lines of the transverse motion components, but these node lines are not lines of zero motion in general because where transverse motion is 0, usually the in-plane motion components are at their maximum; see Sec. 5.3 and 5.5 However, the concept that transverse motion node lines must divide positive and negative areas of transverse deflection is applicable.

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6

Simplified Shell Equations

Except for a few special cases, many of which were discussed in Chapter 5, explicit solutions are not available for Love's equations. Thus the investigator often has no choice but to use approximate solution approaches. The approximate methods are discussed in later chapters. In this chapter, we offer the alternative of using simplified versions of Love's equations.

6.1. MEMBRANE APPROXIMATION

A common approximation in statics of shells, but also used in the analysis of shell vibrations, is to assume that the bending stiffness in Love's equations can be neglected. Obviously, this assumption leads to disaster for transversely vibrating plates and beams but has some justification for shells and arches vibrating in shapes where the stretching of the neutral surface is a dominating contributor to the motion resistance. The approximation is also called the *extensional approximation* and Lord Rayleigh (1889) is commonly credited with it.

Setting

$$D = 0 \tag{6.1.1}$$

implies that

$$M_{11} = M_{22} = M_{12} = Q_{13} = Q_{23} = 0 \tag{6.1.2}$$

The equations of motion are therefore

$$\frac{\partial(N_{11}A_2)}{\partial\alpha_1} + \frac{\partial(N_{12}A_1)}{\partial\alpha_2} + N_{12}\frac{\partial A_1}{\partial\alpha_2} - N_{22}\frac{\partial A_2}{\partial\alpha_1} + A_1A_2q_1 = A_1A_2\rho\ddot{u}_1, \tag{6.1.3}$$

$$\frac{\partial(N_{12}A_2)}{\partial\alpha_1} + \frac{\partial(N_{22}A_1)}{\partial\alpha_2} + N_{12}\frac{\partial A_2}{\partial\alpha_1} - N_{11}\frac{\partial A_1}{\partial\alpha_2} + A_1A_2q_2 = A_1A_2\rho h\ddot{u}_2 \tag{6.1.4}$$

$$-A_1A_2\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}\right) + A_1A_2q_3 = A_1A_2\rho h\ddot{u}_3 \tag{6.1.5}$$

Boundary conditions reduce to the type

$$N_{nn} = N_{nn}^* \quad \text{or} \quad u_n = u_n^* \tag{6.1.6}$$

and

$$u_3 = u_3^* \tag{6.1.7}$$

Note that in general only two boundary conditions can now be satisfied at each edge, as compared to four for the general case.

6.2. AXISYMMETRIC EIGENVALUES OF A SPHERICAL SHELL

Axisymmetric eigenvalues of a spherical shell were first treated by Lamb (1882) by reduction from the solution for a vibrating solid sphere. Here we start with the simplified Love equations (6.1.3)–(6.1.5). We are only interested in axisymmetric vibrations. All derivatives with respect to θ vanish. Since $\alpha_1 = \phi$, $\alpha_2 = \theta$, $A_1 = a$, $A_2 = a\sin\phi$, and $R_1 = R_2 = a$, Eqs. (6.1.3)–(6.1.5) become

$$\frac{\partial}{\partial\phi}(N_{\phi\phi}\sin\phi) - N_{\theta\theta}\cos\phi + aq_\phi\sin\phi = a\rho h\frac{\partial^2 u_\phi}{\partial t^2}\sin\phi \tag{6.2.1}$$

$$-(N_{\phi\phi} + N_{\theta\theta}) + aq_3 = a\rho h\frac{\partial^2 u_3}{\partial t^2} \tag{6.2.2}$$

The strain–displacement relations become

$$\varepsilon_{\phi\phi}^0 = \frac{1}{a}\left(\frac{\partial u_\phi}{\partial\phi} + u_3\right) \tag{6.2.3}$$

$$\varepsilon_{\theta\theta}^0 = \frac{1}{a\sin\phi}(u_\phi\cos\phi + u_3\sin\phi) \tag{6.2.4}$$

This gives

$$N_{\phi\phi}=\frac{K}{a}\left[\frac{\partial u_\phi}{\partial\phi}+\mu u_\phi\cot\phi+(1+\mu)u_3\right] \tag{6.2.5}$$

$$N_{\theta\theta}=\frac{K}{a}\left[u_\phi\cot\phi+\mu\frac{\partial u_\phi}{\partial\phi}+(1+\mu)u_3\right] \tag{6.2.6}$$

We therefore get

$$\frac{\partial^2 u_\phi}{\partial\phi^2}+\frac{\partial}{\partial\phi}(u_\phi\cot\phi)+(1+\mu)\frac{\partial u_3}{\partial\phi}+\frac{a^2 q_\phi}{K}=\frac{a^2\rho h}{K}\frac{\partial^2 u_\phi}{\partial t^2} \tag{6.2.7}$$

$$-\frac{\partial u_\phi}{\partial\phi}-u_\phi\cot\phi-2u_3+\frac{a^2}{K(1+\mu)}q_3=\frac{a^2\rho h}{K(1+\mu)}\frac{\partial^2 u_3}{\partial t^2} \tag{6.2.8}$$

To find the eigenvalues, we set $q_\phi=0$ and $q_3=0$ and substitute:

$$\begin{Bmatrix} u_\phi(\phi,t) \\ u_3(\phi,t) \end{Bmatrix}=\begin{Bmatrix} U_\phi(\phi) \\ U_3(\phi) \end{Bmatrix}e^{j\omega t} \tag{6.2.9}$$

We obtain

$$\frac{d^2U_\phi}{d\phi^2}+\frac{d}{d\phi}(U_\phi\cot\phi)+(1+\mu)\frac{dU_3}{d\phi}+(1-\mu^2)\Omega^2 U_\phi=0 \tag{6.2.10}$$

$$\frac{dU_\phi}{d\phi}+U_\phi\cot\phi+2U_3-(1-\mu)\Omega^2 U_3=0 \tag{6.2.11}$$

where

$$\Omega^2=\frac{a^2\rho\omega^2}{E} \tag{6.2.12}$$

and solving Eq. (6.2.11) for U_3 and differentiating with respect to ϕ gives

$$\frac{dU_3}{d\phi}=\frac{1}{(1-\mu)\Omega^2-2}\left[\frac{d^2U_\phi}{d\phi^2}+\frac{d}{d\phi}(U_\phi\cot\phi)\right] \tag{6.2.13}$$

Substituting this equation in Eq. (6.2.10) gives

$$\frac{d^2U_\phi}{d\phi^2}+\frac{d}{d\phi}(U_\phi\cot\phi)+\lambda(\lambda+1)U_\phi=0 \tag{6.2.14}$$

where

$$\lambda(\lambda+1)=2+\frac{(1+\mu)\Omega^2[3-(1-\mu)\Omega^2]}{1-\Omega^2} \tag{6.2.15}$$

Let us now define a function Φ such that

$$U_\phi=\frac{d\Phi}{d\phi} \tag{6.2.16}$$

Substituting this in Eq. (6.2.14) gives

$$\frac{d}{d\phi}\left[\frac{d^2\Phi}{d\phi^2}+\cot\phi\frac{d\Phi}{d\phi}+\lambda(\lambda+1)\Phi\right]=0 \tag{6.2.17}$$

or, integrating,

$$\frac{d^2\Phi}{d\phi^2}+\cot\phi\frac{d\Phi}{d\phi}+\lambda(\lambda+1)\Phi=C \tag{6.2.18}$$

where C is an integration constant. This equation has a homogeneous solution and a particular solution. The latter is

$$\Phi=\frac{C}{\lambda(\lambda+1)} \tag{6.2.19}$$

and does not contribute anything to U_ϕ according to Eq. (6.2.16). The homogeneous solution is obtained from

$$\frac{d^2\Phi}{d\phi^2}+\cot\phi\frac{d\Phi}{d\phi}+\lambda(\lambda+1)\Phi=0 \tag{6.2.20}$$

We recognize this equation to be Legendre's differential equation. Its general solution is

$$\Phi=AP_\lambda(\cos\phi)+BQ_\lambda(\cos\phi) \tag{6.2.21}$$

or

$$U_\phi=\frac{d\Phi}{d\phi}=A\frac{dP_\lambda(\cos\phi)}{d\phi}+B\frac{dQ_\lambda(\cos\phi)}{d\phi} \tag{6.2.22}$$

To obtain the solution for U_3, we substitute Eq. (6.2.16) in Eq. (6.2.11):

$$-U_3[2-(1-\mu)\Omega^2]=\frac{d^2\Phi}{d\phi^2}+\cot\phi\frac{d\Phi}{d\phi} \tag{6.2.23}$$

Substituting this in Eq. (6.2.20) and utilizing Eq. (6.2.15) gives

$$U_3=\frac{1+(1+\mu)\Omega^2}{1-\Omega^2}\Phi \tag{6.2.24}$$

The functions $P_\lambda(\cos\phi)$ and $Q_\lambda(\cos\phi)$ are the Legendre function of the first and second kind and of fractional order λ. The Legendre function $Q_\lambda(\cos\phi)$ is singular at $\phi=0$. Thus, whenever we have a spherical shell that is closed at the apex $\phi=0$, we set $B=0$ in Eqs. (6.2.21) and (6.2.22). The Legendre function $P_\lambda(\cos\phi)$ is singular at $\phi=\pi$ unless

$$\lambda=n \quad (n=0,1,2,\ldots) \tag{6.2.25}$$

In this case the $P_\lambda(\cos\phi)$ reduces to $P_n(\cos\phi)$, called *Legendre polynomials*. This fact provides us with a very simple solution for the special case of a closed spherical shell since Eq. (6.2.25) allows us an immediate formulation of the natural frequencies, using definition (6.2.15). We get

$$\Omega^4(1-\mu^2)-\Omega^2[n(n+1)+1+3\mu]+[n(n+1)-2]=0 \tag{6.2.26}$$

or

$$\Omega^2_{n1,2} = \frac{1}{2(1-\mu^2)}\{n(n+1)+1+3\mu \pm\sqrt{[n(n+1)+1+3\mu]^2-4(1-\mu^2)[n(n+1)-2]}\} \tag{6.2.27}$$

The natural modes are

$$U_\phi=A\frac{\mathrm{d}P_n(\cos\phi)}{\mathrm{d}\phi} \tag{6.2.28}$$

$$U_3=A\frac{1+(1+\mu)\Omega^2}{1-\Omega^2}P_n(\cos\phi) \tag{6.2.29}$$

For instance, at $n=0$, which is the breathing mode,

$$\Omega^2_{01}=\frac{2}{1-\mu} \tag{6.2.30}$$

Ω^2_{02} is negative, giving rise to an imaginary frequency and is therefore physically meaningless. Since $P_0(\cos\phi)=1$, we get

$$U_3=\frac{1+(1+\mu)\Omega^2_{01}}{1-\Omega^2_{01}} \tag{6.2.31}$$

and

$$U_\phi=0 \tag{6.2.32}$$

The natural frequency in radians per second is obtained from Eq. (6.2.12) to be

$$\omega^2_{01}=\frac{E}{a^2\rho}\Omega^2_{01} \tag{6.2.33}$$

We recognize this case as the "breathing" mode of the sphere. The calculated frequency agrees well with experimental reality.

Natural frequencies for other n values are plotted in Fig. 1. The Ω_{n1} branch is dominated by in-plane motion and agrees well with reality for all thickness-to-radius ratios. The Ω_{n2} branch is dominated by transverse motion. The zero value for $n=1$ defines a rigid-body motion. For thickness-to-radius ratios up to approximately $h/a=0.01$, the membrane approximation gives very good results, as shown in Fig. 1, where the results for the full theory are superimposed. Only when the shell starts to become a "thick" shell do we see a pronounced effect of bending on the transverse

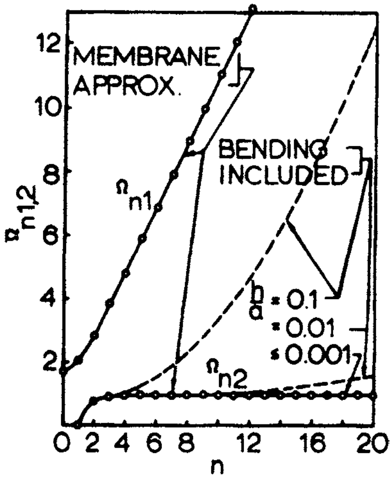

Fig. 1 Natural frequencies of an axisymmetrically vibrating spherical shell.

natural frequencies. This is shown for $h/a=0.1$. The results for the full theory where bending is considered are given by Kraus (1967)

$$\Omega_{n1,2}^2=\frac{1}{2(1-\mu^2)}(A\pm\sqrt{A^2-4mB}) \tag{6.2.34}$$

where

$$A=3(1+\mu)+m+\frac{1}{12}\left(\frac{h}{a}\right)^2(m+3)(m+1+\mu) \tag{6.2.35}$$

$$B=1-\mu^2+\frac{1}{12}\left(\frac{h}{a}\right)^2[(m+1)^2-\mu^2] \tag{6.2.36}$$

$$m=n(n+1)-2 \qquad (n=0,1,2,\ldots) \tag{6.2.37}$$

When h/a is set to 0, this equation reduces to Eq. (6.2.27).

An interesting fact is that if the spherical shell is very thin, all transverse frequencies above approximately $n=4$ exist in a very narrow frequency band.

To aid in the plotting of mode shapes, the following identities are useful: $P_0(\cos\phi)=1, P_1(\cos\phi)=\cos\phi, P_2(\cos\phi)=(3\cos 2\phi+1)/4$, $P_3(\cos\phi)=(5\cos 3\phi+3\cos\phi)/8$. A few transverse mode shapes are shown in Fig. 2. For more work on spherical shell vibrations, see Kalnins (1964); Naghdi (1962); Wilkinson (1965); Kalnins and Kraus (1966).

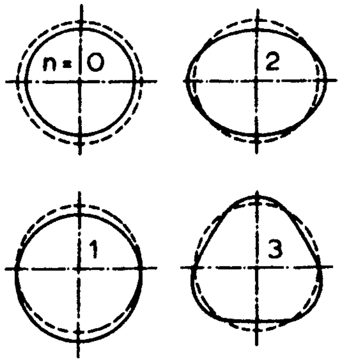

FIG. 2 A few axisymmetrical natural modes of a spherical shell.

6.3. BENDING APPROXIMATION

In one of its versions, the bending approximation is called the *inextensional approximation.* It was first employed by Lord Rayleigh (1881). The simplification sometimes applies to shells that have developable surfaces, but mainly to any shell with transverse modes of a wavelength one order of magnitude smaller than the smallest shell surface dimension. The assumption is that

$$\varepsilon_{11}^0 = \varepsilon_{22}^0 = \varepsilon_{12}^0 = 0 \tag{6.3.1}$$

Thus all membrane force resultants are 0. The equivalent effect is sometimes reached if we set $K = 0$. In the latter case, we obtain

$$\frac{\partial(Q_{13}A_2)}{\partial\alpha_1} + \frac{\partial(Q_{23}A_1)}{\partial\alpha_2} + A_1A_2q_3 = A_1A_2\rho\ddot{u}_3 \tag{6.3.2}$$

where

$$Q_{13} = \frac{1}{A_1A_2}\left[\frac{\partial(M_{11}A_2)}{\partial\alpha_1} + \frac{\partial(M_{12}A_1)}{\partial\alpha_2} + M_{12}\frac{\partial A_1}{\partial\alpha_2} - M_{22}\frac{\partial A_2}{\partial\alpha_1}\right] \tag{6.3.3}$$

$$Q_{23} = \frac{1}{A_1A_2}\left[\frac{\partial(M_{12}A_2)}{\partial\alpha_1} + \frac{\partial(M_{22}A_1)}{\partial\alpha_2} + M_{12}\frac{\partial A_2}{\partial\alpha_1} - M_{11}\frac{\partial A_1}{\partial\alpha_2}\right] \tag{6.3.4}$$

Note that u_1 and u_2 are not 0 but are related to the transverse displacement by two of the three member strain equations:

$$\frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} = -\frac{u_3}{R_1} \tag{6.3.5}$$

$$\frac{1}{A_2}\frac{\partial u_2}{\partial\alpha_2} + \frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} = -\frac{u_3}{R_2} \tag{6.3.6}$$

$$\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_1}{A_1}\right) = 0 \tag{6.3.7}$$

The bending strain expressions are the same as in Eqs. (2.4.22)–(2.4.24), with β_1 and β_2 given by Eqs. (2.4.7) and (2.4.8).

6.4. CIRCULAR CYLINDRICAL SHELL

For the circular cylindrical shell, $A_1=1, d\alpha_1=dx, A_2=a, d\alpha_2=d\theta, R_1=\infty$, and $R_2=a$. This gives the following relationships between transverse and in-plane displacements:

$$\frac{\partial u_x}{\partial x}=0 \tag{6.4.1}$$

$$\frac{\partial u_\theta}{\partial \theta}=-u_3 \tag{6.4.2}$$

$$\frac{\partial u_\theta}{\partial x}+\frac{1}{a}\frac{\partial u_x}{\partial \theta}=0 \tag{6.4.3}$$

For instance, for the simply supported shell vibrating at a natural frequency

$$u_3=A\sin\frac{m\pi x}{L}\cos n(\theta-\phi)e^{j\omega t} \tag{6.4.4}$$

Thus, from Eq. (6.4.2),

$$u_\theta=-\int u_3\,d\theta+C_1=-\frac{A}{n}\sin\frac{m\pi x}{L}\sin n(\theta-\phi)e^{j\omega t} \tag{6.4.5}$$

Selecting (6.4.3) as our second equation gives

$$u_x=-a\int\frac{\partial u_\theta}{\partial x}d\theta+C_2 \tag{6.4.6}$$

or

$$u_x=-\frac{Aa}{n^2}\frac{m\pi}{L}\cos\frac{m\pi x}{L}\cos n(\theta-\phi)e^{j\omega t} \tag{6.4.7}$$

Note that the three displacement mode components are in functional character identical to the exact solution, but they are no longer independent of each other. Thus constants A, B, C of Sec. 5.5 are now replaced by $Ama\pi/Ln^2, -A/n$, and A. Substituting Eqs. (6.4.4), (6.4.5), and (6.4.7) in the bending strain expressions and these in Eqs. (6.3.2)–(6.3.4) gives

$$\omega_{mn}^2=\frac{E}{12\rho(1-\mu^2)}\left(\frac{h}{a}\right)^2\frac{1}{a^2}\left[\left(\frac{m\pi a}{L}\right)^2+n^2\right]\left[\left(\frac{m\pi a}{L}\right)^2+n^2-1\right] \tag{6.4.8}$$

This equation is of special interest to the acoustical engineer since in the higher-mode-number range, the influence of the boundary conditions disappears and any closed circular cylindrical shell will be governed by it.

As a matter of fact, the –1 in the second set of brackets is negligible for higher m and n combinations and the equation simplifies to

$$\omega_{mn}^2 = \frac{E}{12\rho(1-\mu)^2}\left(\frac{h}{a}\right)^2 \frac{1}{a^2}\left[\left(\frac{m\pi a}{L}\right)^2 + n^2\right]^2 \tag{6.4.9}$$

6.5. ZERO IN-PLANE DEFLECTION APPROXIMATION

For modes associated primarily with transverse motion, the contributions of u_1 and u_2 on strain are assumed to be negligible. This seems to work for very shallow shells and bending-dominated modes. The shell equation derived by Sophie Germaine (see Chapter 1) seems to have used this assumption.

The strain–displacement relationships become

$$\varepsilon_{11}^0 = \frac{u_3}{R_1} \tag{6.5.1}$$

$$\varepsilon_{22}^0 = \frac{u_3}{R_2} \tag{6.5.2}$$

$$\varepsilon_{12}^0 = 0 \tag{6.5.3}$$

$$k_{11} = -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1}\right) - \frac{1}{A_1A_2^2}\frac{\partial u_3}{\partial\alpha_2}\frac{\partial A_1}{\partial\alpha_2} \tag{6.5.4}$$

$$k_{22} = -\frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2}\right) - \frac{1}{A_2A_1^2}\frac{\partial u_3}{\partial\alpha_1}\frac{\partial A_2}{\partial\alpha_1} \tag{6.5.5}$$

$$k_{12} = -\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1^2}\frac{\partial u_3}{\partial\alpha_2}\right) - \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2^2}\frac{\partial u_3}{\partial\alpha_1}\right) \tag{6.5.6}$$

Substituting these relationships into Love's equations gives

$$D\nabla^4 u_3 + Ku_3\left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{2\mu}{R_1R_2}\right) + \rho h\frac{\partial^2 u_3}{\partial t^2} = q_3 \tag{6.5.7}$$

where

$$\nabla^2(\cdot) = \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{A_1}\frac{\partial(\cdot)}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{A_2}\frac{\partial(\cdot)}{\partial\alpha_2}\right)\right] \tag{6.5.8}$$

Equation (6.5.7) can be used to estimate quickly effects of curvature in relatively shallow shells. However, the accuracy of prediction leaves a lot to be desired.

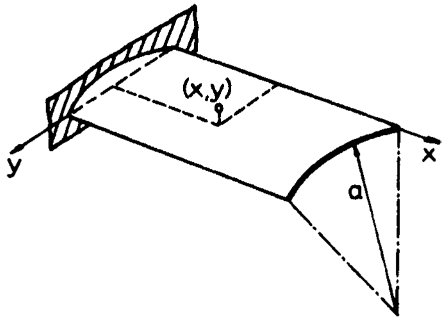

FIG. 3 Simplified curved fan blade.

6.6. EXAMPLE: CURVED FAN BLADE

For some low-speed fan blades, where the centrifugal stiffening effect can be neglected, we have $R_t = \infty$ and $R_2 = a$ (see Fig. 3) Let us use a Cartesian coordinate system in the plane of projection. At a natural frequency,

$$u_3(x, y, t) = U_3(x, y)e^{j\omega t} \tag{6.6.1}$$

we get

$$D\nabla^4 U_3(x, y) + \left(\frac{K}{a^2} - \rho h\omega^2\right) U_3(x, y) = 0 \tag{6.6.2}$$

We therefore notice immediately that

$$\omega_2^2 = \omega_1^2 + \frac{K}{a^2 \rho h} \tag{6.6.3}$$

where ω_2 is the natural frequencies of the curved blade and ω_1 is the natural frequencies of the flat blade. The formula will apply approximately only to the first few beam-type modes but allows quick estimates of the curvature effect.

We may now generalize this finding. According to Eq. (6.5.7), whenever the shell is so shallow that we may use plate coordinates as an approximation,

$$\omega_2^2 = \omega_1^2 + \frac{K}{\rho h}\left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{2\mu}{R_1 R_2}\right) \tag{6.6.4}$$

It is implied that both curvatures are constant over the surface.

6.7. DONNELL–MUSTHTARI–VLASOV EQUATIONS

Of all the simplifications presented, that of Donnell, Mushtari, and Vlasov is used most widely in shell vibrations. It neglects neither bending nor membrane effects. It applies to shells that are loaded normal to their surface and concentrates on transverse deflection behaviour. The approach was developed, apparently independently, by Donnell (1933, 1938) and Mushtari (1938). Donnell derived it for the circular cylindrical shell. The approach was generalized for any geometry by Vlasov (1951). Because Vlasov pointed out that the approach gives particularly good results for shallow shells, the equations are often referred to as shallow shell equations. This is, however, an unnecessarily severe restriction, as we will see when we develop the equations.

The first basic assumption is that contributions of in-plane deflections can be neglected in the bending strain expressions but not in the membrane strain expressions. The bending strains are therefore

$$k_{11} = -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1}\right) - \frac{1}{A_1A_2^2}\frac{\partial u_3}{\partial\alpha_2}\frac{\partial A_1}{\partial\alpha_2} \tag{6.7.1}$$

$$k_{22} = \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2}\right) - \frac{1}{A_2A_1^2}\frac{\partial u_3}{\partial\alpha_1}\frac{\partial A_2}{\partial\alpha_1} \tag{6.7.2}$$

$$k_{12} = -\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_2^2}\frac{\partial u_3}{\partial\alpha_2}\right) - \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_1^2}\frac{\partial u_3}{\partial\alpha_1}\right) \tag{6.7.3}$$

The membrane strain expressions remain the same. The next assumption is that the influence of inertia in the in-plane direction is neglected. Needless to say, the theory is restricted to normal loading. Finally, we neglect the shear terms Q_{31}/R_1 and Q_{32}/R_2. The equations of motion are, therefore,

$$\frac{\partial(A_2N_{11})}{\partial\alpha_1} + \frac{\partial(A_1N_{12})}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}N_{12} - \frac{\partial A_2}{\partial\alpha_1}N_{22} = 0 \tag{6.7.4}$$

$$\frac{\partial(A_2N_{12})}{\partial\alpha_1} + \frac{\partial(A_1N_{22})}{\partial\alpha_2} + \frac{\partial A_2}{\partial\alpha_1}N_{12} - \frac{\partial A_1}{\partial\alpha_2}N_{11} = 0 \tag{6.7.5}$$

$$D\nabla^4 u_3 + \frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} + \rho h\frac{\partial^2 u_3}{\partial t^2} = q_3 \tag{6.7.6}$$

Let us now introduce a function ϕ that we define as

$$N_{11} = \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial\phi}{\partial\alpha_2}\right) + \frac{1}{A_1^2A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\phi}{\partial\alpha_1} \tag{6.7.7}$$

$$N_{22} = \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial\phi}{\partial\alpha_1}\right) + \frac{1}{A_1A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial\phi}{\partial\alpha_2} \tag{6.7.8}$$

$$N_{12} = -\frac{1}{A_1A_2}\left(\frac{\partial^2\phi}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial\phi}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\phi}{\partial\alpha_2}\right) \tag{6.7.9}$$

If we substitute these definitions into Eqs. (6.7.4)–(6.7.6), we find that the first two equations are satisfied and the third equation becomes

$$D\nabla^4 u_3 + \nabla_k^2 \phi + \rho h \frac{\partial^2 u_3}{\partial t^2} = q_3 \tag{6.7.10}$$

where

$$\nabla_k^2(\cdot) = \frac{1}{A_1 A_2}\left\{\frac{\partial}{\partial \alpha_1}\left[\frac{1}{R_2}\frac{A_2}{A_1}\frac{\partial(\cdot)}{\partial \alpha_1}\right] + \frac{\partial}{\partial \alpha_2}\left[\frac{1}{R_1}\frac{A_1}{A_2}\frac{\partial(\cdot)}{\partial \alpha_2}\right]\right\} \tag{6.7.11}$$

This type of function was first introduced by Airy (1863) for the two-dimensional treatment of a beam in bending and is in general known as *Airy's stress function*. With it we have in effect eliminated u_1 and u_2 but still have ϕ and u_3 to contend with. To obtain a second equation, we follow the standard procedure with Airy's stress function, namely to generate the compatibility equation. The way to do this is to take the six strain displacement relationships and eliminate from them the displacement by substitutions, additions, and subtraction. This is shown in detail in Novozhilov (1964) and Nowacki (1963). The result is

$$\frac{k_{11}}{R_1} + \frac{k_{22}}{R_2} + \frac{1}{A_1 A_2}\left\{\frac{\partial}{\partial \alpha_1}\frac{1}{A_1}\left[A_2\frac{\partial \varepsilon_{22}^0}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1}(\varepsilon_{22}^0 - \varepsilon_{11}^0) - \frac{A_1}{2}\frac{\partial \varepsilon_{12}^0}{\partial \alpha_2} - \frac{\partial A_1}{\partial \alpha_2}\varepsilon_{12}^0\right]\right.$$
$$\left. + \frac{\partial}{\partial \alpha_2}\frac{1}{A_2}\left[A_1\frac{\partial \varepsilon_{11}^0}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}(\varepsilon_{11}^0 - \varepsilon_{22}^0) - \frac{A_2}{2}\frac{\partial \varepsilon_{12}^0}{\partial \alpha_1} - \frac{\partial A_2}{\partial \alpha_1}\varepsilon_{12}^0\right]\right\} = 0 \tag{6.7.12}$$

We now substitute the fact that

$$\varepsilon_{11}^0 = \frac{1}{Eh}(N_{11} - \mu N_{22}) \tag{6.7.13}$$

$$\varepsilon_{22}^0 = \frac{1}{Eh}(N_{22} - \mu N_{11}) \tag{6.7.14}$$

$$\varepsilon_{12}^0 = \frac{2(1+\mu)}{Eh}N_{12} \tag{6.7.15}$$

where N_{11}, N_{22}, and N_{12} are replaced by the stress function definitions of Eqs. (6.7.7)–(6.7.9). This gives us

$$Eh\nabla_k^2 u_3 - \nabla^4 \phi = 0 \tag{6.7.16}$$

Thus Eqs. (6.7.10) and (6.7.16) are the equations of motion.

There are four necessary boundary conditions at each edge, two in terms of u_3 and two in terms of ϕ. The ϕ conditions pose a problem if they are given in terms of u_1 or u_2, since we then have to solve Eqs. (6.7.7)–(6.7.9) for ϕ However, boundary conditions where N_{11}, N_{12}, or N_{22} are specified directly are easier to handle.

6.8. NATURAL FREQUENCIES AND MODES

To obtain the eigenvalues of Eqs. (6.7.10) and (6.7.16), We substitute

$$q_3 = 0 \tag{6.8.1}$$

$$u_3(\alpha_1, \alpha_2, t) = U_3(\alpha_1, \alpha_2)e^{j\omega t} \tag{6.8.2}$$

$$\phi(\alpha_1, \alpha_2, t) = \Phi(\alpha_1, \alpha_2)e^{j\omega t} \tag{6.8.3}$$

and get

$$D\nabla^4 U_3 + \nabla_k^2 \Phi - \rho h \omega^2 U_3 = 0 \tag{6.8.4}$$

$$Eh\nabla_k^2 U_3 - \nabla^4 \Phi = 0 \tag{6.8.5}$$

Defining a function $F(\alpha_1, \alpha_2)$ such that

$$U_3 = \nabla^4 F \tag{6.8.6}$$

$$\Phi = Eh\nabla_k^2 F \tag{6.8.7}$$

we obtain, by proper substitution into Eqs. (6.8.4) and (6.8.5),

$$D\nabla^8 F + Eh\nabla_k^4 F - \rho h \omega^2 \nabla^4 F = 0 \tag{6.8.8}$$

The alternative choice is to operate on Eq. (6.8.4) with ∇^4 and on Eq. (6.8.5) with ∇_k^2. Combining the two equations then gives

$$D\nabla^8 U_3 + Eh\nabla_k^4 U_3 - \rho h \omega^2 \nabla^4 U_3 = 0 \tag{6.8.9}$$

This form is probably preferable.

6.9. CIRCULAR CYLINDRICAL SHELL

For a circular cylindrical shell, $A_1 = 1, \alpha_1 = x, A_2 = a$, and $\alpha_2 = \theta$. This gives

$$\nabla^4(\cdot) = \frac{1}{a^4}\frac{\partial^4(\cdot)}{\partial\theta^4} + \frac{\partial^4(\cdot)}{\partial x^4} + \frac{2}{a^2}\frac{\partial^4(\cdot)}{\partial x^2 \partial\theta^2} \tag{6.9.1}$$

$$\nabla_k^4(\cdot) = \frac{1}{a^2}\frac{\partial^4(\cdot)}{\partial x^4} \tag{6.9.2}$$

For the special category where the shell is closed in θ direction, the solution will be of the form

$$U_3(x, \theta) = U_{3n}(x)\cos n(\theta - \phi) \tag{6.9.3}$$

where ϕ is an arbitrary angle accounting for the fact that there is no preferential direction of the mode shape in circumferential direction. The equation of motion becomes

$$D\left(\frac{n^2}{a^2}-\frac{d^2}{dx^2}\right)^4 U_{3n}(x)+\frac{Eh}{a^2}\frac{d^4U_{3n}(x)}{dx^4}-\rho h\omega^2\left(\frac{n^2}{a^2}-\frac{d^2}{dx^2}\right)^2 U_{3n}(x)=0 \tag{6.9.4}$$

Solutions must be of the form

$$U_{3n}(x)=e^{\lambda(x/L)} \tag{6.9.5}$$

This gives

$$D\left[\frac{n^2}{a^2}-\left(\frac{\lambda}{L}\right)^2\right]^4+\frac{Eh}{a^2}\left(\frac{\lambda}{L}\right)^4-\rho h\omega^2\left[\frac{n^2}{a^2}-\left(\frac{\lambda}{L}\right)^2\right]^2=0 \tag{6.9.6}$$

This equation has the following roots: $\lambda_i=\pm\eta_1$, $\pm j\eta_2$, $\pm(\eta_3+j\eta_4)$, $\pm(\eta_3-j\eta_4)$. The general solution must be of the form

$$\begin{aligned}U_{3n}(x) &= A_1\sinh\eta_1\frac{x}{L}+A_2\cos\eta_1\frac{x}{L}+A_3\sin\eta_2\frac{x}{L}+A_4\cos\eta_2\frac{x}{L}\\ &\quad +A_5e^{\eta_3(x/L)}\cos\eta_4\frac{x}{L}+A_6e^{\eta_3(x/L)}\sin\eta_4\frac{x}{L}\\ &\quad +A_7e_3^{-\eta_3(x/L)}\cos\eta_4\frac{x}{L}+A_8e^{\eta_3(x/L)}\sin\eta_4\frac{x}{L}\end{aligned} \tag{6.9.7}$$

We have to enforce four boundary conditions on each edge. However, to enforce boundary conditions involving $N_{xx}, N_{x\theta}, u_x$, or u_θ, we have to translate these conditions into a condition of the stress function ϕ. This is quite complicated and the advantage of a presumably simple theory is lost. Let us therefore follow at first a simplification introduced by Yu (1955). He argued that for shells and modes where

$$\frac{n^2}{a^2}\gg\left(\frac{\lambda}{L}\right)^2 \tag{6.9.8}$$

we may simplify the characteristic equation to

$$D\left(\frac{n}{a}\right)^8+\frac{Eh}{a^2}\left(\frac{\lambda}{L}\right)^4-\rho h\omega^2\left(\frac{n}{a}\right)^4=0 \tag{6.9.9}$$

This gives

$$\lambda_i=\frac{nL}{a}\sqrt[4]{\frac{a^2}{Eh}\left[\rho h\omega^2-D\left(\frac{n}{a}\right)^4\right]} \tag{6.9.10}$$

The roots are therefore $\lambda_i = \pm\eta, \pm j\eta$, where

$$\eta = \frac{nL}{a} \sqrt[4]{\frac{a^2}{Eh}\left|\rho h\omega^2 - D\left(\frac{n}{a}\right)^4\right|} \tag{6.9.11}$$

Thus the general solution for this case is

$$U_{3n}(x) = A_1 \sin\eta\frac{x}{L} + A_2 \cos\eta\frac{x}{L} + A_3 \sinh\eta\frac{x}{L} + A_4 \cosh\eta\frac{x}{L} \tag{6.9.12}$$

The admissible boundary conditions are

$$M_{xx} = M_{xx}^* \text{ or } \beta_x = \beta_x^* \tag{6.9.13}$$

$$V_{x3} = V_{x3}^* \text{ or } u_3 = u_3^* \tag{6.9.14}$$

Equation (6.9.12) is applied to the two appropriate boundary conditions at each end. The determinant of the resulting 4×4 matrix equations will give the roots η_m. The natural frequencies are then obtained from Eq. (6.9.11) as

$$\omega_{mn} = \sqrt{\frac{1}{\rho h}\left[\frac{Eha^2}{L^4 n^4}\eta_m^4 + D\left(\frac{n}{a}\right)^4\right]} \tag{6.9.15}$$

Note that this equation is definitely not valid for $n = 0$. It improves in accuracy as n increases. Note also that the roots η_m will be equal to the roots of the analogous beam case since Eq. (6.9.12) is of the same form as the general beam solution. This implies, of course, that the moment and shear boundary conditions of Eqs. (6.9.13) and (6.9.14) are simplified to be functions of $U_{3n}(x)$ only.

6.10. CIRCULAR DUCT CLAMPED AT BOTH ENDS

For a circular duct clamped at both ends, we do not use the analogy to beams directly, but work with Eq. (6.9.12). The duct is shown in Fig. 4. The boundary conditions are that at $x = 0$,

$$\beta_x = 0 \tag{6.10.1}$$

$$u_3 = 0 \tag{6.10.2}$$

and at $x = L$,

$$\beta_x = 0 \tag{6.10.3}$$

$$u_3 = 0 \tag{6.10.4}$$

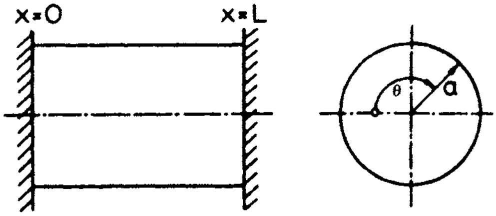

FIG. 4 Clamped circular cylindrical shell.

This translates to

$$\frac{dU_{3n}(0)}{dx}=0 \tag{6.10.5}$$

$$U_{3n}(0)=0 \tag{6.10.6}$$

$$\frac{dU_{3n}}{dx}(L)=0 \tag{6.10.7}$$

$$U_{3n}(L)=0 \tag{6.10.8}$$

Substituting Eq. (6.9.12) in these conditions gives

$$\begin{bmatrix} \eta & 0 & \eta & 0 \\ 0 & 1 & 0 & 1 \\ \eta\cos\eta & -\eta\sin\eta & \eta\cosh\eta & \sinh\eta \\ \sin\eta & \cos\eta & \sinh\eta & \cosh\eta \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = 0 \tag{6.10.9}$$

Setting the determinant of this equation to 0 gives

$$\cos\eta\cosh\eta - 1 = 0 \tag{6.10.10}$$

The roots of this equation are $\eta_1=4.730, \eta_2=7.853, \eta_3=10.996, \eta_4=14.137$, and so on.

Substituting the roots η_m in Eq. (6.10.9) allows us to evaluate A_2, A_3, and A_4 in terms of A_1. Utilizing Eqs. (6.9.12) and (6.9.3) gives

$$U_{3mn}(x,\theta) = \left[H\left(\cosh\eta_m\frac{x}{L} - \cos\eta_m\frac{x}{L}\right) - J\left(\sinh\eta_m\frac{x}{L} - \sin\eta_m\frac{x}{L}\right)\right]\cos n(\theta-\phi) \tag{6.10.11}$$

where

$$H = \sinh\eta_m - \sin\eta_m \tag{6.10.12}$$

$$J = \cosh\eta_m - \cos\eta_m \tag{6.10.13}$$

The solutions given here are best for pipes whose length is large as compared to their diameter. But even for short and stubby pipes, the solution agrees well with experimental evidence in the higher-n range (Koval and Cranch, 1962). Natural frequencies are given by Eq. (6.9.15).

6.11. VIBRATIONS OF A FREESTANDING SMOKESTACK

Let us assume that a smokestack can be approximated as a clamped-free cylindrical shell, as shown in Fig. 5. Initial stresses introduced by its own weight are neglected. The boundary conditions are

$$\beta_x(0)=0 \tag{6.11.1}$$

$$u_3(0)=0 \tag{6.11.2}$$

$$M_{xx}(L)=0 \tag{6.11.3}$$

$$V_{3x}(L)=0 \tag{6.11.4}$$

Let us now utilize the analogy to beams. From Flugge (1962), we find that the characteristic equation is

$$\cos\eta\cosh\eta+1=0 \tag{6.11.5}$$

The first few roots of this equation are $\eta_1=1.875$, $\eta_2=4.694$, $\eta_3=7.855$,

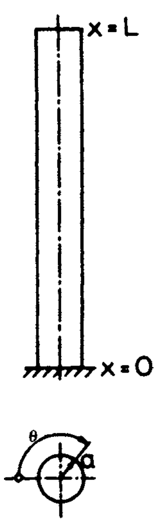

Fig. 5 Slender circular cylindrical shell clamped at one end and free at the other.

$\eta_4 = 10.996$, and so on. The mode shape becomes

$$U_{3n\eta}(x,0) = \left[F\left(\cosh\eta_m\frac{x}{L} - \cos\eta_m\frac{x}{L}\right)\right.$$
$$\left. - G\left(\sinh\eta_m\frac{x}{L} - \sin\eta_m\frac{x}{L}\right)\right]\cos n(\theta-\phi) \qquad (6.11.6)$$

where

$$F = \sinh\eta_m + \sin\eta_m \qquad (6.11.7)$$

$$G = \cosh\eta_m + \cos\eta_m \qquad (6.11.8)$$

6.12. SPECIAL CASES OF THE SIMPLY SUPPORTED CLOSED SHELL AND CURVED PANEL

Cases involving a simply supported closed shell and curved panel are special because we do not need to submit to the approximation of Eq. (6.9.8). It is possible to guess the solution to Eq. (6.9.4) directly. We let the mode shapes be

$$U_{3mn}(x,\theta) = \sin\frac{m\pi x}{L}\cos n(\theta-\phi) \qquad (6.12.1)$$

for the simply supported circular cylindrical shell shown in Fig. 6. Upon substitution in Eq. (6.9.4), we obtain

$$D\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m\pi}{L}\right)^2\right]^4 + \frac{Eh}{a^2}\left(\frac{m\pi}{L}\right)^4 - \rho h\omega^2\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m\pi}{L}\right)^2\right]^2 = 0 \qquad (6.12.2)$$

We may solve this equation directly for the natural frequencies. Writing ω_{mn} instead of ω to indicate the dependency on m and n, we get

$$\omega_{mn} = \frac{1}{a}\sqrt{\frac{(m\pi a/L)^4}{[(m\pi a/L)^2 + n^2]^2} + \frac{(h/a)^2}{12(1-\mu^2)}[(\frac{m\pi a}{L})^2 + n^2]^2}\sqrt{\frac{E}{\rho}} \qquad (6.12.3)$$

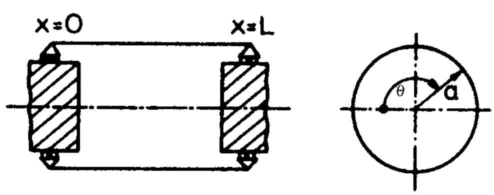

FIG. 6 Simply supported circular cylindrical shell.

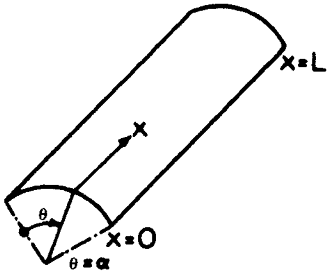

Fig. 7 Simply supported, circular cylindrical shell panel.

Let us now treat a cylindrical panel as shown in Fig. 7. It is simply supported on all four sides and could represent an engine cover, airplane door, and so on. The boundary conditions are simple support on all four edges. We guess the mode shape to be

$$U_{3mn}(x,\theta) = \sin\frac{m\pi x}{L}\sin\frac{n\pi\theta}{\alpha} \tag{6.12.4}$$

It satisfies all boundary conditions. Substituting it in Eq. (6.9.4) gives

$$\omega_{mn} = \frac{1}{a}\sqrt{\frac{(m\pi a/L)^4}{[(m\pi a/L)^2+(n\pi/\alpha)^2]^2} + \frac{(h/a)^2}{12(1-\mu^2)}\left[\left(\frac{m\pi a}{L}\right)^2+\left(\frac{n\pi}{\alpha}\right)^2\right]^2} \times\sqrt{\frac{E}{\rho}} \tag{6.12.5}$$

These two cases illustrate well the influence of the bending and membrane stiffness. The first term under the square root of Eqs. (6.12.3) and (6.12.5) is due to membrane stiffness and reduces to 0 as n increases. The second term is due to the bending stiffness and increases in relative importance as n increases (Soedel, 1971).

6.13. BARREL-SHAPED SHELL

A barrel shape is a very common form, found in shells that seal hermetic refrigeration compressors, pump housings, and so on. It is sketched in Fig. 8. If the curvature is not too pronounced, we may use cylindrical coordinates as an approximation:

$$(\mathrm{d}s)^2 \cong (\mathrm{d}x)^2 + a^2(\mathrm{d}\theta)^2 \tag{6.13.1}$$

This gives $A_1 = 1, A_2 = a, \alpha_1 = x$, and $\alpha_2 = \theta$. Also, we get $R_1 = R$ and $R_2 \cong a$.

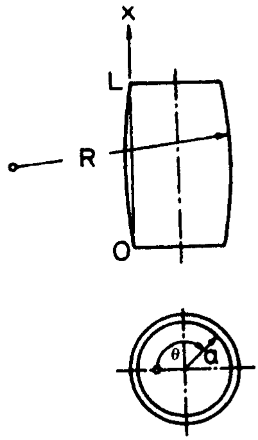

FIG. 8 Barrel shell.

Thus we have

$$\nabla^2(\cdot) = \frac{1}{a^2}\frac{\partial^2(\cdot)}{\partial\theta^2} + \frac{\partial^2(\cdot)}{\partial x^2} \tag{6.13.2}$$

$$\nabla_k^2(\cdot) = \frac{1}{a}\frac{\partial^2(\cdot)}{\partial x^2} + \frac{1}{Ra^2}\frac{\partial^2(\cdot)}{\partial\theta^2} \tag{6.13.3}$$

Boundary conditions are similar to those for cylindrical shells.

For the special case of simple supports on both ends of the barrel shell, we find that the mode function

$$U_3(x,\theta) = A\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{6.13.4}$$

satisfies the boundary condition and Eq. (6.8.9). We obtain

$$\omega_{Bmn}^2 = \omega_{Cmn}^2 + \frac{n^2[n^2(a/R)^2 + 2(a/R)(m\pi a/L)^2]}{a^2[(m\pi a/L)^2 + n^2]^2}\frac{E}{\rho} \tag{6.13.5}$$

where ω_{Cmn} is the natural frequency of the cylindrical shell of radius a given by Eq. (6.12.3) and ω_{Bmn} is the natural frequency of the barrel shell. Note that a negative value of R has a stiffening effect only if

$$n^2\left(\frac{a}{R}\right)^2 > 2\frac{a}{|R|}\left(\frac{m\pi a}{L}\right)^2 \tag{6.13.6}$$

where $|R|$ denote the magnitude of the radius of curvature in the x direction. Otherwise, a negative value of R will reduce the natural frequency. Barrel shells of negative R occur, for instance, as cooling towers and appear as shown in (Fig. 9).

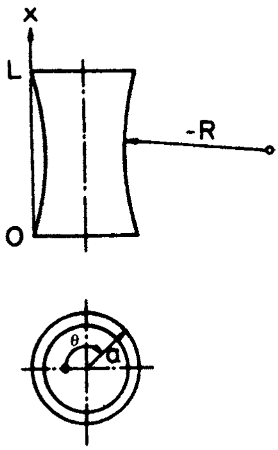

Fig. 9 Hourglass shell which is a barrel shell with negative axial curvature.

6.14. SPHERICAL CAP

If a spherical shell is shallow, it can be interpreted as a circular plate panel with spherical curvature. This allows us to formulate as an approximate fundamental form (Fig. 10)

$$(\mathrm{d}s)^2 \cong (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 \tag{6.14.1}$$

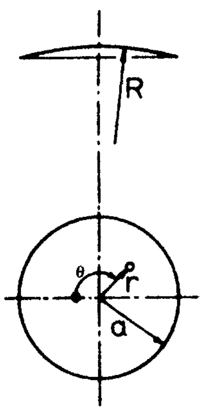

Fig. 10 Shallow spherical cap.

which gives $A_1=1, A_2=r, \alpha_1=r$, and $\alpha_2=\theta$. Furthermore, $R_1=R_2=R$. This gives

$$\nabla^2(\cdot)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(\cdot)}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial(\cdot)}{\partial\theta}\right) \tag{6.14.2}$$

and

$$\nabla_k^2(\cdot)=\frac{1}{R}\nabla^2(\cdot) \tag{6.14.3}$$

Substituting this into Eq. (6.8.9) gives

$$\nabla^4\left[D\nabla^4+\left(\frac{Eh}{R^2}-\rho h\omega^2\right)\right]U_3=0 \tag{6.14.4}$$

Solutions of

$$\nabla^4 U_3=0 \tag{6.14.5}$$

and

$$D\nabla^4 U_3+\left(\frac{Eh}{R^2}-\rho \mathrm{h}\omega^2\right)U_3=0 \tag{6.14.6}$$

are solutions to the problem posed. The first equation has little physical significance. The second equation is recognized as being similar to the equation for the circular plate. The solution must be of the form of the circular plate solution,

$$U_3=[AJ_n(\lambda r)+BK_n(\lambda r)+CY_n(\lambda r)+DI_n(\lambda r)]\cos n(\theta-\phi) \tag{6.14.7}$$

where

$$\lambda^4=\frac{1}{D}\left(\rho h\omega^2-\frac{Eh}{R^2}\right) \tag{6.14.8}$$

The possible boundary conditions are identical to that of a flat plate. Therefore, all circular plate solutions apply to the spherical cap as long as the boundary conditions are the same. Therefore,

$$\omega_{smn}^2=\omega_{pmn}^2+\frac{E}{\rho R^2} \tag{6.14.9}$$

where ω_{pmn} is the natural frequency of the circular plate and ω_{smn} is the natural frequency of the spherical cap. It has been shown in Soedel (1973) that this formula also applies to spherical caps of any shape of boundary, not only to the circular shape. Any boundary is permissible: triangular, square, and so on. ω_{pwn} is the natural frequency of a plate of the same boundary shape and conditions.

6.15. INEXTENSIONAL APPROXIMATION: RING

The inextensional approximation is related to the bending approximation in that we again argue that the extension of the reference surface is negligible:

$$\varepsilon_{11}^0 = \varepsilon_{22}^0 = \varepsilon_{12}^0 = 0 \tag{6.15.1}$$

However, since a consequence of this, namely that membrane force resultants vanish, is in most cases inadmissible, we try to eliminate the membrane force resultants from the equations of motion before applying Eq. (6.15.1). This is best illustrated by way of a ring. From $\varepsilon_{\theta\theta}^0 = 0$, we obtain

$$\frac{\partial u_\theta}{\partial \theta} = -u_3 \tag{6.15.2}$$

utilizing Eq. (4.1.9) with $s = a\theta$. Since a consequential application would mean that $N_{\theta\theta} = 0$, which, as mentioned earlier, cannot be tolerated, we start with the ring equations in force and moment resultant form in order to eliminate $N_{\theta\theta}$. From Eqs. (4.1.6)–(4.1.8)

$$\frac{1}{a}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{1}{a^2}\frac{\partial M_{\theta\theta}}{\partial \theta} - \rho A \frac{\partial^2 u_\theta}{\partial t^2} = -q'_\theta \tag{6.15.3}$$

$$\frac{1}{a^2}\frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} - \frac{N_{\theta\theta}}{a} - \rho A \frac{\partial^2 u_3}{\partial t^2} = -q'_3 \tag{6.15.4}$$

where q'_3 and q'_θ are forces per unit length (N/m). From Eq. (6.15.4),

$$N_{\theta\theta} = \frac{1}{a}\frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} - \rho A a \frac{\partial^2 u_3}{\partial t^2} + q'_3 a \tag{6.15.5}$$

Substituting this expression into Eq. (6.15.3) gives

$$\frac{1}{a^2}\frac{\partial^3 M_{\theta\theta}}{\partial \theta^3} + \frac{1}{a^2}\frac{\partial M_{\theta\theta}}{\partial \theta} - \rho A \frac{\partial^2 u_\theta}{\partial t^2} - \rho A \frac{\partial^3 u_3}{\partial \theta \partial t^2} = -q'_\theta - \frac{\partial q'_3}{\partial \theta} \tag{6.15.6}$$

Now the inextensional assumptions may be applied. From Eq. (4.1.17),

$$M_{\theta\theta} = \frac{EI}{a^2}\left(\frac{\partial u_\theta}{\partial \theta} - \frac{\partial^2 u_3}{\partial \theta^2}\right) = -\frac{EI}{a^2}\left(u_3 + \frac{\partial^2 u_3}{\partial \theta^2}\right) \tag{6.15.7}$$

Substituting this into Eq. (6.15.6) gives $[p = I/(Aa^2), \omega_0^2 = E/(\rho a^2)]$

$$\frac{\partial^6 u_3}{\partial \theta^6} + 2\frac{\partial^4 u_3}{\partial \theta^4} + \frac{\partial^2 u_3}{\partial \theta^2} + \frac{1}{p\omega_0^2}\frac{\partial^4 u_3}{\partial \theta^2 \partial t^2} - \frac{1}{p\omega_0^2}\frac{\partial^2 u_3}{\partial t^2} = \frac{a^4}{EI}\frac{\partial}{\partial \theta}\left(q'_\theta + \frac{\partial q'_3}{\partial \theta}\right) \tag{6.15.8}$$

This is the governing equation of motion for the ring. To obtain the natural frequencies, one substitutes in the homogeneous equation

$$u_3(\theta, t) = U_3(\theta) e^{j\omega t} \tag{6.15.9}$$

which results in

$$\frac{d^6U_3}{d\theta^6}+2\frac{d^4U_3}{d\theta^4}+\frac{d^2U_3}{d\theta^2}\left(1-\frac{\omega^2}{p\omega_0^2}\right)+\frac{\omega^2}{p\omega_0^2}U_3=0 \tag{6.15.10}$$

This equation may be solved in a general way. Selecting as an example the closed ring,

$$U_3(\theta)=A_n\cos(n\theta-\phi) \tag{6.15.11}$$

is substituted, giving

$$-n^6+2n^4-n^2+\omega^2(n^2+1)\frac{1}{p\omega_0^2}=0 \tag{6.15.12}$$

Solving for ω gives the natural frequency for the nth mode, ω_n:

$$\omega_n^2=\frac{n^2(n^2-1)^2}{n^2+1}p\omega_0^2 \tag{6.15.13}$$

This solution was first obtained by Love (1927). It is a good approximation for natural modes where the transverse motion is dominant. It does not give hoop modes since the elasticity in circumferential direction was removed by the inextensional assumption.

6.16. Toroidal Shell

To apply the Donnell–Mushtari–Vlasov equations to the toroidal shell means pushing the boundary of what is permissible, since the toroidal shell is not a developable shell. Still, something can be learned from it.

If the toroidal shell of Fig. 5 of Chapter 3 is such that $a/R \ll 1$, we may simplify the fundamental form of Eq. (3.5.1) to

$$(ds)^2=R^2(d\theta)^2+a^2(d\phi)^2 \tag{6.16.1}$$

This gives $A_1=A_\theta=R$ and $A_2=A_\phi=a$ for $\alpha_1=\theta$ and $\alpha_2=\phi$. The radii of curvature are $R_2=R_\phi=a$, and the curvature $1/R_1=1/R_\theta$ can be averaged utilizing Eq. (3.5.3):

$$\left(\frac{1}{R_1}\right)_{av}=\left(\frac{1}{R_\theta}\right)_{av}=\frac{1}{2\pi}\int_{\phi=0}^{2\pi}\frac{1}{R_1}d\phi=\frac{1}{2\pi}\int_{\phi=0}^{2\pi}\frac{\sin\phi}{R+a\sin\phi}d\phi$$

$$\cong\frac{1}{2\pi}\int_{\phi=0}^{2\pi}\frac{\sin\phi}{R}d\phi \tag{6.16.2}$$

because of our assumption that $R\gg a\sin\phi$ The integral gives

$$\left(\frac{1}{R_1}\right)_{av}=0 \tag{6.16.3}$$

Equation (6.16.3) is a severe approximation but not entirely without logic. Equation (6.16.3) becomes

$$\nabla_k^2(\cdot)=\frac{1}{aR^2}\frac{\partial^2(\cdot)}{\partial\theta^2} \tag{6.16.4}$$

and Eq. (6.5.8) becomes

$$\nabla^2(\cdot)=\frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}+\frac{1}{R^2}\frac{\partial^2(\cdot)}{\partial\theta^2} \tag{6.16.5}$$

and thus,

$$\nabla_k^4(\cdot)=\frac{1}{a^2R^4}\frac{\partial^4(\cdot)}{\partial\theta^4} \tag{6.16.6}$$

$$\nabla^4(\cdot)=\frac{1}{a^4}\frac{\partial^4(\cdot)}{\partial\phi^4}+\frac{1}{R^4}\frac{\partial^4(\cdot)}{\partial\theta^4}+\frac{2}{a^2R^2}\frac{\partial^4(\cdot)}{\partial\phi^2\partial\theta^2} \tag{6.16.7}$$

Substituting this into Eq. (6.8.9) gives

$$\begin{aligned}&D\left(\frac{1}{a^4}\frac{\partial^4}{\partial\phi^4}+\frac{1}{R^4}\frac{\partial^4}{\partial\theta^4}+\frac{2}{a^2R^2}\frac{\partial^4}{\partial\phi^2\partial\theta^2}\right)\\&\times\left(\frac{1}{a^4}\frac{\partial^4U_3}{\partial\phi^4}+\frac{1}{R^4}\frac{\partial^4U_3}{\partial\theta^4}+\frac{2}{a^2R^2}\frac{\partial^4U_3}{\partial\phi^2\partial\theta^2}\right)+\frac{Eh}{a^2R^4}\frac{\partial^4U_3}{\partial\theta^4}\\&-\rho h\omega^2\left(\frac{1}{a^4}\frac{\partial^4U_3}{\partial\phi^4}+\frac{1}{R^4}\frac{\partial^4U_3}{\partial\theta^4}+\frac{2}{a^2R^2}\frac{\partial^4U_3}{\partial\phi^2\partial\theta^2}\right)=0\end{aligned} \tag{6.16.8}$$

For the closed toroidal shell, we find by inspection a solution which satisfies Eq. (6.16.8) and the continuity conditions, namely:

$$U_3(\phi,\theta)=\cos n\phi\cos m(\theta-\xi) \tag{6.16.9}$$

where ξ is an arbitrary phase angle that takes care of the non-preferential direction of the mode in θ direction. Because of the severe simplification, there is no apparent preferential direction in the ϕ direction either since $\sin n\phi$ is also a solution. (This is not quite true if a more precise analysis of a torodial shell is performed.) This means that if this simplified solution is to be used in a forced vibration analysis, we have to remember that we have to account for four modes of the type

$$U_3(\phi,\theta)=\cos n(\phi-\eta)\cos m(\theta-\xi) \tag{6.16.10}$$

where for both η and ξ two values leading to orthogonal modes have to be picked, usually $\xi=0$ and $\xi=\frac{\pi}{2m}$, and we have to pick also $\eta=0$ and $\eta=\frac{\pi}{2n}$ (see the discussions in Chapter 8).

Substituting Eq. (6.16.9) or Eq. (6.16.10) into Eq. (6.16.8) gives

$$D\left[\left(\frac{n}{a}\right)^2+\left(\frac{m}{R}\right)^2\right]^4+\frac{Eh}{a^2}\left(\frac{m}{R}\right)^4-\rho h\omega^2\left[\left(\frac{n}{a}\right)^2+\left(\frac{m}{R}\right)^2\right]^2=0 \tag{6.16.11}$$

or

$$\omega_{mn}=\frac{1}{a}\sqrt{\frac{\left(\frac{ma}{R}\right)^4}{\left[n^2+\left(\frac{ma}{R}\right)^2\right]^2}+\frac{\left(\frac{h}{a}\right)^2}{12(1-\mu^2)}\left[n^2+\left(\frac{ma}{R}\right)^2\right]^2}\sqrt{\frac{E}{\rho}} \tag{6.16.12}$$

Equation (6.16.12) looks similar to Eq. (6.12.3). If we substitute $L=2\pi R$ in Eq. (6.12.3) we obtain Eq. (6.16.12). The reason is that assumptions (6.16.1) and (6.16.3) amount to assuming that we can think of the toroidal shell as being a circular cylindrical shell (tube) that is bent into a ring. This model, of course, makes only sense if the radius of the tube, a, is significantly less than the radius R of the ring formed by the tube.

This means that this chapter is only applicable to toroidal shells that have the approximate proportions of an inner tube for a bicycle tire. The results of this chapter should not be used unless $a/R \ll 1$!

Still, the model is valuable in a general sense because it illustrates at least approximately the parameters that are important to the natural frequencies at a toroidal shell. It even illustrates that we have to expect, for a general, closed toroidal shell, four sets of natural modes that are either identical (the natural frequencies for $\xi=0$ and $\xi=\frac{\pi}{2m}$ are identical), or relatively close pairs (in our simplified example, the natural frequencies for $\eta=0$ and $\mu=\frac{\pi}{2n}$ are the same, but in a "real" toroidal shell they are different, with the mode component shapes in the ϕ direction deviating from $\cos n\phi$ and $\sin n\phi$ but maintaining the same approximate character).

6.17. THE BARREL-SHAPED SHELL USING MODIFIED LOVE EQUATIONS

An argument can be made in many applications that it is desirable to raise the minimum natural frequency of a circular cylindrical shell above a certain critical value which is to be avoided for one reason or the other.

The solution of a circular, cylindrical shell case in Sec. 5.5 shows that thickness changes will raise the natural frequencies of the $i=1$ set (dominant transverse mode components) for n values well to the right of the minimum natural frequency where bending strain effects dominate, but will not change the natural frequencies for modes appreciably where the membrane strain effects are dominant, namely to the left of the minimum natural frequency. In other words, natural frequencies are proportional to shell thickness changes for high n values as in plates, but are virtually unaffected by a thickness change for low n values. The minimum natural frequency is, approximately, only proportional to the square root of the ratio of the increased thickness to the original thickness. Even doubling the thickness will raise the minimum natural frequency of the $i=1, m=1$ curve only by about 40%. See also Secs. 19.6 and 19.7.

A more effective way, if feasible from a design viewpoint, is to avoid pure cylindrical shells but to introduce at least a slight curvature in the direction of the cylinder axis. A circular cylindrical shell, if feasible, should take on the shape of a barrel. This will lift the minimum natural frequencies more effectively than a thickness change, without an appreciable increase in weight.

While the beneficial influence of barreling was already shown by the results of Sec. 6.13, it is here investigated again using a more complete equation set. This case is used to illustrate that one can, with judicious assumptions, generate from case to case approximate Love equations which are tailor-made for only one set of problems. Also, the more complete analysis will demonstrate that the higher natural frequency branches of the $i=1$ and 2 type are virtually unaffected by barreling. This time, we do not start with the Donnell–Mushtari–Vlasov simplifications as in Secs. 6.7–6.16, but utilize Love's equations (2.7.20)–(2.7.24). The only assumption will be the one of Eq. (6.13.1).

Therefore, selecting an approximate base cylinder for the coordinate system, as shown in Fig. 8, with $\alpha_1=x, \alpha_2=\theta, A_1=1, A_2=a, R_1=R$ and $R_2=a$, Love's equations (2.7.20)–(2.7.24) become

$$\frac{\partial N_{xx}}{\partial x}+\frac{1}{a}\frac{\partial N_{\theta x}}{\partial \theta}+\frac{Q_{x3}}{R}-\rho h\ddot{u}_x=0 \tag{6.17.1}$$

$$\frac{\partial N_{x\theta}}{\partial x}+\frac{1}{a}\frac{\partial N_{\theta\theta}}{\partial \theta}+\frac{Q_{\theta 3}}{a}-\rho h\ddot{u}_\theta=0 \tag{6.17.2}$$

$$\frac{\partial Q_{x3}}{\partial x}+\frac{1}{a}\frac{\partial Q_{\theta 3}}{\partial \theta}-\frac{N_{\theta\theta}}{a}-\frac{N_{xx}}{R}-\rho h\ddot{u}_3=0 \tag{6.17.3}$$

and

$$Q_{x3}=\frac{\partial M_{xx}}{\partial x}+\frac{1}{a}\frac{\partial M_{\theta x}}{\partial \theta} \tag{6.17.4}$$

$$Q_{\theta 3}=\frac{\partial M_{x\theta}}{\partial x}+\frac{1}{a}\frac{\partial M_{\theta\theta}}{\partial \theta} \tag{6.17.5}$$

The only difference between Love's equations for the true circular cylindrical shell and these equations is that the third term in Eq. (6.17.1) and the fourth term in Eq. (6.17.3) are now present.

The strain–displacement relationships are

$$\varepsilon^0_{xx}=\frac{\partial u_x}{\partial x}+\frac{u_3}{R} \tag{6.17.6}$$

$$\varepsilon^0_{\theta\theta}=\frac{1}{a}\frac{\partial u_\theta}{\partial \theta}+\frac{u_3}{a} \tag{6.17.7}$$

$$\varepsilon_{x\theta}^{0} = \frac{\partial u_\theta}{\partial x} + \frac{1}{a}\frac{\partial u_x}{\partial \theta} \tag{6.17.8}$$

$$k_{xx} = \frac{\partial \beta_x}{\partial x} \tag{6.17.9}$$

$$k_{\theta\theta} = \frac{1}{a}\frac{\partial \beta_\theta}{\partial \theta} \tag{6.17.10}$$

$$k_{x\theta} = \frac{\partial \beta_\theta}{\partial x} + \frac{1}{a}\frac{\partial \beta_x}{\partial \theta} \tag{6.17.11}$$

where

$$\beta_x = -\frac{\partial u_3}{\partial_x} + \frac{u_x}{R} \tag{6.17.12}$$

$$\beta_\theta = -\frac{1}{a}\frac{\partial u_3}{\partial \theta} + \frac{u_\theta}{a} \tag{6.17.13}$$

The only difference between the equations for the true circular cylindrical shell and these equations is that Eqs. (6.17.6) and (6.17.12) are different. Equations (6.17.1)–(6.17.13) are satisfied by solutions (5.5.42)–(5.5.44) and if the same simply supported boundary conditions of the circular cylindrical shell example of Sec. 5.5 are considered, they also satisfy Eqs. (5.5.1)–(5.5.8), having separated time:

$$U_x(x,\theta) = A\cos\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{6.17.14}$$

$$U_\theta(x,\theta) = B\sin\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{6.17.15}$$

$$U_3(x,\theta) = C\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{6.17.16}$$

Proceeding as in Sec. 5.5, we obtain again the matrix equation

$$\begin{bmatrix} \rho h\omega^2 - k_{11} & k_{12} & k_{13} \\ k_{21} & \rho h\omega^2 - k_{22} & k_{23} \\ k_{31} & k_{32} & \rho h\omega^2 - k_{33} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0 \tag{6.17.17}$$

except that the k_{ij} terms contain now terms involving the radius R

$$k_{11} = K\left[\left(\frac{m\pi}{L}\right)^2 + \frac{1-\mu}{2}\left(\frac{n}{a}\right)^2\right] + \left(\frac{n}{a}\right)^2\frac{D(1-\mu)}{2R^2} \tag{6.17.18}$$

$$k_{12} = \left(\frac{n}{a}\right)\left(\frac{m\pi}{L}\right)\left[\frac{K(1+\mu)}{2} + \frac{D(1-\mu)}{2aR}\right] \tag{6.17.19}$$

$$k_{13} = \left(\frac{m\pi}{L}\right)\left[K\left(\frac{1}{R} + \frac{\mu}{a}\right) + \left(\frac{n}{a}\right)^2\frac{D(1-\mu)}{R}\right] \tag{6.17.20}$$

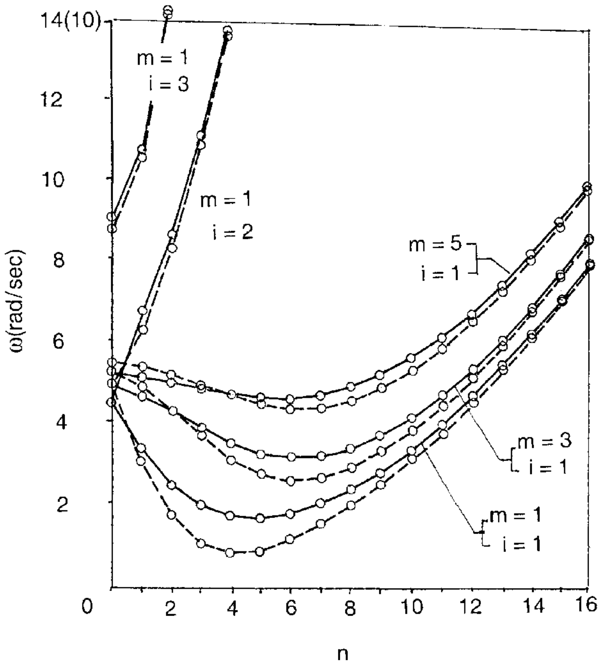

FIG. 11 The stiffening effect of introducing axial curvature in a circular cylindrical shell (dashed lines define the circular cylindrical shell natural frequencies, solid lines define the natural frequencies after axial curvature has been introduced).

$$k_{21} = \left(\frac{m\pi}{L}\right)\left(\frac{n}{a}\right)\left(\frac{1+\mu}{2}\right)\left[K+\frac{D}{aR}\right] \quad \textbf{(6.17.21)}$$

$$k_{22} = \left(K+\frac{D}{a^2}\right)\left[\left(\frac{1-\mu}{2}\right)\left(\frac{m\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right] \quad \textbf{(6.17.22)}$$

$$k_{23} = -\frac{n}{a}\left(\frac{K}{a}+\frac{K\mu}{R}\right)-\left(\frac{n}{a}\right)\left(\frac{D}{a}\right)\left[\left(\frac{m\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right] \quad \textbf{(6.17.23)}$$

$$k_{31} = \left(\frac{m\pi}{L}\right)\left\{\frac{K\mu}{a}+\frac{K}{R}+\frac{D}{R}\left[\left(\frac{m\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]\right\} \quad \textbf{(6.17.24)}$$

$$k_{32} = \left(\frac{n}{a}\right)\left\{\frac{K\mu}{R}-\frac{K}{a}-\left(\frac{D}{a}\right)\left[\left(\frac{m\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]\right\} \quad \textbf{(6.17.25)}$$

$$k_{33} = K\left(\frac{1}{a^2}-\frac{1}{R^2}\right)+D\left[\left(\frac{m\pi}{L}\right)^2+\left(\frac{n}{a}\right)^2\right]^2 \quad \textbf{(6.17.26)}$$

The resulting frequency equation is that of Eq. (5.5.73), and all subsequent Eqs. (5.5.74)–(5.5.87) apply, with the k_{ij} values of Eqs. (6.17.18)–(6.17.26), of course.

The same example as in Sec. 5.5, but with a radius $R = 500\,\text{mm}$, is illustrated in Fig. 11. The effectiveness of the barreling of raising the minimum values of the natural frequency curves for the set $i = 1$ are demonstrated. The solid lines represent the barreled shell, the dashed lines duplicate the original cylindrical shell results of Fig. 5.5.2 As one would expect, the barreling does not appreciably affect the $i = 1$ and $i = 2$ branches of the natural frequency curves.

6.18. DOUBLY CURVED RECTANGULAR PLATE

A simply supported rectangular plate is slightly curved in order to increase its natural frequencies over what they would have been for a flat plate. This was investigated in Sec. 6.14 for a circular plate having spherical, constant curvature, and generalized to all plates having spherical curvature in Eq. (6.14.9).

Here, the plate has a constant radius of curvature R_x in the x direction, and a constant radius of curvature R_y in the y direction, see Fig. 12. Similar to before, we select as coordinate system the x, y coordinates of the flat base plate, assuming that the fundamental form is approximately

$$(\mathrm{d}s)^2 \cong (\mathrm{d}x)^2 + (\mathrm{d}y)^2 \tag{6.18.1}$$

Therefore, for $\alpha_1 = x$ and $\alpha_2 = y$, we obtain $A_1 = 1$ and $A_2 = 1$, and $R_1 = R_x =$ constant and $R_2 = R_y =$ constant. Using the Donnell–Mushtari–Vlasor simplification, we obtain

$$\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2} \tag{6.18.2}$$

and from Eq. (6.7.11),

$$\nabla_k^2(\cdot) = \frac{1}{R_y}\frac{\partial^2(\cdot)}{\partial x^2} + \frac{1}{R_x}\frac{\partial^2(\cdot)}{\partial y^2} \tag{6.18.3}$$

Substituting this into Eq. (6.8.9),

$$D\nabla^8 U_3 + Eh\nabla_k^4 U_3 - \rho h \omega^2 \nabla^4 U_3 = 0 \tag{6.18.4}$$

allows us to obtain the natural frequencies and modes. Note that a general formula for the natural frequencies that is valid for all boundary conditions, as obtained in Eq. (6.14.9) for all spherically curved plates, is not possible.

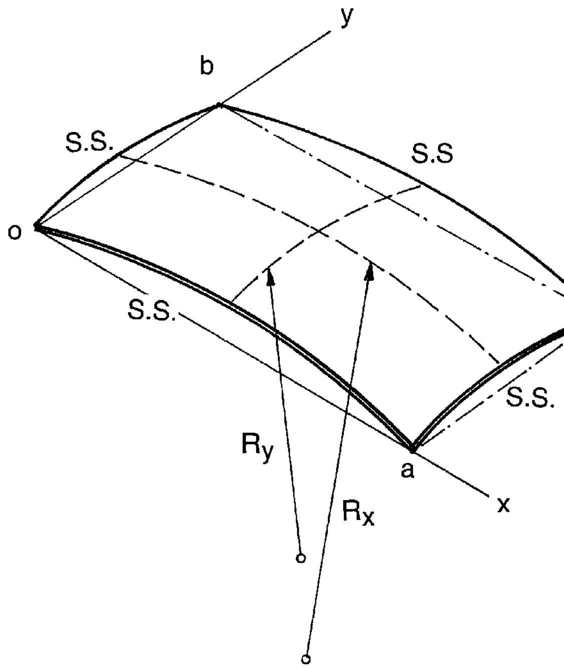

Fig. 12 Doubly curved rectangular plate.

For example, let us analyze the doubly curved, simply supported rectangular plate. By inspection, the boundary conditions (taken to be approximately the same as for the equivalent flat plate) and Eq. (6.18.4) are satisfied by

$$U_3 = A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6.18.5}$$

Substituting this into Eq. (6.18.4) gives

$$D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^4 + Eh\left[\frac{1}{R_y}\left(\frac{m\pi}{a}\right)^2\right] + \frac{1}{R_x}\left(\frac{n\pi}{b}\right)^2\Bigg]^2$$

$$-\rho h\omega^2\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 = 0 \tag{6.18.6}$$

or, solving for the square of the natural frequencies ω_{mnc}^2 of the curved plate,

$$\omega_{mnc}^2 = \omega_{mnf}^2 + \frac{\left[\frac{1}{R_y}\left(\frac{m\pi}{a}\right)^2 + \frac{1}{R_x}\left(\frac{n\pi}{b}\right)^2\right]^2}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}\left(\frac{E}{\rho}\right) \tag{6.18.7}$$

where ω_{mnf} is the equivalent frequency for the flat plate, given by Eq. (5.4.71), namely

$$\omega_{mnf} = \pi^2 \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right] \sqrt{\frac{D}{\rho h}} \tag{6.18.8}$$

Note that if we make the radii of curvature equal, $R_x = R_y = R$, we recover the case of the spherically curved rectangular plate since Eq. (6.18.7) reduces to Eq. (6.14.9), namely

$$\omega_{mnc}^2 = \omega_{mnf}^2 + \frac{E}{\rho R^2} \tag{6.18.9}$$

It must be noted that we have obtained an approximate solution to the Donnell–Mushtari–Vlasov equations which are themselves approximate equations. Therefore, while the results can be used with great benefit to establish trends when parameters are to be changed, solutions to the full theory have to be obtained if results are required that have to be relatively exact.

Equation (6.18.7) is, however, an improvement over Eq. (6.5.7) which was based on a simplification that is not recommended.

If we set $R_x = \infty$, we obtain an approximate version of the solution for the simply supported, cylindrical panel treated in Sec. 6.12, Eq. (6.12.5). This approximate version is

$$\omega_{mnc}^2 = \omega_{mnf}^2 + \frac{E}{\rho R_y^2} \frac{\left(\frac{m\pi}{a}\right)^4}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2} \tag{6.18.10}$$

and is only valid for a very shallow, simply supported cylindrical panel while Eq. (6.12.5) is valid also for deep cylindrical panels that are simply supported cylindrical shells.

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7

Approximate Solution Techniques

Compared to the large number of possible shell configurations, very few exact solutions of plate and shell eigenvalue problems are possible. A representative sample was presented in earlier chapters. Included in this sample were exact solutions to the simplified equations of motion. The exact solutions are very valuable because they are the measure with which the accuracy of the approximation approaches is evaluated. They also allow an accurate and usually elegant and conclusive investigation of the various fundamental phenomena in shell vibrations. However, it is important for engineering applications to have approaches available that give numerical solutions for cases that cannot be solved exactly. To discuss these cases is the purpose of this chapter.

The approximate approaches divide roughly into two categories. In the first category, a minimization of energy approach is used. The variational integral method, the Galerkin method, and the Rayleigh–Ritz method are of this type. They are discussed in this chapter. In the second category we find the finite difference and the finite element method. They are outlined in a later chapter.

7.1. APPROXIMATE SOLUTIONS BY WAY OF THE VARIATIONAL INTEGRAL

Let us start with Eq. (2.7.19). For $q_1 = q_2 = q_3 = 0$ and

$$u_i(\alpha_1, \alpha_2, t) = U_i(\alpha_1, \alpha_2)e^{j\omega t} \tag{7.1.1}$$

it becomes, with time now removed from consideration and all boundary conditions assumed to be satisfied,

$$\int_{\alpha_1}\int_{\alpha_2}([L_1\{U_1,U_2,U_3\}+\rho h\omega^2 U_1]\delta U_1+[L_2\{U_1,U_2,U_3\}+\rho h\omega^2 U_2]\delta U_2 + [L_3\{U_1,U_2,U_3\}+\rho h\omega^2 U_3]\delta U_3)A_1A_2\ d\alpha_1 d\alpha_2 = 0 \tag{7.1.2}$$

or, in short,

$$\int_{\alpha_1}\int_{\alpha_2}\sum_{i=1}^{3}[L_i\{U_1,U_2,U_3\}+\rho h\omega^2 U_i]\delta U_i A_1A_2\ d\alpha_1 d\alpha_2 = 0 \tag{7.1.3}$$

where $i = 1, 2, 3$ and where

$$L_1\{U_1,U_2,U_3\} = \frac{1}{A_1A_2}\left[\frac{\partial(N_{11}A_2)}{\partial\alpha_1}+\frac{\partial(N_{21}A_1)}{\partial\alpha_2}+N_{12}\frac{\partial A_1}{\partial\alpha_2}-N_{22}\frac{\partial A_2}{\partial\alpha_1}\right]+\frac{Q_{13}}{R_1} \tag{7.1.4}$$

$$L_2\{U_1,U_2,U_3\} = \frac{1}{A_1A_2}\left[\frac{\partial(N_{12}A_2)}{\partial\alpha_1}+\frac{\partial(N_{22}A_1)}{\partial\alpha_2}+N_{21}\frac{\partial A_2}{\partial\alpha_1}-N_{11}\frac{\partial A_1}{\partial\alpha_2}\right]+\frac{Q_{23}}{R_2} \tag{7.1.5}$$

$$L_3\{U_1,U_2,U_3\} = \frac{1}{A_1A_2}\left[\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}+\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}\right]-\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right) \tag{7.1.6}$$

We now assume functions $f_{ij}(\alpha_1, \alpha_2)$ that satisfy the boundary conditions and can represent a reasonable-looking mode shape. There may be one function ($j = 1$) or many ($j = 1, 2, \ldots, n$). In general,

$$U_1 = a_{11}f_{11} + a_{12}f_{12} + \cdots \tag{7.1.7}$$

$$U_2 = a_{21}f_{21} + a_{22}f_{22} + \cdots \tag{7.1.8}$$

$$U_3 = a_{31}f_{31} + a_{32}f_{32} + \cdots \tag{7.1.9}$$

or, in short,

$$U_i = \sum_{j=1}^{n} a_{ij}f_{ij} \tag{7.1.10}$$

and also

$$\delta U_i = \sum_{j=1}^{n} f_{ij} \delta a_{ij} \tag{7.1.11}$$

Note that the a_{ij} are coefficients that have to be determined.

Upon the substitution, the integral becomes

$$\int_{\alpha_1}\int_{\alpha_2} \sum_{i=1}^{3} \left[L_i \left\{ \sum_{j=1}^{n} a_{1j} f_{1j}, \sum_{j=1}^{n} a_{2j} f_{2j}, \sum_{j=1}^{n} a_{3j} f_{3j} \right\} + \rho h \omega^2 \sum_{j=1}^{n} a_{ij} f_{ij} \right] \times \sum_{j=1}^{n} f_{ij} \delta a_{ij} A_1 A_2 \, d\alpha_1 \, d\alpha_2 = 0 \tag{7.1.12}$$

where $i = 1,2,3$ and $j = 1,2,\ldots,n$. Next, we collect coefficients of δa_{ij}. Since each δa_{ij} is independent and arbitrary, the equation can only be satisfied if each coefficient is equal to zero.

$$\int_{\alpha_1}\int_{\alpha_2} \left[L_i \left\{ \sum_{j=1}^{n} a_{1j} f_{1j}, \sum_{j=1}^{n} a_{2j} f_{2j}, \sum_{j=1}^{n} a_{3j} f_{3j} \right\} + \rho h \omega^2 \sum_{j=1}^{n} a_{ij} f_{ij} \right] f_{ij} A_1 A_2 \, d\alpha_1 \, d\alpha_2 = 0 \tag{7.1.13}$$

where $i = 1,2,3$ and $j = 1,2,\ldots,n$. The approach is now to carry out the integration and to bring the result into the form

$$[A]\{a_{ij}\} = 0 \tag{7.1.14}$$

where

$$[a_{ij}] = [a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{2n}, a_{31}, a_{32}, \ldots, a_{3n}] \tag{7.1.15}$$

Setting the determinant of A equal to zero,

$$|A| = 0 \tag{7.1.16}$$

gives a characteristic equation for ω^2. There will be $3n$ roots. These are the natural frequencies. Resubstituting a natural frequency into the matrix equation allows us to solve for $3n-1$ values of a_{ij} in terms of one arbitrary a_{ij}. This establishes the mode shape that is associated with this natural frequency. This method works well, provided that the assumed functions are of sufficient variety to give good mode approximations. If $n = 1$ only, the mode shape is fixed by the assumption.

It can be shown that natural frequencies will always be of larger value than obtained from exact solutions, provided that the mode functions satisfy all boundary conditions. This is not necessarily true if moment and shear conditions are not satisfied, which is a common approach.

The rule of thumb is that the *primary boundary conditions*, the deflection and slope conditions, should always be satisfied, but that is possible to ignore the *secondary boundary conditions*, the shear and moment conditions, provided that one is interested only in natural frequency predictions. If stress calculations are the objective, ignoring moment and shear boundary conditions may not be permissible, depending on the circumstances and the accuracy requirement.

7.2. USE OF BEAM FUNCTIONS

A common selection of functions for rectangular plates and cylindrical and conical shells are beam mode shapes, also called *beam functions*. The boundary conditions of the beam and the shell have to be of the same type. The argument is that, for example, the behavior of an axial strip of cylindrical shell should be similar to that of a beam of the same type of boundary conditions.

Since the beam function has the higher mode shapes already built in, it is often sufficient to use $n=1$. The matrix$[A]$ is then a 3×3 matrix only. The characteristics equation is a cubic equation in ω^2. The advantage of using orthogonal beam functions is that the shell or plate mode shapes are orthogonal also.

For plates, the variational integral uncouples into one integral for transverse deflections,

$$\int_{\alpha_1}\int_{\alpha_2}\left[L_3\left\{\sum_{j=1}^{n}a_{3j}f_{3j}\right\}+\rho h\omega^2\sum_{j=1}^{n}a_{3j}f_{3j}\right]f_{3j}A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2=0 \tag{7.2.1}$$

where

$$L_3\{\cdot\}=-D\nabla^4(\cdot) \tag{7.2.2}$$

and two integrals that govern in-plane deflections:

$$\int_{\alpha_1}\int_{\alpha_2}\left[L_i\left\{\sum_{j=1}^{n}a_{1j}f_{1j},\sum_{j=1}^{n}a_{2j}f_{2j}\right\}+\rho h\omega^2\sum_{j=1}^{n}a_{ij}f_{ij}\right]f_{ij}A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2=0 \tag{7.2.3}$$

where $i=1,2$.

Let us now examine further the integral for transverse motion. If $n=1$, we get

$$\int_{\alpha_1}\int_{\alpha_2}[L_3\{a_{31}f_{31}\}f_{31}+\rho h\omega^2a_{31}f_{31}^2]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2=0 \tag{7.2.4}$$

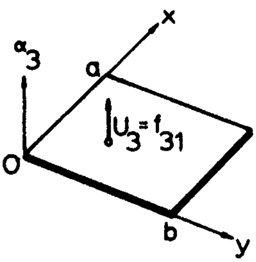

FIG. 1 Rectangular plate.

or

$$\omega^2 = -\frac{1}{\rho h}\frac{\int_{\alpha_1}\int_{\alpha_2} L_3\{f_{31}\} f_{31} A_1 A_2 \,d\alpha_1\, d\alpha_2}{\int_{\alpha_1}\int_{\alpha_2} f_{31}^2 A_1 A_2 \,d\alpha_1\, d\alpha_2} \tag{7.2.5}$$

For the case of a rectangular plate as sketched in Fig. 1, this becomes

$$\omega^2 = \frac{D}{\rho h}\frac{\int_0^a\int_0^b (\partial^4 f_{31}/\partial x^4 + 2\partial^4 f_{31}/\partial x^2\partial y^2 + \partial^4 f_{31}/\partial y^4) f_{31}\,dx\,dy}{\int_0^a\int_0^b f_{31}^2\,dx\,dy} \tag{7.2.6}$$

Furthermore, if we use beam functions,

$$f_{31}(x,y) = \alpha(x)\beta(y) \tag{7.2.7}$$

where $\alpha(x)$ is the beam function in the x direction and $\beta(y)$ is the beam function in the y direction. Thus Eq. (7.2.6) becomes

$$\begin{aligned}\omega^2 = \frac{(D/\rho h)}{\int_0^a \alpha^2 dx \int_0^b \beta^2 dy}&\left[\int_0^a \alpha(\partial^4\alpha/\partial x^4)dx\int_0^b \beta^2 dy + \int_0^a \alpha^2 dx\right.\\ &\left.\times\int_0^b \beta(\partial^4\beta/\partial y^4)dy + 2\int_0^a \alpha(\partial^2\alpha/\partial x^2)dx\int_0^b \beta(\partial^2\beta/\partial y^2)dy\right]\end{aligned} \tag{7.2.8}$$

For beam functions in general,

$$\frac{\partial^4\alpha}{\partial x^4} = \lambda_m^4\alpha \tag{7.2.9}$$

$$\frac{\partial^4\beta}{\partial y^4} = \lambda_n^4\beta \tag{7.2.10}$$

where λ was defined when the beam solution was discussed in Sec. 5.2. After substitution we obtain

$$\omega^2 = \frac{D}{\rho h}\left[\lambda_m^4 + \lambda_n^4 + 2\frac{\int_0^a \alpha(\partial^2\alpha/\partial x^2)dx\int_0^b \beta(\partial^2\beta/\partial y^2)dy}{\int_0^a \alpha^2 dx\int_0^b \beta^2 dy}\right] \tag{7.2.11}$$

Let us, for instance, obtain the natural frequencies of a rectangular plate clamped on all four edges. From Sec. 5.2 the mode shape of a clamped–clamped beam is

$$\alpha(x) = C(\lambda_m x) - \frac{C(\lambda_m a)}{D(\lambda_m a)} D(\lambda_m x) \tag{7.2.12}$$

where

$$C(\lambda_m x) = \cosh \lambda_m x - \cos \lambda_m x \tag{7.2.13}$$

$$D(\lambda_m x) = \sinh \lambda_m x - \sin \lambda_m x \tag{7.2.14}$$

and where $\lambda_1 a = 4.73, \lambda_2 a = 7.85, \lambda_3 a = 11.00, \lambda_4 a = 14.14$, and so on. Also,

$$\beta(y) = C(\lambda_n y) - \frac{C(\lambda_n b)}{D(\lambda_n b)} D(\lambda_n y) \tag{7.2.15}$$

where

$$C(\lambda_n y) = \cosh \lambda_n y - \cos \lambda_n y \tag{7.2.16}$$

$$D(\lambda_n y) = \sinh \lambda_n y - \sin \lambda_n y \tag{7.2.17}$$

and where $\lambda_1 b = 4.73, \lambda_2 b = 7.85, \lambda_3 b = 11.00, \lambda_4 b = 14.14$, and so on.

The function $f_{31}(x, y)$ now satisfies all boundary conditions for the clamped–clamped plate:

$$u_3(0, y, t) = u_3(a, y, t) = u_3(x, 0, t) = u_3(x, b, t) = 0 \tag{7.2.18}$$

$$\frac{\partial u_3}{\partial x}(0, y, t) = \frac{\partial u_3}{\partial x}(a, y, t) = \frac{\partial u_3}{\partial y}(x, 0, t) = \frac{\partial u_3}{\partial y}(x, b, t) = 0 \tag{7.2.19}$$

The results of Eq. (7.2.11) are provided in Table 1 in terms of $\omega a^2 \sqrt{\rho h / D}$ values.

Table 1 Natural Frequencies for Clamped Square Plate

	m		
n	1	2	3
1	36.1	73.7	132.0
2	73.7	108.9	165.8
3	132.0	165.8	221.4

7.3. GALERKIN'S METHOD APPLIED TO SHELL EQUATIONS

Restarting the result of Eq. (7.1.13) in terms of a Galerkin algorithm, we assume as solutions

$$U_1 = \sum_{j=1}^{n} a_{1j} f_{1j} \tag{7.3.1}$$

$$U_2 = \sum_{j=1}^{n} a_{2j} f_{2j} \tag{7.3.2}$$

$$U_3 = \sum_{j=1}^{n} a_{3j} f_{3j} \tag{7.3.3}$$

where the f_{ij} functions satisfy at least the geometric boundary conditions, and substitute this into the equations of motion,

$$L_1\{U_1, U_2, U_3\} + \rho h \omega^2 U_1 = 0 \tag{7.3.4}$$

$$L_2\{U_1, U_2, U_3\} + \rho h \omega^2 U_2 = 0 \tag{7.3.5}$$

$$L_3\{U_1, U_2, U_3\} + \rho h \omega^2 U_3 = 0 \tag{7.3.6}$$

Then we multiply the first equation by U_1, the second by U_2, and the third by U_3, where the U_i serve as weighting functions, and integrate the equations over the domain, to arrive at

$$\int_{\alpha_1}\int_{\alpha_2} [L_1\{U_1, U_2, U_3\} + \rho h \omega^2 U_1] U_1 A_1 A_2 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 = 0 \tag{7.3.7}$$

$$\int_{\alpha_1}\int_{\alpha_2} [L_2\{U_1, U_2, U_3\} + \rho h \omega^2 U_2] U_2 A_1 A_2 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 = 0 \tag{7.3.8}$$

$$\int_{\alpha_1}\int_{\alpha_2} [L_3\{U_1, U_2, U_3\} + \rho h \omega^2 U_3] U_3 A_1 A_2 \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 = 0 \tag{7.3.9}$$

Performing the integration, we obtain a matrix equation

$$[A]\{a_{ij}\} = 0, \quad i = 1, 2, 3; \; j = 1, 2, \ldots, n \tag{7.3.10}$$

The roots of its determinant will furnish the natural frequencies, which in turn, after resubstitution into the matrix equation, will allow determination of all the a_{ij} but one.

It is useful to consider the special case where we approximate the solution by three functions only:

$$U_1 = a_{11} f_{11} \tag{7.3.11}$$

$$U_2 = a_{21} f_{21} \tag{7.3.12}$$

$$U_3 = a_{31} f_{31} \tag{7.3.13}$$

Since all linear equations can be divided such that

$$L_1\{U_1, U_2, U_3\} = L_{11}\{U_1\} + L_{12}\{U_2\} + L_{13}\{U_3\} \tag{7.3.14}$$

$$L_2\{U_1, U_2, U_3\} = L_{21}\{U_1\} + L_{22}\{U_2\} + L_{23}\{U_3\} \tag{7.3.15}$$

$$L_3\{U_1, U_2, U_3\} = L_{31}\{U_1\} + L_{32}\{U_2\} + L_{33}\{U_3\} \tag{7.3.16}$$

we obtain the matrix equation

$$\begin{bmatrix} \rho h\omega^2 + k_{11} & k_{12} & k_{13} \\ k_{21} & \rho h\omega^2 + k_{22} & k_{23} \\ k_{31} & k_{32} & \rho h\omega^2 + k_{33} \end{bmatrix} \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{Bmatrix} = 0 \tag{7.3.17}$$

where

$$k_{11} = \frac{\int_{\alpha_1}\int_{\alpha_2} L_{11}\{U_1\} f_{11} A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2}{\int_{\alpha_1}\int_{\alpha_2} f_{11}^2 A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2} \tag{7.3.18}$$

$$k_{12} = \frac{\int_{\alpha_1}\int_{\alpha_2} L_{12}\{U_2\} f_{11} A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2}{\int_{\alpha_1}\int_{\alpha_2} f_{11}^2 A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2} \tag{7.3.19}$$

$$\vdots$$

or, in general,

$$k_{rs} = \frac{\int_{\alpha_1}\int_{\alpha_2} L_{rs}\{U_r\} f_{r1} A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2}{\int_{\alpha_1}\int_{\alpha_2} f_{r1}^2 A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2} \tag{7.3.20}$$

The determinant becomes

$$\omega^6 + a_1\omega^4 + a_2\omega^2 + a_3 = 0 \tag{7.3.21}$$

where

$$a_1 = \frac{1}{\rho h}(k_{11} + k_{22} + k_{33}) \tag{7.3.22}$$

$$a_2 = \frac{1}{(\rho h)^2}(k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33} - k_{23}^2 - k_{12}^2 - k_{13}^2) \tag{7.3.23}$$

$$a_3 = -\frac{1}{(\rho h)^3}(k_{11}k_{23}^2 - k_{11}k_{22}k_{33} + k_{22}k_{13}^2 + k_{33}k_{12}^2 - 2k_{12}k_{23}k_{13}) \tag{7.3.24}$$

Thus we obtain three natural frequencies that correspond, if f_{11}, f_{21}, and f_{31} were selected properly, to the three frequency branch values.

For example, if we would perform the required integrations using the exact solution for the simply supported shell of Sec. 5.5, so that $a_{11}=A$, $a_{21}=B$, $a_{31}=C$ and

$$f_{11}=\cos\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{7.3.25}$$

$$f_{21}=\sin\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{7.3.26}$$

$$f_{31}=\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{7.3.27}$$

the results of Sec. 5.5, would be obtained. For other boundary conditions, a good approximation for a cylindrical shell closed in the θ direction would be

$$f_{11}=\alpha(x)\cos n(\theta-\phi) \tag{7.3.28}$$

$$f_{12}=\beta(x)\sin n(\theta-\phi) \tag{7.3.29}$$

$$f_{31}=\gamma(x)\cos n(\theta-\phi) \tag{7.3.30}$$

where $\gamma(x)$ is the mode shape of a transversely vibrating beam of the same boundary conditions and $\alpha(x)$ is the mode shape of a longitudinally vibrating rod of the same boundary conditions. The function $\beta(x)$ has to be chosen on the basis of the boundary conditions in the θ direction. This will not satisfy all terms of the real boundary conditions but will still give reasonable approximations.

In conclusion, Galerkin's method is an algorithmic statement of the variational approach, as summarized for shells in Eq. (7.1.13). The algorithm is that the functions f_{ij} that satisfy the boundary conditions are substituted into the equations of motion and the resulting expressions are multiplied by their respective f_{ij} as *weighing functions.* Finally, the product is integrated over the domain. This is exactly what Eq. (7.1.13) does. The Galerkin approach has become a general algorithm for solving a variety of equations and problems and its variational birthmark has disappeared.

7.3.1. Galerkin's Method Applied to Donnell–Mushtari–Vlasov Equation

Let us solve an example of the Donnell–Mushtari–Vlasov equation for closed circular cylindrical shells directly (Soedel, 1980) without the Yu simplification. In this case we need to consider only the transverse deflections. The Galerkin algorithm demands that we assume a function $U_{3m}(x)$ that satisfies the boundary conditions, multiply the equation with the same function as a weighing function, and integrate over the shell length. Starting with Eq. (6.9.4), this gives

$$\int_0^L \left[D\left(\frac{n^2}{a^2}-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^4 U_{3m}(x)+\frac{Eh}{a^2}\frac{\mathrm{d}^4}{\mathrm{d}x^4}U_{3m}(x) - \rho h\omega^2\left(\frac{n^2}{a^2}-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^2 U_{3m}(x)\right] U_{3m}(x)\,\mathrm{d}x=0 \tag{7.3.31}$$

Making use of the fact that for beam functions

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}U_{3m}(x)=\lambda_m^4 U_{3m}(x) \tag{7.3.32}$$

where λ_m are the roots of the beam eigenvalue problem, we get

$$\omega_{mn}^2=\frac{(Eh/a^2)\lambda_m^4+D[(n/a)^8-4(n/a)^6R_m+6(n/a)^4\lambda_m^4 -4(n/a)^2\lambda_m^4R_m+\lambda_m^8]}{\rho h[(n/a)^4-2(n/a)^2R_m+\lambda_m^4]} \tag{7.3.33}$$

where

$$R_m=\frac{\int_0^L (d^2U_{3m}/\mathrm{d}x^2)U_{3m}\mathrm{d}x}{\int_0^L U_{3m}^2\mathrm{d}x} \tag{7.3.34}$$

This result can now be applied to various boundary conditions. However, a further simplification is often possible. Since

$$U_{3m}=U_{3m}(\lambda_m x) \tag{7.3.35}$$

we get

$$R_m=\lambda_m^2\frac{\int_0^L [\mathrm{d}^2U_{3m}/\mathrm{d}(\lambda_m x)^2]U_{3m}\mathrm{d}x}{\int_0^L U_{3m}^2\mathrm{d}x} \tag{7.3.36}$$

The ratio of the integrals is close to -1 for many sets of boundary conditions. For the simply supported beam function and some other cases, it is exactly equal to -1. Thus as an approximation, we may set

$$R_m\cong-\lambda_m^2 \tag{7.3.37}$$

This results in

$$\omega_{mn}^2=\frac{1}{\rho h}\left\{\frac{Eh\lambda_m^4}{a^2[(n/a)^2+\lambda_m^2]^2}+D\left[\left(\frac{n}{a}\right)^2+\lambda_m^2\right]^2\right\} \tag{7.3.38}$$

For the simply supported case, $\lambda_m=m\pi/L$, and Eq. (7.3.38) is exact. Equations (7.3.33) and (7.3.38) are compared in Fig. 2 for the clamped-clamped circular cylindrical shell for $E=20.6\times10^4\,\mathrm{N/mm^2}$, $\rho=7.85\times10^{-9}\ \mathrm{N\cdot s^2/mm^4}$, $\mu=0.3, h=2\mathrm{mm}$, $a=100\mathrm{mm}$, and $L=200\mathrm{mm}$. We see

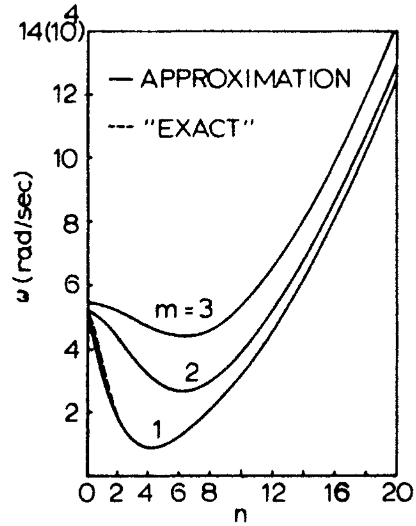

Fig. 2 Natural frequencies of a clamped–clamped circular cylindrical shell.

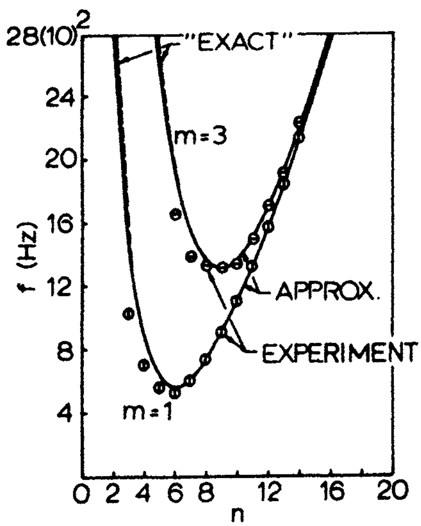

Fig. 3 Typical comparison of an approximate natural frequency solution with experimental data.

that agreement is excellent. Equation (7.3.38) is very easy to use since $\lambda_m L$ values for the majority of the conceivable boundary conditions are available in handbooks (Flügge, 1962). To illustrate how this theory agrees typically with reality, results for a clamped–clamped circular cylindrical shell are compared in Fig. 3 with experimental data collected by Koval and Cranch (1962). The parameters are $E = 20.6 \times 10^4\,\mathrm{N/mm^2}$, $\rho = 7.85 \times 10^{-9}\,\mathrm{N \cdot s^2/mm^4}$, $\mu = 0.3$, $h = 0.254\,\mathrm{mm}$, $a = 76.2\,\mathrm{mm}$, and $L = 304.8\,\mathrm{mm}$. The reason that the experimental frequencies are in general lower than the theoretical frequencies is that (1) the Galerkin method gives upper bound results and (2) it is virtually impossible to have a truly clamped boundary. The elasticity of the clamping device will tend to lower the experimental frequencies.

7.3.2. Approximate Lower Natural Frequencies of Inextensional Ring Segments of Arbitrary Boundary Conditions

Of interest are cases where the inextensional simplification of the ring theory is applicable, which is whenever the ring segments are not restrained in tangential direction. For example, ring-shaped flapper valves in rotary vane compressors are mounted in cantilever fashion, and it is of interest to determine the fundamental natural frequency, since it was found experimentally to be significantly lower than in the equivalent cantilever beam case.

We start with the inextensional equation of the form of Eq. (6.15.10) which is

$$\frac{d^6 U_3}{d\theta^6} + 2\frac{d^4 U_3}{d\theta^4} + \frac{d^2 U_3}{d\theta^2}\left(1 - \frac{\omega^2}{p\omega_0^2}\right) + \frac{\omega^2}{p\omega_0^2} U_3 = 0 \tag{7.3.39}$$

where

$$p = \frac{1}{Aa^2} \tag{7.3.40}$$

$$\omega_0^2 = \frac{E}{\rho a^2} \tag{7.3.41}$$

The ring segment of radius a has length $a\gamma$, where γ is the enclosing angle. Assuming beam modes $U_{3m}(\lambda_m x)$, as for the case of the circular cylindrical shell treated in Sec. 7.3.1, they become $U_{3m}(\lambda_m a\theta)$. Applying Galerkin's method to Eq. (7.3.39) gives

$$\int_{\theta=0}^{\gamma} \left[\frac{d^6 U_{3m}}{d\theta^6} + 2\frac{d^4 U_{3m}}{d\theta^4} + \frac{d^2 U_{3m}}{d\theta^2} + \frac{\omega^2}{p\omega_0^2}\left(U_{3m} - \frac{d^2 U_{3m}}{d\theta^2}\right)\right] U_{3m}\, d\theta = 0 \tag{7.3.42}$$

Solving this equation for the natural frequencies and recognizing that beam modes satisfy

$$\frac{\mathrm{d}^4 U_{3m}(\lambda_m a\theta)}{\mathrm{d}\theta^4} = (\lambda_m a)^4 U_{3m}(\lambda_m a\theta) \tag{7.3.43}$$

we obtain

$$\frac{\omega^2}{p\omega_0^2} = \frac{\int_0^\gamma [[1+(\lambda_m a)^4](\mathrm{d}^2 U_{3m}/\mathrm{d}\theta^2) + 2(\lambda_m a)^4 U_{3m}] U_{3m}\,\mathrm{d}\theta}{\int_0^L [(\mathrm{d}^2 U_{3m}/\mathrm{d}\theta^2) - U_{3m}] U_{3m}\,\mathrm{d}\theta} \tag{7.3.44}$$

This can also be written as

$$\frac{\omega^2}{p\omega_0^2} = \frac{[1+(\lambda_m a)^4]R_m + 2(\lambda_m a)^4}{R_m - 1} \tag{7.3.45}$$

where

$$R_m = \frac{\int_0^\gamma (\mathrm{d}^2 U_{3m}/\mathrm{d}\theta^2) U_{3m}\,\mathrm{d}\theta}{\int_0^\gamma U_{3m}^2\,\mathrm{d}\theta} \tag{7.3.46}$$

Since

$$\frac{\mathrm{d}^2 U_{3m}(\lambda_m a\theta)}{\mathrm{d}\theta^2} = (\lambda_m a)^2 \frac{\mathrm{d}^2 U_{3m}(\lambda_m a\theta)}{\mathrm{d}(\lambda_m a\theta)^2} \tag{7.3.47}$$

we obtain

$$R_m \cong -(\lambda_m a)^2 \tag{7.3.48}$$

since the integral ratio

$$\frac{\int_0^\gamma [\mathrm{d}U_{3m}(\lambda_m a\theta)/\mathrm{d}(\lambda_m a\theta)^2] U_{3m}(\lambda_m a\theta)\,\mathrm{d}\theta}{\int_0^\gamma U_{3m}^2(\lambda_m a\theta)\,\mathrm{d}\theta} \cong -1 \tag{7.3.49}$$

for most beam modes, as discussed in Sec. 7.3.1. Labelling ω as ω_m, Eq. (7.3.45) thus becomes

$$\omega_m^2 = \frac{(\lambda_m a)^2[(\lambda_m a)^2 - 1]^2}{1+(\lambda_m a)^2} p\omega_0^2 \tag{7.3.50}$$

where the λ_m are obtained from the roots of the frequency equation of the equivalent beam, $\lambda_m L$, where $L = a\gamma$. Since

$$\lambda_m a\gamma = \lambda_m L = k_m \tag{7.3.51}$$

we obtain

$$\lambda_m a = \frac{k_m}{\gamma} \tag{7.3.52}$$

Therefore,

$$\omega_m^2 = \frac{(k_m/\gamma)^2[(k_m/\gamma)^2 - 1]^2}{1+(k_m/\gamma)^2} p\omega_0^2 \tag{7.3.53}$$

For example, for the simply supported ring segment, $k_m = m\pi, m = 1, 2, \ldots$ Thus

$$\omega_m^2 = \frac{(m\pi/\gamma)^2[(m\pi/\gamma)^2 - 1]^2}{1 + (m\pi/\gamma)^2} p\omega_0^2 \tag{7.3.54}$$

For the cantilever circular flapper valve, $k_1 = 1.875$, $k_2 = 4.694$, $k_3 = 7.855$, and so on. Thus the fundamental frequency for the case where $\gamma = \pi$ is

$$\omega_1^2 = 0.113 p\omega_0^2 = 0.113 \frac{EI}{a^4 \rho A} \tag{7.3.55}$$

For a straight beam of length $L = a\pi$,

$$\omega_1^2 = \frac{(1.875)^4}{\pi^4} \frac{EI}{a^4 \rho A} = 0.127 \frac{EI}{a^4 \rho A} \tag{7.3.56}$$

That the beam of the same equivalent length has a higher natural frequency is a expected, since the ring segment has more inertia to overcome because there is also tangential displacement. The tangential motion can be calculated from Eq. (6.15.2). At a natural frequency,

$$\frac{\mathrm{d}U_{\theta m}}{\mathrm{d}\theta} = -U_{3m} \tag{7.3.57}$$

or

$$U_{\theta m} = -\int U_{3m} \mathrm{d}\theta \tag{7.3.58}$$

7.4. RAYLEIGH–RITZ METHOD

Rayleigh used the argument that an undamped linear structure, vibrating at its natural frequency, interchanges its vibratory energy from a purely potential state at its maximum amplitude to a purely kinetic state when all vibration amplitudes are zero. At a natural frequency, we have

$$u_i = U_i \mathrm{e}^{j\omega t} \tag{7.4.1}$$

Substituting this in the strain energy expression of Eq. (2.6.3), we get the expression for maximum potential energy $U_{\max}$ upon taking for $\mathrm{e}^{j\omega t}$ the maximum, namely unity:

$$U_{\max} = \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} \left[\frac{E}{2(1-\mu^2)} (\varepsilon_{11}^{*2} + \varepsilon_{22}^{*2} + 2\mu\varepsilon_{11}^{*}\varepsilon_{22}^{*}) + \frac{G}{2}\varepsilon_{12}^{*2} \right] \times A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \tag{7.4.2}$$

where

$$\varepsilon_{11}^* = \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}(U_1+\alpha_3\beta_1^*)+\frac{U_2+\alpha_3\beta_2^*}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{U_3}{R_1} \tag{7.4.3}$$

$$\varepsilon_{22}^* = \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}(U_2+\alpha_3\beta_2^*)+\frac{U_1+\alpha_3\beta_1^*}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{U_3}{R_2} \tag{7.4.4}$$

$$\varepsilon_{12}^* = \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{U_2+\alpha_3\beta_2^*}{A_2}\right)+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{U_1+\alpha_3\beta_1^*}{A_1}\right) \tag{7.4.5}$$

and where

$$\beta_1^*=\frac{U_1}{R_1}-\frac{1}{A_1}\frac{\partial U_3}{\partial\alpha_1} \tag{7.4.6}$$

$$\beta_2^*=\frac{U_2}{R_2}-\frac{1}{A_2}\frac{\partial U_3}{\partial\alpha_2} \tag{7.4.7}$$

Substituting Eqs. (7.4.3)–(7.4.5) in the kinetic energy expression and selecting the maximum value gives

$$K_{\max}=\frac{\omega^2\rho h}{2}\int_{\alpha_1}\int_{\alpha_2}(U_1^2+U_2^2+U_3^2)A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{7.4.8}$$

Equating (7.4.8) and (7.4.2) gives

$$\omega^2=\frac{\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\left[\frac{E}{(1-\mu^2)}(\varepsilon_{11}^{*2}+\varepsilon_{22}^{*2}+2\mu\varepsilon_{11}^{*2}\varepsilon_{22}^{*2})+G\varepsilon_{12}^{*2}\right]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3}{\rho h\int_{\alpha_1}\int_{\alpha_2}(U_1^2+U_2^2+U_3^2)A_1A_2\ \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2} \tag{7.4.9}$$

This formula is exact if the exact mode-shape expression is substituted. However, the same formula results if we argue with Rayleigh that instead of using the unknown exact mode shape, an approximate mode-shape expression with not more than one arbitrary constant can be used that satisfies the boundary conditions and resembles to a reasonable degree the actual mode shape. In this case we try to minimize the difference between the maximum potential energy and the maximum kinetic energy since only in the exact case will it be zero as required. If the assumed mode shape contains only one constant C that can be minimized,

$$\frac{\mathrm{d}}{\mathrm{d}C}(U_{\max}-K_{\max})=0 \tag{7.4.10}$$

also results in Eq. (7.4.9). This equation is also called *Rayleigh's quotient*. If the assumed mode shapes satisfy all boundary conditions, it can be shown that Eq. (7.4.9) results in an upper bound approximation. If ω_R is Rayleigh's frequency and ω is the exact frequency, then

$$\omega\le\omega_R \tag{7.4.11}$$

The reason for this that any deviation from the true mode shape is equivalent to an additional constraint, resulting in a higher value for potential energy.

It was later established by various investigators that it is possible to relax the requirement that all boundary conditions have to be satisfied by the assumed mode shape. Most of the time it is sufficient to satisfy deflection and slope conditions and to neglect moment and shear boundary conditions, and still achieve acceptable approximations of the natural frequency.

Rayleigh's quotient can be used to investigate all natural frequencies of a plate or shell but works best for the determination of the fundamental frequency in case of plates. For higher frequencies even small deviations between the true and assumed mode shape can easily cause large errors in the calculated results. For shells, Rayleigh's quotient is not particularly useful because of the complexity of the modes. An extension is needed that improves accuracy.

This extension was provided by Ritz (1908). The contribution of Ritz was to allow estimated mode shapes of more than one arbitrary constant. In the Rayleigh–Ritz method we minimize with respect to each of the constants $C_1, C_2, \ldots, C_r$.

$$\begin{aligned}
&\frac{\partial}{\partial C_1}(U_{\max} - K_{\max}) = 0 \\
&\frac{\partial}{\partial C_2}(U_{\max} - K_{\max}) = 0 \\
&\qquad\vdots \\
&\frac{\partial}{\partial C_r}(U_{\max} - K_{\max}) = 0
\end{aligned} \tag{7.4.12}$$

If we have r constants C_r, we obtain r homogeneous equations: $r-1$ equations can be solved to express $r-1$ constants in terms of one arbitrarily selected constant. The requirement that the boundary conditions have to be satisfied is the same as before.

Let us illustrate all this on the example of a clamped circular plate (Volterra and Zachmanoglou, 1965). First, let us use the Rayleigh quotient to find the first natural frequency. We assume as a fundamental mode shape

$$U_3 = C_1\left[1 - \left(\frac{r}{a}\right)^2\right]^2 \tag{7.4.13}$$

This satisfies the two boundary conditions

$$U_3(a, \theta) = 0 \tag{7.4.14}$$

$$\frac{dU_3}{dr}(a,\theta)=0 \tag{7.4.15}$$

Setting $U_1=U_2=0, A_1=1, A_2=r, \alpha_1=r, \alpha_2=\theta$, and $d(\cdot)/d\theta=0$, the Rayleigh quotient expression becomes

$$\omega_R^2=\frac{D}{\rho h}\frac{\int_0^a f(r)r\,dr}{\int_0^a U_3^2 r\,dr} \tag{7.4.16}$$

where

$$f(r)=\left(\frac{\partial^2 U_3}{\partial r^2}+\frac{1}{r}\frac{\partial U_3}{\partial r}\right)^2-2(1-\mu)\frac{\partial^2 U_3}{\partial r^2}\frac{1}{r}\frac{\partial U_3}{\partial r} \tag{7.4.17}$$

The result is

$$\omega_R=\frac{(\lambda a)^2}{a^2}\sqrt{\frac{D}{\rho h}} \tag{7.4.18}$$

where $\lambda a=3.214$. This compares fairly well with the exact value of $\lambda a=3.196$.

Next, let us assume that

$$U_3=C_1\left[1-\left(\frac{r}{a}\right)^2\right]^2+C_2\left[1-\left(\frac{r}{a}\right)^2\right]^3 \tag{7.4.19}$$

which we expect will furnish us a better approximation of the first mode and also an approximation of the second axisymmetric mode. In this case we have to use the Rayleigh–Ritz method. We get, on substitution in Eqs. (7.4.2) and (7.4.8),

$$\begin{aligned}U_{max}-K_{max} &= \pi D\frac{32}{3a^2}\left(C_1^2+\frac{3}{2}C_1C_2+\frac{9}{10}C_2^2\right)\\ &\quad -\pi\rho h\omega^2\frac{a^2}{10}\left(C_1^2+\frac{5}{3}C_1C_2+\frac{5}{7}C_2^2\right)\end{aligned} \tag{7.4.20}$$

To minimize this expression with respect to C_1 and C_2, we formulate

$$\frac{\partial}{\partial C_1}(U_{max}-K_{max})=0 \tag{7.4.21}$$

$$\frac{\partial}{\partial C_2}(U_{max}-K_{max})=0 \tag{7.4.22}$$

or

$$\begin{bmatrix}\frac{64}{3}-a^4\omega^2\frac{\rho h}{5D} & \frac{64}{3}-a^4\omega^2\frac{\rho h}{6D}\\ \frac{64}{3}-a^4\omega^2\frac{\rho h}{6D} & \frac{96}{5}-a^4\omega^2\frac{\rho h}{7D}\end{bmatrix}\begin{Bmatrix}C_1\\ C_2\end{Bmatrix}=0 \tag{7.4.23}$$

Setting the determinant equal to zero and solving for ω gives two solutions:

$$\omega_{R1} = \frac{(\lambda a)_1^2}{a^2}\sqrt{\frac{D}{\rho h}} \tag{7.4.24}$$

$$\omega_{R2} = \frac{(\lambda a)_2^2}{a^2}\sqrt{\frac{D}{\rho h}} \tag{7.4.25}$$

where $(\lambda a)_1 = 3.197$ and $(\lambda a)_2 = 6.565$. This compares to the exact values of $(\lambda a)_1 = 3.196$, which is very close, and $(\lambda a)_2 = 6.306$, which is not an entirely unacceptable agreement as far as the second axisymmetric natural frequency is concerned.

The mode shapes are obtained with the help of Eq. (7.4.24), from which we get

$$C_2 = -C_1 \frac{64/3 - a^4\omega^2(\rho h/5D)}{64/3 - a^4\omega^2(\rho h/6D)} \tag{7.4.26}$$

or

$$C_2 = -C_1 \frac{64/3 - (\lambda a)^4/5}{64/3 - (\lambda a^4)/6} \tag{7.4.27}$$

Substituting $(\lambda a)_1 = 3.197$ gives $C_2 = -0.112C_1$. Thus

$$U_{31} = C_1 \left\{ \left[1 - \left(\frac{r}{a}\right)^2\right]^2 - 0.112\left[1 - \left(\frac{r}{a}\right)^2\right]^3 \right\} \tag{7.4.28}$$

Substituting $(\lambda a)_2 = 6.565$ gives $C_2 = -1.215C_1$, and therefore

$$U_{32} = C_1 \left\{ \left[1 - \left(\frac{r}{a}\right)^2\right]^2 - 1.215\left[1 - \left(\frac{r}{a}\right)^2\right]^3 \right\} \tag{7.4.29}$$

We could keep adding terms to the series describing U_3 and get better and better agreement, describing more and more axisymmetric natural frequencies. We could also add the θ dependency into our approximate mode shape and proceed with

$$U_3 = \sum_{i=1}^{p} C_i \left[1 - \left(\frac{r}{a}\right)^2\right]^{i+1} \cos n\theta \tag{7.4.30}$$

For a good example of the Rayleigh–Ritz application, see a paper by Ritz (1909), where he obtains the natural frequencies and modes of a square plate with free edges. Young (1950) applied the method to the square plate clamped on all edges.

There are many applications of the Rayleigh–Ritz method to shells. Let us select as example the paper by Federhofer (1938), who solved the completely clamped conical shell case by this method. As shown in Fig. 4,

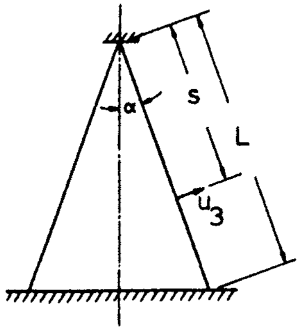

FIG. 4 Clamped conical shell.

the cone is clamped at its base and at its apex. The assumed solution is

$$U_1 = As^2(s-L)^2 \cos n(\theta-\phi)$$
$$U_2 = Bs^2(s-L)^2 \sin n(\theta-\phi)$$
$$U_3 = Cs^2(s-L)^2 \cos n(\theta-\phi)$$

It satisfies all boundary conditions but is limited to the equivalent of the first beam mode in the axial direction.

The Rayleigh–Ritz procedure results in a cubic equation whose three roots for each value of n define the three natural frequencies. As α approaches $\pi/2$, the results approach those of a clamped plate that is also clamped at the center. As α approaches zero and $n=1$, the result approaches that of a clamped–clamped beam of tubular cross-section.

An example of a rectangular plate with cut-outs can be found in Kristiansen and Soedel (1971).

7.5. SOUTHWELL'S PRINCIPLE

Southwell is credited with a formula (Southwell, 1922; Collatz, 1948) that can be applied to the problem of finding natural frequencies of shells and other structures whose stiffness is controlled by several superimposed effects with benefit. Consider the eigenvalue problem

$$L\{U\} - \omega^2 N\{U\} = 0 \tag{7.5.1}$$

If it is possible to separate the operator $L\{U\}$ into

$$L\{U\} = \sum_{r=1}^{n} L_r\{U\} \tag{7.5.2}$$

and if solutions exist for each partial problem

$$L_r\{U\} - \omega_r^2 N\{U\} = 0 \tag{7.5.3}$$

we obtain for the fundamental frequency the fact that

$$\omega_s^2 \leq \omega_1^2 \tag{7.5.4}$$

where ω_s is Southwell's frequency and is given by

$$\omega_s^2 = \sum_{r=1}^{n} \omega_{1r}^2 \tag{7.5.5}$$

The symbol ω_1 designates the true fundamental frequency of the total problem. The symbol ω_{1_r} designates the true fundamental frequency of the rth partial problem.

We prove this by applying Galerkin's approach to Eq. (7.5.1). If U_1 is the true fundamental mode of the total problem, the true fundamental frequency is

$$\omega_1^2 = \frac{\int_A \int U_1 L\{U_1\} \mathrm{d}A}{\int_A \int U_1 N\{U_1\} \mathrm{d}A} \tag{7.5.6}$$

Let us now apply Galerkin's approach to the partial problem. If U_1 is the true mode for the total problem, we obtain the inequality

$$\omega_{1r}^2 \leq \frac{\int_A \int U_1 L_r\{U_1\} \mathrm{d}A}{\int_A \int U_1 N\{U_1\} \mathrm{d}A} \tag{7.5.7}$$

If it happens that U_1 is also the true mode for the partial problem, the equality applies. However, in general, it cannot be expected that U_1 is the true mode for the partial problem.

Let us now sum both sides of the equation from $r = 1$ to n. This gives

$$\sum_{r=1}^{n} \omega_{1r}^2 \leq \frac{\int_A \int U_1 \sum_{r=1}^{n} L_r\{U_1\} \mathrm{d}A}{\int_A \int U_1 N\{U_1\} \mathrm{d}A} \tag{7.5.8}$$

The left side is ω_s^2 by definition of Eq. (7.5.5) and the right side is ω_1^2 because of Eq. (7.5.6). Thus

$$\omega_s^2 \leq \omega_1^2 \tag{7.5.9}$$

and the proof is completed.

Let us illustrate this for the closed circular cylindrical shell, as described by Eq. (6.9.4). By Southwell's principle, we may formulate two partial problems:

$$D\left(\frac{n^2}{a^2} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^4 U_{3n}(x) - \rho h \omega_1^2 \left(\frac{n^2}{a^2} - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^2 U_{3n}(x) = 0 \tag{7.5.10}$$

and

$$\frac{Eh}{a^2}\frac{\mathrm{d}^4U_{3n}(x)}{\mathrm{d}x^4}-\rho h\omega_2^2\left(\frac{n^2}{a^2}-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^2U_{3n}(x)=0 \tag{7.5.11}$$

If the shell is simply supported, all boundary conditions and both equations are satisfied by

$$U_{3n}(x)=\sin\frac{m\pi x}{L} \tag{7.5.12}$$

This gives, upon substitution in Eq. (7.5.10),

$$D\left[\left(\frac{n}{a}\right)^2+\left(\frac{m\pi}{L}\right)^2\right]^4-\rho h\omega_1^2\left[\left(\frac{n}{a}\right)^2+\left(\frac{m\pi}{L}\right)^2\right]^2=0 \tag{7.5.13}$$

or

$$\omega_1^2=\frac{D}{\rho h}\left[\left(\frac{n}{a}\right)^2+\left(\frac{m\pi}{L}\right)^2\right]^2 \tag{7.5.14}$$

This equation defines the natural frequencies due to the bending effect only. Substituting Eq. (7.5.12) in Eq. (7.5.11) gives

$$\frac{Eh}{a^2}\left(\frac{m\pi}{L}\right)^4-\rho h\omega_2^2\left[\left(\frac{n}{a}\right)^2+\left(\frac{\pi}{L}\right)^2\right]^2=0 \tag{7.5.15}$$

or

$$\omega_2^2=\frac{E}{\rho a^2}\frac{(m\pi/L)^4}{[(n/a)^2+(m\pi/L^2)]^2} \tag{7.5.16}$$

This equation defines the natural frequencies due to the membrane effect only. Thus, according to Southwell's principle, we obtain

$$\omega^2\geq\omega_1^2+\omega_2^2 \tag{7.5.17}$$

or

$$\omega^2\geq\frac{1}{a^2}\left\{\frac{(m\pi a/L)^4}{[n^2+(m\pi a/L)^2]^2}+\frac{1}{12(1-\mu^2)}\left(\frac{h}{a}\right)^2\left[n^2+\left(\frac{m\pi a}{L}\right)^2\right]^2\right\}\frac{E}{\rho} \tag{7.5.18}$$

In this case, Southwell's principle gives the exact result. The reason is that for the case chosen, the fundamental mode of the total problem is equal to the fundamental mode for both partial problems. However, in general, exact results cannot be expected. Note also that we have obtained a result that can be applied for all *m, n* combinations, yet only for the fundamental mode is the inequality of Eq. (7.5.9) valid. For frequencies, other than the lowest, nothing can be said in general about boundedness, and the quality of the prediction will have to be verified from case to case.

7.6. DUNKERLEY'S PRINCIPLE

Dunkerley (1894) discovered a method that would allow him to estimate the fundamental frequency of a multidegree-of-freedom system. In the following, a development of Dunkerley's method is shown following essentially Collatz (1948). Consider again the eigenvalue problem

$$L\{U\}-\omega^2N\{U\}=0 \tag{7.6.1}$$

If it is possible to separate the operator $N\{U\}$, which describes the mass effect, into

$$N\{U\}=\sum_{r=1}^{n}N_r\{U\} \tag{7.6.2}$$

and if we know the fundamental frequency of the partial problem, the partial problem having to be self-adjoint and fully defined,

$$L\{U\}-\omega_r^2N_r\{U\}=0 \tag{7.6.3}$$

we obtain

$$\omega_{\mathrm{D}}^2\le\omega_1^2 \tag{7.6.4}$$

where ω_{D} is Dunkerley's frequency, given by

$$\frac{1}{\omega_{\mathrm{D}}^2}=\sum_{r=1}^{n}\frac{1}{\omega_{1r}^2} \tag{7.6.5}$$

and where ω_1 is the actual fundamental frequency of the total problem and the ω_{1r} are the fundamental frequencies of the n partial problems.

To prove Eq. (7.6.4), we again use Galerkin's method. If U_1 is the true fundamental mode, the true fundamental frequency is

$$\omega_1^2=\frac{\int_A\int U_1L\{U_1\}\mathrm{d}A}{\int_A\int U_1N\{U_1\}\mathrm{d}A} \tag{7.6.6}$$

Let us now apply Galerkin's approach to the partial problem. If U_1 is the true mode for the total problem, we obtain

$$\omega_{1r}^2\le\frac{\int_A\int U_1L\{U_1\}\mathrm{d}A}{\int_A\int U_1N_r\{U_1\}\mathrm{d}A} \tag{7.6.7}$$

or

$$\frac{1}{\omega_{1r}^2}\ge\frac{\int_A\int U_1N_r\{U_1\}\mathrm{d}A}{\int_A\int U_1L\{U_1\}\mathrm{d}A} \tag{7.6.8}$$

Let us sum both sides from $r=1$ to n. This gives

$$\sum_{r=1}^{n}=\frac{1}{\omega_{1r}^2}\ge\frac{\int_A\int U_1\sum_{r=1}^{n}N_r\{U_1\}\mathrm{d}A}{\int_A\int U_1L\{U_1\}\mathrm{d}A} \tag{7.6.9}$$

But because of Eq. (7.6.2), the right side of the equation is equal to $1/\omega_1^2$. The left side is $1/\omega_D^2$ according to Eq. (7.6.5). Therefore,

$$\omega_D^2 \leq \omega_1^2 \tag{7.6.10}$$

This completes the proof. The right side is the square of the true first natural frequency. The square root of the left side is also called *Dunkerley's frequency* and given the symbol ω_D.

As example, let us consider a simply supported plate with an attached mass at location x^*, y^*, as shown in Fig. 5. We may express the distributed mass of the plate as $\rho h + M\delta(x-x^*)\delta(y-y^*)$. The equation for free transverse motion may therefore be written as

$$D\nabla^4 u_3 + [\rho h + M\delta(x-x^*)\delta(y-y^*)]\frac{\partial^2 u_3}{\partial t^2} = 0 \tag{7.6.11}$$

Arguing that at a natural frequency

$$u_3(x,y,t) = U_3(x,y)e^{j\omega t} \tag{7.6.12}$$

we obtain upon substitution

$$D\nabla^4 U_3 - [\rho h + M\delta(x-x^*)\delta(y-y^*)]\omega^2 U_3 = 0 \tag{7.6.13}$$

Let us now split this problem into two partial problems,

$$D\nabla^4 U_3 - \rho h\omega^2 U_3 = 0 \tag{7.6.14}$$

and

$$D\nabla^4 U_3 - M\omega^2 U_3\delta(x-x^*)\delta(y-y^*) = 0 \tag{7.6.15}$$

For the simply supported plate, the solution of Eq. (7.6.14) is $(r=1)$

$$U_3 = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{7.6.16}$$

$$\omega_{mn1}^2 = \pi^4\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 \frac{D}{\rho h} \tag{7.6.17}$$

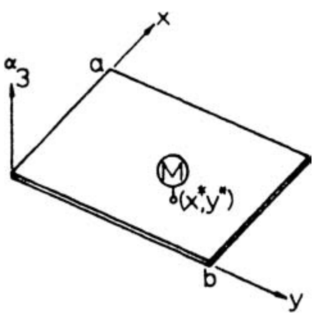

Fig. 5 Rectangular plate with attached point mass.

To solve the second equation, we use Galerkin's method as shown in Eq. (7.5.6), with Eq. (7.6.16) as the approximate mode expression. We obtain $(r=2)$

$$D\pi^4\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2\int_0^b\int_0^a \sin^2\frac{m\pi x}{a}\sin^2\frac{m\pi y}{b}\,dx\,dy$$
$$-M\omega_{mn2}^2\sin^2\frac{m\pi x^*}{a}\sin^2\frac{n\pi y^*}{b}=0 \qquad (7.6.18)$$

This gives

$$\omega_{mn2}^2=\frac{D\pi^4[(m/a)^2+(n/b)^2]^2ab}{4M\sin^2(m\pi x^*/a)\sin^2(n\pi y^*/b)} \qquad (7.6.19)$$

Thus according to Dunkerley's formula, Eq. (7.6.5),

$$\omega_D^2=\frac{1}{1/\omega_{mn1}^2+1/\omega_{mn2}^2}=\frac{\omega_{mn1}^2}{1+\omega_{mn1}^2/\omega_{mn2}^2} \qquad (7.6.20)$$

Therefore (Soedel, 1976),

$$\omega_D^2=\frac{\pi^4[(m/a)^2+(n/b)^2]^2(D/\rho h)}{1+4(M/M_p)\sin^2(m\pi x^*/a)\sin^2(n\pi y^*/b)} \qquad (7.6.21)$$

where M_p is the mass of the total plate,

$$M_p=\rho hab \qquad (7.6.22)$$

This result is identical to the result of Eq. (13.2.8) when curvature in Eq. (13.2.8) is set to zero. Note that we obtain frequency results for all m,n combinations, except that only for (m,n) = (1,1) is the inequality of Eq. (7.6.10) valid. Also, the accuracy of prediction suffers as m,n increase. A similar, yet somewhat different result was obtained in Soedel (1976), where the static Green's function of the simply supported plate was used to obtain ω_{mn2}^2.

7.7. STRAIN ENERGY EXPRESSIONS

Because of the importance of strain energy expressions to the variational and Rayleigh–Ritz methods, and therefore the finite element method, a few are derived in the following starting from the general relationship in Chapter 2.

Introducing assumptions (2.2.8) and (2.2.9) into Eqs. (2.6.3) and (2.6.4), the strain energy expression for any thin shell is (if we neglect σ_{33} and also set $\varepsilon_{13}=\varepsilon_{23}=0$):

$$U=\frac{1}{2}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}(\sigma_{11}\varepsilon_{11}+\sigma_{22}\varepsilon_{22}+\sigma_{12}\varepsilon_{12})A_1A_2\,d\alpha_3\,d\alpha_2\,d\alpha_1 \qquad (7.7.1)$$

Note that it was assumed that $\alpha_3/R_1 \ll 1$ and $\alpha_3/R_2 \ll 1$. Utilizing Eqs. (2.2.10)–(2.2.12), respectively, their inverses

$$\sigma_{11} = \frac{E}{1-\mu^2}(\varepsilon_{11}+\mu\varepsilon_{22}) \tag{7.7.2}$$

$$\sigma_{22} = \frac{E}{1-\mu^2}(\varepsilon_{22}+\mu\varepsilon_{11}) \tag{7.7.3}$$

$$\sigma_{12} = G\varepsilon_{12} \tag{7.7.4}$$

we obtain

$$U=\frac{1}{2}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\left[\frac{E}{1-\mu^2}\left(\varepsilon_{11}^2+\varepsilon_{22}^2+2\mu\varepsilon_{11}\varepsilon_{22}\right)+G\varepsilon_{12}^2\right]A_1A_2\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_1 \tag{7.7.5}$$

Next, we substitute

$$\varepsilon_{11}=\varepsilon_{11}^0+\alpha_3k_{11} \tag{7.7.6}$$

$$\varepsilon_{22}=\varepsilon_{22}^0+\alpha_3k_{22} \tag{7.7.7}$$

$$\varepsilon_{12}=\varepsilon_{12}^0+\alpha_3k_{12} \tag{7.7.8}$$

and $G=E/2(1+\mu)$, and integrate over the shell thickness from $\alpha_3=-h/2$ to $\alpha_3=h/2$.

This gives

$$U=\frac{1}{2}\int_{\alpha_1}\int_{\alpha_2}\left[K\left\{\varepsilon_{11}^{0^2}+2\mu\varepsilon_{11}^0\varepsilon_{22}^0+\varepsilon_{22}^{0^2}+\left(\frac{1-\mu}{2}\right)\varepsilon_{12}^{0^2}\right\}\right.$$
$$\left.+D\left\{k_{11}^2+2\mu k_{11}k_{22}+k_{22}^2+\left(\frac{1-\mu}{2}\right)k_{12}^2\right\}\right]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{7.7.9}$$

where

$$K=\frac{Eh}{1-\mu^2} \tag{7.7.10}$$

$$D=\frac{Eh^3}{12(1-\mu^2)} \tag{7.7.11}$$

Equation (7.7.9) can be further specialized. For example, for the circular cylindrical shell, we substitute Eqs. (3.3.7)–(3.3.14) into Eq. (7.7.9) and obtain

$$U = \frac{a}{2}\int_x\int_\theta\left[K\left\{\left(\frac{\partial u_x}{\partial x}\right)^2+2\mu\left(\frac{\partial u_x}{\partial x}\right)\frac{1}{a}\left(\frac{\partial u_\theta}{\partial\theta}+u_3\right)+\frac{1}{a^2}\left(\frac{\partial u_\theta}{\partial\theta}+u_3\right)^2\right.\right.$$
$$\left.+\frac{1-\mu}{2}\left(\frac{\partial u_\theta}{\partial x}+\frac{1}{a}\frac{\partial u_x}{\partial\theta}\right)^2\right\}$$

$$+D\left\{\left(\frac{\partial^2 u_3}{\partial x^2}\right)^2 - 2\mu\left(\frac{\partial^2 u_3}{\partial x^2}\right)\frac{1}{a^2}\left(\frac{\partial u_\theta}{\partial \theta} - \frac{\partial^2 u_3}{\partial \theta^2}\right) + \frac{1}{a^4}\left(\frac{\partial u_\theta}{\partial \theta} - \frac{\partial^2 u_3}{\partial \theta^2}\right)^2 + \left(\frac{1-\mu}{2}\right)\frac{1}{a^2}\left(\frac{\partial u_\theta}{\partial x} - 2\frac{\partial^2 u_3}{\partial x \partial \theta}\right)^2\right\}\Bigg]\mathrm{d}\theta\, \mathrm{d}x \tag{7.7.12}$$

For transversely vibrating plates, there is no membrane strain energy and Eq. (7.7.9) reduces to

$$U = \frac{D}{2}\int_{\alpha_1}\int_{\alpha_2}\left[k_{11}^2 + 2\mu k_{11}k_{22} + k_{22}^2 + \left(\frac{1-\mu}{2}\right)k_{12}^2\right]A_1 A_2\ \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{7.7.13}$$

Therefore, for Cartesian coordinates, as for example used for rectangular plates, we substitute $A_1 = 1$, $A_2 = 1$, $\alpha_1 = x$, $\alpha_2 = y$ and obtain from Eqs. (4.4.13) to (4.4.15)

$$k_{11} = k_{xx} = -\frac{\partial^2 u_3}{\partial x^2} \tag{7.7.14}$$

$$k_{22} = k_{yy} = -\frac{\partial^2 u_3}{\partial y^2} \tag{7.7.15}$$

$$k_{12} = k_{xy} = -2\frac{\partial^2 u_3}{\partial x\, \partial y} \tag{7.7.16}$$

Equation (7.7.13) becomes

$$U = \frac{D}{2}\int_x\int_y\left[\left(\frac{\partial^2 u_3}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u_3}{\partial y^2}\right)^2 + 2\mu\left(\frac{\partial^2 u_3}{\partial x^2}\right)\left(\frac{\partial^2 u_3}{\partial y^2}\right) + 2(1-\mu)\left(\frac{\partial^2 u_3}{\partial x\, \partial y}\right)^2\right]\mathrm{d}x\ \mathrm{d}y \tag{7.7.17}$$

For polar coordinates as used for example for circular plates, we substitute $A_1 = 1, A_2 = r, \alpha_1 = r, \alpha_2 = \theta$ and obtain from Eqs. (4.4.13)–(4.4.15),

$$k_{11} = k_{rr} = -\frac{\partial^2 u_3}{\partial r^2} \tag{7.7.18}$$

$$k_{22} = k_{\theta\theta} = -\frac{1}{r}\left(\frac{\partial u_3}{\partial r} + \frac{1}{r}\frac{\partial^2 u_3}{\partial \theta^2}\right) \tag{7.7.19}$$

$$k_{12} = k_{r\theta} = -r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial u_3}{\partial \theta}\right) - \frac{1}{r}\frac{\partial^2 u_3}{\partial r\, \partial \theta} \tag{7.7.20}$$

Equation (7.7.13) becomes

$$U = \frac{D}{2}\int_r\int_\theta\left[\left(\frac{\partial^2 u_3}{\partial r^2}+\frac{1}{r}\frac{\partial u_3}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_3}{\partial\theta^2}\right)^2 -2(1-\mu)\left(\frac{\partial^2 u_3}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial u_3}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_3}{\partial\theta^2}\right) +2(1-\mu)\left(\frac{1}{r}\frac{\partial^2 u_3}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial u_3}{\partial\theta}\right)^2\right] r\,\mathrm{d}r\,\mathrm{d}\theta \tag{7.7.21}$$

Next, we consider a circular ring of radius a. If the ring vibrates in its plane without twisting, we may set $\varepsilon_{12}^0=0, \varepsilon_{22}^0=0, k_{12}=0, k_{22}=0$. If the width of the ring is b and the thickness is h, Eq. (7.7.9) becomes

$$U=\frac{b}{2}\int_\theta\left(K\varepsilon_{\theta\theta}^{0^2}+Dk_{\theta\theta}^2\right)a\,\mathrm{d}\theta \tag{7.7.22}$$

Substituting $R_s=a$ and $s=a\theta$ into Eqs. (4.1.9)–(4.1.11) gives

$$\varepsilon_{\theta\theta}^0=\frac{1}{a}\left(\frac{\partial u_\theta}{\partial\theta}+u_3\right) \tag{7.7.23}$$

$$k_{\theta\theta}=\frac{1}{a^2}\left(\frac{\partial u_\theta}{\partial\theta}-\frac{\partial^2 u_3}{\partial\theta^2}\right) \tag{7.7.24}$$

We obtain, therefore,

$$U=\frac{b}{2}\int_\theta\left[\frac{K}{a}\left(\frac{\partial u_\theta}{\partial\theta}+u_3\right)^2+\frac{D}{a^3}\left(\frac{\partial u_\theta}{\partial\theta}-\frac{\partial^2 u_3}{\partial\theta^2}\right)^2\right]\mathrm{d}\theta \tag{7.7.25}$$

Recognizing that for rings of relatively narrow with b the Poisson effect disappears $(1-\mu^2)\to 1$, and generalizing Eq. (7.7.25) to any planar symmetric cross-section of area A and area moment I, Kb becomes EA and Db becomes EI. Equation (7.7.25) becomes

$$U=\frac{E}{2}\int_\theta\left[\frac{A}{a}\left(\frac{\partial u_\theta}{\partial\theta}+u_3\right)^2+\frac{I}{a^3}\left(\frac{\partial u_\theta}{\partial\theta}-\frac{\partial^2 u_3}{\partial\theta^2}\right)^2\right]\mathrm{d}\theta \tag{7.7.26}$$

For transversely vibrating beams, we may set $\varepsilon_{11}^0=0, \varepsilon_{22}^0=0, \varepsilon_{12}^0=0, k_{22}=0, k_{12}=0$, and

$$k_{11}=k_{xx}=-\frac{\partial^2 u_3}{\partial x^2} \tag{7.7.27}$$

If the beam is of rectangular cross-section of width b and height, h, Eq. (7.7.9) becomes

$$U=\frac{bD}{2}\int_x k_{11}^2\,\mathrm{d}x \tag{7.7.28}$$

with $D \to Eh^3/12$ because cutting a strip from a shell eliminates the Poisson effect in the denominator, $(1-\mu^2) \to 1$. Thus, we obtain

$$U = \frac{EI}{2}\int_x \left(\frac{\partial^2 u_3}{\partial x^2}\right)^2 \mathrm{d}x \tag{7.7.29}$$

where $I = bh^3/12$. For other cross-sections that are planar symmetric (we do not allow twisting of the beam), the respective I is used.

Finally, we may just as well obtain also strain energy for a longitudinally vibrating rod by setting $\varepsilon_{22}^0 = 0, \varepsilon_{12}^0 = 0, k_{11} = 0, k_{22} = 0, k_{12} = 0$, and

$$\varepsilon_{11}^0 = \varepsilon_{xx}^0 = \frac{\partial u_x}{\partial x} \tag{7.7.30}$$

Equation (7.7.9) becomes

$$U = \frac{bK}{2}\int_x \left(\frac{\partial u_x}{\partial x}\right)^2 \mathrm{d}x \tag{7.7.31}$$

with $bK \to EA$. Thus,

$$U = \frac{EA}{2}\int_x \left(\frac{\partial u_x}{\partial x}\right)^2 \mathrm{d}x \tag{7.7.32}$$

The strain energy expression for plates that vibrate in-plane is obtained by setting $k_{11} = k_{22} = k_{12} = 0$ and thus, for all plates of uniform thickness:

$$U = \frac{K}{2}\int_{\alpha_1}\int_{\alpha_2}\left[\varepsilon_{11}^{0^2} + 2\mu\varepsilon_{11}^0\varepsilon_{22}^0 + \varepsilon_{22}^{0^2} + \left(\frac{1-\mu}{2}\right)\varepsilon_{12}^{0^2}\right] A_1 A_2 \mathrm{d}\alpha_a \mathrm{d}\alpha_2 \tag{7.7.33}$$

Specializing this expression to Cartesian coordinates with $\alpha_1 = x, \alpha_2 = y, A_1 = A_2 = 1$, using

$$\varepsilon_{xx}^0 = \frac{\partial u_x}{\partial x} \tag{7.7.34}$$

$$\varepsilon_{yy}^0 = \frac{\partial u_y}{\partial y} \tag{7.7.35}$$

$$\varepsilon_{xy}^0 = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \tag{7.7.36}$$

gives the strain energy expression for the in-plane vibration of rectangular plates:

$$U = \frac{K}{2}\int_y\int_x \left[\left(\frac{\partial u_x}{\partial x}\right)^2 + 2\mu\left(\frac{\partial u_x}{\partial x}\right)\left(\frac{\partial u_y}{\partial y}\right) + \left(\frac{\partial u_y}{\partial y}\right)^2 + \frac{(1-\mu)}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)^2\right] \mathrm{d}x\,\mathrm{d}y \tag{7.7.37}$$

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8

Forced Vibrations of Shells by Modal Expansion

So far we have been concerned with the natural frequencies of shells and plates. For the engineer, the ultimate reason for this preoccupation is found in the study of the forced response of shells. For instance, knowing the eigenvalues makes it possible to obtain the forced solution in terms of these eigenvalues. This approach is called *spectral representation* or *modal expansion* and dates back to Bernoulli's work (Bernoulli, 1755). It will be the major point of discussion in this chapter.

Forces will be assumed to be independent of the motion of the shell. This is an admissible approximation for most engineering shell vibration cases. For instance, a fluid impinging on a shell can be thought to be not affected by the relatively small vibration response amplitudes. Thermodynamic forces on the cylinder liner of a combustion engine can be thought to be independent of the motion of this liner. Many other examples can be listed (Timoshenko, 1955; Meirovitch, 1967; Nowacki, 1963).

8.1. MODAL PARTICIPATION FACTOR

A disturbance will excite the various natural modes of a shell in various amounts. The amount of participation of each mode in the total dynamic response is defined by the modal participation factor. This factor may turn out to be 0 for certain modes and may approach large values for others, depending on the nature of the excitation.

In a mathematical sense, the natural modes of a shell structure represent orthogonal vectors that satisfy the boundary conditions of the structure. This vector space can be used to represent any response of the structure. In cases of finite-degree-of-freedom systems, the vector space is of finite dimension and the number of vectors or natural modes is equal to the number of degrees of freedom. For continuous systems, such as shells, the number of degrees of freedom is infinite. This means that the general solution will be an infinite series:

$$u_i(\alpha_1,\alpha_2,t)=\sum_{k=1}^{\infty}\eta_k(\int)U_{ik}(\alpha_1,\alpha_2) \tag{8.1.1}$$

where $i=1,2,3$. The U_{ik} are the natural mode components in the three principal directions. The modal participation factors η_k are unknown and have to be determined in the following.

The Love equations are of the form

$$L_i\{u_1,u_2,u_3\}-\lambda\dot{u}_i-\rho h\ddot{u}_i=-q_i \tag{8.1.2}$$

where λ is an equivalent viscous damping factor. The viscous damping term was introduced through the forcing term, replacing the original q_i by $q_i-\lambda\dot{u}_i$. Also note that the damping factor is assumed to be the same in all three principal directions. This is not necessarily true, but since damping values are notoriously difficult to determine theoretically, and thus have more qualitative than quantitative value, and since a uniform damping factor offers computational advantages, it was decided to adopt the uniform factor here. How this factor relates to the structural damping description in the literature that uses a complex modulus will be discussed later.

The operators L_i are defined, from Love's equation, as

$$L_1\{u_1,u_2,u_3\} = \frac{1}{A_1A_2}\left[\frac{\partial(N_{11}A_2)}{\partial\alpha_1}+\frac{\partial(N_{21}A_1)}{\partial\alpha_2}+N_{12}\frac{\partial A_1}{\partial\alpha_2}\right.$$
$$\left.-N_{22}\frac{\partial A_2}{\partial\alpha_1}+A_1A_2\frac{Q_{13}}{R_1}\right] \tag{8.1.3}$$

$$L_2\{u_1,u_2,u_3\} = \frac{1}{A_1A_2}\left[\frac{\partial(N_{12}A_2)}{\partial\alpha_1}+\frac{\partial(N_{22}A_1)}{\partial\alpha_2}+N_{21}\frac{\partial A_2}{\partial\alpha_1}\right.$$
$$\left.-N_{11}\frac{\partial A_1}{\partial\alpha_2}+A_1A_2\frac{Q_{23}}{R_2}\right] \tag{8.1.4}$$

$$L_3\{u_1,u_2,u_3\} = \frac{1}{A_1A_2}\left[\frac{\partial(Q_{13}A_2)}{\partial\alpha_1} + \frac{\partial(Q_{23}A_1)}{\partial\alpha_2} - A_1A_2\left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}\right)\right] \tag{8.1.5}$$

Of course, any of the simplifications discussed previously can be applied. The important point is that Eq. (8.1.2) is general and will apply for all geometries and simplifications. Substituting Eq. (8.1.1) in Eq. (8.1.2) gives

$$\sum_{k=1}^{\infty}[\eta_k L_i\{U_{1k},U_{2k},U_{3k}\} - \lambda\dot{\eta}_k U_{ik} - \rho h\ddot{\eta}_k U_{ik}] = -q_i \tag{8.1.6}$$

However, from our eigenvalue analysis, we know that

$$L_i\{U_{1k},U_{2k},U_{3k}\} = -\rho h\omega_k^2 U_{ik} \tag{8.1.7}$$

Substituting this in Eq. (8.1.6) gives

$$\sum_{k=1}^{\infty}(\rho h\ddot{\eta}_k + \lambda\dot{\eta}_k + \rho h\omega_k^2\eta_k)U_{ik} = q_i \tag{8.1.8}$$

Since we know that the natural modes U_{ik} are orthogonal, we may proceed as in a Fourier analysis, where we take advantage of the orthogonality of the sine and cosine functions. We multiply the equation on both sides by a mode U_{ip} where p, in general, is either equal to k or not equal:

$$\sum_{k=1}^{\infty}(\rho h\ddot{\eta}_k + \lambda\dot{\eta}_k + \rho h\omega_k^2\eta_k)U_{ik}U_{ip} = q_i U_{ip} \tag{8.1.9}$$

In expanded form this is

$$\sum_{k=1}^{\infty}(\rho h\ddot{\eta}_k + \lambda\dot{\eta}_k + \rho h\omega_k^2\eta_k)U_{1k}U_{1p} = q_1 U_{1p} \tag{8.1.10}$$

$$\sum_{k=1}^{\infty}(\rho h\ddot{\eta}_k + \lambda\dot{\eta}_k + \rho h\omega_k^2\eta_k)U_{2k}U_{2p} = q_2 U_{2p} \tag{8.1.11}$$

$$\sum_{k=1}^{\infty}(\rho h\ddot{\eta}_k + \lambda\dot{\eta}_k + \rho h\omega_k^2\eta_k)U_{3k}U_{3p} = q_3 U_{3p} \tag{8.1.12}$$

Adding Eqs. (8.1.10) to (8.1.12) and integrating over the shell surface gives

$$\sum_{k=1}^{\infty}(\rho h\ddot{\eta}_k + \lambda\dot{\eta}_k + \rho h\omega_k^2\eta_k)\int_{\alpha_2}\int_{\alpha_1}(U_{1k}U_{1p} + U_{2k}U_{2p} + U_{3k}U_{3p})A_1A_2\,d\alpha_1\,d\alpha_2 = \int_{\alpha_2}\int_{\alpha_1}(q_1U_{1p} + q_2U_{2p} + q_3U_{3p})A_1A_2\,d\alpha_1\,d\alpha_2 \tag{8.1.13}$$

Using the orthogonality conditions as defined by Eq. (5.8.22) we are able to remove the summation by realizing that all terms but the term for which $p=k$ vanish. We get

$$\ddot{\eta}_k + \frac{\lambda}{\rho h}\dot{\eta}_k + \omega_k^2\eta_k = F_k \tag{8.1.14}$$

where

$$F_k = \frac{1}{\rho h N_k}\int_{\alpha_2}\int_{\alpha_1}(q_1 U_{1k} + q_2 U_{2k} + q_3 U_{3k})A_1 A_2 \,d\alpha_1 d\alpha_2$$

$$N_k = \int_{\alpha_2}\int_{\alpha_1}(U_{1k}^2 + U_{2k}^2 + U_{3k}^2)A_1 A_2 \,d\alpha_1 d\alpha_2 \tag{8.1.15}$$

Thus, if we take k terms of the modal expansion series as approximation to an infinite number, we have to solve the equation defining the modal participation factors k times. There is no principal difficulty connected with this. The forcing functions q_1, q_2, and q_3 have to be given and the mode components U_{1k}, U_{2k}, and U_{3k} and the natural frequencies ω_k have to be known, either as direct functional or numerical theoretical solutions of the eigenvalue problem or as experimental data in functional or numerical form. The mass density per unit shell surface ρh is obviously also known and the damping factor λ has to be given or has to be estimated. For shells of nonuniform thickness, h has to be moved inside the integrals.

8.2. INITIAL CONDITIONS

For the complete solution of Eq. (8.1.14), two initial conditions for each modal participation factor are required. They are the initial displacements $u_i(\alpha_1, \alpha_2, 0)$ and the initial velocities $\dot{u}_i(\alpha_1, \alpha_2, 0)$. They must be specified for every point of the shell. Initial velocities are in many practical cases zero, except for problems where a periodic switch of boundary conditions occurs. Transient responses to initial conditions die down as time progresses because of damping, as will be shown. Therefore, when the steady-state solution alone is important, the initial conditions are set to 0.

When knowledge of the transient response is required and initial conditions are specified, we have to convert these initial conditions into initial conditions of the modal participation factor, which are η_k and $\dot{\eta}_k$ at $t=0$. Because of Eq. (8.1.1), we may write

$$u_i(\alpha_1, \alpha_2, 0) = \sum_{k=1}^{\infty}\eta(0)U_{ik}(\alpha_1, \alpha_2) \tag{8.2.1}$$

$$\dot{u}_i(\alpha_1, \alpha_2, 0) = \sum_{k=1}^{\infty}\dot{\eta}_k(0)U_{ik}(\alpha_1, \alpha_2) \tag{8.2.2}$$

These equations have to be solved for $\eta_k(0)$ and $\dot{\eta}_k(0)$. For instance, let us multiply Eq. (8.2.1) by $U_{ip}(\alpha_1, \alpha_2)$, where $p \neq k$ or $p = k$. We get

$$u_i(\alpha_1, \alpha_2, 0)U_{ip} = \sum_{k=1}^{\infty} \eta_k(0) U_{ik} U_{ip} \tag{8.2.3}$$

In expanded form, for $i = 1, 2$ and 3, this equation becomes

$$u_1(\alpha_1, \alpha_2, 0)U_{1p} = \sum_{k=1}^{\infty} \eta_k(0) U_{1k} U_{1p} \tag{8.2.4}$$

$$u_2(\alpha_1, \alpha_2, 0)U_{2p} = \sum_{k=1}^{\infty} \eta_k(0) U_{2k} U_{2p} \tag{8.2.5}$$

$$u_3(\alpha_1, \alpha_2, 0)U_{3p} = \sum_{k=1}^{\infty} \eta_k(0) U_{3k} U_{3p} \tag{8.2.6}$$

Summing these equations and integrating over the shell surface gives

$$\begin{aligned}\int_{\alpha_2}\int_{\alpha_1} &[u_1(\alpha_1, \alpha_2, 0)U_{1p} + u_2(\alpha_1, \alpha_2, 0)U_{2p} + u_3(\alpha_1, \alpha_2, 0)U_{3p}] A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \\ &= \sum_{k=1}^{\infty} \eta_k(0) \int_{\alpha_2}\int_{\alpha_1} (U_{1k}U_{1p} + U_{2k}U_{2p} + U_{3k}U_{3p}) A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \end{aligned} \tag{8.2.7}$$

Evoking the orthogonality condition of Eq. (5.8.22) eliminates the summation since the right side of the equation is 0 for any p except $p = k$. We obtain

$$\begin{aligned}\eta_k(0) = \frac{1}{N_K} \int_{\alpha_2}\int_{\alpha_1} &[u_1(\alpha_1, \alpha_2, 0)U_{1k} + u_2(\alpha_1 \alpha_2, 0)U_{2k} + u_3(\alpha_1, \alpha_2, 0)U_{3k}] \\ &\times A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \end{aligned} \tag{8.2.8}$$

where N_k is given by Eq. (8.1.15).

Following the same procedure, we solve Eq. (8.2.2) for the second initial condition:

$$\begin{aligned}\dot{\eta}_k(0) = \frac{1}{N_k} \int_{\alpha_2}\int_{\alpha_1} &[\dot{u}_1(\alpha_1, \alpha_2, 0)U_{1k} + \dot{u}(\alpha_1, \alpha_2, 0)U_{2k} + \dot{u}_3(\alpha_1, \alpha_2, 0)U_{3k}] \\ &\times A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \end{aligned} \tag{8.2.9}$$

8.3. SOLUTION OF THE MODAL PARTICIPATION FACTOR EQUATION

The modal participation factor equation is a simple oscillator equation. Thus we may interpret the forced vibration of shells by considering the shell as composed of simple oscillators, where each oscillator consists of the shell restricted to vibrate in one of its natural modes. All these oscillators respond simultaneously, and the total shell vibration is simply the result of addition (superposition) of all the individual vibrations.

The simple oscillator equation is solved in the following by the Laplace transformation technique. We derive the solution for subcritical, critical, and supercritical damping, even though only the first case is of real importance in shell vibration applications.

The modal participation factor equation can be written as

$$\ddot{\eta}_k + 2\zeta_k\omega_k\dot{\eta}_k + \omega_k^2\eta_k = F_k(t) \tag{8.3.1}$$

where

$$F_k(t) = \frac{\int_{\alpha_2}\int_{\alpha_1}(q_1U_{1k} + q_2U_{2k} + q_3U_{3k})A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{\rho h N_k} \tag{8.3.2}$$

$$\zeta_k = \frac{\lambda}{2\rho h\omega_k} \tag{8.3.3}$$

Note that ζ_k is called the *modal damping coefficient*. It is analogous to the damping coefficient in the simple oscillator problem.

Taking the Laplace transformation of Eq. (8.3.1) allows us to solve for the modal participation factor in the Laplace domain:

$$\eta_k(s) = \frac{F_k(s) + \eta_k(0)(s + 2\zeta_k\omega_k) + \dot{\eta}_k(0)}{(s + \zeta_k\omega_k)^2 + \omega_k^2(1 - \zeta_k^2)} \tag{8.3.4}$$

The inverse transformation depends on whether the term $1 - \zeta_k^2$ is positive, zero, or negative. The positive case, when $\zeta_k < 1$, is the most common since it is very difficult to dampen shells more than that. It is called the *subcritical case*. The critical case occurs when $\zeta_k = 1$ and has no practical significance other than that it defines the damping that causes an initial modal displacement to decay in the fastest possible time without an oscillation. Supercritical damping ($\zeta_k > 1$) occurs only if a shell has such a high damping that it creeps back from an initial modal displacement without overshooting the equilibrium position.

For the subcritical case ($\zeta_k < 1$), we define a real and positive number γ_k:

$$\gamma_k = \omega_k\sqrt{1 - \zeta_k^2} \tag{8.3.5}$$

The inverse Laplace transformation of Eq. (8.3.4) then gives

$$\eta_k(t) = \mathrm{e}^{-\zeta_k\omega_k t}\left\{\eta_k(0)\cos\gamma_k t + [\eta_k(0)\zeta_k\omega_k + \dot{\eta}_k(0)]\frac{\sin\gamma_k t}{\gamma_k}\right\} + \frac{1}{\gamma_k}\int_0^1 F_k(\tau)\mathrm{e}^{-\zeta_k\omega_k(t-\tau)}\sin\gamma_k(t-\tau)\,\mathrm{d}\tau \tag{8.3.6}$$

The solution is given in the form of the convolution integral since the forcing function $F_k(t)$ is at this point arbitrary. Once it is known, the convolution integral can be evaluated. It is also possible to take the inverse Laplace transformation of Eq. (8.3.4) with a known forcing function directly.

We note that vibrations caused by initial conditions will be oscillatory but will decay exponentially with time. The convolution integral, when evaluated for a specific forcing, will divide into a transient part and possibly a steady-state part if the forcing is periodic. The transient part will decay exponentially with time.

A special case of considerable technical interest is when damping is 0. The solution reduces to

$$\eta_k(t) = \eta_k(0)\cos\omega_k t + \dot{\eta}_k(0)\frac{\sin\omega_k t}{\omega_k} + \frac{1}{\omega_k}\int_0^t F_k(\tau)\sin\omega_k(t-\tau)\,\mathrm{d}\tau \tag{8.3.7}$$

Since most structures are very lightly damped, Eq. (8.3.7) in often used to get an approximate response since it is much simpler to use.

Next, let us investigate the supercritical case ($\zeta_k > 1$). In this case, the value of $1-\zeta_k^2$ is negative. Defining a real and positive number ε_k,

$$\varepsilon_k = \omega_k\sqrt{\zeta_k^2 - 1} \tag{8.3.8}$$

we obtain, taking the inverse Laplace transformation,

$$\eta_k(t) = \mathrm{e}^{-\zeta_k\omega_k t}\left\{\eta_k(0)\cosh\varepsilon_k t + [\eta_k(0)\zeta_k\omega_k + \dot{\eta}_k(0)]\frac{\sinh\varepsilon_k t}{\varepsilon_k}\right\} + \frac{1}{\varepsilon_k}\int_0^t F_k(\tau)\mathrm{e}^{-\zeta_k\omega_k(t-\tau)}\sinh\varepsilon_k(t-\tau)\,\mathrm{d}\tau \tag{8.3.9}$$

The vibrations caused by initial conditions are now nonoscillatory, however, if the forcing is periodic, an oscillatory steady-state solution will still result.

As a special case, we obtain the critical damping solution ($\zeta_k = 1$) by reduction:

$$\eta_k(t) = \mathrm{e}^{-\omega_k t}\{\eta_k(0) + [\eta_k(0)\omega_k + \dot{\eta}_k(0)]t\} + \int_0^t F_k(\tau)\mathrm{e}^{-\omega_k(t-\tau)}(t-\tau)\,\mathrm{d}\tau \tag{8.3.10}$$

8.4. REDUCED SYSTEMS

In the special case of a plate, the problem simplifies to a transverse solution with

$$F_k = \frac{1}{\rho h N_k} \int_{\alpha_2} \int_{\alpha_1} q_3 U_{3k} A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{8.4.1}$$

where

$$N_k = \int_{\alpha_2} \int_{\alpha_1} U_{3k}^2 A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{8.4.2}$$

and an in-plane solution with

$$F_k = \frac{1}{\rho h N_k} \int_{\alpha_2} \int_{\alpha_1} (q_1 U_{1k} + q_2 U_{2k}) A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{8.4.3}$$

where

$$N_k = \int_{\alpha_2} \int_{\alpha_1} (U_{1k}^2 + U_{2k}^2) A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{8.4.4}$$

In the special case of a ring, we get

$$F_k = \frac{1}{\rho h N_k} \int_{\alpha_2} \int_{\alpha_1} (q_1 U_{1k} + q_3 U_{3k}) A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{8.4.5}$$

where

$$N_k = \int_{\alpha_2} \int_{\alpha_1} (U_{1k}^2 + U_{3k}^2) A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{8.4.6}$$

For shell approximations, where only the transverse modes are considered, Eqs. (8.4.1) and (8.4.2) apply. Note that this is a good approximation since $|U_{3k}| \gg |U_{1k}|, |U_{2k}|$ for transverse motion-dominated modes. For the transversely vibrating beam

$$F_k = \frac{1}{\rho' N_k} \int_0^L q_3' U_{3k} \, \mathrm{d}x \tag{8.4.7}$$

where

$$N_k = \int_0^L U_{3k}^2 \, \mathrm{d}x \tag{8.4.8}$$

For the longitudinal vibration of a rod,

$$F_k = \frac{1}{\rho' N_k} \int_0^L q_1' U_{1k} \, \mathrm{d}x \tag{8.4.9}$$

where

$$N_k = \int_0^L U_{1k}^2 \, \mathrm{d}x \tag{8.4.10}$$

8.5. STEADY-STATE HARMONIC RESPONSE

A technically important case occurs when the load on the shell varies harmonically with time and when the onset of vibrations (the transient part) is of no interest. Using a complex notation to get the response to both sine and cosine loading, we may write the load as $(j=\sqrt{-1})$

$$q_i(\alpha_1,\alpha_2,t)=q_i^*(\alpha_1,\alpha_2)\mathrm{e}^{-j\omega t} \tag{8.5.1}$$

We could utilize the convolution integral, but in this case it is simpler to use Eq. (8.3.1) directly. It becomes

$$\ddot{\eta}_k+2\zeta_k\omega_k\dot{\eta}_k+\omega_k^2\eta_k=F_k^*\mathrm{e}^{j\omega t} \tag{8.5.2}$$

where

$$F_k^*=\frac{1}{\rho h N_k}\int_{\alpha_2}\int_{\alpha_1}(q_1^*U_{1k}+q_2^*U_{2k}+q_3^*U_{3k})A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{8.5.3}$$

At steady state, the response will be harmonic also but lagging behind by a phase angle ϕ_k:

$$\eta_k=\Lambda_k\mathrm{e}^{j(\omega t-\phi_k)} \tag{8.5.4}$$

Substituting this gives

$$\Lambda_k\mathrm{e}^{-j\phi_k}=\frac{F_k^*}{(\omega_k^2-\omega^2)+2j\zeta_k\omega_k\omega} \tag{8.5.5}$$

The magnitude of the response is, therefore,

$$\Lambda_k=\frac{F_k^*}{\omega_k^2\sqrt{[1-(\omega/\omega_k)^2]^2+4\zeta_k^2(\omega/\omega_k)^2}} \tag{8.5.6}$$

The phase lag is

$$\phi_k=\tan^{-1}\frac{2\zeta_k(\omega/\omega_k)}{1-(\omega/\omega_k)^2} \tag{8.5.7}$$

As expected, a shell will behave similarly to a collection of simple oscillators. Whenever the excitation frequency coincides with one of the natural frequencies, a peak in the response curve will occur.

It has to be noted that the harmonic response solution is the same for subcritical and supercritical damping, except that for equal forcing, the response amplitudes at resonance become less and less pronounced as damping is increased until they are indistinguishable from the off-resonance response.

8.6. STEP AND IMPULSE RESPONSE

Often there is a sudden onset at time $t=t_1$ of an otherwise purely static load. This can be expressed by

$$q_i(\alpha_1,\alpha_2,t)=q_i^*(\alpha_1,\alpha_2)U(t-t_1) \tag{8.6.1}$$

where

$$U(t-t_1)=\begin{cases}0, & t<t_1\\ 1, & t\geq t_1\end{cases} \tag{8.6.2}$$

and where F_k^* is given by Eq. (8.5.3).

Neglecting the initial conditions by setting them equal to 0 gives for the subcritical damping case,

$$\eta_k(t)=\frac{F_k^*}{\gamma_k}\int_0^t U(\tau-t_1)\mathrm{e}^{-\zeta_k\omega_k(t-\tau)}\sin\gamma_k(t-\tau)\,\mathrm{d}\tau \tag{8.6.3}$$

Integrating this gives for $t\geq t_1$,

$$\eta_k(t)=\frac{F_k^*}{\gamma_k^2}\left\{1-\zeta_k^2-\sqrt{1-\zeta_k^2}\,\mathrm{e}^{-\zeta_k\omega_k(t-t_1)}\cos[\gamma_k(t-t_1)-\phi_k]\right\} \tag{8.6.4}$$

where

$$\phi_k=\tan^{-1}\frac{\zeta_k}{\sqrt{1-\zeta_k^2}} \tag{8.6.5}$$

We see that the step response decays exponentially until a static value of F_k^*/ω_k^2 is reached (see also Sec. 8.16). It can also be seen that the maximum step response approaches, as damping decreases, twice the static value. From this, it follows that the sudden application of a static load will produce in the limit twice the stress magnitude that a slow, careful application of the load will produce. This was first established by Krylov in 1898 during his investigations of the bursting strength of canons.

An impulse response occurs when the shell is impacted by a mass. Piston slap on the cylinder liner of a diesel engine is a representative problem of this kind. Impulse loading is a conversion of momentum process. The forcing can be expressed as (see also Sec. 8.11)

$$q_i(\alpha_1,\alpha_2,t)=M_i^*(\alpha_1,\alpha_2)\delta(t-t_1) \tag{8.6.6}$$

where M_i^* is a distributed momentum change per unit area, with the units newton-second per square meter. $\delta(t-t_1)$ is the Dirac delta function which defines the occurrence of impact at time $t=t_1$. Its definitions are

$$\delta(t-t_1)=0 \quad \text{if} \quad t\neq t_1 \tag{8.6.7}$$

$$\int_{t=-\infty}^{t=\infty}\delta(t-t_1)\,\mathrm{d}t=1 \tag{8.6.8}$$

From these equations, we obtain the following integration rule (Sec. 8.12):

$$\int_{t=-\infty}^{t=\infty} F(t)\delta(t-t_1)\,\mathrm{d}t = F(t_1) \tag{8.6.9}$$

The unit of $\delta(t-t_1)$ is second^{-t}.

The value of F_k^* is

$$F_k^* = \frac{1}{\rho h N_k}\int_{\alpha_2}\int_{\alpha_1}(M_1^* U_{1k} + M_2^* U_{2k} + M_3^* U_{3k})A_1 A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{8.6.10}$$

Substituting this in Eq. (8.3.6) gives for subcritical damping and zero initial conditions,

$$\eta_k(t) = \frac{F_k^*}{\gamma_k}\int_0^t \delta(\tau - t_1)\mathrm{e}^{-\zeta_k\omega_k(t-\tau)}\sin\gamma_k(t-\tau)\,\mathrm{d}\tau \tag{8.6.11}$$

Applying the integration rule of Eq. (8.6.9) gives for $t \geq t_1$,

$$\eta_k(t) = \frac{F_k^*}{\lambda_k}\mathrm{e}^{-\zeta_k\omega_k(t-t_1)}\sin\lambda_k(t-t_1) \tag{8.6.12}$$

This result shows that no matter what the spatial distribution of the impulsive load, the response for each mode decays exponentially to 0 and the oscillation is sinusoidal at the associated natural frequency.

8.7. INFLUENCE OF LOAD DISTRIBUTION

Among the many possible load distributions, there are a few that occur frequently in engineering applications. It is worthwhile to single out these for a detailed discussion. The most common is a spatially uniform pressure load normal to the surface

$$q_1^* = 0, \quad q_2^* = 0, \quad q_3^* = p_3 \tag{8.7.1}$$

where p_3 is the uniform pressure amplitude. In this case F_k^* of Eq. (8.5.3) becomes

$$F_k^* = \frac{p_3}{\rho h N_k}\int_{\alpha_1}\int_{\alpha_2} U_{3k} A_1 A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{8.7.2}$$

The interesting feature is that, if the shell, plate, ring, beam, and so on, have modes that are symmetric or skew-symmetric about a line or lines of symmetry, none of the skew-symmetric modes are excited because they integrate out. For example, for a simply supported rectangular plate as shown in Fig. 1, we get

$$F_k^* = \frac{4p_3}{\rho h m n \pi^2}(1-\cos m\pi)(1-\cos n\pi) \tag{8.7.3}$$

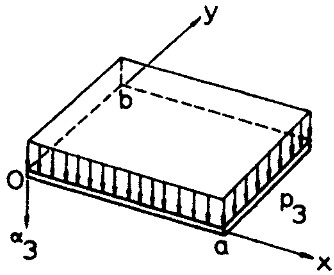

Fig. 1 Uniformly distributed pressure loading on a rectangular plate.

For instance, for the special case where the plate is loaded in time by a step function, the solution is

$$u_3(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\eta_{mn}U_{3mn} \tag{8.7.4}$$

where

$$U_{3mn}=\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{8.7.5}$$

and where $\eta_{mn}=\eta_k$ is given by Eq. (8.6.4).

In this case, only modes that are combinations of $m=1,3,5,\ldots$ and $n=1,3,5,\ldots$ participate in the response. Similarly, we can show that for a uniformly loaded axisymmetric circular plate, only the axisymmetric modes are excited (the $n=0$ modes).

Next, let us look at loads that are uniform along one coordinate only, let us say α_2 for the purpose of this discussion:

$$q_3^*=p_3(\alpha_1) \tag{8.7.6}$$

In this case, we get

$$F_k^*=\frac{1}{\rho h N_k}\int_{\alpha_1}p_3(\alpha_1)\int_{\alpha_2}U_{3k}A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{8.7.7}$$

For example, for the simply supported plate that has a pressure amplitude distribution as shown in Fig. 2, we have $\alpha_1=x, \alpha_2=y$, and

$$P_3=P\frac{x}{a} \tag{8.7.8}$$

This gives

$$F_k^*=-\frac{4P}{\rho h m n \pi^2}\cos m\pi(1-\cos n\pi) \tag{8.7.9}$$

which shows that modes that have $n=2,4,6,\ldots$ do not participate in the solution. But any m number will.

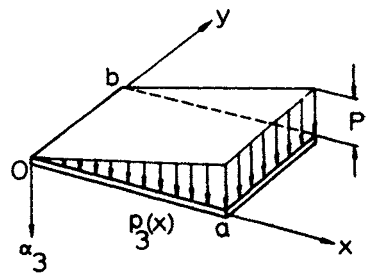

FIG. 2 Pressure loading that is uniform only in the y-direction.

Another example is the cylindrical duct shown in Fig. 3. Let the pressure distribution be axisymmetric and linearly varying with x:

$$q_3^* = P\frac{x}{L} \tag{8.7.10}$$

and the duct be a simply supported cylindrical shell of a transverse mode shape:

$$U_{3k} = \sin\frac{m\pi x}{L}\cos n(\theta - \phi) \tag{8.7.11}$$

We obtain, with $\alpha_1 = x, A_1 = 1, \alpha_2 = \theta, A_2 = a$, for $n = 0$,

$$F_k^* = -\frac{2P}{\rho h m\pi}\cos m\pi \tag{8.7.12}$$

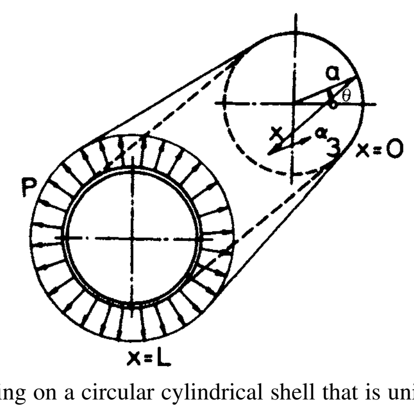

FIG. 3 Pressure loading on a circular cylindrical shell that is uniform in circumferential direction.

None of the modes $n=1,2,3,\ldots$ exist in the response. Let us take the example where the pressure distribution varies harmonically in time. The solution is then, in steady state,

$$u_3(x,\theta,t)=\sum_{m=1}^{\infty}\eta_{m0}\sin\frac{m\pi x}{L} \tag{8.7.13}$$

where $\eta_{m0}=\eta_k$ is given by Eq. (8.5.4).

It is not necessary to recognize the elimination of certain modes in advance. Including all the modes in a computer program will certainly give the correct response because the program will automatically integrate out the nonparticipating modes. The value of considering symmetry conditions in advance is more an economical one. If modes have to be generated experimentally, for instance, it will save experimental time if it is recognized that only the symmetric modes have to be excited and measured. There is a corollary to this since we can show equally well that skew-symmetric load distributions will not excite symmetric modes. Again, this is useful for the experimenter to know.

All of this assumes, of course, that the plate or shell is perfect in its symmetry and that the load does not deviate from symmetry. Since this is never exactly true for engineering applications, we find that in actual engineering, systems, modes other than the theoretically predicted ones will also be present, but with much reduced magnitudes. Such a tendency for modes to be present when they should not be is especially pronounced in cylindrical ducts, where the excitation pressure may be axisymmetric for all practical purposes yet other than $n=0$ modes are excited. The reason here is that, much less energy is required to excite to an equal amplitude, modes with higher n numbers than the $n=0$ modes because the lowest natural frequencies occur at the higher n numbers, as we have seen in Chapter 5. Thus a slight imperfection in either pressure distribution or shell construction will be enough to bring the $n=1,2,\ldots$ modes into the measured response. All that the analyst can do is to allow for a small deviation of axisymmetry in the model to allow for the imperfections of manufacturing.

8.8. POINT LOADS

A type of load that is very common in engineering applications is the point load. In the immediate vicinity of the point load, some of the basic assumptions of thin shell theory are violated (e.g., that $\sigma_{33}=0$). However, outside the immediate vicinity of the point load, the assumptions are not affected and overall vibration responses can be calculated with excellent accuracy.

Since we do not expect to acquire exact results right under the point load anyway, keeping in mind that these results would require three-dimensional elasticity and possibly plasticity analysis and a detailed knowledge of the load application mechanism, it seems reasonable to use the Dirac delta function (Dirac, 1926; Schwartz, 1950, 1951) to define the point load. This function locates the load at the desired point and assures that it is of the desired magnitude, but does not define the actual mechanism and microdistribution of application.

The point load $P_i S(t)$ in newtons can be expressed as a distributed load q_i in newtons per square meter by

$$q_i = P_i \frac{1}{A_1 A_2} \delta(\alpha_1 - \alpha_1^*) \delta(\alpha_2 - \alpha_2^*) S(t) \tag{8.8.1}$$

where α_1^* and α_2^* define the location of the point force. This is discussed in Sec. 8.11. The function $S(t)$ represents the time dependency. For harmonic forcing, it is $e^{j\omega t}$, and for a step loading, $U(t-t_1)$.

This allows us to write Eq. (8.3.1) as

$$\ddot{\eta}_k + 2\zeta_k \omega_k \dot{\eta}_k + \omega_k^2 \eta_k = F_k^* S(t) \tag{8.8.2}$$

where

$$\begin{aligned} F_k^* = {} & \frac{1}{\rho h N_k} \int_{\alpha_2} \int_{\alpha_1} [P_1 U_{1k}(\alpha_1, \alpha_2) + P_2 U_{2k}(\alpha_1, \alpha_2) + P_3 U_{3k}(\alpha_1, \alpha_2)] \\ & \times \delta(\alpha_1 - \alpha_1^*) \delta(\alpha_2 - \alpha_2^*) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \end{aligned} \tag{8.8.3}$$

Applying the integration rule gives

$$F_k^* = \frac{1}{\rho h N_k} [P_1 U_{1k}(\alpha_1^*, \alpha_2^*) + P_2 U_{2k}(\alpha_1^*, \alpha_2^*) + P_3 U_{3k}(\alpha_1^*, \alpha_2^*)] \tag{8.8.4}$$

This can now, for instance, be substituted in the specific modal participation factor solutions such as Eq. (8.5.4) or (8.6.4). It points out that if the point force is located on a node line of a mode component, this particular mode component will not participate in the response because $U_{ik}(\alpha_1^*, \alpha_2^*)$ is 0 if (α_1^*, α_2^*) is on a node line.

Let us take as an example a harmonically varying point force acting on a circular cylindrical shell panel as shown in Fig. 4. In this case

$$P_1 = 0, \quad P_2 = 0, \quad P_3 = F \tag{8.8.5}$$

and

$$U_{3mn} = \sin \frac{m\pi x}{L} \sin \frac{n\pi\theta}{\alpha} \tag{8.8.6}$$

Thus

$$F_k^* = \frac{4F}{\rho h L a \alpha} \sin \frac{m\pi x^*}{L} \sin \frac{n\pi\theta^*}{\alpha} \tag{8.8.7}$$

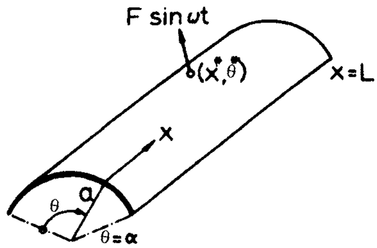

Fig. 4 Point force on a simply supported cylindrical panel.

The value of $\eta_k = \eta_{mn}$ is given by Eq. (8.5.4) and the transverse deflection solution is

$$u_3 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \eta_{mn} \sin\frac{m\pi x}{L}\sin\frac{n\pi\theta}{\alpha} \tag{8.8.8}$$

An interesting special case occurs when a point load acts on a shell of revolution that is closed in the θ direction. The transverse mode components, for example, are of the general form

$$U_{3mn}(\phi,\theta) = H_{3m}(\phi)\cos n(\theta - \eta) \tag{8.8.9}$$

where η is an arbitrary angle. We have chosen η instead of the customary ϕ of Chapter 5 because ϕ is already used as a coordinate. To express any shape in the θ direction, we need two orthogonal components. These we get if one time we let $\eta = 0$ and the other, $\eta = \pi/2n$. This gives a set of modes

$$U_{3mn1}(\phi,\theta) = H_{3m}(\phi)\sin n\theta \tag{8.8.10}$$

and a second set at

$$U_{3mn2}(\phi,\theta) = H_{3m}(\phi)\cos n\theta \tag{8.8.11}$$

A discussion is given in Sec. 8.13.

For a point load acting at (ϕ^*, θ^*), as shown in Fig. 5, we obtain from Eq. (8.8.4) for the first set

$$F_{mn1} = F^*_{mn1} S(t) \tag{8.8.12}$$

where

$$F^*_{mn1} = C_m(\phi^*)\sin n\theta^* \tag{8.8.13}$$

and where for transverse loading only,

$$C_m(\phi^*) = \frac{1}{\rho h N_{mn}} P_3 H_{3m}(\phi^*) \tag{8.8.14}$$

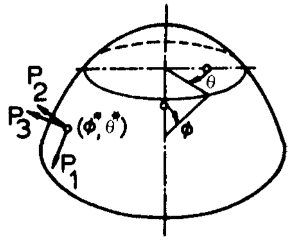

FIG. 5 Point loads on a shell of revolution.

and where for dominating transverse mode components,

$$N_{mn} = \pi \int_{\phi} H_{3m}^2(\phi) A_\phi A_\theta \, d\phi \tag{8.8.15}$$

For the second set, we get

$$F_{mn2} = F_{mn2}^* S(t) \tag{8.8.16}$$

where

$$F_{mn2}^* = C_m(\phi^*) \cos n\theta^* \tag{8.8.17}$$

The modal participation factor solutions are therefore

$$\eta_{mn1} = T_{mn}(\phi^*, t) \sin n\theta^* \tag{8.8.18}$$

$$\eta_{mn2} = T_{mn}(\phi^*, t) \cos n\theta^* \tag{8.8.19}$$

where

$$T_{mn}(\phi^*, t) = \frac{C_m(\phi^*)}{\gamma_{mn}} \int_0^1 S(\tau) e^{-\zeta_{mn}\omega_{mn}(t-\tau)} \sin \gamma_{mn}(t-\tau) \, d\tau \tag{8.8.20}$$

The total solution is, therefore, by superposition,

$$u_3(\phi, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} T_{mn}(\phi^*, t) H_{3m}(\phi) [\sin n\theta \sin n\theta^* + \cos n\theta \cos n\theta^*] \tag{8.8.21}$$

However, since the bracketed quantity is equal to $\cos n(\theta - \theta^*)$, we get

$$u_3(\phi, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} T_{mn}(\phi^*, t) H_{3m}(\phi) \cos n(\theta - \theta^*) \tag{8.8.22}$$

This is an interesting result since it proves that each mode will orient itself such that its maximum deflection occurs at $\theta = \theta^*$. This example also illustrates that for closed shells of revolution the $\cos n(\theta - \eta)$ term has to be thought of as representing the two orthogonal terms $\sin n\theta$ and $\cos n\theta$.

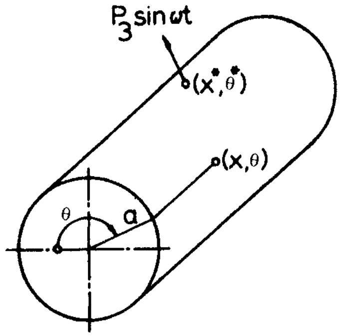

FIG. 6 Point load on a circular cylindrical shell.

When all mode components are considered, the approach is similar, as illustrated in the examples of Secs. 8.13–8.15.

Let us take as specific example, the simply supported cylindrical shell, with loading as shown in Fig. 6, again in the transverse mode approximation. We let $A_\phi \mathrm{d}\phi = \mathrm{d}x, A_\theta = a$. Since

$$H_{3m}(x) = \sin\frac{m\pi x}{L} \tag{8.8.23}$$

we obtain

$$N_{mn} = \frac{\pi a L \varepsilon_n}{2} \tag{8.8.24}$$

where

$$\varepsilon_n = \begin{cases} 1, & n \neq 0 \\ 2, & n = 0 \end{cases}$$

and thus

$$C_m(x^*) = \frac{2P_3}{\rho h a L \pi \varepsilon_n} \sin\frac{m\pi x^*}{L} \tag{8.8.25}$$

Since in this example

$$S(t) = \sin\omega t \tag{8.8.26}$$

we get for steady state,

$$T_{mn}(t, x^*) = \frac{C_m(x^*)\sin(\omega t - \phi_{mn})}{\omega_{mn}^2\sqrt{[1-(\omega/\omega_{mn})^2]^2 + 4\zeta_{mn}^2(\omega/\omega_{mn})^2}} \tag{8.8.27}$$

where

$$\phi_{mn} = \tan^{-1}\frac{2\zeta_{mn}(\omega/\omega_{mn})}{1-(\omega/\omega_{mn})^2} \tag{8.8.28}$$

The total solution is, therefore,

$$u_3(x,\theta,t) = \frac{2P_3}{\rho haL\pi}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\sin(m\pi x^*/L)\sin(m\pi x/L)\cos n(\theta-\theta^*)}{\varepsilon_n\omega_{mn}^2\sqrt{[1-(\omega/\omega_{mn})^2]^2+4\zeta_{mn}^2(\omega/\omega_{mn})^2}}\cdot \times\sin(\omega t-\phi_{mn}) \quad (8.8.29)$$

8.9. LINE LOADS

Another type of loading that is relatively important in engineering is the line load. In the general discussion, we confine ourselves to line loads that occur along coordinate lines. This restriction allows us to utilize the Dirac delta function. If there is a line load along the α_1 coordinate at $\alpha_2=\alpha_2^*$ of amplitude $Q_1^*(\alpha_1)$ in newtons per meter, as shown in Fig. 7, we may express it as

$$q_1=q_1^*S(t) \quad (8.9.1)$$

where

$$q_1^*=Q_1^*\frac{1}{A_2}\delta(\alpha_2-\alpha_2^*) \quad (8.9.2)$$

Therefore, Eq. (8.5.3) becomes

$$F_k^* = \frac{1}{\rho hN_k}\int_{\alpha_1}[Q_1^*(\alpha_1)U_{1k}(\alpha_1,\alpha_2^*)+Q_2^*(\alpha_1)U_{2k}(\alpha_1,\alpha_2^*) +Q_3^*(\alpha_1)U_{3k}(\alpha_1,\alpha_2^*)]A_1\,d\alpha_1 \quad (8.9.3)$$

Similarly, a line load along an α_2 coordinate can be treated.

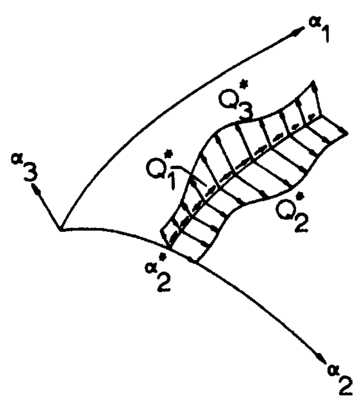

FIG. 7 Three directional line loading on the reference surface of a shell.

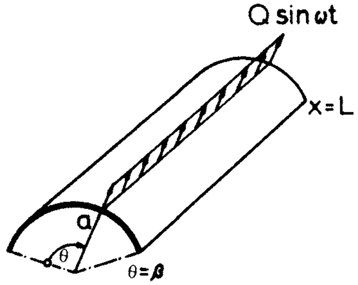

FIG. 8 Line load on a circular cylindrical panel that is uniformly distributed in the axial direction.

As an example, let us investigate a line load on a circular cylindrical panel as shown in Fig. 8. In this case

$$Q_1^*(\alpha_1)=0;\ \ Q_2^*(\alpha_1)=0;\ \ Q_3^*(\alpha_1)=Q \tag{8.9.4}$$

and

$$U_{3k}=\sin\frac{m\pi x}{L}\sin\frac{n\pi\theta}{\beta} \tag{8.9.5}$$

This gives

$$N_k=\frac{\beta aL}{4} \tag{8.9.6}$$

where we have again neglected the U_{1k} and U_{2k} contributions to N_k since they are small. This therefore, gives

$$F_k^*=\frac{4Q}{\rho h\beta m\pi a}(1-\cos m\pi)\sin\frac{n\pi\theta^*}{\beta} \tag{8.9.7}$$

and the rest of the solution follows.

To illustrate why we have restricted the line load discussion to loads that are distributed along coordinate lines, let us look at the example of a simply supported rectangular plate with a transverse diagonal line load of constant magnitude as shown in Fig. 9. The load expression is in this case, Stanisic (1977),

$$q_3=q_3^*S(t) \tag{8.9.8}$$

where

$$q_3^*=\frac{Q_3^*}{\cos\beta}\delta\left(y-\frac{b}{a}x\right) \tag{8.9.9}$$

and where

$$\cos\beta=\frac{a}{\sqrt{a^2+b^2}} \tag{8.9.10}$$

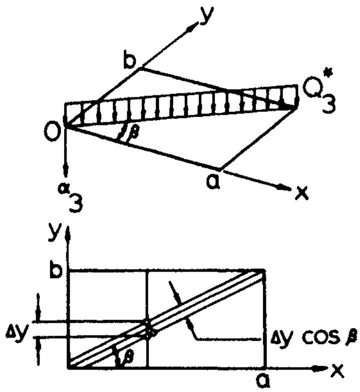

FIG. 9 Uniformly distributed line load that acts diagonally on a rectangular plate.

The reason for the $\cos\beta$ term is that the line load Q_3^* is not distributed over a strip of width $\triangle y$, with $\triangle y \to 0$, but over a narrower strip $\triangle y \cos\beta$, with $\triangle y \to 0$. This means that

$$q_3^* = \lim_{\Delta y \to 0} \frac{Q_3^*}{\Delta y \cos\beta}\left[U\left(y - \frac{b}{a}x\right) - U\left(y - \frac{b}{a}x - \Delta y\right)\right] \tag{8.9.11}$$

From this, Eq. (8.9.9) results. A more general description for loads that act along curved lines can be found in Soedel and Powder (1979).

To finish this particular individual case, let us substitute the load description into the expression for F_k^*. This gives

$$F_k^* = \frac{Q_3^*}{\rho h N_k} \frac{\sqrt{a^2 + b^2}}{a} \int_0^a \int_0^b \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \delta\left(y - \frac{b}{a}x\right) \mathrm{d}x\,\mathrm{d}y \tag{8.9.12}$$

or

$$F_k^* = \frac{Q_3^*}{2\rho h N_k} \sqrt{a^2 + b^2}\,\delta_{mn} \tag{8.9.13}$$

where

$$\delta_{mn} = \begin{cases} 0 & if\, m \neq n \\ 1 & if\, m = n \end{cases} \tag{8.9.14}$$

This indicates that in this special case only the modes where $m = n$ are excited.

8.10. POINT IMPACT

Another problem of general engineering interest is the point impact. In contrast to the distributed impulse load treated in general in Sec. 8.6, point impact can be described by an impulse concentrated at a point, as occurs, for example, when a shell is struck by a projectile or hammer. Piston slap in combustion engines is of this type.

The description of Eq. (8.6.6) suffices, which is

$$q_i(\alpha_1, \alpha_2, t) = M_i^*(\alpha_1, \alpha_2)\delta(t - t_1) \tag{8.10.1}$$

except that we write in this case

$$M_i^*(\alpha_1, \alpha_2) = \frac{M_i}{A_1 A_2}\delta(\alpha_1 - \alpha_1^*)\delta(\alpha_2 - \alpha_2^*) \tag{8.10.2}$$

where M_i is the momentum that is transferred to the shell by the impact. In this case Eq. (8.6.11) becomes

$$F_k^* = \frac{1}{\rho h N_k}[M_1 U_{1k}(\alpha_1^*, \alpha_2^*) + M_2 U_{2k}(\alpha_1^*, \alpha_2^*) + M_3 U_{3k}(\alpha_1^*, \alpha_2^*)] \tag{8.10.3}$$

As an example, let us solve the problem where a mass m of velocity v impacts a spherical shell as shown in Fig 10. In this case $\alpha_1 = \phi$, $A_1 = a, \alpha_2 = \theta, A_2 = a \sin\phi, \alpha_1^* = \phi_1^* = 0$. The mode shapes that have to be considered are given in Sec. 6.2.

$$U_{\phi n} = \frac{\mathrm{d}}{\mathrm{d}\phi} P_n(\cos\phi) \tag{8.10.4}$$

$$U_{3n} = \frac{1 + (1 + \mu)\Omega_n^2}{1 - \Omega_n^2} P_n(\cos\phi) \tag{8.10.5}$$

We assume that the momentum of the projectile

$$M_3 = -mv; \quad M_1 = M_2 = 0 \tag{8.10.6}$$

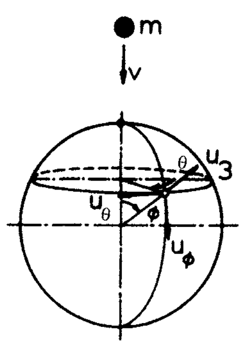

Fig. 10 Point mass impacting a free floating spherical shell.

is completely transferred to the shell in such a brief time span that the process can be described by the Dirac delta function. Equation (8.10.3) applies, therefore, and we get $(k=n)$

$$F_n^* = \frac{-mv}{\rho h N_n} \frac{1+(1+\mu)\Omega_n^2}{1-\Omega_n^2} P_n(1) \tag{8.10.7}$$

where

$$N_n = 2a^2\pi \int_0^\pi \left\{ \left[\frac{d}{d\phi} P_n(\cos\phi) \right]^2 + \left[\frac{1+(1+\mu)\Omega_n^2}{1-\Omega_n^2} P_n(\cos\phi) \right]^2 \right\} \sin\phi \, d\phi \tag{8.10.8}$$

From Eq. (8.6.12), we obtain η_n and the total solution is, since $P_n(1)=1$,

$$\begin{Bmatrix} u_\phi(\phi,\theta,t) \\ u_3(\phi,\theta,t) \end{Bmatrix} = -\sum_{n=0}^{\infty} \frac{1+(1+\mu)\Omega_n^2}{1-\Omega_n^2} \frac{mv}{\rho h \gamma_n N_n} e^{-\zeta_x \omega_k t} \times \sin\gamma_n t \begin{Bmatrix} \frac{d}{d\phi} P_n(\cos\phi) \\ \frac{1+(1+\mu)\Omega_n^2}{1-\Omega_n^2} P_n(\cos\phi) \end{Bmatrix} \tag{8.10.9}$$

Note that the $n=1$ term describes a rigid-body translation. Since $\omega_1 = \gamma_1 = \Omega_1 = 0$ for the transversely dominated mode and since

$$\lim_{\gamma_1 \to 0} \frac{\sin\gamma_1 t}{\gamma_1} = t \tag{8.10.10}$$

we obtain, at $\phi=0$ and for $n=1$,

$$u_3(0,\theta,t) = -\frac{mv}{\rho h N_1} t \tag{8.10.11}$$

Furthermore, for $n=1$,

$$N_1 = 2\pi a^2 \int_0^\pi (\sin^2\phi + \cos^2\phi) \sin\phi \, d\phi = 4\pi a^2 \tag{8.10.12}$$

This gives, taking the time derivative,

$$\dot{u}_3(0,\theta,t) = -\frac{mv}{4\pi\rho h a^2} \tag{8.10.13}$$

This is the velocity with which the spherical shell as a whole moves away from the impact. The minus sign gives the direction of the motion at $\phi=0$, namely downward in Fig. 10.

Note that if the spherical shell would be treated as an elastic body undergoing impact by the classical impact theory, we obtain the same result as given by Eq. (8.10.13), but are unable, of course, to say anything about the resulting vibration.

8.11. IMPULSIVE FORCES AND POINT FORCES DESCRIBED BY DIRAC DELTA FUNCTIONS

Let us assume that a point mass m impacts a shell with velocity $\bar{v}_a$. After impact, it has a velocity $\bar{v}_b$. Writing the linear momentum–impulse law for this mass, we obtain

$$\Delta(m\bar{v}) = m\bar{v}_a - m\bar{v}_b = \int_{t_a}^{t_b} \bar{F}\,\mathrm{d}t \tag{8.11.1}$$

This may be written as

$$\Delta(mv_1) = \int_{t_a}^{t_b} F_1\,\mathrm{d}t, \quad \Delta(mv_2) = \int_{t_a}^{t_b} F_2\,\mathrm{d}t, \quad \Delta(mv_3) = \int_{t_a}^{t_b} F_3\,\mathrm{d}t \tag{8.11.2}$$

For example, Fig. 11 (a) shows a typical impulse in the α_3 direction. If the impulse duration is much less than the period of the highest frequency of interest, impulses of the same magnitude $\Delta(mv_3)$ excite approximately the same dynamic response, even if the shapes differ. Thus we may replace the actual impulse by a rectangular shape, as shown in Fig. 11 (b). The width is Δt and the height is $\Delta(mv_3)/\Delta t$, so the area is $\Delta(mv_3)$. We may describe this impulse by

$$F_3 = \frac{\Delta(mv_3)}{\Delta t}[U(t-t^*) - U(t-t^*-\Delta t)] \tag{8.11.3}$$

Allowing Δt to approach 0, which increases the height of the impulse, we obtain in the limit

$$F_3 = \lim_{\Delta t \to 0} \frac{\Delta(mv_3)}{\Delta t}[U(t-t^*) - U(t-t^*-\Delta t)] \tag{8.11.4}$$

or

$$F_3 = \Delta(mv_3)\delta(t-t^*) \tag{8.11.5}$$

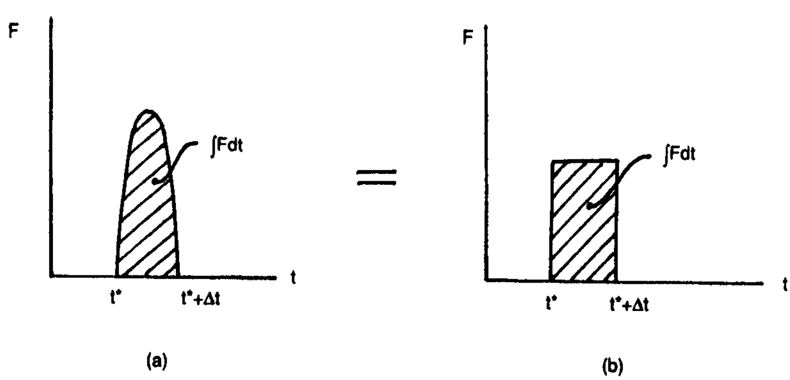

FIG. 11 Illustration of an impulse.

In general, the three components F_i of the force vector are

$$F_i = \Delta(mv_i)\delta(t-t^*) \tag{8.11.6}$$

It should be noted that this force description cannot be used to obtain the actual magnitude of the force. At best, we can determine an average force if the actual impact duration is known. In this case $F_i \cong \Delta(mv_i)/\Delta t$. It should also be noted that the change in momentum expressions cannot easily be determined precisely. While the mass and velocity $\bar{v}_a$ are usually well definable, the exit velocity $\bar{v}_b$ is influenced by the local response of the structure and cannot be calculated without solving a contact problem. However, in many practical cases, we can make approximations based on observation. Often, $\bar{v}_b \cong 0$, for example if no significant rebound is observed.

A special situation occurs if the impacting mass lodges in the structure. In this case, it is often permissable to set $\bar{v}_b = 0$, but if the mass is relatively large, the subsequent structural response problem has to be solved with the lodged mass as part of the structure.

A point force on the shell can always be approximated as a pressure load by dividing F by a small area. If the dimensions of the small area are less than approximately a quarter of the "wavelength" of the shortest wavelength of the mode of interest, the precise size of the distribution area does not matter as long as the load adds up to the same force. Since mode shapes are only approximately sinusoidal, wavelength has to be understood as the average distance between two node lines.

For example, a transverse point force can be approximated by a cylinder of cross-section $A_1A_2, \mathrm{d}\alpha_1, \mathrm{d}\alpha_2$ and height $F_3/A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2$, as illustrated in Fig. 12. We may write

$$q_3(\alpha_1, \alpha_2, t) = \frac{\begin{array}{c} F_3[U(\alpha_1-\alpha_1^*) - U(\alpha_1-\alpha_1^*-\Delta\alpha_1)] \\ \times[U(\alpha_2-\alpha_2^*) - U(\alpha_2-\alpha_2^*-\Delta\alpha_2)] \end{array}}{A_1A_2\Delta\alpha_1\Delta\alpha_2} \tag{8.11.7}$$

Taking the limit as $\Delta\alpha_1, \Delta\alpha_2 \to 0$ gives

$$q_3(\alpha_1, \alpha_2, t) = \frac{F_3}{A_1A_2}\delta(\alpha_1-\alpha_1^*)\delta(\alpha_2-\alpha_2^*) \tag{8.11.8}$$

A general point load consisting of three components may be described as

$$q_1(\alpha_1, \alpha_2, t) = \frac{F_1}{A_1A_2}\delta(\alpha_1-\alpha_1^*)\delta(\alpha_2-\alpha_2^*) \tag{8.11.9}$$

Note that F_1 may or may not be impulsive. Thus a sinusoidal point force is described by $F_i = F_{0i}\sin\omega t$, a step point force by $F_i = F_{0i}U(t-t_1)$, and the force due to an impacting mass by $F_i = \Delta(mv_i)\delta(t-t_1)$.

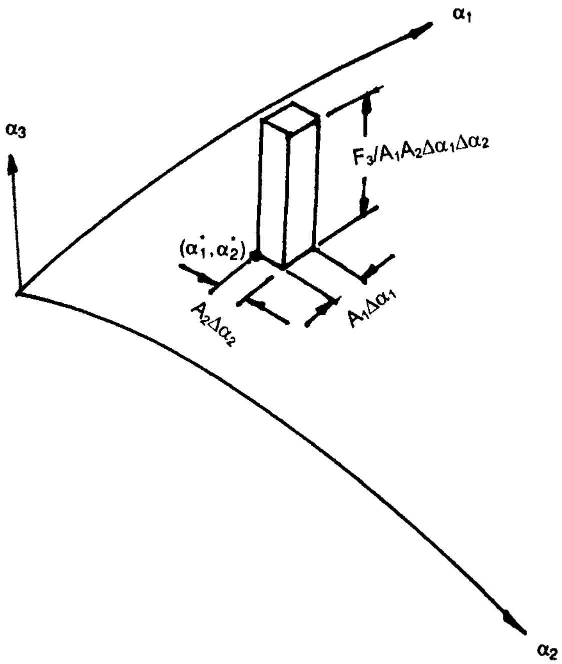

FIG. 12 Approximation of a point load as a column of rectangular cross-section.

8.12. DEFINITIONS AND INTEGRATION PROPERTY OF THE DIRAC DELTA FUNCTION

The definition of the Dirac delta function is based on the concept that one wishes to have a function that locates an event in time or space and has an integral of unity. The obvious way to accomplish this is to create a gate function of width $\Delta\xi$, where, depending on the application, ξ may be a time or space variable, and height $1/\Delta\xi$, so that the integrated area under this gate function is unity and dimensionless. Mathematically, the gate function can be expressed as

$$F(\xi) = \frac{1}{\Delta\xi}[U(\xi - \xi^*) - U(\xi - \xi^* - \Delta\xi)] \quad (8.12.1)$$

For most practical problems, we can stop at this point, but it will be observed that as $\Delta\xi$ becomes very small it no longer matters what the $\Delta\xi$ magnitude is. It can also be shown that the shape, which we have assumed to be a rectangular gate, does not matter and could be triangular, semicircular, or whatever, as long as the enclosed area (the integral of the function) remains unity. Therefore, it can be argued, why not take the limit allowing $\Delta\xi$ to approach zero? This limiting case of the function $F(\xi)$ is the Dirac delta function $\delta(\xi-\xi^*)$ and is defined mathematically as

$$\delta(\xi-\xi^*)=\lim_{\Delta\xi\to 0}\frac{1}{\Delta\xi}[U(\xi-\xi^*)-U(\xi-\xi^*-\Delta\xi)] \tag{8.12.2}$$

Because of this, we may now state that

$$\delta(\xi-\xi^*)=0,\ \ \xi\neq\xi^* \tag{8.12.3}$$

and because the original concept of a unit area has not changed,

$$\int_{-\infty}^{+\infty}\delta(\xi-\xi^*)\mathrm{d}\xi=1 \tag{8.12.4}$$

It has to be noted that $\delta(\xi-\xi^*)$ by itself remains undefined at $\xi=\xi^*$, since to maintain an integral of unity it obviously had to approach infinity as $\Delta\xi$ approached 0.

The mathematical power of the use of the Dirac delta function is largely due to the integration property:

$$\int_{-\infty}^{+\infty}f(\xi)\delta(\xi-\xi^*)\mathrm{d}\xi=f(\xi^*) \tag{8.12.5}$$

This can be proven with the visual help of Fig. 13. The Dirac delta function is shown in its gate function form before letting $\Delta\xi\to 0$. Multiplying the gate function with an arbitrary function $f(\xi)$ whose value at ξ^* is $f(\xi^*)$ and at $\xi^*+\Delta\xi$ is $f(\xi^*+\Delta\xi)$, one obtains a trapezoid whose area is the desired integral as $\Delta\xi$ is allowed to approach 0.

$$\int_{0}^{\xi}f(\xi)\delta(\xi-\xi^*)\delta\xi=\lim_{\Delta\xi\to 0}\frac{f(\xi^*)+f(\xi^*+\Delta\xi)}{2} \tag{8.12.6}$$

Taking the limit results in Eq. (8.12.5).

8.13. SELECTION OF MODE PHASE ANGLES FOR SHELLS OF REVOLUTION

The selection rational is illustrated by the example of a closed ring. The natural modes are given in Sec. 5.3 as

$$\begin{Bmatrix} U_{3n} \\ U_{\theta n} \end{Bmatrix}_i = A_{ni}\begin{Bmatrix} \cos n(\theta-\phi) \\ \dfrac{B_{ni}}{A_{ni}}\sin n(\theta-\phi) \end{Bmatrix} \tag{8.13.1}$$

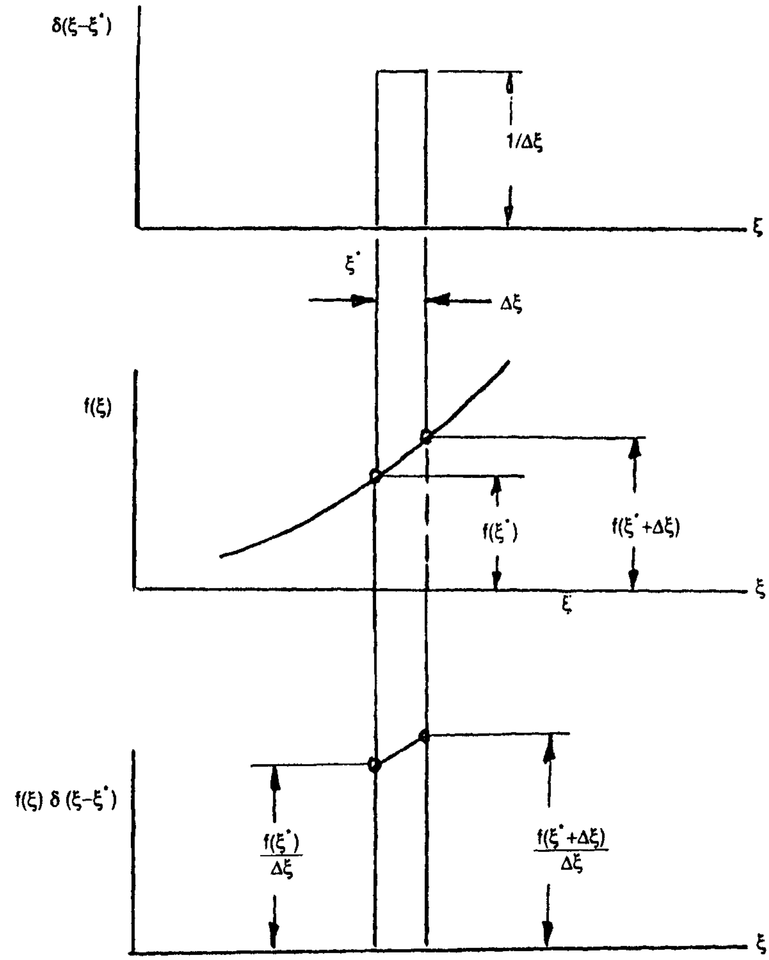

Fig. 13 Illustration of the derivation of the integration property of the Dirac delta function.

Since $i=1,2$, we actually have two sets of mode shapes. We have to select two phase angles ϕ in such a way that the mode shapes for each ϕ are orthogonal to each other. The requirement is that

$$\int_0^{2\pi}\left[\cos n(\theta-\phi_1)\cos n(\theta-\phi_2)+\left(\frac{B_{ni}}{A_{ni}}\right)^2 \times \sin n(\theta-\phi_1)\sin n(\theta-\phi_2)\right]\mathrm{d}\theta=0 \tag{8.13.2}$$

From this, we see that if we select ϕ_1, ϕ_2 must be such that the integral is satisfied. Thus, it must be that

$$\phi_2 = \phi_1 + \frac{\pi}{2n} \tag{8.13.3}$$

This will make

$$\cos n(\theta - \phi_2) = \cos\left[n(\theta - \phi_1) - \frac{\pi}{2}\right] = \sin n(\theta - \phi_1) \tag{8.13.4}$$

and

$$\sin n(\theta - \phi_2) = \sin\left[n(\theta - \phi_1) - \frac{\pi}{2}\right] = -\cos n(\theta - \phi_1) \tag{8.13.5}$$

The orthogonality integral then becomes

$$\int_0^{2\pi} \left[\cos n(\theta - \phi_1) \sin n(\theta - \phi_1) - \left(\frac{B_{ni}}{A_{ni}}\right)^2 \sin n(\theta - \phi_1) \times \cos n(\theta - \phi_1)\right] d\theta = 0 \tag{8.13.6}$$

which is indeed satisfied because of the orthogonality properties of the sin and cos functions.

Note that ϕ_1, can still be arbitrarily selected, but ϕ_2 must satisfy Eq. (8.13.3). It is of advantage to use

$$\phi_1 = 0 \tag{8.13.7}$$

so that

$$\phi_2 = \frac{\pi}{2n} \tag{8.13.8}$$

For our example, we have to consider all together four sets of modes when formulating the modal series solution:

$$\left\{\begin{matrix} \cos n\theta \\ \frac{B_{n1}}{A_{n1}} \sin n\theta \end{matrix}\right\}, \left\{\begin{matrix} \cos n\theta \\ \frac{B_{n2}}{A_{n2}} \sin n\theta \end{matrix}\right\}, \left\{\begin{matrix} \sin n\theta \\ -\frac{B_{n1}}{A_{n1}} \cos n\theta \end{matrix}\right\}, \left\{\begin{matrix} \sin n\theta \\ -\frac{B_{n2}}{A_{n2}} \cos n\theta \end{matrix}\right\} \tag{8.13.9}$$

The example of the ring can easily be generalized to any closed shell of revolution.

8.14. STEADY-STATE CIRCULAR CYLINDRICAL SHELL RESPONSE TO HARMONIC POINT LOAD WITH ALL MODE COMPONENTS CONSIDERED

For simply supported boundary conditions of the type of Sec. 5.5, we have the following two sets of natural modes to consider: For $\phi=0, n=0,1,2,\ldots, m=1,2,3,\ldots,$ we have

$$U_{xmni(1)} = \frac{A_{mni}}{C_{mni}} \cos\frac{m\pi x}{L}\cos n\theta \tag{8.14.1}$$

$$U_{\theta mni(1)} = \frac{B_{mni}}{C_{mni}} \sin\frac{m\pi x}{L}\sin n\theta \tag{8.14.2}$$

$$U_{3mni(1)} = \sin\frac{m\pi x}{L}\cos n\theta \tag{8.14.3}$$

and for $\phi=\pi/(2n)$, the set of natural modes, which is orthogonal to the modes of Eqs. (8.14.1)–(8.14.3), is

$$U_{xmni(2)} = \frac{A_{mni}}{C_{mni}} \cos\frac{m\pi x}{L}\sin n\theta \tag{8.14.4}$$

$$U_{\theta mni(2)} = -\frac{B_{mni}}{C_{mni}} \sin\frac{m\pi x}{L}\cos n\theta \tag{8.14.5}$$

$$U_{3mni(2)} = \sin\frac{m\pi x}{L}\sin n\theta \tag{8.14.6}$$

Note that $i=1,2,3$, corresponding to each of the natural frequencies for a given (*m,n*) combination. In most engineering applications, the natural frequencies associated with $i=2$ and 3 are so high that the contribution of these modes can be neglected. But we consider them here.

The force shown in Fig. 14 results in the loading description

$$q_x(x,\theta,t) = 0 \tag{8.14.7}$$

$$q_\theta(x,\theta,t) = 0 \tag{8.14.8}$$

$$q_3(x,\theta,t) = F_3 e^{j\omega t}\frac{1}{a}\delta(\theta-\theta^*)\delta(x-x^*) \tag{8.14.9}$$

where the imaginary part of $e^{j\omega t}$ represents $\sin\omega t$.

Let us solve the problem for the first set of modes. Equation (8.1.15) becomes

$$N_k = \int_{\theta=0}^{2\pi}\int_{x=0}^{L}\left[\left(\frac{A_{mni}}{C_{mni}}\right)^2\cos^2\frac{m\pi x}{L}\cos^2 n\theta + \left(\frac{B_{mni}}{C_{mni}}\right)^2 \times\sin^2\frac{m\pi x}{L}\sin^2 n\theta + \sin^2\frac{m\pi x}{L}\cos^2 n\theta\right] a\,dx\,d\theta \tag{8.14.10}$$

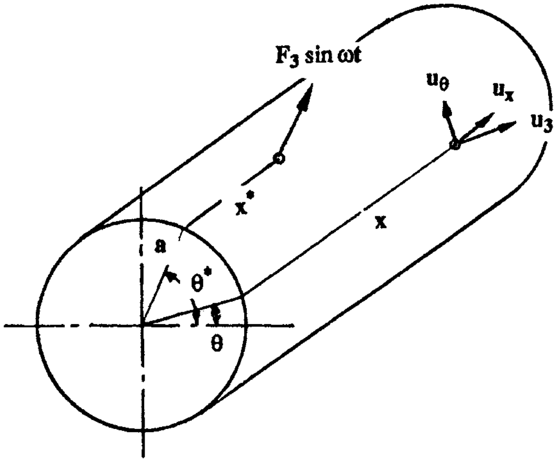

FIG. 14 Circular cylindrical shell acted on by a point force.

This becomes for $n \neq 0$ and $m \neq 0$,

$$N_k = \left[\left(\frac{A_{mni}}{C_{mni}}\right)^2 + \left(\frac{B_{mni}}{C_{mni}}\right)^2 + 1\right]\frac{La\pi}{2} = N_{mni(1)} \tag{8.14.11}$$

For $n = 0, m \neq 0$,

$$N_k = \left[\left(\frac{A_{mni}}{C_{mni}}\right)^2 + 1\right]La\pi = N_{m0i(1)} \tag{8.14.12}$$

No other cases exist since $m \neq 0$. Therefore, for $n \neq 0$,

$$F_{mni(1)} = \frac{1}{\rho h N_{mni(1)}}\int_0^{2\pi}\int_0^L F_3 e^{j\omega t}\frac{1}{a}\delta(\theta-\theta^*)\delta(x-x^*) \cdot \sin\frac{m\pi x}{L}\cos n\theta a \, dx \, d\theta$$

or

$$F_{mni(1)} = \frac{F_3 e^{j\omega t}}{\rho h N_{mni(1)}}\sin\frac{m\pi x^*}{L}\cos n\theta^* \tag{8.14.13}$$

For $n = 0$,

$$\begin{aligned} F_{m0i(1)} &= \frac{1}{\rho h N_{m0i(1)}}\int_0^{2\pi}\int_0^L F_3 e^{j\omega t}\frac{1}{a}\delta(\theta-\theta^*)\delta(x-x^*)\sin\frac{m\pi x}{L} a \, dx \, d\theta \\ &= \frac{F_3 e^{j\omega t}}{\rho h N_{m0i(1)}}\sin\frac{m\pi x^*}{L} \end{aligned} \tag{8.14.14}$$

Therefore,

$$F_k^* = F_{mni(1)}^* = \frac{F_3}{\rho h N_{mni(1)}} \sin\frac{m\pi x^*}{L} \cos n\theta^* \tag{8.14.15}$$

where we remember that for $n=0$, the definition of $N_{m0i(1)}$ is given by Eq. (8.14.12), while for $n\neq 0$, the definition of $N_{nmi(1)}$ is given by Eq. (8.14.11). Utilizing Sec. 8.5, the steady-state response is

$$u_{x(1)} = \sum_{i=1}^{3}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{F_3 \sin(m\pi x^*/L)\cos n\theta^* (A_{mni}/C_{mni})\cos(m\pi x/L) \times \cos n\theta \sin(\omega t - \phi_{mni})}{\rho h N_{mni(1)} f(\omega)} \tag{8.14.16}$$

$$u_{\theta(1)} = \sum_{i=1}^{3}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{F_3 \sin(m\pi x^*/L)\cos n\theta^* (B_{mni}/C_{mni})\sin(m\pi x/L) \times \sin n\theta \sin(\omega t - \phi_{mni})}{\rho h N_{mni(1)} f(\omega)} \tag{8.14.17}$$

$$u_{3(1)} = \sum_{i=1}^{3}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{F_3 \sin(m\pi x^*/L)\cos n\theta^* \sin(m\pi x/L)\cos n\theta \times \sin(\omega t - \phi_{mni})}{\rho h N_{mni(1)} f(\omega)} \tag{8.14.18}$$

where

$$f(\omega) = \omega_{mni}^2 \sqrt{\left[1 - \left(\frac{\omega}{\omega_{mni}}\right)^2\right]^2 + 4\zeta_{mni}^2 \left(\frac{\omega}{\omega_{mni}}\right)^2}$$

The next step is to obtain the solution for the second set of modes. In this case

$$N_k = \int_0^{2\pi}\int_0^L \left[\left(\frac{A_{mni}}{C_{mni}}\right)^2 \cos^2\frac{m\pi x}{L}\sin^2 n\theta + \left(\frac{B_{mni}}{C_{mni}}\right)^2 \sin^2\frac{m\pi x}{L}\cos^2 n\theta + \sin^2\frac{m\pi x}{L}\sin^2 n\theta\right] a\,\mathrm{d}x\,\mathrm{d}\theta \tag{8.14.19}$$

which becomes, for $n\neq 0$,

$$N_k = \left[\left(\frac{A_{mni}}{C_{mni}}\right)^2 + \left(\frac{B_{mni}}{C_{mni}}\right)^2 + 1\right]\frac{La\pi}{2} = N_{mni(2)} \tag{8.14.20}$$

and for $n=0$,

$$N_k = \left(\frac{B_{mni}}{C_{mni}}\right)^2 La\pi = N_{m0i(2)} \tag{8.14.21}$$

Therefore,

$$F_{mni(2)} = \frac{1}{\rho h N_{mni(2)}} \int_0^{2\pi} \int_0^L F_2 e^{j\omega t} \frac{1}{a} \delta(\theta-\theta^*)\delta(x-x^*) \cdot \sin\frac{m\pi x}{L} \times \sin n\theta a \, dx d\theta = \frac{F_3 e^{j\omega t}}{\rho h N_{mni(2)}} \sin\frac{m\pi x^*}{L} \sin n\theta^* \quad (8.14.22)$$

which also covers the $n=0$ case, which results in $F_{m0i(2)}=0$. The steady-state response solution is, therefore,

$$u_{x(2)} = \sum_{i=1}^{3}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \cdot \frac{F_3 \sin(m\pi x^*/L) \sin n\theta^* (A_{mni}/C_{mni}) \cos(m\pi x/L) \times \sin n\theta \sin(\omega t - \phi_{mni})}{\rho h N_{mni(2)} f(\omega)} \quad (8.14.23)$$

$$u_{\theta(2)} = -\sum_{i=1}^{3}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \cdot \frac{F_3 \sin(m\pi x^*/L) \sin n\theta^* (B_{mni}/C_{mni}) \sin(m\pi x/L) \times \cos n\theta \sin(\omega t - \phi_{mni})}{\rho h N_{mni(2)} f(\omega)} \quad (8.14.24)$$

$$u_{3(2)} = \sum_{i=1}^{3}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \cdot \frac{F_3 \sin(m\pi x^*/L) \sin n\theta^* \sin(m\pi x/L) \sin n\theta \times \sin(\omega t - \phi_{mni})}{\rho h N_{mni(2)} f(\omega)} \quad (8.14.25)$$

The total solution is the addition of the two sets of solutions. Note that for $n \neq 0$, $N_{mni(1)} = N_{mni(2)} = N_{mni}$. Needless to state that the ω_{mnl} are the same for the two solutions. Since

$$\cos n\theta^* \sin n\theta - \sin n\theta^* \cos n\theta = \sin n(\theta - \theta^*)$$

$$\sin n\theta^* \sin n\theta + \cos n\theta^* \cos n\theta = \cos n(\theta - \theta^*) \quad (8.14.26)$$

we obtain, combining the solutions for all n,

$$u_x = \sum_{i=1}^{3}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \cdot \frac{F_3 (A_{mni}/C_{mni}) \sin(m\pi x^*/L) \cos(m\pi x/L) \cos n(\theta-\theta^*) \times \sin(\omega t - \phi_{mni})}{\rho h N_{mni} f(\omega)} \quad (8.14.27)$$

$$u_\theta = \sum_{i=1}^{3}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \cdot \frac{F_3 (B_{mni}/C_{mni}) \sin(m\pi x^*/L) \sin(m\pi x/L) \sin n(\theta-\theta^*) \times \sin(\omega t - \phi_{mni})}{\rho h N_{mni} f(\omega)} \quad (8.14.28)$$

$$u_3 = \sum_{i=1}^{3}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \cdot \frac{F_3 \sin(m\pi x^*/L)\sin(m\pi x/L)\cos n(\theta-\theta^*) \times \sin(\omega t - \phi_{mni})}{\rho h N_{mni} f(\omega)} \tag{8.14.29}$$

where

$$N_{mni} = \begin{cases} \left[\left(\frac{A_{mni}}{C_{mni}}\right)^2 + \left(\frac{B_{mni}}{C_{mni}}\right)^2 + 1\right]\frac{La\pi}{2} & \text{if } n \neq 0 \\ \left[\left(\frac{A_{mni}}{C_{mni}}\right)^2 + 1\right] La\pi & \text{if } n = 0 \end{cases} \tag{8.14.30}$$

8.15. INITIAL VELOCITY EXCITATION OF A SIMPLY SUPPORTED CYLINDRICAL SHELL

A circular cylindrical shell structure (Fig. 15) attached to a stiff shaft by simple supports of a kind that permits axial deflections but does not permit tangential deflections at the supports (slot and key arrangement) experiences an initial velocity because the entire system moves with a velocity $\dot{y} = -V$ when the shaft comes suddenly to rest. Measuring time from this moment and neglecting gravitational sag (if the shell is in a position other than vertical), the initial conditions become

$$u_x(x,\theta,0) = 0, \; \dot{u}_x(x,\theta,0) = 0 \tag{8.15.1}$$

$$u_\theta(x,\theta,0) = 0, \; \dot{u}_\theta(x,\theta,0) = -V\cos\theta \tag{8.15.2}$$

$$u_3(x,\theta,0) = 0, \; \dot{u}_3(x,\theta,0) = -V\sin\theta \tag{8.15.3}$$

The boundary conditions are those of Sec.5.5; therefore, the natural modes are gives by Eq. (5.5.87). Selecting $\phi = 0$ and $\phi = \pi/2n$, we obtain

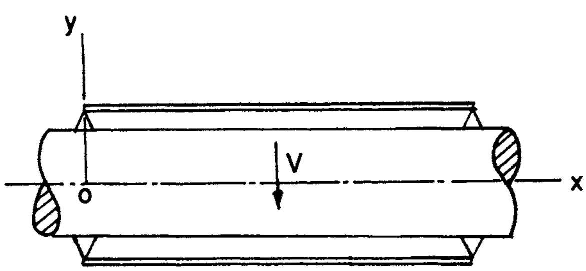

Fig. 15 Simply supported circular cylindrical shell that moves initially with a uniform downward velocity.

two sets of orthogonal modes:

$$U_{xmni(1)} = \frac{A_{mni}}{C_{mni}} \cos\frac{m\pi x}{L} \cos n\theta \tag{8.15.4}$$

$$U_{\theta mni(1)} = \frac{B_{mni}}{C_{mni}} \sin\frac{m\pi x}{L} \sin n\theta \tag{8.15.5}$$

$$U_{3mni(1)} = \sin\frac{m\pi x}{L} \cos n\theta \tag{8.15.6}$$

and

$$U_{xmni(2)} = \frac{A_{mni}}{C_{mni}} \cos\frac{m\pi x}{L} \sin n\theta \tag{8.15.7}$$

$$U_{\theta mni(2)} = -\frac{B_{mni}}{C_{mni}} \sin\frac{m\pi x}{L} \cos n\theta \tag{8.15.8}$$

$$U_{3mni(2)} = \sin\frac{m\pi x}{L} \sin n\theta \tag{8.15.9}$$

From Eq. (8.2.8), we obtain for the first set of modes,

$$\eta_k(0) = 0 \tag{8.15.10}$$

and

$$\dot{\eta}_k(0) = \frac{1}{N_k}\int_0^L \int_0^{2\pi} \left[(-V\cos\theta)\frac{B_{mni}}{C_{mni}} \sin\frac{m\pi x}{L} \sin n\theta + (-V\sin\theta)\sin\frac{m\pi x}{L} \cos n\theta \right] a\,\mathrm{d}\theta\,\mathrm{d}x \tag{8.15.11}$$

Because of the orthogonality of the integrals of $\cos\theta\sin n\theta$ and $\sin\theta\cos n\theta$ for $n \geq 1$, and because for $n=0$ the integral of $\sin n\theta$ is 0, we obtain

$$\dot{\eta}_k(0) = 0 \tag{8.15.12}$$

This partial result simply indicates that modes not symmetric to the yx plane cannot be excited.

For the second set of modes, again

$$\eta_k(0) = 0 \tag{8.15.13}$$

$$\dot{\eta}_k(0) = \frac{1}{N_k}\int_k^L \int_0^{2\pi} \left[-(-V\cos\theta)\frac{B_{mni}}{C_{mni}} \sin\frac{m\pi x}{L} \cos n\theta + (-V\sin\theta)\sin\frac{m\pi x}{L} \sin n\theta \right] a\,\mathrm{d}\theta\,\mathrm{d}x \tag{8.15.14}$$

This integral is 0 for $n>1$ and $n=0$, but for $n=1$, we obtain

$$\begin{aligned}\dot{\eta}_k(0) &= -\frac{VL}{m}(\cos m\pi - 1)\left(\frac{B_{m1i}}{C_{m1i}} - 1\right)\frac{1}{N_k} = \dot{\eta}_{m1i}(0)\\ &= \begin{cases}\dfrac{2VL}{m}\left(\dfrac{B_{m1i}}{C_{m1i}} - 1\right)\dfrac{1}{N_k}, & m=1,3,5,\ldots\\ 0, & m=0,2,4,\ldots\end{cases}\end{aligned} \tag{8.15.15}$$

To evaluate N_λ, we only need to evaluate Eq. (8.1.15) for $n=1$ and the second set of modes:

$$\begin{aligned}N_k &= \int_0^L \int_0^{2\pi} \left[\left(\frac{A_{m1i}}{C_{m1i}}\right)^2 \cos^2\frac{m\pi x}{L}\sin^2\theta + \left(\frac{B_{m1i}}{C_{m1i}}\right)^2 \sin^2\frac{m\pi x}{L}\cos^2\theta\right.\\ &\qquad \left. + \sin^2\frac{m\pi x}{L}\sin^2\theta\right] a\mathrm{d}\theta\mathrm{d}x\\ &= \frac{aL\pi}{2}\left[\left(\frac{A_{m1i}}{C_{m1i}}\right)^2 + \left(\frac{B_{m1i}}{C_{m1i}}\right)^2 + 1\right]\end{aligned} \tag{8.15.16}$$

Normally, we would add the solutions due to the first set of modes to those of the second set of modes. But it is 0 for the former and therefore we obtain, utilizing Eq. (8.3.6) for subcritical damping, with zero forcing, and Eq. (8.1.1)

$$u_x(x,\theta,t) = \sum_{i=1}^{3}\sum_{m=1,3,5,\cdots}^{\infty} \frac{4Vf_{m1i}}{m\pi} \mathrm{e}^{-\zeta_{m1i}\omega_{m1i}t} \frac{\sin\gamma_{m1i}t}{\gamma_{m1i}} \frac{A_{m1i}}{C_{m1i}} \cos\frac{m\pi x}{L}\sin\theta \tag{8.15.17}$$

$$u_\theta(x,\theta,t) = \sum_{i=1}^{3}\sum_{m=1,3,5,\cdots}^{\infty} \frac{4Vf_{m1i}}{m\pi} \mathrm{e}^{-\zeta_{m1i}\omega_{m1i}t} \frac{\sin\gamma_{m1i}t}{\gamma_{m1i}} \frac{B_{m1i}}{C_{m1i}} \sin\frac{m\pi x}{L}\cos\theta \tag{8.15.18}$$

$$u_3(x,\theta,t) = \sum_{i=1}^{3}\sum_{m=1,3,5,\cdots}^{\infty} \frac{4Vf_{m1i}}{m\pi} \mathrm{e}^{-\zeta_{m1i}\omega_{m1i}t} \frac{\sin\gamma_{m1i}t}{\gamma_{m1i}} \sin\frac{m\pi x}{L}\sin\theta \tag{8.15.19}$$

where

$$f_{m1i} = \frac{(B_{m1i}/C_{m1i}) - 1}{(A_{m1i}/C_{m1i})^2 + (B_{m1i}/C_{m1i})^2 + 1} \tag{8.15.20}$$

In conclusion, we see that due to the symmetry of the problem about the yx plane, only modes symmetric to this plane are excited, and of those only the $n=1$ modes. That this is so can be explained by physical intuition if one considers how the inertial effect would tend to deflect the shell.

8.16. STATIC DEFLECTIONS

The static deflection is obtained from the step response of Eq. (8.6.4). It may be written as

$$\eta_k(t)=\frac{F_K^*}{\omega_k^2}\left\{1-\frac{e^{-\zeta_1\omega_k(t-t_1)}}{\sqrt{1-\zeta_k^2}}\cos[\gamma_k(t-t_1)-\phi_k]\right\} \tag{8.16.1}$$

In steady state, oscillations will have decayed to

$$\eta_k(t)=\frac{F_k^*}{\omega_k^2} \tag{8.16.2}$$

Thus the deflections due to static loading are, in general,

$$u_i(\alpha_1,\alpha_2)=\sum_{k=1}^{\infty}\frac{F_k^*}{\omega_k^2}U_{ik}(\alpha_1,\alpha_2) \tag{8.16.3}$$

where

$$F_k^* = \frac{1}{\rho h N_k}\int_{\alpha_2}\int_{\alpha_1}(q_{1s}U_{1k}+q_{2s}U_{2k}+q_{3s}U_{3k})A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{8.16.4}$$

$$N_k = \int_{\alpha_2}\int_{\alpha_1}(U_{1k}^2+U_{2k}^2+U_{3k}^2)A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{8.16.5}$$

The static loads in the three directions are q_{1s}, q_{2s}, and q_{3s}.

8.17. RECTANGULAR PLATE RESPONSE TO INITIAL DISPLACEMENT CAUSED BY STATIC SAG

A rectangular plate is held in an exactly horizontal position on simple supports. The plate is then released, at $t=0$, with zero velocity. Since the static equilibrium position is the static deflection caused by the weight of the plate, the plate is initially displaced equal to the static deflection. What is the subsequent oscillation about the static equilibrium position? The positive direction is taken to be upward.

8.17.1. Static equilibrium position

At static equilibrium, Eq. (8.1.14) becomes

$$\omega_k^2\eta_k=F_k \tag{8.17.1}$$

where, for the transverse deflection of a simply supported rectangular plate,

$$N_k = \int_{y=0}^{b}\int_{x=0}^{a} U_{3k}^2 \mathrm{d}x\mathrm{d}y = \int_{y=0}^{b}\int_{x=0}^{a} \sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)\mathrm{d}x\mathrm{d}y = \frac{ab}{4} \quad (8.17.2)$$

$$F_k = \frac{1}{\rho h N_k}\int_{y=0}^{b}\int_{x=0}^{a} q_3 U_{3k}\mathrm{d}x\mathrm{d}y \quad (8.17.3)$$

Since q_3 is the distributed weight of the plate

$$q_3 = -\rho h g \quad (8.17.4)$$

Thus

$$\begin{aligned} F_k &= -\frac{4g}{ab}\int_{y=0}^{b}\int_{x=0}^{a} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}x\mathrm{d}y \\ &= -\frac{4g}{mn\pi^2}(1-\cos m\pi)(1-\cos n\pi) \end{aligned} \quad (8.17.5)$$

Therefore,

$$\eta_k = \eta_{mn} = -\frac{4g}{mn\pi^2\omega_{mn}^2}(1-\cos m\pi)(1-\cos n\pi) \quad (8.17.6)$$

and the static deflection is

$$u_{3s} = -\frac{4g}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1-\cos m\pi)(1-\cos n\pi)}{mn \quad \omega_{mn}^2}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad (8.17.7)$$

where $\omega_{nn} = \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}}$

8.17.2. Initial conditions with respect to the static equilibrium position

The initial conditions are therefore (measured from the static equilibrium u_{3s})

$$u_3(x,y,0) = \frac{4g}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1-\cos m\pi)(1-\cos n\pi)}{mn \quad \omega_{mn}^2}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad (8.17.8)$$

$$\dot{u}_3(x,y,0) = 0 \quad (8.17.9)$$

This is converted into initial conditions of the modal participation factors by eqs. (8.2.8) and (8.2.9):

$$\begin{aligned}\eta_k(0) &= \eta_{mn}(0) = \frac{1}{N_{mn}}\int_{y=0}^{b}\int_{x=0}^{a} u_3(x,y,0)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{N_{mn}}\left(\frac{4g}{\pi^2}\right)\frac{(1-\cos m\pi)(1-\cos n\pi)}{mn \quad \omega_{mn}^2}\int_{y=0}^{b}\int_{x=0}^{a}\sin^2\left(\frac{m\pi x}{a}\right) \\ &\quad \times\sin^2\left(\frac{n\pi y}{b}\right)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{8.17.10}$$

or

$$\eta_{mn}^{(0)} = \frac{4g}{mn\pi^2\omega_{mn}^2}(1-\cos m\pi)(1-\cos n\pi) \tag{8.17.11}$$

Also, we obtain

$$\dot{\eta}_k(0) = \dot{\eta}_{mn}(0) = 0 \tag{8.17.12}$$

8.17.3. Vibration response about the static equilibrium condition

Assuming a subcritically damped plate, the modal participation factors become, from eq. (8.3.6)

$$\eta_k(t) = \mathrm{e}^{-\zeta_k\omega_k t}\left[\eta_k(0)\cos\gamma_k t + \eta_k(0)\zeta_k\omega_k\frac{\sin\gamma_k t}{\gamma_k}\right] \tag{8.17.13}$$

or

$$\eta_{mn}(t) = \mathrm{e}^{-\zeta_{mn}\omega_{mn}t}\eta_{mn}(0)\left(\cos\gamma_{mn}t + \frac{\zeta_{mn}}{\sqrt{1-\zeta_{mn}^2}}\sin\gamma_{mn}t\right) \tag{8.17.14}$$

The solution is, therefore,

$$\begin{aligned}u_3(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\eta_{mn}(t)U_{3mn}(x,y) \\ &= \frac{4g}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1-\cos m\pi)(1-\cos n\pi)}{mn \quad \omega_{mn}^2} \\ &\quad \times\mathrm{e}^{-\zeta_{mn}\omega_{mn}t}\left(\cos\gamma_{mn}t + \frac{\zeta_{mn}}{\sqrt{1-\zeta_{mn}^2}}\sin\gamma_{mn}t\right)\sin\left(\frac{m\pi x}{a}\right) \\ &\quad \times\sin\left(\frac{n\pi y}{b}\right)\end{aligned} \tag{8.17.15}$$

Note that for $m, n = 2, 4, 6, \ldots$.

$$(1-\cos m\pi)(1-\cos n\pi) = 0 \tag{8.17.16}$$

This means that only the symmetric modes (combinations of m, n = 1, 3, 5, …) will participate in the solution, as one would expect because of the symmetry of the initial condition.

8.18. THE CONCEPTS OF MODAL MASS, STIFFNESS, DAMPING AND FORCING

We may write Eq. (8.1.14) in the form

$$\rho h N_k \ddot{\eta}_k + \lambda N_k \dot{\eta}_k + \omega_k^2 \rho h N_k \eta_k = f_k \tag{8.18.1}$$

where N_k is defined by Eq. (8.1.15), and f_k is

$$f_k = \int_{\alpha_2}\int_{\alpha_1} (q_1 U_{1k} + q_2 U_{2k} + q_3 U_{3k}) A_1 A_2 \, d\alpha_1 \, d\alpha_2 \tag{8.18.2}$$

Since Eq. (8.18.1) is of the form of a one degree of freedom oscillator equation, it has become customary to view this equation in terms of modal mass, stiffness, damping, and forcing. Therefore, the so-called modal mass is

$$M_k = \rho h N_k \tag{8.18.3}$$

the modal stiffness (spring rate) is

$$K_k = \omega_k^2 \rho h N_k = \omega_k^2 M_k \tag{8.18.4}$$

and the modal forcing is defined by Eq. (8.18.2).
We may also think of a modal damping constant

$$C_k = \lambda N_k \tag{8.18.5}$$

If we define the mode components U_{1k}, U_{2k}, U_{3k} as dimensionless ratios (we have a choice—they can also be defined as displacements having units of [m] as long as we are consistent), the unit of the modal mass is [kg], the unit of the modal stiffness is [N/m] and the unit of the modal damping constant is [Ns/m]. The modal forcing term is f_k. The unit of the modal forcing term is [N]. Therefore, Eq. (8.18.1) can be written as

$$M_k \ddot{\eta}_k + C_k \dot{\eta}_k + K_k \eta_k = f_k \tag{8.18.6}$$

Thus, the interpretation of the modal mass is that of an adjusted or corrected mass of the structure for a particular mode, or an "equivalent" mass. The same is true for the modal stiffness. It is an "equivalent" spring rate.

Note again that here we have taken U_{1k}, U_{2k}, U_{3k} as dimensionless ratios; this means that η_k has the dimension [m], even while the preference of this author is to take U_{1k}, U_{2k}, U_{3k} as having dimension [m], which will make η_k dimensionless. (The product $\eta_k U_{ik}$ always has the dimension of [m]).

It can be concluded that for structures with curvature, a correct description of

$$N_k = \int_{\alpha_2}\int_{\alpha_1} \left(U_{1k}^2 + U_{2k}^2 + U_{3k}^2\right) A_1 A_2 \, d\alpha_1 \, d\alpha_2 \tag{8.18.7}$$

is important. One should not be tempted to ignore the contribution of U_{2k} and U_{1k} when evaluating the transverse vibration response $u_3(\alpha_1, \alpha_2, t)$ for example, unless they are truly negligible:

$$U_{2k} \ll U_{3k} \quad \text{and} \quad U_{1k} \ll U_{3k} \tag{8.18.8}$$

Let us analyze the error that may be caused if we ignore the U_{1k} and U_{2k} contributions. If, for a particular mode k,

$$\frac{\int_{\alpha_2}\int_{\alpha_1} \left(U_{1k}^2 + U_{2k}^2\right) A_1 A_2 \, d\alpha_1 \, d\alpha_2}{\int_{\alpha_2}\int_{\alpha_1} U_{3k}^2 A_1 A_2 \, d\alpha_1 \, d\alpha_2} = \xi_k \ , \tag{8.18.9}$$

we get

$$N_k = (1 + \xi_k) \int_{\alpha_2}\int_{\alpha_1} U_{3k}^2 A_1 A_2 \, d\alpha_1 \, d\alpha_2 \tag{8.18.10}$$

Equation (8.18.1) becomes

$$\ddot{\eta}_k + \frac{\lambda}{\rho h}\dot{\eta}_k + \omega_k^2 \eta_k = \frac{f_k}{\rho h N_k} = \frac{f_k}{(1 + \xi_k)\rho h \int_{\alpha_2}\int_{\alpha_1} U_{3k}^2 A_1 A_2 \, d\alpha_1 \, d\alpha_2} \tag{8.18.11}$$

Thus, we can see that our calculated modal participation factor η_k is inversely proportional to $(1 + \xi_k)$:

$$\eta_k \propto \frac{1}{1 + \xi_k} \tag{8.18.12}$$

For example, if $\xi_k \neq 0$, but we take it as 0, our response will be larger by the ratio $(1 + \xi_k)$ to (1). For example, if $\xi_k = 0.2$, our error for the kth mode participation will be 20%.

This type of error sometimes occurs, for example, when experimentally obtained natural modes of structures with curvatures are to be used in forced response calculations. Frequently, only the transverse mode components are measured, and the tangential mode components are ignored (either because they are very difficult to measure, or because they are assumed away). The forced response calculations will than be based on reduced modal masses.

8.19. STEADY-STATE RESPONSE OF SHELLS TO PERIODIC FORCING

The forcing is assumed here to be such that the spatial distribution does not change in time, but its amplitude is periodic in time. While this covers a large number of practical situations, it excludes cases where the spatial distribution itself changes periodically with time. Here, it is assumed that we may write

$$q_1(\alpha_1, \alpha_2, t) = q_1^*(\alpha_1, \alpha_2) f(t) \tag{8.19.1}$$

$$q_2(\alpha_1, \alpha_2, t) = q_2^*(\alpha_1, \alpha_2) f(t) \tag{8.19.2}$$

$$q_3(\alpha_1, \alpha_2, t) = q_3^*(\alpha_1, \alpha_2) f(t) \tag{8.19.3}$$

or in short

$$q_i(\alpha_1, \alpha_2, t) = q_i^*(\alpha_1, \alpha_2) f(t) \tag{8.19.4}$$

where $f(t)$ is a function that is periodic in time; see Fig. 16. The period T of this periodic function is related to the frequency at which the function repeats, Ω, by

$$T = \frac{2\pi}{\Omega} \tag{8.19.5}$$

Expressing the function $f(t)$ as a Fourier series gives

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\Omega t + b_n \sin n\Omega t) \tag{8.19.6}$$

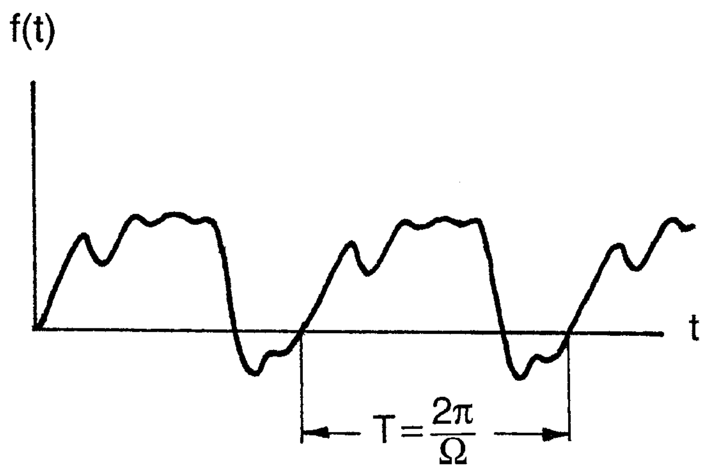

Fig. 16 Periodic forcing.

where

$$a_0 = \frac{1}{T}\int_0^T f(t)\mathrm{d}t \tag{8.19.7}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\Omega t\,\mathrm{d}t \tag{8.19.8}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\Omega t\,\mathrm{d}t \tag{8.19.9}$$

This allows us to write

$$q_i(\alpha_1,\alpha_2,t) = q_i^*(\alpha_1,\alpha_2)\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos n\Omega t + b_n\sin n\Omega t)\right] \tag{8.19.10}$$

The solution due to the a_0 constant is a constant static deflection about which the oscillatory response takes place and is given by Eq. (8.16.3):

$$u_i(\alpha_1,\alpha_2) = \sum_{k=1}^{\infty}\frac{F_k^* a_0}{\omega_k^2}U_{ik}(\alpha_1,\alpha_2) \tag{8.19.11}$$

where from Eq. (8.16.4)

$$F_k^* = \frac{1}{\rho h N_k}\int_{\alpha_2}\int_{\alpha_1}[q_1^*(\alpha_1,\alpha_2)U_{1k} + q_2^*(\alpha_1,\alpha_2)U_{2k} + q_3^*(\alpha_1,\alpha_2)U_{3k}] \times A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{8.19.12}$$

and where N_k is given by Eq. (8.16.5). Also note that the F_k^* term in Eq. (8.16.4) is here replaced by $F_k^* a_0$, for convenience. The meaning is the same.

The solution to each $a_n \cos n\Omega t$ term is

$$u_i^a(\alpha_1,\alpha_2,t) = \sum_{k=1}^{\infty}\eta_{kn}^a(t)U_{ik}(\alpha_1,\alpha_2) \tag{8.19.13}$$

and where, from Sec. 8.5, assuming subcritical damping for all modes,

$$\eta_{kn}^a = \Lambda_{kn}^a \cos(n\Omega t - \phi_{kn}) \tag{8.19.14}$$

Similarly, the solution to each $b_n \sin n\Omega t$ term is

$$u_i^b(\alpha_1,\alpha_2,t) = \sum_{k=1}^{\infty}\eta_{kn}^b(t)U_{ik}(\alpha_1,\alpha_2) \tag{8.19.15}$$

where

$$\eta_{kn}^b = \Lambda_{kn}^b \sin(n\Omega t - \phi_{kn}) \tag{8.19.16}$$

The superscripts a and b define the solutions to the $a_n \cos n\Omega t$ and $b_n \sin n\Omega t$ terms.

The amplitudes and phase lags of the model participation factors are, for the η_{kn}^a set,

$$\Lambda_{kn}^a = \frac{F_k^* a_n}{\omega_k^2 \sqrt{\left[1-\left(\frac{n\Omega}{\omega_k}\right)^2\right]^2 + 4\zeta_k^2 \left(\frac{n\Omega}{\omega_k}\right)^2}} \tag{8.19.17}$$

$$\phi_{kn} = \tan^{-1} \frac{2\zeta_k \left(\frac{n\Omega}{\omega_k}\right)}{1-\left(\frac{n\Omega}{\omega_k}\right)^2} \tag{8.19.18}$$

and for the η_{kn}^b set,

$$\Lambda_{kn}^b = \frac{F_k^* b_n}{\omega_k^2 \sqrt{\left[1-\left(\frac{n\Omega}{\omega_k}\right)^2\right]^2 + 4\zeta_k^2 \left(\frac{n\Omega}{\omega_k}\right)^2}} \tag{8.19.19}$$

with ϕ_{kn} of Eq. (8.19.18) being the same for both sets.

Thus, the total solution is

$$u_i(\alpha_1, \alpha_2, t) = \sum_{k=1}^{\infty} \frac{F_k^* a_0}{\omega_k^2} U_{ik}(\alpha_1, \alpha_2) + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\eta_{kn}^a(t) + \eta_{kn}^b(t)\right) U_{ik}(\alpha_1, \alpha_2) \tag{8.19.20}$$

or, in expanded form,

$$u_i(\alpha_1, \alpha_2, t) = \sum_{k=1}^{\infty} \frac{F_k^* a_0}{\omega_k^2} U_{ik}(\alpha_1, \alpha_2) + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \times \frac{F_k^* [a_n \cos(n\Omega t - \phi_{kn}) + b_n \sin(n\Omega t - \phi_{kn})] U_{ik}(\alpha_1, \alpha_2)}{\omega_k^2 \sqrt{\left[1-\left(\frac{n\Omega}{\omega_k}\right)^2\right]^2 + 4\zeta_k^2 \left(\frac{n\Omega}{\omega_k}\right)^2}} \tag{8.19.21}$$

where ϕ_{kn} is given by Eq. (8.19.18) and F_k^* is given by Eq. (8.19.12). Resonance occurs whenever

$$n\Omega = \omega_k \tag{8.19.22}$$

where $u = 1, 2, 3, \ldots, \infty$. This is the reason that for many types of rotating machinery, for example, reciprocating piston machines such as compressors that have a shaft speed of Ω [rad/sec], with most higher harmonics present in the periodic, mechanical excitation due to the kinematics, resonance in shell-like housings or other structural elements are often difficult to avoid.

8.20. PLATE RESPONSE TO A PERODIC SQUARE WAVE FORCING

As an example that illustrates the reduction of the general shell response to periodic forcing described in Sec. 8.19 to the special case of plates, we consider a uniformly distributed pressure load on a simply supported, rectangular plate that varies in time according to Fig. 17, A being the amplitude of the square wave:

$$q_3(x,y,t)=q_3^*(x,y)f(t) \tag{8.20.1}$$

where $q_3^*(x,y)=\text{I}$ and where

$$f(t)=a_0+\sum_{p=1}^{\infty}(a_p\cos p\Omega t+b_p\sin p\Omega t) \tag{8.20.2}$$

In the Fourier series of Eq. (8.19.6), n is here replaced by p in order to avoid confusion with the mode identification number $n(k=mn)$. For the square wave shown in Fig. 17,

$$a_0=\frac{1}{T}\int_0^T f(t)\mathrm{d}t=\frac{1}{T}\int_0^{T_1}A\mathrm{d}t=A\frac{T_1}{T}=\frac{A\Omega T_1}{2\pi} \tag{8.20.3}$$

$$a_p=\frac{2}{T}\int_0^T f(t)\cos(p\Omega t)\mathrm{d}t=\frac{2}{T}\int_0^{T_1}A\cos(p\Omega t)\mathrm{d}t=\frac{A}{\pi p}\sin p\Omega T_1 \tag{8.20.4}$$

$$b_p=\frac{2}{T}\int_0^T f(t)\sin(p\Omega t)\mathrm{d}t=\frac{2}{T}\int_0^{T_1}A\sin(p\Omega t)\mathrm{d}t=\frac{A}{\pi p}(1-\cos p\Omega t) \tag{8.20.5}$$

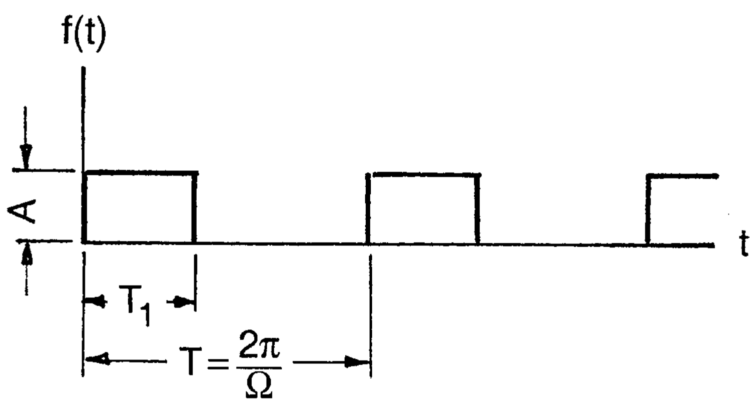

FIG. 17 Example of Periodic forcing.

For a simply supported plate,

$$U_{3k}=U_{3mn}=\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{8.20.6}$$

and

$$\omega_k=\omega_{mn}=\pi^2\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}} \tag{8.20.7}$$

Furthermore,

$$F_k^*=F_{mn}^*=\frac{1}{\rho h N_{mn}}\int_0^a\int_0^b\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}x\,\mathrm{d}y \tag{8.20.8}$$

where

$$N_{mn}=\int_0^a\int_0^b\sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)=\frac{ab}{4} \tag{8.20.9}$$

so that

$$F_k^*=F_{mn}^*=\frac{4}{\rho h m n\pi^2}(1-\cos m\pi)(1-\cos n\pi) \tag{8.20.10}$$

Therefore, Eq. (8.19.21) becomes, for $i=3$:

$$u_3(x,y,t)$$
$$=\frac{4A}{\rho h\pi^2}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1-\cos m\pi)(1-\cos n\pi)}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[\frac{\Omega T_1}{2\pi\omega_{mn}^2}\right.\right.$$
$$\left.\left.+\sum_{p=1}^{\infty}\frac{\dfrac{\sin p\Omega T_1}{\pi p}\cos(p\Omega t-\phi_{mnp})+\dfrac{(1-\cos p\Omega t)}{\pi p}\sin(p\Omega t-\phi_{mnp})}{\omega_{mn}^2\sqrt{\left[1-\left(\frac{p\Omega}{\omega_{mn}}\right)^2\right]^2+4\zeta_{mn}^2\left(\frac{p\Omega}{\omega_{mn}}\right)^2}}\right]\right\} \tag{8.20.11}$$

where

$$\phi_{mnp}=\tan^{-1}\frac{2\zeta_{mn}\left(\frac{p\Omega}{\omega_{mn}}\right)}{1-\left(\frac{p\Omega}{\omega_{mn}}\right)^2} \tag{8.20.12}$$

Because of the uniform load distribution, only modes where $m=1,3,5,\ldots$ and $n=1,3,5,\ldots$ participate in the solution.

8.21. BEATING RESPONSE TO STEADY STATE HARMONIC FORCING

To illustrate beating, the example of a rectangular, undamped, simply supported plate is used. Let the plate be excited by two harmonic point forces as shown in Fig. 18. The excitation frequencies ω_1 and ω_2 are relatively close to each other.

Solving the problem one force at a time, the response to point force $F_1 \sin\omega_1 t$ is

$$u_3(x,y,t) = A_1(x,y)\sin\omega_1 t \tag{8.21.1}$$

where

$$A_1(x,y) = \frac{4F_1}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b}}{\omega_{mn}^2-\omega_1^2} \tag{8.21.2}$$

The steady-state response to point force $F_2 \sin\omega_2 t$ is

$$u_3(x,y,t) = A_2(x,y)\sin\omega_2 t \tag{8.21.3}$$

where

$$A_2(x,y) = \frac{4F_2}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m\pi x_2}{a}\sin\frac{n\pi y_2}{b}}{\omega_{mn}^2-\omega_2^2} \tag{8.21.4}$$

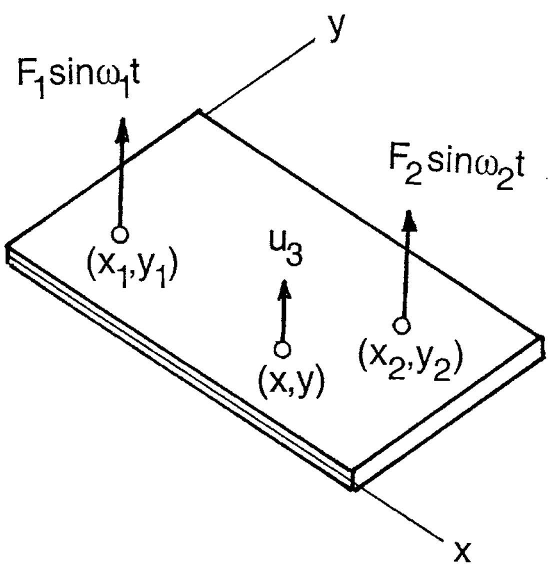

FIG. 18 Steady state forcing of a rectangular plate that will produce beating.

The total solution to both forces is, therefore,

$$u_3(x,y,t)=A_1\sin\omega_1 t+A_2\sin\omega_2 t \tag{8.21.5}$$

This equation can also be written as (using ω_1 as carrier frequency)

$$u_3(x,y,t)=U_3(x,y)\sin(\omega_1 t+\Psi) \tag{8.21.6}$$

where

$$U_3(x,y)=\sqrt{A_1^2+A_2^2+2A_1A_2\cos(\omega_2-\omega_1)t} \tag{8.21.7}$$

$$\Psi=\tan^{-1}\left[\frac{A_2\sin(\omega_2-\omega_1)t}{A_1+A_2\cos(\omega_2-\omega_1)t}\right] \tag{8.21.8}$$

Equation (8.21.6) is a beating response. The amplitude of the sine wave of frequency ω_1 is modulated according to Eq. (8.21.7). The maximum modulation amplitude occurs when

$$\cos(\omega_2-\omega_1)t=1 \tag{8.21.9}$$

or when

$$t=\frac{2n\pi}{\omega_2-\omega_1},\quad n=1,2,\ldots \tag{8.21.10}$$

In this case,

$$U_3(x,y)_{\max}=A_2+A_1 \tag{8.21.11}$$

The minimal modulation amplitude occurs when

$$\cos(\omega_2-\omega_1)t=-1 \tag{8.21.12}$$

or when

$$t=\frac{(2n+1)\pi}{\omega_2-\omega_1},\quad n=1,2,\ldots \tag{8.21.13}$$

In this case,

$$U_3(x,y)_{\min}=A_2-A_1 \tag{8.21.14}$$

The period of the beat frequency (from maximum modulation amplitude to the next maximum modulation amplitude) is

$$T_{\text{Beat}}=\frac{2\pi}{\omega_2-\omega_1} \tag{8.21.15}$$

or the beat (modulation) frequency in [rad/sec] is

$$\Omega_{\text{Beat}}=\omega_2-\omega_1 \tag{8.21.16}$$

The closer the two excitation frequencies ω_2 and ω_1 are to each other, the lower is the frequency of beating. Of course, when $\omega_1=\omega_2$, then $\Omega_{\text{Beat}}=0$.

When ω_2 and ω_1 are not close to each other, the beat frequency becomes unrecognizable and is just a part of the regular response.

Beating can be relatively annoying from an acoustic viewpoint, and occurs, for example, if structures support two or more pieces of rotating machinery whose rotation speeds are not quite synchronized (for example, in aircraft).

The most severe beating in terms of modulation amplitude change occurs when

$$A_1 = A_2 = A \tag{8.21.17}$$

which gives

$$U_3(x,y)_{\max} = 2A \tag{8.21.18}$$

and

$$U_3(x,y)_{\min} = 0 \tag{8.21.19}$$

Beating can also occur after impact on a structure when exciting pairs of natural modes whose natural frequencies are close to each other. It is interesting to note that ancient oriental bells (Korea, China, etc.) were designed to have a beating response, either because of aesthetic or signaling reasons; see Kim et al. (1994).

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9

Dynamic Influence (Green's) Function

The dynamic influence function of a shell describes the response of each point of the shell to a unit impulse applied at some other point. For simple structures, such as transversely vibrating beams, the influence function may be unidirectional. That is, the unit impulse is applied in the transverse direction and the response is in the transverse direction. Also for plates, where the in-plane response is uncoupled from the transverse response for small oscillations, a unidirectional dynamic influence function is applicable to the transverse vibration problem. However, in shell dynamics, coupling between the transverse response and the response in planes tangential to the shell surface has to be considered. Thus a unit impulse applied transversely at a point produces a response in two principal tangential directions as well as in the transverse direction at any point of the shell. The same is true for unit impulses applied tangentially to the shell in the two principal directions. Thus, to be complete, the dynamic influence function for the general shell case has to have nine components. It can be viewed as a field of response vectors due to unit impulse vectors applied at each point of the shell.

Note that the dynamic influence function is a Green's function and is often referred to as the dynamic Green's function of the shell. The following development follows the approach taken in Wilken and Soedel (1971).

9.1. FORMULATION OF THE INFLUENCE FUNCTION

A unit impulse applied at location (α_1^*, α_2^*) at time t^* in the direction of α_1 may be expressed as

$$q_1(\alpha_1, \alpha_2, t) = \frac{1}{A_1 A_2}\delta(\alpha_1 - \alpha_1^*)\delta(\alpha_2 - \alpha_2^*)\delta(t - t^*) \tag{9.1.1}$$

$$q_2(\alpha_1, \alpha_2, t) = 0 \tag{9.1.2}$$

$$q_3(\alpha_1, \alpha_2, t) = 0 \tag{9.1.3}$$

This will produce a displacement response of the shell with the three components

$$u_1(\alpha_1, \alpha_2, t) = G_{11}(\alpha_1, \alpha_2, t; \alpha_1^*, \alpha_2^*, t^*) \tag{9.1.4}$$

$$u_2(\alpha_1, \alpha_2, t) = G_{21}(\alpha_1, \alpha_2, t; \alpha_1^*, \alpha_2^*, t^*) \tag{9.1.5}$$

$$u_3(\alpha_1, \alpha_2, t) = G_{31}(\alpha_1, \alpha_2, t; \alpha_1^*, \alpha_2^*, t^*) \tag{9.1.6}$$

The symbol G_{ij} signifies *Green's function* and represents the response in the i direction at α_1, α_2, t to a unit impulse in the j direction at $\alpha_1^*, \alpha_2^*, t^*$. Equations (8.1.2)–(8.1.5) become

$$L_1(G_{11}, G_{21}, G_{31}) - \lambda\dot{G}_{11} - \rho h\ddot{G}_{11} = -\frac{1}{A_1 A_2}\delta(\alpha_1 - \alpha_1^*)\delta(\alpha_2 - \alpha_2^*)\delta(t - t^*) \tag{9.1.7}$$

$$L_2(G_{11}, G_{21}, G_{31}) - \lambda\dot{G}_{21} - \rho h\ddot{G}_{21} = 0 \tag{9.1.8}$$

$$L_3(G_{11}, G_{21}, G_{31}) - \lambda\dot{G}_{31} - \rho h\ddot{G}_{31} = 0 \tag{9.1.9}$$

Next, applying a unit impulse to the shell at location (α_1^*, α_2^*) at time t^* in the α_2 direction, we obtain G_{12}, G_{22}, G_{32} from the equations

$$L_1(G_{11}, G_{22}, G_{32}) - \lambda\dot{G}_{12} - \rho h\ddot{G}_{12} = 0 \tag{9.1.10}$$

$$L_2(G_{12}, G_{22}, G_{32}) - \lambda\dot{G}_{22} - \rho h\ddot{G}_{22} = -\frac{1}{A_1 A_2}\delta(\alpha_1 - \alpha_1^*)\delta(\alpha_2 - \alpha_2^*)\delta(t - t^*) \tag{9.1.11}$$

$$L_3(G_{12}, G_{22}, G_{32}) - \lambda\dot{G}_{32} - \rho h\ddot{G}_{32} = 0 \tag{9.1.12}$$

We may formulate the equations for G_{13}, G_{23}, G_{33} similarly. All nine equations may be written in short notation:

$$L_i(G_{1j}, G_{2j}, G_{3j}) - \lambda\dot{G}_{ij} - \rho h\ddot{G}_{ij} = -\frac{\delta_{ij}}{A_1 A_2}\delta(\alpha_1 - \alpha_1^*)\delta(\alpha_2 - \alpha_2^*)\delta(t - t^*) \tag{9.1.13}$$

where $i, j = 1, 2, 3$ and where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{9.1.14}$$

Assuming that it is always possible to obtain the natural frequencies and modes of any shell, plate, and so on, we may use modal expansion analysis to find the G_{ij} components. For instance, the solution of Eqs. (9.1.7)–(9.1.9)

$$G_{i1}(\alpha_1, \alpha_2, t; \alpha_1^*, \alpha_2^*, t^*) = \sum_{k=1}^{\infty} \eta_{1k}(t; \alpha_1^*, \alpha_2^*, t^*) U_{ik}(\alpha_1, \alpha_2) \tag{9.1.15}$$

We formulate G_{i2} and G_{i3} similarly. In general, the solution to Eq. (9.1.13) is

$$G_{ij}(\alpha_1, \alpha_2, t; \alpha_1^*, \alpha_2^*, t^*) = \sum_{k=1}^{\infty} \eta_{jk}(t; \alpha_1^*, \alpha_2^*, t^*) U_{ik}(\alpha_1, \alpha_2) \tag{9.1.16}$$

where $i, j = 1, 2, 3$. The term η_{jk} is the modal participation factor of the kth mode due to a unit impulse in the j direction. Note that all influence function components will automatically satisfy all boundary conditions that the natural modes satisfy.

Substituting Eq. (9.1.16) in Eq. (9.1.13) and proceeding in the usual way gives

$$\ddot{\eta}_{jk} + 2\zeta_k \omega_k \dot{\eta}_{jk} + \omega_k^2 \eta_{jk} = F_{jk}(t; \alpha_1^*, \alpha_2^*, t^*) \tag{9.1.17}$$

where

$$F_{jk}(t; \alpha_1^*, \alpha_2^*, t^*) = \frac{1}{\rho h N_k} U_{jk}(\alpha_1^*, \alpha_2^*) \delta(t - t^*) \tag{9.1.18}$$

and where N_k is given by Eq. (8.1.15). Confining ourselves to the subcritical damping case in the following, we solve for η_{jk} and obtain, for $t \geq t^*$,

$$\eta_{jk} = \frac{U(t - t^*)}{\rho h N_k \gamma_k} U_{jk}(\alpha_1^*, \alpha_2^*) e^{-\zeta_k \omega_k (t - t^*)} \sin \gamma_k (t - t^*) \tag{9.1.19}$$

Note that $\eta_{jk} = 0$ when $t < t^*$. To keep track of this causality the unit step function $U(t - t^*)$ was introduced. The dynamic influence function is obtained by substituting Eq. (9.1.19) in Eq. (9.1.16):

$$G_{ij}(\alpha_1, \alpha_2, t; \alpha_1^*, \alpha_2^*, t^*) = \frac{1}{\rho h} \sum_{k=1}^{\infty} \frac{U_{ik}(\alpha_1, \alpha_2) U_{jk}(\alpha_1^*, \alpha_2^*) S(t - t^*)}{N_k} \tag{9.1.20}$$

where for subcritical damping,

$$S(t - t^*) = \frac{1}{\gamma_k} e^{-\zeta_k \omega_k (t - t^*)} \sin \gamma_k (t - t^*) U(t - t^*) \tag{9.1.21}$$

Equation (9.1.20) is also valid for the other damping cases. For critical damping, it is

$$S(t-t^*)=(t-t^*)e^{-\omega_k(t-t^*)}U(t-t^*) \tag{9.1.22}$$

For supercritical damping, it is

$$S(t-t^*)=\frac{1}{\varepsilon_k}e^{-\zeta_k\omega_k(t-t^*)}\sin h\varepsilon_k U(t-t^*) \tag{9.1.23}$$

Note that in Eq. (9.1.20), we may interchange α_1^* and α_2^* and α_1 and α_2 and prove that

$$G_{ij}(\alpha_1,\alpha_2,t;\alpha_1^*,\alpha_2^*,t^*)=G_{ji}(\alpha_1^*,\alpha_2^*,t;\alpha_1,\alpha_2,t^*) \tag{9.1.24}$$

This also follows, of course, from the Maxwell reciprocity theorem.

9.2. SOLUTION TO GENERAL FORCING USING THE DYNAMIC INFLUENCE FUNCTION

From physical reasoning, the response in the i direction has to be equal to the summation in space and time of all loads in the α_1 direction multiplied by G_{i1}, plus all loads in the α_2 direction multiplied by G_{i2}, plus all loads in the normal direction multiplied by G_{i3}. This superposition reasoning leads immediately to the solution integral

$$u_i(\alpha_1,\alpha_2,t) = \int_0^t\int_{\alpha_2}\int_{\alpha_1}\sum_{j=1}^{3}G_{ij}(\alpha_1,\alpha_2,t;\alpha_1^*,\alpha_2^*,t^*)q_j(\alpha_1^*,\alpha_2^*,t^*) \times A_1^*A_2^*\,d\alpha_1^*\,d\alpha_2^*\,dt^* \tag{9.2.1}$$

Let us prove that this is true. Let us substitute Eq. (9.1.20) in Eq. (9.2.1). This gives

$$u_i(\alpha_1,\alpha_2,t) = \sum_{k=1}^{\infty}\frac{U_{ik}(\alpha_1,\alpha_2)}{\rho hN_k}\int_0^t\int_{\alpha_1}\int_{\alpha_2}\sum_{j=1}^{3}U_{jk}(\alpha_1^*,\alpha_2^*) \times q_j(\alpha_1^*,\alpha_2^*,t^*)S(t-t^*)\cdot A_1^*A_2^*\,d\alpha_1^*\,d\alpha_2^*\,dt^* \tag{9.2.2}$$

This expression is identical to the general modal expansion solution for zero initial conditions. From Eqs. (8.1.1),(8.3.2), and (8.3.6), we get

$$u_i(\alpha_1,\alpha_2,t) = \sum_{k=1}^{\infty}\frac{U_{ik}(\alpha_1,\alpha_2)}{\rho hN_k}\int_0^t\int_{\alpha_1}\int_{\alpha_2}\sum_{j=1}^{3}U_{jk}(\alpha_1,\alpha_2)q_j(\alpha_1,\alpha_2,\tau) \times S(t-\tau)A_1A_2\,d\alpha_1\,d\alpha_2\,d\tau \tag{9.2.3}$$

where $S(t-\tau)$ is identical to $S(t-t^*)$except that t^* is replaced by τ. However, we recognize that the integration variables are interchangeable.

We may set $\alpha_1 = \alpha_1^*$, $\alpha_2 = \alpha_2^*$, $\tau = t^*$ without changing the result of the integration. This proves that Eqs. (9.2.2) and (9.2.3) are identical and that Eq. (9.2.1) is the general solution in terms of the dynamic influence function.

9.3. REDUCED SYSTEMS

The total definition of the dynamic influence function for a shell requires in general nine components. For reduced systems, we do not need as many because of the uncoupling of the governing equations. For the in-plane vibration of a plate, we need only

$$[G_{ij}] = \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9.3.1}$$

and for the transverse vibration of a plate

$$[G_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G_{33} \end{bmatrix} \tag{9.3.2}$$

The components G_{13}, G_{23}, G_{31}, G_{32} do not exist because an excitation in the α_1 or α_2 direction does not produce a response in the transverse direction, and vice versa, at least according to the linear approximation.

This means that for in-plane vibration,

$$G_{ij} = \frac{1}{\rho h} \sum_{k=1}^{\infty} \frac{1}{N_{12k}} U_{jk}(\alpha_1^*, \alpha_2^*) U_{jk}(\alpha_1, \alpha_2) S(t - t^*) \tag{9.3.3}$$

where $i, j = 1, 2$ and

$$N_{12k} = \int_{\alpha_2} \int_{\alpha_1} (U_{1k}^2 + U_{2k}^2) A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2 \tag{9.3.4}$$

The solution for a general load is

$$\begin{aligned} u_i = \int_0^t \int_{\alpha_2} \int_{\alpha_1} & [G_{i1} q_1(\alpha_1^*, \alpha_2^*, t^*) \\ & + G_{i2} q_2(\alpha_1^*, \alpha_2^*, t^*)] A_1^* A_2^* \,\mathrm{d}\alpha_1^* \,\mathrm{d}\alpha_2^* \,\mathrm{d}t^* \end{aligned} \tag{9.3.5}$$

For transverse vibrations

$$G_{33} = \frac{1}{\rho h} \sum_{k=1}^{\infty} \frac{1}{N_{3k}} U_{3k}(\alpha_1^*, \alpha_2^*) U_{3k}(\alpha_1, \alpha_2) S(t - t^*) \tag{9.3.6}$$

where

$$N_{3k} = \int_{\alpha_2} \int_{\alpha_1} U_{3k}^2 A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2 \tag{9.3.7}$$

The solution for general transverse loads is

$$u_3 = \int_0^t \int_{\alpha_2} \int_{\alpha_1} G_{33} q_3(\alpha_1^*, \alpha_2^*, t^*) A_1^* A_2^* \mathrm{d}\alpha_1^* \mathrm{d}\alpha_2^* \mathrm{d}t^* \tag{9.3.8}$$

Another interesting case is the ring, where we have

$$[G_{ij}] = \begin{bmatrix} G_{11} & 0 & G_{13} \\ 0 & 0 & 0 \\ G_{31} & 0 & G_{33} \end{bmatrix} \tag{9.3.9}$$

where

$$G_{ij} = \frac{1}{\rho h} \sum_{k=1}^{\infty} \frac{1}{N_k} U_{jk}(\theta^*) U_{ik}(\theta) S(t - t^*) \tag{9.3.10}$$

$$N_k = ba \int_0^{\alpha} (U_{1k}^2 + U_{3k}^2) \mathrm{d}\theta \tag{9.3.11}$$

and where b is the width of the ring, a is the radius, and α defines the size of the segment. The general solution is

$$u_i = ba \int_0^t \int_0^{\alpha} [G_{i1} q_1(\theta^*, t^*) + G_{i3} q_3(\theta^*, t^*)] \mathrm{d}\theta^* \mathrm{d}t^* \tag{9.3.12}$$

9.4. DYNAMIC INFLUENCE FUNCTION FOR THE SIMPLY SUPPORTED SHELL

For the case treated in Chapter 5 that has simply supported ends and no axial end restraints the natural modes are ($\alpha_1 = x$, $\alpha_2 = \theta$)

$$U_{1k}(x, \theta) = A_{mnp} \cos \frac{m\pi x}{L} \cos n(\theta - \phi) \tag{9.4.1}$$

$$U_{2k}(x, \theta) = B_{mnp} \sin \frac{m\pi x}{L} \sin n(\theta - \phi) \tag{9.4.2}$$

$$U_{3k}(x, \theta) = C_{mnp} \sin \frac{m\pi x}{L} \cos n(\theta - \phi) \tag{9.4.3}$$

where $m = 1, 2, \ldots$, $n = 0, 1, 2, \ldots$, and $p = 1, 2, 3$. The index k implies again a certain combination of m, n, and p. We remember from Sec. 5.5 that for any m, n combination there are three natural frequencies and mode combinations.

The angle ϕ is again arbitrary and indicates the nonpreferential nature of the modes of free vibration of the closed axisymmetric shell. For the sake of identifying orthogonal modes that can be used to define the response as function of θ, one set of modes may be formed by letting $\phi = 0$ and a second set by letting $n\phi = \pi/2$.

The $n=0$ modes are axisymmetric. For $n=0$ and $\phi=0$, only the mode components that have axial and transverse motion exist, namely U_{1k} and U_{3k}. For $n=0$ and $n\phi=\pi/2$, this is reversed and only U_{2k} exists. The value of N_k becomes

$$N_k = N_{mnp} = \begin{cases} \dfrac{aL\pi}{2}(A_{mnp}^2 + B_{mnp}^2 + C_{mnp}^2), & n \neq 0 \\ aL\pi(A_{m0p}^2 + C_{m0p}^2), & n=0,\ \phi=0 \\ aL\pi B_{m0p}^2, & n=0,\ \phi=\pi/2n \end{cases} \tag{9.4.4}$$

Components of the influence function may be found simply by substituting in Eq. (9.1.20). For example, the component describing the transverse response to a transverse impulse is

$$\begin{aligned} &G_{33}(x,\theta,t;x^*,\theta^*,t^*) \\ &\quad = \frac{1}{\rho h}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{p=1}^{3}\sum_{n\phi=0,\pi/2}\frac{1}{N_{mnp}}C_{mnp}^2 \sin\frac{m\pi x}{L}\sin\frac{m\pi x^*}{L} \\ &\qquad \times \cos n(\theta-\phi)\cos n(\theta^*-\phi)S(t-t^*) \end{aligned} \tag{9.4.5}$$

The angle ϕ and its associated summation may be eliminated from Eq. (9.4.5) and from the other components of this influence function by noting that each component contains the sum of two products of sine or cosine functions when the summation over ϕ is written out. For example, G_{33} contains the term

$$\cos n\theta \cos n\theta^* + \cos\left(n\theta - \frac{\pi}{2}\right)\cos\left(n\theta^* - \frac{\pi}{2}\right) \tag{9.4.6}$$

This term may be reduced by trigonometric identities to $\cos n(\theta-\theta^*)$. For G_{32}, for instance, the corresponding term (and its reduction) is

$$\cos n\theta \sin n\theta^* + \cos\left(n\theta - \frac{\pi}{2}\right)\sin\left(n\theta^* - \frac{\pi}{2}\right) = \sin n(\theta^* - \theta) \tag{9.4.7}$$

On the other hand, for G_{23}, the corresponding term is

$$\sin n\theta \cos n\theta^* + \sin\left(n\theta - \frac{\pi}{2}\right)\cos\left(n\theta^* - \frac{\pi}{2}\right) = -\sin n(\theta^* - \theta) \tag{9.4.8}$$

The three-directional dynamic influence function for a thin cylindrical shell with simply supported ends is then

$$G_{ij} = \frac{1}{\rho h}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{p=1}^{3}\frac{1}{N_{mnp}}A_{ij}S(t-t^*) \tag{9.4.9}$$

where

$$\begin{aligned} A_{11} &= A_{mnp}^2 \cos\frac{m\pi x}{L}\cos\frac{m\pi x^*}{L}\cos n(\theta-\theta^*) \\ A_{12} &= A_{mnp}B_{mnp}\cos\frac{m\pi x}{L}\sin\frac{m\pi x^*}{L}\sin n(\theta^*-\theta) \end{aligned}$$

$$
\begin{aligned}
A_{13} &= A_{mnp}C_{mnp}\cos\frac{m\pi x}{L}\sin\frac{m\pi x^*}{L}\cos n(\theta-\theta^*)\\
A_{21} &= B_{mnp}A_{mnp}\sin\frac{m\pi x}{L}\cos\frac{m\pi x^*}{L}\sin n(\theta-\theta^*)\\
A_{22} &= B^2_{mnp}\sin\frac{m\pi x}{L}\sin\frac{m\pi x^*}{L}\cos n(\theta-\theta^*)\\
A_{23} &= B_{mnp}C_{mnp}\sin\frac{m\pi x}{L}\sin\frac{m\pi x^*}{L}\sin n(\theta-\theta^*)\\
A_{31} &= C_{mnp}A_{mnp}\sin\frac{m\pi x}{L}\cos\frac{m\pi x^*}{L}\cos n(\theta-\theta^*)\\
A_{32} &= C_{mnp}B_{mnp}\sin\frac{m\pi x}{L}\sin\frac{m\pi x^*}{L}\sin(\theta^*-\theta)\\
A_{33} &= C^2_{mnp}\sin\frac{m\pi x}{L}\sin\frac{m\pi x^*}{L}\cos n(\theta-\theta^*)
\end{aligned}
\tag{9.4.10}
$$

and where for subcritical damping, for instance, $S(t-t^*)$ is given by Eq. (9.1.21). The solution to any other type of loading is now given by the integral of Eq. (9.2.1). Note that

$$A_{ij}(x,\theta;x^*,\theta^*)=A_{ji}(x^*,\theta^*;x,\theta) \tag{9.4.11}$$

This verifies Eq. (9.1.24).

9.5. DYNAMIC INFLUENCE FUNCTION FOR THE CLOSED CIRCULAR RING

If we assume that the ring deforms only in its plane, the modes obtained in Sec. 5.3 apply. They are $(\alpha_1=\theta, A_1=a, k=n)$

$$U_{1k}(\theta) = V_{np}\sin n(\theta-\phi) \tag{9.5.1}$$

$$U_{3k}(\theta) = W_{np}\cos n(\theta-\phi) \tag{9.5.2}$$

where $n=0,1,2,\ldots$ and $p=1,2$, and where k implies a combination of n and p. As seen in Sec. 5.3, for any n number, there exist two separate natural frequencies and mode combinations. In one case, the motion is primarily circumferential and in the other case primarily transversal.

According to Eq. (9.3.11), we obtain

$$N_k=N_{np}=\begin{cases} ab\pi(V^2_{np}+W^2_{np}), & n\neq 0\\ 2ab\pi W^2_{0p}, & n=0,\quad \phi=0\\ 2ab\pi V^2_{0p}, & n=0,\quad \phi=\pi/2n \end{cases} \tag{9.5.3}$$

The four components of the dynamic influence function become $(i, j=1,3)$

$$G_{ij} = \frac{1}{\rho h} \sum_{n=0}^{\infty} \sum_{p=1}^{2} \frac{1}{N_{np}} A_{ij} S(t-t^*) \tag{9.5.4}$$

where

$$\begin{aligned} A_{11} &= V_{np}^2 \cos n(\theta-\theta^*) \\ A_{13} &= V_{np} W_{np} \sin n(\theta-\theta^*) \\ A_{31} &= V_{np} W_{np} \sin n(\theta^*-\theta) \\ A_{33} &= W_{np}^2 \cos n(\theta-\theta^*) \end{aligned} \tag{9.5.5}$$

This dynamic influence function automatically includes the rigid rotating mode, which is described by $n=0$ and $n\phi=\pi/2$ and also the rigid translating mode, which is described by $n=1$ and $n\phi=0$.

9.6. TRAVELING POINT LOAD ON SIMPLY SUPPORTED CYLINDRICAL SHELL

In general, the advantage of the dynamic influence function in the analysis of structures is that once the function is known, the structure is defined and an infinite variety of loading combinations can be treated by a relatively simply integration process.

One application for which the dynamic influence function is particularly useful is the traveling load. Early investigations of traveling loads were made by Krylov and Timoshenko, see Timoshenko (1953), who were concerned with the response of railroad bridges to traveling locomotives. The problem surfaced in Russia before World War I when trains started to travel with high speeds over bridges that were designed for static loads only. Kryloff and Timoshenko used modal expansion without using an influence function approach, but the formulation of such a problem in terms of the dynamic influence function is particularly simple. One of the first to use this technique was Cottis (1965), who calculated the response of a shell to a traveling pressure wave.

Let us as example assume that a force of constant magnitude travels along a $\theta=\psi$ line in the positive x direction with constant velocity v, touching the shell surface at $(x,\theta)=(0,\psi)$ at $t=0$ and leaving it at $(x,\theta)=(L,\psi)$ at $t=L/v$. This is sketched in Fig. 1 Physically, this is a very rough approximation of the action of a piston on a cylinder liner.

Since this example falls into the category of loads moving along a coordinate line, we may express the load as

$$q_1 = q_2 = 0 \tag{9.6.1}$$

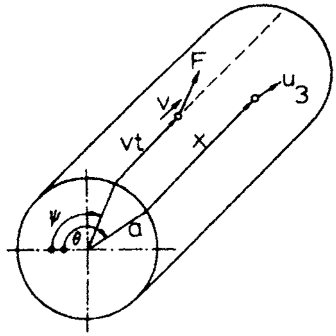

FIG. 1 Circular cylindrical shell with a constant force F that travels in axial direction.

$$q_3(x,\theta,t)=\frac{F}{a}\delta(\theta-\psi)\delta(x-vt)[1-U(vt-L)] \tag{9.6.2}$$

since the instantaneous position is $x=vt$ in this case. The solution is given by Eq. (9.2.1), with G_{ij} for the simply supported shell given by Eq. (9.4.9). For zero initial conditions, this results in

$$u_i(x,\theta,t) = \frac{F}{\rho h}\sum_{k=1}^{\infty}\int_0^t\int_0^L\int_0^{2\pi}\frac{1}{N_k}A_{i3}S(t-t^*)\delta(\theta^*-\psi)\delta(x^*-vt^*) \times[1-U(vt^*-L)]\mathrm{d}\theta^*\mathrm{d}x^*\mathrm{d}t^* \tag{9.6.3}$$

This gives

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{F}{\rho h}\sum_{k=1}^{\infty}\frac{1}{\gamma_k N_k}\begin{Bmatrix} A_kC_k\cos\frac{m\pi x}{L} \\ B_kC_k\sin\frac{m\pi x}{L} \\ C_k^2\sin\frac{m\pi x}{L} \end{Bmatrix}\int_0^{2\pi}\begin{Bmatrix} \cos n(\theta-\theta^*) \\ \sin n(\theta-\theta^*) \\ \cos n(\theta-\theta^*) \end{Bmatrix} \times\delta(\theta^*-\psi)\mathrm{d}\theta^*\int_0^t[1-U(vt^*-L)]\mathrm{e}^{-\zeta_k\omega_k(t-t^*)} \times\sin\gamma_k(t-t^*)\int_0^L\sin\frac{m\pi x^*}{L}\delta(x^*-vt^*)\mathrm{d}x^*\mathrm{d}t^* \tag{9.6.4}$$

Evaluating the integrals one by one results in

$$\int_0^{2\pi}\begin{Bmatrix} \cos n(\theta-\theta^*) \\ \sin n(\theta-\theta^*) \\ \cos n(\theta-\theta^*) \end{Bmatrix}\delta(\theta^*-\psi)\mathrm{d}\theta^*=\begin{Bmatrix} \cos n(\theta-\psi) \\ \sin n(\theta-\psi) \\ \cos n(\theta-\psi) \end{Bmatrix} \tag{9.6.5}$$

$$\int_0^L\sin\frac{m\pi x^*}{L}\delta(x^*-vt^*)\mathrm{d}x^*=\sin\frac{m\pi vt^*}{L} \tag{9.6.6}$$

Let us now solve the time integral. If $0 \le vt < L$, the gate function is $[1-U(vt-L)]=1$ and the integral becomes

$$\int_{t^*=0}^{t} F(t^*,t)[1-U(vt^*-L)]\,\mathrm{d}t^* = \int_{t^*=0}^{t} F(t^*,t)\,\mathrm{d}t^* \tag{9.6.7}$$

If $vt \ge L$, the gate function is 0 and the integral becomes

$$\int_{t^*=0}^{t} F(t^*,t)[1-U(vt^*-L)]\,\mathrm{d}t^* = \int_{t^*=0}^{L/v} F(t^*,t)\,\mathrm{d}t^* \tag{9.6.8}$$

Thus, we get, for the time the load is on the shell,

$$\begin{aligned} I_k(t) &= \int_0^t \mathrm{e}^{-a_k(t-t^*)} \sin\gamma_k(t-t^*)\sin\alpha t^*\,\mathrm{d}t^* \\ &= \frac{1}{2[a_k^4+2a_k^2(\alpha^2+\gamma_k^2)+(\alpha^2-\gamma_k^2)^2]} \\ &\quad \times\{[\mathrm{e}^{-\alpha_k t}(\xi_k\sin\gamma_k t+a_k\cos\gamma_k t)-a_k\cos\alpha t-\xi_k\sin\alpha t](a_k^2+\eta_k^2) \\ &\quad +[0\mathrm{e}^{-\alpha_k t}(\eta_k\sin\gamma_k t-a_k\cos\gamma_k t)+a_k\cos\alpha t+\eta_k\sin\alpha t](a_k^2+\xi_k^2)\} \end{aligned} \tag{9.6.9}$$

where

$$\alpha = \frac{m\pi v}{L} \tag{9.6.10}$$

$$a_k = \zeta_k\omega_k \tag{9.6.11}$$

$$\xi_k = \alpha-\gamma_k \tag{9.6.12}$$

$$\eta_k = \alpha+\gamma_k \tag{9.6.13}$$

During the time the load is traveling on the shell ($0 \le vt < L$), the response of the shell is

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{F}{\rho h}\sum_{k=1}^{\infty}\frac{1}{\gamma_k N_k}\begin{Bmatrix} A_k C_k \cos\dfrac{m\pi x}{L}\cos n(\theta-\psi) \\ B_k C_k \sin\dfrac{m\pi x}{L}\sin n(\theta-\psi) \\ C_k^2 \sin\dfrac{m\pi x}{L}\cos n(\theta-\psi) \end{Bmatrix} I_k(t) \tag{9.6.14}$$

When the load has left the shell $(L \le vt), I_k(t)$ is replaced by $I_k(L/v,t)$

$$I_k(L/v,t) = \int_{t^*=0}^{L/v} \mathrm{e}^{-a_k(t-t^*)}\sin\gamma_k(t-t^*)\sin\alpha t^*\,\mathrm{d}t^* \tag{9.6.15}$$

This integral will give a decaying motion of the shell and will not be evaluated here, since it is of no particular interest.

What is of more interest is what happens as the load is traversing the shell. We see from the denominator of $I_k(t)$ that we will have a developing resonance whenever

$$\alpha = \gamma_k \tag{9.6.16}$$

which means that all traversing velocities, setting $\gamma_k \approx \omega_k$ for small damping,

$$v = \frac{L\omega_k}{m\pi} \tag{9.6.17}$$

have to be avoided.

Another way of looking at it is to recognize that the time it takes for the moving load to traverse the shell is

$$T = \frac{L}{v} \tag{9.6.18}$$

and that α can be looked upon as an excitation frequency. Thus the period of excitation is

$$T_\alpha = \frac{2}{m} T \tag{9.6.19}$$

This means that if it takes the load T seconds to traverse the shell, the periods of possible resonance are $2T, T, 2/3T, 2/4T$, and so on, provided that any of the resonance periods

$$T_k = \frac{2\pi}{\omega_k} \tag{9.6.20}$$

agree with these values.

9.7. POINT LOAD TRAVELING AROUND A CLOSED CIRCULAR CYLINDRICAL SHELL IN CIRCUMFERENTIAL DIRECTION

This case will demonstrate another interesting resonance phenomenon. The traveling load is described in this case by

$$q_1 = q_2 = o \tag{9.7.1}$$

$$q_3(x, \theta, t) = \frac{F}{a} \delta(x - \xi) \delta(\theta - \Omega t) \tag{9.7.2}$$

where Ω is the angular velocity of load travel. The load travels continuously around the shell as shown in Fig. 2. For zero initial conditions, we obtain

$$u_i(x, \theta, t) = \frac{F}{ph} \sum_{k=1}^{\infty} \int_0^t \int_o^L \int_0^{2\pi} \frac{1}{N_{mnp}} A_{i3} S(t - t^*) \delta(x^* - \xi) \times \delta(\theta^* - \Omega t^*) \mathrm{d}\theta^* \mathrm{d}x^* \mathrm{d}t^* \tag{9.7.3}$$

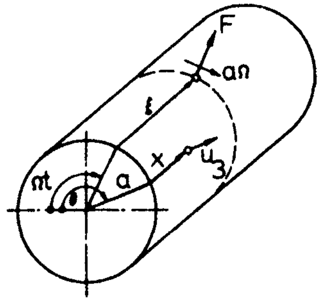

Fig. 2 Circular cylindrical shell with a constant force F that travels in circumferential direction.

This gives, with A_{13}, A_{23}, and A_{33} given by Eq. (9.4.10),

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{F}{\rho h} \sum_{k=1}^{\infty} \frac{1}{\gamma_k N_k} \begin{Bmatrix} A_k C_k \cos \dfrac{m\pi x}{L} \\ B_k C_k \sin \dfrac{m\pi x}{L} \\ C_k^2 \sin \dfrac{m\pi x}{L} \end{Bmatrix} \sin \frac{m\pi \xi}{L}$$

$$\int_0^t e^{-a_k(t-t^*)} \sin \gamma_k (t-t^*) \begin{Bmatrix} \cos n(\theta - \Omega t^*) \\ \sin n(\theta - \Omega t^*) \\ \cos n(\theta - \Omega t^*) \end{Bmatrix} dt^* \tag{9.7.4}$$

Let us single out for further discussion the transverse response u_3. The integral becomes

$$\begin{aligned} J_k(t) &= \int_0^t e^{-a_k(t-t^*)} \sin \gamma_k (t-t^*) \cos n(\theta - \Omega t^*) dt^* \\ &= Y_k \cos[n(\theta - \Omega t) - \phi_k] + e^{-a_k t} T_k(t) \end{aligned} \tag{9.7.5}$$

where

$$Y_k = \frac{\sqrt{a_k^2(\xi_k^2 - \eta_k^2)^2 + [\eta_k(a_k^2 + \xi_k^2) - \xi_k(a_k^2 + \eta_k^2)]^2}}{2[a_k^4 + 2a_k^2(n^2\Omega^2 + \gamma_k^2) + (n^2\Omega^2 - \gamma_k^2)^2]} \tag{9.7.6}$$

$$\phi_k = \tan^{-1} \frac{a_k(\xi_k^2 - \eta_k^2)}{\eta_k(a_k^2 + \xi_k^2) - \xi_k(a_k^2 + \eta_k^2)} \tag{9.7.7}$$

$$\begin{aligned} T_k(t) = {} & \frac{a_k \sin(n\theta - \gamma_k t) + \xi_k \cos(n\theta - \gamma_k t)}{2(a_k^2 + \xi_k^2)} \\ & - \frac{a_k \sin(n\theta + \gamma_k t) + \eta_k \cos(n\theta + \gamma_k t)}{2(a_k^2 + \eta_k^2)} \end{aligned} \tag{9.7.8}$$

and where

$$\xi_k = n\Omega - \gamma_k \tag{9.7.9}$$

$$\eta_k = n\Omega + \gamma_k \tag{9.7.10}$$

$$a_k = \xi_k \omega_k \tag{9.7.11}$$

The transient term due to the sudden onset of travel at $t = 0$ disappears with time. Of primary interest is the steady-state part of the solution. Examining the denominator of the response amplitude Y_k, we observe a resonance condition that exists whenever

$$\Omega = \frac{\gamma_k}{n} \tag{9.7.12}$$

For small damping, $\gamma_k \approx \omega_k$. Therefore, the first critical speed Ω_c occurs when

$$\Omega_c = \left(\frac{\omega_k}{n}\right)_{\min} \tag{9.7.13}$$

This can be illustrated, for example, by the shell described in Sec. 5.5, for which natural frequencies were obtained. The lowest set of ω_k occurs when $m = 1$. Thus we plot ω_{ln}/n as a function of n in Fig. 3 and obtain, as the minimum value, $\Omega_c = 1800$ rad/s at $n = 5$. This is the rotational speed at which the first resonance occurs. Below this speed, we have no resonance. Above this speed, other resonances occur.

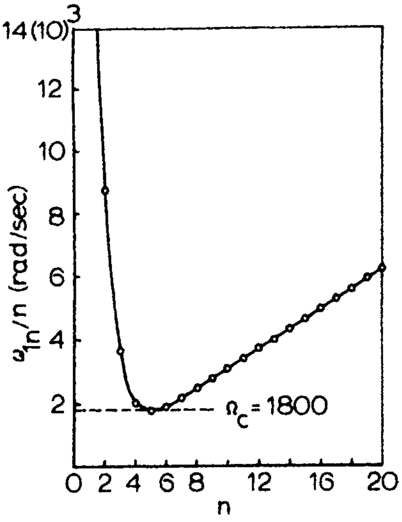

Fig. 3 Illustration of the critical speed of a constant force that travels with constant velocity in circumferential direction.

Another interesting result is indicated by Eq. (9.7.5). An observer who travels alongside the traveling load will, in steady state, see standing wave forms only and will not see an oscillation. These waves are quasi-static with respect to the load, but an observer located on the shell surface will experience oscillations.

These results can be shown to be true for all closed shells of revolution when loads travel in a circumferential direction. In Soedel (1975) these effects were shown to exist for an automobile tire rolling on a smooth surface. The tire was treated as a shell of revolution, nonhomogeneous and nonisentropic. The contact-region pressure resultant was in this case the traveling load. Critical rolling speeds for typical passenger car tires were predicted to occur at about 140 km/h, which agreed well with experimental observation.

Note also that when the critical speed is reached, the mode that dominates the response is not the simplest mode (by *simplest* the $n=0$ or $n=1$ mode is meant), but for the example treated here, is the $n=5$ mode, in combination with neighboring modes. The wave crest is not necessarily directly under the traveling load but may lag or lead the load depending on the damping and the modal participation.

A physical interpretation of the critical speed formula is given by Fig. 4. If we consider as an example the $n=3$ mode, we sense intuitively that the shell will go into resonance if the load can travel from point A to point B (the distance AB is the wavelength λ) in the same time that it takes the mode to go through one oscillation. The time of travel from A to B is

$$T=\frac{\lambda}{\Omega a} \tag{9.7.14}$$

However, the wavelength is given by

$$\lambda=\frac{2\pi a}{n} \tag{9.7.15}$$

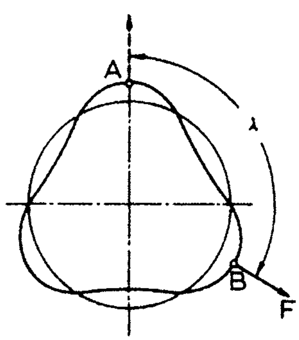

FIG. 4 Physical interpretation of the critical speed resonance.

Thus

$$T=\frac{2\pi}{n\Omega} \tag{9.7.16}$$

Equating this with the period of oscillation,

$$T=\frac{2\pi}{\omega_k} \tag{9.7.17}$$

gives as the resonance condition

$$\Omega=\frac{\omega_k}{n} \tag{9.7.18}$$

9.8. STEADY-STATE HARMONIC GREEN'S FUNCTION

In many applications, it is an advantage to formulate the steady-state response at one point due to a harmonic point load at another. This concept is similar to that of the true Green's or influence function, except that one is dealing now with a harmonic influence function. The loading due to a harmonic unit point force in the j direction can in general be described by use of the Kronecker delta:

$$q_i(\alpha_1,\alpha_2,t)=\frac{\delta_{ij}}{A_1A_2}\delta(\alpha_1-\alpha_1^*)\delta(\alpha_2-\alpha_2^*)\mathrm{e}^{j\omega t} \tag{9.8.1}$$

This will produce a steady-state harmonic displacement response of the shell that can be written

$$u_i(\alpha_1,\alpha_2,t)=T_{ij}(\alpha_1,\alpha_2;\alpha_1^*,\alpha_2^*,\omega)\mathrm{e}^{j\omega t} \tag{9.8.2}$$

where T_{ij} will be a complex number if damping is present in the shell because of the expected phase lag. The symbol $T_{ij}(\alpha_1,\alpha_2;\alpha_1^*,\alpha_2^*,\omega)$ signifies a steady-state harmonic response function and represents the complex response amplitude in the i direction at location α_1,α_2 due to a harmonic unit point load at frequency ω in the j direction at locations α_1^* and α_2^*.

By modal expansion, the solution is given by

$$u_i(\alpha_1,\alpha_2,t)=\sum_{k=1}^{\infty}\eta_{ik}(t)U_{ik}(\alpha_1,\alpha_2) \tag{9.8.3}$$

where, from Sec. 8.5,

$$\eta_{ik}=\Lambda_k\mathrm{e}^{j(\omega t-\phi_k)} \tag{9.8.4}$$

and where

$$\Lambda_k=\frac{F_k^*}{\omega_k^2\sqrt{[1-(\omega/\omega_k)^2]^2+4\zeta_n^2(\omega/\omega_k)^2}} \tag{9.8.5}$$

$$\phi_k = \tan^{-1}\frac{2\zeta_k(\omega/\omega_k)}{1-(\omega/\omega_k)^2} \tag{9.8.6}$$

$$F_k^* = \frac{1}{\rho h N_k} U_{jk}(\alpha_1^*, \alpha_2^*) \tag{9.8.7}$$

This means that

$$T_{ij}(\alpha_1, \alpha_2; \alpha_1^*, \alpha_2^*, \omega) = \frac{1}{\rho h}\sum_{k=1}^{\infty}\frac{U_{ik}(\alpha_1, \alpha_2)U_{jk}(\alpha_1^*, \alpha_2^*)\mathrm{e}^{-j\phi_k}}{N_k\omega_k^2\sqrt{[1-(\omega/\omega_k)^2]^2+4\zeta_k^2(\omega/\omega_k)^2}} \tag{9.8.8}$$

It is sometimes of advantage to plot the amplitude of T_{ij} as a function of the excitation frequency ω, thus obtaining the transfer function response spectrum.

The steady-state harmonic Green's function of a shell has in general nine components, just as does the true dynamic Green's function. To utilize the harmonic Green's function for general distributed harmonic forcing, the harmonic Green's function is multiplied by the loading and the result is integrated in space. Specifically, since the load is (note that the j in $j\omega t$ is not the same as the subscript j)

$$q_j(\alpha_1, \alpha_2, t) = q_j^*(\alpha_1, \alpha_2)\mathrm{e}^{j\omega t}, \quad j = 1, 2, 3 \tag{9.8.9}$$

and the harmonic transfer function (harmonic Green's function) is known, the steady-state solution is

$$u_i(\alpha_1, \alpha_2, t) = \left[\int_{\alpha_1}\int_{\alpha_1}\sum_{j=1}^{3} T_{ij}(\alpha_1, \alpha_2; \alpha_1^*, \alpha_2^*, \omega)q_j(\alpha_1^*, \alpha_2^*)A_1^*A_2^*\mathrm{d}\alpha_1^*\mathrm{d}\alpha_2^*\right]\mathrm{e}^{j\omega t} \tag{9.8.10}$$

The steady-state harmonic Green's function will appear again in Chapter 13, under the name *receptance*. For other applications of Green's functions, see also Greenberg (1971).

9.9. RECTANGULAR PLATE EXAMPLES

9.9.1. Dynamic Green's Function of a Simply Supported Plate

For the simply supported plate, the dynamic Green's function is given by Eq. (9.3.6):

$$G_{33}(x, y, t; x^*, y^*, t^*) = \frac{1}{\rho h}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{N_{mn}}U_{3mn}(x^*, y^*)U_{3mn}(x, y)S(t-t^*) \tag{9.9.1}$$

where

$$U_{3mn}(x,y)=A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{9.9.2}$$

$$U_{3mn}(x^*,y^*)=A_{mn}\sin\frac{m\pi x^*}{a}\sin\frac{m\pi y^*}{b} \tag{9.9.3}$$

$$N_{mn}=\int_{y=0}^{b}\int_{x=0}^{a}A_{mn}^2\sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)\mathrm{d}x\,\mathrm{d}y=\frac{ab}{4}A_{mn}^2 \tag{9.9.4}$$

and where, for subcritical damping, Eq. (9.1.21) applies:

$$S(t-t^*)=\frac{1}{\gamma_{mn}}\mathrm{e}^{-\zeta_{mn}\omega_{mn}(t-t^*)}\sin\gamma_{mn}(t-t^*),\quad t\geq t_1^* \tag{9.9.5}$$

The natural frequencies are, from Eq. (5.4.71),

$$\omega_{mn}=\pi^2\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}} \tag{9.9.6}$$

and, from Eq. (8.3.5),

$$\gamma_{mn}=\omega_{mn}\sqrt{1-\zeta_{mn}^2} \tag{9.9.7}$$

where

$$\zeta_{mn}=\frac{\lambda}{2\rho h\omega_{mn}} \tag{9.9.8}$$

Equation (9.9.1) becomes

$$G_{33}(x,y,t;x^*,y^*,t^*)=\frac{4}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x^*}{a}\sin\frac{n\pi y^*}{b}$$
$$\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{e^{-\zeta_{mn}\omega_{mn}(t-t^*)}}{\gamma_{mn}}\sin\gamma_{mn}(t-t^*)U(t-t^*) \tag{9.9.9}$$

The arbitrary modal amplitudes A_{mn} have canceled as expected.

9.9.2. Response to an Impacting Mass

If a mass strikes the plate at time t_1 at x_1, y_1 with a velocity v_1 and rebounds in opposite direction with a velocity v_2 (assuming that this was measured), then the change of momentum is

$$\Delta(mv)=mv_1-m(-v_2)=m(v_1+v_2) \tag{9.9.10}$$

Therefore, the loading is

$$q_3(x,y,t)=m(v_1+v_2)\delta(x-x_1)\delta(y-y_1)\delta(t-t_1) \tag{9.9.11}$$

Equation (9.3.8) becomes

$$u_3(x,y,t) = \int_{t^*=0}^{t}\int_{x^*=0}^{a}\int_{y^*=0}^{b} G_{33}(x,y,t;x^*,y^*,t^*)m(v_1+v_2) \times\delta(x^*-x_1)\delta(y^*-y_1)\delta(t^*-t_1)\,\mathrm{d}x^*\,\mathrm{d}y^*\,\mathrm{d}t^* \tag{9.9.12}$$

or

$$u_3(x,y,t)=m(v_1+v_2)G_{33}(x,y,t;x_1,y_1,t_1) \tag{9.9.13}$$

or finally

$$u_3(x,y,t) = \frac{4m(v_1+v_2)}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \times\frac{\mathrm{e}^{-\zeta_{mn}\omega_{mn}(t-t_1)}}{\gamma_{mn}}\sin\gamma_{mn}(t-t_1), \quad t\geq t_1 \tag{9.9.14}$$

9.9.3. Response to a Uniformly Distributed Pressure that is Suddenly Applied

In this example, the uniform pressure loading $Q\left(\mathrm{N/m^2}\right)$ that is suddenly applied at $t=t_1$ is described by

$$q_3(x,y,t)=QU(t-t_1) \tag{9.9.15}$$

Equation (9.3.8) becomes

$$\begin{aligned} u_3(x,y,t) &= \int_{t^*=0}^{t}\int_{x^*=0}^{a}\int_{y^*=0}^{b} G_{33}(x,y,t;x^*,y^*,t^*)QU(t^*-t_1)\,\mathrm{d}x^*\,\mathrm{d}y^*\,\mathrm{d}t^* \\ &= \frac{4Q}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[\int_{x^*=0}^{a}\sin\frac{m\pi x^*}{a}\,\mathrm{d}x^*\right] \\ &\quad\times\left[\int_{y^*=0}^{b}\sin\frac{n\pi y^*}{b}\,\mathrm{d}y^*\right]\frac{1}{\gamma_{mn}}\int_{t^*=0}^{t}U(t^*-t_1)\mathrm{e}^{-\zeta_{mn}\omega_{mn}(t-t^*)} \\ &\quad\times\sin\gamma_{mn}(t-t^*)\,\mathrm{d}t^*, \quad t\geq t_1 \end{aligned} \tag{9.9.16}$$

The product of the spatial integrals becomes

$$\left[\int_{x^*=0}^{a}\sin\frac{m\pi x^*}{a}\,\mathrm{d}x^*\right]\left[\int_{y^*=0}^{b}\sin\frac{n\pi y^*}{b}\,\mathrm{d}y^*\right] = \frac{ab}{mn\pi^2}(1-\cos m\pi)(1-\cos n\pi) \tag{9.9.17}$$

which means that only symmetric modes participate in the solution. ($m=1,3,5,\ldots$and $n=1,3,5,\ldots$).

The time integral becomes

$$\int_{t^*=0}^{t} U(t^*-t_1)\mathrm{e}^{-\zeta_{mn}\omega_{mn}(t-t^*)}\sin\gamma_{mn}(t-t^*)\,\mathrm{d}t^*$$
$$=\int_{t^*=t_1}^{t} e^{-\zeta_{mn}\omega_{mn}(t-t^*)}\sin\gamma_{mn}(t-t^*)\,\mathrm{d}t^*$$
$$=\frac{1}{\gamma_{mn}}\Big\{(1-\zeta_{mn}^2)-\sqrt{1-\zeta_{mn}^2}\,\mathrm{e}^{-\zeta_{mn}(t-t_1)}$$
$$\times\cos[\gamma_{mn}(t-t_1)-\phi'_{mn}]\Big\},\quad t\geq t_1 \tag{9.9.18}$$

where

$$\phi'_{mn}=\tan^{-1}\frac{\zeta_{mn}}{\sqrt{1-\zeta_{mn}^2}} \tag{9.9.19}$$

Therefore, the solution becomes

$$u_3(x,y,t)\;=\frac{4Q}{\rho h\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1-\cos m\pi)(1-\cos n\pi)}{mn\gamma_{mn}^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
$$\times\Big\{(1-\zeta_{mn}^2)-\sqrt{1-\zeta_{mn}^2}\,\mathrm{e}^{-\zeta_{mn}(t-t_1)}$$
$$\times\cos[\gamma_{mn}(t-t_1)-\phi'_{mn}]\Big\},\quad t>t_1 \tag{9.9.20}$$

9.9.4. Steady-State Harmonic Green's Function

For the simply supported plate, the steady-state harmonic Green's function is, from Eq. (9.8.8)

$$T_{33}(x,y;x^*,y^*,\omega)=\frac{1}{\rho h}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{U_{3mn}(x,y)U_{3mn}(x^*,y^*)\mathrm{e}^{-j\phi_{mn}}}{N_{mn}\omega_{mn}^2\sqrt{\left[1-\left(\frac{\omega}{\omega_{mn}}\right)^2\right]^2+4\zeta_{mn}^2\left(\frac{\omega}{\omega_{mn}}\right)^2}} \tag{9.9.21}$$

where $U_{3mn}(x,y), U_{3mn}(x^*,y^*)$ are given by Eqs. (9.9.2) and (9.9.3) and where N_{mn} is given by Eq. (9.9.4); ω_{mn} is given by Eq. (9.9.6) and ζ_{mn} by Eq. (9.9.8). The phase angle ϕ_{mn} is given by Eq. (9.8.6) and is

$$\phi_{mn}=\tan^{-1}\frac{2\zeta_{mn}(\omega/\omega_{mn})}{1-(\omega/\omega_{mn})^2} \tag{9.9.22}$$

Thus, Eq. (9.9.21) becomes

$$T_{33}(x,y;x^*,y^*,\omega) = \frac{4}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x^*}{a}\sin\frac{n\pi y^*}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{e}^{-j\phi_{mn}}}{\omega_{mn}^2\sqrt{\left[1-\left(\frac{\omega}{\omega_{mn}}\right)^2\right]^2+4\zeta_{mn}^2\left(\frac{\omega}{\omega_{mn}}\right)^2}} \tag{9.9.23}$$

Note again that the arbitrary modal amplitudes A_{mn} have canceled.

9.9.5. Response to a Harmonic Point Load

In this case, the pressure distribution to a harmonic point force applied at (x_1, y_1) is

$$q_3^*(x,y,t) = q_3^*(x,y)\mathrm{e}^{j\omega t} \tag{9.9.24}$$

where

$$q_3^*(x,y) = F\delta(x-x_1)\delta(y-y_1) \tag{9.9.25}$$

The solution is given by Eq. (9.8.10) and is for this case

$$u_3(x,y,t) = F\mathrm{e}^{j\omega t}\int_{x^*=0}^{a}\int_{y^*=0}^{a} T_{33}(x,y;x^*,y^*,\omega)\delta(x^*-x_1)\delta(y^*-y_1)\,\mathrm{d}x^*\mathrm{d}y^* \tag{9.9.26}$$

or

$$u_3(x,y,t) = F\mathrm{e}^{j\omega t}T_{33}(x,y;x_1,y_1,\omega) \tag{9.9.27}$$

or, finally

$$u_3(x,y,t) = \frac{4F}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{e}^{j(\omega t-\phi_{mn})}}{\omega_{mn}^2\sqrt{\left[1-\left(\frac{\omega}{\omega_{mn}}\right)^2\right]^2+4\zeta_{mn}^2\left(\frac{\omega}{\omega_{mn}}\right)^2}} \tag{9.9.28}$$

9.9.6. Response to a Uniformity Distributed Harmonic Pressure

Here, we have

$$q_3^* = Q \tag{9.9.29}$$

where Q is the amplitude of the uniformly distributed pressure acting on the plate. The solution is again given by Eq. (9.8.10) and is

$$u_3(x,y,t) = Q \int_{x^*=0}^{a} \int_{y^*=0}^{b} T_{33}(x,y;x^*,y^*,\omega)\,\mathrm{d}x^*\,\mathrm{d}y^* \tag{9.9.30}$$

or

$$u_3(x,y,t) = \frac{4Q}{\rho hab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \times \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \left[\int_{x^*=0}^{a} \sin\frac{m\pi x^*}{a}\,\mathrm{d}x^* \right] \left[\int_{y^*=0}^{b} \sin\frac{n\pi y^*}{b}\,\mathrm{d}y^* \right] e^{-j\phi_{mn}}}{\omega_{mn}^2 \sqrt{\left[1 - \left(\frac{\omega}{\omega_{mn}}\right)^2\right]^2 + 4\zeta_{mn}^2 \left(\frac{\omega}{\omega_{mn}}\right)^2}} \tag{9.9.31}$$

Utilizing Eq. (9.9.17) gives

$$u_3(x,y,t) = \frac{4Q}{\rho h \pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \times \frac{(1-\cos m\pi)(1-\cos n\pi) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \mathrm{e}^{j(\omega t - \phi_{mn})}}{mn\omega_{mn}^2 \sqrt{\left[1 - \left(\frac{\omega}{\omega_{mn}}\right)^2\right]^2 + 4\zeta_{mn}^2 \left(\frac{\omega}{\omega_{mn}}\right)^2}} \tag{9.9.32}$$

Again, because of the symmetry of the loading, only symmetric modes participate in the solution ($m = 1,3,5,\ldots$ and $n = 1,3,5,\ldots$).

9.10. FLOATING RING IMPACTED BY A POINT MASS

Let us consider a free floating ring that is impacted at $\theta = 0$ and time $t = t_1$ by a mass m travelling at velocity v in such a way that the mass after impact neither sticks to the ring nor rebounds, but has a rebound velocity of 0 and falls off. Thus, the loading functions in terms of force per unit length are $q_\theta = 0$ and

$$q_3(\theta,t) = -\frac{mv}{a}\delta(\theta - \theta_1)\delta(t - t_1) \tag{9.10.1}$$

where $\theta_1 = 0$ for this case. The Green's function components for a floating ring are, from Eq. (9.5.4),

$$G_{\theta\theta}(\theta, t; \theta^*, t^*) = \frac{1}{\rho h} \sum_{n=0}^{\infty} \sum_{p=1}^{2} \frac{1}{N_{np}} V_{np}^2 \cos n(\theta - \theta^*) S(t - t^*) \tag{9.10.2}$$

$$G_{\theta 3}(\theta, t; \theta^*, t^*) = \frac{1}{\rho h} \sum_{n=0}^{\infty} \sum_{p=1}^{2} \frac{1}{N_{np}} V_{np} W_{np} \sin n(\theta - \theta^*) S(t - t^*) \tag{9.10.3}$$

$$G_{3\theta}(\theta, t; \theta^*, t^*) = \frac{1}{\rho h} \sum_{n=0}^{\infty} \sum_{p=1}^{2} \frac{1}{N_{np}} V_{np} W_{np} \sin n(\theta^* - \theta) S(t - t^*) \tag{9.10.4}$$

$$G_{33}(\theta, t; \theta^*, t^*) = \frac{1}{\rho h} \sum_{n=0}^{\infty} \sum_{p=1}^{2} \frac{1}{N_{np}} W_{np}^2 \cos n(\theta - \theta^*) S(t - t^*) \tag{9.10.5}$$

The response to the impact described by Eq. (9.10.1) is

$$u_3(\theta, t) = \int_{t^*=0}^{t} \int_{\theta^*=0}^{2\pi} G_{33}(\theta, t; \theta^*, t^*) q_3(\theta^*, t^*) a \, d\theta^* \, dt^* \tag{9.10.6}$$

$$u_\theta(\theta, t) = \int_{t^*=0}^{t} \int_{\theta^*=0}^{2\pi} G_{\theta 3}(\theta, t; \theta^*, t^*) q_3(\theta^*, t^*) a \, d\theta^* \, dt^* \tag{9.10.7}$$

Since $q_\theta = 0$, components $G_{\theta\theta}$ and $G_{3\theta}$ do not enter the results integrals. Substituting Eqs. (9.10.3) and (9.10.5) in Eqs. (9.10.6) and (9.10.7) gives

$$\begin{aligned} u_3(\theta, t) &= -\frac{mv}{\rho h} \int_{t^*=0}^{t} \int_{\theta^*=0}^{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{2} \frac{1}{N_{np}} W_{np}^2 \\ &\quad \times \cos n(\theta - \theta^*) \mathrm{S}(t - t^*) \delta(\theta^* - \theta_1) \delta(t^* - t_1) \, d\theta^* \, dt^* \\ &= -\frac{mv}{\rho h} \sum_{n=0}^{\infty} \sum_{p=1}^{2} \frac{1}{N_{np}} W_{np}^2 \cos n(\theta - \theta_1) \frac{1}{\gamma_{np}} e^{-\zeta_{np}\omega_{np}(t - t_1)} \\ &\quad \times \sin \gamma_{np}(t - t_1), \quad t \geq t_1 \end{aligned} \tag{9.10.8}$$

and

$$\begin{aligned} u_\theta(\theta, t) &= -\frac{mv}{\rho h} \int_{t^*=0}^{t} \int_{\theta^*=0}^{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{V_{np} W_{np}}{N_{np}} \sin n(\theta - \theta^*) \mathrm{S}(t - t^*) \delta(\theta^* - \theta_1) \\ &\quad \times \delta(t^* - t_1) \, d\theta^* \, dt^* \\ &= -\frac{mv}{\rho h} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{V_{np} W_{np}}{N_{np}} \sin n(\theta - \theta_1) \frac{1}{\gamma_{np}} e^{-\zeta_{np}\omega_{np}(t - t_1)} \\ &\quad \times \sin \gamma_{np}(t - t_1), \quad t \geq t_1 \end{aligned} \tag{9.10.9}$$

Note that the $n=0$ mode does not participate in the solution for u_θ. N_{np} is given by Eq. (9.5.3).

Now, at $\theta=0$ and $\theta_1=0$, Eq. (9.10.9) gives

$$u_\theta(0,t)=0 \tag{9.10.10}$$

and Eq. (9.10.8) gives

$$u_3(0,t)=-\frac{mv}{\rho h}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\frac{W_{np}^2}{N_{np}}\frac{1}{\gamma_{np}}e^{-\zeta_{np}\omega_{np}(t-t_1)}\sin\gamma_{np}(t-t_1) \tag{9.10.11}$$

For $p=1$ and $n=1$, this equation describes the rigid body motion of the ring away from the impact. Since $\gamma_1=0$,

$$u_3(0,t)=-\frac{mv}{\rho h}\frac{W_{11}^2}{N_{11}}\lim_{\gamma_{11}\to 0}\left[\frac{\sin\gamma_{11}(t-t_1)}{\gamma_{11}}\right] \tag{9.10.12}$$

Since $N_{11}=ab\pi(V_{11}^2+W_{11}^2)$, where b is the width of the ring, we obtain

$$u_3(0,t)=-\frac{mv}{\rho gab\pi}\frac{1}{\left[\left(\frac{V_{11}}{W_{11}}\right)^2+1\right]}t \tag{9.10.13}$$

From Sec. 5.3, $V_{11}/W_{11}=1$, and thus

$$u_3(0,t)=-\frac{m}{M}vt \tag{9.10.14}$$

where $M=2\rho hab\pi$, which is the total mass of the ring. Thus, the velocity $\dot{u}_3(0,t)$ with which the ring moves away from impact is

$$\dot{u}_3(0,t)=-\frac{m}{M}v \tag{9.10.15}$$

which agrees with the conservation of momentum result for impact of two bodies: $-mv=M\dot{u}_3(0,t)$.

Equations (9.10.8) and (9.10.9) therefore predict the vibratory response of the ring to impact which includes the average (rigid body) motion away from the impact. This is similar to the result of Sec. 8.10 for impact applied to a spherical shell.

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10

Moment Loading

Let us now consider the response of shells to the excitation by moments. The formulation presented so far allows only treatment of cases where the excitation can be expressed in terms of pressure loadings q_1, q_2, and q_3. By use of the Dirac delta function, we were also able to treat line and point loads. However, in engineering practice, we often meet problems where the excitation is a moment. For instance, consider rotating machinery with an imbalance whose plane is parallel to the surface of the shell on which the machinery is mounted. Obviously, in addition to the in-plane forces, we have also a force couple. Any type of connection to a shell that is acted on by forces not transverse to the shell surface will produce a moment.

One approach to this problem is to consider, for instance, two transverse point forces equal in magnitude but opposed in direction, a small distance apart. They form a moment. As we let the distance approach 0 in the limit, we have created a true point moment. For example, Bolleter and Soedel (1971) indicate that this approach has been used with great success in special cases, but it is difficult to formulate the general case without rather complicated notation.

In this chapter, an approach is followed that is taken from Soedel (1976). The idea of a distributed moment is used (moment per unit area). Line and point moments, the usual engineering cases, are then formulated using the Dirac delta function.

10.1. FORMULATION OF SHELL EQUATIONS THAT INCLUDE MOMENT LOADING

Let us consider three distributed moment components, T_1 in the α_1 direction, T_2, in the α_2 direction, and a twisting moment about the normal, as shown in Fig. 1. The units are newton-meters per square meter.

To consider moment loading in addition to the classical pressure loading in the three orthogonal directions, one has to add the virtual work due to the applied moments. It is

$$\delta E_m = \int_{\alpha_1}\int_{\alpha_2} (T_1\delta\beta_1 + T_2\delta\beta_2 + T_n\delta\beta_n)A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{10.1.1}$$

where β_1 and β_2 are defined by Eqs. (2.4.7) and (2.4.8) and where

$$\beta_n = \frac{1}{2A_1A_2}\left[\frac{\partial(A_2u_2)}{\partial\alpha_1} - \frac{\partial(A_1u_1)}{\partial\alpha_2}\right] \tag{10.1.2}$$

This is the expression for a twisting angle. Let us derive it for Cartesian coordinates by considering Fig. 2. First, the rotation of the diagonal of the infinitesimal element is β_n. It is

$$\beta_n = \frac{1}{2}\left(\frac{\pi}{2} - \phi_2 - \phi_1\right) + \phi_2 - \frac{\pi}{4} = \frac{\phi_2 - \phi_1}{2} \tag{10.1.3}$$

But since

$$\phi_2 = \frac{\mathrm{d}u_2}{\mathrm{d}x} \tag{10.1.4}$$

$$\phi_1 = \frac{\mathrm{d}u_1}{\mathrm{d}y} \tag{10.1.5}$$

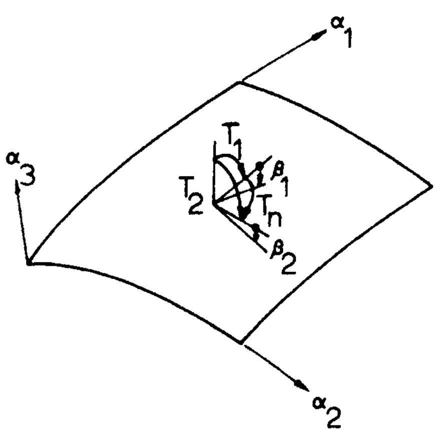

FIG. 1 Moments per unit area acting on the reference surface of a shell.

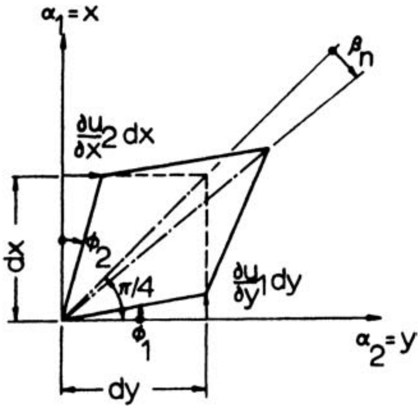

FIG. 2 Twist of an element of a rectangular plate.

we obtain

$$\beta_n = \frac{1}{2}\left(\frac{\mathrm{d}u_2}{\mathrm{d}x} - \frac{\mathrm{d}u_1}{\mathrm{d}y}\right) \tag{10.1.6}$$

If we set $A_1 = A_2 = 1$ and $\alpha_1 = x$, $\alpha_2 = y$ in Eq. (10.1.2), we obtain the same result.

Therefore, in general, we have

$$\delta E_m = \int_{\alpha_1}\int_{\alpha_1} (T_1\delta\beta_1 + T_2\delta\beta_2 + T_n\delta\beta_n)A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{10.1.7}$$

where, from Eq. (10.1.2),

$$\delta\beta_n = \frac{1}{2A_1A_2}\left[\frac{\partial(A_2\delta u_2)}{\partial\alpha_1} - \frac{\partial(A_1\delta u_1)}{\partial\alpha_2}\right] \tag{10.1.8}$$

Let us integrate the third term, as an example:

$$\int_{\alpha_1}\int_{\alpha_2} T_n\delta\beta_n A_1A_2\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_1 = \int_{\alpha_2}\left[\int_{\alpha_1}\frac{T_n}{2A_1A_2}\frac{\partial(A_2\delta u_2)}{\partial\alpha_1}A_1\,\mathrm{d}\alpha_1\right]A_2\,\mathrm{d}\alpha_2 - \int_{\alpha_1}\left[\int_{\alpha_2}\frac{T_n}{2A_1A_2}\frac{\partial(A_1\delta u_1)}{\partial\alpha_2}A_2\,\mathrm{d}\alpha_2\right]A_1\,\mathrm{d}\alpha_1 \tag{10.1.9}$$

Integrating by parts gives

$$\int_{\alpha_1}\int_{\alpha_2} T_n\delta\beta_n A_1A_2\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_1 \tag{10.1.10}$$

$$= \frac{1}{2}\int_{\alpha_1}\int_{\alpha_2}\left[\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{T_n}{A_1}\right)\delta u_1 - \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{T_n}{A_2}\right)\delta u_2\right]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{10.1.11}$$

since the resulting line integrals also vanish at the boundary. Substituting Eq. (10.1.10) in Eq. (10.1.7) and this equation, in turn, in the development of Sec. 2.7, we obtain as equations of motion

$$-\frac{\partial(N_{11}A_2)}{\partial\alpha_1}-\frac{\partial(N_{12}A_1)}{\partial\alpha_2}-N_{12}\frac{\partial A_1}{\partial\alpha_2}+N_{22}\frac{\partial A_2}{\partial\alpha_1}-A_1A_2\frac{Q_{13}}{R_1}$$
$$+A_1A_2\rho h\ddot{u}_1=A_1A_2\left(q_1+\frac{1}{2A_2}\frac{\partial T_n}{\partial\alpha_2}\right) \tag{10.1.12}$$

$$-\frac{\partial(N_{12}A_2)}{\partial\alpha_1}-\frac{\partial(N_{22}A_1)}{\partial\alpha_2}-N_{21}\frac{\partial A_2}{\partial\alpha_1}+N_{11}\frac{\partial A_1}{\partial\alpha_2}-A_1A_2\frac{Q_{23}}{R_2}$$
$$+A_1A_2\rho h\ddot{u}_2=A_1A_2\left(q_2-\frac{1}{2A_1}\frac{\partial T_n}{\partial\alpha_1}\right) \tag{10.1.13}$$

$$-\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}+A_1A_1\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)+A_1A_2\rho h\ddot{u}_3$$
$$=A_1A_2\left\{q_3+\frac{1}{A_1A_2}\left[\frac{\partial(T_1A_2)}{\partial\alpha_1}+\frac{\partial(T_2A_1)}{\partial\alpha_2}\right]\right\} \tag{10.1.14}$$

where Q_{13} and Q_{23} are defined by Eqs. (2.7.23) and (2.7.24). The admissible boundary conditions are the same.

10.2. MODAL EXPANSION SOLUTION

The modal expansion equations may be written in terms of displacement u_i as

$$L_1\{u_1,u_2,u_3\}-\lambda\dot{u}_1-\rho h\ddot{u}_1 = -q_1-\frac{1}{2A_2}\frac{\partial T_n}{\partial\alpha_2} \tag{10.2.1}$$

$$L_2\{u_1,u_2,u_3\}-\lambda\dot{u}_2-\rho h\ddot{u}_2 = -q_2+\frac{1}{2A_1}\frac{\partial T_n}{\partial\alpha_1} \tag{10.2.2}$$

$$L_3\{u_1,u_2,u_3\}-\lambda\dot{u}_3-\rho h\ddot{u}_3 = -q_3-\frac{1}{A_1A_2}\cdot\left[\frac{\partial(T_1A_2)}{\partial\alpha_1}+\frac{\partial(T_2A_1)}{\partial\alpha_2}\right] \tag{10.2.3}$$

The modal expansion series solution is

$$u_i=\sum_{k=1}^{\infty}\eta_kU_{ik} \tag{10.2.4}$$

This gives, after the usual operations,

$$\ddot{\eta}_k+2\zeta_k\omega_k\dot{\eta}_k+\omega_k^2\eta_k=F_k \tag{10.2.5}$$

where

$$F_k = \frac{1}{\rho h N_k} \int\limits_{\alpha_2} \int\limits_{\alpha_1} \left\{ q_1 U_{1k} + q_2 U_{2k} + q_3 U_{3k} + \frac{U_{1k}}{2A_2} \frac{\partial T_n}{\partial \alpha_2} - \frac{U_{2k}}{2A_1} \frac{\partial T_n}{\partial \alpha_1} + \frac{U_{3k}}{A_1 A_2} \left[\frac{\partial (T_1 A_2)}{\partial \alpha_1} + \frac{\partial (T_2 A_1)}{\partial \alpha_2} \right] \right\} A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \quad (10.2.6)$$

$$\zeta_k = \frac{\lambda}{2\rho h \omega_k} \quad (10.2.7)$$

$$N_k = \int\limits_{\alpha_2} \int\limits_{\alpha_1} (U_{1k}^2 + U_{2k}^2 + U_{3k}^2) A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \quad (10.2.8)$$

Solutions of Eq. (10.2.5) are given in Sec. 8.3.

10.3. ROTATING POINT MOMENT ON A PLATE

For the transverse motion of a plate, the solution to Eq. (10.2.3) reduces to

$$u_3 = \sum_{k=1}^{\infty} \eta_k U_{3k} \quad (10.3.1)$$

where η_k is given by Eq. (10.2.5) and where

$$F_k = \frac{1}{\rho h N_k} \int\limits_{\alpha_1} \int\limits_{\alpha_2} U_{3k} \left\{ q_3 + \frac{1}{A_1 A_2} \left[\frac{\partial (T_1 A_2)}{\partial \alpha_1} + \frac{\partial (T_2 A_1)}{\partial \alpha_2} \right] \right\} A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \quad (10.3.2)$$

and

$$N_k = \int\limits_{\alpha_2} \int\limits_{\alpha_1} U_{3k}^2 A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \quad (10.3.3)$$

Let us take as an example a simply supported rectangular plate on which a point moment of magnitude M_0, acting normal to the plate surface, rotates with a constant velocity ω_0 (rad/sec). This is shown in Fig. 3. In this case $\alpha_1 = x$, $\alpha_2 = y$, $A_1 = A_2 = 1$, and therefore,

$$T_1 = T_x = M_0 \cos \omega_0 t \delta(x - x^*) \delta(y - y^*) \quad (10.3.4)$$

$$T_2 = T_y = M_0 \sin \omega_0 t \delta(x - x^*) \delta(y - y^*) \quad (10.3.5)$$

$$T_n = 0 \quad (10.3.6)$$

Also, let us assume that $q_3 = 0$. Since the mode shape is given by

$$U_{3k} = U_{3mn} = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (10.3.7)$$

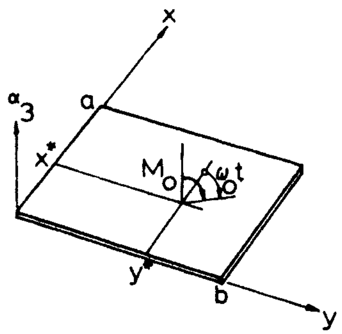

Fig. 3 Rotating point moment acting on a rectangular plate.

we obtain

$$F_k = \frac{M_0}{\rho h N_k}\left[\cos\omega_0 t \sin\frac{n\pi y^*}{b}\int_0^a \sin\frac{m\pi x}{a}\frac{\partial}{\partial x}\delta(x-x^*)\mathrm{d}x\right.$$
$$\left. + \sin\omega_0 t \sin\frac{m\pi x^*}{a}\int_0^b \sin\frac{n\pi y}{b}\frac{\partial}{\partial y}\delta(y-y^*)\mathrm{d}y\right] \quad (10.3.8)$$

Since it can be shown, using integration by parts, that

$$\int_{\alpha_i} F(\alpha_i)\frac{\partial}{\partial\alpha_i}\left[\frac{1}{A_i}\delta(\alpha_i-\alpha_i^*)\right]\mathrm{d}\alpha_i = -\frac{1}{A_i}\frac{\partial F(\alpha_i)}{\partial\alpha_i}\bigg|_{\alpha_i=\alpha_i^*} \quad (10.3.9)$$

we get

$$F_k = -\frac{M_0}{\rho h N_k}\left(\cos\omega_0 t\frac{m\pi}{a}\sin\frac{n\pi y^*}{b}\cos\frac{m\pi x^*}{a}\right.$$
$$\left. + \sin\omega_0 t\frac{n\pi}{b}\sin\frac{m\pi x^*}{a}\cos\frac{n\pi y^*}{b}\right) \quad (10.3.10)$$

where

$$N_k = \frac{ab}{4} \quad (10.3.11)$$

Following Sec. 8.5, we obtain as a steady-state solution

$$u_3 = -\frac{4M_0}{ab\rho h}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cdot\frac{f(m,n)\sin(m\pi x/a)\sin(n\pi y/b)\sin(\omega_0 t+\phi_1-\phi_2)}{\omega_{mn}^2\sqrt{[1-(\omega/\omega_{mn})^2]^2+4\zeta_{mn}^2(\omega/\omega_{mn})^2}}$$
(10.3.12)

where

$$f(m,n) = \sqrt{\left(\frac{m\pi}{a}\right)^2 \sin^2\frac{n\pi y^*}{b}\cos^2\frac{m\pi x^*}{a} + \left(\frac{n\pi}{b}\right)^2 \sin^2\frac{m\pi x^*}{a}\cos^2\frac{n\pi y^*}{b}} \tag{10.3.13}$$

$$\phi_1 = \tan^{-1}\left[\frac{m}{n}\frac{b}{a}\frac{\tan(n\pi y^*/b)}{\tan(m\pi x^*/a)}\right] \tag{10.3.14}$$

$$\phi_2 = \tan^{-1}\frac{2\zeta_{mn}(\omega/\omega_{mn})}{1-(\omega/\omega_{mn})^2} \tag{10.3.15}$$

We see that to minimize the response of a plate to a rotating moment, one ought to avoid having a natural frequency coincide with the rotational speed, $\omega_{mn} \neq \omega_0$. Also, to minimize the response of that mode, the location of the rotating moment should be on an antinode point of the plate. By *antinode points*, we mean those points where the mode has zero slope in all directions. Location of the rotating moment at a point where two node lines cross maximizes the response of that mode.

At the location of the point moment application, certain assumptions of our theory are again violated, and solutions that consider details of the actual hardware have to be found (Dyer, 1960). At a distance removed from the application point by more than twice the thickness, our solution is excellent.

10.4. ROTATING POINT MOMENT ON A SHELL

Let us now consider the case where a point moment M_0 acting normal to the shell surface rotates with a rotational velocity ω_0 as shown in Fig. 4. For cylindrical coordinates, $R_1=\infty$, $R_2=a$, $A_1=1$, $A_2=a$, $\alpha_1=x$, and $\alpha_2=\theta$. Thus the moment distribution is given by

$$T_1 = T_x = \frac{M_0}{a}\cos\omega_0 t\,\delta(x-x^*)\delta(\theta-\theta^*) \tag{10.4.1}$$

$$T_2 = T_\theta = \frac{M_0}{a}\sin\omega_0 t\,\delta(x-x^*)\delta(\theta-\theta^*) \tag{10.4.2}$$

Again, $q_1=q_2=q_3=0$. The natural modes for the simply supported circular cylindrical shell treated in Chapter 5 are

$$U_{1k}(x,\theta) = A_{mnp}\cos\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{10.4.3}$$

$$U_{2k}(x,\theta) = B_{mnp}\sin\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{10.4.4}$$

$$U_{3k}(x,\theta) = C_{mnp}\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{10.4.5}$$

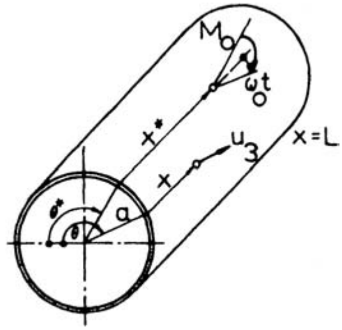

Fig. 4 Rotating point moment acting on a circular cylindrical shell.

where $m=1,2,\ldots,$ $n=0,1,2,\ldots,$ and $p=1,2,3$. The natural frequencies ω_{mnp} are given in chapter 5. For Eq. (10.4.1), we obtain

$$F_k=-C_{mnp}\frac{M_0\cos\omega_0 t}{\rho h N_k}\frac{m\pi}{L}\cos\frac{m\pi x^*}{L} \tag{10.4.6}$$

Since

$$N_k=N_{mnp}=\frac{aL\pi}{2}C_{mnp}^2 k_{mnp} \tag{10.4.7}$$

where

$$k_{mnp}=\begin{cases}\left(\dfrac{A_{mnp}}{C_{mnp}}\right)^2+\left(\dfrac{B_{mnp}}{C_{mnp}}\right)^2+1, & n\neq 0\\ 2\left(\dfrac{A_{mop}}{C_{mop}}\right)^2+2, & n=0\end{cases} \tag{10.4.8}$$

we obtain as the solution for the steady-state response

$$\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}=-\frac{2M_0}{aL\pi\rho h}\sum_{p=1}^{3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{f(m)}{k_{mnp}\omega_{mnp}^2[1-(\omega_0/\omega_{mnp})^2]}$$
$$\times\begin{Bmatrix}\dfrac{A_{mnp}}{C_{mnp}}\cos\dfrac{m\pi x}{L}\cos n(\theta-\theta^*)\\ \dfrac{B_{mnp}}{C_{mnp}}\sin\dfrac{m\pi x}{L}\sin n(\theta-\theta^*)\\ \sin\dfrac{m\pi x}{L}\cos n(\theta-\theta^*)\end{Bmatrix}\cos\omega_0 t \tag{10.4.9}$$

where

$$f(m)=\frac{m\pi}{L}\cos\frac{m\pi x^*}{L} \tag{10.4.10}$$

Similarly, Eq. (10.4.2) has to be considered. In general, the function f is now dependent only on m. The reason is that the response will orient itself on the location of the rotating moment no matter where it is located in the circumferential direction. It is, in general, not possible to have a true antinode at the point of moment application for closed shells of revolution. The modes have no preference as far as circumferential direction is concerned and orient themselves such that they present the least resistance to the action of a rotating moment. However, for selective removal of certain modes from the response, it is possible to locate the moment such that

$$f(m)=0 \tag{10.4.11}$$

In the example case, we have to make

$$\frac{m\pi x^*}{L}=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\ldots \tag{10.4.12}$$

10.5. RECTANGULAR PLATE EXCITED BY A LINE MOMENT

Let us consider the case where a line moment varying sinusoidally with time and of uniform magnitude M_0' in newton-meters per meter is distributed across a simply supported rectangular plate along the line $x=x^*$ as shown in Fig. 5. Letting $\alpha_1=x$, $A_1=1$, $\alpha_2=y$, $A_2=1$, we express the distributed moments as

$$T_x = M_0^1\delta(x-x^*)\sin\omega t \tag{10.5.1}$$

$$T_y = 0 \tag{10.5.2}$$

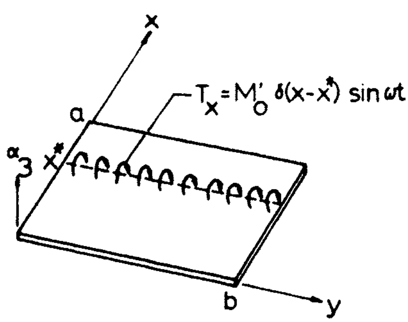

FIG. 5 Rectangular plate excited by a line moment.

This gives

$$F_k = \frac{4M_0'}{ab\rho h}\int_0^a\int_0^b \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{\partial}{\partial x}\delta(x-x^*)\,\mathrm{d}x\,\mathrm{d}y\sin\omega t \tag{10.5.3}$$

or

$$F_k = \frac{-4M_0'm}{a^2\rho hn}(1-\cos n\pi)\cos\frac{m\pi x^*}{a}\sin\omega t \tag{10.5.4}$$

This means that only the modes where $n=1,3,5,\ldots$ are excited. For instance, the steady-state response becomes, using Eqs. (8.5.4)–(8.5.7)

$$u_3(x,y,t) = -\frac{8M_0'}{a^2\rho h}\sum_{n=1,3,\ldots}^{\infty}\sum_{m=1}^{\infty} \cdot\frac{(m/n)\cos(m\pi x^*/a)\sin(n\pi y/b)\sin(m\pi x/a)}{\omega_{mn}^2\sqrt{[1-(\omega/\omega_{mn})^2]^2+4\zeta_{mn}^2(\omega/\omega_{mn})^2}}\cdot\sin(\omega t-\phi_{mn}) \tag{10.5.5}$$

where

$$\phi_{mn} = \tan^{-1}\frac{2\zeta_{mn}(\omega/\omega_{mn})}{1-(\omega/\omega_{mn})^2} \tag{10.5.6}$$

Note that if we let $x^*=0$ or $x^*=a$, we obtain the solutions for harmonic edge moments.

Solutions to harmonic edge moments are important when investigating composite structures such as two plates joined together at one edge, except that for this type of application we have to evaluate the solution for a line distribution where the moment magnitude varies sinusoidally along the line

$$T_x = M_0'\sin\frac{p\pi y}{b}\delta(x-x^*)\sin\omega t \tag{10.5.7}$$

$$T_y = 0 \tag{10.5.8}$$

We obtain

$$F_k = -\frac{4M_0'\pi m}{a^2b\rho h}\left(\int_0^b \sin\frac{n\pi y}{b}\sin\frac{p\pi y}{b}\mathrm{d}y\right)\cos\frac{m\pi x^*}{a}\sin\omega t \tag{10.5.9}$$

Since this expression is 0 if $p\neq n$, we obtain for $n=p$,

$$F_k = F_{mp} = -\frac{2M_0'\pi m}{a^2\rho h}\cos\frac{m\pi x^*}{a}\sin\omega t \tag{10.5.10}$$

and the steady-state solution is then

$$u_3(x,y,t) = -\frac{2M_0'\pi}{a^2\rho h}\sum_{m=1}^{\infty}\frac{m\cos(m\pi x^*/a)\sin(p\pi y/b)\sin(m\pi x/a)}{\omega_{mp}^2\sqrt{[1-(\omega/\omega_{mp})^2]^2+4\zeta_{mn}^2(\omega/\omega_{mp})^2}}$$
$$\cdot\sin(\omega t-\phi_{mp}) \tag{10.5.11}$$

This is an interesting result since it shows that a sinusoidally distributed line moment will only excite the modes that have shapes that match the distribution shape of the line moment. Using a Fourier sine series description, we can use this solution to generate the solutions for all other line distribution shapes. Such an approach can, of course, also be used for line force distributions.

10.6. RESPONSE OF A RING ON AN ELASTIC FOUNDATION TO A HARMONIC POINT MOMENT

The point moment of magnitude M[Nm] is applied at location $\theta=\theta^*$ of the reference line, as shown in Fig. 6. Using the Dirac-delta function, the distributed moment per unit area is (b is the width of the ring):

$$T_\theta=\frac{M}{ab}\delta(\theta-\theta^*)e^{j\omega t} \tag{10.6.1}$$

From Sec. 18.4, the natural modes of a ring on an elastic foundation are, for $\phi=0$,

$$U_{3ni(1)}(\theta)=A_{ni}\cos n\theta \tag{10.6.2}$$

$$U_{\theta ni(1)}(\theta)=B_{ni}\sin n\theta \tag{10.6.3}$$

and for $\phi=\pi/2n$,

$$U_{3ni(2)}(\theta) = A_{ni}\sin n\theta \tag{10.6.4}$$

$$U_{\theta ni(2)}(\theta) = -B_{ni}\cos n\theta \tag{10.6.5}$$

where $i=1,2$. The natural frequencies are given by Eqs. (18.4.15) and (18.4.16), and the mode component amplitude ratios are given by Eq. (18.4.19).

In the following, the harmonic response solutions will be obtained for the first set of natural modes (10.6.2) and (10.6.3). Next, they will be obtained for the second set of natural modes (10.6.4) and (10.6.5), and then the total solution set will be the addition of the two subsolution sets.

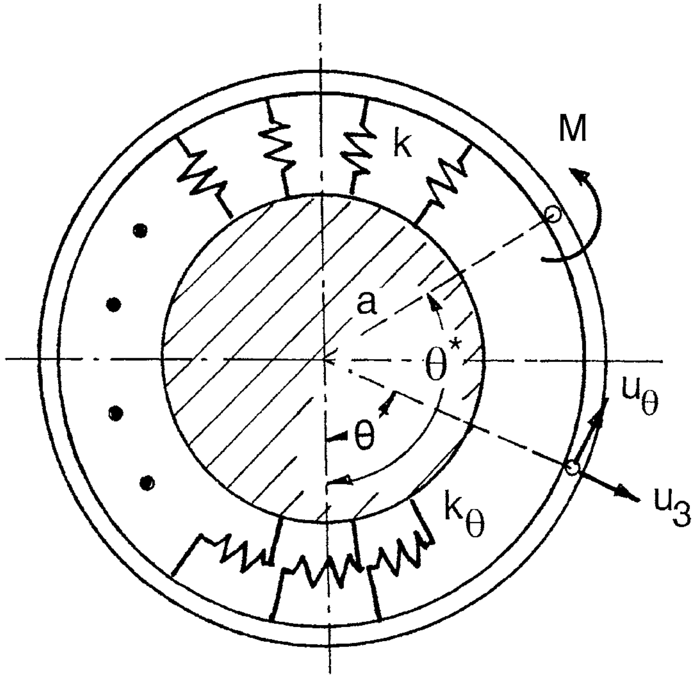

Fig. 6 Harmonic point moment acting on a ring on an elastic foundation.

For the first set of natural modes,

$$N_{k(1)} = N_{ni(1)} = b\int_{\theta=0}^{2\pi} \left(B_{ni}^2 \sin^2 n\theta + A_{ni}^2 \cos^2 n\theta\right) a\,d\theta \tag{10.6.6}$$

or

$$N_{ni(1)} = \begin{cases} ab\pi(B_{ni}^2 + A_{ni}^2), & n \neq 0 \\ 2ab\pi A_{oi}^2, & n = 0 \end{cases} \tag{10.6.7}$$

The modal force F_k, as given by Eq. (10.2.6), becomes (see also Section 18)

$$F_{k(1)} = F_{ni(1)} = \frac{A_{ni} M e^{j\omega t}}{(\rho h + 1/3\rho_F h_F) N_{ni(1)} a} \int_{\theta=0}^{2\pi} \cos n\theta \frac{\partial}{\partial\theta}[\delta(\theta - \theta^*)]\,d\theta \tag{10.6.8}$$

Utiling Eq. (10.3.9), we obtain

$$\int_{\theta=0}^{2\pi} \cos n\theta \frac{\partial}{\partial\theta}[\delta(\theta - \theta^*)]\,d\theta = -\frac{\partial}{\partial\theta}(\cos n\theta)\bigg|_{\theta=\theta^*} = n \sin n\theta^* \tag{10.6.9}$$

Therefore,

$$F_{ni(1)} = \left[\frac{MnA_{ni}\sin n\theta^*}{(\rho h + 1/3\rho_F h_F)N_{ni(1)}a} \right] \mathrm{e}^{j\omega t} = F^*_{k(1)}\mathrm{e}^{j\omega t} \tag{10.6.10}$$

Utilizing Sec. 8.5, we obtain for the modal participation coefficients, in steady state,

$$\eta_{k(1)} = \eta_{ni(1)} = \Lambda_{ni(1)}\mathrm{e}^{j(\omega t - \phi_{ni(1)})} \tag{10.6.11}$$

where

$$\Lambda_{ni(1)} = \frac{F_{ni(1)}}{\omega^2_{ni}\sqrt{[1-(\omega/\omega_{ni})^2]^2 + 4\zeta^2_{ni}(\omega/\omega_{ni})^2}} \tag{10.6.12}$$

$$\phi_{ni(1)} = \tan^{-1}\frac{2\zeta_{ni}(\omega/\omega_{ni})}{1-(\omega/\omega_{ni})^2} = \phi_{ni(2)} = \phi_{ni} \tag{10.6.13}$$

Note again that the natural frequencies for the two sets of modes are the same: $\omega_{ni(1)} = \omega_{ni(2)} = \omega_{ni}$.

The steady-state harmonic response considering the first set of natural modes only is, therefore,

$$u_{3(1)}(\theta, t) = \frac{M}{(\rho h + 1/3\rho_F h_F)a}\sum_{i=1}^{2}\sum_{n=0}^{\infty} \times \frac{nA^2_{ni}\sin n\theta^*\cos n\theta\mathrm{e}^{j(\omega t-\phi_{ni})}}{N_{ni(1)}\omega^2_{ni}\sqrt{[1-(\omega/\omega_{ni})^2]^2 + 4\zeta^2_{ni}(\omega/\omega_{ni})^2}} \tag{10.6.14}$$

$$u_{\theta(1)}(\theta, t) = \frac{M}{(\rho h + 1/3\rho_F h_F)a}\sum_{i=1}^{2}\sum_{n=0}^{\infty} \times \frac{nA_{ni}B_{ni}\sin n\theta^*\sin n\theta\mathrm{e}^{j(\omega t-\phi_{ni})}}{N_{ni(1)}\omega^2_{ni}\sqrt{[1-(\omega/\omega_{ni})^2]^2 + 4\zeta^2_{ni}(\omega/\omega_{ni})^2}} \tag{10.6.15}$$

Note that the $n = 0$ term is 0 because of the n in the numerator.

For the second set of natural modes (10.6.4) and (10.6.5), we repeat the process.

$$N_{k(2)} = N_{ni(2)} = b\int_{\theta=0}^{2\pi}(B^2_{ni}\cos^2 n\theta + A^2_{ni}\sin^2 n\theta)a\,\mathrm{d}\theta \tag{10.6.16}$$

or

$$N_{ni(2)} = \begin{cases} ab\pi(A^2_{ni} + B^2_{ni}), & n \neq 0 \\ 2ab\pi B^2_{oi}, & n = 0 \end{cases} \tag{10.6.17}$$

The modal force given by Eq. (10.2.6) becomes

$$F_{k(2)}=F_{ni(2)}=\frac{A_{ni}Me^{j\omega t}}{(\rho h+1/3\rho_F h_F)N_{ni(2)}a}\int_{\theta=0}^{2\pi}\sin n\theta\frac{\partial}{\partial\theta}[\delta(\theta-\theta^*)]d\theta \tag{10.6.18}$$

Utilizing Eq. (10.3.9),

$$\int_{\theta=0}^{2\pi}\sin n\theta\,\frac{\partial}{\partial\theta}[\delta(\theta-\theta^*)]d\theta=-\frac{\partial}{\partial\theta}(\sin n\theta)\bigg|_{\theta=\theta^*}=-n\sin n\theta^* \tag{10.6.19}$$

Therefore,

$$F_{ni(2)}=-\left[\frac{MnA_{ni}\cos n\theta^*}{(\rho h+1/3\rho_F h_F)N_{ni(2)}a}\right]e^{j\omega t}=F^*_{k(2)}e^{j\omega t} \tag{10.6.20}$$

From Sec. 8.5, the modal participation coefficients in steady state are

$$\eta_{k(2)}=\eta_{ni(2)}=\Lambda_{ni(2)}e^{j(\omega t-\phi_{ni(2)})} \tag{10.6.21}$$

where

$$\Lambda_{ni(2)}=\frac{F_{ni(2)}}{\omega_{ni}^2\sqrt{[1-(\omega/\omega_{ni})^2]^2+4\zeta_{ni}^2(\omega/\omega_{ni})^2}} \tag{10.6.22}$$

and where $\phi_{ni(2)}=\phi_{ni(1)}=\phi_{ni}$, given by Eq. (10.6.13).

The steady-state harmonic response considering the second set of natural modes only is, therefore,

$$u_{3(2)}(\theta,t)=\frac{M}{(\rho h+1/3\rho_F h_F)a}\sum_{i=1}^{2}\sum_{n=0}^{\infty}\times\frac{nA_{ni}^2(-\cos n\theta^*\sin n\theta)e^{j(\omega t-\phi_{ni})}}{N_{ni(2)}\omega_{ni}^2\sqrt{[1-(\omega/\omega_{ni})^2]^2+4\zeta_{ni}^2(\omega/\omega_{ni})^2}} \tag{10.6.23}$$

$$u_{\theta(2)}(\theta,t)=\frac{M}{(\rho h+1/3\rho_F h_F)a}\sum_{i=1}^{2}\sum_{n=0}^{\infty}\times\frac{nA_{ni}B_{ni}\cos n\theta^*\cos n\theta e^{j(\omega t-\phi_{ni})}}{N_{ni(1)}\omega_{ni}^2\sqrt{[1-(\omega/\omega_{ni})^2]^2+4\zeta_{ni}^2(\omega/\omega_{ni})^2}} \tag{10.6.24}$$

Note that because n is in the numerator of each series term, modes of the $n=0$ type do not participate in the solutions for both mode sets and we need only sum from $n=1$ to ∞. Also, from Eqs. (10.6.7) and (10.6.17), for $n\neq 0$:

$$N_{ni(1)}=N_{ni(2)}=ab\pi(B_{ni}^2+A_{ni}^2) \tag{10.6.25}$$

Thus, adding Eqs. (10.6.14) and (10.6.23) results in the addition

$$\sin n\theta^* \cos n\theta - \cos n\theta^* \sin n\theta = -\sin n(\theta - \theta^*) \tag{10.6.26}$$

Adding Eq. (10.6.15) and (10.6.24) gives the addition

$$\sin n\theta^* \sin n\theta + \cos n\theta^* \cos n\theta = \cos n(\theta - \theta^*) \tag{10.6.27}$$

The total steady state response solution to a harmonic point moment is, therefore,

$$u_3(\theta,t) = \frac{-M}{(\rho h + 1/3\rho_F h_F)a^2 b\pi} \times \sum_{i=1}^{2}\sum_{n=1}^{\infty} \frac{nA_{ni}^2 \sin n(\theta-\theta^*) e^{j(\omega t - \phi_{ni})}}{(A_{ni}^2 + B_{ni}^2)\omega_{ni}^2 \sqrt{[1-(\omega/\omega_{ni})^2]^2 + 4\zeta_{ni}^2(\omega/\omega_{ni})^2}} \tag{10.6.28}$$

$$u_\theta(\theta,t) = \frac{M}{(\rho h + 1/3\rho_F h_F)a^2 b\pi} \times \sum_{i=1}^{2}\sum_{n=1}^{\infty} \frac{nA_{ni}B_{ni} \cos n(\theta-\theta^*) e^{j(\omega t - \phi_{ni})}}{(A_{ni}^2 + B_{ni}^2)\omega_{ni} \sqrt{[1-(\omega/\omega_{ni})^2]^2 + 4\zeta_{ni}(\omega/\omega_{ni})^2}} \tag{10.6.29}$$

From this result, we see that the natural modes will orient themselves in such a way that they present their transverse nodes to the point moment. (This is opposite to the case where a transverse point force excites the ring where the modes present their transverse anti-nodes to the point force.) The tangential motion will be a maximum at the point moment location because $\cos n(\theta-\theta^*) = 1$ when $\theta = \theta^*$, while the transverse motion will be 0 because $\sin n(\theta - \theta^*) = 0$ when $\theta = \theta^*$. Again, as for shells of revolution in general, shifting a single point force or point moment in θ-direction will not improve the averaged response amplitudes because the modes follow the forcing because of the nonpreferential direction behavior of the mode components in θ-direction.

10.7. MOMENT GREEN'S FUNCTION

As in Chapter 9, a set of Green's function components can be formulated for moment loading. It requires us to solve the equations of motion (10.2.1)–(10.2.3) for $q_1 = q_2 = q_3 = 0$ and unit angular impulse loading applied at location (α_1^*, α_2^*) at time t^*, first for T_1, then for T_2, and finally for T_n.

For the first case, $T_2 = T_n = 0$ and

$$T_1 = \frac{1}{A_1 A_2}\delta(\alpha_1 - \alpha_1^*)\delta(\alpha_2 - \alpha_2^*)\delta(t - t^*) \tag{10.7.1}$$

Note that the dimension of the unit angular impulse is [Nms]. The unit angular impulse corresponds to a unit angular momentum change of

$$\Delta(J\omega) = 1 \tag{10.7.2}$$

where J is a mass moment of inertia [Nms^2] and ω is an angular velocity [rad/sec].

Therefore, Eq. (10.2.6) becomes

$$F_k = \frac{1}{\rho h N_k}\int_{\alpha_2}\int_{\alpha_1} U_{3k}\frac{\partial(T_1 A_2)}{\partial\alpha_1}\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{10.7.3}$$

Where N_k is given by Eq. (10.2.8). Substituting Eq. (10.7.1) gives

$$F_k = \frac{1}{\rho h N_k}\int_{\alpha_2}\int_{\alpha_1} U_{3k}\frac{\partial}{\partial\alpha_1}\left[\frac{1}{A_1}\delta(\alpha_1 - \alpha_1^*)\right]\delta(\alpha_2 - \alpha_2^*)\delta(t - t^*)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \tag{10.7.4}$$

Applying the integration properties (10.3.9) and (8.12.5) gives

$$F_k = \frac{-1}{\rho h N_k}\left[\frac{1}{A_1}\frac{\partial U_{3k}(\alpha_1, \alpha_2)}{\partial\alpha_1}\right]_{\substack{\alpha_1=\alpha_1^*\\ \alpha_2=\alpha_2^*}}\delta(t - t^*) \tag{10.7.5}$$

For zero initial conditions, the modal expansion coefficient becomes, therefore,

$$\eta_k(t) = -\frac{1}{\rho h \gamma_k N_k}\left[\frac{1}{A_1}\frac{\partial U_{3k}(\alpha_1, \alpha_2)}{\partial\alpha_1}\right]_{\substack{\alpha_1=\alpha_1^*\\ \alpha_2=\alpha_2^*}} \mathrm{e}^{-\zeta_k\omega_k(t-t^*)}\sin\gamma_k(t - t^*) \tag{10.7.6}$$

and the displacement solutions in α_1, α_2 and α_3 directions are the displacement Green's function components for a unit impulse moment loading T_1:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_1 = \begin{Bmatrix} G_{11}^{\mathrm{Md}}(\alpha_1, \alpha_2, t; \alpha_1^*, \alpha_2^*, t^*) \\ G_{21}^{\mathrm{Md}}(\alpha_1, \alpha_2, t; \alpha_1^*, \alpha_2^*, t^*) \\ G_{31}^{\mathrm{Md}}(\alpha_1, \alpha_2, t; \alpha_1^*, \alpha_2^*, t^*) \end{Bmatrix}$$

$$= -\frac{1}{\rho h}\sum_{k=1}^{\infty}\frac{1}{\gamma_k N_k}\left[\frac{1}{A_1}\frac{\partial U_{3k}(\alpha_1, \alpha_2)}{\partial\alpha_1}\right]_{\substack{\alpha_1=\alpha_1^*\\ \alpha_2=\alpha_2^*}}\begin{Bmatrix} U_{1k}(\alpha_1, \alpha_2) \\ U_{2k}(\alpha_1, \alpha_2) \\ U_{3k}(\alpha_1, \alpha_2) \end{Bmatrix} S(t - t^*) \tag{10.7.7}$$

where for subcritical damping,

$$S(t-t^*)=\mathrm{e}^{-\zeta_k\omega_k(t-t^*)}\sin\gamma_k(t-t^*) \tag{10.7.8}$$

The superscript Md means that it is a displacement response to moment loading.

In a similar way, we evaluate the response to a moment loading $T_1=T_n=0$ and

$$T_2=\frac{1}{A_1A_2}\delta(\alpha_1-\alpha_1^*)\delta(\alpha_2-\alpha_2^*)\delta(t-t^*) \tag{10.7.9}$$

The result is

$$\begin{aligned}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}_2&=\begin{Bmatrix}G_{12}^{\mathrm{Md}}(\alpha_1,\alpha_2,t;\alpha_1^*,\alpha_2^*,t^*)\\G_{22}^{\mathrm{Md}}(\alpha_1,\alpha_2,t;\alpha_1^*,\alpha_2^*,t^*)\\G_{32}^{\mathrm{Md}}(\alpha_1,\alpha_2,t;\alpha_1^*,\alpha_2^*,t^*)\end{Bmatrix}\\&=-\frac{1}{\rho h}\sum_{k=1}^{\infty}\frac{1}{\gamma_kN_k}\left[\frac{1}{A_2}\frac{\partial U_{3k}(\alpha_1,\alpha_2)}{\partial\alpha_2}\right]_{\substack{\alpha_1=\alpha_1^*\\\alpha_2=\alpha_2^*}}\begin{Bmatrix}U_{1k}(\alpha_1,\alpha_2)\\U_{2k}(\alpha_1,\alpha_2)\\U_{3k}(\alpha_1,\alpha_2)\end{Bmatrix}S(t-t^*)\end{aligned} \tag{10.7.10}$$

Finally, we evaluate the displacement response to a moment loading $T_1=T_2=0$ and

$$T_n=\frac{1}{A_1A_2}\delta(\alpha_1-\alpha_1^*)\delta(\alpha_2-\alpha_2^*)\delta(t-t^*) \tag{10.7.11}$$

For this case,

$$\begin{aligned}F_k&=\frac{1}{\rho hN_k}\int_{\alpha_2}\int_{\alpha_1}\left(\frac{U_{1k}}{2A_2}\frac{\partial T_n}{\partial\alpha_2}-\frac{U_{2k}}{2A_1}\frac{\partial T_n}{\partial\alpha_1}\right)A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\\&=\frac{1}{2\rho hN_k}\int_{\alpha_2}\int_{\alpha_1}\frac{U_{1k}}{A_2}\delta(\alpha_1-\alpha_1^*)\frac{\partial}{\partial\alpha_2}\left[\frac{1}{A_1A_2}\delta(\alpha_2-\alpha_2^*)\right]\\&\quad\times\delta(t-t^*)A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\\&\quad-\frac{1}{2\rho hN_k}\int_{\alpha_2}\int_{\alpha_1}\frac{U_{2k}}{A_1}\frac{\partial}{\partial\alpha_1}\left[\frac{1}{A_1A_2}\delta(\alpha_1-\alpha_1^*)\right]\delta(\alpha_2-\alpha_2^*)\delta(t-t^*)\\&\quad\times A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\end{aligned} \tag{10.7.12}$$

Integrating by parts

$$\int_{\alpha_1}F(\alpha_1,\alpha_2)\frac{\partial}{\partial\alpha_1}\left[\frac{1}{A_1A_2}\delta(\alpha_1-\alpha_1^*)\right]\mathrm{d}\alpha_1=-\left[\frac{1}{A_1A_2}\frac{\partial F(\alpha_1,\alpha_2)}{\partial\alpha_1}\right]_{\alpha_1=\alpha_1^*} \tag{10.7.13}$$

and

$$\int_{\alpha_2} F(\alpha_1,\alpha_2)\frac{\partial}{\partial\alpha_2}\left[\frac{1}{A_1A_2}\delta(\alpha_2-\alpha_2^*)\right]\mathrm{d}\alpha_2 = -\left[\frac{1}{A_1A_2}\frac{\partial F(\alpha_1,\alpha_2)}{\partial\alpha_2}\right]_{\alpha_2=\alpha_2^*} \tag{10.7.14}$$

Equation (10.7.12) becomes

$$F_k = -\frac{1}{2\rho h N_k}\left\{\left[\frac{1}{A_1A_2}\frac{\partial U_{2k}(\alpha_1,\alpha_2)}{\partial\alpha_2}\right]_{\substack{\alpha_1=\alpha_1^*\\ \alpha_2=\alpha_2^*}} - \left[\frac{1}{A_1A_2}\frac{\partial U_{1k}(\alpha_1,\alpha_2)}{\partial\alpha_1}\right]_{\substack{\alpha_1=\alpha_1^*\\ \alpha_2=\alpha_2^*}}\right\}\delta(t-t^*) \tag{10.7.15}$$

For zero initial condition, the modal expansion coefficient becomes, therefore,

$$\eta_k = -\frac{1}{2\rho h\gamma_k N_k}\left\{\left[\frac{1}{A_1A_2}\frac{\partial U_{2k}(\alpha_1,\alpha_2)}{\partial\alpha_2}\right]_{\substack{\alpha_1=\alpha_1^*\\ \alpha_2=\alpha_2^*}} - \left[\frac{1}{A_1A_2}\frac{\partial U_{1k}(\alpha_1,\alpha_2)}{\partial\alpha_1}\right]_{\substack{\alpha_1=\alpha_1^*\\ \alpha_2=\alpha_2^*}}\right\}\mathrm{e}^{-\zeta_k\omega_k(t-t^*)}\sin\gamma_k(t-t^*) \tag{10.7.16}$$

and the displacement solutions u_1,u_2 and u_3 are the displacement Green's function components for a unit impulse moment loading T_n:

$$\begin{Bmatrix} u_1\\ u_2\\ u_3\end{Bmatrix}_n = \begin{Bmatrix} G_{1n}^{\mathrm{Md}}(\alpha_1,\alpha_2,t;\alpha_1^*,\alpha_2^*,t^*)\\ G_{2n}^{\mathrm{Md}}(\alpha_1,\alpha_2,t;\alpha_1^*,\alpha_2^*,t^*)\\ G_{3n}^{\mathrm{Md}}(\alpha_1,\alpha_2,t;\alpha_1^*,\alpha_2^*,t^*)\end{Bmatrix} = \frac{1}{2\rho h}\sum_{k=1}^{\infty}\frac{1}{\gamma_k N_k}\left\{\left[\frac{1}{A_1A_2}\frac{\partial U_{2k}(\alpha_1,\alpha_2)}{\partial\alpha_2}\right]_{\substack{\alpha_1=\alpha_1^*\\ \alpha_2=\alpha_2^*}} - \left[\frac{1}{A_1A_2}\frac{\partial U_{1k}(\alpha_1,\alpha_2)}{\partial\alpha_1}\right]_{\substack{\alpha_1=\alpha_1^*\\ \alpha_2=\alpha_2^*}}\right\}\begin{Bmatrix} U_{1k}(\alpha_1,\alpha_2)\\ U_{2k}(\alpha_1,\alpha_2)\\ U_{3k}(\alpha_1,\alpha_2)\end{Bmatrix}S(t-t^*) \tag{10.7.17}$$

where $S(t-t^*)$ is given by Eq. (10.7.8). Thus, to describe a shell in general by a moment Green's function, the nine components of Eqs. (10.7.7), (10.7.10) and (10.7.17) have to be obtained. Usually, it suffices, however, to work with Eqs. (10.7.7) and (10.7.10) since structures loaded by twisting moments acting in the tangent plane, T_n, are relatively rare.

The response to a general moment loading can now be obtained by integration. For example, for the case of a point moment that is suddenly applied at $x = x_1, y = y_1$ and $t = t_1$ to a rectangular, simply supported plate,

$$T_1 = M_1 \delta(x - x_1)\delta(y - y_1)U(t - t_1) \tag{10.7.18}$$

where M_1 has the units [Nm], so that T_1 has the units [Nm/m^2]. We may write

$$u_3(x, y, t) = \int_0^t \int_{x=0}^a \int_{y=0}^b G_{31}^{\mathrm{Md}}(x, y, t; x^*, y^*, t^*) T_1(x^*, y^*, t^*) \mathrm{d}x^* \mathrm{d}y^* \mathrm{d}t^* \tag{10.7.19}$$

where, from Eq. (10.7.7) for $(k) = (m, n)$,

$$G_{31}^{\mathrm{Md}}(x, y, t; x^*, y^*, t^*) = \frac{1}{\rho h} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\gamma_{mn} N_{mn}} \left[\frac{\partial U_{3mn}(x, y)}{\partial x} \right]_{\substack{x=x^* \\ y=y^*}} \times U_{3mn}(x, y) S(t - t^*) \tag{10.7.20}$$

where

$$U_{3mn}(x, y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{10.7.21}$$

$$\frac{\partial U_{3mn}(x, y)}{\partial x} = \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{10.7.22}$$

$$N_{mn} = \int_{x=0}^a \int_{y=0}^b U_{3mn}(x, y) \mathrm{d}x \mathrm{d}y = \frac{ab}{4} \tag{10.7.23}$$

and where $S(t - t^*)$ is given by Eq. (10.7.8).

The moment Green's function of Eq. (10.7.20) becomes

$$G_{31}^{\mathrm{Md}}(x, y, t; x^*, y^*, t^*) = \frac{4\pi}{\rho h a^2 b} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m}{\gamma_{mn}} \cos \frac{m\pi x^*}{a} \times \sin \frac{n\pi y^*}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} S(t - t^*) \tag{10.7.24}$$

Therefore, Eq. (10.7.19) becomes

$$u_3(x, y, t) = \frac{4\pi M_1}{\rho h a^3 b} \int_{t^*=0}^{t} \int_{x^*=0}^{a} \int_{y^*=0}^{b} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m}{\gamma_{mn}} \cos \frac{m\pi x^*}{a} \times \sin \frac{n\pi y^*}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \times \delta(x^* - x_1)\delta(y^* - y_1)U(t - t_1) \mathrm{e}^{-\zeta_{mn}\omega_{mn}(t - t^*)} \times \sin \gamma_{mn}(t - t^*) \mathrm{d}x^* \mathrm{d}y^* \mathrm{d}t^* \tag{10.7.25}$$

The step function in time simply changes the integration limit from t_1 to t and we obtain from Eq. (9.9.18)

$$\int_{t^*=t_1}^{t} e^{-\zeta_{mn}\omega_{mn}(t-t^*)} \sin\gamma_{mn}(t-t^*)\,dt^*$$
$$= \frac{1}{\gamma_{mn}}\left\{(1-\zeta_{mn}^2) - \sqrt{1-\zeta_{mn}^2}\,e^{-\zeta_{mn}(t-t_1)}\cos[\gamma_{mn}(t-t_1)-\phi'_{mn}]\right\}, \quad t \geq t_1 \tag{10.7.26}$$

where ϕ'_{mn} is given by Eq. (9.9.19).
The surface integrals become

$$\int_{x^*=0}^{a}\int_{y^*=0}^{b} \cos\frac{m\pi x^*}{a}\sin\frac{n\pi y^*}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\delta(x^*-x_1)$$
$$\delta(y^*-y_1)\,dx^*\,dy^* = \cos\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{10.7.27}$$

Therefore, the final result is

$$u_3(x,y,t) = \frac{4\pi M_1}{\rho h a^2 b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m}{\gamma_{mn}^2}\cos\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
$$\times\left\{(1-\zeta_{mn}^2) - \sqrt{1-\zeta_{mn}^2}\,e^{-\zeta_{mn}(t-t_1)}\cos[\gamma_{mn}(t-t_1)-\phi'_{mn}]\right\}, \quad t \geq t_1 \tag{10.7.28}$$

This result shows again that for a particular plate mode (m,n) to vanish from the response series, the point moment has to be applied at an antinode of this mode so that $\cos(m\pi x_1/a)=0$.

Finally, the relatively slow convergence of the moment Green's function needs to be mentioned because of the m number in the numerator of Eq. (10.7.28).

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11

Vibration of Shells and Membranes Under the Influence of Initial Stresses

All cases treated so far were bending resistant structures which, when in their equilibrium position, have a 0 or negligible stress level. In a very different category are skin structures, where the forces that restore the displaced skin to its equilibrium position are caused by stresses initially present that can be assumed to be independent of motion. Such skin structures are referred to as *membranes*. They can be flat, such as classical membranes, stretched over three-dimensional frames, or supported in shape and tension by internal air pressure, such as tires (Kung et al., 1985; Saigal et al., 1986). Tension may also be provided by gravity, as in hanging curtains or nets (see Sec. 15.8). Note that the restoring effect is entirely different from the membrane approximation for shells discussed earlier, where the restoring forces are caused by the changing membrane stresses as the deflection occurs. There is a slight problem in semantics, of course, since the word *membrane* is used in the context of both categories.

A third possibility and a very likely one in engineering is that a combination of the two restoring effects has to be accounted for. Every time a structure is spinning like a turbine blade, a circular saw, and so on, the centrifugal forces will introduce an initial stress field that is always present, vibration or no vibration, and which acts as an additional restoring mechanism. Shells loaded by high static pressures have a static or initial stress field that resists or aids deflection. For instance, a spherical shell loaded internally by static pressure, such as a boiler, will experience a gain in effective stiffness that will increase natural frequencies, whereas a spherical shell that is loaded externally by static pressure, such as a diving vessel, will experience a decrease of effective stiffness, lowering the natural frequencies. Shells that were fabricated by deep drawing or welding and were not annealed afterward can have large equilibrium stresses that are referred to as *residual stresses*. These stresses can often be so high that pronounced static buckling or warping occurs. In the following, Love's equation is extended to account for the initial stress effect, then reduced to the category of pure membrane or skin structures.

Note that the earliest theoretical investigation of the initial stress influence, going beyond that of a string or pure membrane, is probably due to Lamb (1921) for a circular plate under initial tension. One of the first shell solutions of this type was given by Federhofer (1936) for the axially compressed circular cylindrical shell. A good discussion of more recent work on initial stress problems is given by Leissa (1973).

11.1. STRAIN–DISPLACEMENT RELATIONSHIPS

To investigate how shells vibrate under the influence of initial stresses, we have to include some terms in the strain–displacement equation that were neglected in Sec. 2.3. In Eq. (2.3.9) we retain all terms:

$$(d\alpha_i + d\xi_i)^2 = (d\alpha_i)^2 + 2d\alpha_i d\xi_i + (d\xi_i)^2 \tag{11.1.1}$$

Thus, instead of Eq. (2.3.12), we obtain

$$(d\alpha_i + d\xi_i)^2 = (d\alpha_i)^2 + 2d\alpha_i \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j + \left(\sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j \right)^2 \tag{11.1.2}$$

This means that Eq. (2.3.15) becomes

$$\begin{aligned}(ds')^2 = \sum_{i=1}^{3} \Bigg[& \left(g_{ii} + \sum_{j=1}^{3} \frac{\partial g_{ii}}{\partial \alpha_j} \xi_j \right) (d\alpha_i)^2 + 2d\alpha_i g_{ii} \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j \\ & + 2d\alpha_i \sum_{j=1}^{3} \frac{\partial g_{ii}}{\partial \alpha_j} \xi_j \sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j + g_{ii} \left(\sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j \right)^2 \\ & + \sum_{j=1}^{3} \frac{\partial g_{ii}}{\partial \alpha_j} \xi_j \left(\sum_{j=1}^{3} \frac{\partial \xi_i}{\partial \alpha_j} d\alpha_j \right)^2 \Bigg] \end{aligned} \tag{11.1.3}$$

Neglecting, as before, the third term, and of the two new terms the last one, gives, after introducing the Kronecker delta notation,

$$(\mathrm{d}s')^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\left[\left(g_{ii}+\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k\right)\delta_{ij}\mathrm{d}\alpha_i\mathrm{d}\alpha_j+\left(g_{ii}\frac{\partial \xi_i}{\partial \alpha_j}+g_{ii}\frac{\partial \xi_j}{\partial \alpha_i}\right)\mathrm{d}\alpha_i\mathrm{d}\alpha_j\right.$$
$$\left.+\left(\sum_{k=1}^{3}g_{kk}\frac{\partial \xi_k}{\partial \alpha_i}\frac{\partial \xi_k}{\partial \alpha_j}\right)\mathrm{d}\alpha_i\mathrm{d}\alpha_j\right] \tag{11.1.4}$$

This equation replaces Eq. (2.3.20). We see that our new G_{ij} is

$$G_{ij}=\left(g_{ii}+\sum_{k=1}^{3}\frac{\partial g_{ii}}{\partial \alpha_k}\xi_k\right)\delta_{ij}+g_{ii}\frac{\partial \xi_i}{\partial \alpha_j}+g_{ii}\frac{\partial \xi_j}{\partial \alpha_i}+\sum_{k=1}^{3}g_{kk}\frac{\partial \xi_k}{\partial \alpha_i}\frac{\partial \xi_k}{\partial \alpha_j} \tag{11.1.5}$$

From here on, the derivation proceeds as outlined in Sec. 2.3. We get

$$\varepsilon_{11} = \frac{1}{A_1(1+\alpha_3/R_1)}\left(\frac{\partial U_1}{\partial \alpha_2}+\frac{U_2}{A_2}\frac{\partial A_1}{\partial \alpha_2}+U_3\frac{A_1}{R_1}\right)$$
$$+\frac{1}{2A_1^2}\left[\left(\frac{\partial U_1}{\partial \alpha_1}-\frac{U_1}{A_1}\frac{\partial A_1}{\partial \alpha_1}\right)^2+\left(\frac{\partial U_2}{\partial \alpha_1}-\frac{U_2}{A_2}\frac{\partial A_2}{\partial \alpha_1}\right)^2+\left(\frac{\partial U_3}{\partial \alpha_1}\right)^2\right] \tag{11.1.6}$$

$$\varepsilon_{22} = \frac{1}{A_2(1+\alpha_3/R_2)}\left(\frac{\partial U_2}{\partial \alpha_2}+\frac{U_1}{A_1}\frac{\partial A_2}{\partial \alpha_1}+U_3\frac{A_2}{R_2}\right)$$
$$+\frac{1}{2A_2^2}\left[\left(\frac{\partial U_2}{\partial \alpha_2}-\frac{U_2}{A_2}\frac{\partial A_2}{\partial \alpha_2}\right)^2+\left(\frac{\partial U_1}{\partial \alpha_2}-\frac{U_1}{A_1}\frac{\partial A_1}{\partial \alpha_2}\right)^2+\left(\frac{\partial U_3}{\partial \alpha_2}\right)^2\right] \tag{11.1.7}$$

$$\varepsilon_{33} = \frac{\partial U_3}{\partial \alpha_3} \tag{11.1.8}$$

$$\varepsilon_{12} = \frac{A_1(1+\alpha_3/R_1)}{A_2(1+\alpha_3/R_2)}\frac{\partial}{\partial \alpha_2}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)+\frac{A_2(1+\alpha_3/R_2)}{A_1(1+\alpha_3/R_1)}$$
$$\times\frac{\partial}{\partial \alpha_1}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)+\frac{A_1(1+\alpha_3/R_1)}{A_2(1+\alpha_3/R_2)}$$
$$\times\frac{\partial}{\partial \alpha_1}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)\cdot\frac{\partial}{\partial \alpha_2}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)+\frac{A_2(1+\alpha_3/R_2)}{A_1(1+\alpha_3/R_1)}$$
$$\times\frac{\partial}{\partial \alpha_1}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)\frac{\partial}{\partial \alpha_2}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)$$
$$+\frac{1}{A_1A_2(1+\alpha_3/R_1)(1+\alpha_3/R_2)}\frac{\partial U_3}{\partial \alpha_1}\frac{\partial U_3}{\partial \alpha_2} \tag{11.1.9}$$

$$\varepsilon_{13} = A_1\left(1+\frac{\alpha_3}{R_1}\right)\frac{\partial}{\partial\alpha_3}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)+\frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial U_3}{\partial\alpha_1}$$
$$+A_1\left(1+\frac{\alpha_3}{R_1}\right)\cdot\frac{\partial}{\partial\alpha_1}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)\frac{\partial}{\partial\alpha_3}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)$$
$$+\frac{A_2^2(1+\alpha_3/R_2)^2}{A_1(1+\alpha_3/R_1)}\frac{\partial}{\partial\alpha_1}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)\cdot\frac{\partial}{\partial\alpha_3}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)$$
$$+\frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial U_3}{\partial\alpha_1}\frac{\partial U_3}{\partial\alpha_3} \tag{11.1.10}$$

$$\varepsilon_{23} = A_2\left(1+\frac{\alpha_3}{R_2}\right)\frac{\partial}{\partial\alpha_3}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)+\frac{1}{A_2(1+\alpha_3/R_2)}\frac{\partial U_3}{\partial\alpha_2}$$
$$+A_2\left(1+\frac{\alpha_3}{R_2}\right)\cdot\frac{\partial}{\partial\alpha_2}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)$$
$$\times\frac{\partial}{\partial\alpha_3}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)+\frac{A^2(1+\alpha_3/R_1)^2}{A_2(1+\alpha_3/R_2)}$$
$$\times\frac{\partial}{\partial\alpha_2}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)\cdot\frac{\partial}{\partial\alpha_3}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)$$
$$+\frac{1}{A_2(1+\alpha_3/R_2)}\frac{\partial U_3}{\partial\alpha_2}\frac{\partial U_3}{\partial\alpha_3} \tag{11.1.11}$$

Note that U_3 is not a function of α_3 and each last term in Eqs. (11.1.10) and (11.1.11) is therefore 0. We simplify further by neglecting all square terms involving U_1 and U_2 since it is reasonable to be expected that they will always be small compared to the transverse deflections U_3. Equations (11.1.6)–(11.1.11) thus become

$$\varepsilon_{11} = \frac{1}{A_1(1+\alpha_3/R_1)}\left(\frac{\partial U_1}{\partial\alpha_1}+\frac{U_2}{A_2}\frac{\partial A_1}{\partial\alpha_2}+U_3\frac{A_1}{R_1}\right)+\frac{1}{2A_1^2}\left(\frac{\partial U_3}{\partial\alpha_1}\right)^2 \tag{11.1.12}$$

$$\varepsilon_{22} = \frac{1}{A_2(1+\alpha_3/R_2)}\left(\frac{\partial U_2}{\partial\alpha_2}+\frac{U_1}{A_1}\frac{\partial A_2}{\partial\alpha_1}+U_3\frac{A_2}{R_2}\right)+\frac{1}{2A_2^2}\left(\frac{\partial U_3}{\partial\alpha_2}\right)^2 \tag{11.1.13}$$

$$\varepsilon_{33} = 0 \tag{11.1.14}$$

$$\varepsilon_{12} = \frac{A_1(1+\alpha_3/R_1)}{A_2(1+\alpha_3/R_2)}\frac{\partial}{\partial\alpha_2}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)+\frac{A_2(1+\alpha_3/R_2)}{A_1(1+\alpha_3/R_1)} \times\frac{\partial}{\partial\alpha_1}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)+\frac{1}{A_1A_2(1+\alpha_3/R_1)(1+\alpha_3/R_2)}\frac{\partial U_3}{\partial\alpha_1}\frac{\partial U_3}{\partial\alpha_2} \tag{11.1.15}$$

$$\varepsilon_{13} = A_1\left(1+\frac{\alpha_3}{R_1}\right)\frac{\partial}{\partial\alpha_3}\left(\frac{U_1}{A_1(1+\alpha_3/R_1)}\right)+\frac{1}{A_1(1+\alpha_3/R_1)}\frac{\partial U_3}{\partial\alpha_1} \tag{11.1.16}$$

$$\varepsilon_{23} = A_2\left(1+\frac{\alpha_3}{R_2}\right)\frac{\partial}{\partial\alpha_3}\left(\frac{U_2}{A_2(1+\alpha_3/R_2)}\right)+\frac{1}{A_2(1+\alpha_3/R_2)}\frac{\partial U_3}{\partial\alpha_2} \tag{11.1.17}$$

Next, if we follow Sec. 2.4, where we have introduced the assumptions that displacements U_1 and U_2 are a linear function of α_3, we obtain

$$\varepsilon_{11} = \varepsilon_{11}^0+\alpha_3 k_{11} \tag{11.1.18}$$

$$\varepsilon_{22} = \varepsilon_{22}^0+\alpha_3 k_{22} \tag{11.1.19}$$

$$\varepsilon_{12} = \varepsilon_{12}^0+\alpha_3 k_{12} \tag{11.1.20}$$

where the membrane strains become

$$\varepsilon_{11}^0 = \frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1}+\frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{u_3}{R_1}+\frac{1}{2A_1^2}\left(\frac{\partial u_3}{\partial\alpha_1}\right)^2 \tag{11.1.21}$$

$$\varepsilon_{22}^0 = \frac{1}{A_2}\frac{\partial u_2}{\partial\alpha_2}+\frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{u_3}{R_2}+\frac{1}{2A_2^2}\left(\frac{\partial u_3}{\partial\alpha_2}\right)^2 \tag{11.1.22}$$

$$\varepsilon_{12}^0 = \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2}{A_2}\right)+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_1}{A_1}\right)+\frac{1}{A_1A_2}\left(\frac{\partial u_3}{\partial\alpha_1}\right)\left(\frac{\partial u_3}{\partial\alpha_2}\right) \tag{11.1.23}$$

The bending strains are the same as given by Eqs. (2.4.22)–(2.4.24). The definitions of β_1 and β_2 remain the same also.

11.2. EQUATIONS OF MOTION

While the general expression for strain energy given in Eq. (2.6.3) is still true, the value of F now becomes (the superscript r stands for "residual")

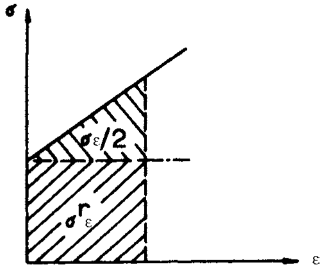

FIG. 1 Illustration of the residual strain energy in a shell.

$$F = \tfrac{1}{2}(\sigma_{11}\varepsilon_{11}+\sigma_{22}\varepsilon_{22}+\sigma_{12}\varepsilon_{12}+\sigma_{13}\varepsilon_{13}+\sigma_{23}\varepsilon_{23})+\sigma^r_{11}\varepsilon_{11}+\sigma^r_{22}\varepsilon_{22} +\sigma^r_{12}\varepsilon_{12}+\sigma^r_{13}\varepsilon_{13}+\sigma^r_{23}\varepsilon_{23} \tag{11.2.1}$$

This is illustrated in Fig. 1. The residual stress σ^r is constant, independent of the vibration-induced strain ε, but energy is stored by the deflection and is proportional to the rectangular area $\sigma^r\varepsilon$. The variation δF is therefore

$$\delta F = (\sigma_{11}+\sigma^r_{11})\delta\varepsilon_{11}+(\sigma_{22}+\sigma^r_{22})\delta\varepsilon_{22}+(\sigma_{12}+\sigma^r_{12})\delta\varepsilon_{12} +(\sigma_{13}+\sigma^r_{13})\delta\varepsilon_{13}+(\sigma_{23}+\sigma^r_{23})\delta\varepsilon_{23} \tag{11.2.2}$$

Recognizing the fact that the integration over the shell thickness of the residual stresses gives rise to force and moment resultants

$$N^r_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^r_{11}\,\mathrm{d}\alpha_3 \tag{11.2.3}$$

$$M^r_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^r_{11}\alpha_3\,\mathrm{d}\alpha_3 \tag{11.2.4}$$

and so on, for $N^r_{12}, N^r_{22}, M^r_{12}, M^r_{22}, Q^r_{13}$, and Q^r_{23}, we obtain the equations of motion in a way similar to that described in Sec. 2.7. Let us consider, for instance, the first term in the strain energy expression. Since

$$\varepsilon_{11} = \cdots + \frac{1}{2A_1^2}\left(\frac{\partial u_3}{\partial \alpha_1}\right)^2 \tag{11.2.5}$$

or

$$\delta\varepsilon_{11} = \cdots + \frac{1}{A_1^2}\frac{\partial u_3}{\partial \alpha_1}\frac{\partial(\delta u_3)}{\partial \alpha_1} \tag{11.2.6}$$

this first term becomes

$$\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}(\sigma_{11}+\sigma^r_{11})\left[\cdots+\frac{A_2}{A_1}\frac{\partial u_3}{\partial \alpha_1}\frac{\partial(\delta u_3)}{\partial \alpha_1}\right]\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{11.2.7}$$

Integrating over the thickness gives

$$\int_{\alpha_1}\int_{\alpha_2}(N_{11}+N_{11}^r)\left[\cdots+\frac{A_2}{A_1}\frac{\partial u_3}{\partial\alpha_1}\frac{\partial(\delta u_3)}{\partial\alpha_1}\right]\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 + \int_{\alpha_1}\int_{\alpha_2}(M_{11}+M_{11}^r)(\cdots)\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{11.2.8}$$

The notation $(\cdots)$ indicates that all other terms are the same as those in Chapter 2.

If we integrate by parts with respect to α_1, we get

$$\cdots\int_{\alpha_1}(N_{11}+N_{11}^r)\frac{A_2}{A_1}\frac{\partial u_3}{\partial\alpha_1}\delta u_3\,\mathrm{d}\alpha_2 - \int_{\alpha_1}\int_{\alpha_2}\frac{\partial}{\partial\alpha_1}\left[(N_{11}+N_{11}^r)\frac{A_2}{A_1}\frac{\partial u_3}{\partial\alpha_1}\right]\delta u_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\cdots \tag{11.2.9}$$

Proceeding similarly with the other new terms and collecting coefficients of the virtual displacements gives the new set of equations of motion.

$$-\frac{\partial}{\partial\alpha_1}[(N_{11}+N_{11}^r)A_2]-\frac{\partial}{\partial\alpha_2}[(N_{21}+N_{21}^r)A_1]-(N_{12}+N_{12}^r)\frac{\partial A_1}{\partial\alpha_2} + (N_{22}+N_{22}^r)\frac{\partial A_2}{\partial\alpha_1}-(Q_{13}+Q_{13}^r)\frac{A_1A_2}{R_1}+A_1A_2\rho h\ddot{u}_1=A_1A_2q_1 \tag{11.2.10}$$

$$-\frac{\partial}{\partial\alpha_1}[(N_{12}+N_{12}^r)A_2]-\frac{\partial}{\partial\alpha_2}[(N_{22}+N_{22}^r)A_1]-(N_{21}+N_{21}^r)\frac{\partial A_2}{\partial\alpha_1} + (N_{11}+N_{11}^r)\frac{\partial A_1}{\partial\alpha_2}-(Q_{23}+Q_{23}^r)\frac{A_1A_2}{R_2}+A_1A_2\rho h\ddot{u}_2=A_1A_2q_2 \tag{11.2.11}$$

$$-\frac{\partial}{\partial\alpha_1}[(Q_{13}+Q_{13}^r)A_2]-\frac{\partial}{\partial\alpha_2}[(Q_{23}+Q_{23}^r)A_1]+(N_{11}+N_{11}^r)\frac{A_1A_2}{R_1} + (N_{22}+N_{22}^r)\frac{A_1A_2}{R_2}-\frac{\partial}{\partial\alpha_1}\left[(N_{11}+N_{11}^r)\frac{A_2}{A_1}\frac{\partial u_3}{\partial\alpha_1}\right] - \frac{\partial}{\partial\alpha_2}\left[(N_{22}+N_{22}^r)\frac{A_1}{A_2}\frac{\partial u_3}{\partial\alpha_2}\right]-\frac{\partial}{\partial\alpha_1}\left[(N_{21}+N_{21}^r)\frac{\partial u_3}{\partial\alpha_2}\right] - \frac{\partial}{\partial\alpha_2}\left[(N_{12}+N_{12}^r)\frac{\partial u_3}{\partial\alpha_1}\right]+A_1A_2\rho h\ddot{u}_3=A_1A_2q_3 \tag{11.2.12}$$

where

$$(Q_{13}+Q_{13}^r)A_1A_2=\frac{\partial}{\partial\alpha_1}[(M_{11}+M_{11}^r)A_2]+\frac{\partial}{\partial\alpha_2}[(M_{21}+M_{21}^r)A_1]$$
$$+(M_{12}+M_{12}^r)\frac{\partial A_1}{\partial\alpha_2}-(M_{22}+M_{22}^r)\frac{\partial A_2}{\partial\alpha_1} \quad (11.2.13)$$

$$(Q_{23}+Q_{23}^r)A_1A_2=\frac{\partial}{\partial\alpha_1}[(M_{12}+M_{12}^r)A_2]+\frac{\partial}{\partial\alpha_2}[(M_{22}+M_{22}^r)A_1]$$
$$+(M_{21}+M_{21}^r)\frac{\partial A_2}{\partial\alpha_1}-(M_{11}+M_{11}^r)\frac{\partial A_1}{\partial\alpha_2} \quad (11.2.14)$$

We recognize that the equations divide into a set that defines initial stresses as function of static loads q_1^r, q_2^r, q_3^r and a set that defines the vibration behavior as a function of dynamic loads q_1, q_2, q_3. The first set becomes

$$-\frac{\partial}{\partial\alpha_1}(N_{11}^rA_2)-\frac{\partial}{\partial\alpha_2}(N_{21}^rA_1)-N_{12}^r\frac{\partial A_1}{\partial\alpha_2}+N_{22}^r\frac{\partial A_2}{\partial\alpha_1}-Q_{13}^r\frac{A_1A_2}{R_1}=A_1A_2q_1^r \quad (11.2.15)$$

$$-\frac{\partial}{\partial\alpha_1}(N_{12}^rA_2)-\frac{\partial}{\partial\alpha_2}(N_{22}^rA_1)-N_{21}^r\frac{\partial A_2}{\partial\alpha_1}+N_{11}^r\frac{\partial A_1}{\partial\alpha_2}-Q_{23}^r\frac{A_1A_2}{R_2}=A_1A_2q_2^r \quad (11.2.16)$$

$$-\frac{\partial}{\partial\alpha_1}(Q_{13}^rA_2)-\frac{\partial}{\partial\alpha_2}(Q_{23}^rA_1)+A_1A_2\left(\frac{N_{11}^r}{R_1}+\frac{N_{22}^r}{R_2}\right)=A_1A_2q_3^r \quad (11.2.17)$$

where

$$Q_{13}^rA_1A_2 = \frac{\partial}{\partial\alpha_1}(M_{11}^rA_2)+\frac{\partial}{\partial\alpha_2}(M_{21}^rA_1)+M_{12}^r\frac{\partial A_1}{\partial\alpha_2}-M_{22}^r\frac{\partial A_2}{\partial\alpha_1} \quad (11.2.18)$$

$$Q_{23}^rA_1A_2 = \frac{\partial}{\partial\alpha_1}(M_{12}^rA_2)+\frac{\partial}{\partial\alpha_2}(M_{22}^rA_1)+M_{21}^r\frac{\partial A_2}{\partial\alpha_1}-M_{11}^r\frac{\partial A_1}{\partial\alpha_2} \quad (11.2.19)$$

This set is usually solved by way of stress functions for the unknown residual stresses. Once these are known, the following vibration equations are solved:

$$-\frac{\partial}{\partial\alpha_1}(N_{11}A_2)-\frac{\partial}{\partial\alpha_2}(N_{21}A_1)-N_{12}\frac{\partial A_1}{\partial\alpha_2}+N_{22}\frac{\partial A_2}{\partial\alpha_1}$$

$$-Q_{13}\frac{A_1A_2}{R_1}+A_1A_2\rho h\ddot{u}_1=A_1A_2q_1 \tag{11.2.20}$$

$$-\frac{\partial}{\partial\alpha_1}(N_{12}A_2)-\frac{\partial}{\partial\alpha_2}(N_{22}A_1)-N_{21}\frac{\partial A_2}{\partial\alpha_1}+N_{11}\frac{\partial A_1}{\partial\alpha_2}$$

$$-Q_{23}\frac{A_1A_2}{R_2}+A_1A_2\rho h\ddot{u}_2=A_1A_2q_2 \tag{11.2.21}$$

$$-\frac{\partial}{\partial\alpha_1}(Q_{13}A_2)-\frac{\partial}{\partial\alpha_2}(Q_{23}A_1)+A_1A_2\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)-\frac{\partial}{\partial\alpha_1}\left(N_{11}^r\frac{A_2}{A_1}\frac{\partial u_3}{\partial\alpha_1}\right)$$

$$-\frac{\partial}{\partial\alpha_2}\left(N_{22}^r\frac{A_1}{A_2}\frac{\partial u_3}{\partial\alpha_2}\right)-\frac{\partial}{\partial\alpha_1}\left(N_{21}^r\frac{\partial u_3}{\partial\alpha_2}\right)$$

$$-\frac{\partial}{\partial\alpha_2}\left(N_{12}^r\frac{\partial u_3}{\partial\alpha_1}\right)+A_1A_2\rho h\ddot{u}_3=A_1A_2q_3 \tag{11.2.22}$$

and where

$$Q_{13}A_1A_2=\frac{\partial}{\partial\alpha_1}(M_{11}A_2)+\frac{\partial}{\partial\alpha_2}(M_{21}A_1)+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1} \tag{11.2.23}$$

$$Q_{23}A_1A_2=\frac{\partial}{\partial\alpha_1}(M_{12}A_2)+\frac{\partial}{\partial\alpha_2}(M_{22}A_1)+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2} \tag{11.2.24}$$

Note that in the fourth to seventh terms of Eq. (11.2.22), we have neglected N_{11}, N_{22}, and N_{12} since they are small when compared to N_{11}^r, N_{22}^r, and N_{12}^r. Not only is this a good assumption for most engineering problems of this type, but also a necessary one if we wish to preserve the linearity of our equations. If this assumption is not possible, for instance, in dynamic buckling problems, a nonlinear set of equations results.

11.3. PURE MEMBRANES

Equations (11.2.10)–(11.2.12) give us the opportunity to obtain the equations of motion of pure membranes. By *pure membrane*, we mean that the structure is a skin stretched over a frame under initial tension N_{11}^r, N_{22}^r, and N_{12}^r. There is a school of thought that holds that pure membranes are not able to support shear forces ($N_{12}^r=0$), but this is technically not supportable. That shear can exist without buckling the membrane can be shown by stretching a rubber skin over a rectangular frame and stretching it at different nonuniform amounts. In special cases such as a uniformly stretched drumhead (circular membrane), shear is 0. The only restriction on a pure membrane is that it cannot support compressive membrane stresses.

A membrane skin has negligible bending resistance ($D=0$). This implies that all bending moments are 0. Equations (11.2.20) and (11.2.21) become

$$-\frac{\partial}{\partial\alpha_1}(N_{11}A_2)-\frac{\partial}{\partial\alpha_2}(N_{21}A_1)-N_{12}\frac{\partial A_1}{\partial\alpha_2}+N_{22}\frac{\partial A_2}{\partial\alpha_1}+A_1A_2\rho h\ddot{u}_1=A_1A_2q_1 \tag{11.3.1}$$

$$-\frac{\partial}{\partial\alpha_1}(N_{12}A_2)-\frac{\partial}{\partial\alpha_2}(N_{22}A_1)-N_{21}\frac{\partial A_2}{\partial\alpha_1}+N_{11}\frac{\partial A_1}{\partial\alpha_2}+A_1A_2\rho h\ddot{u}_2=A_1A_2q_2 \tag{11.3.2}$$

These equations describe the motion of the membrane material in the tangent plane. This motion is independent of the initial stress state. Equation (11.2.22) becomes

$$A_1A_2\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)-\frac{\partial}{\partial\alpha_1}\left(N_{11}^r\frac{A_2}{A_1}\frac{\partial u_3}{\partial\alpha_1}\right)-\frac{\partial}{\partial\alpha_2}\left(N_{22}^r\frac{A_1}{A_2}\frac{\partial u_3}{\partial\alpha_2}\right)$$
$$-\frac{\partial}{\partial\alpha_1}\left(N_{21}^r\frac{\partial u_3}{\partial\alpha_2}\right)-\frac{\partial}{\partial\alpha_2}\left(N_{12}^r\frac{\partial u_3}{\partial\alpha_1}\right)+A_1A_2\rho h\ddot{u}_3=A_1A_2q_3 \tag{11.3.3}$$

This is the membrane equation for transverse vibration. For membranes that are curved surface, the three equations of motion are coupled. For flat membranes, Eq. (11.3.3) is independent of Eqs. (11.3.1) and (11.3.2).

The initial stresses in the membrane are calculated from Eqs. (11.2.15)–(11.2.17), which become (q_1^r, q_2^r, and q_3^r are in this case static loads)

$$-\frac{\partial}{\partial\alpha_1}(N_{11}^rA_2)-\frac{\partial}{\partial\alpha_2}(N_{21}^rA_1)-N_{12}^r\frac{\partial A_1}{\partial\alpha_2}+N_{22}^r\frac{\partial A_2}{\partial\alpha_1}=A_1A_2q_1^r \tag{11.3.4}$$

$$-\frac{\partial}{\partial\alpha_1}(N_{12}^rA_2)-\frac{\partial}{\partial\alpha_2}(N_{22}^rA_1)-N_{21}^r\frac{\partial A_2}{\partial\alpha_1}+N_{11}^r\frac{\partial A_1}{\partial\alpha_2}=A_1A_2q_2^r \tag{11.3.5}$$

$$A_1A_2\left(\frac{N_{11}^r}{R_1}+\frac{N_{22}^r}{R_2}\right)=A_1A_2q_3^r \tag{11.3.6}$$

11.4. EXAMPLE: THE CIRCULAR MEMBRANE

Let us look at the classical membrane problem. It is the case of a drum skin shown in Fig. 2, stretched uniformly around the periphery such that it creates a uniform boundary tension:

$$N_{rr}^{r}=N_{rr}^{*} \tag{11.4.1}$$

At this point, we could state that the resulting tension is uniform through the entire membrane and proceed to the vibration problem, but let us prove this using the static equations.

The fundamental form is

$$(\mathrm{d}s)^2=(\mathrm{d}r)^2+r^2(\mathrm{d}\theta)^2 \tag{11.4.2}$$

Thus $A_1=1$, $A_2=r$, $\mathrm{d}\alpha_1=\mathrm{d}r$, and $\mathrm{d}\alpha_2=\mathrm{d}\theta$. Let us now calculate the initial stress distribution in the interior of the membrane. Equations (11.3.4) and (11.3.5) become [Eq. (11.3.6) is inapplicable since a flat membrane has no curvature]

$$-\frac{\partial}{\partial r}(N_{rr}^{r}r)-\frac{\partial}{\partial\theta}(N_{\theta r}^{r})+N_{\theta\theta}^{r}=0 \tag{11.4.3}$$

$$-\frac{\partial}{\partial r}(N_{r\theta}^{r}r)-\frac{\partial}{\partial\theta}(N_{\theta\theta}^{r})-N_{\theta r}^{r}=0 \tag{11.4.4}$$

with the boundary condition, at $r=a$,

$$N_{rr}^{r}=N_{rr}^{*} \tag{11.4.5}$$

Since the loading is axisymmetric, the stress state has to be axisymmetric. Thus

$$\frac{\partial(\cdot)}{\partial\theta}=0 \tag{11.4.6}$$

and

$$N_{r\theta}^{r}=0 \tag{11.4.7}$$

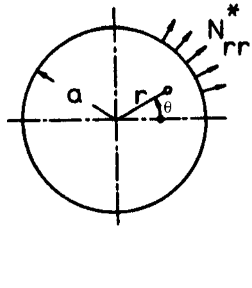

FIG. 2 Circular membrane.

This gives

$$-\frac{\partial}{\partial r}(N_{rr}^r r)+N_{\theta\theta}^r=0 \tag{11.4.8}$$

This equation is satisfied if

$$F=N_{rr}^r r \tag{11.4.9}$$

$$\frac{dF}{dr}=N_{\theta\theta}^r \tag{11.4.10}$$

Since

$$\varepsilon_{rr}^r=\frac{du_r}{dr} \tag{11.4.11}$$

and

$$\varepsilon_{\theta\theta}^r=\frac{u_r}{r} \tag{11.4.12}$$

or

$$\frac{d}{dr}(r\varepsilon_{\theta\theta}^r)=\frac{du_r}{dr} \tag{11.4.13}$$

equating Eqs. (11.4.13) and (11.4.11) gives

$$\frac{d}{dr}(r\varepsilon_{\theta\theta}^r)-\varepsilon_{rr}^r=0 \tag{11.4.14}$$

This equation is known as the *compatibility equation*. Furthermore, from

$$N_{rr}^r = K(\varepsilon_{rr}^r+\mu\varepsilon_{\theta\theta}^r) \tag{11.4.15}$$

$$N_{\theta\theta}^r = K(\varepsilon_{\theta\theta}^r+\mu\varepsilon_{rr}^r) \tag{11.4.16}$$

we get

$$\varepsilon_{rr}^r=\frac{1}{K}(N_{rr}^r+\mu N_{\theta\theta}^r)=\frac{1}{K}\left(\frac{F}{r}+\mu\frac{dF}{dr}\right) \tag{11.4.17}$$

and

$$\varepsilon_{\theta\theta}^r=\frac{1}{K}(N_{\theta\theta}^r+\mu N_{rr}^r)=\frac{1}{K}\left(\frac{dF}{dr}+\mu\frac{F}{r}\right) \tag{11.4.18}$$

Equation (11.4.14) thus becomes

$$\frac{d^2F}{dr^2}+\frac{1}{r}\frac{dF}{dr}-\frac{F}{r^2}=0 \tag{11.4.19}$$

or

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(Fr)}{dr}\right]=0 \tag{11.4.20}$$

Integrating gives

$$F = C_1 r + C_2 \frac{1}{r} \tag{11.4.21}$$

We have, at r $= a$,

$$F = N_{rr}^* a \tag{11.4.22}$$

Also, at $r = 0$, a singularity cannot exist, which implies that $C_2 = 0$. Thus

$$C_1 = N_{rr}^* \tag{11.4.23}$$

Therefore,

$$F = N_{rr}^* r \tag{11.4.24}$$

or

$$N_{rr}^r = \frac{F}{r} = N_{rr}^* \tag{11.4.25}$$

$$N_{\theta\theta}^r = \frac{\mathrm{d}F}{\mathrm{d}r} = N_{rr}^* \tag{11.4.26}$$

This result implies that a circular membrane under uniform boundary tension has equal and constant stress resultants in the radial and circumferential directions at any point in its interior:

$$N_{rr}^r = N_{\theta\theta}^r = N_{rr}^* \tag{11.4.27}$$

Note that this is the case only for uniform boundary tension. If the boundary tension is nonuniform, nonuniform interior membrane forces result that include shear forces.

We may now proceed to Eq. (11.3.3), which describes the transverse vibration of the membrane. It becomes, for our example,

$$-N_{rr}^* \left(\frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r} \frac{\partial u_3}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_3}{\partial \theta^2} \right) + \rho h \ddot{u}_3 = q_3 \tag{11.4.28}$$

To solve the eigenvalue problem, we set $q_3 = 0$ and write

$$u_3(r, \theta, t) = U_{3k} \mathrm{e}^{j\omega_k t} \tag{11.4.29}$$

Substituting this in the equation of motion gives

$$-N_{rr}^* \left(\frac{\partial^2 U_{3k}}{\partial r^2} + \frac{1}{r} \frac{\partial U_{3k}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U_{3k}}{\partial \theta^2} \right) - \rho h \omega_k^2 U_{3k} = 0 \tag{11.4.30}$$

Suspecting that we may be able to separate variables, we write

$$U_{3k}(r, \theta) = R(r)\Theta(\theta) \tag{11.4.31}$$

Substituting this gives

$$\frac{r^2}{R}\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{r}{R}\frac{\mathrm{d}R}{\mathrm{d}r}+r^2\frac{\omega_k^2\rho h}{N_{rr}^*}=-\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} \tag{11.4.32}$$

The left and right sides of Eq. (11.4.32) must be equal to the same constant, which we call p^2. This gives

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}+p^2\Theta=0 \tag{11.4.33}$$

and

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+r\frac{\mathrm{d}R}{\mathrm{d}r}+\left(\frac{\omega_k^2\rho h}{N_{rr}^*}r^2-\rho^2\right)R=0 \tag{11.4.34}$$

The solution of Eq. (11.4.33) is

$$\Theta=A\cos p(\theta-\phi) \tag{11.4.35}$$

where ϕ is an arbitrary angle and A is an arbitrary constant. However, for the closed drumhead, Θ must be a periodic function of period 2π to preserve continuity of deflection. Thus

$$p=n \quad (n=0,1,2,\ldots) \tag{11.4.36}$$

Defining

$$\lambda^2=\frac{\omega_k^2\rho h}{N_{rr}^*} \tag{11.4.37}$$

and a new variable

$$\psi^2=\lambda^2r^2 \tag{11.4.38}$$

we may write Eq. (11.4.34) as

$$\frac{\mathrm{d}^2R}{\mathrm{d}\psi^2}+\frac{1}{\psi}\frac{\mathrm{d}R}{\mathrm{d}\psi}+\left(1-\frac{n^2}{\psi^2}\right)R=0 \tag{11.4.39}$$

This type of equation is called *Bessel's differential equation of integer order n*, as we saw in Chapter 5 for the circular plate. It can be solved using a power series, which can be formulated into what are known as *Bessel functions*. For each value of the integer n there are two linearly independent solutions of Bessel's equation. One of them is the Bessel function of the first kind of order n, denoted as $J_n(\psi)$. The other is the Bessel function of the second kind of order n, denoted $Y_n(\psi)$. Thus the solution of Eq. (11.4.39) is, for each n,

$$R_n=B_nJ_n(\psi)+C_nY_n(\psi) \tag{11.4.40}$$

Note that

$$Y_n(0)=\infty \tag{11.4.41}$$

Table 1 Values for $(\lambda a)_{mn}$

	n			
m	0	1	2	3
0	2.404	5.520	8.654	11.792
1	3.832	7.016	10.173	13.323
2	5.135	8.417	11.620	14.796
3	6.379	9.760	13.017	16.224

Thus, since it is physically impossible for the drumhead to have an infinite deflection at its center, it follows that

$$C_n = 0 \tag{11.4.42}$$

The other condition that has to be satisfied is that at $r = a$,

$$u_3 = 0 \tag{11.4.43}$$

which implies that

$$R(\psi = \lambda a) = 0 \tag{11.4.44}$$

or that

$$J_n(\lambda a) = 0 \tag{11.4.45}$$

For a given n, this equation has an infinite number of roots $(\lambda a)_{mn}$, identified by $m = 0, 1, 2, \ldots$ in ascending order. A few of these are listed in Table 1. The natural frequencies of the membrane are therefore given by

$$\omega_k = \omega_{mn} = \frac{(\lambda a)_{mn}}{a}\sqrt{\frac{N_{rr}^*}{\rho h}} \tag{11.4.46}$$

To obtain the natural mode, we substitute in turn each natural frequency back into Eq. (11.4.31) and get, utilizing Eqs. (11.4.35) and (11.4.40),

$$U_k = U_{mn} = J_n(\lambda_{mn} r)\cos n(\theta - \phi) \tag{11.4.47}$$

This problem was first solved by Pagani (1829).

11.5. SPINNING SAW BLADE

When a circular saw blade (Fig. 3) is spinning with a constant rotational speed Ω in radians per second, centrifugal forces create a stress field that acts like an initial stress in raising natural frequencies. The first treatment of a saw-blade-like case was given by Southwell (1922) . Other

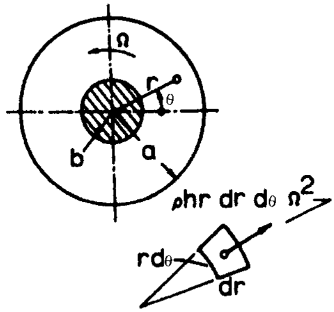

FIG. 3 Spinning saw blade.

variations include the effect of temperature distribution and purposely induced residual stresses, as typically studied by Mote (1965, 1966).

Let us first obtain the centrifugal stresses. Since we have in this case an axisymmetric situation, all derivations with respect to θ vanish. For $\alpha_1 = r$, $A_1 = 1$, $\alpha_2 = \theta$, $A_2 = r$, Eq. (11.2.15) becomes

$$-\frac{\mathrm{d}}{\mathrm{d}r}(rN_{rr}^r) + N_{\theta\theta}^r = rq_r \tag{11.5.1}$$

where the load q_r is the centrifugal force created by a mass element as it is spinning at radius r divided by the area $r\mathrm{d}\theta\mathrm{d}r$:

$$q_r = \rho h\Omega^2 r \tag{11.5.2}$$

Since the membrane strains are

$$\varepsilon_{rr}^0 = \frac{\mathrm{d}u_r^r}{\mathrm{d}r} \tag{11.5.3}$$

$$\varepsilon_{\theta\theta}^0 = \frac{u_r^r}{r} \tag{11.5.4}$$

where the superscript r indicates again residual, we obtain

$$r\frac{d^2u_r^r}{dr^2} + \frac{du_r^r}{dr} - \frac{u_r^r}{r} = -\frac{\rho h\Omega^2 r^2}{K} \tag{11.5.5}$$

This can be written as

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru_r^r)\right] = -\frac{\rho h\Omega^2 r^2}{K} \tag{11.5.6}$$

Integrating this gives the radial displacement due to the centrifugal force as

$$u_r^r = -\frac{\rho h\Omega^2 r^3}{8K} + C_1 r + \frac{C_2}{r} \tag{11.5.7}$$

The integration constants have to be evaluated from the boundary conditions. In case of the freely spinning saw blade, these boundary conditions are

$$u_r^r(r=b)=0 \tag{11.5.8}$$

$$N_{rr}^r(r=a)=K\left(\frac{\mathrm{d}u_r^r}{\mathrm{d}r}+\mu\frac{u_r^r}{r}\right)_{r=a}=0 \tag{11.5.9}$$

Since

$$N_{rr}^r=(1+\mu)KC_1-\frac{(1-\mu)K}{r^2}C_2-\frac{\rho h\Omega^2(3+\mu)}{8}r^2 \tag{11.5.10}$$

we obtain

$$\begin{bmatrix}(1+\mu)a^2 & \mu-1\\ b^2 & 1\end{bmatrix}\begin{Bmatrix}C_1\\ C_2\end{Bmatrix}=\frac{\rho h\Omega^2}{8K}\begin{Bmatrix}a^4(3+\mu)\\ b^4\end{Bmatrix} \tag{11.5.11}$$

From this, we find that

$$C_1=\frac{\rho h\Omega^2}{8K}\frac{a^4(3+\mu)+(1-\mu)b^4}{(1+\mu)a^2+(1-\mu)b^2} \tag{11.5.12}$$

$$C_2=\frac{\rho h\Omega^2}{8K}\frac{(1+\mu)a^2b^4-(3+\mu)a^4b^2}{(1+\mu)a^2+(1-\mu)b^2} \tag{11.5.13}$$

The membrane stress resultant N_{rr}^t is therefore defined. The membrane stress resultant $N_{\theta\theta}^r$ is given by

$$N_{\theta\theta}^r=(1+\mu)KC_1+\frac{(1-\mu)K}{r^2}C_2-\frac{\rho h\Omega^2(3\mu+1)}{8}r^2 \tag{11.5.14}$$

The equation of motion is defined by Eq. (11.2.22). From it, we obtain as the equation of motion of the spinning saw blade,

$$D\nabla^4u_3-\frac{1}{r}\frac{\partial}{\partial r}\left(N_{rr}^r r\frac{\partial u_3}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(N_{\theta\theta}^r\frac{1}{r}\frac{\partial u_3}{\partial\theta}\right)+\rho h\ddot{u}_3=q_3 \tag{11.5.15}$$

To obtain the natural frequencies, we set $q_3=0$ and let

$$u_3=U_3e^{j\omega t} \tag{11.5.16}$$

This gives

$$D\nabla^4U_3-\frac{1}{r}\frac{\partial}{\partial r}\left(N_{rr}^r r\frac{\partial U_3}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(N_{\theta\theta}^r\frac{1}{r}\frac{\partial U_3}{\partial\theta}\right)-\rho h\omega^2U_3=0 \tag{11.5.17}$$

A general exact solution in closed form has not yet been found for the spinning saw problem. Both Southwell (1922) and Mote (1965) approached it with variational methods. Mote used the Raleigh–Ritz technique with the function

$$U_3(r,\theta)=(r-b)\sum_{i=0}^{i=k}a_i(r-b)^i\cos n\theta \tag{11.5.18}$$

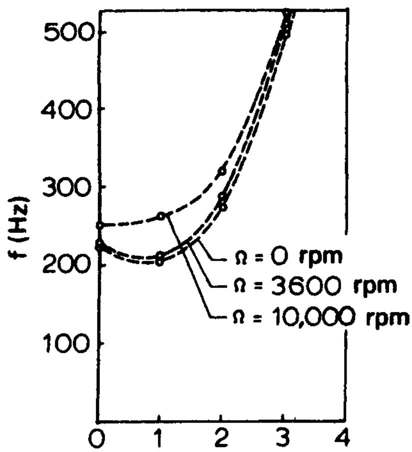

FIG. 4 Change in natural frequencies of a spinning saw blade as function of spin speed.

Numerical values for the case where $a=80$mm, $b=20$mm, $h=1$mm, the material is carbon steel, and the rotational speed is in rotations per minute are shown in Fig. 4. All results are for modes that have no nodal circles, only nodal diameters. Nodal circles occur only at still higher frequencies for this case. It is interesting to note that the influence of the centrifugal force can be ignored at two-pole-induction-motor speed (3600rpm). This seems also to be the general finding for saw blades of other normally used dimensions. However, as the speed increases beyond that, the centrifugal effect gains importance because its influence grows with the square of the rotational speed.

Treatment of other membrane stress-producing effects, such as heating (thermal stresses) or "tensioning" (a process where beneficial residual stresses are hammered into the blade), can be found in Mote (1966).

11.6. DONNELL–MUSHTARI–VLASOV EQUATIONS EXTENDED TO INCLUDE INITIAL STRESSES

Since we have already seen that the Donnell–Mushtari–Vlasov equations are a useful simplification of Love's equations, let us proceed through an identical simplification process with Eqs. (11.2.20)–(11.2.24). We obtain

$$D\nabla^4 u_3 + \nabla_k^2 \phi - \nabla_r^2 u_3 + \rho h \ddot{u}_3 = q_3 \qquad (11.6.1)$$

and

$$Eh\nabla_k^2 u_3 - \nabla^4 \phi = 0 \tag{11.6.2}$$

where

$$\nabla_r^2(\cdot) = \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}\left(N_{11}^r \frac{A_2}{A_1}\frac{\partial(\cdot)}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(N_{22}^r \frac{A_1}{A_2}\frac{\partial(\cdot)}{\partial\alpha_2}\right) + \frac{\partial}{\partial\alpha_1}\left(N_{21}^r \frac{\partial(\cdot)}{\partial\alpha_2}\right) + \frac{\partial}{\partial\alpha_2}\left(N_{12}^r \frac{\partial(\cdot)}{\partial\alpha_1}\right)\right] \tag{11.6.3}$$

and where $\nabla_k^2(\cdot)$ is defined by Eq. (6.7.11) and $\nabla^2(\cdot)$ by Eq. (4.4.21).

To find the natural frequencies and modes, we set $q_3 = 0$ and

$$u_3(\alpha_1, \alpha_2, t) = U_3(\alpha_1, \alpha_2)e^{j\omega t} \tag{11.6.4}$$

$$\phi(\alpha_1, \alpha_2, t) = \Phi(\alpha_1, \alpha_2)e^{j\omega t} \tag{11.6.5}$$

This gives

$$D\nabla^4 U_3 + \nabla_k^2 \Phi - \nabla_r^2 U_3 - \rho h \omega^2 U_3 = 0 \tag{11.6.6}$$

and

$$Eh\nabla_k^2 U_3 - \nabla^4 \Phi = 0 \tag{11.6.7}$$

Operating with ∇^4 and ∇_k^2 on these equations again allows a combination

$$D\nabla^8 U_3 + Eh\nabla_k^4 U_3 - \nabla^4\nabla_r^2 U_3 - \rho h\omega^2 \nabla^4 U_3 = 0 \tag{11.6.8}$$

Let us now investigate, as an example, a closed circular cylindrical shell that is under a uniform boundary tension T in newtons per meter as shown in Fig. 5. By solving the static set of equations, it can be shown that in this case we have throughout the shell, $N_{xx}^r = T$ and $N_{\theta\theta}^r = N_{x\theta}^r = 0$.

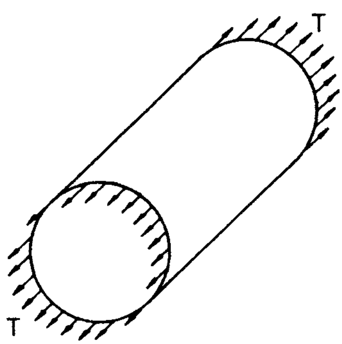

FIG. 5 Circular cylindrical shell under a constant axial tension.

Let us first evaluate the operators. They become

$$\nabla_r^2(\cdot) = T\frac{\partial^2(\cdot)}{\partial x^2} \tag{11.6.9}$$

$$\nabla^2(\cdot) = \frac{1}{a^2}\frac{\partial^2(\cdot)}{\partial\theta^2} + \frac{\partial^2(\cdot)}{\partial x^2} \tag{11.6.10}$$

$$\nabla_k^2(\cdot) = \frac{1}{a^2}\frac{\partial^2(\cdot)}{\partial x^2} \tag{11.6.11}$$

If the shell is simply supported at both ends, we find that the mode shape

$$U_3(x,\theta) = \sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{11.6.12}$$

satisfies both the boundary conditions and Eq. (11.6.8). This equation becomes

$$D\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^4 + \frac{Eh}{a^2}\left(\frac{m\pi}{L}\right)^4 + \left[T\left(\frac{m\pi}{L}\right)^2 - \rho h\omega^2\right]\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2 = 0 \tag{11.6.13}$$

Solving for ω, we may write the result as

$$\omega_{mnT}^2 = \omega_{mn0}^2 + \frac{T}{\rho h}\left(\frac{m\pi}{L}\right)^2 \tag{11.6.14}$$

where ω_{mn0} is the natural frequency of the shell when there is no boundary tension, as given by Eq. (6.12.3), and ω_{mnT} is the natural frequency when tension or compression (since T could be negative) is present. Note that the natural frequency increases as the tension T is increased. If there is compression, $T < 0$, the natural frequency is decreased from that of the compression-free shell. As a matter of fact, when

$$T = -\left(\frac{L}{m\pi}\right)^2 \omega_{mn0}^2 \rho h \tag{11.6.15}$$

we reach a point where the natural frequency for a particular m, n mode becomes 0. This is because at that value of T, the critical buckling load for a buckling mode identical in shape to this particular m, n mode has been reached and the shell has zero stiffness for this particular mode.

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12

Shell Equations with Shear Deformation and Rotatory Inertia

In all previous developments, we have taken the shear deformation to be 0. This assumption allowed us to obtain β_1 and β_2 as functions of the displacements u_1, u_2, and u_3. However, for shells where the thickness is large as compared to either the overall dimension or to the modal wavelength of the highest frequency of interest, shear deformation can no longer be neglected. We must allow for the fact that $\varepsilon_{13} \neq 0$ and $\varepsilon_{23} \neq 0$. This means that we will have two additional unknowns, β_1 and β_2.

12.1. EQUATIONS OF MOTION

From Eqs. (2.3.52) and (2.3.53), we have

$$\varepsilon_{13} = A_1 \frac{\partial}{\partial \alpha_3}\left(\frac{U_1}{A_1}\right) + \frac{1}{A_1}\frac{\partial U_3}{\partial \alpha_1} \tag{12.1.1}$$

$$\varepsilon_{23} = A_2 \frac{\partial}{\partial \alpha_3}\left(\frac{U_2}{A_2}\right) + \frac{1}{A_2}\frac{\partial U_3}{\partial \alpha_2} \tag{12.1.2}$$

Substituting Eqs. (2.4.1) to (2.4.3) gives

$$\varepsilon_{13} = \beta_1 - \frac{u_1}{R_1} + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1} \tag{12.1.3}$$

$$\varepsilon_{23}=\beta_2-\frac{u_2}{R_2}+\frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2} \tag{12.1.4}$$

In Sec. 2.4, we were able to solve these two equations for β_1 and β_2. This is now not possible. β_1 and β_2 have to be treated as unknowns.

All this implies, of course, that we should now use the definitions of shear stress in a more direct way:

$$\sigma_{13}=G\varepsilon_{13} \tag{12.1.5}$$

$$\sigma_{23}=G\varepsilon_{23} \tag{12.1.6}$$

We have to consider ε_{13} and ε_{23}, and therefore σ_{13} and σ_{23}, to be the values at the neutral surface. Since the free surfaces of the shell can clearly not support a shear stress, the average values of σ_{13} and σ_{23} are less. If σ_{13}^a and σ_{23}^a are the average shear stress, then

$$\sigma_{13}^a=k'\sigma_{13} \tag{12.1.7}$$

$$\sigma_{23}^a=k'\sigma_{23} \tag{12.1.8}$$

The factor k' depends on the actual distribution of shear stress in the α_3 direction. If the distribution is parabolic as sketched in Fig. 1, $k'=\frac{2}{3}$.

Summing up over the shell thickness, we obtain the shear force resultants Q_{13} and Q_{23}:

$$Q_{13}=\int_{-h/2}^{h/2}\sigma_{13}^a\left(1+\frac{\alpha_3}{R_2}\right)d\alpha_3 \tag{12.1.9}$$

or

$$Q_{13}=k'\sigma_{13}h=k'\varepsilon_{13}Gh \tag{12.1.10}$$

Similarly,

$$Q_{23}=k'\varepsilon_{23}Gh \tag{12.1.11}$$

Since rotatory inertia effects become noticeable at frequencies where shear deflections have to be considered, it is advisable to include rotatory

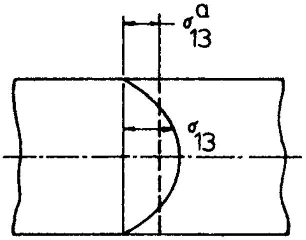

Fig. 1 Shear stress distribution.

inertia at this point. For this purpose, we use as kinetic energy expression Eq. (2.6.7), with the $\dot{\beta}_1$ and $\dot{\beta}_2$ terms not neglected. The equations of motion (Soedel, 1982) thus become

$$-\frac{\partial(N_{11}A_2)}{\partial\alpha_2}-\frac{\partial(N_{21}A_1)}{\partial\alpha_2}-N_{12}\frac{\partial A_1}{\partial\alpha_2}+N_{22}\frac{\partial A_2}{\partial\alpha_1}$$

$$-A_1A_2\frac{k'\varepsilon_{13}Gh}{R_1}+A_1A_2\rho h\ddot{u}_1=A_1A_2q_1 \tag{12.1.12}$$

$$-\frac{\partial(N_{12}A_2)}{\partial\alpha_1}-\frac{\partial(N_{22}A_1)}{\partial\alpha_2}-N_{21}\frac{\partial A_2}{\partial\alpha_1}+N_{11}\frac{\partial A_1}{\partial\alpha_2}$$

$$-A_1A_2\frac{k'\varepsilon_{23}Gh}{R_2}+A_1A_2\rho h\ddot{u}_2=A_1A_2q_2 \tag{12.1.13}$$

$$-k'Gh\frac{\partial(\varepsilon_{13}A_2)}{\partial\alpha_1}-k'Gh\frac{\partial(\varepsilon_{23}A_1)}{\partial\alpha_2}+A_1A_2\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)$$

$$+A_1A_2\rho h\ddot{u}_3=A_1A_2q_3 \tag{12.1.14}$$

$$\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(M_{21}A_1)}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}$$

$$-Ghk'\varepsilon_{13}A_1A_2-A_1A_2\frac{\rho h^3}{12}\ddot{\beta}_1=0 \tag{12.1.15}$$

$$\frac{\partial(M_{12}A_2)}{\partial\alpha_1}+\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}$$

$$-Ghk'\varepsilon_{23}A_1A_2-A_1A_2\frac{\rho h^3}{12}\ddot{\beta}_2=0 \tag{12.1.16}$$

We have five equations and five unknowns: u_1, u_2, u_3, β_1, and β_2.

Summarizing the strain–displacement relationships, we have now

$$\varepsilon_{11}^0=\frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1}+\frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{u_3}{R_1} \tag{12.1.17}$$

$$\varepsilon_{22}^0=\frac{1}{A_2}\frac{\partial u_2}{\partial\alpha_2}+\frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{u_3}{R_2} \tag{12.1.18}$$

$$\varepsilon_{12}^0=\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2}{A_2}\right)+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_1}{A_1}\right) \tag{12.1.19}$$

$$k_{11}=\frac{1}{A_1}\frac{\partial\beta_1}{\partial\alpha_1}+\frac{\beta_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} \tag{12.1.20}$$

$$k_{22}=\frac{1}{A_2}\frac{\partial\beta_2}{\partial\alpha_2}+\frac{\beta_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} \tag{12.1.21}$$

$$k_{12}=\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\beta_2}{A_2}\right)+\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\beta_1}{A_1}\right) \tag{12.1.22}$$

$$\varepsilon_{13}=\frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1}-\frac{u_1}{R_1}+\beta_1 \tag{12.1.23}$$

$$\varepsilon_{23}=\frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2}-\frac{u_2}{R_2}+\beta_2 \tag{12.1.24}$$

The necessary boundary conditions become (the subscripts n and t denote normal to boundary and tangential to the boundary, respectively)

$$\begin{aligned}&N_{nn}=N_{nn}^* \quad \text{or} \quad u_n=u_n^*\\&N_{nt}=N_{nt}^* \quad \text{or} \quad u_t=u_t^*\\&Q_{n3}=Q_{n3}^* \quad \text{or} \quad u_3=u_3^*\\&M_{nn}=M_{nn}^* \quad \text{or} \quad \beta_n=\beta_n^*\\&M_{nt}=M_{nt}^* \quad \text{or} \quad \beta_t=\beta_t^*\end{aligned} \tag{12.1.25}$$

12.2. BEAMS WITH SHEAR DEFLECTION AND ROTATORY INERITA

As our first reduction, let us derive the equation of motion of a transversely vibrating beam, commonly known as the *Timoshenko beam equation* (Timoshenko, 1921). We set $\alpha_1=x$, $A_1=1$, $\alpha_2=y$, $A_2=1$, $\partial(\cdot)/\partial\alpha_2=0$. The equations of motion then reduce to

$$-Ghk'\frac{\partial\varepsilon_{x3}}{\partial x}+\rho h\ddot{u}_3=q_3 \tag{12.2.1}$$

$$\frac{\partial M_{xx}}{\partial x}-Ghk'\varepsilon_{x3}-\frac{\rho h^3}{12}\ddot{\beta}_x=0 \tag{12.2.2}$$

Stress–displacement relationships reduce to

$$k_{xx}=\frac{\partial\beta_x}{\partial x} \tag{12.2.3}$$

$$\varepsilon_{x3}=\frac{\partial u_3}{\partial x}+\beta_x \tag{12.2.4}$$

Since the equation for M_{xx} becomes

$$M_{xx}=D\frac{\partial\beta_x}{\partial x} \tag{12.2.5}$$

Eqs. (12.2.1) and (12.2.2) become

$$-Ghk'\frac{\partial^2 u_3}{\partial x^2}-Ghk'\frac{\partial \beta_x}{\partial x}+\rho h\ddot{u}_3=q_3 \tag{12.2.6}$$

$$D\frac{\partial^2 \beta_x}{\partial x^2}-Ghk'\beta_x-Ghk'\frac{\partial u_3}{\partial x}-\frac{\rho h^3}{12}\ddot{\beta}_x=0 \tag{12.2.7}$$

Differentiating Eq. (12.2.7) with respect to x gives

$$D\frac{\partial^3 \beta_x}{\partial x^3}-Ghk'\frac{\partial \beta_x}{\partial x}-Ghk'\frac{\partial^2 u_3}{\partial x^2}-\frac{\rho h^3}{12}\frac{\partial(\ddot{\beta}_x)}{\partial x}=0 \tag{12.2.8}$$

From Eq. (12.2.6), we obtain

$$\frac{\partial \beta_x}{\partial x}=\frac{\rho}{Gk'}\ddot{u}_3-\frac{\partial^2 u_3}{\partial x^2}-\frac{q_3}{Ghk'} \tag{12.2.9}$$

Substituting this into Eq. (12.2.8) gives

$$D\frac{\partial^4 u_3}{\partial x^4}+\rho h\frac{\partial^2 u_3}{\partial t^2}-\left(\frac{D\rho}{Gk'}+\frac{\rho h^3}{12}\right)\frac{\partial^4 u_3}{\partial x^2\partial t^2}+\frac{\rho^2 h^3}{12Gk'}\frac{\partial^4 u_3}{\partial t^4}$$
$$=q_3+\frac{\rho h^2}{12Gk'}\frac{\partial^2 q_3}{\partial t^2}-\frac{D}{Ghk'}\frac{\partial^2 q_3}{\partial x^2} \tag{12.2.10}$$

Multiplying the equation by the width of the beam, we recognize that

$$Db=EI \tag{12.2.11}$$

$$\rho hb=\rho A \tag{12.2.12}$$

$$q_3 b=p \tag{12.2.13}$$

where I is the area moment of inertia, A the cross-sectional area, and p is the force per unit length. Therefore, Eq. (12.2.10) becomes

$$EI\frac{\partial^4 u_3}{\partial x^4}+\rho A\frac{\partial^2 u_3}{\partial t^2}-\left(\frac{EI\rho}{Gk'}+\rho I\right)\frac{\partial^4 u_3}{\partial x^2\partial t^2}+\frac{\rho^2 I}{Gk'}\frac{\partial^4 u_3}{\partial t^4}$$
$$=p+\frac{\rho I}{Ghk'}\frac{\partial^2 p}{\partial t^2}-\frac{EI}{Ghk'}\frac{\partial^2 p}{\partial x^2} \tag{12.2.14}$$

This is Timoshenko's beam equation (Timoshenko, 1921).

However, it is probably better to work directly with Eqs. (12.2.6) and (12.2.7), which become

$$-GAk'\left(\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial \beta_x}{\partial x}\right)+\rho A\ddot{u}_3=p \tag{12.2.15}$$

$$EI\frac{\partial^2 \beta_x}{\partial x^2}-GAk'\left(\frac{\partial u_3}{\partial x}+\beta_x\right)-\frac{\rho Ah^2}{12}\ddot{\beta}_x=0 \tag{12.2.16}$$

For example, let us solve this equation for its natural frequencies and modes for the simply supported beam. Setting $p=0$ and

$$u_3(x,t)=U_3(x)e^{j\omega t} \tag{12.2.17}$$

$$\beta_x(x,t)=B_x(x)e^{j\omega t} \tag{12.2.18}$$

where

$$U_3=A\sin\frac{m\pi x}{L} \tag{12.2.19}$$

$$B_x=B\cos\frac{m\pi x}{L} \tag{12.2.20}$$

satisfy the boundary conditions, which are at $x=0$ and L,

$$U_3=0 \tag{12.2.21}$$

$$M_{xx}=0 \tag{12.2.22}$$

Substituting this in Eqs. (12.2.15) and (12.2.16) gives

$$\begin{bmatrix} a_{11}-\rho A\omega^2 & a_{12} \\ a_{21} & a_{22}-\frac{\rho A h^2}{12}\omega^2 \end{bmatrix}\begin{Bmatrix} A \\ B \end{Bmatrix}=0 \tag{12.2.23}$$

where

$$a_{11}=GAk'\left(\frac{m\pi}{L}\right)^2 \tag{12.2.24}$$

$$a_{12}=a_{21}=GAk'\left(\frac{m\pi}{L}\right) \tag{12.2.25}$$

$$a_{22}=GAk'+EI\left(\frac{m\pi}{L}\right)^2 \tag{12.2.26}$$

Since A and B cannot be equal to 0, the determinant has to be 0 to satisfy the equation. This gives a second-order algebraic equation in ω^2:

$$A_1\omega^4+A_2\omega^2+A_3=0 \tag{12.2.27}$$

where

$$A_1=(\rho A)^2\frac{h^2}{12} \tag{12.2.28}$$

$$A_2=-\rho A\left(a_{11}\frac{h^2}{12}+a_{22}\right) \tag{12.2.29}$$

$$A_3=a_{11}a_{22}-a_{12}a_{21} \tag{12.2.30}$$

and thus we obtain two natural frequencies for every value of m. The lower one is associated with a mode that is dominated by transverse deflection and can be compared to the natural frequency obtained from the classical beam equation without shear deflection. The higher one is associated with a shear vibration and is of much lesser technical interest.

Let us define

$$\omega_{ms} = \xi \omega_m \tag{12.2.31}$$

where ω_m is the mth natural frequency when shear and rotatory inertia are not considered and ω_{ms} is the mth natural frequency when they are considered. ξ is a correction factor. It turns out that $\xi < 1.0$ since the addition of a shear deflection makes the system behave as if it were less stiff and the addition of a rotatory inertia increases the mass effect. Both tend to decrease the calculated natural frequency.

Utilizing our solution, we may plot the correction factor ξ as shown in Fig. 2. It shows that the error that is introduced when we neglect shear and rotatory inertia is sizable for low values of L/h and for high values of m. In summary, what is required is that the thickness h be small compared to the length between nodes of the highest mode of interest. The length between nodes is, exactly for the simply supported case and approximately for other case, $\lambda = L/m$. Thus only if

$$h \ll \frac{L}{m} \tag{12.2.32}$$

can we ignore shear and rotatory inertia safely. For instance, if h is 10% of L/m, the frequency error is approximately 2% for $m = 1$. In Kristiansen et al. (1972), this question is discussed at length for both beams and plates.

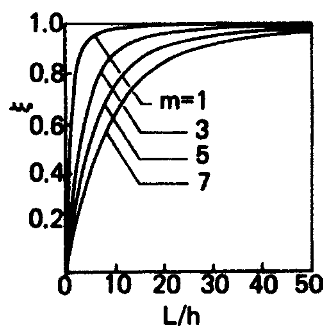

FIG. 2 Correction factor plotted as function of slenderness ratio and mode number.

12.3. PLATES WITH TRANSVERSE SHEAR DEFLECTION AND ROTATORY INERTIA

For plates, we set $1/R_1 = 1/R_2 = 0$. Equations (12.1.12) and (12.1.13) uncouple from the other equations. They are of little interest in this chapter. Equation (12.1.14) becomes

$$-k'Gh\frac{\partial(\varepsilon_{13}A_2)}{\partial\alpha_1} - k'Gh\frac{\partial(\varepsilon_{23}A_1)}{\partial\alpha_2} + A_1A_2\rho h\ddot{u}_3 = A_1A_2q_3 \tag{12.3.1}$$

and Eqs. (12.1.15) and (12.1.16) remain the same. The strain–displacement relations needed are Eqs. (12.1.20)–(12.1.22) and Eqs. (12.1.23) and (12.1.24), which become

$$\varepsilon_{13} = \frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1} + \beta_1 \tag{12.3.2}$$

$$\varepsilon_{23} = \frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2} + \beta_2 \tag{12.3.3}$$

Substituting these in Eqs. (12.3.1), (12.1.15), and (12.1.16) gives

$$-k'Gh\left[\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{A_1}\frac{\partial u_3}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{A_2}\frac{\partial u_3}{\partial\alpha_2}\right) + \frac{\partial}{\partial\alpha_1}(A_2\beta_1 + A_1\beta_2)\right]$$
$$+A_1A_2\rho h\ddot{u}_3 = A_1A_2q_3 \tag{12.3.4}$$

$$D(1-\mu)\left[(k_{11}-k_{22})\frac{\partial A_2}{\partial\alpha_1} + k_{12}\frac{\partial A_1}{\partial\alpha_2} + \frac{A_1}{2}\frac{\partial k_{21}}{\partial\alpha_2} + \frac{A_2}{1-\mu}\left(\frac{\partial k_{11}}{\partial\alpha_1} + \mu\frac{\partial k_{12}}{\partial\alpha_1}\right)\right]$$
$$-k'GhA_1A_2\left(\frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1} + \beta_1\right) - A_1A_2\frac{\rho h^3}{12}\ddot{\beta}_1 = 0 \tag{12.3.5}$$

$$D(1-\mu)\left[(k_{22}-k_{11})\frac{\partial A_1}{\partial\alpha_2} + k_{21}\frac{\partial A_2}{\partial\alpha_1} + \frac{A_2}{2}\frac{\partial k_{12}}{\partial\alpha_1} + \frac{A_1}{1-\mu}\left(\frac{\partial k_{22}}{\partial\alpha_2} + \mu\frac{\partial k_{11}}{\partial\alpha_2}\right)\right]$$
$$-k'GhA_1A_2\left(\frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2} + \beta_2\right) - A_1A_2\frac{\rho h^3}{12}\ddot{\beta}_2 = 0 \tag{12.3.6}$$

where

$$k_{11} = \frac{1}{A_1}\frac{\partial\beta_1}{\partial\alpha_1} + \frac{\beta_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} \tag{12.3.7}$$

$$k_{22} = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \tag{12.3.8}$$

$$k_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\beta_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\beta_1}{A_1}\right) \tag{12.3.9}$$

These are three equations and three unknowns: β_1, β_2, and u_3. The necessary boundary conditions are

$$Q_{n3} = Q_{n3}^* \quad \text{or} \quad u_3 = u_3^* \tag{12.3.10}$$

$$M_{nn} = M_{nn}^* \quad \text{or} \quad \beta_n = \beta_n^* \tag{12.3.11}$$

$$M_{nt} = M_{nt}^* \quad \text{or} \quad \beta_t = \beta_t^* \tag{12.3.12}$$

Let us now look at the special case of a rectangular plate. Since $A_1 = 1, A_2 = 1, \mathrm{d}\alpha_1 = \mathrm{d}x$, and $\mathrm{d}\alpha_2 = \mathrm{d}y$, we obtain

$$-k'Gh\left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} + \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y}\right) + \rho h \ddot{u}_3 = q_3 \tag{12.3.13}$$

$$D(1-\mu)\left[\frac{1}{2}\frac{\partial k_{21}}{\partial y} + \frac{1}{1-\mu}\left(\frac{\partial k_{11}}{\partial x} + \mu \frac{\partial k_{22}}{\partial x}\right)\right]$$
$$-k'Gh\left(\frac{\partial u_3}{\partial x} + \beta_x\right) - \frac{\rho h^3}{12}\ddot{\beta}_x = 0 \tag{12.3.14}$$

$$D(1-\mu)\left[\frac{1}{2}\frac{\partial k_{12}}{\partial x} + \frac{1}{1-\mu}\left(\frac{\partial k_{22}}{\partial y} + \mu \frac{\partial k_{11}}{\partial y}\right)\right]$$
$$-k'Gh\left(\frac{\partial u_3}{\partial y} + \beta_y\right) - \frac{\rho h^3}{12}\ddot{\beta}_y = 0 \tag{12.3.15}$$

$$k_{11} = \frac{\partial \beta_x}{\partial x} \tag{12.3.16}$$

$$k_{22} = \frac{\partial \beta_y}{\partial y} \tag{12.3.17}$$

$$k_{12} = \frac{\partial \beta_y}{\partial x} + \frac{\partial \beta_x}{\partial y} \tag{12.3.18}$$

This gives

$$-k'Gh\left(\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} + \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y}\right) + \rho h \ddot{u}_3 = q_3 \tag{12.3.19}$$

$$D\left(\frac{1+\mu}{2}\frac{\partial^2\beta_y}{\partial x\partial y}+\frac{1-\mu}{2}\frac{\partial^2\beta_x}{\partial y^2}+\frac{\partial^2\beta_x}{\partial x^2}\right)-k'Gh\left(\frac{\partial u_3}{\partial x}+\beta_x\right)$$
$$-\frac{\rho h^3}{12}\ddot{\beta}_x=0 \tag{12.3.20}$$
$$D\left(\frac{1+\mu}{2}\frac{\partial^2\beta_x}{\partial x\partial y}+\frac{1-\mu}{2}\frac{\partial^2\beta_y}{\partial x^2}+\frac{\partial^2\beta_y}{\partial y^2}\right)-k'Gh\left(\frac{\partial u_3}{\partial y}+\beta_y\right)$$
$$-\frac{\rho h^3}{12}\ddot{\beta}_y=0 \tag{12.3.21}$$

These equations are consistent with the Timoshenko beam equation since they reduce to it if we set $\partial(\cdot)/y=0$ and $\beta_y=0$.

By introducing the Laplacian operator

$$\nabla^2(\cdot)=\frac{\partial(\cdot)}{\partial x^2}+\frac{\partial(\cdot)}{\partial y^2} \tag{12.3.22}$$

we may also write these equations as

$$-k'Gh\left(\nabla^2 u_3+\frac{\partial\beta_x}{\partial x}+\frac{\partial\beta_y}{\partial y}\right)+\rho h\ddot{u}_3=q_3 \tag{12.3.23}$$

$$\frac{D}{2}\left[(1-\mu)\nabla^2\beta_x+(1+\mu)\frac{\partial}{\partial x}\left(\frac{\partial\beta_x}{\partial x}+\frac{\partial\beta_y}{\partial y}\right)\right]$$
$$-k'Gh\left(\frac{\partial u_3}{\partial x}+\beta_x\right)-\frac{\rho h^3}{12}\ddot{\beta}_x=0 \tag{12.3.24}$$

$$\frac{D}{2}\left[(1-\mu)\nabla^2\beta_y+(1+\mu)\frac{\partial}{\partial y}\left(\frac{\partial\beta_x}{\partial x}+\frac{\partial\beta_y}{\partial y}\right)\right]$$
$$-k'Gh\left(\frac{\partial u_3}{\partial y}+\beta_y\right)-\frac{\rho h^3}{12}\ddot{\beta}_y=0 \tag{12.3.25}$$

Equations (12.3.23)–(12.3.25) are the equations of motion for the plate in Cartesian coordinates. They were first derived by Mindlin (1951).

Let us, for example, investigate the natural frequencies of a simply supported plate. We let

$$u_3=U_3e^{j\omega t} \tag{12.3.26}$$
$$\beta_x=B_xe^{j\omega t} \tag{12.3.27}$$
$$\beta_y=B_ye^{j\omega t} \tag{12.3.28}$$

where, by inspection, we find that

$$U_3 = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{12.3.29}$$

$$B_x = B \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{12.3.30}$$

$$B_y = C \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{12.3.31}$$

satisfy the boundary conditions that at $x = 0$ and a,

$$U_3 = 0 \tag{12.3.32}$$

$$B_y = 0 \tag{12.3.33}$$

$$M_{xx} = 0 \tag{12.3.34}$$

and at $y = 0$ and b,

$$U_3 = 0 \tag{12.3.35}$$

$$B_x = 0 \tag{12.3.36}$$

$$M_{yy} = 0 \tag{12.3.37}$$

Substituting this in Eqs. (12.3.23)–(12.3.25) gives

$$\begin{bmatrix} a_{11} - \rho h \omega^2 & a_{12} & a_{13} \\ a_{21} & a_{22} - \frac{\rho h^3}{12}\omega^2 & a_{23} \\ a_{31} & a_{32} & a_{33} - \frac{\rho h^3}{12}\omega^2 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0 \tag{12.3.38}$$

where

$$a_{11} = k' G h \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] \tag{12.3.39}$$

$$a_{22} = D \left[\left(\frac{m\pi}{a} \right)^2 + \frac{1-\mu}{2} \left(\frac{n\pi}{b} \right)^2 \right] + k' G h \tag{12.3.40}$$

$$a_{33} = D \left[\frac{1-\mu}{2} \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] + k' G h \tag{12.3.41}$$

$$a_{12} = a_{21} = k' G h \frac{m\pi}{a} \tag{12.3.42}$$

$$a_{13} = a_{31} = k' G h \frac{n\pi}{b} \tag{12.3.43}$$

$$a_{23} = a_{32} = D \frac{1+\mu}{2} \frac{m\pi}{a} \frac{n\pi}{b} \tag{12.3.44}$$

Setting the determinant to 0 will give us a cubic equation in ω^2. For each m, n combination, we obtain three natural frequencies. The lowest of these is the one of most interest since it corresponds to the mode where the transverse deflection dominates. The other two frequencies are much higher and correspond to shear modes.

The error that is introduced by neglecting shear and rotatory inertia is again a function of the thickness and the distance between nodal lines. It is similar in magnitude to the beam analysis error (Kristiansen et al., 1972).

12.4. CIRCULAR CYLINDRICAL SHELLS WITH TRANSVERSE SHEAR DEFLECTION AND ROTATORY INERTIA

In this case $A_1 = 1, A_2 = a, \mathrm{d}\alpha_1 = \mathrm{d}x, \mathrm{d}\alpha_2 = \mathrm{d}\theta, 1/R_1 = 0$, and $R_2 = a$. The equations of motion thus become

$$-a\frac{\partial N_{xx}}{\partial x} - \frac{\partial N_{\theta x}}{\partial \theta} + a\rho h\ddot{u}_x = aq_x \tag{12.4.1}$$

$$-a\frac{\partial N_{x\theta}}{\partial x} - \frac{\partial N_{\theta\theta}}{\partial \theta} - k'Gh\epsilon_{\theta 3} + a\rho h\ddot{u}_\theta = aq_\theta \tag{12.4.2}$$

$$-ak'Gh\frac{\partial \epsilon_{x3}}{\partial x} - k'Gh\frac{\partial \epsilon_{\theta 3}}{\partial \theta} + N_{\theta\theta} + a\rho h\ddot{u}_3 = aq_3 \tag{12.4.3}$$

$$a\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{\theta x}}{\partial \theta} - k'Gha\epsilon_{x3} - \frac{a\rho h^3}{12}\ddot{\beta}_x = 0 \tag{12.4.4}$$

$$a\frac{\partial M_{x\theta}}{\partial x} + \frac{\partial M_{\theta\theta}}{\partial \theta} - k'Gha\epsilon_{\theta 3} - \frac{a\rho h^3}{12}\ddot{\beta}_\theta = 0 \tag{12.4.5}$$

where

$$\epsilon_{x3} = \frac{\partial u_3}{\partial x} + \beta_x \tag{12.4.6}$$

$$\epsilon_{\theta 3} = \frac{1}{a}\frac{\partial u_3}{\partial \theta} - \frac{u_\theta}{a} + \beta_\theta \tag{12.4.7}$$

$$\epsilon^0_{xx} = \frac{\partial u_x}{\partial x} \tag{12.4.8}$$

$$\epsilon^0_{\theta\theta} = \frac{1}{a}\frac{\partial u_\theta}{\partial \theta} + \frac{u_3}{a} \tag{12.4.9}$$

$$\epsilon^0_{x\theta} = \frac{\partial u_\theta}{\partial x} + \frac{1}{a}\frac{\partial u_x}{\partial \theta} \tag{12.4.10}$$

$$k_{xx} = \frac{\partial \beta_x}{\partial x} \tag{12.4.11}$$

$$K_{\theta\theta} = \frac{1}{a}\frac{\partial \beta_\theta}{\partial \theta} \tag{12.4.12}$$

$$k_{x\theta} = \frac{\partial \beta_\theta}{\partial x} + \frac{1}{a}\frac{\partial \beta_x}{\partial \theta} \tag{12.4.13}$$

Let us examine the simply supported shell. We assume the following solution for the vibration at a natural frequency:

$$u_x(x,\theta,t) = U_x(x,\theta)\mathrm{e}^{j\omega t} \tag{12.4.14}$$

$$u_\theta(x,\theta,t) = U_\theta(x,\theta)\mathrm{e}^{j\omega t} \tag{12.4.15}$$

$$u_3(x,\theta,t) = U_3(x,\theta)\mathrm{e}^{j\omega t} \tag{12.4.16}$$

$$\beta_x(x,\theta,t) = B_x(x,\theta)\mathrm{e}^{j\omega t} \tag{12.4.17}$$

$$\beta_\theta(x,\theta,t) = B_\theta(x,\theta)\mathrm{e}^{j\omega t} \tag{12.4.18}$$

where

$$U_x(x,\theta) = A\cos\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{12.4.19}$$

$$U_\theta(x,\theta) = B\sin\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{12.4.20}$$

$$U_3(x,\theta) = C\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{12.4.21}$$

$$B_x(x,\theta) = F\cos\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{12.4.22}$$

$$B_\theta(x,\theta) = G\sin\frac{m\pi x}{L}\sin n(\theta-\phi) \tag{12.4.23}$$

These equations satisfy the 10 boundary conditions

$$u_3(0,\theta,t) = 0 \tag{12.4.24}$$

$$u_\theta(0,\theta,t) = 0 \tag{12.4.25}$$

$$M_{xx}(0,\theta,t) = 0 \tag{12.4.26}$$

$$N_{xx}(0,\theta,t) = 0 \tag{12.4.27}$$

$$\beta_\theta(0,\theta,t) = 0 \tag{12.4.28}$$

and

$$u_3(L,\theta,t) = 0 \tag{12.4.29}$$

$$u_\theta(L,\theta,t) = 0 \tag{12.4.30}$$

$$M_{xx}(L,\theta,t) = 0 \tag{12.4.31}$$

$$N_{xx}(L,\theta,t) = 0 \tag{12.4.32}$$

$$\beta_\theta(L,\theta,t) = 0 \tag{12.4.33}$$

Substituting these equations in Eqs. (12.4.1)–(12.4.5) gives

$$\begin{bmatrix} a_{11}-\rho h\omega^2 & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22}-\rho h\omega^2 & a_{23} & 0 & a_{25} \\ a_{31} & a_{32} & a_{33}-\rho h\omega^2 & a_{34} & a_{35} \\ 0 & 0 & a_{43} & a_{44}-\frac{\rho h^3}{12}\omega^2 & a_{45} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55}-\frac{\rho h^3}{12}\omega^2 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ F \\ G \end{Bmatrix} = 0 \tag{12.4.34}$$

where

$$a_{11} = K\left[\left(\frac{m\pi}{L}\right)^2 + \frac{1-\mu}{2}\left(\frac{n}{a}\right)^2\right] \tag{12.4.35}$$

$$a_{12} = a_{21} = -K\frac{1+\mu}{2}\frac{n}{a}\frac{m\pi}{L} \tag{12.4.36}$$

$$a_{13} = a_{31} = -K\frac{\mu}{a}\frac{m\pi}{L} \tag{12.4.37}$$

$$a_{22} = K\left[\frac{1-\mu}{2}\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right] + k'\frac{Gh}{a^2} \tag{12.4.38}$$

$$a_{23} = a_{32} = \frac{K}{a}\frac{n}{a} + \frac{n}{a}\frac{k'Gh}{a} \tag{12.4.39}$$

$$a_{25} = a_{52} = -\frac{k'Gh}{a} \tag{12.4.40}$$

$$a_{33} = k'Gh\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right] + \frac{K}{a^2} \tag{12.4.41}$$

$$a_{34} = a_{43} = k'Gh\frac{m\pi}{L} \tag{12.4.42}$$

$$a_{35} = a_{53} = -k'Gh\frac{n}{a} \tag{12.4.43}$$

$$a_{44} = k'Gh + D\left[\frac{1-\mu}{2}\left(\frac{n}{a}\right)^2 + \left(\frac{m\pi}{L}\right)^2\right] \tag{12.4.44}$$

$$a_{45} = a_{54} = -D\frac{1-\mu}{2}\frac{n}{a}\frac{m\pi}{L} \tag{12.4.45}$$

$$a_{55} = k'Gh + D\left[\frac{1-\mu}{2}\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{a}\right)^2\right] \tag{12.4.46}$$

Since the matrix equation is, in general, satisfied only if the determinant of the matrix is 0, we obtain a fifth-order algebraic equation in ω^2. The roots of this equation are the natural frequencies. There will be five distinct frequencies for every m and n combination. The mode shapes are obtained by substituting each root back into the matrix equations and by solving for four of the constants in terms of the fifth. For instance, we may solve for $A/C, B/C, F/C$, and G/C. We find, as before for the cylindrical shell in Chapter 5, that we have modes where transverse deflections dominate and modes where in-plane deflections dominate. In addition, there will now be two modes where shear deflections dominate. The influence of the inclusion of shear deformation and rotatory inertia on natural frequencies for modes where transverse motion dominates is similar to that discussed for the rectangular plate and the beam.

Other shell geometries have been analyzed for the influence of rotatory inertia and shear, and conclusions are similar. For conical shells, see for instance, Garnet and Kempner (1964) and Naghdi (1957); for spherical shells see, for instance, Kalnins and Kraus (1966). Summarizing discussions are given by Leissa (1969, 1973).

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13

Combinations of Structures

Most shell structures are combinations of basic shell elements such as cylindrical and conical shells, spherical caps, shells that carry stiffeners, lumped masses, and so on. This has prompted the development of several methods of calculating the eigenvalues of such combinations. One way is, of course, the finite element method. See also Jaquot and Soedel (1970).

A useful approach, discussed for cases other than plates and shells by Bishop and Johnson (1960), is the receptance method. With the receptance method, vibrational characteristics of a combined system, for instance, a shell stiffened by a rib, are calculated from characteristics of the component systems, in this case the shell and the stiffening rib. The number of variables is minimized, since only the displacements at the points of interaction of the subsystems are part of the solution.

A feature of the receptance method of practical value to the design process is that the receptances of the component systems may be determined by any method that is sufficiently accurate. In the following, the receptances are, when appropriate, written in terms of the natural frequencies and modes. These can be obtained experimentally, from finite element programs, and so on.

Even in finite element programming, the receptance method opens a way to reduce program size since each component can be evaluated by itself for its eigenvalues. The eigenvalues of the combined structure are given by the receptance method. For example, a rectangular box can be thought of as being assembled from six rectangular plates. A missile structure can be thought of as being composed of a circular cylindrical shell, a circular conical shell, and a spherical cap.

13.1. RECEPTANCE METHOD

Basically, a *receptance* is defined as the ratio of a deflection response at a certain point to a harmonic force or moment input at the same or at a different point:

$$\alpha_{ij} = \frac{\text{deflection response of system } A \text{ at location } i}{\text{harmonic force or moment input to system } A \text{ at location } j}$$

The response may be either a line deflection or a slope. Usually, the subsystems are labeled A, B, C, and so on, and the receptances are labeled α, β, γ, and so on.

Note that it follows from Maxwell's reciprocity theorem that $\alpha_{ij} = \alpha_{ji}$. Note also that a receptance can be interpreted as the inverse of a mobility. The method could be set up in terms of such mobilities, which is especially interesting to acousticians since they are used to working with mobilities, but there is an advantage in using the receptance notation since it makes some of the operations simpler to describe.

Let us now take the simplest conceivable case of two systems being joined through a single displacement. This is shown schematically in Fig. 1. For instance, a spring attached to a shell panel falls into this category. The receptance of system A at the attachment point is α_{11} and is evaluated first. The input is a harmonic force of amplitude F_{A1}.

$$f_{A1} = F_{A1} e^{j\omega t} \tag{13.1.1}$$

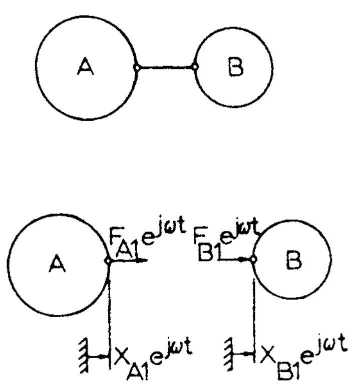

FIG. 1 Two systems connected by one displacement.

The output of the undamped system is

$$x_{A1} = X_{A1} e^{j\omega t} \tag{13.1.2}$$

Note that it is essential that we treat the undamped system in order to obtain the undamped eigenvalues of the combined system.

The receptance is

$$\alpha_{11} = \frac{x_{A1}}{f_{A1}} = \frac{X_{A1}}{F_{A1}} \tag{13.1.3}$$

The receptance of system B at the point of attachment, by similar reasoning, is

$$\beta_{11} = \frac{x_{B1}}{f_{B1}} = \frac{X_{B1}}{F_{B1}} \tag{13.1.4}$$

Now if we join system A and system B, we obtain the condition that

$$X_{A1} = X_{B1} \tag{13.1.5}$$

and

$$F_{A1} + F_{B1} = 0 \tag{13.1.6}$$

Combining these equations and applying the definitions of the receptances gives

$$\alpha_{11} + \beta_{11} = 0 \tag{13.1.7}$$

Frequencies at which this equation is satisfied are natural frequencies of the combined system.

13.2. MASS ATTACHED TO CYLINDRICAL PANEL

The receptance of a mass is found by finding the steady-state response of a mass to a harmonic force input. From Fig. 2 we obtain

$$M\ddot{x}_{A1} = F_{A1} e^{j\omega t} \tag{13.2.1}$$

or

$$-M\omega^2 X_{A1} = F_{A1} \tag{13.2.2}$$

or

$$\alpha_{11} = -\frac{1}{M\omega^2} \tag{13.2.3}$$

The harmonic response at point (x^*, θ^*) due to a point force on the panel is given by Eq. (8.8.8) and is ($\zeta_k = 0$)

$$u_3(x^*, \theta^*, t) = \frac{4F_{B1} e^{j\omega t}}{\rho h L a \alpha} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(\omega_{mn}^2 - \omega^2)} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi\theta^*}{\alpha} \tag{13.2.4}$$

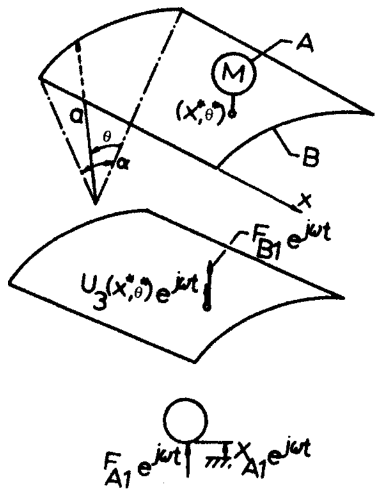

FIG. 2 Point mass connected transversely to a cylindrical panel.

where ω_{mn} is given by Eq. (6.12.5). Thus the receptance is

$$\beta_{11} = \frac{4}{\rho h L a \alpha} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi\theta^*}{\alpha} \tag{13.2.5}$$

The characteristic equation for the total system is, according to Eq. (13.1.7).

$$\frac{4}{\rho h L a \alpha} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi\theta^*}{\alpha} - \frac{1}{M\omega^2} = 0 \tag{13.2.6}$$

This equation has to be solved for its roots, $\omega = \omega_k$, by a numerical procedure.

We can also do it graphically. We plot α_{11} as a function of ω and then β_{11} as a function of ω. According to Eq. (13.1.7), it is required that $\alpha_{11} = -\beta_{11}$. A typical plot is shown in Fig. 3. As expected, the natural frequencies are in general lowered by the attached mass.

To discuss the characteristic behavior of such an equation, let us assume the special case where the influence of the mass is such that the new natural frequencies ω_k are not too different from the original natural frequencies ω_{mn}. In this case, we will find that for a particular root ω_k, one term in the series dominates all the others, so that the equation can be written approximately

$$\frac{4}{\rho h L a \alpha (\omega_{mn}^2 - \omega_k^2)} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi\theta^*}{\alpha} - \frac{1}{M\omega_k^2} = 0 \tag{13.2.7}$$

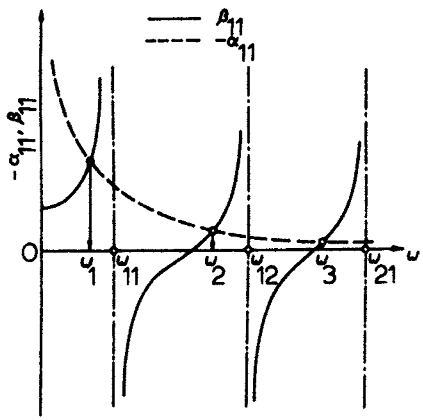

FIG. 3 Illustration of the influence of the receptances of the point mass and the panel on the natural frequencies of the combined system.

or, solving for ω_k,

$$\omega_k^2 = \frac{\omega_{mn}^2}{(4M/M_s)\sin^2(m\pi x^*/L)\sin^2(n\pi\theta^*/\alpha)+1} \tag{13.2.8}$$

where M_s is the mass of the entire panel:

$$M_s = \rho h L a \alpha \tag{13.2.9}$$

First we notice the obvious, namely that if we set $M=0$, nothing changes. Second, if the mass happens to be attached to what is a node line of the panel, the mass has no influence on that particular frequency. Finally, if the mass is located on an antinode, we have

$$\omega_k^2 = \frac{\omega_{mn}^2}{1+4(M/M_s)} \tag{13.2.10}$$

This would be the largest influence the mass can have on a particular mode, subject to the restrictions imposed by the simplifying assumption.

What does the new mode shape look like? Obviously, the mode shape of the panel with a mass attached must be different from that of the panel without mass. The solution is found by arguing that the response of the plate to a harmonic point input of frequency ω_k is given by

$$u_3(x,\theta,t) = \frac{4F_{B1}e^{j\omega t}}{\rho h L a \alpha}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(m\pi x^*/L)\sin(n\pi\theta^*/\alpha)}{\omega_{mn}^2-\omega_k^2}\sin\frac{m\pi x}{L}\sin\frac{n\pi\theta}{\alpha} \tag{13.2.11}$$

Since ω_k is the natural frequency of the combined system, this equation must describe the new mode shape. Since the amplitude is arbitrary, we obtain

$$U_{3k}(x,\theta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(m\pi x^*/L)\sin(n\pi\theta^*/\alpha)}{\omega_{mn}^2-\omega_k^2}\sin\frac{m\pi x}{L}\sin\frac{n\pi\theta}{\alpha} \tag{13.2.12}$$

We see that the more different ω_k is from the original ω_{mn}, the more the new mode shape is a distortion of the original mode shape. We also recognize that if the mass is attached to a node line of an original mode shape m,n, this particular shape is preserved intact in its original form, since in this case the double series becomes dominated by the one particular m,n combination as $\omega_k\to\omega_{mn}$. The foregoing approach was applied to rings and tires in Allaei et al. (1986, 1988).

13.3. SPRING ATTACHED TO SHALLOW CYLINDRICAL PANEL

The receptance of a spring attached to ground is obtained from (Fig. 4)

$$Kx_{A1}=F_{A1}e^{j\omega t} \tag{13.3.1}$$

or

$$KX_{A1}=F_{A1} \tag{13.3.2}$$

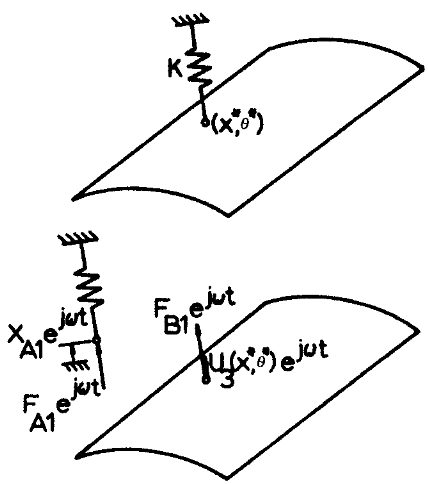

Fig. 4 Transversely grounded spring connected to a cylindrical panel.

This gives

$$\alpha_{11} = \frac{1}{K} \tag{13.3.3}$$

Using the receptance β_{11} of Eq. (13.2.5) for a simply supported circular cylindrical panel gives

$$\frac{4}{M_s} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi\theta^*}{\alpha} + \frac{1}{K} = 0 \tag{13.3.4}$$

where M_s is the mass of the panel and is given by Eq. (13.2.9). The expected solution of this equation is illustrated graphically in Fig. 5, where the solution $\alpha_{11} = -\beta_{11}$ is shown. As expected, we observe that the presence of a grounded spring will in general increase all natural frequencies, provided that the spring can be approximated as massless.

If we make the same assumption as in Sec. 13.2—that for small deviations the series is dominated by the term for which $\omega_{mn}^2 - \omega^2$ is a minimum—we obtain approximately

$$\frac{4}{M_s(\omega_{mn}^2 - \omega^2)} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi\theta^*}{\alpha} + \frac{1}{K} = 0 \tag{13.3.5}$$

or

$$\omega_k^2 = \omega_{mn}^2 + \frac{4K}{M_s} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi\theta^*}{\alpha} \tag{13.3.6}$$

Note that this case allows us to solve the problem of finding the natural frequencies and modes of a simply supported panel that has a point support at (x^*, θ^*), as shown in Fig. 6. All that we have to do is let

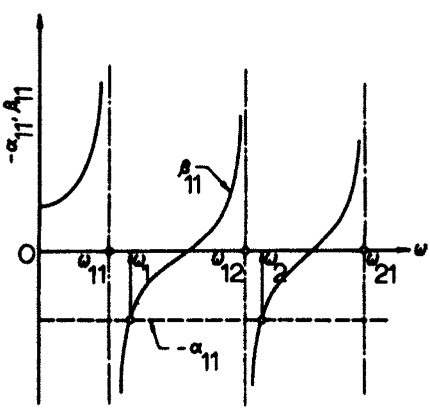

FIG. 5 Illustration of the influence of the receptances of the transverse spring and the panel on the natural frequencies of the combined system.

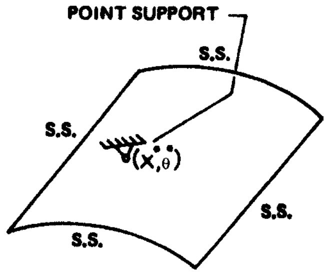

FIG. 6 Simply supported, circular cylindrical panel with a grounded point support.

K approach infinity. The single-term solution cannot any longer be used; rather, we have to solve Eq. (13.3.4) for its roots with $1/K = 0$

$$\frac{4}{M_s} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi \theta^*}{\alpha} = 0 \tag{13.3.7}$$

The solutions are given by the points where curve β_{11} in Fig. 5 crosses the abscissa. As excepted, we notice in general a substantial increase in natural frequency, except for cases where the point support location coincides with a nodal line of one of the original modes. This method was applied to tires in Soedel and Prasad (1980) and Allaei et al. (1987).

13.4. HARMONIC RESPONSE OF A SYSTEM IN TERMS OF ITS COMPONENT RECEPTANCES

Let us take the simplest conceivable case: that of two systems joined through a single displacement, as shown in Fig. 7. The receptance of system A at the attachment point is α_{11} and is evaluated, as described in Sec. 13.1, by considering a force

$$f_{A1} = F_{A1} e^{j\omega t} \tag{13.4.1}$$

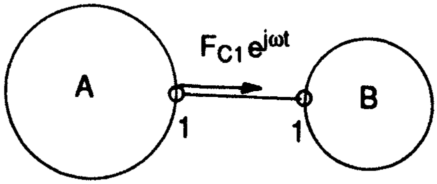

FIG. 7 Harmonic forcing applied to two systems at the displacement location where they are joined.

This causes a vibration response, in steady state, of

$$x_{A1} = X_{A1}e^{j\omega t} \tag{13.4.2}$$

X_{A1} is real if the system is undamped and is a complex number if it is damped. The receptance is

$$\alpha_{11} = \frac{x_{A1}}{f_{A1}} = \frac{X_{A1}}{F_{A1}} \tag{13.4.3}$$

Similarly, the receptance of system B at the point of attachment is

$$\beta_{11} = \frac{x_{B1}}{f_{B1}} = \frac{X_{B1}}{F_{B1}} \tag{13.4.4}$$

If system A and system B are joined, it must be true that

$$X_{A1} = X_{B1} = X_{C1} \tag{13.4.5}$$

where X_{C1} is the amplitude of the motion of the combined system.

Assuming the possibility that an external force $f_{C1} = F_{C1}e^{j\omega t}$ acts on the combined system at junction 1, we must also enforce the force balance

$$F_{A1} + F_{B1} = F_{C1} \tag{13.4.6}$$

Dividing this equation by the displacement gives

$$\frac{1}{\alpha_{11}} + \frac{1}{\beta_{11}} = \frac{1}{\gamma_{11}} \tag{13.4.7}$$

or

$$\gamma_{11} = \frac{\alpha_{11}\beta_{11}}{\alpha_{11} + \beta_{11}} \tag{13.4.8}$$

where $\gamma_{11} = X_{C1}/F_{C1}$. The harmonic response of the system to a force $F_{C1}e^{j\omega t}$ at coordinate 1 is

$$X_{C1} = \gamma_{11}F_{C1}e^{j\omega t} = \frac{\alpha_{11}\beta_{11}}{\alpha_{11} + \beta_{11}}F_{C1}e^{j\omega t} \tag{13.4.9}$$

Let us next consider the case of Fig. 8, where a force $F_{C1}e^{j\omega t}$ is applied to the system at location 1, which is not at the junction of the two subsystems. In this case

$$X_{A1} = X_{C1} = \alpha_{11}F_{A1} + \alpha_{12}F_{A2} \tag{13.4.10}$$

$$X_{A2} = X_{C2} = \alpha_{21}F_{A1} + \alpha_{22}F_{A2} \tag{13.4.11}$$

Since

$$F_{A2} = -F_{B2} \tag{13.4.12}$$

$$X_{C2} = X_{B2} = \beta_{22}F_{B2} \tag{13.4.13}$$

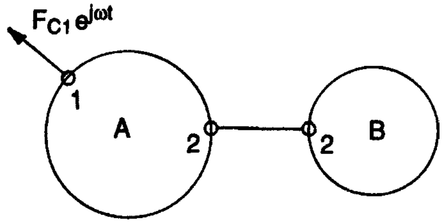

Fig. 8 Harmonic forcing applied to a system to which a second system is attached through one displacement.

We obtain

$$F_{A2} = -\frac{X_{C2}}{\beta_{22}} \tag{13.4.14}$$

Since

$$F_{A1} = F_{C1} \tag{13.4.15}$$

Equations (13.4.10) and (13.4.11) become

$$X_{C1} = \alpha_{11} F_{C1} - \frac{\alpha_{12}}{\beta_{22}} X_{C2} \tag{13.4.16}$$

$$X_{C2} = \alpha_{21} F_{C1} - \frac{\alpha_{22}}{\beta_{22}} X_{C2} \tag{13.4.17}$$

Solving these two equations results in

$$X_{C1} = \left(\alpha_{11} - \frac{\alpha_{21}\alpha_{12}}{\alpha_{22} + \beta_{22}} \right) F_{C1} \tag{13.4.18}$$

$$X_{C2} = \frac{\beta_{22}\alpha_{21}}{\alpha_{22} + \beta_{22}} F_{C1} \tag{13.4.19}$$

or

$$\gamma_{11} = \alpha_{11} - \frac{\alpha_{21}\alpha_{12}}{\alpha_{22} + \beta_{22}} \tag{13.4.20}$$

$$\gamma_{21} = \frac{\beta_{22}\alpha_{21}}{\alpha_{22} + \beta_{22}} \tag{13.4.21}$$

so that, recognizing that $\alpha_{21} = \alpha_{12}$,

$$x_{C1} = \gamma_{11} F_{C1} e^{j\omega t} = \left(\alpha_{11} - \frac{\alpha_{21}^2}{\alpha_{22} + \beta_{22}} \right) F_{C1} e^{j\omega t} \tag{13.4.22}$$

$$x_{C2} = \gamma_{21} F_{C1} e^{j\omega t} = \frac{\beta_{22}\alpha_{21}}{\alpha_{22} + \beta_{22}} F_{C1} e^{j\omega t} \tag{13.4.23}$$

This approach was applied to the forced response of tires and suspension systems in Kung et al. (1987).

13.5. DYNAMIC ABSORBER

Let us now examine the case where a spring–mass system is attached to a panel as shown in Fig. 9. To formulate the receptance α_{11}, we write the equations of motion of system B (the absorber):

$$M\ddot{y}+Ky=Kx_{B2} \tag{13.5.1}$$

and

$$(y-x_{B2})K+F_{B2}e^{j\omega t}=0 \tag{13.5.2}$$

Since

$$y=Ye^{j\omega t} \tag{13.5.3}$$

and

$$x_{B2}=X_{B2}e^{j\omega t} \tag{13.5.4}$$

We obtain

$$\begin{bmatrix} (K-M\omega^2) & -K \\ -K & K \end{bmatrix} \begin{Bmatrix} Y \\ X_{B2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_{B2} \end{Bmatrix} \tag{13.5.5}$$

or

$$X_{B2}=\frac{F_{B2}(K-M\omega^2)}{-KM\omega^2} \tag{13.5.6}$$

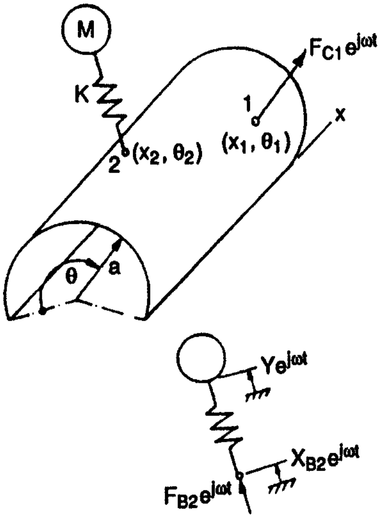

FIG. 9 A transversely vibrating dynamic absorber that is attached to a circular cylindrical shell.

Thus the receptance β_{22} is

$$\beta_{22} = -\frac{1}{M\omega^2}\left[1-\left(\frac{\omega}{\omega_1}\right)^2\right] \tag{13.5.7}$$

where $\omega_1^2 = K/M$. The receptance of the shell α_{22} is given by Eq. (13.2.5). Note here that the shell is system A, the dynamic absorber is attached at point $2(x_2, \theta_2)$ in the transverse direction, and the excitation force is located at point $1(x_1, \theta_1)$. The frequency equation thus becomes

$$\frac{4}{M_s}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\omega_{mn}^2-\omega^2}\sin^2\frac{m\pi x_2}{L}\sin^2\frac{n\pi\theta_2}{\alpha} - \frac{\omega_1^2-\omega^2}{M\omega^2\omega_1^2} = 0 \tag{13.5.8}$$

Let us again examine the approximate case where the roots of this equation are perturbations of the original ω_{mn}^2 values. We obtain

$$\frac{4}{M_s}\frac{1}{\omega_{mn}^2-\omega^2}\sin^2\frac{m\pi x_2}{L}\sin^2\frac{n\pi\theta_2}{\alpha} - \frac{\omega_1^2-\omega^2}{M\omega^2\omega_1^2} = 0 \tag{13.5.9}$$

The new natural frequencies $\omega = \omega_k$ are therefore

$$\omega_k^2 = \frac{\omega_{mn}^2}{2}\left[\left(\frac{\omega_1}{\omega_{mn}}\right)^2(1+A)+1\right]\left(1\pm\sqrt{1-\varepsilon_{mn}}\right) \tag{13.5.10}$$

where

$$\varepsilon_{mn} = \frac{4(\omega_1/\omega_{mn})^2}{[(\omega_1/\omega_{mn})^2(1+A)+1]^2} \tag{13.5.11}$$

$$A = \frac{4M}{M_s}\sin^2\frac{m\pi x_2}{L}\sin^2\frac{n\pi\theta_2}{\alpha} \tag{13.5.12}$$

As a check, when $\omega_1 = 0$, which means that $K = 0$ and thus that there is no attachment, the equation gives $\omega_k^2 = \omega_{mn}^2$, 0, as expected. In this case the zero has no physical meaning. When $K \to \infty$, which implies that $\omega_1 \to \infty$, we obtain from Eq. (13.5.9) the result of Eq. (13.2.8). When $A = 0$, which means that either $M = 0$ or the spring–mass system is attached to a node line for the particular m, n mode, we obtain $\omega_k^2 = \omega_{mn}^2, \omega_1^2$. This is correct because in this case the two subsystems uncouple.

In the general case, we get for every m, n combination two natural frequencies. Let us, for instance, take the case where $\omega_1 = \omega_{mn}$. We get

$$\omega_k^2 = \omega_{mn}^2\left[\left(1+\frac{A}{2}\right)\pm\sqrt{\left(\frac{A}{2}\right)^2+A}\right] \tag{13.5.13}$$

This relationship is plotted in Fig. 10. It can be shown that in general the higher-frequency branch will correspond to a mode where the motion

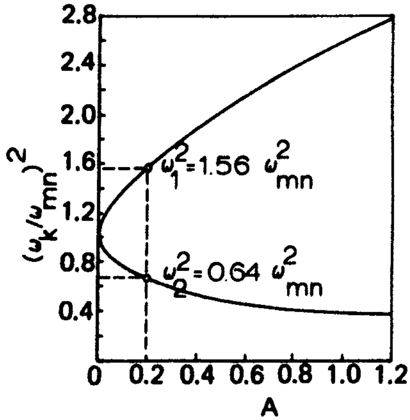

Fig. 10 Splitting of the shell natural frequencies as function of the dynamic absorber mass.

of the mass will be out of phase with the motion of the panel and that the lower-frequency branch will correspond to a system mode where the motion of the mass will be in phase with the motion of the panel. This is sketched in Fig. 11.

Next, we examine the forced response of the dynamic absorber. First, we obtain the other necessary receptances, α_{11} and α_{21}. They are, from Eq. (8.8.8),

$$\alpha_{11} = \frac{4}{M_s} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin^2 \frac{m\pi x_1}{L} \sin^2 \frac{n\pi\theta_1}{\alpha} \tag{13.5.14}$$

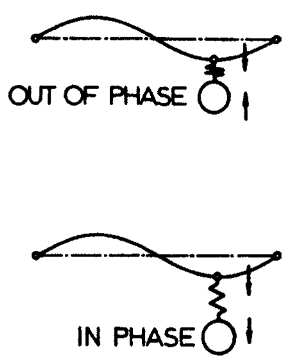

Fig. 11 Typical in-phase and out-of-phase absorber motion at a split natural frequency pair.

$$\alpha_{21} = \alpha_{12} = \frac{4}{M_s} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin \frac{m\pi x_2}{L} \sin \frac{n\pi \theta_2}{\alpha} \sin \frac{m\pi x_1}{L} \sin \frac{n\pi \theta_1}{L} \tag{13.5.15}$$

Equation (13.4.23) gives, at the point of absorber attachment,

$$x_{C2} = \frac{\beta_{22}\alpha_{21}}{\alpha_{22} + \beta_{22}} F_{C1} e^{j\omega t} \tag{13.5.16}$$

This response can now be evaluated as a function of excitation frequency ω. It becomes 0 if

$$\beta_{22} = 0 \tag{13.5.17}$$

which is satisfied if $\omega = \omega_1 = \sqrt{k/M}$. This is the tuning requirement for the one-degree-of-freedom absorber of this example. However, the result of Eq. (13.5.17) is much more general. Any attached multidegree-of-freedom system can act as a dynamic absorber at the absorber resonance frequencies.

13.6. HARMONIC FORCE APPLIED THROUGH A SPRING

It is not possible to define and obtain useful receptances for a single spring that is not grounded. By itself, a spring has no inertia; a harmonic force will therefore produce harmonic motion of infinite amplitude. Thus it is necessary to consider together the spring and the system to which the spring it attached, as shown in Fig. 12.

$$F_0 e^{j\omega t} = k(x_0 - x_1) \tag{13.6.1}$$

However, since

$$x_1 = \alpha_{11} F_1 e^{j\omega t} \tag{13.6.2}$$

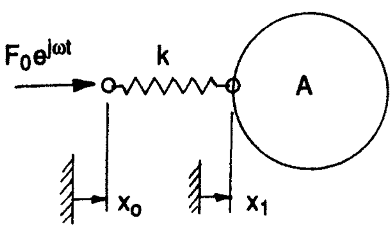

FIG. 12 Harmonic force applied through a spring.

and since

$$F_0 = F_1 \tag{13.6.3}$$

we obtain by substitution

$$x_0 = \frac{1}{k}(1 + k\alpha_{11})F_0 e^{j\omega t} \tag{13.6.4}$$

and therefore,

$$\alpha_{00} = \frac{1}{k} + \alpha_{11} \tag{13.6.5}$$

Consider, for example, the simply supported cylindrical panel of Fig. 13, which is excited through a linear spring. The shell is system A and its point receptance α_{11} is [Eq. (13.2.5)]

$$\alpha_{11} = \frac{4}{\rho h L a \alpha} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi\theta^*}{\alpha} \tag{13.6.6}$$

Therefore,

$$\alpha_{00} = \frac{1}{k} + \frac{4}{\rho h L a \alpha} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin^2 \frac{m\pi x^*}{L} \sin^2 \frac{n\pi\theta^*}{\alpha} \tag{13.6.7}$$

Note that from this, we may obtain the characteristic equation for a grounded spring on a panel as discussed in Sec. 13.3 and given in Eq. (13.3.4) by realizing that for this case $\alpha_{00} = 0$.

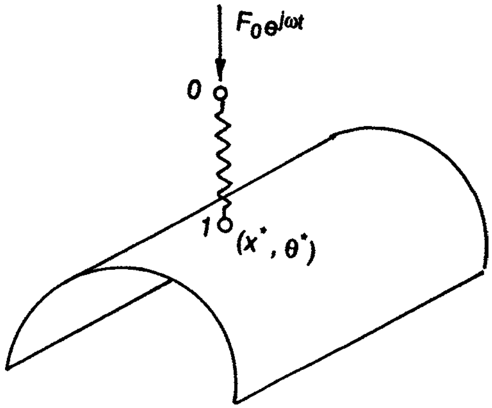

Fig. 13 Example where the spring is transverse to the surface of a circular cylindrical shell.

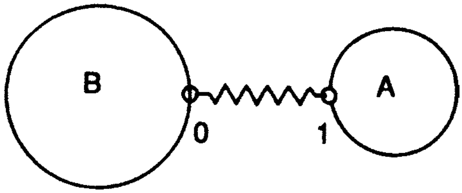

FIG. 14 The two systems are connected through a spring.

We may now use α_{00} to attach a system B to a system A with a spring, as shown in Fig. 14, since the characteristic equation is now simply

$$\alpha_{00}+\beta_{00}=0 \tag{13.6.8}$$

For example, we may consider the case of two cylindrical panels attached through a spring as shown in Fig. 15. Note that in all this discussion, it is assumed that the spring is acting normal to the panel surface. This is not a required restriction, but it allows us, for reasons of simplicity, to consider only transverse deflection receptances. Since β_{00} is also given by Eq. (13.2.5), we obtain the characteristic equation

$$\frac{1}{k}+\frac{4}{\rho_1 h_1 L_1 a_1 \alpha_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\omega_{mn1}^2-\omega^2}\sin^2\frac{m\pi x_1^*}{L_1}\sin^2\frac{n\pi\theta_1^*}{\alpha_1}$$
$$+\frac{4}{\rho_2 h_2 L_2 a_2 \alpha_2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\omega_{mn2}^2-\omega^2}\sin^2\frac{m\pi x_2^*}{L_2}\sin^2\frac{n\pi\theta_2^*}{\alpha_2}=0 \tag{13.6.9}$$

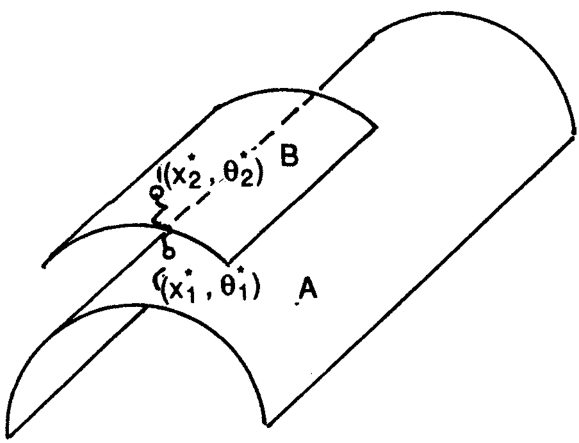

FIG. 15 Two circular cylindrical panels connected by a spring.

13.7. STEADY-STATE RESPONSE TO HARMONIC DISPLACEMENT EXCITATION

System A in Fig. 16 is excited by a displacement

$$x_1 = X_1 e^{j\omega t} \tag{13.7.1}$$

The force at the point of excitation, in steady state, is therefore

$$F_1 = \frac{X_1}{\alpha_{11}} \tag{13.7.2}$$

If the input at point 1 is considered the force F_1, the response at point 2 is

$$X_2 = \alpha_{21} F_1 \tag{13.7.3}$$

Substituting Eq. (13.7.2) gives

$$X_2 = \frac{\alpha_{21}}{\alpha_{11}} X_1 \tag{13.7.4}$$

Equation (13.7.4) illustrates clearly that if point 2 approaches point 1, the response approaches X_1, as one would expect for a displacement input of fixed amplitude. For a force excitation, as point 2 approaches point 1, we approach $X_1 = \alpha_{11} F_1$, which means that point 1 can respond to resonances contained in α_{11}. This difference becomes important, for example, in the response of aircraft or automotive tires, which may experience either displacement or force inputs.

Let us next consider two displacement inputs, one at location 1, the other at location 2, with the response location now designated at 3. In this case

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{13.7.5}$$

or

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}^{-1} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \tag{13.7.6}$$

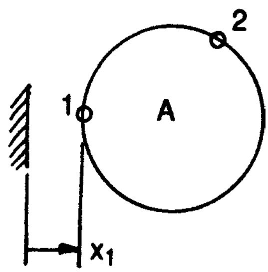

Fig. 16 Displacement excitation at a point.

Because the response at location 3 is given as

$$X_3 = \alpha_{31} F_1 + \alpha_{32} F_2 = \begin{Bmatrix} \alpha_{31} \\ \alpha_{32} \end{Bmatrix}^T \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{13.7.7}$$

we obtain

$$X_3 = \begin{Bmatrix} \alpha_{31} \\ \alpha_{32} \end{Bmatrix}^T \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}^{-1} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \tag{13.7.8}$$

This can easily be generalized to N displacement inputs.

13.8. COMPLEX RECEPTANCES

None of the receptance relationships derived are restricted to zero damping except those used in obtaining the natural frequencies for a combined system. Therefore, receptances may be viewed as having both a magnitude and a phase angle. For a simply supported cylindrical panel, for example, which is viscously damped,

$$\alpha_{11} = \frac{u_3(x_1, \theta_1, t)}{F_1 e^{j\omega t}} = \frac{4}{\rho h L \alpha} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2(m\pi x_1/L)\sin^2(n\pi\theta_1/\alpha)e^{-j\phi_{mn}}}{\sqrt{(\omega_{mn}^2 - \omega^2)^2 + 4\zeta_{mn}^2 \omega^2 \omega_{mn}^2}} \tag{13.8.1}$$

where

$$\phi_{mn} = \tan^{-1} \frac{2\zeta_{mn}(\omega/\omega_{mn})}{1 - (\omega/\omega_{mn})^2} \tag{13.8.2}$$

We may, of course, also combine complex receptances with undamped receptances. Let us, for example, consider an undamped, simply supported cylindrical panel to which is attached a grounded, viscous damper of coefficient C. Designating the plate as system A, the damper as system B, and applying the force input at x_1, y_1 (location 1), the response at location 1 is given by Eq. (13.4.22):

$$x_{C1} = \left(\alpha_{11} - \frac{\alpha_{21}^2}{\alpha_{22} + \beta_{22}} \right) F_1 e^{j\omega t} \tag{13.8.3}$$

where

$$\beta_{22} = \frac{1}{jC\omega} = \frac{1}{C\omega} e^{-j\pi/2} \tag{13.8.4}$$

$$\alpha_{11} = \frac{4}{\rho h L \alpha} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2(m\pi x_1/L)\sin^2(n\pi\theta_1/\alpha)}{\omega_{mn}^2 - \omega^2} \tag{13.8.5}$$

$$\alpha_{22} = \frac{4}{\rho h L \alpha} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2(m\pi x_2/L)\sin^2(n\pi\theta_2/\alpha)}{\omega_{mn}^2 - \omega^2} \tag{13.8.6}$$

$$\alpha_{21} = \frac{4}{\rho h L a \alpha} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \times \frac{\sin(m\pi x_2/L)\sin(m\pi x_1/L)\sin(n\pi\theta_2/\alpha)\sin(n\pi\theta_1/\alpha)}{\omega_{mn}^2 - \omega^2} \tag{13.8.7}$$

The result x_{C1} can easily be converted into magnitude and a phase angle.

Sometimes, it is desirable to represent a complex receptance in terms of a magnitude and a single phase angle. First, for ease of derivation, we replace m, n by k. Equation (13.8.1) can be written

$$\alpha_{11} = \frac{4}{\rho h L a \alpha} \sum_{k=1}^{\infty} A_k(x_1, \theta_1) e^{-j\phi_k} \tag{13.8.8}$$

where

$$A_k(x_1, \theta_1) = \frac{U_{3k}^2(x_1, \theta_1)}{\sqrt{(\omega_k^2 - \omega^2)^2 + 4\zeta_k^2 \omega^2 \omega_k^2}} \tag{13.8.9}$$

and where

$$U_{3k}(x_1, \theta_1) = \sin(m\pi x_1/L)\sin(n\pi\theta_1/\alpha) \tag{13.8.10}$$

Since

$$e^{-j\phi_k} = \cos\phi_k - j\sin\phi_k \tag{13.8.11}$$

we may write

$$\alpha_{11} = \frac{4}{\rho h L a \alpha} \left[\sum_{k=1}^{\infty} A_k(x_1, \theta_1)\cos\phi_k - j \sum_{k=1}^{\infty} A_k(x_1, \theta_1)\sin\phi_k \right] \tag{13.8.12}$$

or, in terms of a magnitude A_{11} and a single-phase angle Ψ_{11},

$$\alpha_{11} = A_{11} e^{-\Psi_{11}} \tag{13.8.13}$$

where

$$A_{11} = \frac{4}{\rho h L a \alpha} \sqrt{ \left[\sum_{k=1}^{\infty} A_k(x_1, \theta_1)\cos\phi_k \right]^2 + \left[\sum_{k=1}^{\infty} A_k(x_1, \theta_1)\sin\phi_k \right]^2 } \tag{13.8.14}$$

$$\Psi_{11} = \tan^{-1} \left[\frac{\sum_{k=1}^{\infty} A_k(x_1, \theta_1)\sin\phi_k}{\sum_{k=1}^{\infty} A_k(x_1, \theta_1)\cos\phi_k} \right] \tag{13.8.15}$$

13.9. STIFFENING OF SHELLS

Let us now investigate requirements for stiffening a shell on the example of a simply supported circular cylindrical panel as shown in Fig. 17.

We are faced here with the requirement that we have to join system A, the panel, to system B, the ring, along a continuous line. A way to do this and still utilize Eq. (13.1.7) was worked out by Sakharov (1962) for the case of a closed circular cylindrical shell with stiffening rings at both ends. It requires that the two systems have the same mode shapes along the line at which they are joined. In our case the mode shape of the panel is

$$U_3(x,\theta)=\sin\frac{m\pi x}{L}\sin\frac{n\pi\theta}{\alpha} \tag{13.9.1}$$

while the mode shape of the stiffening ring is

$$U_3(\theta)=\sin\frac{n\pi\theta}{\alpha} \tag{13.9.2}$$

This allows us to formulate a line receptance that is defined as the response along the line to a harmonic line load that is distributed sinusoidally along the line

$$q_3(x,\theta,t)=P\sin\frac{r\pi\theta}{\alpha}\delta(x-x^*)e^{j\omega t} \tag{13.9.3}$$

where P is the line load amplitude in newtons per meter.

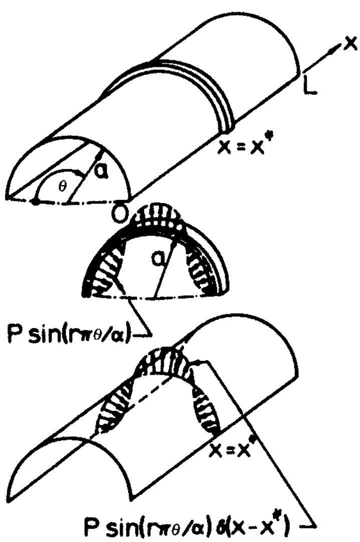

Fig. 17 Simply supported circular cylindrical panel reinforced by a ring stiffener.

Taking advantage of the work described in Sec. 8.9, we note that $Q_3^* = P\sin(r\pi\theta/\alpha), Q_1^* = 0, Q_2^* = 0$. Thus

$$F_k^* = F_{mn}^* = \frac{P}{\rho h N_k}\sin\frac{m\pi x^*}{L}\int_0^\alpha \sin\frac{r\pi\theta}{\alpha}\sin\frac{n\pi\theta}{\alpha}a\ d\theta \tag{13.9.4}$$

where

$$N_k = \frac{a\alpha L}{4} \tag{13.9.5}$$

Evaluating the integral gives 0 unless $n = r$. In that case, we obtain

$$F_{mn}^* = \begin{cases} \dfrac{2P}{\rho hL}\sin\dfrac{m\pi x^*}{L}, & n = r \\ 0, & n \neq r \end{cases} \tag{13.9.6}$$

Since in this case

$$S(t) = e^{j\omega t} \tag{13.9.7}$$

we obtain the steady-state solution of the modal participation factor from Eq. (8.5.4) as

$$\eta_k = \frac{F_{mn}^*}{\omega_{mn}^2 - \omega^2}e^{j\omega t} \tag{13.9.8}$$

and therefore

$$u_3(x,\theta,t) = \frac{2Pe^{j\omega t}}{\rho hL}\sum_{m=1}^{\infty}\frac{\sin(m\pi x^*/L)\sin(m\pi x/L)\sin(r\pi\theta/\alpha)}{\omega_{mn}^2 - \omega^2} \tag{13.9.9}$$

and at $x = x^*$,

$$u_3(x^*,\theta,t) = \frac{2Pe^{j\omega t}}{\rho hL}\sum_{m=1}^{\infty}\frac{1}{\omega_{mn}^2 - \omega^2}\sin^2\frac{m\pi x^*}{L}\sin\frac{r\pi\theta}{\alpha} \tag{13.9.10}$$

Formulating as receptance

$$\alpha_{11} = \frac{u_3(x^*,\theta,t)}{P\sin(r\pi\theta/\alpha)e^{j\omega t}} \tag{13.9.11}$$

gives

$$\alpha_{11} = \frac{2}{\rho hL}\sum_{m=1}^{\infty}\frac{1}{\omega_{mn}^2 - \omega^2}\sin^2\frac{m\pi x^*}{L} \tag{13.9.12}$$

Next, to obtain the receptance β_{11} for the ring, we have to solve for the response of a simply supported ring segment to a load

$$q_3'(\theta,t) = P\sin\frac{r\pi\theta}{\alpha}e^{j\omega t} \tag{13.9.13}$$

where q_3' and P both have the unit newtons per meter. Note that we assume that the systems are joined along their midsurfaces. In reality, the ring may be joined to the panel either above or below. This influence is generally small but was investigated in Wilken and Soedel (1976a).

The transverse mode-shape expression is

$$U_{3n} = \sin\frac{n\pi\theta}{\alpha} \tag{13.9.14}$$

The associated natural frequencies ω_n were obtained in Sec. 5.4.

Multiplying and dividing Eq. (8.4.5) by the width b of the ring segment and performing the indicated integration with respect to the width, we obtain

$$F_k = \frac{2P\mathrm{e}^{j\omega t}}{\rho_s A\alpha}\int_0^\alpha \sin\frac{r\pi\theta}{\alpha}\sin\frac{n\pi\theta}{\alpha}\,\mathrm{d}\theta \tag{13.9.15}$$

where A is the cross-sectional area of the ring and ρ_s is the mass density of the ring material. Evaluating the integral, we obtain

$$F_k = F_n = \begin{cases} \dfrac{P\mathrm{e}^{j\omega t}}{\rho_s A}, & n = r \\ 0, & n \neq r \end{cases} \tag{13.9.16}$$

The response is, therefore,

$$u_3(\theta, t) = \frac{P\mathrm{e}^{j\omega t}}{\rho_s A(\omega_n^2 - \omega^2)}\sin\frac{r\pi\theta}{\alpha} \tag{13.9.17}$$

and the receptance, defined as

$$\beta_{11} = \frac{u_3(\theta, t)}{P\sin(r\pi\theta/\alpha)\mathrm{e}^{j\omega t}} \tag{13.9.18}$$

becomes

$$\beta_{11} = \frac{1}{\rho_s A(\omega_n^2 - \omega^2)} \tag{13.9.19}$$

Note that the receptances α_{11} and β_{11} are compatible. Both describe the same displacement divided by the same input. Thus the characteristic equation whose roots furnish the combined system natural frequencies becomes

$$\frac{2}{\rho h L}\sum_{m=1}^{\infty}\frac{1}{\omega_{mn}^2 - \omega^2}\sin^2\frac{m\pi x^*}{L} + \frac{1}{\rho_s A(\omega_n^2 - \omega^2)} = 0 \tag{13.9.20}$$

The graphical solution is shown in Fig. 18 for a stiffener whose natural frequency corresponding to the nth mode, ω_n, is lower than the panel natural frequencies $\omega_{3n}, \omega_{4n}, \ldots$ corresponding to the same circumferential mode shape but higher than ω_{1n} and ω_{2n}. We see that the natural frequencies of the combined system $\omega_{\mathrm{III}}, \omega_{\mathrm{IV}}, \ldots$, are lower

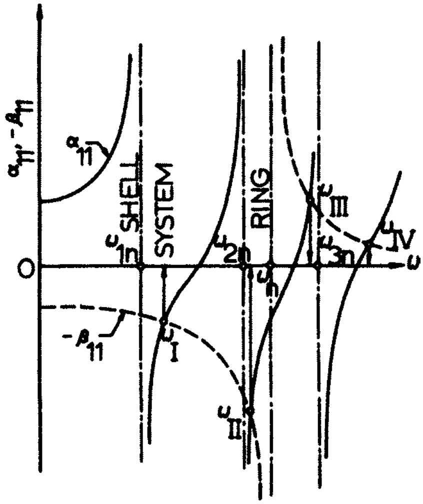

FIG. 18 Illustration of the law of stiffening described by Eqs. (13.9.25) and (13.9.26).

than the corresponding unstiffened panel frequencies $\omega_{3_n}, \omega_{4_n}, \ldots$, while the frequencies ω_{I} and ω_{II} are higher than the corresponding frequencies ω_{1_n} and ω_{2_n}. This allows us to formulate the rule that for a circumferential mode shape corresponding to n, all panel frequencies that are lower than the stiffener frequency are raised and all panel frequencies that are higher than the stiffener frequency are lowered.

We can show this by again assuming that the system frequency is only a small perturbation of the original frequency. In this case, we obtain approximately

$$\frac{2}{\rho h L(\omega_{mn}^2 - \omega^2)} \sin^2 \frac{m\pi x^*}{L} + \frac{1}{\rho_s A(\omega_n^2 - \omega^2)} = 0 \tag{13.9.21}$$

The solutions of this equation are $\omega = \omega_k$ and are given by

$$\omega_k^2 = \omega_{mn}^2 \frac{1 + (2M_s/M)(\omega_n/\omega_{mn})^2 \sin^2(m\pi x^*/L)}{1 + (2M_s/M)\sin^2(m\pi x^*/L)} \tag{13.9.22}$$

where M_s is the mass of the stiffener,

$$M_s = \rho_s A a \alpha \tag{13.9.23}$$

and where M is the mass of the panel,

$$M = \rho h L a \alpha \tag{13.9.24}$$

The approximate solution shows immediately that

$$\omega_k > \omega_{mn} \quad \text{if } \omega_n > \omega_{mn} \tag{13.9.25}$$

and

$$\omega_k < \omega_{mn} \quad \text{if } \omega_n < \omega_{mn} \tag{13.9.26}$$

Based on experimental evidence, these results are generally valid for any kind of shell or plate and any kind of stiffener. We also notice the expected result that the stiffener will have no effect on modes whose node lines coincide with the stiffener location.

Similar results where obtained in Wilken and Soedel (1976b) and Weissenburger (1968), where the eigenvalues of circular cylindrical shells with multiring stiffeners are explored. Stiffening with stringers follows the same rules. Other examples of line receptance applications can be found in Azimi et al. (1984, 1986).

13.10. TWO SYSTEMS JOINED BY TWO OR MORE DISPLACEMENTS

If we want to join a beam to a beam, we need as a maximum to enforce continuity of transverse deflection and slope (two displacements). In cases where axial vibration in each beam member is also of concern, we need to enforce continuity for three displacements. Two shells may be attached to each other at n points; in the case, we will have to enforce n displacements.

In the following, let us take the two-displacement case as a specific example, shown in Fig. 19, but generalize immediately after each step for the case where two components are joined by n coordinates. From the basic definition of a receptance, which is like an influence function, we

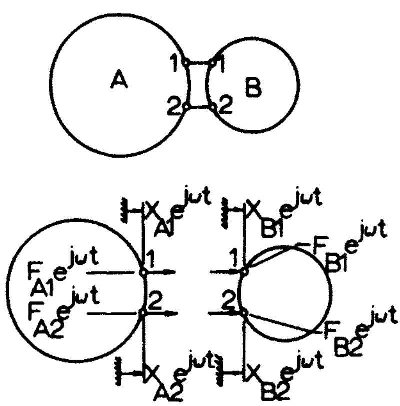

FIG. 19 Two systems joined through two displacements.

obtain the displacement amplitudes of system A as a function of harmonic force inputs at these locations as

$$X_{A1} = \alpha_{11} F_{A1} + \alpha_{12} F_{A2} \tag{13.10.1}$$

$$X_{A2} = \alpha_{21} F_{A1} + \alpha_{22} F_{A2} \tag{13.10.2}$$

In general, for n displacements,

$$\{X_A\} = [\alpha]\{F_A\} \tag{13.10.3}$$

Note that we now require that the cross receptances α_{ij}, where $i \neq j$, also have to be known. Similarly, for system B,

$$X_{B1} = \beta_{11} F_{B1} + \beta_{12} F_{B2} \tag{13.10.4}$$

$$X_{B2} = \beta_{21} F_{B1} + \beta_{22} F_{B2} \tag{13.10.5}$$

In general,

$$\{X_B\} = [\beta]\{F_B\} \tag{13.10.6}$$

When the two systems are joined, the forces at each displacement junction have to add up to 0, or

$$F_{A1} = -F_{B1} \tag{13.10.7}$$

$$F_{A2} = -F_{B2} \tag{13.10.8}$$

In general,

$$\{F_A\} = -\{F_B\} \tag{13.10.9}$$

Also, the displacements have to be equal because of continuity,

$$X_{A1} = X_{B1} \tag{13.10.10}$$

$$X_{A2} = X_{B2} \tag{13.10.11}$$

or, in general,

$$\{X_A\} = \{X_B\} \tag{13.10.12}$$

We may now combine the equation and obtain

$$(\alpha_{11} + \beta_{11}) F_{A1} + (\alpha_{12} + \beta_{12}) F_{A2} = 0 \tag{13.10.13}$$

$$(\alpha_{21} + \beta_{21}) F_{A1} + (\alpha_{22} + \beta_{22}) F_{A2} = 0 \tag{13.10.14}$$

In general, this can be written

$$[[\alpha] + [\beta]]\{F_A\} = 0 \tag{13.10.15}$$

Since $F_{A1}=F_{A2}=0$ would be the trivial solution, it must be that

$$\begin{vmatrix} \alpha_{11}+\beta_{11} & \alpha_{12}+\beta_{12} \\ \alpha_{21}+\beta_{21} & \alpha_{22}+\beta_{22} \end{vmatrix} = 0 \tag{13.10.16}$$

In general, this can be written as

$$|[\alpha]+[\beta]|=0 \tag{13.10.17}$$

In expanded form, the two-displacement case becomes

$$(\alpha_{11}+\beta_{11})(\alpha_{22}+\beta_{22})-(\alpha_{12}+\beta_{12})^2=0 \tag{13.10.18}$$

13.11. SUSPENSION OF AN INSTRUMENT PACKAGE IN A SHELL

To illustrate the case of two systems joined by two displacements, let us treat a circular cylindrical shell inside of which an instrument package is supported by way of two equal springs as shown in Fig. 20. Let us take the case where the springs are attached at opposite points at locations (x^*, θ^*) and $(x^*, \theta^*+\pi)$.

The receptances of system B are obtained by first considering the force F_{B1}, with $F_{B2}=0$, Fig. 20 and evaluating X_{B1} and X_{B2}. This gives,

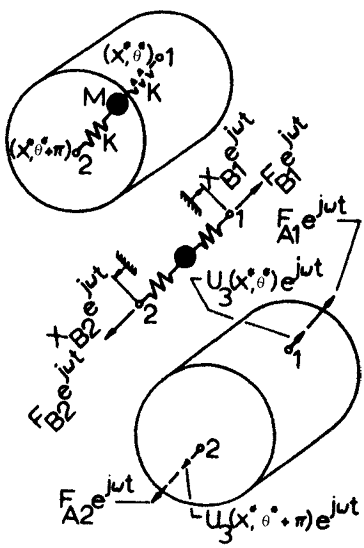

Fig. 20 A mass-spring system joined transversely at two locations to a circular cylindrical shell.

utilizing Eq. (13.10.5),

$$\beta_{11} = \frac{1}{k} - \frac{1}{M\omega^2} \tag{13.11.1}$$

$$\beta_{21} = \frac{1}{M\omega^2} \tag{13.11.2}$$

Next, we consider F_{B2}, with $F_{B1}=0$, and evaluate X_{B1} and X_{B2}. As expected because of symmetry, $\beta_{22}=\beta_{11}$ and $\beta_{12}=\beta_{21}$.

The receptances for the shell are obtained from the solution for a harmonic point force acting on a circular cylindrical shell as obtained in Eq. (8.8.29). We evaluate u_3 at (x^*, θ^*) and let $P_3=F_{A1}$. This gives

$$\alpha_{11}=\frac{4}{M_s}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\sin^2(m\pi x^*/L)}{\varepsilon_n(\omega_{mn}^2-\omega^2)} \tag{13.11.3}$$

and evaluating u_3 at $(x^*, \theta^*+\pi)$, we obtain

$$\alpha_{21}=\frac{4}{M_s}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\sin^2(m\pi x^*/L)\cos n\pi}{\varepsilon_n(\omega_{mn}^2-\omega^2)} \tag{13.11.4}$$

where

$$M_s=2\pi\rho haL \tag{13.11.5}$$

Next, applying the load $P_3=F_{A2}$ at $x^*, \theta^*+\pi$ and evaluating u_3 at $(x^*, \theta^*+\pi)$ gives

$$\alpha_{22}=\frac{4}{M_s}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\sin^2(m\pi x^*/L)}{\varepsilon_n(\omega_{mn}^2-\omega^2)} \tag{13.11.6}$$

and evaluating u_3 at (x^*, θ^*), we obtain, as expected,

$$\alpha_{12}=\frac{4}{M_s}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\sin^2(m\pi x^*/L)\cos n\pi}{\varepsilon_n(\omega_{mn}^2-\omega^2)} \tag{13.11.7}$$

Since in this case $\alpha_{11}=\alpha_{22}$ and $\beta_{11}=\beta_{22}$, the characteristic equation is

$$(\alpha_{11}+\beta_{11})^2-(\alpha_{12}+\beta_{12})^2=0 \tag{13.11.8}$$

or

$$\alpha_{11}+\beta_{11}=\pm(\alpha_{12}+\beta_{12}) \tag{13.11.9}$$

This gives

$$\frac{4}{M_s}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\sin^2(m\pi x^*/L)}{\varepsilon_n(\omega_{mn}^2-\omega^2)}(1\mp\cos n\pi)+\frac{1}{k}-\frac{1}{M\omega^2}(1\pm1)=0 \tag{13.11.10}$$

and may be written in terms of two equations. The first is

$$\frac{4}{M_s}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\sin^2(m\pi x^*/L)}{\varepsilon_n(\omega_{mn}^2-\omega^2)}(1-\cos n\pi)+\frac{1}{k}-\frac{2}{M\omega^2}=0 \tag{13.11.11}$$

and the second is

$$\frac{4}{M_s}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{\sin^2(m\pi x^*/L)}{\varepsilon_n(\omega_{mn}^2-\omega^2)}(1+\cos n\pi)+\frac{1}{k}=0 \tag{13.11.12}$$

The roots $\omega=\omega_k$ of these equations will be the natural frequencies of the system. Substitution of these into the displacement solution of the shell will give the new mode shapes.

Equation (13.11.11), for $n=1,3,5,\ldots$, gives two sets of natural frequencies and modes. The in-phase motion of center mass with the shell is shown in Fig. 21. The other set describes the out-of-phase motion.

In addition, for $n=0,2,4,\ldots$, Eq. (13.11.11) will give the natural frequency of the mass on its two springs when the springs are attached to node points.

Let us next assume that the influence of the spring is small, so that we can utilize a one-term solution. This gives, for Eq. (13.11.12),

$$\omega_k^2=\omega_{mn}^2+\frac{4k}{M_s\varepsilon_n}\sin^2\frac{m\pi x^*}{L}(1+\cos n\theta) \tag{13.11.13}$$

If $n=1,3,5,\ldots$, this equation gives

$$\omega_k^2=\omega_{mn}^2 \tag{13.11.14}$$

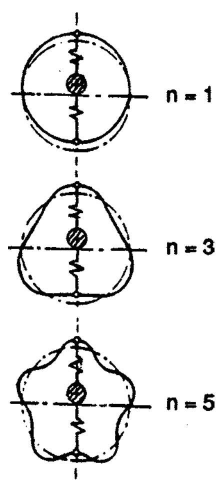

Fig. 21 In-phase motion of the mass-spring system with the shell motion at the attachment points.

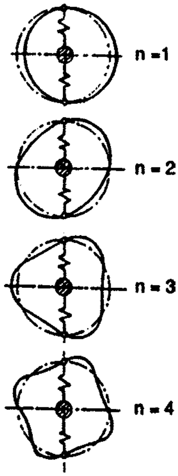

FIG. 22 Natural modes of the combined system where the center mass is motionless because the attachment locations and node line locations coincide.

It means that the spring is not active for $n = 1, 3, 5, \ldots$ since the displacement at each end of the spring is 0, as illustrated in Fig. 22 (see also Soedel, 1987). For $n = 0, 2, 4, \ldots$, Eq. (13.11.13) gives

$$\omega_k^2 = \omega_{mn}^2 + \frac{8k}{M_s \varepsilon_n} \sin^2 \frac{m\pi x^*}{L} \tag{13.11.15}$$

In this case, the springs are compressed equally from both ends, as shown in Fig. 23. In neither case does the center mass experience motion.

13.12. SUBTRACTING STRUCTURAL SUBSYSTEMS

In the foregoing, we have treated the addition of structural subsystems such as masses, springs, dampers, stiffeners, etc., to structural systems by the receptance technique. But at times, it becomes necessary to subtract systems. For example, we may want to make transfer function measurements on a shell structure alone, which for one reason or another, has masses, springs or other systems attached to it which we are unable to remove during the measurement. If the receptances of the attachments are known, is it possible to mathematically subtract the influence of the attachments from the data for the shell with attachments to obtain the transfer function of the shell alone?

Let us first pretend that we would like to add systems A and B. For the case of Fig. 24, where a force $F_1 e^{j\omega t}$ is applied to the system at

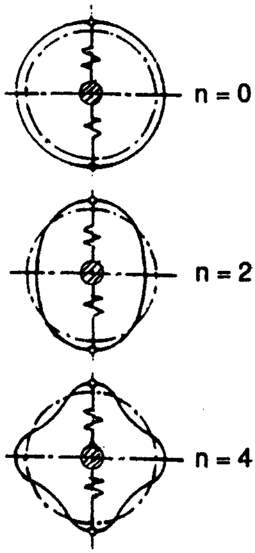

Fig. 23 The center mass is motionless for the set of natural modes that is symmetric with respect to the center mass.

location 1,

$$X_{A1} = X_{C1} = \alpha_{11} F_{A1} + \alpha_{12} F_{A2} \tag{13.12.1}$$

$$X_{A2} = X_{C2} = \alpha_{21} F_{A1} + \alpha_{22} F_{A2} \tag{13.12.2}$$

where the α_{ij} can be complex numbers. Since

$$F_{A2} = -F_{B2} \tag{13.12.3}$$

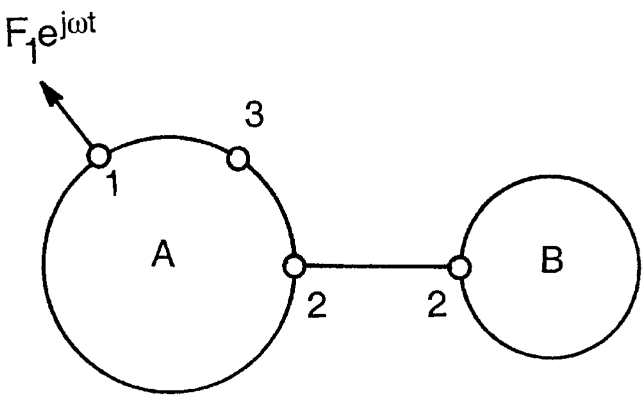

Fig. 24 Illustrative example of subtracting a subsystem.

and

$$X_{C2} = X_{B2} = \beta_{22} F_{B2} \tag{13.12.4}$$

we obtain

$$F_{A2} = -X_{C2}/\beta_{22} \tag{13.12.5}$$

Since

$$F_{A1} = F_1 \tag{13.12.6}$$

Equations. (13.12.1) and (13.12.2) become

$$X_{C1} = \alpha_{11} F_1 - \frac{\alpha_{12}}{\beta_{22}} X_{C2} \tag{13.12.7}$$

and

$$X_{C2} = \alpha_{21} F_1 - \frac{\alpha_{22}}{\beta_{22}} X_{C2} \tag{13.12.8}$$

Solving these equations for the total system receptances γ_{11} and γ_{21} gives

$$\gamma_{11} = \frac{X_{C1}}{F_1} = \alpha_{11} - \frac{\alpha_{21}\alpha_{12}}{\alpha_{22} + \beta_{22}} \tag{13.12.9}$$

and

$$\gamma_{21} = \frac{X_{C2}}{F_1} = \frac{\beta_{22}\alpha_{21}}{\alpha_{22} + \beta_{22}} \tag{13.12.10}$$

Here, γ designates receptances of the total system C(A and B added), α designates receptances of the original system A and β designates receptances of the attached system B.

Next, we assume that we know all γ receptances of system C, and we know the β_{22} receptance of the attached system B. This leaves us with three unknowns: $\alpha_{11}, \alpha_{12} = \alpha_{21}$, and α_{22}. This means that in addition of Eqs. (13.12.9) and (13.12.10) we need to generate a third equation.

This third equation is found by placing a force at the point of system C where subsystem B is attached to the original system A. Similar to the above analysis, we find

$$\gamma_{22} = \frac{\beta_{22}\alpha_{22}}{\alpha_{22} + \beta_{22}} \tag{13.12.11}$$

Note that Eq. (13.12.11) can be obtained by replacing α_{12} by α_{22} in Eq. (13.12.10), but if γ_{22} is to be obtained experimentally, it means that a shaker will have to be attached at point 2 of system C.

Solving Eqs. (13.12.9) to (13.12.11) for $\alpha_{11}, \alpha_{12} = \alpha_{21}$, and α_{22} gives

$$\alpha_{11} = \frac{\gamma_{11}(\gamma_{22} - \beta_{22}) - \gamma_{21}^2}{\gamma_{22} - \beta_{22}} \tag{13.12.12}$$

$$\alpha_{21} = \frac{-\gamma_{21}\beta_{22}}{\gamma_{22} - \beta_{22}} \tag{13.12.13}$$

$$\alpha_{22} = \frac{-\gamma_{22}\beta_{22}}{\gamma_{22} - \beta_{22}} \tag{13.12.14}$$

where the α receptances are the transfer function of the original system without the attached system B. The response of the original systems A at forcing point 1 is, therefore:

$$x_1 = \alpha_{11} F_{A1} e^{j\omega t} \tag{13.12.15}$$

13.12.1. Natural Frequencies and Modes of the Original System *A*

From Eqs. (13.12.12) to (13.12.14), we see that for zero damping, the α receptances approach infinity whenever

$$\gamma_{22} - \beta_{22} = 0 \tag{13.12.16}$$

Values of ω which satisfy Eq. (13.12.16) are the natural frequencies ω_k of the original, or “reduced" system A.

Intuitively, it seems possible to think of the system B subtraction in terms of adding a “negative receptance" $(-\beta_{22})$ to the total system receptance γ_{22}, very much like the standard frequency equation for the total system C which is, from Eqs. (13.12.9) or (13.12.10), $\alpha_{22} + \beta_{22} = 0$. This is not unreasonable for simple systems B, such as a mass or a spring. However, for more complicated systems B interpretation difficulties are encountered. Also, it is not possible to obtain the natural modes of the original system A by such an intuitive approach.

To find the natural modes of the original system A, it is necessary to formulate the response at an arbitrary point 3 on system A, which requires auxiliary measurement or calculations of receptances γ_{31} and γ_{32}. In terms of the subsystem receptances,

$$\gamma_{31} = \alpha_{31} - \frac{\alpha_{32}\alpha_{21}}{\alpha_{22} + \beta_{22}} \tag{13.12.17}$$

$$\gamma_{32} = \frac{\alpha_{32}\beta_{22}}{\alpha_{22} + \beta_{22}} \tag{13.12.18}$$

These equations are solved for α_{31} and α_{32}:

$$\alpha_{31} = \gamma_{31} - \frac{\gamma_{32}\gamma_{21}}{\gamma_{22} - \beta_{22}} \tag{13.12.19}$$

$$\alpha_{32} = -\frac{\gamma_{32}\beta_{22}}{\gamma_{22} - \beta_{22}} \tag{13.12.20}$$

Here, α_{31} defines the natural modes of the original system A whenever $\omega = \omega_k$ in the equation. Thus, α_{31} is evaluated at various points 3 for each natural frequency ω_k obtained from Eq. (13.12.16). In practice, this must be done for damping small enough to allow the kth mode to dominate the response, but large enough to avoid the potential singularity at $\omega = \omega_k$.

Note that the foregoing is also applicable to discrete systems. Soedel and Soedel (1994) derived the general approach discussed here, but applied it to a measurement problem involving an automotive suspension modeled by lumped parameters.

13.12.2. Natural Frequencies, Modes, and Receptances of a Rectangular Plate with a Small Mass Void

The plate in this case is a rectangular, simply supported plate. A relatively concentrated amount of mass, M, is missing. The mass void is located at (x_2, y_2). We assume that the mass void is not accompanied by a loss of stiffness because either the void (hole) is small and located near the reference plane without an appreciable loss of stiffness, or the plate is reinforced there to compensate for the loss of stiffness due to the mass void.

In this case, the plate with the mass void is system A, the full plate is the total system C, and the point mass which is missing is system B. Subtracting system B from the total system C gives system A.

In this case, the receptances of the full plate of system C are

$$\gamma_{11} = \frac{4}{\rho hab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin^2 \frac{m\pi x_1}{a} \sin^2 \frac{n\pi y_1}{b} \tag{13.12.21}$$

$$\gamma_{21} = \gamma_{12} = \frac{4}{\rho hab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin \frac{m\pi x_2}{a} \sin \frac{n\pi y_2}{b} \sin \frac{m\pi x_1}{a} \sin \frac{n\pi y_1}{b} \tag{13.12.22}$$

$$\gamma_{22} = \frac{4}{\rho hab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \sin^2 \frac{m\pi x_2}{a} \sin^2 \frac{n\pi y_2}{b} \tag{13.12.23}$$

The receptance β_{22} of the mass is

$$\beta_{22} = -\frac{1}{M\omega^2} \tag{13.12.24}$$

The natural frequencies are obtained from Eq. (13.12.16)

$$\gamma_{22} - \beta_{22} = 0 \tag{13.12.25}$$

or

$$\frac{4}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\omega_{mn}^2 - \omega^2}\sin^2\frac{m\pi x_2}{a}\sin^2\frac{n\pi y_2}{b} + \frac{1}{M\omega^2} = 0 \tag{13.12.26}$$

The values of ω that satisfy this equation are the natural frequencies ω_k of the plate with the mass void. Or approximately, following a similar thought process that lead to Eq. (13.2.8),

$$\omega_k^2 = \frac{\omega_{mn}^2}{1 - \left(\frac{4m}{M_s}\right)\sin^2\frac{m\pi x_2}{a}\sin^2\frac{n\pi y_2}{b}} \tag{13.12.27}$$

where $M_s = \rho hab$, the mass of the plate without mass void. Thus, the subtraction of a point mass without loss of stiffness leads to an increase of the natural frequencies of the plate as expected.

The natural modes can be obtained either from Eq. (13.12.19) or, in this simple case, by substituting ω_k for ω in the undamped displacement response $u_3(x, y, t)$ of the full plate, similar to the panel example in Eqs. (13.2.11) and (13.2.12). This gives

$$U_{3k}(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin((m\pi x_2)/a)\sin((n\pi y_2)/b)}{\omega_{mn}^2 - \omega_k^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{13.12.28}$$

The receptances of the plate with mass void, $\alpha_{11}, \alpha_{21} = \alpha_{12}$ and α_{22}, are given by Eqs. (13.12.12)–(13.12.14), with $\gamma_{11}, \gamma_{21} = \gamma_{12}, \gamma_{22}$ and β_{22} defined by Eqs. (13.12.21)–(13.12.24).

13.13. THREE AND MORE SYSTEMS CONNECTED

As another example of system connections, we consider systems A,B, and C joined as shown in Fig. 25. A and B, and B and C are joined by one displacement each. The derivation below can easily be extended to multi-displacement connections.

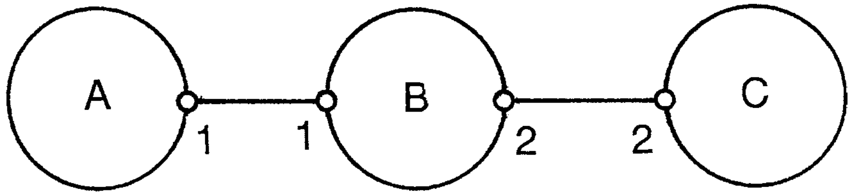

Fig. 25 Three systems connected to each other by one displacement.

13.13.1. Natural Frequencies of the Joined System

The displacement amplitude X_{A1} of system A at location 1 as a function of a harmonic force F_A, is

$$X_{A1} = \alpha_{11} F_{A1} \tag{13.13.1}$$

For system B, the displacement amplitudes at locations 1 and 2 as functions of the harmonic force amplitudes F_{B1} and F_{B2} are

$$X_{B1} = \beta_{11} F_{B1} + \beta_{12} F_{B2} \tag{13.13.2}$$

$$X_{B2} = \beta_{22} F_{B2} + \beta_{21} F_{B1} \tag{13.13.3}$$

Finally, the displacement amplitude of system C at location 2 is, as function of a harmonic force amplitude at location 2,

$$X_{C2} = \gamma_{22} F_{C2} \tag{13.13.4}$$

Joining systems A,B, and C, the continuity conditions are

$$X_{A1} = X_{B1} \tag{13.13.5}$$

$$X_{B2} = X_{C2} \tag{13.13.6}$$

and force equilibrium demands that

$$F_{A1} + F_{B1} = 0 \tag{13.13.7}$$

$$F_{B2} + F_{C2} = 0 \tag{13.13.8}$$

Combining Eqs. (13.12.1)–(13.12.8) gives

$$\begin{bmatrix} \alpha_{11} + \beta_{11} & -\beta_{12} \\ -\beta_{21} & \gamma_{22} + \beta_{22} \end{bmatrix} \begin{Bmatrix} F_{B1} \\ F_{B2} \end{Bmatrix} = 0 \tag{13.13.9}$$

Unless the systems are connect at node points, in general

$$\begin{Bmatrix} F_{B1} \\ F_{B2} \end{Bmatrix} \neq 0 \tag{13.13.10}$$

Therefore, it must be that

$$\begin{vmatrix} \alpha_{11}+\beta_{11} & -\beta_{12} \\ -\beta_{21} & \gamma_{22}+\beta_{22} \end{vmatrix} = 0 \tag{13.13.11}$$

or in expanded form, setting $\beta_{12}=\beta_{21}$,

$$(\alpha_{11}+\beta_{11})(\beta_{22}+\gamma_{22})-\beta_{12}^2=0 \tag{13.13.12}$$

The values of ω, which satisfy this equation, are the natural frequencies of the system, ω_k.

13.13.2. The Steady State Response of a Point on System C to a Harmonic Force Input to System A

The example solved here is shown in Fig. 26. In this case, the force acts at point 0 of system A and the response will be obtained at point 3 located on system C.

For system A, we may write

$$X_{A0}=\alpha_{00}F_{A0}+\alpha_{01}F_{A1} \tag{13.13.13}$$

$$X_{A1}=\alpha_{10}F_{A0}+\alpha_{11}F_{A1} \tag{13.13.14}$$

For system B, the displacement amplitudes are given by Eqs. (13.13.2) and (13.13.3) and the displacement amplitude for system C at point 2 is given by Eq. (13.13.4). The new equation needed is the displacement response at point 3 of system C:

$$X_{C3}=\gamma_{32}F_{C2} \tag{13.13.15}$$

The continuity equations (13.13.5) and (13.13.6) still apply, as do the force equilibrium relationships (13.13.7) and (13.13.8). Combining all of these equations and solving for X_{C3} gives

$$X_{C3}=\frac{\alpha_{01}\beta_{12}\gamma_{23}}{(\alpha_{11}+\beta_{11})(\beta_{22}+\gamma_{22})-\beta_{12}^2}F_{A0} \tag{13.13.16}$$

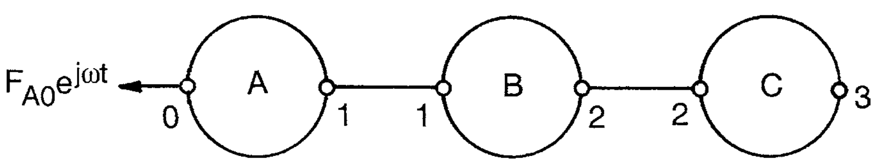

FIG. 26 The connected three systems are forced at system A.

Therefore, the response to $F_{A0}e^{j\omega t}$ is

$$x_{C3} = X_{C3}e^{j\omega t} \tag{13.13.17}$$

Note that X_{C3} is a complex number, as are all or some of the α, β, and γ receptances, if there is damping in the system.

13.13.3. Natural Frequencies if Systems *A*, *B* and *C* are Connected by Springs

If the connections are springs as shown in Fig. 27, we may either redefine the receptances as in Sec. 13.6, or follow a more direct approach by redefining F_{B1} and F_{C2} (this assumes that the left spring of rate k_1 is considered to be part of system B and the right spring of rate k_2 is considered to be part of system C):

$$F_{B1} = k_1(X_{A1} - X_{B1}) \tag{13.13.18}$$

$$F_{C2} = k_2(X_{B2} - X_{C2}) \tag{13.13.19}$$

These equations replace the continuity conditions, since now $X_{A1} \neq X_{B1}$ and $X_{B2} \neq X_{C2}$! Equation (13.13.1)–(13.13.4) still apply. Utilizing the force equilibrium conditions (13.13.7) and (13.13.8) and combining all equations give

$$\begin{bmatrix} [1+k_1(\alpha_{11}+\beta_{11})] & -k_1\beta_{12} \\ -k_2\beta_{21} & [1+k_2(\beta_{22}+\gamma_{22})] \end{bmatrix} \begin{Bmatrix} F_{B1} \\ F_{B2} \end{Bmatrix} = 0 \tag{13.13.20}$$

Again, the argument that leads to Eq. (13.13.11) applies

$$\begin{bmatrix} [1+k_1(\alpha_{11}+\beta_{11})] & -k_1\beta_{12} \\ -k_2\beta_{21} & [1+k_2(\beta_{22}+\gamma_{22})] \end{bmatrix} = 0 \tag{13.13.21}$$

or in expanded form, since $\beta_{12} = \beta_{21}$,

$$[1+k_1(\alpha_{11}+\beta_{11})][1+k_2(\beta_{22}+\gamma_{22})] - k_1k_2\beta_{12}^2 = 0 \tag{13.13.22}$$

The values of ω that satisfy this equation are the natural frequencies ω_k of the system.

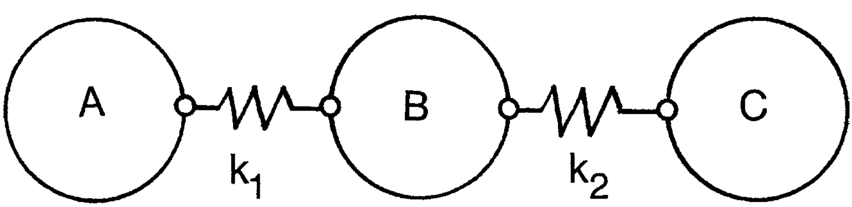

Fig. 27 Three systems connected to each other by springs.

Note that if we factor out $k_1 k_2$, we may write

$$\left[\frac{1}{k_1}+(\alpha_{11}+\beta_{11})\right]\left[\frac{1}{k_2}+(\beta_{22}+\gamma_{22})\right]-\beta_{12}^2=0 \tag{13.13.23}$$

and if we simulate stiff connections by letting $k_1 \to \infty$ and $k_2 \to \infty$, we obtain Eq. (13.13.12) as expected.

13.13.4. Extension to Multiple System Connections

For example, the frequency equation (13.13.11) in determinant form can easily be extrapolated to connected systems A, B, C, D, E, etc. Designating the receptances as $\alpha, \beta, \gamma, \delta, \varepsilon$, etc., we obtain

$$\begin{vmatrix} \alpha_{11}+\beta_{11} & -\beta_{12} & 0 & 0 & \cdot \\ -\beta_{21} & \beta_{22}+\gamma_{22} & -\gamma_{12} & 0 & \cdot \\ 0 & -\gamma_{21} & \gamma_{33}+\delta_{33} & -\delta_{12} & \cdot \\ 0 & 0 & -\delta_{21} & \delta_{44}+\varepsilon_{44} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}=0 \tag{13.13.24}$$

13.14. EXAMPLES OF THREE SYSTEMS CONNECTED TO EACH OTHER

13.14.1. Three Plates Connected to Each Other by Transverse Point Connections

Three identical, simply supported rectangular plates are connected to each other by mass less, rigid links as shown in Fig. 28.

From Eq. (9.9.28), the steady state, harmonic response of a simply supported plate at a location (x_i, y_i) to a harmonic point force at location (x_j, y_j) is, for zero damping (the j in the exponential term is $\sqrt{-1}$),

$$u_3(x_i, y_i, t)=\frac{4F}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x_i}{a}\sin\frac{n\pi y_i}{b}\sin\frac{m\pi x_j}{a}\sin\frac{n\pi y_j}{b}}{\omega_{mn}^2-\omega^2}e^{j\omega t} \tag{13.14.1}$$

Thus, for the identical plates A,B, and C, we obtain from this equation the receptances

$$\alpha_{11}=\beta_{11}=\frac{4}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2\frac{m\pi x_1}{a}\sin^2\frac{n\pi y_1}{b}}{\omega_{mn}^2-\omega^2} \tag{13.14.2}$$

$$\beta_{12}=\beta_{21}=\frac{4}{\rho hab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b}\sin\frac{m\pi x_2}{a}\sin\frac{n\pi y_2}{b}}{\omega_{mn}^2-\omega^2} \tag{13.14.3}$$

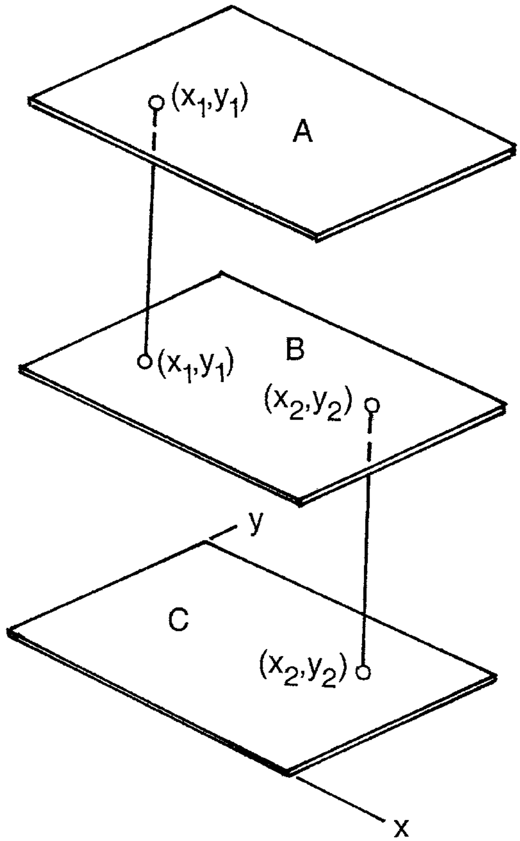

FIG. 28 Three plates connected to each other by transverse connections.

$$\beta_{22} = \gamma_{22} = \frac{4}{\rho hab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2 \frac{m\pi x_2}{a} \sin^2 \frac{n\pi y_2}{b}}{\omega_{mn}^2 - \omega^2} \tag{13.14.4}$$

The frequency equation (13.13.12) becomes, therefore

$$4\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2\frac{m\pi x_1}{a}\sin^2\frac{n\pi y_1}{b}}{\omega_{mn}^2-\omega^2}\right]\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2\frac{m\pi x_2}{a}\sin^2\frac{n\pi y_2}{b}}{\omega_{mn}^2-\omega^2}\right]$$
$$-\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b}\sin\frac{m\pi x_2}{a}\sin\frac{n\pi y_2}{b}}{\omega_{mn}^2-\omega^2}\right]^2 = 0 \tag{13.14.5}$$

The ω that satisfy this equation are the natural frequencies of the system that are affected by the connection.

For the special case that the connectors are in line so that $(x_1, y_1) = (x_2, y_2)$, we obtain

$$4\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2\frac{m\pi x_1}{a}\sin^2\frac{n\pi y_1}{b}}{\omega_{mn}^2-\omega^2}\right]^2-\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2\frac{m\pi x_1}{a}\sin^2\frac{n\pi y_1}{b}}{\omega_{mn}^2-\omega^2}\right]^2=0 \tag{13.14.6}$$

or

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2\frac{m\pi x_1}{a}\sin^2\frac{n\pi y_1}{b}}{\omega_{mn}^2-\omega^2}=0 \tag{13.14.7}$$

The ω that satisfy this equation are the natural frequencies of the combined system. This can also be shown from Eq. (13.13.12) directly. For this special case where $(x_1, y_1) = (x_2, y_2)$, we obtain from Eqs. (13.14.2)–(13.14.4) that

$$\alpha_{11} = \beta_{11} = \beta_{22} = \gamma_{22} = \beta_{12} \tag{13.14.8}$$

Substituting this in Eq. (13.12.12) gives

$$4\alpha_{11}^2 - \alpha_{11}^2 = 0 \tag{13.14.9}$$

or

$$\alpha_{11} = 0 \tag{13.14.10}$$

Please note that it is possible, especially since we have here identical plates, that the connection locations (x_1, y_1) may fall on node lines of the unconnected plates. In this case, there is a set of modes such that

$$F_{A1} = F_{B1} = F_{B2} = F_{C2} = 0 \tag{13.14.11}$$

In this case, we obtain from Eq. (13.13.1) that

$$\frac{1}{\alpha_{11}} = 0 \tag{13.14.12}$$

so, from Eq. (13.14.2), the system frequencies include also

$$\omega = \omega_{mn} \tag{13.14.13}$$

which are the natural frequencies of the plates as if there is no coupling between them. This argument also holds for the general solution of Eq. (13.14.5) since it is possible that both (x_1, y_1) and (x_2, y_2) could be located on node lines of the individual, unconnected plates.

This example shows that one cannot rely on Eq. (13.14.5) to necessarily give all of the system natural frequencies, but that node line connections have to be explored also. This holds for all systems: beams, plates, shell, etc.

Note also that Eqs. (13.14.11)–(13.14.13) also apply if the identical plates move in unison, since in this case the connection forces would also be conceivably 0.

13.14.2. Three Plates Connected to Each Other Through Moment Coupling

In this example, three identical, simply supported rectangular plates are connected to each other at their boundaries as shown in Fig. 29, forming a continuous single plate. Rewriting Eq. (10.5.11) for zero damping (and switching $L=a$) gives

$$u_3(x,y,t) = -\frac{2M_0'\pi}{a^2\rho h}\sin\frac{\rho\pi y}{b}\sum_{m=1}^{\infty}\frac{m\cos\frac{m\pi x^*}{a}\sin\frac{m\pi x}{a}}{\omega_{m\rho}^2-\omega^2}\sin\omega t \qquad (13.14.14)$$

which is the response of the plate to a line moment $M_0'\sin\frac{\rho\pi y}{b}\sin\omega t$ located at x^*.

The derivative with respect to x is

$$\frac{\partial U_3}{\partial_x}(x,y,t) = -\frac{2M_0'\pi^2}{a^3\rho h}\sin\frac{\rho\pi y}{b}\sum_{m=1}^{\infty}\frac{m^2\cos\frac{m\pi x^*}{a}\cos\frac{m\pi x}{a}}{\omega_{m\rho}^2-\omega^2}\sin\omega t \quad (13.14.15)$$

This gives a line receptance (see also Sec. 13.9) for plate A at connection $1(x=x^*=a)$ of

$$\alpha_{11} = -\frac{2\pi^2}{a^3\rho h}\sum_{m=I}^{\infty}\frac{m^2\cos^2 m\pi}{\omega_{mp}^2-\omega^2} = -\frac{2\pi^2}{a^3\rho h}\sum_{m=1}^{\infty}\frac{m^2}{\omega_{mp}^2-\omega^2} \qquad (13.14.16)$$

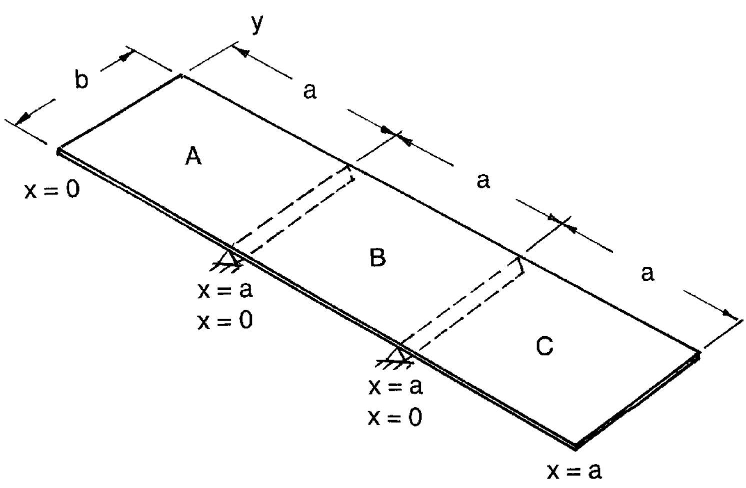

Fig. 29 A continuous plate example (or three plates connected to each other by rotational displacements).

For plate B, we obtain at $x=x^*=0$,

$$\beta_{11}=-\frac{-2\pi^2}{a^3\rho h}\sum_{m=1}^{\infty}\frac{m^2}{\omega_{mn}^2-\omega^2} \tag{13.14.17}$$

which is the same as Eq. (13.14.16), and at $x=x^*=a$ the same again as Eq. (13.14.16), thus

$$\beta_{11}=\beta_{22}=\alpha_{11} \tag{13.14.18}$$

The line receptance β_{12} is obtained by evaluating Eq. (13.14.15) at $x=0$ and $x^*=a$:

$$\beta_{12}=-\frac{2\pi^2}{a^3\rho h}\sum_{m=1}^{\infty}\frac{m^2\cos m\pi}{\omega_{mp}^2-\omega^2}=\beta_{21} \tag{13.14.19}$$

Finally, for plate C, the line receptance at location 2 is the same as β_{11}:

$$\gamma_{22}=\beta_{11}=\beta_{22}=\alpha_{11} \tag{13.14.20}$$

Equation (13.13.12) becomes

$$4\alpha_{11}^2-\beta_{12}^2=0 \tag{13.14.21}$$

Substituting Eqs.(13.14.16) and (13.14.19) gives

$$4\left[\sum_{m=1}^{\infty}\frac{m^2}{\omega_{mp}^2-\omega^2}\right]-\left[\sum_{m=1}^{\infty}\frac{m^2\cos m\pi}{\omega_{mp}^2-\omega^2}\right]^2=0 \tag{13.14.22}$$

The values of ω that satisfy this equation are natural frequencies of the system of three identical plates. Again, it is conceivable that in this special case connecting moments could be 0 if the three identical plates vibrate with mode shapes for unconnected plates, and thus

$$\frac{1}{\alpha_{11}}=0 \tag{13.14.23}$$

or

$$\omega=\omega_{mp} \tag{13.14.24}$$

may also be a solution set. Note the slow convergence of Eq. (13.14.22) because of m^2 in the numerators.

Examples of circular plates connected to circular cylindrical shells can be found in Huang and Soedel (1993a,b,c).

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14

Hysteresis Damping

The equivalent viscous damping coefficient that was used in the chapters on the forced response of shell structures is a function of several effects. While there may truly be a motion-resisting force proportional to velocity, we may also have turbulent damping proportional to velocity squared caused by the surrounding media; boundary damping either because of friction in the boundary joints themselves (rivets, clamps, etc.) or because of the elasticity of the boundary (we have to allow for a certain amount of energy to be converted to wave action of the boundary material which is lost to the system that is being investigated); and internal damping of the material. Internal damping is characterized by a hysteresis loop. There is also the possibility that damping is introduced by friction between two shell surfaces. For example, to dampen the hermetically sealed shells of refrigeration machinery, a ring of the same sheet material is loosely pressed inside the main shell so that the two surfaces can work against each other when vibrating.

Historically, internal damping was first investigated in 1784 by Coulomb (1784). Using his torsional pendulum, he showed experimentally that damping was also caused by a microstructural mechanism and not only by air friction. He recognized that this internal damping, or *hysteresis damping*, as it is often termed, was a function of vibration amplitude. Many investigations of this topic have been carried out since.

14.1. EQUIVALENT VISCOUS DAMPING COEFFICIENT

When equivalent viscous damping is assumed, the forces per unit surface area in the three different directions are given by ($i=1,2,3$)

$$q_i=\lambda\dot{u}_i \tag{14.1.1}$$

and if the motion is harmonic,

$$u_i=U_i\sin\omega t \tag{14.1.2}$$

we get

$$q_i=\lambda U_i\omega\cos\omega t \tag{14.1.3}$$

The average dissipated energy per unit surface area and per cycle of harmonic motion is

$$E_{\mathrm{d}}=\frac{1}{A}\int\int_A\sum_{i=1}^{3}\int_0^{2\pi/\omega}\lambda U_i^2\omega^2\cos^2\omega t\,\mathrm{d}t\,\mathrm{d}A \tag{14.1.4}$$

where A is the surface area of the shell. We get

$$E_{\mathrm{d}}=\lambda\pi\omega\frac{1}{A}\int\int_A(U_1^2+U_2^2+U_3^2)\mathrm{d}A \tag{14.1.5}$$

Thus, if we can identify by theoretical models or by experiment what the energy dissipated per cycle and unit area is, we may solve for λ and obtain

$$\lambda=\frac{E_{\mathrm{d}}A}{\pi\omega\int\int_A(U_1^2+U_2^2+U_3^2)\mathrm{d}A} \tag{14.1.6}$$

When transverse motion is dominant, the most common case, $U_1^2+U_2^2\ll U_3^2$, and we may set

$$\lambda=\frac{E_{\mathrm{d}}A}{\pi\omega\int\int_A U_3^2\mathrm{d}A} \tag{14.1.7}$$

14.2. HYSTERESIS DAMPING

Structural damping is characterized by the fact that if we cycle a tensile test specimen, we obtain a hysteresis loop as shown in Fig. 1. The shaded area of the loop is the total energy dissipated per cycle. If we divide the force by the cross-section of the specimen and the displacement by the length of the specimen, we get a stress–strain plot of the same phenomenon. The area is now equal to the energy dissipated per cycle and volume.

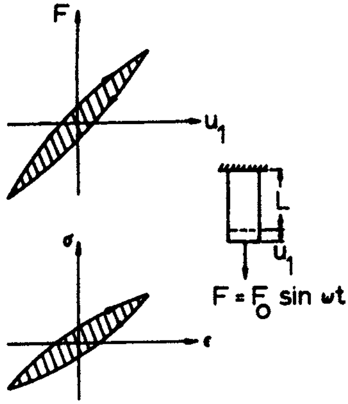

Fig. 1 Typical measured hysteresis loop.

Unfortunately, no generally acceptable way has been found to utilize this information directly. An approximation is the *complex modulus*. We have Hooke's law,

$$\sigma = E\varepsilon \tag{14.2.1}$$

Substitution gives us

$$\sigma = \sigma_{\max} \sin \omega t \tag{14.2.2}$$

where

$$\sigma_{\max} = \frac{F_0}{A} \tag{14.2.3}$$

This gives

$$\varepsilon = \frac{\sigma_{\max}}{E} \sin \omega t \tag{14.2.4}$$

Plotting σ as a function of ε gives, as expected, a straight line. For the line to acquire width so that we obtain a resemblance to a hysteresis loop, we have to replace E by $E(1+j\eta)$, where η is called the hysteresis loss factor. In this case, we obtain

$$\varepsilon = \frac{\sigma_{\max}}{E(1+j\eta)} \sin \omega t \tag{14.2.5}$$

This can be written as

$$\varepsilon = \frac{\sigma_{\max}}{E\sqrt{1+\eta^2}} \sin(\omega t - \phi) \tag{14.2.6}$$

where

$$\phi = \tan^{-1} \eta \tag{14.2.7}$$

For typically small values of η, we obtain

$$\phi \cong \eta \tag{14.2.8}$$

and

$$\sqrt{1+\eta^2} \cong 1 \tag{14.2.9}$$

Thus

$$\varepsilon = \frac{\sigma_{\max}}{E} \sin(\omega t - \eta) \tag{14.2.10}$$

Plotting σ as a function of ε in Fig. 2 gives an ellipse with the approximate half-axes

$$a = \frac{\sigma_{\max}}{\cos\alpha} \tag{14.2.11}$$

$$b = \frac{\sigma_{\max}}{E} \eta \cos\alpha \tag{14.2.12}$$

Since the energy dissipated per cycle and volume is equal to the area of the ellipse, we obtain

$$E_1 = \frac{\pi}{E} \sigma_{\max}^2 \eta = \pi E \varepsilon_{\max}^2 \eta \tag{14.2.13}$$

The total dissipated energy in the specimen is

$$E_T = Lbh\pi E \varepsilon_{\max}^2 \eta \tag{14.2.14}$$

Since the maximum strain energy in the test specimen is

$$U_{max} = \frac{Lbh}{2} \sigma_{\max} \varepsilon_{\max} = \frac{Lbh}{2} E \varepsilon_{\max}^2 \tag{14.2.15}$$

we find that [Ross, Ungar, and Kerwin (1959)]

$$\eta = \frac{1}{2\pi} \frac{E_T}{U_{\max}} \tag{14.2.16}$$

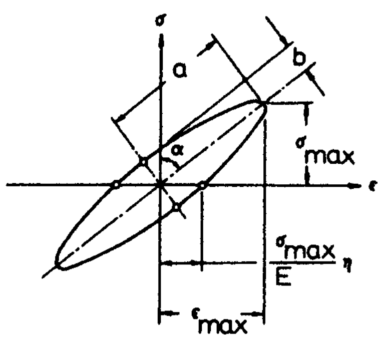

Fig. 2 Hysteresis loop approximated by an ellipse.

This means that $2\pi\eta$ defines the ratio of dissipated energy per cycle to the strain energy at peak amplitude. Thus, in the case of a shell, we can argue that

$$E_T = 2\pi\eta U_{\max} \tag{14.2.17}$$

where $U_{\max}$ is the strain energy of the shell at peak amplitude. The energy dissipated per cycle and unit surface is then

$$E_\mathrm{d} = \frac{2\pi\eta}{A} U_{\max} \tag{14.2.18}$$

where A is the reference surface area of the shell, plate, or beam. Thus the equivalent viscous damping coefficient λ is

$$\lambda = \frac{2U_{\max}}{\omega \int\int_A (U_1^2 + U_2^2 + U_3^2)\,\mathrm{d}A}\eta \tag{14.2.19}$$

Remembering the discussion of Rayleigh's method in Sec.7.4, where it was shown that

$$\omega_k^2 = \frac{2U_{\max}}{\rho h \int\int_A (U_1^2 + U_2^2 + U_3^2)\,\mathrm{d}A} \tag{14.2.20}$$

we may write the equivalent viscous damping coefficient as

$$\lambda = \rho h \omega_k \frac{\omega_k}{\omega}\eta \tag{14.2.21}$$

It may be used directly when the forcing is harmonic. For nonharmonic forcing, some choice about a mean value of ω_k^2/ω will have to be made.

14.3. DIRECT UTILIZATION OF HYSTERESIS MODEL IN ANALYSIS

For the technically significant class of cases where the steady-state response to harmonic excitation is to be obtained, one can work directly with the hysteresis model. Introducing the complex modulus into Love's equation gives

$$(1 + j\eta)L_i\{u_1, u_2, u_3\} - \rho h \ddot{u}_i = -q_i^* \mathrm{e}^{j\omega t} \tag{14.3.1}$$

where $L_i\{u_1, u_2, u_3\}$ represents the same operators as given in Eqs. (8.1.3)–(8.1.5). The general forcing terms in Love's equation are now restricted to harmonic excitation, with q_i^* representing the pressure load distribution.

The modal expansion solution is

$$u_i(\alpha_1, \alpha_2, t) = \sum_{k=1}^{\infty} \eta_k(t) U_{ik}(\alpha_1, \alpha_2) \tag{14.3.2}$$

One should take note that the η_k are the modal participation factors and that η without a subscript is the traditional notation for the hysteresis loss factor. Substitution into Eq. (14.3.1) gives

$$\sum_{k=1}^{\infty}[(1+j\eta)\eta_k L_i\{U_{1k},U_{2k},U_{3k}\}-\rho h\ddot{\eta}_k U_{ik}]=-q_i^* \mathrm{e}^{j\omega t} \tag{14.3.3}$$

From the eigenvalue analysis, where $\eta=0$ and $q_i=0$, we obtain the identity

$$L_i\{U_{1k},U_{2k},U_{3k}\}=-\rho h\omega_k^2 U_{ik} \tag{14.3.4}$$

This gives

$$\sum_{k=i}^{\infty}[\rho h\ddot{\eta}_k+\rho h(1+j\eta)\omega_k^2\eta_k]U_{ik}=q_i^*\mathrm{e}^{j\omega t} \tag{14.3.5}$$

Multiplying both sides by a mode U_{ip}, where p may be either equal to k or unequal, and writing the relationship for every value of $i=1,2,3$, gives

$$\sum_{k=1}^{\infty}[\rho h\ddot{\eta}_k+\rho h(1+j\eta)\omega_k^2\eta_k]U_{1k}U_{1p} = q_1^* U_{1p}\mathrm{e}^{j\omega t} \tag{14.3.6}$$

$$\sum_{k=1}^{\infty}[\rho h\ddot{\eta}_k+\rho h(1+j\eta)\omega_k^2\eta_k]U_{2k}U_{2p} = q_2^* U_{2p}\mathrm{e}^{j\omega t} \tag{14.3.7}$$

$$\sum_{k=1}^{\infty}[\rho h\ddot{\eta}_k+\rho h(1+j\eta)\omega_k^2\eta_k]U_{3k}U_{3p} = q_3^* U_{3p}\mathrm{e}^{j\omega t} \tag{14.3.8}$$

Adding Eqs. (14.3.6)–(14.3.8) and integrating over the reference surface of the shell gives

$$\sum_{k=1}^{\infty}(\rho h\ddot{\eta}_k+\rho h(1+j\eta)\omega_k^2\eta_k)\int_{\alpha_2}\int_{\alpha_1}(U_{1k}U_{1p}+U_{2k}U_{2p}+U_{3k}U_{3p})$$
$$\times A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2=\int_{\alpha_2}\int_{\alpha_1}(q_1^*U_{1p}+q_2^*U_{2p}+q_3^*U_{3p})A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{14.3.9}$$

Utilizing the orthogonality property of natural modes gives

$$\ddot{\eta}_k+(1+j\eta)\omega_k^2\eta_k=F_k^*\mathrm{e}^{j\omega t} \tag{14.3.10}$$

where

$$F_k^* = \frac{1}{\rho h N_k}\int_{\alpha_2}\int_{\alpha_1}(q_1^*U_{1k}+q_2^*U_{2k}+q_3^*U_{3k})A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{14.3.11}$$

$$N_k = \int_{\alpha_2}\int_{\alpha_1}(U_{1k}^2+U_{2k}^2+U_{3k}^2)A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{14.3.12}$$

This result is comparable to the one given by Eqs. (8.5.2) and (8.5.3).

The steady-state solution will be

$$\eta_k = \Lambda_k e^{j(\omega t - \phi_k)} \tag{14.3.13}$$

Substitution in Eq. (14.3.10) gives

$$\Lambda_k e^{-j\phi_k} = \frac{F_k^*}{(\omega_k^2 - \omega^2) + j\eta\omega_k^2} \tag{14.3.14}$$

The magnitude of the response is therefore

$$\Lambda_k = \frac{F_k^*}{\omega_k^2\sqrt{[1-(\omega/\omega_k)^2]^2 + \eta^2}} \tag{14.3.15}$$

The phase angle is

$$\phi_k = \tan^{-1}\frac{\eta}{1-(\omega/\omega_k)^2} \tag{14.3.16}$$

An interesting by-product of this analysis is the relationship between the modal damping coefficient and the hysteresis loss factor. It must be that

$$2\zeta_k\omega_k\omega = \eta\omega_k^2 \tag{14.3.17}$$

Thus the equivalent modal damping coefficient becomes

$$\zeta_k = \frac{1}{2}\frac{\omega_k}{\omega}\eta \tag{14.3.18}$$

The equivalent viscous damping coefficient is therefore

$$\lambda = \rho h \omega_k \frac{\omega_k}{\omega}\eta \tag{14.3.19}$$

This agrees, as expected, with Eq. (14.2.21).

14.4. HYSTERETICALLY DAMPED PLATE EXCITED BY SHAKER

The following illustrates how the hysteresis loss factor can be obtained from a measurement, using as an example the simply supported plate (Fig. 3).

For transverse loading of a simply supported rectangular plate, $q_1^* = q_2^* = 0$. The harmonically varying point load of amplitude F in newtons, representing the harmonic input from a shaker, is described by

$$q_3^* = F\delta(x - x^*)\delta(y - y^*) \tag{14.4.1}$$

The eigenvalues are

$$U_{3k} = U_{3mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{14.4.2}$$

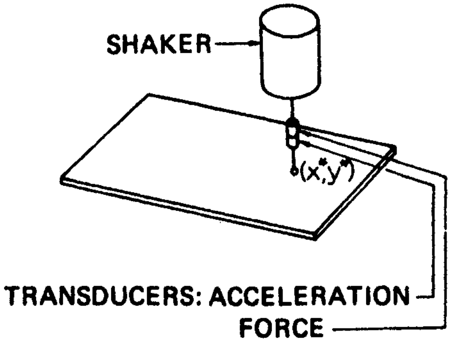

Fig. 3 Hysteretically damped rectangular plate excited by a shaker.

$$\omega_k = \omega_{mn} = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right] \sqrt{\frac{D}{\rho h}} \tag{14.4.3}$$

Equation (14.3.11) becomes

$$F_k^* = \frac{4F}{\rho hab} \sin \frac{m\pi x^*}{a} \sin \frac{n\pi y^*}{b} \tag{14.4.4}$$

Thus the solution is

$$u_3(x, y, t) = \frac{4F}{\rho hab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(m\pi x^*/a) \sin(m\pi x/a) \sin(n\pi y^*/b) \sin(n\pi y/b)}{\omega_{mn}^2 \sqrt{[1-(\omega/\omega_{mn})^2]^2 + \eta^2}} e^{j(\omega t - \phi_{mn})} \tag{14.4.5}$$

where

$$\phi_{mn} = \tan^{-1} \frac{\eta}{1-(\omega/\omega_{mn})^2} \tag{14.4.6}$$

Let us now assume that the acceleration response is measured at the point of attachment of the shaker; also, that the force amplitude is monitored. Furthermore, the measurement is made at each of the natural frequencies ω_{mn}. We have in this case

$$u_3(x^*, y^*, t) = \frac{4F}{\omega^2 \rho hab\eta} \sin^2 \frac{m\pi x^*}{a} \sin^2 \frac{n\pi y^*}{b} e^{j(\omega t - \pi/2)} \tag{14.4.7}$$

or the acceleration is

$$\ddot{u}_3(x^*, y^*, t) = -\frac{4F}{\rho hab\eta} \sin^2 \frac{m\pi x^*}{a} \sin^2 \frac{n\pi y^*}{b} e^{j(\omega t - \pi/2)} \tag{14.4.8}$$

Solving for η in terms of the measured acceleration amplitude $|\ddot{u}_3|$ and force amplitude F yields

$$\eta = \frac{4}{\rho hab} \sin^2 \frac{m\pi x^*}{a} \sin^2 \frac{n\pi y^*}{b} \frac{F}{|\ddot{u}_3|} \tag{14.4.9}$$

In Plunkett (1959), several other methods of defining η are discussed. Typically, η is not constant with frequency.

14.5. STEADY-STATE RESPONSE TO PERIODIC FORCING

As in Sec. 8.19, the forcing is assumed to be such that the spatial distribution does not change with time, but that its amplitude is periodic in time. We may write

$$q_i(\alpha_1, \alpha_2, t) = q_i^*(\alpha_1, \alpha_2) f(t) \tag{14.5.1}$$

where, see Sec. 8.19,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\Omega t + b_n \sin n\Omega t) \tag{14.5.2}$$

and where

$$a_0 = \frac{1}{T} \int_0^T f(t) \mathrm{d}t \tag{14.5.3}$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\Omega t \mathrm{d}t \tag{14.5.4}$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\Omega t \mathrm{d}t \tag{14.5.5}$$

Proceeding as in Sec. 8.19, where we obtained the steady-state harmonic response to each term in the Fourier series for $f(t)$ and then summed to obtain the total response, we obtain here, for the hysteresis damping model:

$$u_i(\alpha_1, \alpha_2, t) = \sum_{k=1}^{\infty} \frac{F_k^* a_0}{\omega_k^2} U_{ik}(\alpha_1, \alpha_2) + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{F_k^* [a_n \cos(n\Omega t - \phi_{kn}) + b_n \sin(n\Omega t - \phi_{kn})] U_{ik}(\alpha_1, \alpha_2)}{\omega_k^2 \sqrt{\left[1 - (n\Omega/\omega_k)^2\right]^2 + \eta^2}} \tag{14.5.6}$$

where

$$F_k^* = \frac{1}{\rho h N_k} \int_{\alpha_2} \int_{\alpha_1} [q_1^*(\alpha_1, \alpha_2) U_{1k} + q_2^*(\alpha_1, \alpha_2) U_{2k} + q_3^*(\alpha_1, \alpha_2) U_{3k}] \times A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{14.5.7}$$

and

$$\phi_{kn} = \tan^{-1} \frac{\eta}{1 - (n\Omega/\omega_k)^2} \tag{14.5.8}$$

Comparing this to the result of Sec. 8.19, the character of the solution is similar. Resonance still occurs whenever $n\Omega = \omega_k$. The difference is that the influence of damping is not anymore proportional to the ratio $n\Omega/\omega_k$ for η = constant. But it should be noted that for most materials, average η values have to be used depending on the frequency bands of interest, because η is typically a function of frequency, and often even of response amplitude; see Ross, Ungar, and Kerwin (1959).

As an example, we evaluate the steady-state response of a hysteretically damped simply supported plate to a periodic, saw tooth type force variation in time, as shown in Fig. 4. In this case, we have

$$F(t) = \frac{F_0}{T} t \tag{14.5.9}$$

for the interval $0 \leq t \leq T$, with $\Omega = 2\pi/T$. From Eqs. (14.5.3)–(14.5.5), we obtain

$$a_0 = \frac{1}{T} \int_0^T \frac{F_0}{T} t \mathrm{d}t = \frac{F_0}{2} \tag{14.5.10}$$

$$a_p = \frac{2}{T} \int_0^T \frac{F_0}{T} t \cos\left(\frac{2p\pi t}{T}\right) \mathrm{d}t = 0 \tag{14.5.11}$$

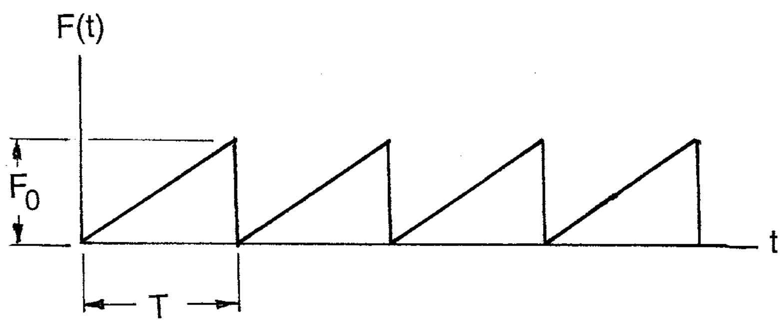

Fig. 4 Example of periodic forcing on a hysteretically clamped plate.

$$b_p = \frac{2}{T}\int_0^T \frac{F_0}{T} t \sin\left(\frac{2p\pi t}{T}\right) \mathrm{d}t = -\frac{F_0}{p\pi} \tag{14.5.12}$$

where the n numbers in Eqs. (14.5.3)–(14.5.5) were replaced by $p(=1,2,\ldots)$ since n will, in this example, be part of the mode description.

For the simply supported, rectangular plate,

$$U_{3mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{14.5.13}$$

$$\omega_{mn} = \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}} \tag{14.5.14}$$

$$N_k = N_{mn} = \frac{ab}{4} \tag{14.5.15}$$

$$\phi_{mnp} = \tan^{-1}\frac{\eta}{1-(p\Omega/\omega_{mn})^2} \tag{14.5.16}$$

and, from Eq. (14.5.7),

$$F_k^* = F_{mn}^* = \frac{4}{\rho hab}\sin\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b} \tag{14.5.17}$$

Therefore, Eq. (14.5.6) becomes

$$u_3(x,y,t) = \frac{4F_0}{\rho hab}\left\{\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sin\frac{m\pi x_1}{a}\sin\frac{n\pi y_1}{b}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{b}\left[\frac{1}{2\omega_{mn}^2} - \sum_{p=1}^{\infty}\frac{\sin(n\Omega t - \phi_{mnp})}{p\pi\omega_{mn}^2\sqrt{\left[1-(p\Omega/\omega_{mn})^2\right]^2+\eta^2}}\right]\right\} \tag{14.5.18}$$

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15

Shells Made of Composite Material

In all of the preceding chapters, the shell material was assumed to be homogeneous and isotropic. Because of the need for lightweight designs (e.g., in space applications) composite shell materials have become more and more common. One of the advantages of composite materials is that one can design directional properties into them almost on demand. The disadvantage is that structures built with composite materials are more difficult to analyze and even to understand in their idiosyncracies of behavior and failure.

15.1. NATURE OF COMPOSITES

In the following, we concentrate on the most common composite arrangement that one finds in thin-walled structures: namely, laminated composite. The composite is in this case built up of sheets (laminae) of uniform thickness. Each lamina may be isotropic, orthotropic, or anisotropic. From a materials composition viewpoint, it may be homogeneous or heterogeneous. Once the lamina are joined to each other, the most general case is what is called *coupled anisotropic*. Some of this is illustrated in Fig. 1.

Usually, a lamina or ply is composed of reinforcing material, most commonly fibers, in a supporting matrix. The fibers usually carry the load.

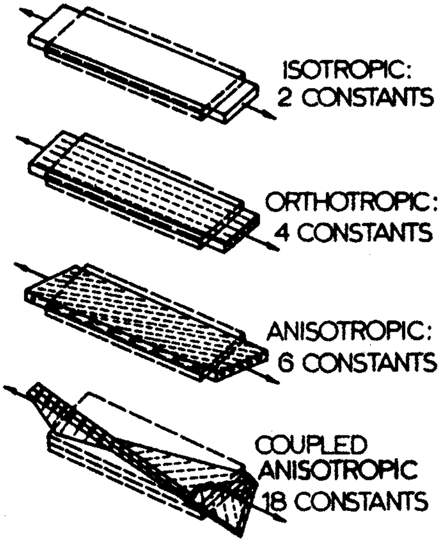

FIG. 1 Distortions of various composite material strips when loaded axially.

The matrix material usually holds the fibers in place so that they are properly spaced, protects them against corrosion, and seals the structure against the escape of gases and liquids. Each lamina usually consists of a set of parallel fibers embedded into the matrix material. The laminae can then be assembled with the fibers of each lamina pointing in different directions such that the desired stiffness properties are obtained.

Most engineering materials are isotropic. This means that the properties are not a function of direction. All planes that pass through a point in the material are planes of material property symmetry. To define the material, we need only two elastic constants: Young's modulus and Poisson's ratio. An axially loaded rectangular strip will remain rectangular as it is distorted, as shown in Fig. 1.

An orthotropic material has three planes of material symmetry. We will see that we need four material constants to describe the plane stress state.

15.2. LAMINA–CONSTITUTIVE RELATIONSHIP

Let us assume that each lamina is in a state of plane stress. For material that is homogeneous and isotropic, we have relations (2.2.10)–(2.2.12).

They may be written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \tag{15.2.1}$$

where

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} \tag{15.2.2}$$

and where

$$Q_{11} = Q_{22} = \frac{E}{1-\mu^2} \tag{15.2.3}$$

$$Q_{12} = Q_{21} = \frac{\mu E}{1-\mu^2} \tag{15.2.4}$$

$$Q_{33} = G = \frac{E}{2(1+\mu)} \tag{15.2.5}$$

The constitutive relationship for a homogeneous orthotropic lamina in a state of plane stress as shown in Fig. 2 is also given by Eq. (15.2.1), except that now (the filament direction is the x direction)

$$Q_{11} = \frac{E_{xx}}{1-\mu_{xy}\mu_{yx}} \tag{15.2.6}$$

$$Q_{22} = \frac{E_{yy}}{1-\mu_{xy}\mu_{yx}} \tag{15.2.7}$$

$$Q_{12} = \frac{\mu_{yx}E_{xx}}{1-\mu_{xy}\mu_{yx}} \tag{15.2.8}$$

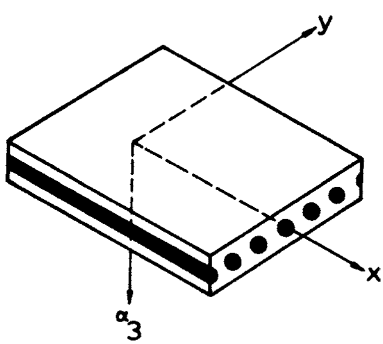

FIG. 2 Orthotropic strip.

$$Q_{21} = \frac{\mu_{yx} E_{yy}}{1 - \mu_{xy}\mu_{yx}} \tag{15.2.9}$$

$$Q_{33} = G_{xy} \tag{15.2.10}$$

Because of the requirement that

$$Q_{12} = Q_{21} \tag{15.2.11}$$

we obtain

$$\mu_{yx} E_{xx} = \mu_{xy} E_{yy} \tag{15.2.12}$$

We note that we have four material constants: $E_{xx}, E_{yy}, \mu_{xy}, \text{and}\, G_{xy}$. For filamentary lamina as shown in Fig. 2, Halpin and Tsai, see Ashton et al. (1969), suggested the following interpolation, based on the volume ratio of filament to matrix material:

$$E_{xx} = E_\mathrm{f} V_\mathrm{f} + E_\mathrm{m} V_\mathrm{m} \tag{15.2.13}$$

$$E_{yy} = E_\mathrm{m} \frac{1 + \zeta\alpha V_\mathrm{f}}{1 - \zeta V_\mathrm{f}} \tag{15.2.14}$$

$$\mu_{xy} = \mu_\mathrm{f} V_\mathrm{f} + \mu_\mathrm{m} V_\mathrm{m} \tag{15.2.15}$$

$$G_{xy} = G_\mathrm{m} \frac{1 + \zeta\beta V_\mathrm{f}}{1 - \zeta V_\mathrm{f}} \tag{15.2.16}$$

where

$$\alpha = \frac{E_\mathrm{f}/E_\mathrm{m} - 1}{E_\mathrm{f}/E_\mathrm{m} + \zeta} \tag{15.2.17}$$

$$\beta = \frac{G_\mathrm{f}/G_\mathrm{m} - 1}{G_\mathrm{f}/G_\mathrm{m} + \zeta} \tag{15.2.18}$$

and where: E_f, modulus of elasticity of fiber (N/m^2); E_m, modulus of elasticity of matrix (N/m^2); μ_f, Poisson's ratio of fiber; μ_m, Poisson's ratio of matrix; V_f, volume fraction of fiber; V_m, volume fraction of material (note: $V_\mathrm{f} + V_\mathrm{m} = 1$); G_f, sher modulus of fiber (N/m^2); G_m, shear modulus of matrix (N/m^2).

The factor ζ is an adjustment factor that depends to some extent on the boundary conditions. It can be taken as $\zeta = 1$ for a first approximation.

A special case occurs when very stiff fibers are embedded in a relatively soft matrix. For instance, if we take pneumatic tires as an example, typically $E_f \gg E_m \text{and}\, G_f \gg G_m$,

$$E_{xx} \cong E_\mathrm{f} V_\mathrm{f} \tag{15.2.19}$$

$$E_{yy} \cong E_\mathrm{m} \tag{15.2.20}$$

$$\mu_{xy} \cong \mu_\mathrm{m} V_\mathrm{m} \tag{15.2.21}$$

$$G_{xy} \cong G_\mathrm{m} \tag{15.2.22}$$

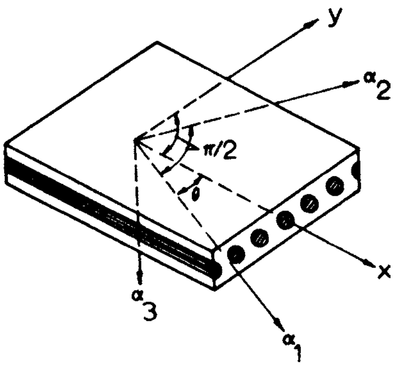

Fig. 3 Orthotropic strip rotated in its plane.

since $\alpha=\beta\cong 1$.

Since the lamina may not always be oriented so that its principal stiffness direction coincides with the coordinates, relation (15.2.1) has to be transformed to account for a possible θ rotation, as shown in Fig. 3. By analyzing the equilibrium of an infinitesimal element as shown in Fig. 4, we obtain

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [T_1] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{15.2.23}$$

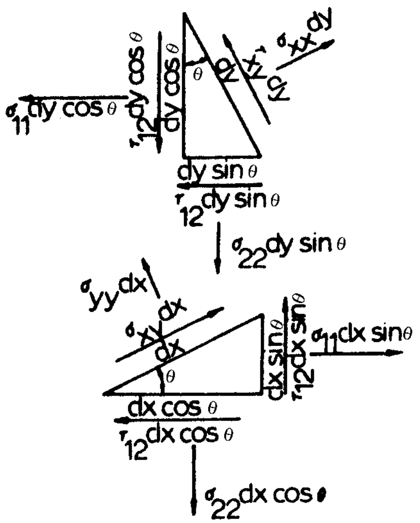

Fig. 4 Equilibrium of an infinitesimal element.

where

$$[T_1]=\begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \tag{15.2.24}$$

Similarly, we obtain for strain

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}=[T_2]\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} \tag{15.2.25}$$

where

$$[T_2]=\begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \tag{15.2.26}$$

Substitution gives

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}=[\overline{Q}]\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} \tag{15.2.27}$$

where

$$[\overline{Q}]=[T_1]^{-1}[Q][T_2] \tag{15.2.28}$$

The coefficients $\overline{Q}_{ij}$ of this matrix are

$$\overline{Q}_{11}=U_1+U_2\cos2\theta+U_3\cos4\theta \tag{15.2.29}$$

$$\overline{Q}_{22}=U_1-U_2\cos2\theta+U_3\cos4\theta \tag{15.2.30}$$

$$\overline{Q}_{12}=U_4-U_3\cos4\theta=\overline{Q}_{21} \tag{15.2.31}$$

$$\overline{Q}_{33}=U_5-U_3\cos4\theta \tag{15.2.32}$$

$$\overline{Q}_{13}=-\tfrac{1}{2}U_2\sin2\theta-U_3\sin4\theta=\overline{Q}_{31} \tag{15.2.33}$$

$$\overline{Q}_{23}=-\tfrac{1}{2}U_2\sin2\theta+U_3\sin4\theta=\overline{Q}_{32} \tag{15.2.34}$$

where

$$U_1=\tfrac{1}{8}(3Q_{11}+3Q_{22}+2Q_{12}+4Q_{33}) \tag{15.2.35}$$

$$U_2=\tfrac{1}{2}(Q_{11}-Q_{22}) \tag{15.2.36}$$

$$U_3=\tfrac{1}{8}(Q_{11}+Q_{22}-2Q_{12}-4Q_{33}) \tag{15.2.37}$$

$$U_4=\tfrac{1}{8}(Q_{11}+Q_{22}+6Q_{12}-4Q_{33}) \tag{15.2.38}$$

$$U_5=\tfrac{1}{8}(Q_{11}+Q_{22}-2Q_{12}-4Q_{33}) \tag{15.2.39}$$

15.3. LAMINATED COMPOSITE

Let us assume again that the shell is thin, even if it is composed of n laminations. Furthermore, we assume again that displacements vary linearly through the shell thickness. This implies that all relationships of Sec. 2.4 hold. Thus

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ \varepsilon_{12}^0 \end{Bmatrix} + \alpha_3 \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} \tag{15.3.1}$$

Introducing the subscript k to denote the kth lamina, we may express the stress in the kth lamina by combining Eqs. (15.3.1) and (15.2.27).

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_k = [\overline{Q}]_k \begin{Bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ \varepsilon_{12}^0 \end{Bmatrix} + \alpha_3 [\overline{Q}]_k \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} \tag{15.3.2}$$

The stress resultants are

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \int_{\alpha_3} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} d\alpha_3 \tag{15.3.3}$$

or since we have n laminae, as shown in Fig. 5,

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \sum_{k=1}^{n} \int_{h_k}^{h_{k+1}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} d\alpha_3 \tag{15.3.4}$$

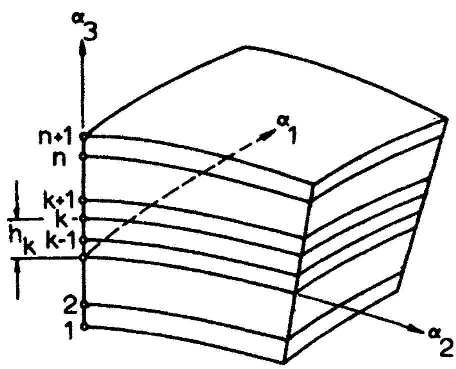

FIG. 5 Laminated shell element.

The subscript k is used here such that h_k defines the distance from the reference surface to the bottom surface of the kth lamina. Substituting Eq. (15.3.2) gives

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \sum_{k=1}^{n} \left\{ [\overline{Q}]_k \begin{Bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ \varepsilon_{12}^0 \end{Bmatrix} \int_{h_k}^{h_{k+1}} \mathrm{d}\alpha_3 + [\overline{Q}]_k \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} \int_{h_k}^{h_{k+1}} \alpha_3 \mathrm{d}\alpha_3 \right\} \tag{15.3.5}$$

This allows us to write

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = [A] \begin{Bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ \varepsilon_{12}^0 \end{Bmatrix} + [B] \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} \tag{15.3.6}$$

where

$$[A] = \sum_{k=1}^{n} [\overline{Q}]_k (h_{k+1} - h_k) \tag{15.3.7}$$

$$[B] = \frac{1}{2} \sum_{k=1}^{n} [\overline{Q}]_k (h_{k+1}^2 - h_k^2) \tag{15.3.8}$$

Each term is therefore given by

$$A_{ij} = \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_{k+1} - h_k) \tag{15.3.9}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_{k+1}^2 - h_k^2) \tag{15.3.10}$$

An interesting result, different from that for the isotropic material equations of Sec. 2.5, is that the stress resultants are in general also a function of the bending strains. Only if it is possible to select the reference plane such that

$$\sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_{k+1}^2 - h_k^2) = 0 \tag{15.3.11}$$

do we have uncoupling. For a single lamina homogeneous and isotropic material (our classical shell case), $n = 1$, and thus we have to satisfy only

$$(\overline{Q}_{ij})_k (h_2^2 - h_1^2) = 0 \tag{15.3.12}$$

This is done by selecting $h_2 = -h_1$, which means that the reference surface is halfway between the inner and outer surfaces, which implies that all B_{ij} are 0.

In general, any composite material whose laminae are homogeneous and isotropic can be made to have a zero $[B]$ matrix. Also, composite materials that are arranged such that each lamina's orthotropic principal directions coincide with the composite material's principal directions can be made to have a zero [B] matrix. In most other cases, we will find that it is impossible to find a location for the reference surface that satisfies this condition. In other words, a neutral surface does not exist for many composites.

The moment resultants are

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \int_{\alpha_3} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \alpha_3 \, \mathrm{d}\alpha_3 \sum_{k=1}^{n} \int_{h_k}^{h_{k+1}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \mathrm{d}\alpha_3 \tag{15.3.13}$$

Substituting Eq. (15.3.2) gives

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = [B] \begin{Bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ \varepsilon_{12}^0 \end{Bmatrix} + [D] \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} \tag{15.3.14}$$

where each term in the two matrices is given by

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_{k+1}^2 - h_k^2) \tag{15.3.15}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_{k+1}^3 - h_k^3) \tag{15.3.16}$$

As expected, the coupling matrix [B] is the same as before and all comments concerning its vanishing apply as before.

It is customary to combine the expressions for force and moment resultants:

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ A_{21} & A_{22} & A_{23} & B_{21} & B_{22} & B_{23} \\ A_{31} & A_{32} & A_{33} & B_{31} & B_{32} & B_{33} \\ \hline B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\ B_{21} & B_{22} & B_{23} & D_{21} & D_{22} & D_{23} \\ B_{31} & B_{32} & B_{33} & D_{31} & D_{32} & D_{33} \end{array} \right] \begin{Bmatrix} \varepsilon_{11}^\circ \\ \varepsilon_{22}^\circ \\ \varepsilon_{12}^\circ \\ k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} \tag{15.3.17}$$

Because of the symmetry of the $\overline{Q}_{ij}$ terms, we also have that

$$\begin{aligned} A_{ij} &= A_{ji} \\ B_{ij} &= B_{ji} \\ D_{ij} &= D_{ji} \end{aligned} \tag{15.3.18}$$

15.4. EQUATION OF MOTION

If we examine the development described in Chapter 2, we note that it is not influenced at all by the fact that we now have a much more complicated relationship between strains and the force and moment resultants. Thus Love's equations [(2.7.20)–(2.7.24)] and boundary condition expressions in force and moment resultant forms are still valid. Mass densities are averaged over the thickness of the shell.

However, as soon as the equations are expressed in terms of displacements, the added complexity becomes apparent. So far, only a few special cases of composite material plate or shell eigenvalues have been obtained analytically. They are almost invariably orthotropic material structures.

15.5. ORTHOTROPIC PLATE

Let us, for instance, find the eigenvalues of a rectangular simply supported orthotropic plate. In this case $\alpha_1 = x$, $\alpha_2 = y$, $A_1 = 1$, $A_2 = 1$, $1/R_1 = 0$, $1/R_2 = 0$. This gives, for transverse deflection, the equations

$$-\frac{\partial Q_{x3}}{\partial x} - \frac{\partial Q_{y3}}{\partial y} + \rho h \ddot{u}_3 = q_3 \tag{15.5.1}$$

where

$$Q_{x3} = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} \tag{15.5.2}$$

$$Q_{y3} = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} \tag{15.5.3}$$

Setting $q_3 = 0$ and substituting gives

$$-\frac{\partial^2 M_{xx}}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial^2 M_{yy}}{\partial y^2} + \rho h \ddot{u}_3 = q_3 \tag{15.5.4}$$

For orthotropic material, with the reference plane coinciding with the neutral plane, we obtain, from Eq. (15.3.17),

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{33} \end{array}\right] \begin{Bmatrix} \varepsilon^0_{xx} \\ \varepsilon^0_{yy} \\ \varepsilon^0_{xy} \\ k_{xx} \\ k_{yy} \\ k_{xy} \end{Bmatrix} \tag{15.5.5}$$

Thus, for our purpose here,

$$M_{xx} = D_{11} k_{xx} + D_{12} k_{yy} \tag{15.5.6}$$

$$M_{yy} = D_{12} k_{xx} + D_{22} k_{yy} \tag{15.5.7}$$

$$M_{xy} = D_{33} k_{xy} \tag{15.5.8}$$

Substituting this in Eq. (15.5.4) gives

$$-\left(D_{11}\frac{\partial^2 k_{xx}}{\partial x^2}+D_{12}\frac{\partial^2 k_{yy}}{\partial x^2}\right)-2D_{33}\frac{\partial^2 k_{xy}}{\partial x\partial y}$$
$$-\left(D_{12}\frac{\partial^2 k_{xx}}{\partial y^2}+D_{22}\frac{\partial^2 k_{yy}}{\partial y^2}\right)+\rho h\ddot{u}_3=q_3 \tag{15.5.9}$$

Since

$$k_{xx}=-\frac{\partial^2 u_3}{\partial x^2} \tag{15.5.10}$$

$$k_{yy}=-\frac{\partial^2 u_3}{\partial y^2} \tag{15.5.11}$$

$$k_{xy}=-2\frac{\partial^2 u_3}{\partial x\,\partial y} \tag{15.5.12}$$

we obtain

$$D_{11}\frac{\partial^4 u_3}{\partial x^4}+2(D_{12}+2D_{33})\frac{\partial^4 u_3}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 u_3}{\partial y^4}+\rho h\bar{u}_3=q_3 \tag{15.5.13}$$

For a simply supported plate, the boundary conditions are

$$u_3(0,y,t)=u_3(a,y,t)=u_3(x,0,t)=u_3(x,b,t)=0 \tag{15.5.14}$$

$$M_{xx}(0,y,t)=M_{xx}(a,y,t)=0 \tag{15.5.15}$$

$$M_{yy}(x,0,t)=M_{yy}(x,b,t)=0 \tag{15.5.16}$$

To solve for the eigenvalues, we set $q_3=0$, and

$$u_3(x,y,t)=U_3(x,y)\mathrm{e}^{j\omega t} \tag{15.5.17}$$

where the mode shape $U_3(x,y)$ is assumed to be

$$U_3(x,y)=\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{15.5.18}$$

This satisfies the partial differential equation and the boundary conditions. The natural frequencies turn out to be (Hearmon, 1959)

$$\omega_{mn}=\pi^2\sqrt{D_{11}\left(\frac{m}{a}\right)^4+2(D_{12}+2D_{33})\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2+D_{22}\left(\frac{n}{b}\right)^4}\sqrt{\frac{1}{\rho h}} \tag{15.5.19}$$

Let us reduce this formula to that for a homogeneous and isotropic plate. In this case $D_{11}=D_{22}=D, D_{12}=\mu D, D_{33}=(1-\mu)D/2$, and the result agrees with that of Sec. (5.4.2).

For a discussion of anisotropic plate vibration, see, for example, Jones (1975). Analytical solutions in this area are primarily iterative in nature.

15.6. CIRCULAR CYLINDRICAL SHELL

Let us utilize the Donnell–Mushtari–Vlasov approximations. In this case, we follow the procedure outlined in Sec. 6.7. After neglecting the influence of inertia in the in-plane direction and the shear term $Q_{3\theta}/a$, we obtain $(A_1=1, A_1 d\alpha_1 = dx, 1/R_1 = 0, A_2 = a, d\alpha_2 = d\theta, R_2 = a)$

$$a\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{x\theta}}{\partial \theta}=0 \tag{15.6.1}$$

$$a\frac{\partial N_{x\theta}}{\partial x}+\frac{\partial N_{\theta\theta}}{\partial \theta}=0 \tag{15.6.2}$$

$$-a\frac{\partial Q_{x3}}{\partial x}-\frac{\partial Q_{\theta 3}}{\partial \theta}+N_{\theta\theta}+a\rho h\ddot{u}_3=aq_3 \tag{15.6.3}$$

where

$$Q_{x3}=\frac{\partial M_{xx}}{\partial x}+\frac{1}{a}\frac{\partial M_{x\theta}}{\partial \theta} \tag{15.6.4}$$

$$Q_{\theta 3}=\frac{\partial M_{x\theta}}{\partial x}+\frac{1}{a}\frac{\partial M_{\theta\theta}}{\partial \theta} \tag{15.6.5}$$

Equations (15.6.1) and (15.6.2) are satisfied by introducing the same stress function as in Sec. 6.7:

$$N_{xx}=\frac{1}{a^2}\frac{\partial^2 \phi}{\partial \theta^2} \tag{15.6.6}$$

$$N_{\theta\theta}=\frac{\partial^2 \phi}{\partial x^2} \tag{15.6.7}$$

$$N_{x\theta}=-\frac{1}{a}\frac{\partial^2 \phi}{\partial x \partial \theta} \tag{15.6.8}$$

Substituting Eqs. (15.6.4), (15.6.5), and (15.6.7) in Eq. (15.6.3) gives

$$-a\frac{\partial^2 M_{xx}}{\partial x^2}-2\frac{\partial^2 M_{xx}}{\partial x \partial \theta}-\frac{1}{a}\frac{\partial^2 M_{\theta\theta}}{\partial \theta^2}+\frac{\partial^2 \phi}{\partial x^2}+a\rho h\ddot{u}_3=aq_3 \tag{15.6.9}$$

Substituting

$$M_{xx}=D_{11}k_{xx}+D_{12}k_{\theta\theta} \tag{15.6.10}$$

$$M_{\theta\theta}=D_{22}k_{\theta\theta}+D_{12}k_{xx} \tag{15.6.11}$$

$$M_{x\theta}=D_{33}k_{x\theta} \tag{15.6.12}$$

gives

$$-\left(D_{11}\frac{\partial^2 k_{xx}}{\partial x^2}+D_{12}\frac{\partial^2 k_{\theta\theta}}{\partial x^2}\right)-\frac{2D_{33}}{a}\frac{\partial^2 k_{xy}}{\partial x\partial\theta}$$

$$-\frac{1}{a^2}\left(D_{12}\frac{\partial^2 k_{xx}}{\partial\theta^2}+D_{22}\frac{\partial^2 k_{\theta\theta}}{\partial\theta^2}\right)+\frac{1}{a}\frac{\partial^2\phi}{\partial x^2}+\rho h\ddot{u}_3=q_3 \tag{15.6.13}$$

A final substitution,

$$k_{xx}=-\frac{\partial^2 u_3}{\partial x^2} \tag{15.6.14}$$

$$k_{\theta\theta}=-\frac{1}{a^2}\frac{\partial^2 u_3}{\partial\theta^2} \tag{15.6.15}$$

$$k_{x\theta}=-\frac{2}{a}\frac{\partial^2 u_3}{\partial x\partial\theta} \tag{15.6.16}$$

results in the equation

$$D_{11}\frac{\partial^4 u_3}{\partial x^4}+2(D_{12}+2D_{33})\frac{1}{a^2}\frac{\partial^4 u_3}{\partial x^2\partial\theta^2}+D_{22}\frac{1}{a^4}\frac{\partial^4 u_3}{\partial\theta^4}+\frac{1}{a}\frac{\partial^2\phi}{\partial x^2}+\rho h\ddot{u}_3=q_3 \tag{15.6.17}$$

Examining as a check the homogeneous and isotropic case, where

$$D_{11}=D_{22}=D \tag{15.6.18}$$

$$D_{12}=\mu D \tag{15.6.19}$$

$$D_{33}=\frac{(1-\mu)D}{2} \tag{15.6.20}$$

and where $D=Eh^3/12(1-\mu^2)$, we obtain the first Donnell–Mushtari–Vlasov equation:

$$D\nabla^4 u_3+\frac{1}{a^2}\frac{\partial^2\phi}{\partial x^2}+\rho h\ddot{u}_3=q_3 \tag{15.6.21}$$

Next, we start with the compatibility equation (6.7.12):

$$\frac{k_{xx}}{a}+\frac{\partial^2\varepsilon^0_{\theta\theta}}{\partial x^2}-\frac{1}{a}\frac{\partial^2\varepsilon^0_{x\theta}}{\partial\theta\partial x}+\frac{1}{a^2}\frac{\partial^2\varepsilon^0_{xx}}{\partial\theta^2}=0 \tag{15.6.22}$$

Since

$$N_{xx}=A_{11}\varepsilon^0_{xx}+A_{12}\varepsilon^0_{\theta\theta} \tag{15.6.23}$$

$$N_{\theta\theta}=A_{12}\varepsilon^0_{xx}+A_{22}\varepsilon^0_{\theta\theta} \tag{15.6.24}$$

$$N_{x\theta}=A_{33}\varepsilon^0_{x\theta} \tag{15.6.25}$$

we obtain

$$\varepsilon^0_{xx} = P_{11}N_{xx} - P_{12}N_{\theta\theta} \tag{15.6.26}$$

$$\varepsilon^0_{\theta\theta} = P_{22}N_{\theta\theta} - P_{12}N_{xx} \tag{15.6.27}$$

$$\varepsilon^0_{x\theta} = P_{33}N_{x\theta} \tag{15.6.28}$$

where

$$P_{11} = \frac{A_{22}}{\alpha} \tag{15.6.29}$$

$$P_{22} = \frac{A_{11}}{\alpha} \tag{15.6.30}$$

$$P_{12} = \frac{A_{12}}{\alpha} \tag{15.6.31}$$

$$\alpha = A_{11}A_{22} - A_{12}^2 \tag{15.6.32}$$

$$P_{33} = \frac{1}{A_{33}} \tag{15.6.33}$$

Substitution gives

$$\frac{k_{xx}}{a} + P_{22}\frac{\partial^2 N_{\theta\theta}}{\partial x^2} - P_{12}\frac{\partial^2 N_{xx}}{\partial x^2} - \frac{1}{a}P_{33}\frac{\partial^2 N_{x\theta}}{\partial\theta\partial x} + \frac{1}{a^2}P_{11}\frac{\partial^2 N_{xx}}{\partial\theta^2} - \frac{1}{a^2}P_{12}\frac{\partial^2 N_{\theta\theta}}{\partial\theta^2} = 0 \tag{15.6.34}$$

Next, we substitute Eqs. (15.6.6)–(15.6.8) and Eq. (15.6.14) and obtain

$$-\frac{1}{a}\frac{\partial^2 u_3}{\partial x^2} + P_{22}\frac{\partial^4\phi}{\partial x^4} + P_{11}\frac{1}{a^4}\frac{\partial^4\phi}{\partial\theta^4} + \frac{1}{a^2}(P_{33} - 2P_{12})\frac{\partial^4\phi}{\partial x^2\partial\theta^2} = 0 \tag{15.6.35}$$

This may also be written as

$$\frac{A_{12}^2 - A_{11}A_{22}}{a}\frac{\partial^2 u_3}{\partial x^2} + A_{11}\frac{\partial^4\phi}{\partial x^4} + \frac{A_{22}}{a^4}\frac{\partial^4\phi}{\partial\theta^4} + \frac{A_{11}A_{22} - A_{12}^2 - 2A_{12}A_{33}}{A_{33}a^2}\frac{\partial^4\phi}{\partial x^2\partial\theta^2} = 0 \tag{15.6.36}$$

To check this equation, we again examine the isotropic case:

$$A_{11} = A_{22} = K \tag{15.6.37}$$

$$A_{12} = \mu K \tag{15.6.38}$$

$$A_{33} = \frac{(1-\mu)K}{2} \tag{15.6.39}$$

where $K = Eh/(1-\mu^2)$. This gives

$$\frac{Eh}{a}\frac{\partial^2 u_3}{\partial x^2} - \nabla^4 \phi = 0 \tag{15.6.40}$$

As expected, this is the second Donnell–Mushtari–Vlasov equation.

Let us now find the natural frequencies and modes of a closed circular shell simply supported at both ends. The boundary conditions are

$$u_3(0,\theta,t) = u_3(L,\theta,t) = 0 \tag{15.6.41}$$

$$M_{xx}(0,\theta,t) = M_{xx}(L,\theta,t) = 0 \tag{15.6.42}$$

We are able to satisfy both of these boundary conditions and the two governing equations, Eqs. (15.6.17) and (15.6.36), by

$$u_3(x,\theta,t) = U_3(x,\theta)\mathrm{e}^{j\omega t} \tag{15.6.43}$$

$$\phi(x,\theta,t) = \Phi(x,\theta)\mathrm{e}^{j\omega t} \tag{15.6.44}$$

where

$$U_3(x,\theta) = U_{mn}\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{15.6.45}$$

$$\Phi(x,\theta) = \Phi_{mn}\sin\frac{m\pi x}{L}\cos n(\theta-\phi) \tag{15.6.46}$$

Equation (15.6.17) becomes

$$\left[D_{11}\left(\frac{m\pi}{L}\right)^4 + 2(D_{12}+2D_{33})\left(\frac{n}{a}\right)^2\left(\frac{m\pi}{L}\right)^2 + D_{22}\left(\frac{n}{a}\right)^4 - \rho h\omega^2\right]U_{mn} - \frac{1}{a}\left(\frac{m\pi}{L}\right)^2\Phi_{mn} = 0 \tag{15.6.47}$$

and Eq. (15.6.36) becomes

$$\frac{A_{11}A_{22}-A_{12}^2}{a}\left(\frac{m\pi}{L}\right)^2 U_{mn} + \left[A_{11}\left(\frac{m\pi}{L}\right)^4 + A_{22}\left(\frac{n}{a}\right)^4 + \frac{A_{11}A_{22}-A_{12}^2-2A_{12}A_{33}}{A_{33}}\left(\frac{n}{a}\right)^2\left(\frac{m\pi}{L}\right)^2\right]\Phi_{mn} = 0 \tag{15.6.48}$$

For these two equations to be satisfied meaningfully, the determinant has to be equal to 0. This gives us the natural frequencies of the orthotropic shell for those modes where transverse deflection components dominate

(Soedel, 1983):

$$\omega^2=\omega_{mn}^2=\frac{1}{\rho h}\left\{\left[D_{11}\left(\frac{m\pi}{L}\right)^4+2(D_{12}+2D_{33})\left(\frac{n}{a}\right)^2\left(\frac{m\pi}{L}\right)^2+D_{22}\left(\frac{n}{a}\right)^4\right]\right.$$
$$\left.+\frac{(A_{11}A_{22}-A_{12}^2)(m\pi/L)^4}{a^2\{A_{11}(m\pi/L)^4+A_{22}(n/a)^4+[(A_{11}A_{22}-A_{12}^2-2A_{12}A_{33})/A_{33}](n/a)^2(m\pi/L)^2\}}\right\} \tag{15.6.49}$$

The result shows that the circumferential bending stiffness component D_{22} gains influence with increasing values of n while the axial bending stiffness component D_{11} increases its influence as m increases. A similar influence division exists for the membrane stiffness terms A_{11} and A_{22}. The structure of the formula shows clearly that if it is desired to raise the natural frequencies of the shell in general, both stringer and ring stiffeners have to be employed, since it is permissible to think of stiffeners as making an isotropic shell orthotropic. Neither stringers nor rings alone can raise the natural frequencies for all m, n combinations.

Let us check Eq. (15.6.49) against the isotropic case treated in Sec. 6.12. Substituting Eqs. (15.6.17)–(15.6.19) and (15.6.37)–(15.6.39) gives, as expected,

$$\omega_{mn}^2=\frac{E}{\rho a^2}\left\{\frac{(m\pi a/L)^4}{[(m\pi a/L)^2+n^2]^2}+\frac{(h/a)^2}{12(1-\mu^2)}\left[\left(\frac{m\pi a}{L}\right)^2+n^2\right]^2\right\} \tag{15.6.50}$$

A treatment of an orthotropic circular cylindrical shell that does not utilize the foregoing simplifications is given in Dong (1968). A literature review of orthotropic cylindrical and conical shell eigenvalue solutions can be found in Leissa (1973).

15.7. ORTHOTROPIC NETS OR TEXTILES UNDER TENSION

Netlike structures, usually with a fine mesh like a textile, are used in engineering because of their high stiffness-to-weight ratio. Net structures derive their stiffness from pretension, since they are usually stretched across frames. It can be assumed that the load is carried entirely by the fiber network, with any possible supporting matrix (whose purpose may also be that of sealing) adding only mass.

In the following, a flat rectangular net under tension is viewed as an equivalent membrane whose motion is defined by Eq. (11.3.3), with $A_1 = A_2 = 1$ and $\alpha_1 = x$ and $\alpha_2 = y$. The averaged mass per unit surface area, $\rho\prime$, is given by

$$\rho' = \frac{1}{ab}(\rho_x'' a n_x + \rho_y'' b n_y) + \rho \tag{15.7.1}$$

It is assumed that all fibers or cables oriented parallel to the x-axis of a Cartesian coordinate system have the same uniform mass per unit length of fiber or cable, ρ_x'', and all fibers or cables oriented parallel to the y axis have the same uniform mass per unit length, ρ_y''. The number of fibres that run parallel to the x axis is n_x, and the number running parallel to the y axis is n_y. The dimensions of the rectangular net are a in the x direction and b in the y direction. The average mass per unit surface area of the supporting matrix (if there is any) is ρ.

Each fiber parallel to x is subjected to the same pretension T_x', and each fiber parallel to y is subjected to the same pretension T_y'. For the equivalent membrane, the averaged pretensions in the x and y directions per unit length are, respectively,

$$T_x = \frac{T_x' n_x}{b}, \quad T_y = \frac{T_y' n_y}{a} \tag{15.7.2}$$

The equation of motion (11.3.3) then becomes

$$-T_x \frac{\partial^2 w}{\partial x^2} - T_y \frac{\partial^2 w}{\partial y^2} + \rho' \ddot{w} = q \tag{15.7.3}$$

where q represents a distributed transverse loading on the net. This equation describes the transverse vibration of the net about its static deflection as equilibrium position. This implies, of course, that the static sag is small, which is the case if the tensions are reasonably large. Another assumption is, of course, that any bending stiffness of the fibres can be neglected.

If the net is stretched over a rigid frame, the boundary conditions are

$$w(0,y,t) = w(a,y,t) = w(x,0,t) = w(x,b,t) = 0 \tag{15.7.4}$$

For free vibrations ($q = 0$), Eqs. (15.7.3) and (15.7.4) are satisfied by ($m, n = 1, 2, \ldots$)

$$w(x,y,t) = W(x,y)\mathrm{e}^{j\omega t}, \quad W(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{15.7.5}$$

Substitution in Eq. (15.7.3) gives the natural frequencies, ω_{mn}, as

$$\omega_{mn}^2 = \frac{1}{\rho'}\left[T_x\left(\frac{m\pi}{a}\right)^2 + T_y\left(\frac{n\pi}{b}\right)^2\right], \tag{15.7.6}$$

or

$$\omega_{mn}^2 = \frac{\pi^2 n_x T_x'}{a^2 b \rho'} \left(m^2 + \frac{T_y'}{T_x'} \frac{n_y}{n_x} \frac{a}{b} n^2 \right) \tag{15.7.7}$$

Equation (15.7.7) gives good results as long as the net is "dense" with respect to the smallest modal "wavelength" of interest.

15.8. HANGING NET OR CURTAIN

It is assumed that a rectangular hanging net or curtain (Fig. 6) is so constructed that all horizontal fibers are under a constant pretension T_x', while the vertical (hanging) fibers have a tension that is a function of the gravitational pull. If one assumes that the hanging net is assembled in its vertical frame in such a way that the horizontal fibers or cables are connected to the frame after the vertical fibers or cables were allowed to elongate due to gravity, the vertical tension correction due to the sag of the horizontal fibers is negligible. The tension in each of the vertical fibers is then proportional to y (measured from the bottom of the curtain):

$$T_y' = \frac{\rho' a g}{n_y} y \tag{15.8.1}$$

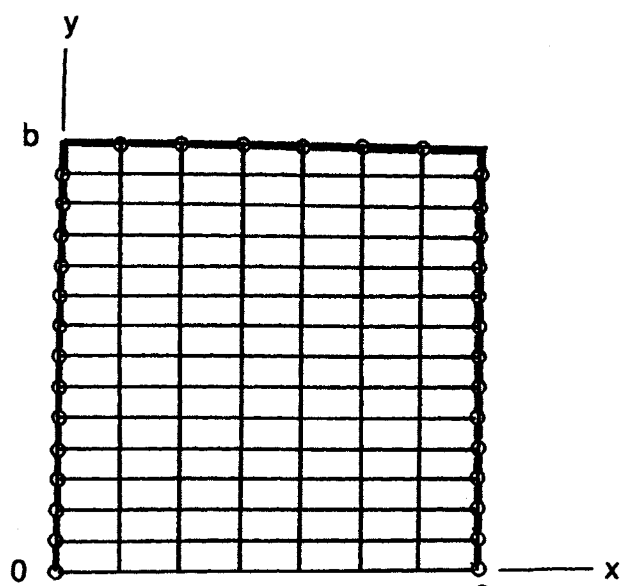

FIG. 6 Hanging net.

The tension in the vertical direction per unit length is, therefore,

$$T_y = t_y y, \quad t_y = \rho' g \tag{15.8.2}$$

In the horizontal direction, as in Eq. (15.7.2),

$$T_x = \frac{T'_x n_x}{b} \tag{15.8.3}$$

Thus the equation of motion (11.3.3) becomes (Soedel et al., 1985)

$$-T_x \frac{\partial^2 w}{\partial x^2} - t_y \frac{\partial}{\partial y}\left(y \frac{\partial w}{\partial y}\right) \rho' \ddot{w} = q \tag{15.8.4}$$

where q is a distributed force transverse to the plane of the handing curtain. It can be seen that as the horizontal fiber tension approaches 0, the equation of motion becomes that of a hanging cable.

The natural frequencies and modes are obtained by setting $q = 0$ and using the solution form

$$w(x, y, t) = W(x, y) e^{j\omega t}, \quad W(x, y) = Y(y) \sin \frac{m\pi x}{a} \tag{15.8.5}$$

This satisfies two of the four boundary conditions of the hanging net or curtain problem:

$$w(0, y, t) = w(a, y, t) = 0 \tag{15.8.6}$$

Substituting Eq. (15.8.5) in Eq.(15.8.4) gives

$$\frac{d^2 Y}{dy^2} + \frac{1}{y}\frac{dY}{dy} + \frac{1}{t_y}\left[\rho' \omega^2 - T_x \left(\frac{m\pi}{a}\right)^2\right] \frac{Y}{y} = 0 \tag{15.8.7}$$

Changing the independent variable y to ψ according to the transformation

$$y = \left\{ \frac{t_y}{4} \left[\rho' \omega^2 - T_x \left(\frac{m\pi}{a}\right)^2\right] \right\} \psi^2 \tag{15.8.8}$$

gives

$$\frac{d^2 Y}{d\psi^2} + \frac{1}{\psi}\frac{dY}{d\psi} + Y = 0 \tag{15.8.9}$$

This is Bessel's equation of zero order. Its solution is

$$Y = A J_0(\psi) + B Y_0(\psi) \tag{15.8.10}$$

The boundary conditions at the top and bottom of the curtain are

$$w(x, b, t) = 0, \quad w(x, 0, t) = \text{finite} \tag{15.8.11}$$

This may be transformed into

$$Y\left\{ 2\sqrt{\left[\rho' \omega^2 - T_x \left(\frac{m\pi}{a}\right)^2\right] \frac{b}{t_y}} \right\} = 0, \quad Y(0) = \text{finite} \tag{15.8.12}$$

Because $Y_0(0) \to \infty$, it must be that $B=0$ in order for Eq. (15.8.12) to be satisfied. Substituting Eq. (15.8.10) gives

$$J_0\left\{2\sqrt{\left[\rho'\omega^2 - T_x\left(\frac{m\pi}{a}\right)^2\right]\frac{b}{t_y}}\right\}=0 \tag{15.8.13}$$

The roots of this equation are $k_1=2.404, k_2=5.520, k_3=8.654, k_4=11.792$, and so on. Labeling these roots $k_n (n=1,2,\ldots)$ and replacing ω by ω_{mn} to signify the dependency on m and n gives (Soedel et al., 1985)

$$\omega_{mn}=\sqrt{\frac{1}{\rho'}\left[\frac{k_n^2 t_y}{4b}+T_x\left(\frac{m\pi}{a}\right)^2\right]} \tag{15.8.14}$$

or, upon substituting Eqs. (15.8.2) and (15.8.3) into this expression, the natural frequencies of the hanging net or curtain are

$$\omega_{mn}=\sqrt{\frac{k_n^2 g}{4b}+\frac{n_x}{b}\left(\frac{m\pi}{a}\right)^2\frac{T_x'}{\rho'}} \tag{15.8.15}$$

As the horizontal fiber or cable tension is decreased, the solution approaches that of the hanging cable. If the vertical fibres or cables are removed, the solution reduces to that of a string in tension.

To obtain the natural modes, one substitutes ω_{mn} of Eq. (15.8.14) for ω in Eq. (15.8.8) and solves for ψ:

$$\psi=2\sqrt{\left[\rho'\omega_{mn}^2 - T_x\left(\frac{m\pi}{a}\right)^2\right]\frac{y}{t_y}}=k_n\sqrt{\frac{y}{b}} \tag{15.8.16}$$

Substituting this Eq. (15.8.10) with $B=0$, and substituting this in Eq. (15.8.5), gives the expression for the natural modes (Soedel et al., 1985)

$$W_{mn}(x,y)=J_0\left(k_n\sqrt{\frac{y}{b}}\right)\sin\frac{m\pi x}{a} \tag{15.8.17}$$

15.9. SHELLS MADE OF HOMOGENEOUS AND ISOTROPIC LAMINA

The constitutive relationships are given by Eqs. (15.2.1)–(15.2.5). Since, for homogeneous and isotropic lamina,

$$|T_1|=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} \tag{15.9.1}$$

Equation (15.2.28) becomes

$$|\bar{Q}|=|Q|=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{21} & Q_{22} & 0\\ 0 & 0 & Q_{33}\end{bmatrix} \tag{15.9.2}$$

for a laminated composite consisting of n isotropic and homogeneous individual lamina made of different materials, Eqs. (15.3.9), (15.3.10), (15.3.15) and (15.3.16) become

$$A_{ij}=\sum_{k=1}^{n}(Q_{ij})_k(h_{k+1}-h_k) \tag{15.9.3}$$

$$B_{ij}=\frac{1}{2}\sum_{k=1}^{n}(Q_{ij})_k(h_{k+1}^2-h_k^2) \tag{15.9.4}$$

$$D_{ij}=\frac{1}{3}\sum_{k=1}^{n}(Q_{ij})_k(h_{k+1}^3-h_k^3) \tag{15.9.5}$$

The force and moment resultant are given by Eq. (15.3.17) where $(Q_{13})_k=(Q_{31})_k=0$, $(Q_{23})_k=(Q_{32})_k=0$, and

$$(Q_{11})_k=(Q_{22})_k=\frac{E_k}{1-\mu_k^2} \tag{15.9.6}$$

$$(Q_{12})_k=(Q_{21})_k=\frac{\mu_k E_k}{1-\mu_k^2} \tag{15.9.7}$$

$$(Q_{33})_k=G_k=\frac{E_k}{2(1+\mu_k)} \tag{15.9.8}$$

The force and moment resultants are from Eq. (15.3.17),

$$\begin{Bmatrix} N_{11}\\ N_{22}\\ N_{12}\\ M_{11}\\ M_{22}\\ M_{12}\end{Bmatrix}=\begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0\\ A_{21} & A_{22} & 0 & B_{21} & B_{22} & 0\\ 0 & 0 & A_{33} & 0 & 0 & B_{33}\\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0\\ B_{21} & B_{22} & 0 & D_{21} & D_{22} & 0\\ 0 & 0 & B_{33} & 0 & 0 & D_{33}\end{bmatrix} \tag{15.9.9}$$

For a laminated composite, where the reference plane is selected to coincide with the neutral plane, this reduces further to

$$\begin{Bmatrix} N_{11}\\ N_{22}\\ N_{12}\end{Bmatrix}=\begin{bmatrix} A_{11} & A_{12} & 0\\ A_{21} & A_{22} & 0\\ 0 & 0 & A_{33}\end{bmatrix}\begin{Bmatrix} \varepsilon_{11}^0\\ \varepsilon_{22}^0\\ \varepsilon_{12}^0\end{Bmatrix} \tag{15.9.10}$$

and

$$\begin{Bmatrix} M_{11}\\ M_{22}\\ M_{12}\end{Bmatrix}=\begin{bmatrix} D_{11} & D_{12} & 0\\ D_{21} & D_{22} & 0\\ 0 & 0 & D_{33}\end{bmatrix}\begin{Bmatrix} k_{11}\\ k_{22}\\ k_{12}\end{Bmatrix} \tag{15.9.11}$$

The effective mass per unit length ρh is given by

$$\rho h=\sum_{k=1}^{n}\rho_k h_k \tag{15.9.12}$$

15.10. SIMPLY SUPPORTED SANDWICH PLATES AND BEAMS COMPOSED OF THREE HOMOGENEOUS AND ISOTROPIC LAMINA

The foregoing results apply, of course, also to cases where the lamina are neither anisotropic nor orthotropic, but simply homogeneous and isotropic.

For the arrangement of Fig. 7, the reference plane is selected to coincide with the neutral plane which results in

$$B_{ij}=0 \tag{15.10.1}$$

The upper and lower layers, each of thickness h_L, are made of the same material, characterized by (E_L, μ_L), and the core layer is of a different material, (E_c, μ_c). The A_{ij} and D_{ij} are given by Eq. (15.9.3) and Eq. (15.9.5). In the following, only the transverse vibrations of the sandwich plate will be investigated; therefore, only the D_{ij} need to be obtained.

We obtain from Eq. (15.9.5)

$$D_{11} = \frac{1}{3}\left\{ \frac{E_L}{1-\mu_L^2}\left[\left(-\frac{h_c}{2}\right)^3 - \left(-\frac{h_c}{2}-h_L\right)^3\right] + \frac{E_c}{1-\mu_c^2}\left[\left(\frac{h_c}{2}\right)^3 - \left(-\frac{h_c}{2}\right)^3\right] + \frac{E_L}{1-\mu_L^2}\left[\left(\frac{h_c}{2}+h_L\right)^3 - \left(\frac{h_c}{2}\right)^3\right]\right\} \tag{15.10.2}$$

where h_c is the core thickness. This simplifies to

$$D_{11}=\frac{E_L}{12(1-\mu_L^2)}\left[(h_c+2h_L)^3-h_c^3\right]+\frac{E_c}{12(1-\mu_c^2)}h_c^3 \tag{15.10.3}$$

Similarly, we find that

$$D_{22}=D_{11} \tag{15.10.4}$$

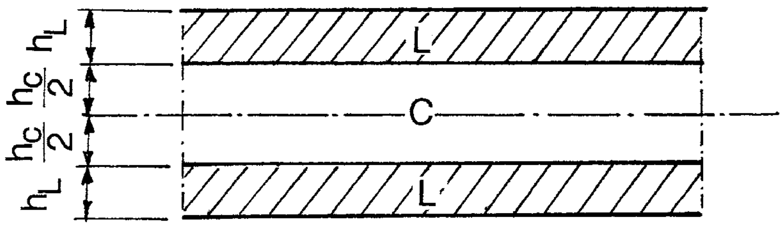

FIG. 7 Sandwich plate or beam.

and

$$D_{12}=D_{21}=\frac{\mu_L E_L}{12(1-\mu_L^2)}\left[(h_c+2h_L)^3-h_c^3\right]+\frac{\mu_c E_c}{12(1-\mu_c^2)}h_c^3 \tag{15.10.5}$$

Finally,

$$D_{33}=\frac{E_L}{24(1+\mu_L)}\left[(h_c+2h_L)^3-h_c^3\right]+\frac{E_c}{24(1+\mu_c)}h_c^3 \tag{15.10.6}$$

Substituting this in Eq. (15.5.19) gives the natural frequencies for this case of a rectangular sandwich plate when all four edges are simply supported. Note again that the product ρh in Eq. (15.5.19) is the mass per unit area of the laminated plate and is

$$\rho h=2\rho_L h_L+\rho_c h_c \tag{15.10.7}$$

Again, note that if the outer layers are removed ($h_L=0$), we obtain

$$D_{11}=D_{22}=\frac{E_c h_c^3}{12(1-\mu_c^2)}=D, \quad D_{12}=D_{21}=\mu_c D,$$

and

$$D_{33}=\frac{1}{2}(1-\mu_c)D$$

and the natural frequencies for the homogeneous and isotropic, single layer plate result.

For a simply supported beam with the same arrangement of three lamina, we obtain when multiplying D_{11} by the width of the beam b and removing the Poisson effect,

$$\overline{EI}=\frac{E_L b}{12}\left[(h_c+2h_L)^3-h_c^3\right]+\frac{E_c b h_c^3}{12} \tag{15.10.8}$$

The natural frequencies are then given by

$$\omega_m=\frac{m^2\pi^2}{L^2}\sqrt{\frac{\overline{EI}}{\overline{\rho'}}} \tag{15.10.9}$$

where the mass per unit length of the beam is given by

$$\overline{\rho'}=2\rho_L'+\rho_c' \tag{15.10.10}$$

where ρ_L' is the mass per unit length of each of the outer layers and ρ_c' is the mass per unit length of the core layer.

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16

Rotating Structures

Rotating structures range from space stations, turbines, and tires to washing machine baskets. When utilizing Hamilton's principle, the effect of rotation is introduced through the kinetic energy expressions. In the following, relatively simple cases are discussed first, working up to shells.

16.1. STRING PARALLEL TO AXIS OF ROTATION

This case is illustrated in Fig. 1. The string is under constant tension T. The rotation is at a constant rotational speed Ω. The string is parallel to the axis of rotation and a distance R removed from it. The radial vibration displacement is u_3 and the vibration displacement tangential to the rotation is u_1. Gravitational influences are neglected.

The kinetic energy of a slice of string of infinitesimal length $\mathrm{d}x$ is

$$\mathrm{d}K = \frac{1}{2}\rho A \mathrm{d}z\, \bar{v}\cdot\bar{v} \tag{16.1.1}$$

where A is the cross-section. The velocity vector is given by the standard formula found in basic texts on dynamics (Ginsberg and Genin, 1984):

$$\bar{v} = \bar{v}_{O'} + (\bar{v}_{P/O'})_{\mathrm{rel}} + \bar{\omega}\times\bar{r}_{P/O'} \tag{16.1.2}$$

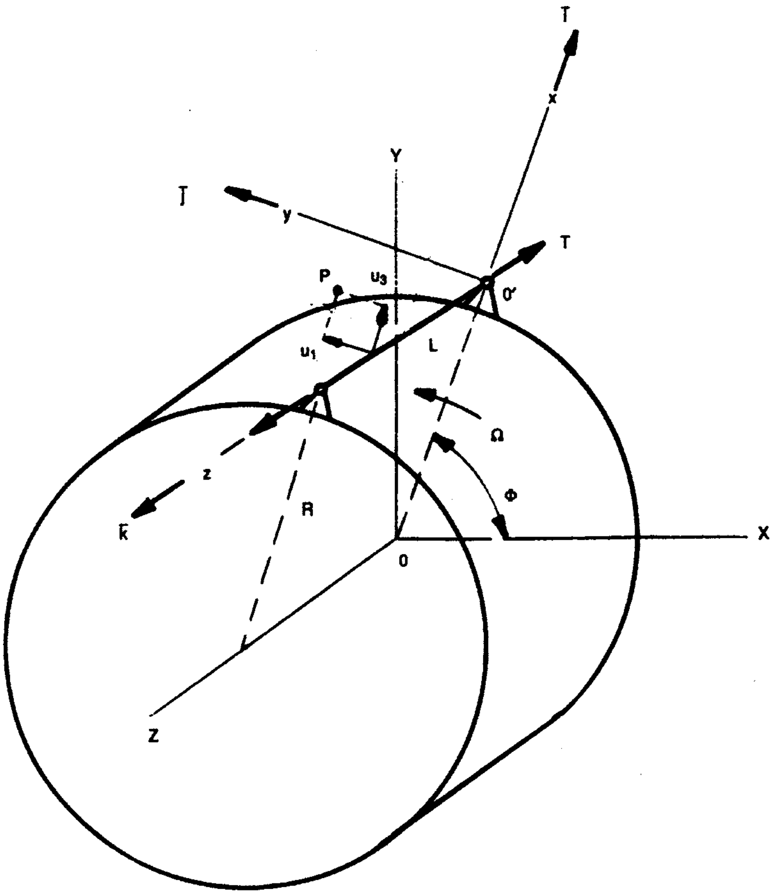

Fig. 1 String mounted on a rotating cylinder.

Since the moving coordinate system is xyz, we have

$$\bar{v}_{O'} = R\Omega\bar{j} \tag{16.1.3}$$

$$\bar{\omega} = \Omega\bar{k} \tag{16.1.4}$$

$$(\bar{v}_{P/O'})_{\text{rel}} = \dot{u}_3\bar{i} + \dot{u}_1\bar{j} \tag{16.1.5}$$

$$\bar{r}_{P/O'} = u_3\bar{i} + u_1\bar{j} + z\bar{k} \tag{16.1.6}$$

We obtain

$$\bar{\omega} \times \bar{r}_{P/O'} = -u_1\Omega\bar{i} + u_3\Omega\bar{j} \tag{16.1.7}$$

Thus

$$\bar{v}=(\dot{u}_3-u_1\Omega)\bar{i}+(\dot{u}_1+u_3\Omega+R\Omega)\bar{j} \tag{16.1.8}$$

and

$$\bar{v}\cdot\bar{v}=(\dot{u}_3-u_1\Omega)^2+(\dot{u}_1+u_3\Omega+R\Omega)^2 \tag{16.1.9}$$

The kinetic energy of the complete string is

$$K=\frac{\rho A}{2}\int_{z=0}^{L}[(\dot{u}_3-u_1\Omega)^2+(\dot{u}_1+u_3\Omega+R\Omega)^2]\,\mathrm{d}z \tag{16.1.10}$$

Evaluating the integral of the variation of K gives

$$\begin{aligned}\int_{t_0}^{t_1}\delta K\,\mathrm{d}t &=\rho A\int_{t_0}^{t_1}\int_{z=0}^{L}[(\dot{u}_3-u_1\Omega)\delta\dot{u}_3+(\dot{u}_1\Omega+u_3\Omega^2+R\Omega^2)\delta u_3\\ &\quad+(\dot{u}_1+u_3\Omega+R\Omega)\delta\dot{u}_1-(\dot{u}_3\Omega-u_1\Omega^2)\delta u_1]\,\mathrm{d}z\,\mathrm{d}t\end{aligned} \tag{16.1.11}$$

Integrating some of the terms by parts gives, finally,

$$\begin{aligned}\int_{t_0}^{t_1}\delta K\,\mathrm{d}t &=-\rho A\int_{t_0}^{t_1}\int_{z=0}^{L}[(\ddot{u}_1+2\Omega\dot{u}_3-u_1\Omega^2)\delta u_1\\ &\quad+(\ddot{u}_3-2\Omega\dot{u}_1-(u_3+R)\Omega^2)\delta u_3]\,\mathrm{d}z\,\mathrm{d}t\end{aligned} \tag{16.1.12}$$

The potential energy is (see Chapter 11)

$$V=\frac{T}{2}\int_0^L\left[\left(\frac{\partial u_1}{\partial z}\right)^2+\left(\frac{\partial u_3}{\partial z}\right)^2\right]\mathrm{d}z \tag{16.1.13}$$

Thus

$$\int_{t_0}^{t_1}\delta V\,\mathrm{d}t=-T\int_{t_0}^{t_1}\int_{z=0}^{L}\left(\frac{\partial^2 u_1}{\partial z^2}\delta u_1+\frac{\partial^2 u_3}{\partial z^2}\delta u_3\right)\mathrm{d}z\,\mathrm{d}t \tag{16.1.14}$$

The load energy is, in terms of a variational integral,

$$\int_{t_0}^{t_1}\delta E_L\,\mathrm{d}t=\int_{t_0}^{t_1}\int_{z=0}^{L}(q_1'\delta u_1+q_3'\delta u_3)\,\mathrm{d}z\,\mathrm{d}t \tag{16.1.15}$$

where q_1' and q_3' are forces per unit string length.

Applying Hamilton's principle, which is in this case

$$\int_{t_0}^{t_1}\delta(V-E_L-K)\,\mathrm{d}t=0 \tag{16.1.16}$$

gives, after collecting coefficients of δu_1 and δu_2 and equating them to 0 in order to satisfy the equation,

$$T\frac{\partial^2 u_1}{\partial z^2} - \rho A(\ddot{u}_1 + 2\Omega\dot{u}_3 - u_1\Omega^2) = -q_1' \tag{16.1.17}$$

$$T\frac{\partial^2 u_3}{\partial z^2} - \rho A(\ddot{u}_3 - 2\Omega\dot{u}_1 - u_3\Omega^2) = -q_3' - \rho A\Omega^2 R \tag{16.1.18}$$

The forcing term $\rho A\Omega^2 R$ causes a static deflection due to centrifugal effects. Taking this static deflection as the equilibrium position, the free vibration of the string about this equilibrium position is described by

$$T\frac{\partial^2 u_1}{\partial z^2} - \rho A(\ddot{u}_1 + 2\Omega\dot{u}_3 - u_1\Omega^2) = 0 \tag{16.1.19}$$

$$T\frac{\partial^2 u_3}{\partial z^2} - \rho A(\ddot{u}_3 + 2\Omega\dot{u}_1 - u_3\Omega^2) = 0 \tag{16.1.20}$$

It should be noted that the radius R has no influence on the vibration response. It influences only the equilibrium position about which this vibration takes place.

If the string vibrates at one of its natural frequencies, the solution must be of the form

$$u_1 = U_1 e^{j\omega t} \tag{16.1.21}$$

$$u_3 = U_3 e^{j\omega t} \tag{16.1.22}$$

This gives

$$T\frac{\partial^2 U_1}{\partial z^2} + \rho A(\omega^2 + \Omega^2)U_1 - 2j\omega\Omega\rho A U_3 = 0 \tag{16.1.23}$$

$$T\frac{\partial^2 U_3}{\partial z^2} + \rho A(\omega^2 + \Omega^2)U_3 + 2j\omega\Omega\rho A U_1 = 0 \tag{16.1.24}$$

For a string supported at both ends, the boundary conditions of zero deflection and the two differential equations are satisfied by

$$U_1 = A_1 \sin\frac{m\pi z}{L} \tag{16.1.25}$$

$$U_3 = A_3 \sin\frac{m\pi z}{L} \tag{16.1.26}$$

This gives

$$\begin{bmatrix} -T\left(\frac{m\pi}{L}\right)^2 + \rho A(\omega^2 + \Omega^2) & -2j\omega\Omega\rho A \\ 2j\omega\Omega\rho A & -T\left(\frac{m\pi}{L}\right)^2 + \rho A(\omega^2 + \Omega^2) \end{bmatrix} \begin{Bmatrix} A1 \\ A3 \end{Bmatrix} = 0 \tag{16.1.27}$$

Discarding the trivial solution $A_1 = A_3 = 0$, this equation is satisfied if the determinant is 0. This gives

$$\omega^4 - 2\omega^2(\omega_{0m}^2 + \Omega^2) + (\omega_{0m}^2 - \Omega^2)^2 = 0 \tag{16.1.28}$$

where

$$\omega_{0m}^2 = \frac{T}{\rho A}\left(\frac{m\pi}{L}\right)^2 \tag{16.1.29}$$

which are the natural frequencies of the string when $\Omega = 0$. Equation (16.1.28) may be written

$$[\omega^2 - (\omega_{0m} - \Omega)^2][\omega^2 - (\omega_{0m} + \Omega)^2] = 0 \tag{16.1.30}$$

Therefore, for every value of m, there are two types of natural frequencies,

$$\omega_1 = \omega_{0m} - \Omega \tag{16.1.31}$$

$$\omega_2 = \omega_{0m} + \Omega \tag{16.1.32}$$

From Eq. (16.1.27), the corresponding amplitude ratios are

$$\frac{A_{11}}{A_{31}} = -j \tag{16.1.33}$$

$$\frac{A_{12}}{A_{32}} = j \tag{16.1.34}$$

Therefore, the mode shape corresponding to the first type of natural frequency is, normalized with respect to A_{31},

$$U_{11} = -j \sin\frac{m\pi x}{L} \tag{16.1.35}$$

$$U_{31} = \sin\frac{m\pi x}{L} \tag{16.1.36}$$

The mode shape corresponding to the second type of natural frequency is

$$U_{12} = j \sin\frac{m\pi x}{L} \tag{16.1.37}$$

$$U_{31} = \sin\frac{m\pi x}{L} \tag{16.1.38}$$

The motion of the string when vibrating at a natural frequency of the first type is

$$u_{11} = \sin\frac{m\pi x}{L} e^{j[(\omega_{0m} - \Omega)t - \pi/2]} \tag{16.1.39}$$

$$u_{31} = \sin\frac{m\pi x}{L} e^{j(\omega_{0m} - \Omega)t} \tag{16.1.40}$$

When vibrating at a natural frequency of the second type, the motion is

$$u_{12} = \sin\frac{m\pi x}{L}\,e^{j[(\omega_{0m}+\Omega)t+\pi/2]} \tag{16.1.41}$$

$$u_{32} = \sin\frac{m\pi x}{L}\,e^{j(\omega_{0m}+\Omega)t} \tag{16.1.42}$$

Plotting the motion of the string at any point of the string as viewed axially results in a clockwise circular motion for vibration at a natural frequency of the first type as shown in Fig. 2(a), and in a counterclockwise circular motion for vibration at a natural frequency of the second type, as shown in Fig. 2(b).

The forced solution is obtained in terms of an infinite series of all natural modes,

$$u_1 = \sum_{m=1}^{\infty}\left(-\eta_{m1} j \sin\frac{m\pi z}{L} + \eta_{m2} j \sin\frac{m\pi z}{L}\right) = j\sum_{m=1}^{\infty}\xi_{m1}\sin\frac{m\pi z}{L} \tag{16.1.43}$$

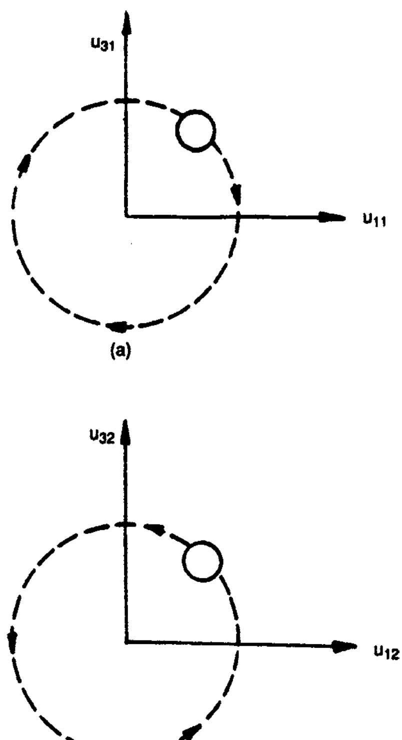

Fig. 2 Motion of the rotating string at a natural frequency pair: (a) clockwise circular, and (b) counter-clockwise circular.

$$u_3 = \sum_{m=1}^{\infty}\left(\eta_{m1}\sin\frac{m\pi z}{L} + \eta_{m2}\sin\frac{m\pi z}{L}\right) = \sum_{m=1}^{\infty}\xi_{m3}\sin\frac{m\pi z}{L} \tag{16.1.44}$$

where

$$\xi_{m1} = -\eta_{m1} + \eta_{m2} \tag{16.1.45}$$

$$\xi_{m3} = \eta_{m1} + \eta_{m2} \tag{16.1.46}$$

Substituting this in Eqs. (16.1.17) and (16.1.18), with equivalent viscous damping added and with the static deflection due to the centrifugal effect subtracted, so that

$$T\frac{\partial^2 u_1}{\partial z^2} - \rho A(\ddot{u}_1 + 2\Omega\dot{u}_3 - u_1\Omega^2) - c\dot{u}_1 = -q_1' \tag{16.1.47}$$

$$T\frac{\partial^2 u_3}{\partial z^2} - \rho A(\ddot{u}_3 - 2\Omega\dot{u}_1 - u_3\Omega^2) - c\dot{u}_3 = -q_3' \tag{16.1.48}$$

gives

$$\sum_{m=1}^{\infty}\{[-\ddot{\xi}_{m1} - \lambda\dot{\xi}_{m1} - (\omega_{0m}^2 - \Omega^2)\xi_{m1}]j - 2\Omega\dot{\xi}_{m3}\}\sin\frac{m\pi z}{L} = -\frac{q_1'}{\rho A} \tag{16.1.49}$$

$$\sum_{m=1}^{\infty}\{[-\ddot{\xi}_{m3} - \lambda\dot{\xi}_{m3} - (\omega_{0m}^2 - \Omega^2)\xi_{m3}] + 2\Omega\dot{\xi}_{m1}j\}\sin\frac{m\pi z}{L} = -\frac{q_3'}{\rho A} \tag{16.1.50}$$

where

$$\lambda = \frac{c}{\rho A} \tag{16.1.51}$$

Multiplying Eqs. (16.1.49) and (16.1.50) by $\sin(p\pi z/L)$, where $p = 1, 2, \ldots$, integrating from $z = 0$ to $z = L$, and utilizing the orthogonality of the sine function eliminates all terms for which $p \neq m$ and results in

$$[\ddot{\xi}_{m1} + \lambda\dot{\xi}_{m1} + (\omega_{0m}^2 - \Omega^2)\xi_{m1}]j + 2\Omega\dot{\xi}_{m3} = F_1(z,t) \tag{16.1.52}$$

$$[\ddot{\xi}_{m3} + \lambda\dot{\xi}_{m3} + (\omega_{0m}^2 - \Omega^2)\xi_{m3}] - 2\Omega\dot{\xi}_{m1}j = F_3(z,t) \tag{16.1.53}$$

where

$$F_1(z,t) = \frac{2}{\rho AL}\int_{z=0}^{L} q_1'\sin\frac{m\pi z}{L}\,dz \tag{16.1.54}$$

$$F_3(z,t) = \frac{2}{\rho AL}\int_{z=0}^{L} q_3'\sin\frac{m\pi z}{L}\,dz \tag{16.1.55}$$

These equations may now be solved for ξ_{m1} and ξ_{m2} in general. But instead of pursuing the general line of investigation, let us treat the special case of steady-state harmonic response. Let us assume that

$$F_1(z,t) = 0 \tag{16.1.56}$$

$$F_3(z,t) = F_3^*(z)e^{j\omega t} \tag{16.1.57}$$

This case represents a distributed load on the string in the direction of u_3, varying harmonically in time. The steady-state response is expected to be of the type

$$\xi_{m1} = \tilde{\tilde{\Xi}}_1 e^{j\omega t} \tag{16.1.58}$$

$$\xi_{m3} = \tilde{\tilde{\Xi}}_3 e^{j\omega t} \tag{16.1.59}$$

where

$$\tilde{\tilde{\Xi}}_1 = \Xi_1 e^{-j\phi_1} \tag{16.1.60}$$

$$\tilde{\tilde{\Xi}}_3 = \Xi_3 e^{-j\phi_1} \tag{16.1.61}$$

Substituting this in Eqs. (16.1.52) and (16.1.53) gives

$$\begin{bmatrix} (\omega_{0m}^2 - \Omega^2 - \omega^2)j - \lambda\omega & 2\Omega\omega j \\ 2\Omega\omega & (\omega_{0m}^2 - \Omega^2 - \omega^2) + j\lambda\omega \end{bmatrix} \begin{Bmatrix} \tilde{\tilde{\Xi}}_1 \\ \tilde{\tilde{\Xi}}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_3^*(z) \end{Bmatrix} \tag{16.1.62}$$

From these expressions, the magnitudes and phase angles of the response can be obtained easily. It is clearly seen that as the rotational speed $\Omega \to 0$, the solution consists only of motion in the xz plane of Fig. 1 and is identical to the modal expansion solution for harmonic vibration of a stationary string. The rotational speed is the coupling factor that will result in an ovaling motion. The ovaling is proportional to Ω. The ovaling motion response will be clockwise as the excitation frequency increases from 0 and passes through the natural frequency of the first type. Before reaching the natural frequency of the second type, where the motion is counter-clockwise, there will be a transition motion where the oval collapses into motion in a plane, inclined from both the vertical and horizontal. This is discussed in Soedel and Soedel (1989) and illustrated in Fig. 3.

16.2. BEAM PARALLEL TO AXIS OF ROTATION

Let us investigate a beam with coordinates as in Fig. 1. The kinetic energy expression is the same as for the string. Thus the development

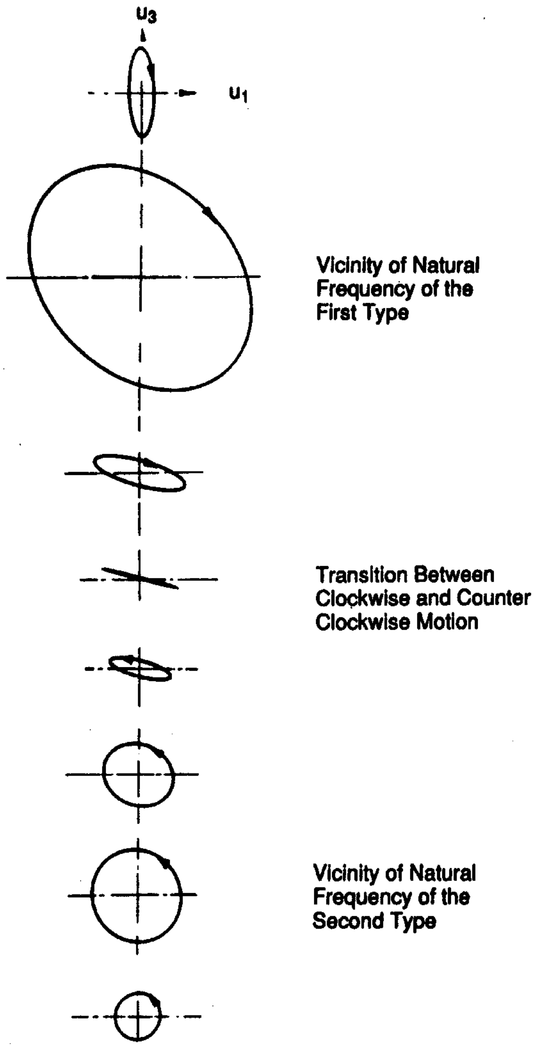

FIG. 3 Transitions of the forced response of a rotating string.

of Eqs. (16.1.1)–(16.1.12) for the string is the same for the beam. The potential energy is, on the other hand,

$$V = \frac{EI_{xx}}{2}\int_0^L \left(\frac{\partial^2 u_1}{\partial z^2}\right)^2 \mathrm{d}z + \frac{EI_{yy}}{2}\int_0^L \left(\frac{\partial^2 u_3}{\partial z^2}\right)^2 \mathrm{d}z \tag{16.2.1}$$

when gravitational influences are neglected (the rotation axis is vertical). Therefore, the equations of motion become

$$EI_{xx}\frac{\partial^4 u_1}{\partial z^4}+\rho A(\ddot{u}_1+2\Omega\dot{u}_3-u_1\Omega^2)=q_1' \tag{16.2.2}$$

$$EI_{yy}\frac{\partial^4 u_3}{\partial z^4}+\rho A(\ddot{u}_3-2\Omega\dot{u}_1-u_3\Omega^2)=q_3'+\rho A\Omega^2 R \tag{16.2.3}$$

It is, of course, assumed that the shear center and load application center coincide with the center of gravity of the prismatic beam.

The forcing term $\rho A\Omega^2 R$ causes a static deflection in the u_3 direction due to the centrifugal effect. The free vibration of the beam about this equilibrium position is described by

$$EI_{xx}\frac{\partial^4 u_1}{\partial z^4}+\rho A(\ddot{u}_1+2\Omega\dot{u}_3-u_1\Omega^2)=0 \tag{16.2.4}$$

$$EI_{yy}\frac{\partial^4 u_3}{\partial z^4}+\rho A(\ddot{u}_3-2\Omega\dot{u}_1-u_3\Omega^2)=0 \tag{16.2.5}$$

At a natural frequency,

$$u_1=U_1 e^{j\omega t} \tag{16.2.6}$$

$$u_3=U_3 e^{j\omega t} \tag{16.2.7}$$

Substitution gives

$$EI_{xx}\frac{d^4 U_1}{dz^4}-\rho A(\omega^2+\Omega^2)U_1+2j\omega\Omega\rho A U_3=0 \tag{16.2.8}$$

$$EI_{yy}\frac{d^4 U_3}{dz^4}-\rho A(\omega^2+\Omega^2)U_3-2j\omega\Omega\rho A U_1=0 \tag{16.2.9}$$

Let us now assume that the beam is supported by a shear diaphragm (simply supported in the x and y directions). For this case, the boundary conditions of zero moments and zero transverse deflections are satisfied by

$$U_1=A_1\sin\frac{m\pi z}{L} \tag{16.2.10}$$

$$U_3=A_3\sin\frac{m\pi z}{L} \tag{16.2.11}$$

Substituting this gives

$$\begin{bmatrix} -EI_{xx}\left(\frac{m\pi}{L}\right)^4+\rho A(\omega^2+\Omega^2) & -2j\omega\Omega\rho A \\ 2j\omega\Omega\rho A & -EI_{yy}\left(\frac{m\pi}{L}\right)^4+\rho A(\omega^2+\Omega^2) \end{bmatrix}\begin{Bmatrix} A_1 \\ A_3 \end{Bmatrix}=0 \tag{16.2.12}$$

This equation is satisfied in a nontrivial fashion if the determinant is 0. This gives.

$$\omega^4 - 2\omega^2(\varepsilon_1^2 + \Omega^2) + (\varepsilon_1^2 - \Omega^2)^2 - \varepsilon_2^2 = 0 \tag{16.2.13}$$

where

$$\varepsilon_1^2 = \frac{1}{2}(\omega_{0mx}^2 + \omega_{0my}^2) \tag{16.2.14}$$

$$\varepsilon_2^2 = \frac{1}{2}(\omega_{0mx}^2 - \omega_{0my}^2) \tag{16.2.15}$$

and where

$$\omega_{0mx}^2 = \left(\frac{m\pi}{L}\right)^4 \frac{EI_{xx}}{\rho A} \tag{16.2.16}$$

$$\omega_{0my}^2 = \left(\frac{m\pi}{L}\right)^4 \frac{EI_{yy}}{\rho A} \tag{16.2.17}$$

The two natural frequencies are, therefore,

$$\omega_1^2 = \varepsilon_1^2 + \Omega^2 - 2\varepsilon_1\Omega\sqrt{1 - \left(\frac{\varepsilon_2}{2\varepsilon_1\Omega}\right)^2} \tag{16.2.18}$$

$$\omega_2^2 = \varepsilon_1^2 + \Omega^2 + 2\varepsilon_1\Omega\sqrt{1 - \left(\frac{\varepsilon_2}{2\varepsilon_1\Omega}\right)^2} \tag{16.2.19}$$

For the special case of a square or circular cross-section of the beam, so that $I_{xx} = I_{yy}$, one obtains $\varepsilon_2 = 0$ since $\omega_{0mx} = \omega_{0my} = \omega_{0m}$. For this case

$$\omega_1 = \omega_{0m} - \Omega \tag{16.2.20}$$

$$\omega_2 = \omega_{0m} + \Omega \tag{16.2.21}$$

where

$$\omega_{0m} = \left(\frac{m\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \tag{16.2.22}$$

and similar to the case of the string, $A_{11}/A_{31} = -j$ and $A_{12}/A_{32} = j$.

16.3. ROTATING RING

The ring rotates with constant angular velocity $\dot{\theta} = \Omega$, as shown in Fig. 4. For a mass element of length $a\mathrm{d}\theta$, the kinetic energy is

$$\mathrm{d}K = \frac{1}{2}\rho A a \,\mathrm{d}\theta\, \bar{v} \cdot \bar{v} \tag{16.3.1}$$

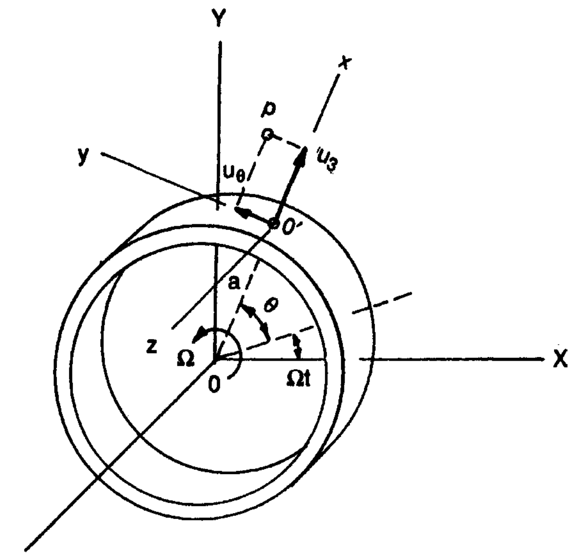

FIG. 4 Rotating ring.

where A is the cross-section of the ring. The velocity vector is given by the standard formula (Ginsberg and Genin, 1984)

$$\bar{v}=\bar{v}_{O'}+(\bar{v}_{P/O'})_{\text{rel}}+\bar{\omega}\times\bar{r}_{P/O'} \tag{16.3.2}$$

where

$$\bar{v}_{O'}=a\Omega\bar{j} \tag{16.3.3}$$

$$\bar{\omega}=\Omega\bar{k} \tag{16.3.4}$$

$$(\bar{v}_{P/O'})_{\text{rel}}=\dot{u}_3\bar{i}+\dot{u}_\theta\bar{j} \tag{16.3.5}$$

$$\bar{r}_{P/O'}=u_3\bar{i}+u_\theta\bar{j} \tag{16.3.6}$$

This gives

$$\bar{\omega}\times\bar{r}_{P/O'}=-u_\theta\Omega\bar{i}+u_3\Omega\bar{j} \tag{16.3.7}$$

and therefore,

$$\bar{v}=(\dot{u}_3-u_\theta\Omega)\bar{i}+(\dot{u}_\theta+u_3\Omega+a\Omega)\bar{j} \tag{16.3.8}$$

Thus the kinetic energy of the ring is

$$K=\frac{\rho A a}{2}\int_{\theta=0}^{2\pi}[(\dot{u}_3-u_\theta\Omega)^2+(\dot{u}_\theta+u_3\Omega+a\Omega)^2]\mathrm{d}\theta \tag{16.3.9}$$

Forming the variation of K and integrating in time from t_0 to t_1 gives

$$\int_{t_0}^{t_1} \delta K \, \mathrm{d}t = \rho A a \int_{t_0}^{t_1} \int_0^{2\pi} [(\dot{u}_3 - u_\theta \Omega)(\delta \dot{u}_3 - \Omega \delta u_\theta) + (\dot{u}_\theta + u_3 \Omega + a\Omega)(\delta u_\theta + \Omega \delta u_3)] \mathrm{d}\theta \, \mathrm{d}t \tag{16.3.10}$$

Integrating by parts in time gives

$$\int_{t_0}^{t_1} \delta K \, \mathrm{d}t = \rho A a \int_{t_0}^{t_1} \int_0^{2\pi} [(-\ddot{u}_3 - 2\dot{u}_\theta \Omega + u_3 \Omega^2 + a\Omega^2)\delta u_3 + (-\ddot{u}_\theta - 2\dot{u}_3 \Omega + u_\theta \Omega^2)\delta u_\theta] \mathrm{d}\theta \, \mathrm{d}t \tag{16.3.11}$$

The strain energy is given by Eq. (2.6.3), after introducing the constraints for a ring as discussed in Chapter 4. The variation of strain energy becomes, integrated in time from t_0 to t_1,

$$\int_{t_0}^{t_1} \delta U \, \mathrm{d}t = a \int_{t_0}^{t_1} \int_0^{2\pi} \left\{ \left[\frac{EI}{a^4} \left(\frac{\partial^3 u_\theta}{\partial \theta^3} - \frac{\partial^4 u_3}{\partial \theta^4} \right) - \frac{EA}{a^2} \left(\frac{\partial u_\theta}{\partial \theta} + u_3 \right) \right] \delta u_3 + \left[\frac{EI}{a^4} \left(\frac{\partial^2 u_\theta}{\partial \theta^2} - \frac{\partial^3 u_3}{\partial \theta^3} \right) + \frac{EA}{a^2} \left(\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial u_3}{\partial \theta} \right) \right] \delta u_\theta \right\} \mathrm{d}\theta \, \mathrm{d}t \tag{16.3.12}$$

Applying Hamilton's principle, with the energy input due to a load considered as in Eq. (2.6.9), gives the following equations of motion:

$$\frac{EI}{a^4} \left(\frac{\partial^3 u_3}{\partial \theta^3} - \frac{\partial^2 u_\theta}{\partial \theta^2} \right) - \frac{EA}{a^2} \left(\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial u_3}{\partial \theta} \right) + \rho A (\ddot{u}_\theta + 2\dot{u}_3 \Omega - u_\theta \Omega^2) = q'_\theta \tag{16.3.13}$$

$$\frac{EI}{a^4} \left(\frac{\partial^4 u_3}{\partial \theta^4} - \frac{\partial^3 u_\theta}{\partial \theta^3} \right) + \frac{EA}{a^2} \left(\frac{\partial u_\theta}{\partial \theta} + u_3 \right) + \rho A (\ddot{u}_3 - 2\dot{u}_\theta \Omega - (a + u_3)\Omega^2) = q'_3 \tag{16.3.14}$$

To obtain natural frequencies and modes, we set $q'_3 = q'_\theta = 0$ and redefine u_θ and u_3 to be the displacements from the equilibrium position determined by the centrifugal pressure term $\rho A a \Omega^2$. We set

$$u_3(\theta, t) = U_3(\theta) \mathrm{e}^{j\omega t} \tag{16.3.15}$$

$$u_\theta(\theta, t) = U_\theta(\theta) \mathrm{e}^{j\omega t} \tag{16.3.16}$$

This gives

$$\frac{EI}{a^4} \left(\frac{\mathrm{d}^2 \mathrm{U}_\theta}{\mathrm{d}\theta^2} - \frac{\mathrm{d}^3 U_3}{\mathrm{d}\theta^3} \right) + \frac{EA}{a^2} \left(\frac{\mathrm{d}^2 U_\theta}{\mathrm{d}\theta^2} + \frac{\mathrm{d} U_3}{\mathrm{d}\theta} \right) + \rho A (\omega^2 + \Omega^2) U_\theta - 2j\omega \Omega \rho A U_3 = 0 \tag{16.3.17}$$

$$\frac{EI}{a^4}\left(\frac{\mathrm{d}^3 \mathrm{U}_\theta}{\mathrm{d}\theta^3}-\frac{\mathrm{d}^4 U_3}{\mathrm{d}\theta^4}\right)-\frac{EA}{a^2}\left(\frac{\mathrm{d}U_\theta}{\mathrm{d}\theta}+U_3\right)$$
$$+\rho A(\omega^2+\Omega^2)U_3+2j\omega\Omega\rho A U_\theta=0 \tag{16.3.18}$$

If the spin velocity Ω is reduced to 0, this equation reduces to Eqs. (5.3.3) and (5.3.4).

For a closed ring, we expect solutions of the form

$$U_3=A_{3n}\mathrm{e}^{j\pi\theta} \tag{16.3.19}$$

$$U_\theta=A_{\theta n}\mathrm{e}^{j\pi\theta} \tag{16.3.20}$$

This gives

$$\begin{bmatrix} \rho A(\omega^2+\Omega^2)-n^4\frac{EI}{a^4}-\frac{EA}{a^2} & j\left(2\omega\Omega\rho A-n^3\frac{EI}{a^4}-n\frac{EA}{a^2}\right) \\ j\left(-2\omega\Omega\rho A+n^3\frac{EI}{a^4}+n\frac{EA}{a^2}\right) & \rho A(\omega^2+\Omega^2)-n^2\left(\frac{EI}{a^4}+\frac{EA}{a^2}\right) \end{bmatrix}\begin{Bmatrix} A_{3n} \\ A_{\theta n}\end{Bmatrix}=0 \tag{16.3.21}$$

For nontrivial solutions, the determinant has to be equal to 0. This results in a fourth-order equation in ω, which is expected since there should be four distinct natural frequencies for every value of n, as opposed to the nonrotating ring, which has only two natural frequencies for every n number. See also Huang and Soedel (1987a,b), and Lin and Soedel (1987, 1988, 1989).

The natural modes can be shown either to rotate in the same direction as the spinning ring or in the opposite direction. This is explained next, using the inextensional approximation.

16.4. ROTATING RING USING INEXTENSIONAL APPROXIMATION

Starting with Eqs. (16.3.13) and (16.3.14) in the force and moment resultant form (see also Sec.6.15), we have

$$\frac{1}{a}\frac{\partial N_{\theta\theta}}{\partial\theta}+\frac{1}{a^2}\frac{\partial M_{\theta\theta}}{\partial\theta}-\rho A(\ddot{u}_\theta+2\dot{u}_3\Omega-u_\theta\Omega^2)=-q'_\theta \tag{16.4.1}$$

$$\frac{1}{a^2}\frac{\partial^2 M_{\theta\theta}}{\partial\theta^2}-\frac{N_{\theta\theta}}{a}-\rho A(\ddot{u}_3-2\dot{u}_\theta\Omega-(a+u_3)\Omega^2)=-q'_3 \tag{16.4.2}$$

We solve Eq. (16.4.2) for $N_{\theta\theta}$:

$$N_{\theta\theta}=\frac{1}{a}\frac{\partial^2 M_{\theta\theta}}{\partial\theta^2}-\rho A a(\ddot{u}_3-2\dot{u}_\theta\Omega-(a+u_3)\Omega^2)+q'_3 a \tag{16.4.3}$$

and substitute it in Eq. (16.4.1):

$$\frac{1}{a^2}\frac{\partial^3 M_{\theta\theta}}{\partial\theta^3}+\frac{1}{a^2}\frac{\partial M_{\theta\theta}}{\partial\theta}-\rho A(\ddot{u}_\theta+2\dot{u}_3\Omega-u_\theta\Omega^2)$$
$$-\rho A\frac{\partial}{\partial\theta}(\ddot{u}_3-2\dot{u}_\theta\Omega-(a+u_3)\Omega^2)=-q'_\theta-\frac{\partial q'_3}{\partial\theta} \quad (16.4.4)$$

Applying the inextensional assumption,

$$\frac{\partial u_\theta}{\partial\theta}=-u_3 \quad (16.4.5)$$

to the expression for the moment resultant [Eq. (4.1.17)],

$$M_{\theta\theta}=\frac{EI}{a^2}\left(\frac{\partial u_\theta}{\partial\theta}-\frac{\partial^2 u_3}{\partial\theta^2}\right) \quad (16.4.6)$$

gives

$$M_{\theta\theta}=-\frac{EI}{a^2}\left(u_3+\frac{\partial^2 u_3}{\partial\theta^2}\right) \quad (16.4.7)$$

Substituting Eqs. (16.4.5) and (16.4.7) in Eq. (16.4.4) results in ($p=I/Aa^2$; $\omega_0^2=E/\rho a^2$)

$$\frac{\partial^6 u_3}{\partial\theta^6}+2\frac{\partial^4 u_3}{\partial\theta^4}+\frac{\partial^2 u_3}{\partial\theta^2}+\frac{1}{p\omega_0^2}\left[\frac{\partial^4 u_3}{\partial\theta^2\partial t^2}-\frac{\partial^2 u_3}{\partial t^2}+4\Omega\frac{\partial^2 u_3}{\partial\theta\partial t}+\Omega^2\left(u_3-\frac{\partial^2 u_3}{\partial\theta^2}\right)\right]$$
$$=\frac{a^4}{EI}\frac{\partial}{\partial\theta}\left(q'_\theta+\frac{\partial q'_3}{\partial\theta}\right) \quad (16.4.8)$$

For the closed ring, we may assume a solution of the form

$$u_3(\theta,t)=A_3 e^{j(n\theta+\omega t)} \quad (16.4.9)$$

for the eigenvalue solution. Equation (16.4.8), with $q'_\theta=q'_3=0$, becomes

$$\omega^2(1+n^2)-4\Omega n\omega+\Omega^2(1+n^2)-p\omega_0^2(n^2-1)^2 n^2=0 \quad (16.4.10)$$

Solving for ω gives

$$\omega_{n1,2}=\frac{2\Omega n}{n^2+1}\pm\sqrt{\left(\frac{2\Omega n}{n^2+1}\right)^2+p\omega_0^2[(n^2-1)^2 n^2/(n^2+1)]-\Omega^2} \quad (16.4.11)$$

Note that when $\Omega=0$, this solution reduces to the result given by Eq. (6.15.13). Note that this solution is not valid for $n=0$, because of the inextensional approximation.

While Eq. (16.4.9) defines the natural mode, it is instructive to use the absolute coordinate

$$\Psi=\Omega t+\theta \quad (16.4.12)$$

instead of θ. This gives

$$u_{3i}(\Psi, t) = A_{3i} e^{j[n\Psi + (\omega_{ni} - n\Omega)t]} \tag{16.4.13}$$

The motion becomes stationary if

$$\Omega = \frac{\omega_{ni}}{n} \tag{16.4.14}$$

This means that if the relationship is satisfied, the mode does not rotate with the rotating ring but appears as a stationary distortion of the ring to an observer who is not rotating with the ring. For modes where $\Omega \neq \omega_{ni}/n$, we may ask at what speeds the mode antinodes rotate (Bryan, 1880). For this purpose, we take the real part of Eq. (16.4.13) and set it to its maximum possible value, unity.

$$\cos[n\Psi_{\max} + (\omega_{ni} - n\Omega)t] = 1 \tag{16.4.15}$$

or

$$n\Psi_{\max} + (\omega_{ni} - n\Omega)t = 0, 2\pi, 4\pi, \ldots, = p\pi \tag{16.4.16}$$

This gives

$$\Psi_{\max} = \frac{p\pi - (\omega_{ni} - n\Omega)t}{n} \tag{16.4.17}$$

or

$$\dot{\Psi}_{\max} = \Omega - \frac{\omega_{ni}}{n} \tag{16.4.18}$$

Thus we see that the antinode lags behind the rotational speed Ω if

$$\Omega > \frac{\omega_{ni}}{n} \tag{16.4.19}$$

For modes where

$$\Omega < \frac{\omega_{ni}}{n} \tag{16.4.20}$$

the mode antinodes rotate in the direction opposite that of the rotational speed. For the case for which Eq. (16.4.14) is satisfied, the rotational speed of the antinode is O, $\Psi_{\max} = 0$; that is, the mode is stationary. Note that the stationary-mode condition corresponds to the resonance condition for a traveling load on a circular cylindrical shell in circumferential direction, as given by Eq. (9.7.13). A rotating circular string can be found in Stutts and Soedel (1992).

16.5. CYLINDRICAL SHELL ROTATING WITH CONSTANT SPIN ABOUT ITS AXIS

The case of a cylindrical shell rotating with constant spin about its axis is shown in Fig. 5. The triad $\bar{e}_1, \bar{e}_2, \bar{e}_3$ of each element rotates with

$$\bar{\omega} = -\Omega\bar{e}_1 \tag{16.5.1}$$

Thus $\omega_1 = -\Omega, \omega_2 = \omega_3 = 0$. The cross product Eq. (16.1.7) therefore reduces to

$$\bar{\omega} \times \bar{r}_{p/o'} = -\omega_1 u_3 \bar{e}_2 + \omega_1 u_2 \bar{e}_3 = u_3 \Omega \bar{e}_2 - u_2 \Omega \bar{e}_3 \tag{16.5.2}$$

The velocity of the origin O' is, everywhere on the shell,

$$\bar{v}_{o'} = a\Omega\bar{e}_2 \tag{16.5.3}$$

Thus we have

$$\bar{v} = \dot{u}_1 \bar{e}_1 + (\dot{u}_2 + a\Omega + u_3\Omega)\bar{e}_2 + (\dot{u}_3 - u_2\Omega)\bar{e}_3 \tag{16.5.4}$$

The total kinetic energy of the cylindrical shell is, therefore,

$$K = \frac{\rho h}{2} \int_{\alpha_1} \int_{\alpha_2} [\dot{u}_1^2 + (\dot{u}_2 + a\Omega + u_3\Omega)^2 + (\dot{u}_3 - u_2\Omega)^2] A_1 A_2 \, \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{16.5.5}$$

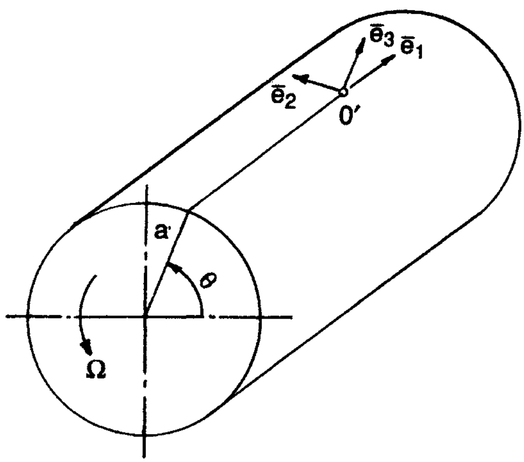

Fig. 5 Circular cylindrical shell rotating about its axis of revolution.

Next, we formulate

$$\int_{t_0}^{t_1} \delta K \,\mathrm{d}t = \frac{\rho h}{2} \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} [2\dot{u}_1 \delta\dot{u}_1 + 2(\dot{u}_2 + \alpha\Omega + u_3\Omega)(\delta\dot{u}_2 + \Omega\delta u_3) + 2(\dot{u}_3 - u_2\Omega)(\delta\dot{u}_3 - \Omega\delta u_2)] A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2 \,\mathrm{d}t \quad (16.5.6)$$

Integrating by parts gives

$$\int_{t_0}^{t_1} \delta K \,\mathrm{d}t = \rho h \int_{t_0}^{t_1} \int_{\alpha_1} \int_{\alpha_2} [(-\ddot{u}_1)\delta u_1 + (-\ddot{u}_2 - 2\dot{u}_3\Omega + \Omega^2 u_2)\delta u_2 + (-\ddot{u}_3 + 2\Omega\dot{u}_2 + \Omega^2 u_3 + a\Omega^2)\delta u_3] A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2 \,\mathrm{d}t \quad (16.5.7)$$

Thus we replace $\rho h\ddot{u}_2$ by $\rho h(\ddot{u}_2 + 2\Omega\dot{u}_3 - \Omega^2 u_2)$ and $\rho h\ddot{u}_3$ by $\rho h(\ddot{u}_3 - 2\dot{u}_2\Omega - u_3\Omega^2 - a\Omega^2)$ in the equation for the cylindrical shell. Recognizing that the term $\rho h a\Omega^2$ causes a static deflection of the shell about which the vibration takes place, this effect can be subtracted and the dynamic equations left are

$$L_1\{u_1, u_2, u_3\} + \rho h\ddot{u}_1 = q_1 \quad (16.5.8)$$

$$L_2\{u_1, u_2, u_3\} + \rho h(\ddot{u}_2 + 2\Omega\dot{u}_3 - \Omega^2 u_2) = q_2 \quad (16.5.9)$$

$$L_3\{u_1, u_2, u_3\} + \rho h(\ddot{u}_3 - 2\Omega\dot{u}_2 - \Omega^2 u_3) = q_3 \quad (16.5.10)$$

The operators L_1, L_2 and L_3 for the circular cylindrical shell can be obtained from Sec. 3.3. The solutions for free and forced vibrations proceed as in Sec. 16.3 for the rotating ring. As a matter of fact, the basic behavior is very similar except that there will be six distinct natural frequencies. The forced solution is developed in Huang and Soedel (1988a).

16.6. GENERAL ROTATIONS OF ELASTIC SYSTEMS

The cases discussed in the previous sections had in common that the rotational axes coincided or were always parallel to the axes of axisymmetry of the structures. While these are arrangements of much practical interest, it is of course possible that shell structures have much more general motion (Fig. 6). The approach in such a case is similar. The kinetic energy has to be formulated:

$$K = \frac{\rho h}{2} \int_{\alpha_1} \int_{\alpha_2} \bar{v} \cdot \bar{v} A_1 A_2 \,\mathrm{d}\alpha_1 \,\mathrm{d}\alpha_2 \quad (16.6.1)$$

where, as in the previous cases,

$$\bar{v} = \bar{v}_{O'} + (\bar{v}_{P'/O'})_{\mathrm{rel}} + \bar{\omega} \times \bar{r}_{P'/O'} \quad (16.6.2)$$

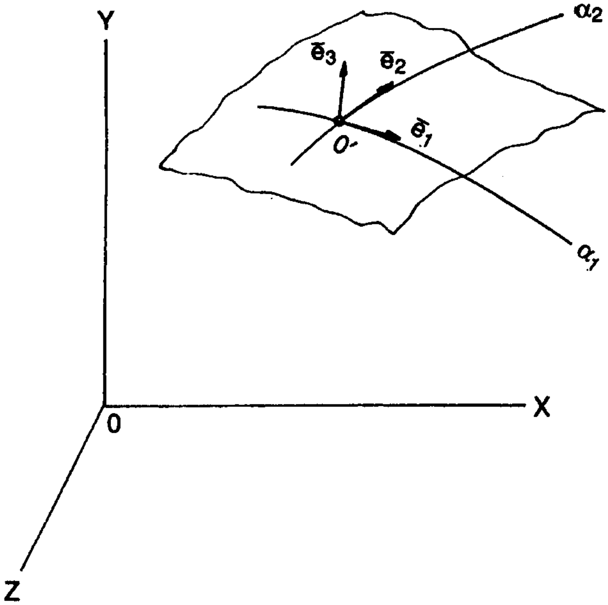

FIG. 6 Coordinate definitions for a general rotating shell.

and where, in general,

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \omega_3 \bar{e}_3 \tag{16.6.3}$$

Equation (16.6.2) has to be specialized to the situation at hand. There is little advantage to continuing beyond this point in a general way. For cases where the spin velocities are not constant, care has to be taken to do the variational operations properly.

16.7. SHELLS OF REVOLUTION WITH CONSTANT SPIN ABOUT THEIR AXES OF REVOLUTION

While rings and circular cylindrical shells belong to the category of shells of revolution and were discussed already, it is of interest to develop the equations of motion for general shells of revolution that spin about their axis with constant speed Ω because this problem is relatively common in engineering.

In Fig. 7, the point of interest on the undeformed shell serves as the origin of a moving, right handed coordinate system defined by the unit vectors $\bar{e}_1, \bar{e}_2$ and $\bar{e}_3$ ($\bar{e}_2$ is into the paper). The particle velocity is, therefore,

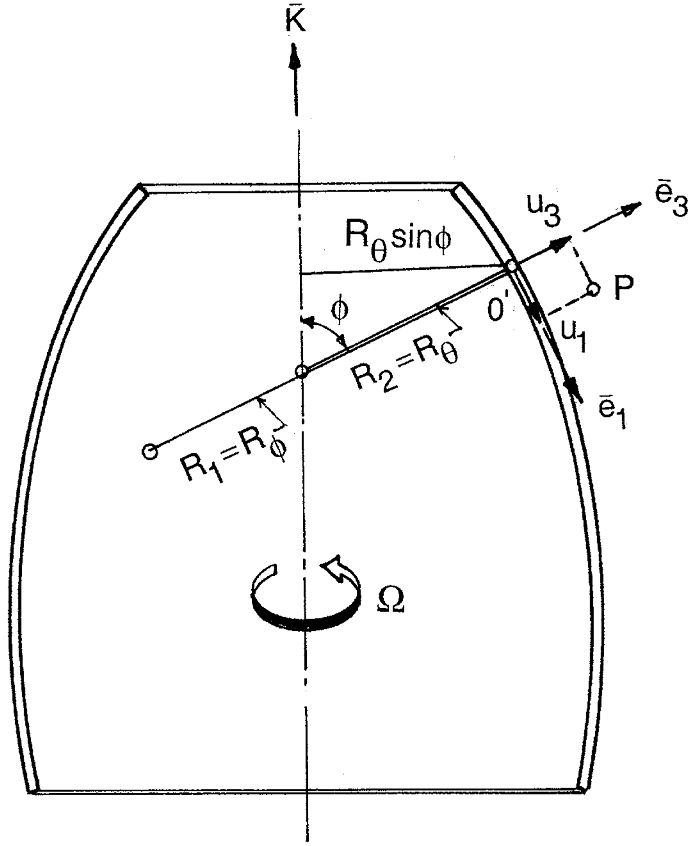

Fig. 7 Coordinate definitions for a general rotating shell.

$$\bar{v} = \bar{v}_{O'} + \left(\bar{v}_{P/O'}\right)_{\text{rel}} + \bar{\omega} \times \bar{r}_{P/O'} \tag{16.7.1}$$

where

$$\bar{v}_{O'} = \Omega R_\theta \sin\phi \bar{e}_2 \tag{16.7.2}$$

$$\bar{\omega} = \Omega \bar{K} = \Omega \cos\phi \bar{e}_3 - \Omega \sin\phi \bar{e}_1 \tag{16.7.3}$$

$$\left(\bar{v}_{P/0'}\right)_{\text{rel}} = \dot{u}_1 \bar{e}_1 + \dot{u}_2 \bar{e}_2 + \dot{u}_3 \bar{e}_3 \tag{16.7.4}$$

$$\bar{r}_{P/O'} = u\bar{e}_1 + u_2 \bar{e}_2 + u_3 \bar{e}_3 \tag{16.7.5}$$

Substituting Eqs. (16.7.2)–(16.7.5) into Eq. (16.7.1) gives

$$\begin{aligned}\bar{v} = {} & (\dot{u}_1 - u_2 \Omega \cos\phi)\bar{e}_1 + (\Omega R_\theta \sin\phi + \dot{u}_2 + u_1 \Omega \cos\phi + u_3 \Omega \sin\phi)\bar{e}_2 \\ & + (\dot{u}_3 - u_2 \Omega \sin\phi)\bar{e}_3\end{aligned} \tag{16.7.6}$$

The kinetic energy is, therefore,

$$K = \frac{\rho h}{2}\int_{\alpha_1}\int_{\alpha 2}(\bar{v}\cdot\bar{v})A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2$$
$$= \frac{\rho h}{2}\int_{\alpha_1}\int_{\alpha_2}[(\dot{u}_1 - u_2\Omega\cos\phi)^2 + (\Omega R_\theta\sin\phi + \dot{u}_2 + u_1\Omega\cos\phi + u_3R_\theta\sin\phi)^2$$
$$+ (\dot{u}_3 - u_2\Omega\sin\phi)^2]A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \qquad (16.7.7)$$

The integral of the variation of the kinetic energy becomes, therefore,

$$\int_{t_1}^{t_2}\delta K\mathrm{d}t = \rho h\int_{t_o}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\{[-\ddot{u}_1 + 2\dot{u}_2\Omega\cos\phi + \Omega^2(R_\theta + u_3)\sin\phi\cos\phi$$
$$+u_1\Omega^2\cos^2\phi]\delta u_1 + [-\ddot{u}_2 - 2\dot{u}_1\Omega\cos\phi - 2\dot{u}_3\Omega\sin\phi + u_2\Omega^2]\delta u_2$$
$$+[-\ddot{u}_3 + 2\dot{u}_2\Omega\sin\phi + \Omega^2(R_\theta + u_3)\sin^2\phi + u_1\Omega^2\sin\phi\cos\phi]\delta u_3)^2\}$$
$$\times A_1A_2\mathrm{d}\alpha_1\mathrm{d}\alpha_2 \qquad (16.7.8)$$

Proceeding as in Chapter 2, we finally obtain the equations of motion for shells of revolution that spin about their axes with a constant angular velocity Ω:

$$L_1\{u_1, u_2, u_3\}\ \rho h[\ddot{u}_1 - 2\dot{u}_2\Omega\cos\phi - (R_\theta + u_3)\Omega^2\sin\phi\cos\phi$$
$$-u_1\Omega^2\cos^2\phi] = -q_1 \qquad (16.7.9)$$

$$L_2\{u_1, u_2, u_3\}\ -\rho h[\ddot{u}_2 + 2\dot{u}_1\Omega\cos\phi + 2\dot{u}_3\sin\phi - u_2\Omega^2]$$
$$= -q_2 \qquad (16.7.10)$$

$$L_3\{u_1, u_2, u_3\}\ -\rho h[\ddot{u}_3 - 2\dot{u}_2\Omega\sin\phi - (R_\theta + u_3)\Omega^2\sin^2\phi$$
$$-u_1\Omega^2\sin\phi\cos\phi] = -q_3 \qquad (16.7.11)$$

where the $L_1\{u_1, u_2, u_3\}, L_2\{u_1, u_2, u_3\}$, and $L_3\{u_1, u_2, u_3\}$ terms are given by Eqs. (8.1.3)–(8.1.5) with $A_1 = R_\phi, A_2 = R_\theta\sin\phi, R_1 = R_\phi$ and $R_2 = R_\theta$. Note that the terms $R_\theta\Omega^2\sin\phi\cos\phi$ in Eq. (16.7.9) and $R_\theta\Omega^2\sin^2\phi$ in Eq. (16.7.11) cause a static deflection because of the centrifugal effect and can be subtracted from the equations. What is left describes the vibratory deflections about static equilibrium.

For example, for the a circular, cyclindrical shell spinning about its axis of revolution with a constant angular velocity Ω, we have $\phi = \frac{\pi}{2}$ = constant $(\sin\phi = 1, \cos\phi = 0)$ and $R_\theta = a$, and Eqs. (16.7.9)–(16.7.11) reduce to Eqs. (16.5.8)–(16.5.10)

16.8. SPINNING DISK

For a disk spinning about its axis with a constant angular velocity Ω, we let $\phi=0$ in Eqs. (16.7.9) and (16.7.10), which describe the inplane vibrations of the disk. They become

$$L_1\{u_r, u_\theta\} - \rho h(\ddot{u}_r - 2\dot{u}_\theta\Omega - u_r\Omega^2) = -q_1 \tag{16.8.1}$$

$$L_2\{u_r, u_\theta\} - \rho h(\ddot{u}_\theta - 2\dot{u}_r\Omega - u_\theta\Omega^2) = -q_2 \tag{16.8.2}$$

and Eq. (16.7.11), which describes the transverse vibration of the disk, becomes the standard plate equation (4.4.19).

$$L_3\{u_3\} - \rho h\ddot{u}_3 = -q_3 \tag{16.8.3}$$

The operators L_1 and L_2 are, because $\alpha_1=r$, $\alpha_2=\theta$, $A_1=1$, $A_2=r$, $R_1=R_2=\infty$,

$$L_1\{u_r, u_\theta\} = \frac{1}{r}\left[\frac{\partial(rN_{rr})}{\partial r} + \frac{\partial N_{r\theta}}{\partial\theta} - N_{\theta\theta}\right] \tag{16.8.4}$$

$$L_2\{u_r, u_\theta\} = \frac{1}{r}\left[\frac{\partial(rN_{r\theta})}{\partial r} + \frac{\partial N_{\theta\theta}}{\partial\theta} + N_{r\theta}\right] \tag{16.8.5}$$

with N_{rr}, $N_{\theta\theta}$ and $N_{r\theta}$ given by Eqs. (2.5.9), (2.5.11) and (2.5.12).

The operator L_3 is

$$L_3\{u_3\} = -D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r}\frac{\partial u_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_3}{\partial\theta^2}\right) \tag{16.8.6}$$

The reason that Eq. (16.8.3) is independent of Ω is (unless Ω is so large that centrifugally caused tensions N_{rr}^r and $N_{\theta\theta}^r$ cannot be neglected; see Chapter 11) that transverse velocities $\dot{u}_3$ do not contribute to a change in radius, which means there is no Coriola's acceleration due to $\dot{u}_3$. The in-plane vibrations, on the other hand, create Coriola's accelerations.

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17

Thermal Effects

Of interest are two basic effects. First, a static, or quasistatic temperature field will cause a static or quasi-static initial stress field. For example, a local hot spot may introduce static compressive stresses in a shell that will lower natural frequencies. Second, time-varying temperatures act as vibration excitation mechanisms. A sudden heating or cooling (thermal shock) will produce an effect similar to a step load. Feedback between motion and heat transfer my cause oscillations of relatively high frequency, as in the classical case of Chladni figure experimentation, where glass plates are excited by an application of dry ice. Additional information can be obtained, for example, from Johns (1965), Boley and Weiner (1960), Jadeja and Loo (1974), Huang and Tauchert (1992).

17.1. STRESS RESULTANTS

Temperature effects are introduced through the stress–strain relationships. Since thermal expansion produces a strain "growth," we have

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \mu\sigma_{22}) + \alpha T \tag{17.1.1}$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \mu\sigma_{11}) + \alpha T \tag{17.1.2}$$

$$\varepsilon_{12} = \frac{1}{G}\sigma_{12} \tag{17.1.3}$$

where α is the coefficient of expansion (m/m°C). For example, for steel it is approximately 11.5×10^{-6} and for aluminum, it is approximately 24×10^{-6}. The temperature distribution

$$T = T(\alpha_1, \alpha_2, \alpha_3, t) \tag{17.1.4}$$

with respect to a reference temperature at which the system is free of temperature effects is assumed to be known at this point. Solving for the stresses, we obtain

$$\sigma_{11} = \frac{E}{1-\mu^2}(\varepsilon_{11} + \mu\varepsilon_{22}) - \frac{E\alpha T}{1-\mu} \tag{17.1.5}$$

$$\sigma_{22} = \frac{E}{1-\mu^2}(\varepsilon_{22} + \mu\varepsilon_{11}) - \frac{E\alpha T}{1-\mu} \tag{17.1.6}$$

$$\sigma_{12} = G\varepsilon_{12} \tag{17.1.7}$$

The strain–displacement relationships are the same. Also, we still may assume a linear variation of U_1 and U_2 displacements across the thickness. Thus

$$\sigma_{11} = \frac{E}{1-\mu^2}[\varepsilon_{11}^0 + \mu\varepsilon_{22}^0 + \alpha_3(k_{11} + \mu k_{22})] - \frac{E\alpha T}{1-\mu} \tag{17.1.8}$$

$$\sigma_{22} = \frac{E}{1-\mu^2}[\varepsilon_{22}^0 + \mu\varepsilon_{11}^0 + \alpha_3(k_{22} + \mu k_{11})] - \frac{E\alpha T}{1-\mu} \tag{17.1.9}$$

$$\sigma_{12} = G(\varepsilon_{12}^0 + \alpha_3 k_{12}) \tag{17.1.10}$$

Using a tilde notation for the force and moment resultants, which now include temperature effects, integrating over the thickness gives

$$\widetilde{N}_{11} = N_{11} - \frac{N_T}{1-\mu} \tag{17.1.11}$$

$$\widetilde{N}_{22} = N_{22} - \frac{N_T}{1-\mu} \tag{17.1.12}$$

$$\widetilde{N}_{12} = \widetilde{N}_{21} = N_{12} = N_{21} \tag{17.1.13}$$

$$\widetilde{M}_{11} = M_{11} - \frac{M_T}{1-\mu} \tag{17.1.14}$$

$$\widetilde{M}_{22} = M_{22} - \frac{M_T}{1-\mu} \tag{17.1.15}$$

$$\widetilde{M}_{12} = \widetilde{M}_{21} = M_{12} = M_{21} \tag{17.1.16}$$

where

$$N_{11} = K(\varepsilon_{11}^0 + \mu\varepsilon_{22}^0) \tag{17.1.17}$$

$$N_{22} = K(\varepsilon_{22}^0 + \mu\varepsilon_{11}^0) \tag{17.1.18}$$

$$N_{12} = N_{21} = K\frac{1-\mu}{2}\varepsilon_{12}^0 \tag{17.1.19}$$

$$M_{11} = D(k_{11} + \mu k_{22}) \tag{17.1.20}$$

$$M_{22} = D(k_{22} + \mu k_{11}) \tag{17.1.21}$$

$$M_{12} = M_{21} = D\frac{1-\mu}{2}k_{12} \tag{17.1.22}$$

and where

$$N_T = E\alpha\int_{-h/2}^{h/2} T\,\mathrm{d}\alpha_3 \tag{17.1.23}$$

$$M_T = E\alpha\int_{-h/2}^{h/2} T\alpha_3\,\mathrm{d}\alpha_3 \tag{17.1.24}$$

For the very common assumption of linear temperature distribution, we may set

$$T(\alpha_1,\alpha_2,\alpha_3,t) = T_0(\alpha_1,\alpha_2,t) + \alpha_3\tau(\alpha_1,\alpha_2,t) \tag{17.1.25}$$

where T_0 is the temperature distribution at the neutral surface (°C) and τ is the rate of temperature change normal to the neutral surface (°C/m). Thus at $\alpha_3 = h/2, T = T_0 + h\tau/2$, and at $\alpha_3 = -h/2, T = T_0 - h\tau/2$. In this case

$$N_T = E\alpha h T_0 \tag{17.1.26}$$

$$M_T = \frac{E\alpha h^3\tau}{12} \tag{17.1.27}$$

17.2. EQUATIONS OF MOTION

Here Love's equations of Chapter 2 are extended to include thermal forcing. The strain energy is

$$U = \int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3} F\,\mathrm{d}V \tag{17.2.1}$$

where

$$F = \frac{1}{2}[\sigma_{11}(\varepsilon_{11} - \alpha T) + \sigma_{22}(\varepsilon_{22} - \alpha T) + \sigma_{12}\varepsilon_{12} + \sigma_{13}\varepsilon_{13} + \sigma_{23}\varepsilon_{23}] \tag{17.2.2}$$

and

$$dV = A_1 A_2 \left(1 + \frac{\alpha_3}{R_1}\right)\left(1 + \frac{\alpha_3}{R_2}\right) d\alpha_1 d\alpha_2 d\alpha_3 \tag{17.2.3}$$

The expression for the variation of strain energy is

$$\delta U = \int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3} \delta F \, dV \tag{17.2.4}$$

where

$$\delta F = \frac{\partial F}{\partial \varepsilon_{11}}\delta\varepsilon_{11} + \frac{\partial F}{\partial \varepsilon_{22}}\delta\varepsilon_{22} + \frac{\partial F}{\partial \varepsilon_{12}}\delta\varepsilon_{12} + \frac{\partial F}{\partial \varepsilon_{13}}\delta\varepsilon_{13} + \frac{\partial F}{\partial \varepsilon_{23}}\delta\varepsilon_{23} \tag{17.2.5}$$

Taking the first term, we obtain

$$\frac{\partial F}{\partial \varepsilon_{11}} = \frac{1}{2}\left[\frac{\partial \sigma_{11}}{\partial \varepsilon_{11}}(\varepsilon_{11} - \alpha T) + \sigma_{11} + \frac{\partial \sigma_{22}}{\partial \varepsilon_{11}}(\varepsilon_{22} - \alpha T)\right] \tag{17.2.6}$$

but since

$$\frac{\partial \sigma_{11}}{\partial \varepsilon_{11}} = \frac{E}{1-\mu^2} \tag{17.2.7}$$

$$\frac{\partial \sigma_{22}}{\partial \varepsilon_{11}} = \frac{\mu E}{1-\mu^2} \tag{17.2.8}$$

we obtain

$$\frac{\partial F}{\partial \varepsilon_{11}} = \sigma_{11} \tag{17.2.9}$$

Thus, in general

$$\delta F = \sigma_{11}\delta\varepsilon_{11} + \sigma_{22}\delta\varepsilon_{22} + \sigma_{12}\delta\varepsilon_{12} + \sigma_{13}\delta\varepsilon_{13} + \sigma_{23}\delta\varepsilon_{23} \tag{17.2.10}$$

From here on, the derivation is identical to the derivation of Love's equations, except that the N_{ij} and M_{ij} resultants are now replaced by the $\widetilde{N}_{ij}$ and $\widetilde{M}_{ij}$ resultants, as they are defined by Eqs. (17.1.11)–(17.1.16). This gives

$$-\frac{\partial(N_{11}A_2)}{\partial\alpha_1} - \frac{\partial(N_{21}A_1)}{\partial\alpha_2} - N_{12}\frac{\partial A_1}{\partial\alpha_2} + N_{22}\frac{\partial A_2}{\partial\alpha_1} - A_1A_2\frac{Q_{13}}{R_1} + A_1A_2\rho h\ddot{u}_1$$
$$= A_1A_2q_1 - \frac{A_2}{1-\mu}\frac{\partial N_T}{\partial\alpha_1} \tag{17.2.11}$$

$$-\frac{\partial(N_{12}A_2)}{\partial\alpha_1} - \frac{\partial(N_{22}A_1)}{\partial\alpha_2} - N_{21}\frac{\partial A_2}{\partial\alpha_1} + N_{11}\frac{\partial A_1}{\partial\alpha_2} - A_1A_2\frac{Q_{23}}{R_2} + A_1A_2\rho h\ddot{u}_2$$
$$= A_1A_2q_1 - \frac{A_1}{1-\mu}\frac{\partial N_T}{\partial\alpha_2} \tag{17.2.12}$$

$$-\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}+A_1A_2\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)+A_1A_2\rho h\ddot{u}_3$$
$$=A_1A_2q_3+\frac{N_T}{1-\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)A_1A_2 \quad (17.2.13)$$

where Q_{13} and Q_{23} are defined by

$$\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(M_{21}A_1)}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}-A_1A_2Q_{13}=\frac{A_2}{1-\mu}\frac{\partial M_T}{\partial\alpha_1} \quad (17.2.14)$$

$$\frac{\partial(M_{12}A_2)}{\partial\alpha_1}+\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}-A_1A_2Q_{23}=\frac{A_1}{1-\mu}\frac{\partial M_T}{\partial\alpha_2} \quad (17.2.15)$$

The necessary boundary conditions have not changed in number or type.

Once the N_{ij} and M_{ij} values are known, stresses may be obtained from

$$\sigma_{11}=\frac{N_{11}}{h}+\frac{12}{h^3}\alpha_3M_{11}-\frac{E\alpha T}{1-\mu} \quad (17.2.16)$$

$$\sigma_{22}=\frac{N_{22}}{h}+\frac{12}{h^3}\alpha_3M_{22}-\frac{E\alpha T}{1-\mu} \quad (17.2.17)$$

$$\sigma_{12}=\frac{N_{12}}{h}+\frac{12}{h^3}\alpha_3M_{12} \quad (17.2.18)$$

Note that Eqs. (17.2.11)–(17.2.15) can be combined. We obtain, from Eqs. (17.2.13) and (17.2.14),

$$-\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}=\frac{1}{1-\mu}\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{A_1}\frac{\partial M_T}{\partial\alpha_1}\right)-\frac{1}{A_1}\left[\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(M_{21}A_1)}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}\right] \quad (17.2.19)$$

$$-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}=\frac{1}{1-\mu}\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{A_2}\frac{\partial M_T}{\partial\alpha_2}\right)-\frac{1}{A_2}\left[\frac{\partial(M_{12}A_2)}{\partial\alpha_1}+\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}\right] \quad (17.2.20)$$

Substituting this into Eq. (17.2.13), for example, gives

$$-\frac{1}{A_1}\left[\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(M_{21}A_1)}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}\right]$$
$$-\frac{1}{A_2}\left[\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+\frac{\partial(M_{12}A_2)}{\partial\alpha_1}+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}\right]$$
$$+A_1A_2\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)+A_1A_2\rho h\ddot{u}_3$$
$$=A_1A_2q_3+\frac{N_T}{1-\mu}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)A_1A_2-\frac{A_1A_2}{1-\mu}\nabla^2M_T \tag{17.2.21}$$

17.3. PLATE

In this case, the radii of curvature are 0 and only transverse motion is of interest. The equation of motion becomes, from Eq. (17.2.21),

$$D\nabla^4u_3+\rho h\bar{u}_3=A_1A_2q_3-\frac{A_1A_2}{1-\mu}\nabla^2M_T \tag{17.3.1}$$

where

$$\nabla^2=\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\left[\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\right]+\frac{\partial}{\partial\alpha_2}\left[\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\right]\right\} \tag{17.3.2}$$

Thus we see that N_T effects do not enter the plate directly. The exception is if the N_T effects are so large that they produce large in-plane stresses in the plate. These stresses have to be treated like initial stresses, as discussed in Chapter 11.

17.4. ARCH, RING, BEAM, AND ROD

In this case, $A_1=1, A_2=1, \mathrm{d}\alpha_1=\mathrm{d}s, \mathrm{d}\alpha_2=\mathrm{d}y, R_1=R_s, 1/R_2=0$, and $\partial(\cdot)/\partial\alpha_2=0$. We obtain, from Eqs. (17.2.11) and (17.2.13),

$$-\frac{\partial N_{ss}}{\partial s}-\frac{Q_{s3}}{R_s}+\rho h\ddot{u}_s=q_s-\frac{1}{1-\mu}\frac{\partial N_T}{\partial s} \tag{17.4.1}$$

$$-\frac{\partial Q_{s3}}{\partial s}+\frac{N_{ss}}{R_s}+\rho h\ddot{u}_3=q_3+\frac{N_T}{(1-\mu)R_s} \tag{17.4.2}$$

where

$$Q_{s3}=\frac{\partial M_{ss}}{\partial s}-\frac{1}{1-\mu}\frac{\partial M_T}{\partial s} \tag{17.4.3}$$

Introducing the strain–displacement relationships, multiplying through by the width b, and eliminating the Poisson effect (see Chapter 4), we obtain

$$-\frac{EI}{R_s}\left[\frac{\partial^2}{\partial s^2}\left(\frac{u_s}{R_s}\right)-\frac{\partial^3 u_3}{\partial s^3}\right]-EA\left[\frac{\partial^2 u_3}{\partial s^2}+\frac{\partial}{\partial s}\left(\frac{u_s}{R_s}\right)\right]$$
$$+\rho A\ddot{u}_s=q'_s-\frac{\partial N'_T}{\partial s}-\frac{1}{R_s}\frac{\partial M_T}{\partial s} \qquad (17.4.4)$$

$$-EI\left[\frac{\partial^3}{\partial s^3}\left(\frac{u_s}{R_s}\right)-\frac{\partial^4 u_3}{\partial s^4}\right]+\frac{EA}{R_s}\left[\frac{\partial u_s}{\partial s}+\frac{u_3}{R_s}\right]$$
$$+\rho A\ddot{u}_3=q'_3+\frac{N'_T}{R_s}-\frac{\partial^2 M'_T}{\partial s^2} \qquad (17.4.5)$$

where $A=bh, I=bh^3/12, q'_3,=bq_3, q'_s=bq_s, N'_T=bN_T$, and $M'_T=bM_T$.

For the ring, $R_s=a$. Furthermore, we replace s by $a\theta$.

$$-\frac{EI}{a^4}\left(\frac{\partial^2 u_\theta}{\partial\theta^2}-\frac{\partial^3 u_3}{\partial\theta^3}\right)-\frac{EA}{a^2}\left(\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{\partial u_3}{\partial\theta}\right)+\rho A\ddot{u}_\theta$$
$$=q'_\theta-\frac{1}{a}\frac{\partial N'_T}{\partial\theta}-\frac{1}{a^2}\frac{\partial M'_T}{\partial\theta} \qquad (17.4.6)$$

$$-\frac{EI}{a^4}\left(\frac{\partial^3 u_\theta}{\partial\theta^3}-\frac{\partial^4 u_3}{\partial\theta^4}\right)+\frac{EA}{a^2}\left(\frac{\partial u_\theta}{\partial\theta}+u_3\right)+\rho A\ddot{u}_3$$
$$=q'_3+\frac{N'_T}{a}-\frac{1}{a^2}\frac{\partial^2 M'_T}{\partial\theta^2} \qquad (17.4.7)$$

For the beam and rod, let $s=x$ and $1/R_s=0$ in Eqs. (17.4.4) and (17.4.5). We obtain

$$-EA\frac{\partial^2 u_x}{\partial x^2}+\rho A\ddot{u}_x=q'_x-\frac{\partial N'_T}{\partial x} \qquad (17.4.8)$$

$$EI\frac{\partial^4 u_3}{\partial x^4}+qA\ddot{u}_3=q'_3-\frac{\partial^2 M'_T}{\partial x^2} \qquad (17.4.9)$$

17.5. LIMITATIONS

The developed equations are valid only if the thermal stresses do not have a large static component which would introduce "initial stresses" of a type that have to be treated by the equations in Chapter 11. The effect of large static thermal stress components is to raise or lower the natural frequencies of the structures. Second, dynamic components (N_T for "flat" structures) have to be small enough so that no dynamic buckling is created.

The analytical treatment of problems of thermal excitation is complicated by the need for a temperature model. A heat transfer solution has to be obtained first. Once the temperature distribution is known, the forced solutions follow the same modal series approach as discussed in Chapter 8.

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18

Elastic Foundations

A shell resting on an elastic foundation can in certain cases be viewed as resting on distributed elastic springs that have spring rates k_1, k_2, and $k_3 (\mathrm{N/m^3})$ and act in α_1, α_2, and α_3 directions. If the foundation is a homogeneous material of uniform thickness $h_F (\mathrm{m})$ that is defined by a modulus of elasticity $E (\mathrm{N/m^2})$ and Poisson's ratio μ, a very approximate first-order estimate is $k_1 = k_2 = G/h_F$, and $k_3 = E/h_F$, where G is the shear modulus of the foundation. Usually, the spring rates are identified experimentally. It should be remembered that the concept of an elastic foundation is an approximation by itself. In cases where a high degree of accuracy is required, or where Young's modulus of the foundation approaches that of the shell, the foundation will have to be modeled as an elastic continuum (see Chapter 20).

The mass effect is taken into account by defining ρ'_F, which is the mass of the foundation per unit area, or in terms of the mass density $\rho_F (\mathrm{kg/m^3})$ and the thickness of the foundation h_F, it becomes $\rho'_F = \rho_F h_F$. From kinetic energy considerations, assuming no surges in the homogeneous foundation itself, one-third of its mass per unit area has to be added to the ρh of the shell. Defining viscous damping coefficients c_1, c_2, and $c_3 (\mathrm{Ns/m^3})$ rounds off the basic model.

18.1. EQUATIONS OF MOTION FOR SHELLS ON ELASTIC FOUNDATIONS

The elastic foundation is included in Love's or any other shell theory through the load terms q_1, q_2, and q_3 by redefining them as

$$q_1 = -k_1 u_1 - c_1 \dot{u}_1 - \tfrac{1}{3}\rho_F h_F \ddot{u}_1 + q_1^* \tag{18.1.1}$$

$$q_2 = -k_2 u_2 - c_2 \dot{u}_2 - \tfrac{1}{3}\rho_F h_F \ddot{u}_2 + q_2^* \tag{18.1.2}$$

$$q_3 = -k_3 u_3 - c_3 \dot{u}_3 - \tfrac{1}{3}\rho_F h_F \ddot{u}_3 + q_3^* \tag{18.1.3}$$

and dropping the $*$ superscript gives

$$L_1\{u_1, u_2, u_3\} - k_1 u_1 - (\lambda_1 + c_1)\dot{u}_1 - (\rho h + \tfrac{1}{3}\rho_F h_F)\ddot{u}_1 = -q_1 \tag{18.1.4}$$

$$L_2\{u_1, u_2, u_3\} - k_2 u_2 - (\lambda_2 + c_2)\dot{u}_2 - (\rho h + \tfrac{1}{3}\rho_F h_F)\ddot{u}_2 = -q_2 \tag{18.1.5}$$

$$L_3\{u_1, u_2, u_3\} - k_3 u_3 - (\lambda_3 + c_3)\dot{u}_3 - (\rho h + \tfrac{1}{3}\rho_F h_F)\ddot{u}_3 = -q_3 \tag{18.1.6}$$

Note that this formulation assumes that the forces from the elastic foundation act at the neutral plane of the shell. This is a fair assumption as far as the transverse motion u_3 is concerned but is strictly speaking in error for u_1 and u_2, which should be replaced by $u_1 + (h\partial u_3/\partial\alpha_1)/2A_1$ and $u_2 + (h\partial u_3/\partial\alpha_2)/2A_2$. The reason that this is usually not done is that this would create an illusion of accuracy that is not warranted by the elastic foundation concept as a whole. The operators L_1, L_2, and L_3 are defined by Eqs. (8.1.3)–(8.1.5).

18.2. NATURAL FREQUENCIES AND MODES

To solve

$$L_1\{u_1, u_2, u_3\} - k_1 u_1 - (\rho h + \tfrac{1}{3}\rho_F h_F)\ddot{u}_1 = 0 \tag{18.2.1}$$

$$L_2\{u_1, u_2, u_3\} - k_2 u_2 - (\rho h + \tfrac{1}{3}\rho_F h_F)\ddot{u}_2 = 0 \tag{18.2.2}$$

$$L_3\{u_1, u_2, u_3\} - k_3 u_3 - (\rho h + \tfrac{1}{3}\rho_F h_F)\ddot{u}_3 = 0 \tag{18.2.3}$$

for natural frequencies and modes, we set

$$u_i(\alpha_1, \alpha_2, t) = U_i(\alpha_1, \alpha_2)e^{j\omega t} \tag{18.2.4}$$

This gives

$$L_i\{U_1, U_2, U_3\} - k_i U_i + \omega^2(\rho h + \tfrac{1}{3}\rho_F h_F)U_i = 0 \tag{18.2.5}$$

and the solution proceeds by whatever method is appropriate for the problem.

For the special situation where $k_1 = k_2 = k_3 = k$, which is not an unreasonable approximation, if one wishes simply to explore the overall influence of an elastic foundation on a shell, the equations simplify to

$$L_1\{U_1, U_2, U_3\} + [\omega^2(\rho h + \tfrac{1}{3}\rho_F h_F) - k]U_1 = 0 \tag{18.2.6}$$

$$L_2\{U_1, U_2, U_3\} + [\omega^2(\rho h + \tfrac{1}{3}\rho_F h_F) - k]U_2 = 0 \tag{18.2.7}$$

$$L_3\{U_1, U_2, U_3\} + [\omega^2(\rho h + \tfrac{1}{3}\rho_F h_F) - k]U_3 = 0 \tag{18.2.8}$$

Comparing this with the standard form of a shell not on an elastic foundation, we recognize that for identical boundary conditions the following similitude condition holds:

$$\omega_F^2 = \frac{\omega_0^2 \rho h + k}{\rho h + \frac{1}{3}\rho_F h_F} \tag{18.2.9}$$

where ω_F are the natural frequencies of the shell on an elastic foundation and ω_0 are the natural frequencies of the same shell without the elastic foundation. Thus, as expected, the natural frequencies increase with k, with the larger percentual increase for the lower natural frequencies. The mass of the elastic foundation tends to lower all natural frequencies, but not enough for a typical foundation that it would override the stiffness effect.

18.3. PLATES ON ELASTIC FOUNDATIONS

The equation of motion for transversely vibrating uniform thickness plates on uniform elastic foundations is

$$D\nabla^4 u_3 + k_3 u_3 + (\lambda_3 + c_3)\dot{u}_3 + (\rho h + \tfrac{1}{3}\rho_F h_F)\ddot{u}_3 = q_3 \tag{18.3.1}$$

where

$$\nabla^2(\cdot) = \frac{1}{A_1 A_2}\left[\frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{A_1}\frac{\partial(\cdot)}{\partial \alpha_1}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{A_2}\frac{\partial(\cdot)}{\partial \alpha_2}\right)\right] \tag{18.3.2}$$

For natural frequencies, the similitude of Eq. (18.2.9) holds exactly:

$$\omega_F^2 = \frac{\omega_0^2 \rho h + k_3}{\rho h + \frac{1}{3}\rho_F h_F} \tag{18.3.3}$$

where the ω_0 are now the natural frequencies of the plate not on an elastic foundation and the ω_F are the natural frequencies of the plate on an elastic foundation.

18.4. RING ON ELASTIC FOUNDATION

Reducing the operators of Eqs. (18.1.4)–(18.1.6) to the case of an elastic ring as discussed in Chapter 4 gives, for zero damping and zero forcing,

$$-\frac{EI}{a^4}\left(\frac{\partial^2 u_\theta}{\partial\theta^2}-\frac{\partial^3 u_3}{\partial\theta^3}\right)-\frac{EA}{a^2}\left(\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{\partial u_3}{\partial\theta}\right)+k'_\theta u_\theta+m\ddot{u}_\theta=0 \tag{18.4.1}$$

$$-\frac{EI}{a^4}\left(\frac{\partial^3 u_\theta}{\partial\theta^3}+\frac{\partial^4 u_3}{\partial\theta^4}\right)+\frac{EA}{a^2}\left(\frac{\partial u_\theta}{\partial\theta}+u_3\right)+k'_3 u_3+m\ddot{u}_3=0 \tag{18.4.2}$$

where

$$k'_\theta=k_\theta b \tag{18.4.3}$$

$$k'_3=k_3 b \tag{18.4.4}$$

$$m=\rho A+\tfrac{1}{3}\rho_F h_F b \tag{18.4.5}$$

and where b is the width of a ring of rectangular cross-section. The area moment of inertia is $I=bh^3/12$. The cross-sectional area is $A=bh$. The elastic foundation is sketched in Fig. 1.

For the case of a closed ring, the solution will be of the form (compare with Sec. 5.3)

$$u_3(\theta,t)=A_n\cos n(\theta-\phi)e^{j\omega t} \tag{18.4.6}$$

$$u_\theta(\theta,t)=B_n\sin n(\theta-\phi)e^{j\omega t} \tag{18.4.7}$$

Substituting this into Eqs. (18.4.1) and (18.4.2) gives

$$\begin{bmatrix}\alpha_{11}-m\omega_n^2 & \alpha_{12}\\ \alpha_{21} & \alpha_{22}-m\omega_n^2\end{bmatrix}\begin{Bmatrix}A_n\\ B_n\end{Bmatrix}=0 \tag{18.4.8}$$

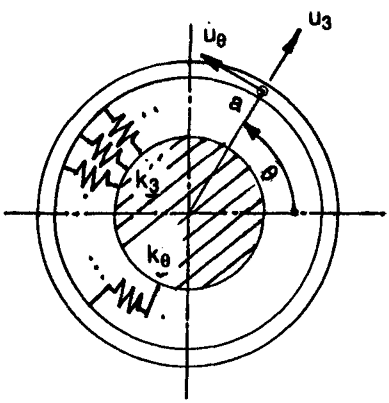

FIG. 1 Ring on an elastic foundation.

where

$$\alpha_{11}=\frac{n^4EI}{a^4}+\frac{EA}{a^2}+k_3' \tag{18.4.9}$$

$$\alpha_{12}=\alpha_{21}=\frac{n^3EI}{a^4}+\frac{nEA}{a^2} \tag{18.4.10}$$

$$\alpha_{22}=\frac{n^2EI}{a^4}+\frac{n^2EA}{a^2}+k_\theta' \tag{18.4.11}$$

For a nontrivial solution, the determinant must be 0. This gives

$$\omega_n^4-K_1\omega_n^2+K_2=0 \tag{18.4.12}$$

where

$$K_1=\frac{n^2+1}{a^2m}\left(\frac{n^2EI}{a^2}+EA\right)+\frac{k_3'+k_\theta'}{m} \tag{18.4.13}$$

$$\begin{aligned}K_2 &= \frac{n^2(n^2-1)^2}{a^6m^2}E^2IA\\ &\quad+\frac{k_3'k_\theta'+k_3'(n^2E/a^2)[(I/a^2)+A]+k_\theta'(E/a^2)[(n^4I/a^2)+A]}{m^2}\end{aligned} \tag{18.4.14}$$

Again, there will be two natural frequencies for every value of n:

$$\omega_{n1}^2=\frac{K_1}{2}\left(1-\sqrt{1-4\frac{K_2}{K_1^2}}\right) \tag{18.4.15}$$

$$\omega_{n2}^2=\frac{K_1}{2}\left(1+\sqrt{1-4\frac{K_2}{K_1^2}}\right) \tag{18.4.16}$$

The lower natural frequency set ω_{n1} belongs in general to ring modes, where transverse motion dominates. The higher natural frequency set ω_{n2} corresponds in general to ring modes where tangential motion dominates. Exceptions are the $n=0$ and $n=1$ cases. Note that the zero natural frequencies of Sec. 5.3 do not any longer exist, as one would expect.

The mode shapes become$(i=1,2)$

$$U_{3ni(\theta)}=A_{ni}\cos n(\theta-\phi) \tag{18.4.17}$$

$$U_{\theta ni(\theta)}=B_{ni}\sin n(\theta-\phi) \tag{18.4.18}$$

where

$$\begin{aligned}\frac{B_{ni}}{A_{ni}} &= \frac{m\omega_{ni}^2-k_3'-n^4EI/a^4-EA/a^2}{n^3EI/a^4+nEA/a^2}\\ &= \frac{n^3EI/a^4+nEA/a^2}{m\omega_{ni}^2-k_\theta'-n^2EI/a^4-n^2EA/a^2}\end{aligned} \tag{18.4.19}$$

For $i=1$ and $n>1$, this ratio is less than unity in magnitude, indicating dominance of transverse motion as discussed in Sec. 5.3. For $i=2$ and $n>1$, it is larger than unity in magnitude, which means that tangential motion is dominant.

If the foundation is such that $k_3'=k_\theta'=k'$, the result of Eq. (18.2.9) applies, which is, when modified for the ring,

$$\omega_{n1F}^2=\frac{\omega_{n10}^2\rho A+k'}{m} \tag{18.4.20}$$

$$\omega_{n2F}^2=\frac{\omega_{n20}^2\rho A+k'}{m} \tag{18.4.21}$$

where ω_{n10}^2 and ω_{n20}^2 are the values for the ring not on an elastic foundation, as given by Eqs. (5.3.13) and (5.3.14).

As to applications, tires are frequently modeled as rings on elastic foundations (Clark, 1975; Kung et al., 1986a, 1986b and 1987). In this treatment, the elastic foundation spring rate is a function of the geometric deformation of the sidewalls and the inflation pressure of the tire. Rails may be viewed as beams on elastic foundations. Soil or liquids supporting plate structures are often simplified in this way.

18.5. DONNELL–MUSHTARI–VLASOV EQUATIONS WITH TRANSVERSE ELASTIC FOUNDATION

Assuming that k_1 and k_2 are 0 and only k_3 exists, the Donnell–Mushtari–Vlasov equations become

$$D\nabla^4 u_3+\nabla_k^2\phi+k_3u_3+(\rho h+\tfrac{1}{3}\rho_F h_F)\ddot{u}_3=q_3 \tag{18.5.1}$$

$$Eh\nabla_k^2 u_3-\nabla^4\phi=0 \tag{18.5.2}$$

Thus the natural frequencies are again related to the natural frequencies of a shell of the same boundary conditions but not on an elastic foundation by

$$\omega_F^2=\frac{\omega_0^2\rho h+k_3}{\rho h+\frac{1}{3}\rho_F h_F} \tag{18.5.3}$$

These frequencies correspond to mode shapes where the transverse motion is dominant, since this is a restriction of this theory.

18.6. FORCES TRANSMITTED INTO THE BASE OF THE ELASTIC FOUNDATION

In the following, it is assumed that the elastic foundation is a relatively thin layer of material that acts directly on the reference surface of the shell.

In this case, the transmitted force per unit area at the base of the foundation is

$$q_{1T} = k_1 u_1 + c_1 \dot{u}_1 \tag{18.6.1}$$

$$q_{2T} = k_2 u_2 + c_2 \dot{u}_2 \tag{18.6.2}$$

$$q_{3T} = k_3 u_3 + c_3 \dot{u}_3 \tag{18.6.3}$$

or, in short,

$$q_{iT} = k_i u_i + c_i \dot{u}_i \tag{18.6.4}$$

In the following, we will investigate the set of cases where the forcing on the shell is harmonic and confine ourselves to investigate the steady-state transmitted forces. We set

$$q_i(\alpha_1, \alpha_2, t) = q_i^*(\alpha_1, \alpha_2)\mathrm{e}^{j\omega t} \tag{18.6.5}$$

and $\lambda_i = \lambda$ and $c_i = c$ in Eqs. (18.1.4)–(18.1.6). Utilizing Sec. 8.5, the solution is

$$u_i(\alpha_1, \alpha_2, t) = \sum_{k=1}^{\infty} \eta_k(t) U_{ik}(\alpha_1, \alpha_2) \tag{18.6.6}$$

where

$$\eta_k(t) = \Lambda_k \mathrm{e}^{j(\omega t - \phi_k)} \tag{18.6.7}$$

and where

$$\Lambda_k = \frac{F_k^*}{\omega_k^2 \sqrt{[1-(\omega/\omega_k)^2]^2 + 4\zeta_k^2(\omega/\omega_k)^2}} \tag{18.6.8}$$

$$F_k^* = \frac{1}{(\rho h + 1/3\rho_F h_F) N_k} \int_{\alpha_2}\int_{\alpha_1} (q_1^* U_{1k} + q_2^* U_{2k} + q_3^* U_{3k}) A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{18.6.9}$$

$$N_k = \int_{\alpha_2}\int_{\alpha_1} \left(U_{1k}^2 + U_{2k}^2 + U_{3k}^2\right) A_1 A_2 \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tag{18.6.10}$$

$$\phi_k = \tan^{-1} \frac{2\zeta_k(\omega/\omega_k)}{1-(\omega/\omega_k)^2} \tag{18.6.11}$$

$$\zeta_k = \frac{(\lambda + c)}{2(\rho h + 1/3\rho_F h_F)\omega_k} \tag{18.6.12}$$

Equation (18.6.6) can be written as

$$u_i(\alpha_1, \alpha_2, t) = \left[\sum_{k=1}^{\infty} \Lambda_k \mathrm{e}^{-j\phi_k} U_{ik}(\alpha_1, \alpha_2)\right] \mathrm{e}^{j\omega t} \tag{18.6.13}$$

Note that the natural frequencies ω_k and natural mode components $U_{ik}(\alpha_1, \alpha_2)$ have to be obtained from Eqs. (18.1.4)–(18.1.6), with $\lambda_i = c_i = 0$ and $q_i = 0$

$$L_i\{u_1, u_2, u_3\} - k_i u_i + \left(\rho h + \frac{1}{3}\rho_F h_F\right)\ddot{u}_i = 0 \tag{18.6.14}$$

The transmitted, steady-state forces per unit area into the elastic foundation are, therefore, from Eq. (18.6.4),

$$q_{iT} = (k_i + j\omega c_i)\left[\sum_{k=1}^{\infty} \Lambda_k e^{-j\phi_k} U_{ik}(\alpha_1, \alpha_2)\right] e^{j\omega t} \tag{18.6.15}$$

Here, it is argued that even while the results for Λ_k and ϕ_k were obtained using average, equivalent damping $\lambda_i = \lambda$ and $c_i = c$, it is from a practical viewpoint less approximate to use the individual c_i in formulation (18.6.15).

Since we may write

$$k_i + j\omega c_i = \sqrt{k_i^2 + \omega^2 c_i^2}\, e^{j\Psi} \tag{18.6.16}$$

where

$$\Psi_i = \tan^{-1}\left(\frac{\omega c_i}{k_i}\right) \tag{18.6.17}$$

Equation (18.6.15) becomes

$$q_{iT} = \sqrt{k_i^2 + \omega^2 c_i^2} \sum_{k=1}^{\infty} \Lambda_k U_{ik}(\alpha_1, \alpha_2) e^{j(\omega t + \Psi_i - \phi_k)} \tag{18.6.18}$$

This result shows that in general, transmitted forces per unit area tend to increase with increasing foundation stiffnesses k_i and damping c_i, even while in specific cases the response may be reduced because of the elastic foundation damping, or changed because the foundation stiffnesses k_i will change the natural frequencies and, thus, tune or detune the response.

18.7. VERTICAL FORCE TRANSMISSION THROUGH THE ELASTIC FOUNDATION OF A RING ON A RIGID WHEEL

A ring on an elastic foundation is harmonically excited by a point force in vertical direction, as shown in Fig. 2. Damping in the foundation is

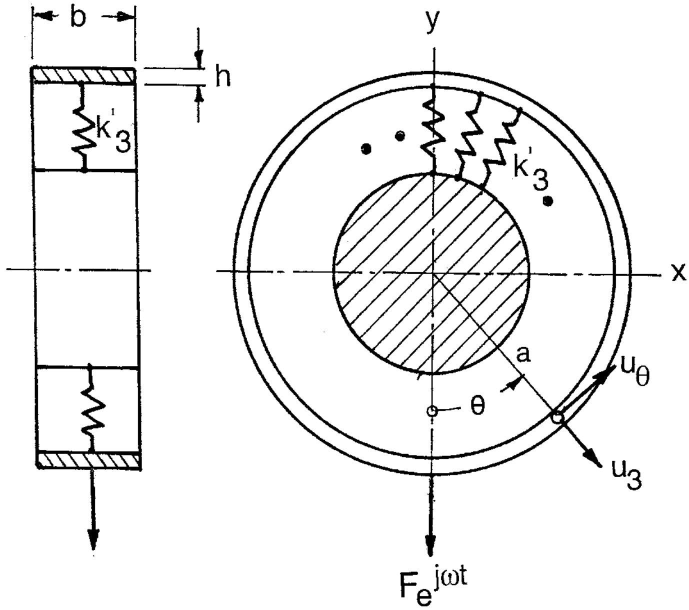

FIG. 2 Ring on an elastic foundation excited by a harmonic point force.

negligible, as is the spring rate in θ direction, $k'_\theta = 0$. Only the spring rate per unit length, $k'_3[N/m^2]$, is present. The base of the elastic foundation is a rigid cylinder.

This arrangement is the simplest model of a tire which is mounted on a rigid wheel–axle assembly which we consider to be ground in this problem. The tire is not rotating and is excited by a shaker at a point. The elastic foundation is provided by the air spring effect of the flexing sidewalls. The tread band of the tire is modeled as the ring. The added stiffness of the "ring" due to the tensile stresses caused by the inflation pressure is neglected (for their effect, see Chapter 11).

In the following, it will be shown that the integrated vertical force components of the transmitted force distribution acting on the rigid wheel will be a function of only the rigid body mode of the ring, namely $n = 1$. All other modes integrate to 0. They are filtered out by the rigid whell–axle assembly.

The natural frequencies are given by Eqs. (18.4.15) and (18.4.16), and the natural modes are obtained from Eqs. (18.4.17) and (18.4.18), with $k'_\theta = 0$:

$$U_{3ni}(\theta) = A_{ni} \cos n\theta \tag{18.7.1}$$

$$U_{\theta ni}(\theta) = B_{ni} \sin n\theta \tag{18.7.2}$$

where $i = 1, 2$ and where we have selected only the set corresponding to $\phi = 0$ because of symmetry (the point force $F e^{j\omega t}$ is located at $\theta = 0$). The nonsymmetrical set of modes corresponding to $\phi = \pi/2n$ in Eqs. (18.4.17) and (18.4.18) will integrate from the solution anyway.

The mode component amplitude ratios B_{ni}/A_{ni} are given by Eq. (18.4.19). The steady-state harmonic response of the ring on its elastic foundation is

$$u_3(\theta, t) = \sum_{i=1}^{2} \sum_{n=0}^{\infty} \eta_{ni}(t) U_{3ni}(\theta) \tag{18.7.3}$$

$$u_\theta(\theta, t) = \sum_{i=1}^{2} \sum_{n=0}^{\infty} \eta_{ni}(t) U_{\theta ni}(\theta) \tag{18.7.4}$$

where

$$\eta_{ni}(t) = \Lambda_{ni} e^{j(\omega t - \phi_{ni})} \tag{18.7.5}$$

and where

$$\Lambda_{ni} = \frac{F^*_{ni}}{\omega^2_{ni} \sqrt{\left[1 - (\omega/\omega_{ni})^2\right]^2 + 4\zeta_{ni}(\omega/\omega_{ni})^2}} \tag{18.7.6}$$

$$\phi_{ni} = \tan^{-1} \frac{2\zeta_{ni}(\omega/\omega_{ni})}{1 - (\omega/\omega_{ni})^2} \tag{18.7.7}$$

$$\zeta_{ni} = \frac{\lambda}{2(\rho h + (1/3)\rho_F h_F)\omega_{ni}} \tag{18.7.8}$$

From Eq. (8.5.3), we obtain

$$F^*_{ni} = \frac{1}{(\rho h + (1/3)\rho_F h_F) N_{ni}} \int_{\theta=0}^{2\pi} q_3 U_{3ni} a \, d\theta \tag{18.7.9}$$

where

$$N_{ni} = b \int_{\theta=0}^{2\pi} \left(U^2_{\theta ni} + U^2_{3ni}\right) a \, d\theta = \varepsilon_n \pi b a (A^2_{ni} + B^2_{ni}) \tag{18.7.10}$$

and where $\varepsilon_n = 1$ if $n \neq 0$, and $\varepsilon_n = 2$ if $n = 0$. Since the force per unit length of ring is, in terms of the point force,

$$q'_3(\theta, t) = \frac{F}{a} e^{j\omega t} \delta(\theta - 0) \tag{18.7.11}$$

we obtain

$$F^*_{ni} = \frac{F}{(\rho h + (1/3)\rho_F h_F) N_{ni}} \int_{\theta=0}^{2\pi} \delta(\theta - 0) A_{ni} \cos n\theta \, d\theta \tag{18.7.12}$$

or

$$F_{ni}^{*}=\frac{F A_{ni}}{(\rho h+(1/3)\rho_F h_F)ab\pi\left(A_{ni}^2+B_{ni}^2\right)\varepsilon_n} \tag{18.7.13}$$

Substituting Eqs. (18.7.13) into Eq. (18.7.6) and (18.7.5), and finally into Eqs. (18.7.3) and (18.7.4) gives

$$u_3(\theta,t)=\sum_{i=1}^{2}\sum_{n=0}^{\infty}\frac{F A_{ni}^2 e^{j(\omega t-\phi_{ni})}\cos n\theta}{\varepsilon_n\pi(\rho h+(1/3)\rho_F h_F)ab\left(A_{ni}^2+B_{ni}^2\right)\omega_{ni}^2\sqrt{\left(1-(\omega/\omega_{ni})^2\right)^2+4\zeta_{ni}(\omega/\omega_{ni})^2}} \tag{18.7.14}$$

$$u_\theta(\theta,t)=\sum_{i=1}^{2}\sum_{n=1}^{\infty}\frac{F A_{ni}B_{ni} e^{j(\omega t-\phi_{ni})}\sin n\theta}{\varepsilon_n\pi(\rho h+(1/3)\rho_F h_F)ab\left(A_{ni}^2+B_{ni}^2\right)\omega_{ni}^2\sqrt{\left(1-(\omega/\omega_{ni})^2\right)^2+4\zeta_{ni}(\omega/\omega_{ni})^2}} \tag{18.7.15}$$

Note that the modal amplitude term can be re-written in terms of the amplitude ratios B_{ni}/A_{ni} which are given by Eq. (18.4.19):

$$\frac{A_{ni}^2}{A_{ni}^2+B_{ni}^2}=\frac{1}{1+\left(B_{ni}^2/A_{ni}^2\right)^2} \tag{18.7.16}$$

and

$$\frac{A_{ni}B_{ni}}{A_{ni}^2+B_{ni}^2}=\frac{(B_{ni}/A_{ni})}{1+(B_{ni}/A_{ni})^2} \tag{18.7.17}$$

The summation over $i=1$ and 2 is necessary if we wish to include in the response both the $i=1$ and $i=2$ natural frequencies and modes. If as an approximation only the lower natural frequency and mode set $i=1$ is deemed important, this summation can be dispensed with. Note that $B_{0i}/A_{0i}=0$ according to Eq. (18.4.19).

The distributed normal loading that is transmitted into the foundation consisting of the rigid wheel is

$$q_{3T}=k_3' u_3(\theta,t) \tag{18.7.18}$$

or

$$q_{3T}=\frac{k_3' F}{\pi(\rho h+(1/3)\rho_F h_F)ab}\sum_{i=1}^{3}\sum_{n=1}^{\infty}P_{ni}(\omega t)\cos n\theta \tag{18.7.19}$$

where

$$P_{ni}(\omega)=\frac{e^{j(\omega t-\phi_{ni})}}{\varepsilon_n\left[1+(B_{ni}/A_{ni})^2\right]\omega_{ni}^2\sqrt{\left(1-(\omega/\omega_{ni})^2\right)^2+4\zeta_{ni}(\omega/\omega_{ni})^2}} \tag{18.7.20}$$

The force component in vertical direction, $\mathrm{d}F_y$, of the transmitted pressure load q_{3T} acting on an infinitesimal area $ab\,\mathrm{d}\theta$ is given by

$$\mathrm{d}F_y=(q_{3T}ab\,\mathrm{d}\theta)\cos\theta \tag{18.7.21}$$

The resultant transmitted force in vertical direction is

$$F_y=\int_{\theta=0}^{2\pi}\frac{k_3'F}{\pi(\rho h+(1/3)\rho_F h_F)}\sum_{i=1}^{3}\sum_{n=1}^{\infty}P_{ni}(\omega t)\cos n\theta\cos\theta\,\mathrm{d}\theta \tag{18.7.22}$$

Since

$$\int_{\theta=0}^{2\pi}\cos n\theta\cos\theta\,\mathrm{d}\theta=\begin{cases}0 & \text{if } n\neq 1\\ \pi & \text{if } n=1\end{cases} \tag{18.7.23}$$

All n terms except $n=1$ in the series expressions are 0.
Therefore

$$F_y=\frac{k_3'F}{(\rho h+(1/3)\rho_F h_F)}\times\sum_{i=1}^{2}\frac{e^{j(\omega t-\phi_{1i})}}{\left[1+(B_{1i}/A_{1i})^2\right]^2\omega_{1i}^2\sqrt{\left(1-(\omega/\omega_{1i})^2\right)^2+4\zeta_{1i}(\omega/\omega_{1i})^2}} \tag{18.7.24}$$

Summing up the transmitted forces in horizontal direction gives, because of the axial symmetry of q_{3T}, a zero resultant in horizontal direction. In conclusion, this example illustrates how a rigid wheel body can filter all mode components except for the $n=1$ mode from the transmitted vertical force response. For a real tire, not using this simplified model, the wheel assembly is, of course, not rigid. Also, the real tire resembles a toroidal form and has mode components other than (ni) which are usually designated (mni) where m defines mode component shapes in a direction that is transverse to the rolling direction. But the filtering effect of a relatively stiff wheel assembly is still present. Only the $mni=m1i$ mode components will be present in the vertical force transmission solutions since it will still turn out that $n=1$.

18.8. RESPONSE OF A SHELL ON AN ELASTIC FOUNDATION TO BASE EXCITATION

If a shell is mounted on an elastic foundation whose base is vibrating with $u_{Bi}(\alpha_1, \alpha_2, t)$, the forces per unit area of Eqs. (18.1.1) become

$$q_i = -k_i(u_i - u_{Bi}) - c_i(\dot{u}_i - \dot{u}_{Bi}) - \frac{1}{3}\rho_F h_F \ddot{u}_i \tag{18.8.1}$$

and Eqs. (18.1.4)–(18.1.6) become

$$L_i\{u_1, u_2, u_3\} - k_i u_i - (\lambda_i + c_i)\dot{u}_i - \left(\rho h + \frac{1}{3}\rho_F h_F\right)\ddot{u}_i = -k_i u_{Bi} - c_i \dot{u}_{Bi} \tag{18.8.2}$$

The implied assumption is here that base motion in one direction, say in the transverse direction, will only generate forces on the shell in the transverse direction. A more complete model of base excitation would have to model the detailed behavior of the elastic foundation material itself since it is conceivable that transverse base motion may also cause forces tangential to the place of the shell surface.

Note that Eq. (18.8.2) reduces to Eqs. (18.1.1)–(18.1.3) if u_{Bi} and $\dot{u}_{Bi}$ are 0.

In the following, we will investigate the case where the base excitation is harmonic:

$$u_{Bi}(\alpha_1, \alpha_2, t) = U_{Bi}(\alpha_1, \alpha_2)e^{j\omega t} \tag{18.8.3}$$

This gives, for the right side of Eq. (18.8.2),

$$-k_i u_{Bi} - c_i \dot{u}_{Bi} = -(k_i U_{Bi} + \omega c_i j U_{Bi})e^{j\omega t} = -\sqrt{k_i^2 + \omega^2 c_i^2}\, U_{Bi} e^{j(\omega t + \Psi_i)} \tag{18.8.4}$$

where

$$\Psi_i = \tan^{-1}\left(\frac{c_i \omega}{k_i}\right) \tag{18.8.5}$$

The natural frequencies ω_k and mode components U_{ik} are obtained from Eq. (18.8.2) by setting $\lambda_i = c_i = 0$, and the right hand side is also taken as 0:

$$L_i\{u_1, u_2, u_3\} - k_i u_i - \left(\rho h + \frac{1}{3}\rho_F h_F\right)\ddot{u}_i = 0 \tag{18.8.6}$$

Having done this, we proceed to the steady-state harmonic response solution:

$$u_i(\alpha_1, \alpha_2, t) = \sum_{k=1}^{\infty} \eta_k(t) U_{ik}(\alpha_1, \alpha_2) \tag{18.8.7}$$

where

$$\eta_k(t) = \Lambda_k e^{j(\omega t + \Psi_i - \phi_k)} \tag{18.8.8}$$

and where

$$\Lambda_k = \frac{F_k^*}{\omega_k^2 \sqrt{\left[1 - (\omega/\omega_k)^2\right]^2 + 4\zeta_k^2 (\omega/\omega_k)^2}} \tag{18.8.9}$$

$$\phi_k = \tan^{-1} \frac{2\zeta_k (\omega/\omega_k)}{1 - (\omega/\omega_k)^2} \tag{18.8.10}$$

Note that we have set $\lambda_i = \lambda$ and $c_i = c$, which gives

$$\zeta_k = \frac{\lambda + c}{2(\rho h + (1/3)\rho_F h_F)\omega_k} \tag{18.8.11}$$

From Eq. (8.5.3), we obtain approximately (note the inconsistency that we used average values of λ_i and c_i to obtain an averaged ζ_k, but that on the forcing side of the equation we keep the individual λ_i and c_i.)

$$F_k^* = \frac{1}{(\rho h + \frac{1}{3}\rho_F h_F)N_k} \int_{\alpha_2}\int_{\alpha_1} \Big(\sqrt{k_1^2 + \omega^2 c_1^2}\, U_{B1} U_{1k} + \sqrt{k_2^2 + \omega^2 c_2^2}\, U_{B2} U_{2k} + \sqrt{k_3^2 + \omega^2 c_3^2}\, U_{B3} U_{3k}\Big) A_1 A_2 \, d\alpha_1 \, d\alpha_2 \tag{18.8.12}$$

If we wish to be consistent with our assumption that we may set $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and $c_1 = c_2 = c_3 = c$, then this equation reduces to

$$F_k^* = \frac{\sqrt{k^2 + \omega^2 c^2}}{(\rho h + (1/3)\rho_F h_F)N_k} \int_{\alpha_2}\int_{\alpha_1} \big[U_{B1}(\alpha_1, \alpha_2) U_{1k}(\alpha_1, \alpha_2) + U_{B2}(\alpha_1, \alpha_2) U_{2k}(\alpha_1, \alpha_2) + U_{B3}(\alpha_1, \alpha_2) U_{3k}(\alpha_1, \alpha_2)\big] A_1 A_2 \, d\alpha_1 \, d\alpha_2 \tag{18.8.13}$$

Note again that this solution is only valid if the entire surface of the shell is supported by a uniform elastic foundation. Elastic foundations that are only patches have to be approached differently.

The result shows again in general that in order to keep the vibration response low, the formulation stiffnesses and damping, k_i and c_i, should be kept low, keeping in mind of course that the elastic foundation stiffness may tune or detune the resonances which may make the response for a lower k_i actually higher.

18.9. PLATE EXAMPLES OF BASE EXCITATION AND FORCE TRANSMISSION

In the following, the rectangular plate is used to illustrate base excitation and force transmission calculations. For Cartesian coordinates $\alpha_1 = x$, $\alpha_2 = y$, the Lamé parameters are $A_1 = A_2 = 1$.

18.9.1. Uniformly Distributed Harmonic Base Excitation of a Simply Supported Plate

The equation of motion to be solved is, by reduction from Eq. (18.8.2),

$$D\nabla^4 u_3 + k_3 u_3 + (\lambda_3 + c_3)\dot{u}_3 + \left(\rho h + \frac{1}{3}\rho_F h_F\right)\ddot{u}_3 = k_3 u_{B3} + c_3 \dot{u}_{B3} \qquad (18.9.1)$$

where the elastic foundation base motion is taken as being uniformly distributed (for example, the entire base of the elastic foundation is attached to a shaker platform), vibrating harmonically with amplitude U_{B3}:

$$u_{B3}(x, y, t) = U_{B3}(x, y)e^{j\omega t} \qquad (18.9.2)$$

where, for a uniform distribution,

$$U_{B3}(x, y) = U_{B3} = \text{const.} \qquad (18.9.3)$$

The natural frequencies and modes are obtained from

$$D\nabla^4 u_3 + k_3 u_3 + \left(\rho h + \frac{1}{3}\rho_F h_F\right)\ddot{u}_3 = 0 \qquad (18.9.4)$$

For a simply supported plate, the natural modes are

$$U_{3mn}(x, y) = A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad (18.9.5)$$

and the natural frequencies are, from Eq. (18.3.3),

$$\omega_{Fmn} = \sqrt{\frac{\omega_{mn}^2 \rho h + k_3}{\rho h + (1/3)\rho_F h_F}} \qquad (18.9.6)$$

and where

$$\omega_{mn} = \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}} \qquad (18.9.7)$$

The amplitudes of the natural modes, A_{mn}, are arbitrary and could be taken as unity. They will cancel during the subsequent development.

Since

$$N_k = N_{mn} = \int_{y=0}^{b}\int_{x=0}^{a} A_{mn}^2 \sin^2\frac{m\pi x}{a} \sin^2\frac{n\pi y}{b}\, dx\, dy = \frac{ab}{4}A_{mn}^2 \qquad (18.9.8)$$

we obtain from Eq. (18.8.12),

$$F_k^* = F_{mn}^* = \frac{4\sqrt{k_3^2+\omega^2 c_3^2}(1-\cos m\pi)(1-\cos n\pi)}{(\rho \mathrm{h}+(1/3)\rho_F h_F)A_{mn} mn\pi^2} \tag{18.9.9}$$

Utilizing Eqs. (18.8.7)–(18.8.11) gives

$$u_3(x,y,t) = \frac{4\sqrt{k_3^2+\omega^2 c_3^2}U_{B3}}{(\rho h+(1/3)\rho_F h_F)\pi^2} \times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(1-\cos m\pi)(1-\cos n\pi)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{e}^{j(\omega t+\Psi_3-\phi_{Fmn})}}{mn\omega_{Fmn}^2\sqrt{\left[1-(\omega/\omega_{Fmn})^2\right]^2+4\zeta_{Fmn}^2(\omega/\omega_{Fmn})^2}} \tag{18.9.10}$$

where

$$\Psi_3 = \tan^{-1}\left(\frac{c_3\omega}{k_3}\right) \tag{18.9.11}$$

$$\phi_{F\mathrm{mn}} = \tan^{-1}\frac{2\zeta_{mn}(\omega/\omega_{\mathrm{Fmn}})}{1-(\omega/\omega_{\mathrm{Fmn}})^2} \tag{18.9.12}$$

$$\zeta_{\mathrm{Fmn}} = \frac{\lambda_3+c_3}{2(\rho h+(1/3)\rho_F h_F)\omega_{\mathrm{Fmn}}} \tag{18.9.13}$$

As one would expect, because of the symmetry, only natural modes of m,n combinations where $m=1,3,5,..$ and $n=1,3,5,\ldots$ participate in the response.

18.9.2. Pressure Transmission into the Base of the Elastic Foundation

A simply supported plate on an elastic foundation is acted on by a uniformly distributed, harmonically varying pressure:

$$q_3(x,y,t) = q_3^*(x,y)\mathrm{e}^{j\omega t} \tag{18.9.14}$$

where

$$q_3^*(x,y) = Q_3 = \text{const.} \tag{18.9.15}$$

The pressure that is transmitted into the foundation is

$$q_{3T}(x,y,t) = k_3 u_3(x,y,t) + c_3\dot{u}_3(x,y,t) \tag{18.9.16}$$

Since we expect, in steady state (see Sec. 8.5), that

$$u_3(x,y,t) = \sum_{k=1}^{\infty}\Lambda_k U_{3k}(x,y)\mathrm{e}^{j(\omega t-\phi_k)} \tag{18.9.17}$$

we obtain

$$q_{3T}(x,y,t) = (k_3 + j\omega c_3)\sum_{k=1}^{\infty}\Lambda_k U_{3k}(x,y)e^{j(\omega t-\phi_k)}$$
$$= \sqrt{k_3^2+\omega^2 c_3^2}\sum_{k=1}^{\infty}\Lambda_k U_{3k}(x,y)e^{j(\omega t+\Psi_3-\phi_k)} \tag{18.9.18}$$

where

$$\Psi_3 = \tan^{-1}\left(\frac{c_3\omega}{k_3}\right) \tag{18.9.19}$$

In order to obtain the response $u_3(x,y,t)$, the following equation has to be solved:

$$D\nabla^4 u_3 + k_3 u_3 + (\lambda_3 + c_3)\dot{u}_3 + \left(\rho h + \frac{1}{3}\rho_F h_F\right)\ddot{u}_3 = q_3 = Q_3 e^{j\omega t} \tag{18.9.20}$$

For a simply supported plate, Eqs. (18.9.5) and (18.9.6) apply again as before. N_{mn} is given by Eq. (18.9.8). Equation (18.6.9) becomes

$$F_k^* = F_{mn}^* = \frac{4Q_3(1-\cos m\pi)(1-\cos n\pi)}{(\rho h + (1/3)\rho_F h_F)A_{mn} mn\pi^2} \tag{18.9.21}$$

Utilizing Eqs. (18.6.8), Eq. (18.9.18) becomes

$$q_{3T}(x,y,t) = \frac{4\sqrt{k_3^2+\omega^2 c_3^2}\,Q_3}{(\rho h + (1/3)\rho_F h_F)\pi^2} \times \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1-\cos m\pi)(1-\cos n\pi)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{j(\omega t+\Psi_3-\phi_{Fmn})}}{mn\omega_{Fmn}^2\sqrt{[1-(\omega/\omega_{Fmn})^2]^2+4\zeta_{Fmn}^2(\omega/\omega_{Fmn})^2}} \tag{18.9.22}$$

where ϕ_{mn} and ζ_{Fmn} are given by Eqs. (18.9.12) and (18.9.13). Note that $u_3(x,y,t)$ and U_{B3}, and $q_{3T}(x,y,t)$ and Q_3, are related by the same transfer function.

18.10. NATURAL FREQUENCIES AND MODES OF A RING ON AN ELASTIC FOUNDATION IN GROUND CONTACT AT A POINT

The natural frequencies of a ring on an elastic foundation are given by Eqs. (18.4.15) and (18.4.16). The natural modes are given by Eqs. (18.4.17)–(18.4.19).

Let us suppose that the ring is brought into point contact at $\theta = 0$, so that motion in radial direction is 0, but motion in tangential direction is not restricted. In the following, it is shown how the natural frequencies and modes are obtained for the point contact case in terms of the natural frequencies and modes of the noncontacting ring. The ability to do this is, for example, important in the tire industry, where natural frequencies and modes of a free (noncontacting) tire can be relatively easily measured or calculated, and where it is at times of interest to predict from this information the natural frequencies and modes of the same tire in road contact where radial deflection is 0, but for wet roads tangential motion is allowed.

First, we calculate the steady state, undamped harmonic response of the ring at location $\theta = 0$ due to a harmonic point at $\theta = 0$. This was done in Sec. 18.7. Equation (18.7.14) becomes

$$u_3(0,t) = \sum_{i=1}^{2}\sum_{n=1}^{\infty} \frac{F A_{ni}^2 e^{j\omega t}}{\varepsilon_n \pi(\rho h + (1/3)\rho_F h_F) ab (A_{ni}^2 + B_{ni}^2)(\omega_{ni}^2 - \omega^2)} \qquad (18.10.1)$$

This allows us to formulate the point receptance at the point of contact, $\theta = 0$, labeled here point 1:

$$\alpha_{11} = \frac{u_3(0,t)}{F e^{j\omega t}} = \sum_{i=1}^{2}\sum_{n=1}^{\infty} \frac{A_{ni}^2}{\varepsilon_n \pi(\rho h + (1/2)\rho_F h_F) ab (A_{ni}^2 + B_{ni}^2)(\omega_{ni}^2 - \omega^2)} \qquad (18.10.2)$$

Because of the radial restriction of zero motion, the frequency equation is (see also Chapter 13)

$$\alpha_{11} = 0 \qquad (18.10.3)$$

or

$$\sum_{i=1}^{2}\sum_{n=1}^{\infty} \frac{A_{ni}^2}{\varepsilon_n \left[1 + (B_{ni}/A_{ni})^2\right] \left(\omega_{ni}^2 - \omega^2\right)} = 0 \qquad (18.10.4)$$

The values of $\omega = \omega_k$ which satisfy this equation are the natural frequency of the ring on an elastic foundation in point contact with ground. Again, the ω_{ni} are given by Eqs. (18.4.15) and (18.4.16). The ratios (B_{ni}/A_{ni}) are given by Eqs. (18.4.19).

To obtain the natural modes, we write Eqs. (18.7.14) and (18.7.15) for a general location θ and zero damping:

$$U_{3ni}(\theta) = A \sum_{i=1}^{2}\sum_{n=1}^{\infty} \frac{\cos n\theta}{\varepsilon_n \left[1 + (B_{ni}/A_{ni})^2\right] \left(\omega_{ni}^2 - \omega_k^2\right)} \qquad (18.10.5)$$

$$U_{\theta ni} = A \sum_{i=1}^{2}\sum_{n=1}^{\infty} \frac{(B_{ni}/A_{ni}) \sin n\theta}{\varepsilon_n \left[1 + (B_{ni}/A_{ni})^2\right] \left(\omega_{ni}^2 - \omega_k^2\right)} \qquad (18.10.6)$$

where

$$A = \frac{F}{\pi(\rho h + (1/3)\rho_F h_F)ab} \tag{18.10.7}$$

and can be viewed as an arbitrary constant which we may take as $A=1$, for example.

As a partial check, Eqs. (18.10.5) and (18.10.6) can be evaluated at the ground contact point $\theta=0$. We get

$$U_{3ni}(\theta) = A\sum_{i=1}^{2}\sum_{n=1}^{\infty}\frac{1}{\varepsilon_n\left[1+(B_{ni}/A_{ni})^2\right]\left(\omega_{ni}^2-\omega_k^2\right)} \tag{18.10.8}$$

Because Eq. (18.10.4) is satisfied by $\omega=\omega_k$, Eq. (18.10.8) gives

$$U_{3ni}(\theta=0)=0 \tag{18.10.9}$$

and Eq. (18.10.6) gives directly

$$U_{\theta ni}(\theta=0)=0 \tag{18.10.10}$$

as one would expect.

Note that here we have only considered the natural modes of the ring not in point contact because of the symmetry of the problem (all summations over n involve only the symmetric modes ($\phi=0$)). Unsymmetrical modes have a transverse deflection node at the contact point and are not affected. Therefore, beside the new ω_k obtained from Eq. (18.10.4), the ω_{ni} corresponding to natural frequencies of the unsymmetrical modes about $\theta=0$ are also natural frequencies of the ring in contact.

18.11. RESPONSE OF A RING ON AN ELASTIC FOUNDATION TO A HARMONIC POINT DISPLACEMENT

In Sec. 13.7, the steady-state harmonic response amplitude at location 2 of system A, due to a harmonic displacement input at location 1 is given by Eq. (13.7.4) as

$$X_2 = \frac{\alpha_{21}}{\alpha_{11}}X_1 \tag{18.11.1}$$

The ring is not restrained in tangential direction at location 1.

In our case, as illustrated in Fig. 3, we take $X_2=U_3(\theta)$ to be the transverse displacement amplitude at location 2, and of course X_1 is a given displacement input at $\theta=0$, namely $X_1=U_3(\theta=0)$. For this case, the receptance α_{11} is defined as the ratio of the transverse response amplitude

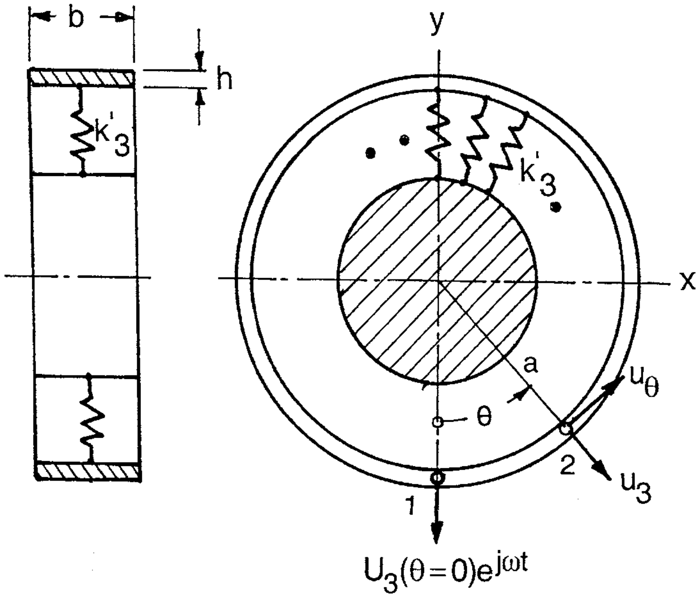

Fig. 3 Ring on an elastic foundation excited by a harmonic point displacement.

at $\theta = 0$ to a transverse harmonic force amplitude at location $\theta = 0$:

$$\alpha_{11} = \frac{U_3(\theta = 0)}{F_3(\theta = 0)} \tag{18.11.2}$$

which is, from Eq. (18.7.14) where the summation over n involves only modes that are symmetric about the $\theta = 0$ axis,

$$\alpha_{11} = \frac{1}{\pi(\rho h + (1/3)\rho_F h_F)ab} \times \sum_{i=1}^{2}\sum_{n=0}^{\infty} \frac{e^{-j\phi_{ni}}}{\varepsilon_n[1+(B_{ni}/A_{ni})^2]\omega_{ni}^2\sqrt{[1-(\omega/\omega_{ni})^2]^2 + 4\zeta_{ni}(\omega/\omega_{ni})^2}} \tag{18.11.3}$$

The receptance α_{21} is defined as the ratio of the transverse response amplitude at a general location θ to a transverse force amplitude at location $\theta = 0$:

$$\alpha_{21} = \frac{U_3(\theta)}{F_3(\theta = 0)} \tag{18.11.4}$$

which is, from Eq. (18.7.14),

$$\alpha_{21} = \frac{1}{\pi(\rho h + (1/3)\rho_F h_F)ab} \times \sum_{i=1}^{2}\sum_{n=0}^{\infty} \frac{e^{-j\phi_{ni}}\cos n\theta}{\varepsilon_n[1+(B_{ni}/A_{ni})^2]\omega_{ni}^2\sqrt{[1-(\omega/\omega_{ni})^2]^2+4\zeta_{ni}(\omega/\omega_{ni})^2}} \tag{18.11.5}$$

and, therefore,

$$U_3(\theta) = \frac{\alpha_{21}}{\alpha_{11}} X_1 \tag{18.11.6}$$

or

$$U_3(\theta) = X_1 \frac{\sum_{i=1}^{2}\sum_{n=0}^{\infty}\left\{\frac{e^{-j\phi_{ni}}\cos n\theta}{\varepsilon_n[1+(B_{ni}/A_{ni})^2]\omega_{ni}^2\sqrt{[1-(\omega/\omega_{ni})^2]^2+4\zeta_{ni}(\omega/\omega_{ni})^2}}\right\}}{\sum_{i=1}^{2}\sum_{n=0}^{\infty}\left\{\frac{e^{-j\phi_{ni}}}{\varepsilon_n[1+(B_{ni}/A_{ni})^2]\omega_{ni}^2\sqrt{[1-(\omega/\omega_{ni})^2]^2+4\zeta_{ni}(\omega/\omega_{ni})^2}}\right\}} \tag{18.11.7}$$

which is the complex response amplitude, which can be rewritten in terms of a magnitude and a phase angle.

As a partial check, when $\theta=0$, Eq. (18.11.7) gives $U_3(\theta=0)=X_1$ as expected.

The transverse response displacement $u_3(\theta, t)$ is then

$$u_3(\theta, t) = U_3(\theta)e^{j\omega t} \tag{18.11.8}$$

We may now do the same thing for $U_\theta(\theta)$, which is now the X_2 of Eq. (18.11.1). In this case, we may write

$$U_\theta(\theta) = \frac{\alpha'_{21}}{\alpha'_{11}} X_1 \tag{18.11.9}$$

where we define the receptance α'_{11} still as the ratio of the transverse response amplitude at $\theta=0$ to a transverse harmonic force at location $\theta=0$, as required by the derivation in Sec. 18.7:

$$\alpha'_{11} = \alpha_{11} \tag{18.11.10}$$

but the receptance α'_{21} is now defined as the ratio of the tangential response at a general location θ to a harmonic transverse force at location $\theta=0$:

$$\alpha'_{21}=\frac{U_\theta(\theta)}{F_3(\theta=0)} \tag{18.11.11}$$

which is, from Eq. (18.7.15),

$$\alpha'_{21} = \frac{1}{\pi(\rho h+(1/3)\rho_F h_F)ab} \times\sum_{i=1}^{2}\sum_{n=1}^{\infty}\frac{(B_{ni}/A_{ni})\mathrm{e}^{-j\phi_{ni}} sin n\theta}{\varepsilon_n\left[1+(B_{ni}/A_{ni})^2\right]\omega_{ni}^2\sqrt{\left[1-(\omega/\omega_{ni})^2\right]^2+4\zeta_{ni}(\omega/\omega_{ni})^2}} \tag{18.11.12}$$

Therefore, the tangential response amplitude (18.11.9) becomes

$$U_\theta(\theta) = X_1\frac{\sum_{i=1}^{2}\sum_{n=1}^{\infty}\left\{\frac{(B_{ni}/A_{ni})\mathrm{e}^{-j\phi_{ni}}\sin n\theta}{\varepsilon_n[1+(B_{ni}/A_{ni})^2]\omega_{ni}^2\sqrt{[1-(\omega/\omega_{ni})^2]^2+4\zeta_{ni}(\omega/\omega_{ni})^2}}\right\}}{\sum_{i=1}^{2}\sum_{n=0}^{\infty}\left\{\frac{\mathrm{e}^{-j\phi_{ni}}}{\varepsilon_n\left[1+(B_{ni}/A_{ni})^2\right]\omega_{ni}^2\sqrt{[1-(\omega/\omega_{ni})^2]^2+4\zeta_{ni}(\omega/\omega_{ni})^2}}\right\}} \tag{18.11.13}$$

Again as a partial check, when $\theta=0$, Eq. (18.11.13) reduces to $U_\theta(\theta=0)=0$, as expected. The tangential response displacement is

$$u_\theta(\theta,t)=U_\theta(\theta)\mathrm{e}^{j\omega t} \tag{18.11.14}$$

We see that resonances will occur if the denominator in Eqs. (18.11.7) and (18.11.13) is minimized. For zero damping, this denominator, when set equal to 0, gives the frequency equation of Sec. 18.10 for a ring in point ground contact ($\alpha_{11}=0$), as one would expect. The natural frequencies ω_k obtained in Sec. 18.10 cause the resonances of the point displacement excitation. This is entirely different from the force excitation case, where resonances occur in the vicinity of $\omega=\omega_{ni}$. All of this is similar to the well-known case where a longitudinally vibrating free–free rod when excited at $x=0$ by a harmonic force has resonance frequencies which are its free–free natural frequencies, but when being forced by a displacement excitation at $x=0$, its resonance frequencies are the clamped-free natural frequencies.

Again, note that the summations over n in the solution include only the natural modes that are symmetric about $\theta=0(\phi=0)$; the $\phi=\pi/2n$ modes cancel); see also Sec. 18.10.

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19

Similitude

It is possible to draw conclusions about the general behavior of structures, without solving specific boundary value problems, by establishing exact or approximate similitude relationships. A second motivation is that in cases of large shells of complicated geometry, vibration analysts have been turning to the use of experimental models. Finally, it allows us to understand how results should be presented in nondimensional form. Similitude arguments were used in Chapter 18 when discussing elastic foundations.

19.1. GENERAL SIMILITUDE

The accepted way of scaling is to scale the shell model faithfully in every respect, shape, and thickness, all as shown in Fig. 1, employing the classical scaling law. This law is usually derived in textbooks by employing the method of dimensional analysis (Focken, 1953), and states that all dimensions have to be scaled in proportion: for example,

$$\frac{h_2}{h_1} = \frac{a_2}{a_1} = \frac{R_2}{R_1} \tag{19.1.1}$$

where the subscripts 1 and 2 designate two similar structures, h a typical thickness, a a typical length or width dimension, and R a typical radius of

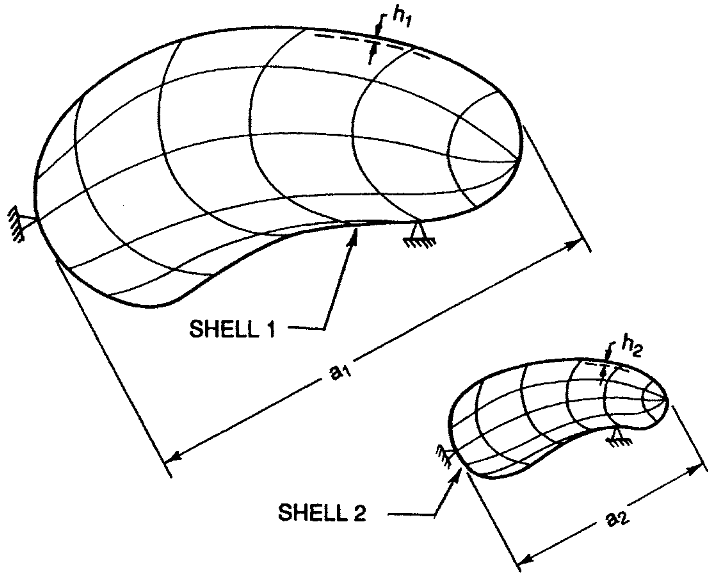

Fig. 1 Similar shells.

curvature. It is also, strictly speaking, required that the Poisson ratios be the same if two different materials are being used:

$$\mu_2 = \mu_1 \tag{19.1.2}$$

But this condition is, in experimental model work, often relaxed as long as Poisson's ratios are approximately equal. The natural frequencies of structure 2 are then related to the natural frequencies of structure 1 by

$$\omega_{k2} = \omega_{k1} \frac{a_1}{a_2} \sqrt{\frac{\rho_1 E_2}{\rho_2 E_1}} = \omega_{k1} \frac{a_1}{a_2} \frac{c_2}{c_1} \tag{19.1.3}$$

where ρ is the mass density, E is Young's modulus, and c is the speed of sound. This scaling law, however, proves to be sometimes too restrictive since it is not always convenient, for instance, to scale the thickness of the shell in proportion to typical surface dimensions. Also, important insights into the influence of shell thickness on the frequency spectrum cannot be obtained. Similar restrictions apply, of course, to all types of elastic structures, not only shells.

19.2. DERIVATION OF EXACT SIMILITUDE RELATIONSHIPS FOR NATURAL FREQUENCIES OF THIN SHELLS

Starting with the equations of motion and boundary conditions given in Chapter 2, the following nondimensional expressions are introduced:

$$u_1^* = \frac{u_1}{a},\quad u_2^* = \frac{u_2}{a},\quad u_3^* = \frac{u_3}{a},\quad t^* = \omega_k t$$
$$A_1^* \mathrm{d}\alpha_1^* = \frac{A_2 \mathrm{d}\alpha_1}{a},\quad A_2^* \mathrm{d}\alpha_2^* = \frac{A_2 \mathrm{d}\alpha_2}{a} \tag{19.2.1}$$

The quantity a is a typical shell dimension on the α_1, α_2 surface. From this, it follows that

$$N_{ij}^* = \frac{N_{ij}}{K}\ (i,j=1,2),\quad M_{ij}^* = \frac{M_{ij} a}{D}$$
$$Q_{i3}^* = \frac{Q_{i3} a^2}{D},\quad R_i^* = \frac{R_i}{a},\quad q_i^* = \frac{q_i a^3}{D} \tag{19.2.2}$$

Substituting these quantities into the equations of motion (2.7.20)–(2.7.22) gives

$$12\left(\frac{a}{h}\right)^2 \left(\frac{\partial N_{11}^* A_2^*}{\partial \alpha_1^*} + \frac{\partial N_{12}^* A_1^*}{\partial \alpha_2^*} + N_{12}^* \frac{\partial A_1^*}{\partial \alpha_2^*} - N_{22}^* \frac{\partial A_2^*}{\partial \alpha_1^*}\right) + A_1^* A_2^* \left(\frac{Q_{13}^*}{R_1^*} + q_1^*\right)$$
$$= \frac{\rho}{E} \omega_k \frac{a^4}{h^2} 12(1-\mu^2) A_1^* A_2^* \frac{\partial^2 u_1^*}{\partial t^{*2}} \tag{19.2.3}$$

$$12\left(\frac{a}{h}\right)^2 \left(\frac{\partial N_{12}^* A_2^*}{\partial \alpha_1^*} + \frac{\partial N_{22}^* A_1^*}{\partial \alpha_2^*} + N_{21}^* \frac{\partial A_2^*}{\partial \alpha_1^*} - N_{11}^* \frac{\partial A_1^*}{\partial \alpha_2^*}\right) + A_1^* A_2^* \left(\frac{Q_{23}^*}{R_2^*} + q_2^*\right)$$
$$= \frac{\rho}{E} \omega_k^2 \frac{a^4}{h^2} 12(1-\mu^2) A_1^* A_2^* \frac{\partial^2 u_2^*}{\partial t^{*2}} \tag{19.2.4}$$

$$\frac{\partial Q_{13}^* A_2^*}{\partial \alpha_1^*} + \frac{\partial Q_{23}^* A_1^*}{\partial \alpha_2^*} - 12\left(\frac{a}{h}\right)^2 \left(\frac{N_{11}^*}{R_1^*} + \frac{N_{22}^*}{R_2^*}\right) A_1^* A_2^* - A_1^* A_2^* q_3^*$$
$$= \frac{\rho}{E} \omega_k^2 \frac{a^4}{h^2} 12(1-\mu^2) A_1^* A_2^* \frac{\partial^2 u_3^*}{\partial t^{*2}} \tag{19.2.5}$$

The typical boundary conditions become [since we use the symbol * to designate nondimensional quantities, we use an overbar to replace the asterisk in Eqs. (2.8.11)–(2.8.14)]

$$N_{11}^* = \overline{N}_{11}^*,\quad T_{12}^* = \overline{T}_{12}^*,\quad V_{13}^* = \overline{V}_{13}^*,\quad M_{11}^* = \overline{M}_{11}^* \tag{19.2.6}$$
$$u_1^* = \overline{u}_1^*,\quad u_2^* = \overline{u}_2^*,\quad u_3^* = \overline{u}_3^*,\quad \overline{\beta}_1^* = \overline{\beta}_1^* \tag{19.2.7}$$

The typical nondimensionalized Kirchhoff effective shear values become

$$T_{12}^* = 12\left(\frac{a}{h}\right)^2 N_{12}^* + \frac{M_{12}^*}{R_2^*} \tag{19.2.8}$$

$$V_{13}^* = Q_{13}^* + \frac{1}{A_2^*}\frac{\partial M_{12}^*}{\partial \alpha_2^*} \tag{19.2.9}$$

For the scaling of eigenvalues, we set $q_k^*(k=1,2,3)=0$, which results in the free oscillation form of Eqs. (19.2.3)–(19.2.5). Examining these equations and typical boundary conditions, we see that we can get identical solutions for two similar shells only if the following conditions are satisfied:

$$\frac{a_1}{h_1} = \frac{a_2}{h_2} \tag{19.2.10}$$

$$\frac{\rho_1}{E_1}\omega_{k1}^2\frac{a_1^4}{h_1^2}(1-\mu_1^2) = \frac{\rho_2}{E_2}\omega_{k2}^2\frac{a_2^4}{h_2^2}(1-\mu_2^2) \tag{19.2.11}$$

From this, it follows that a total geometric scaling is required, as stated in Eq. (19.1.1), and that the scaling law is ($\mu_1 = \mu_2$ is still required, however, because the force and moment resultants contain μ)

$$\omega_{k2} = \omega_{k1}\frac{a_1}{a_2}\sqrt{\frac{\rho_1}{\rho_2}\frac{E_2}{E_1}} \tag{19.2.12}$$

Again, note that $c = \sqrt{E/p}$ is the speed of sound. This equation is the same result as the general similitude of Eq. (19.1.3).

Equations (19.2.3)–(19.2.5) also indicate that nondimensional analytical or experimental results have to be plotted in terms of the two nondimensional numbers Z_1 and Z_2:

$$Z_1 = \frac{a}{h} \tag{19.2.13}$$

$$Z_2 = \frac{\rho}{E}\omega_k^2\frac{a^4}{h^2} \tag{19.2.14}$$

assuming that μ is constant.

19.3. PLATES

For plates, the thickness is separable. Letting the curvatures $1/R_1^*$ and $1/R_2^*$ approach 0 results in an uncoupling of in-plane vibrations and transverse vibrations. The in-plane oscillations are described by

$$\frac{\partial N_{11}^* A_2^*}{\partial \alpha_1^*} + \frac{\partial N_{12}^* A_1^*}{\partial \alpha_2^*} + N_{12}^*\frac{\partial A_1^*}{\partial \alpha_2^*} - N_{22}^*\frac{\partial A_2^*}{\partial \alpha_1^*} = \frac{\rho}{E}\omega_k^2 a^2(1-\mu^2)A_1^* A_2^*\frac{\partial^2 u_1^*}{\partial t^{*2}} \tag{19.3.1}$$

$$\frac{\partial N_{12}^* A_2^*}{\partial \alpha_1^*} + \frac{\partial N_{22}^* A_1^*}{\partial \alpha_2^*} + N_{21}^* \frac{\partial A_2^*}{\partial \alpha_1^*} - N_{11}^* \frac{\partial A_1^*}{\partial \alpha_2^*} = \frac{\rho}{E}\omega_k^2 a^2 (1-\mu^2) A_1^* A_2^* \frac{\partial^2 u_2^*}{\partial t^{*2}} \tag{19.3.2}$$

The transverse oscillations are given by

$$\frac{\partial Q_{13}^* A_2^*}{\partial \alpha_1^*} + \frac{\partial Q_{12}^* A_1^*}{\partial \alpha_2^*} = \frac{\rho}{E}\omega_k^2 \frac{a^4}{h^2} 12(1-\mu^2) A_1^* A_2^* \frac{\partial^2 u_3^*}{\partial t^{*2}} \tag{19.3.3}$$

Boundary conditions in terms of the Kirchhoff effective shear resultant of the second kind are now in terms of $\overline{N}_{ij}^*$ and are applicable to in plane oscillations, while V_{i3}^* is associated with the transverse oscillations.

Two exact but different scaling laws can be defined for the plate, depending if transverse oscillations or in-plane oscillations are to be scaled. The scaling law associated with free in-plane oscillations is

$$\frac{\rho_1}{E_1}\omega_{k1}^2 a_1^2 (1-\mu_1^2) = \frac{\rho_2}{E_2}\omega_{k2}^2 a_2^2 (1-\mu_2^2) \tag{19.3.4}$$

based again on the argument that we will only have identical solutions for two similar plates if this relationship is satisfied, which translates into $(\mu_1 = \mu_2)$

$$\omega_{k2} = \omega_{k1} \frac{a_1}{a_2} \sqrt{\frac{\rho_1}{\rho_2} \frac{E_2}{E_2}} \tag{19.3.5}$$

This appears to be the same as Eq. (19.2.12) but with the important difference that the plate thickness does not enter the derivation of the similitude relationship anywhere. Therefore, the condition (19.2.10) is not required. This means that in-plane natural frequencies of plates are independent of thickness. Thickness changes will not affect the in-plane natural frequency spectrum of a plate. The reason is, of course, that increasing a plate's thickness adds, for in-plane motion, the same stiffness-to-mass ratio.

For transverse vibration, we obtain identical results for two plates similar in shape and boundary conditions if (Kristiansen et al., 1972)

$$\frac{\rho_1}{E_1}\omega_{k1}^2 \frac{a_1^4}{h_1^2}(1-\mu_1^2) = \frac{\rho_2}{E_2}\omega_{k2} \frac{a_2^4}{h_2^2}(1-\mu_2^2) \tag{19.3.6}$$

or $(\mu_1 = \mu_2)$

$$\omega_{k2} = \omega_{k1} \frac{h_2}{h_1} \left(\frac{a_1}{a_2}\right)^2 \sqrt{\frac{\rho_1}{\rho_2} \frac{E_2}{E_1}} \tag{19.3.7}$$

It follows that if one doubles the thickness of any thin plate, all natural frequencies for transverse motion double in magnitude. Increasing the surface dimension of any thin plate (keeping its shape and boundary conditions similar) will change the natural frequencies by the square of this change.

It should be noted that for transversely vibrating plates whose boundary conditions are not of the bending moment or Kirchhoff shear type, Poisson's ratio is not required to be constant, so that Eq. (19.3.7) becomes

$$\omega_{k2}=\omega_{k1}\frac{h_2}{h_1}\left(\frac{a_1}{a_2}\right)^2\sqrt{\frac{\rho_1}{\rho_2}\frac{E_2}{E_1}\frac{1-\mu_1^2}{1-\mu_2^2}} \tag{19.3.8}$$

This can be shown by converting Eq. (19.3.3) into the displacement form.

19.4. SHALLOW SPHERICAL PANELS OF ARBITRARY CONTOURS (INFLUENCE OF CURVATURE)

Specializing Eq. (6.8.9) to $R_1=R_2=R$ gives

$$\nabla^4\left[D\nabla^4U_3+\left(\frac{Eh}{R^2}-\rho h\omega_k^2\right)U_3\right]=0 \tag{19.4.1}$$

or, without loss of generality,

$$D\nabla^4U_3+\left(\frac{Eh}{R^2}-\rho h\omega_k^2\right)U_3=0 \tag{19.4.2}$$

The coordinates α_2 and α_2 are commonly selected to follow the plane projection of the panel and are therefore not spherical coordinates. For example, in the case of a circular or annular contour, one uses polar coordinates; for rectangular or triangular contours, Cartesian coordinates; for elliptical boundary contours, elliptical coordinates; and so on. Upon introduction of the nondimensional expressions

$$A_1^*\mathrm{d}\alpha_1^*=\frac{A_1\mathrm{d}\alpha_1}{a} \tag{19.4.3}$$

$$A_2^*\mathrm{d}\alpha_2^*=\frac{A_2\mathrm{d}\alpha_2}{a} \tag{19.4.4}$$

$$U_3^*=\frac{U_3}{a} \tag{19.4.5}$$

where a is a typical panel projection dimension (Fig. 2), Eq. (19.4.2) becomes

$$\nabla_*^4U_3^*-\frac{a^4}{D}\left(\rho h\omega_k^2-\frac{Eh}{R^2}\right)U_3^*=0 \tag{19.4.6}$$

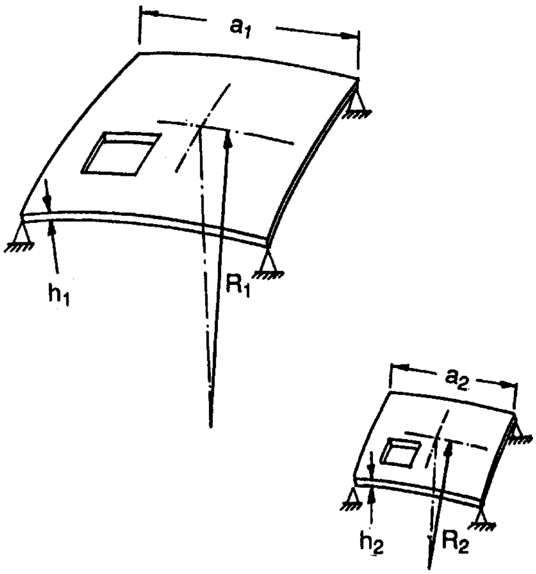

FIG. 2 Similar shallow spherical shells.

where

$$\nabla_*^2 = \frac{1}{A_1^* A_2^*}\left(\frac{\partial}{\partial\alpha_1^*}\frac{A_2^*}{A_1^*}\frac{\partial}{\partial\alpha_1^*} + \frac{\partial}{\partial\alpha_2^*}\frac{A_1^*}{A_2^*}\frac{\partial}{\partial\alpha_2^*}\right) \tag{19.4.7}$$

Examining Eq. (19.4.6) reveals that all spherically curved panels of similar boundary shape and boundary condition distribution (Fig. 2) will have the same solution in u_3^*, provided that they all have the same numerical value of the factor

$$\frac{a^4}{D}\left(\rho h \omega_k^2 - \frac{Eh}{R^2}\right) \tag{19.4.8}$$

and for cases where moments of Kirchhoff shear conditions exist, of μ.

With the subscript 1 assigned to all parameters of spherically curved panel 1 and the subscript 2 to all parameters of spherically curved panel 2, it is required that

$$\frac{a_1^4}{D_1}\left(\rho_1 h_1 \omega_{k1}^2 - \frac{E_1 h_1}{R_1^2}\right) = \frac{a_2^4}{D_2}\left(\rho_2 h_2 \omega_{k2}^2 - \frac{E_2 h_2}{R_2^2}\right) \tag{19.4.9}$$

Similarly to the transversely vibrating plate, only if there are boundary conditions of the bending moment or Kirchhoff shear type is it also required that

$$\mu_1 = \mu_2 \tag{19.4.10}$$

Assuming this to be the case, Eq. (19.4.9) can also be written as

$$\omega_{k2}=\sqrt{\left(\frac{a_1}{a_2}\right)^4\frac{E_2}{E_1}\left(\frac{h_2}{h_1}\right)^2\left(\frac{\rho_1}{\rho_2}\omega_{k1}^2-\frac{E_1}{\rho_2 R_1^2}\right)+\frac{E_2}{\rho_2 R_2^2}} \tag{19.4.11}$$

Since the plate equation is a subcase of Eq. (19.4.2), Eq. (19.4.11) may be written such that panel 1 is a flat plate $(1/R_1=0)$:

$$\omega_{k2}=\sqrt{\left(\frac{a_1}{a_2}\right)^4\frac{E_2}{E_1}\left(\frac{h_2}{h_1}\right)^2\frac{\rho_1}{\rho_2}\omega_{k1}^2+\frac{E_2}{\rho_2 R_2^2}} \tag{19.4.12}$$

The effect of curvature may be isolated by taking the plate and the spherically curved panel 2 to be of the same material, the same thickness, and the same projection dimensions. Equation (19.4.12) then reduces to

$$\omega_{k2}=\sqrt{\omega_{k1}^2+\frac{E}{\rho R^2}} \tag{19.4.13}$$

Therefore, if the natural frequencies ω_{k1} of some plate are known, immediately the solutions for the similar spherically curved panel are also known. For example, if a spherically curved panel whose contour is rectangular and whose boundary conditions are of the simply supported type is considered, then since the natural frequencies for the simply supported rectangular plate are known to be $(\rho, q=1,2,\ldots)$

$$\omega_{k1}^2=\omega_{pq1}^2=\pi^4\left(\frac{p^2}{a^2}+\frac{q^2}{b^2}\right)^2\frac{D}{\rho h} \tag{19.4.14}$$

the solutions for the curved panel are (Soedel, 1973)

$$\omega_{k2}=\omega_{pq^2}=\sqrt{\pi^4\left(\frac{p^2}{a^2}+\frac{q^2}{b^2}\right)^2\frac{D}{\rho h}+\frac{E}{\rho R^2}} \tag{19.4.15}$$

19.5. FORCED RESPONSE

Examining Eqs. (19.2.1)–(19.2.5) shows that we obtain similar solutions for similar excitations in space and time if in addition to satisfying Eqs. (19.2.10), (19.2.11) and (19.2.13), we also scale excitation in amplitude according to

$$q_{i2}=q_{i1}\frac{E_2}{E_1} \tag{19.5.1}$$

in time according to

$$t_2 = t_1 \frac{\omega_{k1}}{\omega_{k2}} \tag{19.5.2}$$

and in space according to the ratio a_2/a_1.

Satisfying these relationships means that the response will be similar. This is of interest in experimental work. The response is then

$$u_{i2} = u_{i1} \frac{a_2}{a_1} \tag{19.5.3}$$

at the time scale

$$t_2 = t_1 \frac{\omega_{k1}}{\omega_{k2}} \tag{19.5.4}$$

Special relationships for special cases (e.g., plates) can be developed that allow the inclusions of other parameters into the scaling relationships (Soedel, 1964).

19.6. APPROXIMATE SCALING OF SHELLS CONTROLLED BY MEMBRANE STIFFNESS

The plate and shallow shell similitude relationships realize what one would like to accomplish for shells in general: namely, to introduce the thickness dependency into the scaling law. This can be done by following the lead of the shell membrane theory where it is assumed that for certain classes of shells and frequencies (or certain frequency bands), it is permissible to neglect the influence of bending resistance on shell oscillations.

For the cases where membrane influences are dominant, Eqs. (19.2.3)–(19.2.5) become

$$\frac{\partial N_{11}^* A_2^*}{\partial \alpha_1^*} + \frac{\partial N_{12}^* A_1^*}{\partial \alpha_2^*} + N_{12}^* \frac{\partial A_1^*}{\partial \alpha_2^*} - N_{22}^* \frac{\partial A_2^*}{\partial \alpha_1^*} + \frac{1}{12}\left(\frac{h}{a}\right)^2 A_1^* A_2^* q_1^*$$

$$= \frac{\rho}{E}\omega_k^2 a^2 (1-\mu^2) A_1^* A_2^* \frac{\partial^2 u_1^*}{\partial t^{*2}} \tag{19.6.1}$$

$$\frac{\partial N_{12}^* A_2^*}{\partial \alpha_1^*} + \frac{\partial N_{22}^* A_1^*}{\partial \alpha_2^*} + N_{21}^* \frac{\partial A_2^*}{\partial \alpha_1^*} - N_{11}^* \frac{\partial A_1^*}{\partial \alpha_2^*} + \frac{1}{12}\left(\frac{h}{a}\right)^2 A_1^* A_2^* q_2^*$$

$$= \frac{\rho}{E}\omega_k^2 a^2 (1-\mu^2) A_1^* A_2^* \frac{\partial^2 u_2^*}{\partial t^{*2}} \tag{19.6.2}$$

$$-\left(\frac{N_{11}^*}{R_1^*} + \frac{N_{22}^*}{R_2^*}\right) A_1^* A_2^* - \frac{1}{12}\left(\frac{h}{a}\right)^2 A_1^* A_2^* q_3^* = \frac{\rho}{E}\omega_k^2 a^2 (1-\mu^2) A_1^* A_2^* \frac{\partial^2 u_3^*}{\partial t^{*2}} \tag{19.6.3}$$

For free vibration, $q_1^* = q_2^* = q_3^* = 0$, and the equations will have the same solutions for two similar shells if

$$\frac{\rho_1}{E_1}\omega_{k1}^2 a_1^2 (1-\mu_1^2) = \frac{\rho_2}{E_2}\omega_{k2}^2 a_2^2 (1-\mu_2^2) \tag{19.6.4}$$

or ($\mu_1 = \mu_2$)

$$\omega_{k2} = \omega_{k1}\frac{a_1}{a_2}\sqrt{\frac{\rho_1}{\rho_2}\frac{E_2}{E_1}} \tag{19.6.5}$$

There is no requirement, for the free-vibrant case, that $h_1/a_1 = h_2/a_2$. This means that in the region where this scaling law is applicable, thickness changes will have no appreciable effects on natural frequencies. This was shown, by a circular cylindrical shell example, in Chapter 5.

19.7. APPROXIMATE SCALING OF SHELLS CONTROLLED BY BENDING STIFFNESS

With increasing mode number, bending influences will eventually start to dominate in most shells. Equations (19.2.3)–(19.2.5) will approach the forms

$$\frac{Q_{13}^*}{R_1^*} + q_1^* = \frac{\rho}{E}\omega_k^2 \frac{a^4}{h^2} 12(1-\mu^2)\frac{\partial^2 u_1^*}{\partial t^{*2}} \tag{19.7.1}$$

$$\frac{Q_{23}^*}{R_2^*} + q_2^* = \frac{\rho}{E}\omega_k^2 \frac{a^4}{h^2} 12(1-\mu^2)\frac{\partial^2 u_2^*}{\partial t^{*2}} \tag{19.7.2}$$

$$\frac{\partial Q_{13}^* A_2^*}{\partial \alpha_1^*} + \frac{\partial Q_{23}^* A_1^*}{\partial \alpha_2^*} - A_1^* A_2^* q_3^* = \frac{\rho}{E}\omega_k^2 \frac{a^4}{h^2} 12(1-\mu^2) A_1^* A_2^* \frac{\partial^2 u_3^*}{\partial t^{*2}} \tag{19.7.3}$$

The scaling law for eigenvalues ($q_1^* = q_2^* = q_3^* = 0$) for this class of cases is, therefore,

$$\frac{\rho_1}{E_1}\omega_{k1}^2 \frac{a_1^4}{h_1^2}(1-\mu_1^2) = \frac{\rho_2}{E_2}\omega_{k2}^2 \frac{a_2^4}{h_2^2}(1-\mu_2^2) \tag{19.7.4}$$

or ($\mu_1 = \mu_2$)

$$\omega_{k2} = \omega_{k1}\left(\frac{a_1}{a_2}\right)^2 \frac{h_2}{h_1}\sqrt{\frac{\rho_1}{\rho_2}\frac{E_2}{E_1}} \tag{19.7.5}$$

Whenever this scaling law is applicable, natural frequencies increase in proportion to thickness changes.

Fig. 3 shows for a typical simply supported circular cylindrical shell, the frequency regions for dominant transverse motion divided into membrane stiffness and bending stiffness controlled regions. Unfortunately,

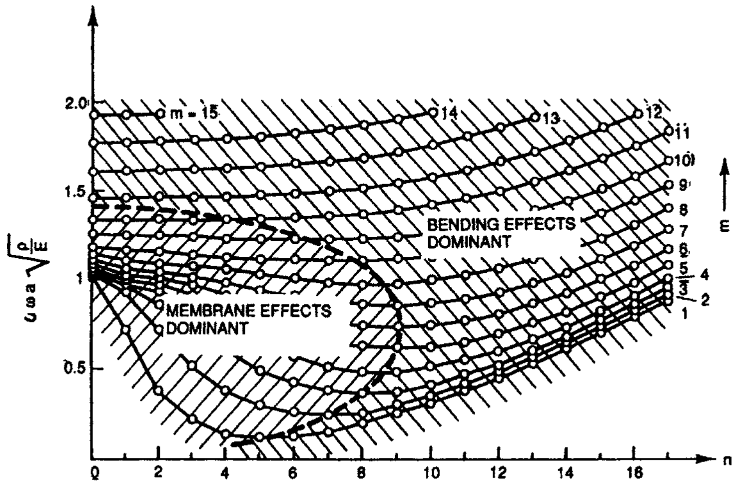

FIG. 3 Membrane and bending effect dominated regions of the natural frequencies of a simply supported, circular cylindrical shell.

the most interesting natural frequencies are usually the very lowest, which are influenced by both membrane and bending effects. Neither one of the two approximate scaling laws applies, except that one may state that in this region, for $h_2 > h_1$,

$$\omega_{k1}\left(\frac{a_1}{a_2}\right)^2\sqrt{\frac{\rho_1}{\rho_2}\frac{E_2}{E_1}} < \omega_{k2} < \omega_{k1}\left(\frac{a_1}{a_2}\right)^2\frac{h_2}{h_1}\sqrt{\frac{\rho_1}{\rho_2}\frac{E_2}{E_1}} \tag{19.7.6}$$

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20

Interactions with Liquids and Gases

In the following, derivation of the vibroelastic equations of motion of three-dimensional solids is presented, utilizing curvilinear coordinates and Hamilton's principle. Similarities to the derivation of Love's equations in Chapter 2 are emphasized for educational purposes. The wave equation forms are also developed. Then shear stresses are eliminated and the three-dimensional wave equation for an inviscid stationary acoustic medium is obtained by reduction. Finally, to describe the behavior of free surface liquids, compressibility is eliminated. Necessary interface boundary conditions are discussed and two examples are given.

20.1. FUNDAMENTAL FORM IN THREE-DIMENSIONAL CURVILINEAR COORDINATES

Obviously, it is now necessary to use

$$x_1 = f_1(\alpha_1, \alpha_2, \alpha_3), \quad x_2 = f_2(\alpha_1, \alpha_2, \alpha_3), \quad x_3 = f_3(\alpha_1, \alpha_2, \alpha_3) \tag{20.1.1}$$

instead of Eq. (2.1.1). The location of a point P can be expressed as (Fig. 1)

$$\bar{r}(\alpha_1, \alpha_2, \alpha_3) = f_1(\alpha_1, \alpha_2, \alpha_3)\bar{e}_1 + f_2(\alpha_1, \alpha_2, \alpha_3)\bar{e}_3 + f_3(\alpha_1, \alpha_2, \alpha_3)\bar{e}_3 \tag{20.1.2}$$

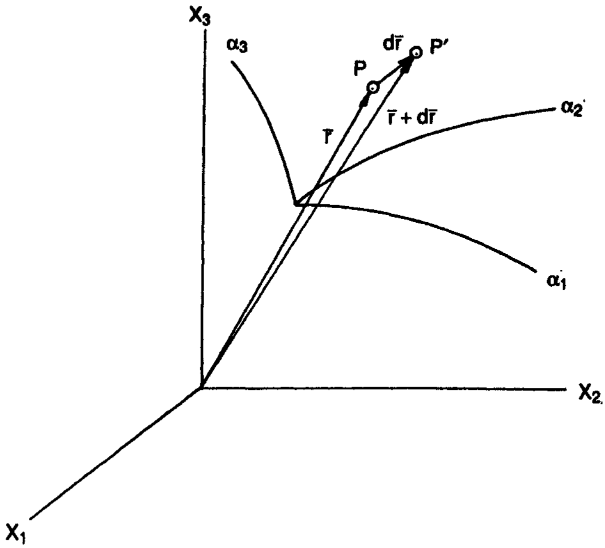

FIG. 1 Curvilinear coordinate definitions for finding Lame parameters for three dimensional solids.

The differential change $d\bar{r}$ is

$$d\bar{r} = \frac{\partial \bar{r}}{\partial \alpha_1} d\alpha_1 + \frac{\partial \bar{r}}{\partial \alpha_2} d\alpha_2 + \frac{\partial \bar{r}}{\partial \alpha_3} d\alpha_3 \tag{20.1.3}$$

The magnitude ds of $d\bar{r}$ is

$$(ds)^2 = d\bar{r} \cdot d\bar{r} = \frac{\partial \bar{r}}{\partial \alpha_1} \cdot \frac{\partial \bar{r}}{\partial \alpha_1} (d\alpha_1)^2 + \frac{\partial \bar{r}}{\partial \alpha_2} \cdot \frac{\partial \bar{r}}{\partial \alpha_2} (d\alpha_2)^2 + \frac{\partial \bar{r}}{\partial \alpha_3} \cdot \frac{\partial \bar{r}}{\partial \alpha_3} (d\alpha_3)^2 \tag{20.1.4}$$

if we specify that the coordinates α_1, α_2 and α_3 are orthogonal. Defining

$$\frac{\partial \bar{r}}{\partial \alpha_1} \cdot \frac{\partial \bar{r}}{\partial \alpha_1} = \left| \frac{\partial \bar{r}}{\partial \alpha_1} \right|^2 = A_1^2 \tag{20.1.5}$$

$$\frac{\partial \bar{r}}{\partial \alpha_2} \cdot \frac{\partial \bar{r}}{\partial \alpha_2} = \left| \frac{\partial \bar{r}}{\partial \alpha_2} \right|^2 = A_2^2 \tag{20.1.6}$$

$$\frac{\partial \bar{r}}{\partial \alpha_3} \cdot \frac{\partial \bar{r}}{\partial \alpha_3} = \left| \frac{\partial \bar{r}}{\partial \alpha_3} \right|^2 = A_3^2 \tag{20.1.7}$$

we obtain the fundamental form equation

$$(ds)^2 = A_1^2 (d\alpha_1)^2 + A_2^2 (d\alpha_2)^2 + A_3^2 (d\alpha_3)^2 \tag{20.1.8}$$

Again, as in Chapter 2, we may obtain the Lamé parameters A_1, A_2 and A_3 either by way of the mathematical definitions (20.1.5)–(20.1.7) or by direct inspection. For instance, in Cartesian coordinates,

$$\alpha_1 = x,\ \ \alpha_2 = y,\ \ \alpha_3 = z,\ \ A_1 = A_2 = A_3 = 1 \tag{20.1.9}$$

For cylindrical coordinates

$$\alpha_1 = r,\ \ \alpha_2 = \theta,\ \ \alpha_3 = x,\ \ A_1 = 1,\ \ A_2 = r,\ \ A_3 = 1 \tag{20.1.10}$$

For spherical coordinates

$$\alpha_1 = r,\ \ \alpha_2 = \phi,\ \ \alpha_3 = 0,\ \ A_1 = 1,\ \ A_2 = r,\ \ A_3 = r\sin\phi \tag{20.1.11}$$

20.2. STRESS-STRAIN-DISPLACEMENT RELATIONSHIPS

The strain-stress relationships are, as in Chapter 2,

$$\varepsilon_{11} = \frac{1}{E}[\sigma_{11} - \mu(\sigma_{22} + \sigma_{33})] \tag{20.2.1}$$

$$\varepsilon_{22} = \frac{1}{E}[\sigma_{22} - \mu(\sigma_{11} + \sigma_{33})] \tag{20.2.2}$$

$$\varepsilon_{33} = \frac{1}{E}[\sigma_{33} - \mu(\sigma_{11} + \sigma_{22})] \tag{20.2.3}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{G} = \varepsilon_{21} \tag{20.2.4}$$

$$\varepsilon_{13} = \frac{\sigma_{13}}{G} = \varepsilon_{31} \tag{20.2.5}$$

$$\varepsilon_{23} = \frac{\sigma_{23}}{G} = \varepsilon_{32} \tag{20.2.6}$$

where

$$\sigma_{12} = \sigma_{21},\ \ \sigma_{13} = \sigma_{31},\ \ \sigma_{13} = \sigma_{32} \tag{20.2.7}$$

This may be inverted to give

$$\sigma_{11} = 2G\varepsilon_{11} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \tag{20.2.8}$$

$$\sigma_{22} = 2G\varepsilon_{22} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \tag{20.2.9}$$

$$\sigma_{33} = 2G\varepsilon_{33} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \tag{20.2.10}$$

$$\sigma_{12} = G\varepsilon_{12} = \sigma_{21} \tag{20.2.11}$$

$$\sigma_{13} = G\varepsilon_{13} = \sigma_{31} \tag{20.2.12}$$

$$\sigma_{23} = G\varepsilon_{23} = \sigma_{32} \tag{20.2.13}$$

where

$$\lambda=\frac{\mu E}{(1+\mu)(1-2\mu)}$$

and

$$\varepsilon_{12}=\varepsilon_{21},\ \ \varepsilon_{13}=\varepsilon_{31},\ \ \varepsilon_{23}=\varepsilon_{32} \tag{20.2.14}$$

It should be noted that shear strain is defined here to be the total strain angle. Let us now consider the fundamental form. It may be written as

$$(\mathrm{d}s)^2=\sum_{i=1}^{3}A_i^2(\alpha_1,\alpha_2,\alpha_3)(\mathrm{d}\alpha_i)^2 \tag{20.2.15}$$

If point P, originally located at $(\alpha_1,\alpha_2,\alpha_3)$, is deflected in the α_1 direction by u_1, in the α_2 direction by u_2, and in the α_3 direction by u_3, it will be located at $(\alpha_1+\xi_1,\alpha_2+\xi_2,\alpha_3+\xi_3)$. Deflections u_i and coordinate changes ξ_i and related by

$$u_i=A_i(\alpha_1,\alpha_2,\alpha_3)\xi_i \tag{20.2.16}$$

A point P', located an infinitesimal distance from point P at $(\alpha_1+\mathrm{d}\alpha_1,\alpha_2+\mathrm{d}\alpha_2,\alpha_3+\mathrm{d}\alpha_3)$ will be located, after deflection, at $(\alpha_1+\mathrm{d}\alpha_1+\xi_1+\mathrm{d}\xi_1,\alpha_2+\mathrm{d}\alpha_2+\xi_2+\mathrm{d}\xi_2,\alpha_3+\mathrm{d}\alpha_3+\xi_3+\mathrm{d}\xi_3)$. The distance $\mathrm{d}s'$ between P and P' in the deflected state will be

$$(\mathrm{d}s')^2=\sum_{i=1}^{3}A_i^2(\alpha_1+\xi_1,\alpha_2+\xi_2,\alpha_3+\xi_3)(\mathrm{d}\alpha_i+\mathrm{d}\xi_i)^2 \tag{20.2.17}$$

Since $A_i(\alpha_1,\alpha_2,\alpha_3)$ varies in a continuous fashion as α_1,α_2, and α_3 change, a Taylor series expansion of $A_i^2(\alpha_1+\xi_1,\alpha_2+\xi_2,\alpha_3+\xi_3)$ about the point $(\alpha_1,\alpha_2,\alpha_3)$ is in order:

$$A_i^2(\alpha_1+\xi_1,\alpha_2+\xi_2,\alpha_3+\xi_3)=A_i^2(\alpha_1,\alpha_2,\alpha_3)+\sum_{j=1}^{3}\frac{\partial A_i^2(\alpha_1,\alpha_2,\alpha_3)}{\partial\alpha_j}\xi_j \tag{20.2.18}$$

One may also expand

$$(\mathrm{d}\alpha_i+\mathrm{d}\xi_i)^2=(\mathrm{d}\alpha_i)^2+2\mathrm{d}\alpha_i\mathrm{d}\xi_i \tag{20.2.19}$$

neglecting the $(\mathrm{d}\xi_i)^2$ term. The differential $\mathrm{d}\xi_i$ may be written as

$$\mathrm{d}\xi_i=\sum_{j=1}^{3}\frac{\partial\xi_i}{\partial\alpha_j}\mathrm{d}\alpha_j \tag{20.2.20}$$

Substituting all this in Eq. (20.2.17) gives, dropping the notation $A_i(\alpha_1, \alpha_2, \alpha_3)$ in favor simply of A_i.

$$(\mathrm{d}s')^2 = \sum_{i=1}^{3}\left[\left(A_i^2 + \sum_{j=1}^{3}\frac{\partial A_i^2}{\partial \alpha_j}\xi_j\right)(\mathrm{d}\alpha_i)^2 + 2\mathrm{d}\alpha_i A_i^2 \sum_{j=1}^{3}\frac{\partial \xi_i}{\partial \alpha_j}\mathrm{d}\alpha_j + 2\mathrm{d}\alpha_i \sum_{j=1}^{3}\frac{\partial A_i^2}{\partial \alpha_j}\xi_j \sum_{j=1}^{3}\frac{\partial \xi_i}{\partial \alpha_j}\mathrm{d}\alpha_j\right] \tag{20.2.21}$$

The last term is negligible except for cases where high initial stresses exist in the solid. Thus

$$2\mathrm{d}\alpha_i \sum_{j=1}^{3}\frac{\partial A_i^2}{\partial \alpha_j}\xi_j \sum_{j=1}^{3}\frac{\partial \xi_i}{\partial \alpha_j}\mathrm{d}\alpha_j \cong 0 \tag{20.2.22}$$

Utilizing the Kronecker delta notation, $\delta_{ij} = 1, i = j$, and $\delta_{ij} = 0, i \neq j$, the first term may be written

$$\sum_{i=1}^{3}\left(A_i^2 + \sum_{j=1}^{3}\frac{\partial A_i^2}{\partial \alpha_j}\xi_j\right)(\mathrm{d}\alpha_i)^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\left(A_i^2 + \sum_{k=1}^{3}\frac{\partial A_i^2}{\partial \alpha_k}\xi_k\right)\delta_{ij}\mathrm{d}\alpha_i\mathrm{d}\alpha_j \tag{20.2.23}$$

The second term of Eq. (20.2.21) may be written

$$\sum_{i=1}^{3}2\mathrm{d}\alpha_i A_i^2 \sum_{j=1}^{3}\frac{\partial \xi_i}{\partial \alpha_j}\mathrm{d}\alpha_j = \sum_{i=1}^{3}\sum_{j=1}^{3}A_i^2\frac{\partial \xi_i}{\partial \alpha_j}\mathrm{d}\alpha_j\mathrm{d}\alpha_i + \sum_{i=1}^{3}\sum_{j=1}^{3}A_j^2\frac{\partial \xi_j}{\partial \alpha_i}\mathrm{d}\alpha_i\mathrm{d}\alpha_j \tag{20.2.24}$$

Therefore, Eq. (20.2.21) may be written as

$$(\mathrm{d}s')^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}B_{ij}^2\mathrm{d}\alpha_i\mathrm{d}\alpha_j \tag{20.2.25}$$

where

$$B_{ij}^2 = \left(A_i^2 + \sum_{k=1}^{3}\frac{\partial A_i^2}{\partial \alpha_k}\xi_k\right)\delta_{ij} + A_i^2\frac{\partial \xi_i}{\partial \alpha_j} + A_j^2\frac{\partial \xi_j}{\partial \alpha_i} \tag{20.2.26}$$

The normal strains ε_{ii} are

$$\varepsilon_{ii} = \frac{(\mathrm{d}s')_{ii} - (\mathrm{d}s)_{ii}}{(\mathrm{d}s_{ii})} \tag{20.2.27}$$

where

$$(\mathrm{d}s)_{ii}^2 = A_i^2(\mathrm{d}\alpha_i)^2 \tag{20.2.28}$$

$$(\mathrm{d}s')_{ii}^2 = B_{ii}^2(\mathrm{d}\alpha_i)^2 \tag{20.2.29}$$

Therefore,

$$\varepsilon_{ii} = \sqrt{\frac{B_{ii}^2}{A_i^2}} - 1 = \sqrt{1 + \frac{B_{ii}^2 - A_i^2}{A_i^2}} - 1 \tag{20.2.30}$$

and nothing that

$$\frac{B_{ii}^2 - A_i^2}{A_i^2} \ll 1 \tag{20.2.31}$$

one obtains after expanding the square root and neglecting higher-order terms of the series,

$$\varepsilon_{ii} = \frac{1}{2}\frac{B_{ii}^2 - A_i^2}{A_i^2} \tag{20.2.32}$$

Thus, we obtain, substituting Eqs.(20.2.16) and (20.2.26) in Eq.(20.2.32),

$$\varepsilon_{11} = \frac{1}{A_1}\left(\frac{\partial A_1}{\partial \alpha_1}\frac{u_1}{A_1} + \frac{\partial A_1}{\partial \alpha_2}\frac{u_2}{A_2} + \frac{\partial A_1}{\partial \alpha_3}\frac{u_3}{A_3}\right) + \frac{\partial}{\partial \alpha_1}\left(\frac{u_1}{A_1}\right) \tag{20.2.33}$$

$$\varepsilon_{22} = \frac{1}{A_2}\left(\frac{\partial A_2}{\partial \alpha_1}\frac{u_1}{A_1} + \frac{\partial A_2}{\partial \alpha_2}\frac{u_2}{A_2} + \frac{\partial A_2}{\partial \alpha_3}\frac{u_3}{A_3}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{u_2}{A_2}\right) \tag{20.2.34}$$

$$\varepsilon_{33} = \frac{1}{A_3}\left(\frac{\partial A_3}{\partial \alpha_1}\frac{u_1}{A_1} + \frac{\partial A_3}{\partial \alpha_2}\frac{u_2}{A_2} + \frac{\partial A_3}{\partial \alpha_3}\frac{u_3}{A_3}\right) + \frac{\partial}{\partial \alpha_3}\left(\frac{u_3}{A_3}\right) \tag{20.2.35}$$

Shear strains are defined as the angular change of an infinitesimal element,

$$\varepsilon_{ij} = \frac{\pi}{2} - \theta_{ij} \tag{20.2.36}$$

where $i \neq j$ and θ_{ij} is the angle between element surfaces normal to the i and j directions after deflection. The angle θ_{ij} may be obtained by applying the cosine formula

$$(\mathrm{d}s')_{ij}^2 = (\mathrm{d}s')_{ii}^2 + (\mathrm{d}s')_{jj}^2 - 2(\mathrm{d}s')_{ii}(\mathrm{d}s')_{jj}\cos\theta_{ij} \tag{20.2.37}$$

Since from Eq.(20.2.25),

$$(\mathrm{d}s')_{ij}^2 = B_{ii}^2(\mathrm{d}\alpha_i)^2 + B_{jj}^2(\mathrm{d}\alpha_j)^2 - 2B_{ij}^2\,\mathrm{d}\alpha_i\,\mathrm{d}\alpha_j \tag{20.2.38}$$

where $i \neq j$, we obtain from Eq.(20.2.37),

$$\cos\theta_{ij} = \frac{B_{ij}^2}{B_{ii}B_{jj}} = \sin\varepsilon_{ij} \tag{20.2.39}$$

Since for reasonably small shear strain magnitudes,

$$\sin\varepsilon_{ij} = \varepsilon_{ij} \tag{20.2.40}$$

and

$$\frac{B_{ij}^2}{B_{ii}B_{jj}} \cong \frac{B_{ij}^2}{A_iA_j} \tag{20.2.41}$$

the shear stain is

$$\varepsilon_{ij} = \frac{B_{ij}^2}{A_i A_j} \tag{20.2.42}$$

Therefore,

$$\varepsilon_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_2}{A_2}\right) \tag{20.2.43}$$

$$\varepsilon_{13} = \frac{A_1}{A_3}\frac{\partial}{\partial\alpha_3}\left(\frac{u_1}{A_1}\right) + \frac{A_3}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{u_3}{A_3}\right) \tag{20.2.44}$$

$$\varepsilon_{23} = \frac{A_2}{A_3}\frac{\partial}{\partial\alpha_3}\left(\frac{u_2}{A_2}\right) + \frac{A_3}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{u_3}{A_3}\right) \tag{20.2.45}$$

20.3. ENERGY EXPRESSIONS

The strain energy stored in an elastic body is

$$U = \frac{1}{2}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + \sigma_{12}\varepsilon_{12} + \sigma_{13}\varepsilon_{13} + \sigma_{23}\varepsilon_{23}) \times A_1 A_2 A_3 \,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.3.1}$$

The kinetic energy is

$$K = \frac{1}{2}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\rho(\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^3) A_1 A_2 A_3 \,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.3.2}$$

The variation of energy input to the elastic body by boundary stresses is, on typical surfaces,

$$\begin{aligned}\delta E_B = &\int_{\alpha_1}\int_{\alpha_2}(\sigma_{13}^*\delta u_1^* + \sigma_{23}^*\delta u_2^* + \sigma_{33}^*\delta u_3^*) A_1 A_2 \,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \\ &+ \int_{\alpha_1}\int_{\alpha_3}(\sigma_{12}^*\delta u_1^* + \sigma_{22}^*\delta u_2^* + \sigma_{23}^*\delta u_3^*) A_1 A_3 \,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_3 \\ &+ \int_{\alpha_2}\int_{\alpha_3}(\sigma_{11}^*\delta u_1^* + \sigma_{12}^*\delta u_2^* + \sigma_{13}^*\delta u_3^*) A_2 A_3 \,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\end{aligned} \tag{20.3.3}$$

The variation of energy introduced by body forces is

$$\delta E_L = \int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}(q_1\delta u_1 + q_2\delta u_2 + q_3\delta u_3) A_1 A_2 A_3 \,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.3.4}$$

where q_i is a force per unit volume in α_i direction.

20.4. EQUATIONS OF MOTION OF VIBROELASTICITY WITH SHEAR

Hamilton's principle may be written as

$$\int_{t_0}^{t_1} (\delta U - \delta K - \delta E_B - \delta E_L)\,\mathrm{d}t = 0 \tag{20.4.1}$$

The times t_1 and t_0 are arbitrary, except that at $t = t_1$ and $t = t_0$, all variations are 0. The variational symbol δ is treated mathematically as a differential symbol. Variational displacements are arbitrary and independent.

In the following, the various terms are examined one by one. The strain energy variation becomes

$$\delta U = \int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\left(\frac{\partial F}{\partial \varepsilon_{11}}\delta\varepsilon_{11} + \frac{\partial F}{\partial \varepsilon_{22}}\delta\varepsilon_{22} + \frac{\partial F}{\partial \varepsilon_{33}}\delta\varepsilon_{33} + \frac{\partial F}{\partial \varepsilon_{12}}\delta\varepsilon_{12} + \frac{\partial F}{\partial \varepsilon_{13}}\delta\varepsilon_{13} + \frac{\partial F}{\partial \varepsilon_{23}}\delta\varepsilon_{23}\right) A_1 A_2 A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.4.2}$$

where

$$F = \frac{1}{2}(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + \sigma_{12}\varepsilon_{12} + \sigma_{13}\varepsilon_{13} + \sigma_{23}\varepsilon_{23}) \tag{20.4.3}$$

Examining the first term of Eq. (20.4.2) gives

$$\frac{\partial F}{\partial \varepsilon_{11}}\delta\varepsilon_{11} = \frac{1}{2}\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial \varepsilon_{11}}\varepsilon_{11} + \frac{\partial \sigma_{22}}{\partial \varepsilon_{11}}\varepsilon_{22} + \frac{\partial \sigma_{33}}{\partial \varepsilon_{11}}\varepsilon_{33}\right)\delta\varepsilon_{11} = \sigma_{11}\delta\varepsilon_{11} \tag{20.4.4}$$

Thus

$$\delta U = \int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}(\sigma_{11}\delta\varepsilon_{11} + \sigma_{22}\delta\varepsilon_{22} + \sigma_{33}\delta\varepsilon_{33} + \sigma_{12}\delta\varepsilon_{12} + \sigma_{13}\delta\varepsilon_{13} + \sigma_{23}\delta\varepsilon_{23}) \times A_1 A_2 A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.4.5}$$

Let us again examine the first term as a typical normal stress term. Since, from Eq.(20.2.33),

$$\delta\varepsilon_{11} = \frac{1}{A_1}\left(\frac{\partial A_1}{\partial \alpha_1}\frac{\delta u_1}{A_1} + \frac{\partial A_1}{\partial \alpha_2}\frac{\delta u_2}{A_2} + \frac{\partial A_1}{\partial \alpha_3}\frac{\delta u_3}{A_3}\right) + \frac{\partial}{\partial \alpha_1}\left(\frac{\delta u_1}{A_1}\right) \tag{20.4.6}$$

the first term of Eq.(20.4.5) becomes

$$\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\sigma_{11}\delta\varepsilon_{11}A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3$$
$$=\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\frac{\sigma_{11}}{A_1}\left(\frac{\partial A_1}{\partial\alpha_1}\frac{\delta u_1}{A_1}+\frac{\partial A_1}{\partial\alpha_2}\frac{\delta u_2}{A_2}+\frac{\partial A_1}{\partial\alpha_3}\frac{\delta u_3}{A_3}\right)$$
$$\times A_1A_2A_3\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3+\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\sigma_{11}\frac{\partial}{\partial\alpha_1}\left(\frac{\delta u_1}{A_1}\right)A_1A_2A_3\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3 \tag{20.4.7}$$

Integrating the second integral by parts gives

$$\int_{\alpha_1}\sigma_{11}\frac{\partial}{\partial\alpha_1}\left(\frac{\delta u_1}{A_1}\right)A_1A_2A_3\,\mathrm{d}\alpha_1=\sigma_{11}\delta u_1A_2A_3-\int_{\alpha_1}\frac{\delta u_1}{A_1}\frac{\partial(\sigma_{11}A_1A_2A_3)}{\partial\alpha_1}\mathrm{d}\alpha_1$$

Therefore,

$$\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\sigma_{11}\delta\varepsilon_{11}A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3$$
$$=\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\left\{\left[\frac{\sigma_{11}}{A_1}\frac{\partial A_1}{\partial\alpha_1}A_2A_3-\frac{1}{A_1}\frac{\partial(\sigma_{11}A_1A_2A_3)}{\partial\alpha_1}\right]\delta u_1\right.$$
$$\left.+\left(\sigma_{11}\frac{\partial A_1}{\partial\alpha_2}A_3\right)\delta u_2+\left(\sigma_{11}\frac{\partial A_1}{\partial\alpha_3}A_2\right)\delta u_3\right\}\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3$$
$$+\int_{\alpha_2}\int_{\alpha_3}\sigma_{11}\delta u_1A_2A_3\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.4.8}$$

Next, let us examine a typical shear term. Since, from Eq. (20.2.43),

$$\delta\varepsilon_{12}=\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\delta u_1}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\delta u_2}{A_2}\right) \tag{20.4.9}$$

the fourth term in Eq. (20.4.5) becomes

$$\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\sigma_{12}\delta\varepsilon_{12}A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3$$
$$=\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\sigma_{12}A_1^2A_3\frac{\partial}{\partial\alpha_2}\left(\frac{\delta u_1}{A_1}\right)\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3$$
$$+\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\sigma_{12}A_2^2A_3\frac{\partial}{\partial\alpha_1}\left(\frac{\delta u_2}{A_2}\right)\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.4.10}$$

Integrating the first integral by parts gives

$$\int_{\alpha_2}\sigma_{12}A_1^2A_3\frac{\partial}{\partial\alpha_2}\left(\frac{\delta u_1}{A_1}\right)\mathrm{d}\alpha_2=\sigma_{12}A_1A_3\delta u_1-\int_{\alpha_2}\frac{\delta u_1}{A_1}\frac{\partial}{\partial\alpha_2}(\sigma_{12}A_1^2A_3)\mathrm{d}\alpha_2$$

For the second integral, one obtains

$$\int_{\alpha_1} \sigma_{12} A_2^2 A_3 \frac{\partial}{\partial \alpha_1}\left(\frac{\delta u_2}{A_2}\right) \mathrm{d}\alpha_1 = \sigma_{12} A_2 A_3 \delta u_2 - \int_{\alpha_1} \frac{\delta u_2}{A_2} \frac{\partial}{\partial \alpha_1}(\sigma_{12} A_2^2 A_3) \mathrm{d}\alpha_1$$

Therefore,

$$\begin{aligned}
&\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3} \sigma_{12} \delta\varepsilon_{11} A_1 A_2 A_3 \,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \\
&\quad = -\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3} \left\{ \left[\frac{1}{A_1} \frac{\partial}{\partial \alpha_2}(\sigma_{12} A_1^2 A_3) \right] \delta u_1 \right. \\
&\qquad\qquad \left. + \left[\frac{1}{A_2} \frac{\partial}{\partial \alpha_1}(\sigma_{12} A_2^2 A_3) \right] \delta u_2 \right\} \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \\
&\quad + \int_{\alpha_1}\int_{\alpha_3} \sigma_{12} A_1 A_3 \delta u_1 \,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_3 \\
&\quad + \int_{\alpha_2}\int_{\alpha_3} \sigma_{12} A_2 A_3 \delta u_2 \,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3
\end{aligned} \tag{20.4.11}$$

All other normal stress and shear terms can be treated similarly. Finally, Eq. (20.4.5) becomes

$$\begin{aligned}
\delta U = \int_{\alpha_1}\int_{\alpha_1}\int_{\alpha_3} \Bigg\{ & \left[\left(\frac{\sigma_{11}}{A_1^2} \frac{\partial A_1}{\partial \alpha_1} + \frac{\sigma_{22}}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} + \frac{\sigma_{33}}{A_1 A_3} \frac{\partial A_3}{\partial \alpha_1} \right) \right. \\
& \qquad \left. - \frac{1}{A_1^2 A_2 A_3} \frac{\partial}{\partial \alpha_1}(\sigma_{11} A_1 A_2 A_3) \right] \delta u_1 \\
& + \left[\left(\frac{\sigma_{11}}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} + \frac{\sigma_{22}}{A_2^2} \frac{\partial A_2}{\partial \alpha_2} + \frac{\sigma_{33}}{A_2 A_3} \frac{\partial A_3}{\partial \alpha_2} \right) \right. \\
& \qquad \left. - \frac{1}{A_2^2 A_1 A_3} \frac{\partial(\sigma_{22} A_1 A_2 A_3)}{\partial \alpha_2} \right] \delta u_2 \\
& + \left[\left(\frac{\sigma_{11}}{A_1 A_3} \frac{\partial A_1}{\partial \alpha_3} + \frac{\sigma_{22}}{A_2 A_3} \frac{\partial A_2}{\partial \alpha_3} + \frac{\sigma_{33}}{A_3^2} \frac{\partial A_3}{\partial \alpha_3} \right) \right. \\
& \qquad \left. - \frac{1}{A_3^2 A_1 A_2} \frac{\partial(\sigma_{33} A_1 A_2 A_3)}{\partial \alpha_3} \right] \delta u_3 \\
& - \left[\frac{1}{A_1^2 A_2 A_3} \frac{\partial}{\partial \alpha_2}(\sigma_{12} A_1^2 A_3) + \frac{1}{A_1^2 A_2 A_3} \frac{\partial}{\partial \alpha_3}(\sigma_{13} A_1^2 A_2) \right] \delta u_1 \\
& - \left[\frac{1}{A_2^2 A_1 A_3} \frac{\partial}{\partial \alpha_1}(\sigma_{12} A_2^2 A_3) + \frac{1}{A_2^2 A_1 A_3} \frac{\partial(\sigma_{23} A_2^2 A_1)}{\partial \alpha_3} \right] \delta u_2
\end{aligned}$$

$$
-\left[\frac{1}{A_3^2A_1A_2}\frac{\partial}{\partial\alpha_1}(\sigma_{13}A_3^2A_2)+\frac{1}{A_3^2A_1A_2}\frac{\partial(\sigma_{23}A_3^2A_1)}{\partial\alpha_2}\right]\delta u_3\Bigg\}
$$
$$
\times A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3
$$
$$
+\int_{\alpha_2}\int_{\alpha_3}(\sigma_{11}\delta u_1+\sigma_{12}\delta u_2+\sigma_{13}\delta u_3)A_2A_3\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3
$$
$$
+\int_{\alpha_1}\int_{\alpha_3}(\sigma_{22}\delta u_2+\sigma_{12}\delta u_1+\sigma_{23}\delta u_3)A_1A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_3
$$
$$
+\int_{\alpha_1}\int_{\alpha_2}(\sigma_{33}\delta u_3+\sigma_{13}\delta u_1+\sigma_{23}\delta u_2)A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{20.4.12}
$$

Next, we evaluate δK, from Eq. (20.3.2),

$$
\delta K=\rho\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}(\dot u_1\delta\dot u_1+\dot u_2\delta\dot u_2+\dot u_3\delta\dot u_3)A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.4.13}
$$

To separate coefficients of $\delta u_1,\delta u_2,\delta u_3$, we integrate by parts, in anticipation of the application of Hamilton's principle, the time integral

$$
\int_{t_0}^{t_1}\delta K\,\mathrm{d}t=\rho\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\int_{t_0}^{t_1}\left[\dot u_1\frac{\partial(\delta u_1)}{\partial t}+\dot u_2\frac{\partial(\delta u_2)}{\partial t}+\dot u_3\frac{\partial(\delta u_3)}{\partial t}\right]
$$
$$
\times\mathrm{d}t A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.4.14}
$$

For example, the first term becomes

$$
\int_{t_0}^{t_1}\dot u_1\frac{\partial(\delta u_1)}{\partial t}\mathrm{d}t=\left[\frac{\partial u_1}{\partial t}\delta u_1\right]_{t_0}^{t_1}-\int_{t_0}^{t_1}\frac{\partial^2u_1}{\partial t^2}\delta u_1\,\mathrm{d}t \tag{20.4.15}
$$

Since the virtual displacement is defined to be 0 at $t=t_0$ and t_1 according to Hamilton's principle, the bracketed quantity disappears and we obtain

$$
\int_{t_0}^{t_1}\dot u_1\frac{\partial(\delta u_1)}{\partial t}\mathrm{d}t=-\int_{t_0}^{t_1}\ddot u_1\delta u_1\,\mathrm{d}t \tag{20.4.16}
$$

Proceeding similarly with the other two terms in Eq. (20.4.14), we obtain

$$
\int_{t_0}^{t_1}\delta K\,\mathrm{d}t=-\rho\int_{t_0}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}(\ddot u_1\delta u_1+\ddot u_2\delta u_2+\ddot u_3\delta u_3)
$$
$$
\times A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.4.17}
$$

Now applying Hamilton's principle as stated in Eq. (20.4.1), one obtains

$$
\int_{t_0}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\int_{\alpha_3}\Bigg\{\Bigg[\rho_1\ddot u_1+q_1-\left(\frac{\sigma_{11}}{A_1^2}\frac{\partial A_1}{\partial\alpha_1}+\frac{\sigma_{22}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{\sigma_{33}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)
$$
$$
+\frac{1}{A_1^2A_2A_3}\frac{\partial}{\partial\alpha_1}(\sigma_{11}A_1A_2A_3)+\frac{1}{A_1^2A_2A_3}\frac{\partial}{\partial\alpha_2}(\sigma_{12}A_1^2A_3)
$$
$$
+\frac{1}{A_1^2A_2A_3}\frac{\partial}{\partial\alpha_3}(\sigma_{13}A_1^2A_2)\Bigg]\delta u_1
$$

$$
\begin{aligned}
&+\Bigg[\rho\ddot{u}_2+q_2-\left(\frac{\sigma_{11}}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{\sigma_{22}}{A_2^2}\frac{\partial A_2}{\partial\alpha_2}+\frac{\sigma_{33}}{A_2A_3}\frac{\partial A_3}{\partial\alpha_2}\right)\\
&+\frac{1}{A_2^2A_1A_3}\frac{\partial}{\partial\alpha_2}(\sigma_{22}A_1A_2A_3)\\
&+\frac{1}{A_2^2A_1A_3}\frac{\partial}{\partial\alpha_1}(\sigma_{12}A_2^2A_3)+\frac{1}{A_2^2A_1A_3}\frac{\partial}{\partial\alpha_3}(\sigma_{23}A_2^2A_1)\Bigg]\delta u_2\\
&+\Bigg[\rho\ddot{u}_3+q_3-\left(\frac{\sigma_{11}}{A_1A_3}\frac{\partial A_1}{\partial\alpha_3}+\frac{\sigma_{22}}{A_2A_3}\frac{\partial A_2}{\partial\alpha_3}+\frac{\sigma_{33}}{A_3^2}\frac{\partial A_3}{\partial\alpha_3}\right)\\
&+\frac{1}{A_3^2A_1A_2}\frac{\partial}{\partial\alpha_3}(\sigma_{33}A_1A_2A_3)\\
&+\frac{1}{A_3^2A_1A_2}\frac{\partial}{\partial\alpha_1}(\sigma_{13}A_3^2A_2)+\frac{1}{A_3^2A_1A_2}\frac{\partial}{\partial\alpha_2}(\sigma_{23}A_3^2A_1)\Bigg]\delta u_3\Bigg\}\\
&\times A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}t\\
&+\int_{t_0}^{t_1}\int_{\alpha_2}\int_{\alpha_3}[(\sigma_{11}^*-\sigma_{11})\delta u_1+(\sigma_{12}^*-\sigma_{12})\delta u_2+(\sigma_{13}^*-\sigma_{13})\delta u_3]\\
&\times A_2A_3\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}t\\
&+\int_{t_0}^{t_1}\int_{\alpha_1}\int_{\alpha_3}[(\sigma_{22}^*-\sigma_{22})\delta u_2+(\sigma_{21}^*-\sigma_{21})\delta u_1+(\sigma_{23}^*-\sigma_{23})\delta u_3]\\
&\times A_1A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}t\\
&+\int_{t_0}^{t_1}\int_{\alpha_1}\int_{\alpha_2}[(\sigma_{33}^*-\sigma_{33})\delta u_3+(\sigma_{31}^*-\sigma_{31})\delta u_1+(\sigma_{32}^*-\sigma_{32})\delta u_2]\\
&\times A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}t=0 \qquad (20.4.18)
\end{aligned}
$$

Employing the argument that this equation can only be satisfied if the volume integral and the surface integrals are 0 individually and that the coefficients of the variational displacements must be 0 because of the independence and arbitrariness of the latter, the equations of motion are obtained as

$$
\begin{aligned}
&\frac{1}{A_1^2A_2A_3}\left[\frac{\partial}{\partial\alpha_1}(\sigma_{11}A_1A_2A_3)+\frac{\partial}{\partial\alpha_2}(\sigma_{12}A_1^2A_3)+\frac{\partial}{\partial\alpha_3}(\sigma_{13}A_1^2A_2)\right]\\
&-\left(\frac{\sigma_{11}}{A_1^2}\frac{\partial A_1}{\partial\alpha_1}+\frac{\sigma_{22}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{\sigma_{33}}{A_1A_3}\frac{\partial A_3}{\partial\alpha_1}\right)-\rho\ddot{u}_1=-q_1 \qquad (20.4.19)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{A_2^2A_1A_3}\left[\frac{\partial}{\partial\alpha_2}(\sigma_{22}A_1A_2A_3)+\frac{\partial}{\partial\alpha_1}(\sigma_{12}A_2^2A_3)+\frac{\partial}{\partial\alpha_3}(\sigma_{23}A_2^2A_1)\right]\\
&-\left(\frac{\sigma_{11}}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{\sigma_{22}}{A_2^2}\frac{\partial A_2}{\partial\alpha_2}+\frac{\sigma_{33}}{A_2A_3}\frac{\partial A_3}{\partial\alpha_2}\right)-\rho\ddot{u}_2=-q_2 \qquad (20.4.20)
\end{aligned}
$$

$$\frac{1}{A_3^2A_1A_2}\left[\frac{\partial}{\partial\alpha_3}(\sigma_{33}A_1A_2A_3)+\frac{\partial}{\partial\alpha_1}(\sigma_{13}A_3^2A_2)+\frac{\partial}{\partial\alpha_2}(\sigma_{23}A_3^2A_1)\right]$$
$$-\left(\frac{\sigma_{11}}{A_1A_3}\frac{\partial A_1}{\partial\alpha_3}+\frac{\sigma_{22}}{A_2A_3}\frac{\partial A_2}{\partial\alpha_3}+\frac{\sigma_{33}}{A_3^2}\frac{\partial A_3}{\partial\alpha_3}\right)-\rho\ddot{u}_3=-q_3 \tag{20.4.21}$$

Admissible boundary conditions are, on a typical α_1–α_2 surface,

$$\begin{aligned}
&\sigma_{33}=\sigma_{33}^* \quad \text{or} \quad u_3=u_3^*\\
&\sigma_{31}=\sigma_{31}^* \quad \text{or} \quad u_1=u_1^*\\
&\sigma_{32}=\sigma_{32}^* \quad \text{or} \quad u_2=u_2^*
\end{aligned} \tag{20.4.22}$$

on a typical α_1–α_3 surface,

$$\begin{aligned}
&\sigma_{22}=\sigma_{22}^* \quad \text{or} \quad u_2=u_2^*\\
&\sigma_{21}=\sigma_{21}^* \quad \text{or} \quad u_1=u_1^*\\
&\sigma_{23}=\sigma_{23}^* \quad \text{or} \quad u_3=u_3^*
\end{aligned} \tag{20.4.23}$$

and on a typical α_2–α_3 surface,

$$\begin{aligned}
&\sigma_{11}=\sigma_{11}^* \quad \text{or} \quad u_1=u_1^*\\
&\sigma_{12}=\sigma_{12}^* \quad \text{or} \quad u_2=u_2^*\\
&\sigma_{13}=\sigma_{13}^* \quad \text{or} \quad u_3=u_3^*
\end{aligned} \tag{20.4.24}$$

Equations (20.4.19)–(20.4.21) are given elsewhere (Love, 1944, Sokolnikoff, 1946) in slightly different form.

20.5. EXAMPLE: CYLINDRICAL COORDINATES

The Lamé parameters are, for cylindrical coordinates,

$$\begin{aligned}
&\alpha_1=z, \quad A_1=1\\
&\alpha_2=r, \quad A_2=1\\
&\alpha_3=\theta, \quad A_3=r
\end{aligned} \tag{20.5.1}$$

Therefore, Eqs. (20.4.19)–(20.4.21) become

$$\frac{1}{r}\left[\frac{\partial}{\partial z}(\sigma_{zz}r)+\frac{1}{\partial r}(\sigma_{zr}r)+\frac{\partial}{\partial\theta}(\sigma_{z\theta})\right]-\rho\ddot{u}_z=-q_z \tag{20.5.2}$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(\sigma_{rr}r)+\frac{\partial}{\partial z}(\sigma_{zr}r)+\frac{\partial}{\partial\theta}(\sigma_{r\theta})\right]-\frac{\sigma_{\theta\theta}}{r}-\rho\ddot{u}_r=-q_r \tag{20.5.3}$$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial\theta}(\sigma_{\theta\theta}r)+\frac{\partial}{\partial z}(\sigma_{z\theta}r^2)+\frac{\partial}{\partial r}(\sigma_{r\theta}r^2)\right]-\rho\ddot{u}_\theta=-q_\theta \tag{20.5.4}$$

or

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\sigma_{zr}}{r} - \rho \ddot{u}_z = -q_z \tag{20.5.5}$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} - \rho \ddot{u}_r = -q_r \tag{20.5.6}$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2}{r}\sigma_{r\theta} - \rho \ddot{u}_\theta = -q_\theta \tag{20.5.7}$$

The strain–displacement relationships (20.2.33)–(20.2.35) become

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{20.5.8}$$

$$\varepsilon_{\theta\theta} = \frac{\partial u_r}{\partial r} \tag{20.5.9}$$

$$\varepsilon_{rr} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) \tag{20.5.10}$$

$$\varepsilon_{zr} = \varepsilon_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \tag{20.5.11}$$

$$\varepsilon_{z\theta} = \varepsilon_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \tag{20.5.12}$$

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \tag{20.5.13}$$

This allows a conversion of Eqs. (20.5.5)–(20.5.7) into a displacement form, if desired.

20.6. EXAMPLE: CARTESIAN COORDINATES

In Cartesian coordinates, the Lamé parameters are

$$\begin{aligned} A_1 &= 1, \quad \alpha_1 = x \\ A_2 &= 1, \quad \alpha_2 = y \\ A_3 &= 1, \quad \alpha_3 = z \end{aligned} \tag{20.6.1}$$

and Eqs. (20.4.15) and (20.4.16) become

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} - \rho \ddot{u}_x = -q_x \tag{20.6.2}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} - \rho \ddot{u}_y = -q_y \tag{20.6.3}$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} - \rho \ddot{u}_z = -q_z \tag{20.6.4}$$

The strain–displacement relations are, from Eqs. (20.2.33) to (20.2.35),

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{20.6.5}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \tag{20.6.6}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{20.6.7}$$

$$\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \tag{20.6.8}$$

$$\varepsilon_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \tag{20.6.9}$$

$$\varepsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \tag{20.6.10}$$

Because of the simplicity of these expressions, it is customary to write Eqs. (20.6.2)–(20.6.4) in displacement form. Substituting Eqs. (20.6.5)–(20.6.10) into Eqs. (20.2.8)–(20.2.13) and these in turn into Eqs. (20.6.2)–(20.6.4) gives

$$G\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + (\lambda + G)\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y \partial x} + \frac{\partial^2 u_z}{\partial z \partial x}\right) - \rho \ddot{u}_x = -q_x \tag{20.6.11}$$

$$G\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + (\lambda + G)\left(\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial z \partial y}\right) - \rho \ddot{u}_y = -q_y \tag{20.6.12}$$

$$G\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + (\lambda + G)\left(\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z} + \frac{\partial^2 u_z}{\partial z^2}\right) - \rho \ddot{u}_z = -q_z \tag{20.6.13}$$

where G is the shear modulus,

$$G = \frac{E}{2(1+\mu)} \tag{20.6.14}$$

and λ is the bulk modulus,

$$\lambda = \frac{\mu E}{(1+\mu)(1-2\mu)} \tag{20.6.15}$$

In abbreviated form, using tensor notation, this may be expressed as

$$Gu_{i\cdot jj}+(\lambda+G)u_{j\cdot ji}-\rho\ddot{u}_i=q_i \tag{20.6.16}$$

where $i, j=x, y, z$.

20.7. ONE-DIMENSIONAL WAVE EQUATIONS FOR SOLIDS

If we set $q_x=q_y=q_z=0$ and assume that u_x, u_y, u_z are only functions of x, we obtain from Eq. (20.6.11)

$$G\frac{\partial^2 u_x}{\partial x^2}+(\lambda+G)\frac{\partial^2 u_x}{\partial x^2}=\rho\ddot{u}_x \tag{20.7.1}$$

or

$$\frac{\partial^2 u_x}{\partial x^2}-\frac{1}{C_1^2}\ddot{u}_x=0 \tag{20.7.2}$$

where

$$C_1^2=\frac{\lambda+2G}{\rho}=\frac{E(1-\mu)}{\rho(1+\mu)(1-2\mu)} \tag{20.7.3}$$

The physical meaning of C_1 is that it is the compression wave velocity. Also, from Eq. (20.6.12),

$$G\frac{\partial^2 u_y}{\partial x^2}=\rho\ddot{u}_y \tag{20.7.4}$$

or

$$\frac{\partial^2 u_y}{\partial x^2}-\frac{1}{C_2^2}\ddot{u}_y=0 \tag{20.7.5}$$

where C_2 is the shear velocity,

$$C_2^2=\frac{G}{\rho} \tag{20.7.6}$$

Similarly, from Eq. (20.6.13),

$$\frac{\partial^2 u_z}{\partial x^2}-\frac{1}{C_2^2}\ddot{u}_z=0 \tag{20.7.7}$$

20.8. THREE-DIMENSIONAL WAVE EQUATIONS FOR SOLIDS

Let us now investigate the three-dimensional model. First, we may write Eqs. (20.6.11)–(20.6.13) as ($q_x = q_y = q_z = 0$)

$$G\nabla^2 \bar{u} + (\lambda + G)\,\mathbf{grad}(\mathbf{div}\,\bar{u}) = \rho \ddot{\bar{u}} \tag{20.8.1}$$

where

$$\bar{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \tag{20.8.2}$$

$$\mathbf{div}\,\bar{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{20.8.3}$$

$$\mathbf{grad}(\mathbf{div}\,\bar{u}) = \begin{Bmatrix} \dfrac{\partial}{\partial x}(\mathbf{div}\,\bar{u}) \\ \dfrac{\partial}{\partial y}(\mathbf{div}\,\bar{u}) \\ \dfrac{\partial}{\partial z}(\mathbf{div}\,\bar{u}) \end{Bmatrix} \tag{20.8.4}$$

Since

$$C_2^2 = \frac{G}{\rho} \tag{20.8.5}$$

and

$$C_1^2 - C_2^2 = \frac{\lambda + G}{\rho} \tag{20.8.6}$$

we obtain

$$C_2^2 \nabla^2 \bar{u} + (C_1^2 - C_2^2)\,\mathbf{grad}(\mathbf{div}\,\bar{u}) = \ddot{\bar{u}} \tag{20.8.7}$$

Let us now divide the response $\bar{u}$ into two parts,

$$\bar{u} = \bar{u}' + \bar{u}'' \tag{20.8.8}$$

This gives

$$C_2^2 \nabla^2 (\bar{u}' + \bar{u}'') + (C_1^2 - C_2^2)\,\mathbf{grad}(\mathbf{div}\,\bar{u}' + \mathbf{div}\,\bar{u}'') = \ddot{\bar{u}}' + \ddot{\bar{u}}'' \tag{20.8.9}$$

Now let us designate $\bar{u}'$ as the compressive component and $\bar{u}''$ as the shear component, postulating that there is no compressibility due to shear,

$$\mathbf{div}\,\bar{u}'' = 0 \tag{20.8.10}$$

and that there is no rotation for the compressive wave components; therefore,

$$\operatorname{curl}\bar{u}' = \begin{vmatrix} \bar{l}_x & \bar{l}_y & \bar{l}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_x^t & u_y^t & u_z^t \end{vmatrix} = 0 \tag{20.8.11}$$

This gives, applying Eq. (20.8.10) to Eq. (20.8.9),

$$C_2^2 \nabla^2 (\bar{u}' + \bar{u}'') + (C_1^2 - C_2^2)\operatorname{grad}(\operatorname{div}\bar{u}') = \ddot{\bar{u}}' + \ddot{\bar{u}}'' \tag{20.8.12}$$

Let us now take the divergence of Eq. (20.8.12):

$$C_2^2 \operatorname{div}\nabla^2 (\bar{u}' + \bar{u}'') + (C_1^2 - C_2^2)\operatorname{div}\operatorname{grad}(\operatorname{div}\bar{u}') = \operatorname{div}\ddot{\bar{u}}' + \operatorname{div}\ddot{\bar{u}}'' \tag{20.8.13}$$

The second term on the right is 0 and the second term in the first set of parentheses on the left is 0. Since $\operatorname{div}(\nabla^2) = \nabla^2(\operatorname{div})$, this gives

$$C_2^2 \operatorname{div}\nabla^2 \bar{u}' + (C_1^2 - C_2^2)\operatorname{div}\nabla^2 \bar{u}' = \operatorname{div}\ddot{\bar{u}}' \tag{20.8.14}$$

or

$$\operatorname{div}(C_1^2 \nabla^2 \bar{u}' - \ddot{\bar{u}}') = 0 \tag{20.8.15}$$

Taking the curl of vector $C_1^2 \nabla^2 \bar{u}' - \ddot{\bar{u}}'$, which is 0 according to Eq. (20.8.11), we argue that if both the divergence and the curl of a vector vanish, the vector must be 0. Thus

$$\nabla^2 \bar{u}' - \frac{1}{C_1^2}\ddot{\bar{u}}' = 0 \tag{20.8.16}$$

This is the compressive wave equation. Next, we take the curl of Eq. (20.8.12):

$$C_2^2 \operatorname{curl}\nabla^2 (\bar{u}' + \bar{u}'') + (C_1^2 - C_2^2)\operatorname{curl}\operatorname{grad}(\operatorname{div}\bar{u}') = \operatorname{curl}\ddot{\bar{u}}' + \operatorname{curl}\ddot{\bar{u}}'' \tag{20.8.17}$$

Applying Eq. (20.8.11) gives

$$\operatorname{curl}(C_2^2 \nabla^2 \bar{u}'' - \ddot{\bar{u}}'') = 0 \tag{20.8.18}$$

Again, since the divergence of this vector vanishes according to Eq. (20.8.10), we obtain

$$\nabla_2 \bar{u}'' - \frac{1}{C_2^2}\ddot{\bar{u}}'' = 0 \tag{20.8.19}$$

This is the shear wave equation.

Wave solutions of Eqs. (20.8.16) and (20.8.19) for three-dimensional solids are not the subject of this discussion. For further study, consult, for example, Kolsky (1953), or for an application, Kim and Soedel (1988). Rather, it will be shown that Eq. (20.8.16) is related to the wave equation governing the acoustics of liquids and gases.

20.9. THREE-DIMENSIONAL WAVE EQUATION FOR INVISCID COMPRESSIBLE LIQUIDS AND GASES (ACOUSTICS)

We can derive the governing equations of an acoustic medium in Cartesian coordinates, without loss of generality. We assume that the gas of fluid is inviscid, so that

$$\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0 \tag{20.9.1}$$

and

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p \tag{20.9.2}$$

where p is a small pressure rise above the static pressure. Note that the σ values are positive for tension, while p is understood to be positive for compression. We obtain (for $q_x = q_y = q_z = 0$)

$$-\frac{\partial p}{\partial x} - \rho \ddot{u}_x = 0 \tag{20.9.3}$$

$$-\frac{\partial p}{\partial y} - \rho \ddot{u}_y = 0 \tag{20.9.4}$$

$$-\frac{\partial p}{\partial z} - \rho \ddot{u}_z = 0 \tag{20.9.5}$$

This can also be written

$$-\frac{\partial^2 p}{\partial x^2} - \rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_x}{\partial x}\right) = 0 \tag{20.9.6}$$

$$-\frac{\partial^2 p}{\partial y^2} - \rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_y}{\partial y}\right) = 0 \tag{20.9.7}$$

$$-\frac{\partial^2 p}{\partial y^2} - \rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_z}{\partial z}\right) = 0 \tag{20.9.8}$$

Adding gives

$$-\nabla^2 p - \rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0 \tag{20.9.9}$$

From Eqs. (20.2.8) to (20.2.11), we obtain (with $G = 0$)

$$-p = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \tag{20.9.10}$$

Thus Eq. (20.9.10) becomes

$$\nabla^2 p - \frac{\rho}{\lambda}\frac{\partial^2 p}{\partial t^2} = 0 \tag{20.9.11}$$

λ is the bulk modulus. Definition (20.6.15) cannot be used here because defining Poisson's ratio for a compressible gas or fluid is rather meaningless. It has to be rederived based on the thermodynamic assumption that compression (or expansion) is adiabatic. The derivation of the bulk modulus is as follows. We assume that in the linear range, the pressure change of a gas is proportional to its density change,

$$\mathrm{d}p = -K\frac{\mathrm{d}V}{V} \tag{20.9.12}$$

where K is liquid or gas bulk modulus. If we argue that in a small control volume, the mass is constant,

$$V\rho = C \tag{20.9.13}$$

and differentiate with respect to pressure, we obtain

$$V\frac{\mathrm{d}\rho}{\mathrm{d}\rho} + \rho\frac{\mathrm{d}V}{\mathrm{d}p} = 0 \tag{20.9.14}$$

or

$$\mathrm{d}p = -\left(\rho\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)\frac{\mathrm{d}V}{V} \tag{20.9.15}$$

Therefore,

$$K = \rho\frac{\mathrm{d}p}{\mathrm{d}\rho} = \rho C^2 \tag{20.9.16}$$

where $C = \sqrt{\mathrm{d}p/\mathrm{d}p}$ is the instantaneous speed of sound, which can, however, be taken as an average value, as is ρ. Equation (20.9.11) therefore becomes

$$\nabla^2 p - \frac{1}{C^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{20.9.17}$$

Equation (20.9.17) can also be derived from Eq. (20.8.16). For either an inviscid liquid or gas, we may assume that no shear stresses exist. Therefore, Eq. (20.8.8) becomes

$$\bar{u} = \bar{u}' \tag{20.9.18}$$

and the governing equation is Eq. (20.8.16)

$$\nabla^2 \bar{u} - \frac{1}{C_1^2}\ddot{\bar{u}} = 0 \tag{20.9.19}$$

where because assuming no shear stresses amounts to setting $G = 0, C_1^2 = \lambda/\rho$. The bulk modulus λ for a solid has to be replaced by the bulk modulus

K for a liquid or gas, and therefore the speed of sound is (the subscript 1 is now dropped)

$$C^2 = \frac{K}{\rho} \tag{20.9.20}$$

Equation (20.9.19) becomes

$$\nabla^2 \bar{u} - \frac{1}{C^2} \ddot{\bar{u}} = 0 \tag{20.9.21}$$

In expanded form, it may be written

$$\nabla^2 u_x - \frac{1}{C^2} \ddot{u}_x = 0 \tag{20.9.22}$$

$$\nabla^2 u_y - \frac{1}{C^2} \ddot{u}_y = 0 \tag{20.9.23}$$

$$\nabla^2 u_x - \frac{1}{C^2} \ddot{u}_z = 0 \tag{20.9.24}$$

or

$$\nabla^2 \frac{\partial u_x}{\partial x} - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_x}{\partial x} \right) = 0 \tag{20.9.25}$$

$$\nabla^2 \frac{\partial u_y}{\partial y} - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_y}{\partial y} \right) = 0 \tag{20.9.26}$$

$$\nabla^2 \frac{\partial u_z}{\partial z} - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_z}{\partial z} \right) = 0 \tag{20.9.27}$$

Adding gives

$$\left(\nabla^2 - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \right) \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) = 0 \tag{20.9.28}$$

Since

$$-p = K \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \tag{20.9.29}$$

one obtains Eq. (20.9.17).

Note that Eq. (20.9.21) can also be written in form of particle velocities by differentiating it with respect to time. In expanded form,

$$\nabla^2 v_x - \frac{1}{C^2} \ddot{v}_x = 0 \tag{20.9.30}$$

$$\nabla^2 v_y - \frac{1}{C^2} \ddot{v}_y = 0 \tag{20.9.31}$$

$$\nabla^2 v_z - \frac{1}{C^2} \ddot{v}_z = 0 \tag{20.9.32}$$

It is often convenient to introduce a potential function ϕ such that

$$v_x = -\frac{\partial \phi}{\partial x}, \quad v_y = -\frac{\partial \phi}{\partial y}, \quad v_z = -\frac{\partial \phi}{\partial z} \tag{20.9.33}$$

Substituting this gives

$$\begin{aligned} \nabla^2 \left(\frac{\partial \phi}{\partial x} \right) - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial \phi}{\partial x} \right) &= 0 \\ \nabla^2 \left(\frac{\partial \phi}{\partial y} \right) - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial \phi}{\partial y} \right) &= 0 \\ \nabla^2 \left(\frac{\partial \phi}{\partial z} \right) - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \left(\frac{\partial \phi}{\partial z} \right) &= 0 \end{aligned} \tag{20.9.34}$$

Adding this results in

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\nabla^2 \phi - \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} \right) = 0 \tag{20.9.35}$$

or, without loss of generality,

$$\nabla^2 \phi - \frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{20.9.36}$$

The potential function ϕ is related to pressure through Eq. (20.9.29). If we differentiate with respect to time

$$-\frac{\partial p}{\partial t} = K \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \tag{20.9.37}$$

and utilizing Eqs. (20.9.33), we obtain

$$\frac{\partial p}{\partial t} = K \nabla^2 \phi \tag{20.9.38}$$

We may replace $\nabla^2 \phi$, using Eq. (20.9.36), and obtain

$$p = \rho \frac{\partial \phi}{\partial t} \tag{20.9.39}$$

In general, using curvilinear coordinates,

$$\nabla^2 = \frac{1}{A_1 A_2 A_3} \left[\frac{\partial}{\partial \alpha_1} \left(\frac{A_2 A_3}{A_1} \frac{\partial}{\partial \alpha_1} \right) + \frac{\partial}{\partial \alpha_2} \left(\frac{A_3 A_1}{A_2} \frac{\partial}{\partial \alpha_2} \right) + \frac{\partial}{\partial \alpha_3} \left(\frac{A_1 A_2}{A_3} \frac{\partial}{\partial \alpha_3} \right) \right] \tag{20.9.40}$$

20.10. INTERFACE BOUNDARY CONDITIONS

The interface conditions for a shell joined to an acoustic medium, be it liquid or gas, are that the normal pressure load q_3 on the shell has to be equal to the boundary pressure p of the acoustic medium,

$$q_3 = p \tag{20.10.1}$$

and that the normal velocity of the shell surface, $\dot{u}_3$, has to be equal to the normal velocity component v_n of the acoustic medium boundary,

$$\dot{u}_3 = v_\mathrm{n} \tag{20.10.2}$$

For example, for a cylindrical shell filled with liquid, using cylindrical coordinates,

$$q_3(x,\theta,t) = p(x,\theta,a,t) \tag{20.10.3}$$

$$\dot{u}_3(x,\theta,t) = v_r(x,\theta,a,t) \tag{20.10.4}$$

These conditions can be translated into displacement or potential function conditions, depending on the preferred form of the wave equation.

Other boundary conditions of the acoustic medium are of a similar type in that one has to interpret the physical situation in terms of pressure or velocity conditions. Because of the second-order nature of the wave equation, only one condition will be required at boundaries other than the interface.

If interaction calculations are not required at the interface, that is, if it is assumed that the back reaction of the acoustic medium is negligible (which is typically the case when one studies acoustic radiation from surfaces), only condition (20.10.2) is required at the interface, with $\dot{u}_3$ being a given quantity because the shell and acoustic medium equations are solved separately. First, the forced response of the shell is evaluated, which gives $\dot{u}_3$; then the wave equation is solved.

20.11. EXAMPLE: ACOUSTIC RADIATION

In this example, we assume that the transverse vibration of a slider-clamped circular cylindrical shell is given as

$$u_3(x,\theta,t) = A_{mn}\cos\frac{m\pi x}{L}\cos n\theta e^{j\omega t} \tag{20.11.1}$$

The transverse velocity is therefore

$$\dot{u}_3(x,\theta,t) = A_{mn} j\omega \cos\frac{m\pi x}{L}\cos n\theta e^{j\omega t} \tag{20.11.2}$$

The implication here is that the shell surface velocity can be specified and is not altered by reaction to the acoustic pressure. This is a very

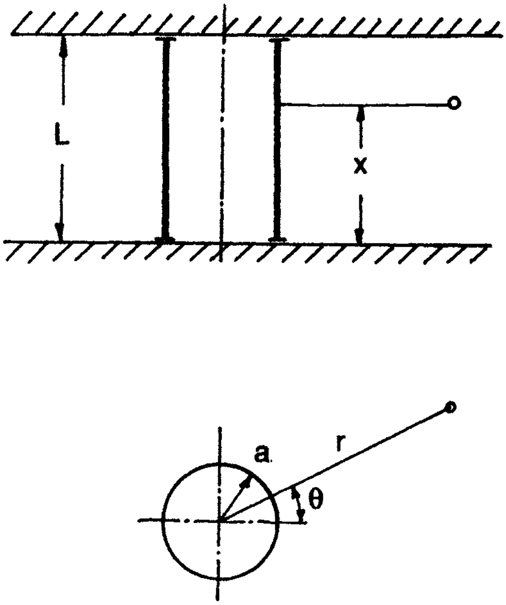

Fig. 2 Acoustic radiation from a circular cylindrical shell.

practical situation if the radiation is to air. Hydroacoustic radiation usually requires that the shell response and the acoustic radiation be investigated as a coupled phenomenon.

In this example, the airspace is confined between two parallel baffles, extending into infinity as shown in Fig. 2. The boundary conditions are, therefore, for the acoustic medium,

$$v_r(r=a, x, \theta, t) = \dot{u}_3(x, \theta, t) \tag{20.11.3}$$

$$v_r(r=\infty, x, \theta, t) = 0 \tag{20.11.4}$$

$$v_x(r, x=0, \theta, t) = 0 \tag{20.11.5}$$

$$v_x(r, x=L, \theta, t) = 0 \tag{20.11.6}$$

The equation of motion, Eq. (20.9.36), is

$$\nabla^2\phi - \frac{1}{C^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{20.11.7}$$

where ∇^2 is, from Eq. (20.9.40), since $\alpha_1 = r, \alpha_2 = \theta, \alpha_3 = x, A_1 = 1, A_2 = r$, and $A_3 = 1$,

$$\nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) + \frac{\partial}{\partial x}\left(r\frac{\partial}{\partial x}\right)\right] \tag{20.11.8}$$

By inspection, we set

$$\phi(r,\theta,x,t)=R(r)\cos\frac{m\pi x}{L}\cos n\theta e^{j\omega t} \tag{20.11.9}$$

Substituting this into Eq. (20.11.7) gives

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\left[\left(k^2-\frac{m^2\pi^2}{L^2}\right)r^2-n^2\right]R=0 \tag{20.11.10}$$

where

$$k=\frac{\omega}{C} \tag{20.11.11}$$

Whenever $k>m\pi/L$, the solution is

$$R(r)=A_nJ_n(\kappa r)+B_nY_n(\kappa r) \tag{20.11.12}$$

where

$$\kappa=k^2-\frac{m^2\pi^2}{L^2} \tag{20.11.13}$$

and J_n and Y_n are Bessel functions of the first and second kind.

A second type of solution exists if $k<m\pi/L$:

$$R(r)=A_nI_n(\chi r)+B_nK_n(\chi r) \tag{20.11.14}$$

where

$$\chi^2=\frac{m^2\pi^2}{L^2}-k^2 \tag{20.11.15}$$

and I_n and K_n are modified Bessel functions of the first and second kind.

Practically, the case of Eq. (20.11.12) is more important. For example, if $L=1.0$m, $C=330$m/s, and $m=1$, this solution applies for all frequencies above 165 Hz. Because of boundary condition (20.11.4), it is convenient to write Eq. (20.11.12) in terms of Hankel functions (Watson, 1948):

$$R(r)=G_nH_n^{(1)}(\kappa r)+D_nH_n^{(2)}(\kappa r) \tag{20.11.16}$$

where

$$H_n^{(1)}(\kappa r) = J_n(\kappa r)+jY_n(\kappa r) \tag{20.11.17}$$

$$H_n^{(2)}(\kappa r) = J_n(\kappa r)-jY_n(\kappa r) \tag{20.11.18}$$

Because $H_n^{(1)}(\kappa r)$ will not approach 0 as $r\to\infty$, it must be that $G_n=0$. Therefore,

$$R(r)=D_nH_n^{(2)}(\kappa r) \tag{20.11.19}$$

Boundary condition (20.11.3) can be written

$$v_r(r=a,\theta,x,t)=-\left.\frac{\partial\phi}{\partial r}\right|_{r=a}=jA_{mn}\omega\cos\frac{m\pi x}{L}\cos n\theta e^{j\omega t} \tag{20.11.20}$$

or, after substituting Eq. (20.11.9),

$$\frac{dR}{dr}(r=a)=-jA_{mn}\omega \tag{20.11.21}$$

Substituting Eq. (20.11.19) gives

$$D_n=-\frac{jA_{mn}\omega}{H_n^{(2)'}}(\kappa a) \tag{20.11.22}$$

where

$$H_n^{(2)'}(\kappa a)=\left.\frac{dH_n^{(2)}(\kappa r)}{dr}\right|_{r=a} \tag{20.11.23}$$

Therefore,

$$\phi(r,\theta,x,t)=-jA_{mn}\omega\frac{H_n^{(2)}(\kappa r)}{H_n^{(2)'}(\kappa a)}\cos\frac{m\pi x}{L}\cos n\theta e^{j\omega t} \tag{20.11.24}$$

The acoustic radiation pressure is, from Eq. (20.9.39),

$$p=A_{mn}\rho\omega^2\frac{H_n^{(2)}(\kappa r)}{H_n^{(2)'}(\kappa a)}\cos\frac{m\pi x}{L}\cos n\theta e^{j\omega t} \tag{20.11.25}$$

A similar solution can be obtained for a cylindrical shell of infinite length (Reynolds, 1981).

20.12. INCOMPRESSIBLE LIQUIDS

If a structure is in contact with a liquid that has a free surface, one may assume that the liquid will act as if it cannot be compressed, generating in the process waves at the free surface. For example, many liquid containers, such as oil pans and liquid storage tanks, are structures interacting with free surface liquids.

A secondary category occurs when the liquid fills the entire space of a flexible shell such that during oscillations of the system, the liquid volume can be assumed to be constant. Typically, this assumption works best for lower natural frequencies and modes; compressibility will become increasingly more important as the frequency of interest increases.

For an incompressible liquid, $1/C=0$, and the wave equation [usually one uses the potential function version, Eq. (20.9.36)] reduces to

$$\nabla^2\phi=0 \tag{20.12.1}$$

The interface conditions (20.10.1) and (20.10.2) become

$$q_3 = \rho \frac{\partial \phi}{\partial t} \tag{20.12.2}$$

$$\dot{u}_3 = -\frac{\partial \phi}{\partial n} \tag{20.12.3}$$

where n indicates a normal coordinate to the shell surface.

In case of free surfaces, gravity is considered, so that Eq. (20.9.39) has to be extended to

$$p = \rho \left(\frac{\partial \phi}{\partial t} - \Omega \right) \tag{20.12.4}$$

where Ω is a body force potential, related to body forces per unit mass by (in Cartesian coordinates, without loss of generality)

$$F_x = -\frac{\partial \Omega}{\partial x}, \quad F_y = -\frac{\partial \Omega}{\partial y}, \quad F_z = -\frac{\partial \Omega}{\partial z} \tag{20.12.5}$$

If gravity acts in the z direction, we have

$$F_x = 0, \quad F_y = 0, \quad F_z = -g \tag{20.12.6}$$

where $g = 9.806\text{m/s}$ at sea level, and as a consequence one obtains

$$\Omega = -\int F_x \, \mathrm{d}z = gz + C_1 \tag{20.12.7}$$

where C_1 is a constant. The boundary condition at the free surface of a liquid is developed next by way of an example. Some references of interest are Lamb (1945), Rapoport (1968), Soedel (1982), Soedel and Soedel (1994), Kito (1970), Rayleigh (1896), Junger (1978), Crighton (1980).

20.13. EXAMPLE: LIQUID ON PLATE

The liquid is on top of a rectangular simply supported plate. It has a uniform average depth h, as shown in Fig. 3, and its top surface is free to form waves. Apart from motion introduced by small oscillations, there is no average overall velocity of the liquid. The liquid is assumed to be incompressible and must satisfy Eq. (20.12.1):

$$\nabla^2 \phi = 0 \tag{20.13.1}$$

For small oscillations, the pressure $p(x, y, z, t)$ is given by [Eq. (20.12.4)]

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - \Omega \tag{20.13.2}$$

when Ω is the body force potential of Eq. (20.12.7):

$$\Omega = gz + C_1 \tag{20.13.3}$$

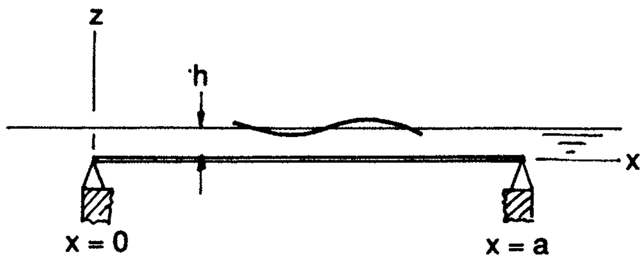

FIG. 3 Liquid on top of a simply supported plate.

where C_1 is a constant. Since at $z=h+\eta$, which describes the free surface of the liquid where η is the oscillatory liquid displacement at the surface, the body force potential must be 0,

$$\Omega(z=h+\eta)=0 \tag{20.13.4}$$

we obtain $C_1=-g(h+\eta)$ and therefore

$$\Omega=-g(h+\eta-z) \tag{20.13.5}$$

This results in

$$p=\rho\frac{\partial\phi}{\partial t}+\rho g(h+\eta-z) \tag{20.13.6}$$

The boundary condition at the free surface of the liquid is that the pressure must be 0. Utilizing Eq. (20.13.6), this gives the relationship

$$\eta(x,y,t)=\frac{1}{g}\frac{\partial\phi}{\partial t}(x,y,h,t) \tag{20.13.7}$$

From the definition of the velocity potential, $v_z=-\partial\phi/\partial z$, it must be that at the free liquid surface, $\partial\eta/\partial t=-\partial\phi(x,y,h,t)/\partial z$. Equation (20.13.7) becomes

$$\frac{1}{g}\frac{\partial^2\phi}{\partial t^2}(x,y,h,t)+\frac{\partial\phi}{\partial z}(x,y,h,t)=0 \tag{20.13.8}$$

The other boundary condition is at the liquid interface with the plate,

$$v_z(x,y,0,t)=\dot{w}(x,y,t) \tag{20.13.9}$$

where $w=u_3(x,y,t)$ is the transverse displacement of the plate. This may be written as

$$-\frac{\partial\phi}{\partial z}(x,y,0,t)=\dot{w}(x,y,t) \tag{20.13.10}$$

The equation of motion of the plate is, from Eq. (11.2.22)

$$D\nabla^4 w-\nabla_T^2 w+\rho_p h_p\ddot{w}=-p(x,y,0,t)+q(x,y,t) \tag{20.13.11}$$

where $\nabla_T^2 = N_{xx}^r(\partial^2/\partial x^2) + N_{yy}^r(\partial^2/\partial y^2) + 2N_{xy}^r(\partial^2/\partial x \partial y), p(x, y, 0, t)$ is the total pressure exerted by the liquid on the plate, $q(x, y, t)$ is a general forcing function, $w(x, y, t) = u_3(x, y, t)$ is the transverse displacement of the plate from the undeflected position, $D = Eh_{\mathrm{p}}^3/12(l - \mu^2)$, ρ_{p} is the mass density of the plate material, and h_{p} is the plate thickness. N_{xx}, N_{yy}, and N_{xy} are constant normal and shear tensions per unit length in the plane of the plate. Utilizing Eq. (20.13.6), Eq. (20.13.11) becomes

$$D\nabla^4 w - \nabla_T^2 w + \rho_{\mathrm{p}} h_{\mathrm{p}} \ddot{w} = -\rho \frac{\partial \phi}{\partial t}(x, y, 0, t) - \rho g h + q(x, y, t) \tag{20.13.12}$$

The static component of the liquid loading may be subtracting by setting $w(x, y, t) = w_s(x, y) + \xi(x, y, t)$, where $\xi(x, y, t)$ is the dynamic transverse displacement, measured from the static equilibrium position, resulting in

$$D\nabla^4 \xi - \nabla_T^2 \xi + \rho_{\mathrm{p}} h_{\mathrm{p}} \ddot{\xi} = -\rho \frac{\partial \phi}{\partial t}(x, y, 0, t) + q(x, y, t) \tag{20.13.13}$$

The equations defining the eigenvalue problem are Eqs. (20.13.1) and (20.13.13), for zero forcing,

$$\nabla^2 \phi(x, y, z, t) = 0 \tag{20.13.14}$$

$$D\nabla^4 \xi(x, y, t) - \nabla_T^2 \xi(x, y, t) + \rho_{\mathrm{p}} h_{\mathrm{p}} \ddot{\xi}(x, y, t) = -\rho \frac{\partial \phi}{\partial t}(x, y, 0, t) \tag{20.13.15}$$

A case that can be solved in closed form is the simply supported rectangular plate loaded by a free surface liquid which is connected to a large reservoir of liquid at the four plate edges. This means that the time derivatives of the velocity potential at the four plate edges $x = 0, a$ and $y = 0, b$ are 0:

$$\left(\frac{\partial \phi}{\partial t}\right)(x, y, z, t) = 0 \quad (x = 0, a;\ y = 0, b) \tag{20.13.16}$$

The plate boundary conditions are for a simply supported plate,

$$\xi(x, y, t) = 0 \quad (x = 0, a;\ y = 0, b)$$

$$\left(\frac{\partial^2 \xi}{\partial x^2}\right)(x, y, t) = 0 \quad (x = 0, a)$$

$$\left(\frac{\partial^2 \xi}{\partial y^2}\right)(x, y, t) = 0 \quad (x = 0, b) \tag{20.13.17}$$

At a natural frequency

$$\phi(x, y, z, t) = \bar{\phi}(x, y, z) \mathrm{e}^{j\omega_{mn} t} \tag{20.13.18}$$

$$\xi(x, y, t) = \bar{\xi}(x, y) \mathrm{e}^{j\omega_{mn} t} \tag{20.13.19}$$

Substituting this into Eqs. (20.13.14) and (20.13.15) gives

$$\nabla^2\bar{\phi}=0 \tag{20.13.20}$$

$$D\nabla^4\bar{\xi}-\nabla_T^2\bar{\xi}-\rho_\mathrm{p}h_\mathrm{p}\omega_{mn}^2\bar{\xi}=-j\rho\omega_{mn}\bar{\phi}(x,y,0) \tag{20.13.21}$$

Boundary conditions (20.13.16) and (20.13.17) are satisfied by

$$\bar{\phi}=jZ_{mn}(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{20.13.22}$$

$$\bar{\xi}=A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{20.13.23}$$

where A_{mn} is a constant and $Z_{mn}(z)$ is an as yet unknown function describing the dependency of $\bar{\phi}$ on z. Substituting Eqs. (20.13.22) and (20.13.23) into Eqs. (20.13.20) and (20.13.21) results in

$$\frac{\mathrm{d}^2Z_{mn}}{\mathrm{d}z^2}-k_{mn}^2Z_{mn}=0 \tag{20.13.24}$$

$$A_{mn}(Dk_{mn}^4+t_{mn}^2-\rho_p h_p\omega_{mn}^2)=\rho\omega_{mn}Z_{mn}(0) \tag{20.13.25}$$

where $k_{mn}^2=(m\pi/a)^2+(n\pi/b)^2$ and $t_{mn}^2=N_{xx}^r(m\pi/a)^2+N_{yy}^r(n\pi/b)^2+2N_{xy}^r(m\pi/a)(n\pi/b)$. The solution of Eq. (20.13.24) is

$$Z_{mn}=B\sinh k_{mn}z+C\cosh k_{mn}z \tag{20.13.26}$$

The integration constants have to be evaluated using Eqs. (20.13.8) and (20.13.10), which become after eliminating time:

$$-\frac{\omega_{mn}^2}{g}\bar{\phi}(x,y,h)+\frac{\partial\bar{\phi}}{\partial z}(x,y,h)=0 \tag{20.13.27}$$

$$-\frac{\partial\bar{\phi}}{\partial z}(x,y,0)=j\omega_{mn}\bar{\xi}(x,y) \tag{20.13.28}$$

the latter because we have subtracted from the plate displacement the equilibrium position and therefore w was displaced by ξ.

Substituting Eqs. (20.13.22) and (20.13.23) gives

$$-\frac{\omega_{mn}^2}{g}Z_{mn}(h)+\frac{\mathrm{d}Z_{mn}}{\mathrm{d}z}(h)=0 \tag{20.13.29}$$

$$\frac{\mathrm{d}Z_{mn}}{\mathrm{d}z}(0)=-\omega_{mn}A_{mn} \tag{20.13.30}$$

Substituting Eq. (20.13.26) and solving for B and C in terms of the unspecified plate amplitude A_{mn} results in

$$B=-\frac{\omega_{mn}}{k_{mn}}A_{mn} \tag{20.13.31}$$

$$C=\frac{\omega_{mn}}{k_{mn}}A_{mn}\frac{k_{mn}g\cosh k_{mn}h-\omega_{mn}^2\sinh k_{mn}h}{k_{mn}g\sinh k_{mn}h-\omega_{mn}^2\cos hk_{mn}h} \tag{20.13.32}$$

Because $Z_{mn}(0)=C$, Eq. (20.13.25) becomes

$$A_1\omega_{mn}^4 - A_2\omega_{mn}^2 + A_3 = 0 \tag{20.13.33}$$

where

$$A_1 = \rho_p h_\rho \cosh k_{mn}h + \frac{\rho}{k_{mn}} \sinh k_{mn}h$$
$$A_2 = (\rho g + Dk_{mn}^4 + t_{mn}^2)\cosh k_{mn}h + \rho_p h_p g k_{mn} \sinh k_{mn}h$$
$$A_3 = (Dk_{mn}^4 + t_{mn}^2)k_{mn} g \sinh k_{mn}h \tag{20.13.34}$$

From this, it follows that there are two natural frequencies for every m, n combination:

$$\omega_{mn12}^2 = \frac{c_1}{2} \pm \sqrt{\left(\frac{c_1}{2}\right)^2 - c_2} \tag{20.13.35}$$

where

$$c_1 = \frac{\omega_{pmn}^2 + (g/h)[(\rho/\rho_p) + (k_{mn}h)\tanh k_{mn}h]}{a_{mn}}$$
$$c_2 = \frac{\omega_{pmn}^2 (g/h)(k_{mn}h)\tanh k_{mn}h}{a_{mn}}$$
$$a_{mn} = 1 + \frac{(\rho/\rho_p)(h/h_p)\tanh k_{mn}h}{k_{mn}h}$$
$$\omega_{pmn}^2 = \frac{k_{mn}^4 D + t_{mn}^2}{\rho_p h_p} \tag{20.13.36}$$

Here the ω_{pmn} are the natural frequencies of the plate if the liquid is entirely removed.

The natural modes in terms of the potential function and the plate motion are $(i=1,2)$

$$\Phi_i(x,y,z) = jA_{mni}\frac{\omega_{mni}}{k_{mn}}[f_1(m,n)\cosh k_{mn}z - \sinh k_{mn}z]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{20.13.37}$$

$$\bar{\xi}_i(x,y) = A_{mni}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{20.13.38}$$

where

$$f_i(m,n) = \frac{k_{mn}g\cosh k_{mn}h - \omega_{mni}^2 \sinh k_{mn}h}{k_{mn}g\sinh k_{mn}h - \omega_{mni}^2\cosh k_{mn}h} \tag{20.13.39}$$

For every m, n combination, there are two natural system frequencies. The higher set of natural frequencies, designated ω_{mn1} and given by Eq. (20.13.35), corresponds to plate-dominated motion. It can be seen that

ω_{mn1} decreases with liquid depth, more so for higher than for lower modes. At $h=0$, the natural frequencies are those of the liquid-free plate. As can be shown from Eq. (20.13.35), for small liquid depth, the results can be approximated by considering the liquid to act simply as a mass loading on the plate, so that the natural frequencies become

$$\omega_{mn1}=\omega_{pmn}\sqrt{\frac{\rho_{\mathrm{p}}h_{\mathrm{p}}}{\rho_{\mathrm{p}}h_{\mathrm{p}}+\rho h}} \tag{20.13.40}$$

That thin sheets of liquid can be approximated in terms of mass loading is well known. However, this approximation will not work for larger liquid depth.

When one obtains natural mode ratios of maximum normal liquid surface amplitude to plate amplitude, one finds that at very low liquid depth the liquid surface amplitudes are more or less equal to the plate amplitude, but diminish as depth increases. The liquid surface amplitudes are in phase with the plate amplitudes.

The second set of natural frequencies, ω_{mn2}, and modes correspond to liquid-dominated motion (wave action of the liquid on the plate). The natural frequencies approach zero with zero depth and tend to approach asymptotically certain constant values as the depth increases. This second set of natural frequencies is much lower than the plate-dominated set. The dominant surface amplitudes are out of phase with the plate amplitudes and tend to increase in magnitude relative to the plate amplitudes as liquid depth increases. As the liquid depth h increases, the liquid starts to slosh as if the plate is motionless.

20.14. ORTHOGONALITY OF NATURAL MODES FOR THREE-DIMENSIONAL SOLIDS, LIQUIDS, AND GASES

The natural modes satisfy all boundary conditions. Thus, following Sec. 5.8, $E_B=0$. Also setting $E_L=0$ gives

$$\delta\int_{t_0}^{t_1}(U-K)\,\mathrm{d}t=0 \tag{20.14.1}$$

or, utilizing Eqs. (20.3.1) and (20.3.2), we have

$$\begin{aligned}\int_{t_0}^{t_1}\int_{\alpha_3}\int_{\alpha_2}\int_{\alpha_1}&(\sigma_{11}\delta\varepsilon_{11}+\sigma_{22}\delta\varepsilon_{22}+\sigma_{33}\delta\varepsilon_{33}+\sigma_{12}\delta\varepsilon_{12}+\sigma_{13}\delta\varepsilon_{13}+\sigma_{23}\delta\varepsilon_{23})\\ \times A_1A_2A_3\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}t&=-\int_{t_0}^{t_1}\int_{\alpha_3}\int_{\alpha_2}\int_{\alpha_1}\rho(\ddot{u}_1\delta u_1+\ddot{u}_2\delta u_2+\ddot{u}_3\delta u_3)\\ &\qquad\times A_1A_2A_3\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}t\end{aligned} \tag{20.14.2}$$

where u_1, u_2, and u_3 are the deflections at location $(\alpha_1, \alpha_2, \alpha_3)$. When the solid is vibrating in its kth mode, displacements are

$$u_i(\alpha_1, \alpha_2, \alpha_3, t) = U_{ik}(\alpha_1, \alpha_2, \alpha_3)e^{j\omega_k t} \tag{20.14.3}$$

and stresses are

$$\sigma_{ij}(\alpha_1, \alpha_2, \alpha_3, t) = \sigma_{ij}^{(k)}(\alpha_1, \alpha_2, \alpha_3)e^{j\omega_k t} \tag{20.14.4}$$

For the virtual displacements, since they are arbitrary but must satisfy boundary conditions, let us select the pth mode. Thus

$$\delta u_i(\alpha_1, \alpha_2, \alpha_3, t) = U_{ip}(\alpha_1, \alpha_2, \alpha_3)e^{j\omega_p t} \tag{20.14.5}$$

from which it follows that

$$\delta\epsilon_{ij}(\alpha_1, \alpha_2, \alpha_3, t) = \epsilon_{ij}^{(p)}(\alpha_1, \alpha_2, \alpha_3)e^{j\omega_p t} \tag{20.14.6}$$

Substitution gives

$$\begin{aligned}\int_{\alpha_3}\int_{\alpha_2}\int_{\alpha_1} &(\sigma_{11}^{(k)}\varepsilon_{11}^{(\rho)} + \sigma_{22}^{(k)}\varepsilon_{22}^{(\rho)} + \sigma_{33}^{(k)}\varepsilon_{33}^{(\rho)} + \sigma_{12}^{(k)}\varepsilon_{12}^{(\rho)} + \sigma_{13}^{(k)}\varepsilon_{13}^{(\rho)} + \sigma_{23}^{(k)}\varepsilon_{23}^{(\rho)}) \\ &\times A_1A_2A_3\,d\alpha_1\,d\alpha_2\,d\alpha_3 = \omega_k^2\int_{\alpha_3}\int_{\alpha_2}\int_{\alpha_1}\rho(U_{1k}U_{1p} + U_{2k}U_{2p} + U_{3k}U_{3p}) \\ &\qquad\times A_1A_2A_3\,d\alpha_1\,d\alpha_2\,d\alpha_3\end{aligned} \tag{20.14.7}$$

Next, we repeat the procedure but assign the pth mode to the displacements and thus stresses

$$u_i(\alpha_1, \alpha_2, \alpha_3, t) = U_{ip}(\alpha_1, \alpha_2, \alpha_3)e^{j\omega_p t} \tag{20.14.8}$$

$$\sigma_{ij}(\sigma_1, \alpha_2, \alpha_3, t) = \sigma_{ij}^{(p)}(\alpha_1, \alpha_2, \alpha_3)e^{j\omega_p t} \tag{20.14.9}$$

and the kth mode to the virtual displacements

$$\delta u_i(\alpha_1, \alpha_2, \alpha_3, t) = U_{ik}(\alpha_1, \alpha_2, \alpha_3)e^{j\omega_k t} \tag{20.14.10}$$

so that

$$\delta\varepsilon_{ij}(\alpha_1, \alpha_2, \alpha_3, t) = \varepsilon_{ij}^{(k)}(\alpha_1, \alpha_2, \alpha_3)e^{j\omega_k t} \tag{20.14.11}$$

This gives

$$\begin{aligned}\int_{\alpha_3}\int_{\alpha_2}\int_{\alpha_1} &(\sigma_{11}^{(p)}\varepsilon_{11}^{(k)} + \sigma_{22}^{(p)}\varepsilon_{22}^{(k)} + \sigma_{33}^{(p)}\varepsilon_{33}^{(k)} + \sigma_{12}^{(p)}\varepsilon_{12}^{(k)} + \sigma_{13}^{(p)}\varepsilon_{13}^{(k)} + \sigma_{23}^{(p)}\varepsilon_{23}^{(k)}) \\ &\times A_1A_2A_3\,d\alpha_1\,d\alpha_2\,d\alpha_3 = \omega_\rho^2\int_{\alpha_3}\int_{\alpha_2}\int_{\alpha_1}\rho(U_{1k}U_{1p} + U_{2k}U_{2p} + U_{3k}U_{3p}) \\ &\qquad\times A_1A_2A_3\,d\alpha_1\,d\alpha_2\,d\alpha_3\end{aligned} \tag{20.14.12}$$

Subtracting the two equations from each other gives

$$(\omega_k^2-\omega_p^2)\int_{\alpha_3}\int_{\alpha_2}\int_{\alpha_1}\rho(U_{1k}U_{1p}+U_{2k}U_{2p}$$
$$+U_{3k}U_{3p})A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3=0 \tag{20.14.13}$$

The equation is satisfied whenever $p=k$ since

$$\omega_k^2-\omega_p^2=0, \quad p=k \tag{20.14.14}$$

In this case, the integral is expected to have some numerical value, which we designate as N_k:

$$N_k=\int_{\alpha_3}\int_{\alpha_2}\int_{\alpha_1}\rho(U_{1k}^2+U_{2k}^2+U_{3k}^3)A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3 \tag{20.14.15}$$

When $p\neq k$, we expect

$$\omega_k^2-\omega_p^2\neq 0, \quad p\neq k \tag{20.14.16}$$

unless there are repeated roots (see Sec. 5.9). Thus it must be that the integral is 0. This may all be summarized as

$$\int_{\alpha_3}\int_{\alpha_2}\int_{\alpha_1}\rho(U_{1k}U_{1p}+U_{2k}U_{2p}+U_{3k}U_{3p})A_1A_2A_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3$$
$$=\begin{cases}0 & \text{if } p\neq k\\ N_k & \text{if } p=k\end{cases} \tag{20.14.17}$$

This is the orthogonality condition for a three-dimensional elastic medium, be it solid, liquid, or gas.

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21

Discretizing Approaches

Two approaches that discretize shell structures are discussed: the finite difference and the finite element techniques (Fenves, 1973; Wah and Calcote, 1970; Yang, 1986). Both result in multidegree-of-freedom matrix equations. How these matrix equations can be solved by the modal series approach is also presented.

21.1. FINITE DIFFERENCES

Then finite differences approach is a purely mathematical technique. The equations of motion have to be known for the structure that is to be investigated. By contrast, the finite element approach will not require knowledge of the differential equations of motion once the element stiffness and mass matrices are defined.

The finite difference approach is based on the argument that a derivative can be approximately replaced by a difference. Let us illustrate this by the plate equation using Cartesian coordinates. Since, at a natural frequency,

$$u_3 = U_3 e^{j\omega t} \tag{21.1.1}$$

the plate equation becomes

$$D\nabla^4 U_3 - \rho h \omega^2 U_3 = 0 \tag{21.1.2}$$

where

$$\nabla^4 = \frac{\partial^4 U_3}{\partial x^4} + 2\frac{\partial^4 U_3}{\partial x^2 \partial y^2} + \frac{\partial^4 U_3}{\partial y^4} \tag{21.1.3}$$

The plate is divided into grids. The grids can be of unequal size, but in this discussion, a square grid is used. Labeling the point at which the equation is to be evaluated the i, j point and counting forward and backward from there as shown in Fig. 1 gives, dropping the subscript 3 to make the notation easier,

$$\left(\frac{\partial U}{\partial x}\right)_{i,j} = \frac{U_{i+1,j} - U_{1-i,j}}{2\Delta} \tag{21.1.4}$$

and

$$\left(\frac{\partial U}{\partial y}\right)_{i,j} = \frac{U_{i,j+1} - U_{i,j-1}}{2\Delta} \tag{21.1.5}$$

where Δ is the grid dimension. The second derivative is obtained from

$$\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} = \frac{(\partial U/\partial x)_{i+1,j} - (\partial U/\partial x)_{i-1,j}}{2\Delta} \tag{21.1.6}$$

or, after substitution of Eq. (21.1.4), and using half-steps,

$$\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta^2} \tag{21.1.7}$$

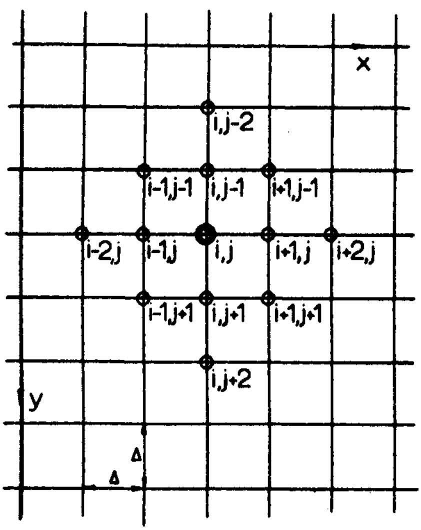

Fig. 1 Illustration of the required grid points spread to evaluate the finite difference form of the plate equation at one location.

We proceed in a similar fashion to define third, fourth, and mixed derivatives. Substitution in the equation of motion gives the equation of motion in finite difference form:

$$\frac{D}{\Delta^4}\big[(U_{i+2,j}+U_{i,j+2}+U_{i-2,j}+U_{i,j-2})$$
$$+2(U_{i+1,j+1}+U_{i-1,j+1}+U_{i-1,j-1}+U_{i+1,j-1})$$
$$-8(U_{i,j+1}+U_{i-1,j}+U_{i,j-1}+U_{i+1,j})+20U_{i,j}\big]-\rho h\omega^2 U_{i,j}=0 \tag{21.1.8}$$

Thus if there are N grid points, we obtain N simultaneous equations. There are more than N unknowns, since as we evaluate the equation of motion along the boundary, points outside the plate boundary will appear in the equation systems. The equations for these additional unknowns are provided by the boundary conditions. For instance, at a clamped edge, the boundary conditions

$$U_3=0 \tag{21.1.9}$$

$$\frac{\partial U_3}{\partial x}=0 \tag{21.1.10}$$

become

$$U_{i,j}=0 \tag{21.1.11}$$

$$U_{i+1,j}-U_{i-1,j}=0 \tag{21.1.12}$$

At a free edge, the boundary conditions

$$\frac{\partial^2 U_3}{\partial x^2}+\mu\frac{\partial^2 U_3}{\partial y^2}=0 \tag{21.1.13}$$

and

$$\frac{\partial^3 U_3}{\partial x^3}+(2-\mu)\frac{\partial^3 U_3}{\partial x\partial y^2}=0 \tag{21.1.14}$$

become

$$U_{i+1,j}-2U_{i,j}+U_{i-1,j}+\mu(U_{i,j+1}-2U_{i,j}+U_{i,j-1})=0 \tag{21.1.15}$$

$$U_{i+2,j}-2U_{i+1,j}+2U_{i-1,j}-U_{i-2,j}+(2-\mu)$$
$$\times(U_{i+1,j+1}+U_{i+1,j-1}-U_{i-1,j-1}-U_{i-1,j+1}+2U_{i-1,j}-2U_{i+1,j})=0 \tag{21.1.16}$$

Let us illustrate this by the example of a simply supported square plate with only four interior grid points as shown in Fig. 2. Because of the

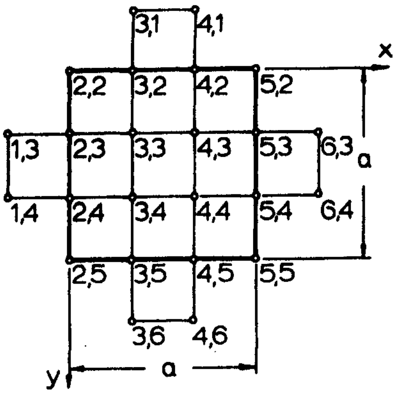

Fig. 2 Illustrative example of a square plate with only four interior grid points.

information requirement that the finite difference form of the plate equation imposes on us, we also have to consider the points at the boundary and dummy points outside the boundary.

From the boundary conditions

$$U_3 = 0 \tag{21.1.17}$$

for $(x, y) = (0, y), (a, y), (x, 0), (x, a)$ and

$$\frac{\partial^2 U_3}{\partial x^2} = 0 \tag{21.1.18}$$

for $(x, y) = (0, y), (a, y)$ and

$$\frac{\partial^2 U_3}{\partial y^2} = 0 \tag{21.1.19}$$

for $(x, y) = (x, 0), (x, a)$, we obtain

$$\begin{aligned} U_{2,2} &= U_{3,2} = U_{4,2} = U_{5,2} = U_{5,3} = U_{5,4} = U_{5,5} = U_{4,5} = U_{3,5} \\ &= U_{2,5} = U_{2,4} = U_{2,3} = 0 \end{aligned} \tag{21.1.20}$$

and

$$\begin{aligned} &U_{1,3} = U_{3,1} = -U_{3,3}, \quad U_{1,4} = U_{3,6} = -U_{3,4}, \\ &U_{4,1} = U_{6,3} = -U_{4,3}, \quad U_{6,4} = U_{4,6} = -U_{4,4} \end{aligned} \tag{21.1.21}$$

Next, evaluating the finite difference form of the plate equation at the four interior points, we get, for instance, for point (3,3), making use of the boundary conditions,

$$\left(18 - \Delta^4 \frac{\rho h \omega^2}{D}\right) U_{3,3} - 8U_{4,3} - 8U_{3,4} + 2U_{4,4} = 0 \tag{21.1.22}$$

where $\Delta = a/3$. Proceeding in a similar manner for points (4,3), (3,4), and (4,4), we have, in matrix form,

$$[A]\{U_{i,j}\} = 0 \tag{21.1.23}$$

where

$$[A] = \begin{bmatrix} 18 - R\omega^2 & -8 & -8 & 2 \\ -8 & 18 - R\omega^2 & 2 & -8 \\ -8 & 2 & 18 - R\omega^2 & -8 \\ 2 & -8 & -8 & 18 - R\omega^2 \end{bmatrix} \tag{21.1.24}$$

$$R = \frac{\rho h \Delta^4}{D} \tag{21.1.25}$$

$$[U_{i,j}] = [U_{3,3}, U_{4,3}, U_{3,4}, U_{4,4}] \tag{21.1.26}$$

The natural frequencies are obtained by setting the determinant of the matrix to 0,

$$|A| = 0 \tag{21.1.27}$$

In this particular case, we obtain approximations to the first four natural frequencies. They are

$$\omega_n = \frac{\lambda_n}{a^2}\sqrt{\frac{D}{\rho h}} \tag{21.1.28}$$

where $\lambda_1 = 18$, $\lambda_2 = \lambda_3 = 36$, and $\lambda_4 = 54$. The exact values are $\lambda_1 = 2\pi^2$, $\lambda_2 = \lambda_3 = 5\pi^2$, and $\lambda_4 = 8\pi^2$. We see that even with only four interior grid points, the first natural frequency is approximated well. For higher mode calculation, we need obviously a finer grid. But this is no difficulty for a computer application.

The mode shapes are obtained by substituting each natural frequency back in Eq. (21.1.23) and solving for three of the four grid points in terms of, say, $U_{3,3}$. This gives the four modes, for $U_{3,3} = 1$, as

$$[U_{i,j}]_1 = [1, 1, 1, 1], \quad [U_{i,j}]_2 = [1, -1, 1, -1],$$
$$[U_{i,j}]_3 = [1, 1, -1, -1], \quad [U_{i,j}]_4 = [1, -1, -1, 1] \tag{21.1.29}$$

Note that for free edges, the equation of motion has to be evaluated at the edge points since these points are not motionless.

For more background on finite difference applications, see, for example, Wah and Calcote (1970). The approach has been used primarily for plate problems. For shells, the equations of motion are of eighth order and many more grid points are involved at every equation evaluation. That makes use of finite differences more involved. For an example, see Fenves (1973).

21.2. FINITE ELEMENTS

The finite element approach is a physical approach. Knowing a solution of a simple element, a shell or plate can be thought to be assembled of these elements. This assembling is done by mathematically enforced continuity and equilibrium conditions at the element node points (points where the elements are joined to each other or to a boundary).

To illustrate the approach, let us select as examples first, a simple beam-bending element, and second, a simple plate-bending element. Steps that have to be taken in deriving the element properties apply essentially to all elements, including curved-shell elements (third example).

There are various ways that can be used to derive the element properties. Here we use Hamilton's principle. For this purpose, we have to generate expressions for strain and kinetic energy in terms of the node displacements.

21.2.1. Beam Elements

First let us obtain the expression for strain energy. From Eq. (2.6.3) it is, for a beam under transverse deflection,

$$U=\frac{EI}{2}\int_0^L\left(\frac{\partial^2 u_3}{\partial x^2}\right)^2 \mathrm{d}x \tag{21.2.1}$$

Next, we have to assume a deflection function that allows enforcement of a transverse deflection condition and a slope condition on each end of the beam element. The four conditions require four constants. We choose

$$u_3(x,t)=a_0+a_1x+a_2x^2+a_3x^3 \tag{21.2.2}$$

This may also be written

$$u_3(x,t)=\{A\}^{\mathrm{T}}\{Z\} \tag{21.2.3}$$

where the superscript T means transpose and where

$$\{A\}^{\mathrm{T}} = \lfloor a_0, a_1, a_2, a_3 \rfloor \tag{21.2.4}$$

$$\{Z\}^{\mathrm{T}} = \lfloor 1, x, x^2, x^3 \rfloor \tag{21.2.5}$$

Therefore,

$$\frac{\partial^2 u_3}{\partial x^2}=\{A\}^{\mathrm{T}}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}=\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^{\mathrm{T}}\{A\} \tag{21.2.6}$$

and therefore,

$$\left(\frac{\partial^2 u_3}{\partial x^2}\right)^2=\{A\}^{\mathrm{T}}[D(x)]\{A\} \tag{21.2.7}$$

where

$$[D(x)] = \left\{ \frac{\partial^2 Z}{\partial x^2} \right\} \left\{ \frac{\partial^2 Z}{\partial x^2} \right\}^{\mathrm{T}} \tag{21.2.8}$$

Substituting Eq. (21.2.5) in Eq. (21.2.8) gives

$$[D(x)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 12x \\ 0 & 0 & 12x & 36x^2 \end{bmatrix} \tag{21.2.9}$$

The strain energy is therefore

$$U = \frac{EI}{2} \{A\}^{\mathrm{T}} \int_0^L [D(x)] \mathrm{d}x \{A\} \tag{21.2.10}$$

Next, let us define the nodal displacements (slopes are also referred to as *displacements*. At the $x=0$ end of the element designated as location k in Fig. 3, we get from Eq. (21.2.2)

$$u_3(0,t) = u_{3k} = a_0 \tag{21.2.11}$$

$$\frac{\partial u_3}{\partial x}(0,t) = \theta_{xk} = a_1 \tag{21.2.12}$$

and on the $x=L$ end of the element designated as location l we obtain

$$u_3(L,t) = u_{3l} = a_0 + a_1 L + a_2 L^2 + a_3 L^3 \tag{21.2.13}$$

$$\frac{\partial u_3}{\partial x}(L,t) = \theta_{xl} = a_1 + 2a_2 L + 3a_3 L^2 \tag{21.2.14}$$

This may be written in the form

$$\{u_3\}_i = [B]\{A\} \tag{21.2.15}$$

where

$$\{u_3\}_i^{\mathrm{T}} = \lfloor u_{3k}, \theta_{xk}, u_{3l}, \theta_{xl} \rfloor \tag{21.2.16}$$

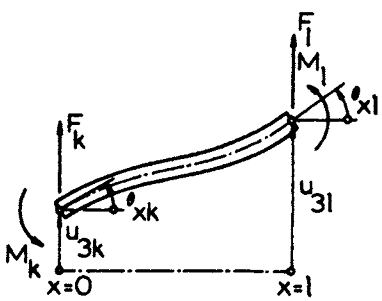

Fig. 3 Finite element for transversely vibrating beams.

and

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \tag{21.2.17}$$

Solving for $\{A\}$ gives

$$\{A\}=[B]^{-1}\{u_3\}_i \tag{21.2.18}$$

Redefining

$$[B]^{-1}=[c] \tag{21.2.19}$$

allows us to write the strain energy equation as

$$U=\frac{EI}{2}\{u_3\}_i^{\mathrm{T}}[c]^{\mathrm{T}}\int_0^L [D(x)]\mathrm{d}x[c]\{u_3\}_i \tag{21.2.20}$$

The variation of U in terms of variations in the node displacement is

$$\delta U=EI\{\delta u_3\}_i^{\mathrm{T}}[c]^{\mathrm{T}}\int_0^L [D(x)]\mathrm{d}x[c]\{u_3\}_i \tag{21.2.21}$$

Next, let us obtain the kinetic energy of the beam element. It is

$$K=\frac{\rho A}{2}\int_0^L \dot{u}_3^2\mathrm{d}x \tag{21.2.22}$$

From Eq. (21.2.3), we get

$$\dot{u}_3^2=\{\dot{A}\}^{\mathrm{T}}\{Z\}\{Z\}^{\mathrm{T}}\{\dot{A}\} \tag{21.2.23}$$

Substituting Eqs. (21.2.18) and (21.2.19) gives

$$\dot{u}_3^2=\{\dot{u}_3\}_i^{\mathrm{T}}[c]^{\mathrm{T}}[F(x)][c]\{\dot{u}_3\}_i \tag{21.2.24}$$

where

$$[F(x)]=\{Z\}\{Z\}^{\mathrm{T}}=\begin{bmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & x^4 \\ x^2 & x^3 & x^4 & x^5 \\ x^3 & x^4 & x^5 & x^6 \end{bmatrix} \tag{21.2.25}$$

The kinetic energy expression is therefore

$$K=\frac{\rho A}{2}\{\dot{u}_3\}_i^{\mathrm{T}}[c]^{\mathrm{T}}\int_0^L [F(x)]\mathrm{d}x[c]\{\dot{u}_3\}_i \tag{21.2.26}$$

The variation in K is

$$\delta K=\rho A\{\delta\dot{u}_3\}_i^{T}[c]^{\mathrm{T}}\int_0^L [F(x)]\mathrm{d}x[c]\{\dot{u}_3\}_i \tag{21.2.27}$$

We also have to consider the virtual work due to boundary forces. At $x=0$, we have a shear force F_k and a bending moment M_k acting on the element.

At $x=L$, the shear force is F_l and the moment is M_l. The virtual work is therefore

$$\delta W=\{F\}_i^T\{\delta u_3\}_i=\{\delta u_3\}_i^T\{F\}_i \tag{21.2.28}$$

where

$$\{F\}_i^T=\lfloor F_k, M_k, F_l, M_l\rfloor \tag{21.2.29}$$

We are now ready to apply Hamilton's principle, which in this case, with boundary forces and moments, becomes

$$\int_{t_0}^{t_1}(\delta K-\delta U+\delta W)\,\mathrm{d}t=0 \tag{21.2.30}$$

Let us examine the kinetic energy part of the integral

$$\int_{t_0}^{t_1}\delta K\,\mathrm{d}t=\rho A\int_{t_0}^{t_1}\{\delta\dot{u}_3\}_i^T[c]^\mathrm{T}\int_0^L[F(x)]\,\mathrm{d}x[c]\{\dot{u}_3\}_i\,\mathrm{d}t \tag{21.2.31}$$

We have to integrate by parts in order to separate the node displacement variations from the time derivative. Since

$$\int u\,\mathrm{d}v=uv-\int v\,\mathrm{d}u \tag{21.2.32}$$

we let

$$\mathrm{d}v=\{\delta\dot{u}_3\}_i^\mathrm{T}\,\mathrm{d}t \tag{21.2.33}$$

$$u=[c]^\mathrm{T}\int_0^L[F(x)]\,\mathrm{d}x[c]\{\dot{u}_3\}_i \tag{21.2.34}$$

This gives

$$\int_{t_0}^{t_1}\delta K\,\mathrm{d}t=\rho A\{\delta u_3\}_i^T[c]^\mathrm{T}\int_0^L[F(x)]\,\mathrm{d}x[c]\{\dot{u}_3\}_i\Big|_{t_0}^{t_1} -\rho A\int_{t_0}^{t_1}\{\delta u_3\}_i^\mathrm{T}[c]^\mathrm{T}\int_0^L[F(x)]\,\mathrm{d}x[c]\{\ddot{u}_3\}_i\,\mathrm{d}t \tag{21.2.35}$$

The first term is 0 because at t_0 and t_1 the variational displacements are 0 by definition of Hamilton's principle. Substituting this expression and Eqs. (21.2.21) and (21.2.28) in Eq. (21.2.30) gives

$$\int_{t_0}^{t_1}\{\delta u_3\}_i^\mathrm{T}\left[A\rho[c]^\mathrm{T}\int_0^L[F(x)]\,\mathrm{d}x[c]\{\ddot{u}_3\}_i +EI[c]^\mathrm{T}\int_0^L[D(x)]\,\mathrm{d}x[c]\{u_3\}_i-\{F\}_i\right]\mathrm{d}t=0 \tag{21.2.36}$$

Because the variational displacements are independent and arbitrary, the equation can be satisfied only if the bracketed quantity is 0. This gives the equation of motion of the element

$$[m]\{\ddot{u}_3\}_i+[K]\{u_3\}_i=\{F\}_i \tag{21.2.37}$$

where

$$[m]=A\rho[c]^{\mathrm{T}}\int_0^L[F(x)]\mathrm{d}x[c] \tag{21.2.38}$$

$$[k]=EI[c]^{\mathrm{T}}\int_0^L[D(x)]\mathrm{d}x[c] \tag{21.2.39}$$

The matrix $[m]$ is the mass matrix and the matrix $[k]$ is the stiffness matrix. We will see that for the plate element, the definition will be similar. The indicated integrations and matrix manipulations can be executed as part of the finite element computer program. However, the beam case is so simple that we can easily do the integration and matrix manipulations by hand. Using Eqs. (21.2.9), (21.2.17), and (21.2.25), we obtain

$$[m]=\frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \tag{21.2.40}$$

$$[k]=\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{21.2.41}$$

21.2.2. Plate Elements

Now let us do not derivation of a rectangular plate element, following the identical procedure. The strain energy for a transversely deflected plate is, from Eq. (2.6.3),

$$U=\frac{D}{2}\int_0^b\int_0^a\left\{\left(\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial^2 u_3}{\partial y^2}\right)^2 \right.$$
$$\left. -2(1-\mu)\left[\frac{\partial^2 u_3}{\partial x^2}\frac{\partial^2 u_3}{\partial y^2}-\left(\frac{\partial^2 u_3}{\partial x\partial y}\right)^2\right]\right\}\mathrm{d}x\mathrm{d}y \tag{21.2.42}$$

In case of the plate element, we have to be able to enforce as a minimum continuity of deflection at each of the four corners and continuity of slope in two orthogonal directions at each corner. It is also actually better to enforce continuity of twisting, but for simplicity's sake the easier example is used. Let us label the four corners k, l, m, and n as shown in Fig. 4.

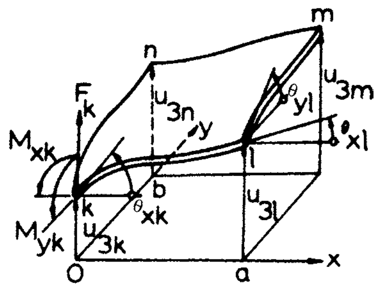

FIG. 4 Finite element for transversely vibrating rectangular plates.

Let us again use the symbol θ to designate slopes. For instance, at the corner l,

$$\theta_{xl} = \frac{\partial u_{3l}}{\partial x} \tag{21.2.43}$$

$$\theta_{yl} = \frac{\partial u_{3l}}{\partial y} \tag{21.2.44}$$

To be able to enforce 12 continuity conditions, the deflection function has to have 12 constants. We choose

$$u_3(x,y,t) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} xy^3 \tag{21.2.45}$$

This may also be written as

$$u_3(x,y,t) = \{A\}^{\mathrm{T}}\{Z\} \tag{21.2.46}$$

where

$$\{A\}^{\mathrm{T}} = \lfloor a_1, a_2, \ldots, a_{12} \rfloor \tag{21.2.47}$$

$$\{Z\}^{\mathrm{T}} = \lfloor 1, x, y, x^2, xy, y^2, x^3, x^2 y, xy^2, y^3, x^3 y, xy^3 \rfloor \tag{21.2.48}$$

The strain energy expression becomes, after substitution,

$$U = \frac{D}{2}\{A\}^{\mathrm{T}} \int_0^b \int_0^a [D(x,y)]\mathrm{d}x\mathrm{d}y\{A\} \tag{21.2.49}$$

where

$$[D(x,y)] = \left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^{\mathrm{T}} + \left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^{\mathrm{T}} + \mu\left\{\frac{\partial^2 Z}{\partial x^2}\right\}\left\{\frac{\partial^2 Z}{\partial y^2}\right\}^{\mathrm{T}} + \mu\left\{\frac{\partial^2 Z}{\partial y^2}\right\}\left\{\frac{\partial^2 Z}{\partial x^2}\right\}^{\mathrm{T}} + 2(1-\mu)\left\{\frac{\partial^2 Z}{\partial x \partial y}\right\}\left\{\frac{\partial^2 Z}{\partial x \partial y}\right\}^{\mathrm{T}} \tag{21.2.50}$$

Next, we formulate the nodal displacements by evaluating Eq. (21.2.45) and its derivatives at the node points, enforcing the conditions that

$$\begin{aligned}
&u_3(0,0,t)=u_{3k}, \quad \frac{\partial u_3}{\partial x}(0,0,t)=\theta_{xk}, \quad \frac{\partial u_3}{\partial y}(0,0,t)=\theta_{yk}\\
&u_3(a,0,t)=u_{3l}, \quad \frac{\partial u_3}{\partial x}(a,0,t)=\theta_{xl}, \quad \frac{\partial u_3}{\partial y}(a,0,t)=\theta_{yl}\\
&u_3(a,b,t)=u_{3m}, \quad \frac{\partial u_3}{\partial x}(a,b,t)=\theta_{xm}, \quad \frac{\partial u_3}{\partial y}(a,b,t)=\theta_{ym}\\
&u_3(0,b,t)=u_{3n}, \quad \frac{\partial u_3}{\partial x}(0,b,t)=\theta_{xn}, \quad \frac{\partial u_3}{\partial y}(0,b,t)=\theta_{yn}
\end{aligned} \tag{21.2.51}$$

This can be written as

$$\{u_3\}_i=[B]\{A\} \tag{21.2.52}$$

where

$$\{u_3\}_i^{\mathrm{T}}=\lfloor u_{3k},\theta_{xk},\theta_{yk},u_{3l},\theta_{xl},\theta_{yl},u_{3m},\theta_{xm},\theta_{ym},u_{3n},\theta_{xn},\theta_{yn}\rfloor \tag{21.2.53}$$

and where

$$[B]=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0\\
1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & a^3b & ab^3\\
0 & 1 & 0 & 2a & b & 0 & 3a^2 & 2ab & b^2 & 0 & 3a^2b & b^3\\
0 & 0 & 1 & 0 & a & 2b & 0 & a^2 & 2ab & 3b^2 & a^3 & 3ab^2\\
1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0\\
0 & 1 & 0 & 0 & b & 0 & 0 & 0 & b^2 & 0 & 0 & b^3\\
0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0
\end{bmatrix} \tag{21.2.54}$$

Solving for $\{A\}$ gives

$$\{A\}=[c]\{u_3\}_i \tag{21.2.55}$$

where

$$[c]=[B]^{-1} \tag{21.2.56}$$

The strain energy expression therefore becomes

$$U=\frac{D_i}{2}\{u_3\}_i^{\mathrm{T}}[c]^{\mathrm{T}}\int_0^b\int_0^a[D(x,y)]\mathrm{d}x\mathrm{d}y[c]\{u_3\}_i \tag{21.2.57}$$

The kinetic energy of the element is

$$K = \frac{\rho h}{2} \int_0^b \int_0^a \dot{u}_3^2 \mathbf{d}x \mathbf{d}y \tag{21.2.58}$$

From Eq. (21.2.46), we obtain

$$\dot{u}_3^2 = \{\dot{A}\}^{\mathrm{T}} \{Z\} \{Z\}^{\mathrm{T}} \{\dot{A}\} \tag{21.2.59}$$

and substituting Eq. (21.2.55) gives

$$\dot{u}_3^2 = \{\dot{u}_3\}_i^T \{c\}^{\mathrm{T}} [F(x,y)][c]\{\dot{u}_3\}_i \tag{21.2.60}$$

where

$$[F(x,y)] = \{Z\}\{Z\}^{\mathrm{T}} \tag{21.2.61}$$

The kinetic energy expression therefore becomes

$$K = \frac{\rho h}{2} \{\dot{u}_3\}_i^T [c]^{\mathrm{T}} \int_0^b \int_0^a [F(x,y)] \mathbf{d}x \mathbf{d}y [c]\{\dot{u}_3\}_i \tag{21.2.62}$$

The virtual work due to the nodal forces and moments is

$$\delta W = \{F\}_i^{\mathrm{T}} \{\delta u_3\}_i = \{\delta u_3\}_i = \{\delta u_3\}_i \{F\}_i \tag{21.2.63}$$

where

$$\{F\}_i^{\mathrm{T}} = \lfloor F_k, M_{xk}, M_{yk}, F_l, M_{xl}, M_{yl}, F_m, M_{xm}, M_{ym}, F_n, M_{xn}, M_{yn} \rfloor \tag{21.2.64}$$

Both the potential and the kinetic energy expressions are similar to the expressions for the beam element. Thus, applying Hamilton's principle, following identical steps, we obtain the equation of motion of the element

$$[m]\{\ddot{u}_3\}_i + [k]\{u_3\}_i = \{F\}_i \tag{21.2.65}$$

where

$$[m] = \rho h [c]^{\mathrm{T}} \int_0^b \int_0^a [F(x,y)] \mathbf{d}x \mathbf{d}y [c] \tag{21.2.66}$$

$$[k] = D[c]^{\mathrm{T}} \int_0^b \int_0^a [D(x,y)] \mathbf{d}x \mathbf{d}y [c] \tag{21.2.67}$$

The stiffness matrix $[k]$ and the mass matrix $[m]$ are usually not given in explicit form but are generated on the computer when needed.

21.2.3. Assembly of Elements into a Global Equation of Motion

The remaining question is how the various elements are joined together. Let us illustrate this on the example of a clamped–clamped uniform beam that is approximated by only two beam elements of the same length $L=a/2$ as shown in Fig. 5. Using the subscript 1 for element 1 and the subscript 2 for element 2, we obtain from the continuity condition

$$u_{3l1}=u_{3k2} \tag{21.2.68}$$

$$\theta_{xl1}=\theta_{xk2} \tag{21.2.69}$$

The forces and moments at the junction have to add up to 0:

$$F_{l1}+F_{k2}=0 \tag{21.2.70}$$

$$M_{l1}+M_{k2}=0 \tag{21.2.71}$$

This allows us to formulate the global equation of motion from the element equations of motion by a simple addition process. We obtain

$$[M]\{\ddot{u}_3\}+[K]\{u_3\}=\{Q\} \tag{21.2.72}$$

where the global matrix is

$$[M]=\frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L & 0 & 0 \\ 22L & 4L^2 & 13L & -3L^2 & 0 & 0 \\ 54 & 13L & 312 & 0 & 54 & -13L \\ -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 \\ 0 & 0 & 54 & 13L & 156 & -22L \\ 0 & 0 & -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \tag{21.2.73}$$

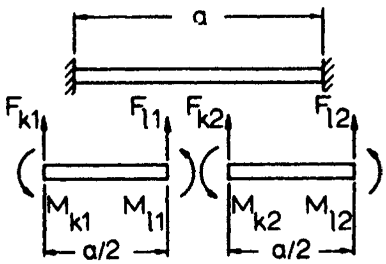

FIG. 5 Illustrative example of a transversely vibrating, clamped–clamped beam described by only two finite elements.

and where the global stiffness matrix is

$$[K]=\frac{EI}{L^3}\left[\begin{array}{cccc|cc} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ \hline -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ \hline 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{array}\right] \tag{21.2.74}$$

The nodal force vector becomes

$$\{Q\}^{\mathrm{T}}=\lfloor F_{k1}, M_{kI}, 0, 0, F_{l2}, M_{l2}\rfloor \tag{21.2.75}$$

The nodal displacement vector becomes

$$\{u_3\}^{\mathrm{T}}=\lfloor u_{3k1}, \theta_{xk1}, u_{3k2}, \theta_{xk2}, u_{3l2}, \theta_{xl2}\rfloor \tag{21.2.76}$$

So far, the boundary conditions at the clamped locations have not been applied yet. If we do so, the displacement vector reduces to

$$\{u_3\}^{\mathrm{T}}=\lfloor 0, 0, u_{3k2}, \theta_{xk2}, 0, 0\rfloor \tag{21.2.77}$$

this means that the equation of motion reduces to

$$\frac{\rho AL}{420}\begin{bmatrix} 312 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{3k2} \\ \ddot{\theta}_{xk2} \end{Bmatrix}+\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{3k2} \\ \theta_{xk2} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{21.2.78}$$

This equation can be solved in the usual way for the first two natural frequencies and modes. The natural frequencies are given by

$$\omega_I=\frac{\lambda_1}{a^2}\sqrt{\frac{EI}{\rho A}} \tag{21.2.79}$$

where $\lambda_1=22.72$ and $\lambda_2=82.0$. This compares to the exact values of $\lambda_1=22.3$ and $\lambda_2=61.67$ and illustrates the need for a larger number of elements if higher modes are to be investigated. The solution of plate and shell problems follows a similar assembly procedure.

21.2.4. Shell Elements

For a shell, the strain energy is given by Eq. (2.6.3), which can be shown after substitution of Eqs. (2.6.16)–(2.4.18) and Eqs. (2.5.4)–(2.5.6), and integration over the shell thickness to reduce to

$$U=\frac{1}{2}\int_{\alpha_1}\int_{\alpha_1}\left[K\left\{\varepsilon_{11}^{0^2}+2\mu\varepsilon_{11}^{0}\varepsilon_{22}^{0}+\varepsilon_{22}^{0^2}+\frac{1-\mu}{2}\varepsilon_{12}^{0^2}\right\}\right.$$
$$\left.+D\left\{k_{11}^2+2\mu k_{11}k_{22}+k_{22}^2+\frac{1-\mu}{2}k_{12}^2\right\}\right]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{21.2.80}$$

where $\varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{12}^0, k_{11}, k_{22}$, and k_{12} are defined in terms of displacements by Eqs. (2.4.19)–(2.4.24).

Deflection functions have to be assumed which, as a minimum, allow enforcement of continuity in u_1, u_2, u_3, β_1, and β_2. The element shapes, orientation, and its curvatures have to be chosen such that there is a continuity in undeflected curvature. This is not easily achieved, but it has to be avoided that the finite element shell has discontinuous original curvatures because it may then behave like a corrugated shell.

As an example, let us discuss a special ring element for free vibrations that is used for structurally axisymmetric shells of revolution such as tires (Chang et al., 1983 and 1984; Hunckler et al., 1983; Kung et al., 1985 and 1986). In this case, one is able analytically to subtract out the θ dependency (the element is shown in Fig. 6). Another advantage is that the lines of principal curvatures of the element coincide with the lines of principal curvature of the shell of revolution. The radius of curvature R_s for each element has to be matched at each element location to the radius R_s of the shell of revolution. Designating $\alpha_1 = s$ and $\alpha_2 = \theta$, with $A_1 = 1$ and $A_z = r$, a workable set of deflection functions is

$$u_3(s, \theta, t) = (a_1 + a_2 s + a_3 s^2 + a_4 s^3)\cos n(\theta - \phi) \tag{21.2.81}$$

$$u_\theta(s, \theta, t) = (a_5 + a_6 s + a_7 s^2 + a_8 s^3)\sin n(\theta - \phi) \tag{21.2.82}$$

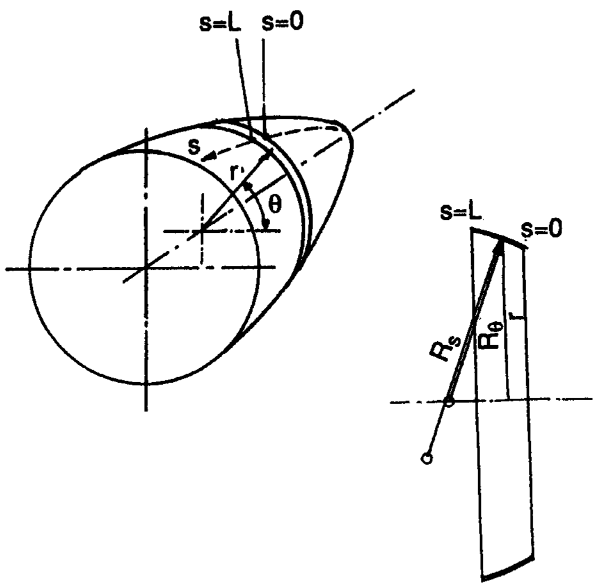

FIG. 6 Circumferential strip element for a vibrating shell of revolution.

$$u_s(s,\theta,t) = (a_9 + a_{10}s + a_{11}s^2 + a_{12}s^3)\cos n(\theta - \phi) \tag{21.2.83}$$

where $n = 0, 1, 2, \ldots$ and $\phi = 0, \pi/2n$ (the reason for the phase angle ϕ is discussed in Chapters 5 and 8). However, to find natural frequencies, only $\phi = 0$ needs to be used. They will be the same for $\phi = \pi/2n$. Eqs. (21.2.81)–(21.2.83) will allow us to satisfy at each ring element edge continuity in $u_3, u_\theta, u_s, \beta_s, \partial u_\theta/\partial s$, and $\partial u_s/\partial s$.

The derivation of the element properties by Hamilton's principle follows the steps of the two previous derivations. Equations (21.2.81)–(21.2.83) can be written

$$u_3(s,\theta,t) = \{A_{1-4}\}^{\mathrm{T}}\{Z\}\cos n(\theta - \phi) \tag{21.2.84}$$

$$u_\theta(s,\theta,t) = \{A_{5-\theta}\}^{\mathrm{T}}\{Z\}\sin n(\theta - \phi) \tag{21.2.85}$$

$$u_s(s,\theta,t) = \{A_{9-12}\}^{\mathrm{T}}\{Z\}\cos n(\theta - \phi) \tag{21.2.86}$$

where

$$\{A_{1-4}\}^{\mathrm{T}} = [a_1, a_2, a_3, a_4] \tag{21.2.87}$$

$$\{A_{5-\theta}\}^{\mathrm{T}} = [a_5, a_6, a_7, a_8] \tag{21.2.88}$$

$$\{A_{9-12}\}^{\mathrm{T}} = [a_9, a_{10}, a_{11}, a_{12}] \tag{21.2.89}$$

$$\{Z\}^{\mathrm{T}} = [1, s, s^2, s^3] \tag{21.2.90}$$

Substituting these equations into Eq. (21.2.80) and integrating with respect to $\alpha_2 = \theta$ will eliminate θ from Eq. (21.2.80) but will make it a function of n (where $2n$ is the number of vibration node lines that one encounters when moving in circumferential direction). Equation (21.2.80) will be of the form

$$U = \{A\}^{\mathrm{T}} \int_{s=0}^{L} [D(s,n)]\,\mathrm{d}s\,\{A\} \tag{21.2.91}$$

where

$$\{A\} = \begin{Bmatrix} A_{1-4} \\ \cdots \\ A_{5-8} \\ \cdots \\ A_{9-12} \end{Bmatrix} \tag{21.2.92}$$

and L is the meridional length of the ring element.

Next, we formulate the boundary displacements and their derivatives by enforcing the conditions

$$\begin{aligned}
u_3(0,t)&=u_{30}, \quad \beta_s(0,t)=\beta_{sv}\\
u_3(L,t)&=u_{3L}, \quad \beta_s(L,t)=\beta_{sL}\\
u_s(0,t)&=u_{s0}, \quad \frac{\partial u_s}{\partial s}(0,t)=u'_{s0}\\
u_s(L,t)&=u_{L0}, \quad \frac{\partial u_s}{\partial s}(L,t)=u'_{sL}\\
u_\theta(0,t)&=u_{\theta 0}, \quad \frac{\partial u_\theta}{\partial s}(0,t)=u'_{\theta 0}\\
u_\theta(L,t)&=u_{\theta L}, \quad \frac{\partial u_\theta}{\partial s}(L,t)=u'_{\theta L}
\end{aligned} \tag{21.2.93}$$

This can be written as

$$\{u\}_i=[B]\{A\} \tag{21.2.94}$$

where

$$\{u\}_i=[u_{30},\beta_{s0},u_{3L},\beta_{sL},\ldots] \tag{21.2.95}$$

Solving for $\{A\}$ gives

$$\{A\}=[C]\{u\}_i \tag{21.2.96}$$

where

$$[C]=[B]^{-1} \tag{21.2.97}$$

Thus the strain energy becomes

$$U=\frac{1}{2}\{u\}_i^{\mathrm{T}}[C]^{\mathrm{T}}\int_{s=0}^{L}[D(s,n)]\mathrm{d}s[C]\{u\}_i \tag{21.2.98}$$

Proceeding similarly, we obtain the kinetic energy in the form

$$K=\frac{\rho h}{2}\{\dot{u}\}_i^{T}[C]^{\mathrm{T}}\int_{s=0}^{L}[F(s,n)]\mathrm{d}s[C]\{\dot{u}\}_i \tag{21.2.99}$$

and applying Hamilton's principle, we obtain

$$[m]\{\ddot{u}\}_i+[k]\{u\}_i=0 \tag{21.2.100}$$

where the element stiffness and mass matrices are

$$[k]=[C]^{\mathrm{T}}\int_{s=0}^{L}[D(s,n)]\mathrm{d}s[C] \tag{21.2.101}$$

$$[m]=\rho h[C]^{\mathrm{T}}\int_{s=0}^{L}[F(s,n)]\mathrm{d}s[C] \tag{21.2.102}$$

These local matrices are then assembled into global matrices and the equations are solved for each value of n. For each value of n, there will be a finite number of roots (natural frequencies).

21.3. FREE AND FORCED VIBRATION SOLUTIONS

Both the finite difference and the finite element method result in a multi-degree-of- freedom global matrix equation

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}=\{Q\} \tag{21.3.1}$$

where u, in general for shells, includes all the u_1, u_2 and u_3 components. Here we have also inserted a damping term, based on the equivalently viscous damping concept. The approach to the solution of this equation as taken here parallels the approach taken in Chapter 8.

We assume that the forced solution can be expressed in terms of a natural mode series. The approach requires that we first find the natural frequencies and modes, impose some restrictive conditions on the form of the damping matrix, and then find the modal participation factors for the forced solution.

21.3.1. Natural Frequencies and Modes

Eliminating [C] and $\{Q\}$ from consideration, we need to solve

$$[M]\{\ddot{u}\}+[K]\{u\}=\{0\} \tag{21.3.2}$$

Time separates out (see Chapter 5), which is equivalent to assuming that

$$\{u\}=\{U\}e^{j\omega t} \tag{21.3.3}$$

As it will turn out, $\{U\}$ is a mode of vibration and ω is the associated natural frequency. Substitution gives

$$[[K]-[M]\omega^2]\{U\}=0 \tag{21.3.4}$$

This equation has $\{U\}=0$ as a solution, which means that zero motion is a solution. When $\{U\}\neq 0$, the only way this equation can have a solution is if the determinant is 0:

$$|[K]-[M]\omega^2|=0 \tag{21.3.5}$$

The roots of this equation are the natural frequencies ω_i, We obtain the associated natural modes by substituting each ω_i, back into Eq. (21.3.4) and solving for $\{U\}_i$.

21.3.2. Orthogonality of Natural Modes

Because the ω_i and $\{U\}_i$ satisfy Eq. (21.3.4) we may write

$$[K]\{U\}_r-\omega_r^2[N]\{U\}_r=0 \tag{21.3.6}$$

where $i=r$ corresponds to a particular pair of natural frequencies and modes. However, Eq. (21.3.4) may also be written

$$[K]\{U\}_s-\omega_s^2[M]\{U\}_s=0 \tag{21.3.7}$$

where s corresponds to another pair. The number s may be equal or not equal to r.

Premultiplying Eq. (21.3.6) by the transpose of $\{U\}_s$, and Eq. (21.3.7) by the transpose of $\{U\}_r$, gives

$$\{U\}_s^{\mathrm{T}}[K]\{U\}_r-\omega_r^2\{U\}_s^{\mathrm{T}}[M]\{U\}_r=0 \tag{21.3.8}$$

$$\{U\}_r^{\mathrm{T}}[K]\{U\}_s-\omega_s^2\{U\}_r^{\mathrm{T}}[M]\{U\}_s=0 \tag{21.3.9}$$

Since the stiffness matrix and the mass matrix are, as a rule, symmetrical,

$$\{U\}_s^{\mathrm{T}}[K]\{U\}_r-\{U\}_r^{\mathrm{T}}[K]\{U\}_s=0 \tag{21.3.10}$$

$$\{U\}_s^{\mathrm{T}}[M]\{U\}_r-\{U\}_r^{\mathrm{T}}[M]\{U\}_s=0 \tag{21.3.11}$$

we obtain, subtracting Eq. (21.3.9) from Eq. (21.3.8),

$$(\omega_s^2-\omega_r^2)\{U\}_r^{\mathrm{T}}[M]\{U\}_s=0 \tag{21.3.12}$$

Unless we have repeated roots, $\omega_s\neq\omega_r$ if $r\neq s$. Therefore, Eq. (21.3.12) is satisfied only if

$$\{U\}_r^{\mathrm{T}}[M]\{U\}_s=0 \tag{21.3.13}$$

This can be summarized as

$$\{U\}_r^{\mathrm{T}}[M]\{U\}_s=\begin{cases}0, & r\neq s\\ M_r, & r=s\end{cases} \tag{21.3.14}$$

where

$$M_r=\{U\}_r^{\mathrm{T}}[M]\{U\}_r \tag{21.3.15}$$

From Eq.(21.3.9),

$$\{U\}_r^{\mathrm{T}}[M]\{U\}_s=\frac{1}{\omega_s^2}\{U\}_r^{\mathrm{T}}[L]\{U\}_s \tag{21.3.16}$$

Substituting this into Eq. (21.3.14) gives

$$\{U\}_r^{\mathrm{T}}[K]\{U\}_s=\begin{cases}0, & r\neq s\\ K_r, & r=s\end{cases} \tag{21.3.17}$$

where

$$K_r=\{U\}_r^{\mathrm{T}}[K]\{U\}_r \tag{21.3.18}$$

21.3.3. Forced Vibration by Modal Series Expansion

In the following, the solution to Eq. (21.3.1) will be expanded in terms of the undamped modes, as in Chapter 8. We make the *Ansatz* that

$$\{u\}=\sum_{i=1}^{n}\{U\}_i h_i(t) \tag{21.3.19}$$

where the $h_i(t)$ are unknown modal participation coefficients and the $\{U\}_i$ are the natural modes. Substituting Eq. (21.3.19) into (21.3.1) gives

$$\sum_{i=1}^{n}\{[M]\{U\}_i\ddot{h}_i(t)+[C]\{U\}_i\dot{h}_i(t)+[K]\{U\}_i h_i(t)\}=\{Q\} \tag{21.3.20}$$

Substituting relationship (21.3.4) gives

$$\sum_{i=1}^{n}\{[M]\{U\}_i\ddot{h}_i(t)+[C]\{U\}_i\dot{h}_i(t)+[M]\{U\}_i\omega_i^2 h_i(t)\}=\{Q\} \tag{21.3.21}$$

Planing to utilize the orthogonality property to dispose of the summation operation, we premultiply both sides by $\{U\}_j^{\mathrm{T}}$:

$$\sum_{i=1}^{n}(\ddot{h}_i(t)+\omega_i^2 h_i(t))\{U\}_j^T[M]\{U\}_i+\{U\}_j^{\mathrm{T}}[C]\{U\}_i\dot{h}_i(t)\omega_i^2=\{U\}_j^{\mathrm{T}}\{Q\} \tag{21.3.22}$$

and realize that we cannot succeed in general except for special cases where

$$\{U\}_j^{\mathrm{T}}[C]\{U\}_i=\begin{Bmatrix}0, & \text{if}\, i\neq j\\ \{U\}_i^T[C]\{U\}_i & \text{if}\, i=j\end{Bmatrix} \tag{21.3.23}$$

This condition is satisfied either

$$[C]=C_M[M] \tag{21.3.24}$$

or

$$[C]=C_K[K] \tag{21.3.25}$$

or

$$[C]=C_M[M]+C_K[K] \tag{21.3.26}$$

where C_M and C_K are proportionality factors. Case (21.3.24) is obvious since

$$\{U\}_i^{\mathrm{T}}[C]\{U\}_i=C_M\{U\}_i^{\mathrm{T}}[M]\{U\}_i \tag{21.3.27}$$

Case (21.3.25) follows from

$$\{U\}_i^{\mathrm{T}}[C]\{U\}_i=C_K\{U\}_i^{\mathrm{T}}[K]\{U\}_i=\omega_i^2 C_K\{U\}_i^{\mathrm{T}}[M]\{U\}_i \tag{21.3.28}$$

and case (21.3.26) is simply a linear combination

$$\{U\}_i^{\mathrm{T}}[C]\{U\}_i = (C_M + \omega_i^2 C_K)\{U\}_i^{\mathrm{T}}[M]\{U\}_i \tag{21.3.29}$$

Let us therefore assume that condition (21.3.23) can be satisfied in engineering practice. This gives

$$\sum_{i=1}^{n}\left[\ddot{h}_i(t) + (C_M + \omega_i^2 C_k)\dot{h}_i(t) + \omega_i^2 h_i(t)\right]\{U\}_j^{\mathrm{T}}[M]\{U\}_i = \{U\}_j^{\mathrm{T}}\{Q\} \tag{21.3.30}$$

and applying the orthogonality condition (21.3.14) results in

$$\ddot{h}_i(t) + (C_M + \omega_i^2 C_K)\dot{h}_i(t) + \omega_i^2 h_i(t) = \frac{\{U\}_i^{\mathrm{T}}\{Q\}}{\{U\}_i^{\mathrm{T}}[M]\{U\}_i} \tag{21.3.31}$$

Defining a modal damping factor

$$\zeta_i = \frac{C_M + \omega_i^2 C_K}{2\omega_i} \tag{21.3.32}$$

allows us to write Eq. (21.3.31) as

$$\ddot{h}_i(t) + 2\zeta_i\omega_i\dot{h}_i(t) + \omega_i^2 h_i(t) = N_i(t) \tag{21.3.33}$$

where

$$N_i(t) = \frac{\{U\}_i^{\mathrm{T}}\{Q(t)\}}{\{U\}_i^{\mathrm{T}}[M]\{U\}_i} \tag{21.3.34}$$

Laplace transforming Eq. (21.3.33) and solving for the modal participation coefficients in the Laplace domain gives

$$h_i(s) = \frac{(s + 2\zeta_i\omega_i)h_i(0)}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} + \frac{\dot{h}_i(0)}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} + \frac{N_i(s)}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \tag{21.3.35}$$

The denominator can be written as

$$s^2 + 2\zeta_i\omega_i s + \omega_i^2 = (s + \zeta_i\omega_i)^2 + \omega_i^2(1 - \zeta_i^2) \tag{21.3.36}$$

and defining

$$a_i = \zeta_i\omega_i \tag{21.3.37}$$

$$\gamma_i^2 = \omega_i^2(1 - \zeta_i^2) = \omega_i^2 - a_i^2 \tag{21.3.38}$$

Equation (21.3.35) takes on the form

$$h_i(s) = \frac{(s + 2a_i)h_i(0)}{(s + a_i)^2 + \gamma_i^2} + \frac{\dot{h}_i(0)}{(s + a_i)^2 + \gamma_i^2} + \frac{N_i(s)}{(s + a_i)^2 + \gamma_i^2} \tag{21.3.39}$$

Taking the inverse Laplace transformation, we have to take into account two basic cases: the under-damped case, where $\zeta_i < 1$, so that γ_i^2 is real and positive, which gives

$$h_i(t) = e^{-a_i t}\left[h_i(0)\cos\gamma_i t + \frac{1}{\gamma_i}(h_i(0)a_i + \dot{h}_i(0))\sin\gamma_i t\right] + \frac{1}{\gamma_i}\int_0^t N_i(\tau)e^{-a_i(t-\tau)}\sin\gamma_i(t-\tau)\,d\tau \tag{21.3.40}$$

and the overdamped case, where $\zeta_i > 1$, so that

$$\gamma_i^2 = k_i^2 \tag{21.3.41}$$

where k_i^2 is real and positive and defined as

$$k_i^2 = a_i^2 - \omega_i^2 \tag{21.3.42}$$

Equation (21.3.35) then reads

$$h_i(s) = \frac{(s+2a_i)h_i(0)}{(s+a_i)^2 - k_i^2} + \frac{\dot{h}_i(0)}{(s+a_i)^2 - k_i^2} + \frac{N_i(s)}{(s+a_i)^2 - k_i^2} \tag{21.3.43}$$

and the inverse transformation results in

$$h_i(t) = e^{-a_i t}\left[h_l(0)\cosh k_i t + \frac{1}{k_t}(h_i(0)a_i + h_i(0))\sinh k_i t\right] + \frac{1}{k_i}\int_0^t N_i(\tau)e^{-a_i(t-\tau)}\sinh k_i(t-\tau)\,d\tau \tag{21.3.44}$$

The critically damped case $\zeta_i = 1$ can be evaluated using either Eq. (21.3.40) or (21.3.44) by taking the limit of $h_i(t)$ as $\gamma_i \to 0 (k_i \to 0)$. This gives

$$h_i(t) = e^{-a_i t}[h_i(0) + (h_i(0)a_i + \dot{h}_i(0))t] + \int_0^t N_i(\tau)e^{-a_i(t-\tau)}(t-\tau)\,d\tau \tag{21.3.45}$$

Note that the possibility exists that in the same modal expansion solution some of the participation factors $h_i(t)$ have to be determined by Eq. (21.3.40) and some by Eq. (21.3.44) since ζ_i, is a function of ω_i according to the definition of Eq. (21.3.32). In engineering practice, it is usually sufficient to work with the subcritical solution (21.3.40) because of typically very low damping.

The forced solutions given here include the initial value solutions. As in Chapter 8, the initial conditions given in terms of displacements $\{u_0\}$ and velocities $\{\dot{u}_0\}$ have to be translated into initial conditions in terms of the modal participation factors, $h_i(0)$ and $\dot{h}_i(0)$.

Starting with Eq. (21.3.19), we take its time derivative and then evaluate both equations at $t = 0$:

$$\{u_0\} = \sum_{i=1}^{\infty} \{U\}_i h_i(0) \tag{21.3.46}$$

$$\{\dot{u}_0\} = \sum_{i=1}^{\infty} \{U\}_i \dot{h}_i(0) \tag{21.3.47}$$

Taking Eq. (21.3.46), as an example, we premultiply both sides of the equation by $\{U\}_j^{\mathrm{T}}[M]$:

$$\{U\}_j^{\mathrm{T}}[M]\{u_0\} = \sum_{i=1}^{\infty} \{U\}_j^{\mathrm{T}}[M]\{U\}_i h_i(0) \tag{21.3.48}$$

Utilizing the orthogonality condition (21.3.14) removes the summation and allows us to solve for $h_i(0)$:

$$h_i(0) = \frac{\{U\}_i^{\mathrm{T}}[M]\{u_0\}}{\{U\}_i^{\mathrm{T}}[M]\{U\}_i} \tag{21.3.49}$$

Proceeding similarly with Eq. (21.3.47) results in

$$\dot{h}_i(0) = \frac{\{U\}_i^{\mathrm{T}}[M]\{\dot{u}_0\}}{\{U\}_i^{\mathrm{T}}[M]\{U\}_i} \tag{21.3.50}$$

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