VARIATIONAL METHODS AND COMPLEMENTARY FORMULATIONS IN DYNAMICS

SOLID MECHANICS AND ITS APPLICATIONS
Volume 31

Series Editor: G.M.L. GLADWELL
Solid Mechanics Division, Faculty of Engineering
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

Aims and Scope of the Series

The fundamental questions arising in mechanics are: *Why?, How?,* and *How much?* The aim of this series is to provide lucid accounts written by authoritative researchers giving vision and insight in answering these questions on the subject of mechanics as it relates to solids.

The scope of the series covers the entire spectrum of solid mechanics. Thus it includes the foundation of mechanics; variational formulations; computational mechanics; statics, kinematics and dynamics of rigid and elastic bodies; vibrations of solids and structures; dynamical systems and chaos; the theories of elasticity, plasticity and viscoelasticity; composite materials; rods, beams, shells and membranes; structural control and stability; soils, rocks and geomechanics; fracture; tribology; experimental mechanics; biomechanics and machine design.

The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.

For a list of related mechanics titles, see final pages.

Variational Methods and Complementary Formulations in Dynamics

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SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-90-481-4422-8 ISBN 978-94-015-8259-9 (eBook)
DOI 10.1007/978-94-015-8259-9

Printed on acid-free paper

Originally published by Kluwer Academic Publishers in 1994
Softcover reprint of the hardcover 1st edition 1994

Dedicated to
Carolyn and Doreen

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Preface

Not many disciplines can claim the richness of creative ideas that make up the subject of analytical mechanics. This is not surprising since the beginnings of analytical mechanics mark also the beginnings of the theoretical treatment of other physical sciences, and contributors to analytical mechanics have been many, including the most brilliant mathematicians and theoreticians in the history of mankind.

As the foundation for theoretical physics and the associated branches of the engineering sciences, an adequate command of analytical mechanics is an essential tool for any engineer, physicist, and mathematician active in dynamics. A fascinating discipline, analytical mechanics is not only indispensable for the solution of certain mechanics problems but also contributes so effectively towards a fundamental understanding of the subject of mechanics and its applications.

In analytical mechanics the fundamental laws are expressed in terms of work done and energy exchanged. The extensive use of mathematics is a consequence of the fact that in analytical mechanics problems can be expressed by variational statements, thus giving rise to the employment of variational methods. Further it can be shown that the independent variables may be either displacements or impulses, thus providing in principle the possibility of two complementary formulations, i.e. a displacement formulation and an impulse formulation, for each problem. This duality is an important characteristic of mechanics problems and is given special emphasis in the present book.

The book begins by discussing fundamental concepts such as coordinate transformation, degrees of freedom, virtual displacement and virtual work, holonomic and nonholonomic constraints, work, potential energy, duality, complementary potential energy, kinetic energy, complementary kinetic energy, and generalized variables, all in the first chapter.

Readers not familiar with variational methods are now directed to Appendix A, which is devoted to the calculus of variations, and introduces concepts such as functions and functionals, extrema, the Euler-Lagrange equations, extrema of functionals subject to some constraints, and functionals with variable end points.

In the second chapter variational methods are used to develop extremum formulations of differential type for problems in dynamics. It begins with D'Alembert's principle, and then introduces Gauss' principle of least constraint, Hertz' principle of least curvature in configuration space, and ends with Lagrange's equation, in dual representation, i.e. in displacement and impulse variables.

The third chapter presents variational methods in integral form. Duality, e.g. formulation in conventional and complementary form, is again emphasized, and Hamilton's principle is discussed, as is the significance of Lagrange multipliers, and the concept of ignorable coordinates.

The fourth chapter deals with canonical transformations, the fifth is devoted to the applications of variational methods in gyrodynamics and the sixth chapter illustrates some of the foregoing theory in special applications.

The text is intended for senior undergraduate or for graduate students, as well as for engineers, physicists, and mathematicians engaged in work in the multifarious branches of theoretical and applied mechanics.

Students using the text are expected to be familiar with the rudiments of mechanics and to have an adequate command of elementary dynamics and of differential and integral calculus.

The derivation of each significant formula is followed by the discussion of a practical example, in order to acquaint the student with typical situations, typical results, and typical numerical values. For the same purpose, there are numerous problems following each chapter. The type of problems chosen covers the whole range from programmed learning type problems to engineering applications.

The nomenclature employed, and the answers to the problems are compiled in appendices. The conventional practice of citing detailed references to the literature has not been followed in this text because of its introductory nature. Chapters end with a list of publications under the heading of Suggested Reading for those interested in tracing work back to its sources, or for further study of the subject matter.

Each chapter is essentially self-contained, allowing teacher and reader flexibility of instruction without loss of coherence.

The summation convention is to apply when subscripts i, j, k and l are used. It does not apply when subscripts m, n, r and s are used.

In Appendix B there is a listing of symbols used in the book. The unfortunate shortage of suitable characters causes certain symbols to be used twice, or even more frequently, sometimes for similar but often for entirely different quantities.

We are grateful Drs. Inna Sharf, Jim Haddow, and Marinos Stylianou for reading the manuscript of this text and making valuable suggestions for improvement of its clarity in the composition. We are also grateful to Ms. Wendy Smith and Mrs. Guida Nene for typing this manuscript and cheerfully agreeing to make the numerous changes as the manuscript was being developed. Finally we wish to thank Mr. Serkan Dost for his help in the preparation of all the drawings for the manuscript.

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Victoria 1993

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Toronto 1993

Chapter I

FUNDAMENTALS

A Historical Note

Mechanics as a separate scholarly discipline is usually considered to have been founded by Galilei (1564-1642), specifically with his Discorsi (1638), which contain the inertial law (now generally referred to as Newton's first law). Laws of mechanics are statements about a wide range of experimental observations. Prior to Newton's (1643-1727) time, many independent and sometimes contrary statements were made in regard to the motion of bodies. Newton (1687), in effect, laid the foundations of present-day mechanics in his three celebrated laws. These laws, based upon the concepts of mass, length and time, aim at relating the motion of a body to the forces acting on the body, at each instant of time. Thus these laws express cause (force) and effect (motion) relationships, at each instant of time, and since these relationships are vectorial in nature, this approach is often referred to as *vectorial mechanics.*

Leibniz (1646-1716) who was a contemporary of Newton proposed another quantity, the *vis viva*† (lt: living force), as the proper gauge for the dynamical action of a force. His living force has the dimensions of energy and is scalar in nature. In this approach, referred to as *analytical mechanics*, the laws of mechanics are expressed in terms of work done and energy expended. Euler (1707-1783), D'Alembert (1717-1783) and Lagrange (1736-1813) made very significant contributions to the further development of mechanics and later Hamilton (1805-1865) expressed the laws of mechanics in a variational statement. Although Newton's and Hamilton's statements are equivalent, they differ in the sense that while the former are in the form of cause and effect relationships at each *instant of time*, the latter is in the form of extremum conditions of a functional over an arbitrary *period of time.* Further contributions to the development of mechanics were made by Jacobi (1804-1851) and in more recent times by Einstein (1879-1955) who, in his theory of relativity, brought the concepts of length, time and simultaneity of events under critical review. Rather new is the application and use of the concepts of *complementary* formulations (Trent, 1952; Toupin,

† *vis viva* = mass times velocity squared = mv^2

1952; Crandall, 1957; Tabarrok, 1984). A special feature of the present book, is the illustration of the complementary formulation whenever feasible.

1.1 Review of Basic Concepts

Newton's†(1687) second law of motion states that when a resultant *force* **F** acts on a particle of constant *mass* m, the particle's *acceleration* **a** will change according to

$$\mathbf{F} = m\,\mathbf{a} \tag{1.1.1}$$

Euler's††(1752) expansion of Newton's second law into

$$\mathbf{F} = \frac{d}{dt}\mathbf{B} \tag{1.1.2}$$

brought the decisive breakthrough for Newtonian mechanics. Now referred to as the *linear momentum law*, equation (1.1.2) states that the resultant force **F** applied to a body equals the time derivative of the body's *linear momentum* **B**, which is defined as

$$\mathbf{B} = \int_m \mathbf{v}\,dm \tag{1.1.3}$$

where **v** is the *velocity* of the mass element dm. The application of Euler's linear momentum law is not always obvious. The examples following this section are intended to demonstrate this.

In 1775 Euler introduced a second postulate, now referred to as the *angular momentum law*,

$$\mathbf{M} = \frac{d}{dt}\mathbf{H} \tag{1.1.4}$$

which states that the resultant *torque* **M**, about some reference point 0, applied to a body equals the time derivative of the body's *angular momentum* **H**. The angular momentum is defined as

$$\mathbf{H} = \int_m \mathbf{r} \times \mathbf{v}\,dm \tag{1.1.5}$$

where **r** is the position of the mass element dm, measured from the same reference point 0, which must be either at rest or it must be the mass centre in case of a finite-sized body.

For a *point mass* (or *particle*), i.e. a body of *finite* mass and *infinitesimal* volume, Euler's momentum laws furnish Newton's second law, provided the mass is constant.

Newton's second law holds in an inertial frame with respect to which the acceleration **a** in (1.1.1) is measured. The same is true for Euler's linear momentum law and **v** in (1.1.3) is measured in the same inertial frame. The existence of such a frame is a fundamental postulate of Newtonian dynamics. The dependency of a

† Isaac Newton (1642-1727), English physicist (Philosophiae naturalis principia mathematica, 1687)

†† Leonhard Euler (1707-1783), Swiss mathematician

physical law upon a specific reference frame raises some doubts, and while Newton was aware of this weakness, he was unable to rectify it. It was left to Einstein† to remove this weakness and show that all physical laws are frame-independent, in his *relativistic mechanics*.

The dependency of a law upon an inertial reference frame implies that this law generally takes different forms when expressed in terms of different frames. For instance, in an accelerating (with respect to the inertial frame) frame the relation between force and acceleration must take into account the acceleration of the moving frame. Note, however, that if the moving frame has only a constant translational velocity with respect to the inertial frame, the equations remain unchanged. Only under this so-called Galilean†† coordinate transformation, does Euler's linear momentum law remain invariant.

In cases where the coordinate frame used is not an inertial frame, one has to ensure that the linear momentum $\mathbf{B}$ and its derivative $\dot{\mathbf{B}}$ are *absolute*, i.e. they are expressed with respect to an inertial frame. In order to achieve this for a rotating frame, one proceeds in the following fashion. For

$$\mathbf{B} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \{\mathbf{e}_i\}^T \{B_i\} \tag{1.1.6}$$

one obtains for its absolute time derivative

$$\dot{\mathbf{B}} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} \dot{B}_x \\ \dot{B}_y \\ \dot{B}_z \end{bmatrix} + [\dot{\mathbf{e}}_x \ \dot{\mathbf{e}}_y \ \dot{\mathbf{e}}_z] \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \tag{1.1.7}$$

The first term on the right side is often called "B circle" and written

$$\overset{\circ}{\mathbf{B}} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} \dot{B}_x \\ \dot{B}_y \\ \dot{B}_z \end{bmatrix} \tag{1.1.8}$$

and represents the change of magnitude of the linear momentum. The second term on the right side of equation (1.1.7) can be shown to be (a derivation is given in Chap V)

$$\boldsymbol{\Omega} \times \mathbf{B} = [\dot{\mathbf{e}}_x \ \dot{\mathbf{e}}_y \ \dot{\mathbf{e}}_z] \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \tag{1.1.9}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the *xyz* coordinate system relative to the inertial frame. Equation (1.1.9) represents the rate of change of the direction of the linear

† Albert Einstein (1879-1955), German physicist

†† Galileo Galilei (1564-1642), Italian astronomer ("Eppure si muòve)

momentum. With the help of equations (1.1.8) and (1.1.9) the absolute time derivative (1.1.7) of the linear momentum (1.1.6) can now be written

$$\dot{\mathbf{B}} = \overset{\circ}{\mathbf{B}} + \boldsymbol{\Omega} \times \mathbf{B} \tag{1.1.10}$$

Example 1.1.8 is intended to illustrate the significance of this equation.

One can now combine equations (1.1.7) and (1.1.10) and obtain

$$\dot{\mathbf{B}} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} \dot{B}_x + \boldsymbol{\Omega} \times \mathbf{B} \cdot \mathbf{e}_x \\ \dot{B}_y + \boldsymbol{\Omega} \times \mathbf{B} \cdot \mathbf{e}_y \\ \dot{B}_z + \boldsymbol{\Omega} \times \mathbf{B} \cdot \mathbf{e}_z \end{bmatrix} \tag{1.1.11}$$

or more compactly

$$\dot{\mathbf{B}} = \{\mathbf{e}_i\}^T \{\dot{B}_i + \boldsymbol{\Omega} \times \mathbf{B} \cdot \mathbf{e}_i\} \tag{1.1.12}$$

A coordinate system for which

$$\boldsymbol{\Omega} \times \mathbf{B} \cdot \mathbf{e}_i = 0 \tag{1.1.13}$$

and

$$\dot{\mathbf{v}}_0 = 0 \tag{1.1.14}$$

is said to be *inertial* (or *Galilean*). Such a system translates at constant velocity.

If equations (1.1.13) and (1.1.14) are satisfied and furthermore the coordinate's origin remains at rest, i.e. if

$$\mathbf{v}_0 = 0 \tag{1.1.15}$$

then the coordinate is said to be *absolute* (or *space–fixed*). Thus a space-fixed coordinate does not change in direction and both the acceleration and the velocity of its origin are zero.

Example 1.1.1

Let us examine a rigid body of finite size carrying out a translatory motion, defined by the same velocity and the same acceleration for each of its mass elements dm, as shown in Figure 1.1.

The body's linear momentum (1.1.3)

$$\mathbf{B} = \int \mathbf{v}\, dm = \mathbf{v} \int dm = m\mathbf{v} \tag{a}$$

and its time derivative

$$\dot{\mathbf{B}} = m\dot{\mathbf{v}} = m\mathbf{a} \tag{b}$$

According to Euler's linear momentum law (1.1.2), $\mathbf{F} = \dot{\mathbf{B}}$ or

$$\mathbf{F} = m\mathbf{a} \tag{c}$$

Thus Newton's second law is obtained. To determine the line of action of the resultant force $\mathbf{F}$ we apply Euler's angular momentum law.

$$\mathbf{r}_C \times \mathbf{F} = \int_m \mathbf{r} \times \dot{\mathbf{v}}\, dm \qquad \text{(d)}$$

where the left hand side is the torque about the refence point C.

Since $\dot{\mathbf{v}} = \mathbf{a}$ is the same for each mass element dm,

$$\mathbf{r}_C \times \mathbf{F} = \left(\int_m \mathbf{r}\, dm \right) \times \mathbf{a} \qquad \text{(e)}$$

The first moment $\int_m \mathbf{r}\, dm$ vanishes if the reference point C is the mass centre of the body. Then

$$\mathbf{r}_C \times \mathbf{F} = 0 \qquad \text{or} \qquad \mathbf{r}_C \text{ is collinear with } \mathbf{F} \qquad \text{(f,g)}$$

i.e. the force F must act through the mass centre C to effect pure translation.

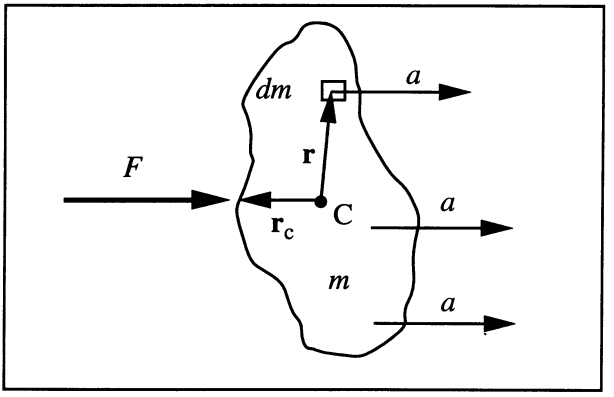


Figure 1.1 Rigid body translatory acceleration

Example 1.1.2

Next let us examine a thin rigid rod of length l and of uniformly distributed mass m, which is to rotate about point 0, as shown in Figure 1.2. The rod is subjected to the force F_y and we wish to determine the point of application of F_y such that the reaction force at 0 would vanish.

The linear momentum in y-direction is

$$B_y = \int_m v_y\, dm = \int_0^l (x\,\dot{\theta})\left(\frac{m}{l}\, dx\right) \qquad \text{(a)}$$

$$B_y = \frac{1}{2}\, m\, l\, \dot{\theta} \qquad \text{(b)}$$

and its time derivative

$$\dot{B}_y = \frac{1}{2}\, m\, l\, \ddot{\theta} \qquad \text{(c)}$$

Since F_y is the only force acting on the body (reaction is to vanish) we have from Euler's linear momentum law (1.1.2), $F_y = \dot{B}_y$, thus

$$F_y = \frac{1}{2}\, m\, l\, \ddot{\theta} \qquad \text{(d)}$$

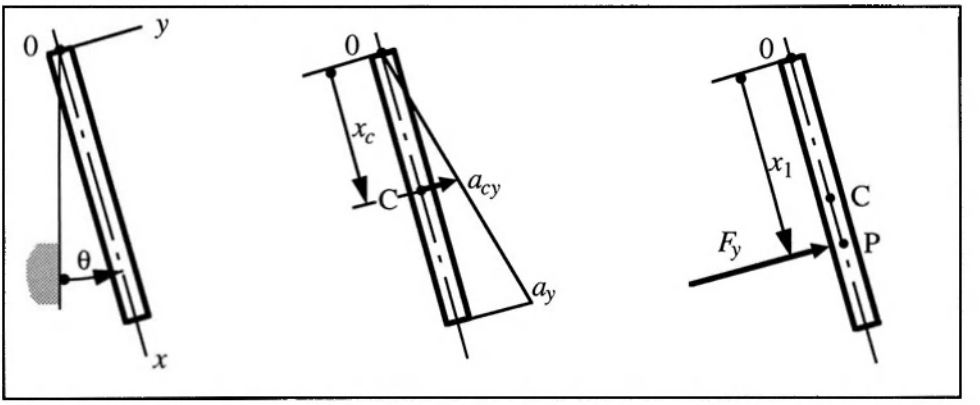


Figure 1.2 Thin rigid rod rotating about point 0
(Forces in x-direction not shown)

Since the mass centre C is located at $x_C = l/2$, and since consequently the mass centre's acceleration component in y-direction is

$$a_{c_y} = l\ddot{\theta}/2$$

one can write

$$F_y = m\, a_{c_y} \tag{e}$$

i.e. Newton's second law applies, with the proviso that of all accelerations, it is the acceleration of the centre of mass that must be used.

Taking moments about the centre 0 of rotation gives, for the resultant force F_y,

$$M_z = x_1\, F_y \tag{f}$$

where a moment arm x_1, yet to be determined, has been introduced.

The angular momentum, about point 0 is

$$H_z = \int_o^l x(x\ \dot{\theta})\,(\frac{m}{l}\,dx) \tag{g}$$

$$H_z = \frac{1}{3}\, m\, l^2\, \dot{\theta} \tag{h}$$

and its time derivative

$$\dot{H}_z = \frac{1}{3}\, m\, l^2\, \ddot{\theta} \tag{i}$$

According to Euler's angular momentum law (1.1.4), $M_z = \dot{H}_z$, thus from equations (f) and (i)

$$x_1\, F_y = \frac{1}{3}\, m\, l^2\, \ddot{\theta} \tag{j}$$

Using F_y from equation (d), the line of action of force F_y is found to be at

$$x_1 = \frac{2}{3}\, l \tag{k}$$

i.e. at the so-called *percussion centre* **P**.

Example 1.1.3

A horizontal water jet impinges upon a deflector as shown in Figure 1.3. What force is acting on the water in order to affect a change in the direction of flow?

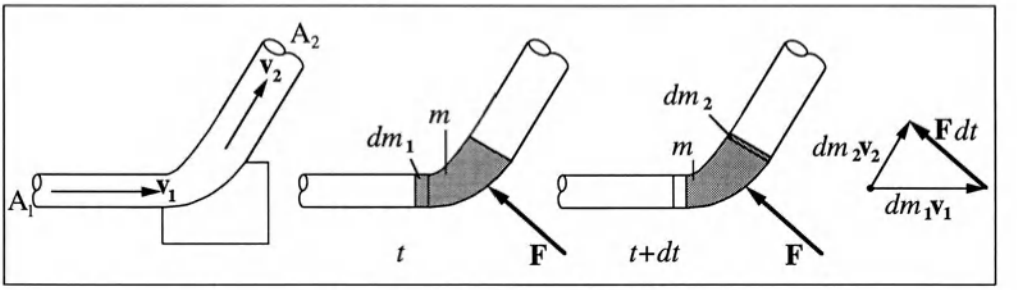


Figure 1.3 Deflected water jet

At time t an amount m of mass is in contact with the deflector, while an amount dm_1 is just about to enter into contact. At time $t + dt$, the same amount m is in contact with the deflector, while an amount dm_2 has just left it. Mass conservation requires that $dm_2 = dm_1 = dm$. While dm_1 has a velocity $\mathbf{v}_1$, mass element dm_2 has a velocity $\mathbf{v}_2$, with $|\mathbf{v}_2| = |\mathbf{v}_1|$ if there is no mechanical energy loss, and $|\mathbf{v}_2| < |\mathbf{v}_1|$ if there is mechanical energy loss during contact with the deflector. Let us rewrite Euler's linear momentum law (1.1.2) into,

$$\mathbf{F} = \dot{\mathbf{B}} = \frac{d\mathbf{B}}{dt} = \frac{(\mathbf{B} + d\mathbf{B}) - \mathbf{B}}{dt} = \frac{\mathbf{B}_{t+dt} - \mathbf{B}_t}{dt} \qquad \text{(a)}$$

and finally into

$$\mathbf{B}_t + \mathbf{F}dt = \mathbf{B}_{t+dt} \qquad \text{(b)}$$

The mass element dm has a linear momentum of $dm\,\mathbf{v}_1$, the mass m in contact with the deflector has a linear momentum of $m\mathbf{v}$, both at time t, such that $\mathbf{B}_t = dm\ \mathbf{v}_1 + m\mathbf{v}$. At time $t + dt$, the same amount of mass $m + dm$ has a linear momentum of $\mathbf{B}_{t+dt} = m\mathbf{v} + dm\,\mathbf{v}_2$, such that equation (b) becomes

$$dm\ \mathbf{v}_1 + m\mathbf{v} + \mathbf{F}dt = m\mathbf{v} + dm\ \mathbf{v}_2 \qquad \text{(c)}$$

leading to

$$\mathbf{F} = \dot{m}(\mathbf{v}_2 - \mathbf{v}_1) \qquad \text{(d)}$$

where $\dot{m} = \frac{dm}{dt} = \rho A_1 v_1 = \rho A_2 v_2$ is the amount of water mass per unit time flowing past the deflector, with ρ(= 1000 kg/m^3) as mass density of the water, and v_1 and v_2 the magnitudes of the corresponding velocity vectors.

Example 1.1.4

An open freight car is being pulled by a force $F = F(t)$, while being loaded continuously with sand as shown in Figure 1.4. For the x-direction, the linear momentum of the amount $m + dm$ of mass at time t is

$$B_t = mv + dm \cdot 0 \tag{a}$$

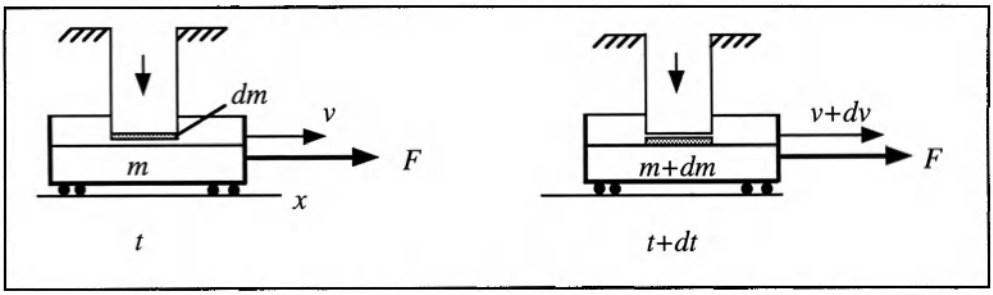


Figure 1.4 Sand falling into open freight car

where mv is the momentum of the sand-filled car, and dm is the mass of sand with a velocity component in x-direction of zero, just about to be added to the car during the time interval . The increment of the linear impulse of the force F is Fdt. After the time interval dt the mass of the sand-filled car is $m + dm$, while the new velocity is $v + dv$. Thus the new linear momentum of the amount $m + dm$ of mass is

$$B_{t+dt} = (m + dm)(v + dv) \tag{b}$$

Euler's linear momentum law (1.1.2), in the form $\mathbf{B}_t + \mathbf{F}dt = \mathbf{B}_{t+dt}$, requires that

$$mv + Fdt = (m + dm)(v + dv) \tag{c}$$

or

$$Fdt = dm\, v + m\, dv + dm\, dv \tag{d}$$

After dropping the second order term $dm\, dv$, Euler's linear momentum law leads to

$$F = m\dot{v} + \dot{m}\, v \tag{e}$$

where $m = m(t)$ is the mass of the sand-filled car. Note that in order to apply Euler's momentum law correctly, the momentum at the beginning of the time interval dt, and the momentum at the end of the time interval dt, must involve one and the *same* amount of mass, which in the case above was $m + dm$.

Example 1.1.5

A sand-filled freight car, pulled by a force $F = F(t)$ as shown in Figure 1.5, loses sand continuously. For the x-direction the linear momentum of the amount m of mass at time t is

$$B_t = mv \tag{a}$$

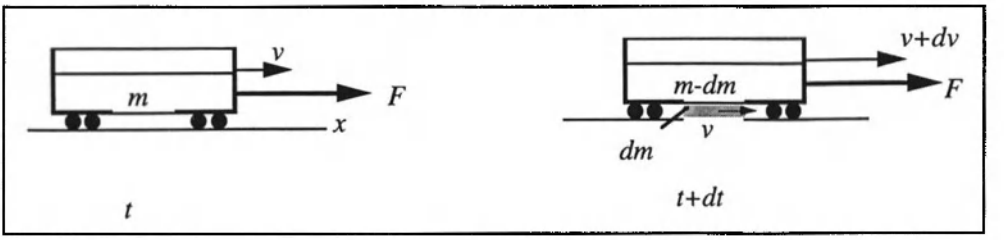


Figure 1.5 Sand falling out of a freight car

At time $t + dt$, the mass of the sand-filled car is only $m - dm$. The car moves now at a velocity $v + dv$. The sand dm, which has left the car, still retains its velocity v in x-direction. Thus the linear momentum in x-direction of the amount m of mass at time $t + dt$ is

$$B_{t+dt} = (m - dm)(v + dv) + dm\ v \tag{b}$$

The increment of the linear impulse in x-direction during the time interval dt is Fdt, such that Euler's linear momentum law (1.1.2), in the form $\mathbf{B}_t + \mathbf{F}dt = \mathbf{B}_{t+dt}$, leads to

$$mv + Fdt = (m - dm)(v + dv) + dm\ v \tag{c}$$

or

$$F = m\dot{v} \tag{d}$$

where $m = m(t)$ is the time-dependent mass of the sand-filled car. Equation (d) is equivalent to Newton's second law, leading to the conclusion that Newton's second law applies also in cases of variable mass, provided the mass added or lost has the same velocity as the body mass at the moment of addition or separation.

Example 1.1.6

A steam locomotive tender of increasing mass m, driven by a force $F = F(t)$ and moving at a velocity v, scoops up water which runs along a channel at a velocity v_w, as shown in Figure 1.6.

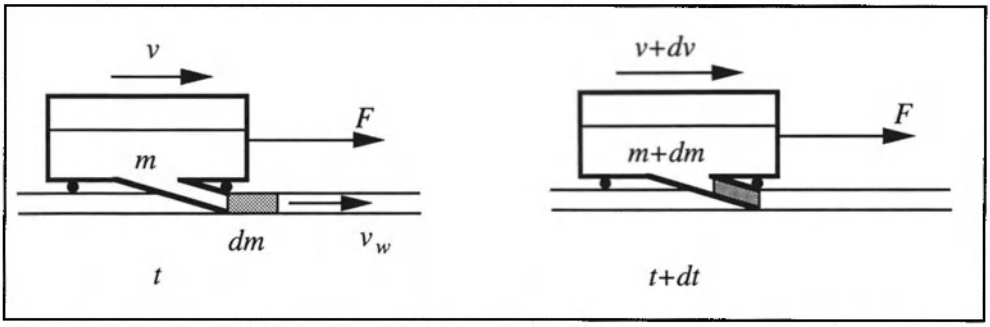


Figure 1.6 Locomotive tender scooping up water

For the horizontal direction, the linear momentum at time t for the mass $m + dm$ is

$$B_t = m\,v + dm\,v_w \tag{a}$$

At time $t + dt$, the mass element dm has joined m and both move at a velocity $v + dv$, such that

$$B_{t+dt} = (m + dm)\,(v + dv) \tag{b}$$

Euler's linear momentum law (1.1.2) in the form $\mathbf{B}_t + \mathbf{F}dt = \mathbf{B}_{t+dt}$, then requires that

$$m\,v + dm\,v_w + Fdt = (m + dm)\,(v + dv) \tag{c}$$

or

$$Fdt = m\,dv + dm(v - v_w) + dm\,dv \tag{d}$$

After dropping the second order term $dm\,dv$, Euler's linear momentum law then provides us with

$$F = m\,\dot{v} + \dot{m}(v - v_w) \tag{e}$$

If we now introduce a relative velocity

$$v_{rel} = v - v_w \tag{f}$$

then

$$F = m\,\dot{v} + \dot{m}\,v_{rel} \tag{g}$$

which reduces to equation (e) of example 1.1.4 if $v_w = 0$, and to equation (d) of example 1.1.5 if $v_w = v$.

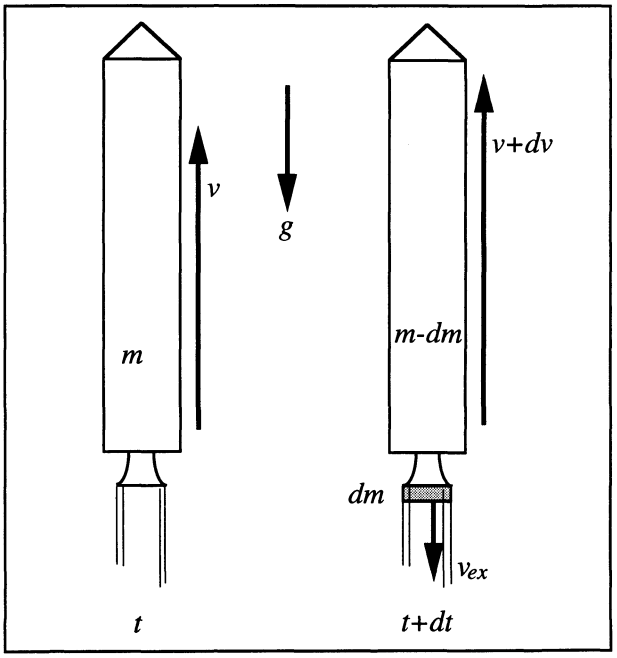


Figure 1.7 Rocket moving vertically

Example 1.1.7

Disregarding atmospheric drag, a rocket of mass $m = m_o - ct$, weight $F = -W = -mg$, and relative velocity of exhaust v_{ex} is moving vertically up as shown in Figure 1.7. For this problem equation (g) of the last example holds and in this case it becomes

$$-mg = m\frac{dv}{dt} - c\,v_{ex} \tag{a}$$

or

$$\frac{dv}{dt} = -g + \frac{cv_{ex}}{m_o - ct} \tag{b}$$

which can be integrated, between the limits $t = 0$ and $t = t$, to give the Tsiolkovsky† rocket equation (1903)

$$v = -gt + v_{ex}\ln\frac{m_o}{m_o - ct} \tag{c}$$

Integrating once again the vertical height is obtained as

$$y = -\frac{1}{2}gt^2 + \frac{m_o v_{ex}}{c}\left(\frac{c}{m_o}t - \frac{m_o - ct}{m_o}\ln\frac{m_o}{m_o - ct}\right) \tag{d}$$

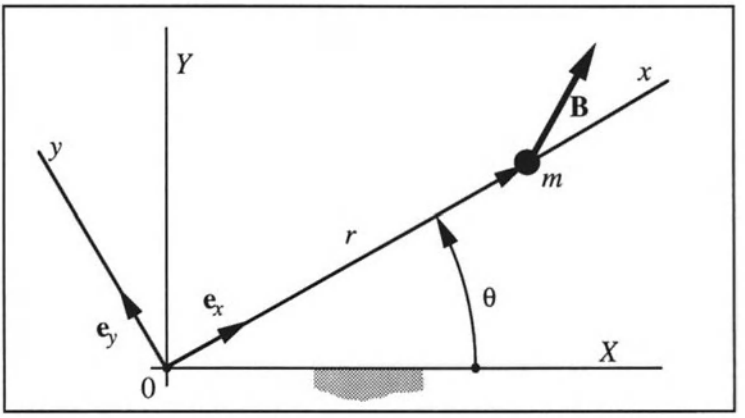


Figure 1.8 A rotating Cartesian coordinate system

Example 1.1.8

The motion of point mass m is to be described in polar coordinates r,θ. Let us do this by introducing a rotating Cartesian reference frame $Oxyz$, such that x coincides with the radial direction and the frame rotates with angular velocity $\dot{\theta}$, with respect to the inertial frame as shown in Figure 1.8.

The position $\mathbf{r}$ of the point mass m can be expressed by

$$\mathbf{r} = [\mathbf{e}_x \;\; \mathbf{e}_y \;\; \mathbf{e}_z]\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} = \{\mathbf{e}\}^T\{r\} \tag{a}$$

† Konstantin Eduardovich Tsiolkovsky (1857-1935), Russian school teacher

The absolute velocity of the point mass m is obtained as

$$\dot{\mathbf{r}} = \{\mathbf{e}\}^T \{\dot{r}\} + \{\dot{\mathbf{e}}\}^T \{r\} = \overset{\circ}{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r} \tag{b}$$

where the first term, $\overset{\circ}{\mathbf{r}} = \{\mathbf{e}\}^T \{\dot{r}\}$, represents the velocity with respect to the rotating frame, and the second term, $\boldsymbol{\Omega} \times \mathbf{r} = \{\dot{\mathbf{e}}\}^T \{r\}$, represents the contribution of the moving frame, which rotates at an angular velocity of

$$\boldsymbol{\Omega} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \tag{c}$$

with the velocity with respect to the moving frame given by

$$\overset{\circ}{\mathbf{r}} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} \dot{r} \\ 0 \\ 0 \end{bmatrix} \tag{d}$$

and

$$\boldsymbol{\Omega} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & \dot{\theta} \\ r & 0 & 0 \end{vmatrix} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} 0 \\ r\dot{\theta} \\ 0 \end{bmatrix} \tag{e}$$

The absolute velocity becomes

$$\dot{\mathbf{r}} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{bmatrix} \tag{f}$$

With the absolute velocity $\mathbf{v} = \dot{\mathbf{r}}$ known, the linear momentum can be established

$$\mathbf{B} = m\mathbf{v} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} m\dot{r} \\ mr\dot{\theta} \\ 0 \end{bmatrix} \tag{g}$$

In order to obtain the time derivative $\dot{\mathbf{B}}$ of the linear momentum, we form

$$\dot{\mathbf{B}} = \overset{\circ}{\mathbf{B}} + \boldsymbol{\Omega} \times \mathbf{B} \tag{h}$$

Since

$$\overset{\circ}{\mathbf{B}} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} m\ddot{r} \\ mr\ddot{\theta} + m\dot{r}\dot{\theta} \\ 0 \end{bmatrix} \tag{i}$$

and since

$$\boldsymbol{\Omega} \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & \dot{\theta} \\ m\dot{r} & mr\dot{\theta} & 0 \end{vmatrix} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} -mr\dot{\theta}^2 \\ m\dot{r}\dot{\theta} \\ 0 \end{bmatrix} \tag{j}$$

we obtain

$$\dot{\mathbf{B}} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_y] \begin{bmatrix} m\ddot{r} - mr\dot{\theta}^2 \\ mr\ddot{\theta} + 2m\dot{r}\dot{\theta} \\ 0 \end{bmatrix} \tag{k}$$

Euler's linear momentum law now gives the force required to move the mass, i.e.

$$\mathbf{F} = \dot{\mathbf{B}} \tag{l}$$

Since it can be shown that the acceleration of the mass is

$$\mathbf{a} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ 0 \end{bmatrix} \tag{m}$$

we conclude, that equation (l) is equivalent to Newton's second law

$$\mathbf{F} = m\,\mathbf{a} \tag{n}$$

Note that, since the O*xyz* coordinate frame is not an inertial frame, special steps are taken to ensure that velocities and accelerations are *absolute*, i.e. taken with respect to an inertial frame.

Problems

1.1.1 A uniform rod of mass m and length l is suspended at one end to form a pendulum. Establish the differential equation of motion.

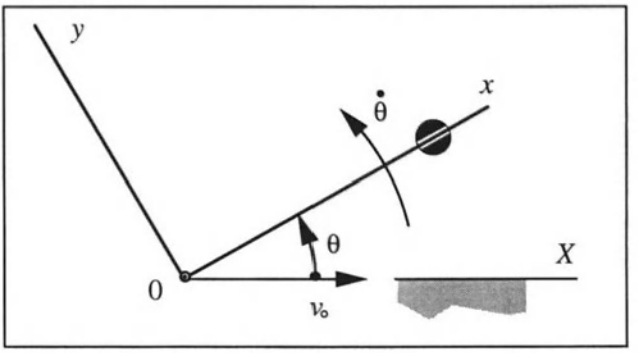
1.1.2 A horizontal water jet of 10 mm diameter is impinging at 12 m/s on a deflector where its direction is changed by 60°. The encounter with the deflector makes the water jet lose some mechanical energy and the jet leaves the deflector at only 10 m/s. What force is transmitted from the deflector to the water?

1.1.3 Sand is falling into an initially empty freight car at the rate of c = 10 kg/s. Initially the mass of the freight car is m_o = 12 000 kg. A constant resultant horizontal force F = 70 N is pulling the car. Initially the car is at rest. How fast will it be moving after 100 seconds?

1.1.4 Sand is falling out of a freight car at a constant rate of c = 10 kg/s. Initially, the mass of car plus sand is m_o = 60 000 kg. The constant resultant horizontal pull force F = 700 N. If starting from rest, how fast is the freight car moving after 100 seconds?

1.1.5 Use the Tsiolkovsky equation, with m_o = 10 000 kg, c = 150 kg/s, v_{ex} = 2500 m/s, and find (a) the time at which the final rocket mass m_f = 1000 kg has been reached, (b) the associated velocity. (c) Is the initial acceleration adequate to overcome the gravitational attraction? (d) What height will the rocket reach?

1.1.6 A point mass can slide along a rod x as shown. The rod itself is rotating in a horizontal plane at an angular velocity $\dot{\theta}$. The origin 0 is subject to a velocity $v_0 = v + at$, where a is a constant acceleration and v is a constant speed. Obtain (a) the linear momentum $\mathbf{B}$ of mass m and (b) the time derivative $\dot{\mathbf{B}}$.



1.2 Generalized Displacement Variables

The position of a particle m in a plane can be specified by its position vector $\mathbf{r}$ or equivalently by the Cartesian† components (r_1, r_2), or in terms of oblique axes by (u_1, u_2) or (u_1', u_2') or even (u_1, r_1) as shown in Figure 1.9. The validity of all these coordinate systems implies that they must be related, and that it should be possible to *transform* from one system to another in a *unique* manner. From Figure 1.9 it can be seen that the (r_1, r_2) and (u_1, u_2) systems are related by

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \cos\beta \\ \sin\alpha & \sin\beta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1.2.1}$$

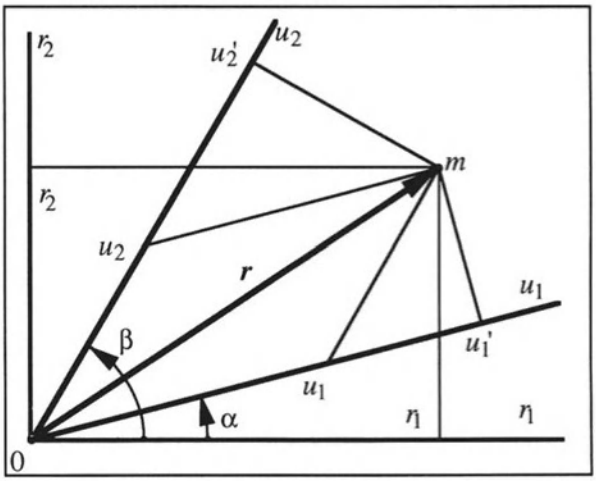


Figure 1.9 Coordinates to locate point m

† Cartesius = René Descartes (1596-1650), French mathematician

In this case, the transformation is linear and, as long as $\alpha \neq \beta$ (or $\beta + n\pi$), the transformation is unique. For incremental changes in the coordinates, we have from equation (1.2.1),

$$\begin{bmatrix} dr_1 \\ dr_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \cos\beta \\ \sin\alpha & \sin\beta \end{bmatrix} \begin{bmatrix} du_1 \\ du_2 \end{bmatrix}$$

i.e. the transformation matrix is not affected, because of the linear character of the transformation.

The coordinates used to locate a particle need not be distances. Thus in Figure 1.10 one can see that the particle may be located by (r, θ) or $(r, \sin\theta)$ or even $(A, \sin\theta)$. The transformation equations relating the Cartesian coordinates $(r_1 = x, r_2 = y)$ to e.g. the coordinate set $(A, \sin\theta)$ are nonlinear,

$$A = \frac{x\,y}{2} \qquad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}} \tag{1.2.2}$$

In order to check that these transformations are valid, we consider small changes in one coordinate system and see if the associated small changes in the other coordinates are unique. To this end, we derive the following differentials

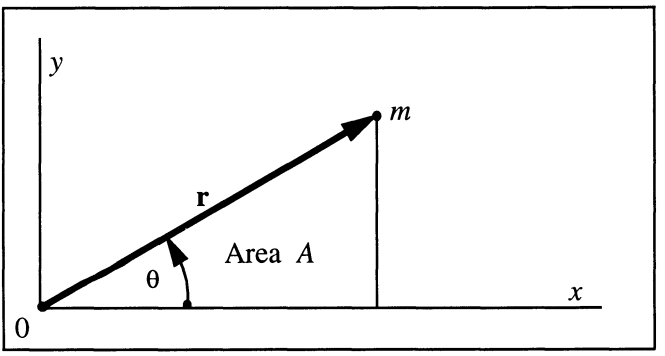


Figure 1.10 Position of particle m

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy$$

$$d(\sin\theta) = \frac{\partial(\sin\theta)}{\partial x}dx + \frac{\partial(\sin\theta)}{\partial y}dy$$

Evaluating the derivatives via equation (1.2.2), we find

$$\begin{bmatrix} dA \\ d(\sin\theta) \end{bmatrix} = \begin{bmatrix} \frac{y}{2} & \frac{x}{2} \\ \frac{-xy}{(x^2+y^2)^{3/2}} & \frac{x^2}{(x^2+y^2)^{3/2}} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \tag{1.2.3}$$

Since the determinant, the so-called *Jacobian*† of the transformation, of the above coefficient matrix given by

$$J = \begin{vmatrix} \frac{y}{2} & \frac{x}{2} \\ \frac{-xy}{(x^2+y^2)^{3/2}} & \frac{x^2}{(x^2+y^2)^{3/2}} \end{vmatrix} = \frac{x^2 y}{(x^2+y^2)^{3/2}} \tag{1.2.4}$$

is definite, i.e. $J \neq 0$, provided $x \neq 0$ and $y \neq 0$, we conclude that the transformation is valid.

As the coordinates used to locate a particle need not be distances, the term *generalized coordinates* is used. It is conventional to denote generalized coordinates by q_i. Thus the area and sine coordinates used above, would be

$$q_1 = A \qquad q_2 = \sin\theta \tag{1.2.5}$$

The Jacobian for the transformation $q_1 = q_1(x,y)$ and $q_2 = q_2(x,y)$ then is

$$J = \begin{vmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_1}{\partial y} \\ \frac{\partial q_2}{\partial x} & \frac{\partial q_2}{\partial y} \end{vmatrix} \tag{1.2.6}$$

In certain problems it may be convenient to use a moving coordinate frame. As a simple example consider the case in Figure 1.11, where the particle can be located by (x, y) in the fixed coordinate system and by (q_1, q_2) in the moving system. If we assume that the linear velocity $\mathbf{v}$ and the angular velocity $\boldsymbol{\omega}$ are constants and that the two coordinate systems were initially coincident, then the following relationship can be established between the two sets of coordinates:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} t + \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Note, that the (x, y) coordinate system represents an inertial frame as does the (x_1,y_1) coordinate system, on account of its constant linear velocity $\mathbf{v}$, while the (q_1, q_2) coordinate system does not, on account of its angular velocity ω. More generally, one can express the transformation from generalized coordinates q_i to Cartesian coordinates r_j by

$$r_j = r_j(q_1, q_2, q_3, \ldots, q_i, \ldots, q_n, t) \qquad \begin{matrix} i = 1,2,\ldots,n \\ j = 1,2,\ldots,m \end{matrix} \tag{1.2.7}$$

with $n \leq m$.

---

† Carl Gustav Jacob Jacobi (1804-1851), German mathematician

Generalized coordinates are employed to describe the configuration, more specifically the *displacements* of certain key points, of a system and provide a convenient means for the study of the system's motion. In the literature, the variables q_i are known as *generalized coordinates*, or *generalized positions*, or *generalized displacements*. Henceforth we shall begin to accord preference to the term *generalized displacements*.

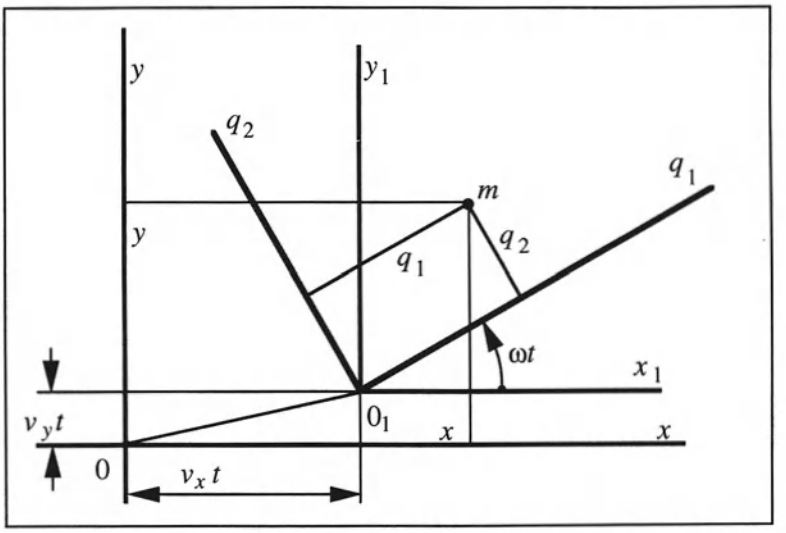
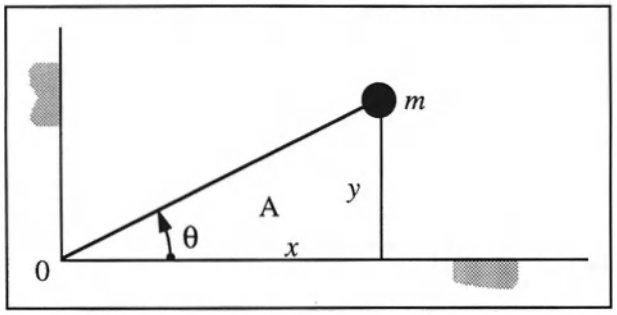


Figure 1.11 Translating and rotating coordinates

Problems

1.2.1 A fly m moves inside a room such that its position at any time in Cartesian coordinates r_j is given by (X, Y, Z). It then alights on the ceiling and walks about, such that $Z = h$ = constant becomes a constraint. Use spherical coordinates as generalized coordinates q_i. (a) Express r_j in terms of q_i. (b) Find the Jacobian of the transformation.

1.2.2 A particle m is located in a plane at (x, y), as shown. Use (a) the coordinates $q_1 = A$ and $q_2 = \theta$ to describe its position, and (b) find the Jacobian. (c) What is the Jacobian for $x = 2, y = 1$?



1.3 Degrees of Freedom and Constraints

The number of *degrees of freedom* for a system of particles depends not only on the number of particles in the system, but also on how the motion of each particle is physically constrained. The degrees of freedom are equal to the number of *independent incremental displacements* required to describe the changes in the system's configuration.

Constraints of a system can be represented by equations of constraint. If these equations relate the position coordinates themselves, the constraints are said to be *holonomic* (gk: holo = whole). If, on the other hand, these relations are amongst the differentials of the position coordinates i.e. amongst incremental displacements, and cannot be integrated to yield relations amongst the position coordinates themselves, the constraints are said to be *nonholonomic*. For systems subjected to holonomic constraints, the number of independent coordinates required to specify the configuration of the system is the same as the number of incremental displacements required to describe the changes in the configuration of the system. On the other hand, for systems subjected to nonholonomic constraints, the number of coordinates, required to describe the configuration of the system, is larger than the number of the independent incremental displacements required to describe the changes in the configuration of the system.

The number d of degrees of freedom and the number of holonomic and nonholonomic constraints of a system of particles are related by

$$d = 3N - h - \nu \tag{1.3.1}$$

with

N = number of particles

h = number of holonomic (position) constraints

ν = number of nonholonomic (velocity) constraints

The degrees of freedom discussed here involve displacements, and are, strictly speaking, *positional* (or *kinematic*) degrees of freedom. Later on, where complementary formulations are discussed, there will occur *impulsive* degrees of freedom.

Holonomic Constraints

In some holonomic constraints the coordinates r_j may be related as

$$f_i(r_1, r_2, r_3, ..., r_j, ..., r_m) = \alpha_i \qquad i = 1,2,..., h \qquad j = 1,2,..., m \tag{1.3.2}$$

when m is the number of original coordinates and h is the number of holonomic constraints. Clearly, $m > h$. The α_i are a set of constants.

Since the coordinates r_j in equation (1.3.2) may be functions of time, one can see that time enters this type of constraint *implicitly*. Such holonomic constraints are classified as *scleronomic* (gk: sclero = hard). As a simple example, consider the case of a bead constrained to move along a straight wire as shown in Figure 1.12. In this case, the

constraint equations may be expressed as

$$f_1(r_1,r_2) = f_1(x,y) = y - \frac{b}{a}x = b \qquad (1.3.3)$$

$$f_2(r_3) = f_2(z) = z = 0$$

with $\alpha_1 = b$ and $\alpha_2 = 0$.

In other holonomic constraints all or some of the constraints may be moving. In such cases, time will appear *explicitly* in the constraint equations. Such holonomic constraints are classified as *rheonomic* (gk: rheo = flow) and may be expressed as

$$f_i(r_1, r_2, r_3, ..., r_j, ..., r_m, t) = \alpha_i \qquad i = 1,2,...,h \qquad (1.3.4)$$

$$j = 1,2,...,m$$

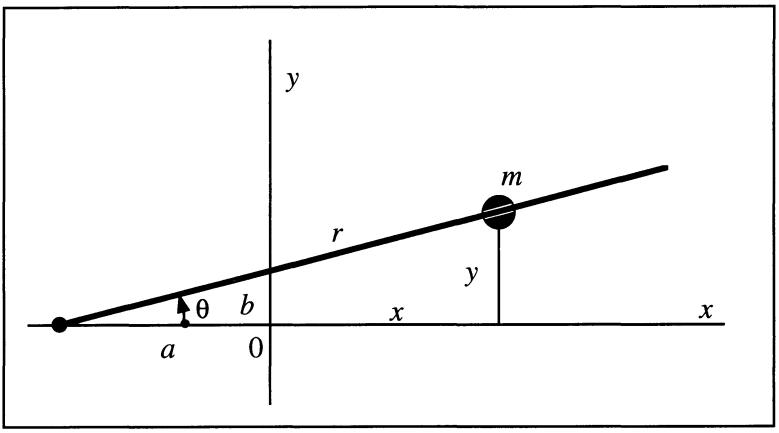


Figure 1.12 Bead m sliding on a straight wire

As a simple example, consider again the case of the bead on a wire, Figure 1.12. This time suppose the wire is rotating about an axis perpendicular to the xy-plane at the intercept of the wire with the x axis, with constant angular velocity ω, such that $\theta = \omega t$. In this case, time will appear in the constraint equation (1.3.4).

$$f(r_1,r_2,t) = f(x,y,t) = y - (x + a)\tan \omega t = 0 \qquad (1.3.5)$$

with $\alpha_1 = 0$. For a system of N point masses subject to h holonomic constraints, it is possible to find $(3N - h)$ independent generalized coordinates to specify the configuration of the system at any instant of time. The number n of these independent generalized coordinates also constitutes the number of degrees of freedom for the system.

$$n = 3N - h \qquad (1.3.6)$$

Before considering nonholonomic constraints, let us derive the differential of a rheonomic holonomic constraint. For a constraint equation of the type in equation (1.3.4), we find

$$df = \frac{\partial f}{\partial r_1} dr_1 + \frac{\partial f}{\partial r_2} dr_2 + \cdots + \frac{\partial f}{\partial r_m} dr_m + \frac{\partial f}{\partial t} dt = 0 \qquad (1.3.7)$$

It is worth noting that while the relation f amongst r_j and time may be nonlinear, the relation df is always a *linear* function of the dr_j and dt. The coefficients $\frac{\partial f}{\partial r_j}$ and $\frac{\partial f}{\partial t}$ may of course be linear or nonlinear functions of the coordinates and time.

Now in some problems, the constraints appear as a set of relations amongst the differentials dr_j and dt as

$$b_1 \, dr_1 + b_2 \, dr_2 + \cdots + b_m \, dr_m + b_t \, dt = 0 \qquad (1.3.8)$$

The coefficients b_j and b_t may be functions of the r_j and of time.

We now ask the question: Is it possible to convert a relation amongst the dr_j and dt to one amongst the r_j and t? On comparing equations (1.3.7) and (1.3.8), one can see that the answer is in the affirmative only if

$$b_1 = \frac{\partial f}{\partial r_1} \qquad b_2 = \frac{\partial f}{\partial r_2} \qquad \cdots \qquad b_t = \frac{\partial f}{\partial t} \qquad (1.3.9)$$

In other words, if equation (1.3.8) represents an exact differential, then the relation amongst the dr_j and dt may be integrated to yield a relation amongst the r_j and t as in equation (1.3.4). The constraints are then holonomic in spite of the fact they are expressed in differential form. Otherwise such constraints in differential form are nonholonomic.

One can check the integrability of a constraint relation amongst the differentials as follows. If a continuous function f exists, such that the relation (1.3.8) represents an exact differential of f, then one can write

$$\frac{\partial}{\partial r_j}\frac{\partial f}{\partial r_k} = \frac{\partial}{\partial r_k}\frac{\partial f}{\partial r_j} \qquad j,k = 1,2,...,m \qquad (1.3.10)$$

$$\frac{\partial}{\partial r_j}\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\frac{\partial f}{\partial r_j}$$

or using equation (1.3.9)

$$\frac{\partial}{\partial r_j} b_k = \frac{\partial}{\partial r_k} b_j \qquad j,k = 1,2,.....,m \qquad (1.3.11)$$

$$\frac{\partial}{\partial r_j} b_t = \frac{\partial}{\partial t} b_j$$

Satisfaction of equations (1.3.11) implies that the required function f exists, i.e. the relation amongst the differentials is immediately integrable. The conditions of integrability are actually somewhat less restrictive than equations (1.3.11) imply, because in some cases equations such as (1.3.8) can be made exact by an *integrating factor* (see problems 1.3.1 and 1.3.2). For a scleronomic system of only two variables, equations (1.3.11) are automatically satisfied. In this case, the constraint equations are always holonomic.

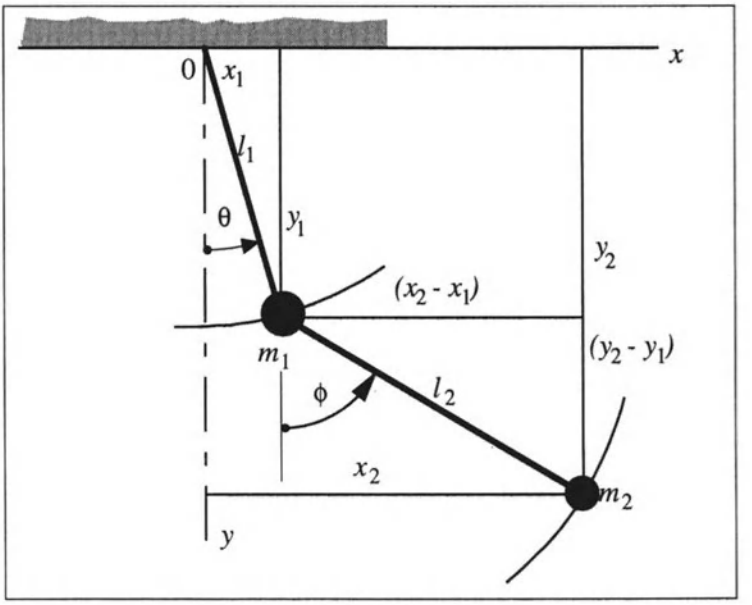


Figure 1.13 Double pendulum

Example 1.3.1

For a double pendulum shown in Figure 1.13, there are two particles involved, thus $N = 2$. Since the pendulum moves in a plane, there are two constraints

$$f_1 = z_1 = 0 \tag{a}$$

$$f_2 = z_2 = 0 \tag{b}$$

Further, particle m_1 is constrained to move on a circle around 0, then

$$f_3 = x_1^2 + y_1^2 = l_1^2 \tag{c}$$

Particle m_2 is constrained to move on a circle around m_1, then

$$f_4 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2 \tag{d}$$

There are consequently $N = 2$ particles, $h = 4$ holonomic constraints, $\nu = 0$ nonholonomic constraints, leaving $d = 3N - h - \nu = 2$ degrees of freedom.

Reduced Transformation Equations

Systems are typically described initially by means of convenient generalised coordinates, foremost amongst these are Cartesian coordinates. Thereafter constraints are considered, and subsequently new generalized coordinates chosen, to satisfy the constraints where possible. The number of coordinates is then *reduced* from that of the m original coordinates to that of the n generalized coordinates.

By way of an example, take the case of the bead sliding on a wire at a fixed angle θ. For this problem, the position of the bead may be specified by a single generalized coordinate r, as shown in Figure 1.12. The constraint relating the two Cartesian coordinates $r_1 = x$ and $r_2 = y$ is

$$f(r_1, r_2) = \frac{y}{x + a} = \tan\theta \tag{1.3.12}$$

with $\alpha_1 = \tan\theta$.

We can satisfy the constraint relation (1.3.12) directly in terms of the generalized coordinate $q = r$ by using equations (1.2.7) and setting

$$r_1 = x = r\cos\theta - a \tag{1.3.13}$$

$$r_2 = y = r\sin\theta \tag{1.3.14}$$

The relation between the Cartesian coordinates $r_1 = x$ and $r_2 = y$, and the generalized coordinate $q = r$ is

$$r^2 = (x + a)^2 + y^2 \tag{1.3.15}$$

This is the reduced transformation equation. From this equation one can see that when x and y are given one can determine r, but the reverse i.e. determination of x,y from r is not possible. However we also have the constraint equation (1.3.12) at our disposal. To check the validity of the transformation we examine incremental changes in x,y and see if they correspond to unique incremental changes in r. Thus from equations (1.3.15) and (1.3.12) we obtain

$$2r\,dr = 2ydy + 2(x+a)dx \tag{1.3.16}$$

$$0 = \frac{dy}{x+a} - \frac{y\,dx}{(x+a)^2} \tag{1.3.17}$$

Now expressing r in terms of x, and y, we may write

$$\begin{bmatrix} dr \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{x+a}{\sqrt{(x+a)^2+y^2}} & \dfrac{y}{\sqrt{(x+a)^2+y^2}} \\ -\dfrac{y}{(x+a)^2} & \dfrac{1}{x+a} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \tag{1.3.18}$$

Since the Jacobian of the transformation is not zero, the transformation is one to one.

Nonholonomic Constraints

As originally pointed out by Voss†, there may be cases where equation (1.3.8) is not an exact differential (and cannot be made so by an integrating factor), then the constraints remain as relations amongst differentials only and are referred to as *nonholonomic*. If $b_t = 0$, and the b_j are not explicit functions of time, the constraints are classified as scleronomic, otherwise they are rheonomic.

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† Aurel Edmund Voss (1845 - 1931), German mathematician

Very often a nonholonomic constraint is divided through by dt and written as

$$b_1\dot{r}_1 + b_2\dot{r}_2 + ... + b_m\dot{r}_m + b_t = 0 \qquad (1.3.19)$$

This equation shows that the nonholonomic constraints are relations amongst speeds. Since these equations are not integrable, the coordinates themselves remain independent. Now to describe the configuration of a system at any instant of time, we require n independent coordinates. If the system is subject to v nonholonomic constraints, the degrees of freedom (1.3.1) of the system are

$$d = n - \mathrm{v} \qquad (1.3.20)$$

But all n independent coordinates will be required for describing the configuration of the system.

Let us suppose that for a system of N particles subjected to h holonomic constraints, n generalized coordinates q_i have been judiciously chosen to satisfy the holonomic constraints. Then

$$n = 3N - h \qquad (1.3.21)$$

If the system is, in addition, subjected to v nonholonomic constraints, as given by equation (1.3.19), then these relations amongst the speeds $\dot{r}_j$ may be also expressed by relations amongst the generalized speeds $\dot{q}_i$ via the reduced transformation equations. Thus there will be v nonholonomic equations of the type

$$a_1\dot{q}_1 + a_2\dot{q}_2 + \cdots + a_n\dot{q}_n + a_t = 0 \qquad (1.3.22)$$

where the coefficients a_j of equation (1.3.22) are related to the coefficients b_i of equation (1.3.19) via the reduced transformation equations.

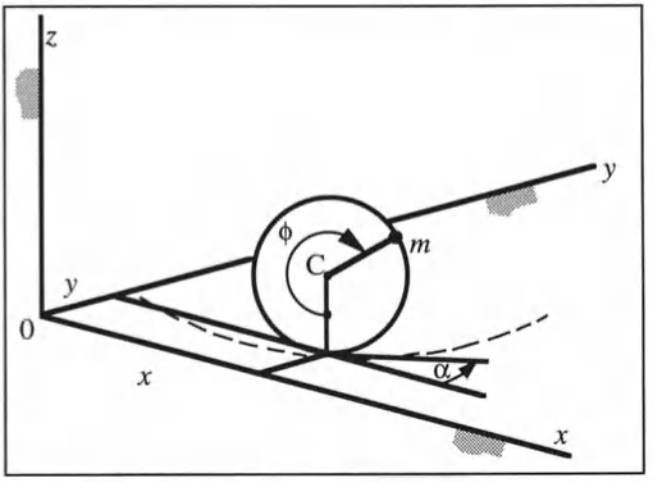


Figure 1.14 Vertical disk rolling on a horizontal plane

Example 1.3.2

Consider a thin disk of radius r, remaining vertical and rolling on a horizontal plane without slipping. The position of the disk may be specified by generalized position coordinates

$q_1 = x, q_2 = y, q_3 = \alpha$ and $q_4 = \phi$, as shown in Figure 1.14. However, it can be seen that the differentials of these variables are related as

$$dx = ds\cos\alpha \qquad dy = ds\sin\alpha \qquad ds = r\,d\phi$$

with α as yaw angle, and ϕ as roll angle.

On using the last equation in the former two equations, we obtain two relations amongst the differentials as

$$dx - r\cos\alpha\, d\phi = 0 \tag{a}$$

$$dy - r\sin\alpha\, d\phi = 0 \tag{b}$$

On checking for integrability of these equations, we find that several of the conditions given in equation (1.3.11) are satisfied but some are not. For instance, with $a_\alpha = 0$, and $a_\phi = -r\cos\alpha$,

$$\frac{\partial a_\phi}{\partial \alpha} \neq \frac{\partial a_\alpha}{\partial \phi}$$

We conclude that equations (a) and (b) are not integrable. This fact can also be readily seen on physical grounds. If the disk starts from a certain position and rolls along two different paths, so that the final position of the point of contact is the same in each case, the two final values for coordinates α, ϕ may be different. If α and ϕ were functions of x and y, then the final values of α and ϕ would have been the same.

Finally, it is worth noting that by dividing equations (a) and (b) by dt, one can express these nonholonomic constraints as a set of relations amongst velocities.

$$\dot{x} - (r\cos\alpha)\dot{\phi} = 0 \tag{c}$$

with $a_1 = a_x = 1, a_2 = a_y = 0, a_3 = a_\alpha = 0, a_4 = a_\phi = -r\cos\alpha$, and

$$\dot{y} - (r\sin\alpha)\dot{\phi} = 0 \tag{d}$$

with $a_1 = a_x = 0, a_2 = a_y = 1, a_3 = a_\alpha = 0, a_4 = a_\phi = -r\sin\alpha$.

In order to establish the degrees of freedom of the system, one may begin by looking at the disk, which is a rigid body, with 6 degrees of freedom when not constrained. The centre of the disk is subject to the holonomic constraint,

$$z = r \tag{e}$$

and the disk is to remain vertical i.e. it is subject to the holonomic constraint

$$\text{pitch angle (of disk)} = 0 \tag{f}$$

thus $h = 2$.

Further, there are the two nonholonomic constraints given by equations (c) and (d), i.e. $\nu = 2$. Thus, from equation (1.3.1), the degrees of freedom are

$$d = 6 - h - \nu = 6 - 2 - 2 = 2 \tag{g}$$

From equation (1.3.6) the number of coordinates required to describe the configuration of the disk at any time t, are

$$n = 6 - h = 6 - 2 = 4 \tag{h}$$

Problems

1.3.1 (a) Is the differential equation

$$\tan\theta\, dx \;-\; dy \;+\; \frac{x}{\cos^2\theta}\, d\theta \;=\; 0$$

integrable? If so, what is the integral?

(b) Is the differential equation

$$\frac{1}{2}\sin 2\theta\, dx \;-\; \cos^2\theta\, dy \;+\; x\, d\theta \;=\; 0$$

integrable? If not, can you find an integrating factor which would make it integrable?

1.3.2 In books in vector calculus it is shown that the necessary and sufficient condition for integrability of a differential form

$$a_x dx \;+\; a_y dy \;+\; a_z dz \;=\; 0$$

is

$$\mathbf{a} \cdot curl\ \mathbf{a} \;=\; 0$$

where

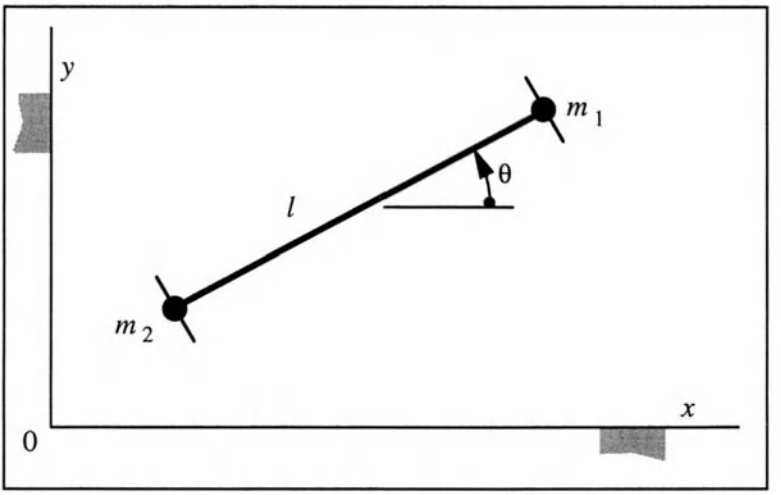
$$\mathbf{a} \;=\; (a_x\ a_y\ a_z)$$

using this show that the constraint

$$yz\ (y+z)\, dx \;+\; zx\ (z+x)\, dy \;+\; xy\ (x+y)dz \;=\; 0$$

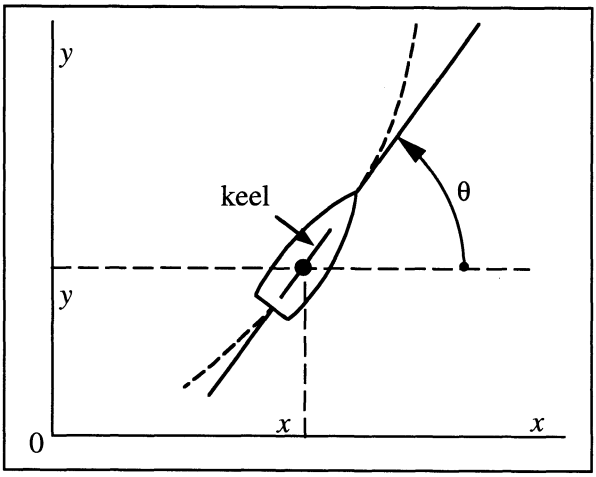
is holonomic.

1.3.3 Two particles, m_1 and m_2, are connected by a rigid rod of constant length l, as shown. Skates are fixed to the particles at right angles to the rod. The assembly is placed on a horizontal sheet of ice. Sideways sliding of the skates is excluded. Establish (a) the number of coordinates required to describe the motion of the system, (b) the constraints and the number of degrees of freedom. (c) Are there nonholonomic constraints? (d) Are they scleronomic or rheonomic, if $\theta = \omega t$?



1.3.4 Find the Jacobian of the transformation (a) from polar (r,θ) to Cartesian (x,y) coordinates, and (b) from Cartesian to polar coordinates, and (c) the numerical value of each Jacobian at the moment when $r = 10$ m, and $\theta = 60°$. (d) Suppose now θ = constant, find the Jacobian of the reduced transformation, i.e. when $d\theta = 0$, from polar (r,θ) to Cartesian (x,y) coordinates, and (e) from Cartesian to polar coordinates, and (f) the numerical value of each Jacobian for $r = 10$ m, $\theta = 60°$.

1.3.5 Assuming that the keel of the boat shown prohibits motion normal to the axis of the boat, write down the constraint equation and identify the nature of the constraint. How many degrees of freedom does the system possess?



1.4 Virtual Displacements and Virtual Work

Let us examine the static equilibrium of a scleronomic system of N particles. For equilibrium of the kth particle, the resultant of *all* forces acting on this particle must vanish i.e. we write

$$(\mathbf{F}_{all})_k = \mathbf{0} \qquad k = 1,2,\ldots, N \tag{1.4.1}$$

Now *imagine* each particle to undergo an infinitesimal displacement $\delta\mathbf{r}_k$. This displacement is imagined in the sense that the forces $(\mathbf{F}_{all})_k$ are considered not to change as a result of $\delta\mathbf{r}_k$ and further, as we shall see later, that $\delta\mathbf{r}_k$ need not necessarily coincide with actual displacements. Such imagined displacements are referred to as *virtual displacements*, i.e. displacements in effect but not in reality. Associated with the virtual displacements $\delta\mathbf{r}_k$ and the forces $(\mathbf{F}_{all})_k$, we can compute the *virtual work* as follows

$$\delta W = (\mathbf{F}_{all})_k \cdot \delta\mathbf{r}_k = 0 \tag{1.4.2}$$

where the summation convention applies.

Clearly, δW will be zero, for all $\delta\mathbf{r}_k$, by virtue of equation (1.4.1). Thus the *vector* equations of equilibrium are equivalent to the vanishing condition of the *scalar* virtual

work, δW.

Now equation (1.4.2) provides considerable flexibility in the choice of the virtual displacements $\delta \mathbf{r}_k$. This flexibility can be used to advantage. For instance, one can split each force $(\mathbf{F}_{all})_k$ into two, namely $\mathbf{R}_k$ as the resultant *reaction* force (due to constraints) and $\mathbf{F}_k$ as the resultant *applied* force, such that equation (1.4.2) becomes

$$\delta W = (\mathbf{F}_k + \mathbf{R}_k) \cdot \delta \mathbf{r}_k = 0 \tag{1.4.3}$$

We can now choose the $\delta \mathbf{r}_k$, such that they are orthogonal to the reaction forces $\mathbf{R}_k$, to obtain

$$\delta W = \mathbf{F}_k \cdot \delta \mathbf{r}_k = 0 \tag{1.4.4}$$

and determine the relations amongst the applied forces only, under the conditions of equilibrium, without involving the reaction forces.

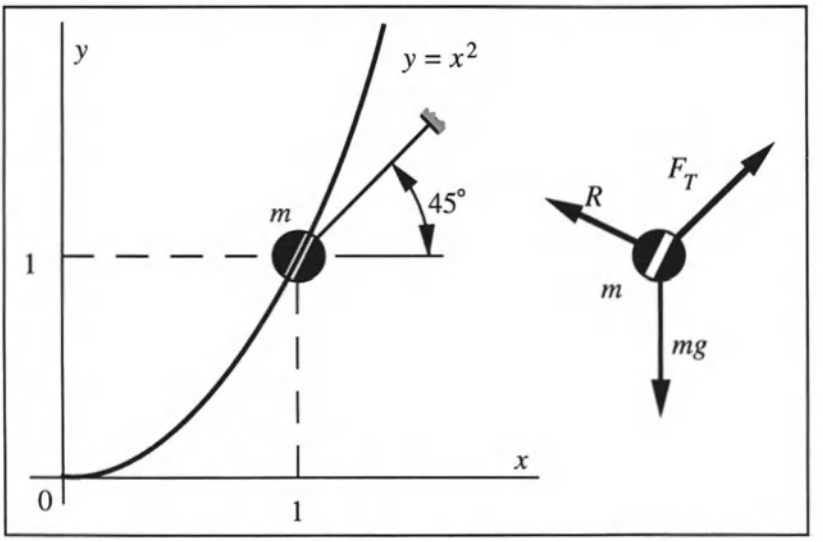


Figure 1.15 Bead on a parabolic wire, held by a string

Example 1.4.1

Suppose we require to find the tension force F_T in a string used to hold the particle at rest, in terms of the weight force mg of the particle m which can slide without friction on a wire as shown in Figure 1.15.

The particle will be in equilibrium under the forces F_T, mg and R, where R is the reaction force. To prevent the reaction from doing virtual work, we will choose a virtual displacement orthogonal to **R** i.e. along the tangent to the wire, at location (1,1). In other words, we simply observe the constraint condition of no displacement of the bead normal to the wire.

Now from $y = x^2$ we have that

$$\delta y = 2x \, \delta x \tag{a}$$

which at location (1,1) becomes

$$\delta y = 2\,\delta x \tag{b}$$

On computing the virtual work done by F_T and mg, we find

$$\delta W = F_T \cos 45° \,\delta x + (F_T \sin 45° - mg)\delta y = 0 \tag{c}$$

Hence, considering equation (b),

$$F_T = \frac{2\sqrt{2}}{3} mg \tag{d}$$

Note that the reaction force R has not appeared in any of the equations.

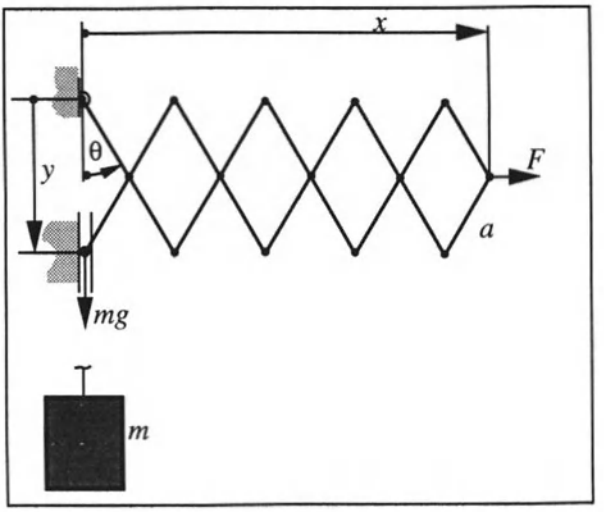


Figure 1.16 Pantograph mechanism

Example 1.4.2

Consider next the pantograph mechanism shown in Figure 1.16. We wish to determine the relation between the forces $F_1 = F$ and $F_2 = mg$, under static equilibrium conditions. We assume that all pin joints are frictionless.

The mechanism members are rigid bars of length $2a$ and can only take axial loads. From Newton's third law, it follows that the forces at the ends of each member are equal and opposite. Hence if we consider some virtual displacements which are consistent with the fixed lengths of every member, then the virtual work done by all member forces will be zero.

Now we may write that under static equilibrium conditions

$$\delta W = F_1\,\delta r_1 + F_2\,\delta r_2 = F\,\delta x_1 + mg\,\delta y_2 = 0 \tag{a}$$

provided δx_1 and δy_2 are consistent with the constraints. From Figure 1.16 it can be seen that the x and y coordinates of the points of application of F and mg are

$$x_1 = 9a\,\sin\theta \qquad y_2 = 2a\,\cos\theta \tag{b,c}$$

respectively. Allowing incremental changes in x_1 and y_2, but not in a, we find

$$\delta x_1 = 9a \cos\theta \, \delta\theta \qquad \delta y_2 = -2a \sin\theta \, \delta\theta \tag{d,e}$$

On substituting into equation (a), we deduce on account of the arbitrariness of $\delta\theta$, that

$$F = \frac{2}{9} mg \tan\theta \tag{f}$$

Rheonomic Constraints

We have seen that in the case of scleronomic constraints, if the virtual displacements satisfy the constraint equations, then the work of the constraint forces will vanish. In the case of rheonomic constraints an additional restriction must be imposed. This restriction requires that during the virtual displacements, *time* shall be held *fixed.* This requirement once again illustrates the imagined nature of virtual displacements since all actual displacements are associated with changes in time. The reason for holding

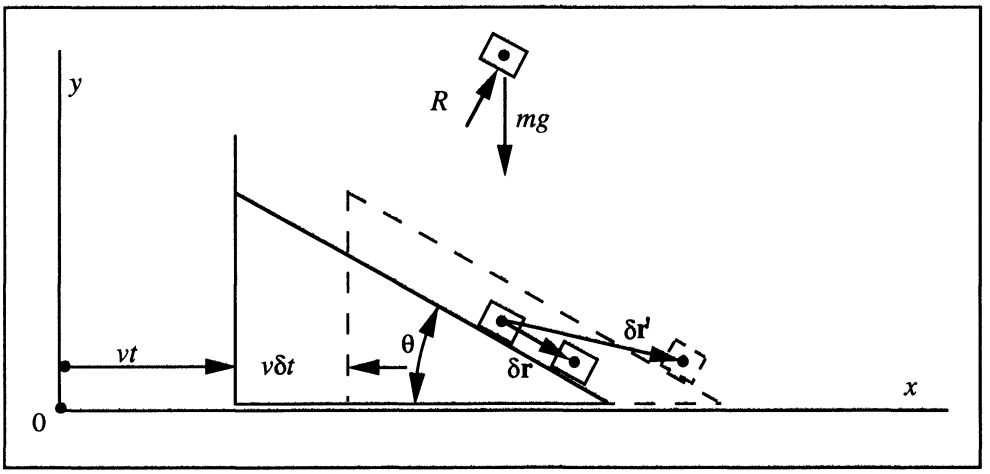


Figure 1.17 Virtual displacement when time is held fixed

time fixed, when virtual displacements are applied, can be readily deduced from the simple example shown in Figure 1.17. In this example of a block sliding without friction on an incline, one can see that even when $\delta\mathbf{r}'$ satisfies the constraint, i.e. the block remains on the incline, the reaction force **R** will do some work since **R** and $\delta\mathbf{r}'$ are not orthogonal. On the other hand, when time is held fixed and the constraint is satisfied, the work of **R** will vanish since $\delta\mathbf{r}$ is orthogonal to **R**.

In summary, we note that if virtual displacements are defined such that

i) constraints are satisfied,

ii) time is held fixed,

then the constraint forces will make no contribution to the associated virtual work and the vanishing of virtual work will yield the equations of equilibrium amongst the

applied forces.

Generalized Forces

Let us now express virtual work in terms of *generalized displacements* and *generalized forces*. To this end, let the Cartesian coordinates r of a system of N particles be expressed in terms of n generalized displacement variables q. That is, we assume that all holonomic constraints are satisfied and the reduced transformation equations take the form

$$r_j = r_j(q_1,q_2,...,q_i,...,q_n,t) \qquad j = 1,2,...,3N \qquad i = 1,2,...,n \tag{1.4.5}$$

From equation (1.4.5), we find the total differential dr_j as

$$dr_j = \frac{\partial r_j}{\partial q_i} dq_i + \frac{\partial r_j}{\partial t} dt \tag{1.4.6}$$

To ensure that the constraint forces do no work, we must hold time fixed in which case the last term in equation (1.4.6) drops out, and the virtual displacements δr_j become

$$\delta r_j = \frac{\partial r_j}{\partial q_i} \delta q_i \tag{1.4.7}$$

Denoting the applied forces on the particles by F_j we thus obtain for the virtual work

$$\delta W = F_j \, \delta r_j = F_j \frac{\partial r_j}{\partial q_i} \delta q_i \tag{1.4.8}$$

or

$$\delta W = Q_i \, \delta q_i \tag{1.4.9}$$

with

$$Q_i = F_j \frac{\partial r_j}{\partial q_i} \tag{1.4.10}$$

as the generalized forces.

If there exist some nonholonomic constraints relating the incremental displacements dr_j, they may be expressed as relations amongst dq_i and dt via equation (1.4.6). Then eliminating the excess dr_j and dq_i, expressions for virtual work may be written in both coordinate systems leading to the determination of generalized forces Q_i.

It is, by the way, quite instructive to retain the last term of equation (1.4.6) and write an expression for real increment of work done. In this case since time is not held fixed, work done by all forces must be taken into account. Thus we write

$$dW = (F_{all})_j dr_j = (F_{all})_j \frac{\partial r_j}{\partial q_i} dq_i + (F_{all})_j \frac{\partial r_j}{\partial t} dt \tag{1.4.11}$$

or

$$dW = (Q_{all})_i \, dq_i + (Q_{all})_t \, dt \tag{1.4.12}$$

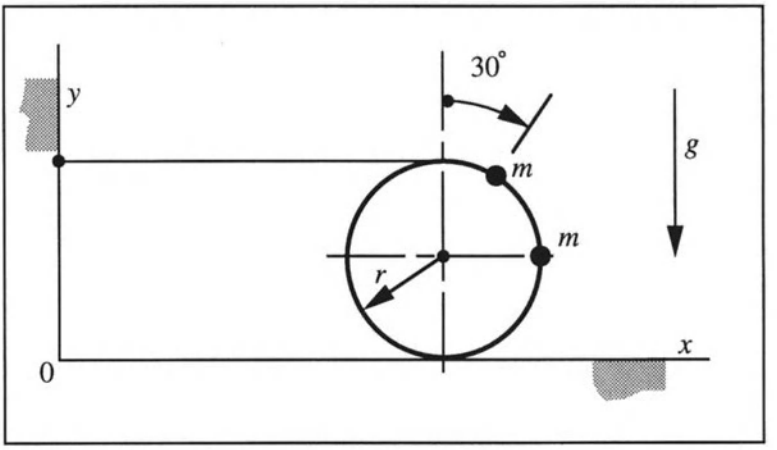
with
$$(Q_{all})_i = (F_{all})_j \frac{\partial r_j}{\partial q_i} \tag{1.4.13}$$

$$(Q_{all})_t = (F_{all})_j \frac{\partial r_j}{\partial q_i} \tag{1.4.14}$$

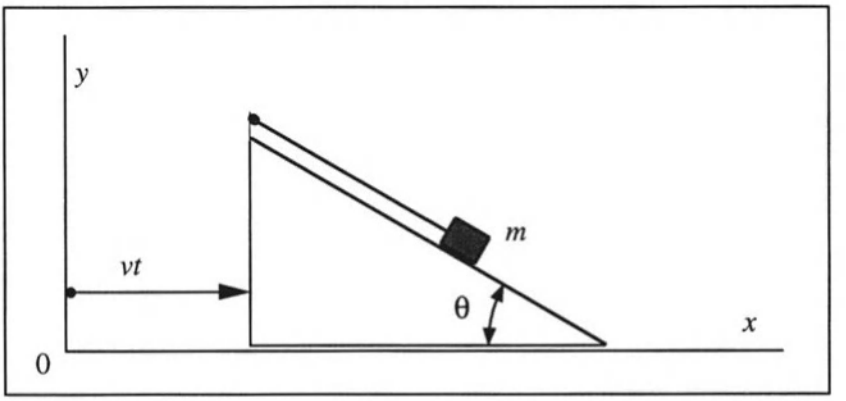
Since generalized displacements need not have dimensions of length, we can see that generalized forces need not have dimensions of force. However, the products $Q_i\ \delta q_i$ and $Q_t\ \delta t$ do have the dimension of work. Example 1.5.3 provides an illustration for cases when time is not held fixed.

Problems

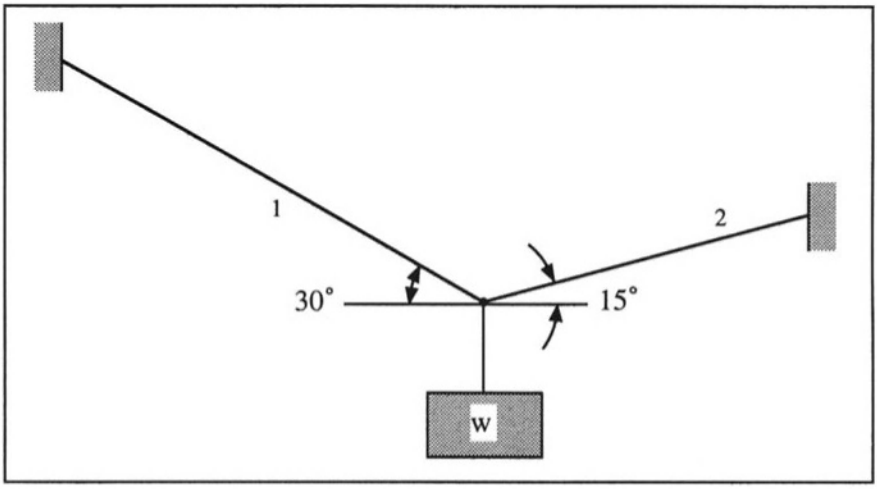
1.4.1 Use virtual work and calculate the tension F_T in the cable of a wheel at rest, which carries two particles m as shown.



1.4.2 A particle m can move along a friction-free incline, which in turn is moving at a constant horizontal speed v of 20 m/s, as shown. Calculate the string force F_S in terms of the weight force mg when $\theta = 30°$.

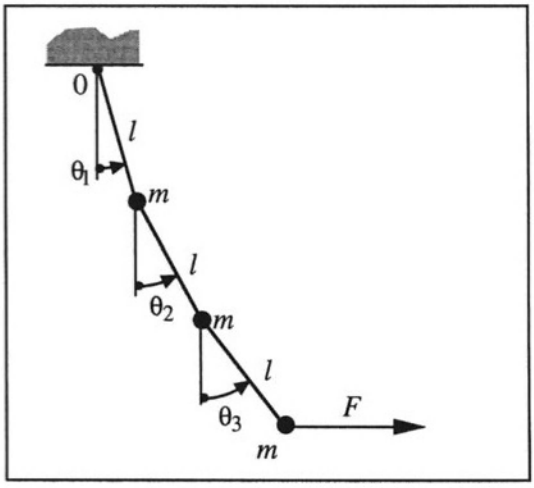


1.4.3 Using the virtual work approach, determine the string tensions in the system shown.



1.4.4 What is the generalized force in a simple pendulum, when the single generalized coordinate is $q_1 = \theta$?

1.4.5 Use virtual work to determine the angles θ_1,θ_2 , and θ_3 of the triple pendulum shown. Find numerical values for F = 120 N, mg = 100 N, and l = 1.5 m.



1.4.6 Determine the generalized force Q_θ for the pantograph mechanism of Figure 1.16.

1.5 Work Function and Potential Energy

The actual *work*

$$W = \int \mathbf{F}_k \cdot d\mathbf{r}_k \tag{1.5.1}$$

done by forces $\mathbf{F}_k$ can be obtained from equation (1.4.12), which we rewrite as

$$dW = Q_i\, dq_i + Q_t\, dt \qquad i = 1,2,3,...,n \tag{1.5.2}$$

with the Q as defined by equations (1.4.8) and (1.4.12). For a certain type of force the expression dW for an increment of work turns out to be an exact differential of a function U, such that

$$dU = \frac{\partial U}{\partial q_i}\, dq_i + \frac{\partial U}{\partial t}\, dt$$

Then

$$Q_i = \frac{\partial U}{\partial q_i} \quad \text{and} \quad Q_t = \frac{\partial U}{\partial t} \tag{1.5.3}$$

This function U, from which all the generalized forces may be generated, is called the *work function*, and forces that possess a work function are classified as *monogenic*, i.e. generated from a single function (lt: mono = single). One important property of monogenic forces is that the work done by such forces is independent of the displacement path. This can be readily seen by evaluating the work done from some initial state *0* to a final state *f*. Writing

$$\int_o^f dW = \int_{U_o}^{U_f} dU$$

and integrating, we find

$$W = U_f - U_o = U(q_f, t_f) - U(q_o, t_o) \tag{1.5.4}$$

Equation (1.5.4) implies that the work done depends only on the initial and final *states*, and is independent of the path between the states. When time does not appear in U explicitly, the work done, over a closed path, i.e. when the initial q_o and the final q_f states are equal, will be zero. Forces generated from such a work function are classified as conservative forces.

It is important to be able to check if a system of forces is monogenic or not. Such a check would test for the existence of a work function or, put another way, it would test for exactness of the differential increment dW in equation (1.5.2). Accordingly, this test is similar to that used for identifying a constraint as holonomic or nonholonomic.

If a continuous function U exists, then

$$\frac{\partial}{\partial q_i}\frac{\partial U}{\partial q_j} = \frac{\partial}{\partial q_j}\frac{\partial U}{\partial q_i} \qquad \begin{matrix} i = 1,2,3,...,n \\ j = 1,2,3,...,n \end{matrix} \tag{1.5.5}$$

and

$$\frac{\partial}{\partial q_i}\frac{\partial U}{\partial t} = \frac{\partial}{\partial t}\frac{\partial U}{\partial q_i}$$

On using equations (1.5.3), we can express these exactness conditions as

$$\begin{aligned} \frac{\partial}{\partial q_i}Q_j &= \frac{\partial}{\partial q_j}Q_i \qquad & i &= 1,2,3,\ldots,n \\ \frac{\partial}{\partial q_i}Q_t &= \frac{\partial}{\partial t}Q_i & j &= 1,2,3,\ldots,n \end{aligned} \tag{1.5.6}$$

If equations (1.5.6) are satisfied for a given system of forces, then these forces are monogenic and they possess a work function.

Instead of working with the work function U, it is usually more convenient to introduce another function called the *potential energy* function V, from which monogenic forces Q may be derived. The potential energy is defined as the negative of the work function, i.e.

$$V = -U \tag{1.5.7}$$

so that

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad Q_t = -\frac{\partial V}{\partial t} \tag{1.5.8}$$

The negative sign has the effect of *decreasing* the potential energy when *positive* work is done by the force, i.e. as the force does some work, its potential for doing further work is reduced. Employing the potential energy function, equation (1.5.4) for the work done becomes

$$W = V_o - V_f \tag{1.5.9}$$

i.e. the final potential energy is subtracted from the initial potential energy.

Example 1.5.1

Consider the work done as the spring shown in Figure 1.18 is stretched. Denoting the spring force by F, we have

$$dW = F\cos\theta\, dx + F\sin\theta\, dy \tag{a}$$

Here we can express $\cos\theta$ and $\sin\theta$ in terms of the end positions of the spring. With the fixed end at the origin, we have $\cos\theta = x/l$ and $\sin\theta = y/l$. Hence

$$dW = F\frac{x}{l}dx + F\frac{y}{l}dy \tag{b}$$

with $dq_1 = dx$ and $dq_2 = dy$, i.e. we use x and y as our generalized coordinates.

If the spring is linear, with stiffness k and unstretched length l_o, then the force F is related to the deformation $(l - l_o)$ as

$$F = -k(l - l_o) \tag{c}$$

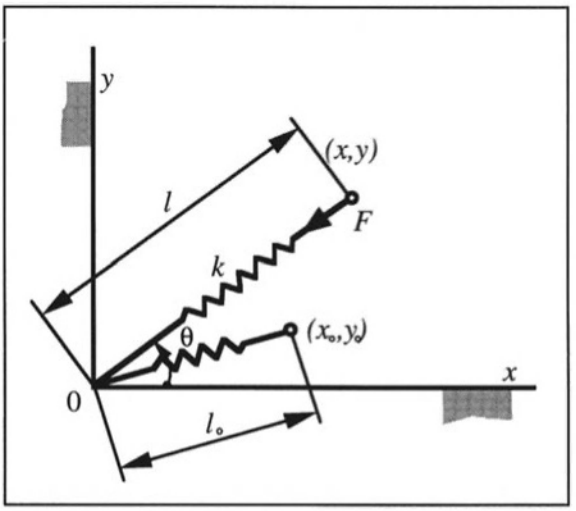


Figure 1.18 Work done on a spring

Substituting, we find

$$dW = -k(l - l_o)\left\{\frac{x}{l}dx + \frac{y}{l}dy\right\} \tag{d}$$

Now since

$$l = (x^2 + y^2)^{1/2} \tag{e}$$

the increment of work done by the spring force may be written as

$$dW = -k\,x\,dx - k\,y\,dy + \frac{kl_o(x\,dx + y\,dy)}{(x^2 + y^2)^{1/2}} \tag{f}$$

or

$$dW = Q_x\,dx + Q_y\,dy \tag{g}$$

where

$$Q_x = -k\,x + \left\{\frac{kl_o x}{(x^2 + y^2)^{1/2}}\right\} \qquad Q_y = -k\,y + \left\{\frac{kl_o y}{(x^2 + y^2)^{1/2}}\right\} \tag{h}$$

It is easy to see that forces Q_x and Q_y satisfy equations (1.5.6). Hence they are monogenic. It is not difficult to show that dW may be written as

$$dW = d\left\{-\frac{1}{2}k(x^2 + y^2) + kl_o\sqrt{x^2 + y^2}\right\} \tag{i}$$

where the term in the winged parentheses is the work function. The potential energy (or strain energy as it is often called) of the spring can be found as

$$V = \int_{x_o, y_o}^{x, y} d\left\{\frac{1}{2} k(x^2 + y^2) - kl_o\sqrt{x^2 + y^2}\right\} \tag{j}$$

or

$$V = \frac{1}{2} k(x^2 + y^2) - kl_o\sqrt{x^2 + y^2} + \frac{1}{2} kl_o^2 = \frac{1}{2} k(l - l_o)^2 \tag{k}$$

including an integration constant chosen such that $V = 0$ when $l = l_o$, i.e. when the spring is unstretched.

The work done by the spring force as the spring is extended from a state V_0 at (x_0, y_0) to a state V_f at (x_f, y_f) is

$$W = V_0 - V_f \tag{l}$$

From this expression, one can see that the work done W is independent of path and it depends on the initial and final values of the potential energies. If the initial and final states coincide, the work done will be equal to zero.

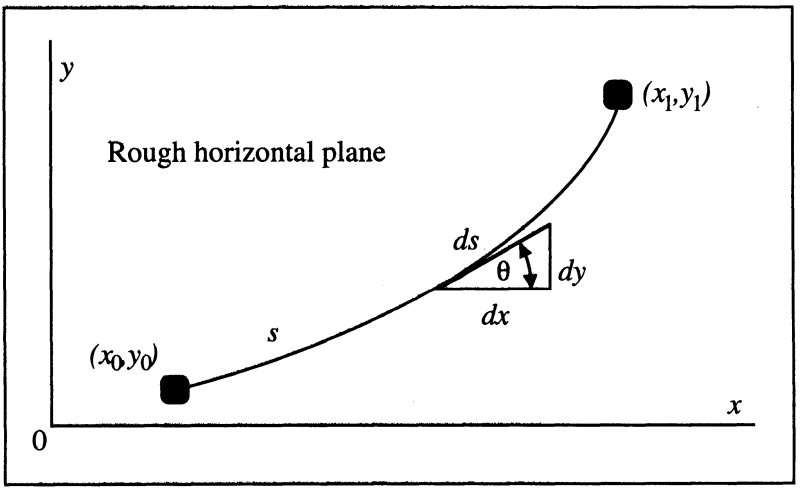


Figure 1.19 Work done by a friction force

Example 1.5.2

As an example of forces which are not monogenic (sometimes called polygenic forces), consider the work done by the force of friction, as a block of weight mg slides from the initial position x_o, y_o, to a final position x_1, y_1, on a rough horizontal plane (Figure 1.19).

Now

$$dW = F_x\, dx + F_y\, dy \tag{a}$$

But $F = \mu mg$, where μ is the coefficient of friction, and the force acts in a direction opposite to that of the motion, hence

$$dW = -\mu mg\cos\theta\, dx - \mu mg\sin\theta\, dy \tag{b}$$

$$= Q_x dx + Q_y dy \tag{c}$$

such that

$$Q_x = -\mu mg\cos\theta \tag{d}$$

$$Q_y = -\mu mg\sin\theta \tag{e}$$

In the present example $\cos\theta$ and $\sin\theta$ are not functions of the initial and the final positions. Thus at any point the angle θ must be known (i.e. the path must be prescribed), before the increment of work may be computed. Hence it is valid to write $dW = Q_x dx + Q_y dy + Q_\theta d\theta$, with $Q_\theta = 0$. Now we can see that since $\frac{\partial Q_\theta}{\partial x} \neq \frac{\partial Q_x}{\partial \theta}$, dW is not an exact differential.

If we denote the path by s, then

$$\cos\theta = \frac{dx}{ds} \qquad \sin\theta = \frac{dy}{ds} \tag{f}$$

and

$$dW = -\mu mg\frac{dx^2 + dy^2}{ds} \tag{g}$$

But

$$ds^2 = dx^2 + dy^2 \tag{h}$$

therefore

$$dW = -\mu mg\left[1 + \left[\frac{dy}{dx}\right]^2\right]^{1/2} dx \tag{i}$$

and the total work done

$$W = -\mu mg \int_{x_o}^{x_1} \left[1 + \left[\frac{dy}{dx}\right]^2\right]^{1/2} dx \tag{j}$$

In the above equation, the quantity under the integral is not an exact differential. Therefore, the integration cannot be performed until $y = y(x)$ is specified, i.e. the integration depends upon the path from the initial to the final location. The force does not possess a potential energy.

Example 1.5.3

The system shown in Figure 1.20 is for the purpose of demonstrating how the explicit appearance of the time t in the potential energy expression affects the virtual work. Note that the upper end of spring 3 moves down at a prescribed constant speed v. The connecting bar A remains horizontal. All four linear springs are unstretched when $q_1 = q_2 = t = 0$. Assume that no vibrations occur.

The potential energy of the system shown, with the generalized displacements q_1 and q_2, is

$$V = \frac{1}{2}\left[k_1(q_2 - q_1)^2 + k_2 q_1^2 + k_3(q_1 - vt)^2 + k_4 q_1^2\right] - m_1 g q_1 - m_2 g q_2 \tag{a}$$

and the generalized forces associated with q_1 and q_2 are

$$Q_1 = -\frac{\partial V}{\partial q_1} = k_1(q_2 - q_1) - k_2 q_1 - k_3(q_1 - vt) - k_4 q_1 + m_1 g \tag{b}$$

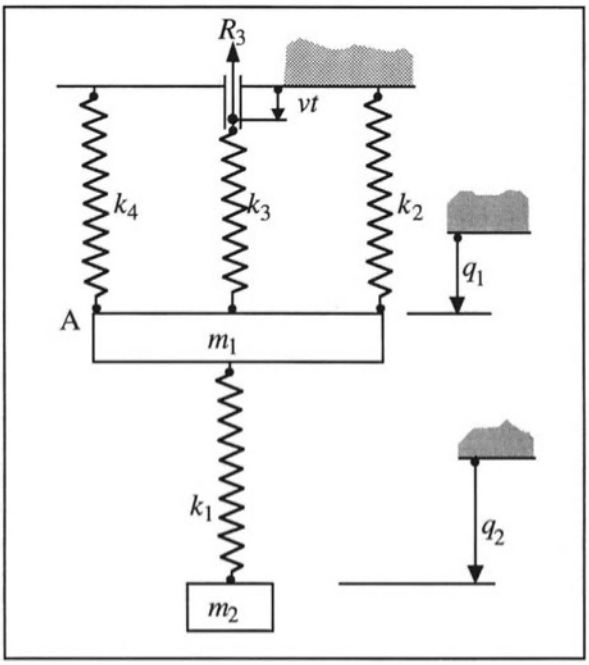


Figure 1.20 Four-spring system with one support moving at a constant speed

$$Q_2 = -\frac{\partial V}{\partial q_2} = -k_1(q_2 - q_1) + m_2 g \tag{c}$$

Now if we do not freeze time, the constraint force R_3 will do some work. This work will be equal to the work done by the generalized force Q_t. The generalized force Q_t is given by

$$Q_t = -\frac{\partial V}{\partial t} = k_3(q_1 - vt)\, v \tag{d}$$

Writing an expression for the increment of work done by all forces including the constraint force, we have for equilibrium

$$dW = Q_1 dq_1 + Q_2 dq_2 + Q_t dt - Rv dt = 0 \tag{e}$$

Since dq_1, dq_2 and dt are arbitrary, satisfaction of equation (e) requires

$$Q_1 = 0, \quad Q_2 = 0 \quad Q_t = -Rv \tag{f}$$

$$Q_1 = k_1(q_2 - q_1) - k_2 q_1 - k_3(q_1 - vt) - k_4 q_1 + m_1 g = 0 \tag{g}$$

$$Q_2 = -k_1(q_2 - q_1) + m_2 g = 0 \tag{h}$$

$$k_3(q_1 - vt) = R \tag{i}$$

Other Forms of Potential Energy

We have seen that potential energy functions, from which generalized forces may be derived, are usually scalar functions of generalized displacements and in some cases also of time. However this form is not unique. That is, it may be possible to define new forms of potential energy for certain classes of forces which are not derivable from the usual scalar form. For instance the forces on a point charge moving in an electromagnetic field cannot be derived from a scalar potential energy function alone, but they may be derived from a combination of scalar and vector potential energy functions. We mention this point here for sake of completeness but will not pursue the matter further (see e.g. Goldstein, 1980).

Problems

1.5.1 The potential energy of a point spacecraft m in the gravitational central force field is given by

$$V = -\frac{\mu m}{r}$$

where $\mu = 398\ 601\ \text{km}^3/\text{s}^2$ for the Earth as master, and r is the distance between Earth centre and spacecraft. (a) How much work is done by the attraction force if a spacecraft ($m = 1\ 000$ kg) is moved from a parking orbit of $r_1 = 7\ 000$ km to a geosynchronous orbit of $r_2 = 42\ 164$ km? (b) What is the potential energy of the same spacecraft when it is located at the surface ($r = 6\ 371$ km) of a (nonrotating!) Earth sphere? (c) What is its potential energy at $r = \infty$?

1.5.2 The potential energy of a mass m located at a height y above the surface of the Earth is

$$V = mgy$$

(a) Show that this formula is in agreement with the potential energy expression for a spacecraft, provided y is small compared to the Earth sphere's radius of $R = 6\ 371$ km. (b) With $R = 6\ 371$ km and $\mu = 398\ 601\ \text{km}^3/\text{s}^2$, what is the value of g? Why does it differ from the standard $g = 9.80665\ \text{m/s}^2$?

1.5.3 Use $V = mgy$ to obtain the potential energy stored in a 10 kg mass as it is raised from sea level to the top of 8 848 m high Jolmo Lungma (Mount Everest). Considering the substantial height, can you suggest an improvement on the value of the potential energy?

1.5.4 Find the work done in moving a particle once around a circle in the xy plane, if the circle has its centre at the origin and its radius is 3 m, and the force field is given by

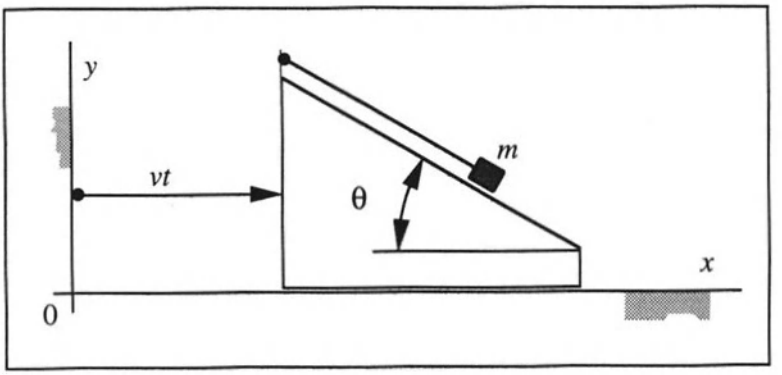
$$\mathbf{F} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} 2x - y + z \\ x + y - z^2 \\ 3x - 2y + 4z \end{bmatrix} \ \text{N}$$

1.5.5 Show that

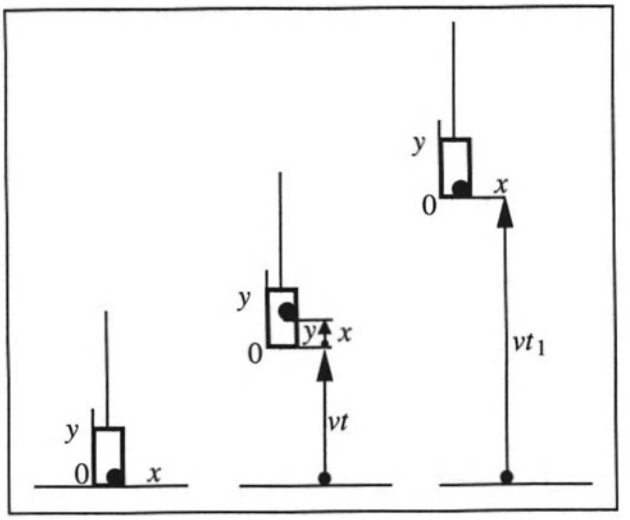
$$\mathbf{F} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} 2xy + z^3 \\ x^2 \\ 3xz^2 \end{bmatrix} \text{ N}$$

is a conservative force field. Find the potential energy of this force field and determine the work done in moving a particle, in this field, from (1, -2, 1) to (3, 1, 4).

1.5.6 Include the time term in the virtual work expression to find the reaction force N between mass point m and the frictionless incline shown. Determine also the tension force F in the cable. The velocity v is constant.



1.5.7 A rock of mass m is lifted by an amount y inside a mine shaft hoist cage, while the cage itself is rising at a constant velocity v. (a) Find the potential energy V. (b) Find the generalized forces Q_y and Q_t. (c) Find the potential energy at some time t_1, when the rock is returned to the cage floor, i.e. $y = 0$.



1.5.8 Determine the potential energy of the spring shown in Figure 1.18 by using the generalized coordinates $q_1 = l$ and $q_2 = \theta$ (instead of the Cartesian coordinates $r_1 = x$ and $r_2 = y$).

1.5.9 Calculate the work W done by a friction force $F = -\mu mg$, with $\mu = 0.2$ and $m = 10$ kg, as it moves from $x_0 = 0$ to $x_1 = (\sqrt{3}/4)$ m along a path given by $y = 2x^2$. Make a sketch.

1.6 Stability of Static Equilibrium

When all the forces in a system are conservative, i.e. derivable from a potential energy function V, the problem of static equilibrium reduces to the problem of extremization of V. The nature of the potential energy's stationary values distinguishes between equilibrium conditions which are stable, neutrally stable, or unstable. To examine this point in some detail, let us follow Dirichlet† and expand the potential energy V into a Taylor†† series, near a point 0 of equilibrium. Thus

$$V = V_0 + \left.\frac{\partial V}{\partial q_j}\right|_0 dq_j + \frac{1}{2!}\left.\frac{\partial^2 V}{\partial q_j \partial q_k}\right|_0 dq_j dq_k + \frac{1}{3!}\left.\frac{\partial^3 V}{\partial q_j \partial q_k \partial q_l}\right|_0 dq_j dq_k dq_l + \cdots \tag{1.6.1}$$

where $V = V(q_1, q_2, q_3, \cdots, q_j, \cdots, q_n)$

and $V_0 = V(q_{10}, q_{20}, q_{30}, \cdots, q_{j0}, \cdots, q_{n0})$

Static equilibrium means that the resultant forces are zero. Thus, each

$$Q_j = -\frac{\partial V}{\partial q_j} = 0 \tag{1.6.2}$$

at equilibrium, i.e. all first derivative terms in equation (1.6.1) vanish. Further, if the equilibrium position 0 is a stable one, then $V - V_0 > 0$ is a requirement, and hence all the terms following must be positive for all possible values of dq_j. Thus the second derivative quadratic terms must be positive definite. If the quadratic terms vanish at 0, one must examine the nature of higher order terms for positive definiteness. If all the higher order terms also vanish at 0, the equilibrium position is neutrally stable.

For the quadratic terms in (1.6.1) to be positive definite it is required that

$$[dq_1\ dq_2 \ldots\ldots dq_n] \begin{bmatrix} V_{,q_1q_1} & V_{,q_1q_2} & \ldots\ldots & V_{,q_1q_n} \\ V_{,q_2q_1} & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ V_{,q_nq_1} & \ldots\ldots & \ldots\ldots & V_{,q_nq_n} \end{bmatrix}_0 \begin{bmatrix} dq_1 \\ dq_2 \\ \vdots \\ dq_n \end{bmatrix} > 0 \tag{1.6.3}$$

† Peter-Lejeune Dirichlet (1805-1859), German mathematician

†† Brook Taylor (1685-1731), English mathematician

for all possible dq_j. This condition is fulfilled, when the determinants

$$D_i = \left| V_{,q_j q_k} \right|_0 > 0 \qquad \text{with} \quad \begin{matrix} i = 1,2,3,...n \\ j,k = 1,2,3,...i \end{matrix} \tag{1.6.4}$$

where n is the number of coordinates and $V_{,jk} = \dfrac{\partial^2 V}{\partial q_j \partial q_k}$. If one or more of D_i are negative no definite statement can be made about the stability of the equilibrium. If one, or more, determinants D_i are zero at the equilibrium state 0, while the others are positive, then the stability of the equilibrium state is still undecided, and the next higher derivatives must be investigated. The following examples illustrate the essential ideas.

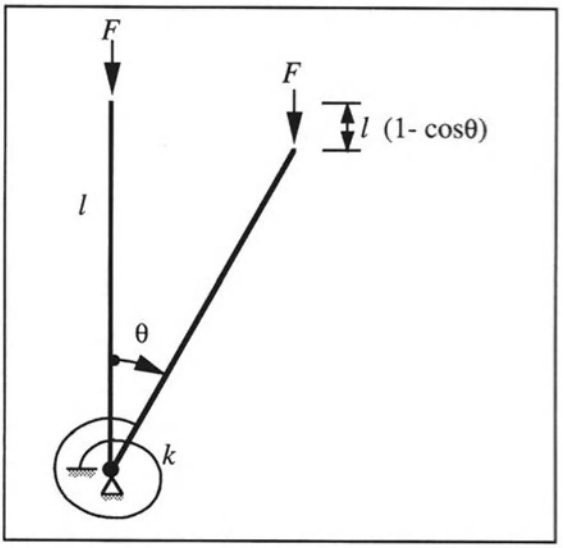


Figure 1.21 Spring-supported column

Example 1.6.1

Let us examine the stability of the idealized column as shown in Figure 1.21. Denoting the torsional stiffness of the (linear) spring by k, we may express the strain energy of the spring as

$$V_s = \frac{1}{2} k \theta^2$$

The potential energy of the constant force F is given by

$$V_F = - F \; l \; (1 - \cos\theta)$$

Hence the total potential energy becomes

$$V = \frac{1}{2} \, k \theta^2 - F \; l \; (1 - \cos\theta)$$

and the equilibrium conditions are found as

$$\frac{\partial V}{\partial \theta} = 0$$

i.e. $$k\theta - F\, l\, \sin\theta = 0 \qquad \text{(a)}$$

Equation (a) shows that the configuration $\theta = 0$ is a possible equilibrium position for all values of F. For non-zero values of θ, equilibrium conditions are governed by

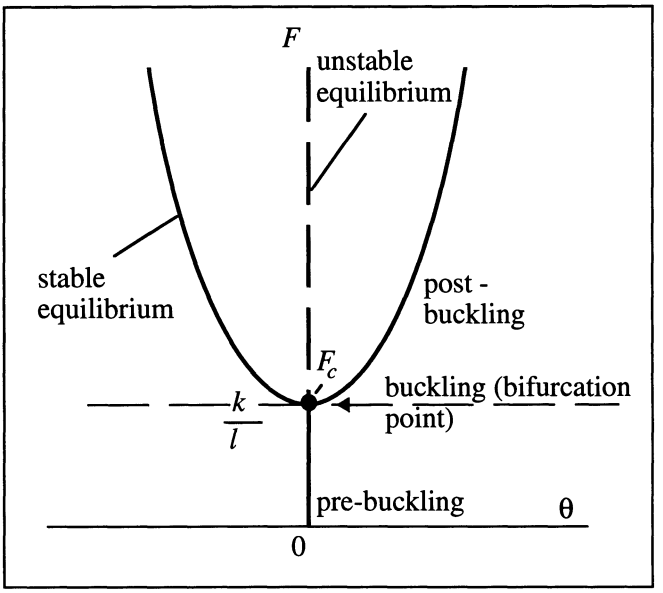


Figure 1.22 Equilibrium force magnitudes

$$F = \frac{k\theta}{l \sin\theta} \qquad \text{(b)}$$

Figure 1.22 shows the possible paths of equilibrium configurations. To study the stability of these equilibrium solutions, we examine the second order derivatives of V.

$$\frac{\partial^2 V}{\partial \theta^2} = k - F\, l\, \cos\theta \qquad \text{(c)}$$

Thus for the fundamental path, for which $\theta = 0$, stability is governed by

$k - F\, l > 0$ or $F < \frac{k}{l}$ stable

$k - F\, l < 0$ or $F > \frac{k}{l}$ unstable

For the post-buckling path, for which $F > k/l$ and $\theta > 0$, we substitute for F, from equation (b) into equation (c) and find that for stability

$$k\left[1 - \frac{\theta}{\tan\theta}\right] > 0 \qquad \text{(d)}$$

Since equation (d) is true for all values of θ, we deduce that the equilibrium given by equation (b) is always stable.

It remains to examine the critical point C. For this point, $F_c = k/l$ and the first and the second derivative terms vanish. Hence we examine the third order derivative

$$\frac{\partial^3 V}{\partial \theta^3} = F\ l\ \sin\theta$$

which also vanishes for $\theta = 0$. Proceeding to the fourth order derivative, we find

$$\frac{\partial^4 V}{\partial \theta^4} = F\ l\ \cos\theta$$

which is positive at $\theta = 0$, and we conclude that at the critical point C, the potential energy has a local minimum point and the equilibrium is stable.

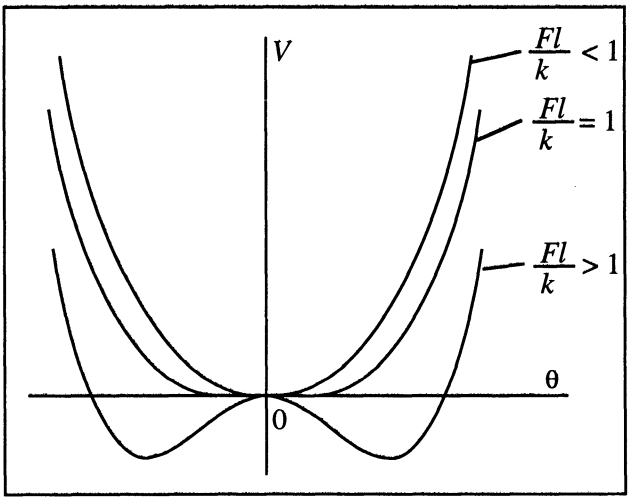


Figure 1.23 Potential energy

In Figure 1.23 the potential energy is plotted against θ, for three different values of F. From this plot, it is evident that for a small load, $Fl/k < 1$, namely below the critical value of F, only the vertical position of the column is stable, and the system is attracted to $\theta = 0°$ whereas above this critical value $F_c = k/l$ the system is attracted to one of the two stable positions on the two sides of the vertical position. The vertical position itself is unstable since at this point V attains a maximum, as can be deduced from equation (c).

Example 1.6.2

Consider a particle m which can slide without friction on a surface $z = \frac{1}{2}(x-2)^2 + \frac{1}{3}(y-1)^3$, as shown in Figure 1.24. The potential energy of this particle is $V = mgz$ or

$$V = \frac{1}{2}\, mg\,(x-2)^2 + \frac{1}{3}\, mg\,(y-1)^3 \qquad \text{(a)}$$

Since in this case there are two generalised coordinates, x and y, equilibrium conditions require that

$$\frac{\partial V}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial V}{\partial y} = 0 \tag{b}$$

i.e.,

$$mg(x-2) = 0 \qquad \text{and} \qquad mg(y-1)^2 = 0 \tag{c}$$

which are satisfied for

$$q_{1o} = x_o = 2 \qquad q_{2o} = y_o = 1 \tag{d}$$

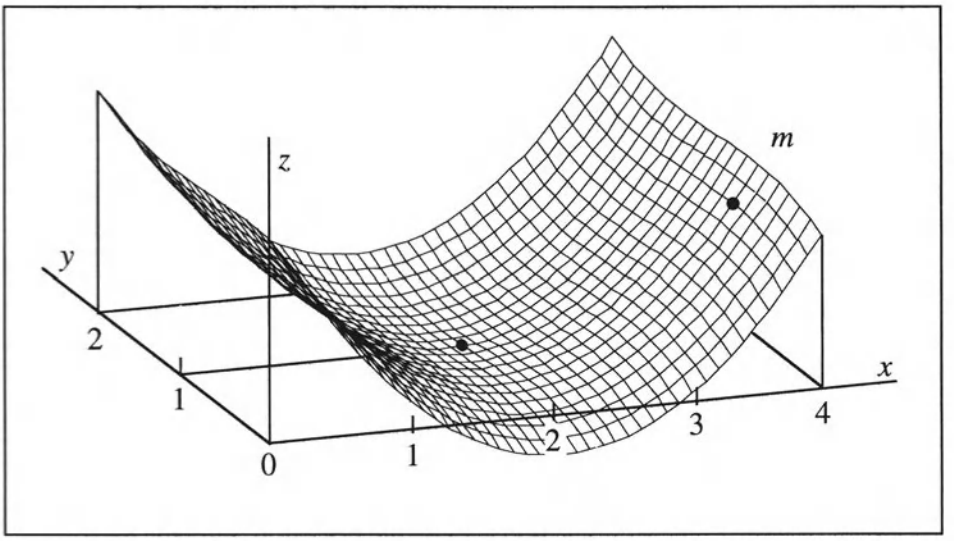


Figure 1.24 Particle sliding on surface

Equations (d) show that $x = 2$ and $y = 1$ is an equilibrium configuration. To explore the stability of this equilibrium configuration, we look at the second order derivatives

$$\delta^2 V = \frac{\partial^2 V}{\partial x^2}dx^2 + \frac{\partial^2 V}{\partial x \partial y}dxdy + \frac{\partial^2 V}{\partial y \partial x}dydx + \frac{\partial^2 V}{\partial y^2}dy^2 \tag{e}$$

for $x_o = 2$ and $y_o = 1$ and get

$$\delta^2 V = mg\,[dx \;\; dy]\begin{bmatrix} 1 & 0 \\ 0 & (y-1) \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix} \tag{f}$$

Invoking the test in equation (1.6.6) for $x = 2$ and $y = 1$ we have

$$D_1 = |1| = 1 \tag{g}$$

and

$$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \tag{h}$$

The vanishing of determinant (h) requires a look at the third order derivatives. Since

$$\frac{\partial^3 V}{\partial x^3} = \frac{\partial^3 V}{\partial x \partial y^2} = \frac{\partial^3 V}{\partial x^2 \partial y} = 0 \tag{i}$$

there remains only

$$\delta^3 V = \frac{\partial^3 V}{\partial y^3} dy^3 = 2mg\ dy^3 \tag{j}$$

which will assume negative values for negative dy. Thus the equilibrium position $x = 2, y = 1$ is unstable.

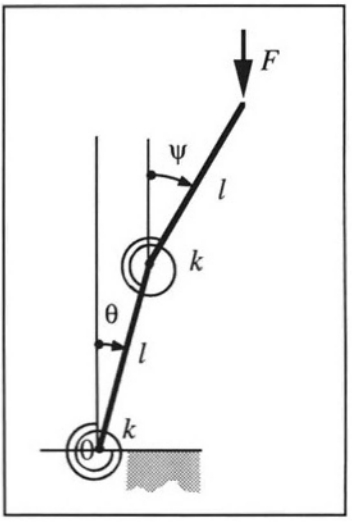


Figure 1.25 Two rod system

Example 1.6.3

Let us examine the static stability of a two degree of freedom system shown in Figure 1.25. The potential energy of the system is

$$V = \frac{1}{2} k\, \theta^2 + \frac{1}{2} k\, (\psi - \theta)^2 - Fl\, (1 - \cos\theta) - Fl\, (1 - \cos\psi) \tag{a}$$

for springs that are unstretched when $\theta = \psi = 0$.

Now

$$\frac{\partial V}{\partial \theta} = k\theta - k\, (\psi - \theta) - Fl \sin\theta = 0 \tag{b}$$

$$\frac{\partial V}{\partial \psi} = k\, (\psi - \theta) - Fl \sin\psi = 0 \tag{c}$$

From equations (b) and (c) one can conclude immediately that

$$\theta = 0 \quad \text{and} \quad \psi = 0 \tag{d,e}$$

is an equilibrium condition. To check whether this equilibrium state is stable, we investigate the

sign definiteness of the matrix

$$\begin{bmatrix} \frac{\partial^2 V}{\partial\theta^2} & \frac{\partial^2 V}{\partial\theta\partial\psi} \\ \frac{\partial^2 V}{\partial\theta\partial\psi} & \frac{\partial^2 V}{\partial\psi^2} \end{bmatrix} = \begin{bmatrix} 2k - Fl\cos\theta & -k \\ -k & k - Fl\cos\psi \end{bmatrix} \tag{f}$$

Thus

$$D_1 = 2k - Fl\cos\theta \tag{g}$$

$$D_2 = (2k - Fl\cos\theta)(k - Fl\sin\psi) - k^2 \tag{h}$$

For $\theta = \psi = 0$, the stability of equilibrium requires that

$$D_1 = 2k - Fl > 0 \tag{i}$$

$$D_2 = (2k - Fl)(k - Fl) - k^2 > 0 \tag{j}$$

Inequality (i) leads to

$$F < 2\frac{k}{l} \tag{k}$$

Inequality (j) leads to either

$$F > 2.618\frac{k}{l} \quad \text{or} \quad F < 0.382\frac{k}{l} \tag{l,m}$$

By inspection, we conclude that, of the three, $F < 0.382\frac{k}{l}$ is the requirement to make the equilibrium state $\theta = \psi = 0$ stable.

Problems

1.6.1 A particle m subjected to its weight mg can slide friction-free along a path whose shape is given by

$$y = 4 - \frac{1}{2}x^3$$

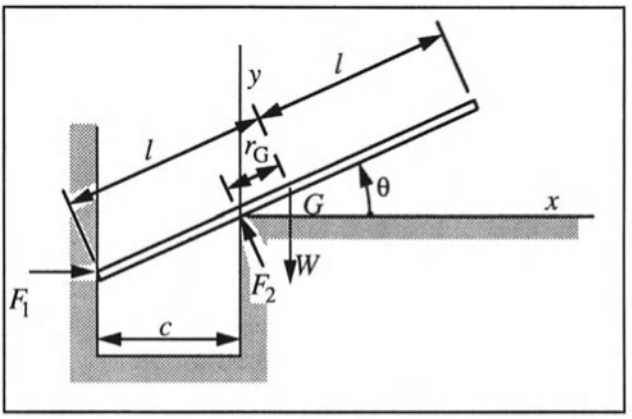
Examine the stability of the particle at $x = 0$, by using its potential energy $V = mgy$.

1.6.2 Plot three potential energy curves

$$V = \frac{1}{2}k\,\theta^2 - F\,l(1 - \cos\theta)$$

in the domain $-180° \le \theta \le 180°$, for $F = k/2l$, $F = k/l$, and $F = \pi k/2l$.

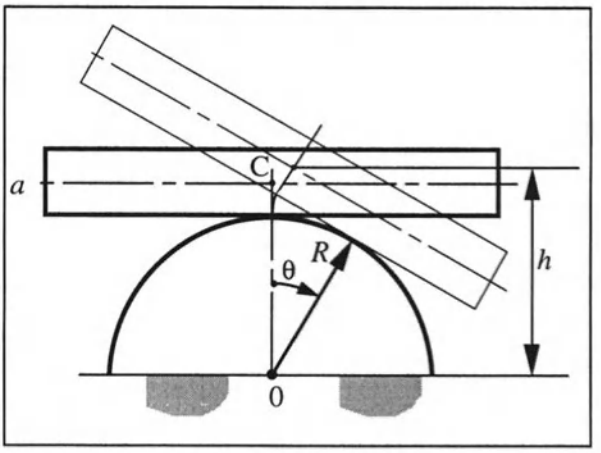
1.6.3 Find the angle θ, for $c = 0.729l$, at which the system shown is in equilibrium, by i) using the virtual work theorem, ii) minimizing the total potential energy. Assume that the contact surfaces are frictionless. Is the equilibrium stable or unstable?



1.6.4 A particle is constrained to move on the wavy smooth surface $z = 5\sin(x + y)$ where z is the vertically upward direction. When can the particle be in equilibrium under its own weight and forces of constraint? Which are stable, and which are unstable equilibria? Sketch the surface.

1.6.5 For the potential energies, i) $V = x^2(6 - x)$, ii) $V = \sin\pi x$, determine the equilibria and find whether they are stable or unstable.

1.6.6 A block of weight mg and of height a as shown, can roll but not slide on a semicylinder of radius R. (a) Establish the block's potential energy, and plot it as function of the angle θ for the case that $a = (\pi/\sqrt{3})\,R$. (b) Find the angles θ at which the system is in equilibrium. (c) Which equilibrium is stable, which unstable?



1.6.7 For the potential energy

$$V = 3(x-1)^2 + 2xy - 3y^2$$

determine the equilibrium, and find whether it is stable or unstable.

1.6.8 For the potential energy

$$V = (x-2)^4 + 2(x-2)^2(y-1)^2 + (y-1)^4$$

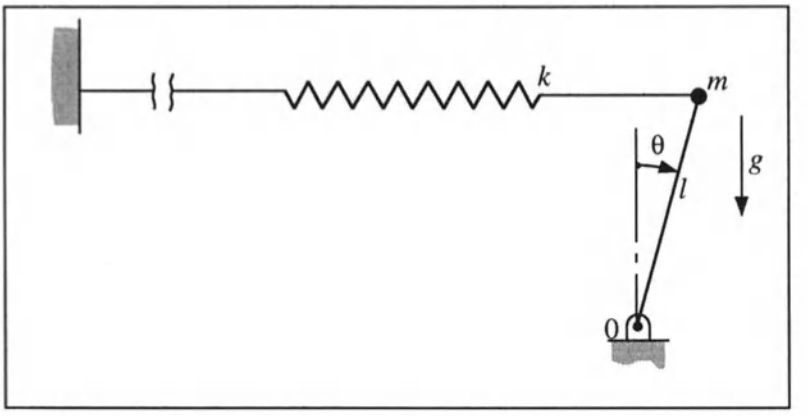
determine the equilibrium, and find whether it is stable or unstable.

1.6.9 For the column shown in Figure 1.21, $k = 1750$ Nm/rad, $F = 5$ kN, and $l = 700$ mm. Determine the stable equilibrium angles.

1.6.10 Find (a) the 5 equilibrium states of the two rod system of Figure 1.25, for the case that $F = 5\,k/l$. Check (b) which of these 5 are stable.

1.6.11 Find the primary and secondary equilibrium paths for the system shown and determine their static stability, and the stability of the critical point. Assume the anchor point of the spring is a long distance away to justify the spring being assumed to remain horizontal throughout. Also assume the spring to be unstretched when $\theta = 0°$, such that the potential energy is

$$V = \frac{1}{2} k l^2 \sin^2\theta - Wl(1 - \cos\theta)$$



1.7 A Second Look at the Expression for Increment of Work

We have expressed an increment of work as

$$dW = \mathbf{F} \cdot d\mathbf{r} \tag{1.7.1}$$

Now if we introduce the *impulse* of the force $\mathbf{F}$ and denote it by $\mathbf{I}$ such that $\dot{\mathbf{I}} = \mathbf{F}$, we may express equation (1.7.1) as

$$dW = \frac{d\mathbf{I}}{dt} \cdot d\mathbf{r} = \frac{d\mathbf{r}}{dt} \cdot d\mathbf{I} = \dot{\mathbf{r}} \cdot d\mathbf{I} = \mathbf{v} \cdot d\mathbf{I} \tag{1.7.2}$$

From the two forms of increment of work, we can define two theorems of virtual work; viz. one of virtual work of the first kind

$$\delta W_I = \mathbf{F}_k \cdot \delta \mathbf{r}_k \quad = \quad \dot{\mathbf{I}}_k \cdot \delta \mathbf{r}_k \tag{1.7.3}$$

and one of virtual work of the second kind

$$\delta W_{II} \quad = \quad \dot{\mathbf{r}}_k \quad \cdot \quad \delta \mathbf{I}_k = \mathbf{v}_k \cdot \delta \mathbf{I}_k \tag{1.7.4}$$

We have seen that the vanishing of δW_I will yield the equilibrium equations amongst the applied forces $\mathbf{F}_k$, providing time is held fixed and virtual displacements $\delta \mathbf{r}_k$ satisfy the constraint equations. In this case the constraint equations are kinematic in nature and express the conditions of geometric fit. By analogy, one can see that the vanishing of δW_{II} will yield the kinematic fit or compatibility equations amongst the velocities $\mathbf{v}_k$, providing time is held fixed and the impulses $\delta \mathbf{I}_k$ satisfy some constraint equations. In this case the constraint equations are relations amongst impulses which ensure that equilibrium conditions are satisfied. Thus the conditions that must be satisfied à priori, for the use of δW_I, appear as the vanishing conditions for δW_{II}, and vice versa. The number of independent virtual displacements forms the number of *positional* degrees of freedom when δW_I is used, and similarly the number of independent virtual impulses forms the number of *impulsive* degrees of freedom when δW_{II} is used. The following example illustrates the dual nature of δW_I and δW_{II}.

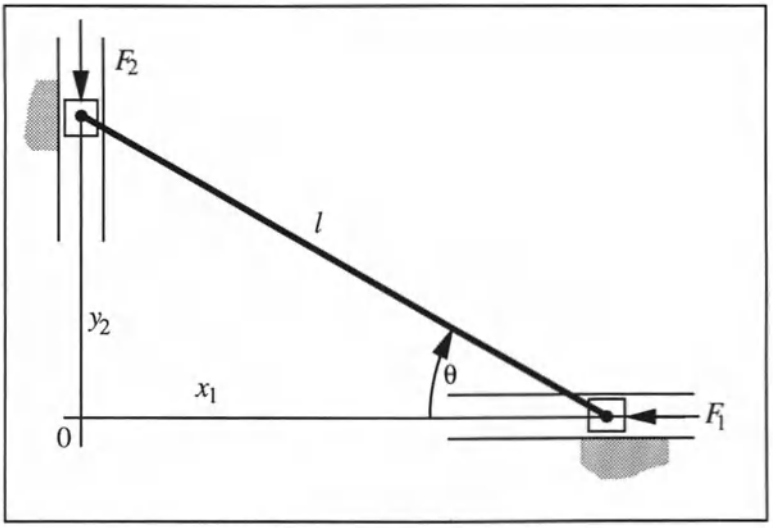


Figure 1.26 Two interconnected sliding beads

Example 1.7.1

Consider the two particles connected by a rigid link as shown in Figure 1.26. The particles can slide without friction, but remain at rest.

To obtain the *equilibrium* conditions for F_1 and F_2, we write down δW_I as follows

$$\delta W_I = -F_2\delta y_2 - F_1\delta x_1 = 0 \tag{a}$$

But x_1 and y_2 are not independent. We can satisfy the displacement constraint equation

$$x_1^2 + y_2^2 = l^2 \tag{b}$$

by choosing (θ) as the single generalized coordinate. Then the reduced transformation equations become

$$y_2 = l\sin\theta \qquad \text{and} \qquad x_1 = l\cos\theta \tag{c}$$

Their variations are

$$\delta y_2 = l\cos\theta\,\delta\theta \qquad \text{and} \qquad \delta x_1 = -l\sin\theta\,\delta\theta \tag{d}$$

Thus we have the constraint

$$\delta x_1 = -\tan\theta\,\delta y_2 \tag{e}$$

Substituting from equation (e) into equation (a), we find

$$\delta W_I = -(F_2 - F_1\tan\theta)\delta y_2 = 0 \tag{f}$$

or

$$F_2 - F_1\tan\theta = 0 \tag{g}$$

To obtain the *kinematic fit* conditions for $\dot{x}_1$ and $\dot{y}_2$, we write

$$\delta W_{II} = -\dot{y}_2\,\delta I_2 - \dot{x}_1\,\delta I_1 \tag{h}$$

But the impulses I_1 and I_2 are not independent, they are required to satisfy the equilibrium equation (g). Thus we obtain the impulse constraint equation

$$\frac{dI_2}{dt} - \frac{dI_1}{dt}\tan\theta = 0 \tag{i}$$

or

$$\delta I_2 - \delta I_1\tan\theta = 0 \tag{j}$$

Substituting from equation (j) into equation (h), we find

$$\delta W_{II} = (\dot{y}_2\tan\theta + \dot{x}_1)\delta I_1 = 0 \tag{k}$$

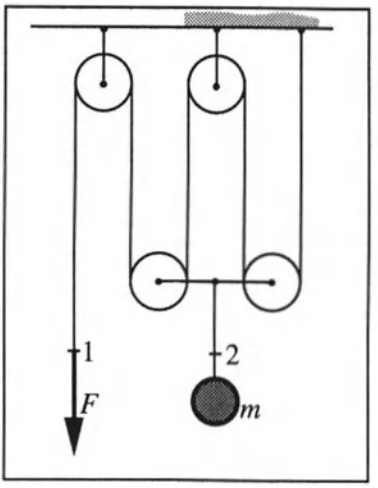
or

$$\dot{x}_1 = -\dot{y}_2\tan\theta \tag{l}$$

Equation (l) can be seen to be essentially the same as the kinematic constraint equation (e). Before leaving this example, it is worth noting that the constraint equations for δW_I, i.e. the kinematic constraint equations, are always in terms of kinematic variables alone. On the other hand, the constraint equations for δW_{II}, namely the force equilibrium equations, are not always in terms of forces alone. In this example for instance, the kinematic quantity θ appears explicitly in the equilibrium equation $F_2 + F_1\tan\theta = 0$. In evaluating the relation amongst the δI_i from equilibrium equations, the kinematic quantities are treated as being fixed.

Problem

1.7.1 Use $\delta W_{II} = \mathbf{v}_I \cdot \delta \mathbf{I}_I = 0$ to obtain the kinematic constraint relation between the velocities $\dot{y}_1$ and $\dot{y}_2$ of points 1 and 2 of the block and tackle mechanism shown.



1.8 Energies and Complementary Energies

Potential Energy* and *Complementary Potential Energy

We have seen that in some cases, the first form of increment of work, in equation (1.7.1) turns out as the negative of an exact differential of the potential energy function V. In this case

$$dV = -\mathbf{F} \cdot d\mathbf{r}$$

and for the *potential energy*

$$V = -\int \mathbf{F} \cdot d\mathbf{r} \tag{1.8.1}$$

Now before the integration in equation (1.8.1) can be carried out, the force vector $\mathbf{F}$ must be expressed in terms of the displacement $\mathbf{r}$. This relation should be single valued, that is, for a given $\mathbf{r}$ there should exist a unique $\mathbf{F}$. Such a relation can often be inverted, yielding $\mathbf{r} = \mathbf{r}(\mathbf{F})$. This inverted relation is unfortunately not always single valued. Nevertheless, if the inverted relation is available, one can, according to Engesser† (1889), define a *complementary potential energy* (or a *potential coenergy*) function as

$$V^* = -\int \mathbf{r} \cdot d\mathbf{F} \tag{1.8.2}$$

† Friedrich Engesser (1848-1931), German engineer

Example 1.8.1

Consider a non-linear spring for which the force displacement relation (Figure 1.27) is given by

$$F = -h\ u^3 \tag{a}$$

where u is the extension of the spring.

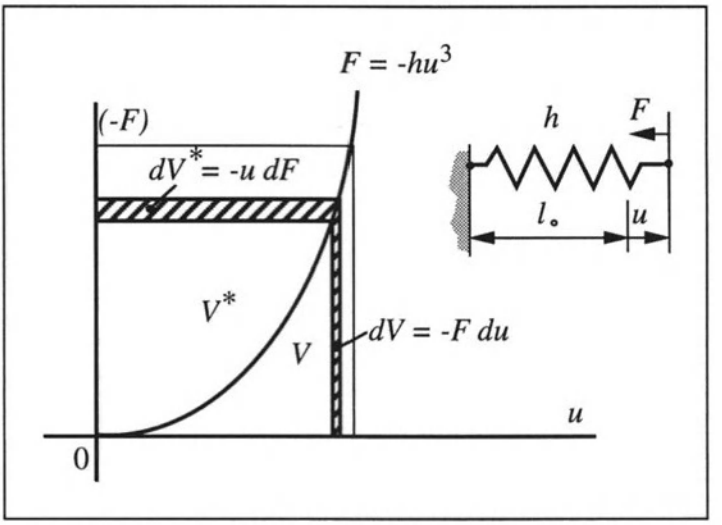


Figure 1.27 Nonlinear cubic stiffening spring

Now we may express the increment of work done by the spring force as

$$dW = F\ du$$

or using equation (a),

$$dW = -h\ u^3\ du$$

Then the potential energy of the spring may be computed as

$$V = -\int dW = \int h\ u^3\ du = \frac{h\ u^4}{4} \tag{b}$$

Now we may invert relation (a) and evaluate the complementary potential energy function as

$$V^* = -\int u\ dF = \int \left[\frac{F}{h}\right]^{1/3} dF = \frac{3}{4}\ \frac{F^{4/3}}{h^{1/3}} \tag{c}$$

From equations (b) and (c), it can be seen that

$$\frac{\partial V}{\partial u} = h\ u^3 = -F \qquad \text{and} \qquad \frac{\partial V^*}{\partial F} = \left[\frac{F}{h}\right]^{1/3} = -u$$

Thus both V and V^* embody the constitutive relation (a) but in terms of different variables. As such, these relations are not independent and from Figure 1.27, it can be seen that

$$dV + dV^* = -(F du + u dF)$$

or

$$d(V + V^*) = -d(Fu)$$

i.e.

$$V + V^* = -Fu \tag{d}$$

In the special case when the force deformation relation is linear and zero deformation corresponds to zero force, $V = V^*$, but in general $V \neq V^*$. In the above example if V^* is expressed in terms of u via equation (a), and the result is compared with V, it will be found that $V^* = 3V$.

Kinetic Energy and *Complementary Kinetic Energy*

Consider again the definition (1.7.1) of an increment of work,

$$dW = \mathbf{F} \cdot d\mathbf{r}$$

Using Euler's linear momentum law (1.1.2), we may write

$$dW = \mathbf{F} \cdot d\mathbf{r} = \frac{d\mathbf{B}}{dt} \cdot d\mathbf{r} = \frac{d\mathbf{r}}{dt} \cdot d\mathbf{B} = \mathbf{v} \cdot d\mathbf{B} \tag{1.8.3}$$

In analogy with the definition of the potential energy V, in equation (1.8.1), we now define a *kinetic energy* T of a single particle as

$$T = \int \mathbf{v} \cdot d\mathbf{B} \tag{1.8.4}$$

To carry out the integration we need to express $\mathbf{v}$ as function of $\mathbf{B}$. This relation can be inverted, and once again in analogy with the definition of the complementary potential energy V^*, in equation (1.8.2), we define a *complementary kinetic* energy (or *kinetic coenergy*) function T^* as

$$T^* = \int \mathbf{B} \cdot d\mathbf{v} \tag{1.8.5}$$

If N point masses are involved in a system, the kinetic energy and coenergy are defined as the sums

$$T = \int \mathbf{v}_k \cdot d\mathbf{B}_k \tag{1.8.6}$$

$$T^* = \int \mathbf{B}_k \cdot d\mathbf{v}_k \tag{1.8.7}$$

A particle m moving in three-dimensional space would have a kinetic energy of

$$T = \int \dot{x}\, dB_x + \int \dot{y}\, dB_y + \int \dot{z}\, dB_z \tag{1.8.8}$$

and, with $\dot{x} = B_x/m$, $\dot{y} = B_y/m$, $\dot{z} = B_z/m$,

$$T = \frac{1}{2m}(B_x^2 + B_y^2 + B_z^2) = \frac{B^2}{2m} \tag{1.8.9}$$

Its kinetic coenergy would be

$$T^* = \int B_x\, d\dot{x} + \int B_y\, d\dot{y} + \int B_z\, d\dot{z} \tag{1.8.10}$$

and, with $B_x = m\dot{x}$, $B_y = m\dot{y}$, $B_z = m\dot{z}$

$$T^* = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} mv^2 \tag{1.8.11}$$

Equation (1.8.11) shows that the function $\frac{1}{2} mv^2$, which is usually referred to as "kinetic energy", is strictly speaking the *complementary* kinetic energy.

Adding equations (1.8.4) and (1.8.5) gives

$$T + T^* = \int (\mathbf{v} \cdot d\mathbf{B} + \mathbf{B} \cdot d\mathbf{v}) = \int d(\mathbf{v} \cdot \mathbf{B}) \tag{1.8.12}$$

The right side of equation (1.8.12) contains a total differential. Thus an integration can be carried out to give

$$T + T^* = \mathbf{v} \cdot \mathbf{B} \tag{1.8.13}$$

Also note that

$$\frac{\partial T^*}{\partial v_j} = B_j \tag{1.8.14}$$

and

$$\frac{\partial T}{\partial B_j} = v_j \tag{1.8.15}$$

For rotational motion the kinetic energy is given by

$$T = \int \boldsymbol{\omega} \cdot d\mathbf{H} \tag{1.8.16}$$

and the kinetic coenergy by

$$T^* = \int \mathbf{H} \cdot d\boldsymbol{\omega} \tag{1.8.17}$$

where $\mathbf{H}$ is the angular momentum vector, and $\boldsymbol{\omega}$ the angular velocity vector.

A point mass m on a circular path with centre 0 in the xy plane, would have an angular momentum of

$$\mathbf{H} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} 0 \\ 0 \\ C\omega \end{bmatrix} \tag{1.8.18}$$

where $C = mr^2$ is the inertia moment, and an angular velocity of

$$\boldsymbol{\omega} = [\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \tag{1.8.19}$$

such that the kinetic energy becomes

$$T = \int \boldsymbol{\omega} \cdot d\mathbf{H} = \int \frac{H}{C} dH = \frac{1}{2} \frac{H^2}{C} \tag{1.8.20}$$

and the kinetic coenergy

$$T^* = \int \mathbf{H} \cdot d\boldsymbol{\omega} = \int C \, \omega \, d\omega = \frac{1}{2} C\omega^2 \tag{1.8.21}$$

Expressions for kinetic energy and kinetic coenergy of a rigid body in 3D space are derived in chapter V.

The examples following are intended to bring to light the physical nature of the functions T and T^*.

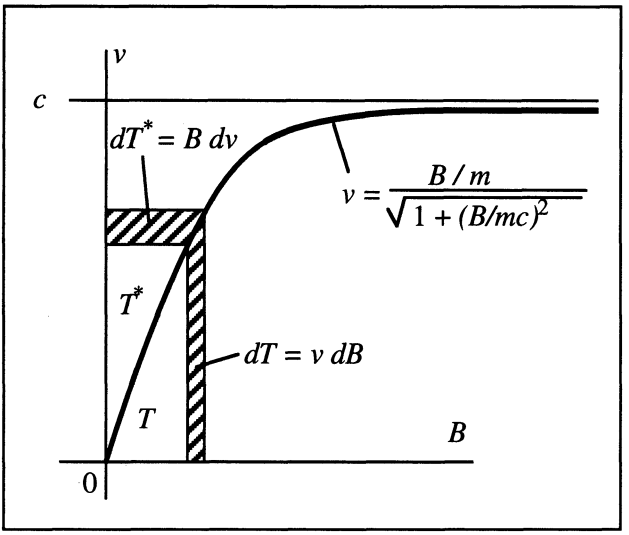


Figure 1.28 Velocity versus momentum plot for relativistic mechanics

Example 1.8.2

Let us have a look at the kinetic energy and the complementary kinetic energy of a single mass point of constant mass m, in Newtonian and relativistic mechanics.

For speeds much lower than the speed of light, linear momentum and linear velocity are related by

$$v = \frac{B}{m} \tag{a}$$

Hence

$$T = \frac{B^2}{2}m \qquad \text{and} \qquad T^* = \frac{1}{2}mv^2 \tag{b,c}$$

Now for the linear relation (a) which holds in Newtonian mechanics, one can see that $T = T^*$, but in relativistic mechanics $T \neq T^*$. In the special theory of relativity (Figure 1.28)

$$v = \frac{B/m}{\sqrt{1 + B^2/m^2c^2}} \tag{d}$$

where c (= 299 792.458 km/s) is the speed of light, and one can easily show that

$$T = \int v\, dB = mc^2\left[\sqrt{1 + \frac{B^2}{m^2c^2}} - 1\right] \tag{e}$$

$$T^* = \int B\, dv = mc^2\left[1 - \sqrt{1 - \frac{v^2}{c^2}}\right] \tag{f}$$

By differentiating equations (b) and (c) or (e) and (f), and using the associated relation (a) or (d), we find

$$\frac{\partial T}{\partial B} = v \qquad \frac{\partial T^*}{\partial v} = B \tag{g,h}$$

At the speed of light, i.e. for $v = c$, the complementary kinetic energy becomes

$$T^* = mc^2 \tag{i}$$

Since, from equation (f),

$$B = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{j}$$

we find $B \to \infty$ for $v \to c$, and thus for $v = c$, the kinetic energy becomes

$$T = \infty \tag{k}$$

Without the integration constant $-mc^2$, introduced to make $T = 0$ when $B = 0$ in equation (e), the integral of equation (e) would give

$$E = mc^2 \sqrt{1 + \frac{B^2}{m^2c^2}} \tag{l}$$

At rest $B = 0$, and the *rest energy* is then given by Einstein's celebrated equation

$$E_o = mc^2 \tag{m}$$

From a mass at rest, energy can, by the way, still be withdrawn by changing the mass.

$$\Delta E_o = \Delta mc^2 \tag{n}$$

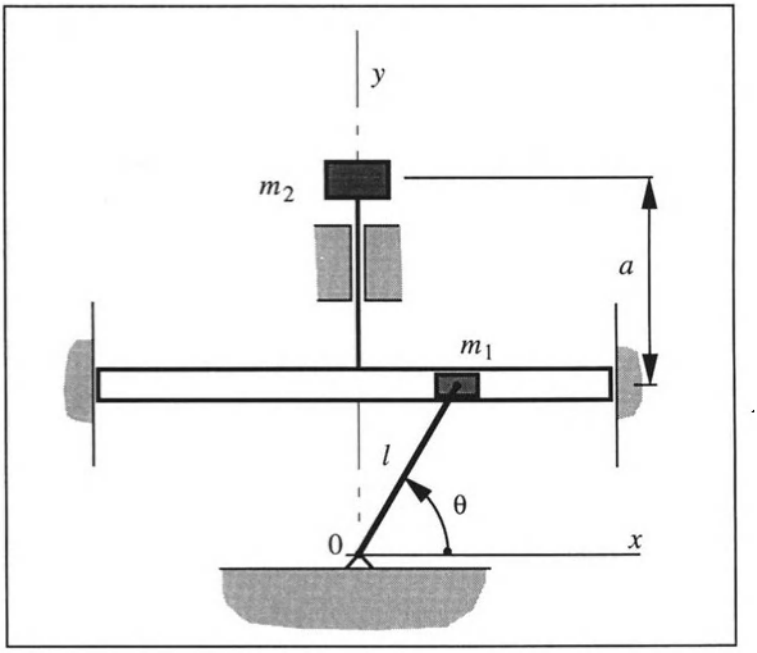
e.g. in a nuclear reaction, a small amount Δm of mass is converted into an amount ΔE_o of energy (which happens to be quite large by virtue of the large value of the speed of light!).

Problems

1.8.1 For a system of particles, show that the complementary kinetic energy may be expressed as the sum of: i) Complementary kinetic energy due to a single particle having a mass equal to the total mass of the system and moving with the velocity of the system's centre of mass, ii) Complementary kinetic energy due to the motion of the particles relative to the system's centre of mass. This is known as König's† theorem.

1.8.2 Assuming that the concentrated masses m_1 and m_2 represent all masses and that consequently the link l is massless, find expressions for the complementary kinetic energy T^* and the kinetic energy T of the crank mechanism shown, i.e. T^* in terms of $\dot{\theta}$, and T in terms of p_θ.

† Samuel König (1712-1757), German mathematician



1.8.3 Establish (a) the potential energy V, and (b) the complementary potential energy V^* of the turning moment M caused by the weight W of the mass at the tip of a simple pendulum of length l, for the generalized coordinate θ. (c) Plot an M versus θ diagram. (d) What are V and V^*, at $\theta = 0°$, at $\theta = 90°$, at $\theta = 180°$?

1.8.4 For the Kepler central force F between two point masses m and M, given by

$$F = -\frac{GMm}{r^2}$$

where G is the universal gravitational constant ($G = 6.672 \times 10^{-11}$ $m^3/kg\ s^2$), determine (a) the potential energy $V(r)$, (b) the complementary potential energy $V^*(F)$, and (c) show that

$$V + V^* = \frac{GMm}{r}$$

1.9 Kinetic Energy and Complementary Kinetic Energy in Terms of Generalized Displacements

Let the Cartesian coordinates r_j of a system of N particles be related to the n generalized coordinates q_i through the transformation equations as

$$r_j = r_j(q_1, q_2, \ldots, q_i, \ldots, q_n, t) \qquad j = 1,2,3,\ldots,3N$$

Time will appear in these transformation equations explicitly if a moving reference frame is used and/or rheonomic constraints are present.

The velocities can now be determined as

$$\dot{r}_j = \frac{\partial r_j}{\partial q_i}\dot{q}_i + \frac{\partial r_j}{\partial t} \qquad i = 1,2,3,\ldots, n \qquad (1.9.1)$$

or in matrix form

$$\{\dot{r}\} = [J]\{\dot{q}\} + \{r_{,t}\} \qquad (1.9.2)$$

where the $\dot{q}_i$ are scalar quantities and represent *generalized speeds*. The term *generalized velocities* for the $\dot{q}_i$ is also in wide-spread use. The complementary kinetic energy is

$$T^* = \frac{1}{2}\{\dot{r}\}^T[m]\{\dot{r}\} \qquad (1.9.3)$$

where $\{\dot{r}\}^T = [\,\dot{x}_1,\ \dot{y}_1\ \dot{z}_1,\ \dot{x}_2,\ \dot{y}_2,\ \dot{z}_2,\ \dot{x}_3,\ldots,\ \dot{z}_N\,]$

From equations (1.9.2) and (1.9.3)

$$T^* = \frac{1}{2}\Big(\{r_{,t}\}^T + \{\dot{q}\}^T[J]^T\Big)[m]\Big([J]\{\dot{q}\} + \{r_{,t}\}\Big)$$

or

$$T^* = \frac{1}{2}\{\dot{q}\}^T[J]^T[m][J]\{\dot{q}\} + \{r_{,t}\}^T[m][J]\{\dot{q}\} + \frac{1}{2}\{r_{,t}\}^T[m]\{r_{,t}\} \qquad (1.9.4)$$

It is of interest to note that the expression for T^*, in terms of generalized coordinates, turns out to be in three parts. These may be designated T_2^*, T_1^* and T_o^*, such that equation (1.9.4) may also be written

$$T^* = T_2^* + T_1^* + T_o^*$$

with

$$\begin{aligned} T_2^* &= \frac{1}{2}\{\dot{q}\}^T[A]\{\dot{q}\} && \text{quadratic in generalized speeds} \\ T_1^* &= \{b\}^T\{\dot{q}\} = \{\dot{q}\}^T\{b\} && \text{linear in generalized speeds} \\ T_o^* &= C && \text{independent of generalized speeds} \end{aligned} \qquad (1.9.5)$$

and

$$\begin{aligned} [A] &= [J]^T[m][J] && \text{a symmetric matrix} \\ \{b\} &= [J]^T[m]\{r_{,t}\} && \text{a column matrix} \\ C &= \frac{1}{2}\{r_{,t}\}^T[m]\{r_{,t}\} && \text{a scalar} \end{aligned} \qquad (1.9.6)$$

It is also of interest to note that T^* is a positive definite function of $\{\dot{r}\}$, as one can see from equation (1.9.3). However, T^* is not a positive definite function of $\{\dot{q}\}$, since T_1^* is linear in $\dot{q}_i$ and hence its sign depends upon the sign of $\dot{q}_i$.

When the transformation equations (1.9.1) do not include time explicitly (scleronomic systems), T_1^* and T_o^* will vanish and in this case, $T^*(=T_2^*)$ will be a positive definite function of $\dot{q}_i$.

As in the case of Cartesian coordinates, for linear momenta (1.8.16), we may define a set of *generalized momenta* p_i as

$$p_i = \frac{\partial T^*}{\partial \dot{q}_i} = \frac{\partial T_2^*}{\partial \dot{q}_i} + \frac{\partial T_1^*}{\partial \dot{q}_i} \tag{1.9.7}$$

Hence

$$\{p\} = [A]\{\dot{q}\} + \{b\} \tag{1.9.8}$$

Equation (1.9.8) shows that the linear relation between the generalized momenta and generalized velocities does not pass through the origin. The term $\{b\}$ can be looked upon as additional momenta due to the motion of the q coordinate system. It is also important to note that both $[A]$ and $\{b\}$ may be functions of the generalized coordinates q_i and time. Now in terms of the Cartesian coordinates r_j, we may define the momenta as

$$\{B\} = [m]\{\dot{r}\} = [m]\Big[[J]\{\dot{q}\} + \{r_{,t}\}\Big] \tag{1.9.9}$$

From equation (1.9.9), we find that

$$[J]\{\dot{q}\} = [m]^{-1}\{B\} - \{r_{,t}\} \tag{1.9.10}$$

On substituting $[J]\{\dot{q}\}$ from this equation into equation (1.9.8) and using equation (1.9.6), we find the transformation equation for the momenta as

$$\{p\} = [J]^T\{B\} \tag{1.9.11}$$

where $\{p\}$ are the *generalized* momenta (1.9.8) and $\{B\}$ are the *Cartesian* linear momenta (1.9.9).

Equations (1.9.2) and (1.9.11) show that apart from the term $\{r_{,t}\}$, the velocities and momenta follow congruent transformation laws. Finally, let us consider the sum of the kinetic energy and complementary kinetic energy functions, i.e.

$$T + T^* = \{B\}^T\{\dot{r}\} \tag{1.9.12}$$

On substituting for $\{\dot{r}\}$ from equation (1.9.2), we may write

$$T + T^* = \{B\}^T\Big[[J]\{\dot{q}\} + \{r_{,t}\}\Big] \tag{1.9.13}$$

which on account of equation (1.9.11) may be written as

$$T + T^* = \{p\}^T\{\dot{q}\} + \{B\}^T\{r_{,t}\} \tag{1.9.14}$$

Equations (1.9.12) and (1.9.14) show that the product of momenta and velocities is not, in general, an invariant of the transformation. Only when time does not appear in the transformation equations explicitly, does this product remain invariant. Finally, let us remove $\{B\}$ from equation (1.9.14) by the use of equations (1.9.9) and (1.9.4). When this is done, we find

$$T + T^* = \{p\}^T\{\dot{q}\} + T_1^* + 2T_o^*$$

Hence

$$\{p\}^T\{\dot{q}\} = T + (T_2^* + T_1^* + T_o^*) - (T_1^* + 2T_o^*)$$

or

$$\{p\}^T\{\dot{q}\} = T + T_2^* - T_o^* \tag{1.9.15}$$

To determine T in terms of $\{p\}$ and $\{r_{,t}\}$, we may eliminate $\dot{q}_i$ from equation (1.9.14) using the momentum-velocity relation of equation (1.9.8). When this is done, we find

$$T = \frac{1}{2}\{p\}^T[A]^{-1}\{p\} - \frac{1}{2}\{b\}^T[A]^{-1}\{b\} + T_o^* \tag{1.9.16}$$

Noting that $[A]$ and $\{b\}$ are functions of q_i and time, one can see that in general, the kinetic energy T is a function of the generalized momenta p_i, generalized coordinates q_i, and time. Since in Newtonian mechanics $T = T^*$, we may also determine T by eliminating $\dot{q}_i$ from T^* via equation (1.9.8).

It is of interest to note that

$$\left\{\frac{\partial T}{\partial p}\right\} = [A]^{-1}\{p\} \tag{1.9.17}$$

is not the relative generalized velocity $\{\dot{q}\}$ but rather the *absolute generalized* velocity $\left(\{\dot{q}\} + [A]^{-1}\{b\}\right)$, as becomes evident from equations (1.9.17) and (1.9.8).

Example 1.9.1

A particle can slide without friction on a rigid wire bent in the form of a circle of radius a. The wire rotates about its vertical diameter with constant angular velocity ω, and it moves along the x axis with constant velocity v as shown in Figure 1.29. Determine the complementary kinetic energy and the kinetic energy of the particle in terms of the generalized variables.

In this problem, we have the following holonomic constraints

$$(x - vt)^2 + y^2 + z^2 = a^2 \tag{a}$$

$$y = (x - vt)\tan\omega t$$

The system has one degree of freedom. The sole generalized coordinate shall be $q_1 = \theta$. The reduced transformation equations for this problem take the form

$$\begin{aligned} r_x &= x = a\sin\theta\cos\omega t + vt \\ r_y &= y = a\sin\theta\sin\omega t \\ r_z &= z = -a\cos\theta \end{aligned} \tag{b}$$

The velocities may now be derived as

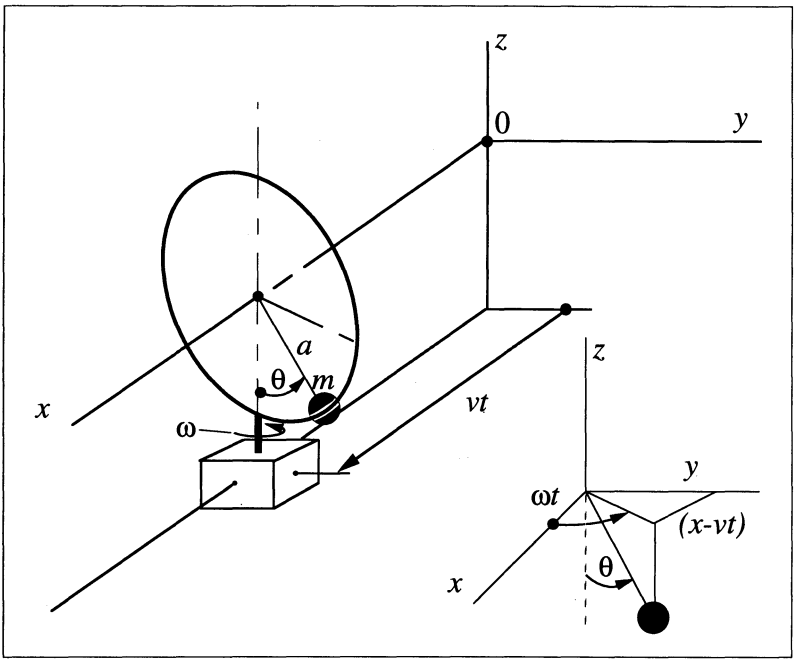


Figure 1.29 Bead on a rotating circular frame

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = a\dot{\theta}\begin{bmatrix} \cos\theta\cos\omega t \\ \cos\theta\sin\omega t \\ \sin\theta \end{bmatrix} + \begin{bmatrix} -a\omega\sin\theta\sin\omega t + v \\ a\omega\sin\theta\cos\omega t \\ 0 \end{bmatrix} \tag{c}$$

Thus

$$[J] = a\begin{bmatrix} \cos\theta\cos\omega t \\ \cos\theta\sin\omega t \\ \sin\theta \end{bmatrix} \quad \text{and} \quad \{r_{,t}\} = \begin{bmatrix} -a\omega\sin\theta\sin\omega t + v \\ a\omega\sin\theta\cos\omega t \\ 0 \end{bmatrix}$$

The Cartesian momenta are given by $B_x = m\dot{x}$, $B_y = m\dot{y}$, $B_z = m\dot{z}$, i.e.

$$\{B\} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} m\dot{x} \\ m\dot{y} \\ m\dot{z} \end{bmatrix} = ma\dot{\theta}\begin{bmatrix} \cos\theta\cos\omega t \\ \cos\theta\sin\omega t \\ \sin\theta \end{bmatrix} + \begin{bmatrix} -ma\omega\sin\theta\sin\omega t + mv \\ ma\omega\sin\theta\cos\omega t \\ 0 \end{bmatrix} \tag{d}$$

Consider next the complementary kinetic energy given in Cartesian coordinates by

$$T^* = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{e}$$

On using the velocities in equation (c), we find $T^* = T^*(\theta, \dot{\theta}, t)$ as

$$T^* = \frac{1}{2}m\,a^2\dot{\theta}^2 + \frac{1}{2}m\,a^2\omega^2\sin^2\theta + \frac{1}{2}mv^2 + mav(\dot{\theta}\cos\theta\cos\omega t - \omega\sin\theta\sin\omega t) \tag{f}$$

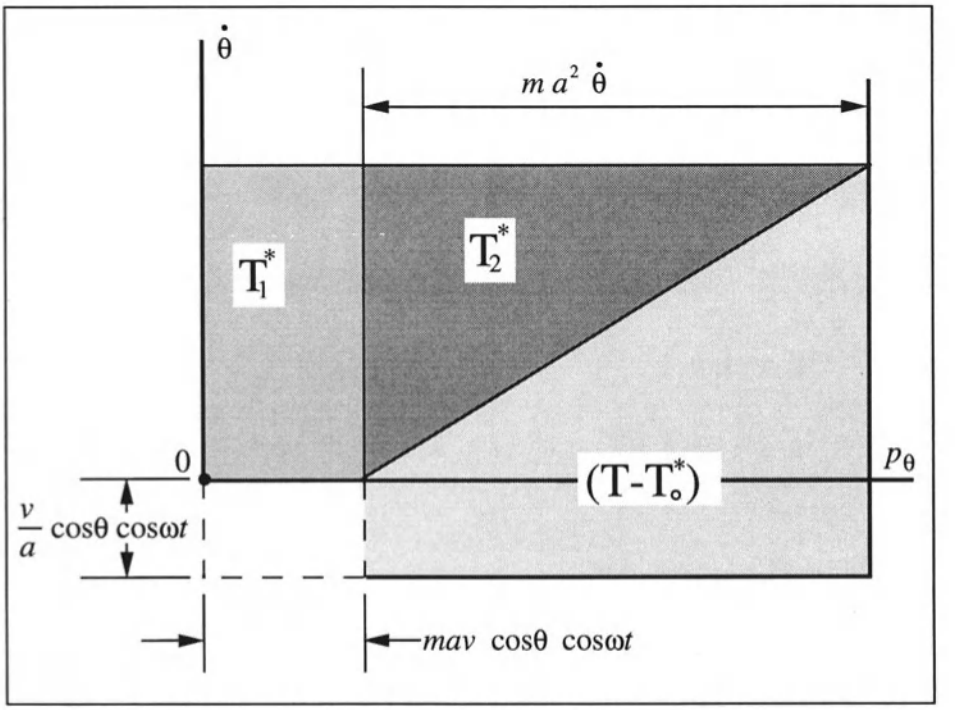


Figure 1.30 Energy and coenergy terms for example 1.9.1

thus

$$T_2^* = \frac{1}{2} ma^2 \dot{\theta}^2 \qquad \text{(a function of } \dot{\theta}^2\text{)} \qquad \text{(g)}$$

$$T_1^* = mav\,\dot{\theta} \cos\theta \cos\omega t \qquad \text{(a function of } \dot{\theta}\text{)} \qquad \text{(h)}$$

$$T_o^* = \frac{1}{2} m\, a^2\, \omega^2 \sin^2\theta + \frac{1}{2} mv^2 - mav\,\omega \sin\theta \sin\omega t \qquad \text{(not a function of } \dot{\theta}\text{)} \qquad \text{(i)}$$

The generalized momentum p_θ can now be derived from T^* as

$$p_\theta = \frac{\partial T^*}{\partial \dot{\theta}} = m\, a^2\, \dot{\theta} + mav \cos\theta \cos\omega t \qquad \text{(j)}$$

or, from $\{p\} = [J]^T \{B\}$, as

$$p_\theta = \begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = m\, a^2\, \dot{\theta} + mav \cos\theta \cos\omega t \qquad \text{(k)}$$

For this example, it is evident that $A = ma^2$ and $b = mav \cos\theta \cos\omega t$. Using equation (1.9.15) the kinetic energy $T = T(\theta, p_\theta, t)$ can be expressed as

$$T = \frac{1}{2} \frac{p_\theta^2}{ma^2} - \frac{1}{2} \frac{(mav \cos\theta \cos\omega t)^2}{ma^2} + \frac{1}{2} ma^2\omega^2 \sin^2\theta + \frac{1}{2} mv^2 - mav\,\omega \sin\theta \sin\omega t \qquad \text{(l)}$$

The absolute generalized velocity

$$\frac{\partial T}{\partial p_\theta} = [A]^{-1}\{p\} = \frac{1}{ma^2} p_\theta = \dot{\theta} + \frac{v}{a}\cos\theta\cos\omega t \qquad \text{(m)}$$

Figure 1.30 shows the various kinetic energy terms in a velocity-momentum diagram.

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Chapter II

DIFFERENTIAL VARIATIONAL FORMULATIONS

2.1 D'Alembert's Principle

D'Alembert† brought the problems of motion under the umbrella of problems of equilibrium. Using Euler's linear momentum law (1.1.2), we have that, when a point mass is acted upon by a set of forces resulting in **F**, it will acquire a linear momentum **B**, such that

$$\mathbf{F} = \dot{\mathbf{B}} \tag{2.1.1}$$

Consider now an increment of work done by the force **F**, as well as by D'Alembert's inertial force $-\dot{\mathbf{B}}$, as the point mass undergoes an imagined virtual displacement $\delta\mathbf{r}$. Then

$$\delta W_I = (\mathbf{F} - \dot{\mathbf{B}}) \cdot \delta\mathbf{r} = 0 \tag{2.1.2}$$

because of relation (2.1.1).

If virtual displacements $\delta\mathbf{r}$ are chosen such that they are kinematically admissible, i.e. they satisfy the constraint equations, and for which time is held fixed, then the constraint forces will not contribute to δW_I, and need not be included in **F**, effecting a noticeable simplification.

For the case of a system of N particles, the virtual work expression (2.1.2) becomes

$$\delta W_I = (\mathbf{F}_k - \dot{\mathbf{B}}_k) \cdot \delta\mathbf{r}_k = 0 \qquad k = 1, 2, 3, \ldots, N \tag{2.1.3}$$

D'Alembert's principle is *inertial element focussed*, i.e. the forces $\mathbf{F}_k$ are those impressed upon the inertial elements (masses) whose momentum derivatives are $\dot{\mathbf{B}}_k$.

For a point mass m, $\dot{\mathbf{B}} = m\,\ddot{\mathbf{r}}$, and D'Alembert's principle (2.1.3) can be expressed as

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† Jean LeRond D'Alembert (1717-1783), French mathematician (Traité de dynamique, 1758)

$$\delta W_I = (\mathbf{F}_k - (m\ddot{\mathbf{r}})_k) \cdot \delta \mathbf{r}_k = 0 \tag{2.1.4}$$

If the virtual displacements $\delta \mathbf{r}_k$ are independent, then their coefficients may be equated to zero for conditions of equilibrium, which in this case, become the equations of motion in terms of displacements.

To use D'Alembert's principle in generalized displacement variables q_i, we need to invoke the transformation equations

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2, q_3, ..., q_i, ..., q_n, t)$$

$$\delta \mathbf{r}_k = \frac{\partial \mathbf{r}_k}{\partial q_i} \delta q_i \tag{2.1.5}$$

$$\dot{\mathbf{r}}_k = \frac{\partial \mathbf{r}_k}{\partial q_i} \dot{q}_i + \frac{\partial \mathbf{r}_k}{\partial t} \qquad k = 1, 2, 3, ..., N \tag{2.1.6}$$

$$\ddot{\mathbf{r}}_k = \frac{\partial \mathbf{r}_k}{\partial q_i} \ddot{q}_i + \frac{\partial^2 \mathbf{r}_k}{\partial q_i \partial q_j} \dot{q}_i \dot{q}_j + 2\frac{\partial^2 \mathbf{r}_k}{\partial q_i \partial t} \dot{q}_i + \frac{\partial^2 \mathbf{r}_k}{\partial t^2} \qquad i, j = 1, 2, 3, ..., n \tag{2.1.7}$$

Hence in terms of generalized displacement variables, we may write D'Alembert's principle as

$$\delta W_I = \left[\mathbf{F}_k - \left(\frac{\partial^2 (m\mathbf{r})_k}{\partial q_i \partial q_j} \dot{q}_i \dot{q}_j + \frac{\partial (m\mathbf{r})_k}{\partial q_i} \ddot{q}_i + 2\frac{\partial^2 (m\mathbf{r})_k}{\partial q_i \partial t} \dot{q}_i + \frac{\partial^2 (m\mathbf{r})_k}{\partial t^2} \right) \right] \cdot \left[\frac{\partial \mathbf{r}_k}{\partial q_i} \delta q_i \right] = 0 \tag{2.1.8}$$

We shall see later, in article 2.6, that above form of virtual work can be expressed in a much more convenient form, given by Lagrange.

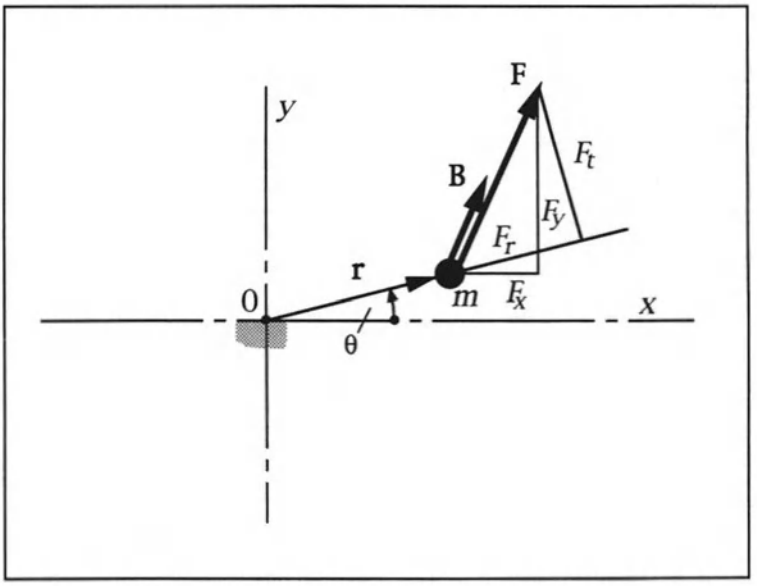


Figure 2.1 Particle moving in a horizontal plane

Example 2.1.1

Consider the point mass m moving in a horizontal plane, as shown in Figure 2.1, and subjected to a force **F**. The equations of motion are to be established in polar coordinates.

The linear momentum components are

$$B_x = m\dot{x} = m(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) \tag{a}$$

$$B_y = m\dot{y} = m(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) \tag{b}$$

The time derivatives are

$$\dot{B}_x = m\ddot{x} = m(\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta) \tag{c}$$

$$\dot{B}_y = m\ddot{y} = m(\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta) \tag{d}$$

D'Alembert's principle (2.1.3) requires that

$$(F_x - m\ddot{x})\delta x + (F_y - m\ddot{y})\delta y = 0 \tag{e}$$

Since

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{f,g}$$

we find

$$\delta x = \cos\theta\,\delta r - r\sin\theta\,\delta\theta \qquad \text{and} \qquad \delta y = \sin\theta\,\delta r + r\cos\theta\,\delta\theta \tag{h,i}$$

Further we may introduce radial and transverse force components given by

$$F_r = F_x\cos\theta + F_y\sin\theta \qquad \text{and} \qquad F_t = -F_x\sin\theta + F_y\cos\theta \tag{j,k}$$

Now expressing equation (e) in terms of generalised coordinates $q_1 = r$ and $q_2 = \theta$, we obtain for D'Alembert's principle

$$(F_r - m(\ddot{r} - r\dot{\theta}^2))\delta r + (F_t - m(r\ddot{\theta} + 2\dot{r}\dot{\theta}))\delta\theta = 0 \tag{l}$$

Since δr and $\delta\theta$ are arbitrary, we conclude that

$$F_r - m(\ddot{r} - r\dot{\theta}^2) = 0 \qquad \text{and} \qquad F_t - m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{m,n}$$

are the equations of motion.

Example 2.1.2

The equations of motion of the oscillator chain of Figure 2.2 are to be obtained via D'Alembert's principle.

Equation (2.1.3) with $k = a, b, c$, requires that

$$\delta W_I = (F_a - \dot{B}_a)\,\delta x_a + (F_b - \dot{B}_b)\,\delta x_b + (F_c - \dot{B}_c)\,\delta x_c = 0 \tag{a}$$

The free-body diagram of Figure 2.2 allows us to write for the virtual work

$$\delta W_I = (F_1 - F_2 - \dot{B}_a)\delta x_a + (F_2 - F_3 - \dot{B}_b)\delta x_b + (F_3 - F_4 - \dot{B}_c)\delta x_c = 0 \tag{b}$$

On comparing with equation (a) we note that in the present case, apart from a change in sign, the forces acting on the masses are:

$$F_a = F_1 - F_2 \qquad F_b = F_2 - F_3 \qquad F_c = F_3 - F_4 \tag{c}$$

Now from constitutive equations for the force elements and for the inertial elements we have that

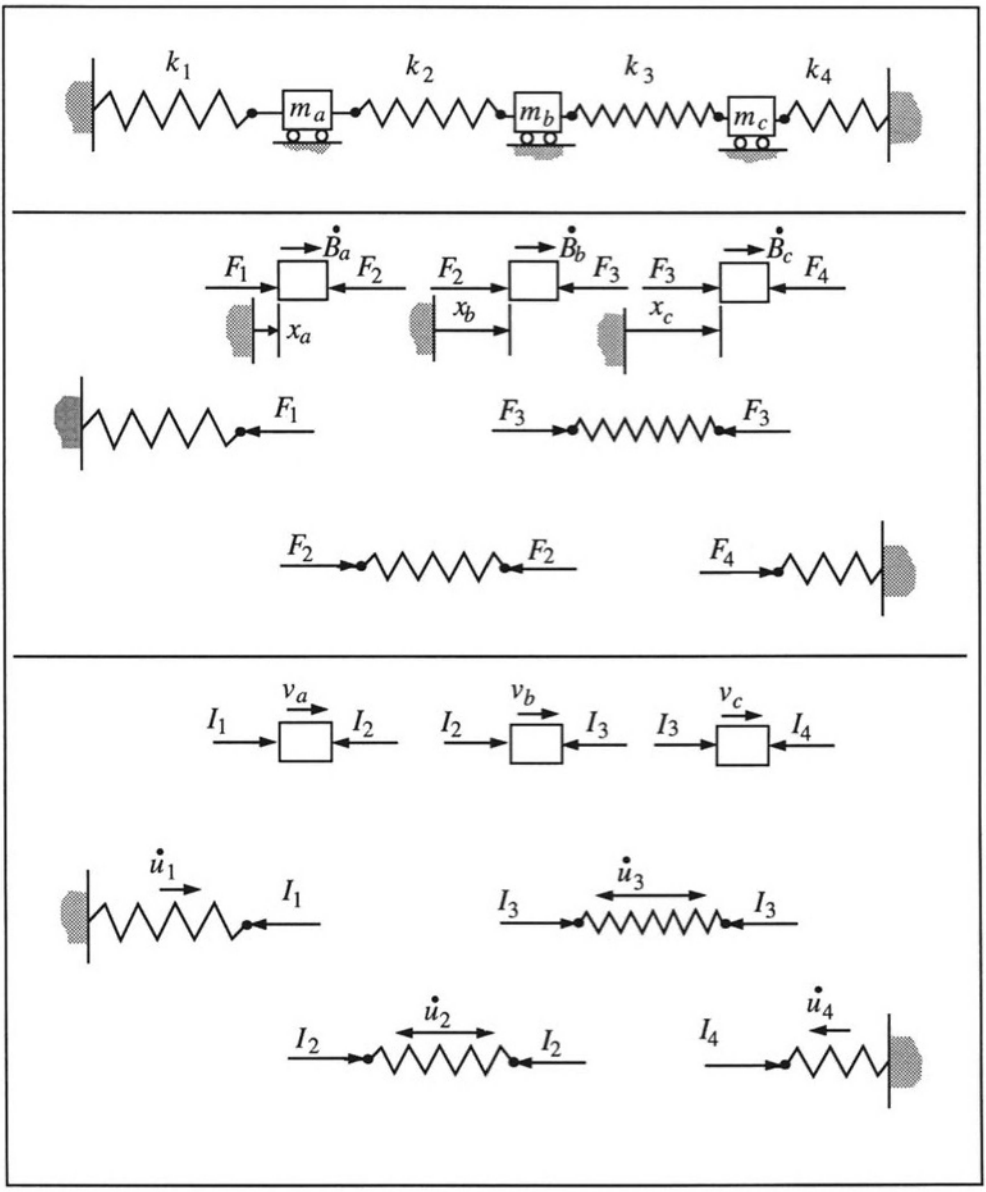


Figure 2.2 A 3-mass, 4-spring oscillator chain

$$F_1 = -k_1 x_a \qquad F_2 = -k_2(x_b - x_a)$$
$$F_3 = -k_3(x_c - x_b) \qquad F_4 = k_4 x_c \tag{d}$$

and

$$B_a = m\dot{x}_a \qquad B_b = m\dot{x}_b \qquad B_c = m\dot{x}_c \tag{e}$$

On substituting from equations (e), (d), and (c) into equation (b) and noting that δx_a, δx_b and δx_c are independent and arbitrary, we obtain the following equations of motion:

$$\begin{bmatrix} m_a & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & m_c \end{bmatrix} \begin{bmatrix} \ddot{x}_a \\ \ddot{x}_b \\ \ddot{x}_c \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{f}$$

Example 2.1.3

Let us derive the equilibrium equation of motion of the particle sliding on the rotating ring as shown in Figure 1.29, by D'Alembert's principle.

For the force of gravity we have the virtual work

$$\mathbf{F} \cdot \delta\mathbf{r} = -mg\,\delta z \tag{a}$$

In this case it is possible to define a potential energy V as

$$V = mg\,(z + a) \tag{b}$$

Introducing the generalized coordinate θ, we have $z = -a\cos\theta$, then

$$V = mga(1 - \cos\theta) \tag{c}$$

and

$$\delta V = mga\sin\theta\,\delta\theta = -\mathbf{F} \cdot \delta\mathbf{r} \tag{d}$$

Next let us derive the inertial forces by differentiating the Cartesian momenta B_x, B_y and B_z with respect to time. Using the expressions we had derived for these momenta in example 1.9.1, we find that

$$\begin{bmatrix} \dot{B}_x \\ \dot{B}_y \\ \dot{B}_z \end{bmatrix} = ma\ddot{\theta}\begin{bmatrix} \cos\theta\cos\omega t \\ \cos\theta\sin\omega t \\ \sin\theta \end{bmatrix} - ma\dot{\theta}^2\begin{bmatrix} \sin\theta\cos\omega t \\ \sin\theta\sin\omega t \\ -\cos\theta \end{bmatrix} - 2\,ma\dot{\theta}\omega\begin{bmatrix} \cos\theta\sin\omega t \\ -\cos\theta\cos\omega t \\ 0 \end{bmatrix} - ma\omega^2\begin{bmatrix} \sin\theta\cos\omega t \\ \sin\theta\sin\omega t \\ 0 \end{bmatrix} \tag{e}$$

Finally, on using D'Alembert's principle (2.1.2) we may write

$$-\delta V - [\dot{B}_x\ \dot{B}_y\ \dot{B}_z]\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = 0 \tag{f}$$

The Cartesian virtual displacements in equation (f) may be expressed in terms of a single generalized $\delta\theta$ via the reduced transformation equations. To prevent the constraint forces from contributing to δW_I, time will be held fixed, i.e. $\delta t = 0$. Then

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\omega t \\ \cos\theta\sin\omega t \\ \sin\theta \end{bmatrix} a\ \delta\theta \tag{g}$$

Finally substituting from equations (e) and (g) into equation (f), and recognizing that $\delta\theta$ is arbitrary, we find the equilibrium equation of motion as

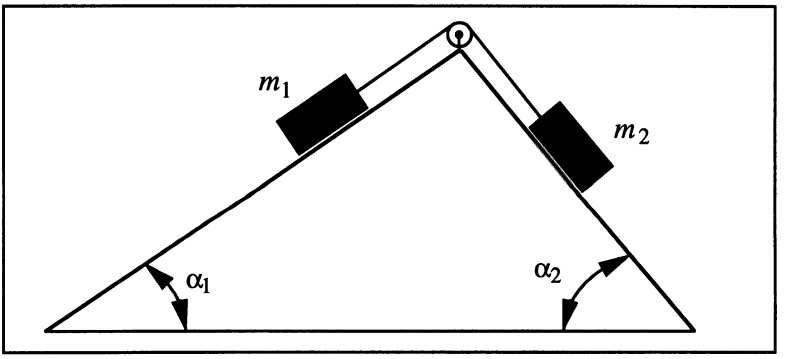
$$m\,a\,\ddot{\theta} \;-\; m\,a\,\omega^2 \sin\theta\cos\theta \;+\; mg\,\sin\theta \;=\; 0 \qquad \text{(h)}$$

It is of interest to note that while the constant translational velocity v appears in the momentum velocity relations, it is absent in the equations of motion. This is because a uniformly translating frame is itself an inertial frame.

In this example one may be tempted to use the form of D'Alembert's principle, as given in equation (2.1.3), for the generalized variables $\delta\theta$ and p_θ. Now we showed earlier in example 1.9.1 that $p_\theta = ma^2\,\dot{\theta} + mav\cos\theta\cos\omega t$. On differentiating the generalized momentum with respect to time and comparing the result with the inertial force in equation (h), we can see that $\dot{p}_\theta$ does not yield all the terms of the inertial force. That is, the time derivative of the momentum is equal to the inertial force only for inertial frames.

Problem

2.1.1 Two particles of masses m_1 and m_2 are attached by an inextensible string and placed on a double incline as shown. Using D'Alembert's principle determine the system's equation of motion. Neglect frictional forces.



2.2 Complementary Form of D'Alembert's Principle

By associating a force with the inertial effects, D'Alembert extended the range of applicability of δW_I from static problems to problems of motion. The inertial force arises as the time derivative of a particle's momentum and the conditions for the application of δW_I remain unchanged. That is, the virtual displacements must satisfy the kinematic constraints and for these displacements time is held fixed. The vanishing of δW_I then yields the equilibrium equations amongst the resultant applied force, and the inertial force.

Now it will be recalled that the second form of virtual work is made up of products of velocities and virtual impulses. In this case the virtual impulses must satisfy the equilibrium equations and for these impulses time is held fixed. The vanishing of δW_{II} then yields the conditions of compatibility or kinematic fit.

For an inertial element, such as a mass particle, we can readily write the expressions for δW_{II} in terms of the particle's velocity $\mathbf{v}$. For an elastic element, which is more naturally described in terms of its deformations, we need to introduce an associated velocity. This velocity is simply the time derivative $\dot{\mathbf{u}}$ of the deformation $\mathbf{u}$. Thus it can be seen that the deformations of elastic elements, in the expression for δW_{II}, are analogous to the momenta of the inertial elements in the expression for δW_I.

The complementary form of D'Alembert's principle thus requires the compatibility of the extension rates of the force elements with the velocities of the inertial elements. To illustrate this let us consider first a simple mass-spring oscillator. To satisfy the equilibrium conditions, which is the prerequisite of the complementary formulation, we set $\mathbf{I}$, the impulse in the spring, equal to $\mathbf{B}$, the momentum of the mass,

$$\mathbf{I} = \mathbf{B} \tag{2.2.1}$$

Hence we may write the complementary form of D'Alembert's principle, in terms of equilibrating impulses, as

$$\delta W_{II} = (\mathbf{v} - \dot{\mathbf{u}}) \cdot \delta \mathbf{I} = 0 \tag{2.2.2}$$

For an arbitrary $\delta \mathbf{I}$,

$$\mathbf{v} = \dot{\mathbf{u}} \tag{2.2.3}$$

For a system of N inertial elements (masses) and M force elements (springs), there will be M independent impulse variables, to write the virtual work of the spring extension rates. For the inertial elements, the net momenta can be expressed in terms of the impulses via the equilibrium conditions. For each of the N inertial elements we may relate the momenta to the impulses in the force elements through the equilibrium equations and write

$$\mathbf{B}_k = \mathbf{B}_k(\mathbf{I}_l) \qquad \begin{matrix} k = 1, 2, ..., N \\ j = 1, 2, ..., m \end{matrix} \tag{2.2.4}$$

where $\mathbf{I}_l$ denote the impulses in the m force elements connected to particle k†. Now we may express the complementary form of D'Alembert's principle

$$\delta W_{II} = \mathbf{v}_k \cdot \delta \mathbf{B}_k - \dot{\mathbf{u}}_l \cdot \delta \mathbf{I}_l = 0 \tag{2.2.5}$$

Expressing the momenta $\mathbf{B}_k$ in terms of impulses $\mathbf{I}_l$ via the equilibrium equations (2.2.4) we obtain

$$\delta W_{II} = \mathbf{v}_k \cdot \frac{\partial \mathbf{B}_k}{\partial \mathbf{I}_l} \cdot \delta \mathbf{I}_l - \dot{\mathbf{u}}_l \cdot \delta \mathbf{I}_l = 0 \tag{2.2.6}$$

To put this expression in the more familiar form shown for a single mass spring oscillator, as given in equation (2.2.2), we introduce a *resultant velocity* $\mathbf{v}_l$ such that

$$\mathbf{v}_k \cdot \left(\frac{\partial \mathbf{B}_k}{\partial \mathbf{I}_l}\right) = \mathbf{v}_l \tag{2.2.7}$$

---

† In the general case the momenta $\mathbf{B}_k$ may be functions of impulses *and* kinematic variables q and $\dot{q}$. We will consider this more general case in section 2.10.

Then the expression for the complementary form of D'Alembert's principle becomes

$$\delta W_{II} = (\mathbf{v}_l - \dot{\mathbf{u}}_l) \cdot \delta \mathbf{I}_l = 0 \qquad (2.2.8)$$

with $l = 1, 2, 3, ..., M$.

Since the virtual impulses $\delta \mathbf{I}_l$ are independent, then their coefficients may be equated to zero for conditions of kinematic fit, which in this case become the compatibility equations of motion. These equations, given by the vanishing condition of the bracketed term in equation (2.2.8) must be expressed in terms of equilibrating impulses. This is done by invoking the constitutive equations for the force and inertial element. Thus in equation (2.2.6) the velocity $\mathbf{v}_k$ of the kth mass is first expressed in terms of momentum $\mathbf{B}_k$ and subsequently in terms of impulses $\mathbf{I}_l$ via equation (2.2.4). For force elements in the constitutive relations are often, though not always, in terms of extensions $\mathbf{u}$. Thus for a linear spring of stiffness k

$$u = -\frac{\dot{I}}{k} \qquad \text{and} \qquad \dot{u} = -\ddot{I}/k \qquad (2.2.9)$$

For a cubic stiffening spring of stiffness h

$$u = -\frac{\dot{I}^{1/3}}{h^{1/3}} \qquad \text{and} \qquad \dot{u} = -\frac{1}{3}\frac{\ddot{I}}{h^{1/3}\,\dot{I}^{2/3}} \qquad (2.2.10)$$

For a linear dashpot of damping c

$$u = -\frac{I}{c} \qquad \text{and} \qquad \dot{u} = -\frac{\dot{I}}{c} \qquad (2.2.11)$$

Example 2.2.1

The equations of motion of the oscillator chain of Figure 2.2 are to be obtained via the complementary form of D'Alembert's principle.

From the general expression in equation (2.2.5) we write

$$\delta W_{II} = v_a \delta B_a + v_b \delta B_b + v_c \delta B_c - \dot{u}_1 \delta I_1 - \dot{u}_2 \delta I_2 - \dot{u}_3 \delta I_3 - \dot{u}_4 \delta I_4 = 0 \qquad (a)$$

But equilibrium conditions require that

$$B_a = (I_1 - I_2) \qquad B_b = (I_2 - I_3) \qquad B_c = (I_3 - I_4) \qquad (b)$$

Substituting from (b) into (a) and collecting terms multiplied by the same virtual impulse we can write this expression as

$$\delta W_{II} = (v_a - \dot{u}_1)\,\delta I_1 + (v_b - v_a - \dot{u}_2)\delta I_2 + (v_c - v_b - \dot{u}_3)\delta I_3 + (-v_c - \dot{u}_4)\delta I_4 = 0 \qquad (c)$$

On comparing this expression with equation (2.2.8) we deduce that the resultant velocities v_l are in this case

$$v_1 = v_a \qquad v_2 = v_b - v_a \qquad v_3 = v_c - v_b \qquad v_4 = -v_c \qquad (d)$$

Now from the constitutive equations for the inertial and force elements we have that

$$v_a = (I_1 - I_2)/m_a \qquad v_b = (I_2 - I_3)/m_b \qquad v_c = (I_3 - I_4)/m_c \qquad (e)$$

$$u_1 = -\dot{I}_1/k_1 \qquad u_2 = -\dot{I}_2/k_2 \qquad u_3 = -\dot{I}_3/k_3 \qquad u_4 = -\dot{I}_4/k_4$$

Substituting from equation (e) into (c) and noting that δI_1, δI_2, δI_3, and δI_4 are arbitrary and

independent we obtain the compatibility equations of motion as

$$\begin{aligned}
&\frac{I_1 - I_2}{m_a} + \frac{\ddot{I}_1}{k_1} = 0 \\
&\frac{(I_2 - I_1)}{m_a} - \frac{(I_3 - I_2)}{m_b} + \frac{\ddot{I}_2}{k_2} = 0 \\
&\frac{(I_3 - I_2)}{m_b} - \frac{(I_4 - I_3)}{m_c} + \frac{\ddot{I}_3}{k_3} = 0 \\
&\frac{(I_4 - I_3)}{m_c} + \frac{\ddot{I}_4}{k_4} = 0
\end{aligned} \tag{f}$$

or in matrix notation

$$\begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} \end{bmatrix} \begin{bmatrix} \ddot{I}_1 \\ \ddot{I}_2 \\ \ddot{I}_3 \\ \ddot{I}_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{m_a} & -\frac{1}{m_a} & 0 & 0 \\ -\frac{1}{m_a} & \frac{1}{m_a} + \frac{1}{m_b} & -\frac{1}{m_b} & 0 \\ 0 & -\frac{1}{m_b} & \frac{1}{m_c} + \frac{1}{m_b} & -\frac{1}{m_c} \\ 0 & 0 & -\frac{1}{m_c} & \frac{1}{m_c} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{g}$$

In passing it is worth noting that we may obtain the equilibrium equations of motion for this system in terms of the three independent displacements of the three masses. Thus this system has three degrees of freedom in the displacement formulation and four degrees of freedom in the impulse formulation.

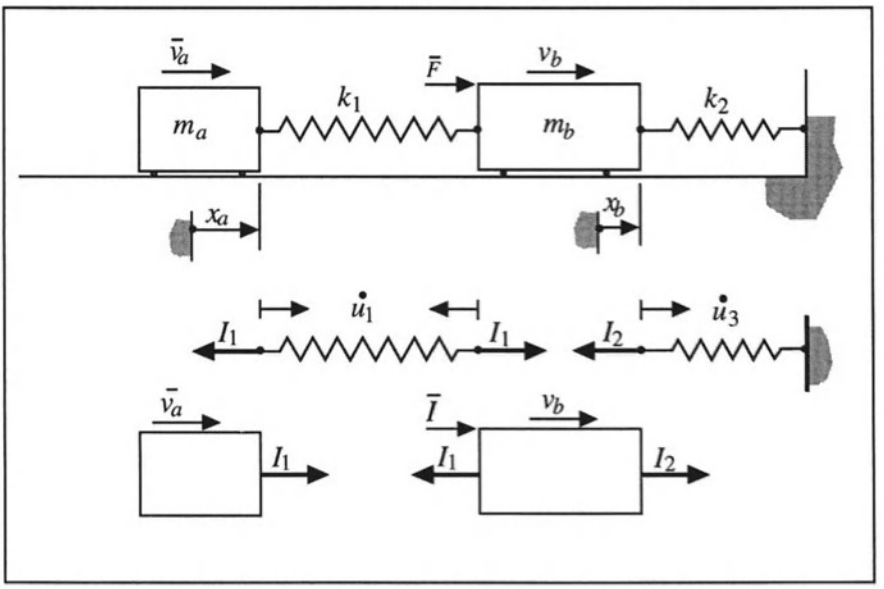


Figure 2.3 Two mass oscillator chain

Example 2.2.2

The system of Figure 2.3 is made up of two masses and two linear springs. Let the velocity $\bar{v}_a$ of the first mass be prescribed, and let a prescribed force $\bar{F}$ act on the second mass. We wish to obtain (a) the (equilibrium) equation of motion of the system in the displacement formulation and (b) the (compatibility) equation of motion in the impulse formulation.

(a) Since the velocity of the first mass m_a is prescribed, the system has only one (positional) degree of freedom in the displacement formulation. We may write the D'Alembert principle for the system as

$$\delta W_I = (F_b - \dot{B}_b)\delta x_b = 0 \tag{a}$$

where F_b includes all forces impressed upon mass m_b, i.e.

$$F_b = \bar{F} + k_1(\bar{x}_a - x_b) - k_2 x_b \tag{b}$$

with

$$\bar{x}_a = \int_0^t \bar{v}_a \, dt \tag{c}$$

and

$$B_b = m\dot{x}_b \tag{d}$$

Thus the (equilibrium) equation of motion emerges as

$$m_b\ddot{x}_b + (k_1 + k_2)x_b = \bar{F} + k_1\bar{x}_a \tag{e}$$

It is worth noting that the virtual work done by the spring forces may be expressed as

$$k_1(\bar{x}_a - x_b)\delta x_b - k_2 x_b \delta x_b = -\delta\left[\frac{k_1}{2}(\bar{x}_a - x_b)^2 + \frac{k_2}{2}x_b^2\right] = \delta V \tag{f}$$

since $\delta\bar{x}_a = 0$. The term in brackets on the right side of equation (f) will be recognized as the work function of the two springs. Thus the prescribed kinematic variable $\bar{x}_a$ can be included in the work function or the potential energy. On the other hand, for the prescribed force a separate virtual work expression must be written unless the force possesses a potential.

(b) Next let us obtain the compatibility equations of motion for the same system from the complementary form of D'Alembert's principle. We note at the outset that the system possesses two independent impulses associated with the two springs. Thus the complementary formulation has two (impulsive) degrees of freedom. Now we can write the virtual work of the various velocities in the expression for δW_{II} as follows:

$$\delta W_{II} = \bar{v}_a \, \delta B_a + v_b \, \delta B_b - \dot{u}_1 \, \delta I_1 - \dot{u}_2 \, \delta I_2 = 0 \tag{g}$$

But equilibrium conditions, as depicted in Figure 2.3, require that

$$B_a = I_1 \qquad B_b = I_2 - I_1 + \bar{I} \tag{h}$$

hence

$$\delta W_{II} = \bar{v}_a \, \delta I_1 + v_b(\delta I_2 - \delta I_1 + \delta\bar{I}) - \dot{u}_1 \, \delta I_1 - \dot{u}_2 \, \delta I_2 = 0 \tag{i}$$

Now noting that $\delta\bar{I} = 0$ and collecting terms multiplied by the same virtual impulse, we can write

$$\delta W_{II} = (\bar{v}_a - v_b - \dot{u}_1)\delta I_1 + (v_b - \dot{u}_2)\delta I_2 = 0 \tag{j}$$

Comparing with the expression in equation (2.2.5) we deduce that for the present problem the resultant velocities are

$$v_1 = \bar{v}_a - v_b \qquad v_2 = v_b \tag{k}$$

Now invoking the constitutive equations for the inertial and force elements we have that

$$v_b = \frac{B_b}{m_b} = \frac{I_2 - I_1 + \bar{I}}{m_b} \tag{l}$$

$$u_1 = -\dot{I}_1/k_1 \qquad u_2 = -\dot{I}_2/k_2 \tag{m}$$

Then on account of the arbitrariness of the virtual impulses we find the compatibility equations of motion as

$$\bar{v}_a - \frac{I_2 - I_1 + \bar{I}}{m_b} + \frac{\ddot{I}_1}{k_1} = 0 \tag{n}$$

$$\frac{I_2 - I_1 + \bar{I}}{m_b} + \frac{\ddot{I}_2}{k_2} = 0 \tag{o}$$

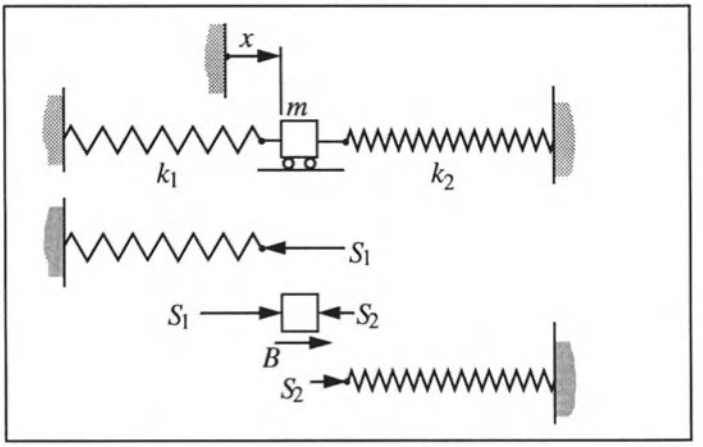
It is of interest to note that when the constitutive equations are used in the expression for δW_{II} the term $v_b\ \delta(I_2 - I_1 + \bar{I})$ may be expressed as

$$v_b\ \delta B_b = \frac{I_2 - I_1 + \bar{I}}{m_b}\ \delta(I_2 - I_1 + \bar{I}) = \delta\frac{(I_2 - I_1 + \bar{I})^2}{2m_b} = \delta T_b \tag{p}$$

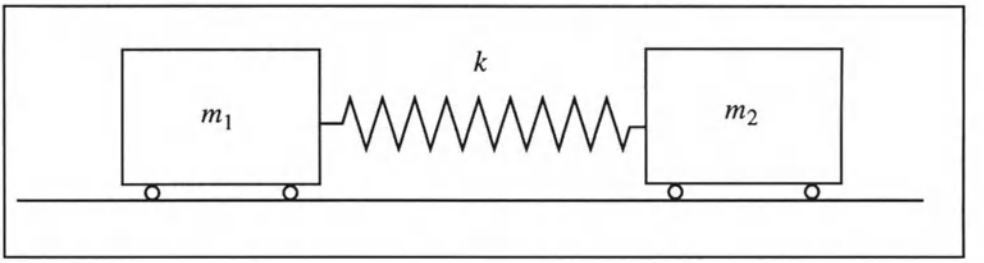
where T_b is the kinetic energy of the second mass. Thus the prescribed impulse $\bar{I}$ can be included in the expression for the kinetic energy. On the other hand for the prescribed velocity $\bar{v}_a$ a separate virtual work expression must be written unless the velocity possesses a kinetic energy.

Problems

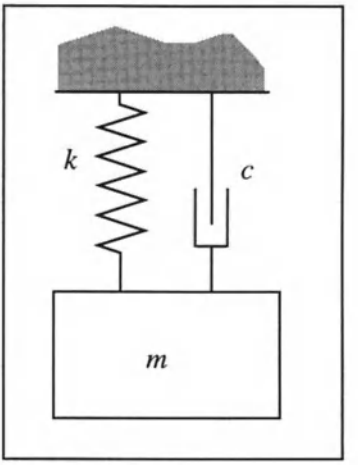
2.2.1 An oscillator chain consists of spring-mass-spring, with ends fixed. Derive the differential equations of motion using (a) the D'Alembert principle and (b) the complementary D'Alembert principle. (c) Find the natural frequencies for (a) and (b).



2.2.2 A free oscillator chain consists of mass-spring-mass, Derive the differential equations of motion using (a) the D'Alembert principle and (b) the complementary D'Alembert principle.



2.2.3 Derive the equations of motion for a simple mass-spring-dashpot oscillator using (a) the D'Alembert principle and (b) the complementary D'Alembert principle.



2.3 Conservation of Energy

Consider a system for which time does not appear explicitly in the reduced transformation equations. For such a scleronomic system we may let the virtual displacements coincide with the real displacements. This can be done because real displacements satisfy the constraint equations and also since there can be no moving constraints or moving reference frames (otherwise time would appear explicitly in the transformation equations) it is not necessary to hold time fixed in order to prevent the constraint forces from doing work. Let us further restrict our case to systems for which all forces, other than the inertial forces, possess potential energy functions. Assuming that time does not appear explicitly in the system's total potential energy we may write D'Alembert's principle (2.1.3) as

$$-dV - \dot{\mathbf{B}}_k \cdot d\mathbf{r}_k = 0 \tag{2.3.1}$$

where the $d\mathbf{r}_k$ are real displacements and $dV = -\mathbf{F}_k \cdot d\mathbf{r}_k$ is the change in potential energy due to changes in real displacements. Now since real displacements occur over a time interval we may write equation (2.3.1) as

$$-dV - \dot{\mathbf{B}}_k \cdot \frac{d\mathbf{r}_k}{dt} dt = 0 \tag{2.3.2}$$

If we now express the velocity vector $\dot{\mathbf{r}}_k$ as $\mathbf{v}_k$, equation (1.12.2) becomes

$$-dV - \mathbf{v}_k \cdot d\mathbf{B}_k = 0 \tag{2.3.3}$$

With $dT = \mathbf{v}_k \cdot d\mathbf{B}_k$, we deduce that

$$-dV - dT = 0 \tag{2.3.4}$$

which, upon integration, becomes

$$V + T = \text{constant.} \tag{2.3.5}$$

Several points are worth noting here. First in deriving the above result for conservation of energy we had to assume the following:

i) All forces, other than the inertial forces, possess potential energy.

ii) Time does not appear explicitly in the reduced transformation equations nor in the total potential energy function.

Second, it is of interest to see that it is the total energy and not a combination of an energy and a complementary energy that is conserved. Now for Newtonian mechanics $T = T^*$ and hence in Newtonian mechanics one can also assert that, under the above conditions i) and ii),

$$V + T^* = \text{constant} \tag{2.3.6}$$

Finally if the force displacement relation, implicit in the expression for V, is linear, then $V = V^*$ and one can also assert that

$$V^* + T = \text{constant} \tag{2.3.7}$$

and

$$V^* + T^* = \text{constant} \tag{2.3.8}$$

If the force-displacement relation is nonlinear, V^* may be some multiple of V , say $V^* = \frac{1}{c}V$, then

$$cV^* + T^* = \text{constant} \tag{2.3.9}$$

The results in equation (2.3.5) to (2.3.9) can also be derived from the complementary form of D'Alembert's principle.

Example 2.3.1

As an example consider a mass-spring system with a nonlinear spring for which the force is given by $F = -hx^3$ (Duffing-type† oscillator). The potential energy is (Example 1.8.1)

$$V = \frac{1}{4} hx^4 \tag{a}$$

There are no energy losses, consequently we conclude that

$$V + T = \text{constant} \tag{b}$$

With $T = T^* = \frac{1}{2} m\dot{x}^2$,

$$V + T = \frac{1}{4} hx^4 + \frac{1}{2} m\dot{x}^2 = \text{constant} \tag{c}$$

Differentiating equation (c) with respect to time, we obtain

$$\dot{x} (m\ddot{x} + hx^3) = 0 \tag{d}$$

The term in the bracket will be recognized as the differential equation for a Duffing-type oscillator.

It will be recalled that for this problem

$$V^* = 3 V \tag{e}$$

Thus in this case we can write

$$\frac{1}{3} V^* + T^* = \text{constant} \tag{f}$$

or

$$V^* + T^* = 3V + T = \text{constant} + 2V \tag{g}$$

i.e. the sum $V^* + T^*$ is not a constant.

Problems

2.3.1 Derive the expression $T + V$ = constant from the complementary form of D'Alembert's principle.

2.3.2 Use the constant mechanical energy $T + V$ of a pendulum and the generalized coordinate $q = \theta$ to (a) derive the (nonlinear) differential equation of motion. (b) Plot the generalized force Q versus the generalized position q. (c) Show that the mechanical coenergy $T^* + V^*$ is not a constant. (d) Write $(T + V)$ and $(T^* + V^*)$ for small displacements and show that for the linearised system the mechanical energy sum and the mechanical coenergy sum are the same.

† Georg Duffing (1861-1944), German engineer

2.4 Gauss' Principle of Least Constraint

D'Alembert's principle is not an extremum principle since the inertial forces in this principle are not derivable from a function. Gauss† gave an ingenious reinterpretation of D'Alembert's principle, which changes it into a minimum principle. Gauss' principle can be derived from D'Alembert's principle as follows.

Suppose that at a given instant, t, we know the real positions and the velocities but not the accelerations of all the particles in a system. At a small increment of time τ later we can determine the positions of the particles by a Taylor series expansion. Thus

$$\mathbf{r}_k(t + \tau) = \mathbf{r}_k(t) + \dot{\mathbf{r}}_k(t)\tau + \frac{1}{2}\ddot{\mathbf{r}}_k(t)\tau^2 + \cdots \tag{2.4.1}$$

Since, the positions and velocities are specified at time t and are not subject to variation, deviations in the position of the particles can arise from variations of accelerations, i.e.

$$\delta\mathbf{r}_k(t + \tau) = \frac{1}{2}\tau^2\,\delta\ddot{\mathbf{r}}_k \tag{2.4.2}$$

This relationship is true no matter what constraints may exist amongst the $\delta\mathbf{r}_k$.

Now let us write D'Alembert's principle (2.1.4) using the virtual displacements at time $(t + \tau)$. Then we have, with $\delta\ddot{\mathbf{r}} = \frac{1}{m}\,\delta\dot{\mathbf{B}}$, that

$$\left(\mathbf{F}_k - \dot{\mathbf{B}}_k\right) \cdot \delta\dot{\mathbf{B}}_k = 0 \tag{2.4.3}$$

Finally, on recognizing that the forces $\mathbf{F}_k$ are also prescribed, i.e. $\delta\mathbf{F}_k = 0$, we may write equation (2.4.3) as

$$\left(\mathbf{F}_k - \dot{\mathbf{B}}_k\right) \cdot \delta\left(\mathbf{F}_k - \dot{\mathbf{B}}_k\right) = 0 \tag{2.4.4}$$

or

$$\delta Z = 0 \tag{2.4.5}$$

where

$$Z = \frac{1}{2}\left(\mathbf{F}_k - \dot{\mathbf{B}}_k\right) \cdot \left(\mathbf{F}_k - \dot{\mathbf{B}}_k\right) \tag{2.4.6}$$

In equation (2.4.4), several points are worthy of note. First, one can see that in this equation, we have a true minimum principle. The minimization is with respect to the momentum derivatives $\dot{\mathbf{B}}_k$. If there exist no constraints, i.e. all $\delta\dot{\mathbf{B}}_k$ are independent, then each coefficient of $\delta\dot{\mathbf{B}}_k$ must vanish and the value of Z will be exactly zero. On the other hand, if some constraints exist, the coefficients of $\delta\dot{\mathbf{B}}_k$ cannot be equated to zero. In this case, the value of Z will be larger than zero. As such, Z is an indicator of constraints and Gauss referred to Z as the *zwang (= constraint)* of the motion

† Carl Friedrich Gauss (1777-1855), German mathematician (Disquisitiones arithmeticae, 1801)

whose variation must vanish - hence the *principle of least constraint.*

Since in Gauss' principle, only momentum derivatives, or equivalently accelerations, are subject to variations, constraints amongst displacement and/or velocities must be converted to a set of relations amongst the accelerations. This is done simply by differentiating the constraint equations with respect to time, twice for holonomic constraints and once for nonholonomic constraints. Thus Gauss' principle can be used for holonomic and nonholonomic systems. It is also of interest to recognize that the constraints amongst accelerations always emerge as linear relationships allowing one to eliminate the superfluous accelerations readily.

Finally, one can interpret equation (2.4.4) as the least squares minimization of $(\mathbf{F}_k - \dot{\mathbf{B}}_k)$. The method of least squares minimization of errors was also discovered by Gauss.

Gauss' principle may be expressed in terms of generalized coordinates q_i, by expressing $\ddot{\mathbf{r}}_k$ in terms of the generalized coordinates as shown in equation (2.1.7).

Starting from the complementary form of D'Alembert's principle and following a similar procedure, it is also possible to find a complementary form of Gauss' principle, namely

$$\delta Z^* = 0 \tag{2.4.7}$$

where

$$Z^* = \frac{1}{2}\left[\mathbf{v}_l - \dot{\mathbf{u}}_l\right] \cdot \left[\mathbf{v}_l - \dot{\mathbf{u}}_l\right] \tag{2.4.8}$$

with $\mathbf{v}_l$ and $\dot{\mathbf{u}}_l$ as velocities and extension rates, as in equation (2.2.8).

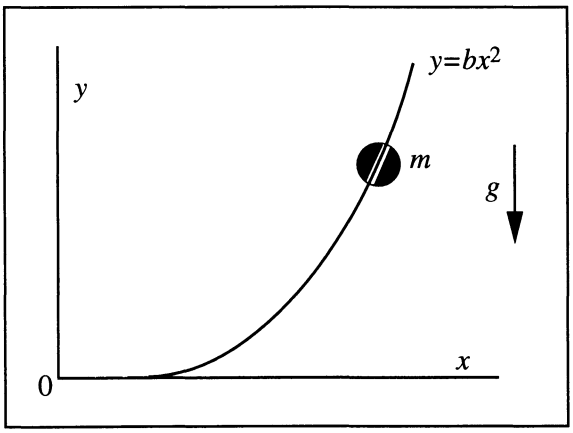


Figure 2.4 Bead on a parabolic wire

Example 2.4.1

A particle is forced to move without friction along a wire bent in the form of a parabola $y = bx^2$, as shown in Figure 2.4. Let us derive the particle's equation of motion via Gauss' principle.

In this example, we have the holonomic constraints

$$y = bx^2 \qquad z = 0 \tag{a}$$

Differentiating twice, we find

$$\dot{y} = 2bx\dot{x} \quad \text{and} \quad \dot{z} = 0 \tag{b}$$

$$\ddot{y} = 2b\dot{x}^2 + 2b\, x\, \ddot{x} \quad \text{and} \quad \ddot{z} = 0 \tag{c}$$

Now since in Gauss' principle the displacements and velocities are prescribed and only accelerations are subject to variations, we find from equation (c) that

$$\delta\ddot{y} = 2bx\ \delta\ddot{x} \tag{d}$$

Using equation (2.4.5), we may write

$$\delta \frac{1}{2}[(F_x - m\ddot{x})^2 + (F_y - m\ddot{y})^2 + (F_z - m\ddot{z})^2] = 0$$

In this example, $F_x = F_z = 0$ and $F_y = -mg$. Hence

$$\delta \frac{1}{2}[(m\ddot{x})^2 + (mg + m\ddot{y})^2] = 0$$

or

$$\ddot{x}\ \delta\ddot{x} + (g + \ddot{y})\ \delta\ddot{y} = 0 \tag{e}$$

On eliminating $\delta\ddot{y}$ from equation (e) via equation (d), we arrive at

$$[\ddot{x} + 2bx(g + \ddot{y})]\ \delta\ddot{x} = 0 \tag{f}$$

Now it remains to recognize that $\delta\ddot{x}$ is arbitrary and hence its coefficient must vanish. It remains to remove $\ddot{y}$ from this coefficient. Using equation (c) for $\ddot{y}$ in equation (f), we find the particle's equation of motion as

$$\ddot{x}(1 + 4b^2x^2) + 4b^2\dot{x}^2x = -2gbx$$

Problems

2.4.1 A particle is forced to stay on the surface

$$z = f(x,y)$$

and is acted upon by the force F. Using Gauss' principle of least constraint, find the equation of motion.

2.4.2 A particle of mass m moves without friction on a horizontal plane. It is subjected to the constraint:

$$\sin at^2\, dx - \cos at^2\, dy = 0$$

where a = constant and where the Cartesian coordinates (x,y) specify its position. Obtain the equations of motion for this particle by means of Gauss' principle of least constraint.

2.4.3 Obtain the equation of motion for a simple mass-dashpot-spring oscillator by means of Gauss' principle of least constraint.

2.4.4 A particle m is forced to stay on the surface $z = \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$ and is acted upon by a force $\mathbf{F}$. Using Gauss' principle of least constraint, find the equations of motion.

2.5 Configuration Space and Hertz' Principle of Least Curvature

It is possible to represent the configuration of a system of N particles by the position vector of a single point in a $3N$ dimensional space. For such a space, called the *configuration space*, we may use a rectilinear coordinate system with $3N$ axes. Now we need to establish a relation between measurements of length in the configuration space and the 3D physical space. To this end, we relate the incremental changes in the position vector $\mathbf{R}$ of the representative point in the configuration space, to the corresponding changes in the position vectors $\mathbf{r}_i$ of the N particles in the 3D space as follows

$$\{dR\}^T = [\sqrt{m_1}dx_1 \;\; \sqrt{m_1}dy_1 \;\; \sqrt{m_1}dz_1 \;\; \sqrt{m_2}dx_2 \;\; \cdots \;\; \sqrt{m_N}dz_N] \qquad (2.5.1)$$

Now the elemental arc length, ds, in the configuration space may be computed as

$$ds^2 = d\mathbf{R} \cdot d\mathbf{R} = (md\mathbf{r})_k \cdot d\mathbf{r}_k \qquad (2.5.2)$$

The advantage of the relation defined in equation (2.5.1) is that when equation (2.5.2) is divided through by $2dt^2$, we find

$$\frac{1}{2}\left[\frac{ds}{dt}\right]^2 = \frac{1}{2}m_k \dot{\mathbf{r}}_k^2 = T^* \qquad (2.5.3)$$

Equation (2.5.3) shows that the total complementary energy of the N particles is equal to the complementary kinetic energy of a single particle of unit mass, moving in the $3N$ dimensional configuration space.

Now if the motion of the N particles is constrained and only n coordinates are required, with $n < 3N$, then not every point of the configuration space of $3N$ dimensions will be accessible and the motion of the representative point will be confined to a manifold of n dimensions. Alternatively, we may define a new configuration space of n dimensions corresponding to the n coordinates. In this case, the n coordinate axes will generally be curved rather than straight. Once again, we may define a convenient metric for this curvilinear space by expressing the elemental arc length of this space as

$$ds^2 = \{dq_i\}^T [m_{ij}] \{dq_j\} \qquad i, j = 1, 2, 3, ..., n \qquad (2.5.4)$$

where $[m_{ij}]$ is the mass matrix of the system. On dividing through by $2dt^2$, we find

$$\frac{1}{2}\left[\frac{ds}{dt}\right]^2 = \frac{1}{2}\{\dot{q}_i\}^T [m_{ij}] \{\dot{q}_j\} \qquad (2.5.5)$$

The right hand side term will be recognized as the complementary kinetic energy of a scleronomic system.

As the configuration of the N particles changes in time, the representative point

traces out a curve in the configuration space. Hertz† showed that, in the absence of impressed forces, the trajectories of the representative point in the configuration space are lines of least curvature.

Hertz' principle of least curvature may be derived from Gauss' principle of least constraint as follows. In the absence of the impressed forces, Gauss' principle reduces to

$$\delta\left(\frac{1}{2} m_k \ddot{\mathbf{r}}_k^2\right) = 0$$

which by virtue of equation (2.5.1), may be written as

$$\delta \frac{\ddot{\mathbf{R}} \cdot \ddot{\mathbf{R}}}{2} = 0 \tag{2.5.6}$$

Also, if there are no impressed forces, the kinetic energy, and in Newtonian mechanics also the complementary kinetic energy, of the system will remain fixed. Thus we also have that

$$\frac{1}{2} m_k \dot{\mathbf{r}}_k^2 = \frac{1}{2}\left[\frac{ds}{dt}\right]^2 = c \qquad \text{a constant}$$

Now dividing equation (2.5.6) by $4c^2$, we obtain

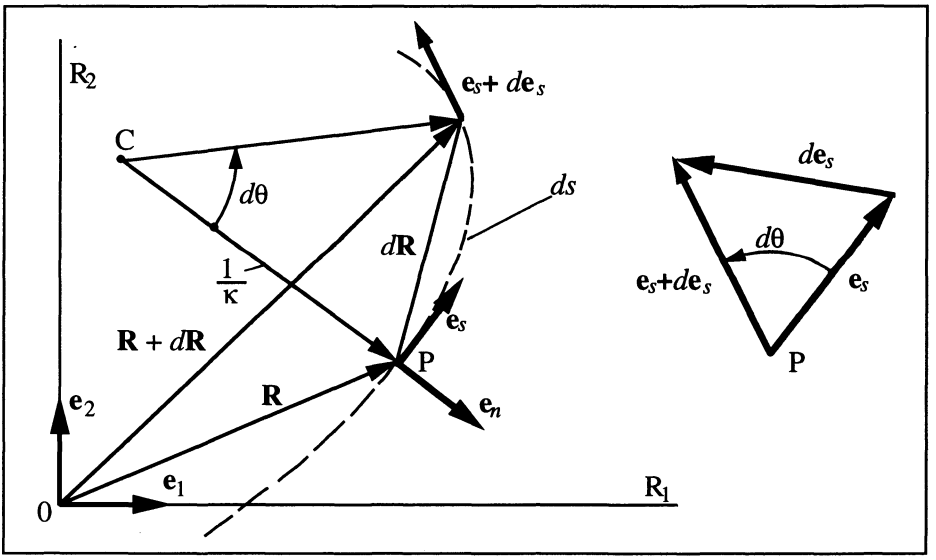


Figure 2.5 Trajectory of the representative point

† Heinrich Hertz (1857-1894), German physicist

$$\delta \frac{\dfrac{d^2\mathbf{R}}{dt^2} \cdot \dfrac{d^2\mathbf{R}}{dt^2}}{2\left[\dfrac{ds}{dt}\right]^2 \left[\dfrac{ds}{dt}\right]^2} = \delta \frac{1}{2}\left[\frac{d^2\mathbf{R}}{ds^2}\right]^2 = 0 \tag{2.5.7}$$

It remains to show that equation (2.5.7) is related to the curvature of the trajectory in the configuration space. We will demonstrate this for the simpler case of a two-dimensional configuration space. In this case,

$$\mathbf{R} = [\mathbf{e}_1 \ \mathbf{e}_2] \begin{bmatrix} R_1 \\ \\ R_2 \end{bmatrix} \tag{2.5.8}$$

as shown in Figure 2.5. The derivative

$$\frac{d\mathbf{R}}{ds} = [\mathbf{e}_1 \ \mathbf{e}_2] \begin{bmatrix} \dfrac{dR_1}{ds} \\ \dfrac{dR_2}{ds} \end{bmatrix} = \mathbf{e}_s \tag{2.5.9}$$

represents a unit vector in tangential direction. The magnitude of a unit vector is constant, so that a change of a unit vector can only be perpendicular to the vector, thus

$$d\mathbf{e}_s = -d\theta\, \mathbf{e}_n \tag{2.5.10}$$

and since

$$d\theta = \kappa\, ds \tag{2.5.11}$$

where κ is the curvature, we obtain

$$\frac{d\mathbf{e}_s}{ds} = -\kappa\, \mathbf{e}_n \tag{2.5.12}$$

Combining equations (2.5.9) and (2.5.12) one finds that

$$\frac{d^2\mathbf{R}}{ds^2} = -\kappa\, \mathbf{e}_n \tag{2.5.13}$$

and

$$\frac{d^2\mathbf{R}}{ds^2} \cdot \frac{d^2\mathbf{R}}{ds^2} = \kappa^2 \tag{2.5.14}$$

On returning to equation (2.5.7), we can now interpret this equation as the minimization of the magnitude of the curvature in the $3N$ dimensional configuration space. One can show that paths of minimum curvature are geodesics on curved surfaces. When there are no constraints, the trajectories in the configuration space are straight lines. It is interesting to note that this finding for a system of N particles represented by the motion of a single particle in the $3N$ configuration space is similar to Newton's first law for a single particle in 3D space. Finally, it is worth noting that Hertz' principle makes a statement about the trajectory of the motion in the configuration space, but it does not provide information on the motion as a function of time. Although of fundamental importance, Hertz' principle does not provide a practical method for

analyzing the dynamics of systems.

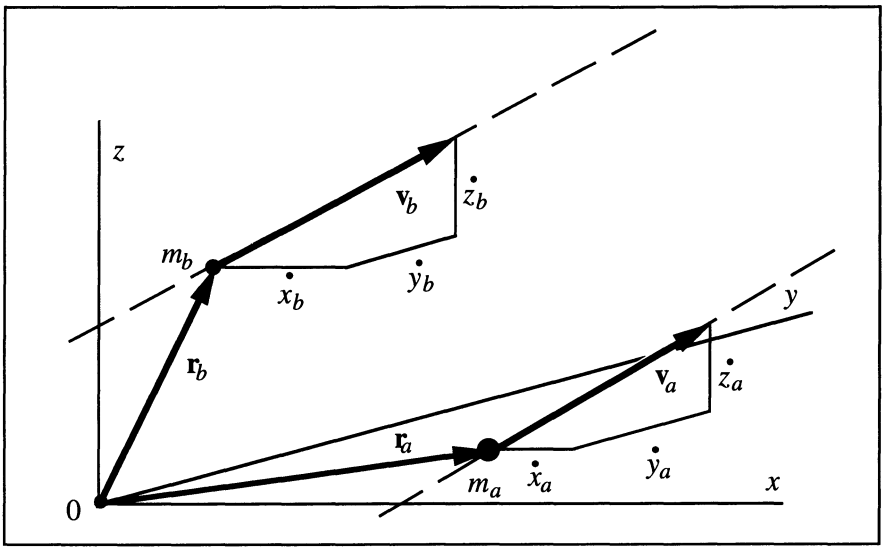


Figure 2.6 A system of two force-free particles

Example 2.5.1

Consider the system involving the two particles shown in Figure 2.6. No forces are acting. As a consequence each particle will continue at a constant speed along a straight path in 3D space.

The changes in position vector in configuration space are obtained from

$$\begin{aligned} dR_1 &= \sqrt{m_a}\, dx_a & dR_4 &= \sqrt{m_b}\, dx_b \\ dR_2 &= \sqrt{m_a}\, dy_a & dR_5 &= \sqrt{m_b}\, dy_b \\ dR_3 &= \sqrt{m_a}\, dz_a & dR_6 &= \sqrt{m_b}\, dz_b \end{aligned} \qquad \text{(a)}$$

The kinetic coenergy is

$$T^* = \frac{1}{2}\,(m_a v_a^2 + m_b v_b^2) = \text{constant} \qquad \text{(b)}$$

With $v_a^2 = \dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2$ and $v_b^2 = \dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2$, it can be expressed in Hertzian coordinates

$$T^* = \frac{1}{2}\,(\dot{R}_1^2 + \dot{R}_2^2 + \dot{R}_3^2 + \dot{R}_4^2 + \dot{R}_5^2 + \dot{R}_6^2) = \text{constant} \qquad \text{(c)}$$

Since there is no external force acting, the velocity $\dot{\mathbf{R}}$ is constant, i.e. the imagined single particle of unit mass moves at a velocity of constant magnitude

$$\| \dot{\mathbf{R}} \| = \sqrt{\dot{R}_1^2 + \dot{R}_2^2 + \dot{R}_3^2 + \dot{R}_4^2 + \dot{R}_5^2 + \dot{R}_6^2} \qquad \text{(d)}$$

along a straight path (minimum curvature) in the 6 dimensional Hertzian configuration space.

2.6 Lagrange's Equations

Let us consider again D'Alembert's principle in terms of generalized coordinates. For this purpose, we require to express the virtual displacement $\delta \mathbf{r}_k$ and the accelerations $\ddot{\mathbf{r}}_k$, in terms of generalized coordinates q_i.

Let the transformation equations be

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2, \ldots, q_i, \ldots, q_n, t)$$

Then

$$\delta \mathbf{r}_k = \frac{\partial \mathbf{r}_k}{\partial q_i} \delta q_i \tag{2.6.1}$$

and

$$\dot{\mathbf{r}}_k = \frac{\partial \mathbf{r}_k}{\partial q_i} \dot{q}_i + \frac{\partial \mathbf{r}_k}{\partial t} \tag{2.6.2}$$

In deriving equation (2.6.1), time is held fixed so as to prevent constraint forces from entering the virtual work expression.

Consider now the virtual work of the applied and the inertial forces. For the former, we may write

$$\mathbf{F}_k \cdot \delta \mathbf{r}_k = \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_i} \delta q_i$$

or

$$\mathbf{F}_k \cdot \delta \mathbf{r}_k = Q_i \, \delta q_i \tag{2.6.3}$$

where

$$Q_i = \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_i} \tag{2.6.4}$$

For the virtual work of the inertial forces, we have that

$$\dot{\mathbf{B}}_k \cdot \delta \mathbf{r}_k = \dot{\mathbf{B}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_i} \delta q_i \tag{2.6.5}$$

It is convenient to write the inertial force on the right hand side of equation (2.6.5) in the following modified form

$$\dot{\mathbf{B}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_i} = \frac{d}{dt}\left[\mathbf{B}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_i}\right] - \mathbf{B}_k \cdot \frac{d}{dt}\left[\frac{\partial \mathbf{r}_k}{\partial q_i}\right] \tag{2.6.6}$$

But

$$\frac{d}{dt}\left[\frac{\partial \mathbf{r}_k}{\partial q_i}\right] = \frac{\partial^2 \mathbf{r}_k}{\partial q_i \partial q_j} \dot{q}_j + \frac{\partial^2 \mathbf{r}_k}{\partial q_i \partial t} \tag{2.6.7}$$

Now an examination of the right hand side of equation (2.6.7) reveals that this term is just

$$\frac{\partial}{\partial q_i} \frac{d}{dt} \mathbf{r}_k = \frac{\partial}{\partial q_i}\left[\frac{\partial \mathbf{r}_k}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_k}{\partial t}\right] \tag{2.6.8}$$

Equations (2.6.7) and (2.6.8) show that

$$\frac{d}{dt}\left[\frac{\partial \mathbf{r}_k}{\partial q_i}\right] = \frac{\partial}{\partial q_i}\left[\frac{d}{dt}\mathbf{r}_k\right] \tag{2.6.9}$$

Furthermore, from equation (2.6.2), we find that

$$\frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_i} = \frac{\partial \mathbf{r}_k}{\partial q_i} \tag{2.6.10}$$

Now making use of equations (2.6.9) and (2.6.10), we may write equations (2.6.6) as

$$\dot{\mathbf{B}}_k \frac{\partial \mathbf{r}_k}{\partial q_i} = \frac{d}{dt}\left[\mathbf{B}_k \frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{q}_i}\right] - \mathbf{B}_k \frac{\partial}{\partial q_i}\dot{\mathbf{r}}_k \tag{2.6.11}$$

Finally, since $\mathbf{B} = m\dot{\mathbf{r}}$, the virtual work of the inertial forces may be written as

$$\dot{\mathbf{B}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_i}\delta q_i = \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\frac{1}{2}\mathbf{B}_k \cdot \dot{\mathbf{r}}_k - \frac{\partial}{\partial q_i}\frac{1}{2}\mathbf{B}_k \cdot \dot{\mathbf{r}}_k\right]\delta q_i \tag{2.6.12}$$

Now recognising the $\frac{1}{2}\mathbf{B}_k \cdot \dot{\mathbf{r}}_k$ terms as the system's kinetic coenergy $T^*(q, \dot{q}, t)$, and using equation (2.6.4), we may express D'Alembert's principle (2.1.5), in terms of generalized coordinates, as

$$\left[\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i} - Q_i\right]\delta q_i = 0 \tag{2.6.13}$$

If the constraints are holonomic, each δq_i will be independent and the term in the brackets of equation (2.6.13) must vanish, leading to the *fundamental form* of the celebrated *Lagrange*† *equations*;

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i} = Q_i \qquad i = 1,2,...,n \tag{2.6.14}$$

For each system there are altogether n such equations, where n is the number of generalized coordinates.

Considering equation (1.8.13), equation (2.6.14) may also be written

$$\frac{d}{dt}p_i - \frac{\partial T^*}{\partial q_i} = Q_i \tag{2.6.15}$$

Equation (2.6.15) shows once again that the time derivative of the generalized momentum does not yield all terms of the inertial force. That is, part of the inertial force is to be obtained from the term $-\frac{\partial T^*}{\partial q_i}$. Only for Galilean coordinate systems in which T^* is not a function of q_i, is the inertial force equal to the time derivative of the

† Joseph Louis Lagrange (1736-1813), French mathematician (Mécanique analytique, 1788)

generalized momentum.

It is worth emphasizing that Lagrange's equations, as derived above, hold for holonomic systems in which the q_i are independent. We shall examine the extension of Lagrange's equations to nonholonomic systems later (see section 3.6).

In the special case when the $\mathbf{F}_k$ are derivable from a potential energy function V, we may write

$$\mathbf{F}_k \cdot d\mathbf{r}_k = - dV$$

or on using equation (2.6.3)

$$Q_i\, dq_i = \mathbf{F}_k \cdot d\mathbf{r}_k = - dV(q_i) = - \frac{\partial V}{\partial q_i} dq_i$$

or
$$Q_i = -\frac{\partial V}{\partial q_i} \tag{2.6.16}$$

For such a case, Lagrange's equations may be expressed as

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i}(T^* - V) = 0 \tag{2.6.17}$$

Since in most cases V is not a function of generalized velocities, it is convenient to define a new function L, called the *Lagrangian*, as

$$L = T^* - V \tag{2.6.18}$$

and express Lagrange's equations as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1,2,...,n \tag{2.6.19}$$

Equation (2.6.19) is the most frequently quoted form of Lagrange's equation, and is called the *standard form*. If in addition to forces which possess a potential, there exist other generalized forces Q_i', which are not derivable from a potential function, Lagrange's equations take the form;

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i' \tag{2.6.20}$$

Example 2.6.1

Consider again the system shown in Figure 1.29. For this system, we found for the kinetic coenergy

$$T^* = \frac{1}{2}m\,a^2(\dot{\theta}^2 + \omega^2\sin^2\theta) + \frac{1}{2}mv^2 + mav(\dot{\theta}\cos\theta\cos\omega t - \omega\sin\theta\sin\omega t) \tag{a}$$

and for the potential energy

$$V = mga(1 - \cos\theta) \tag{b}$$

Thus the Lagrangian (2.6.18) $L = T^* - V$ becomes

$$L = \frac{1}{2}m\,a^2(\dot{\theta}^2 + \omega^2\sin^2\theta) + \frac{1}{2}mv^2 + mav(\dot{\theta}\cos\theta\cos\omega t - \omega\sin\omega t) - mga(1 - \cos\theta) \quad \text{(c)}$$

For Lagrange's equation, we need, with $q_1 = \theta$,

$$\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} + mav\cos\theta\cos\omega t \quad \text{(d)}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ma^2\ddot{\theta} - mav(\dot{\theta}\sin\theta\cos\omega t + \omega\cos\theta\sin\omega t) \quad \text{(e)}$$

$$\frac{\partial L}{\partial \theta} = ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta - mav(\dot{\theta}\sin\theta\cos\omega t + \omega\cos\theta\sin\omega t) \quad \text{(f)}$$

Entering the results of equations (e) and (f) into Lagrange's equation (2.6.19)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \quad \text{(g)}$$

results in

$$ma\ddot{\theta} - ma\omega^2\sin\theta\cos\theta + mg\sin\theta = 0 \quad \text{(h)}$$

as found earlier via D'Alembert's principle.

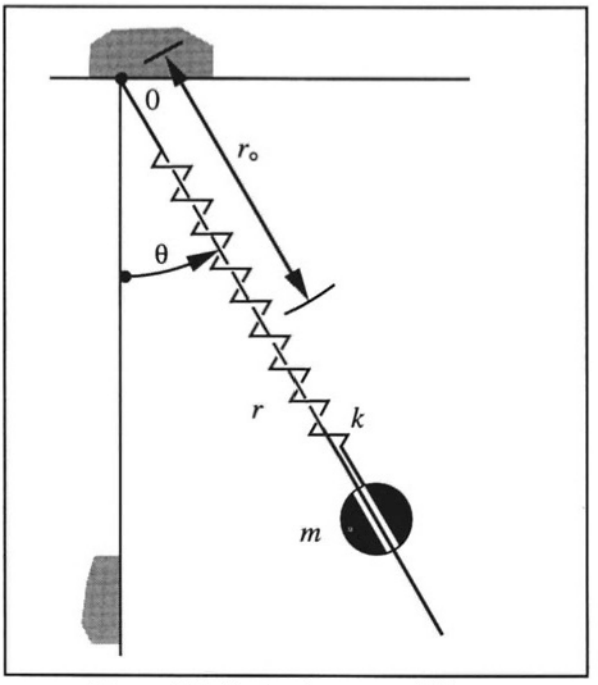


Figure 2.7 Spring-borne pendulum

Example 2.6.2

Consider the two degrees of freedom system shown in Figure 2.7, with point mass m and generalized positions $q_1 = r$ and $q_2 = \theta$. For this system, we have for the complementary kinetic energy

$$T^* = \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2] \tag{a}$$

and for the potential energy

$$V = -mg(r - r_o) + mgr(1 - \cos\theta) + \frac{1}{2}k(r - r_o)^2 \tag{b}$$

where r_o is the original unstretched length of the spring. Forming the Lagrangian $L = T^* - V$, we find for the r coordinate

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \text{and} \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - k(r - r_o) + mg\cos\theta \tag{c,d}$$

and applying the Lagrange equation (2.6.19) in its standard form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + k(r - r_o) - mg\cos\theta = 0 \tag{e}$$

Similarly for the θ coordinate,

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \text{and} \qquad \frac{\partial L}{\partial \theta} = -mgr\sin\theta \tag{f,g}$$

and applying the Lagrange equation (2.6.19) in its standard form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0 \tag{h}$$

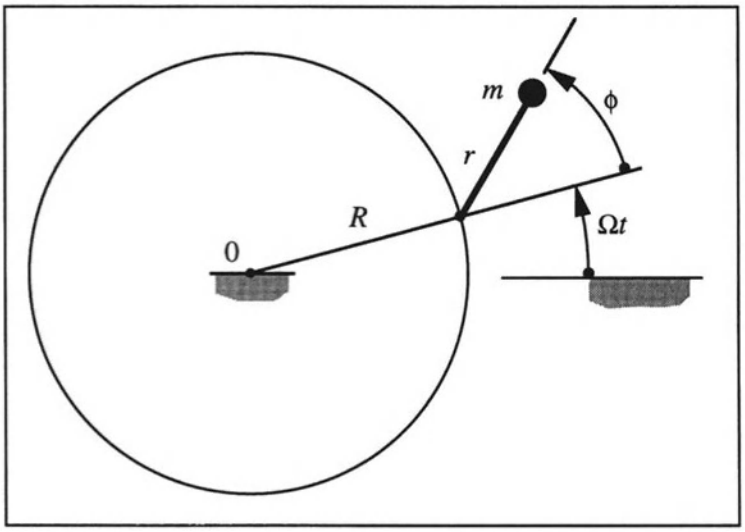
or, after cancelling r,

$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} + mg\sin\theta = 0 \tag{i}$$

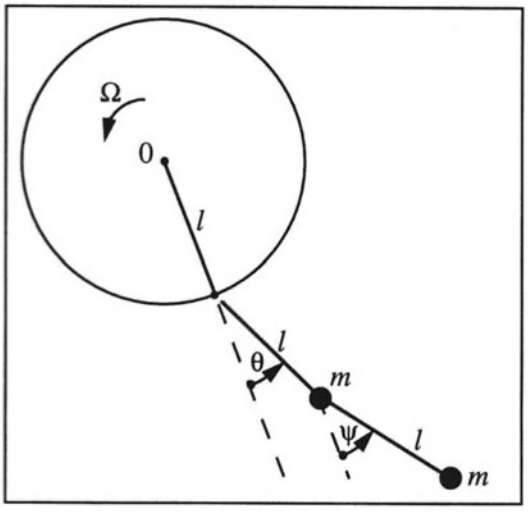
Note that equation (e) involves forces, while equation (h) involves torques.

Problems

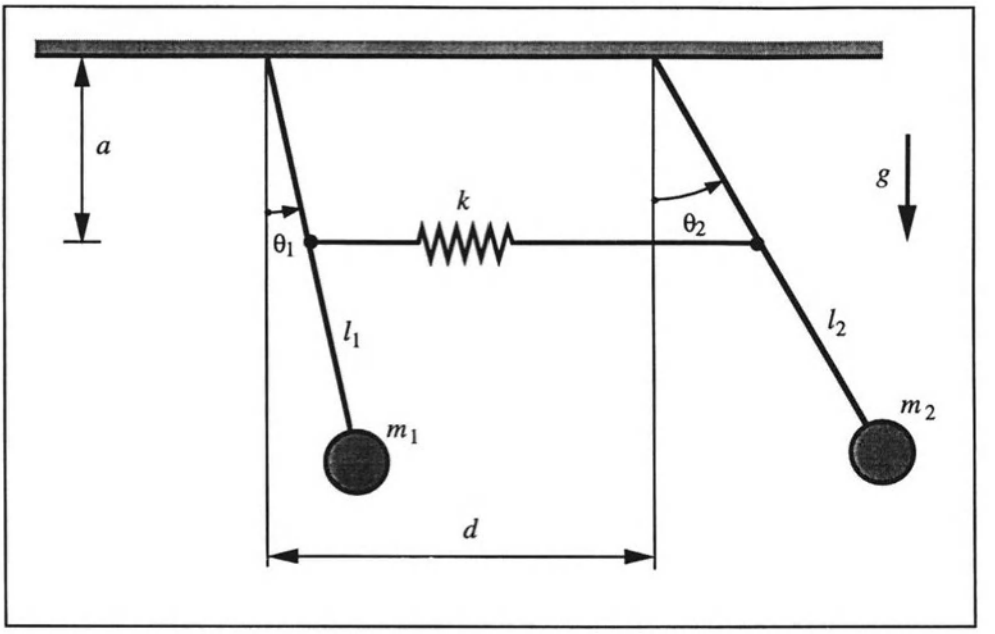
2.6.1 The simple pendulum of length r shown is connected to the edge of a disk of radius R by a pin joint. The link is massless and the horizontal disk rotates about its centre O at a constant angular velocity Ω. (a) Use the Lagrangian equation to obtain the equation of motion in terms of ϕ. (b) What is the natural frequency of the pendulum's vibration for small angles ϕ ?



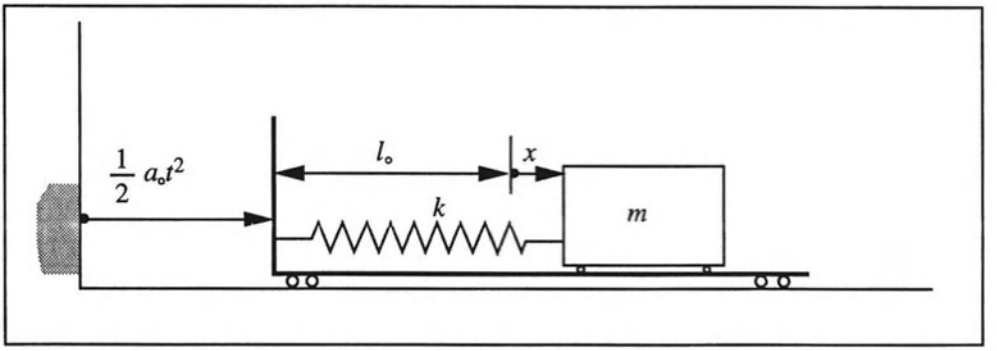
2.6.2 The double pendulum shown is connected to the edge of a horizontal disk by a pin joint. The links are massless and the disk rotates about its centre O, with a constant angular velocity Ω. Form the Lagrangian and obtain the equations of motion for this system in terms of θ and ψ.



2.6.3 Form the Lagrangian and obtain the equations of motion for small amplitudes of the system shown. The links of lengths l_1, l_2 are rigid and massless. The linear spring k is connected at a distance a from the supports. The spring is unstretched when the links are vertical.



2.6.4 Use Lagrange's equation to derive the equation of motion of the spring-borne mass m located on a (massless) car moving at a constant linear acceleration.



2.7 Invariance of Lagrange's Equations

We have pointed out that the *form* of Newton's equation changes when a coordinate transformation is effected. The only exception being the Galilean transformation where the reference frames have relative uniform translational motion. The same is true for D'Alembert's principle. By contrast, the *form* of Lagrange's equation is *invariant* under a coordinate transformation. This can be shown as follows.

Let two possible sets of coordinates be $(q_1,q_2,...,q_i,....,q_n)$ and $(q_1',q_2',...,q_j',....,q_n')$. There exists a one to one relation between these two coordinate systems expressed as

$$q_i = q_i(q_1',q_2',...,q_j',.....,q_n',t) \tag{2.7.1}$$

Then

$$\dot{q}_i = \frac{\partial q_i}{\partial q_j'}\dot{q}_j' + \frac{\partial q_i}{\partial t} \tag{2.7.2}$$

The Lagrangian, being a scalar quantity, has the same value in the two coordinate systems i.e.

$$L(q,\dot{q},t) = L'(q',\dot{q}',t) \tag{2.7.3}$$

Consider now the various terms of Lagrange's equation

$$\frac{\partial L'}{\partial q_j'} = \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial q_j'} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial q_j'} \tag{2.7.4}$$

But from equation (2.7.2), we deduce that

$$\begin{aligned}\frac{\partial \dot{q}_i}{\partial q_j'} &= \frac{\partial^2 q_i}{\partial q_j'\partial q_k'}\dot{q}_k' + \frac{\partial^2 q_i}{\partial q_j'\partial t} \\ &= \frac{d}{dt}\frac{\partial q_i}{\partial q_j'}\end{aligned} \tag{2.7.5}$$

Hence equation (2.7.4) may be written as

$$\frac{\partial L'}{\partial q_j'} = \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial q_j'} + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\frac{\partial q_i}{\partial q_j'}$$

Next consider the term

$$\frac{\partial L'}{\partial \dot{q}_j'} = \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{q}_j'} \tag{2.7.6}$$

Once again from equation (2.7.2), we find that

$$\frac{\partial \dot{q}_i}{\partial \dot{q}_j'} = \frac{\partial q_i}{\partial q_j'}$$

Substituting into equation (2.7.6), we find

$$\frac{\partial L'}{\partial \dot{q}_j'} = \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial q_j'} \tag{2.7.7}$$

Now writing Lagrange's equation for the q_j' system, we have that

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j'} - \frac{\partial L'}{\partial q_j'} = \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial q_j'}\right\} - \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial q_j'} - \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\frac{\partial q_i}{\partial q_j'} \tag{2.7.8}$$

$$= \left\{ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right\} \frac{\partial q_i}{\partial q_j'}$$

Since the Jacobian of the transformation $\frac{\partial q_i}{\partial q_j'}$ is nonvanishing, the invariance of equation (2.7.8) follows.

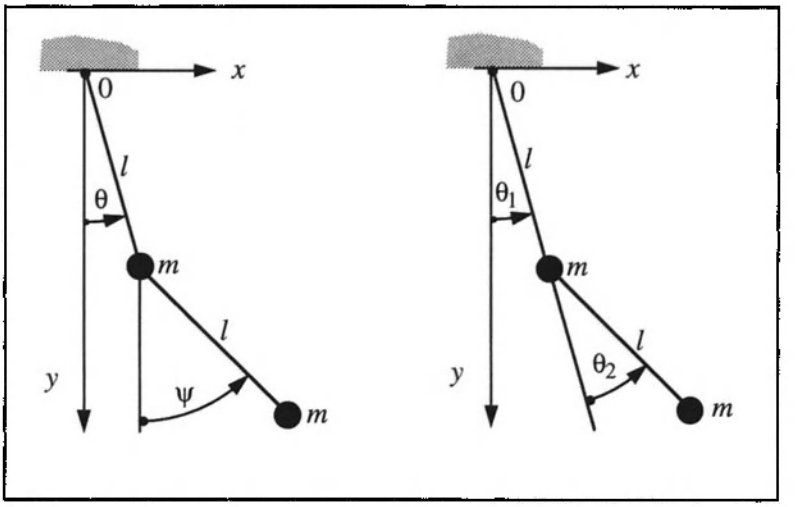


Figure 2.8 Two different generalized coordinate systems for a double pendulum

Example 2.7.1

For the double pendulum shown in Figure 2.8 the complementary kinetic energy and the potential energy may be written as

$$T^* = \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \qquad V = -mg\,(y_1 + y_2) \tag{a}$$

But x_i, y_i etc are not independent and must satisfy the constraint equations

$$x_1^2 + y_1^2 = l^2 \qquad (x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2 \tag{b}$$

These equations may be satisfied directly in terms of the generalized coordinates θ and ψ through the reduced transformation equations

$$x_1 = l\sin\theta \qquad y_1 = l\cos\theta \tag{c}$$

$$x_2 = l\,(\sin\theta + \sin\psi) \qquad y_2 = l\,(\cos\theta + \cos\psi) \tag{d}$$

Using these equations we may express the energy terms in terms of the generalized coordinates as follows

$$T^* = \frac{1}{2}\, ml^2(2\dot{\theta}^2 + 2\dot{\theta}\dot{\psi}\cos(\psi - \theta) + \dot{\psi}^2) \tag{e}$$

$$V = -mgl\,(2\cos\theta + \cos\psi) \tag{f}$$

Using Lagrange's equation (2.6.19) for the coordinate θ yields

$$2\ddot{\theta} + \ddot{\psi}\cos(\psi - \theta) - \dot{\psi}^2 \sin(\psi - \theta) + 2\frac{g}{l}\sin\theta = 0 \quad \text{(g)}$$

and for the coordinate ψ

$$\ddot{\theta}\cos(\psi - \theta) + \ddot{\psi} + \dot{\theta}^2 \sin(\psi - \theta) + \frac{g}{l}\sin\psi = 0 \quad \text{(h)}$$

Next let us use the generalized coordinates θ_1, θ_2. Noting that $\theta_1 = \theta$ and $\theta_2 = \psi - \theta$, we may write the complementary kinetic energy as

$$T^* = \frac{1}{2}ml^2(2\dot{\theta}_1^2 + 2(\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_1\cos\theta_2 + (\dot{\theta}_1 + \dot{\theta}_2)^2) \quad \text{(i)}$$

and the potential energy

$$V = -mgl(2\cos\theta_1 + \cos(\theta_1 + \theta_2)) \quad \text{(j)}$$

Using Lagrange's equation (2.6.19) for the coordinate θ_1 yields

$$2\ddot{\theta}_1 + (2\ddot{\theta}_1 + \ddot{\theta}_2)\cos\theta_2 - (2\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2\sin\theta_2 + \ddot{\theta}_1 + \ddot{\theta}_2 + \frac{g}{l}(2\sin\theta_1 + \sin(\theta_1 + \theta_2)) = 0 \quad \text{(k)}$$

and for the coordinate θ_2

$$\ddot{\theta}_1(1 + \cos\theta_2) + \ddot{\theta}_2 + \dot{\theta}_1^2 \sin\theta_2 + \frac{g}{l}\sin(\theta_1 + \theta_2) = 0 \quad \text{(l)}$$

Now we have

$$\frac{\partial\theta}{\partial\theta_1} = 1 \quad \text{and} \quad \frac{\partial\psi}{\partial\theta_1} = 1; \quad \frac{\partial\theta}{\partial\theta_2} = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\theta_2} = 1 \quad \text{(m)}$$

Equation (l) may be obtained from equations (g) and (h) through use of equation (2.7.7), or

$$\frac{d}{dt}\frac{\partial L'}{\partial\dot{\theta}_2} - \frac{\partial L'}{\partial\theta_2} = \left\{\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta}\right\}\frac{\partial\theta}{\partial\theta_2} + \left\{\frac{d}{dt}\frac{\partial L}{\partial\dot{\psi}} - \frac{\partial L}{\partial\psi}\right\}\frac{\partial\psi}{\partial\theta_2} \quad \text{(n)}$$

Thus multiplication of equation (g) with zero, and equation (h) by one, and introduction of the new coordinates gives

$$\ddot{\theta}_1\cos\theta_2 + \ddot{\theta}_1 + \ddot{\theta}_2 + \dot{\theta}_1^2 \sin\theta_2 + \frac{g}{l}\sin(\theta_1 + \theta_2) = 0 \quad \text{(o)}$$

which is identical to equation (l).

2.8 Complementary Form of Lagrange's Equations

Starting from the complementary form of D'Alembert's principle and following a line of development similar to that used for derivation of Lagrange's equations, we can derive a complementary form of Lagrange's equations. As one may anticipate, various terms in Lagrange's equations have their counterparts in the complementary Lagrange equations. These may be summarized as follows:

Lagrangian Formulation	Complementary Formulation
Cartesian Variables	
r displacements	**I** impulses
F Forces	**v** velocities
B momenta (masses)	**u** stretches (springs)
Generalised Variables	
q_i generalized displacements	S_j generalized impulses
Q_i generalized forces	s_j generalized speed
p_i generalized momenta	e_j generalized stretch
Constraints	
q_i must satisfy kinematic constraints	S_j must satisfy equilibrium equations
Energy terms	
T^* = complementary kinetic energy	T = kinetic energy
V = potential energy	V^* = complementary potential energy
$L = T^* - V$ = Lagrangian (2.6.18)	$L^* = T - V^*$ = complementary Lagrangian (2.8.1)
Virtual work	
of forces	of velocities
$\delta W_I = Q_i' \, \delta q_i$	$\delta W_{II} = s_j' \, \delta S_j$
Lagrange equations	
$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i'$ (2.6.20)	$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}_j} - \frac{\partial L^*}{\partial S_j} = s_j'$ (2.8.2)

Example 2.8.1

As an illustration of an application of the complementary Lagrange equation (2.8.3), consider the motion of a simple pendulum (Figure 2.9). The kinetic energy of the point mass m is

$$T = \frac{B^2}{2m} \tag{a}$$

Invoking relation (1.1.8) and introducing a generalized angular impulse $S = I\, l = Bl$, the kinetic energy becomes

$$T = \frac{S^2}{2ml^2} \tag{b}$$

To evaluate the potential coenergy V^*, let us start from the more familiar expression for the potential energy V.

$$V = mgl(1 - \cos\theta) \tag{c}$$

Then

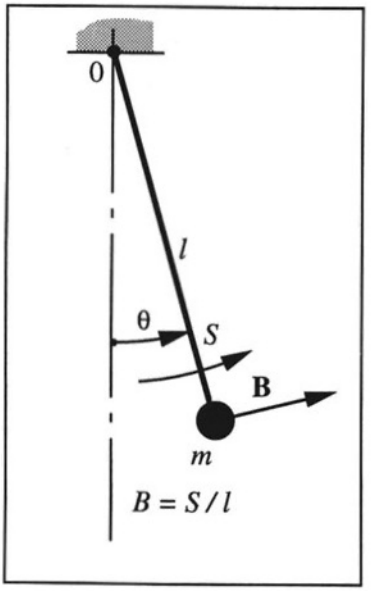


Figure 2.9 Generalized impulse for a simple pendulum

$$\frac{\partial V}{\partial \theta} = mgl \sin\theta = -Q_\theta = -M_\theta \tag{d}$$

where the generalized force $Q_\theta = M_\theta$ is the torque associated with the angle θ. Inverting equation (d), we find

$$\theta = \arcsin\left[-\frac{M_\theta}{mgl}\right] = -\arcsin\frac{M_\theta}{mgl} \tag{e}$$

Then

$$\begin{aligned} V^* &= -\int \theta \, dM_\theta \\ &= \int \arcsin\left[\frac{M_\theta}{mgl}\right] dM_\theta \\ &= M_\theta \arcsin\frac{M_\theta}{mgl} + \sqrt{(mgl)^2 - M_\theta^2} - mgl \end{aligned} \tag{f}$$

Now to ensure that equilibrium condition is satisfied we let

$$\dot{S} = M_\theta \tag{g}$$

Then

$$V^* = \dot{S} \arcsin\frac{\dot{S}}{mgl} + \sqrt{(mgl)^2 - \dot{S}^2} - mgl \tag{h}$$

The complementary Lagrangian (2.8.1) can now be written as follows

$$L^* = T - V^* = \frac{S^2}{2ml^2} - \dot{S}\arcsin\frac{\dot{S}}{mgl} - \left[(mgl)^2 - \dot{S}^2\right]^{1/2} + mgl \tag{i}$$

$$\frac{\partial L^*}{\partial \dot{S}} = -\arcsin\frac{\dot{S}}{mgl} \tag{j}$$

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}} = -\frac{\ddot{S}}{[(mgl)^2 - \dot{S}^2]^{1/2}} \tag{k}$$

$$\frac{\partial L^*}{\partial S} = \frac{S}{ml^2} \tag{l}$$

Hence the equation of motion is given by

$$\frac{\ddot{S}}{[(mgl)^2 - \dot{S}^2]^{1/2}} + \frac{S}{ml^2} = 0 \tag{m}$$

Equation (m) is a statement of compatibility. In effect, it states that the rate $\dot{e}$ of the (angular) extension in the gravitational field must be equal to the (angular) velocity s of the mass, with

$$\dot{e} = \frac{d}{dt}\left[-\frac{\partial V^*}{\partial \dot{S}}\right] = -\frac{\ddot{S}}{[(mgl)^2 - \dot{S}^2]^{1/2}} = \dot{\theta}$$

and

$$s = \frac{\partial T}{\partial S} = \frac{S}{ml^2} = \dot{\theta}$$

For small angles, one can see from equations (d) and (g) that $\dot{S}$ is very small compared to mgl. For such conditions, we may drop $\dot{S}^2$ as compared to $(mgl)^2$ in equation (m), and simplify this equation to

$$\ddot{S} + \frac{g}{l}S = 0 \tag{n}$$

which is identical to the linearized pendulum equation obtained in example 2.2.3.

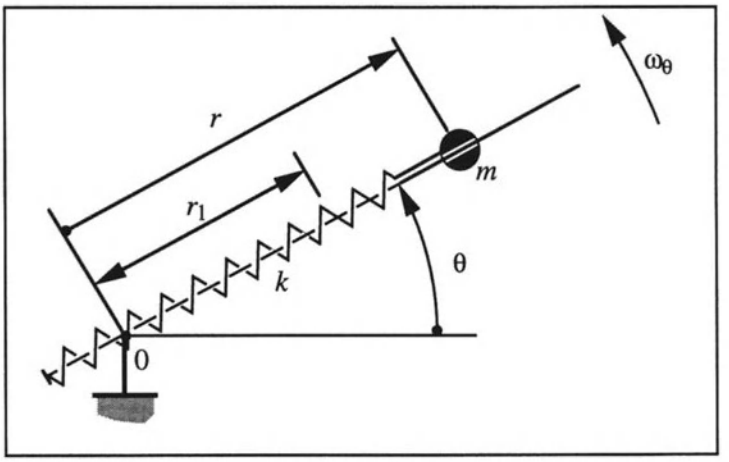


Figure 2.10 Bead moving in horizontal plane at a constant angular speed

Example 2.8.2

The system shown in Figure 2.10 involves a bead m sliding without friction along a massless rod. The bead is attached to a linear spring. The rod rotates at a prescribed constant $\dot{\theta} = \omega_\theta$ in a horizontal plane. The system's equation of motion is to be established (a) by the conventional approach and (b) by the complementary approach.

(a) *Generalized Displacement Formulation:*

With $q = r, \dot{q} = \dot{r}$, the system's kinetic coenergy is

$$T^* = T^*(q, \dot{q}) = \frac{1}{2} m\dot{q}^2 + \frac{1}{2} m\omega_\theta^2 q^2 \tag{a}$$

and its potential energy is

$$V = V(q) = \frac{1}{2} k(q - r_1)^2 \tag{b}$$

For simplicity we choose $V = 0$ at $q = 0$ and obtain the Lagrangian

$$L = L(q, \dot{q}) = T^* - V = \frac{1}{2} m\dot{q}^2 - \frac{1}{2}(k - m\omega_\theta^2)\, q^2 \tag{c}$$

Lagrange's equation (2.6.20)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{d}$$

then leads to

$$m\ddot{q} + (k - m\omega_\theta^2)q = 0 \tag{e}$$

and represents the system's equilibrium equation of motion expressed in terms of a generalized displacement variable.

(b) *Generalized Impulse Formulation:*

We may view equation (e) as that of a nonrotating oscillator with a modified spring constant $(k - m\omega_\theta^2)$. Now letting the impulse in the modified spring be S, we may write for the complementary Lagrangian

$$L^*(S, \dot{S}) = \frac{S^2}{2m} + \frac{\dot{S}^2}{2(k - m\omega_\theta^2)} \tag{f}$$

and we may obtain the compatibility equation of motion from equation (2.8.2)

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}} - \frac{\partial L^*}{\partial S} = 0 \tag{g}$$

as

$$\frac{\ddot{S}}{k - m\omega_\theta^2} + \frac{S}{m} = 0 \tag{h}$$

Example 2.8.3

As an illustration of an application of the complementary Lagrange equation involving speeds s without energy, consider the system in Figure 2.11.

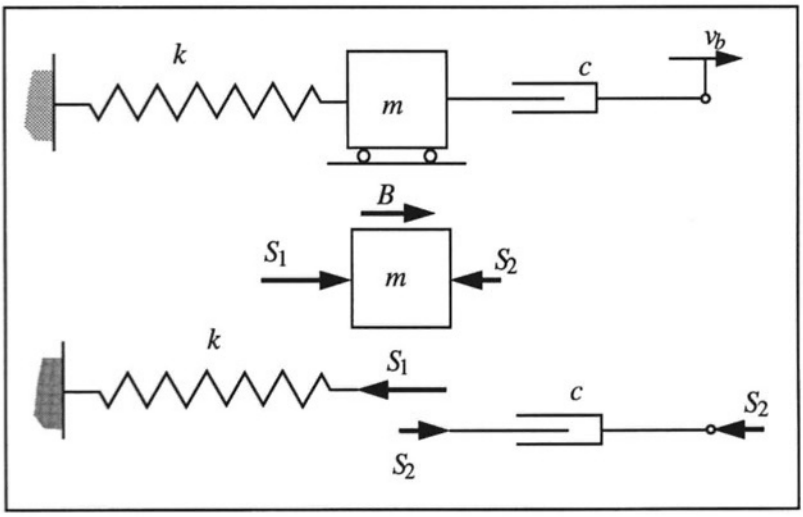


Figure 2.11 System with speed without energy

The kinetic energy is

$$T = \frac{B^2}{2m} \tag{a}$$

The equilibrating impulse relationship is

$$B = S_1 - S_2 \tag{b}$$

Thus

$$T = \frac{(S_1 - S_2)^2}{2m} \tag{c}$$

The potential coenergy is

$$V^* = \frac{\dot{S}_1^2}{2k} \tag{d}$$

The complementary Lagrangian is consequently

$$L^* = T - V^* = \frac{(S_1 - S_2)^2}{2m} - \frac{\dot{S}_1^2}{2k} \tag{e}$$

The complementary Lagrange equation for S_1 is

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}_1} - \frac{\partial L^*}{\partial S_1} = 0 \tag{f}$$

resulting in

$$\frac{\ddot{S}_1}{k} + \frac{S_1 - S_2}{m} = 0 \tag{g}$$

The complementary Lagrange equation for S_2 is

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}_2} - \frac{\partial L^*}{\partial S_2} = s_2' \tag{h}$$

where s_2' is a speed without energy and includes the dashpot's speed $\dot{S}_2/c$ and the prescribed velocity v_b of the free end. Thus

$$s_2' = \frac{\dot{S}_2}{c} + v_b \tag{i}$$

Equation (h) with (i) then results in

$$\frac{S_1 - S_2}{m} = \frac{\dot{S}_2}{c} + v_b \tag{j}$$

It is worth noting that the effect of the dashpot can be taken into account by means of a potential-type function defined as

$$\dot{D}^* = \frac{\dot{S}_2^2}{2c} \tag{k}$$

which can be incorporated in the complementary Lagrange equation

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}_j} - \frac{\partial L^*}{\partial S_j} - \frac{\partial \dot{D}^*}{\partial \dot{S}_j} = s_j'' \tag{l}$$

We shall consider such potentials for dissipative forces in section 2.9.

Rearranging equations (g) and (j) provides us with the compatibility equations of motion

$$\frac{\ddot{S}_1}{k} + \frac{S_1}{m} - \frac{S_2}{m} = 0 \tag{m}$$

$$\frac{\dot{S}_2}{c} + \frac{S_2}{m} - \frac{S_1}{m} = -v_b \tag{n}$$

Problems

2.8.1 Solve Problem 2.6.3 by means of the complementary form of Lagrange's equation.

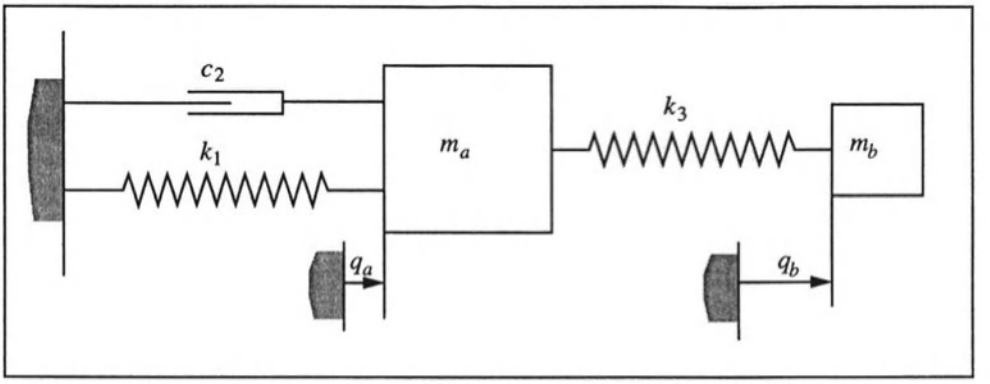
2.8.2 Solve Problem 2.6.4 by means of the complementary form of Lagrange's equation.

2.8.3 A simple Duffing-type oscillator has a differential equation of motion given by

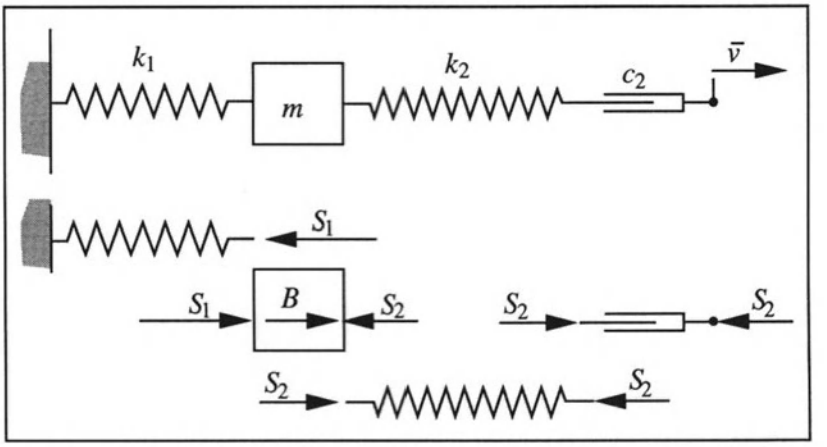
$$m\ddot{x} + hx^3 = 0$$

Use the complementary Lagrange equation to derive the compatibility equation of motion.

2.8.4 Establish the equations of motion for the system shown, by (a) Lagrange equations, and (b) complementary Lagrange equations.

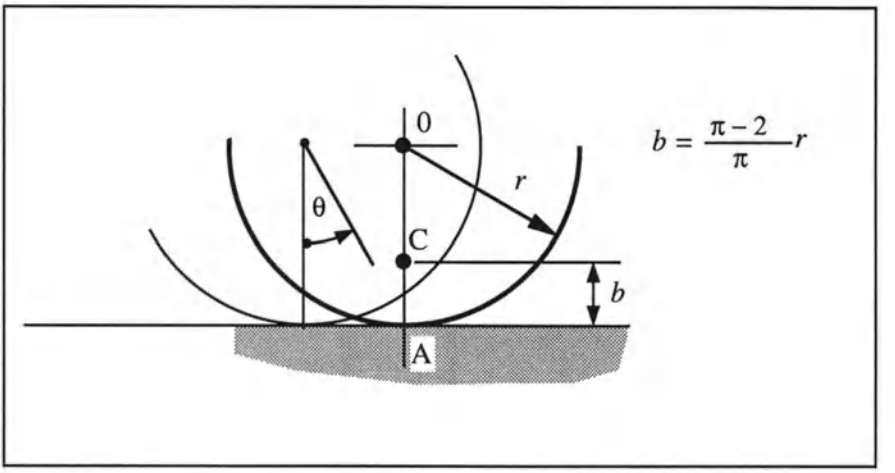


2.8.5 Use complementary Lagrange equations to establish the equations of motion in terms of the generalized impulses S_1 and S_2.



2.8.6 Establish the equation of motion for the system of Figure 2.10 when $r_1 \neq 0$, using a generalized impulse variable S_r, with $\dot{S}_r = -k(r - r_1)$.

2.8.7 For the semicircular thin-walled shell shown assume small rocking amplitudes, and establish the equations of motion, (a) using $q_1 = \theta$ as generalized displacement, and (b) using the linear impulse S in horizontal direction as generalized impulse. (c) Determine the period of rocking motion.



2.9 Dissipation Forces

Invariably, the motion of a system brings into play a set of forces which cause a transformation of system's total mechanical energy into nonmechanical, i.e. non-recoverable, forms. Such *dissipative* forces may be closely represented, at least over certain ranges of velocity v, by

$$F_D = -av^n \tag{2.9.1}$$

Equation (2.9.1) will be recognized as another example of a constitutive relation incorporating a physical property a. Several cases of interest may be immediately identified for various values of n. Thus for $n = 0$, we obtain the classical *Coulomb friction* for which the dissipative force is independent of the relative velocity between the rubbing surfaces but is a function of the normal force between the surfaces and the sign of the relative velocity. For $n = 1$, equation (2.9.1) yields the widely used viscous damping case. For atmospheric drag $n = 2$ is approximately correct. Other values of n may be appropriate in other cases.

To account for dissipative forces in the equilibrium equations, one must write down the virtual work of such forces. In the subsequent discussion, n is an odd integer. If this is not the case, special precaution must be taken to ensure that the virtual work is indeed negative.

Now, let us consider for example, the dissipative force on a single particle moving in the xy plane. If the force, in the direction of v, is given by equation (2.9.1), then the components of F_D along the x and y direction will be

$$F_{Dx} = F_D \cos\theta = -av^n \cos\theta$$

$$F_{Dy} = F_D \sin\theta = -av^n \sin\theta$$

But $\cos\theta = \dot{x}/v$ and $\sin\theta = \dot{y}/v$.

Therefore, the virtual work for such a force may be written as

$$\delta W_I = -av^{n-1}(\dot{x}\ \delta x + \dot{y}\ \delta y) \tag{2.9.2}$$

or since $v = (\dot{x}^2 + \dot{y}^2)^{1/2}$

$$\delta W_I = -a(\dot{x}^2 + \dot{y}^2)^{\frac{n-1}{2}} (\dot{x}\ \delta x + \dot{y}\ \delta y) \tag{2.9.3}$$

For a system of N, such particles moving in the xy plane, the total virtual work becomes

$$\delta W_{I\ total} = -a_i \left[\left(\dot{x}^2 + \dot{y}^2 \right)^{\frac{n-1}{2}} \left[\dot{x}\ \delta x + \dot{y}\ \delta y \right] \right]_i \tag{2.9.4}$$

Finally, it remains to express $\dot{x}_j$, δx_j, etc. in terms of the generalized variables q_i, $\dot{q}_i$ and δq_i, to find the associated generalized forces.

If the reduced transformation equations are

$$x_j = x_j(q,t) \qquad \text{and} \qquad y_j = y_j(q,t)$$

then

$$\dot{x}_j = \frac{\partial x_j}{\partial q_i}\dot{q}_i + \frac{\partial x_j}{\partial t} \qquad \text{and} \qquad \dot{y}_j = \frac{\partial y_j}{\partial q_i}\dot{q}_i + \frac{\partial y_j}{\partial t}$$

and

$$\delta x_j = \frac{\partial x_j}{\partial q_i}\delta q_i \qquad \text{and} \qquad \delta y_j = \frac{\partial y_j}{\partial q_i}\delta q_i$$

On using these equations in equation (2.9.4), we may express the total virtual work as

$$\delta W_{I\ total} = -Q_{Di}\ \delta q_i$$

and thereby identify generalized dissipation forces Q_{Di}.

Given a constitutive relation, as in equation (2.9.1), it is natural to enquire whether such a relation admits the development of a function from which the dissipative forces may be derived, in much the same way as one derives conservative forces from a potential energy function. What is required is a function $\dot{D}(v)$ such that

$$\frac{\partial \dot{D}}{\partial v} = -F_D \tag{2.9.5}$$

From equation (2.9.5), it is clear that $\dot{D}$ has the dimensions of power, and it is therefore referred to as the *power function*. For the constitutive relation in equation (2.9.1), one can see that

$$\dot{D} = \frac{av^{n+1}}{n+1} \tag{2.9.6}$$

For the case of viscous damping when $n = 1$, the power function becomes

$$\dot{D} = \frac{1}{2} av^2 \tag{2.9.7}$$

In this case, $\dot{D}$ $(= -\frac{1}{2} F_D v)$ is known as *Rayleigh's*† dissipation function, and one can see that in this case, the dissipation function is equal to one half the rate of work done by the dissipative force.

Now as in the case of potential and kinetic energy functions, one can also define a function that complements the power function. This can be done whenever the constitutive relation is invertible. In this case, the complementary power function $\dot{D}^*$ is so defined as to express the constitutive relation as

$$\frac{\partial \dot{D}^*}{\partial F_D} = -v \tag{2.9.8}$$

for the relation given in equation (2.10.1), one can show that

$$\dot{D}^* = \frac{n}{n+1} F_D \left[\frac{F_D}{a}\right]^{\frac{1}{n}} \tag{2.9.9}$$

For the case of viscous damping when $n = 1$, $\dot{D} = \dot{D}^*$.

The existence of a power function allows one to incorporate the associated dissipative forces into Lagrange's equations. In this case, we require the generalized dissipative force given by the derivative of $\dot{D}$ with respect to the generalized velocity, i.e.

$$Q_{Di} = -\frac{\partial \dot{D}}{\partial \dot{q}_i} = -\left[\frac{\partial \dot{D}}{\partial \dot{x}_j}\frac{\partial \dot{x}_j}{\partial \dot{q}_i} + \frac{\partial \dot{D}}{\partial \dot{y}_j}\frac{\partial \dot{y}_j}{\partial \dot{q}_i} + \frac{\partial \dot{D}}{\partial \dot{z}_j}\frac{\partial \dot{z}_j}{\partial \dot{q}_i}\right] \tag{2.9.10}$$

Now substituting into Lagrange's equations, we obtain

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial \dot{D}}{\partial \dot{q}_i} = Q_i'' \tag{2.9.11}$$

where the Q_i'' are forces which are not monogenic, i.e. do not possess a potential energy function or a dissipation function. Analogous development for the complementary procedure gives rise to the following equation

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}_j} - \frac{\partial L^*}{\partial S_j} - \frac{\partial \dot{D}^*}{\partial \dot{S}_j} = s_j'' \tag{2.9.12}$$

in terms of generalised impulse variables. The last term on the left hand side provides the rate of stretch for the dashpots i.e.

$$\dot{e}_{Dj} = -\partial \dot{D}^* / \partial \dot{S}_j \tag{2.9.13}$$

and the s_j'' accounts for all other velocities not present on the left hand side, i.e. not derived from a kinetic energy function.

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† John William Strutt, Lord Rayleigh (1842-1919), English physicist

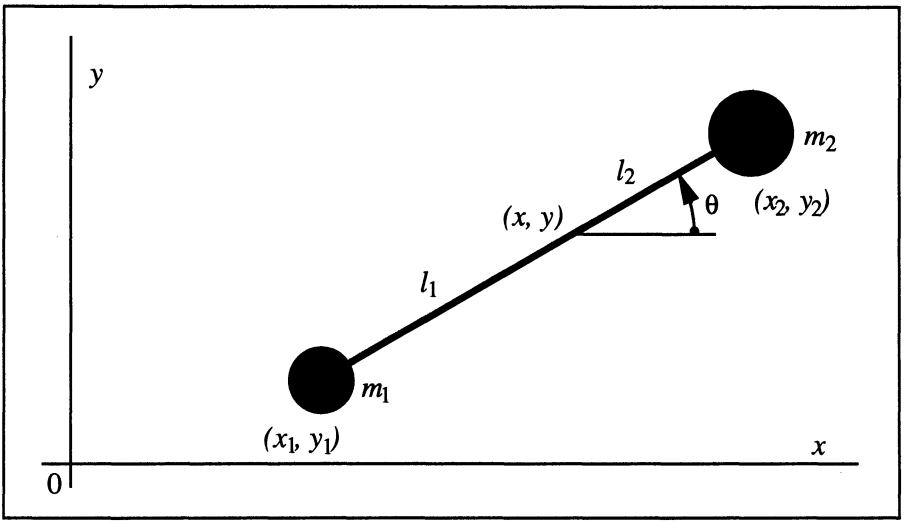


Figure 2.12 Two interconnected particles on a horizontal plane

Example 2.9.1

Consider the two particles in a horizontal plane connected by a massless rigid rod, as shown in Figure 2.12. Let us assume that damping forces on m_1 and m_2 are given by $F_{D_1} = -a_1 v_1$ and $F_{D_2} = -a_2 v_2$. The generalized dissipative forces may be found as follows.

Method 1

Let us write down the virtual work of the two damping forces

$$\delta W_I = -[a_1 \dot{x}_1 \delta x_1 + a_1 \dot{y}_1 \delta y_1 + a_2 \dot{x}_2 \delta x_2 + a_2 \dot{y}_2 \delta y_2] \tag{a}$$

But x_1, y_1, etc. are not independent. If we choose to work with the three degrees of freedom x, y, and θ, as shown in Figure 2.13, we may write the following reduced transformation equations

$$x_1 = x - l_1 \cos\theta \qquad \text{and} \qquad y_1 = y - l_1 \sin\theta \tag{b}$$

$$x_2 = x + l_2 \cos\theta \qquad\qquad y_2 = y + l_2 \sin\theta$$

Evaluating δx_1, etc. and $\dot{x}_1$, etc. in terms of δx, δy, $\delta\theta$ and $\dot{x}$, $\dot{y}$, and $\dot{\theta}$ respectively, from equation (b) and substituting in equation (a), we find

$$\begin{aligned}\delta W_I = &- [(a_1 + a_2)\dot{x} - (a_2 l_2 - a_1 l_1)\dot{\theta} \sin\theta]\delta x \\ &- [(a_1 + a_2)\dot{y} + (a_2 l_2 - a_1 l_1)\dot{\theta} \cos\theta]\delta y \\ &- [(a_2 l_2^2 + a_1 l_1^2)\dot{\theta} - (a_2 l_2 - a_1 l_1)(\dot{x} \sin\theta - \dot{y} \cos\theta)]\delta\theta \end{aligned} \tag{c}$$

The terms multiplied by δx, δy, $\delta\theta$ are then the generalized dissipative forces.

Method 2

We can also derive the generalized forces from Rayleigh's dissipation function, as follows

$$\dot{D} = \frac{1}{2} a_1 v_1^2 + \frac{1}{2} a_2 v_2^2 \tag{d}$$

where

$$v_1^2 = (\dot{x}_1^2 + \dot{y}_1^2) \qquad v_2^2 = (\dot{x}_2^2 + \dot{y}_2^2) \tag{e}$$

But

$$\dot{x}_1 = \dot{x} + l_1\dot{\theta}\sin\theta \qquad \text{and} \qquad \dot{x}_2 = \dot{x} - l_2\dot{\theta}\sin\theta$$
$$\dot{y}_1 = \dot{y} - l_1\dot{\theta}\cos\theta \qquad \dot{y}_2 = \dot{y} + l_2\dot{\theta}\cos\theta \tag{f}$$

Substituting from equation (f) into equation (e) and then into equation (d), we find

$$\dot{D} = \frac{1}{2} a_1[\dot{x}^2 + \dot{y}^2 + l_1^2\,\dot{\theta}^2 - 2l_1\,\dot{\theta}(\dot{y}\cos\theta - \dot{x}\,\sin\theta)]$$
$$+ \frac{1}{2} a_2[\dot{x}^2 + \dot{y}_2 + l_2^2\,\dot{\theta}^2 + 2l_2\dot{\theta}(\dot{y}\,\cos\theta - \dot{x}\,\sin\theta)] \tag{g}$$

The generalized dissipative forces can now be obtained as

$$\frac{\partial \dot{D}}{\partial \dot{x}} = (a_1 + a_2)\dot{x} - (a_2 l_2 - a_1 l_1)\dot{\theta}\sin\theta = -Q_{Dx} \tag{h}$$

$$\frac{\partial \dot{D}}{\partial \dot{y}} = (a_1 + a_2)\dot{y} - (a_1 l_1 - a_2 l_2)\dot{\theta}\cos\theta = -Q_{Dy} \tag{i}$$

$$\frac{\partial \dot{D}}{\partial \dot{\theta}} = (a_1 l_1^2 + a_2 l_2^2)\dot{\theta} - (a_1 l_1 - a_2 l_2)(\dot{y}\,\cos\theta - \dot{x}\,\sin\theta) = -Q_{D\theta} \tag{j}$$

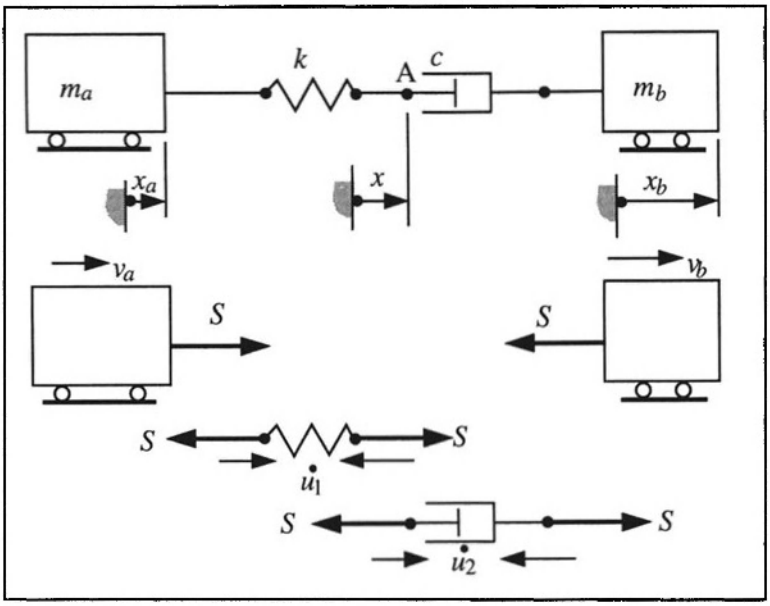


Figure 2.13 A single generalized impulse system

Example 2.9.2

The system shown in Figure 2.13 is comprised of two masses, one linear spring and one viscous damping dashpot. Using the complementary procedure, we may obtain the equations of motion of this system as follows:

Method 1: Using the complementary Lagrange equation

Ensuring the satisfaction of the equilibrium requirements by taking $-B_a = B_b = S$, and $F = \dot{S}$, we may express the energy and polar functions as follows

$$T = S^2/2m_a + S^2/2m_b \tag{a}$$

$$V^* = \dot{S}^2/2k \tag{b}$$

$$D^* = \dot{S}^2/2c \tag{c}$$

Then

$$L^* = S^2\left(\frac{1}{2m_a} + \frac{1}{2m_b}\right) - \frac{\dot{S}^2}{2k} \tag{d}$$

Substituting into the complementary Lagrange equation (2.9.12), we find the equation of motion as

$$\frac{\ddot{S}}{k} + \frac{\dot{S}}{c} + \left[\frac{1}{m_a} + \frac{1}{m_b}\right]S = 0 \tag{e}$$

Method 2: Using the virtual work expression

Using the free body representation of Figure 2.13, where equilibrating impulses are shown, we may write

$$\delta W_{II} = v_a\,\delta S - \dot{e}_1\,\delta S - \dot{e}_2\,\delta S - v_b\,\delta S \tag{a}$$

or

$$\delta W_{II} = (v_a - \dot{e}_1 - \dot{e}_2 - v_b)\,\delta S \tag{b}$$

Now from the equilibrium conditions we have that

$$B_a = -B_b = S \tag{c}$$

and from the constitutive equations

$$v_a = \frac{B_a}{m_a} = \frac{S}{m_a} \tag{d}$$

$$v_b = \frac{B_b}{m_b} = -\frac{S}{m_b} \tag{e}$$

The spring compression rate is

$$\dot{e}_1 = -\frac{\ddot{S}}{k} \tag{f}$$

and the damper compression rate is

$$\dot{e}_2 = -\frac{\dot{S}}{c} \tag{g}$$

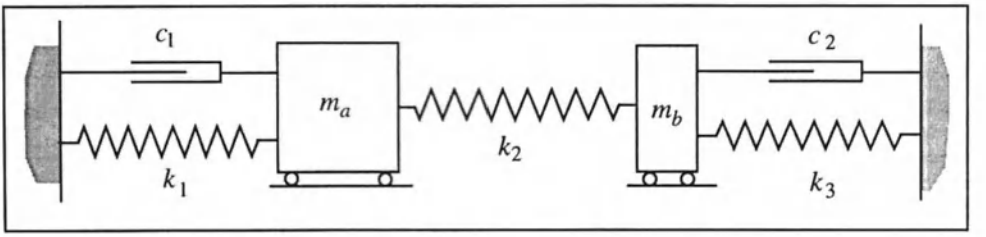
Substituting back into equation (b), we find

$$\frac{\ddot{S}}{k} + \frac{\dot{S}}{c} + \left[\frac{1}{m_a} + \frac{1}{m_b}\right] S = 0 \tag{h}$$

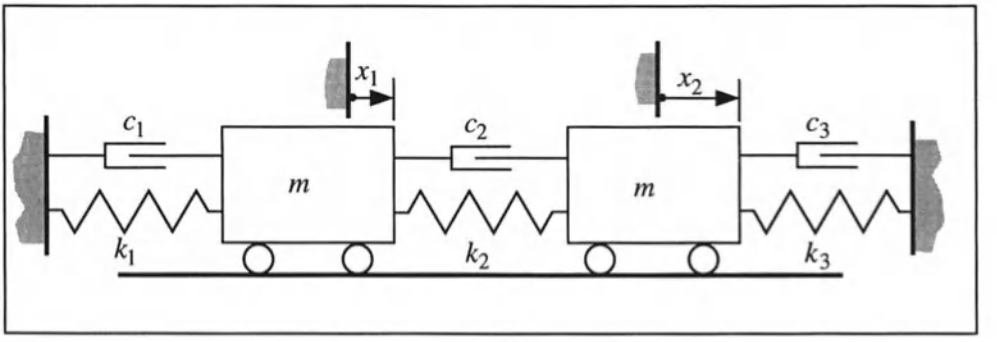
It is of interest to note that if this system is analyzed via the displacement formulation, one would require three variables x_a, x and x_b. However for the coordinate x we would obtain an equilibrium equation for the elastic and dissipative forces only. That is for this coordinate there would be no inertial forces. By contrast in the complementary formulation the system has one variable, namely S.

Problems

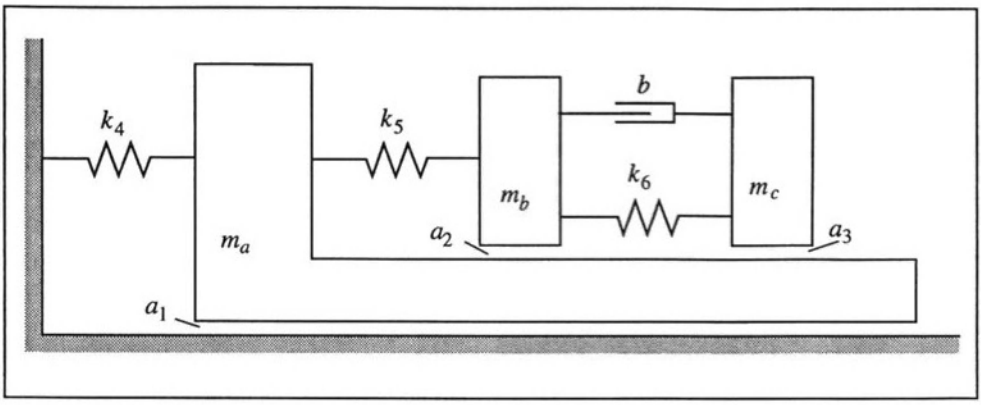
2.9.1 Given is a simple oscillator with dashpot. (a) Find the Lagrangian and the Rayleigh dissipation function, and determine the equation of motion by means of a modified Lagrange equation. (b) Repeat the problem using the complementary formulation.



2.9.2 Use the modified form of Lagrange's equation to derive the equations of motion of the oscillator shown.



2.9.3 For the system shown find the equations of motion using Lagrange's equations, using (a) the direct formulation, and (b) the complementary formulation. For the rubbing surfaces assume viscous friction with coefficients a_1,a_2 and a_3. For the dashpot the coefficient of viscous damping is b.

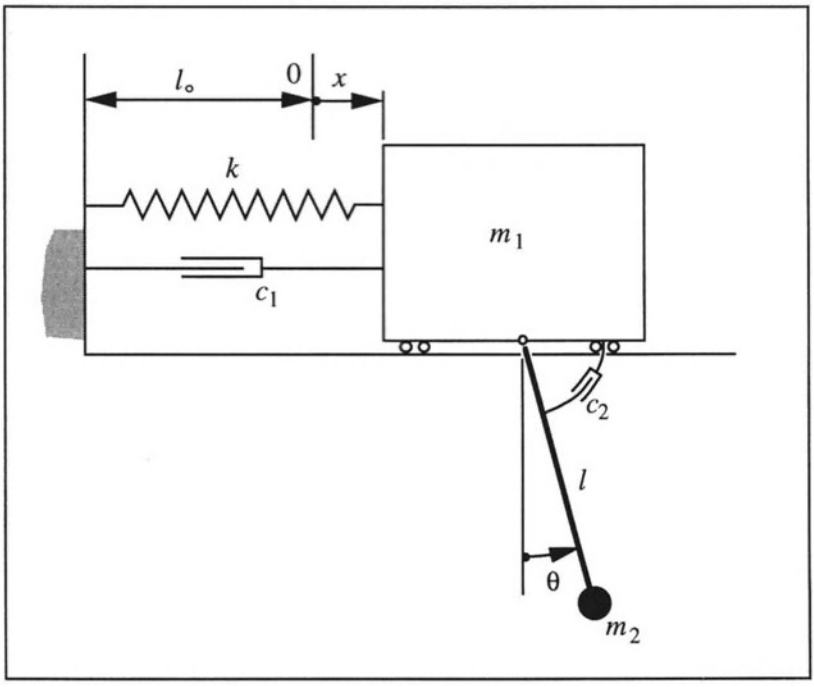


2.9.4 A simple pendulum is subject to air resistance, with the drag given by

$$F_D = - c_d v^2$$

Establish the Lagrangian, the power function, and the differential equation of motion. Pay special attention to the sign of the drag!

2.9.5 Establish the equations of motion for the two-mass system shown, for small angles θ.



2.10 Limitations of the Complementary Formulation

Two difficulties can arise in the application of the complementary formulation. The first concerns the determination of the complementary potential energy function. When the force displacement relation,

$$\mathbf{F} = \mathbf{F}(\mathbf{r}) \tag{2.10.1}$$

say for an elastic element, is nonlinear, it may be difficult and in some cases impossible to invert. In the absence of such an inverse relationship it is not possible to evaluate the complementary potential energy function given by

$$V^*(\mathbf{F}) = -\int \mathbf{r}(\mathbf{F})\, d\mathbf{F} \tag{2.10.2}$$

The source of the nonlinearity in equation (2.10.1) may be the material behaviour of the elastic element and/or large charges in the geometry of the elastic element as it is loaded.

The second difficulty arises when some kinematic variables appear in the equilibrium equations. In this case it is not possible to express the momenta of the inertial elements in terms of impulses of the force elements alone. This prevents the compatibility equations of motion from being expressed in terms of impulses alone. To examine this point in greater detail consider the equilibrium equations of motion given by Lagrange's equations

$$\frac{d}{dt}\left[\frac{\partial T^*}{\partial \dot{q}_i}\right] - \frac{\partial T^*}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \tag{2.10.3}$$

Recognising the first term as the time derivative of momenta, and the third term as the forces S_i with a potential energy V, we can write equation (2.10.3) as

$$\frac{dp_i}{dt} = \dot{S}_i + \frac{\partial T^*}{\partial q_i} + Q_i \tag{2.10.4}$$

Integrating this expression with respect to time we find

$$p_i = S_i + \int \frac{\partial T^*}{\partial q_i}\, dt + \int Q_i\, dt \tag{2.10.5}$$

Now the term $(\partial T^*/\partial q_i)$ is non zero when non inertial coordinate system are used and then this term accounts for such apparent forces as Coriolis and centrifugal, that is, this term is essentially kinematic in nature. It will be recalled that in the general case

$$T^* = \frac{1}{2}\{\dot{q}\}^T [A] \{\dot{q}\} + \{b\}^T \{\dot{q}\} + C \tag{2.10.6}$$

where $[A]$, $\{b\}^T$ and C may be functions of q_i. Hence the first integral in equation (2.10.5) may involve $\dot{q}_i$ and q_i. The nonconservative forces Q_i may also be functions of $\dot{q}_i$ and q_i. Thus in the general case the momenta depend on impulses *and* the kinematic variables. Nevertheless the complementary formulation can still be used. Thus from equation (1.9.16) we determine the kinetic energy function as

$$T = \frac{1}{2}\left[S_i + \int \left(\frac{\partial T^*}{\partial q_i} + Q_i\right) dt \right] [A]^{-1} \left\{ S_i + \int \left(\frac{\partial T^*}{\partial q_i} + Q_i\right) dt \right\}$$

$$- \{b\}^T [A]^{-1} \{b\} + T_o^* \tag{2.10.7}$$

and invoking the complementary Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial V^*}{\partial \dot{S}_i}\right] + \frac{\partial T}{\partial S_i} = 0 \tag{2.10.8}$$

we obtain the compatibility equations of motion as

$$\frac{d}{dt}\left[\frac{\partial V^*}{\partial \dot{S}_i}\right] + A_{ij}^{-1}\left[S_j + \int \left(\frac{\partial T^*}{\partial q_j} + Q_j\right) dt \right] = 0 \tag{2.10.9}$$

Evidently the first term describes the time rate of stretch for the elastic elements and the second term gives the velocity of the inertial terms. We see them that in this general case the equation of motion are integro-differential equations. We may of course remove the integral term through differentiation with respect to time. This, however, will result in a higher order equation requiring one additional initial (or other) condition for evaluation of the integration constants.

Example 2.10.1

Let us obtain the equilibrium and compatibility equations of motion for the Kepler central force problem. In Kepler's problem of planetary motion the gravitational force, directed to a central point, which we take as the origin of our axes, is inversely proportional to the square of the distance of the particle from the centre. Thus using the coordinates r and θ, we have that

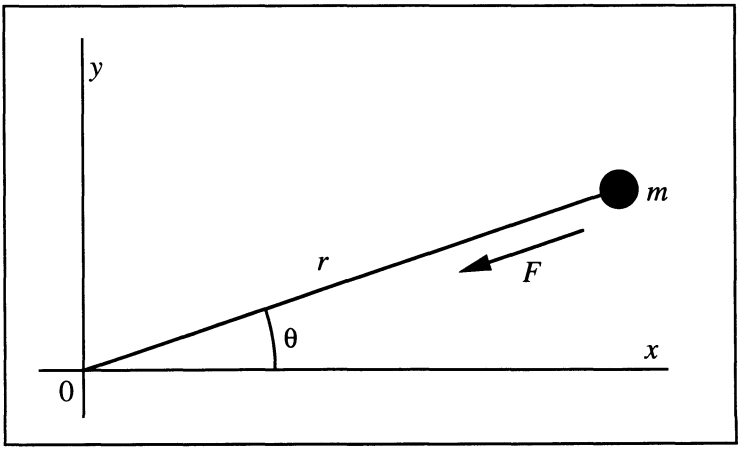


Figure 1.14 The central force problem

$$F = -\frac{\mu m}{r^2} \tag{a}$$

where μ is a constant. This force is conservative and may be obtained from the potential energy

$$V = -\frac{\mu m}{r} \tag{b}$$

The complementary kinetic energy for the particle is given by

$$T^* = \frac{m}{2}\left[\dot{r}^2 + (r\dot{\theta})^2\right] \tag{c}$$

Now evaluating the following terms

$$\frac{\partial T^*}{\partial \dot{\theta}} = mr^2\dot{\theta} = p_\theta \qquad \frac{\partial T^*}{\partial \dot{r}} = m\dot{r} = p_r \tag{d,e}$$

$$\frac{\partial T^*}{\partial r} = mr\dot{\theta}^2 \qquad \frac{\partial T^*}{\partial \theta} = 0 \tag{f,g}$$

$$\frac{\partial V}{\partial \theta} = 0 \qquad \frac{\partial V}{\partial r} = \frac{\mu m}{r^2} \tag{h,i}$$

we can establish the equilibrium equations of motion via Lagrange's equation as

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{j}$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\mu m}{r^2} = 0 \tag{k}$$

The first of these equations shows that the momentum p_θ is constant in time.

To develop the compatibility equations of motion we must satisfy the equilibrium equations (j) and (k). To this end we let

$$mr^2\dot{\theta} = C \tag{e}$$

and rewrite equation (k) as

$$\dot{p}_r - mr\dot{\theta}^2 - \dot{S}_r = 0 \tag{m}$$

where $\dot{S}_r = F$. It should be noted that the centrifugal force, $mr\dot{\theta}^2$ in equation (m) is kinematic in nature and was obtained from $\partial T^*/\partial r$. Integrating equation (m) we can express the momentum p_r as

$$p_r = S_r + \int mr\dot{\theta}^2\, dt \tag{n}$$

We may now obtain the kinetic energy of the particle from the general expression

$$T + T^* = p_r\,\dot{r} + p_\theta\,\dot{\theta} \tag{o}$$

and eliminating $\dot{r}\,(= p_r / m)$ and $\dot{\theta}(= p_\theta / mr^2)$. This leads to

$$T = \frac{p_\theta^2}{2mr^2} + \frac{p_r^2}{2m} = \frac{C^2}{2mr^2} + \frac{p_r^2}{2m} \tag{p}$$

or using equation (n)

$$T = \frac{C^2}{2mr^2} + \frac{\left[S_r + \int mr\dot{\theta}^2\, dt\right]^2}{2m} \tag{q}$$

The force displacement relation in equation (a) is invertible allowing the computation of the complementary potential energy from

$$V^* = \int \left[\frac{\mu m}{F} \right]^{\frac{1}{2}} dF = 2(\mu m F)^{\frac{1}{2}} \tag{r}$$

where we have taken the negative sign outside the square root since its function is simply to account for the opposite signs of r and F. We now express V^* in terms of $\dot{S}_r$ as

$$V^* = 2(\mu m \dot{S}_r)^{\frac{1}{2}} \tag{s}$$

and substitute T and V^* in the complementary Lagrange equation to obtain the compatibility equation of motion as

$$\frac{d}{dt} \left[\frac{\mu m}{\dot{S}_r} \right]^{\frac{1}{2}} + \frac{S_r + \int m r \dot{\theta}^2 \, dt}{m} = 0 \tag{t}$$

Using equation (a) we note that the first term in equation (t) is $(-\dot{r})$. The second can be interpreted via equation (n) and is p_r/m. Hence the compatibility equation of motion, as given in equation (t) is an integro-differential equation and it also involves the kinematic variables r and θ. We may remove the integral term by differentiating through with respect to time. Thus we obtain

$$\frac{d^2}{dt^2} \left[\frac{\mu m}{\dot{S}_r} \right]^{\frac{1}{2}} + \frac{\dot{S}_r + m r \dot{\theta}^2}{m} = 0 \tag{u}$$

Equation (u), being of third order, will require three initial conditions. Alternatively we may relinquish the impulse variable S_r and work with the force variable $F = \dot{S}_r$. Then equation (u) becomes

$$\frac{d^2}{dr^2} \left[\frac{\mu m}{F} \right]^{\frac{1}{2}} + \frac{F + m r \dot{\theta}^2}{m} = 0 \tag{v}$$

Finally we attempt to remove the kinematic variables as follows. First using equation (e) we express the centrifugal force as

$$m r \dot{\theta}^2 = m r \frac{C^2}{m^2 r^4} = \frac{C^2}{m r^3} \tag{x}$$

Next we use equation (a) to remove r and express the centrifugal force

$$\frac{C^2}{m r^3} = \frac{C^2}{-m \left[\frac{\mu m}{F} \right]^{\frac{3}{2}}} = \frac{C^2 F^{\frac{3}{2}}}{-m (\mu m)^{\frac{3}{2}}} \tag{y}$$

Finally we can express the compatibility equation of motion in terms of F alone as

$$\frac{d^2}{dr^2} \left[\frac{\mu m}{F} \right] + \frac{F}{m} - \frac{C^2 F^{\frac{3}{2}}}{m^2 (\mu m)^{\frac{3}{2}}} = 0$$

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2. Greenwood, Donald T.,
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Dover Publications Inc. 4th edition (1986).

Chapter III

INTEGRAL VARIATIONAL FORMULATIONS

D'Alembert's principle and Gauss' principle of least constraint are examples of *differential* variational formulations. These formulations make independent statements at each instant of time during the motion. By contrast *integral* variational formulations, which we shall examine in this chapter, make a single, all inclusive, statement. That is, the motion over an arbitrary period of time is considered as a whole. While the various differential and integral variational formulation take different forms, they are of course related and they make equivalent statements about the motion.

3.1 Hamilton's Principle

In D'Alembert's principle, the component of virtual work done by the inertial forces cannot be derived from a single work function. Hence even if the impressed forces possess a potential, D'Alembert's principle still cannot be expressed as an extremum principle. By means of an integration over time, Hamilton† transformed D'Alembert's principle to a form wherein inertia forces can be derived from a single function. If the impressed forces are also derivable from a single function, then D'Alembert's principle, as transformed by Hamilton, becomes a true extremum principle, albeit an integral rather than a differential variational principle.

To derive Hamilton's principle, we integrate D'Alembert's virtual work expression (2.1.3), over an arbitrary period of time. Since this expression vanishes at each instant of time, its integral from t_1 to t_2 will also vanish. Thus

$$\int_{t_1}^{t_2} (\mathbf{F}_k - \dot{\mathbf{B}}_k) \cdot \delta\mathbf{r}_k \, dt = 0 \tag{3.1.1}$$

Now let us integrate the second term in the bracket, by parts.

† William Rowan Hamilton (1805-1865), Irish mathematician.

$$-\int_{t_1}^{t_2} \dot{\mathbf{B}}_k \cdot \delta\mathbf{r}_k \, dt = -\int_{t_1}^{t_2} \frac{d}{dt}\left(\mathbf{B}_k \cdot \delta\mathbf{r}_k\right) dt + \int_{t_1}^{t_2} \left(\mathbf{B}_k \cdot \frac{d}{dt}\delta\mathbf{r}_k\right) dt \quad (3.1.2)$$

The first part on the right hand side of equation (3.1.2) is integrable and we obtain boundary terms. Thus

$$-\int_{t_1}^{t_2} \frac{d}{dt}\left[\mathbf{B}_k \cdot \delta\mathbf{r}_k\right] dt = -\mathbf{B}_k \cdot \delta\mathbf{r}_k \Big|_{t_1}^{t_2} \quad (3.1.3)$$

The second part, on making use of the interchangeable nature of variation and differentiation operations, may be written as

$$\int_{t_1}^{t_2} \mathbf{B}_k \cdot \frac{d}{dt}\delta\mathbf{r}_k \, dt = \int_{t_1}^{t_2} \mathbf{B}_k \cdot \delta\dot{\mathbf{r}}_k \, dt = \int_{t_1}^{t_2} \mathbf{B}_k \cdot \delta\mathbf{v}_k \, dt = \int_{t_1}^{t_2} \delta T^* dt \quad (3.1.4)$$

Returning to equation (3.1.1) and summing over all particles, we can write it as

$$\int_{t_1}^{t_2} (\delta W_I + \delta T^*) dt - \mathbf{B}_k \cdot \delta\mathbf{r}_k \Big|_{t_1}^{t_2} = 0 \quad (3.1.5)$$

where δW_I denotes the virtual work of all the applied forces. In equation (3.1.5) we have Hamilton's law of varying action. In view of the presence of the time boundary terms in equation (3.1.5), and the virtual work of nonpotential type forces in δW_I, Hamilton's law is *not* an extremum principle.

Now it will be recalled that the virtual displacements $\delta\mathbf{r}_k$ are arbitrary and imagined. However, to ensure that the constraint forces make no contribution to the virtual work expression, we required that time be held fixed for these displacements and all kinematic constraints be satisfied by these displacements. We now impose an additional requirement for the virtual displacements. We require that at times t_1 and t_2 the virtual displacements vanish. The physical implication of this requirement is that the real displacements at times t_1 and t_2 are taken as known and no variations are allowed at these limits. Under this stipulation equation (3.1.5) reduces to

$$\int_{t_1}^{t_2} (\delta W_I + \delta T^*) dt = 0 \quad (3.1.6)$$

In the special case where the impressed forces possess a potential energy function V, equation (3.16) takes the form

$$\int_{t_1}^{t_2} \delta(T^* - V) dt = 0 \quad (3.1.7)$$

In equation (3.17) we have *Hamilton's variational principle*. Since the time limits are fixed, the operations of integration and variation may be changed allowing one to write Hamilton's principle in the following form

$$\delta\Pi_H = 0 \quad (3.1.8)$$

where the functional

$$\Pi_H = \int_{t_1}^{t_2} L \, dt \tag{3.1.9}$$

is the *action*, with units Js, and

$$L = T^* - V$$

We note that Hamilton's principle makes a single statement for the entire motion, from time t_1 to t_2. This statement can be expressed as follows: *Amongst all kinematically possible motions in the interval t_1 to t_2, the actual one is characterized by the stationary condition of the functional Π_H.* The stationary conditions of Π_H emerge as the equilibrium equations of motion.

For the more general case where there exist potential and nonpotential forces, Hamilton's principle takes the form

$$\int_{t_1}^{t_2} \delta L \, dt + \int_{t_1}^{t_2} \mathbf{F}_k' \cdot \delta \mathbf{r}_k \, dt = 0 \tag{3.1.10}$$

where the second term accounts for the virtual work of nonpotential forces $\mathbf{F}_k'$. In this case, the expression in equation (3.1.10) remains an incremental quantity and does not possess extremum character.

In terms of generalized displacements, equation (3.1.5) takes the form

$$\int_{t_1}^{t_2} (\delta W_I + \delta T^*) \, dt = p_i \, \delta q_i \Big|_{t_1}^{t_2} \tag{3.1.11}$$

where the $p_i (= \partial T^* / \partial \dot{q}_i)$ are the generalized momenta. When the generalized forces possess a potential energy V, a Lagrangian function $L (= T^* - V)$ may be defined and equation (3.1.11) may be written as

$$\int_{t_1}^{t_2} \delta L \, dt = p_i \, \delta q_i \Big|_{t_1}^{t_2} \tag{3.1.12}$$

On imposing the condition that all $\delta q_i = 0$ at t_1 and t_2 we arrive once again at equation (3.1.8). If the δq_i do not disappear at t_1 and t_2, and if we write $p_i = \partial L / \partial \dot{q}_i$, then

$$\int_{t_1}^{t_2} \delta L \, dt = \frac{\partial L}{\partial \dot{q}_i} \, \delta q_i \Big|_{t_1}^{t_2} \tag{3.1.13}$$

Example 3.1.1

Consider a simple linear oscillator, with $q_1 = x$, and kinetic coenergy and potential energy given by

$$T^* = \frac{1}{2}m\dot{x}^2 \quad \text{and} \quad V = \frac{1}{2}kx^2 \tag{a,b}$$

Then the Lagrangian is

$$L = T^* - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{c}$$

Hamilton's functional for the problem takes the form

$$\Pi_H = \int_{t_1}^{t_2} [\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2]dt \tag{d}$$

$$\delta\Pi_H = \delta\int_{t_1}^{t_2}[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2]dt = 0 \tag{e}$$

Carrying out the variation, we can write it as

$$m\int_{t_1}^{t_2}\dot{x}\delta\dot{x}\ dt - k\int_{t_1}^{t_2} x\ \delta x\ dt = 0 \tag{f}$$

Now let us take a closer look at the first integral

$$\int_{t_1}^{t_2} \dot{x}\ \delta\dot{x}\ dt = \int_{t_1}^{t_2} \dot{x}\frac{d}{dt}\ \delta x\ dt = \int_{t_1}^{t_2} \frac{d}{dt}\ [\dot{x}\ \delta x\ ]\ dt - \int_{t_1}^{t_2} \ddot{x}\ \delta x\ dt \tag{g}$$

$$= \dot{x}\ \delta x\ \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{x}\ \delta x\ dt$$

Imposing the conditions $\delta x=0$ at the boundaries t_1 and t_2, the integral (g) becomes simply

$$-\int_{t_1}^{t_2} \ddot{x}\ \delta x\ dt \tag{h}$$

Returning to equation (f) we obtain

$$\int_{t_1}^{t_2}(m\ddot{x} + kx)\ \delta x\ dt = 0 \tag{i}$$

Since δx is arbitrary between t_1 and t_2, Hamilton's principle requires

$$m\ddot{x} + kx = 0 \tag{j}$$

Problems

3.1.1 A particle of mass m is thrown upward with initial velocity components $\dot{x}_o$ and $\dot{y}_o$.

(a) Using Hamilton's principle derive the particle's equations of motion.

(b) Solve these equations and determine the particle's trajectory as a function of time.

(c) Use this trajectory in Hamilton's functional and evaluate this functional from the initial time to the time when the particle reaches the horizontal ground again.

(d) Approximate the shape of the trajectory by a half sine curve and determine the value of Hamilton's functional, over the same period of time, for this varied trajectory. Comment on the values of the functional determined in parts (c) and (d).

3.1.2 The motion of a mass-spring oscillator is given by $x = A \sin\omega t$ where $\omega = \sqrt{k/m}$. Assuming a varied path represented by $x = A \sin\omega t + \varepsilon \sin 2\omega t$, where ε is a small quantity, show that for the actual path the value of Hamilton's functional, over $\frac{1}{4}$ of a complete oscillation, is zero while for the varied path, the functional has the positive value of $\frac{3}{8} m\pi\omega\varepsilon^2$.

3.1.3 Use Hamilton's principle to establish the differential equation of motion for a simple pendulum.

3.1.4 The kinetic coenergy of a particle rising at a speed of $v = v_o - gt$ is $T^* = \frac{1}{2} m(v_o - gt)^2$. (a) What is the action Π_H of the particle's kinetic coenergy between $t = t_1$ and $t = t_2$? (b) What is the numerical value of the action, for $m = 2$ kg, $v_o = 3$ m/s, $g = 9.80665$ m/s^2, $t_1 = 0$ and $t_2 = 12$ s?

3.2 Complementary Form of Hamilton's Principle

For the derivation of this principle, we start from the second form of the virtual work principle (see section 1.7). Noting that the virtual work done by the velocities of mass particles is given by δT, we may express the time integral of the total virtual work as

$$\int_{t_1}^{t_2} (\delta W_{II} + \delta T) dt = 0 \qquad (3.2.1)$$

where δW_{II} is the virtual work of the other velocities.

In some cases, the form of δW_{II} is such that its integration over time allows one to express it as $-\delta V^*$. To illustrate this, consider for simplicity the case of a linear spring of stiffness k. Denoting the impulse in the spring by S and the associated deformation by e, we may write the virtual work term for this spring as

$$\delta W_{II} = -\dot{e}\, \delta S \qquad (3.2.2)$$

But

$$e = -\dot{S}/k \qquad (3.2.3)$$

hence

$$\int_{t_1}^{t_2} \delta W_{II}\, dt \;=\; \int_{t_1}^{t_2} \frac{\ddot{S}}{k}\delta S\, dt \tag{3.2.4}$$

We can carry out a partial integration of the right hand side term. To this end we write

$$\int_{t_1}^{t_2} \frac{d}{dt}\left[\frac{\dot{S}}{k}\delta S\right] dt \;-\; \int_{t_1}^{t_2} \frac{\dot{S}}{k}\,\delta\dot{S}\, dt \;=\; \frac{\dot{S}}{k}\delta S \,\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\delta}{2}\left[\frac{\dot{S}^2}{k}\right] dt \tag{3.2.5}$$

Recognizing the integrand of the second term on the right hand side in equation (3.2.5) as δV^* and stipulating that $\delta S = 0$ at t_1, and t_2, we may return to equation (3.2.4) and write it as

$$\int_{t_1}^{t_2} \delta W_{II}\, dt \;=\; -\int_{t_1}^{t_2} \delta V^*\, dt \tag{3.2.6}$$

Finally substituting from equation (3.2.6) into equation (3.2.1), we may write

$$\int_{t_1}^{t_2} \delta(T - V^*)dt \;=\; 0 \tag{3.2.7}$$

In equation (3.2.7) we have the complementary form of Hamilton's principle. Since the times t_1 and t_2 are fixed, we may interchange the operations of variation and integration and express this principle as

$$\delta\Pi_T \;=\; 0 \tag{3.2.8}$$

where

$$\Pi_T \;=\; \int_{t_1}^{t_2} L^*\, dt \tag{3.2.9}$$

and

$$L^* \;=\; T - V^* \tag{3.2.10}$$

Recalling that equilibrium conditions, amongst the impulses S_j, constitute the essential constraints of the complementary form of Hamilton's principle, we may state this principle as follows: *Amongst all possible equilibrating motions in the interval t_1 to t_2 the actual one is characterized by the stationary conditions of the functional Π_T*. The stationary conditions of Π_T emerge as conditions of compatibility or kinematic fit.

In the more general case where there exist speeds derivable from an energy and others that are not derivable from an energy, the complementary form of Hamilton's principle takes the form

$$\int_{t_1}^{t_2} \delta L^*\, dt \;+\; \int_{t_1}^{t_2} s_j'\, \delta S_j\, dt \;=\; 0 \tag{3.2.11}$$

where the second term accounts for the virtual work of velocities s_j' which cannot be derived from an energy function. In this case, the expression in equation (3.2.11) remains an incremental quantity and does not possess extremum character.

Example 3.2.1

Consider a simple linear oscillator, with the momentum of the mass $B = S$, the spring impulse. Having thus satisfied the condition of equilibrium one may write the kinetic energy and potential coenergy of the oscillator as

$$T = \frac{B^2}{2m} = \frac{1}{2m}S^2 \quad \text{and} \quad V^* = \frac{1}{2k}\dot{S}^2 \tag{a,b}$$

Then the complementary Lagrangian is

$$L^* = T - V^* = \frac{1}{2m}S^2 - \frac{1}{2k}\dot{S}^2 \tag{c}$$

The functional

$$\Pi_T = \int_{t_1}^{t_2}\left[\frac{1}{2m}S^2 - \frac{1}{2k}\dot{S}^2\right]dt \tag{d}$$

is the *complementary action* (or *coaction*). Extremising this functional

$$\delta\Pi_T = \delta\int_{t_1}^{t_2}\left[\frac{1}{2m}S^2 - \frac{1}{2k}\dot{S}^2\right]dt = 0 \tag{e}$$

Equation (e) can be rewritten into

$$\frac{1}{m}\int_{t_1}^{t_2} S\,\delta S dt - \frac{1}{k}\int_{t_1}^{t_2}\dot{S}\,\delta\dot{S}dt = 0 \tag{f}$$

We can now integrate the second term by parts. To this end we write

$$\int_{t_1}^{t_2}\dot{S}\frac{d}{dt}\delta S\,dt = \int_{t_1}^{t_2}\frac{d}{dt}(\dot{S}\,\delta S)dt - \int_{t_1}^{t_2}\ddot{S}\,\delta S\,dt = -\int_{t}^{t_2}\ddot{S}\,\delta S\,dt + \dot{S}\delta S\Big|_{t_1}^{t_2} \tag{g}$$

Now imposing the conditions $\delta S = 0$ at t_1 and t_2 and returning to expression (f) we obtain

$$\int_{t_1}^{t_2}\left[\frac{1}{m}S + \frac{1}{k}\ddot{S}\right]\delta S\,dt = 0 \tag{h}$$

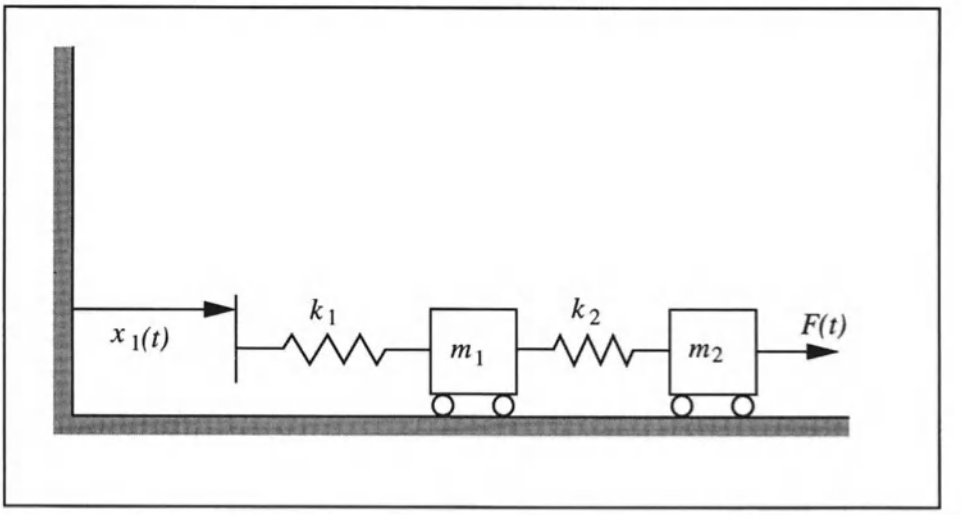
Since δS is arbitrary between t_1 and t_2, equation (h) is satisfied if

$$\frac{1}{m}S + \frac{1}{k}\ddot{S} = 0 \tag{i}$$

Problem

3.2.1 The system shown moves to the right such that the distance x_1 measured from a fixed wall, is given by $x_1 = vt + \frac{1}{2}at^2$. Determine the equations of motion

of this system via (a) Hamilton's principle, and (b) the complementary form of Hamilton's principle.



3.3 Lagrange's Equations from Hamilton's Principle

Our reasoning which led to Hamilton's principle can be pursued in the reversed order. We can start out with the postulate that the first variation of the action Π_H shall vanish, for arbitrary variations of displacements, and thereby deduce the equations of motion. To this end, we first express the Lagrangian $T^* - V$ in terms of generalized displacements q_i, their time derivatives $\dot{q}_i$, and the time t, and then evaluate the first variation of Π_H and equate it to zero. Thus we have

$$\delta\Pi_H = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = 0 \tag{3.3.1}$$

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt = 0 \tag{3.3.2}$$

The second term in the intergrand of equation (3.3.2) may be integrated by parts, giving

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \, dt = \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right] - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right\} dt \tag{3.3.3}$$

$$= -\int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Bigg|_{t_1}^{t_2} \tag{3.3.4}$$

Substituting back into equation (3.3.2), we find the vanishing condition of $\delta\Pi_H$ as

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt + \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Bigg|_{t_1}^{t_2} = 0 \tag{3.3.5}$$

Now if we require that all δq_i should vanish at times t_1 and t_2, the last term drops out and satisfaction of equation (3.3.5) for arbitrary and independent δq_i leads to Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1,2,...,n \tag{3.3.6}$$

By a similar development, we can show that the extremum conditions of the complementary form of Hamilton's principle, namely

$$\delta \Pi_T = \delta \int_{t_1}^{t_2} L^*(S_j, \dot{S}_j, t)dt = 0 \tag{3.3.7}$$

where the complementary Lagrangian L^* is a function of generalized impulses S_j, their time derivatives $\dot{S}_j$ and the time t, emerge as the complementary Lagrange equations

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}_j} - \frac{\partial L^*}{\partial S_j} = 0 \tag{3.3.8}$$

3.4 Conservation of Energy

In Section 2.3 we showed that under certain conditions the total energy of a system will be conserved. This result was obtained by replacing virtual changes of displacements and energy terms, in D'Alembert's principle, by the corresponding real changes. We now follow a similar procedure and consider real changes in displacements in Hamilton's principle, and in so doing, we bring to light the general relation which exists between the total energy of a mechanical system and the Lagrangian function L.

Recognizing that for virtual displacements, δq_i, time is held fixed, whereas real infinitesimal displacements occur during an infinitesimal time $dt = \varepsilon$, we note that for these displacements to coincide, we must choose

$$\delta q_i = dq_i = \varepsilon \dot{q}_i \tag{3.4.1}$$

Consider next the real and virtual changes in the Lagrangian,

$$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \tag{3.4.2}$$

$$dL = \frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial t}dt \tag{3.4.3}$$

From these latter equations, it is evident that dL and δL can coincide only if time does not appear in the Lagrangian explicitly. Under this condition, the last term in equation (3.4.3) drops out and for this case, we may write $\delta L = dL$. Furthermore since $\varepsilon = dt$,

$$dL = \varepsilon \frac{dL}{dt} \tag{3.4.4}$$

Let us now invoke Hamilton's principle. It may be recalled that in deriving this principle, we imposed an additional requirement on the virtual displacements namely that the variations of displacements be limited to the range $t = t_1$ to $t = t_2$ and that at the end points the displacements were not allowed to vary. In this way, we were able to drop the boundary term in Hamilton's law of varying action. For actual displacements, if the variations at time t_1 and t_2 were to vanish, then they would vanish everywhere in the range t_1 to t_2. Hence to compare actual displacements, we must allow variations of displacements at the fixed limits t_1 and t_2. It is not difficult to see that the variation of actual displacements amounts to a vertical shift in the qt diagram.

For the case that variations at end points are allowed, Hamilton's law of varying action, equation (3.1.13), is

$$\int_{t_1}^{t_2} \delta L \, dt = \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Bigg|_{t_1}^{t_2} \tag{3.4.5}$$

Now replacing the virtual changes in L and q_i by real changes, as given by equations (3.4.1) and (3.4.4), above equation becomes

$$\varepsilon \int_{t_1}^{t_2} \frac{dL}{dt} dt = \varepsilon \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \Bigg|_{t_1}^{t_2} \tag{3.4.6}$$

Carrying out the integration and rearranging, we may express equation (3.4.6) as

$$\varepsilon \left[L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right]_{t_1}^{t_2} = 0 \tag{3.4.7}$$

Equation (3.4.7) indicates that under the conditions specified, in particular that time shall not appear in L explicitly, the term in the square brackets is an invariant of the motion. It remains to interpret the physical meaning of this term. Now we have that

$$L = T^* - V \tag{3.4.8}$$

If V is a function of displacements only, then

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T^*}{\partial \dot{q}_i} = p_i \tag{3.4.9}$$

Under this condition, equation (3.4.7) may be written as

$$L - p_i \dot{q}_i = \text{constant} \tag{3.4.10}$$

Now if the system is scleronomic, time will not be present explicitly in L. Further, for scleronomic systems

$$p_i \dot{q}_i = T + T^* \tag{3.4.11}$$

and in this case equation (3.4.10) becomes

$$(T^* - V) - (T + T^*) = \text{constant}$$

or for *conservative scleronomic* systems

$$T + V = \text{constant} \tag{3.4.12}$$

It should be noted that it is the sum of the *energies* that is conserved. This result can also be derived from the complementary form of Hamilton's principle.

In Newtonian mechanics, since $T = T^*$, one may also assert that, under the conditions specified

$$T^* + V = \text{constant} \tag{3.4.13}$$

In rheonomic systems, very often time appears explicitly in the Lagrangian. However, in some rheonomic systems while time is present explicitly in the transformation equations, it drops out of the Lagrangian. For such systems, once again equation (3.4.10) holds, however, in this case the interpretation of the term $p_i \dot{q}_i$ is different.

We have seen in section 1.9 that for rheonomic systems in general $T^* = T_2^* + T_1^* + T_o^*$ where T_2^* is quadratic in generalized velocities, T_1^* is linear in generalized velocities, and T_o^* is independent of generalized velocities. For these systems, we showed in equation (1.9.15) that

$$p_i \dot{q}_i = T + T_2^* - T_o^* \tag{3.4.14}$$

Hence for rheonomic systems for which time does not appear explicitly in the Lagrangian, equation (3.4.10) implies that

$$(T_2^* + T_1^* + T_o^* - V) - (T + T_2^* - T_o^*) = \text{constant}$$

or

$$H = T + V - T_1^* - 2T_o^* = \text{constant} \tag{3.4.15}$$

Thus in this case, the total energy $T + V$ is not conserved yet the expression in equation (3.4.15) is conserved. The quantity H is the *Hamiltonian* of the motion. For the case of Newtonian mechanics, equation (3.4.15) simplifies to

$$(T_2^* + T_1^* + T_o^* + V) - 2T_o^* - T_1^* = \text{constant}$$

or

$$(T_2^* + V) - T_o^* = \text{constant} \tag{3.4.16}$$

Since T_o^* is not a function of the generalized velocity, it is apparent that it may be combined with the potential energy V. Then equation (3.4.16), valid for a *conservative rheonomic* system, becomes

$$T_2^* + V' = \text{constant} \tag{3.4.17}$$

where

$$V' = V - T_o^* \tag{3.4.18}$$

with V' as the *effective* potential energy.

For conservative rheonomic systems for which equation (3.4.17) holds, the rate of change of the total energy $(T^* + V)$ in the system will be

$$\frac{d}{dt}(T_2^* + T_1^* + T_o^* + V) = \frac{d}{dt}(T_2^* + (V - T_o^*)) + \frac{d}{dt}(T_1^* + 2T_o^*) \quad (3.4.19)$$

Now since the first term on the right hand side vanishes, we find

$$\frac{d}{dt}(T^* + V) = \frac{d}{dt}(T_1^* + 2T_o^*) \quad (3.4.20)$$

Example 3.4.1

Consider again the problem shown in Figure 1.29. In section 1.9 we derived the reduced transformation equations for this problem and noted that they were rheonomic both on account of the angular velocity ω and the linear velocity v. Further, in section 2.6, we showed that the Lagrangian for this system may be written as

$$L = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\omega^2\sin^2\theta) + \frac{1}{2}mv^2 + mav(\dot{\theta}\cos\theta\cos\omega t - \omega\sin\theta\sin\omega t) - mga(1 - \cos\theta) \quad (a)$$

We note here that the system is not only rheonomic, but that time appears in the Lagrangian explicitly and hence in this case, energy is not conserved.

Example 3.4.2

If in the problem shown in Figure 1.29, the linear velocity $v = 0$, then the Lagrangian does not contain the time explicitly,

$$L = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta - mga(1 - \cos\theta) \quad (a)$$

Yet the problem is still rheonomic, on account of the angular velocity ω, or more specifically on account of time appearing explicitly in the reduced transformation equations

$$x = a\cos\theta\cos\omega t \qquad y = a\sin\theta\sin\omega t \quad (b)$$

The total energy of the system given by

$$E = T + V = T^* + V = T_2^* + T_1^* + T_o^* + V \quad (c)$$

$$= \frac{1}{2}ma^2\dot{\theta}^2 + 0 + \frac{1}{2}ma^2\omega^2\sin^2\theta + mga(1 - \cos\theta) \quad (d)$$

is not constant. This is easily seen on physical grounds, since energy must be continuously supplied or withdrawn in order to keep ω constant.

The Hamiltonian (3.4.15) is

$$H = T_2^* + V' = T_2^* - T_o^* + V$$

$$= \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\omega^2\sin^2\theta + mga(1 - \cos\theta) \quad (e)$$

Thus

$$\frac{dH}{dt} = \dot{\theta}(ma^2\ddot{\theta} - ma^2\omega^2\sin\theta\cos\theta + mga\sin\theta) \quad (f)$$

Now the term in the bracket vanishes on account of the equation of motion obtained in example 2.6.1, thus the Hamiltonian (e) is constant.

From equation (3.4.20) we may compute the rate at which energy enters the system. Thus

$$\frac{d}{dt}(T_1^* + 2T_o^*) = \frac{d}{dt}(ma^2\omega^2\sin^2\theta)$$

$$= 2ma^2\omega^2\dot{\theta}\sin\theta\cos\theta \qquad \text{(g)}$$

This result may be interpreted as follows. Differentiating T^* with respect to the angular velocity ω, we find the angular momentum about the z axis as follows

$$H_z = ma^2\,\omega\sin^2\theta \qquad \text{(h)}$$

Thus the torque about the z axis given by

$$M_z = \frac{d}{dt}H_z = 2ma^2\omega\dot{\theta}\sin\theta\cos\theta \qquad \text{(i)}$$

Finally we compute the rate at which this torque does work on the system as

$$M_z\omega = \omega\,(2ma^2\omega\dot{\theta}\sin\theta\cos\theta) \qquad \text{(j)}$$

which is identical to the rate of energy entering the system are given in equation (g).

Finally, let us formulate this problem in terms of Cartesian inertial coordinates. Taking the origin of the axes at the centre of the circle, we find

$$L = \frac{m}{2}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2] - mgz \qquad \text{(k)}$$

However, x, y and z are not independent. We have the following holonomic constraints

$$x^2 + y^2 + z^2 = a^2 \qquad \text{(l)}$$

$$y = x\tan\omega t \qquad \text{(m)}$$

It can now be seen that if for instance the coordinates y and z were to be eliminated from L in equation (k), via equations (l) and (m), time would appear in the Lagrangian explicitly and accordingly in this case, neither the total energy $(T + V)$ nor the Hamiltonian $(T_2^* + V')$, in terms of $x, \dot{x}$ and t, will be conserved. If for a given system the total energy is conserved, $(T^* + V)$ will remain an invariant of the motion independent of the coordinate systems used. On the other hand, for a system for which the total energy is not conserved, there may exist a coordinate system in which the Lagrangian will not contain time explicitly. For such a coordinate system the Hamiltonian $(T_2^* + V')$ will be an invariant of the motion.

Whenever the Lagrangian does not include the time explicitly, the invariance of $(L - p_i\,\dot{q}_i)$ provides an additional equation which may be used in integrating the equations of motion. In the case of single degree of freedom problems, this additional equation leads to a complete solution for the problem. This solution may be developed as follows. Given that

$$f(q, \dot{q}) = E = \text{constant} \qquad (3.4.21)$$

then

$$\dot{q} = g(E, q)$$

and

$$t = \int \frac{dq}{g(E, q)} + t_o$$

The integration constant t_o and the constant E, which appear in the solution

$$t = h(q, E, t_o) \tag{3.4.22}$$

may be obtained from the initial conditions. It is worth noting that the solution obtained in this manner is in the inverse form, that is, time is given as a function of q rather than the reverse.

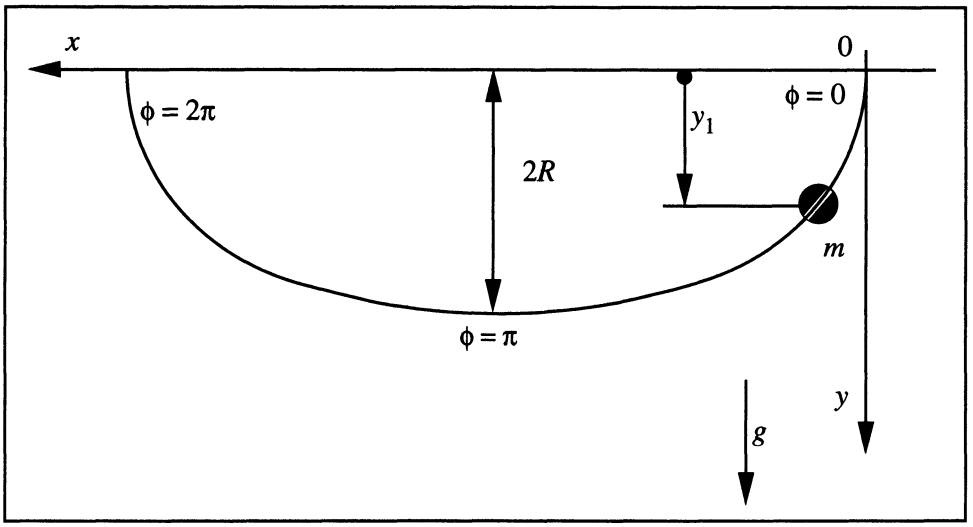


Figure 3.1 Bead on cycloid path

Example 3.4.3

A bead of mass m slides without friction on a wire bent in the shape of a cycloid, as shown in Figure 3.1. The parametric equations of the cycloid in the vertical plane are

$$x = R(\phi - \sin\phi) \qquad y = R(1 - \cos\phi) \tag{a}$$

If the bead starts from point y_1 with zero velocity, determine the time taken for the bead to reach the lowest point on the wire.

The theorem of conservation of energy provides a direct way to determine the time of descent for the bead.

From the position coordinates for the bead, we find

$$\dot{x} = R\dot{\phi}(1 - \cos\phi) \qquad \dot{y} = R\dot{\phi}\sin\phi \tag{b}$$

Then

$$T^* = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = m R^2 \dot{\phi}^2(1 - \cos\phi) \tag{c}$$

The potential energy of the bead is given by

$$V = -mgy = -mgR(1 - \cos\phi) \tag{d}$$

Now recognizing that the initial energy is given by

$$E = -mgy_1 = -mgR(1 - \cos\phi_1) \tag{e}$$

we may express the conservation of energy as

$$f(\phi, \dot{\phi}) = [mR^2\dot{\phi}^2 - mgR](1 - \cos\phi) = -mgR(1 - \cos\phi_1) = E = \text{constant} \tag{f}$$

and on rearranging, we find

$$\dot{\phi} = \sqrt{g/R}\sqrt{\frac{\cos\phi_1 - \cos\phi}{1 - \cos\phi}} \tag{g}$$

Consider for the moment the special case when $\phi_1 = 0$. In this case, $y_1 = 0$ and the bead starts from the origin. If we denote the time of descend in this case by t_o, we find from equation (g)

$$\int_o^{t_o} dt = \int_o^{\pi} \frac{d\phi}{\sqrt{g/R}} \tag{h}$$

or

$$t_o = \frac{\pi}{\sqrt{g/R}} \tag{i}$$

Returning to the general case, and denoting the time of descend by t_1, we find from equation (g)

$$\sqrt{g/R}\int_o^{t_1} dt = \int_{\phi_1}^{\pi} \frac{d\phi}{\sqrt{\frac{\cos\phi_1 - \cos\phi}{1 - \cos\phi}}} \tag{j}$$

To carry out the integration, let

$$\cos\phi = \alpha \tag{k}$$

Then

$$-\sin\phi\, d\phi = d\alpha \tag{l}$$

or

$$-\sqrt{1 - \alpha^2}\, d\phi = d\alpha \tag{m}$$

and

$$\sqrt{g/R}\; t_1 = \int_{\arccos\alpha_1}^{-1} \frac{-d\alpha}{\sqrt{\alpha_1 - \alpha)(1 + \alpha)}} \tag{n}$$

Above integral is of the type

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{-a}}\arcsin\frac{2ax + b}{\sqrt{b^2 - 4ac}} \tag{o}$$

and for our case

$$a = -1 \qquad b = -(1-\alpha_1) \qquad c = \alpha_1 \tag{p}$$

Thus integration of equation (n) yields

$$\sqrt{g/R}\; t_1 = \arcsin \frac{-2\alpha + \alpha_1 - 1}{\sqrt{1 + 2\alpha_1 + \alpha_1^2}} \Bigg|_{\arccos\alpha_1}^{-1} \tag{q}$$

On reverting back to the original variables, we find

$$\sqrt{g/R}\; t_1 = \arcsin \frac{-2\cos\phi + \cos\phi_1 - 1}{\cos\phi_1 + 1} \Bigg|_{\phi_1}^{\pi} \tag{r}$$

and finally, on substituting for the limits, we determine the time of descent as

$$t_1 = \pi\sqrt{\frac{R}{g}} \tag{s}$$

It is of interest to note that $t_1 = t_o$ i.e. the time of descent is independent of the initial position of the bead on the cycloidal wire. This feature of the cycloid curve was used by Huygens†, in 1658, for the construction of pendulum clocks for which the period of pendulum oscillation given by

$$\tau = 4\pi\sqrt{\frac{R}{g}} \tag{t}$$

is independent of amplitude.

It is of interest to compare this period of oscillation with that of a pendulum oscillating with a small amplitude, namely

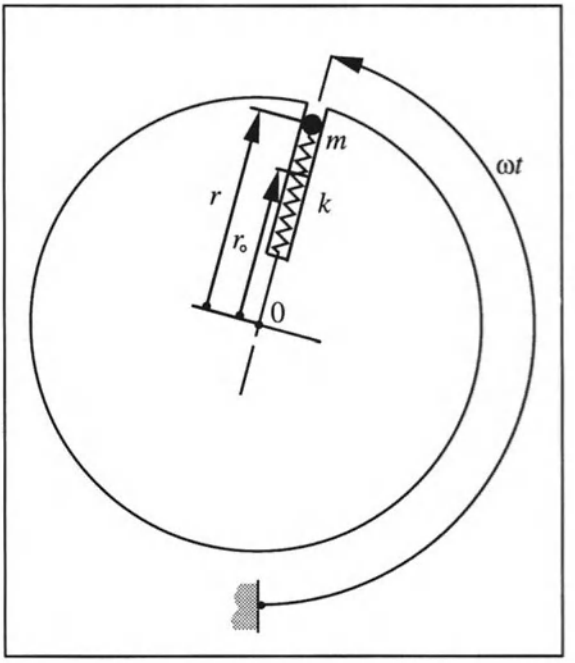
$$\tau_{pendulum} = 2\pi\sqrt{\frac{l}{g}} \tag{u}$$

Thus a cycloid pendulum with $R = \frac{l}{4}$ has the same period as a simple pendulum of length l.

Problems

3.4.1 A bead of mass m is installed at the end of a linear spring of stiffness k and unstretched length r_o, in a horizontal disk rotating at constant angular velocity ω. (a) Write x and y in terms of r and determine whether the system is scleronomic or rheonomic. (b) Write the Lagrangian and establish the equation of motion. (c) Form the effective potential energy V'. (d) Form $T_2^* + V'$ and show that it is constant. (e) How much power, as function of time, must be supplied to the system to keep the angular velocity ω constant?

† Christiaan Huygens (1629-1695), Dutch physicist and mathematician (Horologium oscillatorium, 1673)



3.4.2 Consider the disk in the last example rotating at constant angular velocity in the vertical plane. (a) Express the Cartesian coordinates of the bead and establish whether the system is scleronomic or rheonomic. (b) Write the Lagrangian and establish the equations of motion. (c) Comment on the invariants of the motion.

3.5. Change of Energy Over a Period of Time

Consider now the case wherein time appears explicitly in the Lagrangian. In this case, the relation between δL and dL, as given in equations (3.4.2) and (3.4.3) may be expressed as

$$\delta L = dL - \frac{\partial L}{\partial t} dt = \varepsilon \left(\frac{dL}{dt} - \frac{\partial L}{\partial t} \right) \tag{3.5.1}$$

Once again replacing virtual changes in Hamilton's law of varying action (3.1.12) by the corresponding real changes, and following steps similar to those outlined in the previous section, we find the following relation

$$\left[L - p_i \dot{q}_i \right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \frac{\partial L}{\partial t} dt \tag{3.5.2}$$

Recognizing the term $p_i \dot{q}_i$ as $(T + T_2^* - T_o^*)$, and L as $(T^* - V)$, we may write equation (3.5.2) as

$$\left[-(T+V)+2T_o^*+T_1^*\right]_{t_1}^{t_2} = \int_{t_1}^{t_2}\frac{\partial L}{\partial t}dt \tag{3.5.3}$$

For the special case of Newtonian mechanics with $T = T^*$ $(= T_2^* + T_1^* + T_o^*)$, equation (3.5.3) simplifies to

$$\Delta(T_2^* + V') = -\int_{t_1}^{t_2}\frac{\partial L}{\partial t}dt \tag{3.5.4}$$

where

$$V' = V - T_o^* \tag{3.5.5}$$

We may write equation (3.5.4) as

$$(T_2^* + V')_{t_1} = (T_2^* + V')_{t_2} + \int_{t_1}^{t_2}\frac{\partial L}{\partial t}\,dt \tag{3.5.6}$$

Thus if $(T_2^* + V')_{t_1}$ is given, say as an initial condition, we can assert that

$$(T_2^* + V')_t + \int_{t_1}^{t}\frac{\partial L}{\partial t}\,dt = \text{constant} \tag{3.5.7}$$

for all t.

If the change $\Delta(T^* + V)$ of the total energy is to be determined, then

$$\Delta(T^* + V) = -\int_{t_1}^{t_2}\frac{\partial L}{\partial t}\,dt + \int_{t_1}^{t_2}\frac{d}{dt}(T_1^* + 2T_0^*)\,dt \tag{3.5.8}$$

since $T^* + V = T_2^* + V' + T_1^* + 2T_o^*$.

Example 3.5.1

Consider the problem shown in Figure 3.2. The reduced transformation equations for this problem take the form

$$x = l\sin\theta = (l_o - vt)\sin\theta \qquad \dot{x} = (l_o - vt)\cos\theta\,\dot{\theta} - v\sin\theta \tag{a}$$

$$y = l\cos\theta = (l_o - vt)\cos\theta \qquad \dot{y} = -(l_o - vt)\sin\theta\,\dot{\theta} - v\cos\theta \tag{b}$$

Using these velocities, we find the complementary kinetic energy as

$$T^* = \frac{1}{2}m[v^2 + (l_o - vt)^2\dot{\theta}^2] \tag{c}$$

Likewise the potential energy becomes

$$V = -mg(l_o - vt)\cos\theta \tag{d}$$

From equation (c) and (d) we may form the Lagrangian and find its partial derivative with respect to time as

$$\frac{\partial L}{\partial t} = -mv(l_o - vt)\dot{\theta}^2 - mgv\cos\theta \tag{e}$$

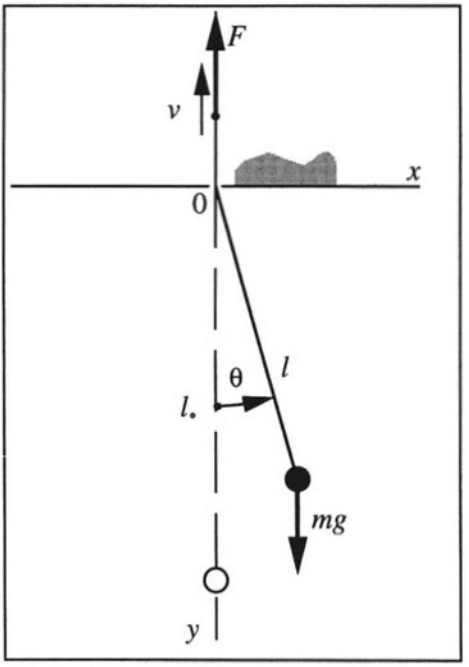


Figure 3.2 Pendulum of varying length

Now the change in the expression $(T_2^* + V')$ (Hamiltonian), for this system, from time t_1 to t_2, can be found as

$$\Delta(T_2^* + V') = \int_{t_1}^{t_2} mv[(l_o - vt)\dot{\theta}^2 + g\cos\theta]dt \tag{f}$$

Since $E = T^* + V = T_2^* + V' + T_1^* + 2T_o^*$, the change in total energy is

$$\Delta(T^* + V) = -\int_{t_1}^{t_2} \frac{\partial L}{\partial t}\, dt + \Delta\left[T_1^* + 2T_o^*\right] \tag{g}$$

or

$$\Delta(T^* + V) = -\int_{t_1}^{t_2} \frac{\partial L}{\partial t}\, dt + \int_{t_1}^{t_2} \frac{d}{dt}(T_1^* + 2T_0^*)\, dt \tag{h}$$

In this example, it is evident that energy is fed into the system as a result of the work done by the tension force F in the string. The tension must be in equilibrium with the component of the weight, along the string, plus the centrifugal force. Since $T_1^* = 0$ and $T_o^* = \frac{1}{2}mv^2$ and consequently $d(T_1^* + 2T_o^*)/dt = 0$ for the present example, the integrands in equations (f) and (h) are the same and can be recognized as the rate of work done by the tension in the string.

Invoking equation (3.5.7) we may write for the example at hand

$$(T_2^* + V')_t - \int_{t_1}^{t} mv[\,(l_o - vt)\,\dot{\theta}^2 + g\,\cos\theta]\,dt \;=\; \text{constant} \qquad \text{(i)}$$

Differentiating through with respect to time we find

$$\frac{d}{dt}\left[\frac{m}{2}(l_o - vt)^2\dot{\theta}^2 - mg(l_o - vt)\cos\theta - \frac{m}{2}v^2\right] - mv(l_o - vt)\dot{\theta}^2 - mg\,v\,\cos\theta \;=\; 0 \qquad \text{(j)}$$

or

$$[\,(l_o - vt)\ddot{\theta} - 2v\dot{\theta} + g\,\sin\theta)\,]\,\dot{\theta} \;=\; 0 \qquad \text{(k)}$$

where the term in square brackets will be recognized as the equilibrium equation of motion for the system.

Problem

3.5.1 Let the string velocity of the problem shown in Figure 3.2 be zero when $t = 0$ and increase linearly according to $v = at$. Form $\Delta(T_2^* + V')$ and $\Delta(T^* + V)$ and show that the string tension force is in equilibrium with the component of the weight along the string, plus the centrifugal force plus the force due to the acceleration of the mass.

3.6 Constraint Equations and Lagrange Multipliers

In deriving Hamilton's principle we did not consider the constraint equations explicitly. We had assumed that these equations were satisfied implicitly via the reduced transformation equations. Now this can be done for holonomic constraints but not for nonholonomic constraints, since in this case only the derivatives of the generalized displacements are related - the generalized displacements themselves remain independent. Accordingly, Hamilton's principle, as given in equation (3.1.8), holds only for holonomic systems for which the constraint equations are satisfied implicitly via the reduced transformation equations.

Holonomic Constraints

Now let us consider Hamilton's principle subject to some side constraints. First we will examine the holonomic constraints. In this case the h constraints amongst $n + h$ generalized coordinates may be expressed as

$$f_i\,(q_1, q_2, q_3, \ldots\ldots, q_j, \ldots\ldots q_{n+h}, t) \;=\; 0 \qquad i = 1,2,\ \ldots,\ h \qquad n > h \quad (3.6.1)$$

We may take these constraint equations into account by the *Lagrange multiplier* method†. To this end we multiply each constraint equation by an undetermined Lagrange multiplier, λ_i, and sum the resulting terms to obtain

† An account of this method is given in Appendix A

$$\lambda_i \; f_i \;\; = \;\; 0 \tag{3.6.2}$$

Since equation (3.6.2) holds at each instant, the integral of $\lambda_i f_i$ over the period t_1 to t_2 will also vanish. Therefore, we may append this integral to Hamilton's functional (3.1.9) without affecting the value of the functional. Thus we write

$$\overline{\Pi}_H \;\; = \;\; \int_{t_1}^{t_2} L dt \;\; - \;\; \int_{t_1}^{t_2} \lambda_i \; f_i \; dt \tag{3.6.3}$$

and for its variation

$$\delta\overline{\Pi}_H \;\; = \;\; \int_{t_1}^{t} \delta(L \;\; - \;\; \lambda_i f_i) \; dt \tag{3.6.4}$$

Now in the modified functional $\overline{\Pi}_H$ we may freely vary the q_j and the λ_i. The Euler-Lagrange equations of the calculus of variations corresponding to these variables emerge as follows:

For δq_j:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \;\; = \;\; - \lambda_i \; \frac{\partial f_i}{\partial q_j} \tag{3.6.5a}$$

$\delta\lambda_i$:

$$f_i \;\; = \;\; 0 \tag{3.6.5b}$$

From equation (3.6.3) it can be seen that above extremum conditions are associated with the *modified* (or *augmented*) Lagrangian $\bar{L}$, given by

$$\bar{L} \;\; = \;\; L \;\; - \;\; \lambda_i \; f_i \tag{3.6.6}$$

or

$$\bar{L} \;\; = \;\; T^* \;\; - \;\; \bar{V} \tag{3.6.7}$$

where

$$\bar{V} \;\; = \;\; V \;\; + \;\; \lambda_i \; f_i \tag{3.6.8}$$

is the *modified* (or *augmented) potential energy.*

Now since the generalized displacements were varied freely without regard to the kinematic constraints, the constraint forces, which maintain the kinematic constraints, will also contribute to the work done. Indeed we can see that these constraint forces possess a potential energy which must be added to the potential energy V of the impressed forces. It now becomes apparent that the addition of the term $\lambda_i \; f_i$, in equation (3.6.8), is not merely for mathematical convenience but it has a clear physical significance. There is however, one important difference between the potential energy V of the impressed forces and the potential energy $\lambda_i \; f_i$ of the constraint forces. The latter can be determined only when the solution is found since the λ_i are determined only as part of the solution. Nevertheless the forces of constraint can be formally derived from their potential energy $\lambda_i \; f_i$. We conclude that at the solution, the generalised constraint forces R_j, associated with q_j, may be determined as follows

$$R_j \;\; = \;\; - \; \frac{\partial \left[\lambda_i f_i \right]}{\partial q_j} \;\; = \;\; - \; \lambda_i \frac{\partial f_i}{\partial q_j} \;\; - \;\; \frac{\partial \lambda_i}{\partial q_j} \; f_i \tag{3.6.9}$$

$$\text{or} \qquad R_j \;=\; -\lambda_i \frac{\partial f_i}{\partial q_j}$$

since the second term on the right hand side of equation (3.6.9) drops out, by virtue of $f_i = 0$ at the solution.

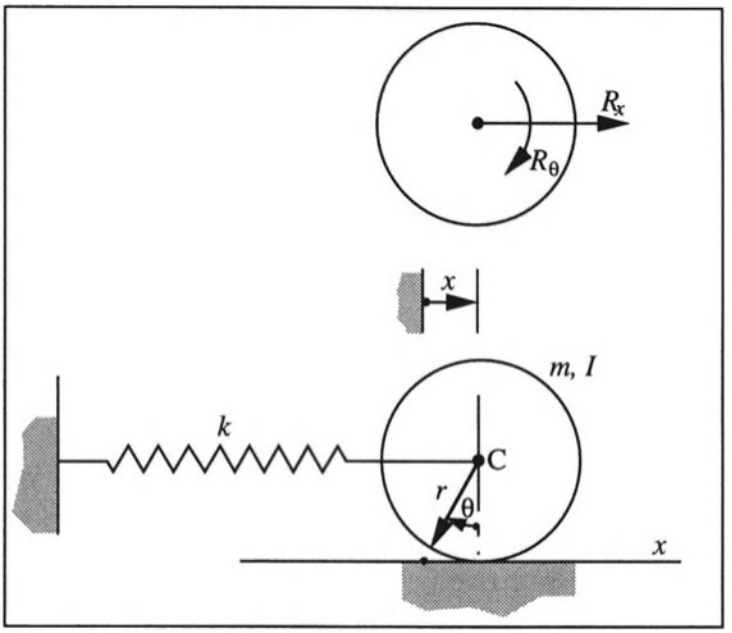


Figure 3.3 Rolling disk

Example 3.6.1

Consider the single degree of freedom system shown in Figure 3.3. The roller's mass and inertia moment are denoted by m and I respectively and the spring is linear with stiffness k. Suppose that the friction force between the roller and the ground is large enough to prevent slipping. Then the coordinates x and θ will be related by the holonomic constraint

$$f \;=\; x - r\theta \;=\; 0 \tag{a}$$

We now keep $q_1 = x$ and $q_2 = \theta$ as generalized coordinates, multiply the constraint equation (a) by λ, and form the modified Lagrangian as follows

$$\bar{L} \;=\; T^* - V - \lambda f \;=\; \frac{1}{2}(m\dot{x}^2 + I\dot{\theta}^2) - \frac{1}{2}kx^2 - \lambda\,(x - r\theta) \tag{b}$$

We introduce the modified action

$$\bar{\Pi}_H \;=\; \int_{t_1}^{t_2} \bar{L}\, dt \tag{c}$$

and, from the calculus of variations, the variation can be written

$$\delta\bar{\Pi}_H = \int_{t_1}^{t_2} \left[\left[\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{x}} - \frac{\partial \bar{L}}{\partial x} \right] \delta x + \left[\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{\theta}} - \frac{\partial \bar{L}}{\partial \theta} \right] \delta\theta + \left[\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{\lambda}} - \frac{\partial \bar{L}}{\partial \lambda} \right] d\lambda \right] dt = 0 \quad \text{(d)}$$

where each term in brackets must vanish since δx, $\delta\theta$ and $\delta\lambda$ are now treated as independent.

For δx:
$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{x}} - \frac{\partial \bar{L}}{\partial x} = 0 \quad \text{(e)}$$

which upon differentiation of the modified Lagrangian (b) results in

$$m\ddot{x} + kx + \lambda = 0 \quad \text{(f)}$$

For $\delta\theta$:
$$I\ddot{\theta} - \lambda r = 0 \quad \text{(g)}$$

For $\delta\lambda$:
$$x - r\theta = 0 \quad \text{(h)}$$

Equations (f), (g), and (h) represent the equations of motion.

On substituting for λ, from equation (g) into equation (f) we obtain

$$-m\ddot{x} - kx - \frac{I}{r}\ddot{\theta} = 0 \quad \text{(i)}$$

But from equation (h), $\ddot{\theta} = \ddot{x}/r$. Hence equation (i) becomes

$$\left(m + \frac{I}{r^2}\right)\ddot{x} + kx = 0 \quad \text{(j)}$$

and represents the equation of motion after elimination of λ. This equation could have been obtained directly by eliminating θ from the Lagrangian, via equation (a).

Next let us evaluate the generalized forces of constraint. The force in x direction, given by equation (3.6.9).

$$R_1 = -\frac{\partial}{\partial q_1}\lambda f = -\frac{\partial}{\partial x}(\lambda (x - r\theta)) = -\lambda \quad \text{(k)}$$

From equation (f), $\lambda = -m\ddot{x} - kx$, and it follows that the magnitude of the required force is given by

$$R_1 = R_x = m\ddot{x} + kx \quad \text{(l)}$$

and it is provided by the friction between the ground and the wheel.

The reaction torque associated with θ is given by

$$R_2 = R_\theta = -\frac{\partial}{\partial \theta}(\lambda(x - r\theta)) = r\lambda \quad \text{(m)}$$

or from equations (g) and (m)

$$R_\theta = I\ddot{\theta} \quad \text{(n)}$$

Nonholonomic Constraints

Let us now examine the case of nonholonomic constraints. These constraints, which may be expressed as

$$a_{ij}\,dq_j \;+\; a_{it}\,dt \;=\; 0 \qquad\qquad i \;=\; 1,2,\ldots,\nu \qquad\qquad j \;=\; 1,2,\ldots,n \tag{3.6.10}$$

can also be taken into account by the Lagrange multiplier method. However, in this case the term to be appended to Hamilton's functional is not a function but rather a nonintegrable differential of first order. As such these terms must be appended to the first variation of the Lagrangian rather than the Lagrangian itself. Before doing so we also note that since in Hamilton's principle time is not varied, the nonholonomic constraints amongst virtual displacements take the form

$$a_{ij}\;\delta q_j \;=\; 0 \qquad\qquad i = 1,2,\ldots,\nu \tag{3.6.11}$$

Now multiplying each constraint equation in equation (3.6.11) by a multiplier λ_i and appending the sum of such terms, integrated from t_1 to t_2, to the first variation of L, we find

$$\int_{t_1}^{t_2} \delta L \;\; dt \;-\; \int_{t_1}^{t_2} \lambda_i \;\; a_{ij} \;\; \delta q_j \;\; dt \;=\; 0 \tag{3.6.12}$$

or

$$\int_{t_1}^{t_2} (\delta L \;-\; \lambda_i \;\; a_{ij} \;\; \delta q_j)\; dt \;=\; 0 \tag{3.6.13}$$

On evaluating the variations of L with respect to q_j and combining all terms multiplied by δq_j, the conditions for satisfaction of equation (3.6.12) are found as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \;-\; \frac{\partial L}{\partial q_j} \;=\; -\;\lambda_i \;\; a_{ij} \qquad\qquad j = 1,2,\ldots,n \tag{3.6.14}$$

In equation (3.6.14) we have n equations for determination of the q, and ν Lagrange multipliers. These equations must be solved along with the ν constraint equations (3.6.10) amongst the displacements. On division through by dt, we may write these equations as

$$a_{ij}\;\; \dot{q}_j \;+\; a_{it} \;=\; 0 \qquad\qquad i = 1,2,\ldots,\nu \tag{3.6.15}$$

Thus in contrast to the case of holonomic constraints, the nonholonomic constraint equations (3.6.14) are not obtained as conditions of extremum of a functional.

Now there are a number of points of interest to note. First we observe that the Lagrange multiplier method once again yields the constraint forces. Thus from equation (3.6.14) we see that the constraint force associated with q_j is given by

$$R_j \;=\; -\,(\,\lambda_1\, a_{1j} + \lambda_2\, a_{2j} \;+\; \cdots \;+\; \lambda_\nu\, a_{\nu j}\,) \;=\; -\;\lambda_i a_{ij} \tag{3.6.16}$$

In this case though, there exists no potential energy for the forces of constraint. This is apparent from equation (3.6.12) wherein the contribution of constraint forces appears

in the form of a virtual work rather than a potential energy. Thus while the constraint forces associated with holonomic constraints are mechanically equivalent to conservative or monogenic forces, the forces associated with nonholonomic constraints may be viewed as being equivalent to polygenic forces.

The n second order equations of motion require $2n$ initial conditions. Since all the q_j are independent, n of the initial conditions may be specified freely. On the other hand the velocities $\dot{q}_j$ are not all independent. We may specify $(n-\nu)$ of such velocities freely but the remainder must be evaluated via the constraint equations (3.6.15).

It is also worth noting that it is always possible to express the holonomic constraints, in equations (3.6.1), in the format of the nonholonomic constraints in equations (3.6.15). Thus on evaluating the time derivative of f_i, we may express the holonomic constraints as

$$\frac{\partial f_i}{\partial q_j}\dot{q}_j + \frac{\partial f_i}{\partial t} = 0 \tag{3.6.17}$$

Accordingly the method of accounting for nonholonomic constraints can also be used for holonomic constraints. The converse, however, is not possible.

It is tempting to append the velocity form of the nonholonomic constraint equations to the Lagrangian by means of Lagrange multipliers. This will lead to incorrect equations of motion. To illustrate consider for simplicity the case of a single nonholonomic constraint equation with the modified Lagrangian as

$$\bar{L} = L - \bar{\lambda}(a_1\dot{q}_1 + a_2\dot{q}_2 + \cdots a_t)$$

The first Lagrange equation now becomes

$$\frac{d}{dt}\left[\frac{\partial \bar{L}}{\partial \dot{q}_1}\right] - \frac{\partial \bar{L}}{\partial q_1} = 0$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_1}\right] - \frac{d}{dt}(\bar{\lambda}a_1) - \frac{\partial L}{\partial q_1} + \bar{\lambda}\frac{\partial}{\partial q_1}(a_1\dot{q}_1 + a_2\dot{q}_2 + \cdots + a_t) = 0$$

$$\text{or} \quad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_1}\right] - \frac{\partial L}{\partial q_1} = \frac{d\bar{\lambda}}{dt}a_1 + \bar{\lambda}\left[\frac{\partial a_1}{\partial q_1}\dot{q}_1 + \frac{\partial a_1}{\partial q_2}\dot{q}_2 + \cdots \frac{\partial a_1}{\partial t}\right]$$

$$- \bar{\lambda}\left[\frac{\partial a_1}{\partial q_1}\dot{q}_1 + \frac{\partial a_2}{\partial q_1}\dot{q}_2 + \cdots + \frac{\partial a_t}{\partial q_1}\right] \tag{3.6.18}$$

on combining the various terms we may write equation (3.6.18) as

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_1}\right] - \frac{\partial L}{\partial q_1} = \frac{d\bar{\lambda}}{dt}a_1 + \bar{\lambda}\left[\frac{\partial a_1}{\partial q_2} - \frac{\partial a_2}{\partial q_1}\right]\dot{q}_2 + \ldots + \bar{\lambda}\left[\frac{\partial a_1}{\partial t} - \frac{\partial a_t}{\partial q_1}\right] \tag{3.6.19}$$

Now comparing with the correct equation of motion in equation (3.6.14) we note that apart from the inconsequential difference between λ and $\bar{\lambda}$ ($\lambda = -d\bar{\lambda}/dt$), the first term on the right hand side of equation (3.6.19) is correct. However the remaining terms should not be present. We can now see that these unwanted terms will drop out if the velocity constraint equation is holonomic, i.e.

$$\frac{\partial a_i}{\partial q_j} = \frac{\partial a_j}{\partial q_i} \quad \text{and} \quad \frac{\partial a_i}{\partial t} = \frac{\partial a_t}{\partial q_i}$$

As a final comment we point out that one is also tempted to satisfy the velocity form of the nonholonomic constraints by eliminating some of the velocities from T^*. Such a procedure tacitly assumes that the nonholonomic constraint forces possess a potential that may be combined with T^*. Once again the resulting equations of motion, obtained in this manner will be incorrect unless of course the constraint equations are integrable, i.e. they are holonomic.

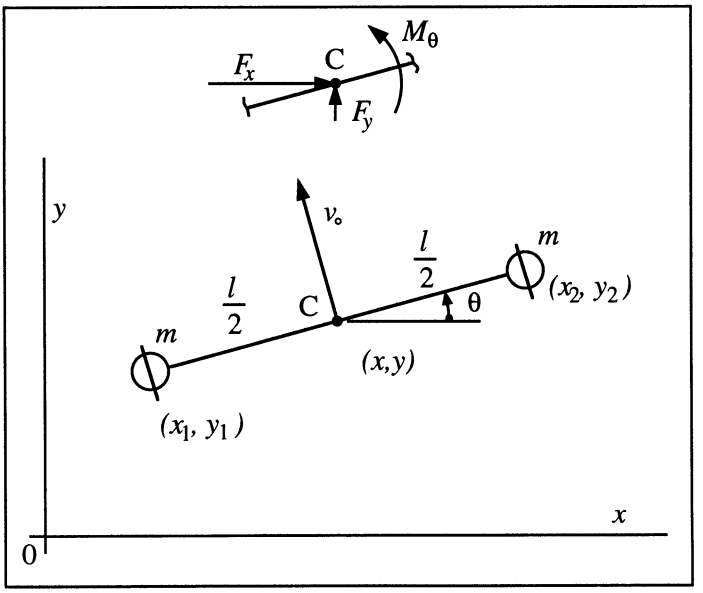


Figure 3.4 Two-mass system with nonholonomic constraint

Example 3.6.2

The two particles shown in Figure 3.4 move on a horizontal plane and are connected by a rigid massless rod of length l. The particles are supported on knife edges which prevent either particle from having a velocity component along the rod. Motion normal to the rod takes place without friction. We wish to analyse the motion of this system.

Using x_1, y_1, x_2, y_2 as the Cartesian displacements we need to satisfy the following holonomic constraint

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2 \qquad \text{(a)}$$

We can satisfy this holonomic equation directly by introducing the generalized coordinates locating the centre of mass of the system and the rod's orientation, namely $q_1 = x$, $q_2 = y$ and $q_3 = \theta$. The reduced transformation equations can now be written as

$$x_1 = x - \frac{1}{2} l \cos\theta \qquad y_1 = y - \frac{1}{2} l \sin\theta \qquad \text{(b)}$$

$$x_2 = x + \frac{1}{2} l \cos\theta \qquad y_2 = y + \frac{1}{2} l \sin\theta$$

The constraint preventing motion normal to the knife edges can also be written in terms of the genralized coordinates as

$$(\cos\theta)\ \delta x + (\sin\theta)\ \delta y + (0)\ \delta\theta = 0 \tag{c}$$

or on dividing through by dt we may express this nonholonomic constraint as

$$\dot{x} \cos\theta + \dot{y} \sin\theta = 0 \tag{d}$$

With the three independent coordinates and one nonholonomic constraint equation the system has two degrees of freedom but all three coordinates are required for describing the configuration of the system.

The complementary kinetic energy of the system can now be written as

$$T^* = m\ (\dot{x}^2 + \dot{y}^2 + \frac{1}{4} l^2\ \dot{\theta}^2) \tag{e}$$

Now we invoke equation (3.6.14)

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_j} = -\lambda_i\ a_{ij} \tag{f}$$

In the problem at hand we have only one nonholonomic constraint for which

$$\begin{aligned} a_{11} &= a_{1x} = \cos\theta \\ a_{12} &= a_{1y} = \sin\theta \\ a_{13} &= a_{1\theta} = 0 \end{aligned} \tag{g}$$

Thus the equations of motion emerge as

$$2\ m\ddot{x} = -\lambda_1 \cos\theta \tag{h}$$

$$2\ m\ddot{y} = -\lambda_1 \sin\theta \tag{i}$$

$$\frac{1}{2} l^2\ \ddot{\theta} = 0 \tag{j}$$

The third equation indicates that the angular momentum is conserved, and hence

$$\dot{\theta} = \omega = \text{constant} \tag{k}$$

and

$$\theta = \omega t + \theta_o \tag{l}$$

We will let $\theta_o = 0$ for convenience. We may substitute from equation (l) into equations (h) and (i) and find two equations in terms of three variables. These equations must now be solved together with the nonholonomic equation (d).

From equations (h), (i) and (l) we find

$$\ddot{y} = \ddot{x} \tan \omega t \tag{m}$$

Also from equation (d)

$$\tan \omega t = -\dot{x}/\dot{y} \tag{n}$$

Equations (m) and (n) combined lead to

$$\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y} = 0 \tag{o}$$

or

$$\frac{d}{dt}(\dot{x}^2 + \dot{y}^2) = 0$$

that is,

$$\dot{x}^2 + \dot{y}^2 = v_o^2 = \text{constant} \tag{p}$$

This equation shows that the mass centre for this system moves with a velocity of constant magnitude v_o, i.e. the system follows a circular path.

Now since the direction of motion is always normal to the rod, we have

$$\dot{x} = -v_o \sin\omega t \tag{q}$$

$$\dot{y} = v_o \cos\omega t \tag{r}$$

Integrating these equations we find

$$x = \frac{v_o}{\omega}\cos\omega t + A \tag{s}$$

$$y = \frac{v_o}{\omega}\sin\omega t + B \tag{t}$$

The integration constants A and B can be determined from initial conditions. If the initial conditions are such that $A = B = 0$, then

$$x^2 + y^2 = \frac{v_o^2}{\omega^2} \tag{u}$$

and the motion is represented by a circle about the origin of the coordinate system. It remains now to find λ_1. Using the solution for, say, $x(t)$ in equations (h) and (l) we find

$$\lambda_1 = 2mv_o\omega \tag{v}$$

The components of the generalized force (3.6.16) associated with the nonholonomic constraint in equation (d) are found as

$$R_1 = R_x = -\lambda_1 a_{1x} = F_x = -\lambda\cos\omega t = -2mv_o\omega\cos\omega t \tag{w}$$

$$R_2 = R_y = -\lambda_1 a_{1y} = F_y = -\lambda\sin\omega t = -2mv_o\omega\sin\omega t \tag{x}$$

$$R_3 = R_\theta = -\lambda_1 a_{1\theta} = M_\theta = 0 \tag{y}$$

Consider now the changes in the energy of the system. Evidently the problem is scleronomic since time does not appear explicitly in the reduced transformation equations and the Lagrangian. Hence energy is conserved in this problem. Since there is no potential energy, then

$$\frac{d}{dt}T^* = 0$$

or

$$\frac{d}{dt}m(\dot{x}^2 + \dot{y}^2 + \frac{1}{4}l^2\dot{\theta}^2) = 0$$

that is

$$\dot{x}\,\ddot{x} + \dot{y}\,\ddot{y} + \frac{1}{2}l^2\,\dot{\theta}\,\ddot{\theta} = 0 \tag{z}$$

But since the angular momentum is conserved the last term drops out and the remaining terms become consistent with equation (o) derived earlier.

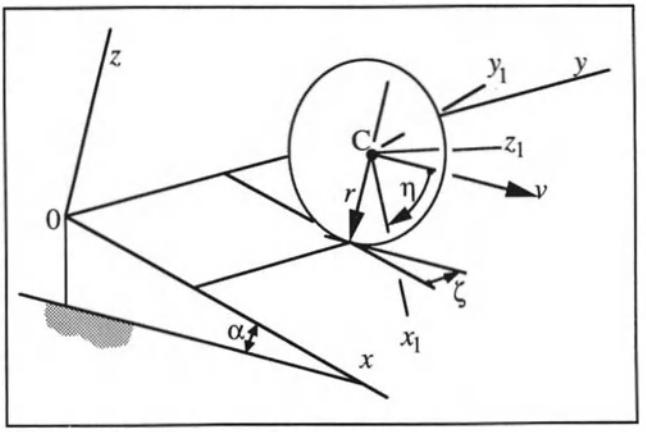


Figure 3.5 Thin disk on inclined plane

Example 3.6.3

A thin disk with a rim wide enough to prevent toppling sideways is to roll down an incline as shown in Figure 3.5. The inertia moments are $B = mr^2/2$, $A = C = mr^2/4$.

The $Cx_1y_1z_1$ frame is body-fixed, the $Oxyz$ frame is space-fixed. The equations of motions are to be obtained. We may describe the configuration of the disk in terms of four coordinates: x, y, η and ζ. However the nonslip condition of the disk's motion imposes the following nonholonomic constraint equations

$$dx - r\,d\eta\cos\zeta = 0 \tag{a}$$

$$dy - r\,d\eta\sin\zeta = 0 \tag{b}$$

Thus the system has two degrees of freedom. However we need to retain all four independent coordinates for deriving the equations of motion. In terms of these variables we have the following energy expressions

$$T^* = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}mr^2\,\dot{\eta}^2 + \frac{1}{8}mr^2\dot{\zeta}^2 \tag{c}$$

$$V = -mg\,x\sin\alpha \tag{d}$$

Appending these constraint equations to δL we obtain for equation (3.6.12)

$$\int_{t_1}^{t_2}\left[m\left(\dot{x}\,\delta\dot{x} + \dot{y}\,\delta\dot{y} + \frac{r^2}{2}\dot{\eta}\,\delta\dot{\eta} + \frac{r^2}{4}\dot{\zeta}\,d\dot{\zeta}\right) + mg\sin\alpha\,\delta x + \lambda_1(r\delta\eta\cos\zeta - \delta x) + \right.$$

$$\left.\lambda_2(rd\eta\sin\zeta - \delta y)\right]dt = 0 \tag{e}$$

Integrating the derivative terms and collecting similar terms we find the following extremum conditions:

$$\text{For } \delta x: \quad m\ddot{x} - mg\ \sin\alpha + \lambda_1 = 0 \tag{f}$$

$$\delta y: \quad m\ddot{y} + \lambda_2 = 0 \tag{g}$$

$$\delta\eta: \quad \frac{mr^2}{2}\ddot{\eta} - \lambda_1 r\ \cos\zeta - \lambda_2 r\ \sin\zeta = 0 \tag{h}$$

$$\delta\zeta: \quad \frac{d}{dt}\left[\frac{mr^2}{4}\dot{\zeta}\right] = 0 \tag{i}$$

These four system equations in terms of the six unknowns must be solved along with the two nonholonomic equations, which we rewrite as follows:

$$\dot{x} = r\dot{\eta}\cos\zeta \tag{j}$$

$$\dot{y} = r\dot{\eta}\sin\zeta \tag{k}$$

Now equation (i) may be readily integrated yielding

$$\dot{\zeta} = \omega_\zeta = \text{constant} \tag{l}$$

$$\text{and} \quad \zeta = \omega_\zeta t + \zeta_0 \tag{m}$$

where ω_ζ and ζ_0 are the constant initial angular velocity and angular position of the disk. From equations (f) and (g) we find the Lagrange multipliers

$$\lambda_1 = mg\ \sin\alpha - m\ddot{x}$$

$$\lambda_2 = -m\ddot{y}$$

Evaluating $\ddot{x}$ and $\ddot{y}$ from equations (j) and (k) we may express the Lagrange multipliers as

$$\lambda_1 = mg\ \sin\alpha - mr\ (\ddot{\eta}\cos\zeta - \dot{\eta}\dot{\zeta}\sin\zeta) \tag{n}$$

$$\lambda_2 = -mr\ (\ddot{\eta}\sin\zeta + \dot{\eta}\dot{\zeta}\cos\zeta) \tag{p}$$

Now substituting from equations (n) and (p) into equation (h) we obtain the governing equation for η as follows

$$\ddot{\eta} = \frac{2}{3}\frac{g}{r}\sin\alpha\cos(\omega_\zeta t + \zeta_o) \tag{q}$$

Integrating this equation we obtain

$$\dot{\eta} = \frac{2}{3}\frac{g}{r}\frac{\sin\alpha}{\omega_\zeta}\sin(\omega_\zeta t + \zeta_o) + \dot{\eta}_o \tag{r}$$

$$\text{and} \quad \eta = -\frac{2}{3}\frac{g}{r}\frac{\sin\alpha}{\omega_\zeta^2}\cos(\omega_\zeta t + \zeta_o) + \dot{\eta}_o t + \eta_o \tag{s}$$

We let $\dot{\eta}_o = 0$ for simplicity. Finally substituting from equations (r) and (m) into the constraint equation (j) we obtain

$$\dot{x} = \frac{g\ \sin\alpha}{3\ \omega_\zeta}\sin 2(\omega_\zeta t + \zeta_o) \tag{t}$$

which on integration yields

$$x = -\frac{g\sin\alpha}{6\ \omega_\zeta^2}(\cos 2(\omega_\zeta t + \zeta_o) - \cos 2\zeta_o) + x_o \tag{u}$$

Also by substituting from equations (r) and (m) into equations (k) we find

$$\dot{y} = \frac{2}{3}\frac{g\sin\alpha}{\omega_\zeta}\sin^2(\omega_\zeta t + \zeta_o) \tag{v}$$

which on integration yields

$$y = \frac{1}{3}\frac{g\sin\alpha}{\omega_\zeta}\left[t - \frac{\sin 2(\omega_\zeta t + \zeta_o)}{2\,\omega_\zeta}\right] + y_o \tag{w}$$

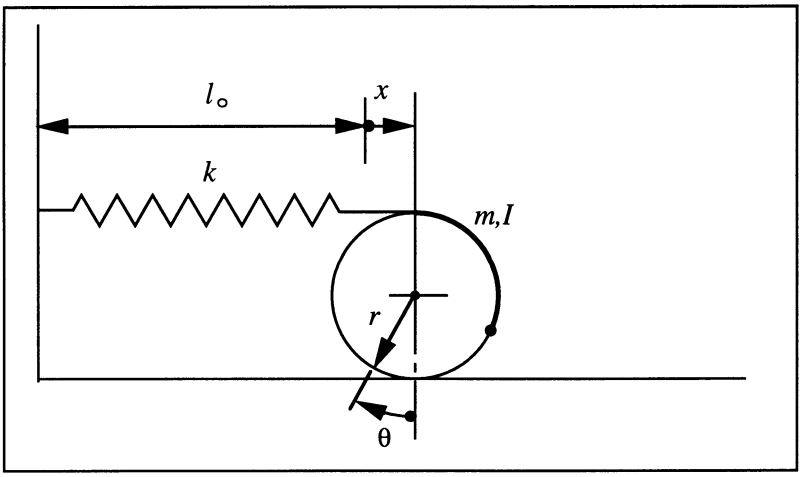
Evidently the motion is fairly complex. It is of interest to note that if the disk's angular velocity $\omega_\zeta \neq 0$ the disk will not move down the incline, but sideways. However if we set $\omega_\zeta = \zeta_o = 0$ we find from equations (u) and (w), through use of L'Hospital's rule that

$$x = \frac{g\sin\alpha}{3}t^2 + x_o \tag{x}$$

$$y = y_o \tag{y}$$

Problems

3.6.1 Use (a) a direct approach, and (b) Lagrange multipliers, to establish the equation of motion of the single degree of freedom system shown. The disk rolls without slippage. (c) Determine the constraint force.

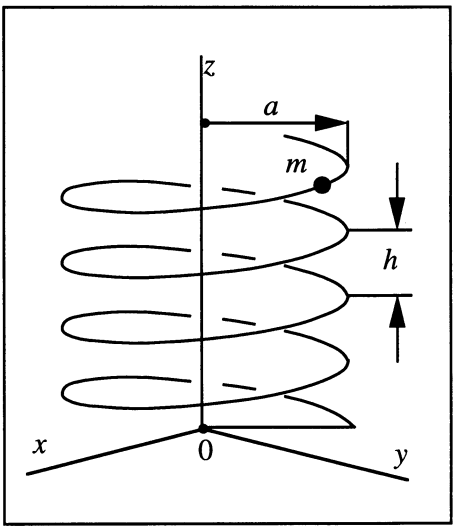


3.6.2 A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$, which is assumed frictionless. Using the Lagrange multiplier method and Hamilton's principle, derive the equations of motion of the particle.

3.6.3 Use a Lagrange multiplier for the constraint equation $f = x^2 + y^2 - l^2 = 0$, and find the equation of motion for a simple pendulum of length l and mass m, in terms of the angle θ.

3.6.4 Use a Lagrange multiplier for the constraint equation $f = x^2 + y^2 + z^2 - l^2 = 0$, and find (a) the equations of motion for a spherical pendulum of length l and point mass m, in terms of x, y, and z. (b) Show that the angular momentum component H_z = constant, (c) that the total energy $E = T^* + V$ = constant, and (d) that $2\lambda l$ represents the constraint force transmitted through the string.

3.6.5 A bead slides down, without friction, a wire in the shape of a helix of radius a and pitch h, as shown. Find the z component of its motion.



3.6.6 In example 3.6.2, using the velocity from the nonholonomic constraint, eliminate one of the velocities from T^* and show that using T^* in this form in Hamilton's principle leads to an incorrect result. Explain why this is the case.

3.6.7 A particle of unit mass is subjected to a force with potential energy $V = V(x,y)$ and is constrained in the xy plane with the slope of its trajectory proportional to time. Determine the particle's equilibrium equation of motion.

3.7 Ignorable Coordinates

Depending upon the generalized coordinates used, it may happen that one or more of the generalized displacements will be absent in the Lagrangian while their velocities are present. Such generalized displacements are called *ignorable* or *cyclic* coordinates. Lagrange's equations, associated with ignorable coordinates, can be partially integrated. To see this, consider for simplicity the case of the single ignorable coordinate which we arrange to be the last generalized displacement q_n. The case for multiple ignorable coordinates can be dealt with in a similar manner. Now since q_n is absent in L, then

$$\frac{\partial L}{\partial q_n} = 0 \tag{3.7.1}$$

and Lagrange's equation associated with q_n reduces to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} = Q_n' \tag{3.7.2}$$

Recognizing $(\partial L/\partial \dot{q}_n)$ as the nth generalized momentum, we may express equation (3.7.2) as

$$\dot{p}_n\ (\ q_1 \ldots q_{n-1}, \dot{q}_1 \cdots \dot{q}_n,\ t\ ) = Q_n' \tag{3.7.3}$$

which, on integration, yields

$$p_n\ (\ q_1 \ldots q_{n-1}, \dot{q}_1 \cdots \dot{q}_n, t\ ) = \int Q_n' dt + c_n \tag{3.7.4}$$

where c_n is an integration constant. In the special case when Q_n' is zero, i.e. when there is no impressed force associated with q_n, equation (3.7.4) shows that the momentum, associated with the ignorable coordinate, remains constant in time. This conservation of momentum allows one to eliminate the ignorable coordinate from the equations of motion. Thus from equation (3.7.4), in the absence of Q_n', we may express $\dot{q}_n$ as

$$\dot{q}_n = f\ (\ q_1, \ldots, q_{n-1}, \dot{q}_1, \ldots, \dot{q}_{n-1}, t, c_n\ ) \tag{3.7.5}$$

Now whenever we encounter $\dot{q}_n$ in the equations of motion (q_n itself is absent) we can replace it by the right hand of equation (3.7.5). In this way the problem of integration of the equations of motion may be reduced to one involving non-ignorable coordinates alone. After obtaining the solution of the reduced problem we may substitute q_i and $\dot{q}_i$, which are now determined functions of time, in equation (3.7.5) and determine q_n as a function of time, by integration.

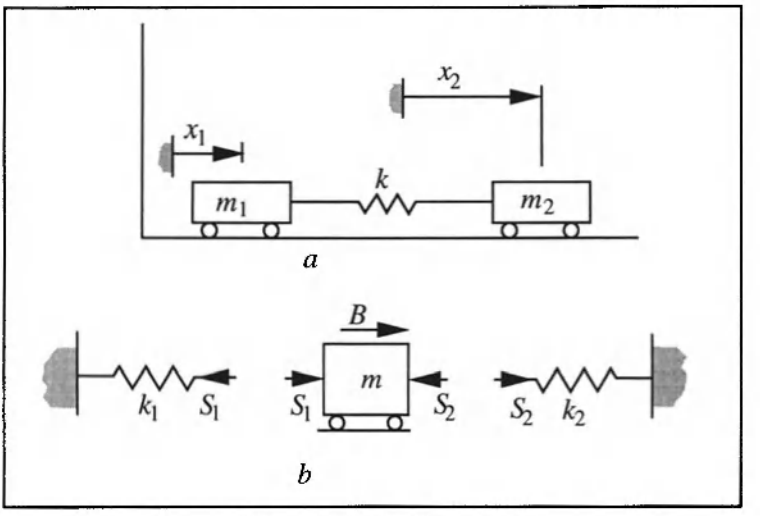


Figure 3.6 Dual systems

Example 3.7.1

Consider the two degree of freedom system shown in Figure 3.6a. Taking the spring as linear, we may use x_1 and x_2 as generalized coordinates and express the Lagrangian of this system as

$$L = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2) - \frac{k}{2}(x_2 - x_1)^2 \tag{a}$$

In this case neither x_1 nor x_2 is an ignorable coordinate. However let us introduce a different set of generalized coordinates, related to x_1 and x_2 as follows

$$x_1 = q_1 + q_2 \tag{b}$$

$$x_2 = q_2 \tag{c}$$

In terms of q_1 and q_2 the Lagrangian takes the form

$$L = \frac{1}{2}m_1(\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2}m_2\dot{q}_2^2 - \frac{1}{2}kq_1^2 \tag{d}$$

Now it is evident that q_2 is an ignorable coordinate since it does not occur in the Lagrangian. The associated generalized momentum p_2 may be derived from $(\partial L/\partial\dot{q}_2)$ and is given by

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = m_1(\dot{q}_1 + \dot{q}_2) + m_2\dot{q}_2 = c_2 = \text{constant} \tag{e}$$

$$\text{or} \qquad m_1\dot{x}_1 + m_2\dot{x}_2 = c_2 \tag{f}$$

The interpretation of equation (f) becomes apparent when we note from the transformation equations (b) and (c) that q_2 is a rigid body displacement, whereas q_1 is the elastic displacement. Thus the momentum associated with q_2 is the momentum of the two masses in a rigid body motion.

Now from equation (e) we have that

$$\dot{q}_2 = \frac{c_2 - m_1\dot{q}_1}{m_1 + m_2} \tag{g}$$

Further the equation of motion associated with q_1 can be obtained from

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0$$

as

$$m_1\frac{d}{dt}(\dot{q}_1 + \dot{q}_2) + kq_1 = 0 \tag{h}$$

It now remains to eliminate $\dot{q}_2$, from equation (h), via equation (g). The equation of motion for q_1, then takes the form

$$\frac{m_1m_2}{m_1 + m_2}\ddot{q}_1 + kq_1 = 0 \tag{i}$$

Example 3.7.2

Let us consider next the system shown in Figure 3.6b and let us analyse this system by the complementary procedure, with the generalized impulses S_1 and S_2 as coordinates. We find, with $B = S_1 - S_2$, for the kinetic energy

$$T = \frac{1}{2}\frac{B^2}{m} = \frac{1}{2}\frac{(S_1 - S_2)^2}{m} \qquad \text{(a)}$$

and for the complementary potential energy of the two linear springs with stiffnesses k_1 and k_2,

$$V^* = \frac{1}{2}\frac{\dot{S}_1^2}{k_1} + \frac{1}{2}\frac{\dot{S}_2^2}{k_2} \qquad \text{(b)}$$

and for the complementary Lagrangian $L^* = T - V^*$,

$$L^* = \frac{1}{2m}(S_1 - S_2)^2 - \frac{1}{2}\left[\frac{\dot{S}_1^2}{k_1} + \frac{\dot{S}_2^2}{k_2}\right] \qquad \text{(c)}$$

where S_1 and S_2 are the spring impulses.

On comparing this expression for L^* with that of the Lagrangian (a) for the system of Example 3.7.1 we note a mathematical similarity. Specifically it is evident that, disregarding signs, the following correspondence exists

$$x \longleftrightarrow S \qquad m \longleftrightarrow \frac{1}{k} \qquad \text{(d)}$$

From this similarity it follows that the (equilibrium) equations of motion for system 3.6a and the (compatibility) equations of motion for system 3.6b are mathematically similar. We may then consider these two systems as *duals* of one another.

Returning to the question of ignorable coordinates it can be seen that by a transformation of impulses, similar in form to the transformations in equations (b) and (c) of Example 3.7.1, we can show that system 3.6b, in its complementary formulation, possesses one ignorable coordinate. It is also worth noting that system 3.6a in its Hamiltonian formulation has two degrees of freedom and one ignorable coordinate. The same system, in its complementary Hamiltonian formulation has one degree of freedom and no ignorable coordinates. The reverse is the case for the system shown in Figure 3.6b.

Finally we note that in the Hamiltonian formulation ignorable coordinates of the q-type do not appear in the expression for the Lagrangian $(T^* - V)$. These ignorable coordinates are generally associated with rigid body motions and they do not give rise to elastic deformations. The momenta, associated with such ignorable coordinates, are conserved. In the complementary Hamiltonian formulation ignorable coordinates are of the S-type, i.e. they do not appear in the complementary Lagrangian $(T - V^*)$ expression. These ignorable coordinates generally do not give rise to kinetic energies. The extensions, associated with such ignorable coordinates, are conserved.

Example 3.7.3

In Kepler's problem of planetary motion (Figure 3.7), with $q_1 = r$ and $q_2 = \theta$, the complementary kinetic energy and potential energies are

$$T^* = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad \text{and} \qquad V = -\frac{\mu m}{r} \qquad \text{(a,b)}$$

The Lagrangian is consequently

$$L = T^* - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\mu m}{r} \qquad \text{(c)}$$

and one equation of motion is

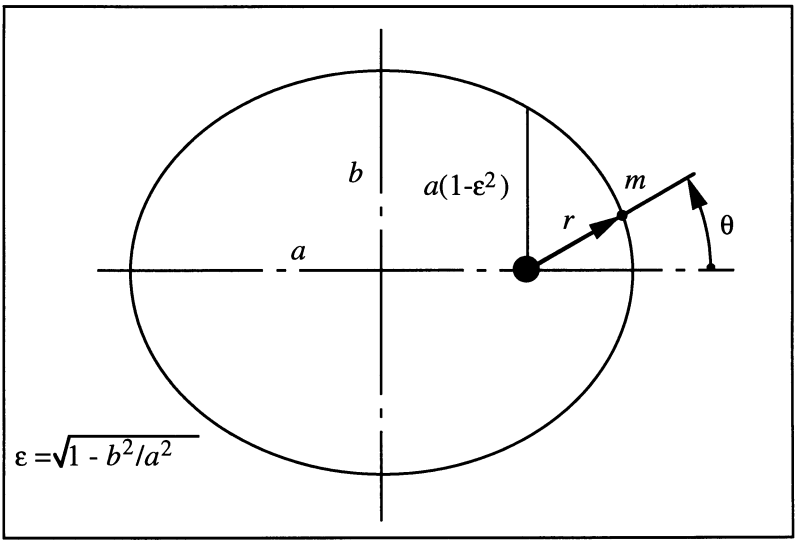


Figure 3.7 The Kepler problem

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\,\dot{\theta}^2 + \frac{\mu m}{r^2} = 0 \tag{d}$$

The coordinate θ, being absent in the Lagrangian, is ignorable. Thus $\frac{\partial L}{\partial \theta} = 0$ and the second equation of motion becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \tag{e}$$

or

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = c_\theta = \text{constant} \tag{f}$$

Solving for $\dot{\theta}$,

$$\dot{\theta} = \frac{c_\theta}{mr^2} \tag{g}$$

We may eliminate it from equation (d) of motion, which then becomes

$$m\ddot{r} + \frac{\mu m}{r^2} - \frac{c_\theta^2}{mr^3} = 0 \tag{h}$$

Since ignorable coordinates can be eliminated from Lagrange equations it is natural to enquire whether it is possible to eliminate them from Hamilton's functional, from the outset. Certainly equation (3.7.5) may be used to eliminate $\dot{q}_n$ from the Lagrangian. However from equation (3.7.5) it can be seen that by prescribing $(q_1, \cdots q_{n-1})$, at the end points t_1 and t_2, we cannot prescribe q_n. Thus at least for q_n we are obliged to retain the boundary terms of Hamilton's functional, as given in equation (3.1.12).

Hence Hamilton's principle must now be expressed as

$$\delta \int_{t_1}^{t_2} L \, dt \;=\; p_n \, \delta q_n \Big|_{t_1}^{t_2} \tag{3.7.6}$$

But p_n, the generalized momentum associated with the ignorable coordinate q_n, we have shown to be a constant. That is,

$$p_n \, \delta q_n \Big|_{t_1}^{t_2} \;=\; p_n \, \delta \int_{t_1}^{t_2} \dot{q}_n dt \;=\; \delta \int_{t_1}^{t_2} c_n \, \dot{q}_n dt \tag{3.7.7}$$

Therefore Hamilton's functional may now be expressed in the following modified form

$$\delta \int_{t_1}^{t_2} L_R \, dt \;=\; 0 \tag{3.7.8}$$

$$\text{where} \qquad L_R \;=\; L \;-\; c_n \dot{q}_n \tag{3.7.9}$$

Now it is permissible to eliminate $\dot{q}_n$ via equation (3.7.5).

For the case of k ignorable coordinates, k equations of the type (3.7.5) can be derived and a modified Lagrangian may be defined as

$$L_R \;=\; L \;-\; c_i \dot{q}_i \qquad\qquad i \;=\; n-k+1,\ldots,n \tag{3.7.10}$$

The modified Lagrangian from which the velocities associated with ignorable coordinates have been *eliminated*, is called the *Routhian*† R of the system. Thus

$$R \;=\; R(q_1,\ldots q_{n-k}, \dot{q}_1,\ldots, \dot{q}_{n-k}, c_{n-k+1},\ldots, c_n) \tag{3.7.11}$$

leading to a modified Lagrange equation

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_j} - \frac{\partial R}{\partial q_j} \;=\; 0 \qquad\qquad j \;=\; 1, 2, \ldots, n-k \tag{3.7.12}$$

The following example 3.7.4 illustrates the application of the Routhian.

Example 3.7.4

Consider Kepler's problem of planetary motion again as shown in Figure 3.7.

The Lagrangian, $L = T^* - V$, is

$$L \;=\; \frac{1}{2} m \left[\dot{r}^2 + (r\dot{\theta})^2 \right] \;+\; \frac{\mu m}{r} \tag{a}$$

Evidently θ is an ignorable coordinate, and consequently the momentum associated with θ is conserved, i.e.

$$\frac{\partial L}{\partial \dot{\theta}} \;=\; p_\theta \;=\; mr^2\dot{\theta} \;=\; c_\theta \;=\; \text{constant}$$

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† Edward John Routh (1831-1907), English mathematician

$$\text{or} \quad \dot{\theta} = \frac{c_\theta}{mr^2} \tag{b}$$

Now we may form the modified Lagrangian (3.7.9) as

$$L_R = L - c_\theta \dot{\theta} = \frac{m}{2}[\dot{r}^2 + (r\dot{\theta})^2] + \frac{\mu m}{r} - c_\theta \dot{\theta} \tag{c}$$

It remains to eliminate $\dot{\theta}$ from L_R, via equation (b). The result takes the form

$$R(r,\dot{r}) = \frac{m}{2}\dot{r}^2 + \frac{\mu m}{r} - \frac{c_\theta^2}{2mr^2} \tag{d}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{r}} - \frac{\partial R}{\partial r} = 0$$

yielding

$$m\ddot{r} + \frac{\mu m}{r^2} - \frac{c_\theta^2}{mr^3} = 0 \tag{e}$$

In principle, once r is determined as a function of t, we may determine θ by integration of equation (b). Unfortunately r cannot be obtained as function of time in closed form in this case, and consequently θ cannot be established in closed form either.

There are, however, a few compact relationships that can be established. They are known as Kepler's laws. The first says that point satellites move on conic sections about their point master, e.g. for an elliptic orbit

$$r = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos\theta} \tag{f}$$

with the integration constants a = semi-major axis and ε = eccentricity.

Then there is Kepler's second law which follows from the constancy of angular momentum as expressed by equation (b), which can be rearranged to state that equal areas $r^2 d\theta$ are swept out in equal time intervals dt.

$$dt = \frac{m}{c_\theta} r^2 \, d\theta \tag{g}$$

And lastly there is Kepler's third law which states that the square of the time τ required for one complete orbit is proportional to the cube of the orbit's major axis.

$$\tau = 2\pi \sqrt{\frac{a^3}{\mu}} \tag{h}$$

It is also of interest to note that the term $c_\theta^2/2mr^2$ may be viewed as an apparent potential energy V_{appa} with an associated force which is proportional to $1/r^3$ and is repulsive. Thus we can write a total potential energy as

$$V + V_{appa} = -\frac{\mu m}{r} + \frac{c_\theta^2}{2mr^2} \tag{i}$$

For a circular orbit, i.e. under steady conditions, when $\dot{r} = 0$, the balance of the attractive and repulsive forces leads to the following equilibrium condition:

$$\mu \frac{m}{r^2} = \frac{c_\theta^2}{mr^3} \tag{j}$$

$$\text{thus} \quad r = \frac{c_\theta^2}{m^2\mu} \qquad \text{or} \qquad \dot{\theta} = \sqrt{\frac{\mu}{r^3}} \tag{k}$$

To see if this equilibrium configuration is stable we can evaluate the second derivative of the combined potential energy and obtain

$$\frac{d^2(V + V_{appa})}{dr^2} = -2\frac{\mu m}{r^3} + \frac{3c_\theta^2}{mr^4} = \frac{\mu m}{r^3} > 0 \tag{l}$$

For the equilibrium configuration in equation (j) one can see that $(V + V_{appa})$ attains a minimum indicating that the steady equilibrium configuration is stable!

Problem

3.7.1 A satellite (m = 500 kg) of the earth (μ = 398 601 km^3/s^2) is on a circular orbit of 7000 km radius. What is (a) the magnitude of its potential energy V, and (b) of its apparent potential energy V_{appa}?

3.8 Hamilton's Canonical Equations

Although in Hamilton's principle both the displacements and velocities are varied, these variations are not independent. Let us relax this requirement and allow independent variations for the velocities $\dot{q}_i$ and the displacements q_i. We do this by letting the velocities be g_i. We can establish the relation between the velocities g_i and the displacements q_i by means of some side constraints of the following form

$$g_i - \dot{q}_i = 0 \qquad i = 1, 2, ..., n \tag{3.8.1}$$

These constraint equations are holonomic and we may take account of them by forming a modified Lagrangian as follows

$$L_\lambda = L(q, g, t) - \lambda_i(g_i - \dot{q}_i) \tag{3.8.2}$$

Now we have a three-variable problem allowing variations of q, g and λ. On determining the extremum conditions of the modified functional,

$$\delta \int_{t_1}^{t_2} L_\lambda(q, g, \lambda, t)\, dt = \delta \int_{t_1}^{t_2} (L(q, g, t) - \lambda_i (g_i - \dot{q}_i))\, dt = 0 \tag{3.8.3}$$

with respect to g_i, q_i, and λ_i, we find the following Euler-Lagrange equations.

For δg_i: $$\frac{d}{dt}\frac{\partial L_\lambda}{\partial \dot{g}_i} - \frac{\partial L_\lambda}{\partial g_i} = 0 \quad \text{or} \quad \frac{\partial L}{\partial g_i} - \lambda_i = 0 \tag{3.8.4}$$

δq_i: $$\frac{d}{dt}\frac{\partial L_\lambda}{\partial \dot{q}_i} - \frac{\partial L_\lambda}{\partial q_i} = 0 \quad \text{or} \quad \frac{\partial L}{\partial q_i} - \dot{\lambda}_i = 0 \tag{3.8.5}$$

$\delta \lambda_i$: $$\frac{d}{dt}\frac{\partial L_\lambda}{\partial \dot{\lambda}_i} - \frac{\partial L_\lambda}{\partial \lambda_i} = 0 \quad \text{or} \quad g_i - \dot{q}_i = 0 \tag{3.8.6}$$

Now from equation (3.8.4) it can be seen that at extremum conditions

$$\lambda_i = \frac{\partial L}{\partial g_i} = \frac{\partial}{\partial g_i}(T^* - V) = p_i \tag{3.8.7}$$

where p_i are the generalized momenta associated with the velocities g_i. Equation (3.8.5) then expresses conditions of equilibrium amongst the inertial and the impressed forces.

Having determined the solution for λ_i, we may return to equation (3.8.2) and replace λ_i by p_i. The modified Lagrangian then becomes

$$L_p = (T^* - V) - p_i (g_i - \dot{q}_i) \tag{3.8.8}$$

Since p_i and g_i are related via equation (3.8.7), in the modified Lagrangian L_p we have a two-variable problem allowing for independent variations of p_i and q_i. We proceed to eliminate the g_i . If the system is scleronomic then

$$p_i g_i = T + T^* \tag{3.8.9}$$

which on substitution into equation (3.8.8), yields

$$L_p = p_i \dot{q}_i - H \tag{3.8.10}$$

where
$$H = T + V \tag{3.8.11}$$

The term H, which for scleronomic systems represents the sum of the kinetic and potential energies, is the Hamiltonian function (3.4.15), with $H = H(q, p)$.

For rheonomic systems we have seen, in section (1.9), that

$$T^* = T_2^* + T_1^* + T_o^* \tag{3.8.12}$$

$$p_i g_i = T + T_2^* - T_o^* \tag{3.8.13}$$

and
$$T = \frac{1}{2} \{p\}^T [A]^{-1} \{p\} - \frac{1}{2} \{b\}^T [A]^{-1} \{b\} + T_o^* \tag{3.8.14}$$

Hence for these systems the modified Lagrangian, in equation (3.8.8), becomes

$$L_p = L + p_i \dot{q}_i - (T + V - T_1^* - 2T_o^*) \tag{3.8.15}$$

Now the presence of T_1^*, in the term in the round brackets, implies that this term is not only a function of p_i and q_i but also of $\dot{q}_i$. However we may remove $\dot{q}_i$ from this term by expressing T_1^* in terms of p_i. This can be done as follows:

We have that

$$T_1^* = \{b\}^T \{\dot{q}\} \tag{3.8.16}$$

and
$$\{\dot{q}\} = [A]^{-1} (\{p\} - \{b\}) \tag{3.8.17}$$

Hence
$$T_1^* = \{b\}^T [A]^{-1} \{p\} - \{b\}^T [A]^{-1} \{b\} \tag{3.8.18}$$

Substituting from equations (3.8.18) and (3.8.14) into equation (3.8.15) we may express the modified Lagrangian as $L_p = p_i \dot{q}_i - (T + V - T_1^* - 2T_o^*) = p_i \dot{q}_i - H$

or
$$L_p = \{p\}^T \{\dot{q}\} - (\frac{1}{2} \{p\}^T [A]^{-1} \{p\} + \frac{1}{2} \{b\}^T [A]^{-1} \{b\} - \{b\}^T [A]^{-1} \{p\} - T_o^* + V) \tag{3.8.19}$$

Once again the term in the round brackets is referred to as the Hamiltonian function (3.4.15), with

$$H = T + V - T_1^* - 2T_o^* \tag{3.8.20}$$

In this more general case the Hamiltonian may be a function of not only p_i and q_i, but also of time t. It is convenient to express this Hamiltonian function in three parts:

$$H(p, q, t) = H_2 + H_1 + H_o \tag{3.8.21}$$

where $$H_2 = \frac{1}{2}\{p\}^T[A]^{-1}\{p\} \quad \text{is quadratic in } p \tag{3.8.22}$$

$$H_1 = -\{b\}^T[A]^{-1}\{p\} \quad \text{is linear in } p \tag{3.8.23}$$

$$H_o = \frac{1}{2}\{b\}^T[A]^{-1}\{b\} - T_o^* + V \quad \text{is not a function of } p \tag{3.8.24}$$

It should be noted that if time does not appear explicitly in the reduced transformation equations, the vector $\{b\}$ would be zero and the Hamiltonian in equation (3.8.20) would simplify to the form given for scleronomic systems, in equation (3.8.11).

No matter whether the system is scleronomic or rheonomic the relation between the modified Lagrangian and the Hamiltonian function takes the following form:

$$L_p = p_i\dot{q}_i - H(q,p,t) \tag{3.8.25}$$

From this relation it follows that

$$\frac{\partial L_p}{\partial q_i} = -\frac{\partial H}{\partial q_i} \tag{3.8.26a}$$

$$\frac{\partial L_p}{\partial t} = -\frac{\partial H}{\partial t} \tag{3.8.26b}$$

Now returning to Hamilton's principle we can write a modified form of this extremum principle as

$$\delta\Pi_H = 0 \tag{3.8.27}$$

where $$\Pi_H(p, q) = \int_{t_1}^{t_2} L_p \, dt = \int_{t_1}^{t_2} \{p_i\dot{q}_i - H\}dt \tag{3.8.28}$$

In the functional in equation (3.8.28) the momenta p_i and displacements q_i are subject to independent variations. The displacements must satisfy the kinematic constraints, whereas p_i may be varied freely.

Consider now the Euler-Lagrange equations of the calculus of variations for Π_H, namely

$$\frac{d}{dt}\frac{\partial L_p}{\partial \dot{q}_i} - \frac{\partial L_p}{\partial q_i} = 0 \tag{3.8.29a}$$

and

$$\frac{d}{dt}\frac{\partial L_p}{\partial \dot{p}_i} - \frac{\partial L_p}{\partial p_i} = 0 \tag{3.8.29b}$$

which lead to

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \tag{3.8.30a}$$

and

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{3.8.30b}$$

In equations (3.8.30) we have Hamilton's celebrated equations, referred to as the *canonical* equations of motion, by Jacobi (gk: canon = general rule). The first of these equations expresses the equilibrium amongst inertial and other forces, whereas the second expresses a generalized velocity-momentum relation for the particles. These equations being first order, are generally easier to integrate than Lagrange's equations which are second order differential equations.

As in its original form the modified form of Hamilton's principle holds for systems which are holonomic and for which all forces possess potentials. If there exist forces Q_i' which are not derivable from potentials then the virtual work of these forces must be added to $\delta\Pi_H$. That is in this case

$$\int_{t_1}^{t_2} \left[\delta\{p_i\dot{q}_i - H\} + Q_i' \delta q_i \right] dt = 0 \tag{3.8.31}$$

The associated system equations now take the following forms:

For δq_i:
$$-\dot{p}_i = \frac{\partial H}{\partial q_i} - Q_i' \tag{3.8.32a}$$

δp_i:
$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{3.8.32b}$$

If nonholonomic constraints are active, they may be appended to the modified functional by means of Lagrange multipliers λ_m as follows:

$$\int_{t_1}^{t_2} \left[\delta \{ p_i \dot{q}_i - H \} + \lambda_m a_{mi} \delta q_i \right] dt = 0 \tag{3.8.33}$$

In this case the system equations take the following forms:

For δq_i:
$$-\dot{p}_i = \frac{\partial H}{\partial q_i} - \lambda_m a_{mi} \tag{3.8.34a}$$

δp_i:
$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{3.8.34b}$$

Example 3.8.1

Consider again the system shown in Figure 3.3. The complementary kinetic energy for this system may be expressed as

$$T^* = \frac{1}{2}(m\dot{x}^2 + I\dot{\theta}^2) \tag{a}$$

By choosing $q = x$ as generalized coordinate and satisfying the kinematic constraint, $x = r\theta$, we may express T^* as

$$T^* = \frac{1}{2}\left[m + \frac{I}{r^2}\right]\dot{q}^2 \tag{b}$$

Then the momentum associated with $\dot{q}$ is found as

$$\frac{\partial T^*}{\partial \dot{q}} = \left[m + \frac{I}{r^2}\right]\dot{q} = p \tag{c}$$

from which $$\dot{q} = \frac{p}{m + \frac{I}{r^2}} \tag{d}$$

We may now evaluate the kinetic energy as

$$T = \int \dot{q}\, dp = \int \frac{p}{m + \frac{I}{r^2}} dp = \frac{1}{2}\frac{p^2}{m + \frac{I}{r^2}}$$

The potential energy is

$$V = \frac{1}{2} kq^2 \tag{f}$$

The Hamiltonian function, $H = T + V$ in the present case, can be expressed as

$$H = \frac{1}{2}\frac{p^2}{m + \frac{I}{r^2}} + \frac{1}{2} kq^2 = H(q,p) \tag{g}$$

The canonical equations become

$$-\dot{p} = \frac{\partial H}{\partial q} = kq \tag{h}$$

and $$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m + \frac{I}{r^2}} \tag{i}$$

These two first order equations are equivalent to the single second order equation

$$\left[m + \frac{I}{r^2}\right]\ddot{q} + kq = 0 \tag{j}$$

Example 3.8.2

Consider the spherical pendulum shown in Figure 3.8.

The kinetic coenergy is

$$T^* = \frac{1}{2} ml^2 (\dot{\phi}^2 + \dot{\psi}^2 \sin^2\phi) \tag{a}$$

and

$$p_\phi = \frac{\partial T^*}{\partial \dot{\phi}} = ml^2\dot{\phi} \qquad \dot{\phi} = \frac{p_\phi}{ml^2} \tag{b,c}$$

$$p_\psi = ml^2\dot{\psi}\sin^2\phi \qquad \dot{\psi} = \frac{p_\psi}{ml^2\sin^2\phi} \tag{d,e}$$

Then the kinetic energy may be expressed as

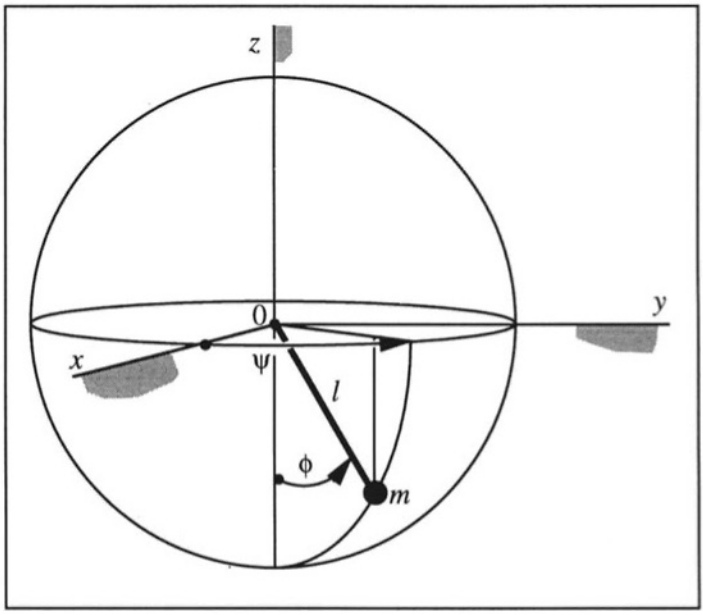


Figure 3.8 Spherical pendulum

$$T = \frac{1}{2ml^2}\left[p_\phi^2 + \frac{p_\psi^2}{\sin^2\phi}\right] \tag{f}$$

The potential energy is

$$V = -mgl\cos\phi \tag{g}$$

For this scleronomic system the Hamiltonian becomes

$$H = T + V = \frac{1}{2ml^2}\left[p_\phi^2 + \frac{p_\psi^2}{\sin^2\phi}\right] - mgl\cos\theta \tag{h}$$

The canonical equations (3.8.30) are now obtained from

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = \frac{1}{ml^2}\frac{p_\psi^2}{\sin^3\phi}\cos\phi - mgl\sin\phi \tag{i}$$

$$\dot{p}_\psi = -\frac{\partial H}{\partial \psi} = 0 \tag{j}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{1}{ml^2}p_\phi \tag{k}$$

$$\dot{\psi} = \frac{\partial H}{\partial p_\psi} = \frac{1}{ml^2\sin^2\phi}p_\psi \tag{l}$$

Evidently p_ψ is conserved, that is ψ is an ignorable coordinate.

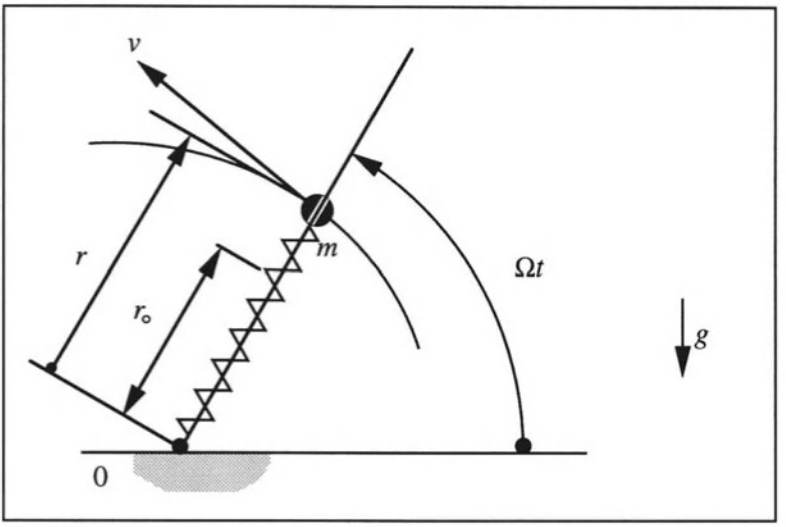


Figure 3.9 Pendulum with spring-borne bead

Example 3.8.3

The pendulum of Figure 3.9, with a bead m which can slide without friction on a rod, is driven at a constant angular velocity Ω. Find the canonical equations (3.8.30) of motion for this system. The sole generalized coordinate is $q_1 = r$.

The kinetic coenergy is

$$T^* = \frac{1}{2} m(\dot{r}^2 + r^2 \Omega^2) \tag{a}$$

The potential energy is

$$V = m g r \sin \Omega t + \frac{1}{2} k(r - r_o)^2 \tag{b}$$

where r_o is the unstretched length of the spring. The kinetic coenergy is given by $T^* = T_2^* + T_1^* + T_o^*$, with

$$T_2^* = \frac{1}{2} m\dot{r}^2 \qquad T_1^* = 0 \qquad T_o^* = \frac{1}{2} mr^2 \Omega^2 \tag{c,d,e}$$

The generalized momentum is

$$p_r = \frac{\partial T^*}{\partial \dot{r}} = m\dot{r} \tag{f}$$

Since in Newtonian mechanics $T = T^*$, we can use equations (a) and (f) to obtain

$$T = \frac{1}{2m} p_r^2 + \frac{1}{2} mr^2 \Omega^2 \tag{g}$$

The Hamiltonian (3.4.15) is

$$H = T + V - T_1^* - 2T_o^* \tag{h}$$

Using equations (g), (b), (d), and (c),

$$H(r, p_r, t) = \frac{1}{2m} p_r^2 + m g r \sin \Omega t + \frac{1}{2} k(r - r_o)^2 - \frac{1}{2} m r^2 \Omega^2 \quad \text{(i)}$$

the canonical equations become

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -mg \sin \Omega t - k(r - r_o) + m r \Omega^2 \quad \text{(j)}$$

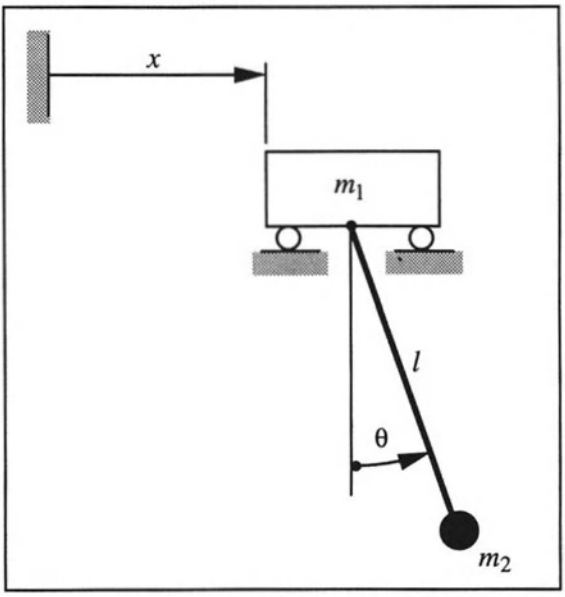
$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{1}{m} p_r \quad \text{(k)}$$

These two first order equations are equivalent to the single second order equation

$$m\ddot{r} - m r \Omega^2 + mg \sin \Omega t + k(r - r_o) = 0 \quad \text{(l)}$$

Problems

3.8.1 Write the Hamiltonian for the system shown and obtain the Hamiltonian equations of motion.



3.8.2 For the problem shown in Figure 1.29 determine the Hamiltonian $H(\theta, p_\theta, t)$ and obtain Hamilton's canonical equations.

3.9 Complementary Canonical Equations

By an analogous procedure we can develop canonical equations from the complementary form of Hamilton's principle. In the complementary formulation the essential variables are the equilibrating impulses S_j (corresponding to the compatible

displacements q_i). The extensions e_j form the second set of variables (corresponding to the momenta p_i). The constraint to be relaxed in this case is the impulse-force relationship. We proceed by letting the forces be K_j. Then we relate the impulses S_j to the force by setting $\dot{S}_j = K_j$, and introduce these equations as constraints. On appending these constraints to the complementary form of Hamilton's principle, by means of Lagrange multipliers, we obtain, analogous to equation (3.8.3),

$$\delta \int_{t_1}^{t_2} [L^*(S,K,t) - \lambda_j(K_j - \dot{S}_j)]\, dt = 0 \tag{3.9.1}$$

Carrying out the variations with respect to S_j, K_j and λ_j, we find the following extremum conditions.

For δK_j: $$\frac{\partial L^*}{\partial K_j} - \lambda_j = 0 \tag{3.9.2}$$

δS_j: $$\frac{\partial L^*}{\partial S_j} - \dot{\lambda}_j = 0 \tag{3.9.3}$$

$\delta \lambda_j$: $$K_j - \dot{S}_j = 0 \tag{3.9.4}$$

From the first of these equations we find

$$\lambda_j = \frac{\partial L^*}{\partial K_j} = \frac{\partial}{\partial K_j}(T - V^*) = -\frac{\partial V^*}{\partial K_j} = e_j \tag{3.9.5}$$

On substituting this extremum value of λ_j in the functional in equation (3.9.1), we obtain the following modified functional

$$\int_{t_1}^{t_2} \delta[(T - V^*) - e_j(K_j - \dot{S}_j)]dt = 0 \tag{3.9.6}$$

But $e_j K_j = -(V + V^*)$ and hence the above functional simplifies to

$$\int_{t_1}^{t_2} \delta[(T + V) + e_j \dot{S}_j]dt = 0$$

or $$\int_{t_1}^{t_2} \delta[H(S, e, t) + e_j \dot{S}_j]dt = 0 \tag{3.9.7}$$

In this functional S_j and e_j are varied independently. The impulse S_j must satisfy the equilibrium equations whereas e_j may be varied freely.

The extremum conditions of the functional in equation (3.9.7) emerge as the following canonical equations

$$\dot{e}_j = \frac{\partial H}{\partial S_j} \tag{3.9.8a}$$

$$-\dot{S}_j = \frac{\partial H}{\partial e_j} \tag{3.9.8b}$$

with

$$H(S,e,t) = T + V \tag{3.9.9}$$

Equations (3.9.8) are valid for scleronomic and rheonomic systems. For the latter case the expression for T will involve time explicitly (see example 3.9.1). When there are speeds s_j' which are not derivable from a kinetic energy function, their virtual work must be appended to the modified Lagrangian in equation (3.9.1) in an analogous manner to that described for the classical Hamilton canonical equation in section 3.8, leading to

$$\dot{e}_j = \frac{\partial H}{\partial S_j} + s_j' \tag{3.9.10a}$$

$$-\dot{S}_j = \frac{\partial H}{\partial e_j} \tag{3.9.10b}$$

Example 3.9.1

Let us obtain the complementary canonical equations of a linear oscillator with stiffness k and mass m. Denoting the spring deformation by e and the corresponding impulse by S, we have

$$H = T + V = \frac{S^2}{2m} + \frac{ke^2}{2} \tag{a}$$

Then the canonical equations (3.9.8) become

$$\dot{e} = \frac{\partial H}{\partial S} = \frac{S}{m} \tag{b}$$

$$-\dot{S} = \frac{\partial H}{\partial e} = ke \tag{c}$$

It is of interest to note that the first of these equations expresses the relation between the velocity (S/m) of the mass and the deformation rate of the spring. This is a compatibility relation. The second equation expresses the force-deformation relation, i.e. the constitutive equation, for the spring.

Now let a force $F \sin \Omega t$ act on the mass. Then from the equilibrium equation we have that

$$\dot{B} = \dot{S} + F \sin \Omega t \tag{d}$$

Where B is the momentum of the mass. Intergrating equation (d) and taking the integration constant as zero, we obtain

$$B = S - \frac{F \cos \Omega t}{\Omega} \tag{e}$$

Now $$T = \frac{B^2}{2m}, \quad V = \frac{ke^2}{2} \tag{f}$$

and $$H = T + V$$

becomes $$H = \frac{1}{2m}\left[S - \frac{F \cos \Omega t}{\Omega}\right]^2 + \frac{ke^2}{2} \tag{g}$$

The explicit appearance of time in the Hamiltonian indicates that the system is rheonomic. Using the canonical equations (3.9.8) we obtain the governing equations as

$$\dot{e} = \frac{\partial H}{\partial S} = \frac{1}{m}\left[S - \frac{F\cos\Omega t}{\Omega}\right]$$

$$-\dot{S} = \frac{\partial H}{\partial e} = ke$$

Example 3.9.2

For the rotating rod of Figure 3.10 with sliding point mass m and linear spring k, the Hamiltonian may be expressed as

$$H = T + V = \frac{1}{2}\frac{S_r^2}{m} + \frac{1}{2}\frac{S_\theta^2}{m(e_r + r_o)^2} + \frac{1}{2}ke_r^2 \tag{a}$$

The canonical equations (3.9.8) become

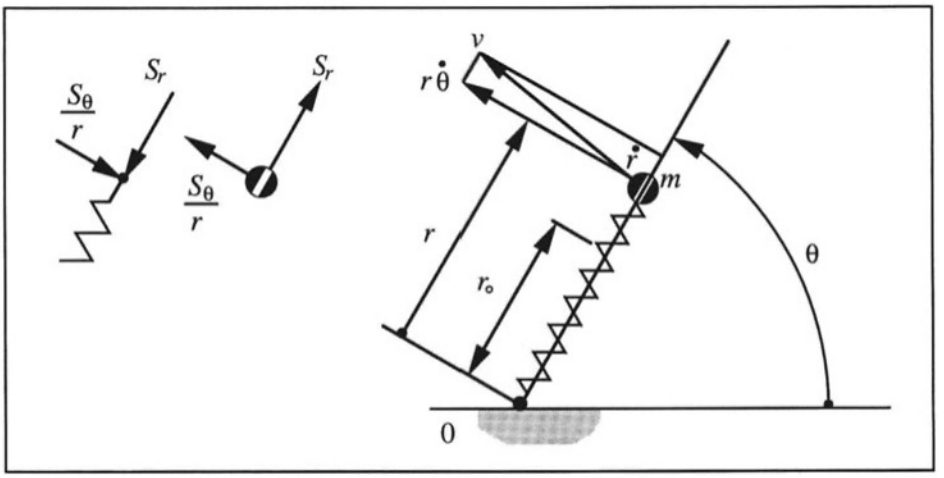


Figure 3.10 Rod rotating in horizontal plane

$$\dot{e}_r = \frac{\partial H}{\partial S_r} = \frac{S_r}{m} \tag{b}$$

$$-\dot{S}_r = \frac{\partial H}{\partial e_r} = -\frac{S_\theta^2}{m(e_r + r_o)^3} + ke_r \tag{c}$$

$$\dot{e}_\theta = \frac{\partial H}{\partial S_\theta} = \frac{S_\theta}{m(e_r + r_o)^2} \tag{d}$$

$$-\dot{S}_\theta = \frac{\partial H}{\partial e_\theta} = 0 \tag{e}$$

From equation (e) we conclude that S_θ is conserved.

Example 3.9.3

For Kepler's problem (Figure 3.7) the Hamiltonian is

$$H = T + V = \frac{1}{2}\frac{S_r^2}{m} + \frac{1}{2}\frac{S_\theta^2}{me_r^2} - \frac{\mu m}{e_r} \tag{a}$$

The canonical equations (3.9.8) give the equations of motion as

$$\dot{e}_r = \frac{\partial H}{\partial S_r} = \frac{S_r}{m} \tag{b}$$

$$-\dot{S}_r = \frac{\partial H}{\partial e_r} = -\frac{S_\theta^2}{me_r^3} + \frac{\mu m}{e_r^2} \tag{c}$$

$$\dot{e}_\theta = \frac{\partial H}{\partial S_\theta} = \frac{S_\theta}{me_r^2} \tag{d}$$

$$-\dot{S}_\theta = \frac{\partial H}{\partial e_\theta} = 0 \tag{e}$$

Equation (e) states that the angular momentum S_θ = constant, i.e. it is conserved.

Problems

3.9.1 At a moment of time, a linear oscillator (m = 10 kg, k = 12 N/m) has an extension of 0.1 m and a velocity of 0.05 m/s. Determine the numerical value of its Hamiltonian.

3.9.2 Given is a simple pendulum. Use S ($= ml^2\dot{\theta}$) and e ($= \theta$) to find (a) the Hamiltonian. Write (b) the complementary canonical equations and (c) combine them into a second order differential equation.

3.10 Elimination of Ignorable Coordinates

We have seen in section 3.7 that the presence of ignorable coordinates admits partial integration of the equations of motion associated with the ignorable coordinates. In this way it is possible to eliminate the ignorable coordinates. The process of elimination is considerably simplified in the Hamiltonian formulation of the equations of motion.

If the Lagrangian function does not contain an ignorable coordinate, say q_n, then the Hamiltonian function will not contain it either. Then from Hamilton's canonical equations (3.8.30) it follows that

$$-\dot{p}_n = \frac{\partial H}{\partial q_n} = 0 \tag{3.10.1}$$

or

$$p_n = c_n = \text{constant} \tag{3.10.2}$$

Thus the conservation of momentum associated with the ignorable coordinate is once again established.

Now since in Π_H, as given in equation (3.8.28), the momenta are varied independently, and p_n is now a constant, then the term in Π_H given by

$$\int_{t_1}^{t_2} p_n \dot{q}_n dt \quad = \quad c_n \int_{t_1}^{t_2} \dot{q}_n dt \quad = \quad c_n \left[q_n \right]_{t_1}^{t_2}$$

leads to a constant which may be dropped as far as the variations of Π_H are concerned. Hence the reduced form of the action Π_H may be expressed as

$$\Pi_r \ (q_{1,\ldots,} q_{n-1}, p_{1,\ldots,} p_{n-1}) = \int_{t_1}^{t_2} (p_i \dot{q}_i \ - \ H) dt \quad i \ = \ 1,2,3,\ldots,n \ - \ 1 \tag{3.10.3}$$

After solving the reduced problem, the ignorable variable may be determined as function of t by direct integration, since the right hand side of the corresponding canonical equation (3.8.30)

$$\dot{q}_n \quad = \quad \frac{\partial H}{\partial p_n} \tag{3.10.4}$$

is then an explicit function of t.

The same procedure holds for any number of ignorable variables.

Example 3.10.1

Consider again Kepler's problem of planetary motion (Figure 3.7). For this problem, with $q_1 = r$ and $q_2 = \theta$, we found for the complementary kinetic energy

$$T^* \quad = \quad \frac{1}{2} m(\dot{r}^2 + (r\dot{\theta})^2) \tag{a}$$

Then $$\frac{\partial T^*}{\partial \dot{r}} \quad = \quad p_r \quad = \quad m\dot{r} \qquad \text{and} \qquad \frac{\partial T^*}{\partial \dot{\theta}} \quad = \quad p_\theta \quad = \quad mr^2\dot{\theta} \tag{b}$$

From the definition of the kinetic energy T, we find that

$$T \quad = \quad \int \frac{p_r}{m} dp_r + \int \frac{p_\theta}{mr^2} dp_\theta \quad = \quad \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} \tag{c}$$

For this system the Hamiltonian, with $V = -\frac{\mu m}{r}$, takes the form

$$H \quad = \quad T + V$$

$$H \quad = \quad \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{\mu m}{r} \tag{d}$$

Since θ is not present, it is an ignorable coordinate.

The equations of motion can now be obtained, from the canonical equations (3.8.30), as

$$\dot{p}_r \quad = \quad -\frac{\partial H}{\partial r} \quad = \quad \frac{p_\theta^2}{mr^3} - \frac{\mu m}{r^2} \tag{f}$$

$$\dot{p}_\theta \quad = \quad -\frac{\partial H}{\partial \theta} \quad = \quad 0 \tag{g}$$

$$\dot{r} \quad = \quad \frac{\partial H}{\partial p_r} \quad = \quad \frac{p_r}{m} \tag{h}$$

$$\dot{\theta} \quad = \quad \frac{\partial H}{\partial p_\theta} \quad = \quad \frac{p_\theta}{mr^2} \tag{i}$$

Equation (g) can be integrated directly and yields a constant value for the angular momentum associated with the ignorable coordinate θ, i.e.

$$p_\theta = c_\theta = \text{constant} \tag{j}$$

Now substituting from equations (h) and (j) into equation (f) we may obtain a single second order equation of motion as

$$m\ddot{r} - \frac{c_\theta^2}{mr^3} + \frac{\mu m}{r^2} = 0 \tag{k}$$

which is in agreement with the equation obtained earlier via the Routhian. The coordinate θ can be eliminated directly from the action Π_H by simply dropping the term $p_\theta\dot{\theta}$ from the functional. Thus the reduced form of the action may be expressed as

$$\Pi_H = \int_{t_1}^{t_2} \left\{ p_i \dot{q}_i - H \right\} dt = \int_{t_1}^{t_2} \left\{ p_r \dot{r} - \left[\frac{p_r^2}{2m} + \frac{c_\theta^2}{2mr^2} - \frac{\mu m}{r} \right] \right\} dt \tag{l}$$

In this functional only r and p_r are subject to variation. The resulting extremum conditions emerge as follows:

$$\text{For } \delta r: \qquad \dot{p}_r - \frac{c_\theta^2}{mr^3} + \frac{\mu m}{r^2} = 0 \tag{m}$$

$$\delta p_r: \qquad -\dot{r} + \frac{p_r}{m} = 0 \tag{n}$$

These results are consistent with equation (k), as can readily be verified.

Problem

3.10.1 A satellite (m = 500 kg) of the earth (μ = 398 601 km^3/s^2) passes its perigee (r_{PE} = 7000 km from earth centre) with a velocity of 28 800 km/h. Its (constant) generalized momentum $p_\theta = c_\theta$ = 28 Gg km^2/s. Make a sketch of the orbit, and calculate (a) the magnitude of its generalized momentum p_r and (b) of its generalized momentum rate $\dot{p}_r$.

3.11 Phase Space and State Space

We have seen that in Lagrangian mechanics the motion of a system, with n coordinates, may be visualized as trajectories of a single representative point in n-dimensional *configuration space*. Now in Hamiltonian mechanics the equations of motion entail $2n$ variables; namely the n displacements and the n momenta. To visualize the motion as trajectories of a representative point, we now need to introduce a space of $2n$ dimensions. This $2n$-dimensional pq-space was named *phase space* by Gibbs†.

Now in discussing the configuration space in section 2.5, we found it advantageous to establish a particular metric for this space, based on the complementary

† Josiah Willard Gibbs (1839-1903), American mathematician and physicochemist

kinetic energy. This metric gave the configuration space a curvilinear geometry. For the phase space there seems no particular advantage in constructing a special metric other than the usual metric of a $2n$ dimensional rectilinear space. Hence in the phase space the q_i and p_i form rectangular coordinates, in a $2n$ dimensional space, of the representative point. We may go one step further and introduce the *time* as an additional dimension. This $(2n+1)$ dimensional space is called the *state space*. While both the extended configuration space (including the time coordinate (Figure 3.11)) and the state space (Figure 3.12) allow one to view the motion of a complex system as that of a single point, the state space (Figure 3.13) has some distinct advantages. To see this consider the solution of Hamilton's first order equations (3.8.30). We may express such solutions formally as

$$\begin{aligned} q_i &= q_i(q_1^o,\ldots,q_n^o, p_1^o,\ldots,p_n^o, t) \qquad i = 1,2,\ldots,n \\ p_i &= p_i(q_1^o,\ldots,q_n^o, p_1^o,\ldots,p_n^o, t) \end{aligned} \tag{3.11.1}$$

where $q_1^o,\ldots,q_n^o, p_1^o,\ldots,p_n^o$ are the constants of integration which we may identify as the $2n$ initial conditions. Equations (3.11.1) show that the motion may be viewed as a mapping from $2n$ coordinates $(q_1^o,\ldots, q_n^o, p_1^o,\ldots, p_n^o, t)$ to the $2n$ coordinates $(q_1,\ldots,q_n,\ldots,p_1,\ldots,p_n,t)$. This mapping is one to one and hence the trajectories of

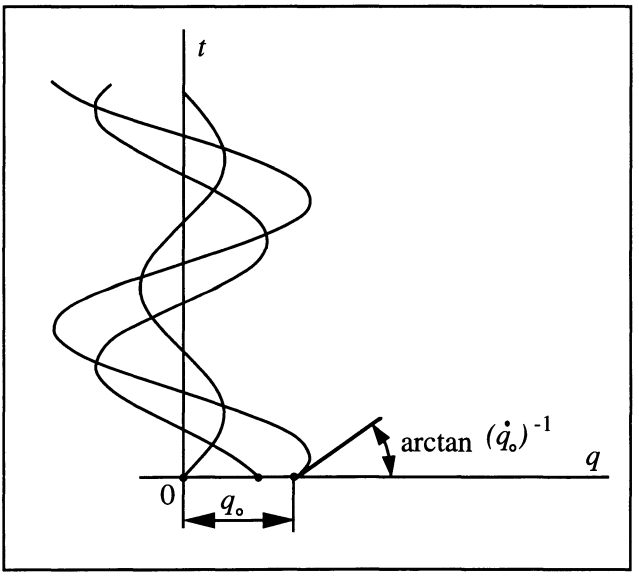


Figure 3.11 The qt-configuration space

the representative point, in state space as well as in phase space, never cross each other. This is assured from the uniqueness of the solutions for the canonical equations. Thus all the trajectories in state space and phase space are well ordered without any overlappings.

Consider next the solution of the second order Lagrangian equations of motion. We may write these solutions formally as

$$q_i = q_i(q_1^o,....,q_n^o, \dot{q}_1^o,....,\dot{q}_n^o,t)$$

In this case one can see that the mapping between the current position (q,t) and the initial position (q_o,t_o) is no longer one to one and it is affected by the infinite choice of initial velocities. This implies that the trajectories in the configuration space are not well ordered and the totality of such trajectories forms a maze of crisscrossing lines.

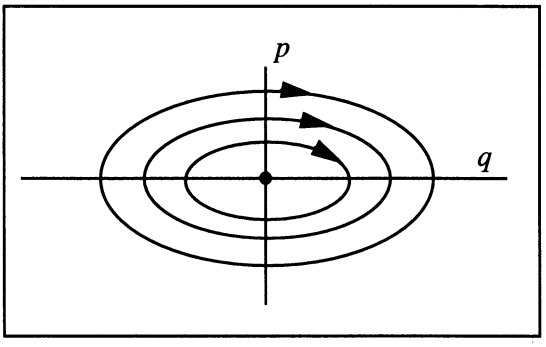


Figure 3.12 Phase space

Example 3.11.1

Consider the motion of a simple mass-spring oscillator. In the Lagrangian description the equation of motion for this system is given by

$$m\ddot{q} + kq = 0 \tag{a}$$

with the solution

$$q = q_o \cos\omega t + \frac{\dot{q}_o}{\omega} \sin\omega t \tag{b}$$

where $\omega = \sqrt{k/m}$ and q_o and $\dot{q}_o$ are the initial displacement and the initial velocity of the mass. The configuration space for this system, i.e. the q-space, is one dimensional and it is clear that in this space all possible motions of the oscillator are superimposed. Let us next represent the motion of the oscillator in the qt-space as shown in Figure 3.11. Evidently the trajectories associated with different initial conditions, which in this case determine the total energy in the system, crisscross in the qt-space.

Consider next the equations of motion in the Hamiltonian formulation. The system's Hamiltonian function is given by

$$H = \frac{1}{2}\left[\frac{p^2}{m} + kq^2\right] \tag{c}$$

yielding the first order differential equations of motion

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq \tag{d}$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{e}$$

with the solutions

$$p = p_o \cos\omega t - q_o\, m\omega \sin\omega t \tag{f}$$

$$q = \frac{p_o}{m\omega} \sin\omega t + q_o \cos\omega t \tag{g}$$

where p_o and q_o are the initial momentum and the initial displacement of the mass. For this example the phase space, i.e. the pq-space, is two-dimensional. In this space the trajectories do not cross. The equation for these trajectories can be found by eliminating the time in equations (f) and (g). Subsequent to some manipulations one can express the trajectory equation as ellipses

$$\frac{p^2}{2Em} + \frac{q^2}{2E/k} = 1 \tag{h}$$

where $E = \frac{p_o^2}{2m} + \frac{kq_o^2}{2}$ is the initial total energy.

Thus for different initial conditions, which also determine the initial energies, different non-intersecting ellipses are obtained in the phase space, i.e. pq-space as shown in Figure 3.12. Finally we may depict the motion in the state space, i.e. pqt-space, as shown in Figure 3.13. In this case we obtain non-intersecting, elliptic helices.

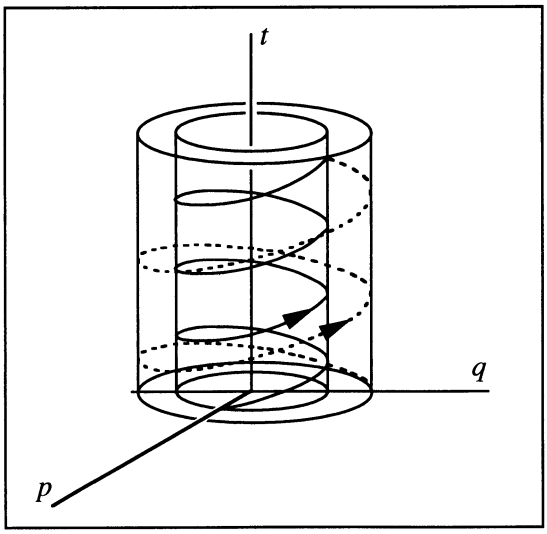


Figure 3.13 State space

Problem

3.11.1 A simple spring-mass oscillator (m = 0.1 kg, k = 12 N/m), has an initial momentum of 2 kg m/s, and an initial displacement of 0.5 m. (a) Plot its trajectory in phase space. (b) Plot its trajectory if the initial displacement is 0.6 m.

3.12 Hydrodynamic Analogy

There exists an interesting analogy between Hamilton's equations of motion and the field description of fluid flow. To bring out this analogy, consider the position of a fluid particle as a function of time. If the particle starts at time t_o from the position (x_o, y_o, z_o) then at time t we may formally give the location of the particle as

$$x = x\,(x_o, y_o, z_o, t) \tag{3.12.1a}$$

$$y = y\,(x_o, y_o, z_o, t) \tag{3.12.1b}$$

$$z = z\,(x_o, y_o, z_o, t) \tag{3.12.1c}$$

Here again we have a one to one mapping for the particle's coordinates, from its initial to its final position. The analogy between equations (3.12.1) and (3.11.1) is apparent. The essential difference between these equations is that for the former we have three coordinates (x, y, z), in three-dimensional space whereas for the latter we have $2n$ coordinates $(q_1,\ldots,q_n, p_1,\ldots,p_n)$ in the phase space. From this analogy further similarities can be deduced. Consider for instance the velocity of a fluid particle located at (x, y, z). We may express this velocity as

$$\dot{x} = \frac{\partial x}{\partial t}(x_o, y_o, z_o) \qquad \dot{y} = \frac{\partial y}{\partial t}(x_o, y_o, z_o) \qquad \dot{z} = \frac{\partial z}{\partial t}(x_o, y_o, z_o) \tag{3.12.2}$$

Now since the mapping in equation (3.12.1) is one to one, it is possible, in principle, to invert these relationships and thereby eliminate (x_o, y_o, z_o) from equations (3.12.2). Then the field velocities may be expressed as

$$\dot{x} = \dot{x}\,(x, y, z, t) \tag{3.12.3a}$$

$$\dot{y} = \dot{y}\,(x, y, z, t) \tag{3.12.3b}$$

$$\dot{z} = \dot{z}\,(x, y, z, t) \tag{3.12.3c}$$

Above equations give the velocity vector, at any point in space, at any time. If the particle description (13.12.1) is given we can obtain the field description (3.12.3) by differentiations and eliminations. On the other hand if the field description is given we can obtain the particle description by integrating equations (3.12.3).

The form of the field velocity equations (3.12.3) is similar to that of Hamilton's canonical equations, (3.8.29) and (3.8.30),

$$\dot{q}_i = \frac{\partial}{\partial p_i} H(q, p, t) = f_i\,(q, p, t) \tag{3.12.4a}$$

$$\dot{p}_i = -\frac{\partial}{\partial q_i} H(q, p, t) = g_i\,(q, p, t) \tag{3.12.4b}$$

This similarity enables us to visualise the motion of a system as the flow of a fluid – the so called phase fluid – in $2n$-dimensional phase space. The flow of this phase fluid has a number of characteristics which can be deduced through the analogy with the flow of real fluids in three-dimensional space. Consider for instance the case of steady flows. In this case the velocity of a real fluid, at any point in space, is independent of time. This implies that for steady flows $\dot{x}, \dot{y}$ and $\dot{z}$, in equations (3.12.3), become independent of time. For the phase fluid an analogous situation arises when

the Hamiltonian becomes independent of time, as can be seen from equations (3.12.4). This occurs for scleronomic systems and for some rheonomic systems [see equations (3.4.15)]. Thus for such systems the flow of the phase fluid is analogous to the steady flow of real fluids.

We have seen that for scleronomic systems the Hamiltonian is equal to the total energy of the system and, if the Hamiltonian is independent of time, then the total energy will remain a constant i.e.

$$H(q, p) = E = \text{constant} \tag{3.12.5}$$

Equation (3.12.5) represents a hypersurface in $2n$-dimensional phase space. Thus we may conclude that for scleronomic systems the representative point, in the phase space, moves on surfaces associated with given energy levels. For each energy level there will be a different surface. The aggregate of all possible motions of the given scleronomic system can be viewed as a set of non-intersecting surfaces in the phase fluid. The analogues of these surfaces are the stream lines for steady flow of real fluids.

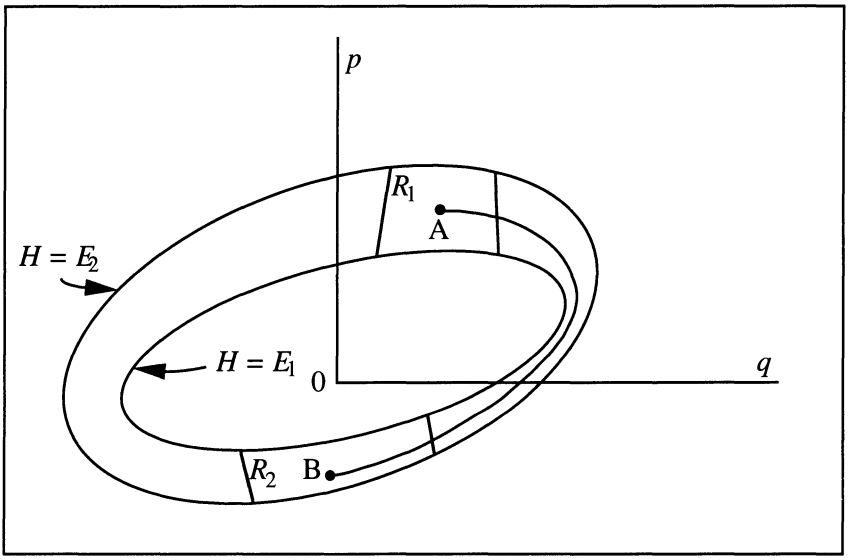


Figure 3.14 Phase space for scleronomic system

Consider next the condition of incompressibility. For the real fluid this condition may be expressed as

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = 0 \tag{3.12.6}$$

The analogue of equation (3.12.6) for the phase fluid takes the form

$$\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = 0 \tag{3.12.7}$$

But from the canonical equations (3.8.29) and (3.8.30)

$$\frac{\partial}{\partial q_i}\dot{q}_i = \frac{\partial}{\partial q_i}\frac{\partial H}{\partial p_i} = \frac{\partial^2 H}{\partial q_i \partial p_i} \tag{3.12.8}$$

and

$$\frac{\partial}{\partial p_i}\dot{p}_i = \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = -\frac{\partial^2 H}{\partial q_i \partial p_i} \tag{3.12.9}$$

On substituting from equations (3.12.8) and (3.12.9) into equation (3.12.7) we note that the latter is identically satisfied. This implies that the phase fluid imitates the behaviour of incompressible real fluids. A more striking analogy can be observed if we restrict ourselves to two-dimensional flows for which we may take $\dot{z} = 0$. In this case the condition of incompressibility may be identically satisfied through the introduction of a stream function ψ defined by

$$\dot{x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \dot{y} = -\frac{\partial \psi}{\partial x} \tag{3.12.10}$$

A comparison of equations (3.12.10) and (3.12.4) reveals that the stream function ψ and the Hamiltonian H are analogous quantities. Thus surfaces of constant H in the phase space correspond to lines of constant ψ for two-dimensional flows, namely the stream lines.

The condition of incompressibility for the phase fluid permits us to add a new conservation law to the previous energy theorem, namely the *constancy of the phase fluid density*. Consider again the representation of the motions of a scleronomic system in phase space. Suppose that for various possible initial conditions the energies of this system lie between E_1 and E_2. Then the paths for various motions will lie between the two surfaces $H = E_1$ and $H = E_2$ as indicated schematically in Figure 3.14. Subsequent to different initial conditions the system will move along different paths in the phase space. Now let us cut out a region R_1 wherein all the initial conditions are contained. As time progresses, region R_1 will move and it may become distorted. At some time t later, all the representative points will occupy a region R_2. For example, the representative point corresponding to one particular set of initial conditions moves from point A to point B. It is clear that the number of representative points in regions R_1 and R_2 is the same and since the volumes of these regions is also the same, due to incompressibility, then the density of the phase fluid remains constant. This conservation of phase fluid density was discovered by Liouville† and is known as *Liouville's theorem.*

† Joseph Liouville (1809-1882), French mathematician

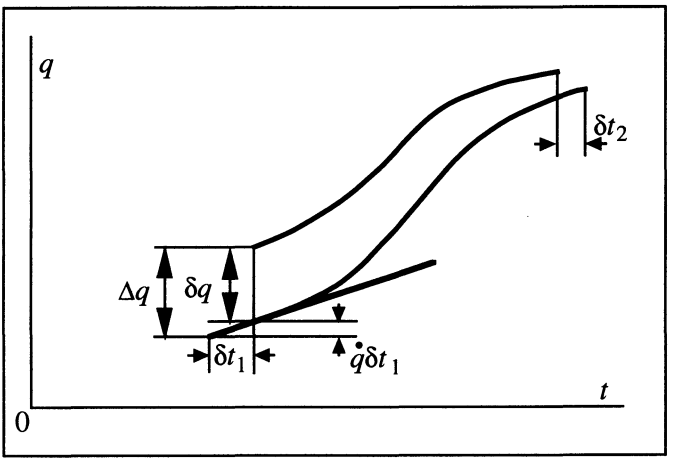


Figure 3.15 Variations in space and time

3.13 Principle of Least Action

In Hamilton's principle, although the interval of time from t_1 to t_2 is arbitrary, the limits t_1 and t_2 themselves are fixed. That is, in this principle all admissible motions are considered over the same period of time; namely from t_1 to t_2 and time is not varied, i.e. for variations of q_i, time is held fixed. We may remove this restriction by considering motions over different periods of time. In this case the limits t_1 and t_2 will not be fixed and from Figure 3.15 it can be seen that to the first order of smallness the total change in the displacements, between motions over different periods of time, may be expressed as

$$\Delta q_i \quad = \quad \dot{q}_i \, \delta t \; + \; \delta q_i \tag{3.13.1}$$

On substituting for δq_i from equation (3.13.1) into Hamilton's principle in equation (3.1.11) we obtain

$$\int_{t_1}^{t_2} \delta(T^* \; - \; V)dt \quad = \quad p_i \; (\Delta q_i \; - \; \dot{q}_i \; \delta t) \Big|_{t_1}^{t_2} \tag{3.13.2}$$

In equation (3.13.2) the time limits are no longer fixed.

Now for any integral with variable limits, we have that

$$\Delta \int_{t_1}^{t_2} g(t) \, dt \; = \; \int_{t_1}^{t_2} \Delta \, [g(t) \, dt] \; = \; \int_{t_1}^{t_2} [\Delta \, g(t) \,] \, dt \; + \int_{t_1}^{t_2} g(t) \, \Delta \, (dt) \tag{3.13.3}$$

and on using equation (3.13.1) we may express this equation as

$$\Delta \int_{t_1}^{t_2} g(t) \, dt \; = \; \int_{t_1}^{t_2} [\dot{g}(t) \, \delta t \; + \; \delta g(t)] \, dt \; + \; \int_{t_1}^{t_2} g(t) \, d(\delta t) \tag{3.13.4}$$

where we have taken note of the fact that for t,

$$\Delta t \equiv \delta t$$

and hence

$$\Delta(dt) = d(\delta t)$$

Now we may carry out a partial integration of the last term in equation (3.13.4) and write this equation as

$$\Delta \int_{t_1}^{t_2} g(t)\, dt = \int_{t_1}^{t_2} [\dot{g}(t)\, \delta t + \delta g(t)]\, dt + \int_{t_1}^{t_2} g(t)\, \frac{d}{dt}(\delta t)\, dt$$

or $$\Delta \int_{t_1}^{t_2} g(t)\, dt = \int_{t_1}^{t_2} [\dot{g}(t)\delta t + \delta g(t)]\, dt + \int_{t_1}^{t_2} \{\frac{d}{dt}[g(t)\delta t] - \dot{g}(t)\delta t\}\, dt$$

i.e. $$\Delta \int_{t_1}^{t_2} g(t)\, dt = \int_{t_1}^{t_2} \delta g(t)\, dt + g(t)\, \delta t \Big|_{t_1}^{t_2} \tag{3.13.5}$$

In equation (3.13.5) we have the Leibniz rule for variation of integrals with variable limits†.

In view of equation (3.13.5) we may express equation (3.13.2) as

$$\Delta \int_{t_1}^{t_2} (T^* - V)dt - [(T^* - V)\delta t] \Big|_{t_1}^{t_2} = p_i\, (\Delta q_i - \dot{q}_i \delta t) \Big|_{t_1}^{t_2} \tag{3.13.6}$$

Let us initially restrict ourselves to scleronomic systems. In this case $T + T^* = p_i\, \dot{q}_i$ and equation (3.13.6) may then be rearranged into the form

$$\Delta \int_{t_1}^{t_2} (T^* - V)dt = [-(V + T)\delta t + p_i \Delta q_i] \Big|_{t_1}^{t_2} \tag{3.13.7}$$

Next consider the total change in the time integral of the Hamiltonian. Once again restricting ourselves to scleronomic systems and making use of equation (3.13.5) we may write

$$\Delta \int_{t_1}^{t_2} (T + V)dt = \int_{t_1}^{t_2} \delta(T + V)dt + (T + V)\delta t \Big|_{t_1}^{t_2} \tag{3.13.8}$$

Finally let us add equations (3.13.7) and (3.13.8) to find

$$\Delta \int_{t_1}^{t_2} (T + T^*)dt = \int_{t_1}^{t_2} \delta(T + V)dt + p_i \Delta q_i \Big|_{t_1}^{t_2} \tag{3.13.9}$$

Now, to simplify, let us impose the following two constraints:

---

† An alternative derivation of this result is given in Appendix A

i) For all admissible motions the total energy shall remain unchanged, i.e. $\delta(T + V) = 0$.

ii) For all motions the trajectories shall be coterminal in space (but not necessarily in time), i.e. $\Delta q_i = 0$ at t_1 and t_2 (Figure 3.16).

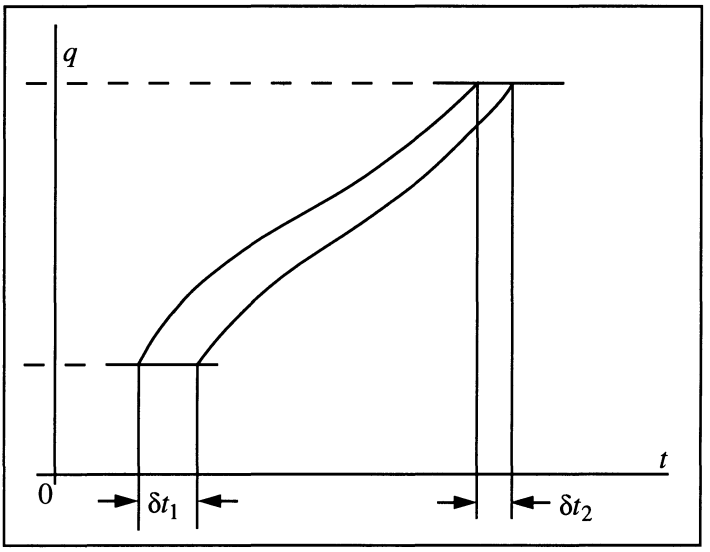


Figure 3.16 Variations with coterminal points in space

Subject to these constraints the variational statement in equation (3.13.9) reduces to

$$\Delta \int_{t_1}^{t_2} (T + T^*)dt = 0 \qquad (3.13.10)$$

In equation (3.13.10) we have the *principle of least action.* In words this principle states that *amongst all possible motions of conservative systems between any two prescribed configurations, the real motion will be such as to render the functional in equation (3.13.10) an extremum.*

It is of interest to compare the principle of least action with Hamilton's principle. For the former the total energy must remain constant while for Hamilton's principle this restriction does not apply. On the other hand in Hamilton's principle time is held fixed whereas for the principle of least action this restriction does not apply. To appreciate these differences consider the following situation. Suppose the motion of a system is recorded on a movie film. Further suppose that during the filming a clock is present so that the configuration of the system, at each instant of time is recorded. Now if part of the film, between times t_1 and t_2, read on the clock, is destroyed one can determine the motion of the system, during this period, via Hamilton's principle since in this case the configuration of the system is known at given times t_1 and t_2. If on the other hand the clock was not in the film, so that t_1 and t_2 would not be known,

then it would not be possible to reconstruct the motion of the system during this period from Hamilton's principle. However one can reconstruct the motion during this period from the principle of least action providing that no energy was added or extracted from the system during this period of time.

It should be mentioned that the principle of least action predates Hamilton's principle by some ninety years. It is possible to arrive at the principle of least action more directly rather than via Hamilton's principle. Such a direct derivation was given by de Maupertuis† in 1744 who expressed this principle as follows: "*Among all possible motions nature chooses that which reaches its goal with the minimum expenditure of action*".

De Maupertuis was not able to satisfactorily establish the quantity to be minimized and he was not aware of the requirement for constancy of energy. Clearer and more rigorous derivations of this principle were given later by Euler and Lagrange††.

It is possible to express the principle of least action in an alternative form by explicitly satisfying the constancy of energy. Thus letting

$$T + V = E$$

where E is a constant, we may remove T from equation (3.13.10) and write it as

$$\Delta \int_{t_1}^{t_2} (E - V + T^*)\, dt = 0$$

$$\text{or } \Delta \int_{t_1}^{t_2} (L + E)\, dt = 0 \qquad (3.13.11)$$

$$\text{since } L = (T^* - V)\,.$$

Although this form of the principle resembles Hamilton's principle, it must be remembered that in this principle time is allowed to vary whereas in Hamilton's principle time is held fixed. With its variable limits, equation (3.13.11) provides a useful means for evaluating the period of oscillation of vibrating systems (see example 3.13.1).

Example 3.13.1

Consider a simple oscillator for which we may write the Lagrangian as

$$L = \frac{1}{2}(\dot{x}^2 - \omega^2 x^2) \qquad \text{with} \qquad \omega^2 = k/m \qquad \text{(a)}$$

Let us approximate one half period of oscillation by a parabola as

$$x(t) = at - bt^2 \qquad \text{(b)}$$

---

† Pierre Louis Moreau de Maupertuis (1698-1759), French physicist, mathematician and philosopher (President of Frederick the Great's Academy of Sciences in Berlin, with academicians Lagrange, Euler, Voltaire, Lambert, etc.)

†† The question of priority for the Principle of Least Action gave rise to a heated and lengthy controversy. A brief account of this controversy may be found in "On the Principle of Least Action and its Complementary Form", Solid Mechanics Archives, Vol. 6, 3, 1981, by B. Tabarrok.

Then the duration of one half period of oscillation is (a/b) and the amplitude of the motion occurs at $t = a/2b$ when $x = a^2/4b$.

Substituting (b) into equation (3.13.11) we find

$$\Delta \int_0^{a/b} \left[\frac{1}{2}(a - 2bt)^2 - \frac{\omega^2}{2}(at - bt^2)^2 + E \right] dt = 0 \qquad \text{(c)}$$

where E is the total energy in the system. Evaluating the integral we find

$$\Delta \left[\frac{a^3}{6b} - \frac{\omega^2 a^5}{60 b^3} + E \frac{a}{b} \right] = 0 \qquad \text{(d)}$$

Now extremizing the terms in the square bracket first with respect to a and then with respect to b we find

$$\frac{a^2}{2b} - \frac{\omega^2 a^4}{12 b^3} + \frac{E}{b} = 0$$

$$-\frac{a^3}{6b^2} + \frac{\omega^2 a^5}{20 b^4} - \frac{Ea}{b^2} = 0$$

Solving for a and b we find

$$a = \sqrt{3E} \qquad b = \omega\sqrt{\frac{3E}{10}}$$

Then the duration of one period $= \frac{2a}{b} = 6.324$. The exact answer is $2\pi = 6.253$. The amplitude of oscillation is

$$\frac{a^2}{4b} = \frac{1.369}{\omega}\sqrt{E}$$

The exact solution is $\frac{\sqrt{2E}}{\omega} = 1.414\frac{\sqrt{E}}{\omega}$. Clearly a better approximation in (b) will yield more accurate results.

Before considering further applications for the principle of least action it is worth expanding this principle to rheonomic systems. For such systems it will be recalled [see equation (3.4.14)] that the product $p_i\,\dot{q}_i$ may be expressed as

$$p_i\,\dot{q}_i = T + T_2^* - T_o^* \qquad (3.13.12)$$

Using this equation we may rewrite equation (3.13.6) as

$$\Delta \int_{t_1}^{t_2} (T^* - V)dt - [(T^* - V)\delta t]\Big|_{t_1}^{t_2} = p_i \Delta q_i \Big|_{t_1}^{t_2} - (T + T_2^* - T_o^*)\delta t \Big|_{t_1}^{t_2} \qquad (3.13.13)$$

Further, noting that $T^* = T_2^* + T_1^* + T_o^*$, we may rearrange equation (3.13.13) and express it as

$$\Delta \int_{t_1}^{t_2} (T_2^* + T_1^* + T_o^* - V)dt = -[(T + V) - T_1^* - 2T_o^*]\delta t \Big|_{t_1}^{t_2} + p_i \Delta q_i \Big|_{t_2}^{t_1} \qquad (3.13.14)$$

Now the term in the square brackets on the right hand side will be recognized as the

Hamiltonian for rheonomic systems (see equation (3.8.20)). As before, we next evaluate the total change in the time integral of the Hamiltonian:

$$\Delta \int_{t_1}^{t_2} H dt = \int_{t_1}^{t_2} \delta H dt + H \delta t \Big|_{t_1}^{t_2} \tag{3.13.15}$$

where for the present rheonomic case $H = (T + V) - T_1^* - 2T_o^*$. It remains to add equations (3.13.15) and (3.13.14) to obtain

$$\Delta \int_{t_1}^{t_2} (T + T_2^* - T_o^*) dt = \int_{t_1}^{t_2} \delta H dt + p_i \Delta q_i \Big|_{t_1}^{t_2} \tag{3.13.16}$$

Finally we impose the following two constraints:

$$\text{i)} \quad H = \text{constant} \tag{3.13.17}$$

$$\text{ii)} \quad \Delta q_i = 0 \qquad \text{at } t_1 \text{ and } t_2 \tag{3.13.18}$$

Under these conditions the functional in equation (3.13.16) simplifies to

$$\Delta \int_{t_1}^{t_2} (T + T_2^* - T_o^*) dt = 0 \tag{3.13.19}$$

A number of points are worthy of note at this juncture. First it will be recognized that in general the Hamiltonian is not a constant for the motion of rheonomic systems. However as we have seen, in section 3.4, for some rheonomic systems it is possible for the Hamiltonian to be conserved while the total energy is not conserved. For such systems the extremum principle in equation (3.13.19) is applicable. We may satisfy the constancy of the Hamiltonian explicitly and write equation (3.13.19) as

$$\Delta \int_{t_1}^{t_2} (L + H)\, dt = 0 \tag{3.13.20}$$

where in this case $L = T_2^* + T_1^* + T_0^* - V$. Consider next the interpretation of the integrand in equation (3.13.19). We showed earlier, in equation (3.4.14), that the expression in the integrand of equation (3.13.19) is simply $p_i \dot{q}_i$. Thus we may finally express the principle of least action as

$$\Delta \int_{t_1}^{t_2} p_i \dot{q}_i \, dt = 0 \tag{3.13.21}$$

Recognizing that for scleronomic systems $p_i \dot{q}_i = T + T^*$, we conclude that the above form of the principle of least action holds for scleronomic as well as rheonomic systems.

In equation (3.13.21) the quantities subject to variation are the generalized displacements q_i. The momenta p_i are to be expressed in terms of $\dot{q}_i$ and q_i. The variations of the displacements are subject to the constraints in equations (3.13.17) and (3.13.18). However the trajectories are not constrained to be coterminal in time.

It can be seen that time may be removed from the functional in equation (3.13.21) and the principle of least action may be expressed as

$$\Delta \int_{q_{io}}^{q_{i1}} p_i(q, \dot{q})\, dq_i \quad = \quad 0 \tag{3.13.22}$$

Once again the quantities to be varied are the displacements q_i.

Example 3.13.2

Let us derive Newton's equations for a system of N particles, from the principle of least action.

Using Cartesian inertial coordinates and expressing the momenta in terms of velocities, we may write the principle of least action (3.13.21) as

$$\Delta \int_{t_1}^{t_2} (m\dot{x})_i\, \dot{x}_i\, dt \;=\; 0 \tag{a}$$

Using Leibniz' rule we may express this functional as

$$\int_{t_1}^{t_2} \delta[\,(m\dot{x})_i\, \dot{x}_i\,]\, dt \;+\; (m\dot{x})_i\, \dot{x}_i \delta t \,\Big|_{t_1}^{t_2} \;=\; 0 \tag{b}$$

$$\text{or} \quad \int_{t_1}^{t_2} 2\,(m\dot{x})_i\, \delta\dot{x}_i\, dt \;+\; (m\dot{x})_i\, \dot{x}_i \delta t \,\Big|_{t_1}^{t_2} \;=\; 0 \tag{c}$$

We must now satisfy the two constraints (3.13.17) and (3.13.18) of the principle of least action. These are:

$$\text{i)} \quad \delta(T + V) \;=\; 0 \tag{d}$$

For Newtonian mechanics

$$\delta\left[\frac{m}{2}\dot{x}_i^2 + V\right] \;=\; 0$$

$$\text{i.e.} \quad (m\,\dot{x})_i\, \delta\dot{x}_i + \frac{\partial V}{\partial x_i}\, \delta x_i \;=\; 0 \tag{e}$$

$$\text{and ii)} \quad \Delta x_i \;=\; \delta x_i + \dot{x}\,\delta t \;=\; 0 \quad \text{at} \quad t_1 \text{ and } t_2 \tag{f}$$

Using equation (e) in (c) we may rearrange the equation to the following form

$$\int_{t_1}^{t_2} [(m\,\dot{x})_i\, \delta\dot{x}_i - \frac{\partial V}{\partial x_i}\, \delta x_i]\, dt + (m\,\dot{x})_i\, \dot{x}_i \delta t \,\Big|_{t_1}^{t_2} \;=\; 0 \tag{g}$$

We can now integrate the first term of the integrand as follows

$$\int_{t_1}^{t_2} \left\{ \frac{d}{dt}\,[(m\,\dot{x})_i\, \delta x_i] - (m\ddot{x})_i\, \delta x_i - \frac{\partial V}{\partial x_i}\, \delta x_i \right\} dt + (m\,\dot{x})_i\, \dot{x}_i\, \delta t \,\Big|_{t_1}^{t_2} \;=\; 0$$

$$\text{or} \quad \int_{t_1}^{t_2} \left\{ (m\,\ddot{x})_i + \frac{\partial V}{\partial x_i} \right\} \delta x_i\, dt - (m\dot{x})_i\, (\delta x_i + \dot{x}_i \delta t) \,\Big|_{t_1}^{t_2} \;=\; 0 \tag{h}$$

The boundary term drops out on account of equation (f) and the vanishing of the first term yields

$$(m\ \ddot{x})_i + \frac{\partial V}{\partial x_i} = 0 \tag{i}$$

Example 3.13.3

Derive the equation of motion for the system described in Example 3.4.2 by means of the principle of least action.

This problem is rheonomic in nature but since time does not appear explicitly in the Lagrangian, the Hamiltonian is conserved and hence the principle of least action applies. Thus in this case we have from equation (3.13.19)

$$\Delta \int_{t_1}^{t_2} (T + T_2^* - T_o^*)\, dt = 0 \tag{a}$$

and since $T = T^* = T_2^* + T_1^* + T_0^*$ we may express this functional as

$$\Delta \int_{t_1}^{t_2} (2T_2^* + T_1^*)\, dt = 0 \tag{b}$$

We found earlier for this problem $T_2^* = \frac{1}{2} m\, a^2\dot{\theta}^2$ and $T_1^* = 0$. Then

$$\Delta \int_{t_1}^{t_2} m\, a^2\dot{\theta}^2\, dt = 0 \tag{c}$$

Using Leibniz' rule we can express this as

$$\int_{t_1}^{t_2} \delta(m\, a^2\, \dot{\theta}^2)dt + m\, a^2\, \dot{\theta}^2\delta t \Big|_{t_1}^{t_2} = 0 \tag{d}$$

Carrying out the variations we find

$$\int_{t_1}^{t_2} 2m\, a^2\, \dot{\theta}\, \delta\dot{\theta}\, dt + m\, a^2\, \dot{\theta}^2\, \delta t \Big|_{t_1}^{t_2} = 0 \tag{e}$$

We must now impose the constraints of this principle. These are

i) constancy of the Hamiltonian, i.e.

$$\delta(T + V - T_1^* - 2T_0^*) = 0$$

or

$$\delta\,(T_2^* + V - T_0^*) = 0$$

which for the problem at hand becomes

$$\delta\,(\frac{1}{2} ma^2\, \dot{\theta}^2 + mg\ a\ (1 - \cos\theta) - \frac{1}{2} ma^2\, \omega^2\, \sin^2\theta) = 0 \tag{f}$$

or

$$m\, a^2\, \dot{\theta}\, \delta\dot{\theta} + mg\ a\ \sin\theta\, \delta\theta - m\, a^2\omega^2\, \sin\theta\, \cos\theta\, \delta\theta = 0 \tag{g}$$

Substituting from equation (g) into equation (e) we obtain

$$\int_{t_1}^{t_2} [m\, a^2\, \dot{\theta}\, \delta\dot{\theta} - ma\sin\theta(g - a\omega^2\cos\theta)\, \delta\theta]\, dt + ma^2\dot{\theta}^2\delta t \Big|_{t_1}^{t_2} = 0 \qquad \text{(h)}$$

On integrating the first term we obtain

$$-\int_{t_1}^{t_2} [m\, a^2\, \ddot{\theta} + m\, a\sin\theta(g - \omega^2 a\cos\theta)\,]\, \delta\theta dt + ma^2\dot{\theta}\,(\delta\theta + \dot{\theta}\delta t) \Big|_{t_1}^{t_2} = 0 \qquad \text{(i)}$$

The last term drops out on account of coterminal space variations at t_1 and t_2, and the vanishing of the integrand yields

$$m\, a^2\, \ddot{\theta} - m\, a^2\omega^2 \sin\theta\cos\theta + mg\, a\, \sin\theta = 0 \qquad \text{(j)}$$

which is the equilibrium equation of motion.

Problems

3.13.1 Derive the differential equation of motion of a simple oscillator, with mass m and spring stiffness k, from the principle of least action.

3.14 Jacobian Form of the Principle of Least Action

The functional for the principle of least action as given in equation (3.13.10) is not easy to use. This is because the limits in this functional are not fixed. The lower limit may be specified without loss in generality, however in order to satisfy the conservation of energy throughout the actual and varied paths, the upper limit must be varied in a definite manner. It is possible to overcome the problem of varying limits by expressing the principle of least action in the form given in equation (3.13.22). The problem of the constancy of the Hamiltonian for actual and varied paths still remains and must be satisfied explicitly.

For scleronomic systems Jacobi showed that the problems of varying limits and constancy of the Hamiltonian, which in this case is the total energy E, may be solved at once by the following procedure. By not recognizing a distinction between T and T^*, Jacobi expressed the integrand of equation (3.13.10) as follows

$$T + T^* = 2T^* = 2T = \sqrt{2T}\ \sqrt{2T^*} = \sqrt{2T}\ \sqrt{\dot{q}_i m_{ij} \dot{q}_j} \qquad (3.14.1)$$

$$= \sqrt{2T}\ \sqrt{dq_i m_{ij} dq_j}/dt \qquad (3.14.2)$$

Now from our discussions on the configuration space, in section 2.5 one can recognize the second square root term, in equation (3.14.2), as the incremental distance, ds, in the configuration space. Furthermore, the condition of constancy of energy may be satisfied by noting that, $T + V = E$, and $\sqrt{2T} = \sqrt{2(E - V)}$. Then equation (3.14.2) may be written as

$$T + T^* = \sqrt{2(E - V)}\, \frac{ds}{dt} \qquad (3.14.3)$$

allowing one to express the functional in equation (3.13.10) as

$$\Delta \int \sqrt{2(E - V)}\, ds \quad = \quad 0 \tag{3.14.4}$$

In equation (3.14.4) we have *Jacobi's functional* for the principle of least action. In the special case when there are no impressed forces, i.e. $V = 0$, Jacobi's principle of least action reduces to

$$\Delta \int ds \quad = \quad 0 \tag{3.14.5}$$

This functional indicates that the representative point in the configuration space moves along paths of shortest distance. Such paths are geodesics of the curvilinear configuration space and are characterized by minimum curvature. It will be recalled that the same result was obtained in Hertz' principle of least curvature discussed in section 2.5.

Returning to the general case, we note that the integrand of the functional in equation (3.14.4) is a function of q_i and not s. Furthermore for different paths in the configuration space the arc lengths will be different and hence the functional in equation (3.14.4) has variable limits. Therefore this functional still does not provide a practical means for solving dynamics problems. Both these difficulties may be resolved by expressing ds in terms of the differential for one of the generalized displacements, via the metric of the configuration space. The following example illustrates the approach.

Example 3.14.1

Consider the motion of a particle of mass m in the field of gravity. For this problem

$$V \quad = \quad mgy \tag{a}$$

and from the metric of the configuration space we have that

$$m(dx^2 + dy^2) \quad = \quad ds^2$$

$$\text{or} \qquad ds \quad = \sqrt{m(1 + y'^2)}\, dx \tag{b}$$

where $y' = dy/dx$.

Using equations (a) and (b) in Jacobi's functional for the principle of least action, we find

$$\Delta \int_{x_1}^{x_2} \sqrt{2m(E - mgy)(1 + y'^2)}\, dx \quad = \quad 0 \tag{c}$$

In equation (c) the independent variable x is not varied, and the limits of the integral are fixed. Therefore there is no distinction between the δ and Δ operators. For simplicity let us take the value of E as zero and then recognizing that the constants m and g do not affect the extremum conditions of the functional, we may express equation (c) as

$$\delta \int_{x_1}^{x_2} \sqrt{y(1 + y'^2)}\, dx \quad = \quad 0 \tag{d}$$

It is worth noting that time has been removed from this functional and the extremum of the functional is expected to yield the equation for the *trajectory* of the motion rather than the

equation of motion in time.

The functional in equation (d) is of standard type with a function $y(x)$ and its first derivative $y'(x)$ in its integrand. Thus the Euler-Lagrange equation for this functional takes the familiar form of

$$\frac{d}{dx}\left[\frac{\partial L}{\partial y'}\right] - \frac{\partial L}{\partial y} = 0 \tag{e}$$

$$\text{where} \qquad L = \sqrt{y(1 + y'^2)} \tag{f}$$

Once again the replacement of t by x is evident in equation (e). We should also note that x does not appear explicitly in the Lagrangian in equation (f). This is similar to the case when t is absent in the Lagrangian in Hamilton's principle. Earlier we showed that when t does not arise explicitly in L, then

$$L - p_i \dot{q}_i = \text{constant} \tag{g}$$

where the momenta $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$. By analogy we can then write for the functional (d) the following conservation statement

$$\sqrt{y(1 + y'^2)} - y' \frac{d}{dy'} \sqrt{y(1 + y'^2)} = \text{constant} = b \tag{h}$$

The differential equation (h) may be rearranged to yield

$$y = b^2(1 + y'^2) \tag{i}$$

where b is a constant. To solve equation (i) let

$$y - b^2 = z^2 \tag{j}$$

Then equation (i) becomes

$$y'^2 = \frac{z^2}{b^2} \qquad \text{or} \qquad \frac{dy}{z} = \frac{dx}{b} \tag{k}$$

But $dy = 2z dz$ which on substitution into equation (k) and integration yields

$$2z + a = \frac{x}{b} \tag{l}$$

where a is another constant of integration. On eliminating z between equations (i) and (l) we find the trajectory of the particle as

$$4(y - b^2) = \left[\frac{x}{b} - a\right]^2 \tag{m}$$

This equation will be recognized as a family of parabolas. For a particular solution the constants a and b may be determined from boundary conditions.

There is a striking similarity between Jacobi's principle of least action and Fermat's† *principle of least time* in geometrical optics. Fermat's principle states that a ray of light, in passing from a point A to a point B, describes a path for which the time of transit is a minimum with respect to all possible paths connecting A to B. This principle may be expressed as

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† Pierre de Fermat (1601-1665), French mathematician

$$\delta \int_A^B \frac{ds}{v} = 0 \qquad (3.14.6)$$

where ds is an element of the path and v is the velocity of light. Comparison of the functionals in equations (3.14.5) and (3.14.6) reveals an analogy between the integrands $\sqrt{2(E - V)}$ and $1/v$. Now in optics the geometrical theory is known to fail in cases where the wavelength of light is of the same order of magnitude as the dimensions of the objects upon which it falls. In such cases the geometrical theory of light must be replaced by the physical, wave theory of light. The transition from the wave theory to the geometrical theory had been well understood. In mechanics too it is known from the work of Bohr†, Einstein††, and Planck††† that the laws of mechanics on a small atomic scale differ from those on the large scale. However, treatment at first consisted in accepting, in general, the known laws of classical mechanics and supplementing them by additional rules or laws which appeared as limitations to the general theory.

It was not until 1924 that the significance of the analogy between optics and mechanics began to be understood. Guided by this analogy de Broglie° put forward the idea that a wave length was to be associated with a particle and that a new theory of mechanics, closely connected with wave optics, should be developed. Such a theory was developed by Schrödinger°° in 1926 when he derived his celebrated wave equation for quantum mechanics.

Problems

3.14.1 Using the principle of least action as

$$\Delta \int_{\theta_1}^{\theta_2} (p_r \frac{dr}{d\theta} + p_\theta)\, d\theta = 0$$

obtain the orbit equation of Kepler's problem.

3.15 Some Generalizations of the Principle of Least Action

In section (3.13) we derived the principle of least action from the form of Hamilton's principle wherein only the generalized displacements are varied. Let us now use the form of Hamilton's principle which admits independent variations of momenta and displacements, for the derivation of the principle of least action. In this case corresponding to equation (3.13.6) we find

† Niels Hendrik David Bohr (1885-1962), Danish physicist

†† Albert Einstein (1879-1955), German physicist

††† Max Planck (1858-1947), German physicist

° Louis-Victor de Broglie (1892-1987), French physicist

°° Erwin Schrödinger (1887-1961), Austrian physicist

$$\Delta \int_{t_1}^{t_2} [p_i \dot{q}_i - H]dt - [p_i \dot{q}_i - H]\delta t \Big|_{t_1}^{t_2} = p_i(\Delta q_i - \dot{q}_i \delta t) \Big|_{t_1}^{t_2} \quad (3.15.1)$$

On adding equations (3.13.15) and (3.15.1) we obtain

$$\Delta \int_{t_1}^{t_2} p_i \dot{q}_i \, dt = \int_{t_1}^{t_2} \delta H + \Delta p_i \Big|_{t_1}^{t_2} \quad (3.15.2)$$

Equation (3.15.2) holds for rheonomic as well as scleronomic systems. Now once again we impose the two constraints associated with the principle of least action, namely

$$\delta H = 0$$

$$\Delta q_i = 0 \qquad \text{at} \quad t_1 \text{ and } t_2$$

Under these conditions the functional in equation (3.15.2) simplifies to

$$\Delta \int_{t_1}^{t_2} p_i \dot{q}_i \, dt = 0 \quad (3.15.3)$$

Equation (3.15.3) is in effect the same as that derived earlier as equation (3.13.21). However in the case of equation (3.13.21) only the q_i were subject to variation whereas now both the p_i and the q_i may be varied independently. This flexibility allows us to remove the time directly from the functional in equation (3.15.3) and write it as

$$\Delta \int p_i \, dq_i = 0 \quad (3.15.4)$$

Example 3.15.1

Let us derive Hamilton's canonical equations from the principle of least action. Through Leibniz' rule we can write equation (3.15.3) as

$$\Delta \int_{t_1}^{t_2} p_i \, \dot{q}_i \, dt = \int_{t_1}^{t_2} \delta(p_i \, \dot{q}_i)dt + p_i \, \dot{q}_i \, \delta t \Big|_{t_1}^{t_2} = 0 \quad \text{(a)}$$

$$\text{or} \quad \int_{t_1}^{t_2} (\dot{q}_i \, \delta p_i + p_i \, \delta \dot{q}_i) \, dt + p_i \dot{q}_i \, \delta t \Big|_{t_1}^{t_2} = 0 \quad \text{(b)}$$

Integrating the second term under the integral we obtain

$$\int_{t_1}^{t_2} (\dot{q}_i \, \delta p_i - \dot{p}_i \, \delta q_i)dt + p_i \, (\delta q_i + \dot{q}_i \, \delta t) \Big|_{t_1}^{t_2} = 0 \quad \text{(c)}$$

Now on imposing the constraint on coterminal end points in space we may drop the last term. We must also impose the constraint on the constancy of the Hamiltonian. Thus

$$\frac{\partial H}{\partial p_i} \delta p_i + \frac{\partial H}{\partial q_i} \delta q_i = 0 \quad \text{(e)}$$

Now one can see that the integrand in equation (c) will vanish, subject to the constraint equation (e), if

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{f}$$

Returning to equation (3.15.4) we can select one of the q_i as our new independent variable, and in this way we obtain a new form for the principle of least action. Without loss of generality we select q_n for this purpose and write equation (3.15.4) as

$$\Delta \int_{q_{no}}^{q_{n1}} (p_i q_i' + p_n)\, dq_n = 0 \qquad i = 1, 2, ..., (n-1) \tag{3.15.5}$$

where $q_i = dq_i/dq_n$. In the functional of equation (3.15.5) all the momenta and the first $(n-1)$ of the generalized displacements are subject to variation. The displacement q_n, which now is the independent variable, is not varied and hence the limits in this functional are fixed. Accordingly we may also express this functional as

$$\delta \int_{q_{no}}^{q_{n1}} (p_i q_i' + p_n) dq_n = \int_{q_{no}}^{q_{n1}} \delta(p_i q_i' + p_n) dq_n = 0 \tag{3.15.6}$$

There is of course a side constraint, namely the constancy of the Hamiltonian, associated with this functional. Restricting ourselves to scleronomic systems we may express the constancy of the Hamiltonian as

$$\frac{1}{2} \left(\frac{p}{m} \right)_i p_i + V = E \tag{3.15.7}$$

From this relation we obtain the last momentum as

$$p_n = \sqrt{2m_n \left[E - V - \frac{1}{2} \left(\frac{p}{m}\right)_j p_j \right]} \qquad j = 1, 2, ...(n-1) \tag{3.15.8}$$

and on substituting from equation (3.15.8) into equation (3.15.6) we obtain the following functional

$$\int_{q_{no}}^{q_{n1}} \delta \left[p_i q_i' + \sqrt{2m_n \left[E - V - \frac{1}{2} \left(\frac{p}{m}\right)_j p_j \right]} \right] dq_n \qquad \begin{matrix} i = 1, 2, 3,...., (n-1) \\ j = 1, 2, 3,...., (n-1) \end{matrix} \tag{3.15.9}$$

Now the momenta and displacements may be varied freely in the functional in equation (3.15.9). It is worth noting that while p_n has been eliminated, q_n is still present in this functional through the potential energy V. However q_n, as pointed out already, is not varied and as such the functional in equation (3.15.9) has $2(n-1)$ dependent variables. The variations of this functional yield the following extremum conditions:

$$\text{For } \delta p_i: \quad q_i' - \frac{m_n p_i / m_i}{\sqrt{2m_n \left[E - V - (\frac{p}{m})_j p_j \right]}} = 0 \quad \text{(sum on } j \text{ only)} \tag{3.15.10}$$

$$\delta q_i: \quad -p_i' - \frac{m_n \partial V / \partial q_i}{\sqrt{2m_n \left[E - V - (\frac{p}{m})_j p_j \right]}} = 0 \tag{3.15.11}$$

On eliminating the square root term between equations (3.15.10) and (3.15.11) we find the following interesting result

$$\left(\frac{p}{m} \right)_j p_j' + \frac{\partial V}{\partial q_j} q_j' = 0 \tag{3.15.12}$$

or

$$\frac{d}{dq_n} \left[\frac{1}{2} (\frac{p}{m})_j p_j + V \right] = \frac{\partial V}{\partial q_n} \tag{3.15.13}$$

Now if q_n is absent in V, i.e. if q_n is an ignorable coordinate, then equation (3.15.13) may be integrated to yield

$$\frac{1}{2} (\frac{p}{m})_j p_j + V = \text{constant} \qquad j = 1,2,...,(n-1) \tag{3.15.14}$$

In equation (3.15.14) we have a statement of conservation of energy. In this statement only $(n-1)$ coordinates are involved. However this statement holds only if q_n is an ignorable coordinate and on comparing equation (3.15.14) with equation (3.15.7), we conclude that for such a case

$$\left(\frac{p^2}{2m} \right)_n = \text{constant}$$

i.e.

$$p_n = \text{constant} \tag{3.15.15}$$

Thus, as found earlier, the momentum associated with an ignorable coordinate is a constant of the motion.

Example 3.15.2

Consider again the motion of a particle in the field of gravity. Let us take y as q_n, then the extremum conditions in equations (3.15.10) and (3.15.11) become,

$$\text{for } \delta p_x: \quad \frac{dx}{dy} = \frac{p_x}{\sqrt{2m \left[E + mgy - \frac{p_x^2}{2m} \right]}} \tag{a}$$

$$\delta x: \quad \frac{dp_x}{dy} = 0 \tag{b}$$

The latter equation shows that p_x is conserved with respect to y. This is consistent with the fact

that x is an ignorable coordinate with respect to the independent variable y. On substituting the solution of equation (b) into equation (a) we may write the latter as

$$\frac{dx}{dy} = \frac{A}{\sqrt{B + Cy}} \tag{c}$$

where A, B and C are constants. Integration of equation (c) yields the following family of parabolic trajectories:

$$x = \frac{2A}{C}\sqrt{B + Cy} \tag{d}$$

It is of interest to note that had we taken x as q_n, then the differential equations would have been more complex since y is not an ignorable coordinate. In this case however we would have the "reduced" energy conservation equation (3.15.13) since x is an ignorable coordinate. This equation becomes

$$\frac{d}{dx}\left[\frac{p_y^2}{2m} - mgy\right] = 0$$

$$\text{i.e.} \quad \frac{p_y^2}{2m} - mgy = \text{constant} \tag{e}$$

Once again comparing the true energy conservation condition with equation (e), one can see that

$$p_x = \text{constant} \tag{f}$$

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Chapter IV

CANONICAL TRANSFORMATIONS AND THE HAMILTON-JACOBI EQUATION

Introduction

We have seen that the equations of motion for dynamical systems can take a variety of forms depending upon the generalized coordinates used. It will also be recalled that when ignorable coordinates arise the associated equations may be partially integrated. Recognizing the equivalence of different forms and attributing the presence of ignorable coordinates to the particular set of generalized coordinates used, it is natural to enquire whether it is possible to transform the equations of motion, expressed in terms of a given set of coordinates, to another set of coordinates for which all the generalized coordinates are ignorable. In this way the problem of solving the equations of motion reduces to a transformation. The researches of Hamilton and Jacobi revealed that the required transformation may be obtained from a single function which is governed by a differential equation referred to as the *Hamilton-Jacobi equation.*

4.1 Canonical Transformations

We have seen in section 2.7 that the form of Lagrange's equation is invariant under a transformation of generalized displacements $Q_i = Q_i(q,t)$. Such a transformation will change the Lagrangian $L(q, \dot{q}, t)$ to a new form $\bar{L}(Q, \dot{Q}, t)$. However from a variational viewpoint the extremum conditions associated with L and $\bar{L}$ are identical in form. The same is not the case for Hamilton's equations. This is because Hamilton's equations are obtained from a particular form of the Lagrangian, $L = p_i \dot{q}_i - H$ wherein the momenta and the displacements may be varied independently. An arbitrary transformation of p_i and q_i to a new set of coordinates P_i, Q_i of the form:

$$Q_i = Q_i(q, p, t) \tag{4.1.1}$$

$$P_i = P_i(q, p, t) \tag{4.1.2}$$

would not preserve the particular form of the Lagrangian from which Hamilton's equations are derived. Accordingly the transformations for p_i and q_i must be restricted to those that preserve the particular form of the Lagrangian from which Hamilton's

equations are derived. This requirement will be satisfied if we introduce a function $K(Q, P, t)$ such that the Lagrangian in the new coordinates takes the form $\bar{L} = P_i \dot{Q}_i - K$. Then, for the new coordinates Hamilton's principle becomes

$$\delta \int (P_i \dot{Q}_i - K(Q, P, t))dt = 0 \tag{4.1.3}$$

the extremum conditions of which yield the equations

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \tag{4.1.4}$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{4.1.5}$$

Evidently the function K plays the role of the Hamiltonian in the new coordinate set. Transformations which preserve the form of Hamilton's canonical equations are called canonical transformations.

Now if the form of Hamilton's equations is to remain invariant then the particular forms of the Lagrangian in the two coordinate systems can differ at most by the total time derivative of an arbitrary action function S, i.e.

$$\int_{t_1}^{t_2} (p_i \dot{q}_i - H)dt - \int_{t_1}^{t_2} (P_i \dot{Q}_i - K)dt = \int_{t_1}^{t_2} \frac{dS}{dt}\, dt = S(t_2) - S(t_1) \tag{4.1.6}$$

or

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{d}{dt} S \tag{4.1.7}$$

In order to affect the transformation between the two sets of variables, S must be a function of both the old and the new variables. That is, besides time t, the function S, called the *generating function*, must depend on $4n$ variables. However only $2n$ of these will be independent since the two coordinate systems are related through the $2n$ transformation equations (4.1.1) and (4.1.2). Thus the generating function may be written as a function of $2n$ independent variables in one of four forms:

$$S_1(q, Q, t), \qquad S_2(q, P, t), \qquad S_1^*(p, P, t), \qquad S_2^*(p, Q, t)$$

By substituting S_1 in Eq. (4.1.7) we obtain

$$(p_i \dot{q}_i - H) - (P_i \dot{Q}_i - K) = \frac{\partial S_1}{\partial q_i} \dot{q}_i + \frac{\partial S_1}{\partial Q_i} \dot{Q}_i + \frac{\partial S_1}{\partial t} \tag{4.1.8}$$

Now balancing the coefficients for $\dot{q}_i$ and $\dot{Q}_i$ we note that the satisfaction of equation (4.1.8) requires that

$$p_i = \frac{\partial S_1 (q, Q, t)}{\partial q_i} \tag{4.1.9}$$

$$P_i = -\frac{\partial S_1 (q, Q, t)}{\partial Q_i} \tag{4.1.10}$$

and

$$K = \frac{\partial S_1(q, Q, t)}{\partial t} + H \tag{4.1.11}$$

Equations (4.1.9) and (4.1.10) provide the information required for the transformations indicated in equations (4.1.1) and (4.1.2). By solving for Q_i in terms of q, p and t, from equation (4.1.9), we may establish the first transformation equation (4.1.1). Substitution of this expression for Q_i into equation (4.1.10) allows one to express P_i in terms of q, p and t and establishes the transformation equation (4.1.2). Thus if S_1 is known the required transformations from (p, q, t) to (P, Q, t) may be determined, and in addition from S_1 the connection between the new Hamiltonian K and the old Hamiltonian H, may be established as indicated in equation (4.1.11).

Now we may express equation (4.1.7) in the following alternative form

$$p_i \dot{q}_i - H = - \dot{P}_i Q_i - K + \frac{d}{dt}\left[S_1(q, Q, t) + P_i Q_i\right] \quad (4.1.12)$$

In the last term of equation (4.1.12) three of the four sets of variables arise. We may eliminate Q_i from this term by expressing Q_i in terms of q_i and P_i through equation (4.1.10). In this way only q_i and P_i will arise in the last term of equation (4.1.12) and we may therefore introduce a second generating function S_2 and write

$$p_i \dot{q}_i - H = - \dot{P}_i Q_i - K + \frac{d}{dt} S_2(q, P, t) \quad (4.1.13)$$

Following the procedure outlined above we can show that equation (4.1.13) will be satisfied under the following conditions:

$$p_i = \frac{\partial S_2(q, P, t)}{\partial q_i} \quad (4.1.14)$$

$$Q_i = \frac{\partial S_2(q, P, t)}{\partial P_i} \quad (4.1.15)$$

$$K = H + \frac{\partial S_2(q, P, t)}{\partial t} \quad (4.1.16)$$

Equations (4.1.14) to (4.1.16) once again provide the information required for the transformations (4.1.1) and (4.1.2). Thus equation (4.1.14) may be solved for P_i in terms of p_i, q_i and t, and thereby the transformation equation (4.1.2) may be established. On substituting this expression for P_i, into equation (4.1.15) we find the transformation equations (4.1.1).

To introduce the S_1^* form of the generating function let us write equation (4.1.13) in the following form

$$\frac{d}{dt}(p_i q_i) - \dot{p}_i q_i - H = - \dot{P}_i Q_i - K + \frac{d}{dt} S_2(q, P, t) \quad (4.1.17)$$

or

$$- \dot{p}_i q_i - H = - \dot{P}_i Q_i - K + \frac{d}{dt}\left[S_2(q, P, t) - p_i q_i\right] \quad (4.1.18)$$

Now eliminating q_i from the last term of equation (4.1.18) via equation (4.1.14) and introducing $S_1^*(p, P, t)$, we may write equation (4.1.18) as

$$- \dot{p}_i q_i - H = - \dot{P}_i Q_i - K + \frac{d}{dt} S_1^*(p, P, t) \quad (4.1.19)$$

The satisfaction of equation (4.1.19) requires that

$$q_i = -\frac{\partial S_1^*(p, P, t)}{\partial p_i} \tag{4.1.20}$$

$$Q_i = \frac{\partial S_1^*(p, P, t)}{\partial P_i} \tag{4.1.21}$$

$$K = \frac{\partial S_1^*(p, P, t)}{\partial t} + H \tag{4.1.22}$$

To complete the cycle we write equation (4.1.19) in the following alternative form

$$-\dot{p}_i q_i - H = -\frac{d}{dt}(P_i Q_i) + P_i \dot{Q}_i - K + \frac{d}{dt} S_1^*(p, P, t) \tag{4.1.23}$$

or

$$-\dot{p}_i q_i - H = P_i \dot{Q}_i - K + \frac{d}{dt}\left[S_1^*(p, P, t) - P_i Q_i\right] \tag{4.1.24}$$

This time we need to remove P_i from the last term. We may do so via equation (4.1.21) and on introducing the generating function $S_2^*(p, Q, t)$, we may write equation (4.1.24) as

$$-\dot{p}_i q_i - H = P_i \dot{Q}_i - K + \frac{d}{dt} S_2^*(p, Q, t) \tag{4.1.25}$$

Equation (4.1.25) may be satisfied under the conditions

$$q_i = -\frac{\partial S_2^*(p, Q, t)}{\partial p_i} \tag{4.1.26}$$

$$P_i = -\frac{\partial S_2^*(p, Q, t)}{\partial Q_i} \tag{4.1.27}$$

$$K = \frac{\partial S_2^*(p, Q, t)}{\partial t} + H \tag{4.1.28}$$

Above equations too, may be used to generate the transformation equations (4.1.1) and (4.1.2).

Although all four generating functions, S_1, S_2, S_1^* and S_2^* facilitate canonical transformations, one or the other of these functions may be more advantageous to use in a particular problem. It should also be noted that when the generating functions do not depend on time explicitly, then the Hamiltonian will be an invariant of the transformation.

Example 4.1.1

Consider the following form of the generating function

$$S_2(q, P, t) = q_i P_i \tag{a}$$

The associated transformation equations (4.1.15) to (4.1.17) yield the relations

$$p_i = P_i \qquad Q_i = q_i \qquad H = K \tag{b,c,d}$$

Thus the above form of S_2 generates the identity transformation.

Consider next the generating function

$$S_1^*(p, P, t) = p_i P_i \tag{e}$$

The associated transformation equations (4.1.21) to (4.1.23) yield the relations

$$q_i = -P_i \qquad Q_i = p_i \qquad H = K \tag{f,g,h}$$

Evidently the above form of S_1^* interchanges the momenta and generalized displacements.

Another class of transformations is characterized by the generating function

$$S_2(q, P, t) = f_i(q, t) P_i \tag{i}$$

where f_i may be any assumed set of functions. The transformation equations associated with S_2 show that the displacements are transformed as

$$Q_i = f_i(q, t) \tag{j}$$

In this case we see that the new displacements are obtained from the old displacements and do not depend on the old momenta. This is the case of point transformation under which, as pointed out earlier, Lagrange's equations remain invariant. It can now be concluded that all point transformations are canonical providing their associated momenta are derived through the appropriate transformations. In this example the momenta transform as

$$p_i = \frac{\partial f_j}{\partial q_i} P_j$$

Example 4.1.2

Let us identify the transformation affected by the following generating function

$$S_2^* = -Q_1 (p_1 \cos Q_2 + p_2 \sin Q_2) \tag{a}$$

Using equations (4.1.27) and (4.1.28) we obtain

$$q_1 = -\frac{\partial S_2^*}{\partial p_1} = Q_1 \cos Q_2 \tag{b}$$

$$q_2 = -\frac{\partial S_2^*}{\partial p_2} = Q_1 \sin Q_2 \tag{c}$$

$$P_1 = -\frac{\partial S_2^*}{\partial Q_1} = p_1 \cos Q_2 + p_2 \sin Q_2 \tag{d}$$

$$P_2 = -\frac{\partial S_2^*}{\partial Q_2} = Q_1 [p_2 \cos Q_2 - p_1 \sin Q_2] \tag{e}$$

This will be recognized as a transformation from Cartesian coordinates $q_1 = x$, $q_2 = y$, $p_1 = p_x$, $p_2 = p_y$ to polar coordinates $Q_1 = r$, $Q_2 = \theta$, $P_1 = P_r$ and $P_2 = P_\theta$.

Example 4.1.3

Finally consider the following generating function

$$S_1(q, Q, t) = \frac{m}{2} \omega q^2 \cot Q \tag{a}$$

where m and ω are constants whose significance will become evident in the following. The transformation equations (4.1.9) and (4.1.10) associated with S_1 yield the relations

$$p = \frac{\partial S_1}{\partial q} = m\omega q \cot Q \tag{b}$$

$$P = -\frac{\partial S_1}{\partial Q} = \frac{m}{2} \frac{\omega q^2}{\sin^2 Q} \tag{c}$$

Through these equations we may express P and Q in terms of p and q. However for our purposes it is desirable to express p and q in terms of P and Q. From equations (b) and (c) we find

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \tag{d}$$

and

$$p = \sqrt{2m\omega P} \cos Q \tag{e}$$

As the generating function does not involve the time explicitly, the Hamiltonian will have the same value in the old and the new coordinates. The constants in S_1 have been chosen with the problem of the simple linear oscillator in mind. For the simple linear oscillator

$$T = \frac{p^2}{2m} \qquad V = \frac{kq^2}{2} \tag{f,g}$$

where k is the stiffness of the spring. Then for the Hamiltonian $H = E = T + V$ in the present case,

$$H = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \tag{h}$$

where $\omega = \sqrt{k/m}$ is the natural frequency of the oscillator. Let us now evaluate the Hamiltonian in the new coordinate system using the relations in equations (d) and (e). Then, since $\partial S_1/\partial t = 0$,

$$K = H = \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P \tag{i}$$

Using this expression for K in equations (4.1.4) and (4.1.5) we find the governing equations in the Q, P coordinates as

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \tag{j}$$

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \tag{k}$$

Equation (j) shows that Q is an ignorable coordinate and the associated momentum P is conserved. Thus from equations (i) we have

$$P = \frac{K}{\omega} = \frac{H}{\omega} = \frac{E}{\omega} \tag{l}$$

Equation (k) shows that

$$Q = \omega t + \alpha \tag{m}$$

where α is a constant of integration fixed by the initial conditions. Finally from equations (d) and (j) we find that

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha) \tag{n}$$

which is the familiar form of the solution for the simple linear oscillator.

Here we have an example of canonical transformation which changes the Hamiltonian to a form in which the generalized displacements are ignorable. It can therefore be seen that, in principle, canonical transformations provide a new approach for the solution of the equations of motion. In section 4.3 we return to this point and consider a systematic method for finding transformations such that all the new generalized coordinates are ignorable.

4.2 Infinitesimal Canonical Transformations

The canonical transformation equations considered in the above section are quite general in nature. Let us now make use of these transformation equations to relate two states of a given system. That is in the state space we identify the system, at two states, by the coordinates (q_i, p_i, t) and (Q_i, P_i, t) and relate these states via canonical transformation equations. Evidently such a transformation describes the evolution of motion from the state (q_i, p_i, t) to the state (Q_i, P_i, t).

Let us initially consider two states close to each other. Then Q_i and P_i will differ very slightly from q_i, p_i. It will be recalled that if the generating function S_2 takes the form $S_2 = q_i P_i$, then the identity transformation results. For our purposes we need another form of $S_2(q, P, t)$ which generates a transformation only slightly different from the identity transformation. Thus let us take

$$S_2(q, P, t) = q_i P_i + \varepsilon G(q, P, t) \tag{4.2.1}$$

where ε is a small parameter and for the present case of the evolution of system's motion we may identify ε with an incremental change in time. We may view G as another generating function similar in form to S_2. Using the transformation equations associated with S_2, namely equations (4.1.14) and (4.1.15), we find

$$p_i = \frac{\partial S_2}{\partial q_i} = P_i + \varepsilon \frac{\partial G}{\partial q_i} \tag{4.2.2}$$

$$Q_i = \frac{\partial S_2}{\partial P_i} = q_i + \varepsilon \frac{\partial G}{\partial P_i} \tag{4.2.3}$$

or

$$Q_i - q_i = \varepsilon \frac{\partial G}{\partial P_i} \tag{4.2.4}$$

$$P_i - p_i = -\varepsilon \frac{\partial G}{\partial q_i} \tag{4.2.5}$$

These equations represent an *infinitesimal canonical transformation.* It should be noted that the small changes in the coordinates and momenta are of first order in ε. Thus if we neglect terms of order ε^2 and higher, we can replace each P_i by p_i in G. The resulting function $G(q, p, t)$ can then be considered as the generator of the infinitesimal canonical transformation. Under these conditions we may write equations (4.2.4) and (4.2.5) as

$$dq_i = \varepsilon \frac{\partial G}{\partial p_i} \tag{4.2.6}$$

$$dp_i = -\varepsilon \frac{\partial G}{\partial q_i} \tag{4.2.7}$$

Since the changes are due to the passage of time, we can let $\varepsilon = dt$ and for this case we denote the generating function G by $H(q, p, t)$. Then equations (4.2.6) and (4.2.7) may be expressed as

$$dt \frac{\partial H}{\partial p_i} = \dot{q}_i \, dt = dq_i \tag{4.2.8}$$

$$-dt \frac{\partial H}{\partial q_i} = dp_i \tag{4.2.9}$$

Above equations will be recognized as Hamilton's canonical equations (3.8.29) and (3.8.30). Hence we conclude that the Hamiltonian function can be considered to generate an infinitesimal canonical transformation at each instant of time. Since the result of two canonical transformations, applied one after the other, is equivalent to a single canonical transformation, the values of q and p at any time t can be obtained from their initial values by a canonical transformation which is a continuous function of time. According to this view the motion of a mechanical system corresponds to the continuous evolution of a canonical transformation. The Hamiltonian can thus be recognized as the generating function for the system's motion in time.

Conversely, there must exist a canonical transformation from the values of the coordinates at time t to their constant initial values. Obtaining such a transformation is obviously equivalent to solving the problem of the system's motion. We examine this point in some detail in the next section.

4.3 Hamilton-Jacobi Equation

Let us seek a canonical transformation from the coordinates q and momenta p at time t, to a new set of constant quantities which may be the $2n$ initial conditions q_o and p_o at $t = 0$. Such a set of transformation equations, relating the old and the new coordinates,

$$q = q(q_o, p_o\, t)$$

$$p = p(q_o, p_o, t)$$

are exactly the desired solution of the equations of motion since they give the coordinates as a function of their initial values and time.

To ensure that the new coordinates are constant in time, we need only require that the transformed Hamiltonian K, shall be identically zero, for then the equations of motion become

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0 \tag{4.3.1}$$

$$-\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0 \tag{4.3.2}$$

Now K is related to the old Hamiltonian and the generating function by the equation

$$K = H + \frac{\partial S}{\partial t} \tag{4.3.3}$$

and hence will be zero if S satisfies the equation

$$H(q, p, t) + \frac{\partial S}{\partial t} = 0 \tag{4.3.4}$$

Depending on the type of generating function used, equation (4.3.4) will take on different but equivalent forms. For instance using $S_2(q, P, t)$ and recalling that the old momenta are related to S_2 by $p_i = \partial S_2/\partial q_i$, we may express equation (4.3.4) as

$$\frac{\partial S_2(q, P, t)}{\partial t} + H(q_1, \ldots q_n, \frac{\partial S_2}{\partial q_1} \ldots \frac{\partial S_2}{\partial q_n}, t) = 0 \tag{4.3.5}$$

In equation (4.3.5) we have the celebrated *Hamilton-Jacobi equation.* It should be noted that the independent variables of this partial differential equation are the generalized displacements q_i and time t. The new momenta P_i are constants by virtue of $K = 0$. The integration of equation (4.3.5) will involve $(n + 1)$ constants. However since in this equation only the derivatives of the generating function S_2 appear and S_2 itself is absent, one can see that one of the integration constants is simply an additive constant tacked on to S_2. This constant has no importance as far as the transformation is concerned, since only the partial derivatives of S_2 appear in the transformation equations. Thus dropping this additive constant we may express a complete solution of equation (4.3.5) as

$$S_2(q_1 \cdots q_n, \beta_1 \cdots \beta_n, t) \tag{4.3.6}$$

where none of the β_i constants are solely additive. Now by definition we have that $S_2 = S_2(q, P, t)$ and through the Hamilton-Jacobi equation we have ensured that the new momenta P_i are constants. Without loss of generality we may identify these constant momenta with the integration constants β_i, i.e. we simply take

$$P_i = \beta_i \tag{4.3.7}$$

Returning to equation (4.3.5) we can see that in the Hamilton-Jacobi equation it is the momenta, and hence also the kinetic energy, that are expressed in terms of the derivatives of the generating function. Finally we note that the Hamiltonian is a quadratic function of the momenta, and that it may also be a nonlinear function of the q_i.

If we use the generating function $S_2^*(p, Q, t)$ in equation (4.3.4) and recall that the old displacements are related to S_2^* by $q_i = -\partial S_2^*/\partial p_i$, we find

$$\frac{\partial S_2^*(p,Q,t)}{\partial t} + H(p_1, \cdots p_n, \frac{\partial S_2^*}{\partial p_1} \cdots \frac{\partial S_2^*}{\partial p_n}, t) = 0 \qquad (4.3.8)$$

Equation (4.3.8) may be viewed as a complementary form of the Hamilton-Jacobi equation. In this equation the momenta and time are the independent variables. The new displacements are constants by virtue of $K = 0$. The solution of equation (4.3.8) may be formally expressed as

$$S_2^* = S_2^*(p_1 \cdots p_n, \alpha_1 \cdots \alpha_n, t)$$

This time we identify the constants α with the new (constant) displacements.

That is, we set

$$Q_i = \alpha_i \qquad (4.3.9)$$

It is worth noting that it is the generalized displacements, and hence also the potential energy, which are expressed in terms of the derivatives of S_2^* in the complementary Hamilton-Jacobi equation. Although equations (4.3.5) and (4.3.8) look formally alike, we shall see in the following examples that depending on the problem, they can take quite different forms.

It can be shown that S_1 and S_1^* give rise to equations (4.3.5) and (4.3.8) respectively.

Some insight into the physical significance of the generating functions can be gained by evaluating the total time derivative of these functions. Thus for $S_2(q, P, t)$, noting that the P_i are constants, we find

$$\frac{dS_2}{dt} = \frac{\partial S_2}{\partial t} + \frac{\partial S_2}{\partial q_i}\dot{q}_i \qquad (4.3.10)$$

Now recognizing that $\partial S_2/\partial t = -H$ and $\partial S_2/\partial q_i = p_i$, we may express Eq. (4.3.10) as

$$\frac{dS_2}{dt} = p_i \dot{q}_i - H = L \qquad (4.3.11)$$

Hence

$$S_2 = \int L\, dt + \text{constant} \qquad (4.3.12)$$

That is, S_2 is the functional for Hamilton's principle. In this functional time is not varied and the Hamiltonian need not be conserved.

In a similar manner we can show that

$$S_2^* = -\int(\dot{p}_i q_i + H)dt + \text{constant} \qquad (4.3.13)$$

Example 4.3.1

Consider the planar vertical motion of a single particle of unit mass in the field of gravity. For this problem the Hamiltonian is $H = T + V$,

$$H = \frac{1}{2}(p_x^2 + p_y^2) + gy \tag{a}$$

and the Hamilton-Jacobi equation becomes

$$\frac{\partial S_2}{\partial t}(x,y,t) + \frac{1}{2}\left[\left(\frac{\partial S_2}{\partial x}\right)^2 + \left(\frac{\partial S_2}{\partial y}\right)^2\right] + gy = 0 \tag{b}$$

This nonlinear partial differential equation admits a solution by separation of variables. We take a solution of the form

$$S_2(x,y,t) = S_x(x) + S_y(y) + S_t(t) \tag{c}$$

On substituting from equation (c) into (b) we find

$$\frac{dS_t}{dt} + \frac{1}{2}\left[\left(\frac{dS_x}{dx}\right)^2 + \left(\frac{dS_y}{dy}\right)^2\right] + gy = 0 \tag{d}$$

Now recognizing that in this equation we have terms which are pure functions of t, x and y, it follows that equation (d) will be satisfied if

$$\frac{dS_t}{dt} = -\beta_1 \tag{e}$$

$$\frac{dS_x}{dx} = \beta_2 \tag{f}$$

$$\frac{1}{2}\left(\frac{dS_y}{dy}\right)^2 + gy = \beta_1 - \frac{\beta_2^2}{2} \tag{g}$$

Where β_1, β_2 are constants.

On rearranging we may express S_y as

$$S_y = \int \sqrt{2\left[\beta_1 - \frac{\beta_2^2}{2} - gy\right]}\, dy \tag{h}$$

Hence the original generating function $S_2(x,y,t)$ may be written as

$$S_2 = -\beta_1 t + \beta_2 x + \int \sqrt{2\left[\beta_1 - \frac{\beta_2^2}{2} - gy\right]}\, dy \tag{i}$$

The constants β_1 and β_2 are as yet unspecified. We may associate these constants with the new (constant) momenta P_x and P_y. It should, however, be noted that $dS_t/dt\,(= \partial S_2/\partial t)$ is equal to the constant Hamiltonian. Thus β_1 has the dimension of energy, but it can nevertheless be associated with the initial momentum p_o.

Then from the relations between Q_i and S_2, namely $Q_i = \partial S_2/\partial P_i$, we find

$$Q_x = \alpha_1 = -t + \frac{\partial}{\partial \beta_1}\int [2(\beta_1 - \frac{\beta_2^2}{2} - gy)]^{1/2}\, dy \tag{j}$$

$$Q_y = \alpha_2 = x + \frac{\partial}{\partial \beta_2}\int [2(\beta_1 - \frac{\beta_2^2}{2} - gy)]^{1/2}\, dy \tag{k}$$

where α_1 and α_2 are constants identified with the new (initial) displacements. Integration of equation (j) yields the motion of the particle as a function of time, while the integration of equation (k) provides the orbit of the particle. Hence for the motion of the particle we find

$$\alpha_1 + t = -\frac{1}{g}\left[2\left(\beta_1 - \frac{\beta_2^2}{2} - gy\right)\right]^{1/2} \tag{l}$$

and for the orbit

$$\alpha_2 - x = \frac{\beta_2}{g}\left[2\left(\beta_1 - \frac{\beta_2^2}{2} - gy\right)\right]^{1/2} \tag{m}$$

The four constants in equations (l) and (m) may be determined from the two initial displacements and the two initial momenta. Consider for instance, the case of a projectile of unit mass ejected horizontally from the top of a hill. Then taking the origin at the hill top we have α_1, and α_2 as the initial displacement which we take as

$$\alpha_1 = \alpha_2 = 0 \tag{n}$$

Recalling that β_1 is the constant energy and β_2 a constant momentum, we take

$$\beta_1 = \dot{x}_O^2/2 \qquad \beta_2 = \dot{x}_O \tag{o}$$

Substituting from equations (n) and (o) into equations (l) and (m) we find from equation (l)

$$y = -\frac{1}{2}gt^2 \tag{p}$$

and from equation (m)

$$x^2 = -\frac{2\dot{x}_o^2}{g}y \tag{q}$$

Example 4.3.2

Next let us analyze the same problem via the complementary Hamilton-Jacobi equation. In this case we have

$$\frac{\partial S_2^*}{\partial t} + \frac{p_x^2}{2} + \frac{p_y^2}{2} + g\frac{\partial S_2^*}{\partial p_y} = 0 \tag{a}$$

where we have expressed y as a derivative of S_2^* with respect to p_y. It is worth noting that for this problem the form of the complementary Hamilton-Jacobi equation is quite different from the direct Hamilton-Jacobi equation. Once again attempting a solution by separation of variables we assume

$$S_2^* = S_t^*(t) + S_{p_x}^*(p_x) + S_{p_y}^*(p_y) \tag{b}$$

On substitution into equation (a) and collecting terms of similar nature we observe that

$$p_x = \text{constant}$$

$$\frac{dS_t^*}{dt} = -\alpha \qquad \text{a constant} \tag{c}$$

$$\frac{dS_{p_y}^*}{dp_y} = \frac{1}{g}\left[\alpha - \frac{p_y^2}{2} - \frac{p_x^2}{2}\right] \tag{d}$$

On integrating these equations we may express

$$S_2^* = -\alpha t + \int \frac{1}{g}\left[\alpha - \frac{p_y^2}{2} - \frac{p_x^2}{2}\right] dp_y \tag{e}$$

This time we identify α with the new (constant) displacement Q_y. Then using the relation between S_2^* and P_i, namely $P_i = -\partial S_2^*/\partial Q_i$, we find

$$P_y = \beta = t - \frac{1}{g}\int dp_y = t - \frac{p_y}{g} \tag{f}$$

or

$$p_y = (t - \beta)\, g \tag{g}$$

where β is again a constant. Above equation implies that the momentum in the y direction is a linear function of time. The x momentum remains constant as noted earlier. Once again β may be evaluated from the initial conditions. The displacement of the particle, as a function of time, may be obtained from the integration of the momentum-velocity relation.

Example 4.3.3

As another example consider the motion of a simple pendulum. For this problem the Hamiltonian takes the form

$$H = \frac{p_\theta^2}{2I} + mgl\,(1 - \cos\theta) \tag{a}$$

where l is the length of the pendulum and θ denotes the angle of the pendulum measured from the vertical. The pendulum's angular momentum is denoted by $p_\theta = I\dot{\theta}$, where I is the moment of inertia about the suspension point. Now recalling that in terms of $S_2(q, P, t)$ the old momenta are given by $p_i = \partial S_2/\partial q_i$, we may write the Hamilton-Jacobi equation for the pendulum as

$$\frac{\partial S_2}{\partial t} + \frac{1}{2I}\left(\frac{\partial S_2}{\partial \theta}\right)^2 + mgl(1 - \cos\theta) = 0 \tag{b}$$

To solve this equation we attempt, once again, a solution by separation of variables. We assume

$$S_2 = S_t(t) + S_\theta(\theta) \tag{c}$$

Upon substitution from equation (c) into (b) and collecting terms involving t only and θ only, we deduce that

$$\frac{dS_t}{dt} = -\beta \qquad \text{a constant} \tag{d}$$

$$\frac{1}{2I}\left(\frac{dS_\theta}{d\theta}\right)^2 + mgl(1 - \cos\theta) = \beta \tag{e}$$

Solving for S_t and S_θ and substituting into equations (c), we find

$$S_2 = -\beta t + \int \sqrt{2I\,[mgl\,(\cos\theta - 1) + \beta]}\, d\theta \tag{f}$$

In equation (f) we may identify the constant β with the new (constant) momentum. The associated new (constant) displacement is obtained from the relation $Q_i = \partial S_2/\partial P_i$. Thus for our case

$$Q = \alpha = -t + \int \frac{I}{\sqrt{2I\,[mgl\,(\cos\theta - 1) + \beta]}}\, d\theta \tag{g}$$

Now to cast the above integral into a more recognizable form we note from equation (e) that β is the total energy of the pendulum. Thus if we denote the amplitude of the pendulum oscillation by ϕ, we may express β as the maximum value of the potential energy namely;

$$\beta = mgl(1 - \cos\phi) \tag{h}$$

Substituting into equation (g) yields

$$t + \alpha = \int \frac{I \, d\theta}{\sqrt{2I \, mgl(\cos\theta - \cos\phi)}} \tag{i}$$

or

$$t + \alpha = \int \frac{I \, d(\frac{\theta}{2})}{\sqrt{I \, mgl(\sin^2\frac{\phi}{2} - \sin^2\frac{\theta}{2})}} \tag{j}$$

Finally by letting

$$\sin\frac{\theta}{2} = \sin\frac{\phi}{2}\sin v \tag{k}$$

above integral may be transformed into

$$t + \alpha = \int \frac{I \, dv}{\sqrt{Imgl(1 - k^2\sin^2 v)}} \tag{l}$$

where $k = \sin\frac{\phi}{2}$

Equation (l) is the well known elliptic integral of the first kind associated with the pendulum problem. For small values of the amplitude ϕ, it is readily shown that equation (l) results in

$$\theta = \phi \sin\sqrt{\frac{mgl}{I}}\,(t + \alpha) \tag{m}$$

and for a bob pendulum, with $I = ml^2$,

$$\theta = \phi \sin\sqrt{\frac{g}{l}}\,(t + \alpha) \tag{n}$$

Example 4.3.4

Now let us analyze the same problem via the complementary Hamilton-Jacobi equation. This time we need to remove the displacement θ from the Hamiltonian. Noting that the old displacement is related to $S_2^*(p, Q, t)$ by $q_i = -\partial S_2^*/\partial p_i$ and noting that $\cos\theta$ is an even function we may express the complementary Hamilton-Jacobi equation for the pendulum as

$$\frac{\partial S_2^*}{\partial t} + \frac{p_\theta^2}{2I} + mgl[1 - \cos(\frac{\partial S_2^*}{\partial p_\theta})] = 0 \tag{a}$$

Attempting a solution by separation of variables we write

$$S_2^* = S_t^*(t) + S_{p_\theta}^*(p_\theta) \tag{b}$$

Substituting into equation (a) and grouping similar terms, we find

$$\frac{dS_t^*}{dt} = -\alpha \tag{c}$$

$$\frac{p_\theta^2}{2I} + mgl[1 - \cos(\frac{dS_{p_\theta}^*}{dp_\theta})] = \alpha \tag{d}$$

Solving for S_t^* and $S_{p_\theta}^*$ and substituting into equation (b) we obtain

$$S_2^* = \int \text{arc cos} \left\{ \frac{1}{mgl} \left[\frac{p_\theta^2}{2I} - \alpha + mgl \right] \right\} dp_\theta - \alpha t$$

Once again we identify the constant α with the new (constant) displacement. The new (constant) momentum β is obtained from the relation $P_i = -\partial S_2^*/\partial Q_i$. For the problem at hand we find, after some manipulations:

$$P = \beta = -\frac{\partial S_2^*}{\partial \alpha} = -\int \frac{dp_\theta}{\sqrt{2mgl(\alpha - \frac{p_\theta^2}{2I}) - (\frac{p_\theta^2}{2I} - \alpha)^2}} + t \qquad \text{(e)}$$

In the above equation we have an elliptic integral, though not in one of the standard forms†. Now from equations (a) and (c) it can be seen that

$$\frac{p_\theta^2}{2I} - \alpha = mgl(\cos\theta - 1) \qquad \text{(f)}$$

For small values of θ, the square of $(\cos\theta - 1)$ may be neglected. In that case equation (e) simplifies to

$$\beta - t = -\int \frac{dp_\theta}{\sqrt{\frac{mgl}{I}(2\alpha I - p_\theta^2)}} = \sqrt{\frac{I}{mgl}} \text{ arc sin } \frac{p_\theta}{\sqrt{2\alpha I}} \qquad \text{(g)}$$

or

$$p_\theta = \sqrt{2\alpha I} \sin \omega (\beta - t) \qquad \text{(h)}$$

where

$$\omega^2 = \frac{mgl}{I} \qquad \text{(i)}$$

In these expressions we have the solution for small amplitude oscillations of the pendulum. The constants α and β may be evaluated from the initial conditions.

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† On rewriting equation (e) as follows

$$\beta - t = -\int \frac{dp_\theta}{\sqrt{(\alpha - \frac{p_\theta^2}{2I})\,[2mgl - (\alpha - \frac{p_\theta^2}{2I})\,]}}$$

and letting

$$2\alpha I = a^2, \qquad 4Imgl = b^2 \qquad \text{and } p_\theta = a \cos v$$

one can cast the above integral into one of the standard elliptic integrals.

$$\beta - t = -\int \frac{2I dv}{\sqrt{b^2 - a^2 \sin^2 v}}$$

Problems

4.3.1 Write the Hamilton-Jacobi equation for a single degree of freedom oscillator and by solving this equation determine the motion of the oscillator.

4.3.2 Use the Hamilton-Jacobi equation to solve Kepler's problem of motion in an inverse square central force field, in terms of S_2 and S_2^*.

4.3.3 A particle of mass m moves in a force field whose potential energy in spherical coordinates is given by

$$V = -K\frac{\cos\theta}{r^2}$$

Write the Hamilton-Jacobi equation for this particle and find an expression for S_2, in integral form.

4.4 Hamilton's Characteristic Function

In the above examples we were able to separate the generating function S_2(or S_2^*) into two parts, one involving q(or p) alone and the other time alone. Such a separation of variables is always possible whenever the old Hamiltonian does not involve time explicitly. In the following we examine this point further but restrict our discussions to the generating function S_2. Similar conclusions hold for the other generating functions.

If H is not an explicit function of time, the Hamilton-Jacobi equation (4.3.5) becomes

$$\frac{\partial S_2}{\partial t} + H(q, \frac{\partial S_2}{\partial q}) = 0 \tag{4.4.1}$$

The first term involves only the explicit appearance of t in S_2, whereas the second term is concerned only with the explicit appearance of q in S_2. The time variable can therefore be separated by assuming a solution for S_2 of the form

$$S_2(q, \beta, t) = W_2(q, \beta) - \beta_1 t \tag{4.4.2}$$

where the β stand for the new (constant) momenta. Upon substituting this form of S_2 into equation (4.4.1), we find

$$\frac{dS_2}{dt} = -\beta_1 \qquad \text{and} \qquad H(q, \frac{\partial W_2}{\partial q}) = \beta_1 \tag{4.4.3}$$

The differential equations (4.4.3), referred to as the *modified Hamilton-Jacobi equation*, do not involve the time. One of the constants of integration appearing in S_2, namely β_1, is now equal to the constant value of H. The function W_2 is known as *Hamilton's characteristic function.* It is not difficult to see that $W_2(q, P)$ is indeed the generating function for canonical transformations when the Hamiltonian is a constant (see section 4.1). For this case we find the transformation equation as

$$H = K = E \tag{4.4.4}$$

$$p_i = \frac{\partial W_2}{\partial q_i} \qquad \text{and} \qquad Q_i = \frac{\partial W_2}{\partial P_i} \tag{4.4.5}$$

Using equation (4.4.5) we may express the constancy of the Hamiltonian as

$$H(q,p) = H(q, \frac{\partial W_2}{\partial q}) = E \tag{4.4.6}$$

which is essentially identical to equation (4.4.3).

Hamilton's characteristic function provides another solution approach - one that was illustrated in Example 4.1.3. In this approach we require that all the new displacements be ignorable. Then

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \qquad \text{or} \qquad P_i = \beta_i \tag{4.4.7}$$

where the β_i are a set of constants. The new displacements can then be determined as follows

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \frac{\partial K}{\partial \beta_i} = \frac{\partial \beta_1}{\partial \beta_i} = \begin{cases} 1 & \text{when } i = 1 \\ 0 & i \neq 1 \end{cases} \tag{4.4.8}$$

Integrating with respect to time we find

$$\begin{aligned} Q_1 &= t + \beta_1 \\ Q_i &= \beta_i \qquad \text{for} \qquad i \neq 1 \end{aligned} \tag{4.4.9}$$

The only coordinate which is not simply a constant of the motion is Q_1, which is equal to the time plus a constant.

The characteristic function W_2 possesses a physical significance similar to that of S_2. As W_2 does not involve time explicitly, its total time derivative is

$$\frac{dW_2}{dt} = \frac{\partial W_2}{\partial q_i}\dot{q}_i = p_i\dot{q}_i \tag{4.4.10}$$

and hence

$$W_2 = \int p_i\dot{q}_i dt = \int p_i dq_i \tag{4.4.11}$$

Equation (4.4.11) reveals that W_2 is the functional for the principle of least action, for which the Hamiltonian is conserved but the time is allowed to vary.

4.5 Hamilton's Optico-Mechanical Analogy

When we examined Jacobi's form of the principle of least action in section 3.14, we pointed out an interesting analogy between this principle and Fermat's principle† of least time, in geometrical optics. Through the Hamilton-Jacobi equation one can gain further insight into this analogy.

Consider a conservative system for which the Hamiltonian is a constant of the motion. Then the relation between the generating functions S_2 and W_2 may be expressed as

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† Pierre de Fermat (1601-1665), French mathematician

$$S_2(q,P,t) = W_2(q,P) - Et \tag{4.5.1}$$

Now a constant value of W_2 describes a fixed hypersurface in the phase space. Thus for $W_2 = a$ and $W_2 = b$ we may envisage two fixed surfaces in the phase space. Likewise for different constant values of S_2 we will have further surfaces in the state space. From equation (4.5.1) it follows that at time $t = 0$, the surfaces $W_2 = a$ and $W_2 = b$ coincide with the $S_2 = a$ and $S_2 = b$ surfaces. However at a time dt later, the surface $S_2 = a$, coincides with the $W_2 = a + Edt$ surface. Likewise the surface $S_2 = b$, now coincides with the $W_2 = b + Edt$ surface. It is evident then that the S_2 surfaces propagate in the state space in an analogous manner to the propagation of waves in 3D space. Since the constant S_2 surfaces change their shape in the course of time, the wave velocity of these surfaces will change from point to point. To determine this wave velocity let us consider the simpler case of a single particle moving in 3D space.

If at a given point the perpendicular distance between two neighbouring S_2 surfaces is denoted by $d\mathbf{s}$, then at this point we may express the wave velocity as

$$\mathbf{u} = \frac{d\mathbf{s}}{dt} \tag{4.5.2}$$

Now in the time dt, the S_2 surface moves from a given W_2 surface to a new surface on which the value of the characteristic function is $W_2 + dW_2$, where

$$dW_2 = E\,dt \tag{4.5.3}$$

We may evaluate the change dW_2 from the geometry of the surface as follows. We have the directional derivative

$$\frac{dW_2}{ds} = (\mathbf{grad}\ W_2)\cdot\mathbf{n} \tag{4.5.4}$$

where $\mathbf{n}$ is the unit normal vector in the direction of $\mathbf{s}$. We also know that $(\mathbf{grad}\ W_2)$ is normal to the surface $W_2 =$ constant. Therefore

$$\mathbf{n} = \frac{\mathbf{grad}\ W_2}{|\ \mathbf{grad}\ W_2\ |} \tag{4.5.5}$$

Substituting into equation (4.5.4) we obtain

$$dW_2 = \mathbf{grad}\ W_2 \cdot d\mathbf{s} \tag{4.5.6}$$

Finally using equation (4.5.6) in conjunction with equations (4.5.3) and (4.5.2) we deduce that

$$|\mathbf{u}| = \frac{E}{|\ \mathbf{grad}\ W_2\ |} \tag{4.5.7}$$

Now the magnitude of the gradient of the W_2 surface is readily obtained from the Hamilton-Jacobi equation. Noting that for a single particle

$$\frac{1}{2m}\left[\left(\frac{\partial W_2}{\partial x}\right)^2 + \left(\frac{\partial W_2}{\partial y}\right)^2 + \left(\frac{\partial W_2}{\partial z}\right)^2\right] = T = E - V \tag{4.5.8}$$

we may express the wave velocity as

$$|\mathbf{u}| = \frac{E}{\sqrt{2m(E - V)}} \tag{4.5.9}$$

It will be recalled that when we compared the integrands of Fermat's principle of least time and Jacobi's principle of least action, we deduced that the expression $1/\sqrt{2(E - V)}$ for the representative particle of unit mass moving in the configuration space was analogous to the velocity of light. This analogy is now reinforced by equation (4.5.9) which also shows that the velocity of light is analogous to the velocity of the S_2 surfaces moving in the configuration space. It is of interest to relate the velocity of the S_2 surfaces to the momentum (and hence the velocity) of the particle. This is readily accomplished by noting that

$$E - V = T = \frac{B^2}{2m} \tag{4.5.10}$$

Substituting into equation (4.5.9), we find

$$|\mathbf{u}| = \frac{E}{B} = \frac{E}{mv} \tag{4.5.11}$$

Thus the particle velocity is inversely proportional to the velocity of the S_2 surface, i.e. as the particle speeds up, the S_2 surface slows down. What about the direction of the particle's motion? Clearly the direction of the particle's motion is tangential to its momentum vector. The momentum vector is given by

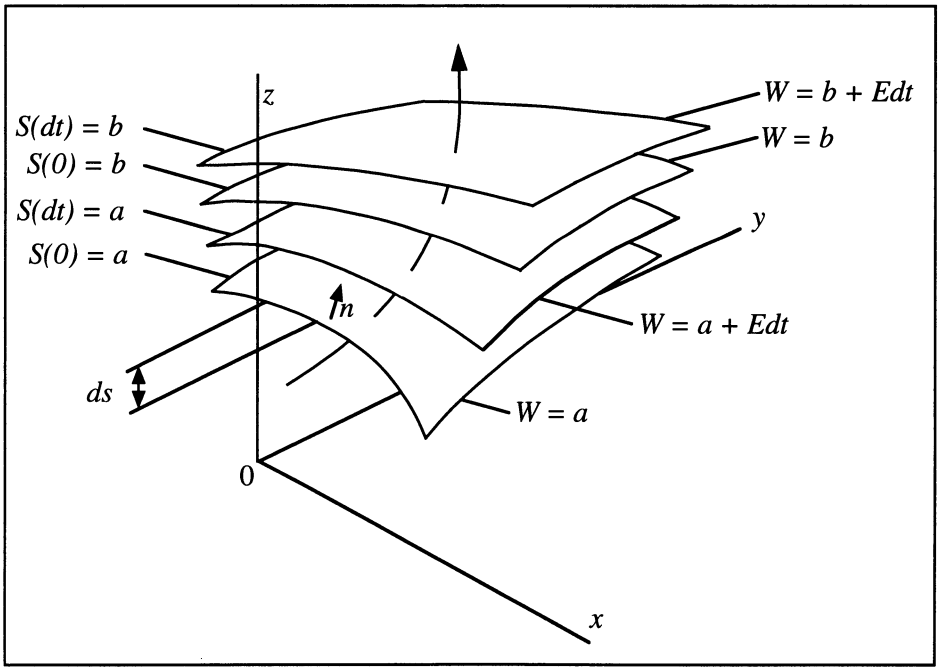

Figure 4.1 Motion of the S_2 surface and the momentum vector of a particle

$$\mathbf{B} = [\,\mathbf{e}_x \;\; \mathbf{e}_y \;\; \mathbf{e}_z\,] \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = [\,\mathbf{e}_x \;\; \mathbf{e}_y \;\; \mathbf{e}_z\,] \begin{bmatrix} \dfrac{\partial W_2}{\partial x} \\ \dfrac{\partial W_2}{\partial y} \\ \dfrac{\partial W_2}{\partial z} \end{bmatrix} \tag{4.5.12}$$

and the gradient of the W_2 surface is normal to the surface. Hence we conclude that the particle's motion is normal to the W_2 (and hence also to the S_2) surfaces. In terms of the analogy with optics we note that the particle's motion is analogous to the directions of light rays. The motion of the S_2 surface and the momentum vector of the particle are depicted in Figure 4.1.

The analogy between optics and mechanics has an interesting history. This analogy was first noted by Johann Bernoulli† who compared the motion of a particle in a given field of force with the propagation of light in an optically heterogeneous medium. Using this approach Johnann Bernoulli solved the famous brachystochrone problem - the curve of quickest descent. Maupertuis†† too was aware of this analogy and he demonstrated how the law of refraction of light can be deduced from his principle of least action. However the realization that problems of mechanics and geometrical optics can be handled from a unified point of view is due to Hamilton†††. The significance of the analogy for the development of laws of mechanics at the atomic scale was not realised until the nineteen twenties when Schrödinger° derived the wave equation from the Hamilton-Jacobi equation and paved the way for the development of quantum mechanics.

† Johann Bernoulli (1667-1748), Swiss mathematician

†† Pierre Louis Moreau de Maupertuis (1698-1759), French physicist

††† William Rowan Hamilton (1805-1865), Irish mathematician

° Erwin Schrödinger (1887-1961), Austrian physicist

Suggested Reading

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"Classical Mechanics"
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2. Park, David
"Classical Dynamics and its Quantum Analogues"
Second Edition, Springer-Verlag, (1989).

3. Sommerfeld, Arnold
"Vorlesungen über Theoretische Physik I: Mechanik"
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Chapter V

RIGID BODY DYNAMICS

Introduction

In the foregoing we have studied the motion of *particles*. Strictly speaking, the term *point mass* should have been used instead of particle, in order to emphasize the theoretical nature of the concept. A point mass represents a finite mass of infinitesimal volume. Reality comes close to it in many cases. Even a huge body such as the earth, can be looked upon as a point mass, for example when its fundamental motion about the sun is investigated.

We will now introduce another useful concept, viz. that of a *rigid body*, i.e. a body of finite size and fixed geometry. There is, of course, no such thing as a *rigid* body in nature, all real bodies deform, to some extent, under loads, but some bodies come close to being rigid, and the benefits derived from making the assumption of rigidity far outweigh the disadvantages.

The translatory motion of a rigid body can be analysed in the same fashion as the motion of a point mass at the center of mass of the rigid body. This has been illustrated in examples 1.1.1 and 1.1.2.

On the other hand, the *rotatory motion* of a rigid body requires a new treatment, which is the subject of the present chapter. A rotating body is often called a *gyroscope*, or simply a *gyro* (gk: gyros = ring), and the field of study of its dynamic behaviour is consequently called *gyrodynamics*.

5.1 Rotating Coordinate System

In gyrodynamics it is often useful to introduce rotating reference axes. The basis vectors of such axes change with time relative to a fixed inertial coordinate system. A vector, such as the position $\mathbf{r}$, may be expressed in terms of its components in the rotating coordinate system as

$$\mathbf{r} = [\, \mathbf{e}_x \; \mathbf{e}_y \; \mathbf{e}_z \,] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{5.1.1}$$

The time derivative of the position vector **r** can be expressed as

$$\dot{\mathbf{r}} = [\,\mathbf{e}_x\ \mathbf{e}_y\ \mathbf{e}_z\,]\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} + [\,\dot{\mathbf{e}}_x\ \dot{\mathbf{e}}_y\ \dot{\mathbf{e}}_z\,]\begin{bmatrix}x\\ y\\ z\end{bmatrix} \tag{5.1.2}$$

Now since $\mathbf{e}_x$ is a unit vector $\dot{\mathbf{e}}_x$ is perpendicular to $\mathbf{e}_x$ and must therefore lie in the plane of $\mathbf{e}_y$ and $\mathbf{e}_z$. Then

$$\dot{\mathbf{e}}_x = a_1\,\mathbf{e}_y + a_2\,\mathbf{e}_z \tag{5.1.3}$$

Similarly

$$\dot{\mathbf{e}}_y = a_3\,\mathbf{e}_z + a_4\,\mathbf{e}_x \tag{5.1.4}$$

$$\dot{\mathbf{e}}_z = a_5\,\mathbf{e}_x + a_6\,\mathbf{e}_y \tag{5.1.5}$$

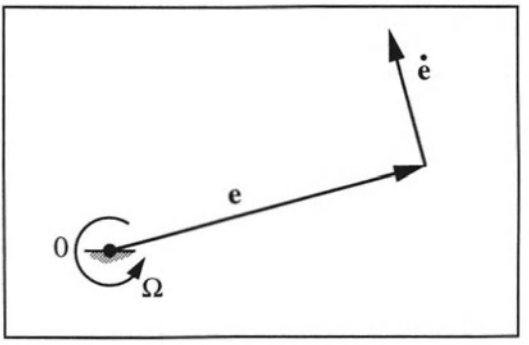


Figure 5.1 Velocity of tip of basis vector

From $\mathbf{e}_x \cdot \mathbf{e}_y = 0$ we obtain, by differentiation, $\mathbf{e}_x \cdot \dot{\mathbf{e}}_y + \dot{\mathbf{e}}_x \cdot \mathbf{e}_y = 0$. But $\mathbf{e}_x \cdot \dot{\mathbf{e}}_y = a_4$ and $\dot{\mathbf{e}}_x \cdot \mathbf{e}_y = a_1$, thus $a_4 = -a_1$. Similarly from differentiation of $\mathbf{e}_x \cdot \mathbf{e}_z = 0$ and $\mathbf{e}_y \cdot \mathbf{e}_z = 0$ we find $a_5 = -a_2$ and $a_6 = -a_3$. With this information we may now write equations (5.1.3) to (5.1.5) as

$$[\,\dot{\mathbf{e}}_x\ \dot{\mathbf{e}}_y\ \dot{\mathbf{e}}_z\,] = [\,\mathbf{e}_x\ \mathbf{e}_y\ \mathbf{e}_z\,]\begin{bmatrix}0 & -a_1 & -a_2\\ a_1 & 0 & -a_3\\ a_2 & a_3 & 0\end{bmatrix} \tag{5.1.6}$$

Then if we choose $a_3 = \Omega_x$, $-a_2 = \Omega_y$ and $a_1 = \Omega_z$, equation (5.1.6) may be written as

$$[\,\dot{\mathbf{e}}_x\ \dot{\mathbf{e}}_y\ \dot{\mathbf{e}}_z\,] = [\,\mathbf{e}_x\ \mathbf{e}_y\ \mathbf{e}_z\,]\begin{bmatrix}0 & -\Omega_z & \Omega_y\\ \Omega_z & 0 & -\Omega_x\\ -\Omega_y & \Omega_x & 0\end{bmatrix} \tag{5.1.7}$$

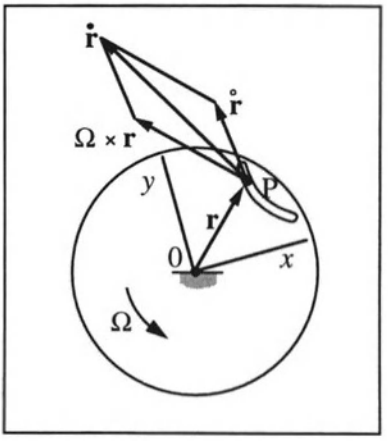


Figure 5.2 Rotating disk with slot

or in vector notation

$$\dot{\mathbf{e}}_i = \boldsymbol{\Omega} \times \mathbf{e}_i \qquad i = 1, 2, 3 \tag{5.1.8}$$

The vector $\boldsymbol{\Omega} = \Omega_x\, \mathbf{e}_x + \Omega_y\, \mathbf{e}_y + \Omega_z\, \mathbf{e}_z$ is the angular velocity vector of the rotating axes relative to the fixed axes. The skew symmetric matrix $[\Omega_{ij}]$ in equation (5.1.7) is singular and the relation in this equation can not be inverted.

Using equation (5.1.7) in (5.1.2) we find the velocity vector as

$$\dot{\mathbf{r}} = [\, \mathbf{e}_x\ \mathbf{e}_y\ \mathbf{e}_z \,] \left(\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \tag{5.1.9}$$

or in vector notation

$$\dot{\mathbf{r}} = \mathring{\mathbf{r}} + \boldsymbol{\Omega} \times \mathbf{r} \tag{5.1.10}$$

where $\mathring{\mathbf{r}}$ (pronounced "r circle") is the apparent velocity vector as seen by an observer in the rotating frame. In the same manner we may express the time derivatives of the linear and the angular momentum vectors as

$$\dot{\mathbf{B}} = \mathring{\mathbf{B}} + \boldsymbol{\Omega} \times \mathbf{B} \tag{5.1.11}$$

$$\dot{\mathbf{H}} = \mathring{\mathbf{H}} + \boldsymbol{\Omega} \times \mathbf{H} \tag{5.1.12}$$

Equation (5.1.10) for the velocity $\dot{\mathbf{r}}$ may be differentiated again to obtain the acceleration,

$$\ddot{\mathbf{r}} = (\dot{\mathbf{r}})^{\circ} + \boldsymbol{\Omega} \times \dot{\mathbf{r}} \tag{5.1.13}$$

With the help of equation (5.1.10) one obtains eventually

$$\ddot{\mathbf{r}} = \overset{\circ\circ}{\mathbf{r}} + \overset{\circ}{\boldsymbol{\Omega}} \times \mathbf{r} + 2\boldsymbol{\Omega} \times \overset{\circ}{\mathbf{r}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{5.1.14}$$

with $\overset{\circ}{\boldsymbol{\Omega}} = \dot{\boldsymbol{\Omega}}$, because $\boldsymbol{\Omega} \times \boldsymbol{\Omega} = 0$.

Example 5.1.1

To illustrate the velocity of the tip of a basis vector, assume that an orthogonal coordinate system $Oxyz$ is subject to an angular velocity given by

$$\boldsymbol{\Omega} = [\,\mathbf{e}_x \;\; \mathbf{e}_y \;\; \mathbf{e}_z\,] \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix} \text{ rad/s} \tag{a}$$

Let us find the velocity of the tip of the basis vector $\mathbf{e}_x$. We may use, either equation (5.1.9) or equation (5.1.10). The first approach, with

$$\mathbf{e}_x = [\,\mathbf{e}_x \;\; \mathbf{e}_y \;\; \mathbf{e}_z\,] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{b}$$

gives

$$\dot{\mathbf{e}}_x = \boldsymbol{\Omega} \times \mathbf{e}_x = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_y \\ 3 & 4 & 12 \\ 1 & 0 & 0 \end{vmatrix} = [\,\mathbf{e}_x \;\; \mathbf{e}_y \;\; \mathbf{e}_z\,] \begin{bmatrix} 0 \\ 12 \\ -4 \end{bmatrix} \tag{c}$$

The second approach, with

$$[\Omega_{ij}] = \begin{bmatrix} 0 & 12 & -4 \\ -12 & 0 & 3 \\ 4 & -3 & 0 \end{bmatrix} \tag{d}$$

and

$$\begin{bmatrix} \dot{\mathbf{e}}_x \\ \dot{\mathbf{e}}_y \\ \dot{\mathbf{e}}_z \end{bmatrix} = \begin{bmatrix} 0 & 12 & -4 \\ -12 & 0 & 3 \\ 4 & -3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix} \tag{e}$$

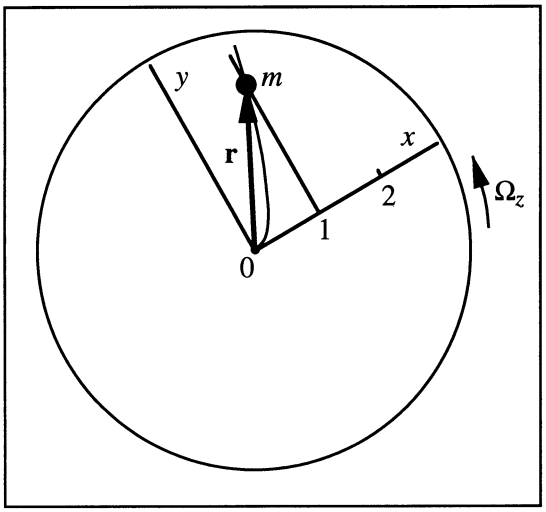
gives

$$\dot{\mathbf{e}}_x = [\,0 \;\; 12 \;\; -4\,] \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix} = [\,\mathbf{e}_x \;\; \mathbf{e}_y \;\; \mathbf{e}_z\,] \begin{bmatrix} 0 \\ 12 \\ -4 \end{bmatrix} \tag{f}$$

as found in equation (c).

Problems

5.1.1 A disk is rotating about its z-axis at $\Omega_z = 0.2$ rad/s = constant. A point mass m slides along a slot in the disk whose shape is given by $y = 2x^2$. At the moment when it passes the point $x = 1$, the point mass has a relative velocity component $\dot{x} = 0.3$ m/s = constant. Determine (a) the absolute velocity $\dot{\mathbf{r}}$, and (b) the absolute acceleration $\ddot{\mathbf{r}}$ of the point mass at that moment, using $\dot{\mathbf{v}} = \overset{\circ}{\mathbf{v}} + \boldsymbol{\Omega} \times \mathbf{v}$.



5.1.2 A rotating refrence frame with its origin coincident with that of a fixed or inertial frame has the angular velocity (relative to the fixed frame) $\boldsymbol{\Omega} = t\,\mathbf{e}_x - t^2\,\mathbf{e}_y + (3t + 2)\,\mathbf{e}_z$, where t is the time. The position vector of a particle of time t as observed in the rotating frame is given by $\mathbf{r} = (t^2 + 1)\,\mathbf{e}_x - 5t\,\mathbf{e}_y + 3t^3\,\mathbf{e}_z$. Find (a) the apparent velocity $\overset{\circ}{\mathbf{r}}$ and acceleration $\overset{\circ\circ}{\mathbf{r}}$ and (b) absolute velocity $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$ at time $t = 1$.

5.2 Momentum Velocity Relation

Suppose one point of a rigid body is fixed in space limiting the body's motion to one of pure rotation. Let a reference frame $Ox_1y_1z_1$ be attached to the body and let the angular velocity of the body-fixed frame be denoted by $\boldsymbol{\omega}$.

Let us now relate the angular momentum vector $\mathbf{H}$, about the fixed point, to the angular velocity vector $\boldsymbol{\omega}$ of the body. By definition

$$\mathbf{H} = \int_m \mathbf{r} \times \dot{\mathbf{r}}\, dm \tag{5.2.1}$$

Now since the position of each material element dm is fixed relative to the frame attached to the body, $\overset{\circ}{\mathbf{r}}$ vanishes in the expression for $\dot{\mathbf{r}}$ as given in equation (5.1.10) and we can write the above expression for angular momentum as

$$\mathbf{H} = \int_m \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) dm \tag{5.2.2}$$

From vector algebra we have that

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega} (\mathbf{r} \cdot \mathbf{r}) - \mathbf{r} (\boldsymbol{\omega} \cdot \mathbf{r}) \tag{5.2.3}$$

which, on expansion allows us to write

$$\begin{aligned} \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = & \left\{ \omega_{x_1} (y_1^2 + z_1^2) - \omega_{y_1} x_1 y_1 - \omega_{z_1} x_1 z_1 \right\} \mathbf{e}_{x_1} \\ & + \left\{ \omega_{y_1} (x_1^2 + z_1^2) - \omega_{x_1} x_1 y_1 - \omega_{z_1} y_1 z_1 \right\} \mathbf{e}_{y_1} \\ & + \left\{ \omega_{z_1} (x_1^2 + y_1^2) - \omega_{x_1} x_1 z_1 - \omega_{y_1} y_1 z_1 \right\} \mathbf{e}_{z_1} \end{aligned} \tag{5.2.4}$$

On substituting into equation (5.2.2) and carrying out the integration we can write

$$\mathbf{H} = [\, \mathbf{e}_{x_1} \;\; \mathbf{e}_{y_1} \;\; \mathbf{e}_{z_1} \,] \begin{bmatrix} I_{x_1x_1} & I_{x_1y_1} & I_{x_1z_1} \\ I_{x_1y_1} & I_{y_1y_1} & I_{y_1z_1} \\ I_{x_1z_1} & I_{y_1z_1} & I_{z_1z_1} \end{bmatrix} \begin{bmatrix} \omega_{x_1} \\ \omega_{y_1} \\ \omega_{z_1} \end{bmatrix} \tag{5.2.5}$$

where

$$\begin{aligned} I_{x_1x_1} &= \int_m (y_1^2 + z_1^2)\, dm & I_{x_1y_1} &= -\int_m x_1 y_1\, dm = I_{y_1x_1} \\ I_{y_1y_1} &= \int_m (z_1^2 + x_1^2)\, dm & I_{y_1z_1} &= -\int_m y_1 z_1\, dm = I_{z_1y_1} \\ I_{z_1z_1} &= \int_m (x_1^2 + y_1^2)\, dm & I_{x_1z_1} &= -\int_m x_1 z_1\, dm = I_{z_1x_1} \end{aligned}$$

or in matrix notation

$$\{H\} = [\, I \,] \{\omega\} \tag{5.2.6}$$

The quantities $I_{x_1x_1}$, $I_{y_1y_1}$, $I_{z_1z_1}$ are called the *moments of inertia* about the x_1, y_1, z_1 axes respectively. The quantities $I_{x_1y_1}$, $I_{y_1z_1}$, $I_{x_1z_1}$ are called *products of inertia.*

Equation (5.2.6) is in essence the generalisation of the more familiar constitutive equation $\mathbf{B} = m\dot{\mathbf{r}}$ for a single particle. We see that the rotational inertia of a body is not scalar but rather tensorial in nature. Hence the inertia matrix $[\, I \,]$ is reference frame dependent and in general the angular velocity and angular momentum vectors have different directions. It is natural to enquire under what conditions will these vectors,

namely $\boldsymbol{\omega}$ and **H**, have the same directions and will differ only in magnitude. Under such conditions $\mathbf{H} = \lambda\boldsymbol{\omega}$, where λ is a scaler and equation (5.2.6) becomes

$$[I]\{\omega\} = \lambda\{\omega\} \tag{5.2.7}$$

This is a linear eigenvalue equation with solutions provided by

$$\begin{vmatrix} I_{x_1x_1}-\lambda & I_{x_1y_1} & I_{x_1z_1} \\ I_{x_1y_1} & I_{y_1y_1}-\lambda & I_{y_1z_1} \\ I_{x_1z_1} & I_{y_1z_1} & I_{z_1z_1}-\lambda \end{vmatrix} = 0 \tag{5.2.8}$$

The vanishing of the above determinant will yield values for the three eigenvalues and the substitution of these into equation (5.2.7) will provide the associated eigenvectors. These latter vectors identify the principal axes of the body and one can show that a transformation from the x_1, y_1, z_1 axes to the principal axes $x\ y\ z$ of the body - diagonalises the inertia matrix allowing one to write the momentum velocity relation for a rotating body in the form

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{5.2.9}$$

Where A, B, C are the principal moments of inertia which are the eigenvalues of equation (5.2.7).

From their definitions one can show that certain restrictive conditions apply to the inertia elements. These include

$$I_{x_1x_1} + I_{y_1y_1} \geq I_{z_1z_1} \tag{5.2.10a}$$

$$I_{y_1y_1} + I_{z_1z_1} \geq I_{x_1x_1} \tag{5.2.10b}$$

$$I_{z_1z_1} + I_{x_1x_1} \geq I_{y_1y_1} \tag{5.2.10c}$$

These conditions require that $I_{x_1x_1} = I_{y_1y_1}$ when $I_{z_1z_1} = 0$. They also require if $I_{z_1z_1}$ is the largest of the three and $I_{x_1x_1} = I_{y_1y_1}$, then $I_{z_1z_1} > I_{x_1x_1} = I_{y_1y_1} > \frac{1}{2}I_{z_1z_1}$. Since $(y_1 - z_1)^2 \geq 0$, we find that $y_1^2 + z_1^2 \geq 2y_1z_1$, and consequently

$$I_{x_1x_1} \geq 2\,|I_{y_1z_1}| \tag{5.2.11a}$$

$$I_{y_1y_1} \geq 2\,|I_{x_1z_1}| \tag{5.2.11b}$$

$$I_{z_1z_1} \geq 2\,|I_{x_1y_1}| \tag{5.2.11c}$$

Equations (5.2.10) also apply of course to the principal inertia moments. Consequently

$$A + B \geq C \tag{5.2.12a}$$

$$B + C \geq A \tag{5.2.12b}$$

$$C + A \geq B \tag{5.2.12c}$$

According to the Huygens-Steiner† parallel axes theorem, the inertia tensor $[\, I \,]_0$ for an $Ox_1y_1z_1$ coordinate system parallel to a $C\xi_1\eta_1\zeta_1$ coordinate system with origin at the mass center C (Figure 5.3), which need not be a principal axes system, can be obtained from

$$[\, I \,]_0 = [\, I \,]_C + m \begin{bmatrix} (\bar{y}_1^2 + \bar{z}_1^2) & -\bar{x}_1 \bar{y}_1 & -\bar{x}_1 \bar{z}_1 \\ -\bar{x}_1 \bar{y}_1 & (\bar{z}_1^2 + \bar{x}_1^2) & -\bar{y}_1 \bar{z}_1 \\ -\bar{x}_1 \bar{z}_1 & -\bar{y}_1 \bar{z}_1 & (\bar{x}_1^2 + \bar{y}_1^2) \end{bmatrix} \tag{5.2.13}$$

For a reference frame attached to the body the inertia tensor will be constant, and then the time derivative of the angular momentum becomes,

$$\dot{\mathbf{H}} = \{ \mathbf{e}_i \}^T [\, I_{ij} \,] \{ \dot{\omega}_j \} + \{ \dot{\mathbf{e}}_i \}^T [\, I_{ij} \,] \{ \omega_j \} \tag{5.2.14}$$

Analogous to equation (5.1.8), we write the first term as $\overset{\circ}{\mathbf{H}}$ ("H circle"), and the second term can be shown to be

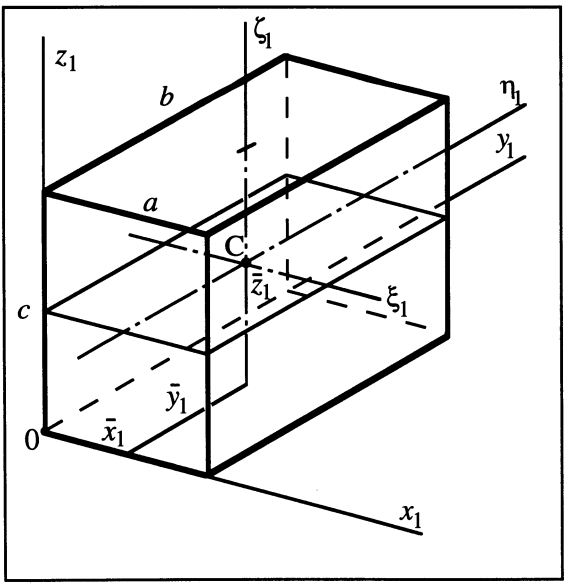


Figure 5.3 Parallelepiped

† Christiaan Huygens (1629-1695), Dutch physicist and mathematician
Jakob Steiner (1796-1863), Swiss mathematician

$$\{ \dot{\mathbf{e}}_i \}^T [I_{ij}] \{ \omega_j \} = \boldsymbol{\omega} \times \mathbf{H} \tag{5.2.15}$$

such that

$$\dot{\mathbf{H}} = \overset{\circ}{\mathbf{H}} + \boldsymbol{\omega} \times \mathbf{H} \tag{5.2.16}$$

Example 5.2.1

Let us establish the inertia moments and inertia products for the parallelepiped of uniform mass distribution of Figure 5.3.

$$\int_m x_1^2 \, dm = \int_0^a x_1^2 \, \rho \, bc \, dx_1 = \frac{1}{3} \rho \, a^3 \, bc = \frac{1}{3} ma^2 \tag{a}$$

$$\int_m y_1^2 \, dm = \frac{1}{3} mb^2 \qquad \int_m z_1^2 \, dm = \frac{1}{3} mc^2 \tag{b,c}$$

$$\int_m x_1 y_1 \, dm = \int_0^b \int_0^a x_1 y_1 \, \rho c \, dx_1 \, dy_1 = \frac{1}{2} \rho a^2 c \int_0^b y_1 \, dy_1 = \frac{1}{4} \rho a^2 b^2 c = \frac{1}{4} m \, ab \tag{d}$$

$$\int_m x_1 z_1 \, dm = \frac{1}{4} mac \qquad \int_m y_1 z_1 \, dm = \frac{1}{4} mbc \tag{e,f}$$

where ρ is the density and $m = \rho abc$ is the mass of the parallelepiped. Collecting terms

$$[I_{ij}] = m \begin{bmatrix} \frac{1}{3}(b^2 + c^2) & -\frac{1}{4} ab & -\frac{1}{4} ac \\ -\frac{1}{4} ab & \frac{1}{3}(a^2 + c^2) & -\frac{1}{4} bc \\ -\frac{1}{4} ac & -\frac{1}{4} bc & \frac{1}{3}(a^2 + b^2) \end{bmatrix} \tag{g}$$

Note that the result applies for the body *and* the coordinate system used.

Example 5.2.2

For the parallelepiped of example 5.4.1 let us determine the principal axes and the principal moments of inertia by taking the dimensions $a = b = c$.

The inertia matrix in equation (g) can now be evaluated numerically as

$$[I_{ij}] = \frac{ma^2}{3} \begin{bmatrix} 2 & -\frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{4} & 2 & -\frac{3}{4} \\ -\frac{3}{4} & -\frac{3}{4} & 2 \end{bmatrix}$$

The eigenvalues and the associated eigenvectors of this matrix can be readily computed as

$$\lambda_1 = 0.917 \, ma^2 \qquad [0.707 \quad -0.707 \quad 0.0 \ ]$$

$$\lambda_2 = 0.917\, ma^2 \qquad [\ 0.408 \quad 0.408 \quad -0.816\ ]$$

$$\lambda_3 = 0.170\, ma^2 \qquad [\ 0.577 \quad 0.577 \quad 0.577\ ]$$

where the vectors have been normalised to have unit lengths. The three eigenvalues are the principal inertias A, B, and C and the eigenvectors identify the principal directions. In this example one principal direction is along the diagonal of the cube and the remaining two are in a plane orthogonal to the diagonal. Since the first two eigenvalues are equal, any linear combination of their eigenvectors is also a possible eigenvector for these eigenvalues. When one eigenvector is fixed in this plane the second one may be constructed so that it is orthogonal to all the other eiganvectors.

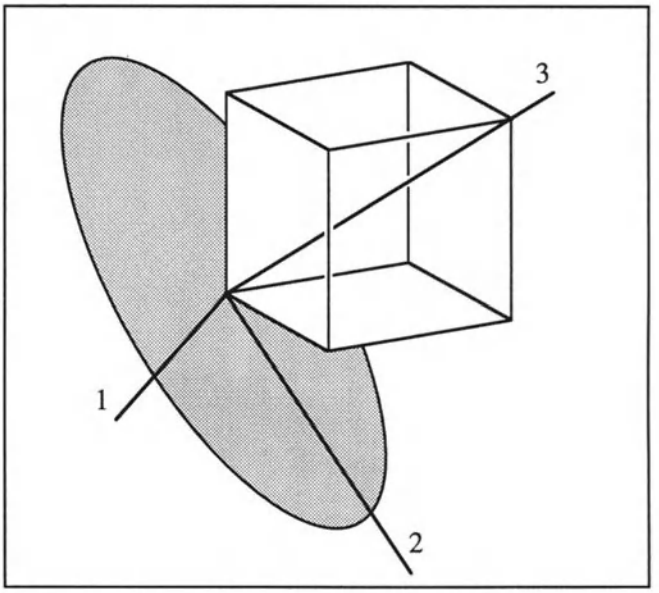


Figure 5.4 Principal axes of a cube

Example 5.2.3

Let us establish the inertia tensor and apply the parallel axes theorem for the parallelepiped shown in Figure 5.3. For the $C\xi\eta\zeta$ coordinate system shown

$$\int \xi^2\, dm = \int_{-a/2}^{+a/2} \xi^2\, \rho\, bc\, d\xi = \frac{1}{3}\, \rho\, \xi^3\, bc \Big|_{-a/2}^{+a/2} = \frac{1}{12}\, ma^2 \qquad \text{(a)}$$

$$\int \eta^2\, dm = \frac{1}{12}\, mb^2 \qquad \text{and} \qquad \int \zeta^2\, dm = \frac{1}{12}\, mc^2 \qquad \text{(b,c)}$$

Collecting terms, and realizing that all inertia products vanish,

$$[I]_C = \begin{bmatrix} \frac{m}{12}(b^2+c^2) & 0 & 0 \\ 0 & \frac{m}{12}(c^2+a^2) & 0 \\ 0 & 0 & \frac{m}{12}(a^2+b^2) \end{bmatrix} \tag{d}$$

The inertia tensor for a parallel coordinate system $Ox_1y_1z_1$ is given by equation (5.2.13). Since $\bar{x}_1 = a/2, \bar{y}_1 = b/2$ and $\bar{z}_1 = c/2$, the second term of equation (5.2.13) becomes

$$\begin{bmatrix} \frac{m}{4}(b^2+c^2) & -\frac{m}{4}ab & -\frac{m}{4}ac \\ -\frac{m}{4}ab & \frac{m}{4}(c^2+a^2) & -\frac{m}{4}bc \\ -\frac{m}{4}ac & -\frac{m}{4}bc & \frac{m}{4}(a^2+b^2) \end{bmatrix} \tag{e}$$

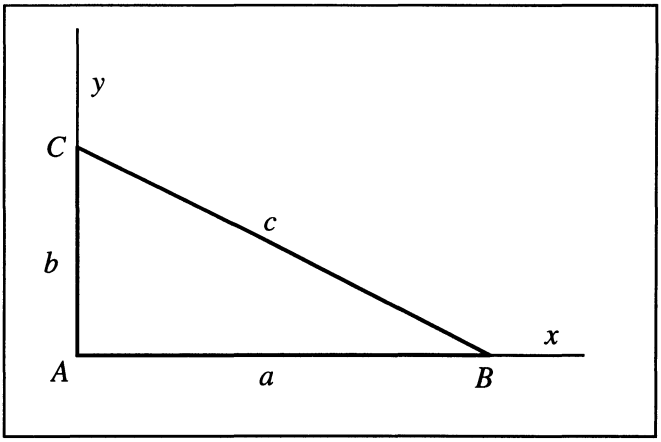
Adding the matrices (e) and (d) results in

$$[I]_0 = m \begin{bmatrix} \frac{1}{3}(b^2+c^2) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}(a^2+c^2) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}(a^2+b^2) \end{bmatrix} \tag{f}$$

i.e. the result (g) of Example 5.2.1.

Problems

5.2.1 Determine the (a) principal moments of inertia and (b) directions of the principal axes for the right triangle ABC.



5.2.2 A block of uniform mass distribution has a mass of 600 kg and side lengths $a = 1$ m, $b = 1.5$ m, $c = 2$ m. Place a $Cxyz$ coordinate system along the

principal axes, and determine the angular momentum if $\omega_x = 3, \omega_y = 4, \omega_z = 12$ rad/s. If this angular velocity (which also represents the axis of rotation) remains constant (in space), what is the time derivative of the angular momentum?

5.3 Rotational Kinetic Energy and Kinetic Coenergy

The complementary kinetic energy of an arbitrary mass element *dm* is

$$dT^* = \frac{1}{2}\, dm\, |\dot{\mathbf{r}}|^2 \tag{5.3.1}$$

For the case that the absolute velocity of the origin O of the coordinate system is zero, the velocity $\dot{\mathbf{r}}$ of the mass element *dm* is given by

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} \tag{5.3.2}$$

The *complementary kinetic energy* (1.8.4) of rotation of the whole body is then

$$T^* = \frac{1}{2} \int_m |\, \boldsymbol{\omega} \times \mathbf{r}\, |^2\, dm\ . \tag{5.3.3}$$

With an angular velocity $\boldsymbol{\omega}$ of the rigid body given by

$$\boldsymbol{\omega} = [\mathbf{e}_{x_1}\ \mathbf{e}_{y_1}\ \mathbf{e}_{z_1}] \begin{bmatrix} \omega_{x_1} \\ \omega_{y_1} \\ \omega_{z_1} \end{bmatrix} \tag{5.3.4}$$

and a position $\mathbf{r}$ of a mass element *dm* given by

$$\mathbf{r} = [\mathbf{e}_{x_1}\ \mathbf{e}_{y_1}\ \mathbf{e}_{z_1}] \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \tag{5.3.5}$$

the complementary kinetic energy becomes

$$T^* = \frac{1}{2} (I_{x_1x_1}\omega_{x_1}^2 + I_{y_1y_1}\omega_{y_1}^2 + I_{z_1z_1}\omega_{z_1}^2 + 2I_{x_1y_1}\omega_{x_1}\omega_{y_1} + 2I_{x_1z_1}\omega_{x_1}\omega_{z_1} + 2I_{y_1z_1}\omega_{y_1}\omega_{z_1})$$

or

$$T^* = \frac{1}{2} \{\omega\}^T [\, I\,] \{\omega\} \tag{5.3.6}$$

In case the $Ox_1y_1z_1$ coordinate system coincides with the principal axes, the inertia matrix diagonalises and the complementary kinetic energy becomes

$$T^* = \frac{1}{2} (A\omega_x^2 + B\omega_y^2 + C\omega_z^2)\ . \tag{5.3.7}$$

Since, $\omega_x = H_x/A$, $\omega_y = H_y/B$, and $\omega_z = H_z/C$, and in Newtonian Mechanics $T = T^*$, we may obtain an expression for the *kinetic energy* from equation (5.3.7) as

$$T = \frac{1}{2} \left(\frac{H_x^2}{A} + \frac{H_y^2}{B} + \frac{H_z^2}{C} \right) \tag{5.3.8}$$

One may also write

$$T + T^* = \boldsymbol{\omega} \cdot \mathbf{H} \tag{5.3.9}$$

analogous to equation (1.8.12).

5.4 Euler Equations

Euler's momentum law is a statement of equilibrium between the torque $\mathbf{M}$ and the angular momentum rate $\dot{\mathbf{H}}$ of a body in rotational motion The vectors $\mathbf{M}$ and $\dot{\mathbf{H}}$ are measured from a fixed point about which the body rotates. Thus

$$\mathbf{M} = \dot{\mathbf{H}} \tag{5.4.1}$$

Using equation (5.2.16) we can express equation (5.4.1) in component form as follows

$$M_{x_1} = \dot{H}_{x_1} + \omega_{y_1} H_{z_1} - \omega_{z_1} H_{y_1} \tag{5.4.2a}$$

$$M_{y_1} = \dot{H}_{y_1} + \omega_{z_1} H_{x_1} - \omega_{x_1} H_{z_1} \tag{5.4.2b}$$

$$M_{z_1} = \dot{H}_{z_1} + \omega_{x_1} H_{y_1} - \omega_{y_1} H_{x_1} \tag{5.4.2c}$$

We now invoke the constitutive equations that relate the components of $\mathbf{H}$ to those of $\boldsymbol{\omega}$. These equations take their simplest form in the body-fixed principal coordinate system. Then

$$H_{x_1} = H_x = A\,\omega_x \quad H_{y_1} = H_y = B\,\omega_y \quad H_{z_1} = H_z = C\,\omega_z \tag{5.4.3a,b,c}$$

On substituting from equation (5.4.3) into equation (5.4.2) and using the components of $\mathbf{M}$ in the principal coordinate ssytem we arrive at Euler's celebrated equations.

$$M_x = A\,\dot{\omega}_x - (B - C)\,\omega_y\,\omega_z \tag{5.4.4a}$$

$$M_y = B\,\dot{\omega}_y - (C - A)\,\omega_z\,\omega_x \tag{5.4.4b}$$

$$M_z = C\,\dot{\omega}_z - (A - B)\,\omega_x\,\omega_y \tag{5.4.4c}$$

Through equation (5.4.3) we may eliminate the angular velocities and express Euler's equations also in terms of angular as

$$M_x = \dot{H}_x - \frac{B - C}{BC} H_y\,H_z \tag{5.4.5a}$$

$$M_y = \dot{H}_y - \frac{C - A}{AC} H_z\,H_x \tag{5.4.5b}$$

$$M_z = \dot{H}_z - \frac{A - B}{AB} H_x\,H_y \tag{5.4.5c}$$

Although Euler's equations do not admit general closed form solutions we can deduce two integrals of motions for the special case of torque-free motions. Thus letting $\mathbf{M} = 0$ and multiplying equations (5.4.5a, b, c) by H_x, H_y, H_z respectively and adding them we find

$$\dot{H}_x H_x + \dot{H}_y H_y + \dot{H}_z H_z = 0 \tag{5.4.6}$$

or
$$\frac{d}{dt}\left[\frac{H_x^2}{2} + \frac{H_y^2}{2} + \frac{H_z^2}{2}\right] = 0 \tag{5.4.7}$$

i.e.
$$H_x^2 + H_y^2 + H_z^2 = \text{constant} \tag{5.4.8}$$

This equation shows that under torque-free conditions the magnitude of the angular momentum vector is constant. The direction of this vector is also constant as one can see from equation (5.4.1) which shows that under torque-free conditions **H** will remain a constant in the inertial or space-fixed coordinate system.

Next multiplying equations (5.4.5 a, b, c) by H_x/A, H_y/B, H_z/C respectively and adding we can deduce in a similar manner that

$$\frac{1}{2}\left[\frac{H_x^2}{A} + \frac{H_y^2}{B} + \frac{H_z^2}{C}\right] = \text{constant} \tag{5.4.9}$$

i.e. the kinetic energy of motion of a rigid body is also conserved under torque-free conditions. Since in Newtonian Mechanics $T = T^*$ we can see that kinetic coenergy is also conserved. It then follows from Equation (5.3.9) that

$$\boldsymbol{\omega} \cdot \mathbf{H} = \text{constant}$$

This means that the projection of $\boldsymbol{\omega}$ on **H** is constant, so that the terminal point of $\boldsymbol{\omega}$ destribes a plane - the so called invariable plane (see Figure 5.5). If the body rotates

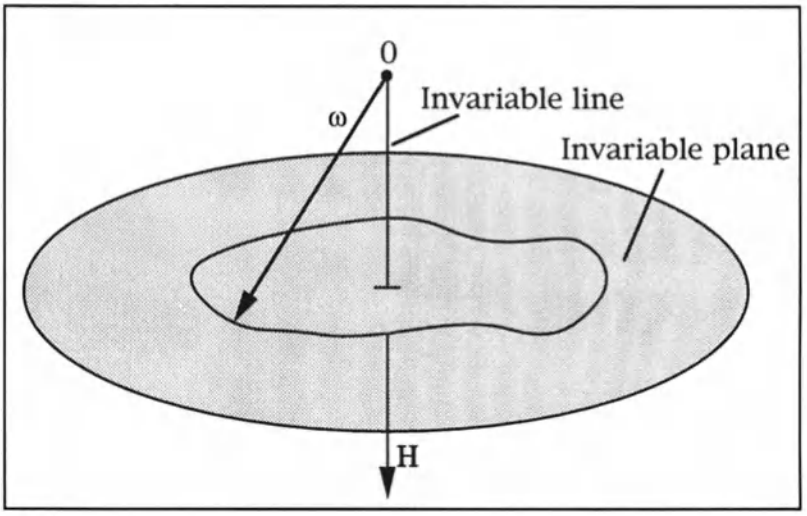


Figure 5.5 Invariable plane

about a principal axis, $\boldsymbol{\omega}$ and **H** will be parallel and both will remain constant as long as no torque acts. However if the initial motion of the body is such that the direction of $\boldsymbol{\omega}$ is different from that of the fixed (in the inertial system) vector **H**, then $\boldsymbol{\omega}$ will

not necessarily be a constant and the terminal point of vector ω will be confined to the invariable plane.

The conservation equations (5.4.8) and (5.4.9) describe spherical and ellipsoidal surfaces respectively. Thus for torque-free motions we can picture the terminal point of the momentum vector **H** to remain on the intersection of the sphere and the ellipsoid described by these equations.

Example 5.4.1

Let us examine the constant angular velocity motion, about the principal axes, under torque-free conditions. The following motions evidently satisfy Euler's equations, under the stated conditions

$$\omega_x = \omega_{xo} \qquad \omega_y = 0 \qquad \omega_z = 0\,, \tag{a}$$

$$\omega_y = \omega_{yo} \qquad \omega_x = 0 \qquad \omega_z = 0\,, \tag{b}$$

$$\omega_z = \omega_{zo} \qquad \omega_x = 0 \qquad \omega_y = 0\,, \tag{c}$$

Where ω_{xo}, ω_{yo}, ω_{zo} are constants. Indeed these are the only possible motions for the case of $\boldsymbol{\omega}$ = constant, when the principal moments of inertia A, B, C are unequal. This can be seen by substituting $\omega_x = \omega_{xo}$, $\omega_y = \omega_{yo}$ and $\omega_z = \omega_{zo}$ in Euler's equations (5.4.4) and noting that under torque-free conditions at least two of the three constant angular velocities must vanish for a solution to exist.

Next let us examine the stability of the motion described in equation (a). To this end we consider small perturbation about the steady motion by letting

$$\omega_x = \omega_{xo} + \alpha \qquad \omega_y = \beta \qquad \omega_z = \gamma \tag{d}$$

Substituting these into Euler's equations for torque-free motions and neglecting quadratic terms in α, β, γ we find

$$A\dot{\alpha} = 0 \tag{e}$$

$$B\dot{\beta} + (A - C)\,\omega_{xo}\,\gamma = 0 \tag{f}$$

$$C\dot{\gamma} + (B - A)\omega_{xo}\,\beta = 0 \tag{g}$$

The first equation shows that α = constant and with $\gamma = \beta = 0$ we obtain, once again a solution of type (a) but with slightly changed angular velocity around the x axis. To solve the last two equations let us substitute for β from equation (g) into equation (f). This will provide the equation

$$BC\ddot{\gamma} + (A - C)(A - B)\,\omega_{xo}^2\,\gamma = 0 \tag{h}$$

This linear equation will be recognised as that of an oscillator with inertial properties BC and stiffness coefficient $(A - C)(A - B)\,\omega_{xo}^2$. Evidently the motion will be bounded (stable) if

$$(A - C)(A - B) > 0 \tag{i}$$

That is when A is either the largest or the smallest of the three principal moments of inertia. If A is the intermediate principal inertia the motion will grow exponentially (unstable).

Example 5.4.2

Consider another special case of torque-free motion for a gyro for which $A = B \neq C$. Then equation (5.4.4c) shows that

$$\omega_z = \omega_{z0} = \text{constant} \tag{a}$$

Equations (5.4.4a,b) when $A = B$ then become

$$A\dot{\omega}_x + (C - A)\,\omega_y\,\omega_{z0} = 0 \tag{b}$$

$$A\dot{\omega}_y + (A - C)\,\omega_x\,\omega_{z0} = 0 \tag{c}$$

These equations also describe an oscillatory motion given by

$$\omega_x = \omega_{x0}\cos[(C - A)\,\omega_{z0}t/A] \tag{d}$$

$$\omega_y = \omega_{y0}\sin[(C - A)\,\omega_{z0}t/A] \tag{e}$$

The motion is that of a rotation of vector ω around the z axis.

The earth is a good example of a body with $A = B$. With $(C - A)/A$ of about 1/300 and ω_{z0} is cycle/day the precessional period is of the order of one year.

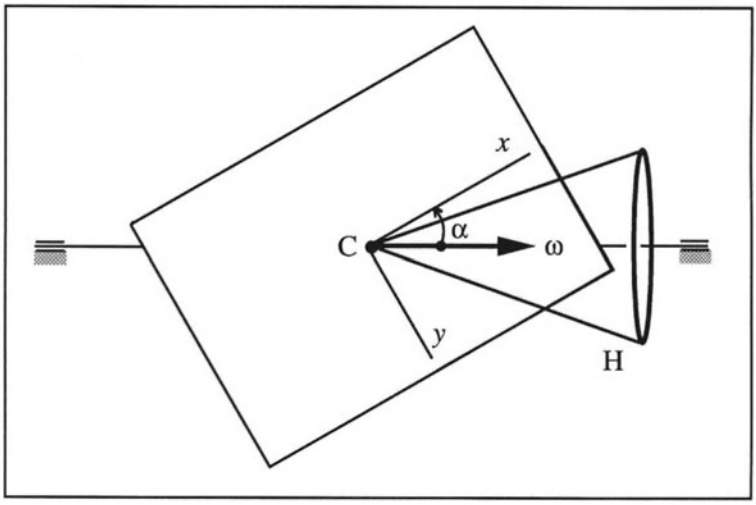


Figure 5.6 Rotating plate

Example 5.4.3

The plate gyro ($B = 2A$) shown in Figure 5.6 has a known constant angular velocity ω. Determine the torque acting on the plate.

The plate's angular velocity is

$$\omega = [\,\mathbf{e}_x\ \mathbf{e}_y\ \mathbf{e}_z\,]\begin{bmatrix}\omega\cos\alpha\\ \omega\sin\alpha\\ 0\end{bmatrix} \tag{a}$$

The angular velocity is constant, thus

$$\dot{\boldsymbol{\omega}} = \mathbf{0} \tag{b}$$

From Euler's equations (5.4.4), we find for the torque components

$$M_x = 0 \tag{c}$$

$$M_y = 0 \tag{d}$$

$$M_z = -(A - B)\,\omega_x\,\omega_y = A\,\omega^2 \sin\alpha \cos\alpha\,. \tag{e}$$

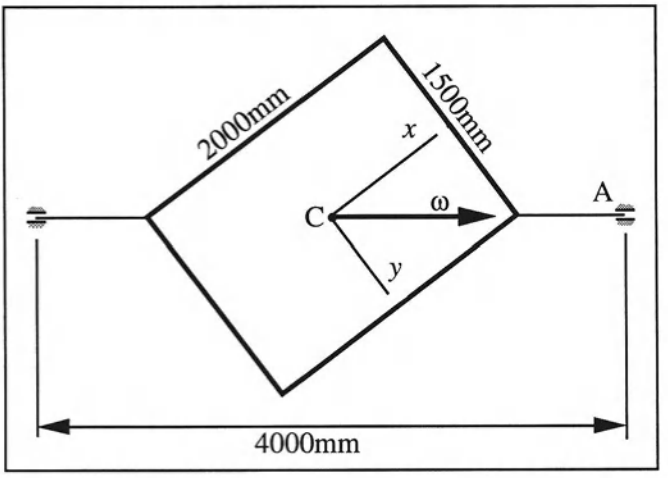
The angular momentum vector is

$$\mathbf{H} = [\mathbf{e}_x\ \mathbf{e}_y\ \mathbf{e}_z] \begin{bmatrix} A\,\omega\cos\alpha \\ B\,\omega\sin\alpha \\ 0 \end{bmatrix} \tag{f}$$

and can be seen to describe a cone.

Problem

5.4.1 A solid oak ($\rho = 800$ kg/m^3) plate is 5 cm thick and is mounted on massless rods as shown. It rotates at 28.65 rev/min. Determine the force applied to the rod at bearing A (disregarding the plate's weight force).



5.5 Euler Angles

Euler equations furnish solutions for angular velocities of a rigid body when subjected to a set of torques. These equations however cannot always provide information on the *angular positions* of a rigid body. It is tempting to think that angular positions may be determined through time integration of angular velocities. This however is only possible for the special case of planar motion. For the general case of three dimensional motion angular velocities are not integrable. We may of course represent the angular velocity vector as

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt} \tag{5.5.1}$$

and note that $d\boldsymbol{\theta}$ is a differential vector but it is not the differential of a vector $\boldsymbol{\theta}$.

A variety of generalised coordinates may be used to describe the angular position of a gyro. The most frequently used coordinates are three angles, the so called *Euler angles*. Since these angular coordinates do not obey the vector addition law, the axes about which these angles are measured must be carefully noted. Unfortunately there exists no uniformity in the literature on the symbols for these angles. We will express the Euler angles in mnemonic symbols as follows

$$\psi = \textit{precession angle}$$

$$\nu = \textit{nutation angle}$$

$$\sigma = \textit{spin angle}$$

The sequence of rotations which carries a frame from $Cx_0y_0z_0$, initially coincident with the space-fixed axes $CXYZ$, to $Cx_3y_3z_3$, may be described as follows (see Figure 5.7). First a rotation ψ is carried out about the Z axis. This brings the axes

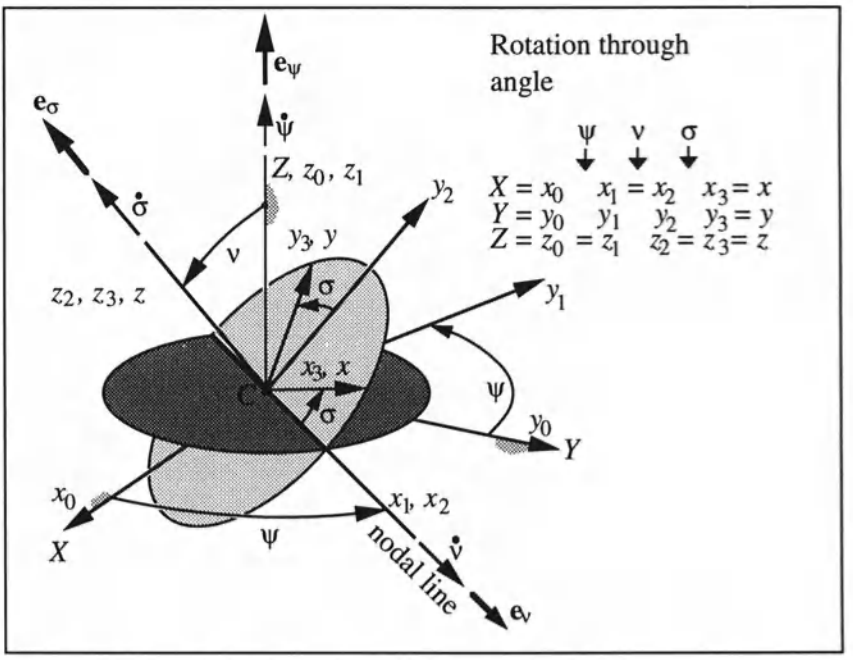


Figure 5.7 Gyro axes $Cxyz$ rotated through Euler angles

in coincidence with the intermediate axes $Cx_1y_1z_1$ with z_1 and Z axis coincident and x_1 along the line of intersection of XY and xy planes - the so called *nodal* line. Next a rotation ν is carried out about the x_1 axis. This sets up $Cx_2y_2z_2$ axes with x_2 and x_1 axes coincident. Finally a rotation σ is carried out about the z_2 axis which brings the

moving frame to $Cx_3y_3z_3$, coincident with the body-fixed axes $Cxyz$.

Conversely, if the body-fixed frame is in a given orientation the corresponding Euler angles may be determined as follows. The angle between the z and Z axis is the nutation angle ν. Once ν is determined, the nodal line may be located and the ψ and σ may then be measured from the nodal line to the X axis and the x axis respectively.

For computational purposes the various rotations are best described through matrix multiplications. Thus a rotation through the precession angle ψ transforms the basis vectors $\mathbf{e}_{x_0}$, $\mathbf{e}_{y_0}$, $\mathbf{e}_{z_0}$ of the $Cx_0y_0z_0$ axes to those of the $Cx_1y_1z_1$ axes as follows

$$\begin{bmatrix} \mathbf{e}_{x_1} \\ \mathbf{e}_{y_1} \\ \mathbf{e}_{z_1} \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_{x_0} \\ \mathbf{e}_{y_0} \\ \mathbf{e}_{z_0} \end{bmatrix} \tag{5.5.2}$$

or

$$\{ \mathbf{e} \}_1 = [R]_{1\leftarrow 0} \{ \mathbf{e} \}_0 \tag{5.5.3}$$

where $[R]$ is called the rotation matrix. For three successive rotations one obtains

$$\{ \mathbf{e} \}_3 = [R]_{3\leftarrow 2} [R]_{2\leftarrow 1} [R]_{1\leftarrow 0} \{ \mathbf{e} \}_0 \tag{5.5.4}$$

with the remaining rotation matrices

$$[R]_{2\leftarrow 1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\nu & \sin\nu \\ 0 & -\sin\nu & \cos\nu \end{bmatrix} \tag{5.5.5}$$

$$[R]_{3\leftarrow 2} = \begin{bmatrix} \cos\sigma & \sin\sigma & 0 \\ -\sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.5.6}$$

We may write equation (5.5.4) as

$$\{ \mathbf{e} \}_3 = [R]_{3\leftarrow 0} \{ \mathbf{e} \}_0 \tag{5.5.7}$$

where

$$[R]_{3\leftarrow 0} = [R]_{3\leftarrow 2} [R]_{2\leftarrow 1} [R]_{1\leftarrow 0} \tag{5.5.8}$$

is given by

$$[R]_{3\leftarrow 0} = \begin{bmatrix} \cos\psi\cos\sigma - \sin\psi\cos\nu\sin\sigma & \sin\psi\cos\sigma + \cos\psi\cos\nu\sin\sigma & \sin\nu\sin\sigma \\ -\cos\psi\sin\sigma - \sin\psi\cos\nu\cos\sigma & -\sin\psi\sin\sigma + \cos\psi\cos\nu\cos\sigma & \sin\nu\cos\sigma \\ \sin\psi\sin\nu & -\cos\psi\sin\nu & \cos\nu \end{bmatrix} \tag{5.5.9}$$

Since the above rotation matrices relate the basis vectors of orthogonal frames they are orthogonal matrices with the property that

$$[R]^T = [R]^{-1} \tag{5.5.10}$$

Thus the inverse relation to (5.5.7) may be expressed as

$$\{ \mathbf{e} \}_0 = [R]_{0\leftarrow 3} \{ \mathbf{e} \}_3 \tag{5.5.11}$$

with $$[R]_{0\leftarrow 3} = [R]^T_{3\leftarrow 0}$$ i.e.

$$[R]_{0\leftarrow 3} = \begin{bmatrix} \cos\psi\cos\sigma - \sin\psi\cos\nu\sin\sigma & -\cos\psi\sin\sigma - \sin\psi\cos\nu\cos\sigma & \sin\psi\sin\nu \\ \sin\psi\cos\sigma + \cos\psi\cos\nu\sin\sigma & -\sin\psi\sin\sigma + \cos\psi\cos\nu\cos\sigma & -\cos\psi\sin\nu \\ \sin\nu\sin\sigma & \sin\nu\cos\sigma & \cos\nu \end{bmatrix} \tag{5.5.12}$$

Above development shows that a sequence of finite rotations may be determined via the laws of matrix multiplication and not those of vector addition.

Example 5.5.1

The book experiment is the most convincing, when it comes to demonstrating that the angles θ_x, θ_y and θ_z do not have vector characteristics. Place a book on a table top. Then carry out a 90° rotation about the book's x axis, followed by a 90° rotation about the book's y axis. Note the final position (a) of the book in Figure 5.8.

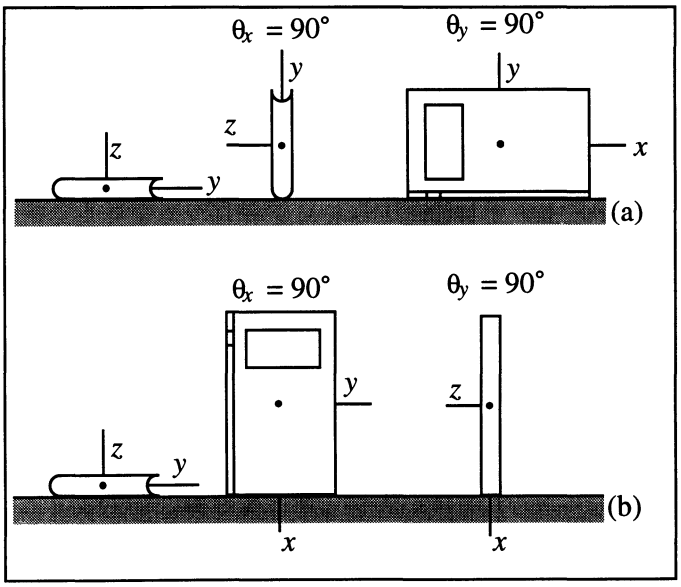


Figure 5.8 Book rotation experiment

Then repeat the experiment beginning with the rotation about the y axis, and note that the final position is now different (b). Thus θ_x and θ_y do not follow the commutative law of vector addition, or in other words θ_x and θ_y are *not* x and y components of a vector.

Example 5.5.2

We may use the vector character of $\boldsymbol{d\theta}$ to obtain the general equations of motion for a rigid body in an inertial frame of reference. To this end we consider a rigid body as a large collection of particles connected together in such a way that the distances between the particles remain constant. With this in mind we can write D'Alembert's principle for the set of particles as

$$(\dot{\mathbf{B}}_i - \mathbf{F}_i) \cdot \delta\mathbf{r}_i = 0 \tag{a}$$

where $\mathbf{B}_i$ are the linear momenta of the particles and $\mathbf{F}_i$ are the externally applied forces acting on the particles. Now we must ensure that $\delta\mathbf{r}_i$ satisfy the kinematic constrains of fixed distances between the particles. This is a very restrictive requirement and allows only two types of virtual displacements:

$$\delta\mathbf{r}_i = \delta\mathbf{r} \qquad \text{pure translation} \tag{b}$$

and

$$\delta\mathbf{r}_i = \delta\boldsymbol{\theta} \times \mathbf{r}_i \qquad \text{pure rotation} \tag{c}$$

substituting from equation (b) into (a) we obtain

$$\left(\sum \dot{\mathbf{B}}_i\right) \cdot \delta\mathbf{r} = \left(\sum \mathbf{F}_i\right) \cdot \delta\mathbf{r} \tag{d}$$

and since $\delta\mathbf{r}$ is arbitrary

$$\sum \dot{\mathbf{B}}_i = \sum \mathbf{F}_i \tag{e}$$

Now we can write this equation as

$$\dot{\mathbf{B}}_{cm} = \mathbf{F}_{res} \tag{f}$$

where

$$\mathbf{B}_{cm} = \sum \mathbf{B}_i = \sum m_i \dot{\mathbf{r}}_i \tag{g}$$

and

$$\mathbf{F}_{res} = \sum \mathbf{F}_i$$

For pure translatory motion $\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_{cm}$ where $\dot{\mathbf{r}}_{cm}$ is the velocity of the body's centre of mass. Then we can express $\mathbf{B}_{cm}$ as

$$\mathbf{B}_{cm} = \left(\sum m_i\right) \dot{\mathbf{r}}_{cm} = \mathrm{M}\, \dot{\mathbf{r}}_{cm} \tag{h}$$

where M is the total mass of the body. It can now be seen that for translatory motion equation (f) expresses the fact that the centre of mass of a rigid body moves as if all the mass was concentrated in that point and all the forces were acting there. Next consider the rotatory motion. We substitute from equation (c) into (a) and obtain

$$\dot{\mathbf{B}}_i \cdot (\delta\boldsymbol{\theta} \times \mathbf{r}_i) = \mathbf{F}_i \cdot (\delta\boldsymbol{\theta} \times \mathbf{r}_i) \tag{i}$$

or

$$(\mathbf{r}_i \times \dot{\mathbf{B}}_i) \cdot \delta\boldsymbol{\theta} = (\mathbf{r}_i \times \mathbf{F}_i) \cdot \delta\boldsymbol{\theta} \tag{j}$$

and again noting that $\delta\boldsymbol{\theta}$ is arbitrary, we can write

$$\dot{\mathbf{H}} = \mathbf{M} \tag{k}$$

where

$$\mathbf{H} = [\mathbf{r}_i \times \mathbf{B}_i] \qquad \text{and} \qquad \mathbf{M} = \mathbf{r}_i \times \mathbf{F}_i \tag{e}$$

H is the angular momentum of the body and **M** is the applied torque. The moment arms $\mathbf{r}_i$ for both **H** and **M**, in equation (e), are from the origin of the inertial axes. Equation (k) will be recognised as Euler's equation of rotatory motion.

Problems

5.5.1 Draw a (constant) **H** vector for a torque-free axisymmetric ($B = A$) gyro. Draw the z axis at an angle ν ("attitude") from the **H** vector. For $H = 120\ \text{kg m}^2/\text{s}$, $A = 24\ \text{m}^2\text{kg}$, $C = 10\ \text{m}^2\text{kg}$, $\nu = 10°$, find ω_z and $\omega_\upsilon = \sqrt{\omega_x^2 + \omega_y^2}$.

5.5.2 A fast-spinning ($\omega_z = \omega_1 =$ constant) axisymmetric ($B = A$) gyro is mounted in massless gimbal rings. The gyro is constrained such that $\psi = \dot{\psi} = 0$ and $\omega_x = -\omega_2 \sin\sigma$ and $\omega_y = \omega_2 \sin\sigma$ with $\omega_2 =$ constant and $\dot{\sigma} = \omega_1$. Determine the torque required to maintain this motion.

5.6 Components of ω, H and M in Different Coordinates Systems

The time derivatives of the Euler angles, ($\dot{\psi}$, $\dot{\nu}$, $\dot{\sigma}$) are called *Euler rates* (or *Euler frequencies*). These rates are also vectorial in nature. If we introduce the basis vectors $\mathbf{e}_\psi$, $\mathbf{e}_\nu$, and $\mathbf{e}_\sigma$ along the directions of these rates we will be able to describe any vector in three dimensional space in terms of these basis vectors. It is however important to note that $\mathbf{e}_\psi$, $\mathbf{e}_\nu$, $\mathbf{e}_\sigma$ are not mutually orthogonal (see Figure 5.7). Now the angular velocity vector $\boldsymbol{\omega}$ may be described in terms of its components in various reference frames. Thus in terms of the body-fixed frame we have

$$\boldsymbol{\omega} = \mathbf{e}_x\,\omega_x + \mathbf{e}_y\,\omega_y + \mathbf{e}_z\,\omega_z \tag{5.6.1}$$

and in terms of the axes defined by the Euler rates we have

$$\boldsymbol{\omega} = \mathbf{e}_\psi\,\dot{\psi} + \mathbf{e}_\nu\,\dot{\nu} + \mathbf{e}_\sigma\,\dot{\sigma} \tag{5.6.2}$$

To relate the components (ω_x, ω_y, ω_z) to the rates ($\dot{\psi}$, $\dot{\nu}$, $\dot{\sigma}$) we express the basis vectors ($\mathbf{e}_\psi$, $\mathbf{e}_\nu$, $\mathbf{e}_\sigma$) in terms of ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$). From Figure 5.7 we can see that

$$\mathbf{e}_\psi = \mathbf{e}_z \cos\nu + \mathbf{e}_{y_2} \sin\nu \tag{5.6.3a}$$

$$\mathbf{e}_\nu = \mathbf{e}_x \cos\sigma - \mathbf{e}_y \sin\sigma \tag{5.6.3b}$$

$$\mathbf{e}_\sigma = \mathbf{e}_z \tag{5.6.3c}$$

where $\mathbf{e}_{y_2}$ is the basis vector along the intermediate axis y_2 and may be related to $\mathbf{e}_x$ $\mathbf{e}_y$ as follows

$$\mathbf{e}_{y_2} = \mathbf{e}_x \sin\sigma + \mathbf{e}_y \cos\sigma \tag{5.6.4}$$

Substituting from equations (5.6.3) and (5.6.4) into equation (5.6.2) and collecting terms associated with $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ and comparing with equation (5.6.1) we find

$$\omega_x = \dot{\psi} \sin\nu \sin\sigma + \dot{\nu} \cos\sigma \tag{5.6.5a}$$

$$\omega_y = \dot{\psi} \sin \nu \cos \sigma - \dot{\nu} \sin \sigma \tag{5.6.5b}$$

$$\omega_z = \dot{\psi} \cos \nu + \dot{\sigma} \tag{5.6.5c}$$

Combined they give the transition relationship

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \sin \nu \sin \sigma & \cos \sigma & 0 \\ \sin \nu \cos \sigma & -\sin \sigma & 0 \\ \cos \nu & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\nu} \\ \dot{\sigma} \end{bmatrix} \tag{5.6.6}$$

Unlike the rotation matrices, the square transition matrix in equation (5.6.6) is not an orthogonal matrix, since as noted earlier ($\mathbf{e}_\psi$, $\mathbf{e}_\nu$, $\mathbf{e}_\sigma$) do not form a mutually orthogonal triad of basis vectors. Nevertheless this transition matrix can be inverted yielding

$$\begin{bmatrix} \dot{\psi} \\ \dot{\nu} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} \dfrac{\sin \sigma}{\sin \nu} & \dfrac{\cos \sigma}{\sin \nu} & 0 \\ \cos \sigma & -\sin \sigma & 0 \\ -\dfrac{\sin \sigma}{\tan \nu} & -\dfrac{\cos \sigma}{\tan \nu} & 1 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{5.6.7}$$

It can be seen that this relationship breaks down when $\nu = 0$, that is when $\mathbf{e}_\sigma$ and $\mathbf{e}_\psi$ coincide and the three basis vectors lie in one plane. This singularity in the Euler angle system can be avoided by defining different angular coordinates but we will not pursue this matter at this juncture.

The angular velocity vector $\boldsymbol{\omega}$ may also be expressed in terms of its components in the space-fixed frame as

$$\boldsymbol{\omega} = \omega_X \, \mathbf{e}_X + \omega_Y \, \mathbf{e}_Y + \omega_Z \, \mathbf{e}_Z \tag{5.6.8}$$

The relationships between the components (ω_X, ω_Y, ω_Z) and the Euler rates ($\dot{\psi}$, $\dot{\nu}$, $\dot{\sigma}$) can be readily established by invoking the rotation matrix $[\, R \,]_{3 \leftarrow 0}$ relating the components of $\boldsymbol{\omega}$ in the body-fixed and space-fixed frames. Thus using the relation

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = [\, R \,]_{3 \leftarrow 0} \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} \tag{5.6.9}$$

in equation (5.6.7) we can deduce that

$$\begin{bmatrix} \dot{\psi} \\ \dot{\nu} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} -\dfrac{\sin \psi}{\tan \nu} & \dfrac{\sin \psi}{\tan \nu} & 1 \\ \cos \psi & \sin \psi & 0 \\ \dfrac{\sin \psi}{\sin \nu} & -\dfrac{\cos \psi}{\sin \nu} & 0 \end{bmatrix} \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} \tag{5.6.10}$$

The inverse of this transition relation is given by

$$\begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} = \begin{bmatrix} 0 & \cos\psi & \sin\psi \sin\nu \\ 0 & \sin\psi & -\cos\psi \sin\nu \\ 1 & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\nu} \\ \dot{\sigma} \end{bmatrix} \tag{5.6.11}$$

The above transformation equations relating the components of the vector $\boldsymbol{\omega}$ in the three coordinate systems:

$(\mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z)$	space - fixed
$(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$	body - fixed
$(\mathbf{e}_\psi, \mathbf{e}_\nu, \mathbf{e}_\sigma)$	Euler axes

apply of course to other vectors such as the angular momentum vector $\mathbf{H}$ and the vector of applied torques $\mathbf{M}$. Thus we can write for instance

$$\begin{bmatrix} H_X \\ H_Y \\ H_Z \end{bmatrix} = \begin{bmatrix} 0 & \cos\psi & \sin\psi \sin\nu \\ 0 & \sin\psi & -\cos\psi \sin\nu \\ 1 & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} H_\psi \\ H_\nu \\ H_\sigma \end{bmatrix} \tag{5.6.12}$$

and

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \sin\nu \sin\sigma & \cos\sigma & 0 \\ \sin\nu \cos\sigma & -\sin\sigma & 0 \\ \cos\nu & 0 & 1 \end{bmatrix} \begin{bmatrix} H_\psi \\ H_\nu \\ H_\sigma \end{bmatrix} \tag{5.6.13}$$

Similarly

$$\begin{bmatrix} M_X \\ M_Y \\ M_Z \end{bmatrix} = \begin{bmatrix} 0 & \cos\psi & \sin\psi \sin\nu \\ 0 & \sin\psi & -\cos\psi \sin\nu \\ 1 & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} M_\psi \\ M_\nu \\ M_\sigma \end{bmatrix} \tag{5.6.14}$$

and

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} \sin\nu \sin\sigma & \cos\sigma & 0 \\ \sin\nu \cos\sigma & -\sin\sigma & 0 \\ \cos\nu & 0 & 1 \end{bmatrix} \begin{bmatrix} M_\psi \\ M_\nu \\ M_\sigma \end{bmatrix} \tag{5.6.15}$$

Inverse relationships are similar to those of the vector $\boldsymbol{\omega}$.

Before closing this section let us reconsider the question of integrability of the angular velocity. To this end we write the first of equation (5.6.6) as

$$\omega_x \, dt = \sin\nu \sin\sigma \, d\psi + \cos\sigma \, d\nu + 0 \cdot d\sigma \tag{5.6.16}$$

and on replacing $\omega_x \, dt$ by $d\theta_x$ we write

$$d\theta_x = a_\psi \, d\psi + a_\nu \, d\nu + a_\sigma \, d\sigma \tag{5.6.17}$$

where $$a_\psi = \sin\nu \sin\sigma \qquad a_\nu = \cos\sigma \qquad a_\sigma = 0 \tag{5.6.18}$$

Now we ask if the incremental relation in equation (5.6.17) is an exact differential which upon integration can yield a relationship of the type

$$\theta_x = \theta_x (\psi, \nu, \sigma) \tag{5.6.19}$$

If the answer is in the affirmative and the other two equations in (5.6.6) can be similarly integrated, then the existence of $(\theta_x, \theta_y, \theta_z)$ can be assured by virtue of the existence of (ψ, ν, σ). However since for instance

$$\frac{\partial a_\psi}{\partial \nu} \neq \frac{\partial a_\nu}{\partial \psi} \tag{5.6.20}$$

we can see that equation (5.6.17) is not an exact differential and a relationship of the type (5.6.20) does not exist.

5.7 Angular Momentum, Kinetic Energy and Coenergy in terms of Euler Angles

We have seen that kinetic energy and coenergy expressions for rotation of the body take the following simple forms

$$T = \frac{1}{2}\left[\frac{H_x^2}{A} + \frac{H_y^2}{B} + \frac{H_z^2}{C}\right] \tag{5.7.1}$$

$$T^* = \frac{1}{2}\left[A\omega_x^2 + B\omega_y^2 + C\omega_z^2\right] \tag{5.7.2}$$

when the components of $\mathbf{H}$ and $\boldsymbol{\omega}$ are expressed in body-fixed principal axes. Evidently the derivatives of T and T^*, with respect to their arguments, yield the associated variable, that is, for example

$$\frac{\partial T^*}{\partial \omega_x} = H_x \qquad \text{and} \qquad \frac{\partial T}{\partial H_x} = \omega_x \qquad \text{etc.} \tag{5.7.3a,b}$$

Now let us express T and T^* in terms of Euler angles and rates. Using equations (5.6.13) and (5.6.5) in equations (5.7.1) and (5.7.2) we find

$$T = \frac{1}{2}\left\{\frac{1}{A}(H_\psi \sin\nu \sin\sigma + H_\nu \cos\sigma)^2 + \frac{1}{B}(H_\psi \sin\nu \cos\sigma - H_\nu \sin\sigma)^2 + \frac{1}{C}(H_\psi \cos\nu + H_\sigma)^2\right\} \tag{5.7.4}$$

$$T^* = \frac{1}{2}\left\{A(\dot\psi \sin\nu \sin\sigma + \dot\nu \cos\sigma)^2 + B(\dot\psi \sin\nu \cos\sigma - \dot\nu \sin\sigma)^2 + C(\dot\psi \cos\nu + \dot\sigma)^2\right\} \tag{5.7.5}$$

We may now obtain the associated generalised momenta from equation (5.7.5) as

$$p_\psi = \frac{\partial T^*}{\partial \dot{\psi}} = A\,(\dot{\psi}\sin\nu\sin\sigma + \dot{\nu}\cos\sigma)\sin\nu\sin\sigma + B\,(\dot{\psi}\sin\nu\cos\sigma - \dot{\nu}\sin\sigma)\sin\nu\cos\sigma + C\,(\dot{\psi}\cos\nu + \dot{\sigma})\cos\nu \quad (5.7.6a)$$

$$p_\nu = \frac{\partial T^*}{\partial \dot{\nu}} = A\,(\dot{\psi}\sin\nu\sin\sigma + \dot{\nu}\cos\sigma)\cos\sigma - B\,(\dot{\psi}\sin\nu\cos\sigma - \dot{\nu}\sin\sigma)\sin\sigma \quad (5.7.6b)$$

$$p_\sigma = \frac{\partial T^*}{\partial \dot{\sigma}} = C\,(\dot{\psi}\cos\nu + \dot{\sigma}) \quad (5.7.6c)$$

The generalised momenta $(p_\psi, p_\nu, p_\sigma)$ differ from the components of $\mathbf{H}$ in the direction of the Euler rates namely $(H_\psi, H_\nu, H_\sigma)$. To establish the relationship between these two sets of components we may derive the generalised momenta in the following indirect manner. For p_ψ for instance we may write

$$p_\psi = \frac{\partial T^*}{\partial \omega_x}\frac{\partial \omega_x}{\partial \dot{\psi}} + \frac{\partial T^*}{\partial \omega_y}\frac{\partial \omega_y}{\partial \dot{\psi}} + \frac{\partial T^*}{\partial \omega_z}\frac{\partial \omega_z}{\partial \dot{\psi}} \quad (5.7.7)$$

Now one can see that

$$\frac{\partial \omega_x}{\partial \dot{\psi}} = \mathbf{e}_x \cdot \mathbf{e}_\psi = \cos(x, \psi) \quad \text{etc.} \quad (5.7.8)$$

and using equation (5.7.3a) we deduce that

$$p_\psi = H_x\,\mathbf{e}_x \cdot \mathbf{e}_\psi + H_y\,\mathbf{e}_y \cdot \mathbf{e}_\psi + H_z\,\mathbf{e}_z \cdot \mathbf{e}_\psi \quad (5.7.9)$$

or $$p_\psi = \mathbf{H} \cdot \mathbf{e}_\psi \quad (5.7.10a)$$

Similarly $$p_\nu = \mathbf{H} \cdot \mathbf{e}_\nu \quad (5.7.10b)$$

and $$p_\sigma = \mathbf{H} \cdot \mathbf{e}_\sigma \quad (5.7.10c)$$

Thus $(p_\psi, p_\nu, p_\sigma)$ are the projections of the $\mathbf{H}$ vector onto the $\mathbf{e}_\psi$, $\mathbf{e}_\nu$ and $\mathbf{e}_\sigma$ axes (see Figure 5.9). Now we may express $\mathbf{H} \cdot \mathbf{e}_\psi$ for instance in terms of components of $\mathbf{H}$ along $(\mathbf{e}_\psi, \mathbf{e}_\nu, \mathbf{e}_\sigma)$ by writing

$$\mathbf{H} \cdot \mathbf{e}_\psi = H_\psi \mathbf{e}_\psi \cdot \mathbf{e}_\psi + H_\nu \mathbf{e}_\nu \cdot \mathbf{e}_\psi + H_\sigma \mathbf{e}_\sigma \cdot \mathbf{e}_\psi \quad (5.7.11)$$

Thus $$\mathbf{H} \cdot \mathbf{e}_\psi = p_\psi = H_\psi + H_\sigma \cos\nu \quad (5.7.12a)$$

and similarly $$\mathbf{H} \cdot \mathbf{e}_\nu = p_\nu = H_\nu \quad (5.7.12b)$$

$$\mathbf{H} \cdot \mathbf{e}_\sigma = p_\sigma = H_\psi \cos\nu + H_\sigma \quad (5.7.12c)$$

These relations may be inverted giving

$$H_\psi = (p_\psi - p_\sigma \cos\nu)\frac{1}{\sin^2\nu} \quad (5.7.13a)$$

$$H_\nu = p_\nu \tag{5.7.13b}$$

$$H_\sigma = (-p_\psi \cos \nu + p_\sigma) \frac{1}{\sin^2 \nu} \tag{5.7.13c}$$

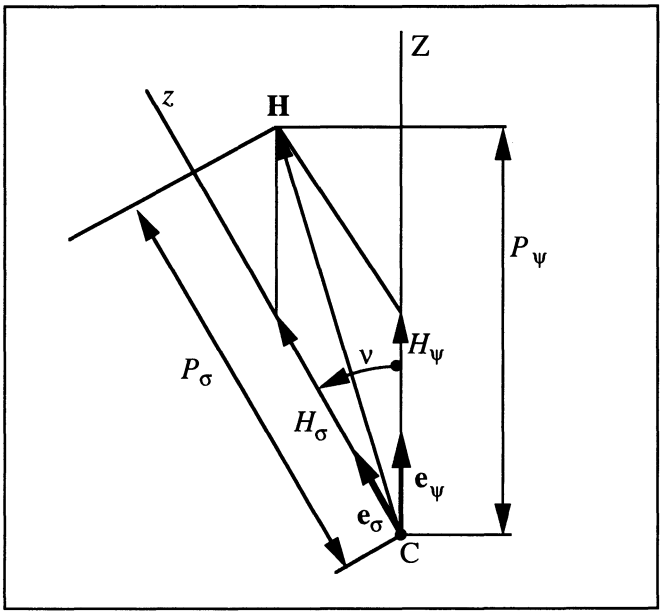


Figure 5.9 Angular momentum components and projections

In like manner we may generate a set of generalised velocities from the kinetic energy expression as follows

$$\omega_\psi = \frac{\partial T}{\partial H_\psi} = \frac{1}{A}(H_\psi \sin \nu \sin \sigma + H_\nu \cos \sigma) \sin \nu \sin \sigma + \frac{1}{B}(H_\psi \sin \nu \cos \sigma - H_\nu \sin \sigma) \sin \nu \cos \sigma + \frac{1}{C}(H_\psi \cos \nu + H_\sigma) \cos \nu \tag{5.7.14a}$$

$$\omega_\nu = \frac{\partial T}{\partial H_\nu} = \frac{1}{A}(H_\psi \sin \nu \sin \sigma + H_\nu \cos \sigma) \cos \sigma - \frac{1}{B}(H_\psi \sin \nu \cos \sigma - H_\nu \sin \sigma) \sin \sigma \tag{5.7.14b}$$

$$\omega_\sigma = \frac{\partial T}{\partial H_\sigma} = \frac{1}{C}(H_\psi \cos \nu + H_\sigma) \tag{5.7.14c}$$

Through a similar development we can show that $(\omega_\psi, \omega_\nu, \omega_\sigma)$ are the projections of $\boldsymbol{\omega}$ onto the $\mathbf{e}_\psi$, $\mathbf{e}_\nu$, and $\mathbf{e}_\sigma$ axes and are related to the components $(\dot{\psi}, \dot{\nu}, \dot{\sigma})$ as follows

$$\omega_\psi = \dot{\psi} + \dot{\sigma}\cos\nu \tag{5.7.15a}$$

$$\omega_\nu = \dot{\nu} \tag{5.7.15b}$$

$$\omega_\sigma = \dot{\sigma}\cos\nu + \dot{\sigma} \tag{5.7.15c}$$

These equations, on inversion yield

$$\dot{\psi} = (\omega_\psi - \omega_\sigma \cos\nu)\frac{1}{\sin^2\nu} \tag{5.7.16a}$$

$$\dot{\nu} = \omega_\nu \tag{5.7.16b}$$

$$\dot{\sigma} = (-\omega_\psi \cos\nu + \omega_\sigma)\frac{1}{\sin^2\nu} \tag{5.7.16c}$$

For the vector **M** the components in the Euler axes are $(M_\psi, M_\nu, M_\sigma)$ and as can be anticipated the projections turn out to be $(Q_\psi, Q_\nu, Q_\sigma)$. This can be shown by writing an expression for the virtual work in the two coordinate systems as

$$M_x\,\delta\theta_x + M_y\,\delta\theta_y + M_z\,\delta\theta_z = Q_\psi\,\delta\psi + Q_\nu\,\delta\nu + Q_\sigma\,\delta\sigma \tag{5.7.17}$$

Now transforming M_x etc. to M_ψ etc. via equation (5.6.15) and $\delta\theta_x$ etc. via equation (5.6.6) and collecting similar terms we can show that

$$Q_\psi = M_\psi + M_\sigma \cos\nu \tag{5.7.17a}$$

$$Q_\nu = M_\nu \tag{5.7.17b}$$

$$Q_\sigma = M_\psi \cos\nu + M_\sigma \tag{5.7.17c}$$

with the inverse relationships

$$M_\psi = (Q_\psi - Q_\sigma \cos\nu)\frac{1}{\sin^2\nu} \tag{5.7.18a}$$

$$M_\nu = Q_\nu \tag{5.7.18b}$$

$$M_\sigma = (-Q_\psi \cos\nu + Q_\sigma)\frac{1}{\sin^2\nu} \tag{5.7.18c}$$

From the foregoing we can see that in the $(\mathbf{e}_\psi, \mathbf{e}_\nu, \mathbf{e}_\sigma)$ frame the corresponding component/projection pairs are

$(\dot{\psi}, \dot{\nu}, \dot{\sigma},)$ – components of $\boldsymbol{\omega}$ <—> $(\omega_\psi, \omega_\nu, \omega_\sigma)$–projections of $\boldsymbol{\omega}$

$(H_\psi, H_\nu, H_\sigma)$ – components of **H** <—> $(p_\psi, p_\nu, p_\sigma)$–projections of **H**

and $(M_\psi, M_\nu, M_\sigma)$ – components of **M** <—> $(Q_\psi, Q_\nu, Q_\sigma)$–projections of **M**

While the components have vector character, the projections do not.

Thus far we have expressed T^* and T in terms of the components of $\boldsymbol{\omega}$ and **H** respectively. Evidently these functions can also be expressed in terms of the projections of $\boldsymbol{\omega}$ and **H**. Thus substituting from equation (5.7.16) into equation (5.7.5) we can write

$$T^* = \frac{1}{2}\left\{ A\left[(\omega_\psi - \omega_\sigma \cos\nu)\frac{\sin\sigma}{\sin\nu} + \omega_\nu \cos\sigma\right]^2 + B\left[(\omega_\psi - \omega_\sigma \cos\nu)\frac{\cos\sigma}{\sin\nu} - \omega_\nu \sin\sigma\right]^2 + C\,\omega_\sigma^2 \right\} \quad (5.7.19)$$

The derivatives of T^* with respect to the projections of $\boldsymbol{\omega}$ namely ω_ψ, ω_ν and ω_σ will yield expressions for the components of the associated momenta H_ψ, H_ν, and H_σ. These expressions are the inverses of those in equations (5.7.14). Finally we express T in terms of the projections of $\mathbf{H}$ namely $(p_\psi, p_\nu, p_\sigma)$. This is readily accomplished by substituting from equations (5.7.13) into equation (5.7.4) and obtaining

$$T = \frac{1}{2}\left\{ \frac{1}{A}\left[(p_\psi - p_\sigma \cos\nu)\frac{\sin\sigma}{\sin\nu} + p_\nu \cos\sigma\right]^2 + \frac{1}{B}\left[(p_\psi - p_\sigma \cos\nu)\frac{\cos\sigma}{\sin\nu} - p_\nu \sin\sigma\right]^2 + \frac{1}{C}\,p_\sigma^2 \right\} \quad (5.7.20)$$

Once again the derivatives of T with respect to the projections of $\mathbf{H}$ namely, $(p_\psi, p_\nu, p_\sigma)$ will yield expressions for the components of the associated rates $(\dot\psi, \dot\nu, \dot\sigma)$, that is the inverse of equations (5.7.6).

Problems

5.7.1 At a point in time, a gyro has an angular momentum of

$$\mathbf{H} = [\,\mathbf{e}_x \;\; \mathbf{e}_y \;\; \mathbf{e}_z\,]\begin{bmatrix} 300 \\ 400 \\ 1200 \end{bmatrix} \text{ kg m}^2/\text{s}$$

(a) Obtain the components $(H_\psi, H_\nu, H_\sigma)$ in Euler axes if $\psi = 90°$, $\nu = 10°$, $\sigma = 60°$.

(b) Obtain the generalized momenta projections $(p_\psi, p_\nu, p_\sigma)$ for the same Euler axes.

5.7.2 A torque-free gyro has an angular momentum of

$$\mathbf{H} = [\,\mathbf{e}_\psi \;\; \mathbf{e}_\nu \;\; \mathbf{e}_\sigma\,]\begin{bmatrix} 120 \\ 0 \\ 0 \end{bmatrix} \text{ kg m}^2/\text{s}$$

and an inertia tensor of

$$[I] = \begin{bmatrix} 24 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 10 \end{bmatrix} \text{ m}^2\text{kg}$$

with respect to a gyro-fixed principal coordinate system Cxyz originating at the gyro's mass centre C.

(a) Calculate the numerical value of the generalized momenta p_i using Euler coordinates with $\nu = 10°$.

(b) Determine the Cartesian components (H_x, H_y, H_z) of the angular momentum, at the moment when $\sigma = 0°$.

(c) Find the kinetic energy.

5.8 Lagrange's Equations

With the expression for T^* in terms of generalised velocities $\dot{q}_1 = \dot{\psi}$, $\dot{q}_2 = \dot{\nu}$ and $\dot{q}_3 = \dot{\sigma}$ and the generalised coordinates we are now in a position to obtain the equation of motion via Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}_i}\right) - \frac{\partial T^*}{\partial q_i} = Q_i \tag{5.8.1}$$

If the generalised forces Q_i (torques in the present case) do not possess a potential energy then may be obtained via the virtual work expression

$$\delta W_I = Q_\psi \, \delta\psi + Q_\nu \, \delta\nu + Q_\sigma \, \delta\sigma \tag{5.8.2}$$

Thus Lagrange's equations of motion for rotatory motion take the form

$$\dot{p}_\psi - \frac{\partial T^*}{\partial \psi} = Q_\psi \tag{5.8.3a}$$

$$\dot{p}_\nu - \frac{\partial T^*}{\partial \nu} = Q_\nu \tag{5.8.3b}$$

$$\dot{p}_\sigma - \frac{\partial T^*}{\partial \sigma} = Q_\sigma \tag{5.8.3c}$$

where T^* and the generalised momenta are given in equations (5.7.7) and (5.7.8). It is worth noting that ψ is absent in T^* and hence it can be considered as an ignorable coordinate.

Although Lagrange's and Euler's equations appear to be different, they describe the same motion of course but in different (and related) variables. To see this consider Lagrange's third equation (5.8.3c) which on expansion becomes

$$C \frac{d}{dt}(\dot{\psi}\cos\nu + \dot{\sigma}) - A(\dot{\psi}\sin\nu\sin\sigma + \dot{\nu}\cos\sigma)(\dot{\psi}\sin\nu\cos\sigma - \dot{\nu}\sin\sigma) + B(\dot{\psi}\sin\nu\cos\sigma - \dot{\nu}\sin\sigma)(-\dot{\psi}\sin\nu\sin\sigma - \dot{\nu}\cos\sigma) = Q_\sigma \tag{5.8.4}$$

By introducing the angular velocities $(\omega_x, \omega_y, \omega_z)$ from equation (5.6.6) into this equation we obtain

$$C\,\dot{\omega}_z + (B - A)\,\omega_x\omega_y = Q_\sigma \tag{5.8.5}$$

which is the third of Euler's equations of motion and it also reveals that $Q_\sigma = M_z$. By systematic permutations of the subscripts and A B and C we can obtain the first two Euler equations from equation (5.8.5). However it should be noted that while the Largrange equations associated with σ coincides with the Euler equation in the principal direction z, the same is not the case for the remaining two equations. One can see

for instance that while Q_ν is a torque along the line of nodes M_y is a torque along the second principal axes.

Example 5.8.1

A fast spinning $(\dot{\sigma} \gg \dot{\psi}, \dot{\nu})$, axisymmetric $(B = A)$ gyro is mounted in massless gimbal rings. It has two linear torsion springs mounted such that $Q_\psi = -k_\psi \psi$, $Q_\nu = -k_\nu \nu$, $Q_\sigma = 0$. Establish the equation of motion for the σ coordinate.

From Lagrange's equations (5.8.3) we have

$$k_\psi \psi + \dot{p}_\psi = 0 \tag{a}$$

$$k_\nu \nu + \dot{p}_\nu - \frac{\partial T^*}{\partial \nu} = 0 \tag{b}$$

$$\dot{p}_\sigma - \frac{\partial T^*}{\partial \sigma} = 0 \tag{c}$$

with

$$p_\sigma = \frac{\partial T^*}{\partial \dot{\sigma}} = C\,(\dot{\psi}\cos\nu + \dot{\sigma}) \tag{d}$$

$$\dot{p}_\sigma = \frac{d}{dt}\frac{\partial T^*}{\partial \dot{\sigma}} = C\,(\ddot{\psi}\cos\nu - \dot{\psi}\dot{\nu}\sin\nu + \ddot{\sigma}) \tag{e}$$

$$\frac{\partial T^*}{\partial \sigma} = A\,(\dot{\psi}\sin\nu\sin\sigma + \dot{\nu}\cos\sigma)(\dot{\psi}\sin\nu\cos\sigma - \dot{\nu}\sin\sigma) + B\,(\dot{\psi}\sin\nu\cos\sigma - \dot{\nu}\sin\sigma)(\dot{\psi}\sin\nu\sin\sigma - \dot{\nu}\cos\nu) \tag{f}$$

so that

$$C(\ddot{\psi}\cos\nu - \dot{\psi}\dot{\nu}\sin\nu + \ddot{\sigma}) - A(\dot{\psi}\sin\nu\sin\sigma + \dot{\nu}\cos\sigma)(\dot{\psi}\sin\nu\cos\sigma - \dot{\nu}\sin\sigma) - B(\dot{\psi}\sin\nu\cos\sigma - \dot{\nu}\sin\sigma)(\dot{\psi}\sin\nu\sin\sigma - \dot{\nu}\cos\sigma) = 0 \tag{g}$$

with $B = A$, equation (c) eventually becomes

$$C(\ddot{\psi}\cos\nu - \dot{\psi}\dot{\nu}\sin\nu + \ddot{\sigma}) - A(\dot{\psi}^2\sin^2\nu\sin 2\sigma - 2\dot{\psi}\dot{\nu}\sin\nu\sin^2\sigma) = 0 \tag{h}$$

Example 5.8.2

A fast spinning $(\omega_1 = \dot{\sigma} \gg \dot{\nu})$, axisymmetric $(B = A)$ gyro is mounted in massless gimbal rings. The gyro is constrained such that $\psi = \dot{\psi} = 0$, and $\dot{\nu} = -\omega_2 = \text{constant}$. Further $\ddot{\sigma} = 0$.

The torques required to maintain this motion are to be determined. From Lagrange's equations (5.8.3) we have

$$Q_\psi = \dot{p}_\psi - \frac{\partial T^*}{\partial \psi} \tag{a}$$

$$Q_\nu = \dot{p}_\nu - \frac{\partial T^*}{\partial \nu} \tag{b}$$

$$Q_\sigma = \dot{p}_\sigma - \frac{\partial T^*}{\partial \sigma} \tag{c}$$

With $\frac{\partial T^*}{\partial \psi} = 0$, and

$$p_\psi = \frac{\partial T^*}{\partial \dot{\psi}} = A\,\dot{\nu}\,\sin\nu\,\sin\sigma\,\cos\sigma - A\,\dot{\nu}\,\sin\nu\,\sin\sigma\,\cos\sigma + C\,\dot{\sigma}\,\cos\nu \quad \text{(d)}$$

$$p_\psi = C\,\dot{\sigma}\,\cos\nu \quad \text{(e)}$$

$$\dot{p}_\psi = \frac{d}{dt}\,p_\psi = C(\ddot{\sigma}\,\cos\nu - \dot{\nu}\dot{\sigma}\,\sin\nu) = -C\,\dot{\nu}\dot{\sigma}\,\sin\nu \quad \text{(f)}$$

Thus, from equation (a), the required torque becomes

$$Q_\psi = C\,\omega_1\,\omega_2\,\sin\nu \quad \text{(g)}$$

With

$$p_\nu = \frac{\partial T^*}{\partial \dot{\nu}} = A\,\dot{\nu}\,\cos^2\sigma + A\,\dot{\nu}\,\sin^2\sigma = A\,\dot{\nu} \quad \text{(h)}$$

$$\dot{p}_\nu = A\,\ddot{\nu} = 0 \quad \text{(i)}$$

$$\frac{\partial T^*}{\partial \nu} = -C\,(\dot{\psi}\,\cos\nu + \dot{\sigma})\,\dot{\psi}\,\sin\nu = 0 \quad \text{(j)}$$

equation (b) gives

$$Q_\nu = 0 \quad \text{(k)}$$

And with

$$p_\sigma = \frac{\partial T^*}{\partial \dot{\sigma}} = C\,(\dot{\psi}\,\cos\nu + \dot{\sigma}) = C\,\dot{\sigma} \quad \text{(l)}$$

$$\dot{p}_\sigma = C\,\ddot{\sigma} = 0 \quad \text{(m)}$$

$$\frac{\partial T^*}{\partial \sigma} = 0 \quad \text{(n)}$$

from equation (c)

$$Q_\sigma = 0 \quad \text{(o)}$$

Example 5.8.3

The case of a torque-free axisymmetric gyro is of particular importance in practical applications (satellites). For a torque-free gyro, the angular momentum **H** is constant, in magnitude as well as direction. It turns out to be advantageous to choose the generalized Euler coordinates such that the angular momentum **H** is directed along the precession axis $\mathbf{e}_\psi$. Thus

$$\mathbf{H} = [\mathbf{e}_\psi\ \mathbf{e}_\nu\ \mathbf{e}_\sigma]\begin{bmatrix} H \\ 0 \\ 0 \end{bmatrix} \quad \text{(a)}$$

As a consequence, the generalized momenta (5.7.10) become

$$p_\psi = H \quad \text{(b)}$$

$$p_\nu = 0 \quad \text{(c)}$$

$$p_\sigma = H \cos \nu \tag{d}$$

Under these conditions the kinetic energy (5.7.20) simplifies to

$$T = \frac{1}{2} H^2 \left[\frac{1}{A} \sin^2 \nu + \frac{1}{C} \cos^2 \nu \right] \tag{e}$$

and the generalized velocities become

$$\dot{\psi} = \frac{H}{A} \tag{f}$$

$$\dot{\nu} = 0 \qquad \text{and} \qquad \nu = \nu_o \tag{g}$$

$$\dot{\sigma} = \frac{A - C}{A} \frac{H}{C} \cos \nu_o \tag{h}$$

The Cartesian velocity components are

$$\omega_x = \frac{H}{A} \sin \nu_o \cos \sigma \tag{i}$$

$$\omega_y = \frac{H}{A} \sin \nu_o \sin \sigma \tag{j}$$

$$\omega_z = \frac{H}{C} \cos \nu_o \tag{k}$$

$$\omega = [\mathbf{e}_u \ \mathbf{e}_v \ \mathbf{e}_z] \begin{bmatrix} 0 \\ \omega_v \\ \omega_z \end{bmatrix} \tag{n}$$

and

$$\mathbf{H} = [\mathbf{e}_u \ \mathbf{e}_v \ \mathbf{e}_z] \begin{bmatrix} 0 \\ H \sin \nu_o \\ H \cos \nu_o \end{bmatrix} \tag{o}$$

Example 5.8.4

Given is an axisymmetric gyro (Figure 5.10) with constant attitude angle ν_o and zero spin torque component i.e. $M_\sigma = 0$. The precession is to increase at a constant rate i.e. $\ddot{\psi} = \ddot{\psi} t + \dot{\psi}_o$. The torque components M_ψ and M_ν required to achieve this motion are to be determined.

For axisymmetric gyros ($A = B$) the expression for T^* simplifies to

$$T^* = \frac{1}{2}[A(\dot{\psi}^2 \sin^2 \nu + \dot{\nu}^2) + C(\dot{\psi} \cos \nu + \dot{\sigma})^2] \tag{a}$$

Then from Lagrange equations (5.8.1) we find the equations of motion as

$$A \ddot{\psi}_o \sin^2 \nu_o + C \ddot{\psi}_o \cos^2 \nu_o + C \ddot{\sigma} \cos \nu_o = Q_\psi \tag{b}$$

$$(C - A)\dot{\psi}^2 \sin \nu_o \cos \nu_o + C \dot{\psi} \dot{\sigma} \sin \nu_o = Q_\nu \tag{c}$$

$$C \ddot{\psi}_o \cos \nu_o + C \ddot{\sigma} = Q_\sigma \tag{d}$$

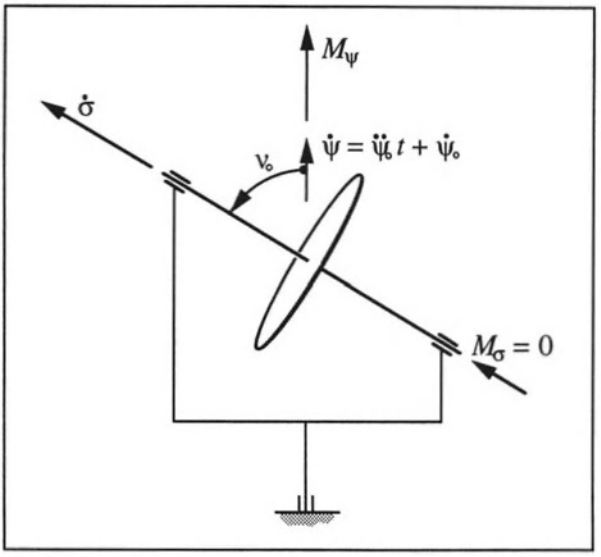


Figure 5.10 Gyro with constant precession acceleration

where we have taken account of the constant nutation angle and constant precession acceleration. Now the torque component M_σ, as given in equation (5.7.18c), is to vanish, thus

$$M_\sigma = \frac{1}{\sin^2 \nu} (-Q_\psi \cos \nu + Q_\sigma) = 0$$

i.e. $$Q_\sigma = Q_\psi \cos \nu_o \tag{e}$$

Using equations (d) and (b) we may express this requirement as

$$C\ddot{\psi}_o \cos \nu_o + C\ddot{\sigma} = A\ddot{\psi}_o \sin^2 \nu_o \cos \nu_o + C\ddot{\psi}_o \cos^3 \nu_o + C\ddot{\sigma} \cos^2 \nu_o \tag{f}$$

or $$C\ddot{\psi}(1 - \cos^2 \nu_o) \cos \nu_o = A\ddot{\psi}_o \sin^2 \nu_o \cos\nu_o - C\ddot{\sigma} \sin^2 \nu_o$$

On simplification we find

$$\ddot{\psi}_o \cos \nu_o (C - A) = -C\ddot{\sigma} \tag{g}$$

or $$\ddot{\sigma} = \frac{(A - C)}{C} \ddot{\psi}_o \cos \nu_o \tag{h}$$

Integration of this expression yields

$$\dot{\sigma} = \frac{(A - C)}{C} \ddot{\psi}_o t \cos \nu_o t + \dot{\sigma}_o \tag{i}$$

Now substituting from equations (h) and (i) into equations (b), (c) and (d) we can evaluate the torque *projections* as

$$Q_\psi = A\ddot{\psi}_o \tag{j}$$

$$Q_\nu = C(\ddot{\psi}_o t + \dot{\psi}_o) \dot{\sigma}_o \sin \nu_o + (C - A)(\ddot{\psi}_o t + \dot{\psi}_o) \dot{\psi}_o \sin \nu_o \cos \nu_o \tag{k}$$

$$Q_\sigma = A\ddot{\psi}_o \cos \nu_o \tag{l}$$

Finnally we can determine the torque *components* as

$$M_\psi = A\ddot{\psi}_o \tag{m}$$

$$M_\nu = C(\ddot{\psi}_o t + \dot{\psi}_o)\,\dot{\sigma}_o \sin \nu_o + (C - A)(\ddot{\psi}_o t + \dot{\psi}_o)\,\dot{\psi}_o \sin \nu_o \cos \nu_o \tag{n}$$

$$M_\sigma = 0 \tag{o}$$

Problems

5.8.1 A spinning ($\dot{\sigma}$ = 10 rad/s = constant) axisymmetric ($B = A$) gyro is mounted in massless gimbal rings. The gyro is to move at a constant nutation rate $\dot{\nu} = -2$ rad/s, while $\dot{\psi} = 0$, and $C = 50$ m^{2}kg. What torque is required to maintain this motion?

5.8.2 A spinning ($\dot{\sigma}$ = constant) axisymmetric gyro is mounted in massless gimbal rings. The gyro is to move at a constant rotation rate $\dot{\nu}$, while the precession $\dot{\psi} = 0$. What torques are required to maintain this motion?

5.8.3 The motion of a torque-free axisymmetric gyro is described in Euler angles, with the angular momentum vector and the precession rate collinear. Find the generalized forces Q_i, and the Eulerian torque components M_j, which are required to give the gyro a constant $\dot{\psi}$, with all the other velocities remaining unchanged, i.e. $\dot{\sigma}$ = constant, $\dot{\nu} = 0$, $\nu = \nu_o$.

5.9 Other Generalised Variables for Describing Angular Positions

The Euler angles are the best known of the generalised coordinates for describing the angular positions of rigid bodies. However for certain applications other coordinates have been proposed. For instance in the aeronautical field the preferred coordinates are the yaw, pitch, and roll angles. Other angular coordinates include the Cardan angles and Bryant angles. Occasionally a constrained set of four coordinates known as the Euler parameters are also employed. Various coordinates have their advantages and drawbacks. For instance the four coordinates of the Euler parameters overcome the singularity of the three angle representations by Euler, Bryant and Cardan angles. Specialised texts on space and rigid body dynamics describe these alternative coordinates and illustrate their use.

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Chapter VI

SPECIAL APPLICATIONS

Introduction

The Lagrangian equations of motion often take the form of second-order nonlinear differential equations. Such equations rarely admit analytical solutions. In some cases it is possible to linearize the equations of motion by imposition of some acceptable approximations. The linearized equations may then be solved analytically. An important class of such problems is the case of small amplitude oscillations about a position of equilibrium or about a predetermined steady motion. The process of linearization may be carried out on the differential equations or one may introduce the pertinent approximations when the energy and virtual work terms are written down. In this chapter we outline the latter procedure and describe methods of solution for the linearized equations.

As a second area of application we consider the special case of spike loads (also called impulsive loads) and spike speeds and determine the changes in the energies of a system when it is subjected to such loading.

6.1 Approximate Expressions for T^*, V, $\dot{D}$ and δW_I

Consider a restricted class of problems which do not involve moving coordinates or moving constraints. For such problems time will not appear explicitly in the transformation equations and for a system of N particles we may write these equations as

$$\mathbf{r}_k = \mathbf{r}_k (q_1, q_2 \ldots, q_i, \ldots, q_n) \qquad k = 1, 2, \ldots, N \quad (6.1.1)$$

where $\mathbf{r}_k$ is the position vector for the kth particle in 3D space. Now let us expand for $\mathbf{r}_k$, by Taylor series, about the position of equilibrium defined by $(\mathbf{r}_k)_o$. Then we have

$$\mathbf{r}_k = (\mathbf{r}_k)_o + (\frac{\partial \mathbf{r}_k}{\partial q_i})_o \, q_i' + \frac{1}{2} (\frac{\partial^2 \mathbf{r}_k}{\partial q_i \partial q_j})_o \, q_i' \, q_j' + \cdots \qquad \begin{matrix} i = 1,2,\ldots,n \\ j = 1,2,\ldots,n \end{matrix} \quad (6.1.2)$$

where the terms $(\ )_o$ are evaluated at the position of equilibrium and q_i' are the perturbations about the position of equilibrium. Since the $(\ )_o$ terms are constants, we

may write an approximation for the velocities as

$$\dot{\mathbf{r}}_{k\ approx} = \left(\frac{\partial \mathbf{r}_k}{\partial q_i}\right)_o \dot{q}_i' \tag{6.1.3}$$

For the sake of simplicity in notation, from now on we drop the prime on $\dot{q}_i'$ and unless otherwise stated we take $\dot{q}_i$ to mean small velocities about the position of equilibrium. With this understanding and using equation (6.1.3) we obtain an approximate expression for the complementary kinetic energy of the system as

$$T^*_{approx} = \frac{1}{2}\left(\frac{\partial (m\,\mathbf{r})_k}{\partial q_i}\right)_o \cdot \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right)_o \dot{q}_i\, \dot{q}_j \tag{6.1.4}$$

or in matrix notation

$$T^*_{approx} = \frac{1}{2}\{\dot{q}\}^T [m]\{\dot{q}\} \tag{6.1.5}$$

where the square symmetric matrix $[m]$, called the mass matrix, has constant elements.

Consider next the Rayleigh dissipation function which may be expressed as

$$\dot{D} = \frac{1}{2}(c\ \dot{\mathbf{r}})_k \cdot \dot{\mathbf{r}}_k \tag{6.1.6}$$

where c is the coefficient of viscous drag on the kth particle. Once again using the approximate expression for the velocities, as given in equation (6.1.3), we may obtain an approximate expression for Rayleigh's dissipation function as

$$\dot{D}_{approx} = \frac{1}{2}\left(\frac{\partial (c\,\mathbf{r})_k}{\partial q_i}\right)_o \cdot \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right)_o \dot{q}_i\, \dot{q}_j \tag{6.1.7}$$

or in matrix form

$$\dot{D}_{approx} = \frac{1}{2}\{\dot{q}\}^T [c]\{\dot{q}\} \tag{6.1.8}$$

where $[c]$, called the damping matrix, is symmetric and has constant elements.

Next let us obtain an approximate expression for the potential energy. Assuming that V is a function of the small displacements about the position of equilibrium, namely q_i, we may expand $V(q)$ into a Taylor series, about the position of equilibrium. Then

$$V = (V)_o + \left(\frac{\partial V}{\partial q_i}\right)_o q_i + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_o q_i\, q_j + \cdots \quad \begin{matrix} i = 1,2,\ldots,n \\ j = 1,2,\ldots,n \end{matrix} \tag{6.1.9}$$

The first term in equation (6.1.9) is a constant. Since it does not influence the equations of motion, we set $(V)_o = 0$. The second term vanishes since by definition $(\partial V/\partial q_i)_o = 0$ at a position of equilibrium. Hence as a first approximation for V we find

$$V_{approx} = \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_o q_i\, q_j \tag{6.1.10}$$

or

$$V_{approx} = \frac{1}{2}\{q\}^T [k] \{q\} \tag{6.1.11}$$

where $[k]$ is a symmetric matrix called the stiffness matrix and its elements are constants.

Finally consider the virtual work terms of forces not included in V nor in $\dot{D}$. Denoting such disturbing forces by $\mathbf{F}_k'$ we may write the associated virtual work term for the N particles as

$$\delta W_I = \mathbf{F}_k' \cdot \delta \mathbf{r}_k \qquad k = 1,2,3,...,N \tag{6.1.12}$$

Now letting $\delta \mathbf{r}_k = \mathbf{r}_k - (\mathbf{r}_k)_o$ and using the linear terms in equation (6.1.2) we obtain an approximate form of δW_I as

$$\delta W_{I\ approx} = \mathbf{F}_k' \left(\frac{\partial \mathbf{r}_k}{\partial q_i}\right)_o \delta q_i \qquad i = 1,2,3,...,n \tag{6.1.13}$$

or

$$\delta W_{I\ approx} = \{Q'\}^T \{\delta q\} \tag{6.1.14}$$

where the column matrix $\{Q'\}$ of generalized disturbing forces has elements given by

$$Q_i' = \mathbf{F}_k' \left(\frac{\partial \mathbf{r}_k}{\partial q_i}\right)_o \tag{6.1.15}$$

Since the approximate forms of T^* and $\dot{D}$ are quadratic in velocities, and approximate forms of V and δW_I are quadratic and linear, respectively, in displacements, their use in Lagrange's equation (2.10.11.) yields a set of linear differential equations of the form

$$[m]\{\ddot{q}\} + [c]\{\dot{q}\} + [k]\{q\} = \{Q'\} \tag{6.1.16}$$

Example 6.1.1

Consider the double pendulum shown in Figure 6.1. Let the two particles be subjected to viscous dissipation forces, in the x direction, with coefficients c_1, c_2 and suppose that a disturbing force $\mathbf{F}_2'$ acts on the second particle, as shown.

The reduced transformation equations for this problem may be written as

$$\begin{aligned} \mathbf{r}_1 &= [\mathbf{e}_x \ \mathbf{e}_y] \begin{bmatrix} l_1 \sin\theta_1 \\ -l_1 \cos\theta_1 \end{bmatrix} \\ \mathbf{r}_2 &= [\mathbf{e}_x \ \mathbf{e}_y] \begin{bmatrix} l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ -l_1 \cos\theta_1 - l_2 \cos\theta_2 \end{bmatrix} \end{aligned} \tag{a}$$

Use of these equations for evaluations of T^* and $\dot{D}$, will yield expressions which are not quadratic in $\dot{\theta}_1$ and $\dot{\theta}_2$. However, if the motion is restricted to the vicinity of the equilibrium position defined by $\theta_1 = \theta_2 = 0$, we may obtain approximate expressions for T^* and $\dot{D}$ which are quadratic in $\dot{\theta}_1$ and $\dot{\theta}_2$, and V which is quadratic in θ_1 and θ_2. Using equation (6.1.3) we find

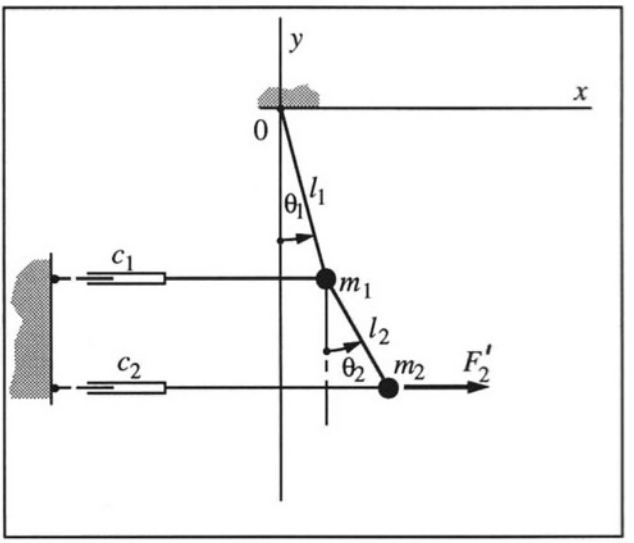


Figure 6.1 Double pendulum

$$\dot{\mathbf{r}}_{1\,approx} = [\mathbf{e}_x \quad \mathbf{e}_y] \begin{bmatrix} l_1 \dot{\theta}_1 \cos \theta_1 \\ l_1 \dot{\theta}_1 \sin \theta_1 \end{bmatrix}_{\theta_1 = 0} \tag{b}$$

or

$$\dot{\mathbf{r}}_{1\,approx} = [\mathbf{e}_x \quad \mathbf{e}_y] \begin{bmatrix} l_1 \dot{\theta}_1 \\ 0 \end{bmatrix} \tag{c}$$

Similarly

$$\dot{\mathbf{r}}_{2\,approx} = [\mathbf{e}_x \quad \mathbf{e}_y] \begin{bmatrix} l_1 \dot{\theta}_1 \cos \theta_1 + l_2 \dot{\theta}_2 \cos \theta_2 \\ l_1 \dot{\theta}_1 \sin \theta_1 + l_2 \dot{\theta}_2 \sin \theta_2 \end{bmatrix}_{\theta_1 = \theta_2 = 0} \tag{d}$$

or

$$\mathbf{r}_{2\,approx} = [\mathbf{e}_x \quad \mathbf{e}_y] \begin{bmatrix} l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 \\ 0 \end{bmatrix} \tag{e}$$

Then

$$T^* = \frac{m_1}{2} (\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1) + \frac{m_2}{2} (\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2) \tag{f}$$

can be approximated as

$$T^*_{approx} = \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} (l_1^2 \dot{\theta}_1^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta}_2^2) \tag{g}$$

$$T^*_{approx} = \frac{1}{2}[\dot{\theta}_1 \ \dot{\theta}_2] \begin{bmatrix} m_1 l_1^2 + m_2 l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \quad \text{(h)}$$

In like manner we find that

$$\dot{D}_{approx} = \frac{c_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{c_2}{2}(l_1^2 \dot{\theta}_1^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta}_2^2) \quad \text{(i)}$$

or

$$\dot{D}_{approx} = \frac{1}{2}[\dot{\theta}_1 \ \dot{\theta}_2] \begin{bmatrix} c_1 l_1^2 + c_2 l_1^2 & c_2 l_1 l_2 \\ c_2 l_1 l_2 & c_2 l_2^2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \quad \text{(j)}$$

Consider next the expression for V. From Figure 6.1 we note that

$$V = m_1 g l_1 (1 - \cos\theta_1) + m_2 g[l_1(1 - \cos\theta_1) + l_2(1 - \cos\theta_2)] \quad \text{(k)}$$

Now using equation (6.1.10) we obtain an approximate expression for the potential energy as

$$V_{approx} = \frac{1}{2}\left[(m_1 g l_1 \cos\theta_1 + m_2 g l_1 \cos\theta_1)_{\theta_1 = 0} \theta_1^2 + (m_2 g l_2 \cos\theta_2)_{\theta_2 = 0} \theta_2^2\right] \quad \text{(l)}$$

or

$$V_{approx} = \frac{1}{2}[\theta_1 \ \theta_2] \begin{bmatrix} m_1 g l_1 + m_2 g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad \text{(m)}$$

Finally let us evaluate the force $\mathbf{F}_2' = \mathbf{e}_x F_2$ acting on m_2. To this end we write the virtual work of this force as

$$\delta W_I = \mathbf{F}_2' \cdot \delta \mathbf{r}_2 \quad \text{(n)}$$

or

$$\delta W_I = (\mathbf{e}_x F_2) \cdot \left[\left(\frac{\partial \mathbf{r}_2}{\partial \theta_1}\right)\delta\theta_1 + \left(\frac{\partial \mathbf{r}_2}{\partial \theta_2}\right)\delta\theta_2\right] \quad \text{(o)}$$

Now evaluating the derivative of $\mathbf{r}_2$, at the position of equilibrium, we find

$$\delta W_{I\,approx} = (\mathbf{e}_x F_2) \cdot \left[\mathbf{e}_x (l_1 \delta\theta_1 + l_2 \delta\theta_2)\right] \quad \text{(p)}$$

we may write this expression as

$$\delta W_{I\,approx} = [Q_1' \quad Q_2'] \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \end{bmatrix} \quad \text{(q)}$$

where

$$Q_1' = F_2 l_1 \qquad Q_2' = F_2 l_2 \tag{r}$$

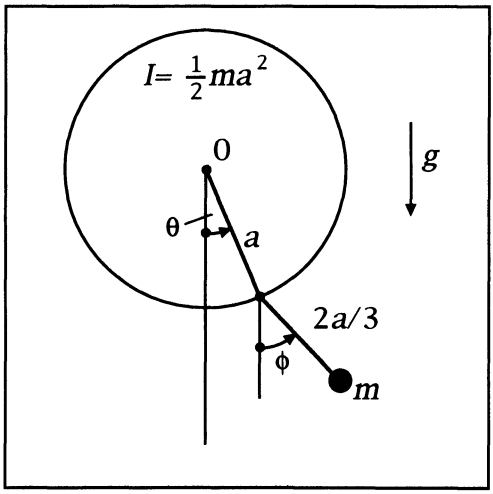
Using the approximate expression derived for T^*, V, $\dot{D}$, and δW_I, we may obtain the linearized equation of motion from Lagrange's equation (2.10.11) as

$$\begin{bmatrix} m_1 l_1^2 + m_2 l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c_1 l_1^2 + c_2 l_1^2 & c_2 l_1 l_2 \\ c_2 l_1 l_2 & c_2 l_2^2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} m_1 g l_1 + m_2 g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F_2 l_1 \\ F_2 l_2 \end{bmatrix} \tag{s}$$

Problem

6.1.1 A uniform circular disc of mass m and radius a turns freely about its axis which is fixed horizontally. A pendulum of mass m and length $2a/3$ is attached to the rim of the disc. Consider small amplitude oscillations about the equilibrium position and obtain the linearised equations of motion.



6.2 Free Undamped Oscillations

In the absence of damping and disturbing forces the linear equations (6.1.16) of motion become

$$[m]\{\ddot{q}\} + [k]\{q\} = \{0\} \tag{6.2.1}$$

To obtain a solution let us assume that all q_i vary in time in a similar manner. That is we let

$$\{q(t)\} = \{\mu\}y(t) \tag{6.2.2}$$

where $\{\mu\}$ is a column matrix of constants μ_i and $y(t)$ is a scalar function. Substituting from equation (6.2.2) into equation (6.2.1) we find

$$[m]\{\mu\}\ddot{y} + [k]\{\mu\}y = \{0\} \tag{6.2.3}$$

On premultiplying equations (6.2.3) by $\{\mu\}^T$ and rearranging, we may write

$$-\frac{\ddot{y}}{y} = \frac{\{\mu\}^T[k]\{\mu\}}{\{\mu\}^T[m]\{\mu\}} \tag{6.2.4}$$

Now noting that the right hand side of equation (6.2.4) is a constant, say ω^2, we have that

$$\omega^2 = \frac{\{\mu\}^T[k]\{\mu\}}{\{\mu\}^T[m]\{\mu\}} \tag{6.2.5}$$

and

$$\ddot{y} + \omega^2 y = 0 \tag{6.2.6}$$

Equation (6.2.6) reveals that

$$y = A\cos\omega t + B\sin\omega t \tag{6.2.7}$$

i.e. the motion is sinusoidal in time with angular frequency ω. The constants A and B may be determined from initial conditions.

Returning to equations (6.2.3) and using the solution (6.2.7) we find that ω^2 is governed by the following eigenvalue equation

$$([k] - \omega^2[m])\{\mu\} = \{0\} \tag{6.2.8}$$

Associated with the roots ω_r^2 that satisfy equation (6.2.8), there are r eigenvectors $\{\mu\}_r$ allowing one to write

$$y_r = A_r\cos\omega_r t + B_r\sin\omega_r t \tag{6.2.9}$$

For instance, for a four degree of freedom system, equation (6.2.2) becomes

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}_1 y_1 + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}_2 y_2 + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}_3 y_3 + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}_4 y_4 \tag{6.2.10}$$

with

$$y_1 = A_1\cos\omega_1 t + B_1\sin\omega_1 t$$
$$y_2 = A_2\cos\omega_2 t + B_2\sin\omega_2 t$$

$$y_3 = A_3 \cos\omega_3 t + B_3 \sin\omega_3 t$$
$$y_4 = A_4 \cos\omega_4 t + B_4 \sin\omega_4 t$$

The eight integration constants A, B, must be determined from initial conditions.

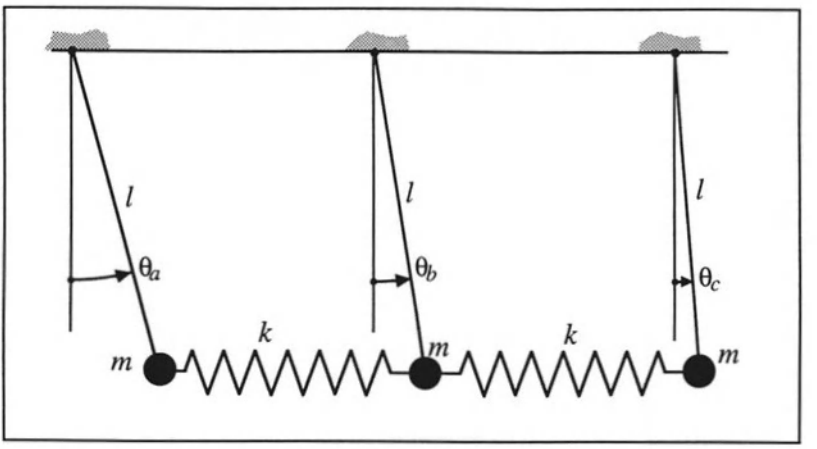


Figure 6.2 Three interconnected pendulums

Example 6.2.1

Consider the three pendulums, coupled by two springs, as shown in Figure 6.2, with the generalized coordinates $q_1 = \theta_a$, $q_2 = \theta_b$, and $q_3 = \theta_c$. Assuming that the springs are not strained when each $\theta_i = 0$, we may take this configuration as the position of equilibrium. Then for small amplitude oscillations we have the kinetic coenergy

$$T^* = \frac{1}{2} m l^2 (\dot{\theta}_a^2 + \dot{\theta}_b^2 + \dot{\theta}_c^2) \tag{a}$$

The potential energy, due to gravity and the springs, becomes

$$V = \frac{1}{2} mg\, l(\theta_a^2 + \theta_b^2 + \theta_c^2) + \frac{k}{2} l^2 [(\theta_b - \theta_a)^2 + (\theta_c - \theta_b)^2] \tag{b}$$

Using the expressions for T^* and V, in Lagrange's equation, or in Hamilton's principle, we find the equations of motion as

$$[m]\{\ddot{\theta}\} + [k]\{\theta\} = \{0\} \tag{c}$$

with

$$[m] = \begin{bmatrix} ml^2 & 0 & 0 \\ 0 & ml^2 & 0 \\ 0 & 0 & ml^2 \end{bmatrix} \tag{d}$$

and

$$[k] = \begin{bmatrix} mgl + kl^2 & -kl^2 & 0 \\ -kl^2 & mgl + 2kl^2 & -kl^2 \\ 0 & -kl^2 & mgl + kl^2 \end{bmatrix} \tag{e}$$

On assuming sinusoidal oscillations with frequency ω, the equations of motion take the form of the eigenvalue equation (6.2.8)

$$([k] - \omega^2[m])\{\mu\} = \{0\} \tag{f}$$

For nontrivial solutions the determinant of the coefficient matrix must vanish, ie. $\det([k] - \omega^2[m]) = 0$. Subsequent to some algebraic manipulations this condition is found as

$$(\omega^2 - \frac{g}{l})\left[\omega^2 - (\frac{g}{l} + \frac{k}{m})\right]\left[\omega^2 - (\frac{g}{l} + 3\frac{k}{m})\right] = 0 \tag{g}$$

The condition for vanishing of the determinant shows that the three natural frequencies of the system can be obtained from

$$\omega_1^2 = \frac{g}{l} \tag{h}$$

$$\omega_2^2 = \frac{g}{l} + \frac{k}{m} \tag{i}$$

$$\omega_3^2 = \frac{g}{l} + \frac{3k}{m} \tag{j}$$

For each of these frequencies we may obtain the associated mode of vibration by substituting the values of ω^2 in equation (f). Evidently then the matrix $([k] - \omega^2[m])$ will be singular, making $\{\mu\}$ indeterminate. That is absolute values of $\{\mu\}$ cannot be determined. We can, however, obtain relative values for the elements of $\{\mu\}$. To this end we may for instance ascribe a unit value for one of the μ_i's and obtain the corresponding values for the others. Thus letting $\mu_a = 1$ for the first mode, i.e. when $\omega = \omega_1 = \frac{g}{l}$, we can solve for μ_b and μ_c, from

$$\left(\begin{bmatrix} mgl + kl^2 & -kl^2 & 0 \\ -kl^2 & mgl + 2kl^2 & -kl^2 \\ 0 & -kl^2 & mgl + kl^2 \end{bmatrix} - \frac{g}{l}\begin{bmatrix} ml^2 & 0 & 0 \\ 0 & ml^2 & 0 \\ 0 & 0 & ml^2 \end{bmatrix}\right)\begin{bmatrix} 1 \\ \mu_b \\ \mu_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{k}$$

The solution shows that $\mu_b = \mu_c = 1$, that is, in the first mode (Figure 6.3) the masses move in unison, under the gravitational force, while the springs do not contribute to this motion. We write for $r = 1$,

$$\{\mu\}_1 = \begin{bmatrix} \mu_a \\ \mu_b \\ \mu_c \end{bmatrix}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{l}$$

Following a similar procedure the second and third modes of the system can be found. For the second mode $r = 2$, we find

$$\{\mu\}_2 = \begin{bmatrix} \mu_a \\ \mu_b \\ \mu_c \end{bmatrix}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{m}$$

Evidently in the second mode the two end masses move in opposite directions, while the middle mass remains at rest. In passing it is worth noting that in this case one cannot choose and ascribe a unit value to μ_b and hope to find μ_a and μ_c. The second mode exhibits one vibration node, that is a point where there is no motion. In this mode the node is located in the middle of the

system.

For the third mode $r = 3$, we find

$$\{\mu\}_3 = \begin{bmatrix} \mu_a \\ \mu_b \\ \mu_c \end{bmatrix}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \tag{n}$$

The end masses move in unison, while the middle mass moves in the opposite direction, with twice the amplitude. In this case there are two vibration nodes, one in between the first and the second mass and the other in between the second and third mass.

The general solution is

$$\begin{bmatrix} \theta_a \\ \theta_b \\ \theta_c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} y_1 + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} y_2 + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} y_3 \tag{o}$$

or

$$\begin{bmatrix} \theta_a \\ \theta_b \\ \theta_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{p}$$

or

$$\{\theta\} = [\mu]\{y\} \tag{q}$$

with

$$y_1 = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t \tag{r}$$

$$y_2 = A_2 \cos \omega_2 t + B_2 \sin \omega_2 t \tag{s}$$

$$y_3 = A_3 \cos \omega_3 t + B_3 \sin \omega_3 t \tag{t}$$

The y coordinates are called the *modal* coordinates as opposed to the *physical* coordinates denoted by θ. The six integration constants, $A_1, A_2, A_3, B_1, B_2, B_3$, may be determined from initial conditions, that is from the initial displacements and velocities of the three masses.

Orthogonality of Modes of Vibration

The eigenmodes of vibration have certain orthogonality properties that can be used to advantage when the equations of motion are to be solved. Consider the equations of motion for the vibration in two distinct modes r and s. According to equation (6.2.8) we can write these as

$$\omega_r^2 [m]\{\mu\}_r = [k]\{\mu\}_r \tag{6.2.11}$$

$$\omega_s^2 [m]\{\mu\}_s = [k]\{\mu\}_s \tag{6.2.12}$$

Premultiplying the first of these by $\{\mu\}_s^T$ and the second by $\{\mu\}_r^T$ we obtain

$$\omega_r^2 \{\mu\}_s^T [m]\{\mu\}_r = \{\mu\}_s^T [k]\{\mu\}_r \tag{6.2.13}$$

$$\omega_s^2 \{\mu\}_r^T [m]\{\mu\}_s = \{\mu\}_r^T [k]\{\mu\}_s \tag{6.2.14}$$

Transposing the last equation and subtracting it from equation (6.2.13) we obtain

$$(\omega_r^2 - \omega_s^2)\ \{\mu\}_s^T\ [m]\ \{\mu\}_r\ =\ 0 \tag{6.2.15}$$

where we have made use of the symmetry of the $[k]$ and $[m]$ matrices.

Now equation (6.2.15) shows that eigenvectors $\{\mu\}_s$ and $\{\mu\}_r$ are orthogonal with respect to the mass matrix since $\omega_r \neq \omega_s$. Using this result in equation (6.2.13) we deduce that this orthogonality property is also valid with respect to the stiffness matrix, i.e.

$$\{\mu\}_s^T\ [k]\ \{\mu\}_r\ =\ 0 \tag{6.2.16}$$

In the special case of modes of vibration with equal frequencies, that is when $\omega_r = \omega_s$, this orthogonality property can again be established. In such cases, when one of the eigenvectors has been computed, a second one can be generated such that it is orthogonal to the first with respect to either the stiffness or the mass matrix.

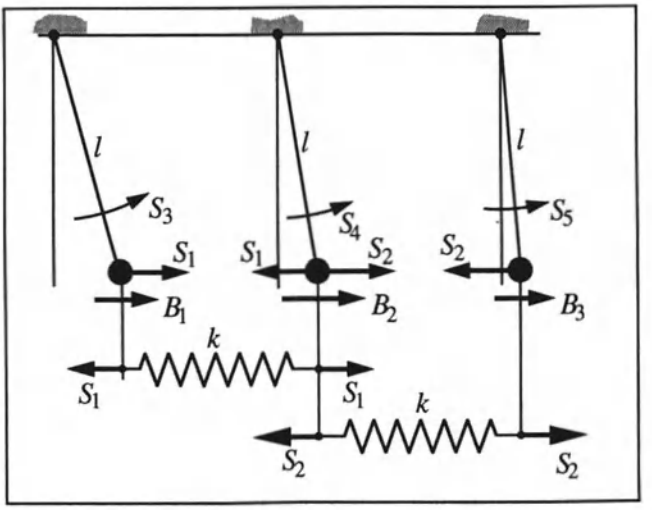


Figure 6.3 Impulse variables

Example 6.2.2

To see the relations between the energy and the complementary energy procedures let us analyse the free vibrations of the 3 pendulum problem by means of the complementary energy method. In this case we need to work with equilibrating impulse variables. Let us denote the linear (spring) impulses by S_1, S_2 and the angular (gravitational) impulses by S_3, S_4, S_5, as shown in Figure 6.3. There are thus 5 impulsive degrees of freedom.

The linear generalized momenta B_i of the 3 masses can now be expressed as

$$B_1 = S_1 + \frac{S_3}{l} \tag{a}$$

$$B_2 = S_2 - S_1 + \frac{S_4}{l} \tag{b}$$

$$B_3 = \frac{S_5}{l} - S_2 \tag{c}$$

Thus the kinetic energy becomes

$$T = \frac{1}{2m}(B_1^2 + B_2^2 + B_3^2)$$

$$\text{or} \quad T = \frac{1}{2m}\left[(S_1 + \frac{S_3}{l})^2 + (S_2 - S_1 + \frac{S_4}{l})^2 + (\frac{S_5}{l} - S_2)^2\right] \tag{d}$$

Consider next the complementary potential energy. For the springs we may write

$$V^*_{\text{spring}} = \frac{1}{2k}\left[\dot{S}_1^2 + \dot{S}_2^2\right] \tag{e}$$

For the gravity component we wrote the approximate potential energy for small amplitudes, for one pendulum, as

$$V_{\text{gravity}} = \frac{1}{2mg} l\theta^2 \tag{f}$$

Then

$$\frac{\partial V_{\text{gravity}}}{\partial \theta} = mgl\theta = -\dot{S} \tag{g}$$

Noting that the force displacement relation is linear and thus the potential energy and the complementary potential energy are equal, we may evaluate V^*_{gravity} for one pendulum as

$$V^*_{\text{gravity}} = V_{\text{gravity}} = \frac{1}{2} mgl \left(-\frac{\dot{S}}{mgl}\right)^2 = \frac{\dot{S}^2}{2mgl} \tag{h}$$

Thus the total complementary potential energy for the three pendulum system may now be expressed as

$$V^* = \frac{1}{2k}\left[\dot{S}_1^2 + \dot{S}_2^2\right] + \frac{1}{2mg\ l}\left[\dot{S}_3^2 + \dot{S}_4^2 + \dot{S}_5^2\right] \tag{i}$$

Using above expressions for T and V^* and $L^* = T - V^*$ in the complementary Lagrange equation

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}_j} - \frac{\partial L^*}{\partial S_j} = 0 \tag{j}$$

we find the equations of motion as

$$\begin{bmatrix} 1/k & 0 & 0 & 0 & 0 \\ 0 & 1/k & 0 & 0 & 0 \\ 0 & 0 & 1/mgl & 0 & 0 \\ 0 & 0 & 0 & 1/mgl & 0 \\ 0 & 0 & 0 & 0 & 1/mgl \end{bmatrix} \begin{bmatrix} \ddot{S}_1 \\ \ddot{S}_2 \\ \ddot{S}_3 \\ \ddot{S}_4 \\ \ddot{S}_5 \end{bmatrix} +$$

$$\begin{bmatrix} 2/m & -1/m & 1/ml & -1/ml & 0 \\ -1/m & 2/m & 0 & 1/ml & -1/ml \\ 1/ml & 0 & 1/ml^2 & 0 & 0 \\ -l/ml & 1/ml & 0 & 1/ml^2 & 0 \\ 0 & -1/ml & 0 & 0 & 1/ml^2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{k}$$

Above equations of motion are statements of compatibility. Noting that

$$\frac{\ddot{S}_1}{k} = -\dot{e}_1 \tag{l}$$

and

$$\left[\frac{S_1}{m} + \frac{S_3}{ml}\right] - \left[\frac{S_2}{m} + \frac{S_4}{ml} - \frac{S_1}{m}\right] = \frac{B_1}{m} - \frac{B_2}{m} \tag{m}$$

we can interpret the first equation of motion as: the rate of compression of the first spring is equal to the rate at which the first two masses move towards each other.

Assuming a harmonic solution with frequency ω, we convert the equations of motion into the following eigenvalue form

$$\begin{bmatrix} 2/m - \omega^2/k & -1/m & 1/ml & -1/ml & 0 \\ -1/m & 2/m - \omega^2/k & 0 & 1/ml & -1/ml \\ 1/ml & 0 & 1/ml^2 - \omega^2/mgl & 0 & 0 \\ -1/ml & 1/ml & 0 & 1/ml^2 - \omega^2/mgl & 0 \\ 0 & -1/ml & 0 & 0 & 1/ml^2 - \omega^2/mgl \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{n}$$

On expanding the determinant and equating it to zero, we find the following characteristic equations for the eigenvalues ω^2:

$$\omega^4 (\omega^2 - \frac{g}{l}) \left[\omega^2 - (\frac{g}{l} + \frac{k}{m})\right] \left[\omega^2 - (\frac{g}{l} + \frac{3k}{m})\right] = 0 \tag{o}$$

Thus we find five frequencies, two of which are equal to zero, pointing to a double degeneracy. The remaining frequencies are, as expected, identical to the frequencies obtained by the direct energy formulation. The two zero frequencies are associated with the two degrees of statical indeterminacy of our system. The corresponding modes describe the nonzero values of S_j which are in self-equilibrium and hence do not give rise to momenta for the masses.

The eigenvalues ω_r^2 and the associated eigenvectors $\{\mu\}_r$ are as follows:

$$r = 1 \qquad \omega_1^2 = \frac{g}{l} \qquad [\,0,\ 0,\ 1,\ 1,\ 1\,] = \{\mu\}_1^T \tag{p}$$

$$r = 2 \qquad \omega_2^2 = \frac{k}{m} + \frac{g}{l} \qquad [\,1,\ 1,\ \frac{mg}{k},\ 0,\ -\frac{mg}{k}\,] = \{\mu\}_2^T \tag{q}$$

$$r = 3 \qquad \omega_3^2 = \frac{3k}{m} + \frac{g}{l} \qquad [\,3,\ -3,\ \frac{mg}{k},\ -2\frac{mg}{k},\ \frac{mg}{k}\,] = \{\mu\}_3^T \tag{r}$$

These vectors, as expected, exhibit orthogonality properties with respect to either of the two system matrices. For the (equal) zero frequencies we may find many (linearly related) vectors. For instance the following eigenvectors are associated with the zero frequencies:

For

$$r = 4 \qquad \omega_4^2 = 0 \qquad [\,1,\ 1,\ -l,\ 0,\ l\,] = \{\mu\}_4^T \tag{s}$$

$$r = 5 \qquad \omega_5^2 = 0 \qquad [\,2,\ 1,\ -2l,\ l,\ l\,] = \{\mu\}_5^T \tag{t}$$

Again each of these vectors is orthogonal (with respect to either system matrix) to the nonzero frequency mode vectors. However the two vectors are not mutually orthogonal (with respect to either system matrix). Nevertheless we may construct a new vector and require that it be orthogonal to one of the zero frequency mode vectors. To this end we write

$$\{\mu\}_4 + b\{\mu\}_5 = \begin{bmatrix} 1 \\ 1 \\ -l \\ 0 \\ l \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ -2l \\ l \\ l \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \{a\} \tag{u}$$

Now we require that the vector $\{a\}$ be orthogonal (with respect to one of the system matrices of equation (k)) to, say, the first zero frequency mode i.e.

$$[1 \;\; 1 \;\; -l \;\; 0 \;\; l] \begin{bmatrix} 1/k & 0 & 0 & 0 & 0 \\ 0 & 1/k & 0 & 0 & 0 \\ 0 & 0 & 1/mgl & 0 & 0 \\ 0 & 0 & 0 & 1/mgl & 0 \\ 0 & 0 & 0 & 0 & 1/mgl \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ -l \\ 0 \\ l \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ -2l \\ l \\ l \end{bmatrix} \right) = 0 \tag{v}$$

or

$$\left(\frac{2}{k} + \frac{2l}{mg}\right) + b\left(\frac{3}{k} + \frac{3l}{mg}\right) = 0 \tag{w}$$

Therefore

$$b = -\frac{2}{3} \tag{x}$$

and from equation (u), after cancelling a multiplicative constant, we find the new eigenvector, associated with the second zero frequency mode as

$$\omega_5^2 = 0 \qquad [-1, \; 1, \; l, \; -2l, \; l,] = \{\mu\}_{5new}^T \tag{y}$$

The general solution of equation (k) is

$$\{S\} = [\mu]\{y\} \tag{z}$$

or

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 1 & -1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & \frac{mg}{k} & \frac{mg}{k} & -l & l \\ 1 & 0 & \frac{-2mg}{k} & 0 & -2l \\ 1 & -\frac{mg}{k} & \frac{mg}{k} & l & l \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

with

$$y_1 = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t$$

$$y_2 = A_2 \cos \omega_2 t + B_2 \sin \omega_2 t$$

$$y_3 = A_3 \cos \omega_3 t + B_3 \sin \omega_3 t$$

$$y_4 = A_4 + B_4 t$$

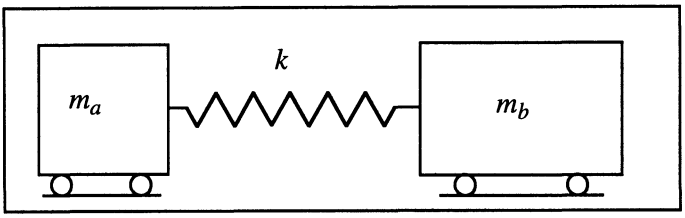
$$y_5 = A_5 + B_5 t$$

Note that in this example the modal coordinates y are impulses, rather than displacements.

Problems

6.2.1 For the triple pendulum of Figure 6.2 find (a) the three natural frequencies, and (b) the integration constants A and B, if $l = 2.5$ m, $k = 12$ N/m, $m = 2$ kg, $\theta_{ao} = 0$, $\theta_{bo} = 0.03$ rad, $\theta_{co} = 0.06$ rad, $\dot{\theta}_{ao} = \dot{\theta}_{bo} = \dot{\theta}_{co} = 0$. Write (c) the solutions for the three angles as function of time.

6.2.2 Given is a two-mass, one-spring system, as shown. Find the complementary Lagrangian, establish the equation of motion in the impulse coordinate, and find the natural frequency.



6.2.3 For the Problem 6.1.1 obtain the natural frequencies and modes of vibration.

6.3 Modal Matrix and Decoupling of Equations of Motion

The n linearly independent eigenvectors of a linear system may be used as basis vectors of a new n dimensional space which we may call the *eigenvector space.* As we have seen these basis vectors exhibit orthogonality properties with respect to the two system matrices. We may establish a transformation from the configuration space to the eigenvector space by means of a new matrix called the *modal matrix.* The modal matrix is set up by stacking the eigenvectors of the system, side by side, as follows

$$[\mu] = [\{\mu\}_1 \{\mu\}_2 \ldots \{\mu\}_r \ldots \{\mu\}_n] \tag{6.3.1}$$

where $\{\mu\}_r$ is the rth eigenvector (or modal vector). Now denoting the coordinates of a point in the eigenvector space by $[y_1 y_2 \ldots y_r \ldots y_n]$, y_r being the coordinate measured along the $\{\mu\}_r$ basis vector, we may write the relation between the coordinates q_i in the configuration space and the coordinates y_r in the eigenvector space as

$$\{q\} = [\mu] \{y\} \tag{6.3.2}$$

This transformation equation may be used to diagonalize the two system matrices, that is in the eigenvector space the equations of motion become decoupled. We illustrate this for the case when the variables are generalized displacements and the equations of motion take the form of equilibrium equations. Similar results hold for the complementary energy formulation.

Noting that $[\mu]$ is a square matrix of constants, we can transform the equilibrium equations of motion for free vibrations

$$[m] \{\ddot{q}\} + [k] \{q\} = \{0\} \tag{6.3.3}$$

to

$$[m]\,[\mu]\,\{\ddot{y}\} + [k]\,[\mu]\,\{y\} = \{0\}$$

Premultiplying through by $[\mu]^T$ we obtain

$$[\mu]^T\,[m]\,[\mu]\,\{\ddot{y}\} + [\mu]^T\,[k]\,[\mu]\,\{y\} = \{0\} \tag{6.3.4}$$

which may be written as

$$[m_r]\,\{\ddot{y}\} + [k_r]\,\{y\} = \{0\} \tag{6.3.5}$$

Recalling the structure of the modal matrix one can see that the triple matrix product $[\mu]^T\,[m]\,[\mu]$ has the following form

$$[m_r] = [\mu]^T\,[m]\,[\mu] = \begin{bmatrix} \{\mu\}_1^T\,[m]\,\{\mu\}_1 & \{\mu\}_1^T\,[m]\,\{\mu\}_2 & \cdots & \{\mu\}_1^T\,[m]\,\{\mu\}_n \\ \vdots & \vdots & & \vdots \\ \{\mu\}_n^T\,[m]\,\{\mu\}_1 & \{\mu\}_n^T\,[m]\,\{\mu\}_2 & \cdots & \{\mu\}_n^T\,[m]\,\{\mu\}_n \end{bmatrix}$$

$$= \begin{bmatrix} \{\mu\}_1^T\,[m]\,\{\mu\}_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \{\mu\}_n^T\,[m]\,\{\mu\}_n \end{bmatrix} \tag{6.3.6}$$

This matrix is diagonal, in view of the orthogonality of the eigenvectors with respect to the mass matrix. Likewise the triple product $[\mu]^T\,[k]\,[\mu]$ yields a diagonal matrix $[k_r]$ and therefore equation (6.3.5) may be written as

$$\begin{bmatrix} m_{r=1} & 0 & & 0 \\ 0 & m_{r=2} & & \\ & & \ddots & 0 \\ 0 & & 0 & m_{r=n} \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_n \end{bmatrix} + \begin{bmatrix} k_{r=1} & 0 & & 0 \\ 0 & k_{r=2} & & \\ & & \ddots & 0 \\ 0 & & 0 & k_{r=n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6.3.7}$$

where m_r and k_r are the modal mass coefficients and the modal stiffness coefficients, given by

$$m_r = \{\mu\}_r^T\,[m]\,\{\mu\}_r \tag{6.3.8}$$

and

$$k_r = \{\mu\}_r^T\,[k]\,\{\mu\}_r \tag{6.3.9}$$

For free vibration

$$[m_r]\,\{\ddot{y}\} + [k_r]\,\{y\} = \{0\} \tag{6.3.10}$$

or

$$m_r \ddot{y}_r + k_r y_r = 0 \tag{6.3.11}$$

with

$$k_r/m_r = \omega_r^2 \tag{6.3.12}$$

Since the eigenvectors are determined to within a multiplicative constant, it is possible to normalize them such that all the modal mass coefficients become equal to unity. Then the modal stiffness coefficients become equal to k_r/m_r, i.e. equal to the eigenvalues ω_r^2.

In the modal coordinates the now decoupled equations of motion can be solved independently. Thus for the rth mode we find

$$y_r = (y_r)_o \cos\omega_r t + \frac{(\dot{y}_r)_o}{\omega_r} \sin\omega_r t \tag{6.3.13}$$

where $(y_r)_o$ and $(\dot{y}_r)_o$ are the initial displacement and initial velocity, in mode r.

Normally the initial conditions are specified in terms of the physical coordinates q_i. From the transformation equation (6.3.2) we may determine the initial conditions in the modal coordinates as follows:

$$\{y\}_o = [\mu]^{-1}\{q\}_o \tag{6.3.14}$$

$$\{\dot{y}\}_o = [\mu]^{-1}\{\dot{q}\}_o \tag{6.3.15}$$

The orthogonality property of the eigenvectors allows one to compute $[\mu]^{-1}$ indirectly and easily as follows. Given that

$$[\mu]^T [m] [\mu] = [m_r] \tag{6.3.16}$$

where $[m_r]$ is the diagonal modal mass matrix, we may obtain $[\mu]^{-1}$ as

$$[\mu]^{-1} = [m_r]^{-1}[\mu]^T [m] \tag{6.3.17}$$

If furthermore the eigenvectors are normalized such that each $m_r = 1$, the computation of $[\mu]^{-1}$ simplifies to a matrix product.

Once the solution for each mode has been obtained in the modal coordinates y_r, the solution in the physical coordinates q_i may be computed via equation (6.3.2). It can be seen that the motion of the system in the physical coordinates is made up of a linear combination of the system's motion in modal coordinates, in each of its modes.

Similar argumentation can be employed when generalized impulse coordinates are used. Instead of equation (6.3.2), one would then begin with

$$\{S\} = [\mu] \{y\} \tag{6.3.18}$$

where $[\mu]$ is the modal matrix made up of impulse eigenvectors stacked side by side. Since the modal matrices are of an order equal to the number of degrees of freedom of the system, it is clear that the displacement and impulse modal matrices need not be of equal order.

Example 6.3.1

Consider the problem of the three pendulums of example 6.2.1 again. Suppose that the motion of the system is started with the initial conditions

$$\begin{aligned} \theta_a &= 0 \qquad \theta_b = 0 \qquad \theta_c = 0 \\ \dot{\theta}_a &= (\dot{\theta}_a)_o \qquad \dot{\theta}_b = 0 \qquad \dot{\theta}_c = 0 \end{aligned} \tag{a}$$

where $(\dot{\theta}_a)_o$ is the initial angular velocity. For this problem the eigenvectors, as determined earlier, can be stacked side by side to form the modal matrix (6.3.1) as

$$[\mu] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \tag{b}$$

The modal mass matrix (6.3.6) can now be computed as

$$[m_r] = [\mu]^T [m] [\mu] \tag{c}$$

or

$$[m_r] = \begin{bmatrix} 3ml^2 & 0 & 0 \\ 0 & 2ml^2 & 0 \\ 0 & 0 & 6ml^2 \end{bmatrix} \tag{d}$$

with an inverse

$$[m_r]^{-1} = \begin{bmatrix} \frac{1}{3ml^2} & 0 & 0 \\ 0 & \frac{1}{2ml^2} & 0 \\ 0 & 0 & \frac{1}{6ml^2} \end{bmatrix} \tag{e}$$

Next let us compute $[\mu]^{-1}$. From equation (6.3.18) we have that

$$[\mu]^{-1} = [m_r]^{-1} [\mu]^T [m] \tag{f}$$

or

$$[\mu]^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix} \tag{g}$$

Now since at $t = 0$, the $q_i = \theta_i = 0$, it can be seen from equation (6.3.15) that all $(y_r)_o$ also vanish. We may determine $(\dot{y}_r)_o$ using equations (6.3.16) and (g). Thus

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix}_o = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & -1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix} \begin{bmatrix} (\dot{\theta}_a)_o \\ 0 \\ 0 \end{bmatrix} \tag{h}$$

Hence

$$(\dot{y}_1)_o = (\dot{\theta}_a)_o/3 \qquad (\dot{y}_2)_o = (\dot{\theta}_a)_o/2 \qquad (\dot{y}_3)_o = (\dot{\theta}_a)_o/6 \tag{i}$$

Accordingly the angular displacements y_r in the 3 modes, as obtained from equation (6.3.13), may be written as follows

$$\begin{aligned} y_1(t) &= B_1 \sin \omega_1 t = \frac{(\dot{\theta}_a)_o}{3\omega_1} \sin \omega_1 t \\ y_2(t) &= B_2 \sin \omega_2 t = \frac{(\dot{\theta}_a)_o}{2\omega_2} \sin \omega_2 t \\ y_3(t) &= B_3 \sin \omega_3 t = \frac{(\dot{\theta}_a)_o}{6\omega_3} \sin \omega_3 t \end{aligned} \tag{j}$$

where the natural frequencies squared, ω_r^2, as found earlier, are given by

$$\omega_1^2 = \frac{g}{l} \qquad \omega_2^2 = \frac{g}{l} + \frac{k}{m} \qquad \omega_3^2 = \frac{g}{l} + 3\frac{k}{m} \tag{k}$$

Finally transforming back to the physical coordinates, we find for $\{q\} = [\mu]\{y\}$,

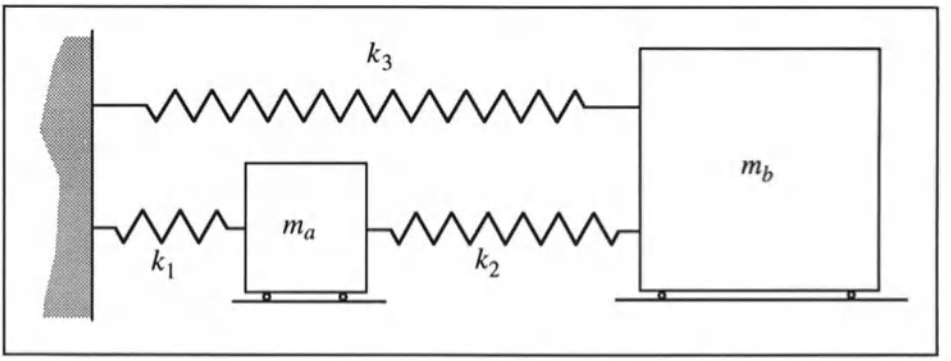
$$\begin{bmatrix} \theta_a \\ \theta_b \\ \theta_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{(\dot{\theta}_a)_o}{3\omega_1} \sin \omega_1 t \\ \frac{(\dot{\theta}_a)_o}{2\omega_2} \sin \omega_2 t \\ \frac{(\dot{\theta}_a)_o}{6\omega_3} \sin \omega_3 t \end{bmatrix} \tag{l}$$

Thus the angular displacements $q_i = \theta_i$ of the three masses can be expressed as a function of time as follows:

$$\begin{aligned} \theta_a &= \frac{1}{3\omega_1}(\dot{\theta}_a)_o \sin \omega_1 t + \frac{1}{2\omega_2}(\dot{\theta}_a)_o \sin \omega_2 t + \frac{1}{6\omega_3}(\dot{\theta}_a)_o \sin \omega_3 t \\ \theta_b &= \frac{1}{3\omega_1}(\dot{\theta}_a)_o \sin \omega_1 t - \frac{1}{3\omega_3}(\dot{\theta}_a)_o \sin \omega_3 t \\ \theta_c &= \frac{1}{3\omega_1}(\dot{\theta}_a)_o \sin \omega_1 t - \frac{1}{2\omega_2}(\dot{\theta}_a)_o \sin \omega_2 t + \frac{1}{6\omega_3}(\dot{\theta}_a)_o \sin \omega_3 t \end{aligned} \tag{m}$$

Problem

6.3.1 Given is a two-mass, three-spring oscillator chain as shown. Use q_a and q_b and find (a) the kinetic coenergy, (b) the potential energy, and (c) the Lagrangian. Establish (d) the equations of motion of free vibration. By letting $k_1 = k_2 = k_3$ and $m_a = 1/2\ m_b$, solve for (e) the natural frequencies, and find (f) the modal eigenvectors. Establish (g) the modal matrix, and (h) write the decoupled equations of motion.



6.4 Forced Undamped Oscillations

When generalized disturbing forces $Q_i'(t)$ are present the equilibrium equations (6.1.15) of motion take the form

$$[m]\{\ddot{q}\} + [k]\{q\} = \{Q'\} \tag{6.4.1}$$

In the interest of notational brevity, we shall use $\{Q\}$ (instead of $\{Q'\}$) for the disturbing forces in the present chapter, and write

$$[m]\{\ddot{q}\} + [k]\{q\} = \{Q\} \tag{6.4.2}$$

The general solution of this equation is made up of a homogeneous part, for which $Q_i = 0$, and a particular part, with $Q_i \neq 0$. To obtain these solutions we transform once again from the physical generalized coordinates q_i to the modal coordinates y_r, via the modal matrix. Thus we let

$$\{q\} = [\mu]\{y\} \tag{6.4.3}$$

Substituting into equation (6.4.1) we find

$$[m][\mu]\{\ddot{y}\} + [k][\mu]\{y\} = \{Q\} \tag{6.4.4}$$

Now by premultiplying equation (6.4.4) by $[\mu]^T$ we diagonalize the mass and stiffness matrices and obtain the following set of decoupled equations

$$m_r \ddot{y}_r + k_r y_r = Y_r \qquad r = 1,2, ..., n \tag{6.4.5}$$

where

$$m_r = \{\mu\}_r^T [m]\{\mu\}_r \qquad \text{and} \qquad k_r = \{\mu\}_r^T [k]\{\mu\}_r$$

and

$$Y_r = \{\mu\}_r^T \{Q\} \tag{6.4.6}$$

The generalized force Y_r represents the sum of the components of Q_i, with $i = 1,2, ..., n$, along the rth coordinate axis of the eigenvector space. Thus Y_r is a measure of the participation of $\{Q\}$ in the rth mode.

Dividing through by m_r and recalling that $k_r/m_r = \omega_r^2$ we may express equation (6.4.5) as

$$\ddot{y}_r + \omega_r^2 y_r = \frac{Y_r}{m_r} \tag{6.4.7}$$

One method of solving this equation is to represent the force $Y(t)$ by its associated elemental impulse $Y\, d\tau$, such that the associated change of momentum is given by

$$Y\, d\tau = m\, d\dot{y} \tag{6.4.8}$$

where $d\dot{y}$ denotes a change in the velocity of the mass. Now we showed in equation (6.3.13) that under free vibrations the response of the system is given by

$$y(t) = (y)_o \cos \omega t + \frac{(\dot{y})_o}{\omega} \sin \omega t$$

Thus when the system is initially at rest, the application of an elemental impulse increment $Y(0)d\tau$ at time $t = 0$ will give rise to an incremental displacement response of

$$dy = \frac{d\dot{y}_o}{\omega} \sin \omega t = \frac{Y(0)\, d\tau}{m\omega} \sin \omega t \tag{6.4.9}$$

Suppose next that the elemental impulse increment is applied at time $t = \tau$ (see Figure 6.4). In that event the response will commence at time $t = \tau$, and at time t we find

$$dy = \frac{Y(\tau)\, d\tau}{m\omega} \sin \omega(t - \tau) \tag{6.4.10}$$

or

$$dy = g(t - \tau)\, Y(\tau)\, d\tau \tag{6.4.11}$$

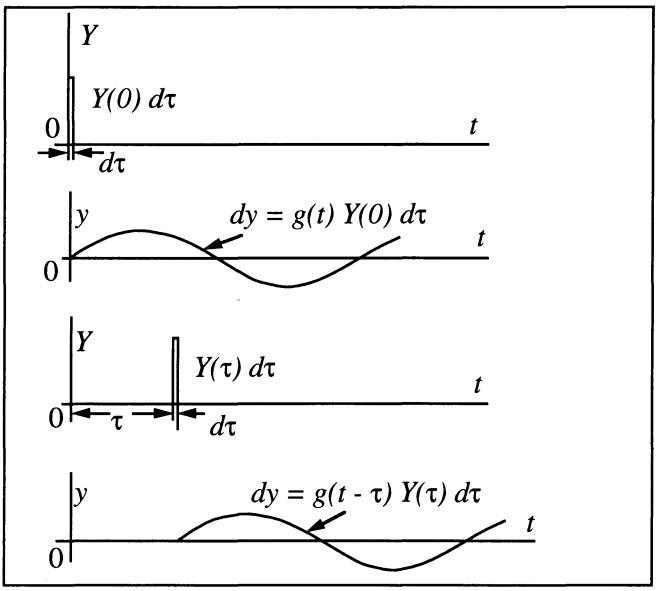


Figure 6.4 Spike force, impulse and response

where

$$g(t-\tau) = \frac{\sin \omega(t-\tau)}{m\omega} \tag{6.4.12}$$

is the so-called *unit impulse response.*

Now we may represent the forcing function $Y(t)$ by a series of incremental impulses acting one after the other each at its time of application τ. Since the system is linear, the total response may be obtained by superposition of responses for each incremental impulse. Thus we may write

$$\int_o^t dy = \int_o^t g(t-\tau)\, Y(\tau)\, d\tau \tag{6.4.13}$$

or

$$y(t) = \int_o^t Y(\tau)\, g(t-\tau)\, d\tau \tag{6.4.14}$$

The integral in equation (6.4.14) is the *Duhamel*† (or *faltungs* or *convolution*) integral.

To illustrate the application of equation (6.4.14) let us consider the case of a harmonically varying forcing function with frequency Ω. That is we write

$$Y(\tau) = Y_o \cos \Omega\tau \tag{6.4.15}$$

On substituting into equations (6.4.14) and (6.4.12) and carrying out the integration, by parts (twice), we find the particular solution as

$$y(t) = \frac{Y_o(\cos \Omega t - \cos \omega t)}{m(\Omega^2 - \omega^2)} \tag{6.4.16}$$

Thus the complete response in the rth mode may be written as

$$y_r(t) = \left[y_r(0) - \frac{Y_o}{m_r(\Omega^2 - \omega_r^2)} \right] \cos \omega_r t + \frac{\dot{y}_r(0)}{\omega_r} \sin \omega_r t + \frac{Y_o}{m_r(\Omega^2 - \omega_r^2)} \cos \Omega t \tag{6.4.17}$$

It is of interest to note that the general motion is composed of oscillations at the natural frquency ω_r and the forcing frequency Ω. In any real system, where some light damping is always present, the oscillations with frequency ω_r will decay while those with frequency Ω will persist. Accordingly the former part of the response is called the *transient* while the latter is the *steady state.*

If one specifies that initial displacement $y_r(0) = 0$ and initial velocity $\dot{y}_r(0) = 0$, then of equation (6.4.17) there remains

$$y_r(t) = \frac{Y_o(\cos \Omega t - \cos \omega_r t)}{m_r(\Omega^2 - \omega_r^2)} \tag{6.4.18}$$

---

† Jean-Marie-Constant Duhamel (1797-1872), French mathematician

In the special case when $\Omega \to \omega_s$, where ω_s is one of the natural frequencies ω_r of the system, the response becomes indeterminate in form. We may however evaluate this term by L'Hospital's† rule as

$$y_s(t) = \frac{Y_o}{m_s} \lim_{\Omega \to \omega_s} \frac{\frac{d}{d\Omega}[\cos \Omega t - \cos \omega_s t]}{\frac{d}{d\Omega}[\Omega^2 - \omega_s^2]}$$

$$y_s(t) = \frac{Y_o}{m_s} \lim_{\Omega \to \omega_s} \frac{t \sin \Omega t}{2\Omega} = \frac{Y_o}{m_s} \frac{t \sin \omega_s t}{2\omega_s} \tag{6.4.19}$$

Thus in this case the response becomes (Figure 6.5)

$$y_s(t) = \frac{Y_o}{2m_s\omega_s} t \sin \omega_s t \tag{6.4.20}$$

This equation shows that in the case of resonance ($\Omega = \omega_s$) in mode s, the modal response $y_s(t)$ increases linearly in time. For most real systems there is always some damping present and also the increase in response amplitude brings into play nonlinear effects which have been neglected. These effects generally limit the growth of the response.

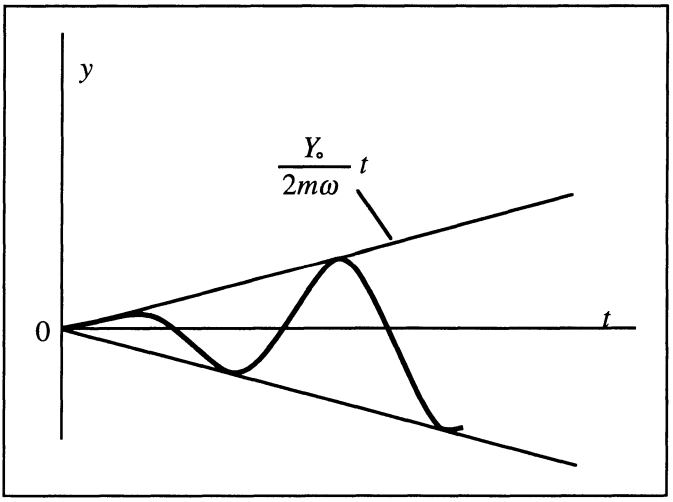


Figure 6.5 Response at resonance

Problems

6.4.1 Solve the Duhamel integral for the step function $Y(\tau) = Y_o = \text{constant}$.

† Guillaume Francois Antoine Marquis de l'Hospital (1661-1704), French mathematician.

6.4.2 Solve the Duhamel integral for the ramp function $Y(\tau) = \frac{Y_1}{t_1}\tau$.

6.5 Free Damped Oscillations

When viscous damping forces are present the linearised equilibrium equations of free motion take the following form

$$[m]\{\ddot{q}\} + [c]\{\dot{q}\} + [k]\{q\} = \{0\} \tag{6.5.1}$$

where $[c]$ is the damping matrix. For light damping, which is the most common case, the amplitudes of damping forces will be much smaller than those of inertial and elastic forces. As such one can anticipate that the motion of a lightly damped system will be very close to that of an undamped system. The major differences arise near resonance. Damping forces typically limit the oscillation amplitudes at resonance.

In attempting to solve equation (6.5.1) by a transformation to modal coordinates we note that the eigenvectors of the undamped system will not, in general, be orthogonal with respect to the damping matrix and hence the modal matrix of the undamped system cannot be used to diagonalize the $[c]$ matrix. Knowing that $[m]$ as well as $[k]$ can be diagonalized, one can approximate the $[c]$ matrix by a linear combination of the mass matrix and the stiffness matrix. That is we let

$$[c]_{approx} = \alpha[m] + \beta[k] \tag{6.5.2}$$

where α and β are some unknown scalar multipliers to be determined later from the damping characteristics of the system, with α in s^{-1} and β in s. On substituting from equation (6.5.2) into equation (6.5.1) and also transforming from the physical generalized coordinates $\{q\}$ to the modal coordinates $\{y\}$, via the modal matrix of the undamped system, we find

$$[m][\mu]\{\ddot{y}\} + (\alpha[m] + \beta[k])[\mu]\{\dot{y}\} + [k][\mu]\{y\} = \{0\} \tag{6.5.3}$$

Now on premultiplying this equation by $[\mu]^T$, as in equation (6.3.4), we obtain decoupled equations of motion, in the modal coordinates. These appear as

$$m_r \ddot{y}_r + (\alpha m_r + \beta k_r)\dot{y}_r + k_r y_r = 0 \tag{6.5.4}$$

or

$$\ddot{y}_r + 2\zeta_r \omega_r \dot{y}_r + \omega_r^2 y_r = 0 \tag{6.5.5}$$

where

$$\omega_r^2 = k_r/m_r \tag{6.5.6}$$

and

$$2\zeta_r \omega_r = \alpha + \beta\omega_r^2 \tag{6.5.7}$$

Note that α and β are constants for the whole system, not just for one mode.

The auxiliary equation for the differential equation (6.5.5) takes the form

$$D^2 + 2\zeta_r \omega_r D + \omega_r^2 = 0 \tag{6.5.8}$$

with roots given by

$$D_{1,2} = -\zeta_r \omega_r \pm \omega_r \sqrt{\zeta_r^2 - 1} \tag{6.5.9}$$

Thus the solution becomes

$$y_r(t) = C_r e^{(-\zeta_r + \sqrt{\zeta_r^2 - 1})\omega_r t} + E_r e^{(-\zeta_r - \sqrt{\zeta_r^2 - 1})\omega_r t} \tag{6.5.10}$$

where C_r and E_r are integration constants to be determined from initial conditions.

Now depending upon the value of ζ_r, three distinct types of motion can be identified:

i) $\zeta_r > 1$. In this case y_r decays exponentially. The fastest decay occurs when $\zeta_r = 1$. This value is called critical damping.

ii) $\zeta_r < 1$. For this case, equation (6.5.10) may be expressed as

$$y_r = e^{-\zeta_r \omega_r t} (A_r \cos \omega_{cr} t + B_r \sin \omega_{cr} t) \tag{6.5.11}$$

where

$$\omega_r \sqrt{1 - \zeta_r^2} = \omega_{cr} \tag{6.5.12}$$

is the *damped natural frequency*. The motion is oscillatory with decaying amplitude.

iii) $\zeta_r < 0$. In this case the motion is again oscillatory, but y_r *grows* exponentially. This case of *negative damping* (or *fanning*) gives rise to dynamic instability.

Denoting the initial conditions in the rth mode by $y_r(0)$ and $\dot{y}_r(0)$, we may express $y_r(t)$ as

$$y_r(t) = e^{-\zeta_r \omega_r t} \left[y_r(0) \cos \omega_{cr} t + \frac{\dot{y}_r(0) + \zeta_r \omega_r y_r(0)}{\omega_{cr}} \sin \omega_{cr} t \right] \tag{6.5.13}$$

From a knowledge of the damping ratio ζ_r for any two modes, the coefficients α and β may be found by means of equation (6.5.7). If data exist for several modes, the coefficients α and β are best determined from a least square fit of equation (6.5.7) to the data available.

Problems

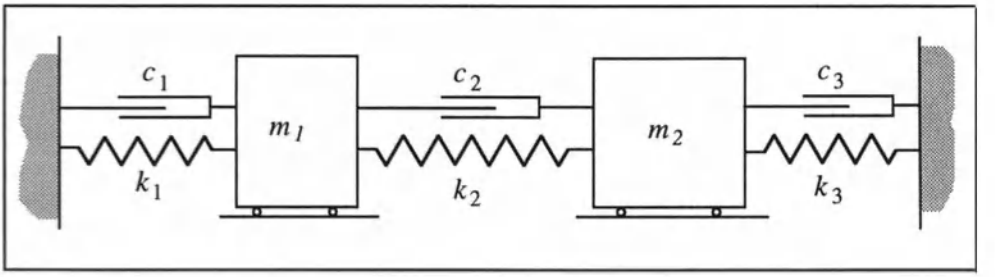
6.5.1 A lightly damped system with two degrees of freedom is measured to have the undamped natural frequencies $\omega_1 = 12$ rad/s and $\omega_2 = 20$ rad/s. The damping ratios associated with these frequencies are $\zeta_1 = 0.09$ and $\zeta_2 = 0.04$. Determine the damping constants α and β for this system.

6.5.2 A lightly damped system's decoupled equations of motion are

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} 0.18 & 0 \\ 0 & 0.19 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find (a) the damping coefficients α and β, (b) the undamped natural frequencies ω_r, (c) the damping ratios ζ_r, and (d) the damped natural frequencies ω_{cr}.

6.5.3 Given is a system with two masses m_i, three springs k_j, and three dashpots c_k. (a) Establish the equations of motion with $q_1 = x_1$ and $q_2 = x_2$. Let $m_1 = 4$ kg, $m_2 = 3$ kg, $c_1 = 0.1$ kg/s, $c_2 = 0.12$ kg/s, $c_3 = 0.13$ kg/s, $k_1 = 100$ N/m, $k_2 = 120$ N/m, and $k_3 = 130$ N/m. (b) Find the natural frequencies of the undamped system. (c) Find α, β and damping ratios ζ_r. (d) Calculate the damped frequencies.



6.6 Forced Damped Oscillations

Assuming a damping matrix as given in equation (6.5.2) and decoupling the equilibrium equations of motion of forced oscillations, as described in section 6.4, we find the equation of motion, in the rth mode, as

$$m_r \ddot{y}_r + (\alpha m_r + \beta k_r)\dot{y}_r + k_r y_r = Y_r \tag{6.6.1}$$

where Y_r is the component of the physical force vector $\{Q\}$, in the rth mode. Dividing through by m_r we may express equation (6.6.1) as follows

$$\ddot{y}_r + 2\zeta_r \omega_r \dot{y}_r + \omega_r^2 y_r = \frac{Y_r}{m_r} \tag{6.6.2}$$

where

$$\omega_r^2 = k_r/m_r$$

and

$$2\zeta_r \omega_r = \alpha + \beta\omega_r^2$$

To solve equation (6.6.2) we will follow the procedure described in section 6.4. For this case we find from equation (6.5.13) that the incremental response due to an incremental initial velocity $d\dot{y}_r(0)$, when the system is at rest for $t<0$, is given by

$$dy_r(t) = e^{-\zeta_r \omega_r t}\left[\frac{d\dot{y}_r(0)}{\omega_{cr}} \sin \omega_{cr} t\right]$$

Thus the incremental response due to an incremental impulse $Y_r(0)\, d\tau$, applied at time $t = 0$, may be written as

$$dy_r(t) = e^{-\zeta_r \omega_r t}\left[\frac{Y_r(0)d\tau}{m_r \omega_{cr}} \sin \omega_{cr} t\right] \tag{6.6.3}$$

Hence in the case of damped forced oscillations the function $g(t)$ of equation (6.4.11) takes the following form for the rth mode

$$g_r(t) = \frac{e^{-\zeta_r \omega_r t}}{m_r \omega_{cr}} \sin \omega_{cr} t \tag{6.6.4}$$

and the total response in the rth mode may be determined from

$$y_r(t) = \int_0^t Y_r(\tau)\ g_r(t-\tau)\, d\tau \tag{6.6.5}$$

Example 6.6.1

Let us determine the response in mode r when a constant step force Y_o is applied at time $t = 0$, in this mode. Assuming that initial values of y and $\dot{y}$ are equal to zero, the complete response may be derived from equation (6.6.5) as follows

$$y_r(t) = \frac{Y_o}{m_r \omega_{cr}} \int_o^t e^{-\zeta_r \omega_r (t-\tau)} \sin \omega_{cr}\ (t-\tau)\, d\tau \tag{a}$$

Carrying out the integration by parts (twice) we find the response as

$$y_r(t) = \frac{Y_o}{m_r \omega_r^2} \left[1 - e^{-\zeta_r \omega_r t} \left(\cos \omega_{cr}\ t + \frac{\zeta_r}{\sqrt{1-\zeta_r^2}} \sin \omega_{cr}\ t \right) \right] \tag{b}$$

Evidently the transient component, with oscillations at frequency ω_{cr}, will die out due to the exponential term.

Problem

6.6.1 Find the steady-state response of a damped single degree of freedom system, subject to a forcing function $Y = Y_o \cos \Omega t$.

6.7 Small Oscillations About Steady Motion

In the last section we considered small oscillations about the position of equilibrium which is considered to be fixed. In some systems a dominant steady motion can prevail and a small perturbation may then bring about small oscillations about the dominant steady motion. Consider for instance the motion of a conical pendulum rotating in a horizontal circle. Under steady motion the orientation of the pendulum from the vertical and the angular velocity of the pendulum about the vertical will remain constant. Indeed we can see that the angular momentum, in the circular orbit, will be conserved. Satellites orbiting the earth and tops spinning on horizontal plates provide other examples of steady motion for which certain coordinates and velocities remain constant in time. In such systems generally some momenta are conserved which implies that the coordinates associated with those momenta are ignorable coordinates. If the steady motion of such systems is perturbed, oscillatory motions about the original steady motion will result. In this oscillatory motion the variables which had

constant values in the steady motion will of course change in time, however for small amplitude oscillations such changes in the dominant or gross motion of the system are often neglected.

As noted in Section 3.7 the equations of motion for systems with ignorable coordinates can be conveniently obtained from the Routhian of the system. The following example illustrates this procedure.

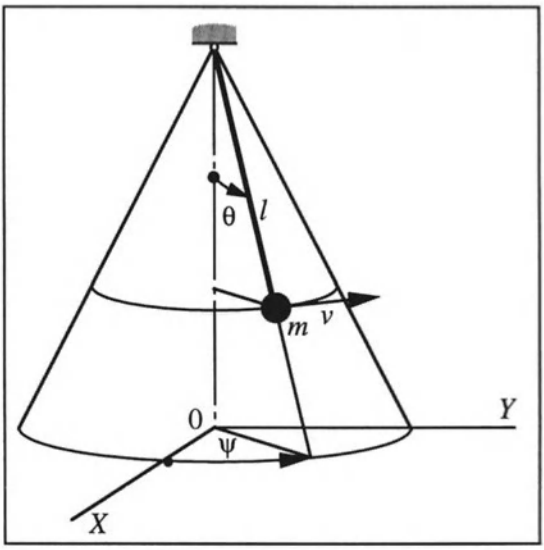


Figure 6.6 Conical pendulum

Example 6.7.1

Consider the motion of the conical pendulum shown in Figure 6.6. For this system we have

$$T^* = \frac{1}{2} m ((l\dot{\theta})^2 + (l\dot{\psi}\sin\theta)^2) \tag{a}$$

$$V = mg\, l\cos\theta \tag{b}$$

It is now evident that in the Lagrangian $L = T^* - V$ the coordinate ψ is absent while $\dot{\psi}$ is present. Hence ψ is an ignorable coordinate and the momentum associated with ψ, namely

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = ml^2\dot{\psi}\sin^2\theta = c_\psi = \text{constant} \tag{c}$$

Now we can write the Routhian for this system, using the relation

$$R = L - p_\psi \dot{\psi} \tag{d}$$

and eliminating $\dot{\psi}$ through equation (c), we find

$$R = \frac{m}{2}\left[l^2\dot{\theta}^2 + l^2\sin^2\theta \left[\frac{c_\psi}{ml^2\sin^2\theta} \right]^2 + mgl\cos\theta \right] - \frac{c_\psi^2}{ml^2\sin^2\theta}$$

or

$$R = \frac{m}{2}(l^2\dot{\theta}^2) - \frac{1}{2}\frac{c_\psi^2}{ml^2\sin^2\theta} + mg\,l\cos\theta \tag{e}$$

The general equation of motion for θ can now be obtained from

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{\theta}} - \frac{\partial R}{\partial \theta} = 0$$

and yields

$$\ddot{\theta} - \frac{c_\psi^2\cos\theta}{m^2 l^4 \sin^3\theta} + \frac{g}{l}\sin\theta = 0 \tag{f}$$

For steady motion $\ddot{\theta} = \dot{\theta} = 0$. By taking the steady state values of θ and $\dot{\psi}$ as θ_0 and $\dot{\psi}_0$ respectively we can write for the steady state motion

$$\frac{c_\psi^2\cos\theta_0}{m^2 l^4 \sin^3\theta_o} = \frac{g}{l}\sin\theta_0 \tag{g}$$

using equation (c) we can simplify this expression to

$$\cos\theta_0 = \frac{g}{l\dot{\psi}_0^2} \tag{h}$$

Now suppose the bob m of the pendulum is tapped lightly with a hammer and assume for the moment that p_ψ remains unchanged. This will give rise to oscillatory motion about the steady motion. To obtain linear equations for this secondary motion we may expand the Routhian about the position of steady motion.

Taylor Series Expansion of Routhian

In the general case of the Routhian we consider small deviations q_i' from the steady state motion and write the Routhian as $R(q_i, \dot{q}_i) = R(q_{i0} + q_i', \dot{q}_{i0} + \dot{q}_i')$ where $(q_{i0}, \dot{q}_{i0})$ are steady state values of the generalized coordinates and velocities. Now expanding by Taylor series up to the second order terms about the steady state motion we have

$$R_{approx} = (R)_0 + \left[\frac{\partial R}{\partial q_i}\right]_0 q_i' + \left[\frac{\partial R}{\partial \dot{q}_i}\right]_0 \dot{q}_i' + \frac{1}{2}\left[\frac{\partial^2 R}{\partial q_i \partial q_j}\right]_0 q_i' q_j' + \frac{1}{2}\left[\frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_j}\right]_0 \dot{q}_i' \dot{q}_j' + \left[\frac{\partial^2 R}{\partial \dot{q}_i \partial q_j}\right]_0 \dot{q}_i' q_j' + \cdots \tag{6.7.1}$$

where $(\,)_0$ terms are constants to be evaluated for the steady state motion. The linearized equations of motion can now be obtained from

$$\frac{d}{dt}\frac{\partial R_{approx}}{\partial \dot{q}_j'} - \frac{\partial R_{approx}}{\partial q_j'} = 0 \tag{6.7.2}$$

and they take the form

$$\left[\frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_j}\right]_0 \ddot{q}_j' + \left[\left[\frac{\partial^2 R}{\partial \dot{q}_i \partial q_j}\right]_0 - \left[\frac{\partial^2 R}{\partial \dot{q}_j \partial q_i}\right]_0\right]\dot{q}_j' - \left[\frac{\partial^2 R}{\partial q_i \partial q_j}\right]_0 q_j' - \left[\frac{\partial R}{\partial q_i}\right]_0 = 0 \tag{6.7.3}$$

Now in these linearised equations of motion the condition of steady (gross) motion requires that $(\partial R/\partial q_i)_0 = 0$, since under steady conditions momentum associated with q_i is conserved i.e. q_i is an ignorable coordinate. The remaining terms may be written in matrix notation as follows

$$[m]\{\ddot{q}\}' + [c]\{\dot{q}\}' + [k]\{q\}' = 0 \tag{6.7.4}$$

where

$$[m] = \left[\frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_j}\right]_0 \qquad \text{symmetric}$$

$$[c] = \left[\left[\frac{\partial^2 R}{\partial \dot{q}_i \partial q_j}\right]_0 - \left[\frac{\partial^2 R}{\partial \dot{q}_j \partial q_i}\right]_0\right] \qquad \text{skew symmetric}$$

$$[k] = \left[\frac{\partial^2 R}{\partial q_i \partial q_j}\right]_0 \qquad \text{symmetric}$$

Equation (6.7.4) resembles the equation for damped oscillations. However in the present formulation no account has been taken of any damping forces. The skew symmetric matrix $[c]$ in equation (6.7.4) arises due to the presence of gyroscopic forces whenever q_i' are measured in a moving reference frame. These forces, in contrast to real damping forces, do no work in the system and do not affect the energy in the system. This can be seen in the skew symmetry of $[c]$ which ensures that

$$\{q\}'^T [c]\{\dot{q}\}' = 0 \tag{6.7.5}$$

Nevertheless the gyroscopic forces can have a marked influence on the stability of the motion as we will see in a subsequent section.

Example 6.7.2

Let us obtain the linearised equation of secondary motion for the conical pendulum of example 6.7.1. We found that under steady motion

$$\ddot{\theta}_0 = \dot{\theta}_0 = 0 \tag{a}$$

and

$$\cos\theta_0 = \frac{g}{r\,\dot{\psi}_0^2} \tag{b}$$

Now expanding the Routhian into a Taylor series, retaining the nonzero terms up to the second order, and setting $\theta = \theta_0 + \xi$, we find

$$R_{approx} = (R)_0 + \left[\frac{c_\psi^2 \cos\theta_0}{ml^2 \sin^3\theta_0} - mgl\sin\theta_0\right]\xi \tag{c}$$

$$- \frac{1}{2}\left[\frac{c_\psi^2 (1 + 2\cos^2\theta_0)}{ml^2 \sin^4\theta_0} + mgl\cos\theta_0\right]\xi^2 + \frac{1}{2} ml^2 \dot{\xi}^2$$

The first term being a constant does not influence the equations of motion and may be dropped. We also note the absence of gyroscopic terms. From the remaining terms of the Routhian we obtain the following equations of motion

$$ml^2\ddot{\xi} + \left[\frac{c_\psi^2(1 + 2\cos^2\theta_0)}{ml^2 \sin^4\theta_0} + mgl\cos\theta_0\right]\xi - \left[\frac{c_\psi^2 \cos\theta_0}{ml^2 \sin^3\theta_0} - mgl\sin\theta_0\right] = 0 \tag{d}$$

Now the last term will be recognised as the condition for steady motion (see example 6.7.1) and vanishes. The equation of motion for small amplitude oscillation then becomes

$$\ddot{\xi} + \left[\frac{c_\psi^2\,(1 + 2\cos^2\theta_0)}{m^2 l^4 \sin^4\theta_0} + \frac{g}{l}\cos\theta_0\right]\xi = 0 \qquad \text{(e)}$$

We can now see that the small secondary oscillations about θ_0 have an angular frequency given by

$$\omega = \left[\frac{c_\psi^2(1 + 2\cos^2\theta_0)}{m^2 l^4 \sin^4\theta_0} + \frac{g}{l}\cos\theta_0\right]^{1/2} \qquad \text{(f)}$$

To determine the ignorable coordinate ψ as a function of time we invoke the general expression for p_ψ derived earlier, which we now write as

$$p_\psi = ml^2\dot{\psi}\sin^2(\theta_0 + \xi) = c_\psi \qquad \text{(g)}$$

Then

$$\dot{\psi} = \frac{c_\psi}{ml^2\sin^2(\theta_0 + \xi)} \qquad \text{(h)}$$

We assumed earlier that the disturbance does not affect the value of the constant momentum p_ψ. If the value of p_ψ is altered but remains a constant, say $\bar{c}_\psi$, then in the above relations c_ψ may be replaced by $\bar{c}_\psi$. It is usually assumed that c_ψ is a good approximation for $\bar{c}_\psi$ i.e. the momentum p_ψ does not change significantly as a result of the disturbance.

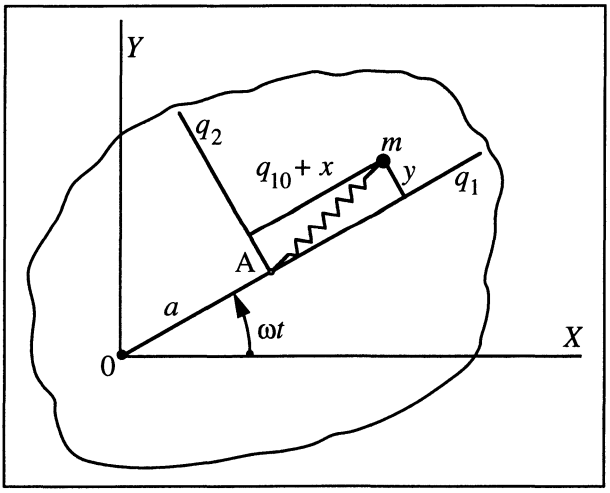


Figure 6.7 Mass-spring system mounted on rotating table

Example 6.7.3

Consider the motion of a mass-spring system on a rotating horizontal table as shown in Figure 6.7. The X,Y axes are inertial and fixed in space, while q_1, q_2 are attached to the table and rotate with a constant angular velocity ω. Let us obtain the equations of motion for small oscillations.

The transformation equations from the X, Y to q_1, q_2 coordinates take the form

$$X = (a + q_1)\cos\omega t - q_2 \sin\omega t \tag{a}$$

$$Y = (a + q_1)\sin\omega t + q_2 \cos\omega t \tag{b}$$

Then

$$\dot{X} = \dot{q}_1 \cos\omega t - \omega (a + q_1)\sin\omega t - \dot{q}_2 \sin\omega t - q_2 \omega \cos\omega t \tag{c}$$

$$\dot{Y} = \dot{q}_1 \sin\omega t + \omega (a + q_1)\cos\omega t + \dot{q}_2 \cos\omega t - q_2 \omega \sin\omega t \tag{d}$$

and

$$T^* = \frac{m}{2} (\dot{X}^2 + \dot{Y}^2) \tag{e}$$

may be expressed in terms of the rotating coordinates as

$$T^* = \frac{m}{2} \left\{ \dot{q}_1^2 + \dot{q}_2^2 + [(a + q_1)^2 + \dot{q}_2^2]\omega^2 + 2[(a + q_1)\dot{q}_2 - 2\dot{q}_1 q_2]\omega \right\} \tag{f}$$

The potential energy of the spring may be written directly in terms of the rotating coordinates as

$$V = \frac{k}{2} (\sqrt{q_1^2 + q_2^2} - l_0)^2 \tag{g}$$

where l_0 is the unstretched length of the spring. Now forming the Lagrangian $L = (T^* - V)$ we may obtain the equations of motion as

$$m(\ddot{q}_1 - 2\omega\dot{q}_2 - (a + q_1)\omega^2) + \frac{\partial V}{\partial q_1} = 0 \tag{h}$$

$$m(\ddot{q}_2 + 2\omega\dot{q}_1 - q_2\omega^2) + \frac{\partial V}{\partial q_2} = 0 \tag{i}$$

Gyroscopic coupling terms are clearly present in these equations. Letting $q_1 = q_{10}$ and $q_2 = 0$, under steady motion, i.e. when $\ddot{q}_1 = \ddot{q}_2 = \dot{q}_1 = \dot{q}_2 = 0$ we have

$$-m (a + q_{10})\omega^2 + \left[\frac{\partial V}{\partial q_1}\right]_0 = 0 \tag{j}$$

$$\left[\frac{\partial V}{\partial q_2}\right]_0 = 0 \tag{k}$$

For small oscillations we can set $q_1 = q_{10} + x$ and $q_2 = y$ and then expanding $V(q_1, q_2)$ in equation (g) into a Taylor series we find

$$V_{approx} = (V)_0 + \left[\frac{\partial V}{\partial q_1}\right]_0 x + \left[\frac{\partial V}{\partial q_2}\right]_0 y + \frac{1}{2} k \left[x^2 + y^2 \left[\frac{q_{10} - l_0}{q_{10}}\right]^2\right] + \cdots \tag{l}$$

The first term being a constant does not affect the equations of motion, that is, we can set $(V)_0 = 0$. The third term vanishes on account of the conditions for steady motion as given in equation (k). By the same reason the second term may be expressed as

$$\left[\frac{\partial V}{\partial q_1}\right]_0 x = m (a + q_{10}) x \omega^2 \tag{m}$$

Thus we can now write the Lagrangian for small oscillations about steady motion as

$$L = \frac{m}{2}\{\dot{x}^2 + \dot{y}^2 + [(a + q_{10} + x)^2 + y^2]\omega^2 + 2[\dot{y}(a + q_{10} + x) - 2\dot{x}\, y]\omega\} - \{\frac{k}{2}[x^2 + y^2\left[\frac{q_{10} - l_0}{q_{10}}\right]^2] + (a + q_{10})x\dot{\omega}^2\} \quad \text{(n)}$$

The equations of motion can be determined via Lagrange's equations as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 0 & -2m\omega \\ 2m\omega & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} k - m\omega^2 & 0 \\ 0 & k\left[\frac{q_{10} - l_0}{q_{10}}\right] - m\omega^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(o)}$$

6.8 Solution of the Linear Equations of Motion

For the general linear equations of the type

$$[m]\{\ddot{q}\} + [c]\{\dot{q}\} + [k]\{q\} = \{0\} \quad (6.8.1)$$

where the matrices have constant elements and the approach used in section 6.5 cannot be used, we may assume a solution of the following form

$$\{q\} = \{A\}e^{\lambda t} \quad (6.8.2)$$

Substituting into equation (6.8.1) we find a set of homogeneous linear equations

$$\left[\lambda^2[m] + \lambda[c] + [k]\right]\{A\} = \{0\} \quad (6.8.3)$$

For non-trivial solutions then

$$\det\,[\lambda^2[m] + \lambda[c] + [k]] = 0 \quad (6.8.4)$$

which leads to a characteristic polynomial of order n in λ^2. Since the coefficients of this polynomial are real, its n roots, namely the n eigenvalues will be in complex conjugate pairs of the type

$$\lambda_r = a_r + i\omega_r \qquad \lambda_s = a_r - i\omega_r$$

where $i = \sqrt{-1}$.

The constants a_r and ω_r are real and they can be positive, zero or negative. We can now make the following observations about the solutions of the equations of motion.

a) If all a are negative and all $\omega \neq 0$, the motion will be damped simple harmonic. If all λ are real and negative, the disturbed motion will gradually settle back to steady motion.

b) If all a are positive with all $\omega \neq 0$, or if all λ are real and positive, the disturbed motion will grow exponentially with time. In this case then the motion is unstable in the vicinity of the steady motion.

The conditions a) always prevail when the matrix $[c]$ is due to physical damping only. When the matrix $[c]$ accounts for gyroscopic terms only, the characteristic polynomial for the eigenvalues will have even terms only, as is the case when gyroscopic terms are absent. Conditions a) and b) can arise when $[c]$ is due to gyroscopic and physical damping terms.

Once the eigenvalues are determined the corresponding eigenvectors $\{A\}$ may be computed via equation (6.8.3). However the elements of the eigenvectors will be in general complex. For an oscillatory mode then the various coordinates will move with the same frequency but will be out of phase.

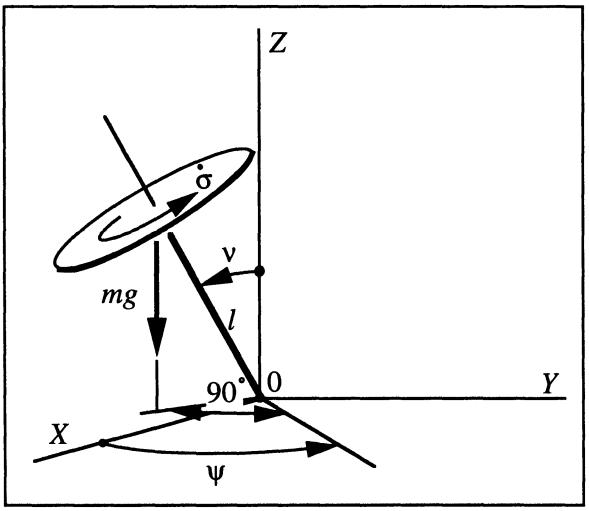


Figure 6.8 Spinning top

Example 6.8.1

Consider the motion of a spinning top as shown in Figure 6.8. Using Euler angles and Euler frequencies we can express the Lagrangian of this system as

$$L = \frac{1}{2}[A(\dot{\sigma}^2 + \dot{\psi}^2 \sin^2\nu) + C(\dot{\sigma} + \dot{\psi}\cos\nu)^2] - mgl\cos\nu \tag{a}$$

where l is the distance from 0 to the centre of mass of the top. It is now clear that both ψ and σ are ignorable and we may remove these coordinates. Let us however remove σ only. We have that

$$p_\sigma = \frac{\partial L}{\partial \dot{\sigma}} = C(\dot{\sigma} + \dot{\psi}\cos\nu) = c_\sigma \tag{b}$$

Now the modified Lagrangian L_R can be expressed as

$$L_R = L - c_\sigma\dot{\sigma} \tag{c}$$

Removing $\dot{\sigma}$ from L_R via equation (b) we obtain the Routhian

$$R = \frac{1}{2}[A(\dot{\sigma}^2 + \dot{\psi}^2 \sin^2\nu) - \frac{c_\sigma^2}{C} + c_\sigma\dot{\psi}\cos\nu] - mgl\cos\nu \tag{d}$$

The equations of motion can now be obtained from Lagrange's equations in the form

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_i} - \frac{\partial R}{\partial q_i} = 0 \tag{e}$$

from which there emerge

for $\dot{q}_1 = \psi$: $$A(\ddot{\psi}\sin^2\nu + 2\dot{\psi}\dot{\theta}\sin\nu\cos\nu) - c_\sigma\dot{\nu}\sin\nu = 0 \quad \text{(f)}$$

for $\dot{q}_2 = \nu$: $$A\ddot{\nu} - A\dot{\psi}^2\sin\nu\cos\nu + c_\sigma\dot{\psi}\sin\nu - mgl\sin\nu = 0 \quad \text{(g)}$$

The presence of the gyroscopic terms ($-c_\sigma\dot{\nu}\sin\nu$) in (f) and ($c_\sigma\dot{\psi}\sin\nu$) in (g) can be observed. For steady motion $\ddot{\sigma} = \ddot{\psi} = 0$. Then from equation (f) we obtain the condition for steady motion (steady precession) as

$$\dot{\psi}_0 = \frac{c_\sigma \pm \sqrt{c_\sigma^2 - 4A\,mgl\cos\nu}}{2A\cos\nu} \quad \text{(h)}$$

There are therefore two possible steady precession rates, provided that

$$c_\sigma^2 > 4A\,mgl\cos\nu \quad \text{(i)}$$

Thus the magnitude of the angular momentum c_σ must be large enough to provide gyroscopic stabilisation. To obtain the equations for small amplitude oscillations about steady motion we set

$$\dot{\psi} = \dot{\psi}_0 + \dot{\alpha}, \qquad \text{and} \qquad \nu = \nu_0 + \beta \text{ and } \dot{\nu} = \dot{\beta}$$

in the Routhian and expand into a Taylor series

$$\begin{aligned} R_{approx} = {} & (R)_0 + (A\dot{\psi}^2\sin\nu\cos\nu + mgl\sin\nu - c_\sigma\dot{\psi}\sin\nu)_0\,\beta + \\ & (A\dot{\nu})_0\dot{\beta} + (A\dot{\psi}\sin^2\nu + c_\sigma\cos\nu)_0\dot{\alpha} + \frac{1}{2}A\dot{\beta}^2 + \\ & \frac{1}{2}(A\dot{\psi}^2\cos 2\nu - mgl\cos\nu - c_\sigma\dot{\psi}\cos\nu)_0\,\beta^2 + \frac{1}{2}(A\sin^2\nu)_0\dot{\alpha}^2 \\ & + (2A\dot{\psi}\sin\nu\cos\nu - c_\sigma\sin nu)_0\,\dot{\alpha}\,\beta \end{aligned} \quad \text{(j)}$$

Now using R_{approx} in Lagrange's equation (e) and taking note of the conditions for steady motion, we obtain the equations of motion for small oscillations about the steady motion as

$$A\ddot{\beta} - (A\dot{\psi}^2\cos 2\nu - mgl\cos\nu - c_\sigma\dot{\psi}\cos\nu)_0\,\beta - (A\dot{\psi}\sin 2\nu - c_\sigma\sin\nu)_0\,\dot{\alpha} = 0 \quad \text{(k)}$$

$$(A\sin^2\nu)_0\,\ddot{\alpha} + (A\dot{\psi}\sin 2\nu - c_\sigma\sin\nu)_0\,\dot{\beta} = 0 \quad \text{(l)}$$

where the $(\,)_0$ are constant terms. For the special case of $\nu_0 = 90°$, these equations simplify to

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} + \begin{bmatrix} 0 & -c_\sigma \\ c_\sigma & 0 \end{bmatrix}\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A\dot{\psi}_0^2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(m)}$$

Now assuming a solution of the form

$$\alpha = A_1 e^{\lambda t} \qquad \text{and} \qquad \beta = A_2 e^{\lambda t} \quad \text{(n)}$$

we find the condition for a nontrivial solution of the above equations as

$$\begin{bmatrix} A\lambda^2 & -c_\sigma\lambda \\ c_\sigma\lambda & A(\lambda^2 + \dot{\psi}_0^2) \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(o)}$$

The vanishing of the determinant of above matrix yields

$$A^2\lambda^4 + (A^2\dot{\psi}_0^2 + c_\sigma^2)\lambda^2 = 0 \quad \text{(p)}$$

which provides us with the eigenvalues

$$\lambda_1 = 0, \qquad \lambda_2 = 0, \qquad \lambda_3 = i\left[\dot{\psi}_0^2 + \frac{c_\sigma^2}{A^2}\right]^{\frac{1}{2}}, \qquad \lambda_4 = -i\left[\dot{\psi}_0^2 + \frac{c_\sigma^2}{A^2}\right]^{\frac{1}{2}} \quad \text{(q)}$$

Substituting these values back into equation (l) and letting $A_1 = 1$, we find the eigenvectors as

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ i(1 + \frac{A\dot{\psi}_0^2}{c_\sigma^2}) \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -i(1 + \frac{A\dot{\psi}_0^2}{c_\sigma^2}) \end{bmatrix} \quad \text{(r)}$$

The presence of i in the computed value for A_2 implies that α lags β by 90°.

Problem

6.8.1 For Kepler's problem of central force motion obtain (a) the condition for steady motion. Then by considering a small disturbance about the steady motion obtain (b) the equation for small oscillations about the steady motion. Show (c) that the period of this oscillation is the same as the orbital period of the steady motion.

6.9 System Response to a Step Impulse

Spike forces, i.e. forces of very large magnitude and very short duration, arise typically in impact problems. For purposes of mathematical modelling such forces are assumed to have the following distribution over time. Prior to impact the force is taken as zero. During the period, Δt, of impact the force is assumed to be very large approaching infinity as $\Delta t \to 0$. However the impulse, i.e. the area under the force time-curve, is assumed to remain finite. After impact the force is again taken as zero. A system subjected to such *spike forces* will normally suffer finite discontinuities in its velocities. On the other hand, given the short time duration for these forces and the finite changes in velocities, one can see that the changes in the system configuration, over the short period of time, will be negligibly small. It follows then that such properties of the system that depend upon its configuration, such as positions or mass distribution, will remain constant during the action of spike forces. Furthermore, the impulse of any other force of finite magnitude, over the small period Δt, can be considered negligibly small compared with the impulse of the very large spike forces.

In the following we use Lagrange's equations and Gauss' principle of least constraint for the analysis of motion resulting from a spike force.

Lagrangian Method

Lagrange's equation in its fundamental form (2.6.20) is

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i} = Q_i$$

Let us now split the generalized force Q_i into the usual components Q_{iu}, i.e. elastic, gravitational, dissipative, etc., and a spike force Q_{is}, such that

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i} - Q_{iu} = Q_{is} \tag{6.9.1}$$

Integrating equation (6.9.1) with respect to time between the limits $t = 0$ and $t = \Delta t$, we find

$$\frac{\partial T^*}{\partial \dot{q}_i}\bigg|_{\Delta t} - \frac{\partial T^*}{\partial \dot{q}_i}\bigg|_{o} - \int_o^{\Delta t}\left[\frac{\partial T^*}{\partial q_i} + Q_{iu}\right] dt = \int_o^{\Delta t} Q_{is}\, dt \tag{6.9.2}$$

We now invoke the definition of generalised momenta $p_i = \dfrac{\partial T^*}{\partial \dot{q}_i}$ and write

$$\Delta p_i = p_i\bigg|_{\Delta t} - p_i\bigg|_{o} = \frac{\partial T^*}{\partial \dot{q}_i}\bigg|_{\Delta t} - \frac{\partial T^*}{\partial \dot{q}_i}\bigg|_{o} \tag{6.9.3}$$

On taking the limit as $\Delta t \to 0$ we note that the integral on the left hand side of equation (6.9.2) vanishes since $(\partial T^*/\partial q_i + Q_{iu})$ is finite in magnitude. On the other hand the integrand on the right hand side approaches infinity as $\Delta t \to 0$, but the integral, which yields the step impulse G_i of the spike force Q_{is}, remains finite. Thus noting equation (6.9.3), we may write equation (6.9.2) as

$$\Delta p_i = G_i \tag{6.9.4}$$

It is useful to put this equation into a virtual work form. To maintain the dimensions of work we write the virtual work in terms of virtual *velocities* rather than virtual displacements. Thus we write

$$\delta W_I = (\Delta p_i - G_i)\delta \dot{q}_i = 0 \tag{6.9.5}$$

Now it will be recalled that in general the transformation relating the Cartesian coordinates r to the generalized coordinates q takes the form

$$r_j = r_j(q_1, q_2, \ldots, q_i, \ldots, q_n, t) \tag{6.9.6}$$

The associated transformation for the forces, when time is held fixed, was shown to take the form

$$Q_i = F_j \frac{\partial r_j}{\partial q_i} \tag{6.9.7}$$

On integrating equation (6.9.7) over the period from 0 to Δt we deduce that for spike forces

$$\int_0^{\Delta t} Q_{is} = \left[\int_0^{\Delta t} F_j\, dt\right]\frac{\partial r_j}{\partial q_i}$$

or

$$G_i = I_j \frac{\partial r_j}{\partial q_i} \tag{6.9.8}$$

where I_j is a step impulse if F_j is a spike force (Figure 6.9). Also it was shown that in general the relation (1.9.8) between the generalized momenta and generalized velocities takes the form

$$\{p\} = [A]\{\dot{q}\} + \{b\} \tag{6.9.9}$$

where the vector $\{b\}$ arises due to the explicit appearance of t, in equation (6.9.6). Again t is taken as a constant over the short period Δt when the spike force is acting. Thus for a spike force the column matrix $\{b\}$ vanishes, and we may invert equation (6.9.9) to obtain

$$\{\dot{q}\} = [A]^{-1}\{p\} \tag{6.9.10}$$

Using equation (6.9.4) we have then

$$\{\Delta\dot{q}\} = [A]^{-1}\{G\} \tag{6.9.11}$$

Since [A] is a positive definite matrix, the above relation, implying a finite change in velocities, can always be obtained.

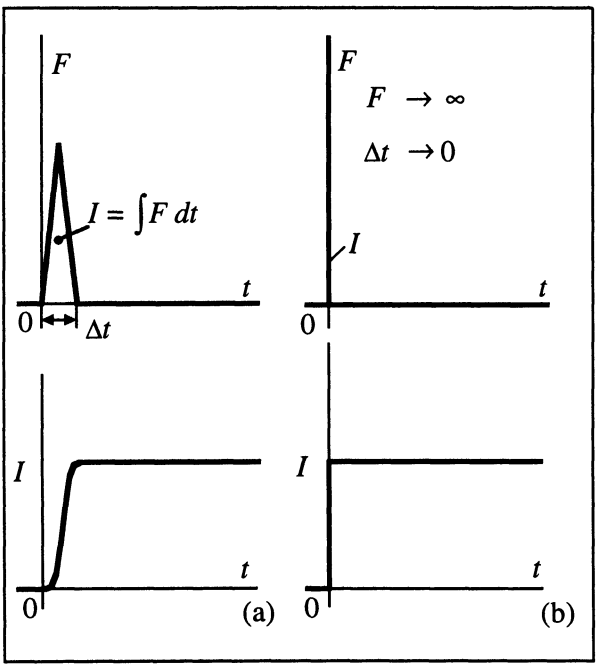


Figure 6.9 A triangular spike force (a) and its idealization (b)

Example 6.9.1

Four rigid bars each of mass m and length l are pin-jointed at their ends to form the rhombus ABDE as shown in Figure 6.10. The system lies on a smooth horizontal plane and it is subjected to a step impulse I, at point A. Determine the angular velocities of the bars and the linear velocity of the mass centre immediately after the application of the step impulse I.

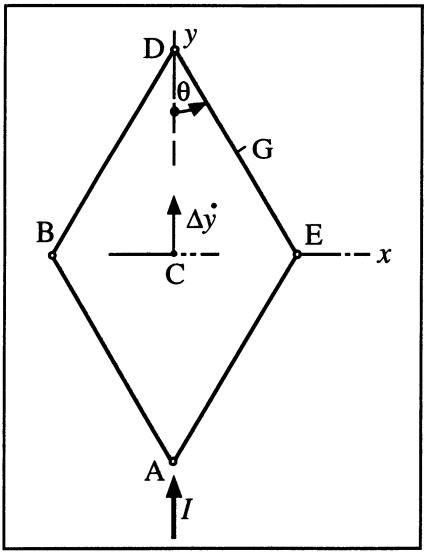


Figure 6.10 Pin-jointed rhombus subjected to step impulse

Evidently after the application of the impulse I the system will acquire some complementary kinetic energy. This energy will be in part due to translation of the four bars and in part due to their rotation.

For a typical bar such as DE, we can write the coordinates of its mass centre as

$$x_G = \frac{1}{2} l \sin\theta \qquad y_G = \frac{1}{2} l \cos\theta \tag{a}$$

Hence the complementary kinetic energy of bar DE becomes

$$T^*_{DE} = \frac{m}{2}\left[\left[\frac{1}{2} l\dot{\theta}\cos\theta\right]^2 + \left[\dot{y}_C - \frac{1}{2} l\dot{\theta}\sin\theta\right]^2\right] + \frac{1}{2} J\,\dot{\theta}^2 \tag{b}$$

where the first two terms account for the translational and the last term for the rotational component of the complementary kinetic energy, with

$$J = \frac{1}{12} ml^2 \tag{c}$$

as the bar's inertia moment, and $\dot{y}_C$ as the velocity component of the system's mass centre in the y direction. Writing the energy terms for the remaining three bars in like manner we find the total complementary kinetic energy as

$$T^* = 2\,m\,\dot{y}_C^2 + 2\,J\,\dot{\theta}^2 \tag{d}$$

The change Δp_i in the generalized momenta can now be determined from equation (6.9.3).

$$\Delta p_y = \Delta\frac{\partial T^*}{\partial \dot{y}_C} = \Delta\,(4\,m\,\dot{y}_C) = 4m\,\Delta\dot{y}_C \tag{e}$$

$$\Delta p_\theta = \Delta\frac{\partial T^*}{\partial \dot{\theta}} = \Delta\,(4\,J\,\dot{\theta}) = 4J\,\Delta\dot{\theta} \tag{f}$$

To evaluate the generalized step impulses G_i we write down the virtual work of the step impulse I. Before doing so we need the displacement and velocity of point A in the y direction

$$y_A = y_C - l \cos \theta \tag{g}$$

with $y_C = 0$, and

$$\dot{y}_A = \dot{y}_C + (l \sin \theta)\, \dot{\theta} \tag{h}$$

Now

$$\delta W_I = I\, \delta \dot{y}_A \tag{i}$$

$$= I\, \delta \dot{y}_C + I\, l \sin \theta\ \delta \dot{\theta} \tag{j}$$

We conclude that

$$G_y = I \tag{k}$$

and

$$G_\theta = I\, l \sin \theta \tag{l}$$

We can now determine the changes in the system's momenta using equation (6.9.4). Thus

$$\Delta p_y - G_y = 4\, m\, \Delta \dot{y}_C - I = 0 \tag{m}$$

and

$$\Delta p_\theta - G_\theta = 4\, J\, \Delta \dot{\theta} - I\, l \sin \theta = 0 \tag{n}$$

Thus the acquired initial velocities are

$$\Delta \dot{y}_C = \frac{I}{4m} \tag{o}$$

$$\Delta \dot{\theta} = \frac{I\, l \sin \theta}{4\, J} \tag{p}$$

If we assume that the system was at rest at the beginning of the impact, then its kinetic coenergy immediately after impact would be

$$\Delta T^* = \frac{1}{2} 4m\, (\Delta \dot{y}_C)^2 + \frac{1}{2} 4J\, (\Delta \dot{\theta})^2 \tag{q}$$

Constraint Consideration

Whether any constraints present are holonomic or nonholonomic, they may be expressed in differential form as follows:

$$a_{ij}\, dr_j + a_{it}\, dt = 0 \tag{6.9.12}$$

or

$$a_{ij}\, \dot{r}_j + a_{it} = 0 \qquad j = 1,2,3,...,3N \tag{6.9.13}$$

where the r_j represent Cartesian coordinates.

Now let us invoke the virtual work expression of equation (6.9.5) and write it for Cartesian coordinates

$$\delta W_I = (\Delta B_j - I_j)\,\delta \dot{r}_j = 0 \tag{6.9.14}$$

Since $\mathbf{B} = m\dot{\mathbf{r}}$ for each mass, we recognize that equation (6.9.14) involves velocities $\dot{r}_j$ by virtue of ΔB_j and *virtual velocities* $\delta \dot{r}_j$. To prevent the constraint forces from doing work, virtual velocities are considered at a fixed time, that is for virtual velocities time is not varied and hence virtual velocities must satisfy the *instantaneous* constraint equation

$$a_{ij}\,\delta \dot{r}_j = 0 \tag{6.9.15}$$

On the other hand, the velocities appearing in ΔB_j must conform to the actual constraint equations as given in equation (6.9.12).

Now let us write the virtual work expression, with $\Delta B_j = B_j - B_{jo}$, as

$$\delta W_I = (B_j - B_{jo} - I_j)\,\delta \dot{r}_j = 0 \tag{6.9.16}$$

Noting that B_{jo} and I_j are given quantities, and consequently $\delta B_{jo} = 0$ and $\delta I_j = 0$, and further noting that $B_j = (m\dot{r})_j$ for each mass m, we may express the virtual velocities as $\delta \dot{r}_j = \delta(\dfrac{\Delta B_j}{m_j} - I_j)$. Then the condition (6.9.16) of δW_I vanishing becomes

$$(\Delta B_j - I_j)\,\delta(\frac{\Delta B_j}{m_j} - I_j) = 0 \tag{6.9.17}$$

Noting that the constant m has no effect on the virtual changes in the second bracket, we may introduce a function

$$z = \frac{1}{2}(\Delta B_j - I_j)(\Delta B_j - I_j) \tag{6.9.18}$$

and minimize it by setting

$$\delta z = 0 \tag{6.9.19}$$

i.e. the vanishing condition (6.9.17) is equivalent to minimization of z with respect to ΔB_j.

The function (6.9.18) may also be written in vector form

$$z = \frac{1}{2}(\Delta \mathbf{B}_k - \mathbf{I}_k) \cdot (\Delta \mathbf{B}_k - \mathbf{I}_k) \tag{6.9.20}$$

which exhibits some resemblance to equation (2.4.6) of Gauss' principle of least constraint. Sometimes this approach for spike forces is indeed referred to by the same name. The point to note, however, is that in the minimum principle presented here the quantities varied are the linear momenta, whereas in Gauss' principle of least constraint proper the quantities varied are the time derivatives of linear momenta.

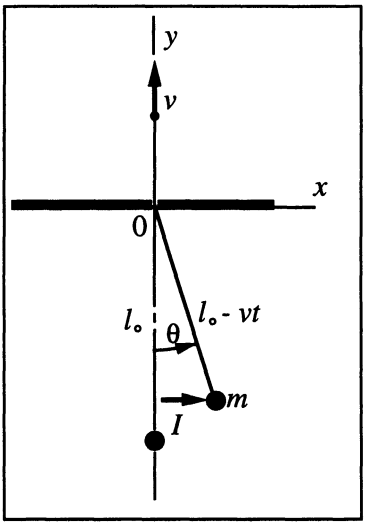


Figure 6.11 Horizontal spike force

Example 6.9.2

The pendulum shown in Figure 6.11 is pulled through the platform at a constant velocity v. At time $t = 0$, the point mass is struck horizontally by a spike force (or step impulse). Determine the angular velocity of the pendulum immediately after impact.

The constraint for this problem is

$$x^2 + y^2 = (l_0 - vt)^2$$

or in differential form

$$xdx + ydy + v(l_0 - vt)\,dt = 0 \tag{a}$$

or

$$x\dot{x} + y\dot{y} + v(l_0 - vt) = 0$$

Thus $a_x = x, a_y = y, a_t = v(l_0 - vt)$. With $B_x = m\dot{x}$ and $B_y = m\dot{y}$, we may now write Gauss' constraint (6.9.18) as

$$z = \frac{1}{2m}\left[(\Delta B_x - I)^2 + (\Delta B_y)^2\right] \tag{b}$$

Minimizing z with respect to B_x and B_y we have

$$(\Delta B_x - I)\delta B_x + (\Delta B_y)\delta B_y = 0 \tag{c}$$

However the virtual velocities are not independent and they must satisfy the *instantaneous* constraint equation, namely

$$\delta\dot{y} = -\frac{x}{y}\,\delta\dot{x} \tag{d}$$

or

$$\delta B_y = -\frac{x}{y}\,\delta B_x \tag{e}$$

Substituting into equation (c)

$$\left[(\Delta B_x - I) - \frac{x}{y} \Delta B_y \right] \delta B_x = 0 \tag{f}$$

The term in the large bracket must therefore vanish. However B_x, B_y and x, y are not independent either, and they must conform to the rheonomic constraints. An expedient way of satisfying the constraints is to use the coordinate θ and let

$$x = (l_o - vt) \sin\theta \tag{g}$$

$$y = -(l_o - vt) \cos\theta \tag{h}$$

$$\dot{x} = (l_o - vt)\,\dot{\theta} \cos\theta - v \sin\theta \tag{i}$$

$$\dot{y} = (l_o - vt)\,\dot{\theta} \sin\theta + v \cos\theta \tag{j}$$

such that

$$B_x = m(l_o - vt)\dot{\theta} \cos\theta - mv \sin\theta \tag{k}$$

$$B_y = m(l_o - vt)\dot{\theta} \sin\theta + mv \cos\theta \tag{l}$$

since over the short time interval Δt, neither the configuration of the system nor the velocity v change

$$\Delta B_x = \left[m(l_o - vt)\cos\theta \right] \Delta\dot{\theta} \tag{m}$$

$$\Delta B_y = \left[m(l_o - vt)\sin\theta \right] \Delta\dot{\theta} \tag{n}$$

since $\Delta v = 0$.

Substituting equations (m) and (n) into equation (f) realizing that δB_x is arbitrary and simplifying we find

$$(l_o - vt)\,\Delta\dot{\theta} - \frac{I}{m} \cos\theta = 0 \tag{o}$$

or

$$\Delta\dot{\theta} = \frac{I \cos\theta}{m(l_o - vt)} \tag{p}$$

Energy Change

Due to the changes $\Delta\dot{q}$ in velocities one can expect a change ΔT^* in the kinetic coenergy of the system, when the system is subjected to a step impulse.

For cases where t is taken as constant over the very short period Δt the kinetic coenergy (1.9.4) becomes simply

$$T^* = \frac{1}{2} \{\dot{q}\}^T [A] \{\dot{q}\} \tag{6.9.21}$$

and the change,

$$\Delta T^* = \frac{1}{2} \{\dot{q}\}_{\Delta t}^T [A] \{\dot{q}\}_{\Delta t} - \frac{1}{2} \{\dot{q}\}_o^T [A] \{\dot{q}\}_o \tag{6.9.22}$$

with $[A]$ as the symmetric matrix (1.9.6). For such a system the generalized momenta are as given by equation (6.9.10), i.e.

$$\{p\} = [A]\{\dot{q}\} \tag{6.9.23}$$

and hence

$$\Delta T^* = \frac{1}{2}\{\dot{q}\}_{\Delta t}^T \{p\}_{\Delta t} - \frac{1}{2}\{\dot{q}\}_o^T \{p\}_o \tag{6.9.24}$$

Now premultiplying equation (6.9.23) by $\{\dot{q}\}_o^T$ we deduce that

$$\{\dot{q}\}_o^T \{p\} = \{\dot{q}\}_o^T [A]\{\dot{q}\} \tag{6.9.25}$$

On transposing the right hand side and recalling the symmetry of $[A]$ we find

$$\{\dot{q}\}_o^T \{p\} = \{\dot{q}\}^T \{p\}_o \tag{6.9.26}$$

On substituting from equation (6.9.26) into (6.9.24) we may express the change in the kinetic coenergy as

$$\Delta T^* = \frac{1}{2}(\{\dot{q}\}_{\Delta t}^T + \{\dot{q}\}_o^T)(\{p\}_{\Delta t} - \{p\}_o) \tag{6.9.27}$$

Next noting that the change in generalized momenta is equal to the generalized impulses according to equation (6.9.4), we may write

$$\Delta T^* = \frac{1}{2}(\{\dot{q}\}_{\Delta t}^T + \{\dot{q}\}_o^T)\{G\} \tag{6.9.28}$$

or

$$\Delta T^* = \frac{1}{2}\{G\}^T(\{\dot{q}\}_{\Delta t} + \{\dot{q}\}_o) \tag{6.9.29}$$

That is, the change in kinetic coenergy is given by the product of the generalized step impulse and the mean of the velocities after and before the impact.

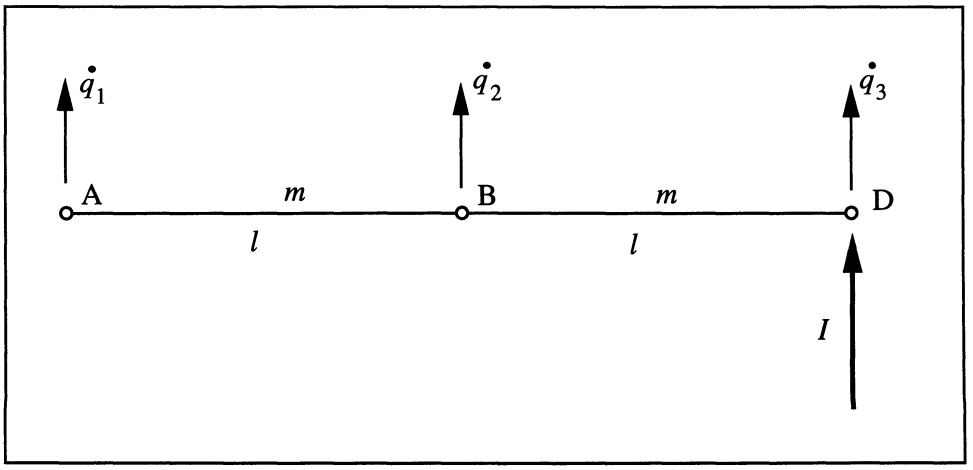


Figure 6.12 Two-rod system subjected to step impulse

Example 6.9.3

Two rods AB and BD, each of mass m and length l, as shown in Figure 6.12, are smoothly hinged at B and rest on a smooth horizontal plane. A step impulse I is applied at D normal to BD giving rise to velocity v_o at D. Find (a) the initial velocities of points A and B, (b) the magnitudes of initial angular velocities of AB and BD about their centres of mass and (c) the change in the complementary kinetic energy of the system.

Take q_1, q_2, q_3 as the generalized displacement coordinates of the system for the transverse motion. The complementary kinetic energy for rod AB may be expressed as

$$T^*_{AB} = \frac{m}{2}\left[\frac{\dot{q}_1 + \dot{q}_2}{2}\right]^2 + \frac{1}{2}\frac{ml^2}{12}\left[\frac{\dot{q}_2 - \dot{q}_1}{l}\right]^2 \tag{a}$$

where the first term describes the complementary kinetic energy for the translational motion of the mass centre and the second term gives the complementary kinetic energy for rotation about the mass centre. We may write this in matrix form as follows

$$T^*_{AB} = \frac{1}{2}[\dot{q}_1\ \dot{q}_2]\begin{bmatrix} \frac{m}{3} & \frac{m}{6} \\ \frac{m}{6} & \frac{m}{3} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{b}$$

For BD the complementary kinetic energy is of the same form but expressed in terms of $\dot{q}_2$ and $\dot{q}_3$. Thus the total complementary kinetic energy of the system becomes

$$T^* = \frac{1}{2}\{\dot{q}\}^T[A]\{\dot{q}\} = \frac{1}{2}[\dot{q}_1\ \dot{q}_2\ \dot{q}_3]\begin{bmatrix} \frac{m}{3} & \frac{m}{6} & 0 \\ \frac{m}{6} & \frac{2m}{3} & \frac{m}{6} \\ 0 & \frac{m}{6} & \frac{m}{3} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \tag{c}$$

The virtual work of the impulse I is given by

$$\delta W_I = G_1\,\delta\dot{q}_1 + G_2\,\delta\dot{q}_2 + G_3\,\delta\dot{q}_3 = I\,\delta\dot{q}_3 \tag{d}$$

such that

$$\{G\}^T = [0\quad 0\quad I] \tag{e}$$

The generalized momenta p_i are

$$\{p\} = [A]\{\dot{q}\} = \begin{bmatrix} \frac{m}{3} & \frac{m}{6} & 0 \\ \frac{m}{6} & \frac{2m}{3} & \frac{m}{6} \\ 0 & \frac{m}{6} & \frac{m}{3} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \tag{f}$$

Equating the changes $\{\Delta p\}$ to the step impulse $\{G\}$ according to equation (6.9.4) gives

$$\begin{bmatrix} \frac{m}{3} & \frac{m}{6} & 0 \\ \frac{m}{6} & \frac{2m}{3} & \frac{m}{6} \\ 0 & \frac{m}{6} & \frac{m}{3} \end{bmatrix} \begin{bmatrix} \Delta\dot{q}_1 \\ \Delta\dot{q}_2 \\ \Delta\dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \tag{g}$$

Solving for the velocity changes we find

$$\Delta\dot{q}_3 = \frac{7}{2}\frac{I}{m} = v_o \qquad \Delta\dot{q}_2 = -\frac{2}{7}v_o \qquad \Delta\dot{q}_1 = \frac{v_o}{7} \tag{h}$$

The initial angular velocity of AB will be

$$\frac{\Delta\dot{q}_2 - \Delta\dot{q}_1}{l} = \frac{\left[-\frac{2}{7} - \frac{1}{7}\right]v_0}{l} = -\frac{3}{7}\frac{v_o}{l} \tag{i}$$

The initial angular velocity of BD will be

$$\frac{\Delta\dot{q}_3 - \Delta\dot{q}_2}{l} = \frac{(1 + \frac{2}{7})\,v_o}{l} = \frac{9}{7}\frac{v_o}{l} \tag{j}$$

and the change in complementary kinetic energy may be computed by using equation (6.9.29)

$$\begin{aligned} \Delta T^* &= \frac{1}{2} I\,\Delta\dot{q}_3 = \frac{1}{2} I\,v_o \\ &= \frac{1}{7} m\,v_o^2 \\ &= \frac{7}{4}\frac{I^2}{m} \end{aligned} \tag{k}$$

The magnitude of step impulse required to produce a $\Delta\dot{q}_3 = v_o$ is, from equation (h)

$$I = \frac{2}{7} m\,v_o \tag{l}$$

Bertrand's Theorem

If a system at rest is subjected to some spike forces, the kinetic coenergy resulting will be greater than if the system had been subjected to additional scleronomic constraints and subjected to the same spike forces.

To prove this theorem, let T^* and T_c^* be the kinetic coenergies of the system before and after the constraints are imposed respectively, both immediately after application of the step impulse. Now writing virtual work expressions for the unconstrained and the constrained systems respectively, we have

$$\{\delta\dot{q}\}^T [A] (\{\dot{q}\} - \{0\}) = \{\delta\dot{q}\}^T \{G\} \tag{6.9.30}$$

$$\{\delta\dot{q}\}_c^T [A] (\{\dot{q}\}_c - \{0\}) = \{\delta\dot{q}\}_c^T \{G\} \tag{6.9.31}$$

The null vectors indicate that the system is initially at rest. The subscript c identifies the constrained system.

As virtual velocities let us take

$$\{\delta\dot{q}\} = \{\delta\dot{q}\}_c = \{\dot{q}\}_c \tag{6.9.32}$$

This can be done for the constrained system since the constraints are scleronomic and $\{\dot{q}\}_c$ satisfies the same constraint equations as $\{\delta\dot{q}\}_c$. For the unconstrained system, $\{\dot{q}\}_c$ is certainly an admissible virtual velocity vector. Using equation (6.9.32) in equations (6.9.30) and (6.9.31) and subtracting the latter from the former, we find

$$\{\dot{q}\}_c^T [A] (\{\dot{q}\} - \{\dot{q}\}_c) = 0 \tag{6.9.33}$$

We may now expand equation (6.9.33) and write it in the following modified form:

$$\frac{1}{2}\{\dot{q}\}^T [A] \{\dot{q}\} - \frac{1}{2}\{\dot{q}\}_c [A] \{\dot{q}\}_c = \frac{1}{2}(\{\dot{q}\}^T - \{\dot{q}\}_c^T) [A] (\{\dot{q}\} - \{\dot{q}\}_c) \tag{6.9.34}$$

Evidently the right hand side of equation (6.9.35) is a positive quantity and the left hand side gives $T^* - T_c^*$. This proves the theorem.

Example 6.9.4

Let us consider the system of example 6.9.3, this time let point A be pinned. Thus $\Delta\dot{q}_1 = 0$. Let us determine $\Delta\dot{q}_2$, $\Delta\dot{q}_3$, and the change in the kinetic coenergy of the system.

Since $\Delta\dot{q}_1 = 0$, the governing equation becomes

$$\{\Delta p\} = [A] \{\Delta\dot{q}\} = \begin{bmatrix} \frac{2m}{3} & \frac{m}{6} \\ \frac{m}{6} & \frac{m}{3} \end{bmatrix} \begin{bmatrix} \Delta\dot{q}_2 \\ \Delta\dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{a}$$

Solving equation (a)

$$\Delta\dot{q}_2 = -\frac{6}{7}\frac{I}{m} \tag{b}$$

$$\Delta\dot{q}_3 = \frac{24}{7}\frac{I}{m} \tag{c}$$

The kinetic coenergy change (6.9.29) is

$$\Delta T^* = \frac{1}{2} G_i (\Delta\dot{q}_i) = \frac{1}{2}\left[0 + I\frac{24}{7}\frac{I}{m}\right] = \frac{12}{7}\frac{I^2}{m} \tag{d}$$

which is smaller than the kinetic energy change (k) of example 6.9.3. Now in this problem if an additional constraint is imposed we would expect the resulting kinetic coenergy to be further reduced. Is this however always the case? From equations (b) and (c) we can see that the resulting motion will have a node (where the transverse velocity $\Delta\dot{q} = 0$) at a point between points B and D. The point, which is the instantaneous centre of rotation, is at a distance of (l/5) to the right of B. Thus if this point is constrained by means of a frictionless pin the value of the resulting kinetic energy will be unchanged since the constraint will be in effect inactive. Based on this perspective we can present Bertrand's theorem as an extremum problem in the magnitude of T^*. That is T^* will assume its maximum value, when the active constraints are least. To illustrate, suppose we introduce a constraint by means of a frictionless pin at some point x to the right of B. Then $\dot{q}_2$ and $\dot{q}_3$ will be related by

$$\dot{q}_2 = -\dot{q}_3 \frac{x}{l-x} \tag{e}$$

Under the additional constraint T^* becomes

$$T^* = \frac{m\dot{q}_3^2}{2}\left[2\left[\frac{x}{l-x}\right]^2 - \left[\frac{x}{l-x}\right] + 1\right] \tag{f}$$

Now setting dT^*/dx to zero we obtain

$$\frac{4x}{l-x} - 1 = 0$$

or

$$x = l/5 \tag{g}$$

That is, T^* will assume its maximum value when the constraint is at the instantaneous centre of rotation and is therefore inactive.

Kelvin's Theorem

This theorem states that if any points of a system are suddenly set in motion with prescribed velocities, the kinetic coenergy of the resulting motion will be less than that of any other kinematically admissible motion in which the given points have the same prescribed velocities.

To prove this, suppose the prescribed velocities are brought about by step impulses G_j. Then from the expression for virtual work in equation (6.9.5) which we now write in terms of velocity, we have that

$$[(m\dot{q})_i - (m\dot{q})_{i0}]\,\delta\dot{q}_i = G_j\,\delta\dot{q}_j \qquad \begin{matrix} i = 1, 2, ..., 3N \\ j = 1, 2, ..., k \end{matrix} \tag{6.9.35}$$

where N is the total number of particles and k is the number of prescribed velocities. Now for simplicity let us assume that the system is initially at rest, i.e. $(m\dot{q})_{i0} = 0$. Suppose we use the actual velocities as virtual velocities. This is admissible since actual velocities also satisfy the kinematic constraints. Then equation (6.9.35) can be simplified and expressed in the following two forms:

$$(m\,\delta\dot{q})_i\,\dot{q}_i = G_j\,\dot{q}_j$$
$$(m\dot{q})_i\,\dot{q}_i = G_j\,\dot{q}_j$$

It follows then that

$$(m\dot{q})_i\,\dot{q}_i = (m\,\delta\dot{q})_i\,\dot{q}_i$$

Let us now add and subtract $\frac{1}{2}(m\dot{q})_i\,\dot{q}_i$ and $\frac{1}{2}(m\,\delta\dot{q}_i)\dot{q}_i$ to the right hand side and write the above equation as

$$(m\dot{q})_i\,\dot{q}_i = \frac{1}{2}(m\dot{q})_i\,\dot{q}_i + \frac{1}{2}(m\,\delta\dot{q})_i\delta\dot{q}_i - \frac{1}{2}[m(\dot{q} - \delta\dot{q})]_i\,[(\dot{q} - \delta\dot{q})]_i \tag{6.9.36}$$

on rearranging

$$\frac{1}{2}(m\,\delta\dot{q})_i\,\dot{q}_i - \frac{1}{2}(m\,\dot{q})_i\,\dot{q}_i = \frac{1}{2}[m(\dot{q} - \delta\dot{q})]_i\,(\dot{q} - \delta\dot{q})_i \tag{6.9.37}$$

In equation (6.9.37) we can recognise the first term as an expression for an admissible kinetic coenergy and the second term as the actual kinetic coenergy. Since the difference between these energies is positive, the actual kinetic coenergy is least. In other words the actual motion corresponds to the minimum value of kinetic coenergy.

Example 6.9.5

Consider example 6.9.4 again and this time suppose $\bar{\dot{q}}_3 (= \Delta \bar{\dot{q}}_3)$ is prescribed. Let us now write the expression for the kinetic coenergy of the system in terms of linear and angular velocities of the centres of mass for the two rods. Then we have

$$T^* = \left[\frac{m}{2} \dot{q}_c^2 + \frac{ml^2}{24} \omega^2 \right]_{AB} + \left[\frac{m}{2} \dot{q}_c^2 + \frac{ml^2}{24} \omega^2 \right]_{BD} \tag{a}$$

But now we need to introduce the kinematic constraints

$$\begin{aligned} (\dot{q}_c)_{BD} + \frac{l}{2}(\omega)_{BD} &= \bar{\dot{q}}_3 \\ (\dot{q}_c)_{AB} - \frac{l}{2}(\omega)_{AB} &= 0 \\ (\dot{q}_c)_{BD} - \frac{l}{2}(\omega)_{BD} &= (\dot{q}_c)_{AB} + \frac{l}{2}(\omega)_{AB} \end{aligned} \tag{b}$$

where we have taken the positive angular velocities in the anticlockwise direction. These three constraint equations in terms of four variables can be satisfied in an infinite number of ways. That is there exist an infinity of kinematically admissible motions. For such admissible motions T^* can be expressed in terms of one of the variables, say $(\omega)_{BD}$. Thus satisfying equations (b) we find

$$(\dot{q}_c)_{BD} = \bar{\dot{q}}_3 - \frac{l}{2}(\omega)_{BD}$$

$$(\dot{q}_c)_{AB} = \frac{\bar{\dot{q}}_3}{2} - \frac{l}{2}(\omega)_{BD}$$

$$(\omega)_{AB} = \frac{\bar{\dot{q}}_3}{l} - (\omega)_{BD}$$

Substituting these results in equation (a) we find

$$T^* = \frac{m}{3} l^2 (\omega^2)_{BD} - \frac{5}{6} m \bar{\dot{q}}_3 l (\omega)_{BD} + \frac{2}{3} m \bar{\dot{q}}_3^2 \tag{c}$$

Now minimizing T^* with respect to $(\omega)_{BD}$ we obtain

$$\frac{dT^*}{d(\omega)_{BD}} = \frac{2m}{3} l^2 (\omega)_{BD} - \frac{5}{6} ml \bar{\dot{q}}_3 = 0$$

$$\text{or} \qquad (\omega)_{BD} = \frac{5}{4l} \bar{\dot{q}}_3 \tag{d}$$

from equation (b) we now find

$$(\dot{q}_c)_{BD} = \frac{3}{8}\,\bar{\dot{q}}_3$$

and since

$$(\dot{q}_c)_{BD} = \frac{1}{2}\,(\bar{\dot{q}}_2 + \bar{\dot{q}}_3)$$

we have

$$\dot{q}_2 = -\frac{1}{4}\,\bar{\dot{q}}_3 \qquad \text{(e)}$$

in agreement with the result found earlier in example 6.9.4.

Carnot's Theorem

This theorem states that when scleronomic constraints are suddenly imposed on a system, part of the system's kinetic coenergy will be lost. Again we invoke the virtual work theorem in equation 6.9.5.

$$((p_i - p_{i0}) - G_i)\,\delta\dot{q}_i = 0 \qquad (6.9.38)$$

For the present case there are no generalized impulses i.e. $G_i = 0$ and we must choose the virtual velocities so that they satisfy the constraints. This being the case the virtual work of the step impulses associated with the constraint forces will vanish. Further we may use actual velocities immediately, after the imposition of constraints, as admissible velocities. Then equation (6.9.38) may be written as

$$[p_i - p_{i0}]\dot{q}_i = 0 \qquad \text{or} \qquad p_i\,\dot{q}_i = p_{i0}\,\dot{q}_i \qquad (6.9.39)$$

Now let us add and subtract $\frac{1}{2}\,p_{i0}\,\dot{q}_{i0}$ and $\frac{1}{2}\,p_i\,\dot{q}_i$ to the right hand side and write the above equation as

$$p_i\dot{q}_i = \frac{1}{2}\,p_{i0}\,\dot{q}_{i0} + \frac{1}{2}\,p_i\,\dot{q}_i - \frac{1}{2}(p_{i0} - p_i)\,(\dot{q}_{i0} - \dot{q}_i)$$

where we have used the fact that $p_{i0} = m\dot{q}_{i0}$ and $p_i = m\dot{q}_i$.
On rearranging we have

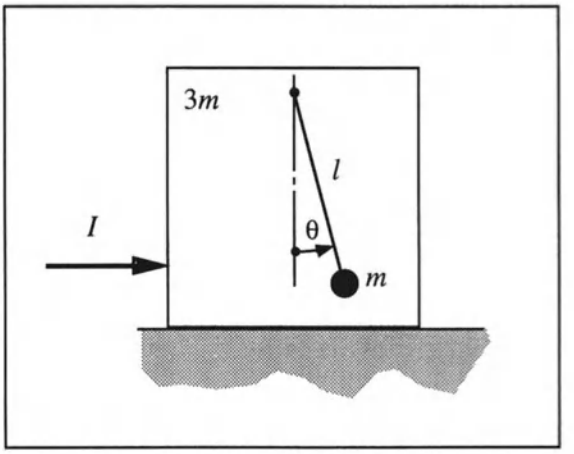
$$\frac{1}{2}p_i\,\dot{q}_i - \frac{1}{2}\,p_{i0}\,\dot{q}_{i0} = -\frac{1}{2}(p_{i0} - p_i)\,(\dot{q}_{i0} - \dot{q}_i) \qquad (6.9.40)$$

Now the first two terms on the left hand side will be recognized as the kinetic coenergy just after and just before the imposition of the constraints. The right hand side term being negative implies that kinetic coenergy will be reduced when the constraints are imposed. The "lost energy" in effect is converted to nonrecoverable energy.

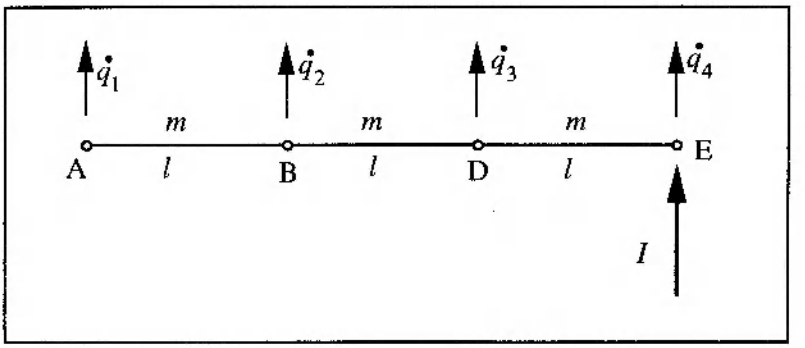
The converse of Carnot's theorem is also true. That is when some constraints are suddenly removed, as in an explosion of a shell, the fragments of the shell will acquire additional kinetic coenergy.

Problems

6.9.1 A pendulum of length l and mass m, is pivoted at a point 0 on a block of mass 3m. There is Coulomb friction μ between block and horizontal surface. The block is struck by a horizontal spike force, producing a step impulse I, at a point 0 in time when $\dot{x} = 3$ m/s and $\dot{\theta} = 0.1$ rad/s. (a) Obtain the velocity changes $\Delta\dot{x}$ and $\Delta\dot{\theta}$ immediately after application of the spike force, assuming small angles θ. (b) What is the kinetic coenergy change? (c) What is the constant magnitude F of the spike force if $\Delta t = 0.01\ s$, $m = 8$ kg, $l = 0.2\ m$, and $\Delta\dot{x} = 5$ m/s?

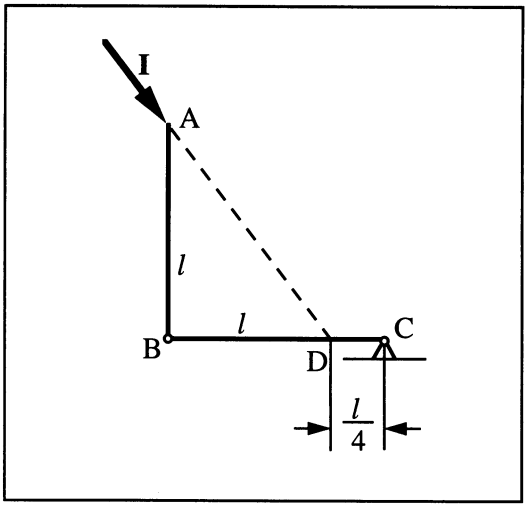


6.9.2 Given is a pin-jointed three-rod system moving at $\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = \dot{q}_4 = v$, when a step impulse is applied at point E. (a) Determine the step increases Δq_i. (b) Find the change ΔT^* of the kinetic coenergy. (c) Repeat the problem taking joint B as rigid.



6.9.3 Two equal uniform rods AB and BC each of length l are freely joined at B and lie on a smooth horizontal plane, as shown, the angle at B being 90°, while BC

is free to rotate about its end C which is pinned. A step impulse is applied to A, towards D, a point in BC such that DC = $\frac{1}{4}$ BC. Find the resulting velocities in the rods.



6.10 System Response to a Step Displacement

It is of interest to enquire what kind of problem, when written in the complementary formulation, would have equations analogous to those obtained for the spike force problem (6.9.1).

As noted in the spike force problem, we find the spike force so dominant, that other forces present become negligibly small. This is a consequence of the high magnitude and short duration of the spike force. Now in the complementary formulation, the place of the spike force will be taken by a spike velocity, and the place of the step impulse will be taken by a step displacement.

The complementary Lagrange equation (2.8.2)

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{S}_j} - \frac{\partial L^*}{\partial S_j} = s_j$$

can be written, with the help of Table 1.11.1,

$$-\frac{d}{dt}\frac{\partial V^*}{\partial \dot{S}_j} = s_j \tag{6.10.1}$$

Now let the generalized speed be split into the usual speed components s_{ju} and the spike speed s_{js}, such that

$$s_j = s_{ju} + s_{js} \tag{6.10.2}$$

and the complementary Lagrange equation (6.10.1) can be rearranged to

$$-\frac{d}{dt}\frac{\partial V^*}{\partial \dot{S}_j} - s_{ju} = s_{js} \tag{6.10.3}$$

Integrating equation (6.10.3) with respect to time between the limits $t = 0$ and $t = \Delta t$ gives

$$- \frac{\partial V^*}{\partial \dot{S}_j}\bigg|_{\Delta t} + \frac{\partial V^*}{\partial \dot{S}_j}\bigg|_{o} + \int_o^{\Delta t} (s_{ju})\, dt = \int_o^{\Delta t} s_{js}\, dt \qquad (6.10.4)$$

and using $e_j = - \dfrac{\partial V^*}{\partial \dot{S}_j}$, we obtain

$$e_j\bigg|_{\Delta t} - e_j\bigg|_{o} + \int_o^{\Delta t} s_{ju}\, dt = \int_o^{\Delta t} s_{js}\, dt \qquad (6.10.5)$$

Taking the limit as $\Delta t \to 0$ we note that the integral on the left hand side of equation (6.10.5) vanishes, since s_{ju} is finite.

On the other hand, the integrand on the right hand side approaches infinity as $\Delta t \to 0$, but the integral, which yields the generalized displacement d_j of the speed s_{js} remains finite. Thus, we may write equation (6.10.5) as

$$\Delta e_j = d_j \qquad (6.10.6)$$

That is the imposed spike velocity s_{js} gives rise to changes in the stretch of the associated force element with potential coenergy (e.g. spring, gravitational force). Equation (6.10.6) is analogous to equation (6.9.4).

It is useful to express equation (6.10.6) in virtual work form. To maintain the dimension of work, we write the virtual work expression in terms of virtual *forces* $\delta \dot{S}$ rather than virtual impulses. Thus

$$\delta W_{II} = (\Delta e_j - d_j)\, \delta \dot{S}_j = 0 \qquad (6.10.7)$$

Equation (6.10.7) is essentially static in nature and is the analogue of equation (6.9.14).

Example 6.10.1

The system shown in Figure (6.13) is subjected to a *displacement* d_1. We wish to determine the spring stretches and the system's complementary strain energy.

For the complementary formulation of this problem we need an equilibrating set of impulses. Such a set is shown in Figure (6.13). However since we are dealing with a static problem, inertial and damping forces may be neglected. That is

$$p_1 = 0 \qquad (a)$$

$$p_2 = 0 \qquad (b)$$

$$S_4 = \textit{constant} \qquad (c)$$

This implies that

$$\dot{S}_1 + \dot{S}_2 = \dot{S}_3 \qquad (d)$$

Thus there are there unknown forces and we have one equilibrium equation (d) relating the three forces. The problem has therefore two degrees of statical indeterminacy.

Now the complementary strain energy of the system, under constraint equation (d), maybe expressed as

$$V^* = \frac{\dot{S}_1^2}{2k_1} + \frac{\dot{S}_2^2}{2k_2} + \frac{(\dot{S}_1 + \dot{S}_2)^2}{2k_3} \tag{e}$$

The virtual work of the applied displacement becomes

$$\delta W_{II} = d_1(\delta\dot{S}_1 + \delta\dot{S}_2) \tag{f}$$

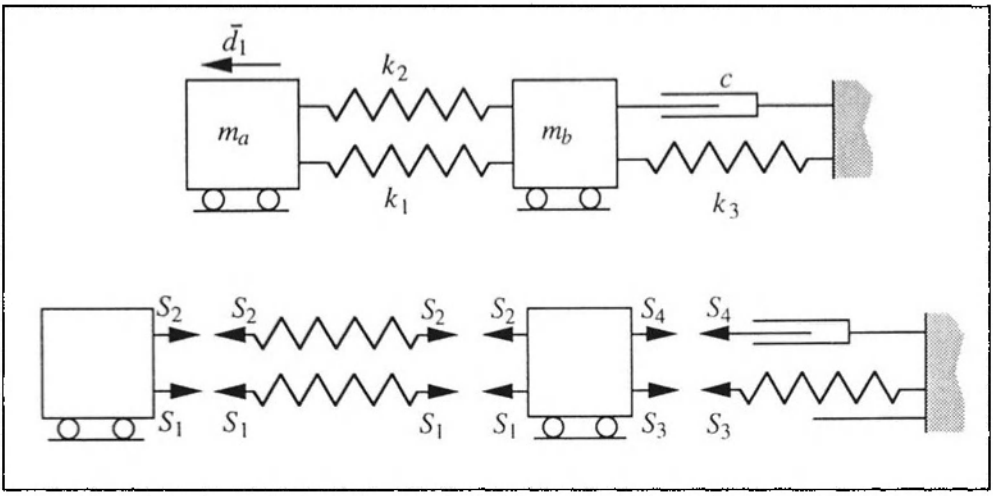


Figure 6.13 A system subjected to a prescribed displacement

Now evaluating the generalised stretches from $(-\partial V^* / \partial \dot{S}_i)$, as in equation (6.10.4) we can write

$$\Delta e_1 + \frac{\dot{S}_1}{k_1} + \frac{\dot{S}_1 + \dot{S}_2}{k_3} = 0 \tag{g}$$

$$\Delta e_2 + \frac{\dot{S}_2}{k_2} + \frac{\dot{S}_1 + \dot{S}_2}{k_3} = 0 \tag{h}$$

Finally using equation (6.10.7) we obtain the compatibility equations of the system as

$$-\left\{\frac{\dot{S}_1}{k_1} + \frac{\dot{S}_1 + \dot{S}_2}{k_3}\right\} = d_1 \tag{i}$$

$$-\left\{\frac{\dot{S}_2}{k_2} + \frac{\dot{S}_1 + \dot{S}_2}{k_3}\right\} = d_1 \tag{j}$$

Solving for $\dot{S}_1$ and $\dot{S}_2$ we find

$$\dot{S}_1 = -\frac{k_3 k_1 d_1}{k_1 + k_2 + k_3} \tag{k}$$

$$\dot{S}_2 = -\frac{k_3 k_2 d_1}{k_1 + k_2 + k_3} \tag{l}$$

Hence the stretches are

$$\Delta e_1 = -\frac{\dot{S}_1}{k_1} = \frac{k_3 d_1}{k_1 + k_2 + k_3} \tag{n}$$

$$\Delta e_2 = -\frac{\dot{S}_2}{k_2} = \frac{k_3 d_1}{k_1 + k_2 + k_3} \tag{o}$$

$$\Delta e_3 = -\frac{\dot{S}_3}{k_3} = \left[\frac{\dot{S}_1 + \dot{S}_2}{k_3}\right] = \frac{(k_1 + k_2) d_1}{k_1 + k_2 + k_3} \tag{p}$$

The complementary strain energy of the system can now be obtained from equations (e), (k) and (l) as

$$V^* = \frac{1}{2} \frac{k_3 (k_1 + k_2)}{(k_1 + k_2 + k_3)} d_1^2 \tag{q}$$

Analogue of Bertrand's Theorem

An analogue of Bertrand's theorem may be developed for elastostatic problems. Consider for instance an elastostatic system with n degrees of statical indeterminacy. Let the system be subjected to a number of prescribed displacements. Under this configuration the elastic system will store a certain amount of complementary potential (strain) energy. Now the analogue of Bertrand's theorem asserts that this complementary potential energy will be reduced if the degrees of statical indeterminacy of the system are reduced, for instance by releasing some of the redundant members.

Example 6.10.2

Consider the last example again but this time let us remove the first spring and thereby reduce the statical indeterminacy of the system from two to one. We wish to compute the complementary strain energy of the new system.

The new value of the V^* can be readily computed from equation (q) of example (6.10.1) by setting $k_1 = 0$. That's if the stiffness of the first spring vanishes the spring will carry no force. Hence for the new system

$$V^*_{new} = \frac{1}{2} \frac{k_3 k_2}{(k_2 + k_3)} d_1^2 \tag{a}$$

Now the analogue of Bertrand's theorem asserts that $V^* > V^*_{new}$. Using equation (a) above and (q) of the last example we can write this inequality as

$$\frac{k_1 + k_2}{k_1 + k_2 + k_3} > \frac{k_2}{k_2 + k_3} \tag{b}$$

or

$$(k_1 + k_2)(k_2 + k_3) > k_2(k_1 + k_2 + k_3) \tag{c}$$

i.e.

$$k_1 k_3 > 0 \tag{d}$$

which evidently is true since k_1 and k_3 are both positive.

Analogue of Kelvin's Theorem

This theorem may be stated as follows. If an elastostatic system is subjected to some forces the complementary strain energy of the system will be least when the compatibility requirements of the system are satisfied. This theorem will be recognised as the classical minimum complementary strain energy principal of elasticity.

Analogue of Carnot's Theorem

This theorem states that that if in a statically indeterminate elastostatic system, subjected to some prescribed displacements, the degrees of statical determinacy are reduced, by removal of some redundant members, part of the systems complementary strain energy will be lost. The "lost energy" is in effect converted to nonrecoverable energy.

Example 6.10.3

Consider once again the system shown in Figure 6.13. We found the complementary strain energy of this system in example (6.10.1) as

$$V^* = \frac{1}{2} \frac{k_3 (k_1 + k_2)}{(k_1 + k_2 + k_3)} d_1^2 \tag{a}$$

In example (6.10.2) we noted that when the statical determinacy of the system was reduced by removal of the first spring, the complementary strain energy reduced to

$$V^*_{new} = \frac{1}{2} \frac{k_3 k_2}{k_2 + k_3} d_1^2 \tag{b}$$

Thus the reduction in V^*, brought about by the removal of the first spring in given by

$$\Delta V^* = \frac{1}{2} \left[\frac{k_3 (k_1 + k_2)}{k_1 + k_2 + k_3} - \frac{k_3 k_2}{k_2 + k_3} \right] d_1^2 \tag{c}$$

$$\text{or } \Delta V^* = \frac{1}{2} \left[\frac{k_3^2 k_1}{(k_1 + k_2 + k_3)(k_2 + k_3)} \right] d_1^2 \tag{d}$$

Now before the first spring was removed, we found, in example (6.10.1), that it had a force

$$\dot{S}_1 = - \frac{k_3 k_1}{k_1 + k_2 + k_3} d_1 \tag{e}$$

Hence the contribution of this spring to V^* is given by

$$V_1^* = \frac{\dot{S}^2}{2k_1} = \frac{k_3^2 k_1}{2(k_1 + k_2 + k_3)^2} d_1^2 \tag{f}$$

Now the analogue of Carnots theorem asserts that

$$\Delta V^* \geq V_1^* \tag{g}$$

That is the reduction in the complementary strain energy is larger than that which was stored in the first spring. To demonstrate, we obtain from equations (d) and (g)

$$\frac{1}{(k_1 + k_2 + k_3)(k_2 + k_3)} > \frac{1}{(k_1 + k_2 + k_3)^2} \quad \text{(h)}$$

The truth of this statement is self evident since $k_1 > 0$.

Suggested Reading

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Appendix A

THE CALCULUS OF VARIATIONS

Introduction

The calculus of variations provides the mathematical foundation for the study of analytical mechanics as well as a number of other branches of physical sciences. In this appendix we study the essentials of the calculus of variations for applications to problems of analytical dynamics.

We begin our study by introducing the notion of a functional and by way of motivation pose a number of classical problems concerned with the extremum of functionals. We then show that the extremum conditions of a functional appear as solutions of a differential equation - the so-called Euler-Lagrange equation. Next we consider the extremization of functionals subject to different types of constraints and show that the Lagrange multiplier method provides a versatile technique for taking account of such constraints. Finally we consider functionals with variable end points and derive associated extremum conditions.

A.1 Functions and Functionals

A *function*, for example $u = u(x,y)$, establishes a relationship between two or more variables e.g. u, and x and y . Generally the dependent variable, u, is expressed in terms of one or more independent variables, in this case x and y, and when the numerical values of the independent variables are specified, that of the dependent variable can be calculated.

A *functional* establishes a relationship between one variable and several *functions*. Consider, for instance,

$$s = \int_{x_1}^{x_2} F\,(x, u(x), u'(x), v(x), v'(x))\,dx \qquad \text{(A.1.1)}$$

Here in order to calculate a numerical value for s one must specify the functions $u(x)$, and $v(x)$.

For only one function, we may write

$$s = \int_{x_1}^{x_2} F\,(x, y(x), y'(x))\,dx \tag{A.1.2}$$

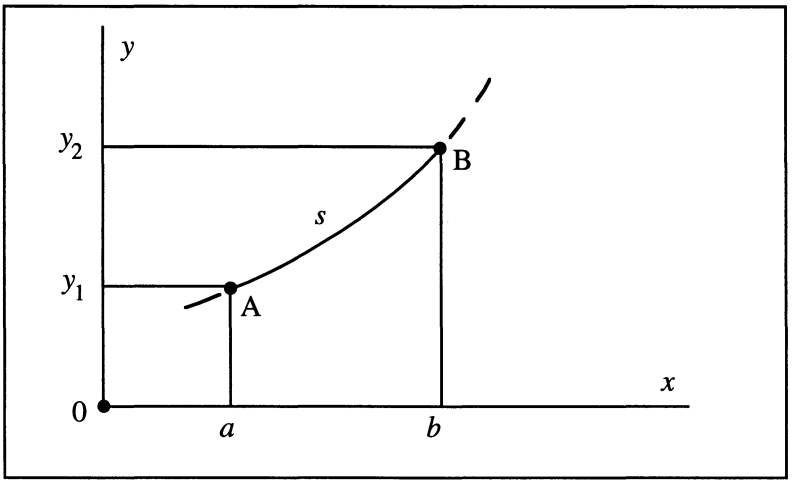


Figure A.1. Distance between two points in a plane

Example A.1.1

What is the shortest distance between two points in a plane? The distance between points A and B may be expressed as (see Figure A.1)

$$s = \int ds \tag{a}$$

But

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{dy}{dx})^2}\,dx \tag{b}$$

Thus

$$s = \int_a^b \sqrt{1 + y'^2}\,dx \tag{c}$$

Clearly when y, as a function of x, takes different forms, the functional s will take different values. Hence in equation (c) we have the *integrand* $F = \sqrt{1 + y'^2}$, and we ask which particular form of y, as a function of x, minimizes s.

The study of finding the minima and/or maxima of functionals is referred to as the *calculus of variations*. We shall develop procedures for finding such extremum conditions of functionals in due course. For example A.1.1 at hand it is worth noting that:

1) Different forms of y which compete to minimize s in (c) are all functions of x, i.e. the independent variable is not altered.

2) All forms of y which are allowed to compete for minimization of s should pass through points A and B, where $x = a$ and $x = b$ respectively.

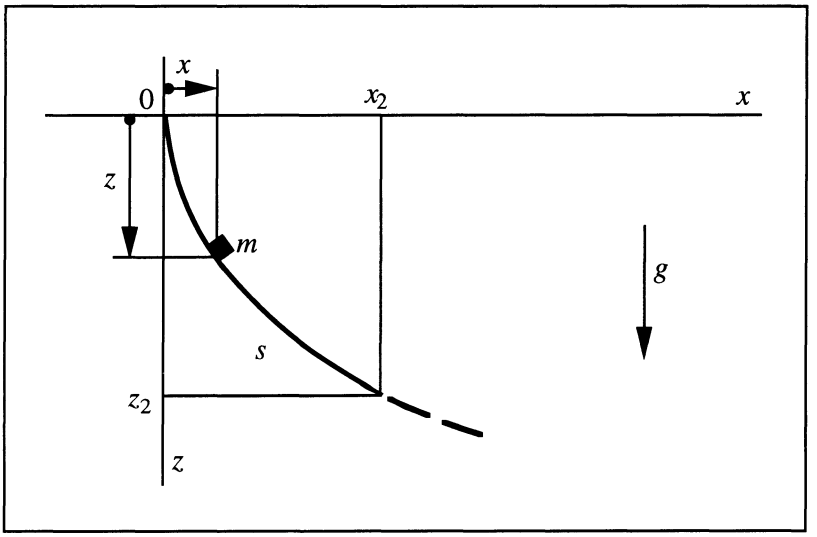
3) The forms of y considered must be continuous in the interval AB so that y in the integrand should remain finite.

Thus there are some restrictions to be imposed on competing functions. In other problems the restrictions may take different forms. Such restrictions on the competing functions are referred to as the *admissibility requirements.*

Problem

A.1.1 The brachystochrone problem: A particle m subject to gravitation g on a frictionless path s of arbitrary shape $z = z(x)$, is to move between points (0,0) and (x_2, z_2) in the shortest time possible. Determine the integrand F in

$$t = \int_0^{x_2} F \, dx$$



A.2 Review of Extremum Values of Functions

The maximum (minimum) points of functions are distinguished by the property that slight changes of the independent variables will result in a decrease (increase) in the value of the dependent variable whether the changes in the independent variable are positive or negative. For instance for a function such as $y = y(x)$, if there exists a minimum point at $x = a$, we have that

$$y(a + \varepsilon) - y(a) > 0 \qquad \text{(A.2.1)}$$

Where ε is a small change in x and it may be positive or negative.

Let us expand $y(a + \varepsilon)$ by Taylor series near the point $x = a$:

$$y(a + \varepsilon) = y(a) + y' \Big|_a \varepsilon + \frac{1}{2!} y'' \Big|_a \varepsilon^2 + \frac{1}{3!} y''' \Big|_a \varepsilon^3 + \text{(A.2.2)}$$

The derivatives y', y'' etc. are all evaluated at $x = a$ and hence the function $y = y(x)$ and its derivatives must be continuous at $x = a$.

From equation (A.2.2) it can be seen that if the sign of $[\, y(a + \varepsilon) - y(a)\,]$ is to be independent of the sign of ε, as ε approaches zero, the coefficient of y' must vanish since the sign of this term does depend upon the sign of ε. Further, it is apparent from equation (A.2.2) that at a maximum point the coefficient of ε^2, namely $\frac{1}{2!} y''$, must be non-positive while at a minimum point it must be non-negative. Thus summarizing

$$\begin{aligned} y' = 0 \quad \text{and} \quad y'' < 0 \quad &\rightarrow \quad \text{maximum point} \\ y' = 0 \quad \text{and} \quad y'' > 0 \quad &\rightarrow \quad \text{minimum point} \end{aligned} \qquad \text{(A.2.3)}$$

It is of course possible for y' to vanish at a point where $y'' = 0$. In that case conditions in relations (A.2.3) are not sufficient to determine the nature of the point and one must examine the coefficients of higher order terms. Thus at $x = 0$ of the function $y = x^3$ the coefficient of ε^3 does not vanish and since the value of this term will depend upon the sign of ε one must conclude that $x = 0$ is neither a maximum nor a minimum point of the function. On the other hand $x = 0$ for the function $y = x^4$ can be readily seen to be a minimum point.

It is to be noted that our definition of a maximum or minimum point, in effect describes a "turning point" of the function. Now a given function may possess several turning points in its range and while the conditions stated in relations (A.2.3) hold locally at every turning point, they do not yield information as to whether a given point is the global maximum or the global minimum point of the function. Thus for determining global extremum points a systematic search must be carried out amongst all the turning points of the function in the domain of the independent variable. It is also possible for a function to attain its extremum value at the boundaries of its range. In that case the extremum point need not be a turning point and such conditions in relations (A.2.3) may not hold.

In the case of a function of two independent variables such as

$$z = z(x, y) \qquad \text{(A.2.4)}$$

we have a surface which can be pictured as a terrain. We are now interested in finding the peaks of the hills and the lowest points of the valleys in this terrain.

At a point $x = a, y = b$ the following possibilities may arise

i) Maximum

Then

$$z(a + \varepsilon, b + \alpha) - z(a, b) < 0$$

independent of the sign of small increments ε and α and as $\varepsilon \to 0$ and $\alpha \to 0$.

ii) Minimum

Then

$$z(a + \varepsilon, b + \alpha) - z(a, b) > o$$

independent of the sign of ε and α and as $\varepsilon \to 0$ and $\alpha \to 0$.

iii) A saddle point (see Figure A.2)

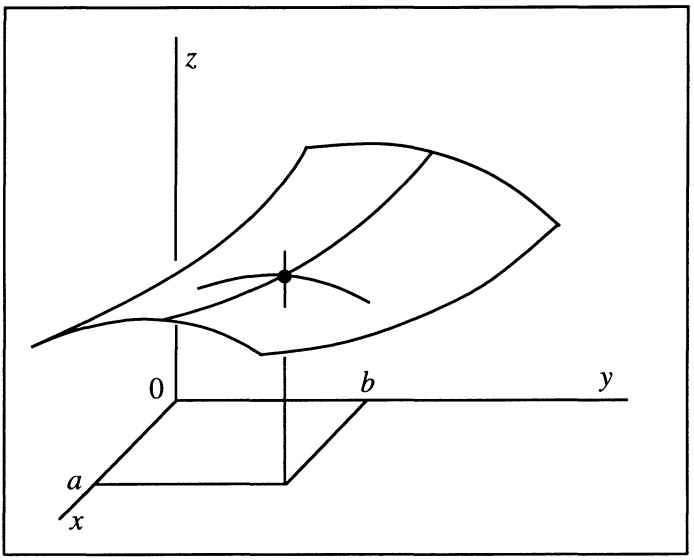


Figure A.2. A saddle point

Then for some values of ε and α, again independent of their signs, we may have

$$z(a + \varepsilon, b + \alpha) - z(a, b) > 0$$
$$z(a + \varepsilon, b + \alpha) - z(a, b) = 0$$
$$z(a + \varepsilon, b + \alpha) - z(a, b) < 0$$

as $\varepsilon \to 0$ and $\alpha \to 0$.

Again let us expand $z(a + \varepsilon, b + \alpha)$ by a Taylor series about $x = a, y = b$.

$$z(a + \varepsilon, b + \alpha) = z(a, b) + z_{,x}\Big|_{a,b} \varepsilon + z_{,y}\Big|_{a,b} \alpha + \frac{1}{2!} z_{,xx}\Big|_{a,b} \varepsilon^2 + \frac{1}{2!} z_{,yy}\Big|_{a,b} \alpha^2 + z_{,xy}\Big|_{a,b} \varepsilon\alpha + \frac{1}{3!} z_{,xxx}\Big|_{a,b} \varepsilon^3 + \cdots \tag{A.2.5}$$

where, $(\ )_{,x}$ implies partial differentiation with respect to x etc., and once again all the partial derivatives are evaluated at $x = a, y = b$. Hence it is assumed that $z(x, y)$ is continuous and possesses continuous derivatives at (a, b).

By a similar reasoning for the independence of the sign of $[z(a + \varepsilon, b + \alpha) - z(a, b)]$ from the signs of ε and α we conclude that for the cases i, ii, and iii, cited above, we must have

$$z_{,x} = 0 \qquad z_{,y} = 0 \qquad \text{at } (a, b) \tag{A.2.6}$$

To find out whether the point (a, b) is a maximum or minimum point we examine the second order terms. These form a quadratic function $Q(\varepsilon, \alpha)$ which we write as

$$Q(\varepsilon, \alpha) = \frac{1}{2!}[\varepsilon \ \alpha]\begin{bmatrix} z_{,xx} & z_{,xy} \\ z_{,xy} & z_{,yy} \end{bmatrix}\begin{bmatrix} \varepsilon \\ \alpha \end{bmatrix} \tag{A.2.7}$$

For a maximum or minimum point the sign of Q must be independent of the signs of ε and α. Thus for a minimum point, Q should remain non-negative for all real values and either signs of ε and α. Likewise, for a maximum point, Q should remain non-positive for all real values and signs of ε and α.

This property of Q, namely possessing only one sign, independent of signs of ε and/or α, depends upon the form of the symmetric coefficient matrix in equation (A.2.7). If this matrix is *positive definite*, then Q will maintain a positive sign for all real values of ε and α. Conversely, if this matrix is *negative definite* then Q will always maintain a negative sign. In the next section we will comment on properties of positive definite and negative definite matrices. For the simple case at hand we merely state that if the 2×2 coefficient matrix is to be positive definite, its determinant and both of its diagonal elements should be positive†. On the other hand, if the 2×2 matrix is to be negative definite, its determinant should be positive and both of its diagonal elements should be negative. Summarizing now we have that sufficient conditions for maximum and minimum points are

a) $$z_{,x} = 0 \qquad z_{,y} = 0 \tag{A.2.8}$$

b) $$\begin{vmatrix} z_{,xx} & z_{,xy} \\ z_{,xy} & z_{,yy} \end{vmatrix} > 0 \tag{A.2.9}$$

Further, at a minimum point

$$z_{,xx} > 0 \qquad \text{or} \qquad z_{,yy} > 0 \tag{A.2.10}$$

and at a maximum point

$$z_{,xx} < 0 \qquad \text{or} \qquad z_{,yy} < 0 \tag{A.2.11}$$

Finally, consider the case when conditions a) above are satisfied but the determinant in b) is negative. In this special case, the sign of $Q(\varepsilon, \alpha)$ will generally depend upon the value and the sign of the ratio (ε/α). In particular, it is possible to find one value of (ε/α) for which Q becomes equal to zero. This is the case of a *saddle point.* If the determinant in b) vanishes identically the nature of the point remains indeterminate and

† Clearly if the determinant and one of its diagonal elements is positive the other diagonal element will necessarily be positive.

one must examine the higher order terms in the Taylor series expansion of the function.

Example A.2.1

Let us determine the maximum and minimum points of

$$z(x, y) = x^2 y + xy^2 - axy \tag{a}$$

where a is a constant.

$$z_{,x} = (2x + y - a)\, y \qquad z_{,y} = (2y + x - a)\, x \tag{b}$$

$$z_{,xx} = 2y \qquad z_{,yy} = 2x$$

$$z_{,xy} = 2x + 2y - a$$

For the vanishing of $z_{,x}$ and $z_{,y}$ the following four solutions exist:

$$x = 0 \quad y = 0 \; ; \qquad x = 0 \quad y = a \tag{c}$$

$$x = a \quad y = 0 \; ; \qquad x = \frac{a}{3} \quad y = \frac{a}{3}$$

For a positive definite coefficient matrix (A.2.9), the requirement of

$$z_{,xx}\, z_{,yy} > z_{,xy}^2 \tag{d}$$

takes the form

$$4xy > [2(x + y) - a]^2 \tag{e}$$

Now on substituting the above four possible sets of x and y in this equation we note that only the case $x = a/3$, $y = a/3$ satisfies the inequality requirement. We also note that if a is a positive constant then $z_{,xx} > 0$ and hence $(a/3, a/3)$ is a true minimum point. If a happens to be negative then the point $(a/3, a/3)$ will be a maximum point.

Finally consider the case of a function of n variables

$$w = f(u_1, u_2, \ldots, u_n) \tag{A.2.12}$$

To examine the point $u_1 = q_1$, $u_2 = q_2$, ..., $u_n = q_n$ we consider the neighbouring points $u_1 = q_1 + \varepsilon_1$, $u_2 = q_2 + \varepsilon_2$ etc., where $\varepsilon_1, \varepsilon_2$,... are small increments. Then expanding into a Taylor series we have

$$w(q_1 + \varepsilon_1, q_2 + \varepsilon_2, \ldots, q_n + \varepsilon_n) = w(q_1, q_2, \ldots, q_n) +$$

$$\frac{\partial w}{\partial u_i}\bigg|_{q_1, q_2, \ldots, q_n} \varepsilon_i + \frac{1}{2!} \frac{\partial^2 w}{\partial u_i\, \partial u_j}\bigg|_{q_1, q_2, \ldots, q_n} \varepsilon_i\, \varepsilon_j + \frac{1}{3!} \frac{\partial^3 w}{\partial u_i\, \partial u_j\, \partial u_k}\bigg|_{q_1, q_2, \ldots, q_n} \varepsilon_i\, \varepsilon_j\, \varepsilon_k + \cdots \tag{A.2.13}$$

where the partial derivatives are evaluated at the point $q_1, q_2, \cdots q_n$. Now the independence of the sign of $[\, w(q_i + \varepsilon_i)\,]$ from that of ε_i requires that at a stationary point

$$\left.\frac{\partial w}{\partial u_i}\right|_{q_1, q_2, \ldots, q_n} = 0 \qquad i = 1, 2, \cdots n \tag{A.2.14}$$

The nature of the point, that is, whether it is a minimum or maximum point, is governed by the quadratic terms. If the matrix

$$\begin{bmatrix} \frac{\partial^2 w}{\partial u_1 \partial u_1} & \frac{\partial^2 w}{\partial u_1 \partial u_2} & \frac{\partial^2 w}{\partial u_1 \partial u_3} & \cdots & \frac{\partial^2 w}{\partial u_1 \partial u_n} \\ \frac{\partial^2 w}{\partial u_2 \partial u_1} & \frac{\partial^2 w}{\partial u_2 \partial u_2} & \frac{\partial^2 w}{\partial u_2 \partial u_3} & \cdots & \frac{\partial^2 w}{\partial u_1 \partial u_n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{\partial^2 w}{\partial u_n \partial u_1} & \frac{\partial^2 w}{\partial u_n \partial u_2} & \frac{\partial^2 w}{\partial u_n \partial u_3} & \cdots & \frac{\partial^2 w}{\partial u_n \partial u_n} \end{bmatrix} \tag{A.2.15}$$

is positive definite, at $(q_1, q_2, \ldots, q_n)$, then we have a minimum point. On the other hand, if the matrix is negative definite, then we have a maximum point.

An essential property of positive definite matrices is that they possess positive determinants and positive *principal minors.* A square matrix of order n has $(n-1)$ principal minors. The first principal minor is a determinant of order $(n-1)$ obtained by removing one row and one column that share a *diagonal* element. The second principal minor, being a determinant of order $(n-2)$, is obtained by removing two such rows and columns. Finally, the $(n-1)$th principal minor will be a diagonal element of the matrix.

A negative definite matrix of order n is also characterized by the sign of its determinant and its principal minors. These should be positive if they are of even order and negative if they are of odd order.

By appropriate transformations a symmetric matrix can be diagonalized. If the given matrix is positive definite, its diagonal form will have all its elements greater than zero. The diagonal form of a negative definite matrix will have all its elements smaller than zero. In their diagonal forms it can be seen that sign definite matrices give rise to quadratic forms which are sums of squared terms multiplied by plus or minus 1.

Problems

A.2.1 Given is the function

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Search for minima, maxima, and saddle points.

A.2.2 Plot the surface

$$z = x^2y + xy^2 - 3xy$$

Using a graphics program, for $-1 < x < 4$ and $-1 < y < 4$.

A.3 Stationary Values of Definite Integrals

In problems involving the calculus of variations we are seeking functions which extremize a definite integral. The determination of the necessary conditions for extremization of a definite integral is relatively simple. However to show that a given function actually minimizes the definite integral, is considerably more involved. Fortunately in physical problems the nature of the extremum state is often apparent from physical considerations.

Let the definite integral be

$$I = \int_a^b F(x, y, y') \, dx \tag{A.3.1}$$

We are given the integrand F and the boundary points a,b and we are seeking $y = y(x)$ which extremizes I.

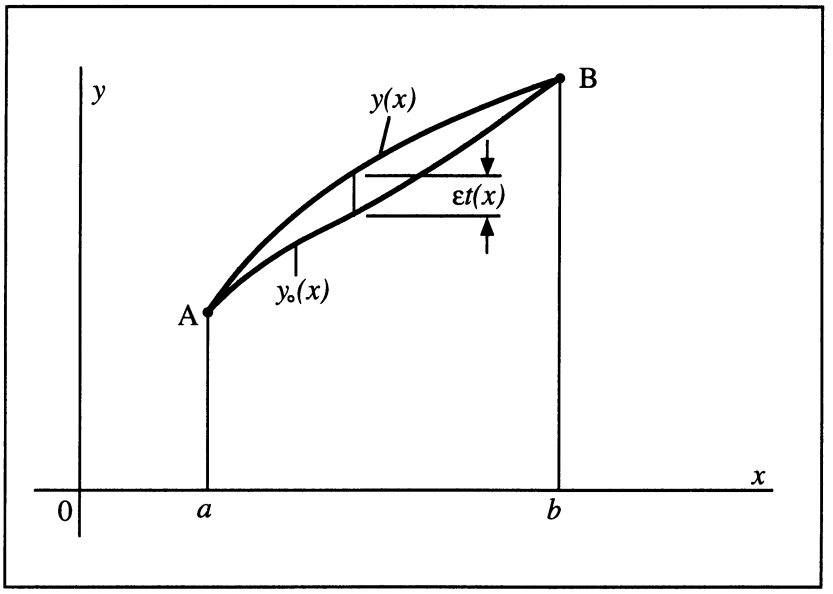


Figure A.3. Extremizing function $y(x)$ and an admissible function $y_O(x)$

Clearly the admissible functions $y(x)$ must:

i) Pass through points $x = a$ and $x = b$.

ii) Be continuous and single valued functions of x in the interval $a \leq x \leq b$.

The last requirement is imposed to ensure that the integrand F is defined in an unambiguous manner.

Suppose now that the required extremizing function is $y_o(x)$ and let $y(x)$ be another admissible function (Figure A.3).

Then we may write

$$y(\varepsilon, x) = y_o(x) + \varepsilon t(x) \tag{A.3.2a}$$

and

$$y'(\varepsilon, x) = y_o'(x) + \varepsilon t'(x) \tag{A.3.2b}$$

where ε is a small parameter which does not depend upon x, and the function $t(x)$ is an arbitrary function independent of ε. Evidently $t(x)$ is continuous in the domain of x between a and b, and further at $x = a$ and $x = b$, it must vanish since $y(x)$ is an admissible function.

Now let the specific value of I at the extremum condition be denoted by I_o. Then

$$I_o = \int_a^b F(x, y_o, y_o')\, dx \tag{A.3.3}$$

For $y(x)$ we have

$$I = I_o + \Delta I = \int_a^b F(x, y, y')\, dx \tag{A.3.4}$$

In equation (A.3.4) it can be seen that to evaluate the integrand we require to know the three quantities x,y, and y'. In other words these quantities can be regarded as three independent variables of F. The fact that one of these variables happens to be the derivative of another, is incidental as far as the changes in the value of F are concerned. The integrand of equation (A.3.4) can then be regarded as the function F when two of its independent variables, namely y_o and y_o', are changed by amounts εt and $\varepsilon t'$ respectively. For a given x we may then expand F by a Taylor series about y_o and y_o'. Thus we write

$$\begin{aligned} F(x, y_o + \varepsilon t, y_o' + \varepsilon t') = {} & F(x, y_o, y_o') + \left.\frac{\partial F}{\partial y}\right|_o \varepsilon t + \left.\frac{\partial F}{\partial y'}\right|_o \varepsilon t' \\ & + \frac{1}{2!}\left.\frac{\partial^2 F}{\partial y^2}\right|_o (\varepsilon t)^2 + \frac{1}{2!}\left.\frac{\partial^2 F}{\partial y'^2}\right|_o (\varepsilon t')^2 + \frac{1}{2!}\left.\frac{2\partial^2 F}{\partial y \partial y'}\right|_o \varepsilon^2 tt' \\ & + \frac{1}{3!}\left.\frac{\partial^3 F}{\partial y^3}\right|_o (\varepsilon t)^3 + \ldots\ldots \end{aligned} \tag{A.3.5}$$

where $\left.\frac{\partial F}{\partial y}\right|_o$ and $\left.\frac{\partial F}{\partial y'}\right|_o$ etc. imply the partial derivative of F with respect to its argument y and y', evaluated at y_o and y_o'. It is assumed that at y_o and y_o' these partial derivatives are continuous. We emphasize that x, the independent variable, is not changed here. It is now apparent that

$$\Delta I = \varepsilon \int_a^b \left[t \left.\frac{\partial F}{\partial y}\right|_o + t' \left.\frac{\partial F}{\partial y'}\right|_o \right] dx$$

$$+ \frac{\varepsilon^2}{2} \int_a^b \left[t^2 \left.\frac{\partial^2 F}{\partial y^2}\right|_o + 2t' \left.\frac{\partial^2 F}{\partial y \partial y'}\right|_o + t'^2 \left.\frac{\partial^2 F}{\partial y'^2}\right|_o \right] dx + \cdots \quad \text{(A.3.6)}$$

or

$$\Delta I = \varepsilon I_1 + \frac{\varepsilon^2}{2} I_2 + \ldots\ldots \quad \text{(A.3.7)}$$

where εI_1 is called the *first variation* of the definite integral, and $\varepsilon^2 I_2$ is its *second variation.*

Evidently if I_o is the maximum value of I, then ΔI must be negative for sufficiently small values of ε, whether ε is positive or negative. Hence sufficient conditions for I_o to be maximum are

$$I_1 = 0 \qquad I_2 < 0 \quad \text{(A.3.8)}$$

Similarly for I_o to be minimum we have as sufficient conditions

$$I_1 = 0 \qquad I_2 > 0 \quad \text{(A.3.9)}$$

Evidently ΔI vanishs when $\varepsilon = 0$ hence at extremum condition $\partial F/\partial y \Big|_o = \partial F/\partial y$, as an inspection of equations (A.3.2) indicates. Thus from here on we need not retain the subscript o.

From equations (A.3.6) and (A.3.7), we have that

$$I_1 = \int_a^b \left[t \frac{\partial F}{\partial y} + t' \frac{\partial F}{\partial y'} \right] dx \quad \text{(A.3.10)}$$

Integrating the second term of the integrand by parts we have

$$I_1 = \int_a^b t \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] dx + \left. \frac{\partial F}{\partial y'} t \right|_{x=a}^{x=b} \quad \text{(A.3.11)}$$

Now since $t(x)$ is equal to zero at $x = a$ and $x = b$ the second term of equation (A.3.11) vanishes. Further, recognizing that $t(x)$ is an arbitrary function we conclude that for I_1 to vanish identically, i.e. for all possible admissible forms of $t(x)$, then we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \quad \text{(A.3.12)}$$

Equation (A.3.12) is referred to as the *Euler-Lagrange equation*† of the functional in

† Leonhard Euler (1707-1783), Swiss mathematician
Joseph Louis Lagrange (1736-1813), French mathematician

equation (A.3.1). Its satisfaction is a necessary condition for the stationarity of the functional. By stationarity we mean the condition $I_1 = 0$ without being specific about the sign of I_2.

The derivatives arising in the Euler-Lagrange equation require some clarification. We have already pointed out that in obtaining the derivatives of F with respect to y and y', the latter is treated as an independent variable. However in carrying out the differentiation of $\partial F/\partial y'$ with respect to x it should be borne in mind that y and y' are functions of x. Thus

$$\frac{d}{dx}\left[\frac{\partial F}{\partial y'}(x,y,y')\right] = \frac{\partial^2 F}{\partial y' \partial x} + \frac{\partial^2 F}{\partial y' \partial y}\frac{dy}{dx} + \frac{\partial^2 F}{\partial y'^2}\frac{dy'}{dx} \qquad \text{(A.3.13)}$$

The origins of the calculus of variations date back to the time of Euler and Lagrange – hence the name Euler-Lagrange equation. The derivation of this equation as outlined here is essentially due to Lagrange. Euler arrived at the same equation via a different scheme of reasoning. It is worth considering Euler's approach briefly.

To find the stationary condition of the functional I in equation (A.3.1) Euler first discretized the functional and thereby he reduced the problem of extremizing the functional to the problem of extremizing a function. His method is as follows:

Let the interval between $x = a$ and $x = b$ be divided into n subintervals and, for convenience, let us take these to be of equal lengths Δx. Thus the set of values for the x and y coordinates become (see Figure A.4)

$$x_o, x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n \qquad \text{with } x_o = a \quad \text{and } x_n = b \qquad \text{(A.3.14a)}$$

$$y_o, y_1, y_2, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n \qquad \text{with } y_o = \alpha \quad \text{and } y_n = \beta \qquad \text{(A.3.14b)}$$

The derivative y' may be approximately represented by the following central difference formula.

$$(y')_{x=x_k} \approx \frac{y_{k+1} - y_{k-1}}{2\Delta x} \qquad \text{(A.3.15)}$$

Then the functional

$$I = \int_a^b F(x, y, y')\, dx \qquad \text{(A.3.16)}$$

may be approximately represented by a function

$$I_{\text{approx}} = \sum_{k=0}^{n} F\left[x_k, y_k, \frac{y_{k+1} - y_{k-1}}{2\Delta x}\right]\Delta x \qquad \text{(A.3.17)}$$

To find the stationary condition of I_{approx} we must now evaluate the derivatives of I_{approx} with respect to y_k, and equate them to zero.

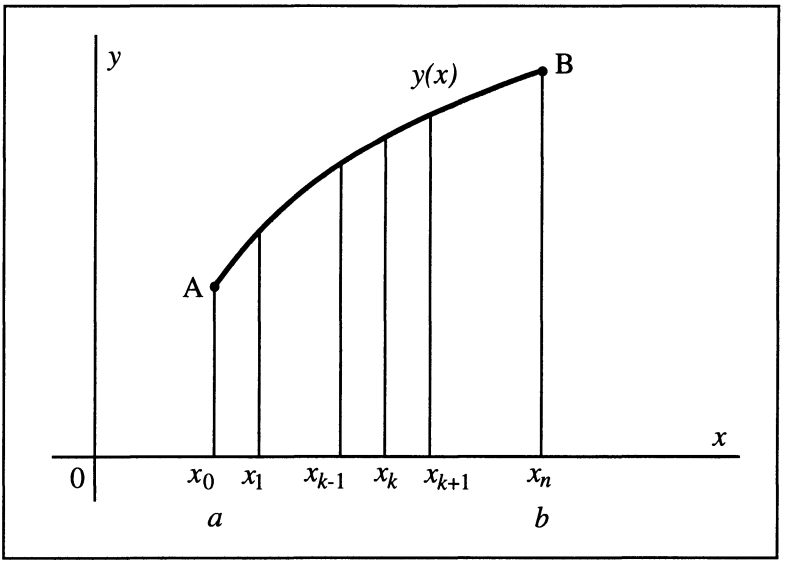


Figure A.4. Discrete representation of $y(x)$

The only terms in I_{approx} that involve y_k are

$$F\left[x_k, y_k, \frac{y_{k+1} - y_{k-1}}{2\Delta x}\right]\Delta x \tag{A.3.18a}$$

$$F\left[x_{k+1}, y_{k+1}, \frac{y_{k+2} - y_k}{2\Delta x}\right]\Delta x \quad \text{and} \tag{A.3.18b}$$

$$F\left[x_{k-1}, y_{k-1}, \frac{y_k - y_{k-2}}{2\Delta x}\right]\Delta x \tag{A.3.18c}$$

Thus we have

$$\frac{\partial I_{\text{approx}}}{\partial y_k} = \left[\frac{\partial F}{\partial y}\right]_{x=x_k}\Delta x + \frac{1}{2\Delta x}\left[\frac{\partial F}{\partial y'}\right]_{x=x_{k-1}}\Delta x - \frac{1}{2\Delta x}\left[\frac{\partial F}{\partial y'}\right]_{x=x_{k+1}}\Delta x = 0 \tag{A.3.19}$$

Now recognizing that the last two terms express the change in $\left[\frac{\partial F}{\partial y'}\right]$ at x_k, i.e.

$$\Delta\left[\frac{\partial F}{\partial y'}\right]_{x=x_k} = \frac{1}{2}\left[\left[\frac{\partial F}{\partial y'}\right]_{x=x_{k+1}} - \left[\frac{\partial F}{\partial y'}\right]_{x=x_{k-1}}\right] \tag{A.3.20}$$

and finally dividing by Δx and equating the whole expression to zero, we have

$$\left[\frac{\partial F}{\partial y'}\right]_{x=x_k} - \frac{\Delta}{\Delta x}\left[\frac{\partial F}{\partial y'}\right]_{x=x_k} = 0 \tag{A.3.21}$$

In the limit, as Δx approaches zero, above equation becomes

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0 \qquad \text{(A.3.22)}$$

Example A.3.1

Let us return to the problem of example A.1.1 of the shortest distance between two points in a plane. The integrand in this case was

$$F = \sqrt{1 + y'^2} \qquad \text{(a)}$$

Since y is absent in the integrand, the Euler-Lagrange equation (A.3.12) becomes simply

$$-\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] = 0 \qquad \text{(b)}$$

or

$$-\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = 0 \qquad \text{(c)}$$

which on integration yields

$$\frac{y'}{\sqrt{1 + y'^2}} = \text{constant}$$

This equation can be satisfied only if y' is equal to a constant, say m, i.e. $y' = m$ or

$$y = mx + c \qquad \text{(d)}$$

where c is a constant of integration.

Alternatively if we differentiate equation (c) we find, after some simplifications

$$\frac{y''}{[1 + y'^2]^{3/2}} = 0 \qquad \text{(e)}$$

Equation (e) states that the extremizing function must have zero curvature.

In this example, it is apparent that the solution corresponds to minimum value of the functional. Either equation (d) or equation (e) then state the shortest distance between two points is a straight line.

Problems

A.3.1 Given is an integrand

$$F(t, y, \dot{y}) = \frac{1}{2} m (\dot{y} + at)^2 - mg (y + \frac{1}{2} at^2)$$

Use the Euler-Lagrange equation to obtain $y = y(t)$. The boundary conditions are $\dot{y} = y = 0$ when $t = 0$.

A.3.2 The shortest distance between the points (0, 0) and (2, 4) in a plane is measured along a line

$$y_o = 2x$$

Use $y = x^2$, and $t = 10x - 5x^2$, such that

$$y = y_o + \varepsilon t$$

(a) What is the magnitude of ε ? (b) What is the distance between (0, 0) and (2, 4) measured along y_o? (c) What is the distance between, (0, 0) and (2, 4) measured along y? (d) Is y an admissible function? Why? (e) Does y satisfy the Euler-Lagrange equation? (f) Does y_o satisfy the Euler-Lagrange equation? (g) In order to satisfy the Euler-Lagrange equation for the shortest distance between two points, what form must y assume?

A.4 A Note about Weak and Strong Variations

Earlier we expressed an admissible function $y(x)$ in terms of the extremizing function $y_o(x)$ as follows

$$y(x) = y_o(x) + \varepsilon t(x) \tag{A.4.1}$$

Further we stipulated that $t(x)$ should be independent of ε. The reason for this requirement is that we wish to ensure that as $\varepsilon \to 0$ not only does $y(x) \to y_o(x)$ but also all derivatives of $y(x)$ approach the corresponding derivatives of $y_o(x)$. By relating $t(x)$ and ε it is possible to relax the requirement for convergence of the derivatives of $y(x)$ to those of $y_o(x)$ as $\varepsilon \to 0$. For example if we let

$$t(x) = \sin \frac{x}{\varepsilon^2} \tag{A.4.2}$$

then

$$t'(x) = \frac{1}{\varepsilon^2} \cos \frac{x}{\varepsilon^2} \tag{A.4.3}$$

Now as ε approaches zero, $\varepsilon t(x)$ vanishes and hence $y(x)$ approaches $y_o(x)$. On the other $\varepsilon t'(x)$ does not vanish and $y'(x)$ does not approach $y_o'(x)$.

Whenever $t(x)$ is independent of ε we speak of a weak variation and when $t(x)$ is related to ε we speak of a strong variation. In this text we shall be concerned with weak variations only.

A.5 An Alternative Expression for a Single Euler-Lagrange Equation

Consider the total derivative of F with respect to x. That is

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' \tag{A.5.1}$$

But from the Euler-Lagrange equation we have that

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] \tag{A.5.2}$$

Substituting this expression for $\partial F/\partial y$ in equation (A.5.1) we obtain

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + y' \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] + y'' \frac{\partial F}{\partial y'} \tag{A.5.3}$$

On rearranging and simplifying this expression we obtain

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = \frac{\partial F}{\partial x} \qquad \text{(A.5.4)}$$

Now equations (A.5.2) and (A.5.4) provide alternative differential equations for the determination of the extremizing function as we can see by carrying out the differentiation indicated in equation (A.5.4). Thus

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' - y''\frac{\partial F}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}$$

or

$$y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right] = 0 \qquad \text{(A.5.5)}$$

Equations (A.5.2) and (A.5.4) allow partial integration under two special conditions. If F does not contain y, then from equation A.5.2 it follows that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{or} \qquad \frac{\partial F}{\partial y'} = \text{constant} \qquad \text{(A.5.6)}$$

On the other hand if F does not contain x explicitly then equation (A.5.4) becomes

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{or} \qquad F - y'\frac{\partial F}{\partial y'} = \text{constant} \qquad \text{(A.5.7)}$$

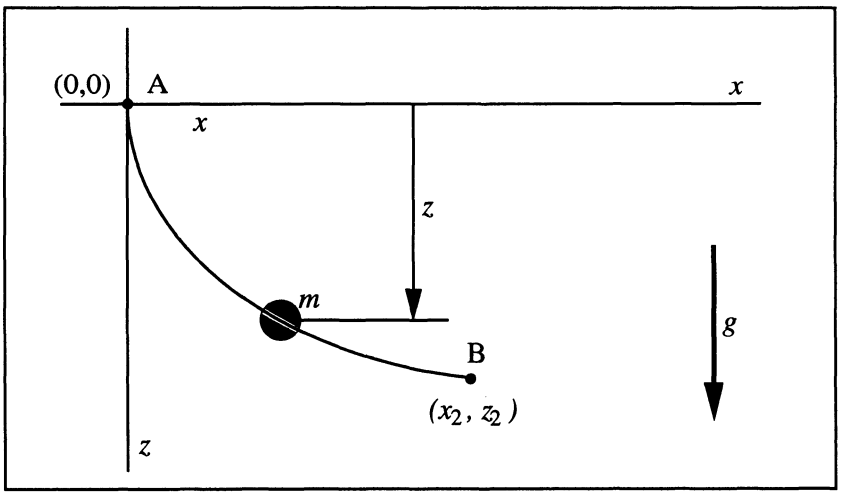


Figure A.5. The brachystochrone problem - a bead m sliding down a smooth wire

A.6 The Brachystochrone Problem

A smooth wire joins the points (0,0) and (x_2,z_2) as shown in Figure A.5. A bead slides along the wire under gravity and without friction. Given the initial velocity of the bead what should be the shape of the wire such that the time of travel for the bead

between the two points is as short as possible? This problem, referred to as the *brachystochrone* problem, was one of the early examples which involved the essential ideas of the calculus of variations. The word brachystochrone is derived from Greek words meaning shortest time. The problem was posed by Johann Bernoulli in 1696 as a challenge to the mathematicians of the time. Johann Bernoulli himself, his elder brother Jakob Bernoulli, Newton, Leibniz and de L'Hospital† gave the correct solution for the problem.

For simplicity let the initial velocity of the bead be zero. Then the kinetic coenergy of the bead at any instant will be equal to the loss of the bead's potential energy, i.e.

$$\frac{1}{2} mv^2 = mgz \tag{A.6.1}$$

Where v is the bead's velocity and g denotes the gravitational attraction constant. From equation (A.6.1) we have that

$$v = \frac{ds}{dt} = \sqrt{2gz}$$

or $$dt = \frac{1}{\sqrt{2gz}} ds$$

But $$ds = \sqrt{dx^2 + dz^2} = \sqrt{1 + z'^2}\, dx$$

and thus $$t = \frac{1}{\sqrt{2g}} \int_o^{x_2} \left[\frac{1 + z'^2}{z} \right]^{1/2} dx \tag{A.6.2}$$

In the functional in equation (A.6.2) the independent variable, x, is absent. It is then convenient to use the alternative form (A.5.4) of the Euler-Lagrange equation. Thus for the extremum condition of the functional in equation (A.6.2) we may write

$$\left[\left[\frac{1 + z'^2}{z} \right]^{1/2} - z' \frac{\partial}{\partial z'} \left[\frac{1 + z'^2}{z} \right]^{1/2} \right] = \text{constant} = 2R$$

After carrying out the differentiation, above expression simplifies to

$$z(1 + z'^2) = 2R \tag{A.6.3}$$

Equation (A.6.3) is a first order nonlinear ordinary differential equation the solution of which provides the path for quickest descent of the bead. To solve let

$$z' = \cot \frac{\alpha}{2}$$

Substituting into equation (A.6.3) and simplifying we find

$$z = 2R \sin^2 \frac{\alpha}{2} = R(1 - \cos \alpha) \tag{A.6.4}$$

---

† Jakob Bernoulli (1654-1705), Swiss mathematician (coined the term "integral")
Johann Bernoulli (1667-1748), Swiss mathematician (Euler's teacher)
Guillaume Francois Antoine Marquis de L'Hospital (1661-1704), French mathematician
Gottfried Wilhelm Leibniz (1646-1716), German philosopher (calculus).

Having obtained z in terms of α, we now consider the relation between x and α.

Since $z' = \frac{dz}{dx} = \cot\frac{\alpha}{2}$ we have $dx = dz \tan\frac{\alpha}{2}$

but $dz = R \sin\alpha \, d\alpha$

Hence $dx = R \sin\alpha \tan\frac{\alpha}{2} d\alpha$ or $dx = R(1 - \cos\alpha) \, d\alpha$

On integrating we obtain

$$x = R(\alpha - \sin\alpha) + c \qquad \text{(A.6.5)}$$

Where c is a constant of integration. Equations (A.6.4) and (A.6.5) are the parametric equations of a cycloid the curve for which is obtained by the locus of a point on a circle rolling along a straight line in the xz plane. To determine the constant of integration we take the initial position of the bead at the origin of the coordinate axes, $x = 0$, $z = 0$ and $\alpha = 0$. From equation (A.6.5) it then follows that $c = 0$. The curve for the cycloid is shown in Figure A.6 from which it is apparent that the constant R is the radius of the rolling circle.

To determine the required value of R for the given points (0,0) and (x_2, z_2) we can write from equations (A.6.4) and (A.6.5)

$$z_2 = R(1 - \cos\alpha_2) \qquad x_2 = R(\alpha_2 - \sin\alpha_2)$$

These two equations can now be solved for R and α_2.

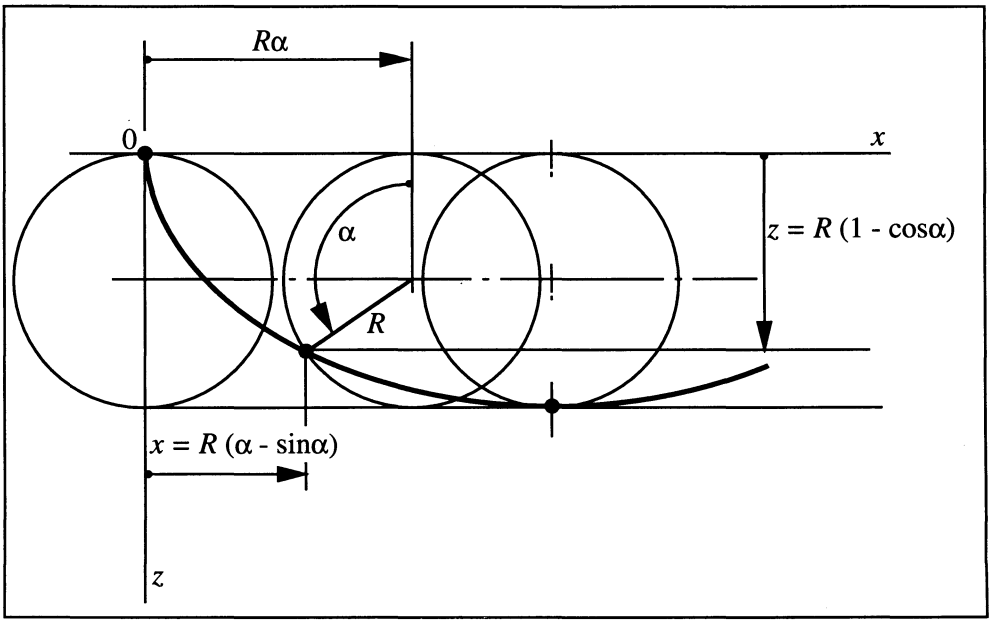


Figure A.6. The cycloid curve solution of the brachystochrone problem

Problems

A.6.1 Determine the stationary function and the value of the following functional with the given end conditions

$$\int_0^{1/2} \frac{\sqrt{1+y'^2}}{y}\,dx \qquad y(0) = 1 \qquad y(1) = 0$$

A.6.2 Given are the initial point (0, 0) and the final point (2, 3) of a brachystochrone. (a) Find the constants R and α_2 and (b) make a sketch.

A.6.3 A bead moves on a frictionless cycloid in the gravitational field from point (0, 0) to point $(\pi R, 2R)$. Calculate the time of travel, if the initial velocity is zero, and $R = 2\text{m}$.

A.6.4 The tautochrone (gk: same time) problem is complementary to the brachystochrone problem in the sense that it asks for the shape of the wire for which the time of descent of a frictionless bead to the lowest point will be a constant, i.e. independent of the starting point. Show that the tautochrone shape is a cycloid.

A.6.5 Fermat's principle states that a ray of light will travel between two fixed points along the curve for which the travel time is a minimum. Obtain the required functional and show that its Euler-Lagrange equation takes the following form

$$\frac{vy''}{1+y'^2} - \frac{\partial v}{\partial x}y' + \frac{\partial v}{\partial y} = 0$$

where $v(x,y)$ is the variable velocity of light.

A.7 Path-independent Functionals

It is possible for the extremum value of a definite integral to be dependent upon the limits of the integral only. Consider for instance the integral

$$I = \int_a^b [2xy + (x^2 + 3y^2)\,y']\,dx \tag{A.7.1}$$

When the integrand $F[= 2xy + (x^2 + 3y^2)y']$ is substituted into the Euler-Lagrange equation it is found that this equation is identically satisfied immaterial of the form of y, as a function of x. Hence an extremum value of the integral I is dependent only on the limits of the integral. To examine this further let us consider the implications of the Euler-Lagrange equation being identically satisfied.

$$\text{If} \quad \frac{\partial F(x,y,y')}{\partial y} - \frac{d}{dx}\frac{\partial F(x,y,y')}{\partial y'} \equiv 0 \tag{A.7.2}$$

$$\text{or} \quad \frac{\partial F(x,y,y')}{\partial y} - \left[\frac{\partial^2 F(x,y,y')}{\partial y'\partial x} + \frac{\partial^2 F(x,y,y')}{\partial y'\partial y}\frac{dy}{dx} + \frac{\partial^2 F(x,y,y')}{\partial y'^2}\frac{d^2y}{dx^2}\right] \equiv 0 \tag{A.7.3}$$

Then $\dfrac{\partial^2 F}{\partial y'^2}$ must vanish since y'' does not appear in any other term in equation (A.7.3).

This implies that F can be at most a linear function of y' or generally

$$F = p(x,y)y' + q(x,y) \tag{A.7.4}$$

where p and q can be functions of x and y but not of y'.

Substituting from equation (A.7.4) into equation (A.7.2) we find

$$\frac{\partial p(x,y)}{\partial y} y' + \frac{\partial q(x,y)}{\partial y} - \frac{d}{dx} p(x,y) = 0$$

or

$$\frac{\partial p}{\partial y} y' + \frac{\partial q}{\partial y} - \frac{\partial p}{\partial x} - \frac{\partial p}{\partial y} y' = 0$$

i.e.

$$\frac{\partial q}{\partial y} = \frac{\partial p}{\partial x} \tag{A.7.5}$$

Returning now to the integral I we may write

$$I = \int_a^b F dx = \int_a^b (py' dx + q dx)$$

or

$$I = \int_{y(a)}^{y(b)} p \, dy + \int_a^b q \, dx \tag{A.7.6}$$

Finally we show that providing equation (A.7.5) holds the integrand of equation (A.7.6) becomes an exact differential. To this end let us consider a continuous function $\phi = \phi(x,y)$, the differential of which is given by

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

On comparing with the integrand of I we note the correspondence between

$$\frac{\partial \phi}{\partial x} \longleftrightarrow q \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} \longleftrightarrow p \tag{A.7.7}$$

However since $\phi(x,y)$ is a continuous function of x and y, we also have the condition of continuity

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x} \tag{A.7.8}$$

now for the correspondence between (p,q) and the derivatives of ϕ to be valid it is further required that from equations (A.7.7) and (A.7.8), we obtain

$$\frac{\partial q}{\partial y} = \frac{\partial p}{\partial x}$$

which is the condition we showed to be true in equation (A.7.5). Finally we may rewrite equation (A.7.6) as

$$I = \int d\phi = \phi(b, y(b)) - \phi(a, y(a)) \tag{A.7.9}$$

It is a simple matter to show that in the example we considered in equation (A.7.1), the integrand is the differential of the following function

$$\phi = x^2 y + y^3 + c$$

A.8 Several Dependent Functions

We now consider the more general case of integrals of the type

$$I = \int_a^b F(x, q_1, q_2, \cdots q_n, q_1', q_2', \cdots q_n')dx \tag{A.8.1}$$

where q_i may be varied independently, and q_i' are the derivatives of q_i with respect to x. For simplicity consider the case of two dependent functions.

$$I = \int_a^b F(x, q_1, q_2, q_1', q_2')dx \tag{A.8.2}$$

Let us take the extremizing functions as $q_{10}(x)$ and $q_{20}(x)$ and as before let us express any pair of admissible functions q_1 and q_2 as

$$q_1 = q_{10} + \varepsilon t_1(x)$$
$$q_2 = q_{20} + \varepsilon t_2(x)$$

For the admissible functions, the value of the integral may be written as

$$I = I_o + \Delta I = \int_a^b F(x, q_{10} + \varepsilon t_1, q_{10}' + \varepsilon t_1', q_{20} + \varepsilon t_2, q_{20}' + \varepsilon t_2')dx \tag{A.8.3}$$

Expanding the integrand by Taylor series about q_{10} and q_{20}, we obtain

$$\Delta I = \varepsilon \int_a^b \left[\frac{\partial F}{\partial q_1} t_1 + \frac{\partial F}{\partial q_1'} t_1'\right]dx + \varepsilon \int_a^b \left[\frac{\partial F}{\partial q_2} t_2 + \frac{\partial F}{\partial q_2'} t_2'\right]dx + O(\varepsilon^2) \tag{A.8.4}$$

Now for $\Delta I = \varepsilon I_1 + \frac{\varepsilon^2}{2} I_2 + \ldots$ and integrating the t' terms by parts we can write the first variation of the integral ΔI as

$$\varepsilon I_1 = \varepsilon \int_a^b t_1 \left[\frac{\partial F}{\partial q_1} - \frac{d}{dx}\frac{\partial F}{\partial q_1}\right]dx + \varepsilon t_1 \frac{\partial F}{\partial q_1'}\Bigg|_a^b +$$
$$\varepsilon \int_a^b t_2 \left[\frac{\partial F}{\partial q_2} - \frac{d}{dx}\frac{\partial F}{\partial q_2}\right]dx + \varepsilon t_2 \frac{\partial F}{\partial q_2'}\Bigg|_a^b \tag{A.8.5}$$

According to admissibility requirements, t_1 and t_2 vanish at points $x = a$ and $x = b$ and hence the terms to be evaluated at these points also vanish. Arguing now on the arbitrariness of t_1 and t_2 (apart from the admissibility requirements) we conclude that for I_1 to be identically zero we must have

$$\frac{\partial F}{\partial q_1} - \frac{d}{dx}\frac{\partial F}{\partial q_1'} = 0 \tag{A.8.6a}$$

and

$$\frac{\partial F}{\partial q_2} - \frac{d}{dx}\frac{\partial F}{\partial q_2'} = 0 \qquad \text{(A.8.6b)}$$

The above pair of Euler-Lagrange equations govern the extremizing solutions $q_1(x)$ and $q_2(x)$.

In this case we can again obtain a useful expression which will be satisfied if the above pair of Euler-Lagrange equations are satisfied. The required expression is given by

$$\frac{d}{dx}\left[F - q_1'\frac{\partial F}{\partial q_1'} - q_2'\frac{\partial F}{\partial q_2'}\right] = \frac{\partial F}{\partial x} \qquad \text{(A.8.7)}$$

On carrying out the differentiations and subsequent simplifications we obtain the following equation

$$q_1'\left[\frac{\partial F}{\partial q_1} - \frac{d}{dx}\frac{\partial F}{\partial q_1'}\right] + q_2'\left[\frac{\partial F}{\partial q_2} - \frac{d}{dx}\frac{\partial F}{\partial q_2'}\right] = 0 \qquad \text{(A.8.8)}$$

Evidently equation (A.8.8) will be satisfied if equations (A.8.6) are satisfied.

Generalization of the above considerations to the case of n dependent variables is self-evident, giving the following Euler-Lagrange equations

$$\frac{\partial F}{\partial q_i} - \frac{d}{dx}\frac{\partial F}{\partial q_i'} = 0 \qquad i = 1,2\cdots n \qquad \text{(A.8.9)}$$

It is relatively easy to show that the following expression will be satisfied if the Euler-Lagrange equations (A.8.9) are satisfied:

$$\frac{d}{dx}\left[F - q_i'\frac{\partial F}{\partial q_i'}\right] = \frac{\partial F}{\partial x} \qquad i = 1,2\cdots n \qquad \text{(A.8.10)}$$

Example A.8.1

Find the stationary functions and stationary value of

$$I(q_1,q_2) = \int_0^{\frac{\pi}{2}}\left[(q_1')^2 + (q_2')^2 + 2q_1q_2\right]dx \qquad \text{(a)}$$

Subject to the boundary conditions

$$q_1(0) = 0 \qquad q_1(\tfrac{\pi}{2}) = 1 \qquad q_2(0) = 0 \qquad q_2(\tfrac{\pi}{2}) = -1 \qquad \text{(b)}$$

We now have

$$F = (q_1')^2 + (q_2')^2 + 2q_1q_2 \qquad \text{(c)}$$

$$\frac{\partial F}{\partial q_1} = 2q_2 \qquad \frac{\partial F}{\partial q_2} = 2q_1 \qquad \text{(d)}$$

$$\frac{\partial F}{\partial q_1'} = 2q_1' \qquad \frac{\partial F}{\partial q_2'} = 2q_2'$$

Hence the Euler-Lagrange equations give

$$q_2 - q_1'' = 0 \qquad q_1 - q_2'' = 0 \tag{e}$$

i.e. a pair of simultaneous differential equations. Eliminating q_2 we obtain

$$q_1^{IV} - q_1 = 0 \tag{f}$$

Hence $$q_1 = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x \tag{g}$$

and from the first of the second order equations we have

$$q_2 = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x \tag{h}$$

Now, imposing the boundary conditions (b) we obtain

$$c_1 = c_2 = c_3 = 0 \qquad c_4 = 1 \tag{i}$$

i.e. the required solutions are

$$q_1 = \sin x \qquad q_2 = -\sin x \tag{j}$$

and the value of the functional is

$$I = \int_0^{\frac{\pi}{2}} [\, 2\cos^2 x - 2\sin^2 x \,] dx = 0 \tag{k}$$

Substituting the solutions obtained into equation (A.8.7) we find $-2\cos^2 x - 2\sin^2 x$ = constant, which is evidently true.

Problems

A.8.1 Given is an integrand

$$F = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2 \dot{\theta}^2 - \frac{1}{2} k\,(r - r_o)^2$$

where t is the independent variable, and $q_1 = r(t)$ and $q_2 = \theta(t)$ are the two dependent variables. Find (a) the two Euler-Lagrange equations that have to be satisfied to make the functional

$$I = \int F\, dt$$

stationary. Also (b) show that $m\, r^2\, \dot{\theta}$ = constant.

A.8.2 Given is an integrand

$$F = \frac{1}{2} m\, \dot{r}^2 + \frac{1}{2} m\, r^2 \dot{\theta}^2 + \frac{\mu m}{r}$$

(a) Find the two Euler-Lagrange equations that have to be satisfied to make the functional

$$I = \int F\, dt$$

stationary. (b) Check whether the integrable combinations

$$r = \frac{p}{1 + \varepsilon \cos\theta}$$

and

$$R^2 \dot{\theta} = h$$

satisfy the Euler-Lagrange equations identically, where p, ε, and h are constants and $h = \sqrt{\mu p}$.

A.9 Variational Notation

Earlier we expressed an admissible function $y(x)$ in terms of the extremizing function $y_o(x)$, as

$$y(x) = y_o(x) + \varepsilon t(x) \tag{A.9.1}$$

The term $\varepsilon t(x)$ is referred to as the *variation* of the function y. Lagrange introduced the notation δy for this variation and so in Lagrange's notation, equation (A.9.1) may be written as

$$y(x) = y_o(x) + \delta y(x) \tag{A.9.2}$$

Likewise

$$y'(x) = y_o'(x) + \delta y'(x)$$

$$y''(x) = y_o''(x) + \delta y''(x) \qquad \text{etc.}$$

Clearly the *differential* of a function y, i.e. dy, and the *variation* of y denoted by δy, are different quantities. The first denotes a small change in the *value* of the given function while the second is a small change in the *function* itself. Also dy is meaningless without a corresponding change in x, namely dx, while for δy there is no change in x (see Figure A.7).

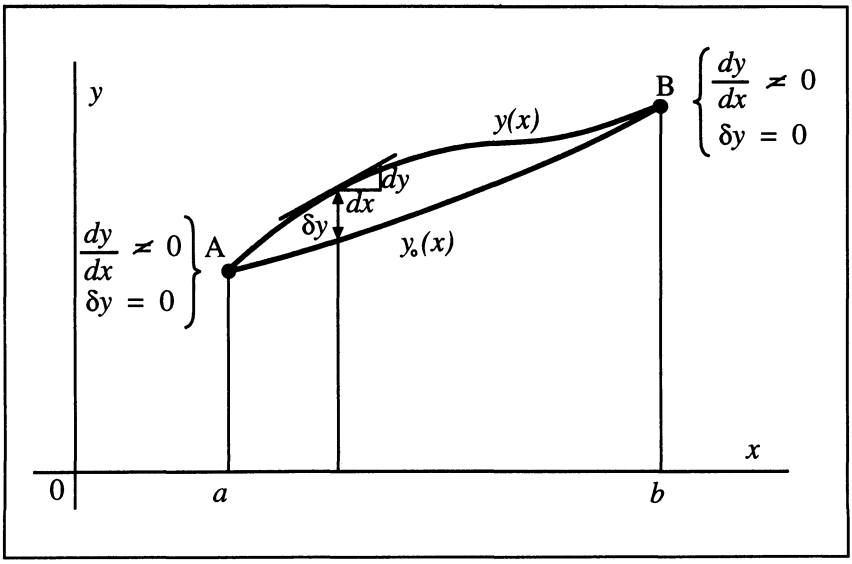


Figure A.7. Variation δy and differential dy of function $y(x)$

We shall see that the δ notation exposes some striking similarities in the *operations* of variation and differentiation and herein lies the main advantage of this notation. However with such similarities there is the risk of overlooking the essential differences between the differential and the variation of a function.

Consider now the following operations on two independent functions F_1 and F_2,

i) $$\delta(F_1 + F_2) = \delta F_1 + \delta F_2 \qquad \text{(A.9.3)}$$

Proof: Let $G = F_1 + F_2$

then
$$\delta G = (G + \delta G) - G$$
$$\delta G = (F_1 + \delta F_1 + F_2 + \delta F_2) - (F_1 + F_2)$$
$$\delta G = \delta F_1 + \delta F_2$$

ii) $$\delta(F_1 \cdot F_2) = F_2\delta F_1 + F_1\delta F_2 \qquad \text{(A.9.4)}$$

Proof: Let $G = F_1 \cdot F_2$

then
$$\delta G = (G + \delta G) - G$$
$$\delta G = (F_1 + \delta F_1)(F_2 + \delta F_2) - F_1F_2$$

now dropping the second-order term we obtain
$$\delta G = F_1\delta F_2 + F_2\delta F_1$$

iii) $$\delta\left[\frac{F_1}{F_2}\right] = \frac{F_2\delta F_1 - F_1\delta F_2}{F_2^2} \qquad \text{(A.9.5)}$$

Proof: Let $G = \dfrac{F_1}{F_2}$

then
$$\delta G = \frac{F_1 + \delta F_1}{F_2 + \delta F_2} - \frac{F_1}{F_2}$$

Rearranging and again dropping the second-order term, we have

$$\delta G = \frac{-F_1\delta F_2 + F_2\delta F_1}{F_2^2}$$

Now let us consider what is meant by

$$\frac{d}{dx}\delta y$$

Clearly since
$$\delta y = (y_o + \varepsilon t) - y_o$$
$$\frac{d}{dx}\delta y = \frac{d}{dx}\varepsilon t = \varepsilon t' = \delta\frac{dy}{dx} \qquad \text{(A.9.6)}$$

i.e. the derivative (*with respect to the independent variable*) of the variation of a function is equivalent to the variation of the derivative of that function.

Consider next the operation $\delta \int_a^b F\,dx$

Since the limits of the integral are fixed, then

$$\delta\int_a^b F\,dx = \int_a^b (F + \delta F)dx - \int_a^b F\,dx = \int_a^b \delta F\,dx \qquad \text{(A.9.7)}$$

i.e. variation of a definite integral (*with fixed limits*) is equivalent to the integration of the variation of the integrand. The above commutative laws hold only when the differentiations and integrations are with respect to the independent variable.

Consider for instance the following case where the two variables x and y are related through a parameter θ,

$$x = x(\theta) \qquad \text{and} \qquad y = y(\theta)$$

Here θ is the independent variable and it is not to be varied. However both $x(\theta)$ and $y(\theta)$ are subject to variation. Suppose we require to determine $\delta\left[\frac{dy}{dx}\right]$. Now

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

denoting the derivatives of x and y, with respect to θ by $\dot{x}$ and $\dot{y}$ respectively we may write

$$\delta\left[\frac{dy}{dx}\right] = \delta\left[\frac{\dot{y}}{\dot{x}}\right] = \frac{\dot{x}\,\delta\dot{y} - \dot{y}\,\delta\dot{x}}{\dot{x}^2}$$

or

$$\delta\left[\frac{dy}{dx}\right] = \frac{\frac{d}{d\theta}\delta y}{\frac{d}{d\theta}x} - (\dot{y}/\dot{x})\frac{\frac{d}{d\theta}\delta x}{\frac{d}{d\theta}x}$$

$$\delta\frac{dy}{dx} = \frac{d}{dx}\delta y - \frac{dy}{dx}\cdot\frac{d}{dx}\delta x \tag{A.9.8}$$

If x had been the independent variable then δx would have vanished and above expression would then have simplified to the commutative form we obtained earlier.

The δ notation can also be used to represent equation (A.3.7) by writing

$$\Delta I = \delta I + \frac{1}{2}\delta^2 I \tag{A.9.9}$$

where the first variation is

$$\delta I = \varepsilon I_1 \tag{A.9.10}$$

and the second variation

$$\delta^2 I = \varepsilon^2 I_2 \tag{A.9.11}$$

Problems

A.9.1 Given is a functional

$$I = \int_a^b \sqrt{1+y'^2}\,dx$$

Use the δ variation symbol and derive the condition which the integrand has to

satisfy in order to make the functional stationary.

A.9.2 Given is a functional

$$I = \int_a^b y \sqrt{1 + y'^2}\, dx$$

Use the δ variation symbol and derive the condition which the integrand has to satisfy in order to make the functional stationary.

A.10 Constraint Equations

Until now we have considered the stationary properties of functions and functionals in the absence of constraints. We now examine the extremum conditions of functions subject to some constraints. Subsequently we will extend the analysis for determination of extremum conditions of functionals subject to various types of constraints.

Suppose we require to determine the extremum points of the function $W = W(x, y)$ subject to the constraint $y = y(x)$. Geometrically this implies that we search for maximum and minimum points on the surface $W = W(x, y)$. However the domain of our search is to be restricted to those points on the surface, whose projection on the xy plane fall on the curve $y = y(x)$ (see Figure A.8).

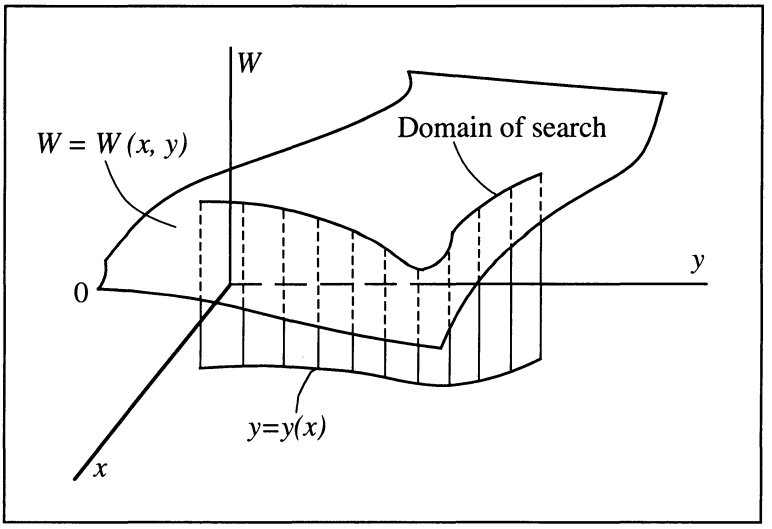


Figure A.8. Search on the surface $W = W(x,y)$ subject to constraint $y = y(x)$

To take a more general case let us examine the extremum condition of the function

$$W = W(x, y, z) \tag{A.10.1}$$

subject to the constraint

$$\phi(x, y, z) = 0 \tag{A.10.2}$$

In principle we can use the constraint equation to eliminate one of the independent variables, say z, from $W = W(x, y, z)$. Then the constraint equation would be explicitly satisfied and we may proceed to determine its extremum points. In practice this elimination may be very cumbersome and sometimes impossible. Further the elimination of one independent variable renders the relation between the dependent variable and the remaining independent variables somewhat artificial.

Now for the extremum points of W we require that

$$dW = \frac{\partial W}{\partial x} dx + \frac{\partial W}{\partial y} dy + \frac{\partial W}{\partial z} dz = 0 \tag{A.10.3}$$

Also by the nature of the constraint equation (A.10.2) we have

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \tag{A.10.4}$$

If no constraints were present we would consider dx, dy and dz as arbitrary and independent and we would equate their coefficients to zero. However in the presence of constraints we are unable to do so since dx, dy and dz are no longer independent. The relation amongst these increments is provided by equation (A.10.4). In contrast to the relation amongst x,y, and z, which may be very complicated, we note that the relation amongst dx, dy, dz is linear. As such one of the increments in equation (A.10.3) can be readily eliminated via equation (A.10.4). Thus we may write, for example

$$dz = - \frac{\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy}{\frac{\partial \phi}{\partial z}} \tag{A.10.5}$$

Substituting this into equation (A.10.3) we obtain

$$dW = (W_{,x} - W_{,z}\, \phi_{,x} / \phi_{,z}) dx + (W_{,y} - W_{,z}\, \phi_{,y} / \phi_{,z}) dy = 0 \tag{A.10.6}$$

where $$( )_{,x} = \frac{\partial ( )}{\partial x} \quad \text{etc.}$$

Having explicitly satisfied the constraint amongst the *increments* we may now consider dx and dy as independent and arbitrary. Thus the extremum conditions of $W = W(x, y, z)$, in the presence of constraint $\phi(x, y, z) = 0$, are

$$W_{,x} - W_{,z}\, \phi_{,x} / \phi_{,z} = 0 \qquad W_{,y} - W_{,z} \phi_{,y} / \phi_{,z} = 0 \tag{A.10.7}$$

It is worth emphasizing that while the elimination of dz was easily accomplished, z itself remains in equations (A.10.7). Thus these two equations have three unknowns and they must be solved simultaneously with the constraint equation (A.10.2). Finally we point out that we could equally well have eliminated dx or dy instead of dz and while then the formal appearance of equations (A.10.7) would be different, the solution of these equations, along with the constraint equation, would yield the same extremum points.

Example A.10.1

Find the rectangular parallepiped of maximum volume V which fits inside a sphere of radius R. Taking the origin of the axes at the centre of the sphere we have that

$$W = V = 8xyz \tag{a}$$

while the constraint equation is

$$\phi = x^2 + y^2 + z^2 - R^2 = 0 \tag{b}$$

For extremum volume we have

$$V_{,x} - V_{,z}\,\phi_{,x}/\phi_{,z} = 0 \qquad V_{,y} - V_{,z}\,\phi_{,y}/\phi_{,z} = 0 \tag{c}$$

and

$$\phi(x, y, z) = 0 \tag{d}$$

In the present problem, these conditions take the following forms

$$8yz - 8x^2y/z = 0 \tag{e}$$

$$8xz - 8xy^2/z = 0 \tag{f}$$

and

$$x^2 + y^2 + z^2 = R^2 \tag{g}$$

From equation (e) $z^2 = x^2$ assuming $y \neq 0$ (h)

From equation (f) $z^2 = y^2$ assuming $x \neq 0$ (i)

Substituting into equation (g) we find $z = \pm R/\sqrt{3}$. Hence the maximum volume is

$$V = \frac{8}{3\sqrt{3}} R^3 \tag{j}$$

and the minimum volume, by the way, from equations (e) and (f) with $x = y = 0$, is

$$V = 0 \tag{k}$$

Lagrange Multipliers

For the extremization of functions, subject to constraints, Lagrange put forward a scheme which at first sight appears to complicate the problem but which usually systematizes the manipulations enough to bring about a substantial net gain.

Consider equations (A.10.3) and (A.10.4) again and let us multiply the latter by a parameter λ, referred to as the *Lagrange multiplier*, and add it to the former. The equations to be satisfied are then

$$dW + \lambda d\phi = (W_{,x} + \lambda\phi_{,x})dx + (W_{,y} + \lambda\phi_{,y})dy + (W_{,z} + \lambda\phi_{,z})dz = 0$$

$$d\phi = \phi_{,x}\,dx + \phi_{,y}\,dy + \phi_{,z}\,dz = 0$$

Since in effect we have added zero to dW, we have not changed the problem. Now proceeding as we did in the previous section, we obtain the extremum conditions as

$$(W_{,x} + \lambda\phi_{,x}) - (W_{,z} + \lambda\phi_{,z})\phi_{,x}/\phi_{,z} = 0 \tag{A.10.8}$$

$$(W_{,y} + \lambda\phi_{,y}) - (W_{,z} + \lambda\phi_{,z})\phi_{,y}/\phi_{,z} = 0 \tag{A.10.9}$$

We have not so far specified the parameter λ. Now suppose we choose λ such that

$$W_{,z} + \lambda\phi_{,z} = 0 \tag{A.10.10}$$

Then the above extremum conditions simplify to

$$W_{,x} + \lambda\phi_{,x} = 0 \tag{A.10.11}$$

$$W_{,y} + \lambda\phi_{,y} = 0 \tag{A.10.12}$$

The systematic form of equations (A.10.10), (A.10.11) and (A.10.12) is quite remarkable since the underlying reason for the first is quite different from that of the last two. For the extremum conditions we must now solve these three equations. However now λ has joined the rank of the unknowns. The fourth equation which must be solved along with the above three is clearly the constraint equation. We note then that λ is obtained only at the solution along with the specific values of x, y, and z. As such λ is a constant.

The above procedure increases the number of unknowns but it also enables one to view the entire problem in a new light. On examination of equations (A.10.10), (A.10.11), (A.10.12) and the constraint equation (A.10.2) it is apparent that these equations can be derived directly as the extremum conditions of a new function F, defined as,

$$F(x, y, z, \lambda) = W(x, y, z) + \lambda\phi(x, y, z) \tag{A.10.13}$$

Treating λ as a new independent variable we obtain the extremum conditions for F, in the absence of any constraints, as follows

$$\frac{\partial F}{\partial x} = 0 \rightarrow \qquad W_{,x} + \lambda\phi_{,x} = 0 \tag{A.10.14a}$$

$$\frac{\partial F}{\partial y} = 0 \rightarrow \qquad W_{,y} + \lambda\phi_{,y} = 0 \tag{A.10.14b}$$

$$\frac{\partial F}{\partial z} = 0 \rightarrow \qquad W_{,z} + \lambda\phi_{,z} = 0 \tag{A.10.14c}$$

$$\frac{\partial F}{\partial \lambda} = 0 \rightarrow \qquad \phi(x, y, z) = 0 \tag{A.10.14d}$$

From the above equations it is apparent that λ cannot be zero unless the constraint is inactive. Therefore instead of F we can equally well search for the extremum conditions of

$$f = \lambda^* W + \phi \tag{A.10.15}$$

in which case $\lambda^* = 1/\lambda$.

The procedure we have outlined is readily generalized for extremization of functions subject to more than one constraint. Thus for a function of n variable such as $W(q_1, q_2, \cdots q_n)$ which is subject to m constraints ($m < n$), of the form $\phi_i(q_1, q_2, \cdots q_n) = 0$, with $i = 1, 2 \cdots m$, we first form a new function

$$F(q_1, q_2, \cdots q_n, \lambda_1, \lambda_2, \cdots \lambda_m) = W + \lambda_i \phi_i \tag{A.10.16}$$

and then seek the extremum conditions of F in the absence of any constraints. The n equations of the form $\partial F/\partial q_n = 0$ and the m constraint equations are then sufficient

for the determination of the n variables, and the m Lagrange multipliers λ_i.

Example A.10.2

Determine the volume of the largest rectangular parallepiped that will fit inside an ellipsoid.

Taking the origin at the centre of the ellipsoid we require to extremize the volume V, given by

$$W = V = 8\,x\,y\,z \tag{a}$$

and the constraint is

$$\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{b}$$

We now form the new function

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) \tag{c}$$

and obtain the necessary conditions for its extremum values as

$$\frac{\partial F}{\partial x} = 8\,y\,z + \frac{2\lambda}{a^2}x = 0 \tag{d}$$

$$\frac{\partial F}{\partial y} = 8\,x\,z + \frac{2\lambda}{b^2}y = 0 \tag{e}$$

$$\frac{\partial F}{\partial z} = 8\,x\,y + \frac{2\lambda}{c^2}z = 0 \tag{f}$$

$$\frac{\partial F}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{g}$$

multiplying the first equation by x, the second one by y and the last one by z, and adding them, we obtain

$$3V + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0 \tag{h}$$

which in view of the constraint equation (g) may be written as

$$3V + 2\lambda = 0 \qquad \text{or} \qquad \lambda = -\frac{3V}{2} \tag{i}$$

Substituting this value into equation (d) and multiplying the result by x we obtain

$$V\left(1 - \frac{3}{a^2}x^2\right) = 0 \qquad \text{or} \qquad x = a/\sqrt{3} \tag{j}$$

Similarly we obtain

$$y = b/\sqrt{3} \qquad z = c/\sqrt{3} \tag{k}$$

Hence the required volume is

$$V = \frac{8}{3\sqrt{3}}\,abc \tag{l}$$

Example A.10.3

As an example of a problem which involves more than one constraint consider the following. Let us determine the maximum and minimum distances from the origin to the curve defined by the intersection of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ and the plane $z = x + y$.

In this case the quantity to be extremized is given by $W = s^2 = x^2 + y^2 + z^2$ and the constraints $\phi_1 = 0$ and $\phi_2 = 0$ are the equations of the ellipsoid and the plane. The intersection curve is clearly an ellipse and we can determine its equation by substituting from the second constraint equation into the first. In that way the number of constraint equations would reduce to one. However let us retain the two constraints and define a new function F, as

$$F = x^2 + y^2 + z^2 + \lambda_1(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1) + \lambda_2(x + y - z) \qquad \text{(a)}$$

The necessary conditions for an extremum of F are

$$\frac{\partial F}{\partial x} = 2x + \frac{\lambda_1 x}{2} + \lambda_2 = 0 \qquad \text{or} \qquad x = -\frac{2\lambda_2}{\lambda_1 + 4} \qquad \text{(b)}$$

$$\frac{\partial F}{\partial y} = 2y + \frac{2\lambda_1 y}{5} + \lambda_2 = 0 \qquad \text{or} \qquad y = -\frac{5\lambda_2}{2\lambda_1 + 10} \qquad \text{(c)}$$

$$\frac{\partial F}{\partial z} = 2z + \frac{2\lambda_1 z}{25} - \lambda_2 = 0 \qquad \text{or} \qquad z = \frac{25\lambda_2}{2\lambda_1 + 50} \qquad \text{(d)}$$

Substituting these values of x, y, and z into the second constraint equation and dividing through by λ_2, we find

$$\frac{2}{\lambda_1 + 4} + \frac{5}{2\lambda_1 + 10} + \frac{25}{2\lambda_1 + 50} = 0 \qquad \text{(e)}$$

or

$$2(2\lambda_1 + 10)(2\lambda_1 + 50) + 5(\lambda_1 + 4)(2\lambda_1 + 50) + 25(\lambda_1 + 4)(2\lambda_1 + 10) = 0$$

This equation simplifies to

$$17\lambda_1^2 + 245\lambda_1 + 750 = 0 \qquad \text{or} \qquad (\lambda_1 + 10)(17\lambda_1 + 75) = 0 \qquad \text{(f)}$$

from which we obtain the roots $\lambda_1 = -10$ or $\lambda_1 = -\frac{75}{17}$ (g,h)

Using the first root in equations (b), (c) and (d) we obtain

$$x = \frac{1}{3}\lambda_2 \qquad y = \frac{1}{2}\lambda_2 \qquad z = \frac{5}{6}\lambda_2 \qquad \text{(i,j,k)}$$

Substituting these values into the first constraint equation we find

$$\frac{\lambda_2^2}{36} + \frac{\lambda_2^2}{20} + \frac{\lambda_2^2}{36} = 1 \qquad \text{(l)}$$

from which

$$\lambda_2 = \pm 6\sqrt{5/19} \qquad \text{(m)}$$

Use of this value of λ_2 in the expressions (i,j,k) for x, y, and z yields

$$x = \pm 2\sqrt{5/19} \qquad y = \pm 3\sqrt{5/19} \qquad \text{and} \qquad z = \pm 5\sqrt{5/19} \qquad \text{(n,o,p)}$$

Therefore the value of $s^2 = x^2 + y^2 + z^2 = (20 + 45 + 125)/19 = 10$. If we employ the second root of λ_1 and proceed in like-manner we find

$$x = \pm\ 40/\sqrt{646} \qquad y = \pm\ 35/\sqrt{646} \qquad z = \pm\ 5/\sqrt{646} \qquad \text{(q,r,s)}$$

The value of s^2 in this case is (1600 + 1225 + 25)/646 = 75/17

Thus the required maximum value is 10 and the minimum value is 75/17.

Algebraic and Differential Equation Constraints

In the functional

$$I = \int_a^b F(x, u, v, w, u', v', w')\, dx \qquad \text{(A.10.17)}$$

let u, v and w be related through a constraint equation which for generality includes the derivatives of u, v and w as well, i.e.

$$G(x, u, v, w, u', v', w') = 0 \qquad \text{(A.10.18)}$$

We will as usual write some varied functions as follows

$$u = u_o + \delta u \qquad v = v_o + \delta v \qquad w = w_o + \delta w$$

where u_o, v_o, and w_o form the required solution.

The first variation of I can now be evaluated and expressed as

$$\delta I = \int_a^b \left\{ \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v'}\delta v' + \frac{\partial F}{\partial w}\delta w + \frac{\partial F}{\partial w'}\delta w' \right\} dx \qquad \text{(A.10.19)}$$

As yet, the varied functions have not been restricted in any way and hence they will bring about a change in the value of G. By constraining this change to zero we can ensure that the varied functions will satisfy the constraint equation $G = 0$. Thus we impose the condition that

$$\delta G = \frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial v}\delta v \quad \frac{\partial G}{\partial w}\delta w + \frac{\partial G}{\partial u'}\delta u' + \frac{\partial G}{\partial v'}\delta v' + \frac{\partial G}{\partial w'}\delta w' = 0 \qquad \text{(A.10.20)}$$

It is worth noting that while x may appear in G in explicit form, its associated increment does not arise in δG since the independent variable is not varied.

Equation (A.10.20) provides the required relation amongst the variations. We will again use the Lagrange multiplier method. This time however, we will integrate the product $\lambda\, \delta G$ from $x = a$ to $x = b$ and then add it to δI. Since δG vanishes at every point x, this addition will not affect the value of δI. Also the vanishing of δG at every point x implies that at each such point the Lagrange multiplier may have a different value, i.e. now the Lagrange multiplier may be a function of x. The modified first variation of I can thus be written as

$$\delta I = \int_a^b \left\{ \left[\frac{\partial F}{\partial u} + \lambda \frac{\partial G}{\partial u} \right] \delta u + \left[\frac{\partial F}{\partial v} + \lambda \frac{\partial G}{\partial v} \right] \delta v + \left[\frac{\partial F}{\partial w} + \lambda \frac{\partial G}{\partial w} \right] \delta w + \left[\frac{\partial F}{\partial u'} + \lambda \frac{\partial G}{\partial u'} \right] \delta u' + \left[\frac{\partial F}{\partial v'} + \lambda \frac{\partial G}{\partial v'} \right] \delta v' + \left[\frac{\partial F}{\partial w'} + \lambda \frac{\partial G}{\partial w'} \right] \delta w' \right\} dx \tag{A.10.21}$$

As usual the terms that involve derivatives of variations can now be integrated and finally for extremum conditions the coefficients of variations δu, and δv, and δw can be equated to zero. It is apparent that the net effect is equivalent to the extremization of a modified functional whose integrand is given by $H = F + \lambda G$. If we have k constraints, then H can be written as $H = F + \lambda_i G_i$ $(i = 1, 2, ...k)$.

In the present case $\lambda(x)$ may be treated as an extra function to be varied independently. The associated Euler-Lagrange equation will then emerge as the equation of constraint.

Example A.10.4

As an illustration consider the functional

$$I = \int_a^b F(x, y, y', y'')dx \tag{a}$$

We may replace this functional by

$$I_1 = \int_a^b F(x, y, z, z')dx \tag{b}$$

subject to the constraint $G = y' - z = 0$. Thus we now form the modified integrand

$$H(x, y, y', z, z', \lambda) = F(x, y, z, z') + \lambda(y' - z) \tag{c}$$

and obtain its Euler-Lagrange equations as follows:

For δy: $$\frac{\partial H}{\partial y} - \frac{d}{dx}\frac{\partial H}{\partial y'} = 0 \quad \rightarrow \quad \frac{\partial F}{\partial y} - \frac{d}{dx}\lambda = 0 \tag{e}$$

δz: $$\frac{\partial H}{\partial z} - \frac{d}{dx}\frac{\partial H}{\partial z'} = 0 \quad \rightarrow \quad \frac{\partial F}{\partial z} - \lambda - \frac{d}{dx}\frac{\partial F}{\partial z'} = 0 \tag{f}$$

$\delta\lambda$: $$\frac{\partial H}{\partial \lambda} - \frac{d}{dx}\frac{\partial H}{\partial \lambda'} = 0 \quad \rightarrow \quad y' - z = 0 \tag{g}$$

From equation (f) we have $$\lambda = \frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} \tag{h}$$

Substitution into equation (e) yields

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} \right] = 0 \tag{i}$$

Now, eliminating z via equation (g) we obtain

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'} \right] + \frac{d^2}{dx^2}\left[\frac{\partial F}{\partial y''} \right] = 0 \tag{j}$$

Before leaving this example it should be noted that the presence of y' and z' in H gives rise to the following boundary terms

$$\frac{\partial H}{\partial y'}\delta y \Big|_{x=a}^{x=b} = 0$$

$$\frac{\partial H}{\partial z'}\delta z \Big|_{x=a}^{x=b} = 0$$

Hence, at a and b we require that either $\delta y = 0$ and $\delta z = \delta y' = 0$ (imposed boundary conditions) or that $\frac{\partial H}{\partial y'} = \lambda = 0$ and $\frac{\partial H}{\partial z'} = \frac{\partial F}{\partial y''} = 0$ (natural boundary conditions). Clearly, if I is extremized directly, the same conditions will be obtained.

Problems

A.10.1 Use the Lagrange multiplier method and find the rectangular parallelepiped of maximum volume which fits inside a sphere of radius R.

A.10.2 Use (a) the Lagrange multiplier method to fit the largest rectangle into an ellipse. (b) What is the area of the rectangle, if the ellipses' semi-major axis is 4 m, and its eccentricity is 0.5?

A.10.3 Determine the maximum and minimum distances from the coordinate origin to the curve defined by the intersection of the ellipsoid

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{3^2} = 1$$

and the plane

$$z = x$$

Also make a sketch.

A.10.4 The end points of a cable of length $L = 6$ m are fastened to the points $x = 0, y = 0$ and $x = 1$ m, $y = 0$ of an xy coordinate system. Determine the shape of the curve assumed by the cable, if the area between cable and x-axis is to be a maximum.

A.10.5 Let an area under a curve and above the x-axis be of a constant value $J = A$. Determine the shape and length $I = L$ of the minimum perimeter curve enclosing the area.

A.11 Variable End Points

Until now we have considered functionals with fixed end points. Thus in the simple functional

$$I = \int_a^b F\,(x, y, y')\,dx \tag{A.11.1}$$

we have required that at the ends $x = a$ and $x = b$ the values of y be prescribed. We now relax this restriction and consider two generalizations. In the first we allow y to vary while x, at the ends, is retained at a and b. In the second

generalization we allow the end values of y and of x to vary.

In the first case we compare admissible functions with end points on the *lines* $x = a$ and $x = b$, as illustrated in Figure A.9. Extremization of I is carried out as before and we find the first variation as

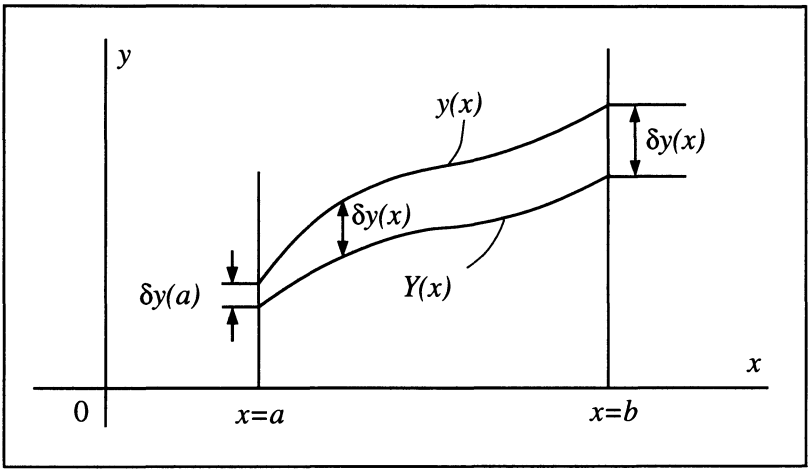


Figure A.9. Variations of y(x) at end points

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right] dx \tag{A.11.2}$$

On integrating the second term of the integrand we may write

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \delta y \, dx + \frac{\partial F}{\partial y'} \delta y \Bigg|_{x=b}^{x=a} \tag{A.11.3}$$

In this case δy does not vanish at the end points and hence for δI to vanish, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \tag{A.11.4}$$

in the domain and

$$\frac{\partial F}{\partial y'} = 0 \tag{A.11.5}$$

at the end points $x = a$ and $x = b$. This last condition is referred to as the *natural* – as against *imposed* – boundary condition. Clearly it is also possible to have an imposed boundary condition at one end and a natural boundary condition at the other.

Example A.11.1

Let us consider the brachystochrone problem again. This time we will seek the curve of the quickest descend from the origin (0,0) to any point on the vertical line $x = b$.

As shown earlier the required functional is given by

$$t = \frac{1}{\sqrt{2g}} \int_0^b \frac{\sqrt{1 + z'^2}}{\sqrt{z}}\, dx \tag{a}$$

The integrand is

$$F = \frac{\sqrt{1 + z'^2}}{\sqrt{z}} \tag{b}$$

The Euler-Lagrange equation and its solution in the form of a cycloid is as derived earlier in Section A.6. The solution that satisfies the imposed boundary condition at the origin (0,0) is given by

$$x = R(\alpha - \sin\alpha) \qquad \text{and} \qquad z = R(1 - \cos\alpha) \tag{c}$$

To determine R we invoke the natural boundary condition at $x = b$, i.e.

$$\frac{\partial F}{\partial z'} = \frac{z'}{\sqrt{z}\,\sqrt{1 + z'^2}} = 0 \tag{d}$$

It is evident then that at this end $z' = 0$, i.e. the solution cuts the line $x = b$ at right angles. Now since

$$z' = \frac{\partial z/\partial\alpha}{\partial x/\partial\alpha} = \frac{\sin\alpha}{1 - \cos\alpha} = 0 \tag{e}$$

we note that at this end $\alpha = \pi$. Further, from the parametric equation for x we find that $R = b/\pi$. Hence the required curve is

$$x = \frac{b}{\pi}(\alpha - \sin\alpha) \qquad \text{and} \qquad z = \frac{b}{\pi}(1 - \cos\alpha) \tag{f}$$

Consider next the general variation of

$$I = \int_{x_a}^{x_b} F(x, y, y')\, dx$$

wherein the end points of the functions competing in the extremization of I are allowed to move in an arbitrary way. In this case the extremizing solution $y_o(x)$, and an admissible function $y(x)$ will be generally defined on different intervals. In order that we may compare these functions we have to consider a large enough interval that contains both $y_o(x)$ and $y(x)$. This can be done by drawing tangents to the curves at their end points as shown in Figure A.10. Now let us denote the difference between $y_o(x)$ and $y(x)$ as

$$\delta y(x) = y(x) - y_o(x) \tag{A.11.6}$$

However, in this case it should be noted that in view of the difference in the intervals of $y(x)$ and $y_o(x)$, the total difference between $y(x)$ and $y_o(x)$, which we denote by $\Delta y(x)$ is not equal to $\delta y(x)$, see Figure A.10.

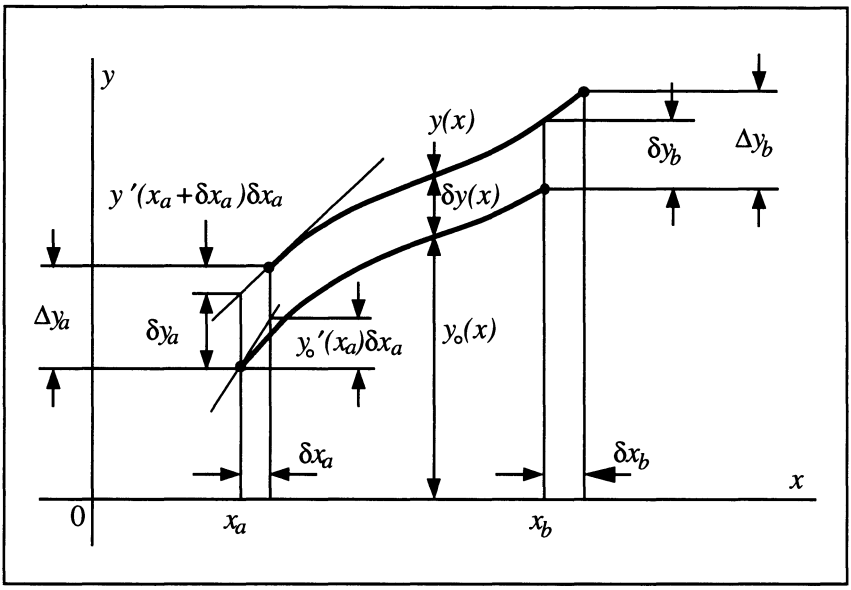


Figure A.10. General variations at end point

The variation of I may now be computed as follows

$$I - I_o = \int_{x_a + \delta x_a}^{x_b + \delta x_b} F(x, y_o + \varepsilon t, y_o' + \varepsilon t')\, dx - \int_{x_a}^{x_b} F(x, y_o, y_o')\, dx$$

$$= \int_{x_a}^{x_b} \{F(x, y_o + \varepsilon t, y_o' + \varepsilon t') - F(x, y, y_o')\}\, dx \qquad \text{(A.11.7)}$$

$$+ \int_{x_b}^{x_b + \delta x_b} F(x, y_o + \varepsilon t, y_o' + \varepsilon t')\, dx - \int_{x_a}^{x_a + \delta x_a} F(x, y_o + \varepsilon t, y_o' + \varepsilon t')\, dx$$

By expanding into Taylor series and retaining terms of the first order of smallness only, we obtain

$$\Delta I = \int_{x_a}^{x_b} \left[\frac{\partial F}{\partial y} \varepsilon t + \frac{\partial F}{\partial y'} \varepsilon t' \right] dx$$

$$+ F(x, y, y') \Big|_{x=x_b} \delta x_b - F(x, y, y') \Big|_{x=x_a} \delta x_a \qquad \text{(A.11.8)}$$

On integrating the second term of the integrand we may write

$$\Delta I = \int_{x_a}^{x_b} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \varepsilon t\, dx + F \Big|_{x=x_b} \delta x_b$$

$$+ \frac{\partial F}{\partial y'} \delta y \Big|_{x=x_b} - F \Big|_{x=x_a} \delta x_a - \frac{\partial F}{\partial y'} \delta y \Big|_{x=x_a} \qquad \text{(A.11.9)}$$

Now from Figure A.10 it is evident that for a small δx_a we can write

$$\delta y \Big|_{x=x_a} \approx \Delta y_a - y'(x_a + \delta x_a)\,\delta x_a \qquad \text{(A.11.10)}$$

But $y'(x_a + \delta x_a)\,\delta x_a$ differs from $y'(x_a)\delta x_a$ by second order quantities only, so that we may write equation (A.11.10) as

$$\varepsilon t \Big|_{x=x_a} \approx \Delta y_a - y'(x_a)\delta x_a \qquad \text{(A.11.11a)}$$

and similarly

$$\varepsilon t \Big|_{x=x_b} \approx \Delta y_b - y'(x_b)\delta x_b \qquad \text{(A.11.11b)}$$

Hence the variation of I may be expressed as

$$\Delta I = \int_{x_a}^{x_b} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right] \delta y\, dx + \frac{\partial F}{\partial y'}\Big|_{x=x_b} \Delta y_b + \left[F - \frac{\partial F}{\partial y'} y' \right]\Big|_{x=x_b} \delta x_b - \frac{\partial F}{\partial y'}\Big|_{x=x_a} \Delta y_a - \left[F - \frac{\partial F}{\partial y'} y' \right]\Big|_{x=x_a} \delta x_a$$

or more compactly as

$$\Delta I = \int_{x_a}^{x_b} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right] \delta y\, dx + \frac{\partial F}{\partial y'} \Delta y \Big|_{x=x_a}^{x=x_b} + \left[F - \frac{\partial F}{\partial y'} y' \right] \delta x \Big|_{x=x_a}^{x=x_b} \qquad \text{(A.11.12)}$$

This is the basic formula for the general variation of I as defined in equation (A.11.5). If the end points of the admissible functions are constrained to lie on the straight lines $x = a, x = b$, i.e. they are coterminal in x as in the simple variable end point problem considered earlier, then $\delta x_a = \delta x_b = 0$, while in the case of the fixed end point problem $\delta x_a = \delta x_b = 0$ and $\delta y_a = \delta y_b = 0$. It is also possible for the admissible functions to be coterminal in y, in which case $\Delta y_a = \Delta y_b = 0$.

Above result can be readily extended for functionals of the form

$$I = \int_{x_a}^{x_b} F(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n')\, dx \qquad \text{(A.11.13)}$$

In this case we obtain the following extremizing conditions:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} = 0 \qquad \text{in the domain} \qquad \text{(A.11.14)}$$

and the boundary conditions

$$\frac{\partial F}{\partial y_i'} = 0 \qquad \text{for an arbitrary } \Delta y_i \qquad \text{(A.11.15)}$$

$$F - y_i' \frac{\partial F}{\partial y_i'} = 0 \qquad \text{for an arbitrary } \delta x_i \qquad \text{(A.11.16)}$$

Suggested Reading

1. Gelfand, I.M.; Fomin, S.V.,
"Calculus of Variations",
Prentice-Hall, (1964).

2. Lanczos, C.,
"The Variational Principles of Mechanics",
University of Toronto Press, (1966).

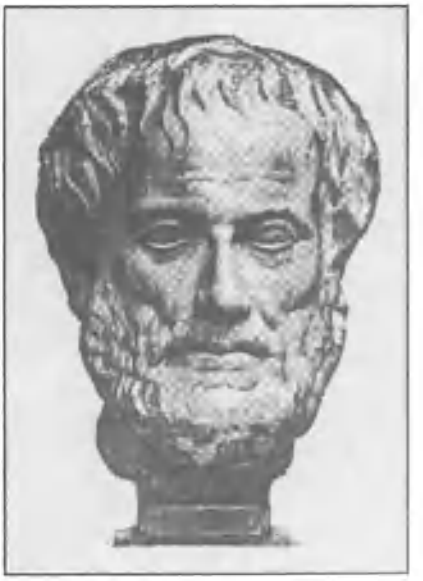
3. Pars, L.A.,
"An Introduction to the Calculus of Variations"
Heinemann, (1962).

4. Weinstock, Robert,
"Calculus of Variations with Applications"
Dover, (1974).

Appendix B

DEVELOPMENTS IN MECHANICS - SOME HISTORICAL PERSPECTIVES

Aristotle (384-322 BC)



Aristotle was a prolific writer producing major works on philosophy (logic and metaphysics), astronomy, natural history, politics, biology and physics. His teachings became the most influential of all the works of the early Greeks.

Aristotle believed that all things on or near the earth were formed from four fundamental elements: Earth, Air, Fire and Water. Each of these elements was thought to have a "natural place". Earth itself occupied the central place and was considered to be in the centre of the entire universe. Water was next, then Air. Fire which seemed to have the ability to rise through Air, was accorded the highest place.

In this scheme of things objects would move to their "natural places" - a rock would fall to the Earth, even if it had to pass through Air and Water to get there. When water boiled, the element Water would be joined by the element Fire, whose higher natural place would cause the mixture to rise, as steam.

According to Aristotle the stars, planets and other heavenly bodies differed fundamentally in composition and behaviour from objects on or near the earth. The heavenly bodies were considered to be in their natural places, not striving to get nearer or farther from the earth, that is, they moved in perfect circles around the centre of the universe (earth). The motions of celestial objects then, were regarded as following laws quite different from those of the terrestrial bodies.

According to Aristotle, all motions in the physical world occurred in order to preserve the general order of the universe. It was more important that the explanation of a particular event should fit into a qualitative philosophical-theological pattern of the time than it was to be quantitatively accurate. Thus it was believed that objects fell at constant speed and that the larger the mass the faster the fall of the object. Aristotle's theory presupposed that all motions occurred in order to achieve some purpose, to satisfy some pre-ordained goal. This view was naturally acceptable to the medieval Church and was one of the reasons why Aristotelian physics was adopted with such a sense of certainty by the Church.

It was not until the fifteenth and sixteenth centuries that the laws of motion began to be better understood. However, the early Greeks had

discovered the law of the lever and Archimedes (287-212 BC) used this law to invent ingenious war machines during the siege of his city, Syracuse, by the Romans. Archimedes also used the principle of the lever to determine the centre of gravity of planar bodies.

Aristotle expressed the law of the lever as: forces balance each other if they are inversely proportional to their velocities. Here we see a veiled application of the principle of virtual work. Virtual velocities instead of virtual displacement were widely used up to the 19th century. The same principle, in another formulation, "what is gained in forces, is lost in velocities", was used by Simon Stevin (1548-1620) in deducing the equilibrium of pulley systems.

Galilei (1564-1642)

Galileo Galilei intended to take a degree in medicine but a taste for Euclid and Archimedes led him instead to become a professor of mathematics, first at Pisa and later at Padua.

Galileo discovered the law of the pendulum and proved the uniform acceleration of all bodies falling to earth. He also made significant contributions to astronomy and strength of materials. Above all, he discovered the laws of motion though he was not able to formulate them quantitatively. It is of interest that in Galileo's time two of the conics, which had been studies 2000 years earlier, found applications: the parabola in trajectory of projectiles and the ellipse in the motion of heavenly bodies.

In 1632 Galileo discussed the merits of the Copernican and Ptolemaic views of the universe, in his book the "Two Chief World Systems". The text is written as a dialogue between three people - Salviati (a scientifically informed scholar), Sagredo (an intelligent layman) and Simplicio (an Aristotelian). Galileo left no doubts about where his preferences lay. For his views, which were in discord with the official views of the Church, Galileo, at the age of 73, was imprisoned for life. Later the sentence was commuted and he died, a blind and broken man, in 1642.

Although the principle of virtual work or virtual power had been used by others, Galileo pointed out that it is the component of the velocity in the direction of the force which must be used in this principle. He applied the principal of virtual power to the equilibrium of a body on an inclined plane and found that his principle yielded the same result as given by Stevin, based on the energy principle.

Newton (1642-1744)

Isaac Newton was born on the Christmas day of 1642, the year that Galileo died. It seems almost as if nature had ordained a continuity in the lives of Galileo and Newton so that her secrets could be brought to light without loss of knowledge. Newton enrolled at Trinity College of Cambridge University in 1661. Chemistry was his first interest and he kept this interest throughout his life. However, he studied Euclid and became familiar with the works of Galileo, Fermat and Huygens. It is therefore not surprising that Newton later wrote to Hooke "if I have seen farther than Descartes it is because I have stood on the shoulders of giants". At Cambridge Newton took lectures from Isaac Barrow (1630-1677), the Lucasion professor of mathematics, and in 1669 he succeeded Barrow.

Immediately after earning his B.A. degree Newton returned home since Trinity College was closed due to the plague. During this period he made four of his major discoveries; the binomial theorem, the differential calculus, the law of gravitation and the nature of colours in light.

Newton had so generalized and unified Galileo's ideas on motion that ever since we refer to this formulation as Newton's laws of motion. That the laws of mechanics apply to the motions of heavenly bodies as well as those of terrestrial objects was an important new idea proposed by Newton. At the time Johannes Kepler (1571-1642) had discovered three empirical laws from Tycho Brahe's (1546-1601) extensive astronomical observations. These laws indicated to Newton that the attraction between two bodies may be inversely proportional to the distance between the bodies. In 1666, when he made a calculation to check his guess for the moon/earth attraction, his knowledge of earth's radius was imperfect and he found his predication in error by about 12 percent. It was not until 1679, when he learnt of Picard's new measurement of the radius of the earth, that he was able to verify his hypothesis. From then on Newton readily developed his law of universal gravitation, namely $F = Gm_a m_b/r^2$ where m_a and m_b are the masses of the two attracting bodies and r is their distance apart, and G is the gravitational constant.

An interesting application of Newton's theory of gravity occurred when a small discrepancy between the calculated and observed orbits of the planet Uranus was noted. It appeared that the sum of the forces on Uranus did not equal its mass times acceleration unless there was another force on the planet that was not included in the calculation. Assuming that this force was exerted by yet unobserved planet, J.E. Adams (1819-1892) and J.J. Leverrier (1811-1877) independently calculated where this unknown planet must be in order to

account for the discrepancy. Pointing their telescopes in the predicted direction, astronomers found the planet we now call Neptune.

It is likely that Newton used calculus to solve his dynamical problems. However since the scientific community at that time was not familiar with calculus, Newton translated all of his proofs into the language of geometry.

Over a disagreement with Robert Hooke (1635-1703, inventor of Hooke's joint, designer of telescopes, discoverer of force-deformation relations for solid materials), concerned with Newton's work on colours of light, Newton had decided not to publish his discoveries. It was only due to the insistence of Edmond Halley (1656-1742, discoverer of the comet named after him) and at Halley's expense that Newton's "Principia" (Philosophiae naturalis principia mathematica), was first published in 1687.

Another disagreement concerned with the discovery of the calculus led to a bitter quarrel between Newton and G.W. Leibniz (1646-1716). In the first edition of the Principia Newton admitted that Leibniz was in possession of a similar method but in the third edition, in 1726, reference to Leibniz was dropped. It is now acknowledged that Leibniz and Newton discovered calculus independently. Leibniz' discovery predated Newton's by about ten years. Another controversy, this time involving Huygens, centred around the nature and propagation of light. Newton favoured the corpuscular theory while Huygens advanced the wave theory.

Later in life Newton was appointed the warden and subsequently the master of the Royal Mint. He died on March 20, 1727, and is buried in Westminister Abbey, a place that had often been refused to the highest nobility.

Euler (1707-1783)

Leonhard Euler, the great mathematician from Switzerland, was born in Basel on April 15, 1707. His father was a clergyman who had hoped that his son would enter the ministry. At Basel University, Euler studied theology, astronomy, physics, oriental languages and mathematics. In mathematics he was sufficiently advanced to attract the attention of Johann Bernoulli (1667-1748) who generously gave young Euler one private lesson per week. Soon his diligence and marked abilities were noticed by Johann's sons, Daniel and Nikolaus Bernoulli, who became Euler's friends. At the age of seventeen Euler earned his master's degree and was much encouraged by the Bernoullis to concentrate on his studies in mathematics. In 1727 Euler learnt of an opening in medicine in the St. Petersburg Academy where the young Bernoullis had gone as professors of

mathematics. This important institution had been established only a few years earlier by Catherine I, along lines laid down by her late husband, Peter the Great, with the advice of Leibniz. On the very day that Euler arrived in Russia, Catherine died and the fledgling academy very nearly collapsed because the new rulers showed little sympathy for learned foreigners. However, the academy survived and in 1730 Euler found himself in the chair of natural philosophy rather than in the medical section. Nikolaus Bernoulli had died, drowning, the year before, and in 1733 Daniel Bernoulli left Russia to occupy the chair of mathematics at Basel, whereupon Euler, at the age of twenty-six, became the academy's chief mathematician.

One of the most important works of this period were Euler's treatises on mechanics. Euler did for mechanics what René Descartes (1596-1650) had done for geometry, that is he freed it from the shackles of synthetic demonstration and made it analytical. For the first time the full power of the calculus was directed towards mechanics and the modern era in that basic science began. In particular Euler made very significant contributions to problems of rigid body dynamics. He also devised approximate methods for the solution of the three body problem (earth, moon, sun) governed by Newton's gravitational law. This problem had important applications in navigation problems. During this period Euler began to lose the sight in his right eye.

In 1740 Euler was glad to accept an invitation from Frederick the Great to join the Berlin Academy. Over the next twenty-four years at Frederick's court Euler worked on many practical problems ranging from coinage, water conduits, navigation, canals and pension systems. At Frederick's court he was not entirely happy. Frederick preferred scholars who scintillated as did Voltaire. Consequently in 1766 at the age of fifty-nine Euler pulled up his stakes and once again migrated back to St. Petersburg, where he was cordially received by Catherine the Great. Soon after he lost the sight in his other eye and became totally blind. Even this tragedy did not abate the flow of his research and publications. He trained himself to write on a slate and his son transcribed his writings for purposes of publication.

Euler is considered the most prolific mathematician in history. He published a total of 886 books and mathematical memoirs and his output averaged 800 printed pages per year. For almost half a century after his death, works by Euler continued to appear in the publications of the St. Petersburg Academy. He contributed to many branches of mathematics and he co-founded (with Lagrange) the calculus of variations. He also wrote in notations we use today, he was the most successful notation-builder of all times. The symbols e, π, i, Σ and $f(x)$ were all introduced by Euler.

All his life Euler had been blessed with an astonishing ability to do mental calculations, and with a phenomenal memory. All the leading formulas of the whole range of mathematics, as it existed in his day, were accurately stowed away in his memory. He was also able to work anywhere under any condition. He was very fond of children (he had thirteen of his own, all but five of whom died very young) and would write his papers with a child on his lap while other children would play around him.

D'Alembert (1717-1783)

Jean le Rond D'Alembert was an illegitimate child and as a new-born he was left on the steps of the church of St. Jean Baptiste Le Rond, near Notre Dame de Paris. His mother was sister of a cardinal and an aristocrat. His father was an artillery general. D'Alembert was brought up by the wife of a glazier and in later years when he became a celebrated mathematician he rejected the overtures of his own mother preferring to be recognised as the son of his impoverished foster parents.

Like Euler and the Bernoullis, D'Alembert was broadly educated in law, medicine, science and mathematics. One of his early accomplishments was his collaboration with Denis Diderot (1713-1784) in preparing a twenty-eight volume encyclopedia on science, arts and the professions. D'Alembert wrote much of the material on science and mathematics. His writings showed strong tendencies towards learning free of the church doctrine. As a result he was attacked by the Jesuits who nicknamed him the "fox of the encyclopedia". D'Alembert was influential in the expulsion of the Jesuit order from France. Also as a result of his activities, and of his friendship with Voltaire and others among the "philosophers", D'Alembert was one of those who paved the way for the French Revolution.

At an early age of twenty-four D'Alembert was elected secretary of the Academy of Sciences and later he became its perpetual secretary. In this role he became the most influential scientist of his time in France. Towards the close of Euler's residency in Berlin, Frederick the Great invited D'Alembert to head the Prussian Academy. D'Alembert declined the offer arguing that it would be inappropriate to place any contemporary in a position of academic superiority over the great Euler. D'Alembert was also invited by Catherine the Great of Russia to serve as an instructor to her own son. This offer too was declined.

Amongst D'Alembert's early discoveries was the logarithm of negative numbers. Before D'Alembert mathematicians had believed that logarithm of positive and negative numbers were equal. Euler and D'Alembert showed that logarithms of negative numbers were complex numbers.

D'Alembert made substantial contributions to the clarifications and rigorous definitions of derivatives in calculus. He viewed derivatives as the limits of the ratio of the finite differences of two variables. This was in opposition to views held by Leibniz, Euler and most mathematicians of the time. D'Alembert's ideas on limits are prevalent in calculus, as taught today. D'Alembert also made major contributions to dynamics and hydrodynamics. Through a concept of "lost motions" he reduced the problem of analysing motion to a problem of equilibrium, via the principle of virtual work. He also contributed to the theory of probability and its applications to life expectancy,

annuities, lotteries and other social sciences.

Lagrange (1736-1813)

Joseph Louis Lagrange was born in Turin of Italian-French ancestry. His parents were wealthy but the father was an incorrigible speculator and by the time his son was ready to inherit the family fortune there was nothing worth inheriting. In later life Lagrange looked back on this disaster as the luckiest thing that had happened to him –"If I had inherited a fortune I should probably not have cast my lot with mathematics".

At school Lagrange was first interested in the classics and it was more or less an accident that he developed a passion for mathematics. An essay by Halley extolling the superiority of the calculus over the synthetic geometrical methods of the Greeks fell into young Lagrange's hands. He was captivated and converted.

At the age of nineteen Lagrange became the professor of mathematics in the artillery school of Turin. In this period he made his contributions to the calculus of variations. Soon after Euler's memoir on this subject, in 1755, Lagrange wrote to Euler about the general methods that he had developed for handling extremals of functionals. In these methods he introduced the notion of variations of functions and employed the symbol δ for variations. The result was Lagrange's purely analytical calculus of variations. Euler generously held up publication of some related works of his own in order that the younger man should receive full credit for the newer methods which Euler regarded as superior.

When in 1766 Euler was to leave Berlin for St. Petersburg, he and D'Alembert advised Frederick the Great to invite Lagrange as Euler's successor at the Berlin Academy. In his invitation Frederick the Great stated, in his customary modest manner "it is necessary that the greatest geometer of Europe should live near the greatest king". Lagrange stayed in Berlin until the death of Frederick the Great (1786), after which he went to Paris. Three years thereafter the French revolution started. In this period Lagrange assisted in the reforming of weights and measures and later he became professor, first at the Ecole Normale then at the Ecole Polytechnique.

Lagrange's "Mechanique analytique" is perhaps his most valuable book. In this book which appeared in 1788, some one hundred years after Newton's "Principia", the full power of the newly developed analysis was applied to the mechanics of particles and rigid bodies. It was by means of the of variations that Lagrange unified the science of mechanics and as Hamilton said, made of

it "a kind of scientific poem".

Notable amongst his other contributions are Lagrange's works on minimum and maximum problems subject to some constraints. Lagrange introduced the multipliers which now bear his name. They provide an elegant and symmetric algorithm. He also developed the method of variation of parameters for non-homogeneous linear differential equations. Other contributions were made in number theory and group theory. In dynamics, Lagrange made a study of one of the standard problems of his day, the theory of the moon. He also derived the biharmonic differential equation for elastic plates.

Gauss (1777-1855)

Johann Friedrich Carl Gauss, son of a labourer, was born on April 30, 1777, in the German city of Brunswick. The Duke of Brunswick recognised in young Gauss an infant prodigy and took charge of his education. From 1795 to 1798 Gauss studied in Göttingen and in 1799 he obtained his doctorate degree at Helmstedt. From 1807 until his death in 1855, he worked quietly and undisturbed as the director of Astronomical Observatory at Göttingen. He never left Germany, not even for a visit.

Gauss' diaries show that already in his seventeenth year he began to make startling discoveries. Some of his early discoveries appear in his doctoral dissertation and in the impressive "Disquisitiones arithmeticae" which he dedicated to the Duke of Brunswick. The dissertation gave the first rigorous proof of the so-called fundamental theorem of algebra – every algebraic equation of n^{th} degree has n roots.

In the period 1821-1848 Gauss was scientific advisor to the Hanoverian and Danish governments, in an extensive geodesic survey. In this work he had to analyse masses of numerical data. He devised a number of ingenious methods for extracting the required information from such data. Amongst these methods was the method of least squares (also studied by Legendre and Laplace). The problems of precise survey for portions of the earth surface, led Gauss to the study of curved surfaces. The subject had been investigated by Euler and Lagrange for certain types of curved surfaces but it remained for Gauss to attack the problem in its full generality and from his investigations the first period of differential geometry development began.

In mechanics Gauss was interested in extremal questions. Using ideas from D'Alembert and Maupertius, Gauss derived a new extremal principle of mechanics, the principle of least constraint. This principle asserts that any

movement of a mechanical system, at an arbitrary moment, is in maximal agreement with the free movement; in other words, that minimal constraints always prevail. Gauss was much attached to this principle because it represented a perfect physical analogy to the method of least squares in the adjustment of errors.

On the first day of 19th century a new planet, Ceres, had been discovered but a few weeks later the tiny body was lost to sight. Gauss took up the challenge of determining the trajectory of this planet. Using the few recorded observations and the method of least squares he found the planet's orbit. He devised a scheme which is still used to track satellites.

Like Newton, Gauss was reluctant to publish much of his discoveries. His motivation in scientific research was driven by his own curiosity and it was a wholly secondary consideration to him whether his discoveries were ever published. He did however keep a record of his work in his diaries. Much of his discoveries surfaced after Gauss' death, through these diaries.

Hamilton (1805-1865)

William Rowan Hamilton was born in Dublin, youngest of three brothers and one sister. His father, a practising attorney, and his mother said to have been intellectually gifted, both died while he was a boy. An extremely precocious youngster, William read Greek, Latin and Hebrew by the time he was five. Under the tutelage of his uncle James, William continued learning other languages. By thirteen he was able to brag that he had mastered one language for each year he had lived (including Persian!).

Hamilton's redemption from his obsession of mastering new languages one after the other began when he met the American calculating boy Zerah Colbourn who was attending school in London. Hamilton soon understood Colbourn's methods and indeed improved on them.

By the age of seventeen Hamilton had mastered mathematics through to integral calculus. He read Newton and Lagrange and soon began to make fundamental discoveries in optics. In 1823 he enrolled at Trinity College in Dublin and quickly became a celebrity. Some declared that a second Newton had arrived. In his undergraduate career he won numerous prizes and obtained the highest honours in both classics and mathematics. The close of Hamilton's undergraduate career at Trinity College was even more spectacular than its beginning. The Professorship in Astronomy had become vacant and several distinguished astronomers including G.B. Airy (1801-1892), of Airy stress function fame, who later became Astronomer Royal of England, sent in their credentials. After some discussions the Board of Governors passed over

all the applicants and unanimously elected Hamilton, then an undergraduate of twenty-two, to the position. Having accepted the professorship Hamilton resolved not to disappoint the hopes of his enthusiastic electors. Within the first year of his appointment Hamilton presented a paper to the Irish Academy on a "System of Rays", the great classic which does for optics what Lagrange's "Mechanique analytique" had done for mechanics, and which, in Hamilton's own hands, was to be extended to dynamics, giving that fundamental science an entirely new perspective.

At the age of thirty Hamilton was knighted. At thirty-two he became the president of the Royal Irish Academy.

Other significant contributions of Hamilton concerned the development of the algebra of quaternions. Quaternions may be regarded as the extension of complex numbers, represented by $(a + ib)$, to quadruples such as $(a + ib + jc + kd)$. Hamilton discovered self-consistent algebra of such forms by abandoning the commutative law of multiplication. He had worked for over ten years on the algebra of n-tuples and the flash of inspiration came when he was walking along the Royal Canal in Dublin. He stopped in his walk, and with a knife he cut the fundamental formula $i^2 = j^2 = k^2 = ijk$ on a stone at Brougham Bridge.

Towards the end of his life, Hamilton was bedevilled by excesses in alcohol. The tragedy of a semi-invalid wife also dogged his later years. The last honour, bestowed in his deathbed and one which pleased Hamilton a great deal, was his election as the first foreign member of the National Academy of Sciences of the United States (founded 1863 during the Civil War). This honour was in recognition of Hamilton's works on quaternions.

Jacobi (1804-1851)

Carl Gustav Jacob Jacobi, the second son of a prosperous banker, secured a good education at the University of Berlin, with concentration in philology and mathematics. Like Gauss, he finally settled for the latter but unlike Gauss, Jacobi was a born teacher and became the most inspiring mathematics teacher of his time. In 1825 Jacobi received his doctorate after which he lectured at the University of Berlin on the application of the calculus to curved surfaces and twisted curves.

In 1842 Jacobi and F.W. Bessel (1784-1846) attended the meeting of the British Association at Manchester, where Jacobi and Hamilton met in person. It was to be one of Jacobi's greatest glories to continue the work of Hamilton in dynamics and, in a sense, to complete what the Irishman had abandoned in favour of quaternions.

In 1848 Jacobi entered the arena of

politics with disastrous results. Not only was he not elected but he annoyed his patron, the King of Prussia, who stopped Jacobi's allowance. Hearing his plight, the university of Vienna began angling for him. However, when a definite and generous offer was tendered, Alexander von Humboldt talked the King around; the allowance was restored and Jacobi stayed in Berlin.

In his researches Jacobi made very important contributions to the theory of elliptic functions and functions of complex variables. He also contributed to the theory of determinants and functional determinants, now referred to as Jacobians of transformation. In dynamics, Jacobi made the first significant advance beyond Lagrange and Hamilton. The celebrated Hamilton-Jacobi equation played an important role in the early days of quantum mechanics.

Jacobi died in his fourty-seventh year from smallpox.

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