# ELEMENTARY THEORY OF STRUCTURES

THIRD EDITION

Prentice-Hall International Editions

YUAN-YU HSIEH

# *Elementary Theory*
# *of Structures*

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## *Third Edition*

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# Preface

The first two editions of this book were written within the scope of linear and statical aspects of structure, and with equal weight on the classical and modern methods of solution. This new third edition is distinguished from the first two by including elements of nonlinear and dynamic behavior of structure and by shifting emphasis to the matrix analysis of structure. As a result, three new chapters have been added in this edition; namely, the direct stiffness method, elastic stability, and structural dynamics, all presented in matrix notation. The direct stiffness method is important in that it formalizes the structural analysis readily for computer programming. The matrix analysis of elastic stability demonstrates the direct stiffness approach to solving buckling problems. And the structural dynamics is arranged so that it leads to the course of earthquake engineering, which has become an increasingly popular topic, especially in California.

In order to accommodate these new materials while at the same time retaining the same length as previous editions, the entire second edition was restructured by deleting some of the original materials and by trimming and merging others without impairing the integrity and continuity of the whole book. With changes here and there, two goals remain intact: the treatment is kept simple but comprehensive, and the text is readable and teachable. The length of little more than four hundred pages is not intimidating to students and the contents are at an elementary level throughout.

As in the second edition, the author wishes to express his sincere appreciation to many university professors and students in various parts of the world, who have read, used, and supported this book. Special thanks are due to Dr. Y. C.

Fung, professor at the University of California at San Diego, and to Mr. D. Humphrey, senior engineering editor at Prentice Hall, for their enthusiastic encouragement and editorial guidance. The author is also grateful to Dr. B. Koo, professor at the University of Toledo, Dr. K. P. Chong, professor at the University of Wyoming, and Dr. Z. A. Lu of the University of California at Berkeley for their helpful suggestions and kind interest in the development of the revision, and to Mr. S. N. Yao of Washington State Highway Department for providing solutions to the problems in Chapters 11 and 13. Finally, the author is indebted to his wife, Nelly, for her careful typing of the manuscript.

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# *Elementary Theory*

# *of Structures*

# 1

# *Introduction*

## 1-1 ENGINEERING STRUCTURES

The word *structure* has various meanings. By an *engineering structure* we mean roughly something constructed or built. The principal structures of concern to civil engineers are bridges, buildings, walls, dams, towers, shells, and cable structures. Such structures are composed of one or more solid elements so arranged that the whole structures as well as their components are capable of holding themselves without appreciable geometric change during loading and unloading.

The design of a structure involves many considerations, among which are four major objectives that must be satisfied:

1. The structure must meet the performance requirement (utility).
2. The structure must carry loads safely (safety).
3. The structure should be economical in material, construction, and overall cost (economy).
4. The structure should have a good appearance (beauty).

Consider, for example, the roof truss resting on columns shown in Fig. 1-1. The purposes of the roof truss and of the columns are, on the one hand, to hold in equilibrium their own weights, the load of roof covering, and the wind and snow (if any) and, on the other hand, to provide rooms for housing a family, for a manufacturing plant, or for other uses. During its development the design is generally optimized to achieve minimum expenditure for materials and construction. Proper attention is also given to the truss formation so that it is both

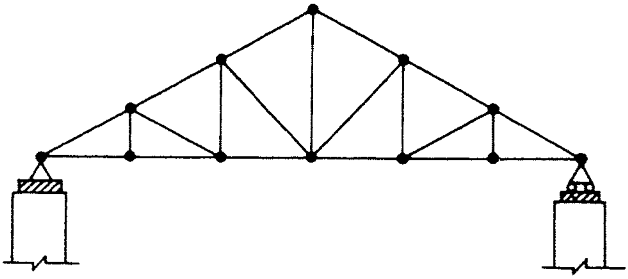


Fig. 1-1

practical and aesthetic. In this book, however, we are concerned only with the load-carrying function of structures.

## 1-2 THEORY OF STRUCTURES DEFINED

The complete design of a structure is outlined in the following stages:

1. *Developing a general layout.* The general layout of a structure is selected from many possible alternatives. The primary concern is the purpose for which the structure is to be built. This stage involves the choice of structural type, the selection of material, and a tentative estimation of cost based on a reasonable analysis of a preliminary structural design. It may also involve selecting the best location or adapting the structure to a site that has not been predetermined. There are many other considerations, including the legal, financial, sociological, cultural, aesthetic, and environmental aspects. It is clear that this stage of design calls for an engineer with a high order of experience, skill, general knowledge, and imagination.
2. *Investigating the loads.* Before a refined structural analysis can be carried out, it is necessary to determine the loads for which a given structure should be designed. General information about the loads imposed on a structure is usually given in the specifications and codes. However, it is part of the designer's responsibility to specify the load conditions and to take care of exceptional cases.

The weight of the structure itself together with the material permanently attached to it is called *dead load* and is regarded as fixed in magnitude and location. Since the dead load must be approximated before the structure is designed, the preliminary data are only tentative. Revision must be made if the initial estimation is not satisfactory.

All loads other than dead load may be called *live loads.* Live loads are generally classified as movable loads and moving loads. *Movable loads* are loads that may be transported from one location to another on a structure without dynamic impact; for example, people, furniture, and goods on a building floor, or snow or ice on a roof. *Moving loads* are loads that move continuously over the structure, such as railway trains or tracks on a bridge, wind on a roof or wall, or hydrostatic pressure on an abutment. Moving loads may also be applied suddenly to the structure—for example, the centrifugal and longitudinal forces induced by the acceleration of vehicles and the dynamic forces generated by earthquakes.

In an ordinary structural design all loads are treated as static loads in order to simplify the analysis. In this way the impact due to a moving live load is expressed as a percentage of the live load, and the earthquake force is commonly considered to be a horizontal force equal to a fraction of the weight of a structure.

Other load considerations may include thermal effects and resistance to bomb blasting.

3. *Stress analysis.* Once the basic form of the structure and the external loads are defined, a structural analysis can be made to determine the internal forces in various members of the structure and the displacements at some controlling points. When live loads are involved, it is important to determine the maximum possible stresses in each member being considered. The principles governing this phase of design are usually discussed in the theory of structures.
4. *Selection of elements.* The selection of suitable sizes and shapes of members and their connections depends on the results of the stress analysis together with the design provisions of the specifications or codes. A trial-and-error approach may be used in the search for a proportioning of elements that will be both economical and adequate. A sound knowledge of strength of material and process of fabrication is also essential.
5. *Drawing and detailing.* Once the makeup of each part of the structure has been determined, the last stage of design can begin. This final stage includes the preparation of contract drawing, detailing, job specification, and final cost; this information is necessary for construction to proceed.

These five stages are interrelated and may be subdivided and modified. In many cases they must be carried out more or less simultaneously. The subject matter of the theory of structures is stress analysis with occasional reference to loadings. The emphasis of structural theory is usually on the fundamentals rather than on the details of design.

## 1-3 THEORIES OF STRUCTURES CLASSIFIED

Structural theories may be classified from various points of view. For convenience of study, we shall characterize them by the following aspects:

1. *Statics versus dynamics.* Ordinary structures are usually designed under static loads. Dead load and snow load are static loads that cause no dynamic effect on structures. Some live loads, such as trucks and locomotives moving on bridges, are also assumed to be concentrated static load systems. They do cause impact on structures; however, the dynamic effects are treated as a fraction of the moving loads to simplify the design.

The specialized branch that deals with the dynamic effects on structures of accelerated moving loads, earthquake loads, wind gusts, or bomb blasts is *structural dynamics*.

2. *Plane versus space*. No structure is really planar, that is, two-dimensional. However, structural analyses for beams, trussed bridges, or rigid frame buildings are usually treated as plane problems. On the other hand, in some structures, such as towers and framing for domes, the stresses between members not lying in a plane are interrelated in such a way that the analysis cannot be simplified in terms of component planar structures. Such structures must be considered as space frameworks under a noncoplanar force system.

3. *Linear versus nonlinear structures*. Linear structure means that a linear relationship is assumed to exist between the applied loads and the resulting displacements in a structure. This assumption is based on the following conditions:

a. The material of the structure is elastic and obeys Hooke's law at all points and throughout the range of loading considered.

b. The changes in the geometry of the structure are so small that they can be neglected when the stresses are calculated.

Note that if the principle of superposition is to apply, a linear relationship must exist, or be assumed to exist, between loads and displacements. The principle of superposition states that the total effect at some point in a structure due to a number of causes (forces or displacements) acting simultaneously is equal to the sum of the effects for the causes acting separately.

A nonlinear relationship between the applied actions and the resulting displacements exists under either of two conditions:

a. The material of the structure is inelastic.

b. The material is within the elastic range, but the geometry of the structure changes significantly during the application of loads.

The study of nonlinear behavior of structures includes *plastic analysis of structures* and *buckling of structures*.

4. *Statically determinate versus statically indeterminate structures*. The term *statically determinate structure* means that structural analysis can be carried out by statics alone. If this is not so, the structure is statically indeterminate.

A statically indeterminate structure is solved by the equations of statics together with the equations furnished by the geometry of the elastic curve of the structure in linear analysis. We note that the elastic deformations of the structure are affected not only by the applied loads on the structure but also by the material properties (e.g, the modulus of elasticity $E$) and by the geometric properties of the member section (e.g., the cross-sectional area $A$ or the moment of inertia $I$). Thus, loads, material properties, and geometric properties are all involved in solving a statically indeterminate structure, while load factor alone dominates in a statically determinate problem.

5. *Force versus displacement*. Force and displacement are two categories of events that affect a structure. The objective of a structural analysis is to determine the forces and displacements pertaining to the structure and to analyze their

relationships as specified by the geometric and material properties of structural elements. Structural analysis in a broader sense can then be divided into two categories: the force method and the displacement method. In the *force method,* we treat the forces as the basic unknowns and express the displacements in terms of forces; whereas in the *displacement method,* we regard the displacements as the fundamental unknowns and express the forces in terms of displacements. In matrix analysis of linear structures, the force method is often referred to as the *flexibility method,* and the displacement method is called the *stiffness method.*

## 1-4 ACTUAL AND IDEAL STRUCTURES

All analyses are based on some assumptions not quite in accordance with the facts. It is impossible for an actual structure to correspond fully to the idealized model on which the analysis is based. The materials of which the structure is built do not exactly follow the assumed properties, and the dimensions of the actual structure do not coincide with their theoretical values.

To illustrate, let us take a simple example. In designing a reinforced concrete beam of rectangular section, the values of $E$ and $I$ are usually assumed to be constant. However, the amount of reinforcing steel placed in the beam varies with the stresses; therefore, the values of $E$ and $I$ are not constant throughout the span. Besides, there is great uncertainty involved even in choosing a constant $E$ or $I$. Even without considering other factors, such as the supports, the connections, and the working dimensions of the structure, we find that the behavior of an actual structure often deviates from that of an idealized structure by a considerable amount. However, it does not follow from this that the results of analysis are not useful for practical purposes. We must set an idealized model in order to carry out practical analysis, and from practical analyses we make the idealization more and more consistent with actuality.

## 1-5 SCOPE OF THIS BOOK

Three major types of basic structures are thoroughly discussed in this book:

1. *Beam*
2. *Truss*
3. *Rigid frame*

A *beam,* in its narrow sense, is a straight member subjected only to transverse loads. A beam is completely analyzed when the values of bending moment and shear are determined.

A *truss* is composed of members connected by frictionless hinges or pins. The loads on a truss are assumed to be concentrated at the joints. Each member of a truss is considered as a two-force member subjected to axial forces only.

A *rigid frame* is built of members connected by rigid joints capable of resisting moment. Members of a rigid frame, in general, are subjected to bending moment, shear, and axial forces.

This book is confined exclusively to the planar aspect of structure. The first four chapters deal with the basic concepts of structure and an analysis of statically determinate structures in which only forces are involved. Chapter 5 deals with elastic deformations, whereas Chapter 6 discusses the method of consistent deformations, a classical force method for analyzing statically indeterminate structures. Chapters 7 and 8 present the slope-deflection method and the moment-distribution method, respectively. They are the classical displacement methods for analyzing statically indeterminate structures. The remaining chapters (9 to 14) are concerned with the matrix analyses of structures, including the matrix force method, the matrix displacement method, the direct stiffness method, elements of elastic stability, and structural dynamics.

# 2

# *Stability and Determinacy*

## 2-1 EQUATIONS OF EQUILIBRIUM FOR A COPLANAR FORCE SYSTEM

The first and major function of a structure is to carry loads. Beams, trusses, and rigid frames all have one element in common: Each sustains the burden of certain loads without showing appreciable distortions. In structural statics all force systems are assumed to act on rigid bodies. Actually, there are always some small deformations that may cause some small change of dimension in structure and a shifting of the action lines of the forces. However, such deviations are neglected in stress analysis.

A structure is said to be *in equilibrium* if, under the action of external forces, it remains at rest relative to the earth. Also, each part of the structure, if taken as a free body isolated from the whole, must be at rest relative to the earth under the action of the internal forces at the cut sections and of the external forces thereabout. If such is the case, the force system is balanced, or in equilibrium, which implies that the resultant of the force system (either a resultant force or a resultant couple) imposed on the structure, or segment thereof, must be zero.

Since this book is confined to planar structures, all the force systems are coplanar. The generally balanced coplanar force system must then satisfy the following three simultaneous equations:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_a = 0 \tag{2-1}$$

where $\Sigma F_x$ = summation of the $x$ components of each force in the system

$\Sigma F_y$ = summation of the $y$ components of each force in the system

The subscripts $x$ and $y$ indicate two mutually perpendicular directions in the Cartesian coordinate system;

$\Sigma M_a$ = summation of moments about any point $a$ in the plane due to each force in the system

Note that $\Sigma F_x$ also represents the $x$ component of the resultant of the force system, $\Sigma F_y$ the $y$ component of the resultant of the force system, and $\Sigma M_a$ the moment about $a$ of the resultant of the force system.

The alternative to Eq. 2-1 may be given by

$$\sum F_y = 0 \qquad \sum M_a = 0 \qquad \sum M_b = 0 \tag{2-2}$$

provided that the line through points $a$ and $b$ is not perpendicular to the $y$ axis, $a$ and $b$ being two arbitrarily chosen points and the $y$ axis being an arbitrarily chosen axis in the plane. Or

$$\sum M_a = 0 \qquad \sum M_b = 0 \qquad \sum M_c = 0 \tag{2-3}$$

provided that points $a$, $b$, and $c$ are not collinear, $a$, $b$, and $c$ being three arbitrarily chosen points in the plane.

The explanation of Eq. 2-2 is as follows:

1. Let $R$ denote the resultant of the force system. Assume that $R \neq 0$. Since $\Sigma M_a = 0$ and $\Sigma M_b = 0$, the resultant $R$ cannot be a couple. It must be a force through $a$ and $b$ and by assumption is not perpendicular to the $y$ axis.
2. By $\Sigma F_y = 0$, we mean that the resultant has no $y$-axis component and must therefore be perpendicular to the $y$ axis.

The foregoing contradictory statements lead to the conclusion that the force is also zero. Therefore, Eq. 2-2 is the condition for $R = 0$.

A similar explanation is given here for Eq. 2-3:

1. Assume that $R \neq 0$. Since $\Sigma M_a = 0$, $\Sigma M_b = 0$, and $\Sigma M_c = 0$, the resultant $R$ cannot be a couple. It must be a force through $a$, $b$, and $c$.
2. But by assumption $a$, $b$, and $c$ are not collinear.

These statements lead us to conclude that the force is also zero. Therefore, Eq. 2-3 is the condition for $R = 0$.

Two special cases of the coplanar force system in equilibrium are worth noting:

1. *Concurrent forces.* If a system of coplanar, concurrent forces is in equilibrium, then the forces of the system must satisfy the following equations:

$$\sum F_x = 0 \qquad \sum F_y = 0 \tag{2-4}$$

Another set of independent equations necessary and sufficient for the equilibrium of the forces of a coplanar, concurrent force system is

$$\sum F_y = 0 \qquad \sum M_a = 0 \tag{2-5}$$

provided that point $a$ is not on the line through the concurrent point of forces and perpendicular to the $y$ axis.

A third set of equations of equilibrium for a coplanar, concurrent force system is

$$\sum M_a = 0 \qquad \sum M_b = 0 \tag{2-6}$$

where $a$ and $b$ are any two points in the plane of the forces, provided that the line through $a$ and $b$ does not pass through the concurrent point of forces.

2. *Parallel forces.* If a coplanar, parallel force system is in equilibrium, the forces of the system must satisfy the equations

$$\sum F_y = 0 \qquad \sum M_a = 0 \tag{2-7}$$

where the $y$ axis is in the direction of the force system and $a$ is any point in the plane.

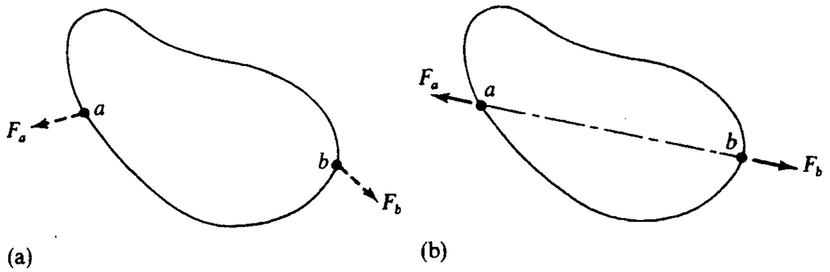
Another set of independent equations of equilibrium for a system of coplanar, parallel forces may be given as

$$\sum M_a = 0 \qquad \sum M_b = 0 \tag{2-8}$$

where $a$ and $b$ are any two points in the plane, provided that the line through $a$ and $b$ is not parallel to the forces of system.

There are two simple, special cases of equilibrium that deserve explicit mention:

1. *Two-force member.* Figure 2-1 shows a body subjected to two external forces applied at $a$ and $b$. If the body is in equilibrium, then the two forces cannot be in random orientation, as shown in Fig. 2-1(a), but must be directed along $ab$, as shown in Fig. 2-1(b). Furthermore, they must be equal in magnitude and opposite in sense. This can be proved by using first the equations $\Sigma\ M_a = 0$ and $\Sigma\ M_b = 0$. In order for the moment $a$ to vanish, the force $F_b$ must pass through $a$. Similarly, the force $F_a$ must pass through $b$. Next, since $\Sigma\ F = 0$, it is readily seen that $F_a = -F_b$.



(a) (b)

Fig. 2-1

2. *Three-force member.* Figure 2-2 shows a body subjected to the action of three external forces applied at $a$, $b$, and $c$. If the body is in equilibrium, then

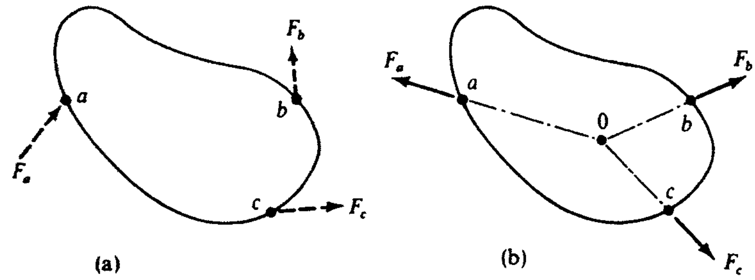


Fig. 2-2

the three cannot be in random orientation, as shown in Fig. 2-2(a). They must be concurrent at a common point $O$, as shown in Fig. 2-2(b); otherwise, the total moment about the intersection of any two forces could not vanish. A limiting case occurs when point $O$ moves off at infinite distance from $a$, $b$, and $c$, in which case the forces $F_a$, $F_b$, and $F_c$ are parallel.

## 2-2 SUPPORT REACTIONS

Structures are either partially or completely restrained so that they cannot move freely in space. Such restraints are provided by supports that connect the structure to some stationary body, such as the ground or another structure. The first step in structural analysis is to take the structure without the supports and calculate the forces, known as *reactions*, exerted on the structure by the supports. The reactions are considered part of the external forces other than the loads on the structure and are to balance the other external loads in a state of equilibrium.

Certain symbols used to designate supports must first be described. There are generally three different types of support: the *hinge*, the *roller*, and the *fixed support*. Some intermediate models of support between the idealized three can be made to respond to the reality. The distribution of the reactive forces of a support may be very complicated, but in an idealized state the resultant of the forces may be represented by a single force completely specified by three elements—the *point of application*, the *direction*, and the *magnitude*. It may be noted that, in analysis, the direction simply means the slope of the action line, while the magnitude of force may be positive or negative, thus indicating not only its numerical size but also the sense of the action line.

**Hinge support.** A hinge support is represented by the symbol  or  . It can resist a general force $P$ in any direction but cannot resist the moment of the force about the connecting point, as illustrated in Fig. 2-3.

The reaction of a hinge support is assumed to be through the center of the connecting pin; its magnitude and slope of action line are yet to be determined.

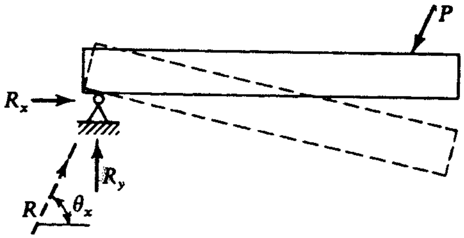


Fig. 2-3

It is therefore a reaction with two unknown elements which could equivalently be represented by the unknown magnitudes of its horizontal and vertical components, both acting through the center of the hinge pin. This representation is justified by the following equations from statics:

$$|R| = \sqrt{R_x^2 + R_y^2} \qquad \theta_x = \tan^{-1}\frac{R_y}{R_x} \tag{2-9}$$

where $|R|$ = magnitude of the reaction $R$
$R_x$ = $x$ component of $R$
$R_y$ = $y$ component of $R$
$\theta_x$ = angle that $R$ makes with the $x$ direction

The magnitude and direction of $R$ can be determined if the unknown magnitudes of $R_x$ and $R_y$ are found.

Thus, a hinge support can also be replaced by two links along the horizontal and vertical directions through the center of the connecting pin, as shown in Fig. 2-4(a). Each link is a two-force member, the axial force of which represents an element of reaction ($R_x$ or $R_y$). In general, a hinge support is equivalent to two supporting links provided in any two different directions, which are not necessarily an orthogonal set, through the connecting point, as shown in Fig. 2-4(b), where $R_1$ and $R_2$ indicate the axial forces in two links. The reaction $R$ at the pin can always be determined if the magnitudes of $R_1$ and $R_2$ are obtained.

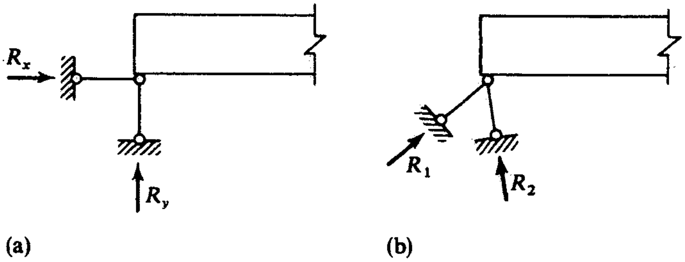
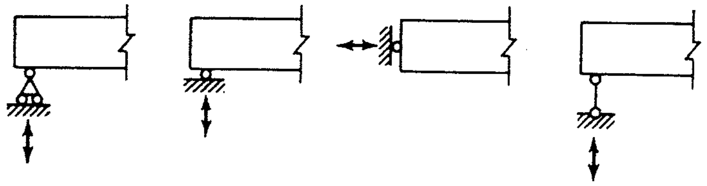


Fig. 2-4

**Roller support.** A roller support is represented by either the symbol  or  . The support mechanism used is such that the reaction acts normal to the supporting surface through the center of the connecting pin, as shown in Fig. 2-5(a)–(c). The reaction may be either away from or toward the supporting surface. As such, the roller support is incapable of resisting moment and lateral force along the surface of support.

A roller support supplies a reactive force, fixed at a known point and in a known direction, the magnitude of which is unknown. It is therefore a reaction with one unknown element.

A link support, shown in Fig. 2-5(d), is also of this type since the link is a two-force member and the reaction must be along the link.

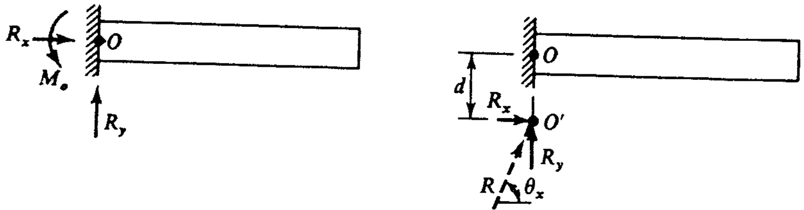


(a) (b) (c) (d)

Fig. 2-5

**Fixed support.** A fixed support is designated by the symbol  . It is capable of resisting force in any direction and moment of force about the connecting end, thus preventing the end of the member from both translation and rotation. The reaction supplied by a fixed support may be represented by the unknown magnitudes of a moment called $M_o$, a horizontal force $R_x$, and a vertical force $R_y$ acting through the centroid of the end cross section $O$, as shown in Fig. 2-6(a). These three unknown elements can be expressed as equal to a single force $R$ with its three elements—the magnitude, direction, and point of application—yet to be determined, as shown in Fig. 2-6(b). Now the magnitude and direction of $R$ can be related to its components $R_x$ and $R_y$ by Eq. 2-9,

$$|R| = \sqrt{R_x^2 + R_y^2} \qquad \theta_x = \tan^{-1}\frac{R_y}{R_x}$$



(a) (b)

Fig. 2-6

and the point of application $O'$ can be located by the distance $d$ from $O$, which, in turn, is related to $M_o$ by

$$d = \frac{M_o}{R_x}$$

Since fixed support provides moment resistance, it is one step beyond the hinge support in rigidity.

Two devices equivalent to the fixed support are shown in Fig. 2-7. Each is composed of a hinge and a roller and represents three elements of reaction capable of resisting both force and moment.

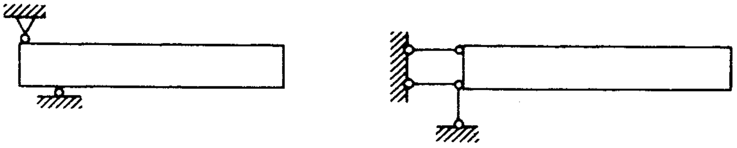
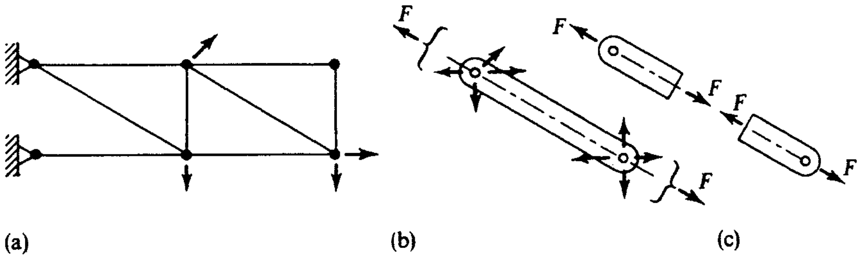


Fig. 2-7

## 2-3 INTERNAL FORCES AT A CUT SECTION OF A STRUCTURE

A truss structure is composed of pin-connected members and is assumed to be pin-loaded, as shown in Fig. 2-8(a). Now if any one of the members is taken from its connecting pins as a free body, the forces exerted on the member must be concentrated at the two ends of the member through the centers of pins. Furthermore, these two systems of concurrent forces can be combined into two resultant forces that must be equal, opposite, and collinear, as indicated in Fig. 2-8(b). In other words, each member of a truss is a *two-force* member. Hence, the internal forces existing in any cut section of a truss member (assumed straight and uniform) must be a pair of equal and opposite axial forces to balance the axial forces exerted on the ends, as shown in Fig. 2-8(c).

The fact that each member of a truss represents an unknown element of internal force enables us to obtain the total number of unknown elements of internal force by counting the total number of members of which the truss is composed.

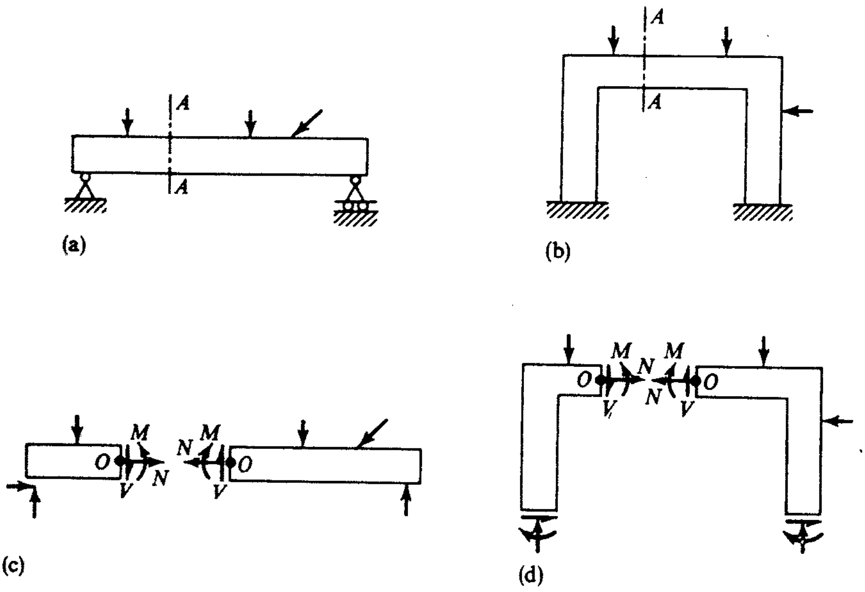


(a) (b) (c)

Fig. 2-8

Members of structures, such as beams and rigid frames, are acted on by more than two forces. Let us investigate the elements of internal force in any cut section *A–A* of the beam in Fig. 2-9(a) or of the rigid frame in Fig. 2-9(b).

We begin by taking the free bodies of the portions to the left and right of section *A–A*, as shown in Fig. 2-9(c) and (d). It is obvious that forces of internal constraint must exist between these two portions in order to hold them together. Such internal forces, of course, always occur in pairs of equal and opposite forces. The actual distribution of these internal forces cannot be easily discovered. To maintain the equilibrium of the free body, however, the internal forces must be statically equal and opposite to the system of forces acting externally on the portion considered, and the internal forces can always be represented by a force applied at the centroid $O$ of the cross section together with a couple of moment $M$. Furthermore, the force can, in turn, be resolved into a normal component $N$ and a tangential component $V$. Thus, in Fig. 2-9(c) and (d) we represent the stress resultant on any section *A–A* by the three unknown magnitudes of $N$, $V$, $M$, called, respectively, the *normal force*, the *shearing force*, and the *resisting moment* at that section.



(a)

(b)

(c)

(d)

Fig. 2-9

From the foregoing discussion, we remember that to take a free body from a beam or rigid frame we must assume three unknown elements of internal force generally existing in the cut section.

## 2-4 EQUATIONS OF CONDITION OR CONSTRUCTION

Structures such as trusses, beams, and rigid frames may sometimes be considered to be one rigid body sustained in space by a number of supports. Out of several such rigid bodies a compound form of structure may be built by means of connecting devices, such as hinges, links, or rollers, and mounted on a number of supports. In either the simple or the compound type of structure the external force system of the entire structure, consisting of the loads on the structure and the support reactions, must satisfy the equations of equilibrium if the structure is to remain at rest. However, in the compound type of structure the connecting devices impose further restrictions on the force system acting on the structure, thus providing additional equations of statics to supplement the equations of equilibrium. Equations supplied by the method of special construction (other than external supports) are called *equations of condition or construction.* We discuss these further in Sec. 2-6.

## 2-5 STABILITY AND DETERMINACY OF A STRUCTURE WITH RESPECT TO SUPPORTS

When one considers the design of a structure, one must give careful thought to the number and arrangement of the supports directly related to the stability and determinacy of the structure. In the following discussions we shall treat the structures as a monolithic rigid body mounted on a number of supports. Thus, there will be no internal condition involved, and the stability and determinacy of the structure will be judged solely by the stability and determinacy of supports.

1. Two elements of reaction supplied by supports, such as two forces each with a definite point of application and direction, are not sufficient to ensure the stability of a rigid body, because the two are either collinear, parallel, or concurrent. In each of these cases, the condition of equilibrium is violated, not because of the lack of strength of supports, but because of the insufficient number of support elements. This situation is referred to as *statical instability.*

If two reactive forces are collinear [see Fig. 2-10(a)], they cannot resist an external load that has a component normal to the line of reactions. If they are parallel [see Fig. 2-10(b)], they cannot prevent the body from lateral sliding. If they are concurrent [see Fig. 2-10(c) or (d)], they cannot resist the moment about the concurrent point $O$ due to any force not through $O$.

Algebraically, in each of the cases above, one equilibrium condition is not satisfied. For instance, in Fig. 2-10(a) or (b) the condition $\Sigma F_x = 0$ is violated ($x$ indicates the direction normal to the line of reaction); whereas in Fig. 2-10(c) or (d) the condition $\Sigma M_o = 0$ is not fulfilled. The body is, therefore, not in equilibrium; it is unstable.

Only under some very special conditions of loading can the body be stable,

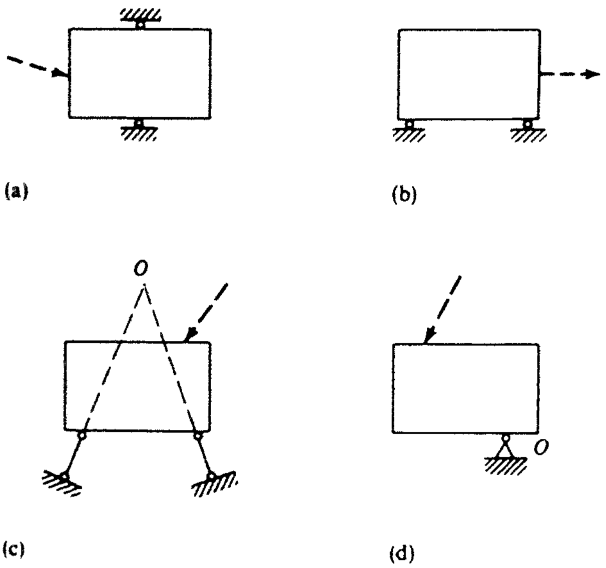


Fig. 2-10

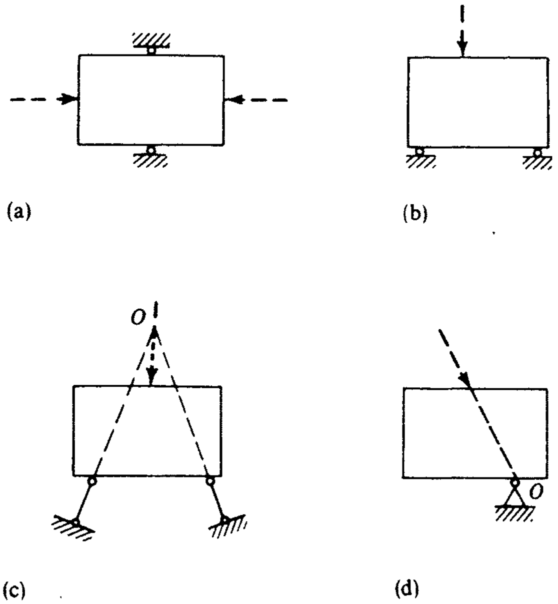
such as those shown in Fig. 2-11. In Fig. 2-11(a) the applied loads acting on the body are themselves in equilibrium; therefore, no reaction is required. In Fig. 2-11(b) the applied load is in the same direction as the reactions, so equilibrium can be maintained for the parallel force system; and in Fig. 2-11(c) or (d) the applied load is through the concurrent point $O$; therefore, equilibrium can also be established.

Structures stable under special conditions of loading but unstable under general conditions of loading are said to be in a state of *unstable equilibrium* and are classified as unstable structures.

2. At least three elements of reaction are necessary to restrain a body in stable equilibrium. Consider each of the cases shown in Fig. 2-12. The rigid body is subjected to restraints by three elements of reaction, and the restraints can be solved by the three available equilibrium equations. The satisfaction of all three equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$, for loads and reactions acting on the body guarantees, respectively, that the body will neither move horizontally or vertically nor rotate. The system is said to be *statically stable and determinate*.

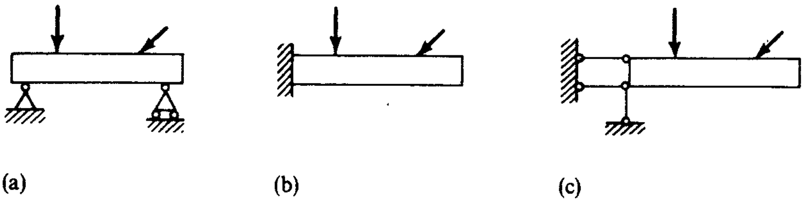
3. If there are more than three elements of reaction, as in each of the cases shown in Fig. 2-13, the body is necessarily more stable, because of the additional restraints. Since the number of unknown elements of reaction is greater than the number of equations for static equilibrium, the system is said to be *statically indeterminate* with regard to the reactions of support.

4. That the number of elements of reaction should be at least three is a necessary but not a sufficient condition for an externally stable structure. There



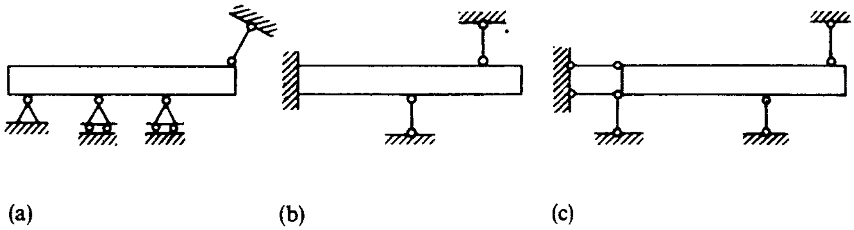
(a) (b) (c) (d)

Fig. 2-11



(a) (b) (c)

Fig. 2-12



(a) (b) (c)

Fig. 2-13

are many cases that are obviously not stable with respect to the support system even though three or more than three elements of reaction are supplied. When, for example, the lines of reaction are all parallel, as in Fig. 2-14(a), the body is unstable, because it is vulnerable to lateral sliding. Another case is shown in Fig. 2-14(b), where the lines of the three reaction elements are originally concurrent

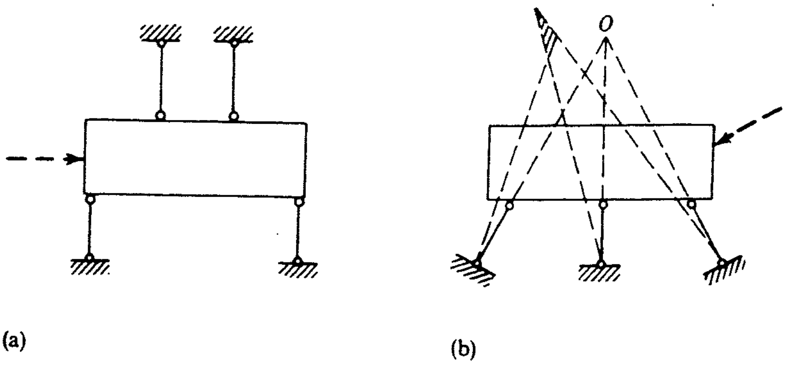


Fig. 2-14

at point $O$. The system is also unstable because, even though complete collapse probably will not result, a small initial rotation about $O$ because of the moment caused by any force not through $O$ will certainly occur until the three reaction lines form the triangle indicated by the crosshatched lines.

The above-mentioned instability, which results from the inadequacy of arrangement of supports, is referred to as *external geometric instability*.

5. A monolithic rigid body is rigid by definition; hence, it will have no problem of internal instability. Furthermore, at any cut section of a monolithic rigid body, the elements of internal force, which are no more than three in number, can always be determined by the equations of equilibrium, once the reactions are completely defined. Therefore, the stability and determinacy of the entire system mentioned in this section are solely determined by the stability and determinacy of supports and reactions.

Let us sum up the main points of the foregoing discussions as follows:

1. If the number of unknown elements of reaction is fewer than three, the equations of equilibrium are generally not satisfied, and the system is said to be statically unstable.
2. If the number of unknown elements of reaction is equal to three and if no external geometric instability is involved, then the system is statically stable and determinate.
3. If the number of unknown elements of reaction is greater than three, then the system is statically indeterminate; it is statically stable provided that no external geometric instability is involved. The excess number $n$ of unknown elements designates the $n$th degree of statical indeterminacy. For example, in each case of Fig. 2-13 there are five unknown elements of reaction. Thus, $5 - 3 = 2$, which indicates a statical indeterminacy of second degree.

## 2-6 GENERAL STABILITY AND DETERMINACY OF STRUCTURES

Structural stability and determinacy must be judged by the number and arrangement of the supports as well as the number and arrangement of the members and the connections of the structure. They are determined by inspection or by formula. For convenience, we shall deal with the general stability and determinacy of beams, trusses, and rigid frames in separate sections.

### 2-6a General Stability and Determinacy of Beams

If a beam is built up without any internal connections (internal hinge, roller, or link), the entire beam may be considered as a single monolithic rigid body placed on a number of supports, and the question of the stability and of the determinacy of the beam is settled solely by the number and arrangement of supports, as discussed in Sec. 2-5.

Now let us investigate what will happen if a certain connecting device is inserted in a beam. Let us suppose that a hinge is introduced into the statically stable and determinate beam of Fig. 2-15(a) or (b). The beam in each case will obviously become unstable under general loading as the result of a relative rotation between the left and the right portions of the beam at the internal hinge, as indicated in Fig. 2-15(c) or (d). That the hinge has no capacity to resist moment constitutes a restriction on the external forces acting on the structure; that is,

$$M = 0$$

about the hinge. In other words, the moment about the hinge calculated from the external forces on either side of the hinge must be zero in order to guarantee that these portions will not rotate about the hinge.

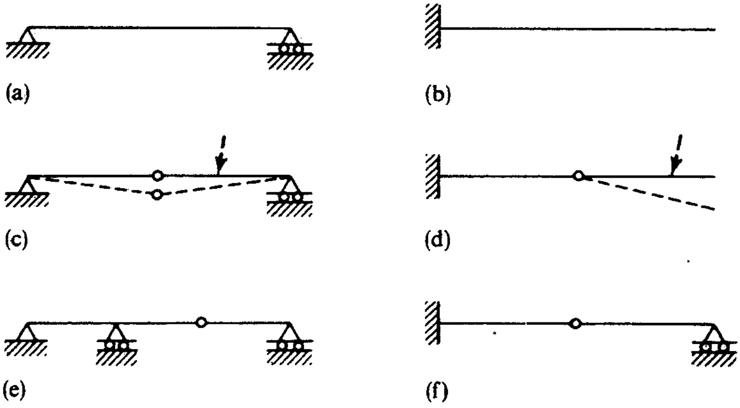


Fig. 2-15

Referring to Fig. 2-15(c) or (d), we see that in each case there are three elements of reaction supplied by supports, whereas there are four conditions of statics to restrict the external forces—three from equilibrium plus one from construction. This means that the number of unknown elements of reaction is one fewer than the independent equations of statics available for their solution. Therefore, the equations of statics for the force system are generally not satisfied. The beam is statically unstable unless we provide at least one additional element of reaction, such as the additional roller support shown in Fig. 2-15(e) or (f), which makes the total number of unknown elements of reaction equal to the total number of independent equations of statics needed to determine the elements. If this is done, the beam will be restored to a statically stable and determinate state.

Next, let us suppose that a link (or a roller) is introduced into a section of the statically stable and determinate beam of Fig. 2-15(a) or (b). We expect that this beam will be less stable than one with a hinged connection because the link (roller) cannot resist both moments about the link pin and forces normal to the link. The beam will collapse under general types of loading as a result of the relative rotation and the lateral translation of the left and right portions of the beam at the link, as indicated in Fig. 2-16(a) or (b).

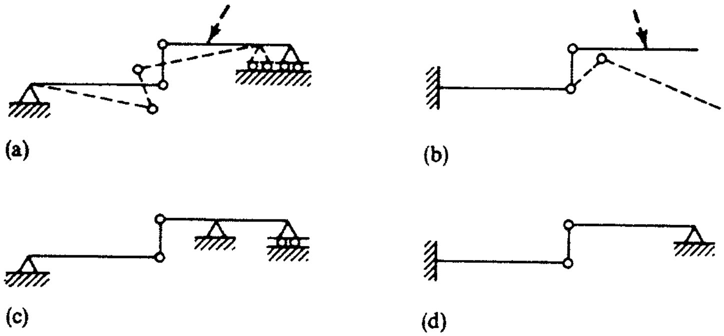


Fig. 2-16

That a link (or roller) has no capacity to resist lateral forces and moments constitutes two restrictions to the external forces acting on the structure, namely,

$$H = 0 \quad \text{and} \quad M = 0$$

about the link (either of two pins). $H$ is the sum of forces on either side of the link in the direction normal to the link. The satisfaction of condition $H = 0$ for the portion of the structure on either side of the link prevents the movement of one portion of the structure relative to the other in the direction normal to the link. Satisfying condition $M = 0$ for the portion of the structure on either side of the link ensures that these portions will not rotate about the pins of the link.

Referring to Fig. 2-16(a) or (b), we find that in each case there are three elements of reaction supplied by the support system, while there are five conditions of statics to restrict them—three from equilibrium and two from construction. Since the number of elements in reaction is two fewer than the number of statical equations to determine them, the beam is, therefore, quite unstable unless we supply at least two more elements of reaction, such as the hinged support shown in Fig. 2-16(c) or (d), to balance the situation. This done, the beam will be restored to its statically stable and determinate state.

There are beams for which the number of reaction elements is greater than the total number of independent equations of statics available. The beams are then classified as *statically indeterminate,* and the excess number of unknown elements indicates the degree of statical indeterminacy.

Geometric instability is most likely to occur whenever internal connections are introduced into an originally stable structure. Consider, for example, Fig. 2-17(a). The beam is statically indeterminate to the first degree. Now if a hinge is inserted into the beam, as shown in Fig. 2-17(b), it seems to be statically determinate. However, when a load is applied, a small initial displacement will result and will not be resisted elastically by the structure. In such a case, the beam is unstable not because of the inadequacy of the supports but because of the inadequacy of the arrangement of members. This situation is referred to as *internal geometric instability.* Very often when this occurs, the structure will collapse. In the present case collapse will not occur; the beam will come to rest in a position such as that marked by the dashed lines shown in Fig. 2-17(b).

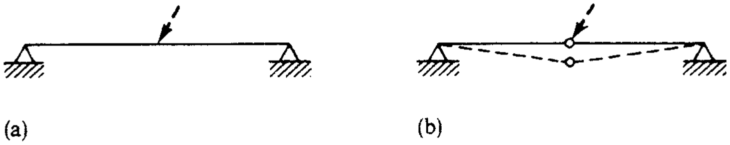

(a) (b)

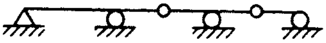
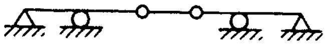
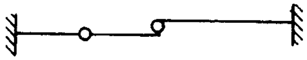
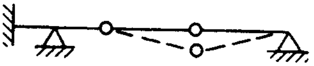
Fig. 2-17

From the foregoing discussions, a criterion may be established for the statical stability and determinacy of beams. Let $r$ denote the number of reaction elements and $c$ the number of equations of condition ($c = 1$ for a hinge; $c = 2$ for a roller; $c = 0$ for a beam without internal connection).

1. If $r < c + 3$, the beam is statically unstable.
2. If $r = c + 3$, the beam is statically determinate provided that no geometric instability (internal and external) is involved.
3. If $r > c + 3$, the beam is statically indeterminate.

Further illustrations are given in Table 2-1.

**TABLE 2-1**

| Beam | $r$ | $c$ | $r \gtreqless c + 3$ | Classification |
|---|---|---|---|---|
|  | 5 | 2 | $5 = 5$ | Stable and determinate |
|  | 6 | 2 | $6 > 5$ | Stable and indeterminate to the first degree |
|  | 5 | 2 | $5 = 5$ | Unstable* |
|  | 4 | 3 | $4 < 6$ | Unstable |
|  | 6 | 3 | $6 = 6$ | Stable and determinate |
|  | 7 | 2 | $7 > 5$ | Unstable* |

* Internal geometric instability; a possible form of displacement is indicated by the dotted lines.

### 2-6b General Stability and Determinacy of Trusses

A truss is composed of a number of bars connected at their ends by a number of pinned joints so as to form a network, usually a series of triangles, and mounted on a number of supports, such as the one shown in Fig. 2-18(a). Each bar of a truss is a two-force member; hence, each represents one unknown element of internal force (see Sec. 2-3). The total number of unknown elements for the entire system is counted by the number of bars (internal) plus the number of independent reaction elements (external). Thus, if we let $b$ denote the number of bars and $r$ the number of reaction components, the total number of unknown elements of the entire system is $b + r$. Now if the truss is in equilibrium, every isolated portion must likewise be in equilibrium. For a truss having $j$ joints, the entire system may be separated into $j$ free bodies, as illustrated in Fig. 2-18(b), in which each joint yields two equilibrium equations, $\Sigma F_x = 0$ and $\Sigma F_y = 0$, for the concurrent force system acting on it. From this a total of $2j$ independent equations, involving $(b + r)$ unknowns, is obtained. We may thus establish

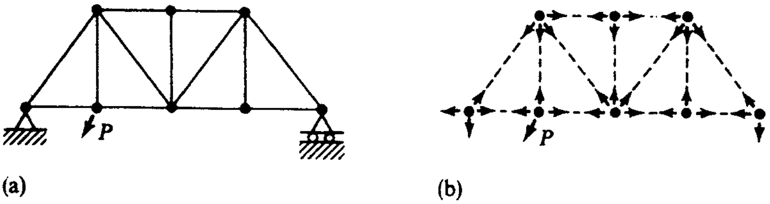


Fig. 2-18

criteria for the statical stability and the determinacy of a truss by counting the total unknowns and the total equations.

1. If $b + r < 2j$, the system is statically unstable.
2. If $b + r = 2j$, the system is statically determinate provided that it is also stable.
3. If $b + r > 2j$, the system is statically indeterminate.

The satisfaction of condition $b + r \geqslant 2j$ does not ensure a stable truss. For the truss to be stable requires fulfillment of further conditions. First, the value of $r$ must be equal to or greater than the three required for statical stability of supports. Next, there must be no inadequacy in the arrangement of supports and bars so as to avoid both external and internal geometric instability.

Basically, a stable truss can usually be obtained by starting with three bars pinned together at their ends in the form of a triangle and then by extending from it by adding two new bars for each new joint, as shown in Fig. 2-18(a). Since this truss satisfies $b + r = 2j (b = 13, r = 3, j = 8)$, it is statically determinate.

Suppose that this truss form is changed, as shown in Fig. 2-19. The number of bars and joints remains the same; the criterion equation is still satisfied. But it is geometrically unstable, since there is no bar to carry the vertical force (shear) in the panel where the diagonal is omitted. Other examples are given in Table 2-2.

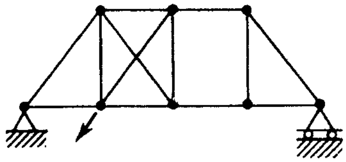
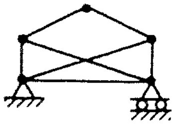
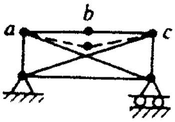
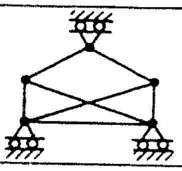
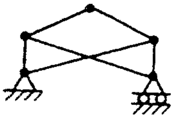
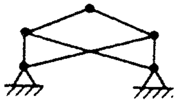
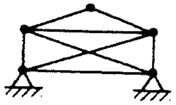
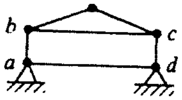


Fig. 2-19

Figure 2-20 shows a long-span trussed bridge, which we may consider to be composed of three rigid trusses connected by a hinge $A$ and a link $BC$ and mounted on a number of supports. These connections are not completely rigid, so certain equations of condition are introduced to restrict the external forces

**TABLE 2-2**

| Truss | $b$ | $r$ | $j$ | $b + r \gtreqless 2j$ | Classification |
|---|---|---|---|---|---|
|  | 7 | 3 | 5 | $10 = 10$ | Stable and determinate |
|  | 7 | 3 | 5 | $10 = 10$ | Unstable* |
|  | 7 | 3 | 5 | $10 = 10$ | Unstable** |
|  | 6 | 3 | 5 | $9 < 10$ | Unstable |
|  | 6 | 4 | 5 | $10 = 10$ | Stable and determinate |
|  | 8 | 4 | 5 | $12 > 10$ | Stable and indeterminate to the second degree |
|  | 6 | 4 | 5 | $10 = 10$ | Unstable*** |

* Internal geometric instability due to three pins *a*, *b*, *c* on a line; possible displacement as indicated by dotted lines.

** External geometric instability due to parallel lines of reaction.

*** Internal geometric instability due to lack of lateral resistance in panel *abcd*.

acting on the structure. In this case the hinge at $A$ provides one condition equation, $M_A = 0$, which means that the moment about $A$ of the forces on either side of $A$ must be zero. The hanger $BC$ provides two condition equations, $M_B = 0$ (or $M_C = 0$) and $H = 0$, which means that the moment about $B$ (or $C$) of the forces on either side of $B$ (or $C$) must be zero and also that the sum

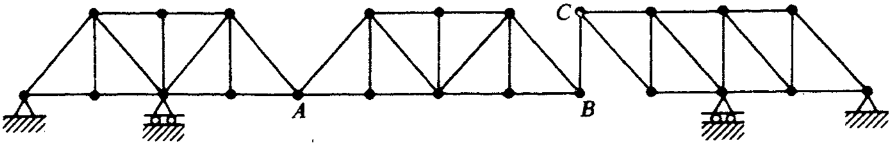


Fig. 2-20

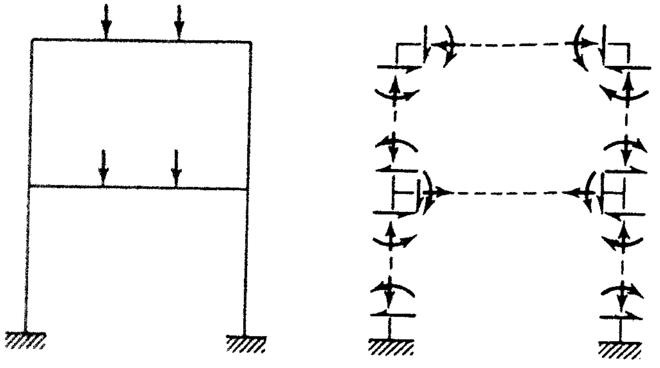
of the horizontal forces on either side of the hanger must be zero, since the vertical hanger is incapable of resisting horizontal forces.

The stability and determinacy of the truss may be investigated by first counting the number of bars, joints, and reaction elements. It is found that the equation $b + r = 2j$ is satisfied by the truss since $b = 40$, $r = 6$, and $j = 23$. Thus, the necessary condition for the system to be statically determinate is fulfilled. Next, there is no obvious instability either in the formation of the truss or in the supports. Both the portions to the left of $A$ and to the right of $BC$ are rigidly formed and adequately supported. The portion in the center span is also rigidly formed. Its connection to the side portions by a hinge and a hanger constitutes three elements of support. In regard to reactions, there is a total of six elements that can just be determined by six statical equations, three from equilibrium and three from construction. Thus, the entire system is stable and statically determinate; furthermore, it is stable and statically determinate as regards support reactions.

There are certain cases in which the stability or instability of a truss is not obvious. One way of determining stability is to attempt a stress analysis and to discover whether the results are consistent or not. An inconsistent result indicates that the answer is not unique but infinite and indeterminate. If such is the case, then the truss is said to be *unstable*. We discuss this further in Sec. 3-5.

### 2-6c General Stability and Determinacy of Rigid Frames

A rigid frame is built of beams and columns connected rigidly, such as the one shown in Fig. 2-21(a). The stability and determinacy of a rigid frame may also be investigated by comparing the number of unknowns (internal unknowns and reaction unknowns) with the number of equations of statics available for their solution. Like a truss, a rigid frame may be separated into a number of free bodies of joints, as shown in Fig. 2-21(b), which requires that every member of the frame be taken apart. As discussed in Sec. 2-3, there are usually three unknown magnitudes ($N$, $V$, $M$) existing in a cut section of a member. However, if these quantities are known at one section of a member, similar quantities for any other section of the same member can be determined. Hence, there are only three independent, internal, unknown elements for each member in a frame. If we let $b$ denote the total number of members and $r$ the reaction elements, then the total number of independent unknowns in a rigid frame is $(3b + r)$.



(a) (b) Fig. 2-21

Next, a rigid joint isolated as a free body will generally be acted upon by a system of forces and couples, as indicated in Fig. 2-21(b), since a rigid joint is capable of resisting moments. For equilibrium of such a joint, this system, therefore, must satisfy three equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$. Thus, if the total number of rigid joints is $j$, then $3j$ independent equilibrium equations may be written for the entire system.

It may happen that hinges or other devices of construction are introduced into the structure so as to provide additional equations of statics, say a total of $c$. Then the total number of equations of statics available for the solution of the $(3b + r)$ unknowns is $(3j + c)$. The criteria for the statical stability and determinacy of the rigid frame are thus established by comparing the number of unknowns $(3b + r)$ with the number of independent equations $(3j + c)$:

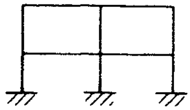
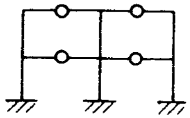
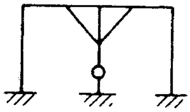
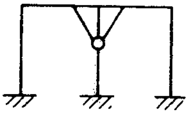
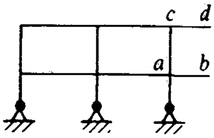
1. If $3b + r < 3j + c$, the frame is statically unstable.
2. If $3b + r = 3j + c$, the frame is statically determinate provided that it is also stable.
3. If $3b + r > 3j + c$, the frame is statically indeterminate.

It should be recalled from the similar discussion dealing with the criteria for trusses that satisfaction of the condition $3b + r \geqslant 3j + c$ does not warrant a stable frame unless $r \geqslant 3$ and, also, that no geometric instability is involved in the system.

Consider the frame in Fig. 2-21(a). There are six joints (including those at supports), six members, and six reaction elements, but no condition of construction. Thus, $3b + r = 18 + 6 > 3j + c = 18 + 0$. The excess number six in unknowns indicates that the frame is statically indeterminate to the sixth degree. Further examples for classifying frame stability and determinacy are given in Table 2-3.

Criteria such as the above are general and useful; but many problems, which may be investigated by a formula, can readily be settled by inspection through cutting frame members and reducing the structure to several simple parts.

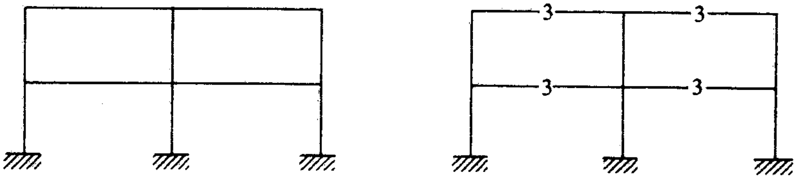
**TABLE 2-3**

| Frame | $b$ | $r$ | $j$ | $c$ | $3b + r \gtreqless 3j + c$ | Classification |
|---|---|---|---|---|---|---|
|  | 10 | 9 | 9 | 0 | $39 > 27$ | Indeterminate to the 12th degree |
|  | 10 | 9 | 9 | 4 | $39 > 31$ | Indeterminate to the eighth degree |
|  | 10 | 9 | 9 | 1 | $39 > 28$ | Indeterminate to the 11th degree |
|  | 10 | 9 | 9 | 3* | $39 > 30$ | Indeterminate to the ninth degree |
|  | 10** | 6 | 9 | 0 | $36 > 27$ | Indeterminate to the ninth degree |

* If a pin is inserted in a rigid frame, generally, $c$ = the number of members meeting at the pin minus one. In this case $c = 4 - 1 = 3$.

** The overhanging portions, such as *ab* and *cd* on the right side of the frame, should not be counted in the number of members.

Suppose that we wish to analyze the degree of indeterminacy of the frame shown in Fig. 2-22(a). The best approach is to cut members, as indicated in Fig. 2-22(b), so that the structure is separated into three statically determinate and stable parts. The number of restraints removed to accomplish this result gives the degree of indeterminacy of the frame. Since each cut involves three



(a) (b)

Fig. 2-22

internal unknown elements, the total number of restraints removed by four cuts is (4)(3) = 12; the frame is statically indeterminate to the 12th degree.

The advantage of this approach over counting the number of bars and joints and reaction elements will easily be seen when we come to determine the degree of indeterminacy of the frame of a tall building, such as the one shown in Fig. 2-23. Since the building can be separated into 12 stable and determinate parts by 77 cuts in the beams, it is statically indeterminate to the 231st degree.

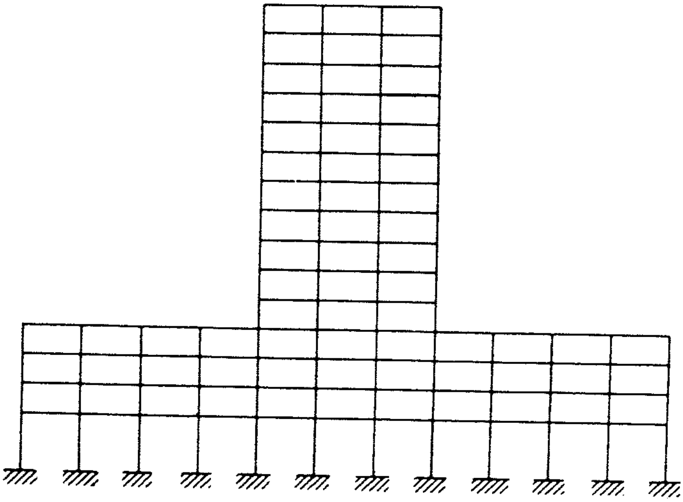


Fig. 2-23

## PROBLEMS

**2-1.** Discuss the stability and determinacy of the beams shown in Fig. 2-24.

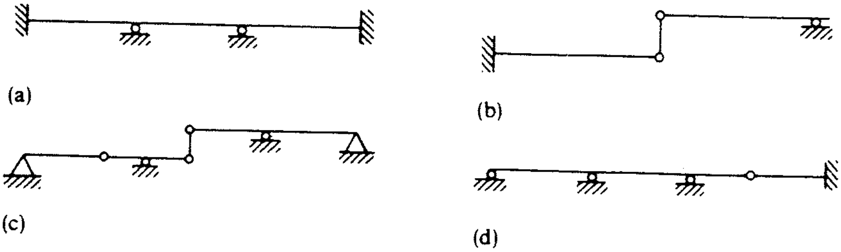


Fig. 2-24

**2-2.** Discuss the stability and determinacy of the trusses shown in Fig. 2-25.

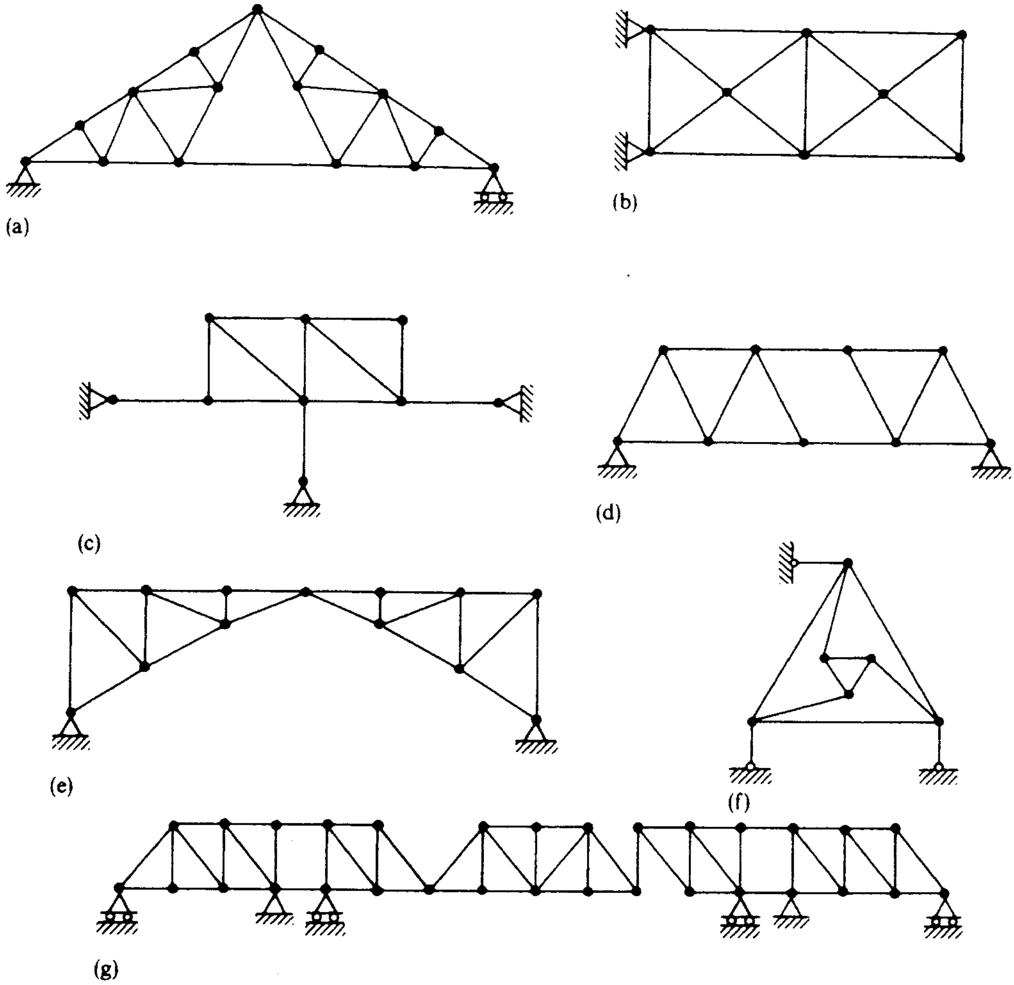


Fig. 2-25

2-3. Discuss the stability and determinacy of the rigid frames shown in Fig. 2-26.

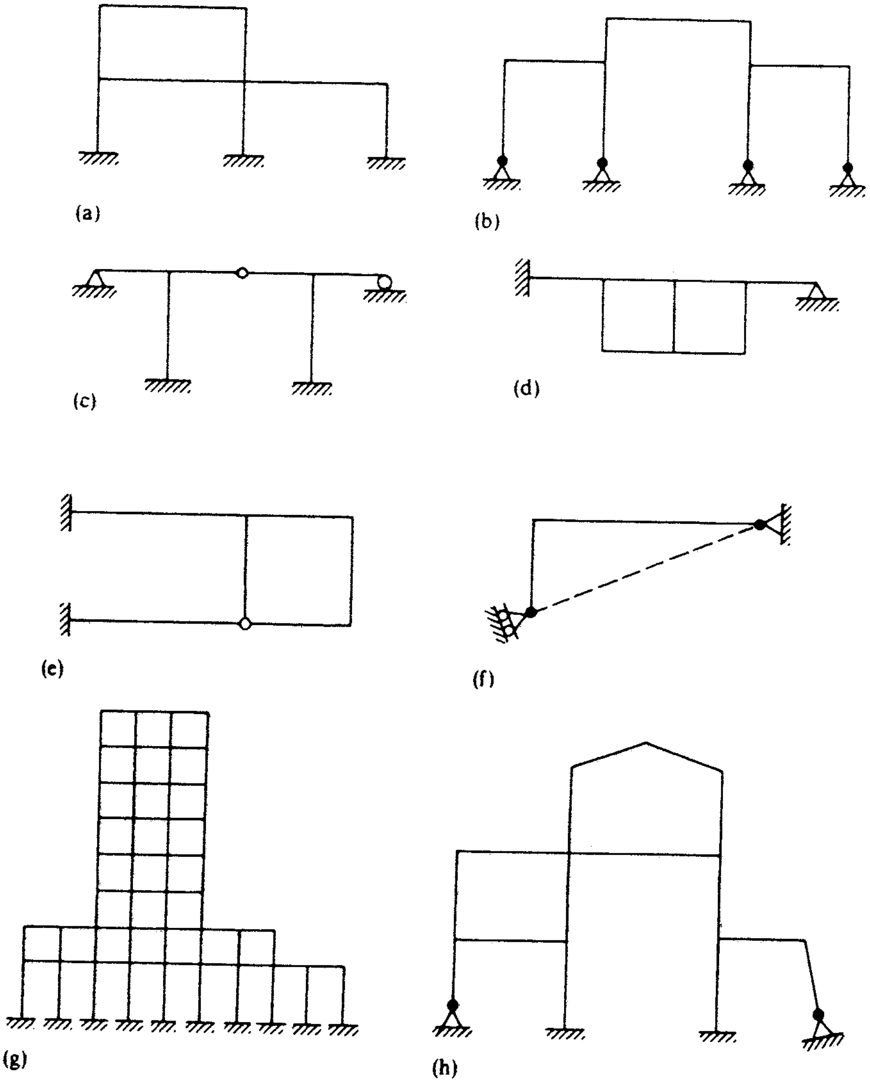


Fig. 2-26

# 3

# Structural Statics

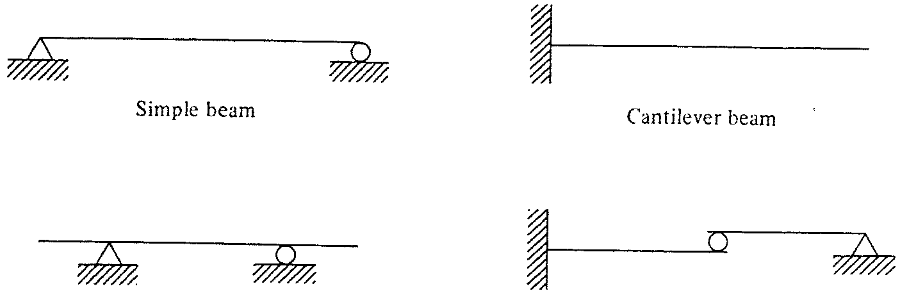
## 3-1 GENERAL

In this chapter we analyze planar statically determinate structures, including beams, trusses, and rigid frames.

A *beam* is defined as a structural member predominantly subjected to bending moment. We limit our discussions to beams of symmetrical section, in which the centroidal axis is a straight line. Furthermore, we assume that the beam is acted on by only transverse loading and moment loading and that all the loads and reactions lie in the plane of symmetry. It thus follows that such a beam will be subjected to bending and shear in the plane of loading without axial stretching and twisting.

The basic types of statically determinate beams are *simple beams* and *cantilever beams*. A beam that is supported at its two ends with a hinge and a roller is called a *simply supported beam,* or *simple beam.* A cantilever beam is fixed or built-in at one end and free at the other end. The end portion (or portions) of a simple beam may extend beyond the support to form a simple beam with overhang. Several beams of different types may be connected by internal hinges or rollers to form a *compound beam.* These are illustrated in Fig. 3-1.

A *truss,* such as the one shown in Fig. 3-2, may be defined as a plane structure composed of a number of members joined together at their ends by smooth pins so as to form a rigid framework the external forces and reactions of which are assumed to lie in the same plane and to act only at the pins. Furthermore, we assume that the centroidal axis of each member coincides with



Simple beam

Cantilever beam

Simple beam with overhang

Compound beam

**Fig. 3-1**

the line connecting the joint centers at the ends of member and that the weight of each member is negligible in comparison to the other external forces acting on the truss. From these conditions it follows that each member in a truss is a *two-force* member and is subjected only to direct axial forces (tension or compression).

A modern truss made of bolted or welded joints is not really a truss by a strict interpretation of this definition. However, since a satisfactory stress analysis may usually be worked out by assuming that such a structure acts as if it were pin-connected, it may still be called a truss.

Common trusses may be classified according to their formations as *simple, compound,* and *complex*. A rigid plane truss can always be formed by beginning with three bars pinned together at their ends in the form of a triangle and then extending from this two new bars for each new joint, as explained in Sec. 2-6. Of course, the new joint and the two joints to which it is connected should never lie along the same straight line, to avoid geometric instability. Trusses whose members have been so arranged are called *simple trusses,* for they are the simplest type of bar arrangement encountered in practice.

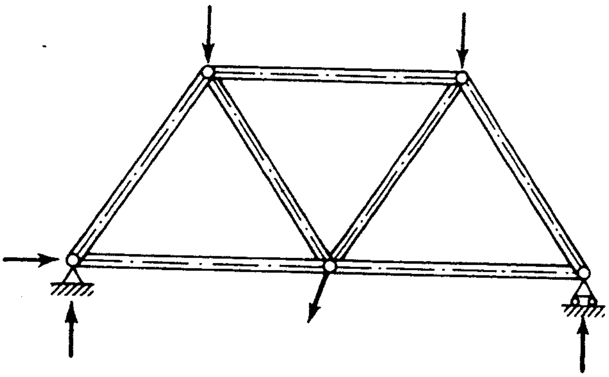


**Fig. 3-2**

The trusses shown in Fig. 3-3 are all simple trusses. The shaded triangle *abc* in each truss diagram is the base figure from which we extended the form by using two additional bars to connect each of the new joints in alphabetical order.

It can easily be shown that there exists a very definite relationship between the number of bars $b$ and the number of joints $j$ in a simple truss. Since the base triangle of a simple truss consists of three bars and three joints, the additional bars and joints required to complete the truss are $(b - 3)$ and $(j - 3)$, respectively. These two numbers should be in a 2:1 ratio. Thus,

$$b - 3 = 2(j - 3)$$

or

$$b + 3 = 2j$$

Comparing the above equation with the necessary condition for a statically determinate truss (see Sec. 2-6) given by

$$b + r = 2j$$

we find that if the supports of a simple truss are so arranged that they are composed of three elements of reaction neither parallel nor concurrent, then the structure is stable and statically determinate under general conditions of loading.

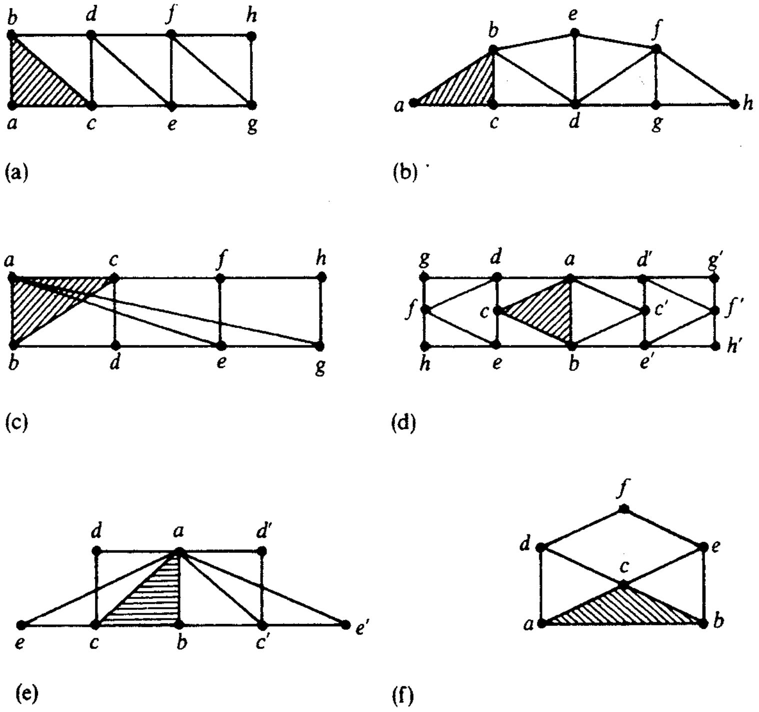
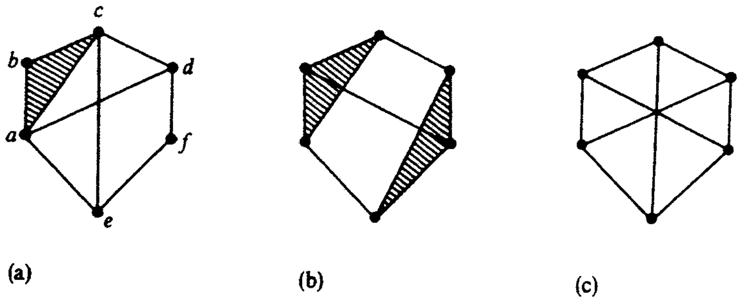


Fig. 3-3

If two or more simple trusses are connected together to form one rigid framework, the composite truss is called a *compound truss*. One simple truss can be rigidly connected to another simple truss at certain joints by three links neither parallel nor concurrent or by the equivalent of this type of connection. Additional simple trusses can be joined in a similar manner to the framework already constructed to obtain a more elaborate compound truss.

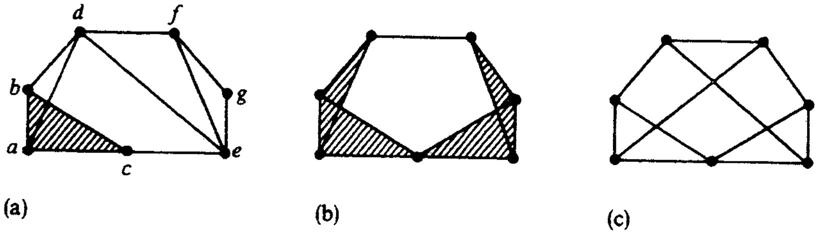
Trusses that cannot be classified as either simple or compound are called *complex trusses*.

Figure 3-4(a) shows a simple truss. Rearranging the bars results in a compound truss such as the one shown in Fig. 3-4(b). However, the truss shown in Fig. 3-4(c), made of the same number of bars and joints, does not belong to either of the above categories and may be termed a complex truss. Similarly, we find that the truss shown in Fig. 3-5(a) is a simple truss, that in Fig. 3-5(b) a compound truss, and that in Fig. 3-5(c) a complex truss.



(a) (b) (c)

Fig. 3-4



(a) (b) (c)

Fig. 3-5

A *rigid frame* may be defined as a structure composed of a number of members connected together by joints some or all of which are rigid, that is, capable of resisting both force and moment as distinguished from a pin-connected joint, which offers no moment resistance. In steel structures, rigid joints may be formed by certain types of riveted, bolted, or welded connections. In reinforced concrete structures, the materials in the joined members are mixed together in one unit so as to be substantially rigid. In the analysis of rigid frames, we assume that the centroidal axis of each member coincides with the line connecting the joint centers of the ends of the member. The so-called joint center is therefore the concurrent point of all centroidal axes of members meeting at the joint. With the joint rigid the ends of all connected members must not only translate but also rotate identical amounts at the joint. Rigid frames are usually built to be highly statically indeterminate. The discussion of statically determinate rigid frames in this chapter is primarily of academic interest rather than of practical use and serves as a prelude to the analysis of statically indeterminate frames.

## 3-2 ANALYSIS OF STATICALLY DETERMINATE BEAMS

To illustrate the general procedure in analyzing a statically determinate beam, let us consider the loaded beam in Fig. 3-6(a).

The first step in the analysis is to find the reactions at ends $a$ and $b$, denoted by $R_a$ and $R_b$, respectively. This can readily be accomplished by applying the equilibrium equations:

$$\sum M_a = 0 \qquad \sum M_b = 0$$

or

$$\sum M_b = 0 \qquad \sum F_y = 0$$

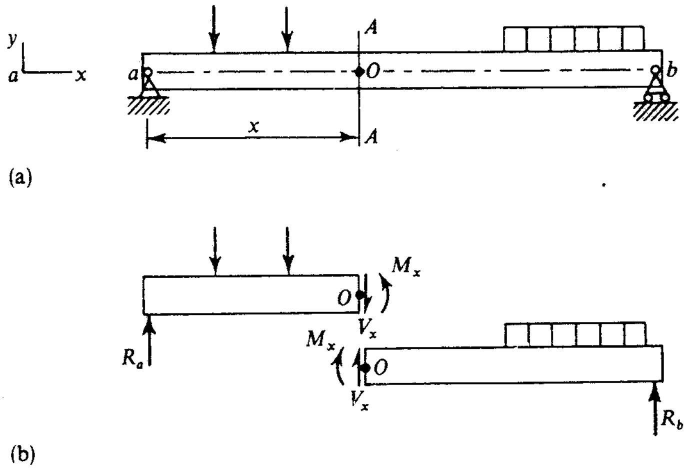


Fig. 3-6

Next we investigate the shear force and bending moment at each transverse cross section of the beam. The *shear force* at any transverse cross section of the beam, say section $A$–$A$, at a distance $x$ from the left end [see Fig. 3-6(a)], is the algebraic sum of the external forces (including those of reaction) applied to the portion of the beam on either side of $A$–$A$. The *bending moment* at section $A$–$A$ of the beam is the algebraic sum of the moments taken about an axis through $O$ (the centroid of section $A–A$) and normal to the plane of loading of all the external forces applied to the portion of the beam on either side of $A–A$. By considering either the left or the right portion as the free body, as shown in Fig. 3-6(b), we readily see that the *shear resisting force* at section $V_x$ is equal and opposite to the shear force for that section just defined; and the *resisting moment* at section $M_x$ is equal and opposite to the bending moment for the section just defined. The values of $V_x$ and $M_x$ can be found from the two equations of equilibrium

$$\sum F_y = 0 \qquad \text{and} \qquad \sum M_o = 0$$

for the portion considered.

Since the shear and bending moment in a transversely loaded beam will, in general, vary with the distance $x$ defining the location of the cross section on which they occur, both are therefore functions of $x$. It is advisable to plot curves or diagrams from which the value of functions ($V_x$ and $M_x$) at any cross section may readily be obtained. To do this, we let one axis, the $x$ axis, coincide with the centroidal axis of the beam, indicating the position of the beam section, and the other axis, the $y$ axis, indicate the value of function $V_x$ or $M_x$. The graphic representation is called *shear* or *moment curve*.

Our sign conventions for beam shear and moment are as follows:

1. Shear is considered positive at a section when it tends to rotate the portion of the beam in the clockwise direction about an axis through a point inside the free body and normal to the plane of loading; otherwise, it is negative [see Fig. 3-7(a)].
2. Bending moment is considered positive at a section when it tends to bend the member concave upward; otherwise, it is negative [see Fig. 3-7(b)].

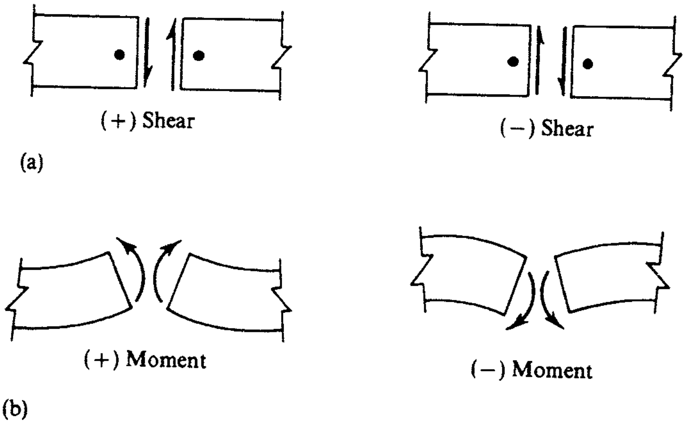


Fig. 3-7

Such sign conventions, although arbitrary, must be carefully observed to avoid confusion.

The analysis of statically determinate beams is illustrated in the following examples.

**Example 3-1**

Figure 3-8(a) shows a simple beam under a concentrated load $P$ acting at $C$.

The reactions $R_A$ and $R_B$ are readily found to be

$$R_A = \frac{Pb}{l} \qquad R_B = \frac{Pa}{l}$$

from $\Sigma M_B = 0$ and $\Sigma M_A = 0$.

The shear at any section to the left of $P$ is equal to $R_A$; that is,

$$V_x = \frac{Pb}{l} \qquad (0 < x < a)$$

The shear at any section to the right of $P$ is found to be equal to $R_B$ but with negative sign, according to our sign convention. Thus,

$$V_x = -\frac{Pa}{l} \qquad (a < x < l)$$

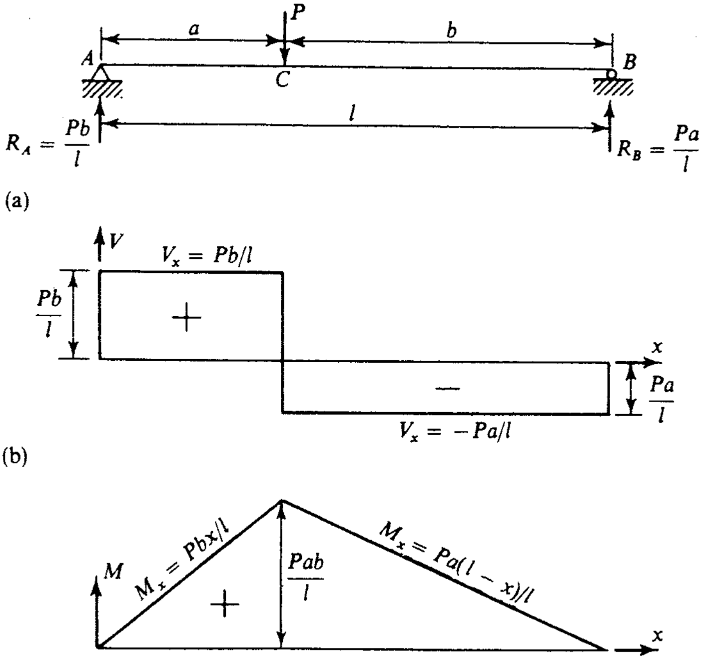


Fig. 3-8

A change process of shear occurs at section $C$, the total change being $-P$ (from $Pb/l$ to $-Pa/l$). In this connection, we note that at a concentrated load (including reaction) there is, in general, an abrupt change in the shear equal to the load. Consider the shear in the immediate vicinity of each support. The shear on a section an infinitesimal distance to the right of point $A$ is $Pb/l$; therefore, the shear curve rises abruptly from zero to $Pb/l$ at $A$. Similarly, the shear goes to zero from the value $-Pa/l$ at $B$. In general, *the shear curve always starts at zero and ends at zero* [see Fig. 3-8(b) for the shear curve].

The bending moment at any section distance $x$ from $A$ is given by

$$M_x = \frac{Pb}{l}x \qquad (0 \leqslant x \leqslant a)$$

$$M_x = \frac{Pa}{l}(l - x) \qquad (a \leqslant x \leqslant l)$$

Both are of linear variation and are plotted in Fig. 3-8(c).

If there are several concentrated loads on the beam, we need as many linear equations to represent the shear or moment as the number of segments involved. The shear or moment diagram is then composed of a series of line segments.

It is customary to drop the coordinate axes in the diagram unless the origin of the coordinate system is otherwise specified.

**Example 3-2**

Figure 3-9(a) shows a simple beam subjected to a uniform load of intensity $w$.

Because of symmetry, the reactions are each equal to $wl/2$, as shown. Then at any section distance $x$ from the left end $A$, we have

$$V_x = \frac{wl}{2} - wx$$

$$M_x = \frac{wl}{2}x - \frac{wx^2}{2}$$

These are shown in Fig. 3-9(b) and (c), respectively.

**Example 3-3**

Figure 3-10(a) shows a simple beam subjected to an external couple of $M$ applied at $C$.

The reactions $R_A$ and $R_B$ must be such as to form a couple to balance $M$. They must be equal to $M/l$ and opposite in sense, as indicated in Fig. 3-10(a).

In this case the shear is of constant value equal to $-M/l$ in the range $0 < x < l$, as shown in Fig. 3-10(b).

The moments vary linearly from $A$ to $C$ and from $C$ to $B$ and are given by

$$M_x = -\frac{M}{l}x \qquad (0 \leqslant x < a)$$

$$M_x = \frac{M}{l}(l - x) \qquad (a < x \leqslant l)$$

The process of moment changes at $C(x = a)$, the total change being $M$ (from $-Ma/l$ to $Mb/l$). The moment diagram is shown in Fig. 3-10(c), and the point of inflection (zero moment) is at $C$ where the moment curve passes the $x$ axis.

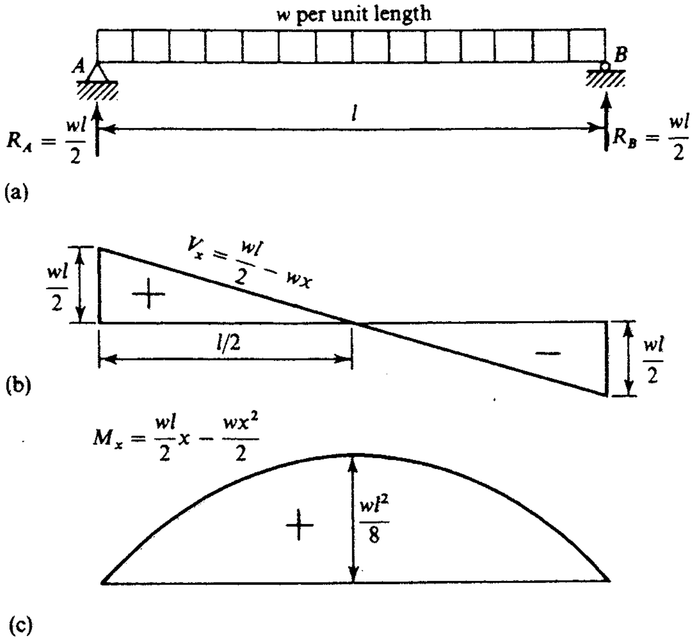


Fig. 3-9

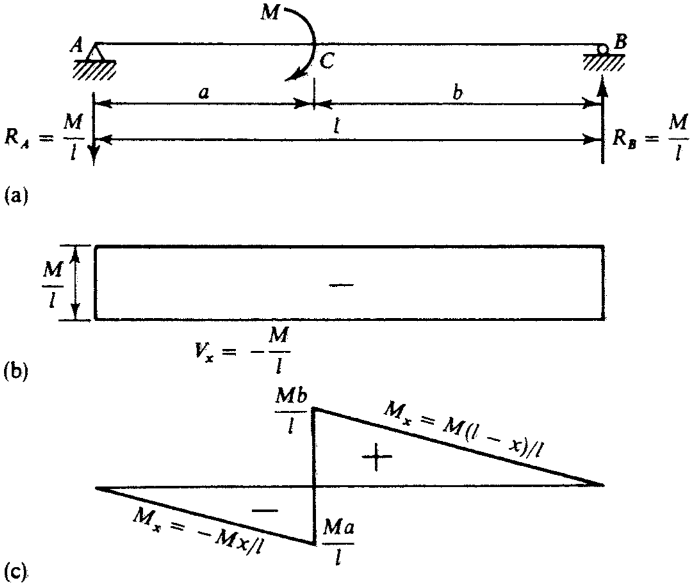
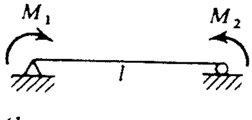
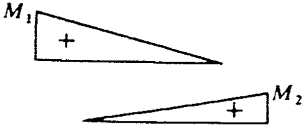
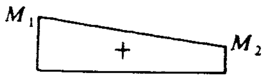
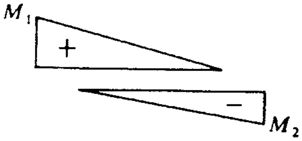
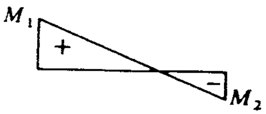
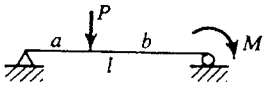
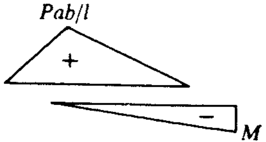
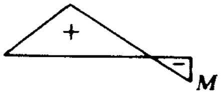
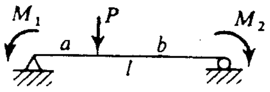
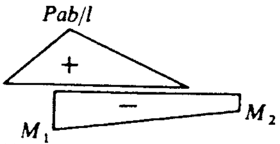
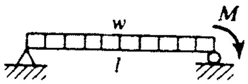
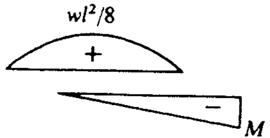
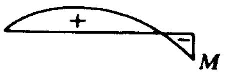
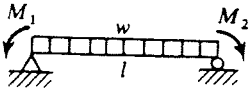
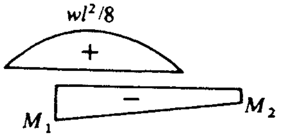
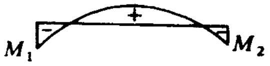


Fig. 3-10

If the beam is subjected to several loads (or several groups of loads), the shear or the moment diagram may be plotted separately for each load (or each group of loads) and then combined into one diagram by the principle of superposition.

Table 3-1 shows that application of this principle in drawing a qualitative moment diagram for a simple beam (or a general member) restrained by end moments.

**TABLE 3-1**

| Case | Separate Moment Diagram | Combined Moment Diagram* |
|---|---|---|
|  (1) |  |  |
|  (2) |  |  |
|  (3) |  |  |
|  (4) |  |  |
|  (5) |  |  |
|  (6) |  |  |

* With large end moment (or moments) the combined moment diagrams for Cases (3), (4), (5), or (6) could be all negative without point of inflection.

From Table 3-1, we see that if the beam carries no load but end moments, the moment curve is a straight line with one point of inflection or no point of inflection. If the beam carries a concentrated load or a uniform load together with the end moment on one end or both ends, the moment curve may pass zero at one, or two, or no point on the beam axis.

It may be worth mentioning here that if the beam is made of elastic material, the beam will be deformed under load; the elastic deformations of beams are primarily caused by bending. With reference to the moment diagram and points of zero moment, we can easily sketch the deflected elastic curve.

**Example 3-4**

Figure 3-11(a) shows a cantilever carrying a distributed load the intensity of which varies linearly from $w$ per unit length at the fixed end to zero at the free end.

At any section distance $x$ from the free end $a$,

$$V_x = -\left(\frac{wx}{l}\right)\left(\frac{x}{2}\right) = -\frac{wx^2}{2l}$$

$$M_x = -\left(\frac{wx^2}{2l}\right)\left(\frac{x}{3}\right) = -\frac{wx^3}{6l}$$

These are plotted in Fig. 3-11(b) and (c), respectively.

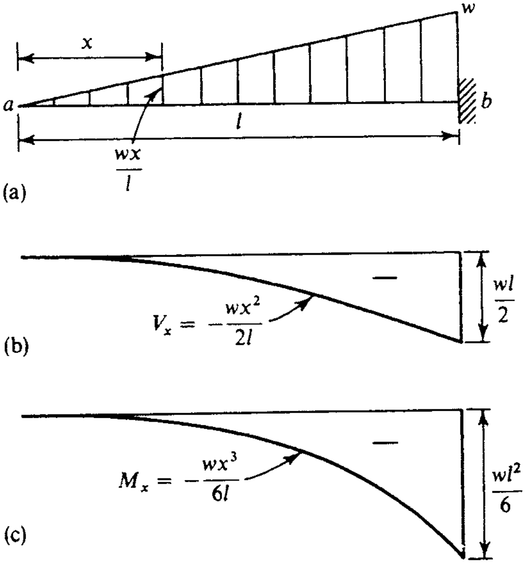


Fig. 3-11

**Example 3-5**

For the loaded compound beam of Fig. 3-12(a), draw the shear and moment diagrams.

Theoretically we can use two equilibrium equations and a condition equation ($M = 0$ at hinge $b$) to find the three reaction elements and then determine the shear and moment of the beam. But in the present case, we can conveniently separate the beam from the connecting hinge into two portions, a simple beam and a cantilever, as shown in Fig. 3-12(b). The shear and moment diagrams are easily drawn for each of the separated portions and then combined, as shown in Fig. 3-12(c) and (d), respectively.

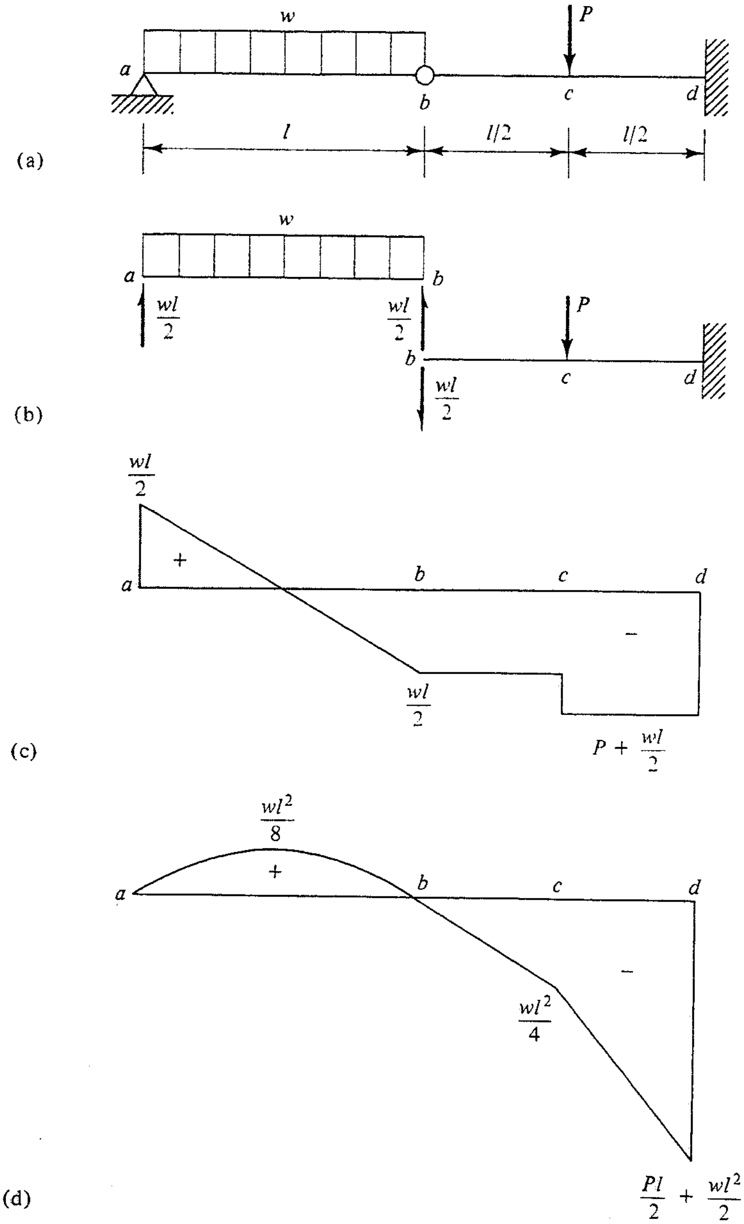


Fig. 3-12

## 3-3 RELATIONSHIPS BETWEEN LOAD, SHEAR, AND BENDING MOMENT

There exist at any cross section of a loaded beam certain relationships between load, shear, and bending moment that are tremendously helpful in constructing the shear and bending moment curves.

Consider a portion of a beam of any type subjected to transverse loading and moment loading, such as the one shown in Fig. 3-13(a). To investigate the relationships among load, shear, and bending moment in a beam, we may classify the beam segments in the following way [as partly illustrated in Fig. 3-13(a)]:

1. Segment under no load
2. Segment under distributed load
3. Segment under concentrated load
4. Segment under moment load

We shall deal with each of these four cases as follows:

1. *Segment under no load.* As indicated in Fig. 3-13(a), a segment between two concentrated loads is an example of a segment under no load. Let us take an element cut out by two adjacent cross sections at a distance $dx$ apart, as shown in Fig. 3-13(b). On the left-hand face of this element, we represent the shear force and bending moment by $V_x$ and $M_x$ and on the right-hand face of

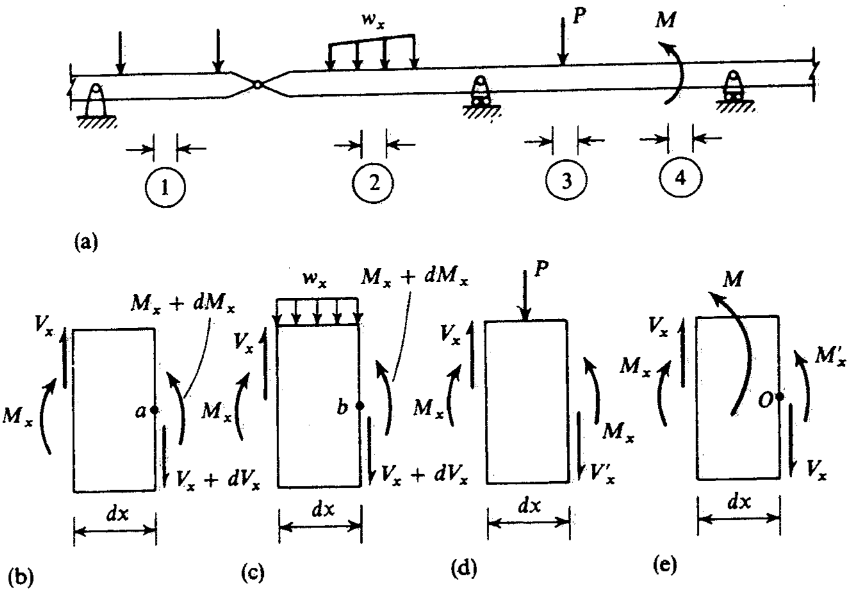


Fig. 3-13

the element, by $V_x + dV_x$ and $M_x + dM_x$, in which $dV_x$ and $dM_x$ are changes of shear and moment in $dx$. We assume that $x$ increases from left to right. Since the element is in equilibrium, we have from $\Sigma F_y = 0$

$$V_x - (V_x + dV_x) = 0$$

that is,

$$dV_x = 0$$

or

$$V_x = \text{constant} \tag{3-1}$$

Also from $\Sigma M_a = 0$,

$$M_x + V_x dx - (M_x + dM_x) = 0$$

Reducing and using Eq. 3-1, we arrive at

$$\frac{dM_x}{dx} = \text{constant} \tag{3-2}$$

Equation 3-1 states that no change of shear takes place, and Eq. 3-2 states that the rate of change of bending moment at any point with respect to $x$ is constant.

2. *Segment under distributed load.* Let us take an element subjected to a distributed load cut out by two adjacent cross sections distance $dx$ apart, as shown in Fig. 3-13(c). Assume a downward distributed load in a positive direction. From $\Sigma F_y = 0$,

$$V_x - (V_x + dV_x) - w_x dx = 0$$

$$dV_x = -w_x dx$$

or

$$\frac{dV_x}{dx} = -w_x \tag{3-3}$$

From $\Sigma M_b = 0$,

$$M_x + V_x dx - w_x dx\frac{dx}{2} - (M_x + dM_x) = 0$$

Neglecting the small term $w_x(dx)^2/2$ and reducing, we find that

$$\frac{dM_x}{dx} = V_x \tag{3-4}$$

Equation 3-3 states that the rate of change of shear with respect to $x$ at any point is equal to the intensity of the load at that point but with the opposite sign. Equation 3-4 states that the rate of change of bending moment with respect to $x$ at any point is equal to the shear force at that point.

3. *Segment under concentrated load.* Figure 3-13(d) shows an element subjected to a concentrated load $P$. Now $P$ is assumed to be acting at a point. As the distance between two adjacent sections becomes infinitesimal, there will be no moment difference between the sections to the immediate left of $P$ and to the immediate right of $P$. However, an abrupt change in the shear force equal to $P$ between the two sections takes place, since from $\Sigma F_y = 0$,

$$V_x - P - V_x' = 0$$

or

$$V'_x = V_x - P \tag{3-5}$$

as indicated in Fig. 3-13(d). Accordingly, there will be an abrupt change in the derivative $dM_x/dx$ at the point of application of concentrated force.

4. *Segment under moment load.* Figure 3-13(e) shows an element subjected to a couple of $M$. Now $M$ is assumed to be acting at a point. As the distance between the two adjacent sections becomes infinitesimal, there will be no shear difference between the sections to the immediate left of $M$ and to the immediate right of $M$. However, there will be an abrupt change of moment equal to $M$ between the two sections, for from $\Sigma M_o = 0$ we have

$$M_x - M - M'_x = 0$$

or

$$M'_x = M_x - M \tag{3-6}$$

as indicated in Fig. 3-13(e).

Construction of the shear and moment diagrams is facilitated by the relationships previously stated. For instance, the equation

$$\frac{dV_x}{dx} = -w_x$$

implies that the slope of the shear curve at any point is equal to the negative value of the ordinate of load diagram applied to the beam at that point. There are cases worth noting:

1. For a segment under no load, the slope of the shear curve is zero (i.e., parallel to the beam axis). The shear curve is therefore a line parallel to the beam axis.
2. For a segment under a uniform load of intensity $w$, the slope of the shear curve is constant. The shear curve is therefore a sloping line.
3. At a point of concentrated load, the intensity of the load is infinite, and the slope of the shear curve will thus be infinite (i.e., vertical to the beam axis). There will be a discontinuity in the shear curve, and a change process of shear equal to the applied force occurs between the two sides of the loaded point.
4. Under distributed load the change in shear between two cross sections a differential distance $dx$ apart is

$$dV_x = -w_x dx$$

Thus, the difference in the ordinates of the shear curve between any two points $a$ and $b$ is given by

$$\begin{aligned} V_b - V_a &= -\int_{x_a}^{x_b} w_x dx \\ &= -(\text{area of load diagram between } a \text{ and } b) \end{aligned} \tag{3-7}$$

Suppose that there are additional concentrated forces $\Sigma P$ acting between $a$ and $b$. The result of the shear difference between the two points must include the effect due to $\Sigma P$:

$$\begin{aligned} V_b - V_a &= -\int_{x_a}^{x_b} w_x \, dx - \sum P \\ &= -(\text{area of load diagram between } a \text{ and } b + \sum P) \end{aligned} \tag{3-8}$$

in which $\Sigma P$ has been assumed to act downward.

Similarly, from the equation

$$\frac{dM_x}{dx} = V_x$$

the slope of the bending moment curve at any point equals the ordinate of the shear curve at that point. We note the following:

1. If the shear is constant in a portion of the beam, the bending moment curve will be a straight line in that portion.
2. If the shear varies in any manner in a portion of the beam, the bending moment curve will be a curved line.
3. At a point where a concentrated force acts, there will be an abrupt change in the ordinate of shear curve and, therefore, an abrupt change in the slope of the bending moment curve at the point. In fact, the moment curve will have two different slopes at that point.
4. Maximum and minimum bending moments occur at the points where a shear curve goes through the $x$ axis—the maximum where shear changes from positive (at the left) to negative (at the right); the minimum in the reverse manner.
5. For a concentrated force system the maximum bending moment must occur under a certain concentrated force, since change of shear from positive to negative must occur at a certain point where a concentrated force is applied.
6. Referring to the equation $dM_x/dx = V_x$, we find that under transverse loading the change in bending moment between two cross sections a differential distance $dx$ apart is given by

$$dM_x = V_x dx$$

Therefore, the difference in the ordinates of the bending moment curve between any two points $a$ and $b$ is given by

$$\begin{aligned} M_b - M_a &= \int_{x_a}^{x_b} V_x dx \\ &= \text{area of shear diagram between } a \text{ and } b \end{aligned} \tag{3-9}$$

If there are external moments $\Sigma M$ acting between $a$ and $b$, then the result of the moment difference between the two points must include the effect due to these moments:

$$\begin{aligned} M_b - M_a &= \int_{x_a}^{x_b} V_x dx - \sum M \\ &= (\text{area of shear diagram between } a \text{ and } b) - \sum M \end{aligned} \tag{3-10}$$

in which $\Sigma M$ has been assumed to act in a counterclockwise direction.

**Example 3-6**

Consider the beam shown in Fig. 3-14(a). From $\Sigma M_b = 0$ and $\Sigma M_d = 0$, the support reactions are found to be

$$R_b = 22 \text{ kN} \qquad R_d = 14 \text{ kN}$$

We may now regard the beam as being in equilibrium under the balanced system

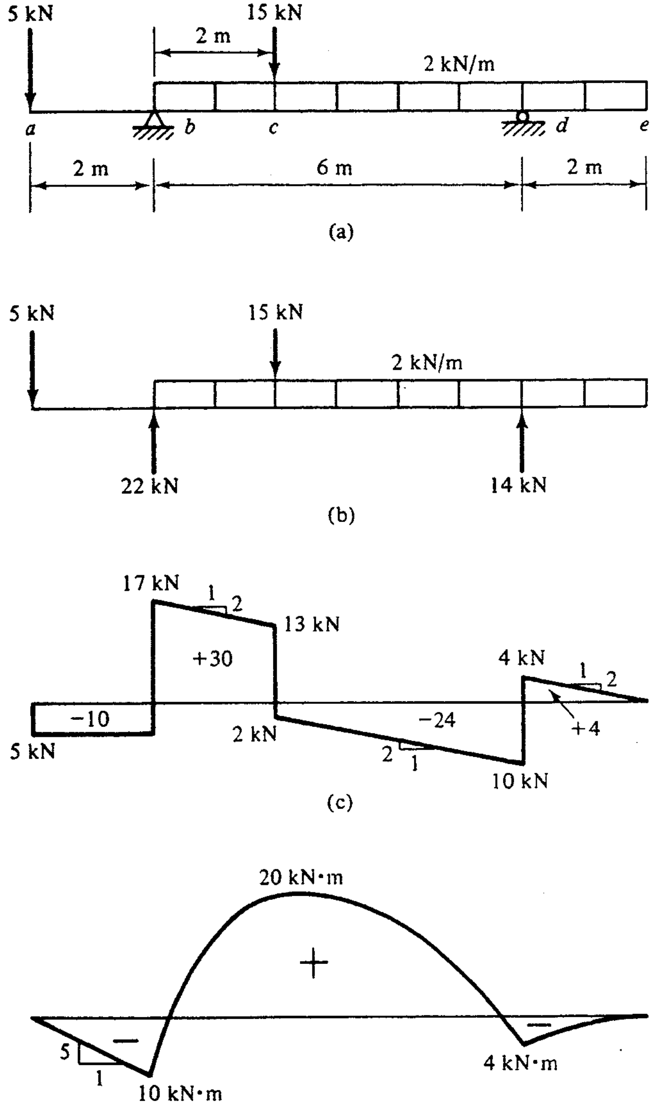


Fig. 3-14

of applied loads and reactions and present the load diagram as shown in Fig. 3-14(b).

A freehand sketch of the shear diagram can then be drawn, as in Fig. 3-14(c). In connection with this diagram, we note the following facts:

1. The shear at *a* goes from 0 dropping to −5 kN; also, the shear at *e* is 0. Recall that the shear curve always starts at zero and ends up at zero.
2. There will be constant shear in portion *ab* since it is not loaded. As a result, the shear curve in this portion is a horizontal line parallel to the beam axis.
3. Except abrupt changes in shear at *b*, *c*, and *d* corresponding to the concentrated forces acting at these points, the shear curves from *b* to *e* are sloping lines, the slope being given by

$$\frac{dV}{dx} = -w = -2$$

that is, 2:1 downward to the right, as indicated in Fig. 3-14(c).

A freehand sketch of a bending moment diagram can be drawn, as in Fig. 3-14(d). In connection with it, we note the following:

1. Moments at *a* and *e* are null. The moment curve from *a* to *b* is a sloping line with the slope given by

$$\frac{dM}{dx} = V = -5$$

that is, 5:1 downward to the right, as indicated in Fig. 3-14(d).

2. There are extreme values of moment at points *b*, *c*, and *d* where the shear curve goes through the *x* axis. Minimum bending moments occur at *b* and *d* since abrupt changes in the slope of moment curve from negative to positive take place at these points, corresponding to the abrupt changes in shear from negative to positive. Maximum bending moment occurs at *c*, where an abrupt change in the slope of the moment curve from positive to negative takes place, corresponding to the shear change from positive to negative at *c*.
3. Since the shear curve between *bc* or *cd* or *de* decreases from left to the right, the slope of the moment curve in each portion also decreases from left to right. This means that the moment curve in each portion is concave downward.
4. One way to obtain the ordinates of the moment diagram at *b*, *c*, and *d* is to compute the areas of the shear diagram [see the values indicated in Fig. 3-14(c)], from which we may find the moment difference between any two points:

$$M_b - M_a = -10\text{ kN}\cdot\text{m} \qquad M_c - M_b = 30\text{ kN}\cdot\text{m}$$

$$M_d - M_c = -24\text{ kN}\cdot\text{m} \qquad M_e - M_d = 4\text{ kN}\cdot\text{m}$$

From the above and using $M_a = M_e = 0$, we find that

$$M_b = -10\text{ kN}\cdot\text{m} \qquad M_c = 20\text{ kN}\cdot\text{m} \qquad M_d = -4\text{ kN}\cdot\text{m}$$

as indicated in Fig. 3-14(d).

The algebraic sum of the total area of the shear diagram for the beam is zero in this example, since $M_a = M_e = 0$ and there is no moment force acting between *a* and *e*.

## 3-4 ANALYSIS OF STATICALLY DETERMINATE TRUSSES

The method of joint and the method of section are the most fundamental tools in the analysis of trusses. These procedures may be explained by considering a specific example, such as the simple truss shown in Fig. 3-15.

**Method of joint.** The reactions

$$R_a = R_d = 12 \text{ kips}$$

are first obtained by taking the whole truss as a free body.

The two equations of equilibrium $\Sigma F_x = 0$ and $\Sigma F_y = 0$ are then applied to the *free body of each joint* in such an order that not more than two unknown forces are involved in each free body. This can always be done for a simple truss. In this example we start with joint $a$ at the left end and proceed in succession to joints $b$ and $B$; then we turn to joint $d$ at the right end and proceed

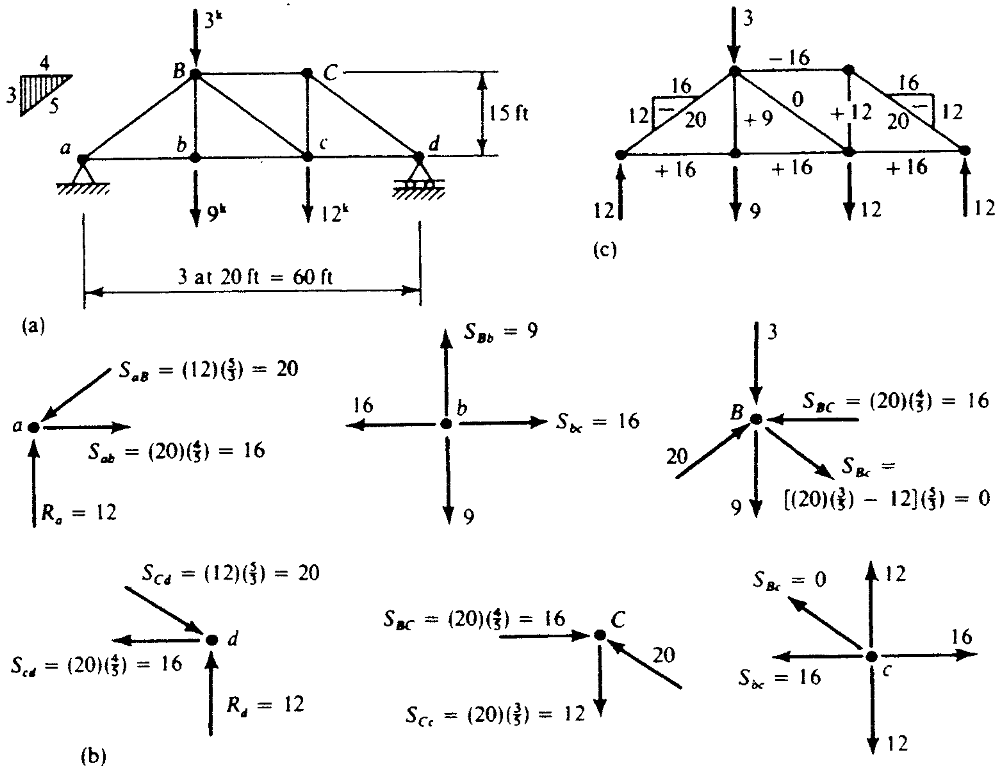


Fig. 3-15

to joints $C$ and $c$. We thus provide three checks for the analysis by obtaining the internal forces in members $BC$, $Bc$, and $bc$ from two directions.

The analysis for each joint is given briefly in Fig. 3-15(b). Usually, when the slopes of the members are in simple ratios, the solution for unknown forces can readily be obtained by inspection rather than by using equations. The arrows in each free body of the joint indicate the directions of the member forces acting on the joint, not the actions of the joint on the member. Note that the internal force in the member is a tensile force if it acts outward such as $S_{ab}$ and that the internal force in the member is a compressive force if it acts toward the joint such as $S_{aB}$.

The *answer diagram* [Fig. 3-15(c)] gives the results obtained from the preceding analysis together with the horizontal and vertical components. A plus sign indicates a tensile force, and a minus sign indicates a compressive force.

**Method of section.** Sometimes when only the forces in certain members are desired or when the method of joint is less convenient for solving forces, it is expedient to use the method of section, which involves isolating a portion of the truss by cutting certain members and then solving the forces on these members with the equilibrium equations. Consider the truss in Fig. 3-15(a). Let us determine the internal forces in the members $BC$, $Bc$, and $bc$.

We start by passing a section $m$–$m$ through these members and treating either side of the truss as a free body (see Fig. 3-16). Note that the sense of the unknown force in each cut member is assumed to be tensile and if this is done, a plus sign in the answer indicates that the assumed sense is correct, and therefore tension; whereas a minus sign indicates that the assumed sense is incorrect, and therefore compression.

Since in each free body only three unknown forces are involved, the unknowns can be solved by three equilibrium equations. In this example, it is convenient

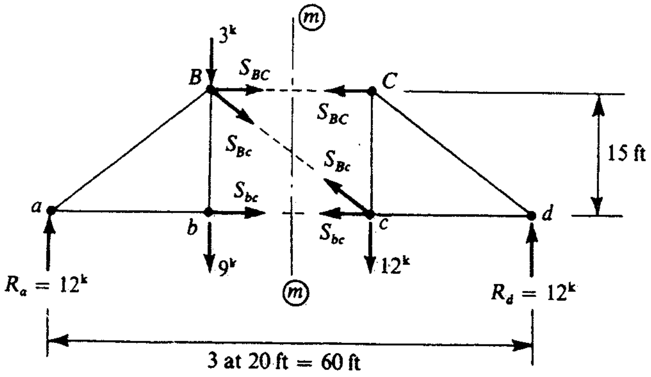


Fig. 3-16

to solve $S_{BC}$ by $\Sigma M_c = 0$; $S_{bc}$ by $\Sigma M_B = 0$; and $S_{Bc}$ by $\Sigma F_y = 0$. Thus, if we consider the left portion of Fig. 3-16 as a free body, we have

$$\sum M_c = 0 \qquad (12)(40) - (9 + 3)(20) + 15S_{BC} = 0$$

$$S_{BC} = -16 \text{ kips} \qquad \text{(compression)}$$

$$\sum M_B = 0 \qquad (12)(20) - 15S_{bc} = 0$$

$$S_{bc} = +16 \text{ kips} \qquad \text{(tension)}$$

$$\sum F_y = 0 \qquad 12 - (9 + 3) - V_{Bc} = 0$$

$$V_{Bc} = 0 \text{ or } S_{Bc} = 0$$

in which $V_{Bc}$ represents the vertical component of $S_{Bc}$. Since $S_{Bc} = (5/3)V_{Bc}$, the zero value of $V_{Bc}$ evidently implies the nonexistence of $S_{Bc}$.

In applying the method of section, we note that by proper choice of moment centers we can often determine the forces on certain members, such as the members $BC$ and $bc$ of Fig. 3-16, directly from the moment equations and avoid solving simultaneous equations. This technique is called the *method of moment* and can best be illustrated in the following example.

**Example 3-7**

In Fig. 3-17(a) is shown a simple nonparallel chord truss. Find the forces in chord members $cd$ and $CD$ and in the diagonal $Cd$.

First, from $\Sigma M_i = 0$ for the entire structure, the reaction at $a$ is found to be

$$R_a = \frac{(5)(60) + (4 + 3 + 2 + 1)(90)}{8} = 150 \text{ kips}$$

Next, to find the internal force in member $cd$, we pass a section $m$–$m$ through members $CD$, $Cd$, and $cd$, as indicated by the dashed line in Fig. 3-17(a), and take the left portion of the truss as a free body, as shown in Fig. 3-17(b).

From $\Sigma M_C = 0$,

$$S_{cd} = \frac{(150)(50)}{30} = 250 \text{ kips} \qquad \text{(tension)}$$

To find the internal force in member $CD$, we use the same free body and resolve $S_{CD}$ into a vertical component $V_{CD}$ and a horizontal component $H_{CD}$ at $D$, as shown in Fig. 3-17(c).

From $\Sigma M_d = 0$,

$$H_{CD} = -\frac{(150)(75)}{33} = -341 \text{ kips} \qquad \text{(compression)}$$

Thus,
$$S_{CD} = (-341)\left(\frac{25.2}{25}\right) = -344 \text{ kips} \qquad \text{(compression)}$$

Similarly, to find the internal force in member $Cd$, we resolve $S_{Cd}$ into a vertical component $V_{Cd}$ and a horizontal component $H_{Cd}$ at $d$, as shown in Fig. 3-17(d). Note that the moment center is chosen at $o$, where the extending lines of members $CD$ and $cd$ intersect. The distance $oa$ is found to be 200 ft.

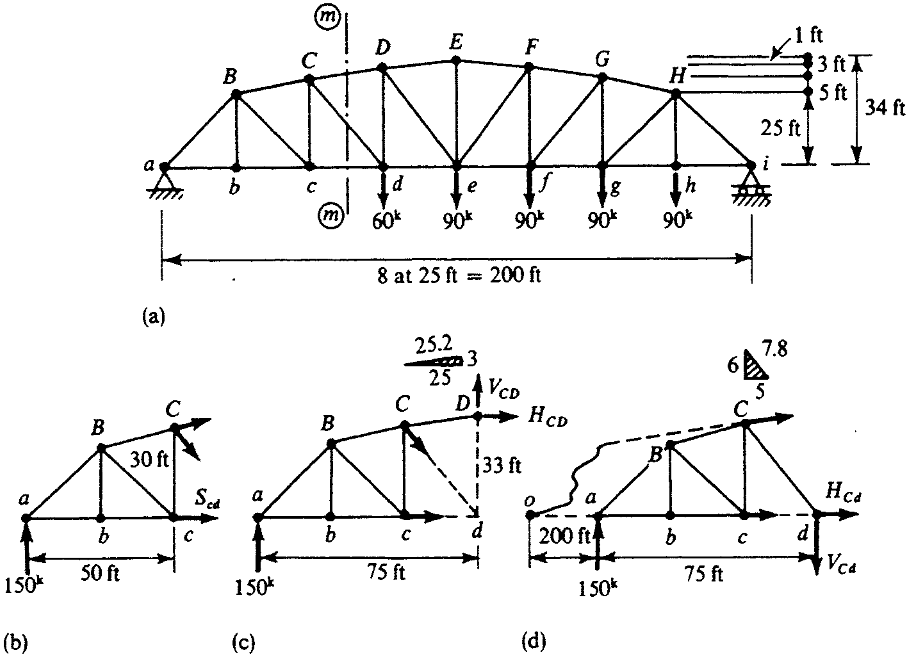


Fig. 3-17

From $\Sigma M_o = 0$,

$$V_{Cd} = \frac{(150)(200)}{275} = 109 \text{ kips} \qquad \text{(tension)}$$

Thus,

$$S_{Cd} = (109)\left(\frac{7.8}{6}\right) = 142 \text{ kips} \qquad \text{(tension)}$$

In general, no truss is analyzed by one method alone. Instead, it is often analyzed by a mixed method based on knowledge from both the joint and section methods combined as illustrated in the following example.

**Example 3-8**

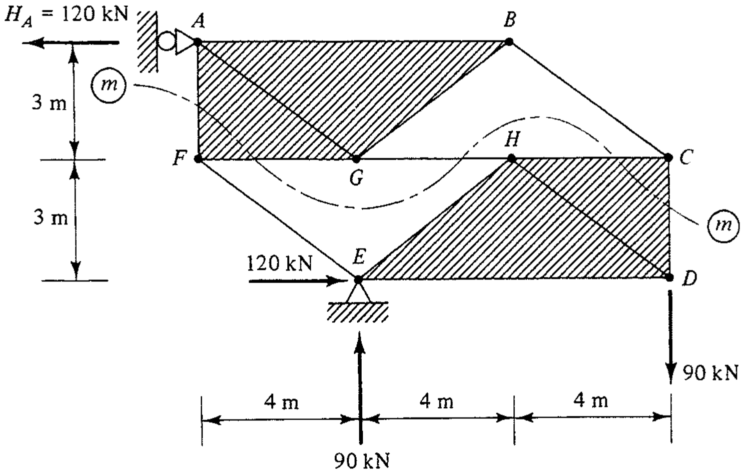
In Fig. 3-18(a) we have a compound truss consisting of two simple trusses (shaded) connected by three bars, *BC*, *EF*, and *GH*. The truss is subjected to a vertical load of 90 kN at joint *D*.

The first step in the analysis is to obtain the reactions at *A* and *E* by considering the entire truss as a free body. Thus,

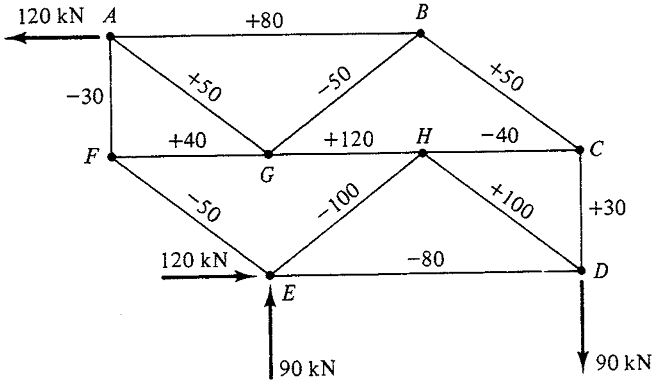
$$H_A = 120 \text{ kN} \qquad H_E = 120 \text{ kN} \qquad V_E = 90 \text{ kN}$$

as indicated.

After this the method of joint fails, since each remaining joint involves more than two unknowns. It also appears at first glance that it is not possible to apply



(a)



(b)

**Fig. 3-18**

the method of section, since we cannot take any section that cuts only three bars that are not concurrent. However, if we pass section *m–m* through five bars, as indicated in Fig. 3-18(a), we can easily obtain from $\Sigma M_C = 0$ that

$$S_{AF} = -\frac{(120)(3)}{12} = -30 \text{ kN}$$

by taking either portion of the truss as a free body and by assuming that $S_{AF}$ acts in a positive direction. Similarly, from $\Sigma M_F = 0$ or $\Sigma F_y = 0$, we obtain that

$$S_{CD} = 30 \text{ kN}$$

Having done this, we can solve the forces for the remaining bars by the method of joint without difficulty. An answer diagram for the analysis is given in Fig. 3-18(b).

In analyzing a complex truss, we frequently find that the method of joint and the method of section, described in previous sections, are not directly applicable. For example, let us consider the loaded complex truss shown in Fig. 3-19(a). After the reactions at $A$ and $E$ are found, we observe that no further progress can be made by either the method of joint or the method of section. One way to handle this is to substitute for the bar $AD$ a bar $AC$ and thus obtain a stable simple truss, as in Fig. 3-19(b), which can be completely analyzed by the method of joint for the given loading. Next, let the same simple truss be loaded with two equal and opposite forces $X$ at $A$ and $D$ representing the internal force in bar $AD$, as shown in Fig. 3-19(c). Again a complete analysis can be carried out by the method of joint such that the internal force for each member will be expressed in terms of the unknown $X$. Or for convenience, we put a pair of unit forces in place of the $X$'s, as given in Fig. 3-19(d). It is apparent that the bar forces obtained from Fig. 3-19(d) times $X$ will give those of Fig. 3-19(c).

Now the analysis of Fig. 3-19(a) can be made equivalent to the superposing effects of Fig. 3-19(b) and (c) if we let the bar force of $AC$ obtained from (b) and (c), or from (b) and (d) times $X$, be equal to zero. Thus, if we let $S_i'$ denote the force in any bar of (b) and $\delta_i$ the corresponding bar force of (d), then the corresponding internal force $S_i$ in any bar of (a) is expressed by

$$S_i = S_i' + \delta_i X$$

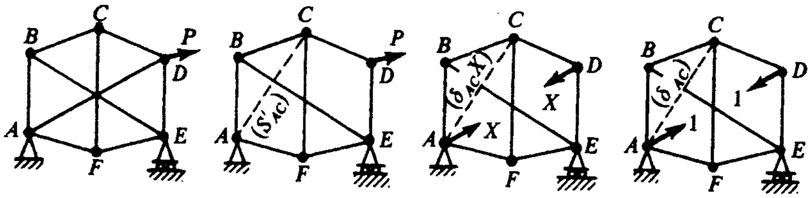
in which $X$ is solved by

$$S_{AC} = S_{AC}' + \delta_{AC} X = 0$$

or

$$X = -\frac{S_{AC}'}{\delta_{AC}}$$

With the value of $X$ determined, the force in any other bar of the given truss can be obtained without difficulty.



(a) (b) (c) (d)

Fig. 3-19

It should be noted that complex trusses may often be arranged so as to be geometrically unstable. However, it is not always possible to see a critical form just by inspection. Detection is based on the principle that if the analysis for the truss yields a unique solution then the truss is stable and statically determinate; on the other hand, if the analysis fails to yield a unique solution, then the truss has a critical form.

## 3-5 A GENERAL METHOD FOR ANALYZING STATICALLY DETERMINATE TRUSSES

Theoretically, we can always solve any statically stable and determinate truss by $2j$ simultaneous equilibrium equations for $j$ joints of the system. The method is perfectly general but must be done with the aid of a modern computer.

As a simple illustration of this process, let us consider the three-hinged truss shown in Fig. 3-20. The unknown elements involved in this truss are the reaction components $H_A$ and $V_A$ at joint $A$, $H_B$ and $V_B$ at joint $B$, and the bar forces $S_a$ and $S_b$. The six unknowns can be solved by six equilibrium equations, two for each of three discrete joints. Thus,

Joint $A$:

$$\sum F_x = 0 \qquad H_A + 0.6S_a = 0$$
$$\sum F_y = 0 \qquad V_A + 0.8S_a = 0$$

Joint $B$:

$$\sum F_x = 0 \qquad H_B - 0.8S_b = 0$$
$$\sum F_y = 0 \qquad V_B + 0.6S_b = 0$$

Joint $C$:

$$\sum F_x = 0 \qquad 0.6S_a - 0.8S_b = 0$$
$$\sum F_y = 0 \qquad 0.8S_a + 0.6S_b = -10$$

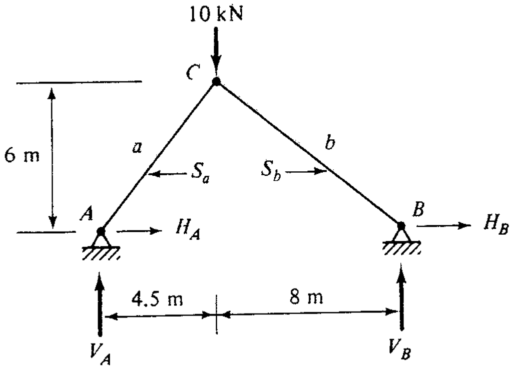


Fig. 3-20

Collecting the preceding six equations in matrix form gives

$$\underset{D}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 1 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 1 & 0 & 0 & -0.8 \\ 0 & 0 & 0 & 1 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0 & 0 & 0.8 & 0.6 \end{bmatrix}} \underset{X}{\begin{Bmatrix} H_A \\ V_A \\ H_B \\ V_B \\ S_a \\ S_b \end{Bmatrix}} = \underset{E}{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \end{Bmatrix}}$$

or

$$DX = E$$

The unknown values are found by

$$X = D^{-1}E$$

The problem is now to find $D^{-1}$. In this case,

$$D^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & -0.36 & -0.48 \\ 0 & 1 & 0 & 0 & -0.48 & -0.64 \\ 0 & 0 & 1 & 0 & -0.64 & 0.48 \\ 0 & 0 & 0 & 1 & 0.48 & -0.36 \\ 0 & 0 & 0 & 0 & 0.6 & 0.8 \\ 0 & 0 & 0 & 0 & -0.8 & 0.6 \end{bmatrix}$$

After the performance of matrix multiplication, we obtain

$$X = \begin{Bmatrix} H_A \\ V_A \\ H_B \\ V_B \\ S_a \\ S_b \end{Bmatrix} = \begin{Bmatrix} 4.8 \\ 6.4 \\ -4.8 \\ 3.6 \\ -8 \\ -6 \end{Bmatrix} \text{kN}$$

It should be noted that the unknowns $X$ are to be uniquely determined by $X = D^{-1}E$ under the condition of the nonsingularity of the square matrix $D$; that is, the determinant containing the same elements as $D$ is not zero. This principle provides a general way of detecting the stability of a structural system; that is, if

$$|D| \neq 0$$

then the system has a unique solution, which indicates that the system is stable; on the other hand, if

$$|D| = 0$$

then the system is unstable, since many solutions are possible. Refer to Fig.

3-20. If we replace the hinge support at $B$ with a roller ($H_B = 0$), then

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 1 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.8 \\ 0 & 0 & 0 & 1 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0.6 & -0.8 \\ 0 & 0 & 0 & 0 & 0.8 & 0.6 \end{bmatrix}$$

Apparently,

$$|D| = 0$$

implying that the truss is unstable.

Although the illustration we have considered concerns a simple truss, the principle and the method of detecting a critical form described above can be applied to other types of structures, such as beams and rigid frames.

## 3-6 DESCRIPTION OF BRIDGE AND ROOF TRUSS FRAMEWORKS

Figure 3-21 shows a typical *through trussed bridge*. The word *through* indicates that the trains (or vehicles) actually travel through the bridge. If the bridge is installed under the floor or deck, then the bridge is called a *deck bridge*. If the

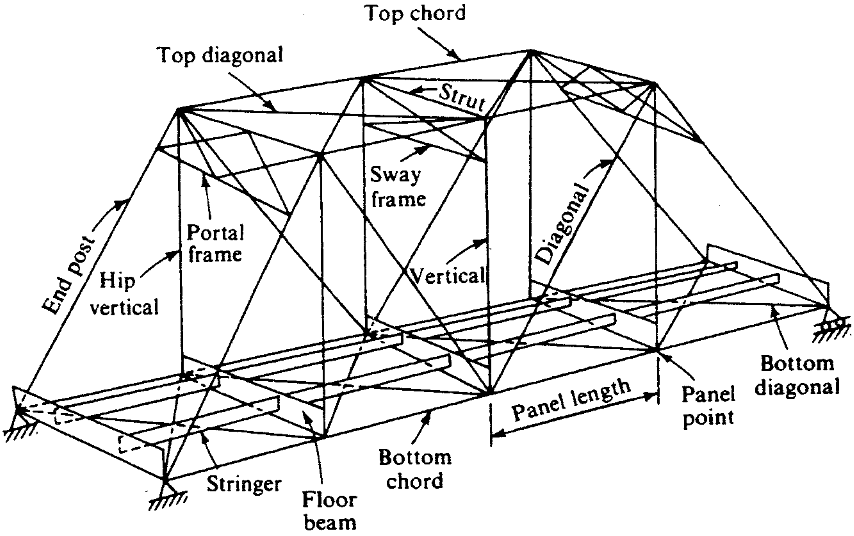


Fig. 3-21

trains pass between trusses but the depth is insufficient to allow the use of a top chord bracing system, the bridge is called *half-through*.

Referring to Fig. 3-21, we place the road surface (or the rail and tie system in railways) on the short longitudinal beams called *stringers,* assumed simply supported on the *floor beams* which in turn are supported by the two *main trusses*. The moving loads on bridge are transmitted to the main trusses through the system of the connection of road surface (or rail and tie), stringer, and floor beam.

The top series of truss members parallel to the stringer are called *top chords;* while the corresponding bottom series of members are called *bottom chords*. The members connecting the top and bottom chords form the web system and are referred to as *diagonals* and *verticals*. The end diagonals are called *end posts,* and the side verticals are called *hip verticals*. The point at which web members connect to a chord is called a *panel point,* and the length between two adjacent panel points on the same chord is called the *panel length*.

The cross struts at corresponding top-chord panel points, together with the top diagonals connecting the adjacent struts, make up the top-chord lateral system. The bottom-chord lateral system is composed of the floor beams and the bottom diagonals connecting the adjacent floor beams.

The two main trusses are also cross braced at each top-chord panel point by *sway frames*. The frame in the plane of each pair of end posts is called a *portal frame*.

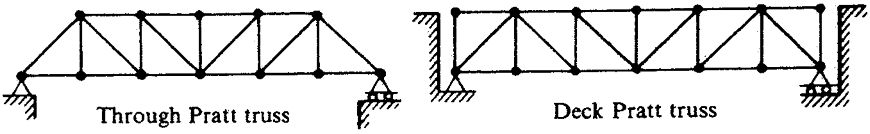
The members of a main truss may be arranged in many different ways. However, the principal types of trusses encountered in bridges are shown in Fig. 3-22. Among these types, the Pratt, Howe, and Warren trusses are more commonly used. We may note that in the Pratt truss the diagonals, except the end posts, are stressed in tension and that the verticals, except the hip verticals, are stressed in compression under dead load. On the other hand, in a Howe truss the diagonals are in compression and the verticals are in tension. Note also that, of all the trusses shown in Fig. 3-22 under dead load, the upper chords are in compression and bottom chords in tension.

A typical roof truss framework supported by columns is shown in Fig. 3-23.

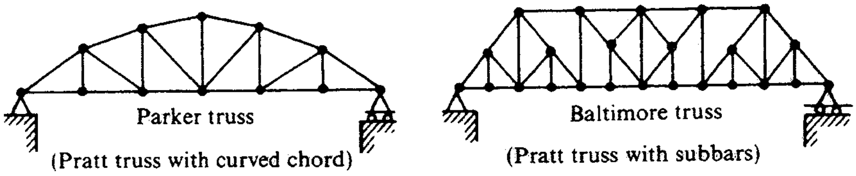
A roof truss with its supporting columns is called a *bent*. The space between adjacent bents is called a *bay*. *Purlins* are longitudinal beams that rest on the top chord and preferably at the joints of the truss, in accordance with the definition of truss. The *roof covering* may be laid directly on the purlins for very short bay lengths but usually is laid on the wood *sheathings* that, in turn, rest either on the purlins or on the *rafters* (if provided). Rafters are the sloping beams extending from the *ridge* to the *eaves* and are supported by the purlins.

For a symmetrical roof truss the ratio of its *rise,* center height, to its *span,* the horizontal distance between the center lines of the supports, is called *pitch*.

The truss consists of top-chord, bottom-chord, and *web members*. Although


Through Pratt truss

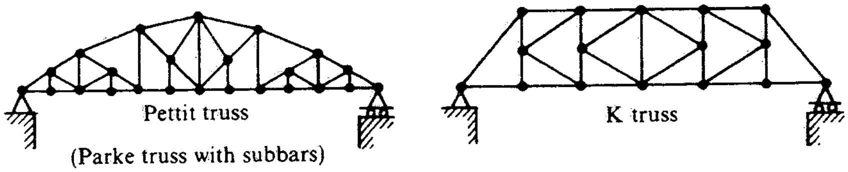
Deck Pratt truss


Parker truss

(Pratt truss with curved chord)

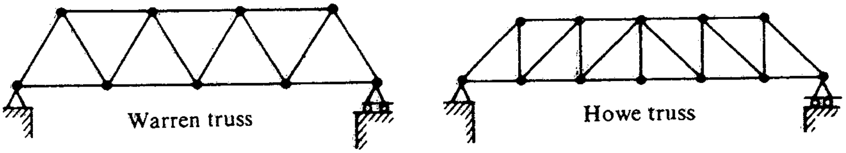
Baltimore truss

(Pratt truss with subbars)


Pettit truss

(Parke truss with subbars)

K truss


Warren truss

Howe truss

Fig. 3-22

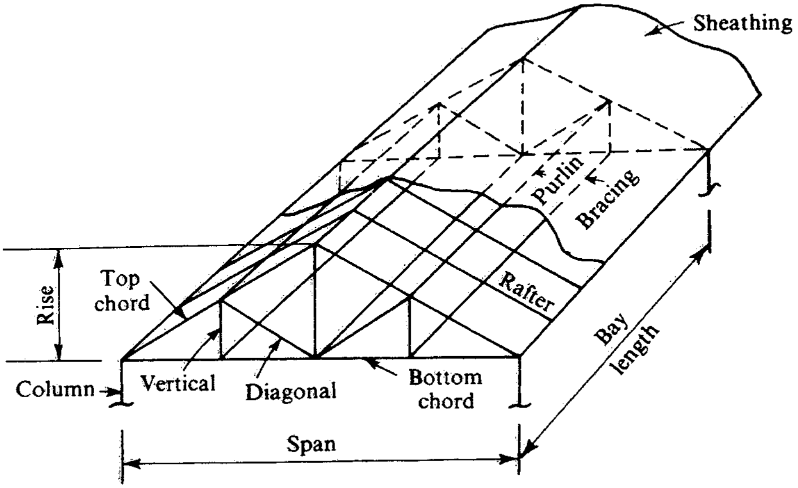


Fig. 3-23

the purlins act to strengthen the longitudinal stability, additional bracing is always necessary. The *bracing members* may run from truss to truss longitudinally or diagonally and may be installed in the plane of the bottom chord, the top chord, or both. Surface loads are transmitted from covering, sheathing, rafter, purlin and distributed to adjacent trusses.

The common types of roof trusses are shown in Fig. 3-24.

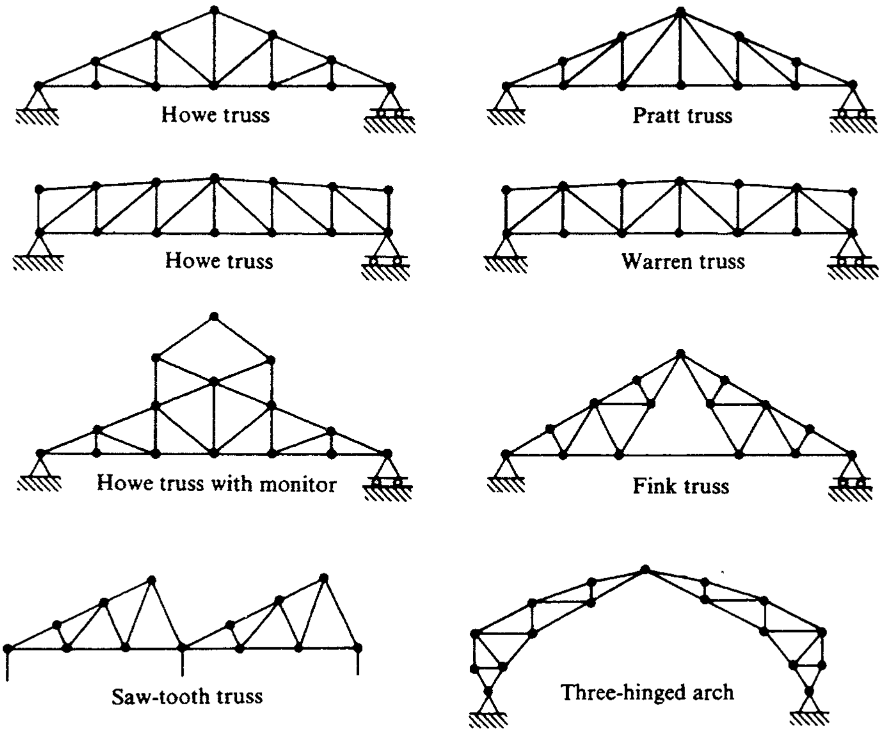


Fig. 3-24

## 3-7 ANALYSIS OF STATICALLY DETERMINATE RIGID FRAMES

To analyze a statically determinate rigid frame, we start by finding the reaction components from statical equations for the entire structure. This done, we are able to determine the shear, moment, and axial force at any cross section of the frame by taking a free body cut through that section and by using the equilibrium equations. Based on the centroidal axis of each member, we can plot the shear, bending moment, and the direct force diagrams for the rigid frame. However, it is the bending moment diagram with which we are mainly concerned in the analysis of a rigid frame.

The following numerical examples will serve to illustrate the procedure.

**Example 3-9**

Analyze the rigid frame in Fig. 3-25(a). Let $H_a$, $V_a$, and $M_a$ denote the horizontal, vertical, and rotational reaction components, respectively, at support $a$, and let $V_e$ be the vertical reaction at support $e$.

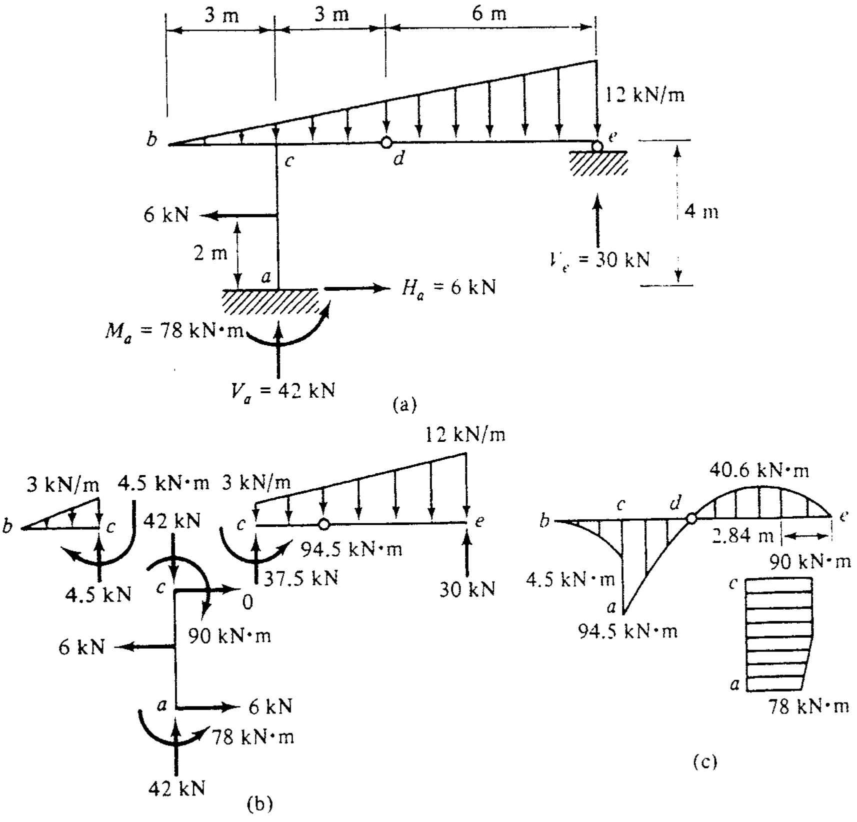


Fig. 3-25

From the condition equation $M_d = 0$, we find that $V_e = 30$ kN; from $\Sigma F_x = 0$, $H_a = 6$ kN; from $\Sigma F_y = 0$, $V_a = 42$ kN; and from $\Sigma M_a = 0$, $M_a = 78$ kN · m, as indicated in Fig. 3-25(a).

After all the external forces acting on the rigid frames are determined, the internal forces at each end of the members can easily be obtained by taking each member as a free body [Fig. 3-25(b)]. Take member *ac*, for instance. At end *c* of *ac*, we find the shear force equal to zero by applying $\Sigma F_x = 0$; axial force equal to 42 kN (down) from $\Sigma F_y = 0$; and the resisting moment equal to 90 kN · m (clockwise) from $\Sigma M_c = 0$. By inspection, we see that the shear and moment at end *c* of the overhanging portion *ac* are 4.5 kN (up) and 4.5 kN · m (clockwise), respectively. Finally, we use the equilibrium of joint *c* to obtain the end forces at *c* of member *ce* as

shear = 37.5 kN (up) moment = 94.5 kN · m (counterclockwise)

With all the end forces for each member found, the shear, bending moment, and axial force in any section of the frame can be obtained by simple statics.

The moment diagrams for beam *be* and column *ac* are shown separately in Fig. 3-25(c).

**Example 3-10**

Analyze the simply supported gable frame shown in Fig. 3-26(a), which is composed of two columns and two sloping members.

From $\Sigma F_x = 0$, $\Sigma M_e = 0$, and $\Sigma F_y = 0$ for the entire frame, the reaction elements are found to be

$$H_a = 8 \text{ kips} \qquad V_a = 11 \text{ kips} \qquad V_e = 21 \text{ kips}$$

as shown in Fig. 3-26(a).

Next, we take member *ab* as a free body. With the end forces known at *a*, we can readily obtain those at the other end *b* from the equilibrium conditions:

shear = 8 kips moment = 80 ft-kips axial force = 11 kips

acting as indicated in Fig. 3-26(b).

Following this, we sketch the free-body diagram for joint *b*, as shown in Fig. 3-26(c). Note that the joint is shown in an exaggerated manner, since theoretically it should be represented by a point and all forces acting on the joint should be concurrent at this point.

Next, let us take member *bc* as a free body subjected to the external load of 2 kips per horizontal unit length. With the internal forces known at end *b*, we can apply $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M_c = 0$ to obtain the internal forces at end *c* as

horizontal force = 0 vertical force = 5 kips moment = 104 ft-kips

These act as indicated in the upper sketch of Fig. 3-26(d). To determine the forces in each section of the member, we resolve all the indicated forces into components normal and tangential to the member section, as shown in the lower sketch of Fig. 3-26(d). For instance, at end *b* we have

$$\text{normal force (axial force)} = (11)(\tfrac{3}{5}) = 6.6 \text{ kips}$$

$$\text{tangential force (shear)} = (11)(\tfrac{4}{5}) = 8.8 \text{ kips}$$

Similarly, at end *c* we have

$$\text{normal force (axial force)} = (5)(\tfrac{3}{5}) = 3 \text{ kips}$$

$$\text{tangential force (shear)} = (5)(\tfrac{4}{5}) = 4 \text{ kips}$$

The total uniform load on member *bc* is 16 kips, of which there are

$(16)(\tfrac{4}{5}) = 12.8$ kips acting transversely to the member axis

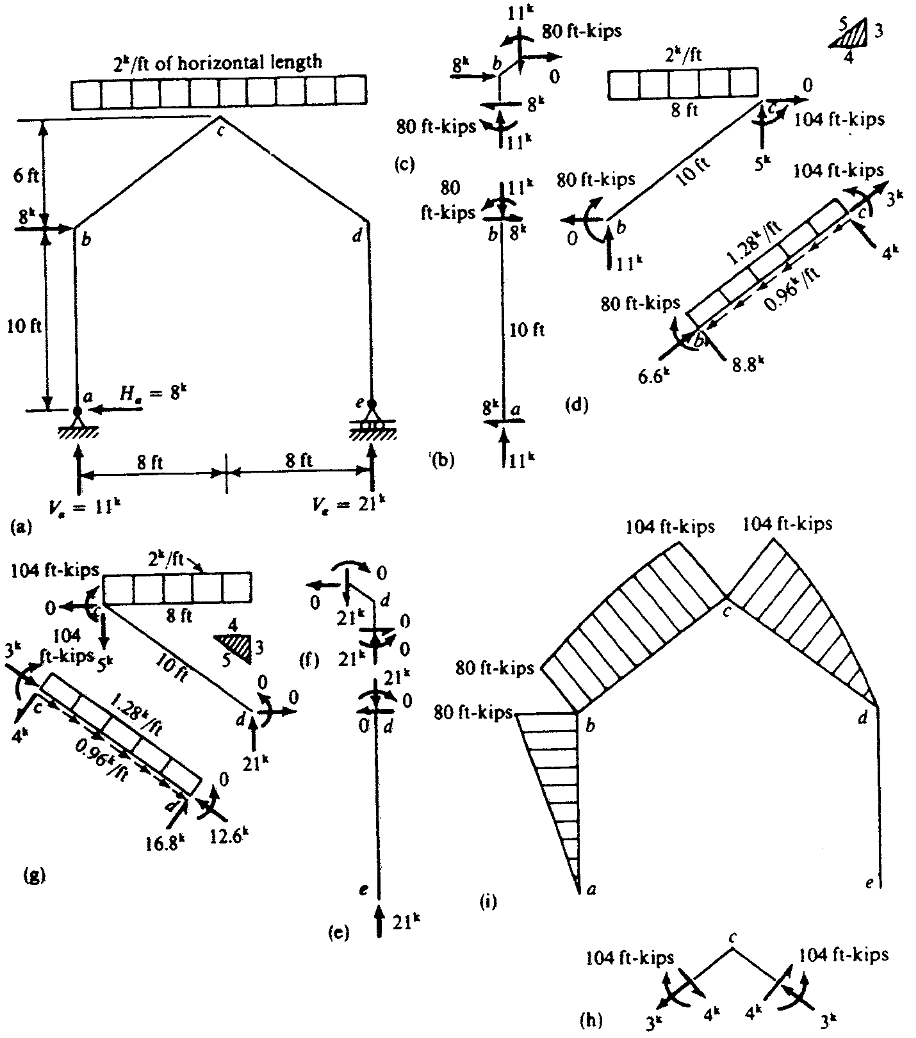


Fig. 3-26

$$(16)(\tfrac{3}{5}) = 9.6 \text{ kips acting axially to the member axis}$$

thus giving a uniform load of intensity:

$$\frac{12.8}{10} = 1.28 \text{ kips/ft acting transversely to the member axis}$$

$$\frac{9.6}{10} = 0.96 \text{ kip/ft acting axially to the member axis}$$

With these determined, the shear, bending moment, and direct force in any section of member *bc* can readily be obtained, as shown in Fig. 3-27.

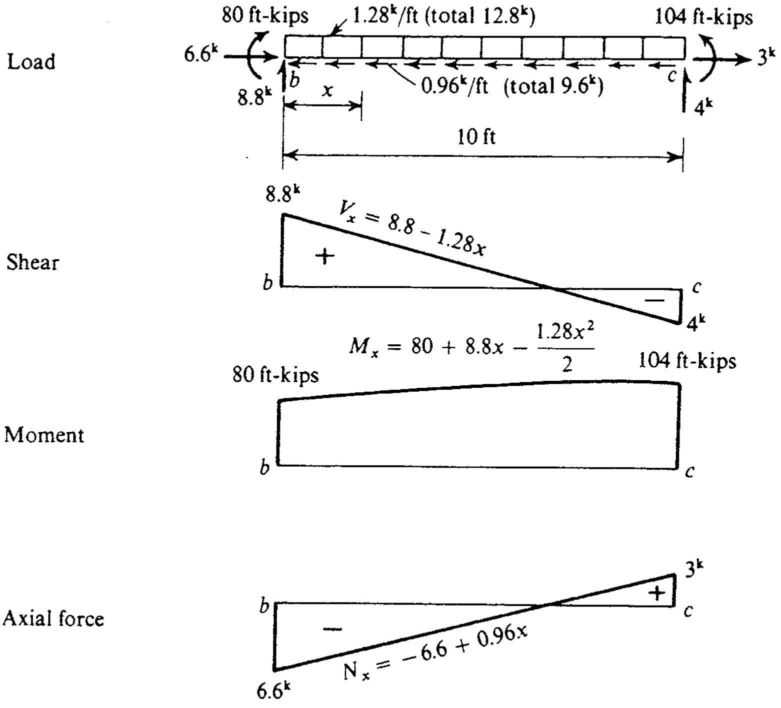


**Fig. 3-27**

In this manner we may proceed from member *bc* to joint *c*, then to member *cd* and joint *d*, and finally to member *de*. However, it seems more convenient to analyze *de* now and then to turn to joint *d* and member *cd*, and to leave the joint *c* as a final check, as shown in Fig. 3-26(e), (f), (g), and (h), respectively. The bending moment diagram for the whole frame is plotted in Fig. 3-26(i).

**Example 3-11**

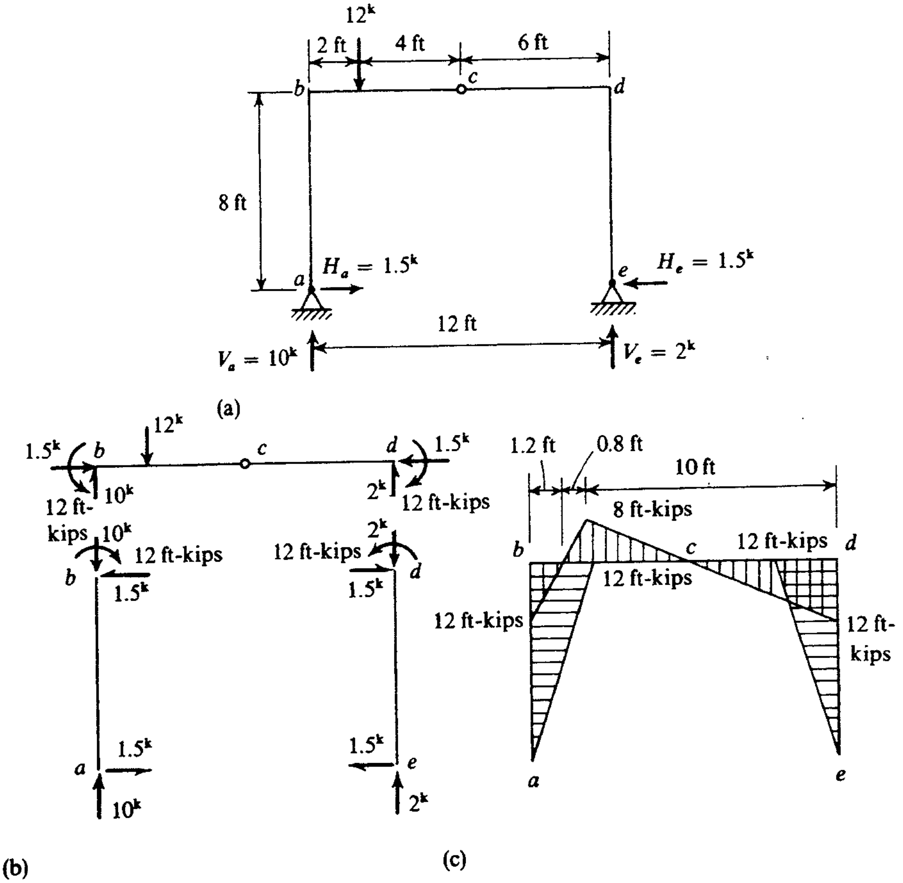
Consider the three-hinged frame loaded as in Fig. 3-28(a). The four reaction elements at supports *a* and *e* are first obtained by solving simultaneous equations, three from equilibrium and one from construction.

$$
\begin{aligned}
\sum F_x &= 0 \qquad H_a - H_e = 0 \\
\sum F_y &= 0 \qquad V_a + V_e - 12 = 0 \\
\sum M_e &= 0 \qquad 12V_a - (12)(10) = 0 \\
M_c &= 0 \qquad 6V_e - 8H_e = 0
\end{aligned}
$$

which give

$$V_a = 10\text{ kips} \qquad V_e = 2\text{ kips} \qquad H_a = H_e = 1.5\text{ kips}$$

The free-body diagrams for members *ab*, *bd*, *de* are then drawn as in Fig. 3-28(b). From these we plot the moment diagram for the frame, as shown in Fig. 3-28(c).



(a)

(b)

(c)

Fig. 3-28

It is interesting to note that in this particular case the portion to the right of hinge *c* (i.e., *cde*) carries no external load and is therefore a two-force member if isolated. The line of reaction at support *e*, called $R_e$, must be through points *e* and *c* and must meet the action line of the applied load at some point *o*, as shown in Fig. 3-29(a). Now if we take the whole frame as a free body, we see that the system constitutes a three-force member subjected to the applied load and support reactions.

Thus, the line of reaction at support *a*, called $R_a$, must be through points *a* and *o* so that the three forces are concurrent at point *o* as required by equilibrium. The vectors $R_a$ and $R_e$ can then be easily determined by the equilibrium triangle, as shown in Fig. 3-29(b).

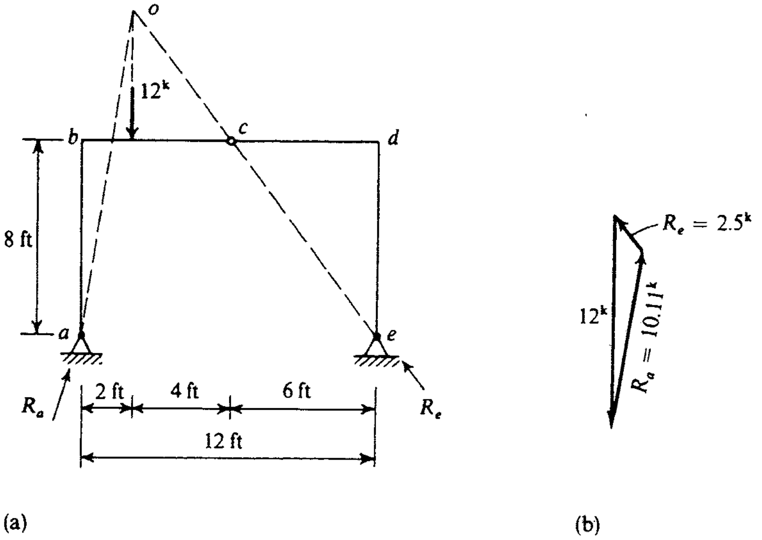


Fig. 3-29

In the case where loads are placed both to the left and to the right of the connecting hinge of a three-hinged frame, one way to analyze this is to use the method of superposition. This is illustrated in Fig. 3-30, in which the case shown in part (a) can be made equivalent to the sum of effects of (b) and (c), each analyzed by the method discussed previously.

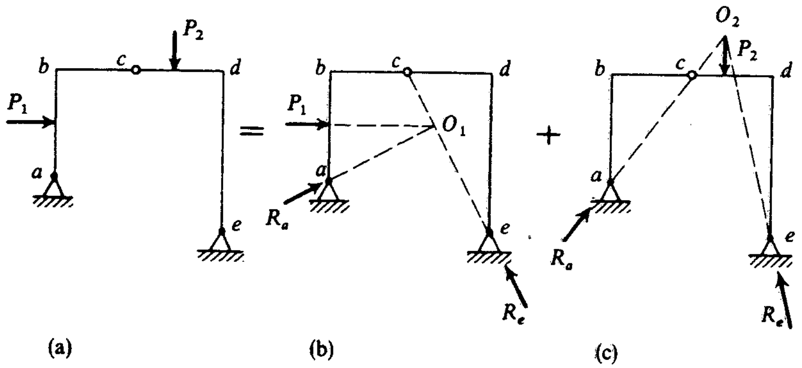
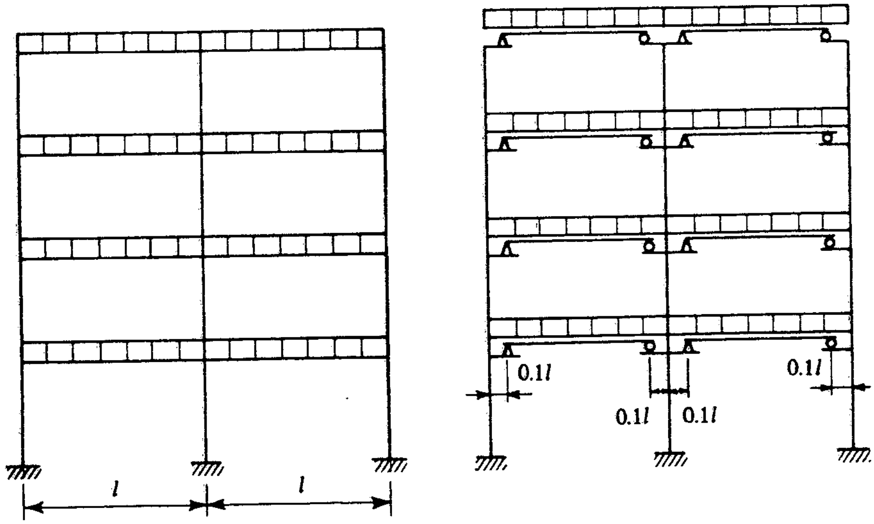


Fig. 3-30

## 3-8 APPROXIMATE ANALYSIS FOR STATICALLY INDETERMINATE RIGID FRAMES

As previously mentioned, the rigid frames of present-day construction are highly indeterminate. It will be seen in the later chapters, which deal with statically indeterminate structures, that to obtain the solution for a building frame based on more exact analyses is often tedious and time consuming. In many cases, we cannot obtain the solution without the aid of modern electronic computers. For this reason empirical rules and approximate methods were often used in the past by structural and architectural engineers in designing various kinds of indeterminate structures. In order to do this, as many independent equations of statics as there are independent unknowns must be available. The additional equations of statics are worked out by reasonable assumptions based on experience and knowledge of the more exact analyses. Even today the approximate methods are still useful in a preliminary design and cost estimation.

To illustrate, consider a frame subjected to uniform floor loads, such as the one shown in Fig. 3-31(a). The frame is indeterminate to the 24th degree since eight cuts in the girders would render the frame into three stable and determinate parts and since each cut involves the removal of three elements of restraint (i.e., bending moment, shear, and axial force). A preliminary survey of stresses may be performed by assuming the following so that the indeterminate



(a) (b)

Fig. 3-31

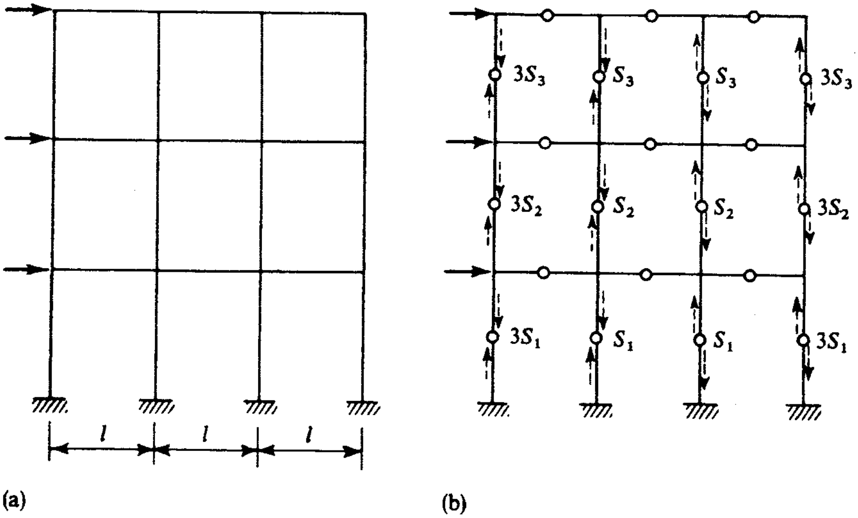
frame can be solved by a determinate approach, that is, by equations of statics alone:

1. The axial force in each girder is small and can be neglected.
2. A point of inflection (zero moment) occurs in each girder at a point one-tenth of the span length from the left end of the girder.
3. A point of inflection occurs in each girder at a point one-tenth span length from the right end of the girder.

This would render the frame equivalent to the one shown in Fig. 3-31(b), which is statically determinate.

Another case that may also be worth brief mention, without going into details, is the approximate analysis for wind stresses in building frames. Consider a frame subjected to lateral forces (equivalent wind) acting at the joints such as the one shown in Fig. 3-32(a). The frame is statically indeterminate to the 27th degree. There are several methods available for dealing with the problem. The method chosen to illustrate this is called the *cantilever method* and is based on the following assumptions:

1. A point of inflection exists at the center of each girder.
2. A point of inflection exists at the center of each column.
3. The unit axial stresses in the columns of a story vary as the horizontal distances of the columns from the center of gravity of the bent. It is usually further assumed that all columns are identical in a story, so that the axial forces



(a) (b)

Fig. 3-32

of the columns in a story will vary in proportion to the distances from the center gravity of the bent.

This would lead the frame to appear as the form shown in Fig. 3-32(b). Note that the last assumption virtually puts the column axial forces in one story in terms of a single unknown [see the dashed arrows in Fig. 3-32(b)]. It is therefore equivalent to making ($n$ − 1) additional assumptions for each story, $n$ being the number of columns in one story. In this case, there are three for each story, or nine in total regarding column axial forces. As a result, the total number of additional equations is 30 (9 from column axial forces and 21 from inserting pins), which is three more than are necessary. However, it happens that a statical analysis for the frame can be carried out without inconsistency on the basis of the foregoing assumptions.

## PROBLEMS

**3-1.** In each part of Fig. 3-33 qualitative loadings are shown. Draw the shear and moment diagrams consistent with these loadings; give the equation for each curve.

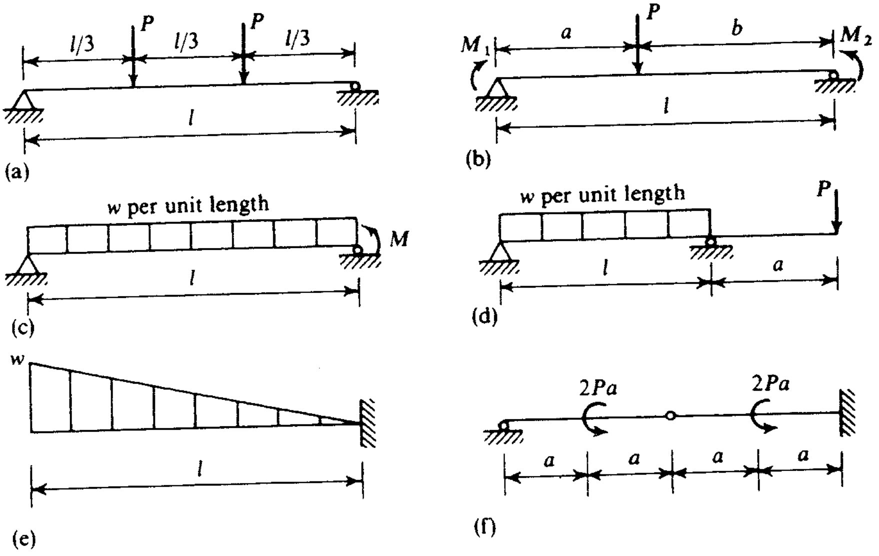


Fig. 3-33

**3.2.** Sketch the shear and moment diagrams for each of the loaded beams shown in Fig. 3-34. Use the relationships between load, shear, and moment.

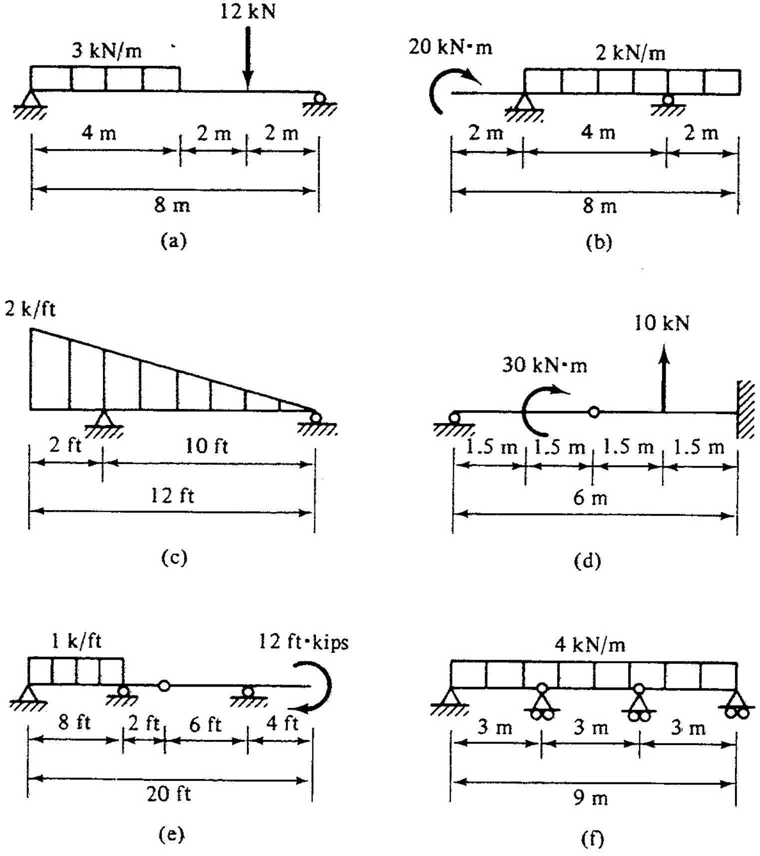


Fig. 3-34

**3-3.** Determine the bar force in each member of the trusses shown in Fig. 3-35 by the method of joint.

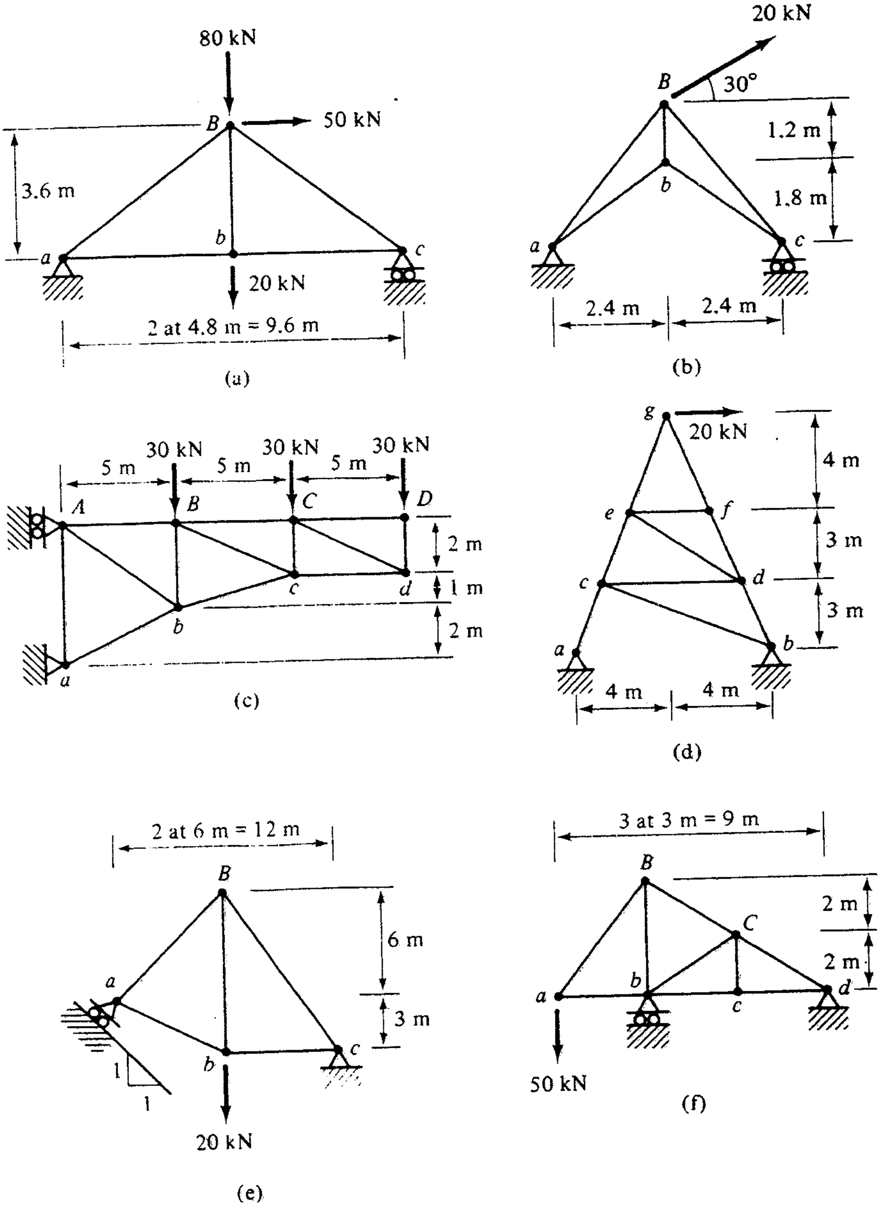


Fig. 3-35

**3-4.** By the method of section, compute the bar forces in the lettered bars of the trusses shown in Fig. 3-36.

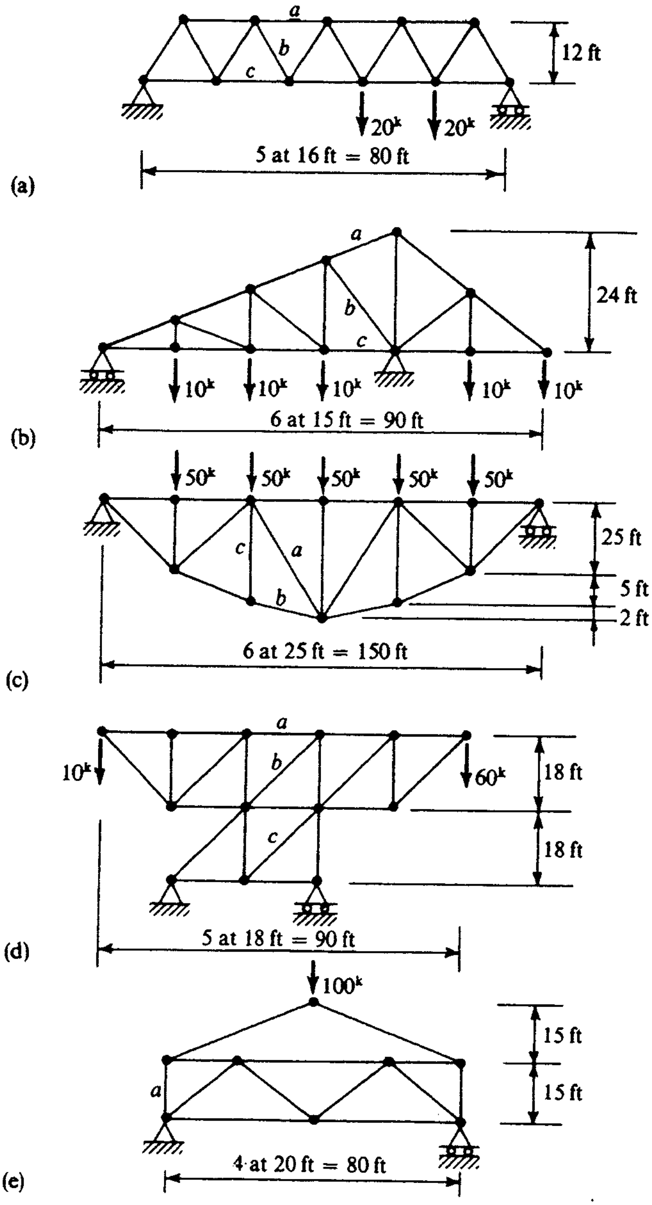


Fig. 3-36

**3-5.** By the mixed method of joint and section, determine the bar forces in the lettered bars of the K truss shown in Fig. 3-37.

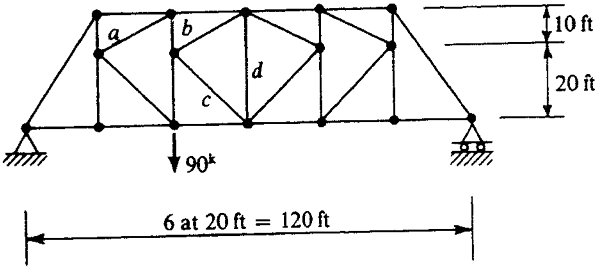


Fig. 3-37

**3-6.** Make a complete analysis of the compound truss shown in Fig. 3-38.

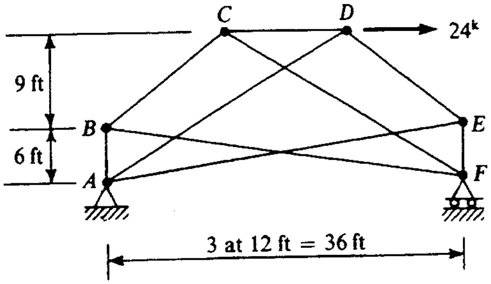


Fig. 3-38

**3-7.** Use the substitute-member method to make a complete analysis of the complex truss shown in Fig. 3-39. Repeat it by solving 12 joint equations if a digital computer is available.

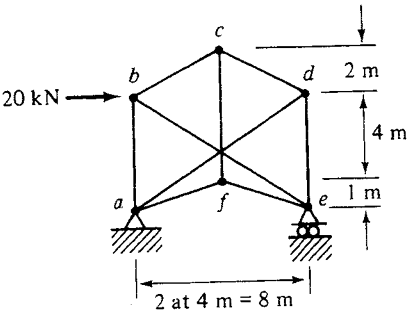


Fig. 3-39

**3-8.** Analyze each of the frames shown in Fig. 3-40, and draw the bending moment diagram.

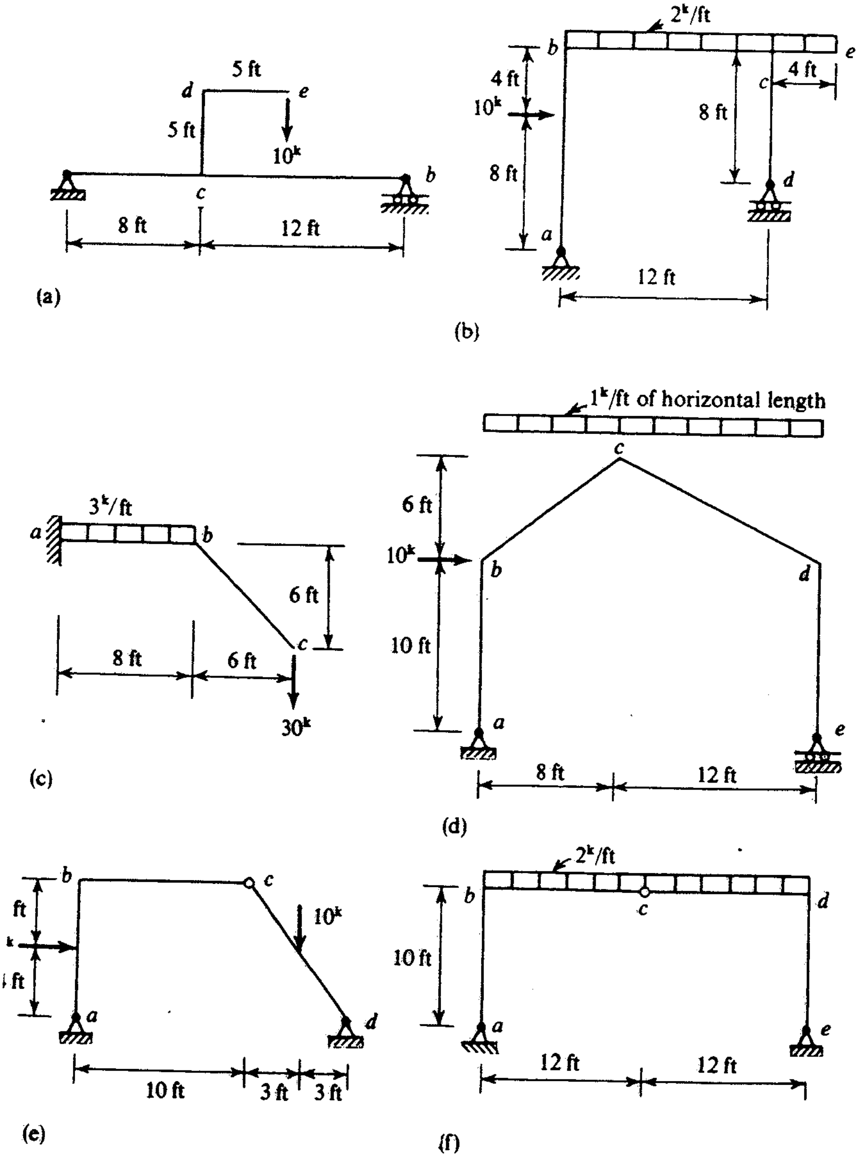
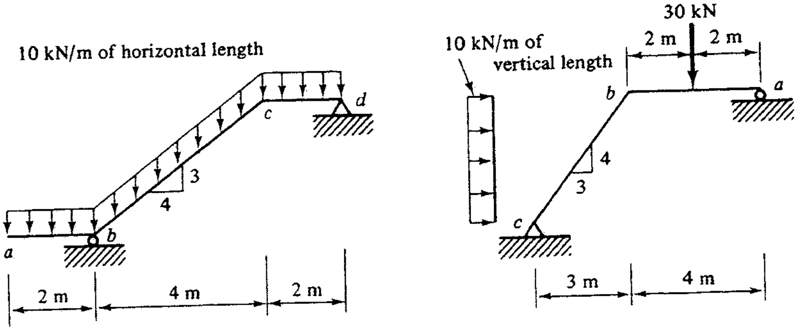


Fig. 3-40

**3-9.** Analyze each of the frames shown in Fig. 3-41 and draw the shear, moment, and axial force diagrams for member *bc*.



(a) (b)

Fig. 3-41

**3-10.** Analyze the frame of Fig. 3-42 by the cantilever method. Assume constant *EI* for all members.

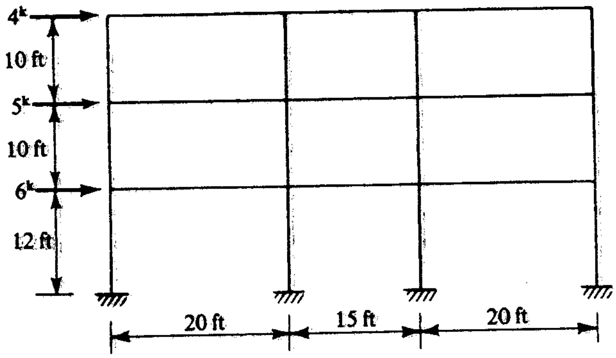


Fig. 3-42

# 4

# *Influence Lines*

## 4-1 THE CONCEPT OF THE INFLUENCE LINE

In the design of a structure, as discussed in Sec. 1-2, the loading conditions for the structure must be established before the stress analysis can be made. For a static structure, we are mainly concerned with two kinds of load, dead load and live load (the impact load being a fraction of the live load). The dead load remains stationary with the structure, whereas the live load, either the moving or the movable load, may vary in position on the structure. When designing any specific part of a structure, we should know where to place the live load so that it will cause the maximum live stresses for the part considered. The part of the structure and the type of stress may be the reaction of a support, the shear or moment of a beam section, or the bar force in a truss. The position of the load that causes the maximum bending moment at a section will not necessarily cause the maximum shear at the same section, and the condition of loading that causes the maximum axial force at one member may not cause the maximum axial force at some other member. To handle these, it is advisable to plot curves that show the individual effect on a desired force element at a certain location of the structure caused by a unit load moving across the structure span. This can be done by taking the $x$ axis to indicate the path of the unit moving load, and the $y$ axis the corresponding force variation at the given location. The graphic representation of the relationship $y = f(x)$ is called the *influence line*.

As an illustration, let us draw a bending moment influence line for the midspan section of a simple beam 10 ft long [Fig. 4-1(a)]. We may first divide the span into equal segments, say 10 segments $AB$, $BC$, . . . , $JK$, to indicate

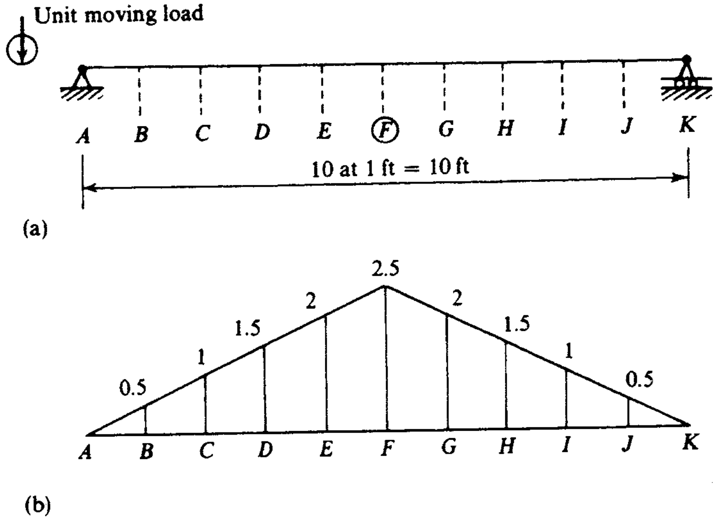


Fig. 4-1

the position of load. As the unit load moves continuously from the left to the right, we focus our attention on the midspan section $F$ and compute the bending moment at $F$ for each 1-ft interval. The results are plotted in Fig. 4-1(b), which gives the bending moment influence line for section $F$. The abscissa coincides with the beam axis, indicating the position of the load, and the ordinate gives the corresponding moment at $F$ due to the single unit load placed at the ordinate. For instance, the ordinate at $D$ is 1.5, which is the value of the moment at $F$ caused by a unit load at $D$.

Of course, we need not always plot the influence line in this fashion, since it is time consuming. In most cases we can find an equation $y = f(x)$ expressing the desired force $y$ at the given section in terms of the load position $x$. The plane curve represented by the equation gives the desired influence line. To illustrate this technique, let us use the same problem but picture it in a different way, as shown in Fig. 4-2(a). The unit load is placed at a distance $x$ from the left end $A$. The reactions at ends $A$ and $K$ are expressed as functions of $x$,

$$R_A = \frac{(10 - x)(1)}{10} = 1 - \frac{x}{10} \qquad R_K = \left(\frac{x}{10}\right)(1) = \frac{x}{10}$$

respectively. When the moving load is confined to the left of section $F$ (as shown), the bending moment at $F$ may be found from $R_K$:

$$M_F = 5R_K = \frac{x}{2}$$

or

$$y = \frac{x}{2}$$

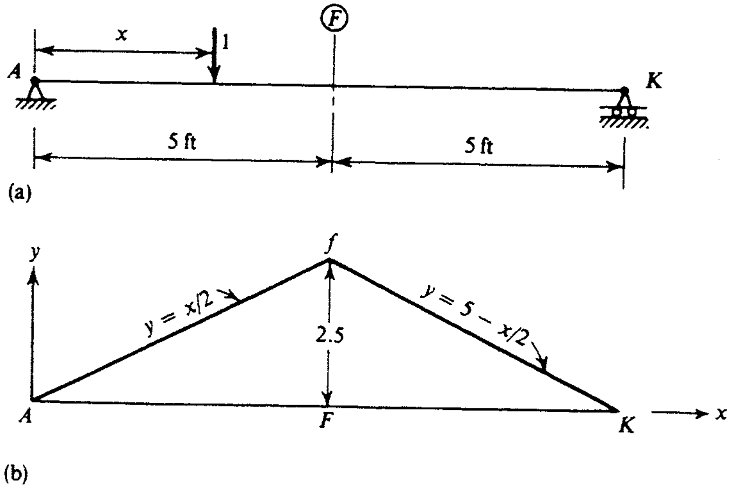


Fig. 4-2

where $y$ denotes $M_F$. As the moving load is confined to the right of section $F$ (not shown), the bending moment at $F$ may be found from $R_A$:

$$M_F = 5R_A = (5)\left(1 - \frac{x}{10}\right) = 5 - \frac{x}{2}$$

or

$$y = 5 - \frac{x}{2}$$

Selecting the coordinate axes as shown in Fig. 4-2(b), we plot $y = x/2$ and $y = 5 - (x/2)$. The curve $AfK$ given in Fig. 4-2(b) is the desired *bending moment influence line* for section $F$, and the corresponding diagram $AFKf$ is called the *bending moment influence diagram* for section $F$.

A generalized definition of the influence line may be given as follows: *An influence line is a curve whose ordinate* ($y$ value) *gives the value of the function* (shear, moment, reaction, bar force, etc.) *in a fixed element* (member section, support, bar in truss, etc.) *when a unit load is at the ordinate.*

Although in this particular case the influence diagram of Fig. 4-2(b) is identical with the moment diagram for the same beam under a unit load at midspan, we must not confuse the influence diagram with a bending moment diagram for the beam. Whereas the ordinate in the latter shows the bending moment at the corresponding section due to a fixed load, the ordinate in the influence diagram shows the bending moment at a fixed section due to a unit load placed at that point.

## 4-2 USE OF THE INFLUENCE LINE

An influence line is a useful tool in stress analysis in two ways:

1. It serves as a criterion in determining the maximum stress—that is, it is a guide for determining what portion of the structure should be loaded in order to cause the maximum effect on the part under consideration.
2. It simplifies the computation.

To illustrate, consider a simple beam 10 ft long subjected to the passage of a moving uniform load of 1 kip/ft without limit in length and a movable concentrated load of 10 kips that may be placed at any point of the span [see Fig. 4-3(a)]. Determine the maximum bending moment at the midspan section $C$.

We start by drawing the bending moment influence line for section $C$, as in Fig. 4-3(b). It is apparent from the influence line that to obtain the maximum $M_C$, the single concentrated load of 10 kips should be placed at the midpoint of the span, where the maximum ordinate of the influence line occurs, and the uniform load should be spread over the entire span.

Next, to compute the bending moment at $C$ due to the live loads so placed, we simply multiply each load by the corresponding influence ordinate and add.

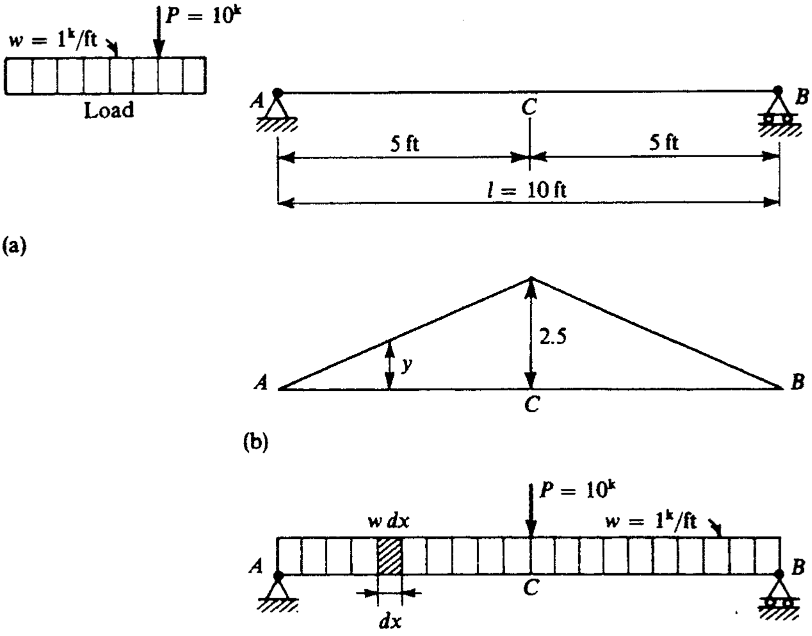


Fig. 4-3

Referring to Fig. 4-3(b) and (c), we obtain

$$M_C = P(2.5) + \sum (w\,dx)y$$

Note that the uniform load is treated as a series of infinitesimal concentrated loads each of magnitude $w\,dx$, and the total effect of the uniform load is the summation

$$\sum (w\,dx)y$$

Now,

$$\sum (w\,dx)y = \int_0^l wy\,dx = w\int_0^l y\,dx$$

$$= \text{(load intensity)} \times \text{(area of influence diagram)}$$

Therefore, the total bending moment at $C$ is

$$M_C = (10)(2.5) + (1)\frac{(2.5)(10)}{2} = 25 + 12.5 = 37.5 \text{ ft-kips}$$

This value may be checked by the conventional method of computing $M_C$:

$$M_C = \left(\frac{10}{2}\right)(5) + \frac{(1)(10)^2}{8} = 25 + 12.5 = 37.5 \text{ ft-kips}$$

In this simple case, such a conclusion may be drawn without the aid of the influence diagram; but for more complicated moving load systems, we find that the influence diagram can be of substantial help, as discussed in Sec. 4-5.

## 4-3 INFLUENCE LINES FOR STATICALLY DETERMINATE BEAMS

The basic approach to drawing influence lines for a statically determinate beam is to apply the equilibrium based on the procedure of taking the appropriate free body as the unit load travels along the beam span. It is often convenient to construct the reaction influence lines first and then deduce the shear and moment influences.

**Example 4-1**

Consider the simple beam with an overhang shown in Fig. 4-4(a). To construct the influence line for $R_B$, we place a unit load distance $x$ from end $A$ and apply $\Sigma M_C = 0$ to obtain

$$R_B = \frac{20 - x}{16}$$

The expression represents a straight line with a maximum ordinate of $\frac{5}{4}$ at $A$ and a minimum ordinate of 0 at $C$, as shown in Fig. 4-4(b). Note that when the unit load is placed at $B$, the influence ordinate for $R_B$ should be equal to unity.

The influence line for $R_C$ may be found by applying $\Sigma F_y = 0$:

$$R_C = 1 - R_B = 1 - \frac{20 - x}{16} = \frac{x - 4}{16}$$

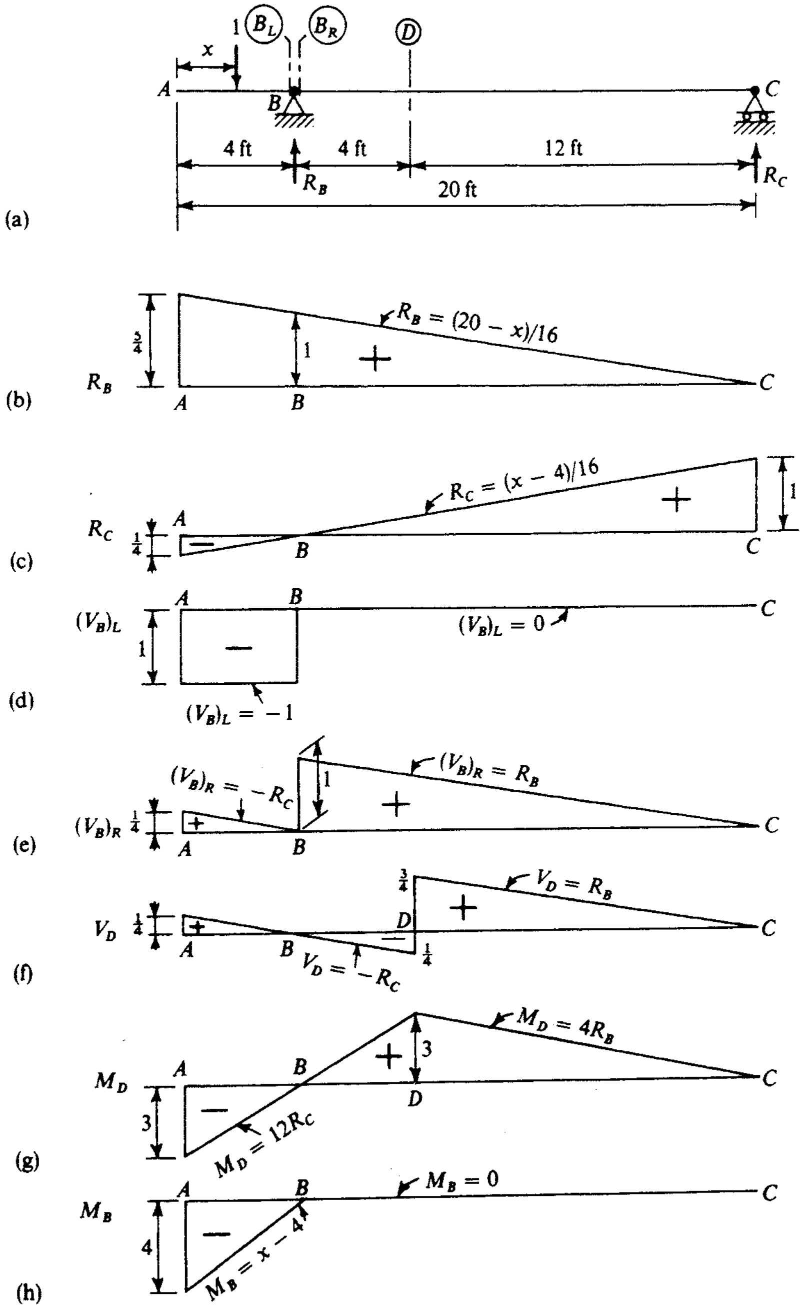

(a)
1
$B_L$
$B_R$
D
x
A
B
C
4 ft
4 ft
12 ft
$R_B$
20 ft
$R_C$
(b)
$R_B$
$\frac{5}{4}$
1
$R_B = (20 - x)/16$
+
A
B
C
(c)
$R_C$
$\frac{1}{4}$
A
−
B
$R_C = (x - 4)/16$
+
1
C
(d)
$(V_B)_L$
1
A
B
−
$(V_B)_L = 0$
C
$(V_B)_L = -1$
(e)
$(V_B)_R$
$\frac{1}{4}$
+
A
B
$(V_B)_R = -R_C$
1
$(V_B)_R = R_B$
+
C
(f)
$V_D$
$\frac{1}{4}$
+
A
B
$V_D = -R_C$
D
−
$\frac{1}{4}$
$\frac{3}{4}$
$V_D = R_B$
+
C
(g)
$M_D$
3
A
B
−
$M_D = 12R_C$
+
3
D
$M_D = 4R_B$
C
(h)
$M_B$
4
A
B
−
$M_B = x - 4$
$M_B = 0$
C

Fig. 4-4

which is also a linear function of $x$, as shown in Fig. 4-4(c). As a check, the ordinate at $C$ should be equal to unity, and that at $B$ to zero.

The influence line for the shear at the section just to the left of $B$, called $(V_B)_L$, is given in Fig. 4-4(d). As long as the unit load is on the overhanging portion of beam, $(V_B)_L = -1$; as the load passes $B$ to the right, $(V_B)_L = 0$.

The influence line for the shear at the section just to the right of $B$, called $(V_B)_R$, is shown in Fig. 4-4(e). As long as the unit load is on the overhanging portion of beam, $(V_B)_R$ equals $R_C$ but with opposite sign. When the load is on the simple-beam portion, $(V_B)_R$ equals $R_B$.

By a similar approach, we construct the influence line for shear at section $D$, as shown in Fig. 4-4(f). As a check, when the unit load passes $D$ from the left to the right, the shear at $D$ increases suddenly from $-\frac{1}{4}$ to $+\frac{3}{4}$ (i.e., there is an abrupt change of shear equal to unity at $D$).

The influence line for the moment at $D$ is shown in Fig. 4-4(g). We note that as long as the unit load is confined to the portion $AD$, the moment at $D$ may be found from $R_C$,

$$M_D = 12R_C = (12)\left(\frac{x - 4}{16}\right) = \frac{3x - 12}{4} \qquad (0 \leq x \leq 8)$$

which represents a straight line from $A$ to $D$ with ordinates of $-3$ at $A$ and $+3$ at $D$. When the load passes $D$ to the right, the moment at $D$ may be found from $R_B$:

$$M_D = 4R_B = (4)\left(\frac{20 - x}{16}\right) = \frac{20 - x}{4} \qquad (8 \leq x \leq 20)$$

which represents a straight line from $D$ to $C$ with ordinates of $+3$ at $D$ and $0$ at $C$.

Finally, we construct the influence line for the moment at $B$, as in Fig. 4-4(h). When the load is placed at $A$, $M_B$ has its greatest negative value of 4. As the load travels from $A$ to $B$, the moment varies linearly from $-4$ to 0. As the load enters the portion $BC$, there is no moment at $B$.

A more simple and elegant way to construct beam influence lines is to apply Müller-Breslau's principle, which can be stated as follows:

1. To obtain an influence line for the reaction of any statically determinate beam, remove the support and make a positive unit displacement of its point of application. The deflected beam is the influence line for the reaction.
2. To obtain an influence line for the shear at a section of any statically determinate beam, cut the section and induce a unit relative transverse sliding displacement between the portion to the left of the section and the portion to the right of the section keeping all other constraints (both external and internal) intact. The deflected beam is the influence line for the shear at the section.
3. To obtain the influence line for the moment at a section of any statically determinate beam, cut the section and induce a unit rotation between the portion to the left of the section and the portion to the right of the section keeping all other constraints (both external and internal) intact. The deflected beam is the influence line for the moment at the section.

Müller-Breslau's principle is based on the theorem of virtual work, which states that if a compatible virtual displacement is induced in an ideal system in equilibrium under balanced forces, the total virtual work $\delta W$ done by all active forces is equal to zero.

To prove the theory, we take the case of a simple beam. The proof is generally applicable to more complicated beams. Figure 4-5(a) shows a simple beam subjected to a single unit moving load. To find the reaction at $A$ by the method of virtual work, we remove the constraint at $A$, substitute $R_A$ for it, and

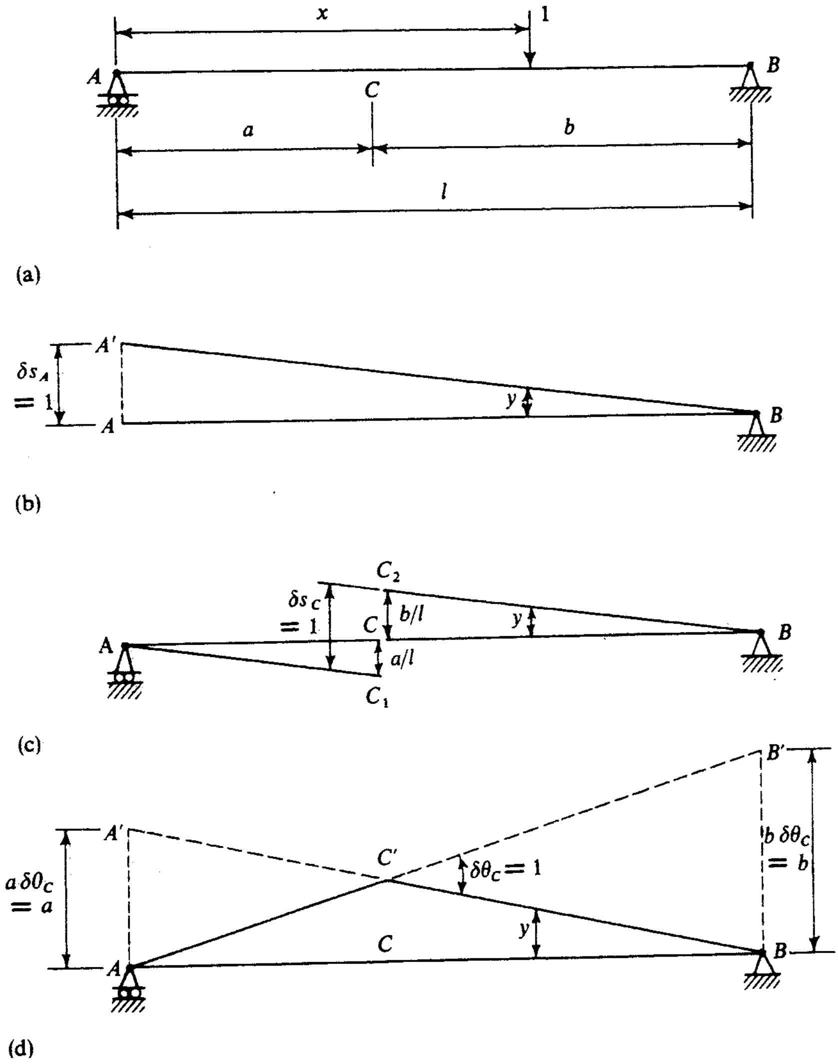


Fig. 4-5

let $A$ travel a small virtual displacement $\delta s_A$ along $R_A$. We then have a deflected beam $A'B$, as shown in Fig. 4-5(b), where $y$ indicates the transverse displacement at the point of unit load. Applying $\delta W = 0$, we obtain

$$(R_A)(\delta s_A) - (1)(y) = 0$$

from which $$R_A = \frac{y}{\delta s_A}$$

If we let $$\delta s_A = 1$$

then $$R_A = y$$

Since $y$ is, on the one hand, the ordinate of the deflected beam at the point where the unit load stands and is, on the other hand, the value of function $R_A$ due to the unit moving load (i.e., the influence ordinate at the point), we conclude that the deflected beam $A'B$ of Fig. 4-5(b) is the influence line for $R_A$ if $\delta s_A$ is set to be unity.

To determine the shearing force at any beam cross section $C$, we cut the beam at $C$ and let the two portions $AC$ and $CB$ have a relative virtual transverse displacement $\delta s_C$ at $C$ without causing relative rotation between the two portions. This is equivalent to rotating $AC$ and $BC$ the same small angle about $A$ and $B$, respectively. Applying $\delta W = 0$, we obtain

$$(V_C)(\delta s_C) - (1)(y) = 0$$

from which $$V_C = \frac{y}{\delta s_C}$$

If we let $$\delta s_C = 1$$

then $$V_C = y$$

This proves that the deflected beam $AC_1C_2B$ of Fig. 4-5(c) is the influence line for $V_C$. It should be pointed out that the virtual displacement is supposed to be vanishingly small and that when we say $\delta s_C = 1$, we do not mean that $\delta s_C =$ 1 ft or 1 in. but one unit of very small distance for which the expressions

$$CC_1 = \frac{a}{l}$$

$$CC_2 = \frac{b}{l}$$

shown in Fig. 4-5(c) are justified.

To determine the moment at any beam cross section $C$ by the method of virtual work, we cut the beam at $C$ and induce a relative virtual rotation between the two portions $AC$ and $CB$ at $C$ without producing relative transverse sliding between the two. Thus, by $\delta W = 0$,

$$(M_C)(\delta\theta_C) - (1)(y) = 0$$

from which $$M_C = \frac{y}{\delta\theta_C}$$

If we let $\delta\theta_C = 1$

then $M_C = y$

This proves that the deflected beam $AC'B$ of Fig. 4-5(d) is the influence line for $M_C$. Note that when we say $\delta\theta_C = 1$, we do not mean that $\delta\theta_C = 1$ radian. One unit of $\delta\theta_C$ may be as small as $\frac{1}{100}$ radian, for which it is justified to write

$$AA' = a \cdot \delta\theta_C = a \text{ units} \qquad BB' = b \cdot \delta\theta_C = b \text{ units}$$

as indicated in Fig. 4-5(d).

**Example 4-2**

Figure 4-6(a) shows a compound beam. Draw influence lines for $R_A$, $V_D$, $M_D$, $V_E$, and $M_E$ by Müller-Breslau's principle.

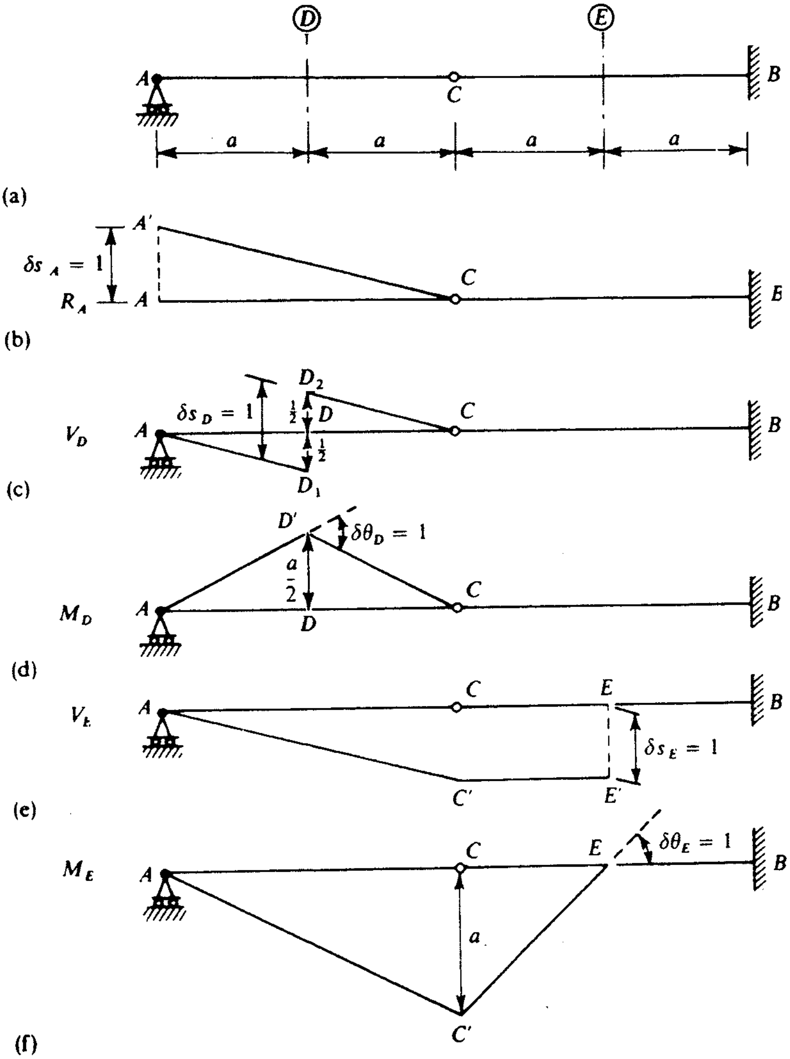


Fig. 4-6

To construct the influence line for $R_A$, we remove support $A$ and move end $A$ up a unit distance. The deflected bean $A'CB$ shown in Fig. 4-6(b) is the influence line for $R_A$. Note that portion $CB$ is a cantilever and will remain unmoved.

To construct the influence line for $V_D$, we cut the beam through $D$ and let the left portion of beam have a relative transverse displacement equal to unity with respect to the right portion of beam at $D$ without causing relative rotation between the two. The deflected beam $AD_1D_2CB$ shown in Fig. 4-6(c) is the influence line for $V_D$.

To construct the influence line for $M_D$, we cut the beam through $D$ and let the left portion of beam rotate a unit angle with respect to the right portion at $D$. The deflected beam $AD'CB$ of Fig. 4-6(d) is the influence line for $M_D$.

The influence line for $V_E$ is shown in Fig. 4-6(e) by $AC'E'EB$, which results from cutting the beam through $E$ and moving the left portion of beam down a unit distance with respect to the right portion of beam at $E$ while keeping the deflected portion $C'E'$ parallel to $BE$.

The influence line for $M_E$ is shown in Fig. 4-6(f) by $AC'EB$, which results from cutting the beam through $E$ and rotating the left portion of beam a unit angle with respect to the right portion of beam at $E$. Point $E$ is kept fixed in the original position.

## *4-4 INFLUENCE LINES FOR STATICALLY DETERMINATE BRIDGE TRUSSES*

As stated in Sec. 3-6, the live loads on the deck of a bridge are transmitted to the loaded chords of main trusses through the system of stringers and floor beams. The stringers running parallel to the main trusses are usually assumed to act as simple beams supported by the adjacent floor beams, which, in turn, are connected to the panel points (truss joints) of the loaded chords (see Fig. 3-21). Any live load on the deck is thus considered as a panel-point load at the loaded chord in a truss analysis.

We can draw influence lines for the bar forces of a bridge truss by placing the unit load at each successive panel point of the loaded chord, computing the bar forces as the influence ordinates, and connecting adjacent influence ordinates by straight lines. The reason that the influence line between consecutive panel points will be a straight line can be explained as follows. Refer to Fig. 4-7(a) for a diagram of a truss chord loaded with cross beams and stringers. Let a unit load travel along a panel $m$–$n$. When the unit load is at $m$, we let $y_m$ be the corresponding influence ordinate for it; when the unit load is at $n$, we let $y_n$ be the corresponding influence ordinate for it [see Fig. 4-7(b)]. Now when the unit load is in any intermediate position, say at distance $x$ from $m$, it will be transmitted to the girder through the floor beams at $m$ and $n$ with values $(l - x)/l$ and $x/l$, respectively. The effect of the load is found by multiplying each of these values by the corresponding value of the influence ordinate, and adding. Thus,

$$(1)(y) = \left(\frac{l - x}{l}\right)(y_m) + \left(\frac{x}{l}\right)(y_n)$$

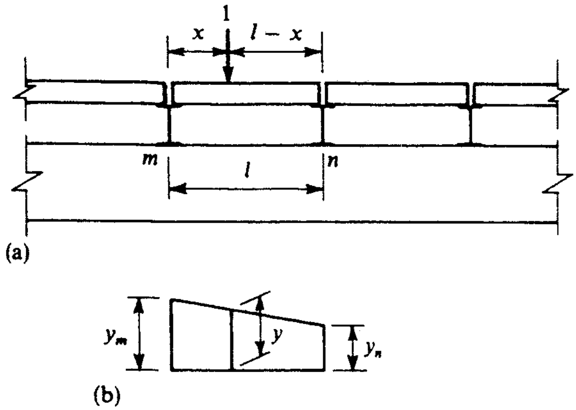


Fig. 4-7

in which $y$ denotes the influence ordinate where the unit load is located as indicated in Fig. 4-7(b). This expression, being linear in $x$, specifies the influence ordinate for the general intermediate position in the panel.

**Example 4-3**

For the truss shown in Fig. 4-8(a), draw the influence lines for forces in members *aB*, *Bb*, *Bc*, and *bc*.

We start with a unit load at joint *a* and then move it to *b*, *c*, and *d* successively. Each time we place the unit load at a joint, we compute the bar forces (or components) in the desired members and we erect the ordinates to the respective influence lines, as shown in Fig. 4-8(b)–(e). Finally we connect the consecutive ordinates by straight lines to complete the influence lines.

Although it is always possible to obtain the ordinates to an influence line for any element for a unit load at each point of a truss, the method may become time consuming when dealing with a truss involving many panels without the aid of a computer. Alternatively we may first seek the influence lines for support reactions since they are related in a simple manner to the unit load of variable position. After that we can deduce the influence lines for bar forces very quickly, as can be seen in the following example.

**Example 4-4**

Figure 4-9(a) shows a Warren truss. Let us draw the influence lines for bar forces (or components) in members *cd* and *Cc*.

The influence lines for reactions $R_a$ and $R_g$ are readily drawn as shown in Fig. 4-9(b) and (c). They are constructed in the same way as the influence lines for the reactions of a simple beam 36 m long, because end reactions of a truss are not affected by the presence of the floor system.

To construct the influence line for the force in bar *cd*, denoted by $S_{cd}$, we pass a section *m–m* through bars *cd*, *Cc*, and *BC*, as shown in Fig. 4-9(a). With the unit load at or to the left of panel point *c*, we find that

$$S_{cd} = \frac{21R_g}{5}$$

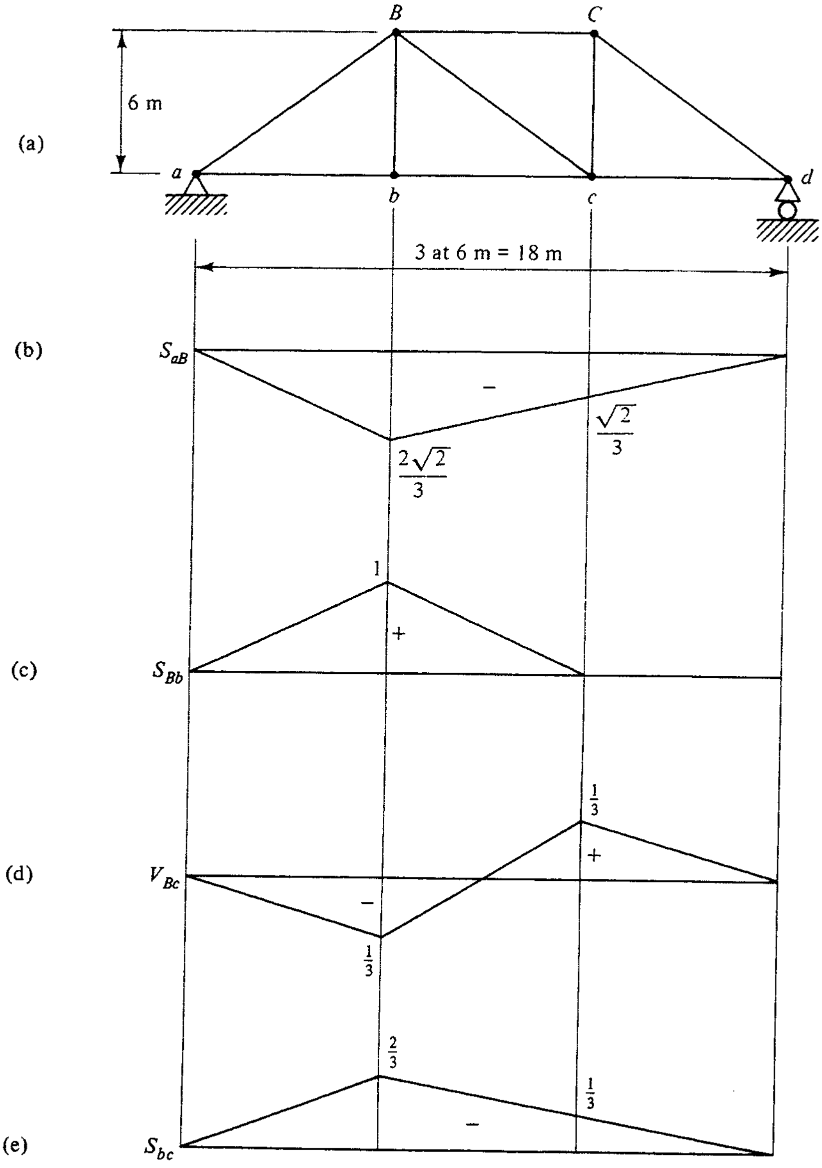

B
C
6 m
(a)
a
b
c
d
3 at 6 m = 18 m
(b)
$S_{aB}$
–
$\frac{\sqrt{2}}{3}$
$\frac{2\sqrt{2}}{3}$
1
+
(c)
$S_{Bb}$
$\frac{1}{3}$
+
(d)
$V_{Bc}$
–
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{1}{3}$
–
(e)
$S_{bc}$

**Fig. 4-8**

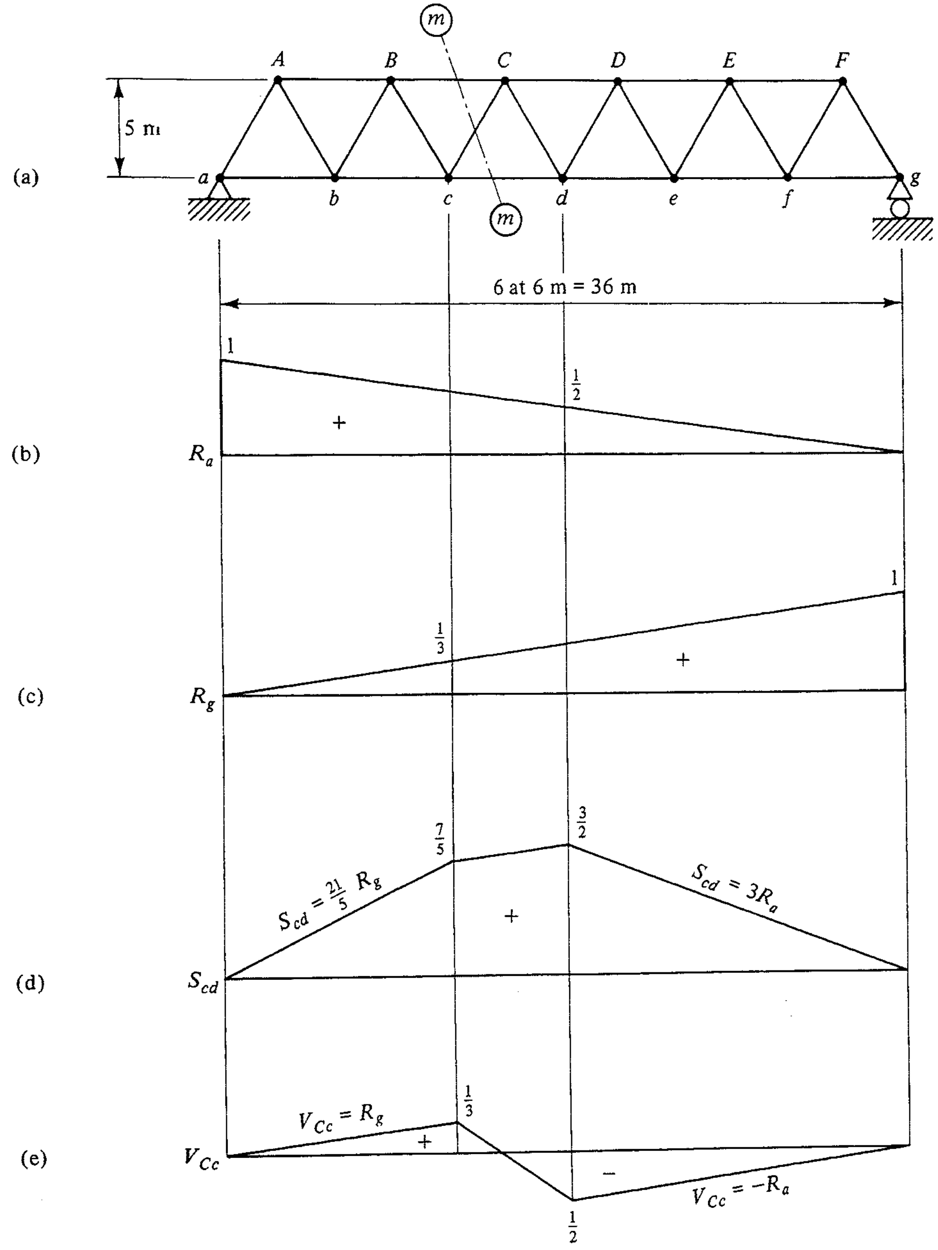

(a)
m
A
B
C
D
E
F
5 m
a
b
c
d
e
f
g
m
6 at 6 m = 36 m
(b)
$R_a$
1
$\frac{1}{2}$
+
(c)
$R_g$
$\frac{1}{3}$
1
+
(d)
$S_{cd}$
$S_{cd} = \frac{21}{5} R_g$
$\frac{7}{5}$
$\frac{3}{2}$
+
$S_{cd} = 3R_a$
(e)
$V_{Cc}$
$V_{Cc} = R_g$
+
$\frac{1}{3}$
−
$\frac{1}{2}$
$V_{Cc} = -R_a$

Fig. 4-9

by using the right portion of truss as a free body and applying $\Sigma M_C = 0$. With a unit load at or to the right of panel point $d$, we find that

$$S_{cd} = \frac{(R_a)(15)}{5} = 3R_a$$

by using the left portion and the same moment equation. This procedure results in two straight lines similar to the respective segments of influence lines for $R_a$ and $R_g$ but with multipliers. Connecting the influence ordinates at $c$ and $d$ by a straight line gives the influence line for $S_{cd}$, as shown in Fig. 4-9(d).

We employ the same procedure to obtain the influence line for the vertical component force in bar $Cc$, denoted by $V_{Cc}$, except that we apply $\Sigma F_y = 0$ instead of the moment equation to compute the bar force. As the unit load travels from $a$ to $c$, we use the right portion of the truss and observe that

$$V_{Cc} = R_g$$

which represents a straight line identical to the segment of influence line for $R_g$. As the unit load travels from $d$ to $g$, we use the left portion of truss and find that

$$V_{Cc} = -R_a$$

which represents a straight line opposite to the segment of the influence line for $R_a$. Connecting the influence ordinates at $c$ and $d$ by a straight line completes the influence line for $V_{Cc}$, as shown in Fig. 4-9(e).

## 4-5 INFLUENCE LINES AND CONCENTRATED LOAD SYSTEMS

As mentioned in Sec. 4-2, the influence line serves a guide for determining the maximum live stresses. Under a single concentrated load or a uniform load, the critical position causing a certain maximum live stress can be spotted at once by inspection of the influence line. For more complicated conditions of loading of various magnitudes and spacings, such as a series of moving wheels on a locomotive, we cannot tell the critical position by just looking at the influence line. The method that should be followed in such cases is essentially one of trial and error with reference to the influence line in order to minimize computations.

**Example 4-5**

Figure 4-10(a) shows a simple beam subjected to the passage of wheel loads. We wish to find the maximum reaction at the left end $A$, the influence line of which is also shown in the same figure.

Since the influence ordinate increases toward the left, the system of wheel loads must not stay in an intermediate position on the beam but should continue to move until wheel load 1 reaches support $A$. The first possible position for producing the maximum $R_A$ therefore has wheel 1 directly over $A$ as shown in Fig. 4-10(b). Next, with wheel 1 leaving the span, the second possible position has wheel 2 over $A$, as shown in Fig. 4-10(c). In each case, we obtain the value of $R_A$ by multiplying each of the wheel loads by the corresponding influence ordinate and adding, as indicated in Fig. 4-10(b) and (c), respectively. The final trial position has wheel 3 over $A$, as shown in Fig. 4-10(d).

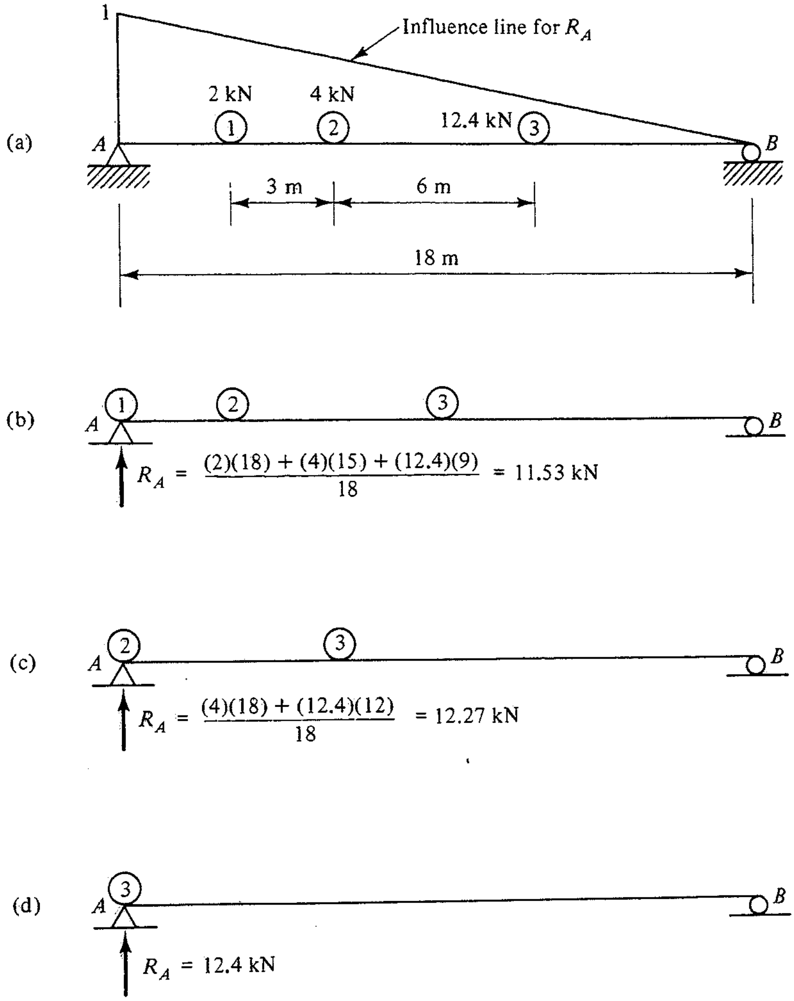


Fig. 4-10

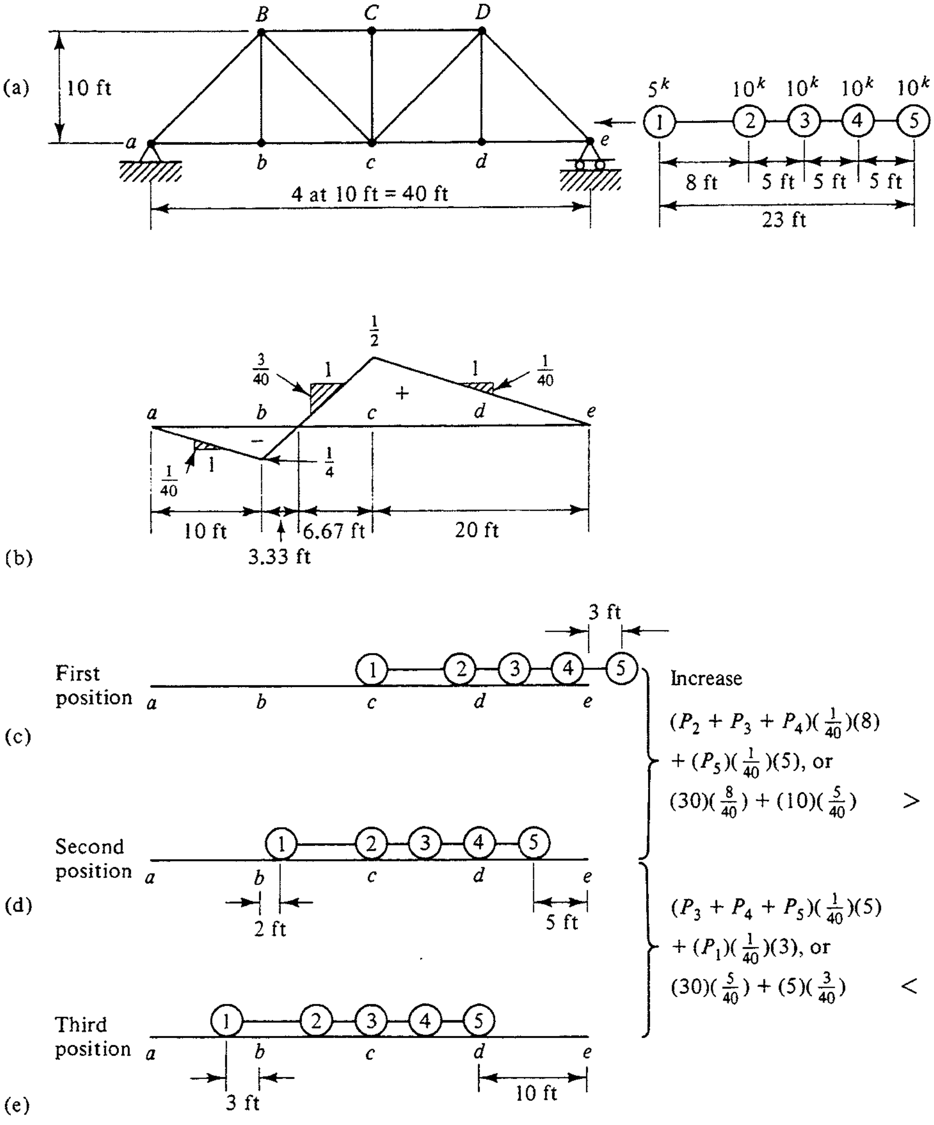
By comparing the results, we conclude that the maximum reaction is 12.4 kN when wheel 3 is directly over support *A*.

**Example 4-6**

For the bridge truss subjected to the passage of the group of wheel loads in Fig. 4-11(a), find the maximum force in the member *Bc*.

To do this, we construct the influence line for the vertical component of the bar force in *Bc* as shown in Fig. 4-11(b).

The best approach is to try several loading positions and to compare the changes in the value of the function because of the movement. The increase or decrease in the value of the function caused by a moving load is determined by the multiplication of three quantities, that is, the load, the slope of the influence



| Increase | | Decrease |
|---|---|---|
| $(P_2 + P_3 + P_4)(\frac{1}{40})(8)$ | | $(P_1)(\frac{3}{40})(8)$, or |
| $+ (P_5)(\frac{1}{40})(5)$, or | | |
| $(30)(\frac{8}{40}) + (10)(\frac{5}{40})$ | $>$ | $(5)(\frac{24}{40})$ |
| $(P_3 + P_4 + P_5)(\frac{1}{40})(5)$ | | $(P_2)(\frac{3}{40})(5)$ |
| $+ (P_1)(\frac{1}{40})(3)$, or | | $+ (P_1)(\frac{3}{40})(2)$, or |
| $(30)(\frac{5}{40}) + (5)(\frac{3}{40})$ | $<$ | $(10)(\frac{15}{40}) + (5)(\frac{6}{40})$ |

**Fig. 4-11**

line, and the distance the load moves. Let us try the first loading position, shown in Fig. 4-11(c), with wheel 1 at the peak ordinate. Next, let the system move to the left until wheel 2 reaches the peak ordinate, as shown in Fig. 4-11(d). The computations to the right of Fig. 4-11(c) and (d) show that the movement results in an increase in the value of the function. Next, let this system move farther to the left until wheel 3 reaches the peak ordinate, as shown in Fig. 4-11(e). The computations to the right of Fig. 4-11(d) and (e) show that this causes a decrease in the value of the function. Thus, the second position of loading, shown in Fig. 4-11(d), produces the maximum tensile force in member *Bc*.

By using the influence diagram, we find the maximum value of $V_{Bc}$ to be

$$V_{Bc} = (10)\left(\frac{20 + 15 + 10 + 5}{40}\right) - (5)\left(\frac{1}{4}\right)\left(\frac{3.33 - 2}{3.33}\right)$$

$$= 12.5 - 0.5 = +12 \text{ kips}$$

or

$$S_{Bc} = +12\sqrt{2} \text{ kips}$$

Note that this method of situating a load system for the maximum effect is perfectly general and may always be employed in cases with more complicated influence diagrams.

## PROBLEMS

**4-1.** Given a simple beam 24 ft long, construct the influence lines for the shear and bending moment at a section 8 ft from the left end, and obtain the maximum shear and bending moment for the section resulting from a moving uniform load of 3 kips/ft and a movable concentrated load of 50 kips.

**4-2.** A cantilever beam 20 ft long is fixed at the right end. Construct the shear and moment influence lines for sections 5 ft, 10 ft, and 20 ft from the free end. Using the same loadings given in Prob. 4-1, compute maximum shears and moments at these sections.

**4-3.** In Fig. 4-12 is shown a simple beam with an overhang. Draw the influence lines for $R_B$, $R_C$, $V_D$, $M_B$, and $M_D$.

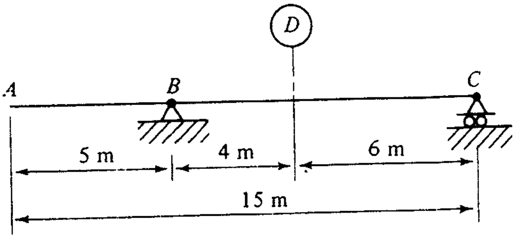


Fig. 4-12

**4-4.** Given a compound beam such as that shown in Fig. 4-13, construct the influence lines for $R_A$, $R_C$, $R_E$, $V_B$, $M_B$, and $M_C$. Compute the maximum value for each of them due to a moving uniform load of 20 kN/m.

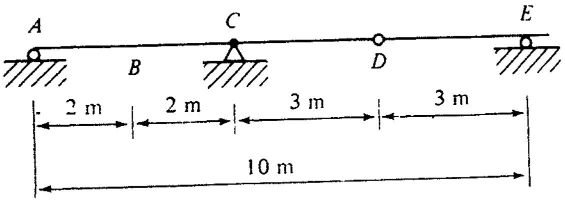


Fig. 4-13

**4-5.** Solve Prob. 4-3 by Müller-Breslau's principle.

**4-6.** Construct the influence lines for Prob. 4-4 by Müller-Breslau's principle.

**4-7.** For the trusses shown in Fig. 4-14, construct the influence lines for the bar force (or component) in each of the lettered bars.

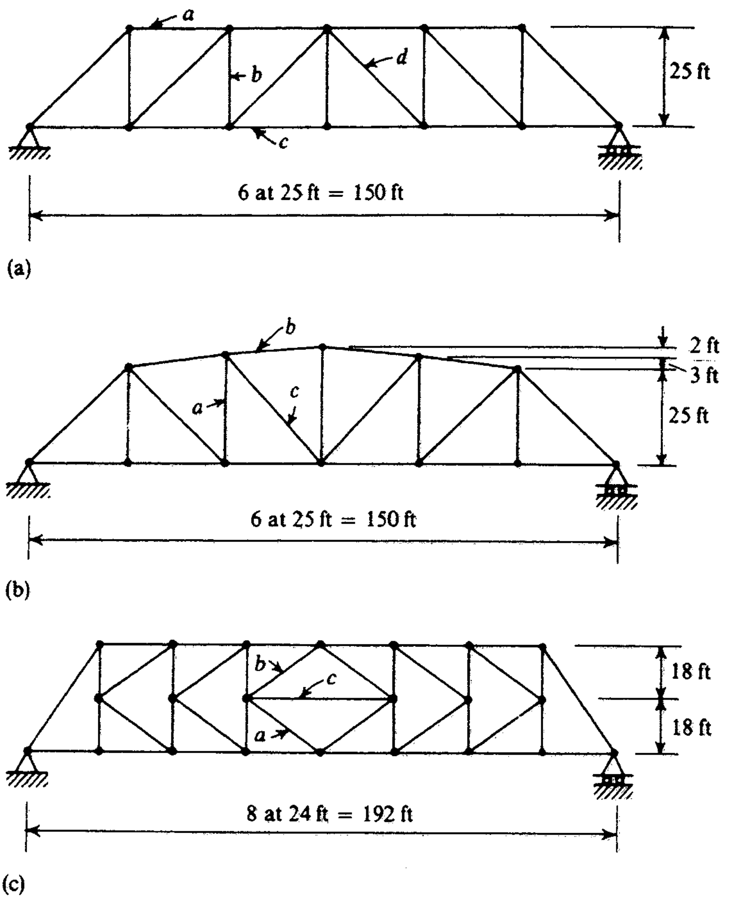


Fig. 4-14

**4-8.** A simple beam 45 ft long carries moving loads of 5 kips, 10 kips, and 10 kips spaced 5 ft apart. Calculate (a) the maximum left reaction and (b) the maximum shear and bending moment at a section 15 ft from the left end.

**4-9.** For the truss and the loading shown in Fig. 4-15, compute the maximum forces in bars *a* and *b*. Consider both tension and compression in bar *a*.

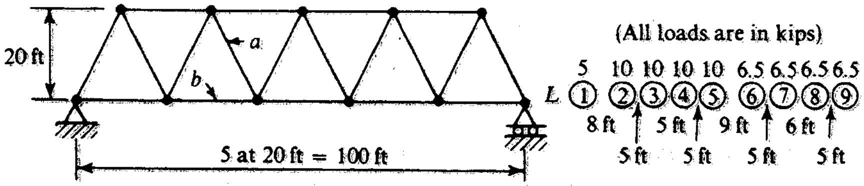


Fig. 4-15

**4-10.** For the compound beam and loads shown in Fig. 4-16, find (a) the maximum reaction at $C$ and (b) the maximum moment at $D$.

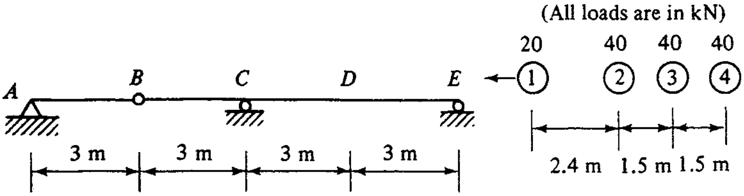


Fig. 4-16

**4-11.** Refer to the frame and loads in Fig. 4-17. Find the maximum moment and vertical reaction at support $E$ due to the passage of loads over the beam.

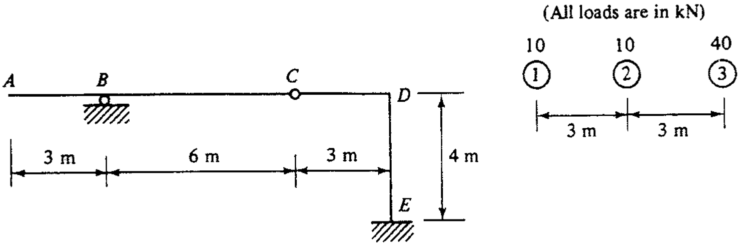


Fig. 4-17

# 5

# *Elastic Deformations*

## 5-1 GENERAL

The calculation of elastic deformations of structures, both the linear displacements of points and the rotational displacements of lines (slopes) from their original positions, is of great importance in the analysis, design, and construction of structures. For instance, in the erection of a bridge structure, especially when the cantilever method is used, the theoretical elevations of some or all joints must be computed for each stage of the work. In building design the sizes of beams and girders are sometimes governed by the allowable deflections. Most important, the stress analysis for statically indeterminate structures is based largely upon an evaluation of their elastic deformations under load. By a statically indeterminate structure we mean a structure in which the number of unknown forces involved is greater than the number of equations of statics available for their solution. If such is the case, there will be an infinite number of solutions that can satisfy the statical equations. In order to reach a *unique* correct solution, the conditions of the *continuity* of structure, which are associated with the geometric and elastic properties of structure, are a necessary supplement.

Numerous methods of computing elastic deformations have been developed. Among them the following are considered the most significant in conventional structural analysis and will, therefore, be discussed in this chapter:

1. The method of virtual work (unit-load method)
2. Castigliano's theorem
3. The conjugate-beam method

## 5-2 CURVATURE OF AN ELASTIC LINE

The mathematical definition for curvature is *the rate at which a curve is changing direction.* To derive the expression for curvature, we shall consider a curve such as the one shown in Fig. 5-1. The average rate of change of direction between points $P_1$ and $P_2$ is $\Delta\phi/\Delta s$. The limiting value of this ratio as $\Delta s$ approaches zero is called *curvature,* and the *radius of curvature* is the reciprocal of the curvature. Thus, if we let $\kappa$ denote the curvature and $\rho$ the radius of curvature, we have

$$\kappa = \frac{1}{\rho} = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds}$$

Now since $\tan\phi = dy/dx$,

$$\frac{d}{dx}\tan\phi = \frac{d^2y}{dx^2}$$

or

$$(1 + \tan^2\phi)\frac{d\phi}{dx} = \frac{d^2y}{dx^2}$$

This gives

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{d\phi}{dx}$$

whereby

$$\frac{d\phi}{dx} = \frac{d^2y/dx^2}{1 + (dy\,/\,dx)^2}$$

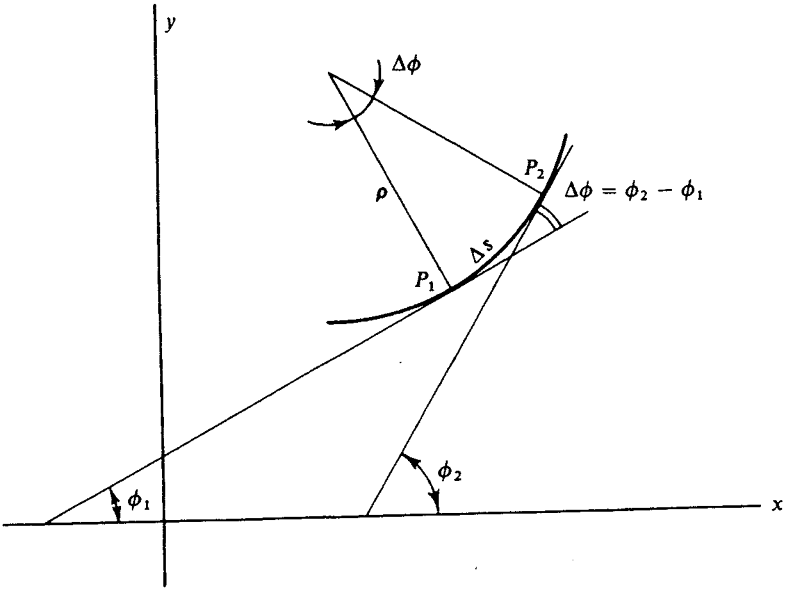


Fig. 5-1

Also,

$$\frac{dx}{ds} = \frac{1}{ds/dx} = \frac{1}{[(dx^2 + dy^2)/dx^2]^{1/2}} = \frac{1}{[1 + (dy/dx)^2]^{1/2}}$$

Hence,

$$\kappa = \frac{d\phi}{ds} = \left(\frac{d\phi}{dx}\right)\left(\frac{dx}{ds}\right) = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}} \tag{5-1}$$

For a loaded beam with its longitudinal axis taken as the $x$ axis, we may set $dy/dx$ in formula 5-1 equal to zero if the deflection of the beam is small. Thus, we obtain.

$$\kappa = \frac{d\phi}{ds} \approx \frac{d^2y}{dx^2} \tag{5-2}$$

In general, except for very deep beams with a short span, the deflection due to the shearing force is negligible and only that due to the bending moment is considered. In order to develop a formula for the curvature due to elastic bending, let us consider a small element of a beam shown in Fig. 5-2. Owing to the action of bending moment $M$, the two originally parallel sections $AB$ and $A'B'$ will change directions. This angle change is denoted by $d\phi$. If the length of the element is $ds$ and the maximum bending stress, which occurs at the extreme fibers, is called $f$, the total elongation at the top or bottom fiber is $c\ d\phi$ (see Fig. 5-2), which equals $f\ ds/E$, $E$ being the modulus of elasticity. Thus,

$$c\ d\phi = \frac{f\ ds}{E}$$

or

$$\frac{d\phi}{ds} = \frac{f}{Ec}$$

Replacing $f$ with $Mc/I$, $I$ being the moment of inertia of the cross-sectional area of the beam about the axis of bending, gives

$$\frac{d\phi}{ds} = \frac{M}{EI} \tag{5-3}$$

which expresses the relationship between the curvature and the bending moment.

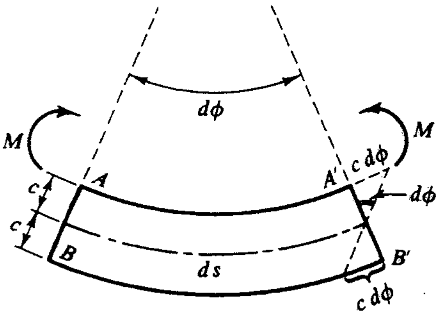


Fig. 5-2

Now equating Eqs. 5-2 and 5-3, we obtain the approximate curvature for a loaded beam:

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{5-4}$$

Note that Eq. 5-4 involves four major assumptions:

1. Small deflection of beam
2. Elastic material
3. Only bending moment considered significant
4. Plane section remaining plane after bending

The curvature, established in the coordinate axes of Fig. 5-1, clearly has the same sign as $M$, but the sign may be reversed if the direction of the $y$ axis is reversed. In that case, we have

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} \tag{5-4a}$$

## 5-3 EXTERNAL WORK AND INTERNAL WORK

If a variable force $F$ moves along its direction a distance $ds$, the work done is $F\,ds$. The total work done by $F$ during a period of movement may be expressed by

$$W = \int_{s_1}^{s_2} F\,ds \tag{5-5}$$

where $s_1$ and $s_2$ are the initial and final values of the position.

Consider a load gradually applied to a structure. Its point of application deflects and reaches a value $\Delta$ as the load increases from 0 to $P$. As long as the principle of superposition holds, a linear relationship exists between the load and the deflection, as represented by the line *oa* in Fig. 5-3. The total work performed by the applied load during this period is given by

$$W = \int_0^\Delta F\,ds = \int_0^\Delta \left(\frac{Ps}{\Delta}\right) ds = \frac{1}{2}P\Delta \tag{5-6}$$

which equals the area of the shaded triangle *oab* in Fig. 5-3.

If further deflection $\delta\Delta$, caused by an agent other than $P$, occurs to the structure in the action line of $P$, then the additional amount of work done by the already existing load $P$ will be $P\,\delta\Delta$, which equals the shaded rectangular area *abcd* shown in Fig. 5-3.

Similarly, the work done by a couple $M$ to turn an angular displacement $d\phi$ is $M\,d\phi$. The total work done by $M$ is

$$W = \int_{\phi_1}^{\phi_2} M\,d\phi \tag{5-7}$$

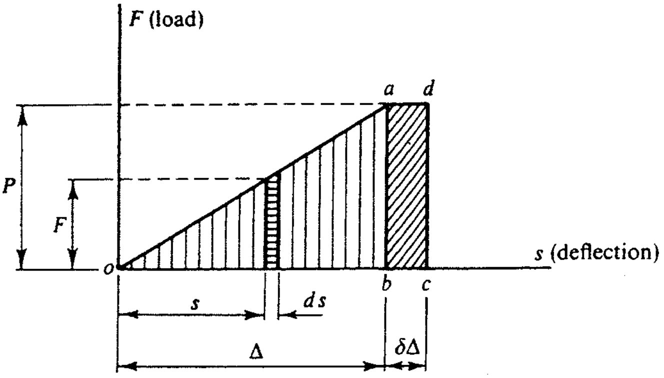


Fig. 5-3

Also, the work performed by a gradually applied couple $C$ accompanied by a rotation increasing from 0 to $\Theta$ is given by

$$W = \tfrac{1}{2}C\Theta \tag{5-8}$$

Now consider a beam subjected to gradually applied forces. As long as the linear relationship between the load and the deflection holds, all the external work will be converted into internal work or elastic strain energy. Let $dW$ be the strain energy restored in an infinitesimal element of the beam (see Fig. 5-2). We have

$$dW = \tfrac{1}{2}M\,d\phi$$

if only the bending moment $M$ produced by the forces on the element is considered significant. Using Eq. 5-3,

$$\frac{d\phi}{ds} = \frac{M}{EI}$$

or

$$d\phi = \frac{M\,ds}{EI}$$

we have

$$dW = \frac{M^2\,ds}{2EI}$$

For a loaded beam with its longitudinal axis taken as the $x$ axis, we let $ds \approx dx$ and obtain

$$dW = \frac{M^2dx}{2EI}$$

The total strain energy restored in the beam of length $l$ is, therefore, given by

$$W = \int_0^l \frac{M^2\,dx}{2EI} \tag{5-9}$$

For a truss subjected to gradually applied loads, the internal work performed by a member with constant cross-sectional area $A$, length $L$, and internal axial force $S$ is $S^2L/2AE$. The total internal work or elastic strain energy for the entire truss is

$$W = \sum \frac{S^2L}{2AE} \tag{5-10}$$

In some special cases deformations of structures can be found by equating external work $W_E$ and internal work (strain energy) $W_I$:

$$W_E = W_I \tag{5-11}$$

For instance, to find the deflection at the free end of the loaded cantilever beam shown in Fig. 5-4, we have

$$W_E = \tfrac{1}{2} P\Delta_b$$

$$W_I = \int_0^l \frac{M^2\,dx}{2EI}$$

$$= \int_0^l \frac{(-Px)^2\,dx}{2EI} = \frac{P^2l^3}{6EI}$$

Setting $W_E = W_I$ gives

$$\Delta_b = \frac{Pl^3}{3EI}$$

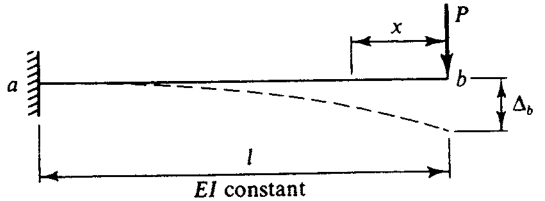


**Fig. 5-4**

Note that the method illustrated is quite limited in application since it is applicable only to deflection at a point of concentrated force. Furthermore, if more than one force is applied simultaneously to a structure, then more than one unknown deformation will appear in one equation, and a solution becomes impossible. Thus, we do not consider this as a general method.

## 5-4 METHOD OF VIRTUAL WORK (UNIT-LOAD METHOD)

Consider the two cases in Fig 5-5. Figure 5-5(a) illustrates a deformed elastic structure (be it a beam, a rigid frame, or a truss) subjected to the gradually applied loads $P_1$, $P_2$, . . . which move their points of application the distances $\Delta_1$, $\Delta_2$, . . . , respectively. In order to find an expression for the deformation at any point of the structure, say the vertical deflection component $\Delta$ at point $C$, we present the case of Fig. 5-5(b), which shows the same structure with all

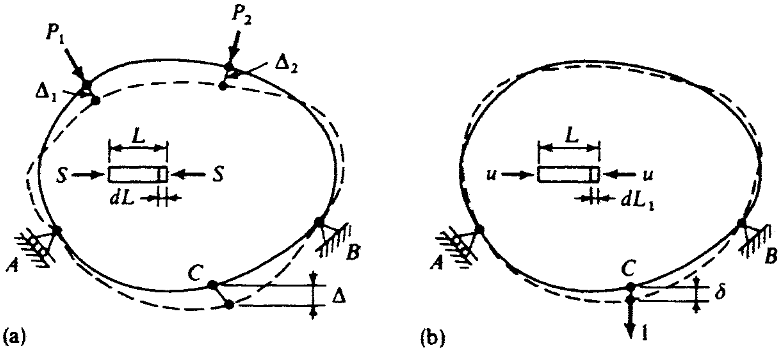


Fig. 5-5

the actual loads removed but a virtual load of unity being gradually applied at point $C$ along the desired deflection. Let $\delta$ denote the distance the unit load moves its point of application. Note that the virtual load is supposed to be vanishingly small and so are the corresponding virtual deformations.

Also shown in Fig. 5-5(a) is one of the typical deformed elements (be it a fiber in a beam or a rigid frame, or a bar in a truss) of length $L$ subjected to internal forces, called $S$, with a corresponding change in length $dL$. In Fig. 5-5(b) the same element is subjected to internal forces, called $u$, with a corresponding change in length $dL_1$.

Since the external work done by the applied loads must equal the internal strain energy of all elements in the structure, we obtain for Fig. 5-5(a),

$$\tfrac{1}{2}P_1\Delta_1 + \tfrac{1}{2}P_2\Delta_2 = \tfrac{1}{2}\sum S \cdot dL \qquad (5\text{-}12)$$

and for Fig. 5-5(b),

$$\tfrac{1}{2}(1)(\delta) = \tfrac{1}{2}\sum u \cdot dL_1 \qquad (5\text{-}13)$$

Now imagine that the case in Fig. 5-5(b) exists first; the actual loads $P_1$ and $P_2$ are then gradually applied to it. Equating the total work done and the total strain energy restored during this period, we have

$$\begin{aligned}\tfrac{1}{2}(1)(\delta) + \tfrac{1}{2}P_1\Delta_1 + \tfrac{1}{2}P_2\Delta_2 + 1 \cdot \Delta \\ = \tfrac{1}{2}\sum u \cdot dL_1 + \tfrac{1}{2}\sum S \cdot dL + \sum u \cdot dL \qquad (5\text{-}14)\end{aligned}$$

Since the strain energy and work done must be the same whether the loads are applied together or separately, we obtain from subtracting the sum of Eqs. 5-12 and 5-13 from Eq. 5-14,

$$\overbrace{\underbrace{1 \cdot \Delta = \sum u}_{\text{virtual}} \cdot dL}^{\text{actual}} \qquad (5\text{-}15)$$

Note that Eq. 5-15 is the basic equation of the unit-load method. When the rotation of tangent at any point in the structure is desired, we need only replace the unit virtual force with a *unit virtual couple* in the procedure described above, and we obtain

$$\overbrace{\underbrace{1 \cdot \theta = \sum u}_{\text{virtual}} \cdot dL}^{\text{actual}} \tag{5-16}$$

where $u$ is the internal force for a typical element caused by the unit couple and $\theta$ is the desired rotation angle.

To find a working formula for solving beam deformations, let us consider a statically determinate beam subjected to loads $P_1$ and $P_2$ as shown in Fig. 5-6(a); the longitudinal axis of the beam is taken as the $x$ axis. To find the vertical deflection $\Delta$ at an arbitrary point $C$, we place a unit vertical force at $C$, as shown in Fig. 5-6(b), and apply Eq. 5-15:

$$1 \cdot \Delta = \sum u \cdot dL$$

To interpret the terms $dL$ and $u$ involved in the equation above, let us first refer to Fig. 5-6(a) and observe that in the present case $dL$ is the change of length of any fiber having length $dx$ and cross-sectional area $dA$ caused by the actual loads $P_1$ and $P_2$. $dL$ equals unit elongation times $dx$ and can, therefore, be expressed by $My\,dx/EI$, in which $M$ is the bending moment at the section considered resulting from the actual loads, $I$ the moment of inertia of the cross-sectional area of the beam about the axis of bending, $y$ the distance from the fiber to the axis of bending, and $E$ the modulus of elasticity. Next, refer to Fig. 5-6(b) and observe that $u$ in this case is the internal force of the same fiber resulting from a fictitious unit load applied at $C$; $u$ equals the bending stress of the fiber times

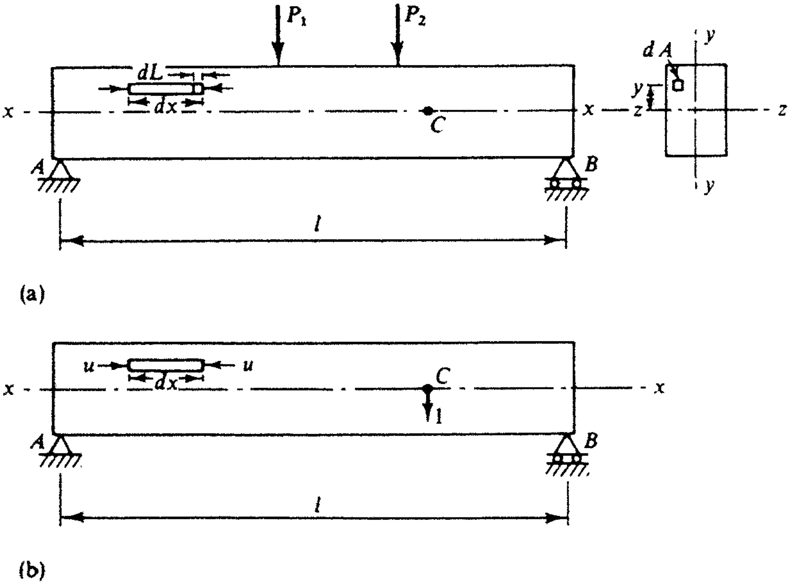


Fig. 5-6

$dA$, that is, $u = my\,dA/I$, where $m$ is the bending moment at the same section due to the unit load.

Substituting $dL = My\,dx/EI$ and $u = my\,dA/I$ in the basic equation gives

$$1 \cdot \Delta = \sum \left(\frac{my}{I}\,dA\right)\left(\frac{My}{EI}\,dx\right)$$

$$= \int_0^l \frac{Mm\,dx}{EI^2} \int_A y^2\,dA$$

Using $\int_A y^2\,dA = I$, we obtain

$$1 \cdot \Delta = \int_0^l \frac{Mm\,dx}{EI} \tag{5-17}$$

Equation 5-17 is the working formula for the determination of the deflection at any point of a beam. If rotation of the tangent at $C$ is desired, we place a unit couple at $C$ and apply the basic formula

$$1 \cdot \theta = \sum u \cdot dL$$

In a similar manner, we obtain

$$1 \cdot \theta = \int_0^l \frac{Mm\,dx}{EI} \tag{5-18}$$

where $m$ is the bending moment at any section due to a unit couple at $C$.

**Example 5-1**

Find the deflection and slope at the free end of a cantilever beam subjected to a uniform load [Fig. 5-7(a)].

To find $\Delta_b$, we place a unit vertical downward load at $b$ [Fig. 5-7(b)].

$$\Delta_b = \int_0^l \frac{Mm\,dx}{EI} = \int_0^l \frac{(-wx^2/2)(-x)\,dx}{EI} = \frac{wl^4}{8EI}$$

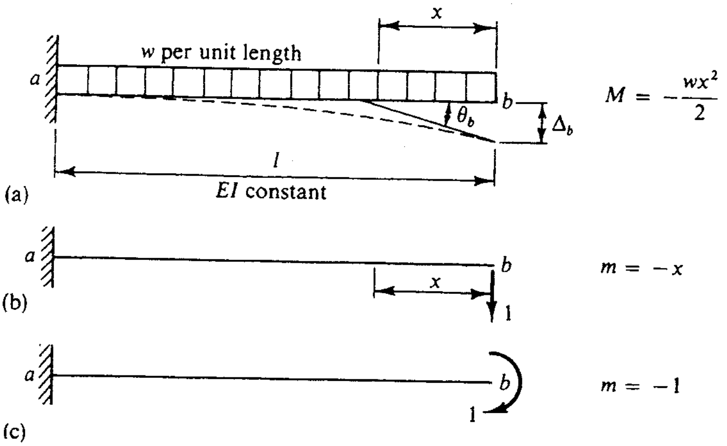


Fig. 5-7

To find $\theta_b$, we place a unit clockwise couple at $b$ [Fig. 5-7(c)].

$$\theta_b = \int_0^l \frac{Mm\,dx}{EI} = \int_0^l \frac{(-wx^2/2)(-1)\,dx}{EI} = \frac{wl^3}{6EI}$$

The positive results indicate that $\Delta_b$ and $\theta_b$ are in the directions assumed.

**Example 5-2**

Find $\theta_A$, $\theta_C$, and $\Delta_C$ of the loaded beam in Fig. 5-8(a). Assume constant $EI$.

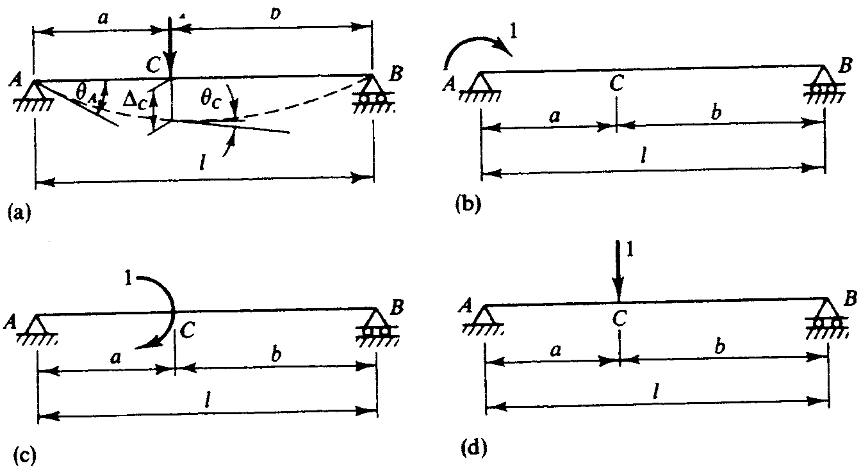


Fig. 5-8

To do this, we find it is advantageous to use double origins to perform the integration. That is,

$$\int_0^l \frac{Mm\,dx}{EI} = \int_0^a \frac{Mm\,dx}{EI} + \int_0^b \frac{Mm\,dx}{EI}$$

The terms of $M$ and $m$ in the expression above, solving for $\theta_A$, $\theta_C$, and $\Delta_C$, are evaluated as shown in Table 5-1.

$$\theta_A = \int_0^a \frac{(Pbx/l)[1-(x/l)]\,dx}{EI} + \int_0^b \frac{(Pax/l)(x/l)\,dx}{EI}$$

$$= \frac{1}{EI}\left(\frac{Pa^2b}{2l} - \frac{Pa^3b}{3l^2} + \frac{Pab^3}{3l^2}\right) = \frac{Pab}{6EIl}\left(3a - \frac{2a^2}{l} + \frac{2b^2}{l}\right) = \frac{Pab(l+b)}{6EIl}$$

$$\theta_C = \int_0^a \frac{(Pbx/l)(-x/l)\,dx}{EI} + \int_0^b \frac{(Pax/l)(x/l)\,dx}{EI}$$

$$= \frac{1}{EI}\left(-\frac{Pa^3b}{3l^2} + \frac{Pab^3}{3l^2}\right) = \frac{Pab(b-a)}{3EIl}$$

$$\Delta_C = \int_0^a \frac{(Pbx/l)(bx/l)\,dx}{EI} + \int_0^b \frac{(Pax/l)(ax/l)\,dx}{EI}$$

$$= \frac{1}{EI}\left(\frac{Pa^3b^2}{3l^2} + \frac{Pa^2b^3}{3l^2}\right) = \frac{Pa^2b^2}{3EIl}$$

**TABLE 5-1**

| | | | $M$ | $m$ for $\theta_A$ | $m$ for $\theta_C$ | $m$ for $\Delta_C$ |
|---|---|---|---|---|---|---|
| Section | Origin | Limit | Fig. 5-8(a) | Fig. 5-8(b) | Fig. 5-8(c) | Fig. 5-8(d) |
| $AC$ | $A$ | 0 to $a$ | $\frac{Pb}{l}x$ | $1 - \frac{x}{l}$ | $-\frac{x}{l}$ | $\frac{b}{l}x$ |
| $BC$ | $B$ | 0 to $b$ | $\frac{Pa}{l}x$ | $\frac{x}{l}$ | $\frac{x}{l}$ | $\frac{a}{l}x$ |

If $a = b = l/2$, then

$$\theta_C = 0 \qquad \theta_A = \frac{Pl^2}{16EI} \qquad \Delta_C = \frac{Pl^3}{48EI}$$

**Example 5-3**

Find the deflection at the center of the beam in Fig. 5-9. Use $E = 30{,}000$ kips/in.$^2$.

Refer to Table 5-2 and obtain

$$E\Delta_C = \int_0^l \frac{Mm\,dx}{I}$$

$$= 2\left[\frac{1}{1{,}000}\int_0^{10} (5x)\left(\frac{x}{2}\right) dx + \frac{1}{1{,}500}\int_0^{10} \frac{5(10 + x)^2}{2} dx\right]$$

$$= 2\left\{\frac{1}{1{,}000}\left[\frac{5}{6}x^3\right]_0^{10} + \frac{1}{1{,}500}\left(\frac{5}{2}\right)\left[100x + 10x^2 + \frac{x^3}{3}\right]_0^{10}\right\}$$

$$= 2\left(\frac{5}{6} + \frac{70}{18}\right) = 9.44$$

Now let us check the dimensions of both sides of the preceding expression. Note that a unit load of 1 kip must be included in the left side of the expression.

$$30{,}000\left(\frac{\text{kips}}{\text{in.}^2}\right)(1\text{ kip})(\Delta_C) = 9.44\,\frac{\text{ft-kips ft-kips ft}}{\text{in.}^4}$$

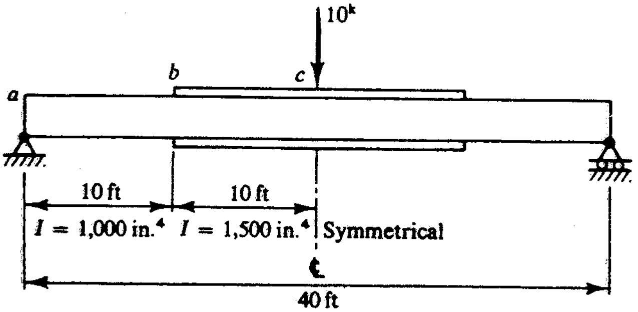


Fig. 5-9

**TABLE 5-2**

| Section | Origin | Limit (ft) | $M$ (ft-kips) | $m$ (ft-kips)* | $I$ (in.$^4$) |
|---|---|---|---|---|---|
| $ab$ | $a$ | 0 to 10 | $5x$ | $\frac{x}{2}$ | 1,000 |
| $bc$ | $b$ | 0 to 10 | $5(10 + x)$ | $\frac{1}{2}(10 + x)$ | 1,500 |

* We use a unit load of 1 kip.

Thus,

$$\Delta_C = \frac{9.44}{30{,}000}\,\frac{\text{ft}^3}{\text{in.}^2}$$

or

$$\Delta_C = \frac{(9.44)(1{,}728)\text{ in.}}{30{,}000} = 0.544 \text{ in.} \qquad \text{(down)}$$

In a rigid frame the strain energy due to axial forces and shearing forces is usually much smaller than that due to bending moment and can, therefore, be neglected. The formula

$$\int \frac{Mm\,dx}{EI}$$

previously derived for beam deformations is also good for finding the elastic deformations for a rigid frame, as illustrated in the following examples.

**Example 5-4**

Determine the horizontal, vertical, and rotational deflection components at end $a$ of rigid frame shown in Fig. 5-10(a). Assume that all members have the same value of $EI$.

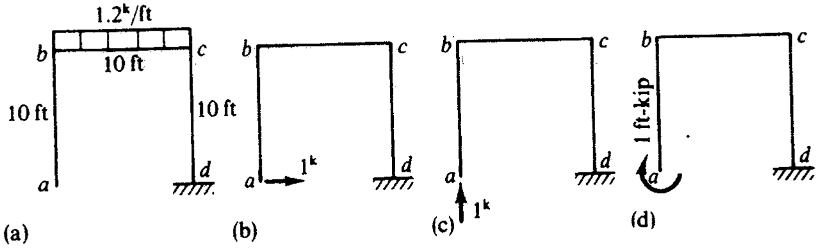


Fig. 5-10

To perform the integration for the entire frame denoted by

$$\int_F$$

we must consider each member as a unit, the centroidal axis of the member being taken as the $x$ axis. Thus,

$$\int_F \frac{Mm\,dx}{EI} = \int_{ab} \frac{Mm\,dx}{EI} + \int_{bc} \frac{Mm\,dx}{EI} + \int_{cd} \frac{Mm\,dx}{EI}$$

The terms of $M$ and $m$ in this expression, when we solve for each of the deflection components at $a$, are listed in Table 5-3, in which we use $m_1$ to denote

**TABLE 5-3**

| Member | Origin | Limit (ft) | $M$ (ft-kips)<br>Fig. 5-10(a) | $m_1$ (ft-kips)<br>Fig. 5-10(b) | $m_2$ (ft-kips)<br>Fig. 5-10(c) | $m_3$ (ft-kips)<br>Fig. 5-10(d) |
|---|---|---|---|---|---|---|
| $ab$ | $a$ | 0 to 10 | 0 | $-x$ | 0 | 1 |
| $bc$ | $b$ | 0 to 10 | $-\dfrac{1.2x^2}{2}$ | $-10$ | $x$ | 1 |
| $cd$ | $c$ | 0 to 10 | $-60$ | $x - 10$ | 10 | 1 |

the bending moment at any section due to a unit horizontal force applied at $a$; $m_2$ that due to a unit vertical force at $a$; and $m_3$ that due to a unit couple at $a$. Note that the bending moment resulting in compression on the outside fibers of the frame is assumed to be positive.

To find the horizontal deflection at $a$, called $\Delta_1$, we apply

$$\Delta_1 = \int_F \frac{Mm_1\,dx}{EI}$$

$$= \frac{1}{EI}\left[0 + \int_0^{10}\left(-\frac{1.2x^2}{2}\right)(-10)\,dx + \int_0^{10}(-60)(x - 10)\,dx\right]$$

$$= 5{,}000\,\frac{\text{kips-ft}^3}{EI} \qquad \text{(right)}$$

Similarly, we have the vertical deflection at $a$, called $\Delta_2$,

$$\Delta_2 = \int_F \frac{Mm_2\,dx}{EI}$$

$$= \frac{1}{EI}\left[0 + \int_0^{10}\left(-\frac{1.2x^2}{2}\right)(x)\,dx + \int_0^{10}(-60)(10)\,dx\right]$$

$$= -7{,}500\,\frac{\text{kips-ft}^3}{EI} \qquad \text{(down)}$$

and the rotational displacement at $a$, called $\Delta_3$,

$$\Delta_3 = \int_F \frac{Mm_3\,dx}{EI}$$

$$= \frac{1}{EI}\left[0 + \int_0^{10}\left(-\frac{1.2x^2}{2}\right)(1)\,dx + \int_0^{10}(-60)(1)\,dx\right]$$

$$= -800\,\frac{\text{kips-ft}^2}{EI} \qquad \text{(counterclockwise)}$$

**Example 5-5**

Find the deflection components at $a$ of the same frame for each of three loading cases shown in Fig. 5-10(b)–(d). Let

$\delta_{11}$ = horizontal deflection at $a$ due to a unit horizontal force at $a$
$\delta_{21}$ = vertical deflection at $a$ due to a unit horizontal force at $a$
$\delta_{31}$ = rotational displacement at $a$ due to a unit horizontal force at $a$

Then

$$\delta_{11} = \int_F \frac{(m_1)^2\,dx}{EI} \qquad \delta_{21} = \int_F \frac{m_1 m_2\,dx}{EI} \qquad \delta_{31} = \int_F \frac{m_1 m_3\,dx}{EI}$$

since in this case [see Fig. 5-10(b)] $M = m_1$.

Likewise, if the frame is subjected only to a unit vertical force at $a$ [see Fig. 5-10(c)], the three deflection components at $a$ are found to be

$$\delta_{12} = \int_F \frac{m_2 m_1\,dx}{EI} \qquad \delta_{22} = \int_F \frac{(m_2)^2\,dx}{EI} \qquad \delta_{32} = \int_F \frac{m_2 m_3\,dx}{EI}$$

And if the frame is subjected only to a unit couple at $a$ [see Fig. 5-10(d)], the three deflection components at $a$ are found to be

$$\delta_{13} = \int_F \frac{m_3 m_1\,dx}{EI}. \qquad \delta_{23} = \int_F \frac{m_3 m_2\,dx}{EI} \qquad \delta_{33} = \int_F \frac{(m_3)^2\,dx}{EI}$$

Taking numerical values from Table 5-3 and substituting in each of the expressions above, we find

$$\delta_{11} = \frac{1}{EI}\left[\int_0^{10} x^2\,dx + \int_0^{10} (-10)(-10)\,dx + \int_0^{10} (x-10)^2\,dx\right]$$

$$= 1{,}667\,\frac{\text{kips-ft}^3}{EI} \qquad \text{(right)}$$

$$\delta_{21} = \frac{1}{EI}\left[0 + \int_0^{10} (-10)(x)\,dx + \int_0^{10} (x-10)(10)\,dx\right]$$

$$= -1{,}000\,\frac{\text{kips-ft}^3}{EI} \qquad \text{(down)}$$

$$\delta_{31} = \frac{1}{EI}\left[\int_0^{10} (-x)(1)\,dx + \int_0^{10} (-10)(1)\,dx + \int_0^{10} (x-10)(1)\,dx\right]$$

$$= -200\,\frac{\text{kips-ft}^2}{EI} \qquad \text{(counterclockwise)}$$

$$\delta_{12} = \delta_{21} = -1{,}000\,\frac{\text{kips-ft}^3}{EI} \qquad \text{(left)}$$

$$\delta_{22} = \frac{1}{EI}\left[0 + \int_0^{10} x^2\,dx + \int_0^{10} (10)^2\,dx\right] = 1{,}333\,\frac{\text{kips-ft}^3}{EI} \qquad \text{(up)}$$

$$\delta_{32} = \frac{1}{EI}\left[0 + \int_0^{10} (x)(1)\,dx + \int_0^{10} (10)(1)\,dx\right] = 150\,\frac{\text{kips-ft}^2}{EI} \qquad \text{(clockwise)}$$

$$\delta_{13} = -200\,\frac{\text{kips-ft}^3}{EI} \qquad \text{(left)}$$

$$\delta_{23} = 150\,\frac{\text{kips-ft}^3}{EI} \qquad \text{(up)}$$

$$\delta_{33} = \frac{1}{EI}\left[\int_0^{10} dx + \int_0^{10} dx + \int_0^{10} dx\right] = 30\,\frac{\text{kips-ft}^2}{EI} \qquad \text{(clockwise)}$$

Note that $\delta_{13}$ has the same values as $\delta_{31}$, but they differ by one dimension of length. The same is true for $\delta_{23}$ and $\delta_{32}$.

In matrix form the result shown on the previous page is

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} \int_F \frac{(m_1)^2\,dx}{EI} & \int_F \frac{m_1 m_2\,dx}{EI} & \int_F \frac{m_1 m_3\,dx}{EI} \\ \int_F \frac{m_1 m_2\,dx}{EI} & \int_F \frac{(m_2)^2\,dx}{EI} & \int_F \frac{m_2 m_3\,dx}{EI} \\ \int_F \frac{m_1 m_3\,dx}{EI} & \int_F \frac{m_2 m_3\,dx}{EI} & \int_F \frac{(m_3)^2\,dx}{EI} \end{bmatrix}$$

$$= \begin{bmatrix} 1{,}667\text{ ft} & -1{,}000\text{ ft} & -200\text{ ft} \\ -1{,}000\text{ ft} & 1{,}333\text{ ft} & 150\text{ ft} \\ -200 & 150 & 30 \end{bmatrix} \frac{\text{kips-ft}^2}{EI}$$

The working formula for the deflection of any joint of a loaded truss can be evaluated from the basic equation, Eq. 5-15,

$$1 \cdot \Delta = \sum u \cdot dL$$

by considering each member of the truss as an element. Thus, the term $dL$ is the shortening or lengthening of a bar due to applied loads and can be expressed by $SL/AE$. The equation above becomes

$$1 \cdot \Delta = \sum_{1}^{m} \frac{SuL}{AE} \tag{5-19}$$

where $S$ = internal force in any member due to actual loads
$u$ = internal force in the same member due to a fictitious unit load at the point where the deflection is sought, acting along the desired direction
$L$ = length of the member
$A$ = cross-sectional area of the member
$E$ = modulus of elasticity of the member
$m$ = total number of members

Equation 5-19 in matrix form is

$$\Delta = [u_1 \quad u_2 \cdots u_m] \begin{bmatrix} \frac{L_1}{A_1 E} & & & \\ & \frac{L_2}{A_2 E} & & \\ & & \ddots & \\ & & & \frac{L_m}{A_m E} \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_m \end{Bmatrix} \tag{5-19a}$$

Sometimes the change of bar length $dL$ is not caused by any external force but is due to the effect of temperature. If this is the case, we let $dL = \alpha t L$ and the working formula for finding deflection due to temperature change is given by

$$1 \cdot \Delta = \sum u \cdot \alpha t L \tag{5-20}$$

where $\alpha$ = coefficient of linear thermal expansion
$t$ = temperature rise in degrees

**Example 5-6**

Find the vertical deflection of joint $b$ of the loaded truss shown in Fig. 5-11(a). Assume that $L$ (ft)/$A$ (in.$^2$) = 1 and that $E$ = 30,000 kips/in.$^2$ for all members.

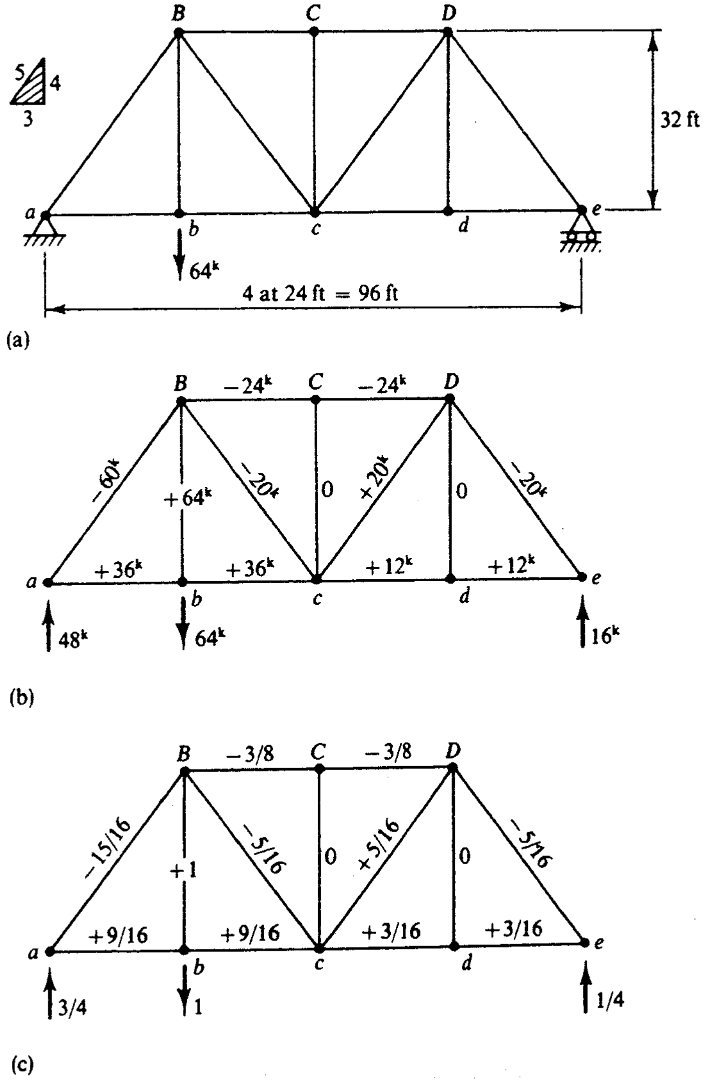


Fig. 5-11

We begin with the evaluation of $S$ and $u$. The answer diagrams for them are shown in Fig. 5-11(b) and (c), respectively. Next we apply Eq. 5-19. The complete solution is given in Table 5-4.

**TABLE 5-4**

| Member | $\frac{L}{A}\left(\frac{\text{ft}}{\text{in.}^2}\right)$ | $S$(kips) | $u$* | $\frac{SuL}{A}\left(\frac{\text{ft-kips}}{\text{in.}^2}\right)$ |
|---|---|---|---|---|
| $ab$ | 1 | +36 | +9/16 | + 20.25 |
| $bc$ | 1 | +36 | +9/16 | + 20.25 |
| $cd$ | 1 | +12 | +3/16 | + 2.25 |
| $de$ | 1 | +12 | +3/16 | + 2.25 |
| $BC$ | 1 | −24 | −3/8 | + 9.0 |
| $CD$ | 1 | −24 | −3/8 | + 9.0 |
| $aB$ | 1 | −60 | −15/16 | + 56.25 |
| $Bb$ | 1 | +64 | +1 | + 64.0 |
| $Bc$ | 1 | −20 | −5/16 | + 6.25 |
| $Cc$ | 1 | 0 | 0 | 0 |
| $cD$ | 1 | +20 | +5/16 | + 6.25 |
| $Dd$ | 1 | 0 | 0 | 0 |
| $De$ | 1 | −20 | −5/16 | + 6.25 |
| | | | $\Sigma$ | +202.0 |

*We use a fictitious load of 1 (not 1 kip) for determining $u$ values.

$$\Delta_v = \sum \frac{SuL}{AE} = \frac{+202}{30{,}000} = +0.00673 \text{ ft} \qquad \text{(down)}$$

**Example 5-7**

For the loaded structure in Example 5-6, find the absolute deflection of joint $b$.

To do this, we have to obtain the horizontal deflection of joint $b$ in addition to the vertical deflection of that joint already found. The vector sum of these two displacement components is the solution.

When a unit horizontal load is applied at joint $b$ to the right, only the member $ab$ is under the stress of tension (i.e., $u = 1$); all other members are unstressed. The horizontal movement, called $\Delta_h$, at joint $b$ is thus given by

$$\Delta_h = \left(\frac{SuL}{AE}\right)_{ab} = \frac{(36)(1)}{30{,}000} = +0.0012 \text{ ft} \qquad \text{(right)}$$

and the absolute deflection of joint $b$ is given by

$$\Delta = \sqrt{(\Delta_v)^2 + (\Delta_h)^2} = \sqrt{(0.00673)^2 + (0.0012)^2} = 0.00684 \text{ ft}$$

moving down to the right and making an angle $\phi$ with the horizontal direction,

$$\phi = \tan^{-1}\frac{0.00673}{0.00120} = \tan^{-1} 5.72 \approx 80^\circ$$

**Example 5-8**

For the same loaded truss [Fig. 5-11(a)], find the rotation of member $bc$.

Finding the rotation of member $bc$ is equivalent to finding the relative displacement between ends *b* and *c* (in the direction perpendicular to *bc*) divided by the length of *bc*. Assume counterclockwise rotation. We then apply a pair of unit fictitious loads to joints *b* and *c* and evaluate the *u* value for each member, as shown in Fig. 5-12. The computation leading to the solution of the relative displacement between joints *b* and *c* perpendicular to the original line of *bc* is contained in Table 5-5. The rotation of the member, denoted by $\theta$, is then determined:

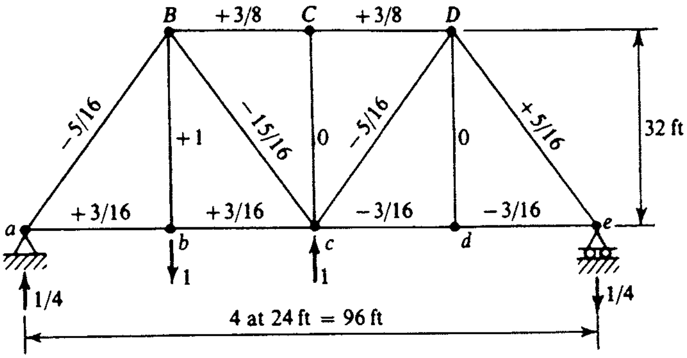


Fig. 5-12

$$\theta = \frac{80}{24E} = \frac{80}{(24)(30{,}000)} = \frac{1}{9{,}000}\text{ rad}$$

The positive value of the angle indicates a counterclockwise rotation.

**TABLE 5-5**

| Member | $\frac{L}{A}\left(\frac{\text{ft}}{\text{in.}^2}\right)$ | $S$(kips) | $u$ | $\frac{SuL}{A}\left(\frac{\text{ft-kips}}{\text{in.}^2}\right)$ |
|---|---|---|---|---|
| *ab* | 1 | +36 | +3/16 | +27/4 |
| *bc* | 1 | +36 | +3/16 | +27/4 |
| *cd* | 1 | +12 | −3/16 | − 9/4 |
| *de* | 1 | +12 | −3/16 | − 9/4 |
| *BC* | 1 | −24 | +3/8 | − 9 |
| *CD* | 1 | −24 | +3/8 | − 9 |
| *aB* | 1 | −60 | −5/16 | +75/4 |
| *Bb* | 1 | +64 | +1 | +64 |
| *Bc* | 1 | −20 | −15/16 | +75/4 |
| *Cc* | 1 | 0 | 0 | 0 |
| *cD* | 1 | +20 | −5/16 | −25/4 |
| *Dd* | 1 | 0 | 0 | 0 |
| *De* | 1 | −20 | +5/16 | −25/4 |
|  |  |  | Σ | +80 |

**Example 5-9**

Find the vertical deflection at joint $b$ resulting from a rise in temperature of 50°F in the top chords $BC$ and $CD$ (Fig. 5-13). $\alpha = 0.0000065$ in./in./1°F.

On a statically determinate truss, no reactions or internal forces can be developed because of a temperature rise or drop in truss members. However, certain changes of bar length will take place if the temperature rises or drops in a bar. This in turn will cause the distortion of the whole truss.

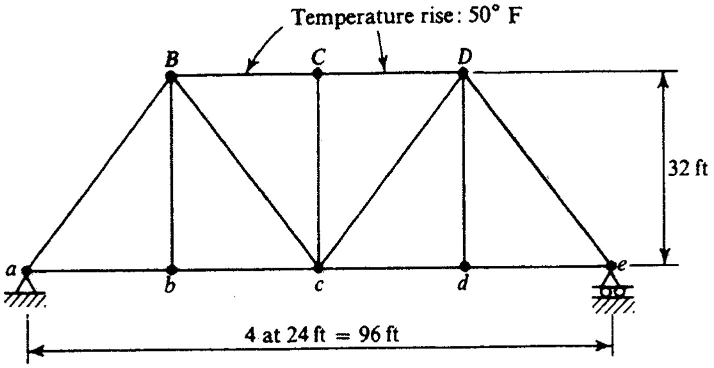


Fig. 5-13

To find the vertical deflection of joint $b$, we apply Eq. 5-20,

$$\Delta_b = \sum u \cdot \alpha t L$$

Note that, in this problem, only bars $BC$ and $CD$ are involved in computation since the rest of the members undergo no change of length. Now $u = -\frac{3}{8}$ [see Fig. 5-11(c)] and $\alpha t L = (0.0000065)(50)(24) = 0.0078$ for $BC$ and $CD$. Thus,

$$\Delta_b = 2(-\tfrac{3}{8})(0.0078) = -0.00585 \text{ ft}$$

The negative sign indicates an upward movement of joint $b$.

## 5-5 CASTIGLIANO'S THEOREM

In 1876, Alberto Castigliano published a notable paper in which he presented a general method for determining the deformations of linear structures, namely *the first partial derivative of the total strain energy of the structure with respect to one of the applied actions gives the displacement along that action.*

If we use $P$ to denote the action (force or couple), $\Delta_p$ the corresponding displacement (deflection or rotation) along $P$, and $W$ the total strain energy, the statement can be expressed by

$$\Delta_p = \frac{\partial W}{\partial P} \tag{5-21}$$

To demonstrate the theorem, consider the loaded beam in Fig. 5-14. The deflected position is represented by the dashed line.

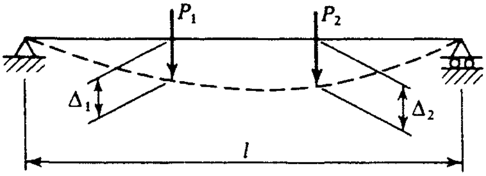


Fig. 5-14

If we consider only the internal work resulting from the bending moment, we have the total strain energy of the beam (see Eq. 5-9):

$$W = \int_0^l \frac{M^2 dx}{2EI}$$

Now let $M_1$ be the bending moment at any section due to the gradually applied load $P_1$, and let $M_2$ be the bending moment at the same section due to the gradually applied load $P_2$. The total bending moment at any section is given by

$$M = M_1 + M_2 = m_1 P_1 + m_2 P_2$$

where $m_1$ = bending moment at any section due to a unit load in place of $P_1$

$m_2$ = bending moment of the same section due to a unit load in place of $P_2$

Thus,
$$\frac{\partial W}{\partial P_1} = \frac{\partial}{\partial P_1}\int_0^l \frac{M^2\,dx}{2EI} = \int_0^l \frac{M(\partial M/\partial P_1)\,dx}{EI} = \int_0^l \frac{Mm_1\,dx}{EI} = \Delta_1$$

and
$$\frac{\partial W}{\partial P_2} = \frac{\partial}{\partial P_2}\int_0^l \frac{M^2\,dx}{2EI} = \int_0^l \frac{M(\partial M/\partial P_2)\,dx}{EI} = \int_0^l \frac{Mm_2\,dx}{EI} = \Delta_2$$

The last equality in each of the two expressions above is based on Eq. 5-17 from virtual work.

Let us now turn to the loaded truss in Fig. 5-15. The total strain energy (see Eq. 5-10) is

$$W = \sum \frac{S^2 L}{2AE}$$

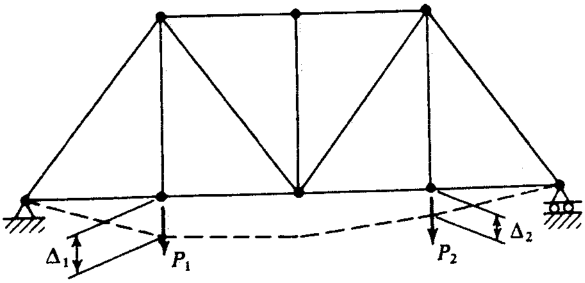


Fig. 5-15

If we let

$S_1$ = internal force in any bar due to the gradually applied load $P_1$

$S_2$ = internal force in the same bar due to the gradually applied load $P_2$

then the total internal force in any bar is given by

$$S = S_1 + S_2 = P_1 u_1 + P_2 u_2$$

where $u_1$ = internal force in any bar due to a unit load in place of $P_1$

$u_2$ = internal force in the same bar due to a unit load in place of $P_2$

Thus, $$\frac{\partial W}{\partial P_1} = \frac{\partial}{\partial P_1}\sum\frac{S^2L}{2AE} = \sum\frac{S(\partial S/\partial P_1)L}{AE} = \sum\frac{Su_1L}{AE} = \Delta_1$$

and $$\frac{\partial W}{\partial P_2} = \frac{\partial}{\partial P_2}\sum\frac{S_2L}{2AE} = \sum\frac{S(\partial S/\partial P_2)L}{AE} = \sum\frac{Su_2L}{AE} = \Delta_2$$

The last equality in each of the expressions above is based on Eq. 5-19 from virtual work.

It is interesting to point out that the Castigliano's theorem basically does not differ from the method of virtual work for the analysis of linear structures subjected to external forces. The difference is only a matter of the arrangement of calculation. Using the method of virtual work, we have from Eqs. 5-17 and 5-19,

$$\Delta = \int\frac{Mm\,dx}{EI} \qquad \text{for a beam or rigid frame}$$

and $$\Delta = \sum\frac{SuL}{AE} \qquad \text{for a truss}$$

while applying Castigliano's theorem, we have

$$\Delta = \int\frac{M(\partial M/\partial P)\,dx}{EI} \qquad \text{for a beam or rigid frame} \tag{5-22}$$

and $$\Delta = \sum\frac{S(\partial S/\partial P)\,L}{AE} \qquad \text{for a truss} \tag{5-23}$$

**Example 5-10**

Find the vertical deflection at the free end $b$ of a cantilever beam $ab$ subjected to a concentrated load $P$ at $b$ (see Fig. 5-4).

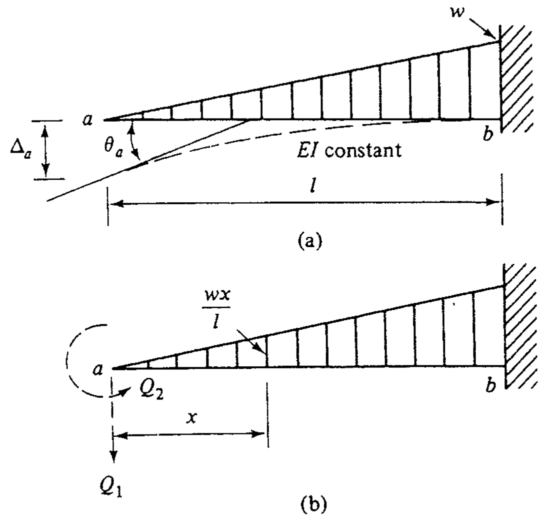
$$\Delta_b = \frac{\partial W}{\partial P} = \int_0^l \frac{M(\partial M/\partial P)\,dx}{EI} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial P}\,dx$$

Now $$M = -Px \qquad \frac{\partial M}{\partial P} = -x$$

Therefore, $$\Delta_b = \frac{1}{EI}\int_0^l(-Px)(-x)\,dx = \frac{Pl^3}{3EI} \qquad \text{(down)}$$

**Example 5-11**

For the beam and load shown in Fig. 5-16(a), find the vertical and rotational displacements at the free end $a$. Given $w$ = 15 kN/m, $l$ = 5 m, $E$ = 20,000 kN/cm$^2$, and $I$ = 12,000 cm$^4$, determine the magnitude of the displacements.



(a)

(b)

**Fig. 5-16**

To find the displacements at the free end, we note that in this case no vertical force or moment force actually acts at the free end; thus Castigliano's theorem cannot be directly applied. In order to carry out the partial derivative, we must first assume the imaginary forces $Q_1$, $Q_2$, corresponding respectively to the vertical deflection $\Delta_a$ and the rotation $\theta_a$ at the free end, and then set $Q_1 = Q_2 = 0$ in the final operation.

Refer to Fig. 5-16(b). The moment at any section is given by

$$M = -Q_1 x - Q_2 - \frac{wx^3}{6l}$$

It follows that

$$\frac{\partial M}{\partial Q_1} = -x \qquad \text{and} \qquad \frac{\partial M}{\partial Q_2} = -1$$

$$\Delta_a = \int_0^l \frac{M(\partial M/\partial Q_1)\,dx}{EI}$$

$$= \int_0^l \frac{(-wx^3/6l)(-x)\,dx}{EI} = \frac{wl^4}{30EI} \qquad \text{(down)}$$

$$\theta_a = \int_0^l \frac{M(\partial M/\partial Q_2)\,dx}{EI}$$

$$= \int_0^l \frac{(-wx^3/6l)(-1)\,dx}{EI} = \frac{wl^3}{24EI} \qquad \text{(counterclockwise)}$$

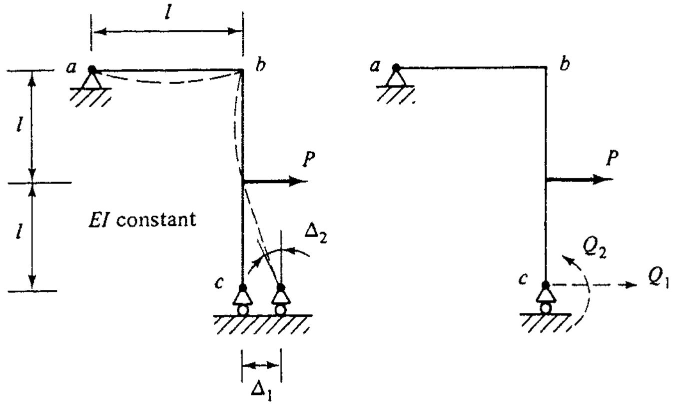
Substituting $w = 15$ kN/m, $l = 5$ m, $E = 20{,}000$ kN/cm$^2$, and $I = 12{,}000$ cm$^4$ into the foregoing expressions, we have

$$\Delta_a = \frac{(15)(5)^4(10)^6}{(30)(20{,}000)(12{,}000)} = 1.3 \text{ cm}$$

$$\theta_a = \frac{(15)(5)^3(10)^4}{(24)(20{,}000)(12{,}000)} = 0.003 \text{ rad}$$

**Example 5-12**

Find, by Castigliano's theorem, the horizontal displacement $\Delta_1$ and the rotational displacement $\Delta_2$ at support $c$ for the rigid frame shown in Fig. 5-17(a). Consider the bending effect only.



(a) (b)

Fig. 5-17

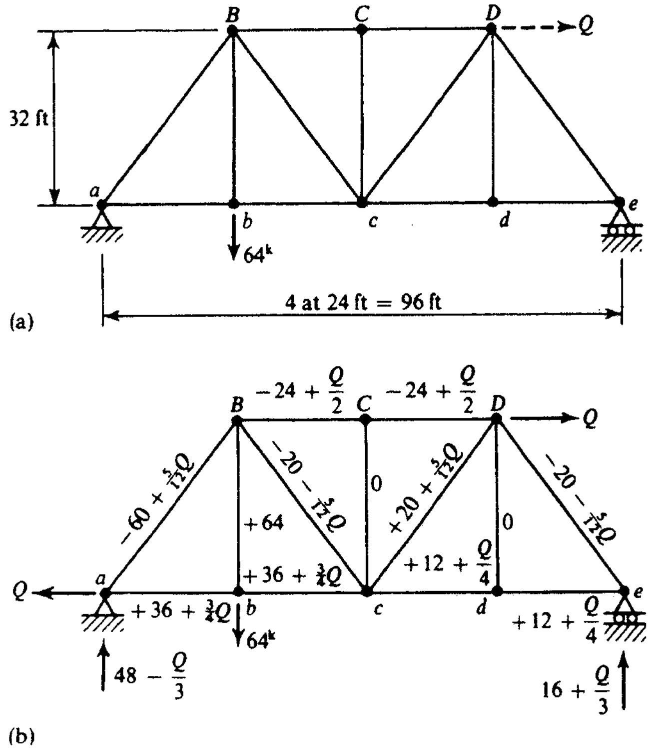


Fig. 5-18

**TABLE 5-6**

| Member | Origin | Limit | $M$ | $\frac{\partial M}{\partial Q_1}$ | $\frac{\partial M}{\partial Q_2}$ | $\int \frac{M(\partial M/\partial Q_1)\,dx}{EI}$ | $\int \frac{M(\partial M/\partial Q_2)\,dx}{EI}$ |
|---|---|---|---|---|---|---|---|
| $ab$ | $a$ | 0 to $l$ | $\left(P + 2Q_1 + \frac{Q_2}{l}\right)x$ | $2x$ | $\frac{x}{l}$ | $\int_0^l \frac{(Px)(2x)\,dx}{EI}$ | $\int_0^l \frac{(Px)(x/l)\,dx}{EI}$ |
| $bc$ | $c$ | $l$ to $2l$ | $P(x - l) + Q_1x + Q_2$ | $x$ | 1 | $\int_l^{2l} \frac{P(x - l)(x)\,dx}{EI}$ | $\int_l^{2l} \frac{P(x - l)(1)\,dx}{EI}$ |
| | | | | | | $\Delta_1 = \frac{3Pl^3}{2EI}$ | $\Delta_2 = \frac{5Pl^2}{6EI}$ |

Since no horizontal force or moment actually acts at support $c$, we must assume forces $Q_1$ and $Q_2$ of zero value, corresponding respectively to the desired displacements $\Delta_1$ and $\Delta_2$ at $c$ [Fig. 5-17(b)], in order to carry out the partial derivatives such that

$$\Delta_1 = \int_F \frac{M(\partial M/\partial Q_1)\,dx}{EI}$$

$$\Delta_2 = \int_F \frac{M(\partial M/\partial Q_2)\,dx}{EI}$$

where $\int_F$ indicates the sign of integration carried through the entire frame. The complete solution is given in Table 5-6.

**Example 5-13**

Given the loaded truss in Fig. 5-11(a), find the horizontal deflection at $D$.

Assume that joint $D$ will move to the right. To apply the theorem we place an imaginary horizontal force $Q$ acting at $D$, as shown in Fig. 5-18(a). The bar forces thus obtained are shown in Fig. 5-18(b). The complete solution is shown in Table 5-7.

$$\Delta = \sum \frac{S(\partial S/\partial Q)L}{AE} = +\frac{36}{30{,}000} = +0.0012\text{ ft} \qquad \text{(right)}$$

**TABLE 5-7**

| Member | $\frac{L}{A}$ (ft/in.$^2$) | $S$(kips) | $\frac{\partial S}{\partial Q}$ | $\frac{S(\partial S/\partial Q)L}{A}$ (ft-kips/in.$^2$) |
|---|---|---|---|---|
| $ab$ | 1 | $+36 + 3/4\ Q$ | $+3/4$ | $+27$ |
| $bc$ | 1 | $+36 + 3/4\ Q$ | $+3/4$ | $+27$ |
| $cd$ | 1 | $+12 + 1/4\ Q$ | $+1/4$ | $+\ 3$ |
| $de$ | 1 | $+12 + 1/4\ Q$ | $+1/4$ | $+\ 3$ |
| $BC$ | 1 | $-24 + 1/2\ Q$ | $+1/2$ | $-12$ |
| $CD$ | 1 | $-24 + 1/2\ Q$ | $+1/2$ | $-12$ |
| $aB$ | 1 | $-60 + 5/12Q$ | $+5/12$ | $-25$ |
| $Bb$ | 1 | $+\ 64$ | 0 | 0 |
| $Bc$ | 1 | $-20 - 5/12Q$ | $-5/12$ | $+\ 8.33$ |
| $Cc$ | 1 | 0 | 0 | 0 |
| $cD$ | 1 | $+20 + 5/12Q$ | $+5/12$ | $+\ 8.33$ |
| $Dd$ | 1 | 0 | 0 | 0 |
| $De$ | 1 | $-20 - 5/12Q$ | $-5/12$ | $+\ 8.33$ |
| | | | | $\Sigma$ $+36.0$ |

## 5-6 CONJUGATE-BEAM METHOD

The purpose of this method is to transform the problem of solving the slopes and deflections of a structure resulting from the applied loads (actual loads) to a problem of solving the shears and moments of a conjugate beam due to the elastic load derived from the angle changes of structural elements. The advantages of this method over the method of virtual work or Castigliano's theorem are as follows:

1. Unlike the previous methods, which are used to find one item of deformation at one point of the structure in an operation, this method enables us to find deformations at many points of the structure in a single setup.
2. It is generally acknowledged that structural engineers prefer to deal with shear and bending moment rather than to do tedious integral calculus.

Consider a typical loaded beam, such as the one shown in Fig. 5-19(a), for which we may plot the moment diagram and, therefore, the $M/EI$ diagram as in Fig. 5-19(b).

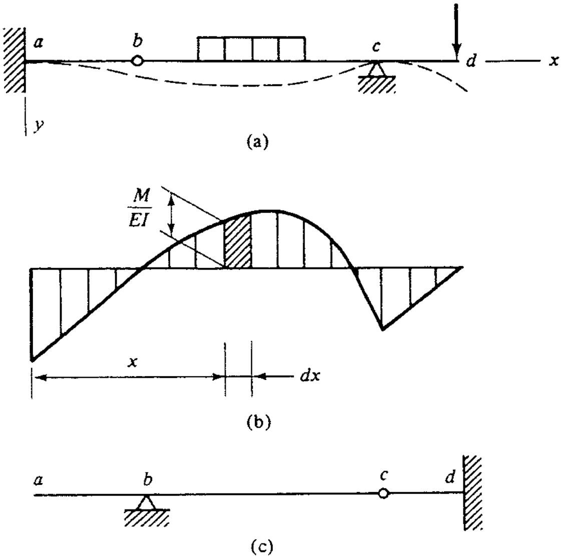


Fig. 5-19

We recall that the curvature at any point of the beam of Fig. 5-19(a) is given by Eq. 5-4a:

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}$$

Now since the slope at any point of the beam is expressed by

$$\frac{dy}{dx} = \tan\theta \approx \theta$$

for small deformation, we have

$$\frac{d\theta}{dx} = -\frac{M}{EI}$$

or

$$d\theta = -\frac{M\,dx}{EI}$$

Integrate.

$$\theta = -\int \frac{M}{EI}\,dx \tag{5-24}$$

Substituting for $\theta$ by $dy/dx$ and integrating again gives

$$y = \int \theta\,dx = -\int\int \frac{M}{EI}\,dx\,dx \tag{5-25}$$

Next, for a beam under a distributed load of intensity $w(x)$, the relationships between the load, the shear, and the bending moment at any point are given by (see Sec. 3-3)

$$\frac{dV}{dx} = -w$$

and

$$\frac{dM}{dx} = V$$

Thus, over a portion of the beam

$$V = -\int w\,dx \tag{5-26}$$

and

$$M = \int V\,dx = -\int\int w\,dx\,dx \tag{5-27}$$

Now suppose that we have a beam, called a *conjugate beam*, whose length equals that of the actual beam in Fig. 5-19(a). Let this beam be subjected to the so-called *elastic load* of intensity $M/EI$ given in Fig. 5-19(b). [Elastic load is sometimes referred to as the *angle load*, a term obviously associated with $d\theta = M\,(dx/EI)$.] The integral expressions for the shear and moment over a portion of the conjugate beam, denoted by $\overline{V}$ and $\overline{M}$, respectively, can be obtained by replacing $w$ in Eqs. 5-26 and 5-27 with $M/EI$:

$$\overline{V} = -\int \frac{M}{EI}\,dx \tag{5-28}$$

and

$$\overline{M} = -\int\int \frac{M}{EI}\,dx\,dx \tag{5-29}$$

When we compare Eqs. 5-24 and 5-25 with Eqs. 5-28 and 5-29, it follows logically that, with properly prescribed boundary conditions for the conjugate beam, we may reach the following results:

1. The slope at a given section of a loaded beam (actual beam) equals the shear in the corresponding section of the conjugate beam subjected to the elastic load.
2. The deflection at a given section of a loaded beam equals the bending

moment in the corresponding section of the conjugate beam subjected to the elastic load.

Thus far we have stated only that the conjugate beam is identical to the actual beam with regard to the length of the beam. In order that the above-stated identities be possible, the setup of the support and connection of the conjugate beam must be such as to induce shear and moment in the conjugate beam in conformity to the slope and deflection induced by the counterparts in the actual beam. These requirements are given in Table 5-8 and can be briefly summarized as follows:

fixed end ⟷ free end
simple end ⟷ simple end
interior connection ⟷ interior support

The symbols between the two groups indicate conjugation.

**TABLE 5-8**

| Actual Beam<br>Subjected to Applied Load | | | Conjugate Beam<br>Subjected to Elastic Load |
|---|---|---|---|
| Fixed end | $\begin{cases}\theta = 0\\ y = 0\end{cases}$ | $\left.\begin{matrix}\overline{V} = 0\\ \overline{M} = 0\end{matrix}\right\}$ | Free end |
| Free end | $\begin{cases}\theta \neq 0\\ y \neq 0\end{cases}$ | $\left.\begin{matrix}\overline{V} \neq 0\\ \overline{M} \neq 0\end{matrix}\right\}$ | Fixed end |
| Simple end<br>(hinge or roller) | $\begin{cases}\theta \neq 0\\ y = 0\end{cases}$ | $\left.\begin{matrix}\overline{V} \neq 0\\ \overline{M} = 0\end{matrix}\right\}$ | Simple end<br>(hinge or roller) |
| Interior support<br>(hinge or roller) | $\begin{cases}\theta \neq 0\\ y = 0\end{cases}$ | $\left.\begin{matrix}\overline{V} \neq 0\\ \overline{M} = 0\end{matrix}\right\}$ | Interior connection<br>(hinge or roller) |
| Interior connection<br>(hinge or roller) | $\begin{cases}\theta \neq 0\\ y \neq 0\end{cases}$ | $\left.\begin{matrix}\overline{V} \neq 0\\ \overline{M} \neq 0\end{matrix}\right\}$ | Interior support<br>(hinge or roller) |

Thus, the conjugate beam for the beam in Fig. 5-19(a) is the one shown in Fig. 5-19(c). If we use the $M/EI$ diagram of Fig. 5-19(b) as the load to put on the beam in Fig. 5-19(c), the resulting shear and bending moment for any section of this beam will give the slope and deflection for the corresponding section of the original beam. Other examples of conjugate beams are shown in Fig. 5-20.

Unlike the actual beams, the conjugate beams may be unstable in themselves. For instance, the conjugate beams shown in Fig. 5-20(c) and (f) are unstable beams. However, they will maintain an unstable equilibrium under the action of an elastic load.

The same figure indicates that the conjugate-beam method is not limited

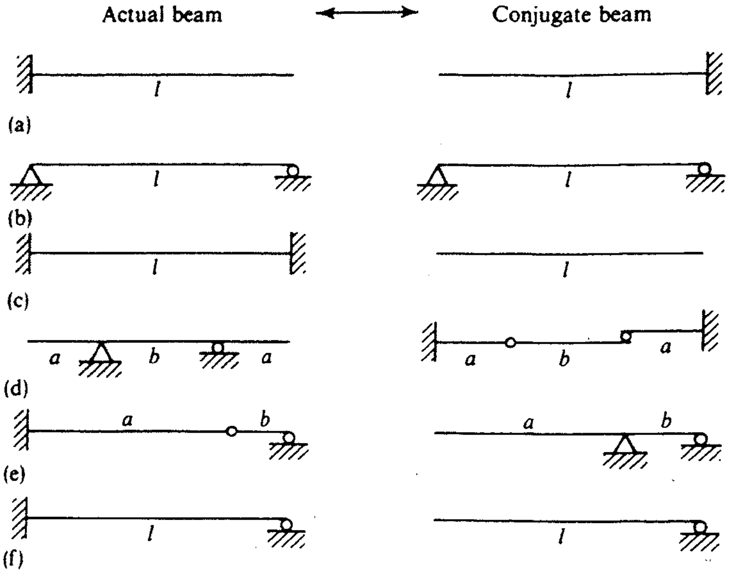


Fig. 5-20

to the analysis of statically determinate beams; in fact, the conjugate-beam method is also applicable to statically indeterminate beams.

The sign convention we use may be stated as follows. The origin of the loaded beam is taken at the left end of the beam with $y$ positive downward and $x$ positive to the right. As a result, a positive deflection means a downward deflection and a positive slope means a clockwise rotation of the beam section. Recall that the derivation from the relationships among load, shear, and bending moment is based on taking the downward load as positive. Therefore, a positive $M/EI$ should be taken as a downward elastic load.

Having defined these, we readily see that a positive shear at a section of the conjugate beam corresponds to a clockwise rotation at the section of the actual beam. A positive moment at a section of the conjugate beam corresponds to a downward deflection at the section of the actual beam.

**Example 5-14**

Find, by the conjugate-beam method, the vertical deflection at the free end $c$ of the cantilever beam shown in Fig. 5-21(a). Assume constant $EI$.

To do this, we place an elastic load on the conjugate beam, as shown in Fig. 5-21(b). The vertical deflection at $c$ of the actual beam is the moment at $c$ of the conjugate beam. Thus,

$$\Delta_c = \left(\frac{wk^3}{6EI}\right)\left(l - \frac{1}{4}k\right) \qquad \text{(down)}$$

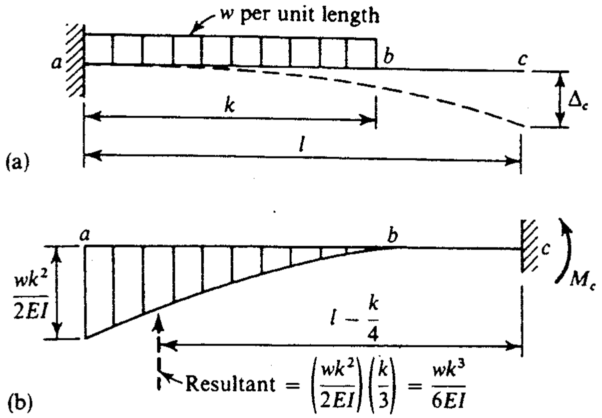


Fig. 5-21

**Example 5-15**

Find $\theta_A$, $\theta_C$, and $\Delta_C$ for the loaded beam shown in Fig. 5-22(a) by the conjugate-beam method. Assume constant $EI$. Note that the deformations were solved by the method of virtual work in Example 5-2.

The conjugate beam together with the elastic load is shown in Fig. 5-22(b), in which the resultant of the loading is found to be $Pab/2EI$ acting at a distance $(l + b)/3$ from the right end as indicated. Thus,

$$\theta_A = \frac{Pab(l + b)}{6EIl} \qquad \text{(clockwise)}$$

$$\theta_C = \frac{Pab(l + b)}{6EIl} - \left(\frac{Pab}{EIl}\right)\left(\frac{a}{2}\right) = \frac{Pab(b - a)}{3EIl} \qquad \text{(clockwise, if } b > a\text{)}$$

$$\Delta_C = \frac{Pa^2b(l + b)}{6EIl} - \left(\frac{Pab}{EIl}\right)\left(\frac{a}{2}\right)\left(\frac{a}{3}\right) = \frac{Pa^2b^2}{3EIl} \qquad \text{(down)}$$

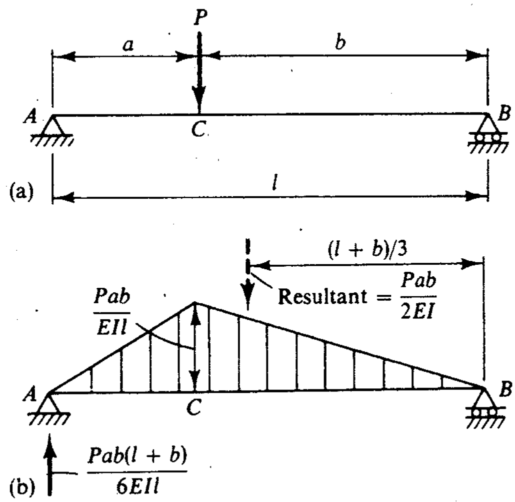


Fig. 5-22

**Example 5-16**

Use the conjugate-beam method to determine the deflection and rotation at point $b$ in Fig. 5-23(a). $E = 20{,}000\ \text{kN/cm}^2$.

To do this, we first plot the moment diagram as shown in Fig. 5-23(b). The diagram for the elastic load and the conjugate beam is then given in Fig. 5-23(c).

The deflection at $b$ of the original beam is the bending moment about $b$ of the conjugate beam subjected to the elastic load. Using the overhanging portion of the conjugate beam, we obtain

$$\Delta_b = \frac{(180)(6)(4)}{2EI_1} = \frac{2{,}160}{EI_1}$$

$$= \frac{(2{,}160)(10)^6}{(20{,}000)(60{,}000)} = 1.8 \text{ cm} \qquad \text{(down)}$$

Because of the hinge connection provided at $b$, a change of slope takes place

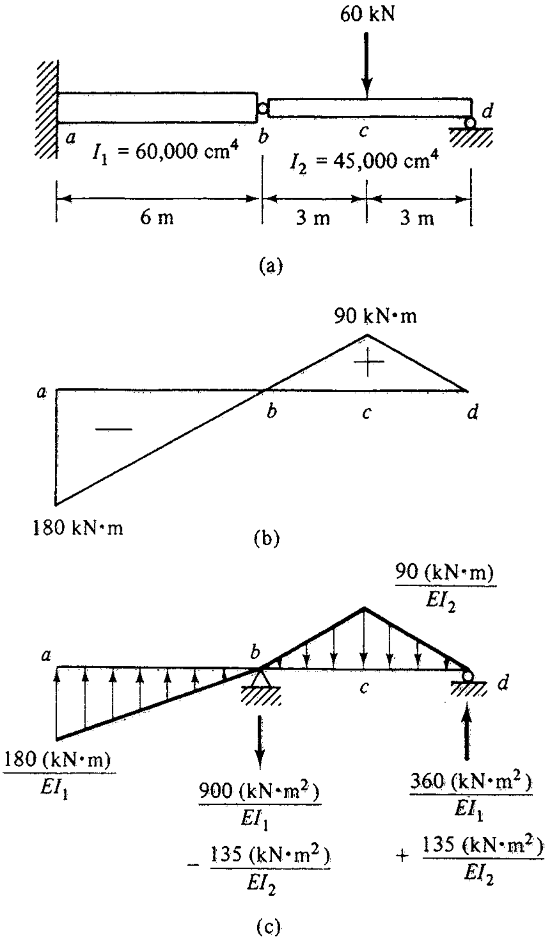


Fig. 5-23

at that point. In fact, we have different slopes to the immediate left and right of $b$, corresponding to the shearing forces in the conjugate beam. Thus,

$$\begin{aligned}(\theta_b)_{\text{left}} &= \frac{(180)(6)}{2EI_1} = \frac{540}{EI_1}\\ &= \frac{(540)(10)^4}{(20{,}000)(60{,}000)} = 0.0045 \text{ rad} \qquad \text{(clockwise)}\\ (\theta_b)_{\text{right}} &= \frac{270}{EI_2} - \left(\frac{360}{EI_1} + \frac{135}{EI_2}\right) = -\frac{360}{EI_1} + \frac{135}{EI_2}\\ &= -\frac{(360)(10)^4}{(20{,}000)(60{,}000)} + \frac{(135)(10)^4}{(20{,}000)(45{,}000)}\\ &= -0.0015 \text{ rad} \qquad \text{(counterclockwise)}\end{aligned}$$

The relative rotation between the left and right sides of $b$ is the reaction at support $b$ of the conjugate beam:

$$\frac{900}{EI_1} - \frac{135}{EI_2} = \frac{(900)(10)^4}{(20{,}000)(60{,}000)} - \frac{(135)(10)^4}{(20{,}000)(45{,}000)} = 0.006 \text{ rad}$$

## 5-7 MAXWELL'S LAW OF RECIPROCAL DEFLECTIONS

Referring to Fig. 5-24, we note that Maxwell's law simply states that

$$\Delta_{21} = \Delta_{12} \tag{5-30}$$

where $\Delta_{21}$ = deflection at point 2 due to the load $P$ applied at point 1

$\Delta_{12}$ = deflection at point 1 along the original line of action of $P$ due to the same load applied at point 2 along the original deflection $\Delta_{21}$

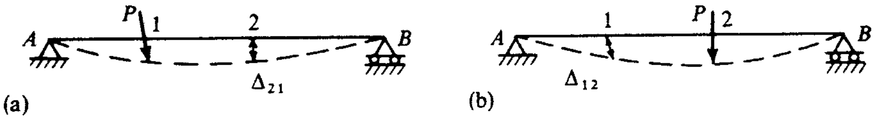


Fig. 5-24

To prove this statement, both deflections are evaluated by the method of virtual work. Thus, from Fig. 5-24(a) we have

$$\Delta_{21} = \int_0^l \frac{M_1 m_2\, dx}{EI}$$

where $M_1$ = moment at any section due to load $P$ applied at point 1

$m_2$ = moment at the same section due to a unit load applied at point 2 along the desired deflection

Similarly, from Fig. 5-24(b) we obtain

$$\Delta_{12} = \int_0^l \frac{M_2 m_1\, dx}{EI}$$

but $\qquad M_1 = Pm_1 \qquad$ and $\qquad M_2 = Pm_2$

It is readily seen that

$$\Delta_{21} = \int_0^l \frac{(Pm_1)m_2\, dx}{EI} = \int_0^l \frac{(Pm_2)m_1\, dx}{EI} = \int_0^l \frac{M_2 m_1\, dx}{EI} = \Delta_{12}$$

The special case is that $P = 1$, for which we can write

$$\delta_{21} = \delta_{12} \tag{5-31}$$

where $\delta_{21}$ = deflection at point 2 resulting from a unit load applied at point 1

$\delta_{12}$ = deflection at point 1 along the original line of action due to a unit load applied at point 2 along the original deflection $\delta_{21}$

We have hitherto demonstrated the law in regard to applied forces and their corresponding linear deflections. However, the reciprocity extends also to rotational displacement. For the case of two unit couples applied separately to any two points of a structure, the law is: *The rotational deflection at point 2 on a structure caused by a unit couple at point 1 is equal to the rotational deflection at point 1 due to a unit couple at point 2.*

Because of virtual work we also observe that the *rotational deflection at point 2 due to a unit force at point 1 is equal in magnitude to the linear deflection at point 1 along the original force due to a unit couple at point 2.*

Maxwell's law is perfectly general and is applicable to any type of structure as long as the material of the structure is elastic and follows Hooke's law.

## PROBLEMS

**5-1.** Use the method of virtual work to determine the vertical deflection at center and the slope at left end for a simply supported beam subjected to a uniform load over the entire span. Assume constant *EI*.

**5-2.** Use the method of virtual work to determine the vertical deflections at the load point and at the center of the beam in Fig. 5-25. *E* and *I* are both constant.

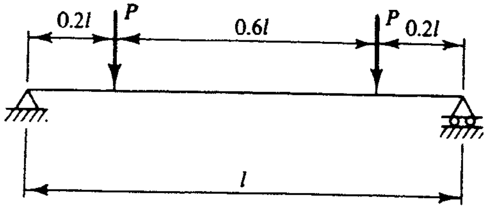


Fig. 5-25

**5-3.** Use the method of virtual work to determine the slope and deflection at the load point of the beam in Fig. 5-26. Use $E = 30{,}000$ kips/in.$^2$.

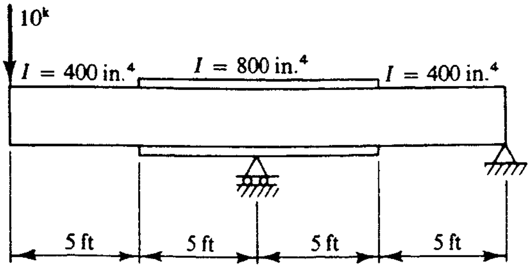


Fig. 5-26

**5-4.** By the method of virtual work, find the horizontal, vertical, and rotational displacement components at point $a$ of the frame shown in Fig. 5-27. Use $E = 30{,}000$ kips/in.$^2$ and $I = 500$ in.$^4$.

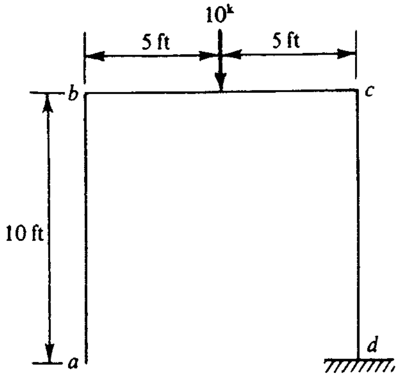


Fig. 5-27

**5-5.** For the load and beam in Fig. 5-28, use the method of virtual work to determine the slope and deflection at points $b$ and $c$.

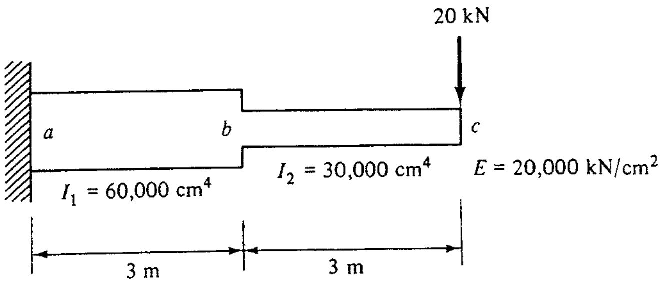


Fig. 5-28

5-6. For the loads and frame in Fig. 5-29, use the method of virtual work to find the horizontal, vertical, and rotational displacement components at point *a*. $EI = 2 \times 10^8$ kN · cm$^2$.

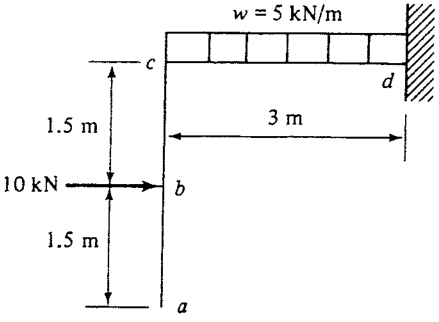


**Fig. 5-29**

5-7. For the loads and truss in Fig. 5-30, use the method of virtual work to find the displacement components corresponding to the applied loads at joint *B*. $E = 20{,}000$ kN/cm$^2$ and $A = 20$ cm$^2$.

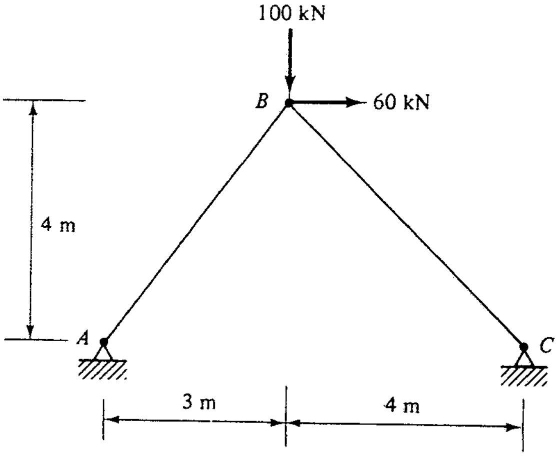


**Fig. 5-30**

5-8. For the truss in Fig. 5-31, the area of each bar in square inches equals one-half its length in feet. $E = 30{,}000$ kips/in.$^2$. Use the method of virtual work to compute (a) the vertical deflection at point *B*, (b) the horizontal deflection at point *C*, (c) the relative deflection between points *b* and *C* along the line joining them, and (d) the rotation of member *bc*.

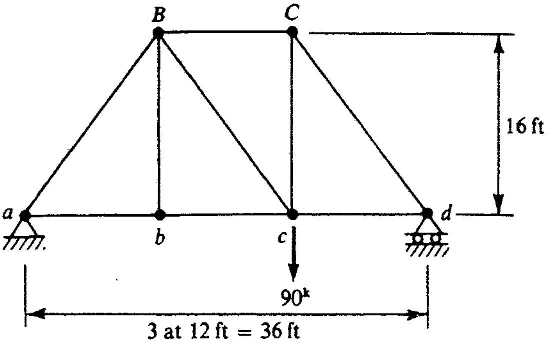


Fig. 5-31

**5-9.** Solve Prob. 5-1 by Castigliano's theorem.

**5-10.** Solve Prob. 5-2 by Castigliano's theorem.

**5-11.** Solve Prob. 5-3 by Castigliano's theorem.

**5-12.** Solve Prob. 5-4 by Castigliano's theorem.

**5-13.** Solve Prob. 5-5 by Castigliano's theorem.

**5-14.** Solve Prob. 5-6 by Castigliano's theorem.

**5-15.** Solve Prob. 5-7 by Castigliano's theorem.

**5-16.** Solve Prob. 5-8 by Castigliano's theorem.

**5-17.** Solve Prob. 5-1 by the conjugate-beam method.

**5-18.** Solve Prob. 5-2 by the conjugate-beam method.

**5-19.** Solve Prob. 5-3 by the conjugate-beam method.

**5-20.** For the load and beam shown in Fig. 5-32, use the conjugate-beam method to find the deflection at $b$ and the rotation at $c$. $E = 20{,}000\ \text{kN/cm}^2$ and $I = 5{,}000\ \text{cm}^4$.

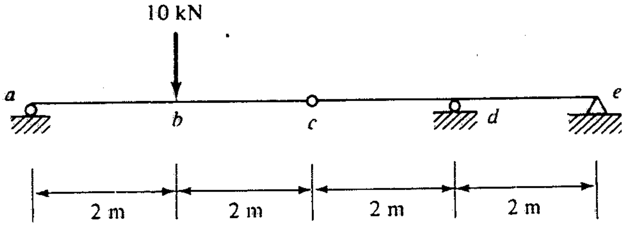


Fig. 5-32

# 6

# *Method of Consistent Deformations*

## 6-1 GENERAL

Statically indeterminate structures can be analyzed by direct use of the theory of elastic deformations developed in Chapter 5. Any statically indeterminate structure can be made statically determinate and stable by removing the extra restraints called *redundant forces* or *statical redundants,* that is, the force elements that are more than the minimum necessary for the static equilibrium of the structure. The number of redundant forces therefore represents the degrees of statical indeterminacy of the original structure. The statically determinate and stable structure that remains after removal of the extra restraints is called the *primary,* or *released, structure.* The choice of the redundant forces is arbitrary. They may be external support reactions or internal member forces or both. In all cases, the statical redundants should be so chosen that the resulting primary structure is stable.

The original structure is then equivalent to the primary structure subjected to the combined action of the original loads plus the unknown redundants. The conditional equations for geometric consistence of the original structure at redundant points (releases), called the *compatibility equations,* are then obtained from the primary structure by superposition of the deformations caused separately by the original loads and redundants. There can be as many compatibility equations as the number of unknown redundants so that the redundants can be determined by solving these simultaneous equations. This method, known as *consistent deformations,* is generally applicable to the analysis of any structure, whether it is being analyzed for the effect of loads, support settlement, temperature change, or any other case. However, there is one restriction on the use of this method: the principle of superposition must hold.

As an illustration, consider the loaded continuous beam with nonyielding supports shown in Fig. 6-1(a). It is statically indeterminate to the second degree, that is to say, with two redundants. The first step in the application of the method is to remove, say, the two interior supports and to introduce in these releases the redundant actions called $X_1$ and $X_2$, respectively, and by so doing to reduce or cut back the structure to a condition of determinateness and stability. The original structure is now considered as a simple beam (the primary structure) subjected to the combined action of a number of external forces and two redundants $X_1$ and $X_2$, as shown in Fig. 6-1(b).

The resulting structure in Fig. 6-1(b) can be regarded as the superposition of those shown in Fig. 6-1(c)–(e). Consequently, any deformation of the structure can be obtained by the superposition of these effects.

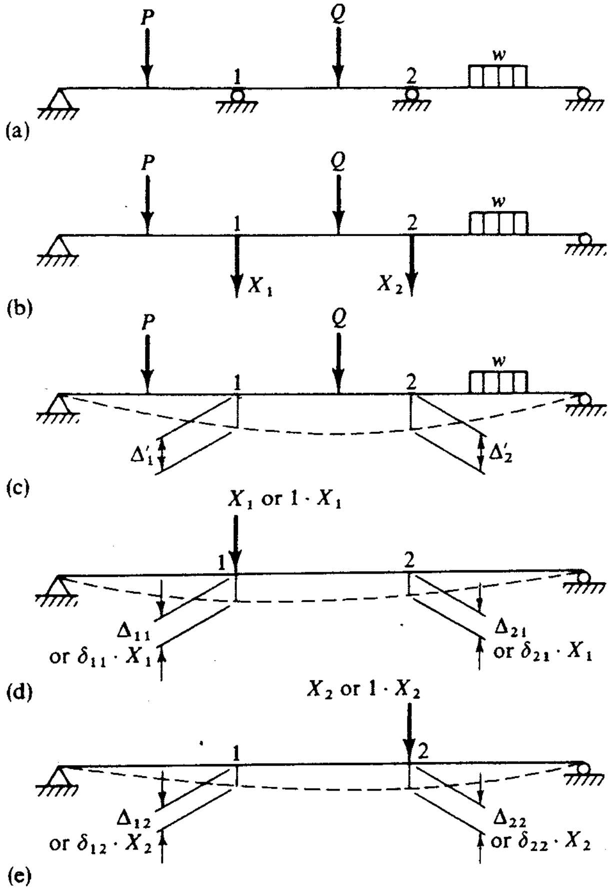


Fig. 6-1

Referring to Fig. 6-1(b), for unyielding supports we find that compatibility requires

$$\Delta_1 = 0 \tag{6-1}$$

$$\Delta_2 = 0 \tag{6-2}$$

where $\Delta_1$ = deflection at redundant point 1 (in the line of redundant force $X_1$)
$\Delta_2$ = deflection at redundant point 2 (in the line of redundant force $X_2$)

By the principle of superposition we may expand Eqs. 6-1 and 6-2:

$$\Delta_1' + \Delta_{11} + \Delta_{12} = 0 \tag{6-3}$$

$$\Delta_2' + \Delta_{21} + \Delta_{22} = 0 \tag{6-4}$$

where $\Delta_1'$ = deflection at redundant point 1 due to external loads [see Fig. 6-1(c)]
$\Delta_{11}$ = deflection at redundant point 1 due to redundant force $X_1$ [see Fig. 6-1(d)]
$\Delta_{12}$ = deflection at redundant point 1 due to redundant force $X_2$ [see Fig. 6-1(e)]

The rest are similar.

Equations 6-3 and 6-4 may be expressed in terms of the *flexibility coefficients*. A typical flexibility coefficient $\delta_{ij}$ is defined by

$\delta_{ij}$ = displacement at point $i$ due to a unit action at $j$, all other points being assumed unloaded

Thus, Eqs. 6-3 and 6-4 may be written as

$$\Delta_1' + \delta_{11}X_1 + \delta_{12}X_2 = 0 \tag{6-5}$$

$$\Delta_2' + \delta_{21}X_1 + \delta_{22}X_2 = 0 \tag{6-6}$$

Apparently,

$\delta_{11}$ = deflection at point 1 due to a unit force at point 1 [see Fig. 6-1(d)]
$\delta_{12}$ = deflection at point 1 due to a unit force at point 2 [see Fig. 6-1(e)]

and so on.

Both the deflections resulting from the original external loads and the flexibility coefficients for the primary structure can be obtained by any method described in Chapter 5. The remaining redundant unknowns are then solved by simultaneous equations. In general, for a structure with $n$ redundants, we have

$$\begin{aligned}
\Delta_1' + \delta_{11}X_1 + \delta_{12}X_2 + \cdots + \delta_{1n}X_n &= 0 \\
\Delta_2' + \delta_{21}X_1 + \delta_{22}X_2 + \cdots + \delta_{2n}X_n &= 0 \\
&\cdot \\
&\cdot \\
&\cdot \\
\Delta_n' + \delta_{n1}X_1 + \delta_{n2}X_2 + \cdots + \delta_{nn}X_n &= 0
\end{aligned} \tag{6-7}$$

Equation 6-7 in matrix form is

$$\begin{Bmatrix} \Delta'_1 \\ \Delta'_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta'_n \end{Bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{6-8}$$

or simply

$$\Delta' + FX = 0 \tag{6-9}$$

In a more general form, we may include the prescribed displacements (other than zeros) occurring at the releases of the original structures. Then these values $\Delta_1$, $\Delta_2$, . . . must be substituted for the zeros on the right-hand side of Eq. 6-8. Thus,

$$\begin{Bmatrix} \Delta'_1 \\ \Delta'_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta'_n \end{Bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{Bmatrix} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n \end{Bmatrix} \tag{6-10}$$

or simply

$$\Delta' + FX = \Delta \tag{6-11}$$

in which the column matrix $\Delta'$ on the left-hand side represents the displacements at redundant points of the released structure due to the original loads; the square matrix $F$ represents the structure flexibility, each column of which gives various displacements at redundant points due to a certain unit redundant force; and the column matrix $\Delta$ on the right-hand side contains the actual displacements at redundant points of the original structure. Equation 6-11 expresses the compatibility at redundant points in terms of unknown redundant forces.

## 6-2 ANALYSIS OF STATICALLY INDETERMINATE BEAMS BY THE METHOD OF CONSISTENT DEFORMATIONS

The method of consistent deformations is quite easy to understand and can be most effectively demonstrated by a series of illustrations. In all the following examples we assume that only the bending distortion is significant.

**Example 6-1**

Analyze the propped beam shown in Fig. 6-2(a), which is statically indeterminate to the first degree. Assume constant $EI$.

**Solution 1** One of the reactions may be considered as being extra. In this case let us first choose the vertical reaction at $b$ as the redundant assumed to be acting downward, as shown in Fig. 6-2(b). By the principle of superposition we may

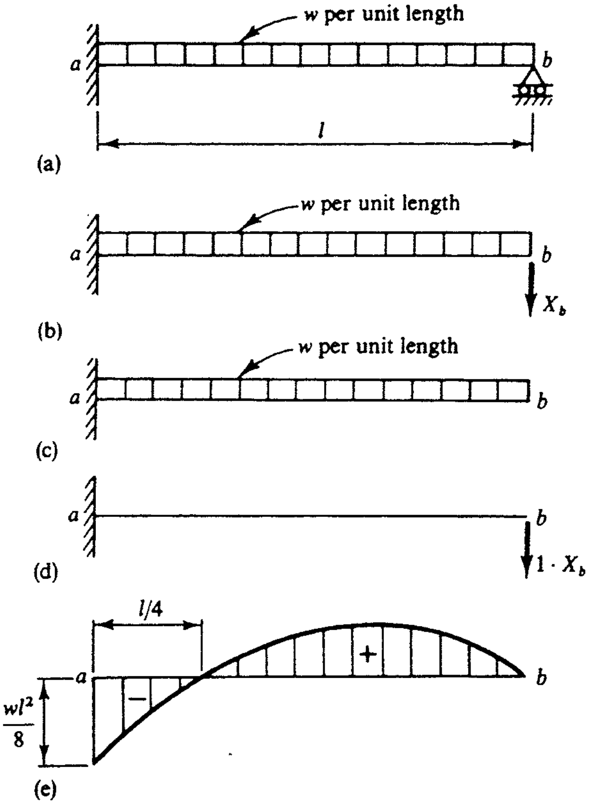


Fig. 6-2

consider the beam as being subjected to the sum of the effects of th uniform loading and the unknown redundant $X_b$, as shown in Fig. 6-2(c respectively.

Next, we find that the vertical deflection at $b$ resulting from the uni [Fig. 6-2(c)] is given by

$$\Delta_b' = \frac{wl^4}{8EI}$$

and that the vertical deflection at $b$ because of a unit load applied at $b$ i $X_b$ [Fig. 6-2(d)] is given by

$$\delta_{bb} = \frac{l^3}{3EI}$$

Note that $\Delta_b'$ and the flexibility coefficient $\delta_{bb}$ may be found by ar described in Chapter 5.

Applying compatibility equation

$$\Delta_b = \Delta_b' + \delta_{bb}X_b = 0$$

we obtain

$$\frac{wl^4}{8EI} + \left(\frac{l^3}{3EI}\right)X_b = 0$$

from which $$X_b = -\frac{3wl}{8}$$

The minus sign indicates an upward reaction.

With reaction at $b$ determined, we find that the beam reduces to a statically determinate one. We can readily obtain reaction components at $a$ from the equilibrium equations:

$$\sum F_y = 0 \qquad V_a = wl - \tfrac{3}{8}wl = \tfrac{5}{8}wl \qquad \text{(upward)}$$

$$\sum M_a = 0 \qquad M_a = \tfrac{1}{2}wl^2 - \tfrac{3}{8}wl^2 = \tfrac{1}{8}wl^2 \qquad \text{(counterclockwise)}$$

The moment diagram for the beam is shown in Fig. 6-2(e).

**Solution 2** The beam in Fig. 6-2(a) can be rendered statically determinate by removing the fixed support and replacing it with a hinged support. In addition to the original uniform loading, a redundant moment $M_a$ is then applied to the primary structure, a simple beam, as shown in Fig. 6-3(a). The unknown $M_a$ can be solved by the condition of compatibility that the rotation at end $a$ must be zero.

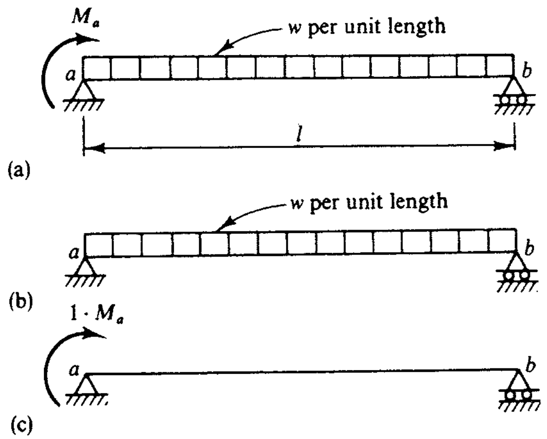


Fig. 6-3

The rotation at end $a$ for the primary structure due to the uniform loading alone [Fig. 6-3(b)] is given by

$$\theta_a' = \frac{wl^3}{24EI}$$

and that due to a unit couple applied at end $a$ [Fig. 6-3(c)] is given by

$$\delta_{aa} = \frac{l}{3EI}$$

Using the compatibility equation

$$\theta_a = \theta_a' + \delta_{aa}M_a = \frac{wl^3}{24EI} + \frac{M_a l}{3EI} = 0$$

we solve for $$M_a = -\tfrac{1}{8}wl^2$$

The minus sign indicates a counterclockwise moment. After $M_a$ is determined, the rest of the analysis can be carried out without difficulty.

**Solution 3** From the previous solutions we recognize that we are free to select redundants in analyzing a statically indeterminate structure, the only restriction being that the redundants should be so selected that a *stable* cut structure remains. Figure 6-4 will serve as an illustration. Let us cut the beam at midspan section $c$ and introduce in its place a hinge so that the beam is stable and determinate. A pair of redundant couples, called $M_c$, together with the original loading are then applied to the primary structure, as shown in Fig. 6-4(a).

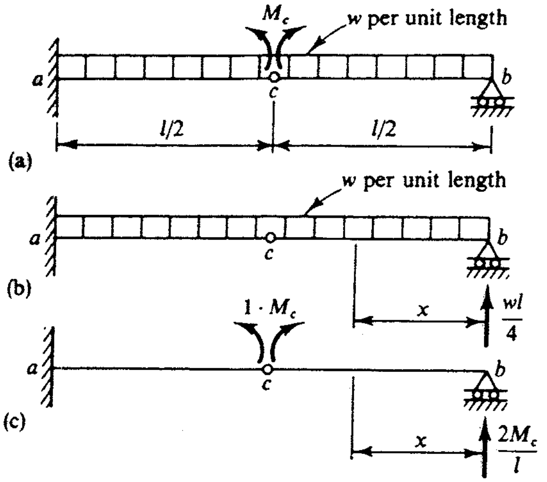


Fig. 6-4

The redundant $M_c$ is solved by the condition of compatibility that the rotation of the left side relative to the right side at section $c$ must be zero.

Using the method of virtual work we evaluate the relative rotation at $c$ due to the external loading alone [Fig. 6-4(b)] as

$$\theta'_c = \int_0^l \frac{Mm\,dx}{EI} = \int_0^l \frac{[(wlx/4) - (wx^2/2)](2x/l)\,dx}{EI} = -\frac{wl^3}{12EI}$$

and that due to a pair of unit couples acting at $c$ [Fig. 6-4(c)] as

$$\delta_{cc} = \int_0^l \frac{m^2\,dx}{EI} = \int_0^l \frac{(2x/l)^2\,dx}{EI} = \frac{4l}{3EI}$$

Setting the total relative angular displacement at $c$ equal to zero, we have

$$-\frac{wl^3}{12EI} + M_c\left(\frac{4l}{3EI}\right) = 0$$

from which

$$M_c = +\frac{wl^2}{16}$$

After $M_c$ is determined the rest of the analysis can easily be carried out.

**Example 6-2**

Suppose that the support at $b$ of Example 6-1 is elastic and the spring flexibility is $f$ (displacement per unit force), as shown in Fig. 6-5. Determine reaction at $b$ (the spring force), denoted by $X_b$.

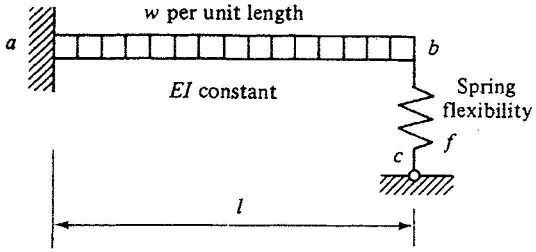


Fig. 6-5

Assume downward $X_b$ (i.e., tension in the spring) as positive. The compatibility is

$$\Delta_b' + \delta_{bb}X_b + fX_b = 0$$

This equation can be explained by putting it in the form

$$\Delta_b' - \delta_{bb}(-X_b) = f(-X_b)$$

Since $(-X_b)$ represents the compression in the spring, the equation indicates that the downward deflection at $b$ caused by the beam load minus that caused by upward reaction should be equal to the spring contraction.

By substituting $\Delta_b' = wl^4/8EI$, $\delta_{bb} = l^3/3EI$ in the preceding equation, we obtain

$$\frac{wl^4}{8EI} + \frac{X_b l^3}{3EI} + fX_b = 0$$

from which

$$X_b = -\frac{3}{8}wl\left[\frac{1}{1 + (3fEI/l^3)}\right]$$

The minus sign indicates an upward reaction.

For a nonyielding support, $f = 0$, the preceding equation gives

$$X_b = -\tfrac{3}{8}wl$$

as found previously.

If a beam is provided with $n$ redundant elastic supports having spring flexibilities $f_1, f_2, \ldots, f_n$, respectively, then the general compatibility equation is

$$\begin{Bmatrix} \Delta_1' \\ \Delta_2' \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n' \end{Bmatrix} + \begin{bmatrix} \delta_{11} + f_1 & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} + f_2 & \cdots & \delta_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} + f_n \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \qquad (6\text{-}12)$$

**Example 6-3**

Find the reactions for the beam with two sections shown in Fig. 6-6(a).

In this problem it may be convenient to select the vertical reaction at support $b$ as redundant. The beam is then considered as a simple beam subject to the original loading and the redundant $R_b$, as shown in Fig. 6-6(b) and (c), respectively.

The compatibility requires

$$\Delta_b = \Delta_b' + \delta_{bb}R_b = 0$$

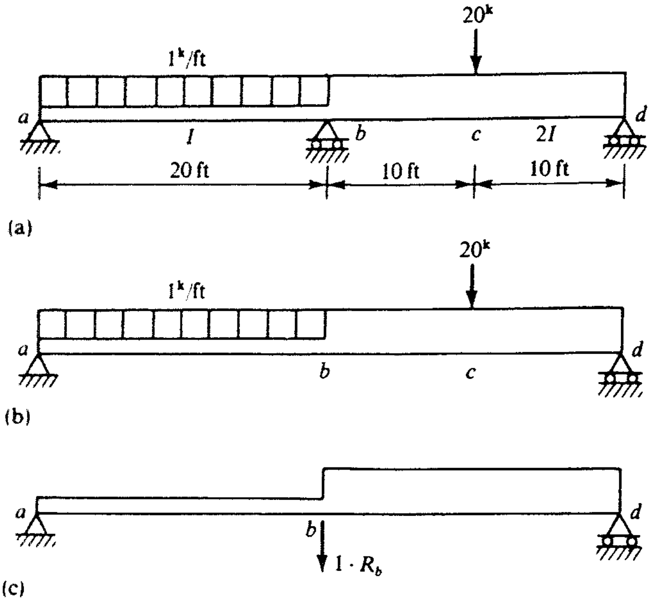


Fig. 6-6

Using the method of virtual work, we have

$$\int \frac{Mm_b\,dx}{EI} + R_b \int \frac{(m_b)^2\,dx}{EI} = 0$$

from which

$$R_b = -\frac{\int Mm_b\,dx/EI}{\int (m_b)^2\,dx/EI}$$

where $M$ = bending moment at any section of the primary beam caused by the original loading [Fig. 6-6(b)]

$m_b$ = bending moment at the same section of the primary beam caused by a unit load in place of the redundant $R_b$ [Fig. 6-6(c)]

The solution is completely shown in Table 6-1.

$$R_b = -\frac{\int_0^{20} \frac{(20x - x^2/2)(x/2)\,dx}{EI} + \int_0^{10} \frac{(20x)(x/2)\,dx}{2EI} + \int_{10}^{20} \frac{(200)(x/2)\,dx}{2EI}}{\int_0^{20} \frac{(x/2)^2\,dx}{EI} + \int_0^{10} \frac{(x/2)^2\,dx}{2EI} + \int_{10}^{20} \frac{(x/2)^2\,dx}{2EI}}$$

$$= -25.84 \text{ kips}$$

The negative sign indicates an upward reaction at support $b$.

After $R_b$ is obtained, we can readily find the reactions at the other two supports by statics. That is,

$$R_a = R_d = 20 - (\tfrac{1}{2})(25.84) = 7.08 \text{ kips}$$

acting upward.

**TABLE 6-1**

| Section | Origin | Limit (ft) | $M$ (ft-kips) | $m_b$ (ft-kips) | $I$ |
|---|---|---|---|---|---|
| $ab$ | $a$ | 0 to 20 | $20x - \frac{(1)(x)^2}{2}$ | $\frac{x}{2}$ | $I$ |
| $dc$ | $d$ | 0 to 10 | $20x$ | $\frac{x}{2}$ | $2I$ |
| $cb$ | $d$ | 10 to 20 | $20x - 20(x - 10)$<br>or 200 | $\frac{x}{2}$ | $2I$ |

The end moments for a fixed-end beam, called *fixed-end moments*, are important in the methods of slope deflection and of moment distribution, which are discussed in later chapters. The following examples are attempts to solve fixed-end moments due to common types of loading by the method of consistent deformations.

**Example 6-4**

The fixed-end beam of uniform cross section subjected to a single concentrated load shown in Fig. 6-7(a) is statically indeterminate to the second degree since the horizontal force does not exist. End moments $M_A$ and $M_B$ are selected as redundants. The original beam is then considered as equivalent to a simple beam (not shown) under the combined action of a concentrated force $P$ and redundant moments $M_A$

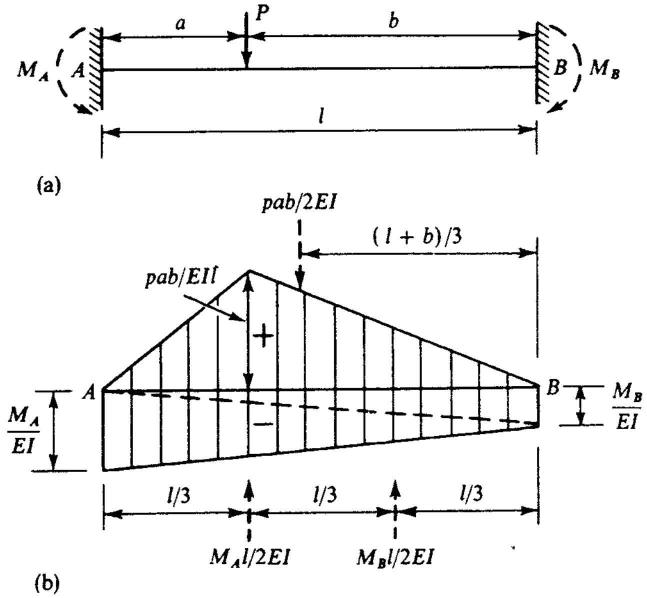


Fig. 6-7

and $M_B$. It is convenient to apply the conjugate-beam method to determine $M_A$ and $M_B$ based on the condition that the slope and deflection at either end of the fixed-end beam must be zero. In other words, there will be no support reactions for the conjugate beam, and the positive and negative $M/EI$ diagrams (elastic loads) given in Fig. 6-7(b) must form a balanced system. Thus, from $\Sigma F_y = 0$,

$$\frac{Pab}{2EI} - \frac{M_A l}{2EI} - \frac{M_B l}{2EI} = 0$$

or

$$M_A + M_B = \frac{Pab}{l} \tag{6-13}$$

From $\Sigma M_B = 0$,

$$\left(\frac{Pab}{2EI}\right)\left(\frac{l+b}{3}\right) - \left(\frac{M_A l}{2EI}\right)\left(\frac{2l}{3}\right) - \left(\frac{M_B l}{2EI}\right)\left(\frac{l}{3}\right) = 0$$

or

$$2M_A + M_B = \frac{Pab}{l} + \frac{Pab^2}{l^2} \tag{6-14}$$

Solving Eqs. 6-13 and 6-14 simultaneously, we obtain

$$M_A = \frac{Pab^2}{l^2} \qquad M_B = \frac{Pa^2b}{l^2} \tag{6-15}$$

**Example 6-5**

Find the end moments of a fixed-end beam of constant $EI$ caused by a uniform load, as shown in Fig. 6-8(a).

Because of symmetry, the beam is statically indeterminate to the first degree, since $M_A = M_B = M$, as indicated in Fig. 6-8(a). By the method of conjugate beam [Fig. 6-8(b)], $\Sigma F_y = 0$,

$$\left(\frac{wl^2}{8EI}\right)\left(\frac{2l}{3}\right) - \frac{Ml}{EI} = 0$$

from which

$$M = \tfrac{1}{12} wl^2 \tag{6-16}$$

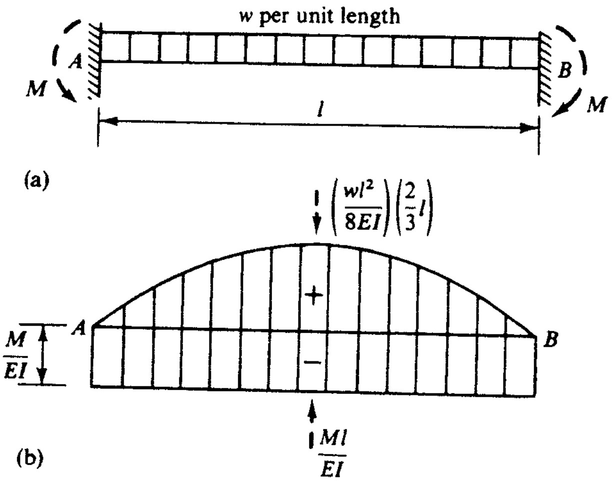


Fig. 6-8

**Example 6-6**

If the fixed-end beam is loaded with an external couple $M$ as shown in Fig. 6-9(a), the deflected elastic shape will be somewhat like that shown by the dotted line, which gives the sense of the end moments as indicated.

As before, end moments $M_A$ and $M_B$ are chosen as redundants. The elastic loads based on the moment diagrams divided by $EI$ plotted for external moment $M$ and redundants $M_A$ and $M_B$, as given in Fig. 6-9(b) and (c), must be in equilibrium themselves. From $\Sigma F_y = 0$,

$$\frac{M_A l}{2EI} + \frac{Mb^2}{2EIl} - \frac{M_B l}{2EI} - \frac{Ma^2}{2EIl} = 0$$

or

$$M_A - M_B = \frac{M(a^2 - b^2)}{l^2} \tag{6-17}$$

From $\Sigma M_B = 0$,

$$\left(\frac{M_A l}{2EI}\right)\left(\frac{2l}{3}\right) + \left(\frac{Mb^2}{2EIl}\right)\left(\frac{2b}{3}\right) - \left(\frac{M_B l}{2EI}\right)\left(\frac{l}{3}\right) - \left(\frac{Ma^2}{2EIl}\right)\left(b + \frac{a}{3}\right) = 0$$

or

$$2M_A - M_B = \frac{M[a^2 + 2b(a - b)]}{l^2} \tag{6-18}$$

Solving Eqs. 6-17 and 6-18 simultaneously, we obtain

$$M_A = \frac{Mb}{l^2}(2a - b) \qquad M_B = \frac{Ma}{l^2}(2b - a) \tag{6-19}$$

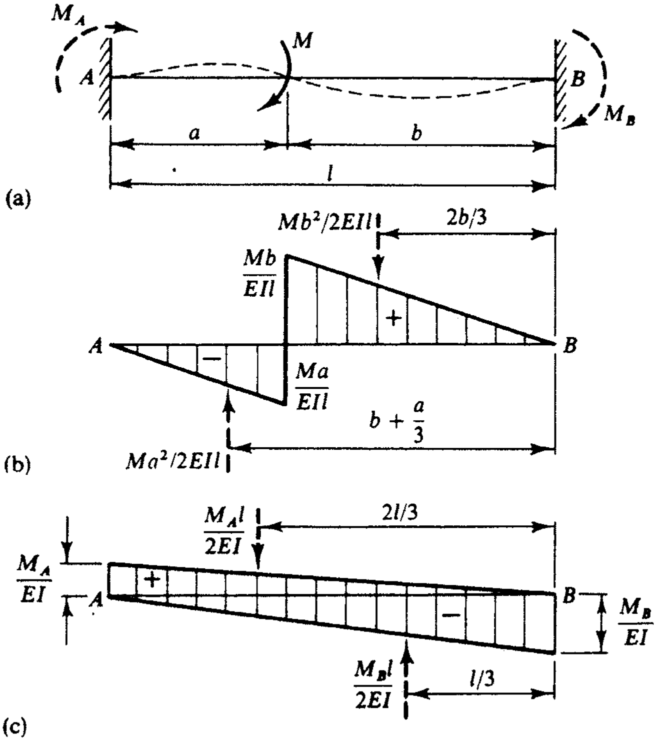


Fig. 6-9

Note that $M_A$ and $M_B$ bear the same sense as the externally applied $M$, as indicated in Fig. 6-9(a), if $a > l/3$ and $b > l/3$.

## 6-3 ANALYSIS OF STATICALLY INDETERMINATE RIGID FRAMES BY THE METHOD OF CONSISTENT DEFORMATIONS

The general procedure illustrated in Sec. 6-2 in solving statically indeterminate beams can be applied equally well to the analysis of statically indeterminate rigid frames, as in the following example.

**Example 6-7**

For the loaded rigid frame shown in Fig. 6-10(a), find the reaction components at the fixed end $a$, and plot the moment diagram for the entire frame. Assume the same $EI$ for all members.

To do this, we start by removing support $a$ and introducing in its place three redundant reaction components $X_1$, $X_2$, and $X_3$, as shown in Fig. 6-10(b). These can be taken as the superposition of four basic cases, as shown in Fig. 6-10(c), (d), (e), and (f), respectively. Since end $a$ is fixed, compatibility requires that

$$\begin{Bmatrix} \Delta_1' \\ \Delta_2' \\ \Delta_3' \end{Bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6-20}$$

Taking advantage of Examples 5-4 and 5-5, we note that

$$\begin{Bmatrix} \Delta_1' \\ \Delta_2' \\ \Delta_3' \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 5{,}000 \\ -7{,}500 \\ -800 \end{Bmatrix}$$

and

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 1{,}667 & -1{,}000 & -200 \\ -1{,}000 & 1{,}333 & 150 \\ -200 & 150 & 30 \end{bmatrix}$$

Substituting these values in Eq. 6-20, we obtain

$$\begin{Bmatrix} 5{,}000 \\ -7{,}500 \\ -800 \end{Bmatrix} + \begin{bmatrix} 1{,}667 & -1{,}000 & -200 \\ -1{,}000 & 1{,}333 & 150 \\ -200 & 150 & 30 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6-21}$$

Solving, we obtain

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 6 \\ 3.33\text{ ft} \end{Bmatrix} \text{kips}$$

Note that the solution of this problem could be simplified by setting $X_2 = 6$ kips in Eq. 6-20, since we know this value beforehand because of the symmetry of the loaded frame.

The final results are shown in Fig. 6-10(g); the moment diagram for the whole frame is shown in Fig. 6-10(h). A sketch of the elastic deformation of the frame due to bending distortion is shown by the dashed line in Fig. 6-10(i). Note that in

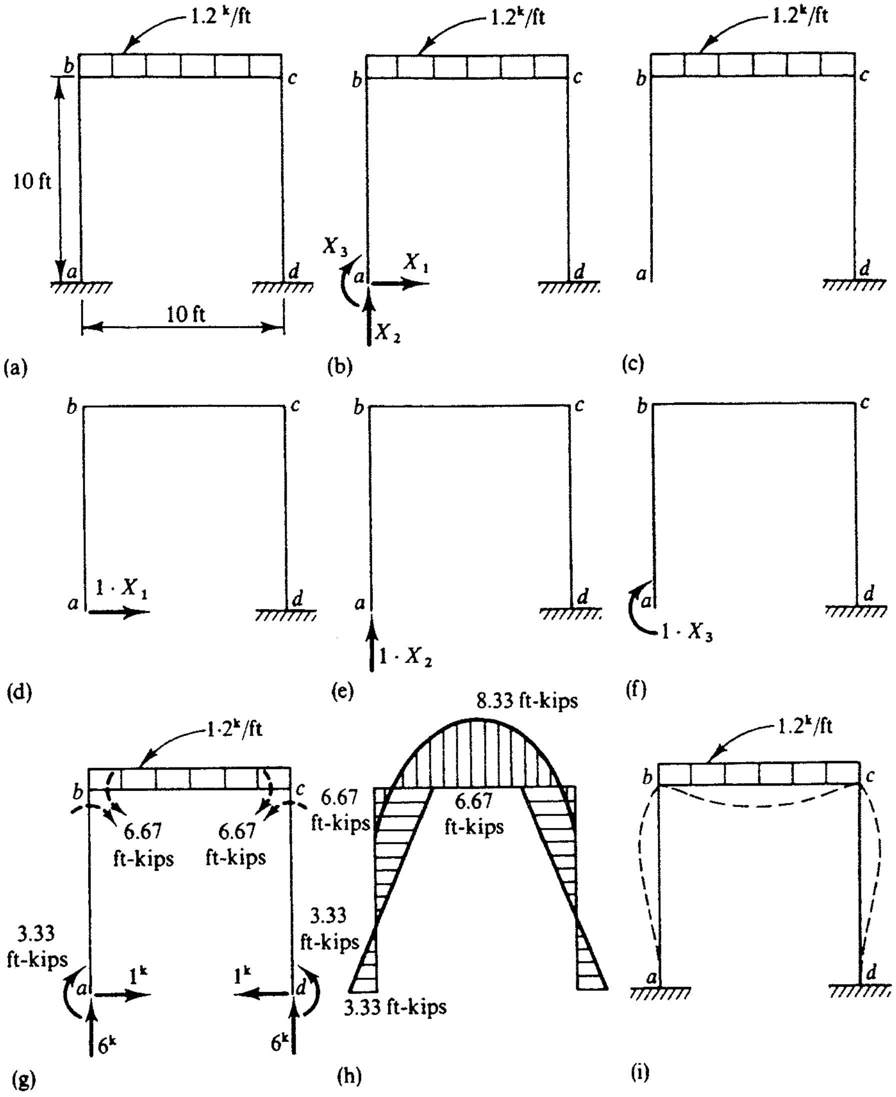


Fig. 6-10

this case there is one point of inflection in each column and two points of inflection in the beam.

By referring to Example 6-7 we see that by using the method of consistent deformations in analyzing a rigid frame, we encounter tedious calculations of the flexibility coefficients. The work, if done by hand, will become intolerable if the problem involves as many redundants as a rigid frame usually does. As a matter of fact, the method of consistent deformations is seldom used for analysis of rigid frames by hand calculation, since a solution can be much more easily obtained by the method of slope deflection or of moment distribution. However, with the development of high-speed *electronic computers,* this method has regained considerable strength in the scope of structural analysis.

## 6-4 ANALYSIS OF STATICALLY INDETERMINATE TRUSSES BY THE METHOD OF CONSISTENT DEFORMATIONS

The indeterminateness of a truss may be due to redundant supports or redundant bars or both. If it results from redundant supports, the procedure for attack is the same as that described for a continuous beam. If the superfluous element is a bar, the bar is considered to be cut at a section and replaced by two equal and opposite axial redundant forces representing the internal action for that bar. The condition equation is such that the relative axial displacement between the two sides at the cut section caused by the combined effect of the original loading and the redundants should be zero.

**Example 6-8**

Analyze the continuous truss in Fig. 6-11(a). Assume that $E = 30{,}000$ kips/in.$^2$ and $L(\text{ft})/A(\text{in.}^2) = 1$ for all members.

In this problem it is convenient to select the central support as the redundant element. We begin by removing support $c$ and introducing in its place a redundant reaction $X_c$, as shown in Fig. 6-11(b). The primary structure is then a simply supported truss subjected to an external load of 64 kips at joint $b$ and a redundant $X_c$. The effects can be separated, respectively, as shown in Fig. 6-11(c) and (d).

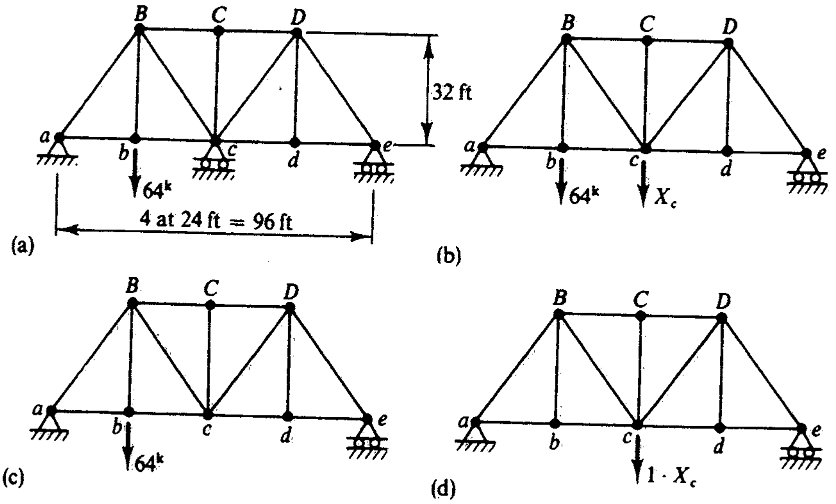


Fig. 6-11

Since support $c$ is on a rigid foundation, the compatibility equation can be expressed by

$$\Delta_c = \Delta_c' + \delta_{cc}X_c = 0$$

Using virtual work gives

$$\sum\frac{S'u_cL}{AE} + X_c\sum\frac{u_c^2L}{AE} = 0$$

from which

$$X_c = -\frac{\sum(S'u_cL/AE)}{\sum(u_c^2L/AE)}$$

where $S'$ = internal force in any member of the primary truss due to the original loading [Fig. 6-11(c)]

$u_c$ = internal force in the same member of the primary truss due to a unit force at $c$ [Fig. 6-11(d)]

The solution is shown completely in Table 6-2.

**TABLE 6-2**

| Member | $\frac{L}{A}$ (ft/in.$^2$) | $S'$ (kips) | $u_c$ | $\frac{S'u_cL}{A}$ (ft-kips/in.$^2$) | $\frac{u_c^2L}{A}$ (ft/in.$^2$) | $S = S' + u_cX_c$ (kips) |
|---|---|---|---|---|---|---|
| $ab$ | 1 | +36 | +3/8 | +13.5 | +9/64 | 36 − 14.1 = +21.9 |
| $bc$ | 1 | +36 | +3/8 | +13.5 | +9/64 | 36 − 14.1 = +21.9 |
| $cd$ | 1 | +12 | +3/8 | + 4.5 | +9/64 | 12 − 14.1 = − 2.1 |
| $de$ | 1 | +12 | +3/8 | + 4.5 | +9/64 | 12 − 14.1 = − 2.1 |
| $BC$ | 1 | −24 | −3/4 | +18 | +36/64 | −24 + 28.2 = + 4.2 |
| $CD$ | 1 | −24 | −3/4 | +18 | +36/64 | −24 + 28.2 = + 4.2 |
| $aB$ | 1 | −60 | −5/8 | +37.5 | +25/64 | −60 + 23.4 = −36.6 |
| $Bb$ | 1 | +64 | 0 | 0 | 0 | +64 + 0 = +64 |
| $Bc$ | 1 | −20 | +5/8 | −12.5 | +25/64 | −20 − 23.4 = −43.4 |
| $Cc$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $cD$ | 1 | +20 | +5/8 | +12.5 | +25/64 | +20 − 23.4 = − 3.4 |
| $Dd$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $De$ | 1 | −20 | −5/8 | +12.5 | +25/64 | −20 + 23.4 = + 3.4 |
| | | | Σ | +122 | +3.25 | |

$$X_c = -\frac{122}{3.25} = -37.5 \text{ kips}$$

The negative sign indicates an upward reaction at support $c$.

After $X_c$ is determined, we can readily obtain each bar force $S$ from

$$S = S' + u_cX_c$$

as given in the last column of Table 6-2.

**Example 6-9**

Analyze the truss in Fig. 6-12(a). Assume that $E = 30{,}000$ kips/in.$^2$ and $L$(ft)/$A$(in.$^2$) = 1 for all members.

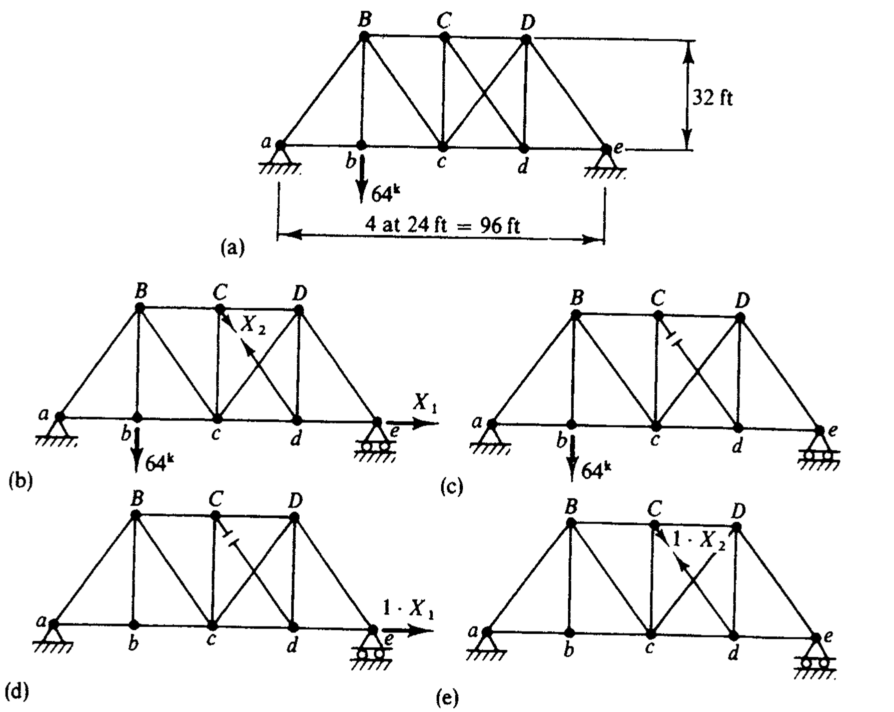


Fig. 6-12

The truss in Fig. 6-12(a) has two redundant elements, one in the reaction component and the other in the bar. Let us select the horizontal component of reaction at the right-end hinge and the internal force in bar *Cd* as redundants. We then have a primary truss loaded, as shown in Fig. 6-12(b), in which the original hinged support *e* is replaced by a roller acted on by a redundant horizontal reaction $X_1$, and the bar *Cd* is cut and a pair of redundant forces $X_2$ applied to it. This may again be replaced by the three basic cases shown in Fig. 6-12(c)–(e). Since both the horizontal movement at support *e* and the relative axial displacement between the cut ends of bar *Cd* are zero, we have

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1' \\ \Delta_2' \end{Bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6-22}$$

or

$$\begin{Bmatrix} \sum \frac{S'u_1L}{AE} \\ \sum \frac{S'u_2L}{AE} \end{Bmatrix} + \begin{bmatrix} \sum \frac{u_1^2L}{AE} & \sum \frac{u_1u_2L}{AE} \\ \sum \frac{u_1u_2L}{AE} & \sum \frac{u_2^2L}{AE} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6-23}$$

Note that

$S'$ = internal force in any bar of the primary truss due to the original loading [Fig. 6-12(c)]

$u_1$ = internal force in the same bar of the primary truss due to a unit horizontal force acting at $e$ [Fig. 6-12(d)]

$u_2$ = internal force in the same bar of the primary truss due to a pair of unit axial forces acting at the cut ends of bar $Cd$ [Fig. 6-12(e)]

Using the values summed up in Table 6-3, we reduce Eq. 6-23 to

$$\begin{Bmatrix} 96 \\ 27.2 \end{Bmatrix} + \begin{bmatrix} 4 & -\frac{3}{5} \\ -\frac{3}{5} & 4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{6-24}$$

Solving, we obtain

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} -25.6 \\ -10.6 \end{Bmatrix} \text{kips}$$

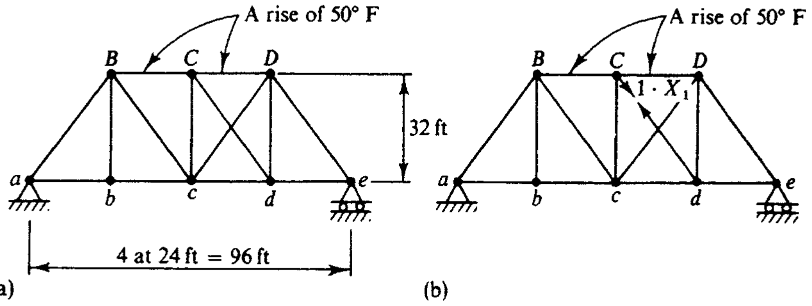
The negative signs indicate that the horizontal reaction at hinge $e$ acts to the left and that the axial force in member $Cd$ is compressive. The rest of the member forces are obtained by

$$S = S' + u_1X_1 + u_2X_2$$

The complete solution is shown in Table 6-3.

**Example 6-10**

Analyze the truss in Fig. 6-13(a) subject to a rise of 50°F at the top chords $BC$ and $CD$. Assume $\alpha = 0.0000065$ in./in./1°F; $E = 30{,}000$ kips/in.$^2$; and $L(\text{ft})/A(\text{in.}^2) = 1$ for all members.



(a) (b)

Fig. 6-13

The truss is statically indeterminate to the first degree. Cut bar $Cd$ and select its bar force $X_1$ as the redundant as shown in Fig. 6-13(b). The primary truss is then a simply supported truss subjected to the temperature rise in the top chords and the redundant axial force $X_1$. Since the relative axial displacement between the cut ends due to the combined effect of temperature rise and $X_1$ must be zero, we have

$$\Delta_1 = \Delta_1' + \delta_{11}X_1 = 0$$

or

$$\sum u_1(\alpha t^\circ L) + X_1 \sum \frac{u_1^2L}{AE} = 0$$

**TABLE 6-3**

| Member | $\frac{L}{A}$ (ft/in.$^2$) | $S'$ (kips) | $u_1$ | $u_2$ | $\frac{S' u_1 L}{A}$ (ft-kips/in.$^2$) | $\frac{S' u_2 L}{A}$ (ft-kips/in.$^2$) | $\frac{u_1^2 L}{A}$ (ft/in.$^2$) | $\frac{u_2^2 L}{A}$ (ft/in.$^2$) | $\frac{u_1 u_2 L}{A}$ (ft/in.$^2$) | $S = S' + u_1X_1 + u_2X_2$ (kips) |
|---|---|---|---|---|---|---|---|---|---|---|
| *ab* | 1 | +36 | +1 | 0 | +36 | 0 | +1 | 0 | 0 | 36 − 25.6 + 0 = +10.4 |
| *bc* | 1 | +36 | +1 | 0 | +36 | 0 | +1 | 0 | 0 | 36 − 25.6 + 0 = +10.4 |
| *cd* | 1 | +12 | +1 | $-\frac{3}{5}$ | +12 | − 7.2 | +1 | $+\frac{9}{25}$ | $-\frac{3}{5}$ | 12 − 25.6 + 6.4 = − 7.0 |
| *de* | 1 | +12 | +1 | 0 | +12 | 0 | +1 | 0 | 0 | 12 − 25.6 + 0 = −13.6 |
| *BC* | 1 | −24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −24 + 0 + 0 = −24 |
| *CD* | 1 | −24 | 0 | $-\frac{3}{5}$ | 0 | +14.4 | 0 | $+\frac{9}{25}$ | 0 | −24 + 0 + 6.4 = −17.6 |
| *aB* | 1 | −60 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −60 + 0 + 0 = −60 |
| *Bb* | 1 | +64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | +64 + 0 + 0 = +64 |
| *Bc* | 1 | −20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −20 + 0 + 0 = −20 |
| *Cc* | 1 | 0 | 0 | $-\frac{4}{5}$ | 0 | 0 | 0 | $+\frac{16}{25}$ | 0 | 0 + 0 + 8.5 = + 8.5 |
| *Cd* | 1 | 0 | 0 | +1 | 0 | 0 | 0 | + 1 | 0 | 0 + 0 − 10.6 = −10.6 |
| *cD* | 1 | +20 | 0 | +1 | 0 | +20 | 0 | + 1 | 0 | 20 + 0 − 10.6 = + 9.4 |
| *Dd* | 1 | 0 | 0 | $-\frac{4}{5}$ | 0 | 0 | 0 | $+\frac{16}{25}$ | 0 | 0 + 0 + 8.5 = + 8.5 |
| *De* | 1 | −20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −20 + 0 + 0 = −20 |
| | | | | Σ | +96 | +27.2 | +4 | +4 | $-\frac{3}{5}$ | |

where $\Delta_1'$ = relative displacement between the cut ends of the primary truss due to the temperature rise = $\Sigma u_1(\alpha t^\circ L)$ (see Eq. 5-20)

$u_1$ = internal force in any member of the primary truss due to a pair of unit axial forces acting at the ends of the cut section

The solution is shown in Table 6-4.

$$\frac{4X_1}{30{,}000} - 0.00468 = 0$$

$$X_1 = 35.1 \text{ kips} \qquad \text{(tension)}$$

**TABLE 6-4**

| Member | $\frac{L}{A}$ (ft/in.$^2$) | $u_1$ | $\alpha t^0 L$ (ft) | $u_1 \alpha t^0 L$ (ft) | $\frac{u_1^2 L}{A}$ (ft/in.$^2$) | $S = u_1 X_1$ (kips) |
|---|---|---|---|---|---|---|
| $ab$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $bc$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $cd$ | 1 | $-\frac{3}{5}$ | 0 | 0 | $+\frac{9}{25}$ | $-21.1$ |
| $de$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $BC$ | 1 | 0 | $+0.0078$ | 0 | 0 | 0 |
| $CD$ | 1 | $-\frac{3}{5}$ | $+0.0078$ | $-0.00468$ | $+\frac{9}{25}$ | $-21.1$ |
| $aB$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $Bb$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $Bc$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $Cc$ | 1 | $-\frac{4}{5}$ | 0 | 0 | $+\frac{16}{25}$ | $-28.1$ |
| $Cd$ | 1 | $+1$ | 0 | 0 | $+1$ | $+35.1$ |
| $cD$ | 1 | $+1$ | 0 | 0 | $+1$ | $+35.1$ |
| $Dd$ | 1 | $-\frac{4}{5}$ | 0 | 0 | $+\frac{16}{25}$ | $-28.1$ |
| $De$ | 1 | 0 | 0 | 0 | 0 | 0 |
| | | | $\Sigma$ | $-0.00468$ | $+4$ | |

Although these illustrations are aimed at statically indeterminate trusses with one or two redundants, the procedure described can be extended to trusses with many degrees of redundancy.

## 6-5 CASTIGLIANO'S COMPATIBILITY EQUATION (METHOD OF LEAST WORK)

The method of consistent deformations hitherto discussed involves superposition equations for the elastic deformations of the primary structure at the points of application of the redundants $X_1, X_2, \ldots, X_n$, the primary structure being stable and determinate and subjected to external actions, together with $n$ redundant forces. The expressions that the displacement at each of $n$ redundants equals zero for a loaded structure with nonyielding supports may be set up by the use of Castigliano's theorem as

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial W}{\partial X_1} \\ \dfrac{\partial W}{\partial X_2} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial W}{\partial X_n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{6-25}$$

where $W$ is the *total strain energy* of the primary structure and is therefore a function of the external loads and the unknown redundant forces $X_1, X_2, \ldots, X_n$. There are as many simultaneous equations as the number of unknown redundants involved. Equation 6-25,

$$\frac{\partial W}{\partial X_1} = \frac{\partial W}{\partial X_2} = \cdots = \frac{\partial W}{\partial X_n} = 0$$

is known as Castigliano's compatibility equation and it may be stated as follows: *The redundants must have such value that the total strain energy of the structure is a minimum consistent with equilibrium.* For this reason it is sometimes referred to as the *theorem of least work.* Note that Castigliano's compatibility equation is limited to the computation of redundant forces produced only by external loads on a structure mounted on unyielding supports. It cannot be used to determine stresses caused by temperature change, support movements, fabrication errors, and the like.

In the analysis of statically indeterminate beams or rigid frames, we consider bending moment to be the only significant factor contributing to the internal energy. Therefore, the total strain energy can be expressed by

$$W = \int \frac{M^2\,dx}{2EI}$$

Setting the derivative of this expression with respect to any redundant $X_i$ equal to zero gives

$$\int \frac{M(\partial M/\partial X_i)\,dx}{EI} = 0$$

Therefore, for a statically indeterminate beam (or rigid frame) with $n$ redundants, we can write a set of $n$ simultaneous compatibility equations:

$$\begin{Bmatrix} \dfrac{\partial W}{\partial X_1} \\ \dfrac{\partial W}{\partial X_2} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial W}{\partial X_n} \end{Bmatrix} = \begin{Bmatrix} \displaystyle\int \frac{M(\partial M/\partial X_1)\,dx}{EI} \\ \displaystyle\int \frac{M(\partial M/\partial X_2)\,dx}{EI} \\ \cdot \\ \cdot \\ \cdot \\ \displaystyle\int \frac{M(\partial M/\partial X_n)\,dx}{EI} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{6-26}$$

to solve all the unknown redundants.

In the analysis of statically indeterminate trusses, the total strain energy can be expressed by

$$W = \sum \frac{S^2L}{2AE}$$

Setting the derivative of this expression with respect to any redundant $X_i$ equal to zero gives

$$\sum \frac{S(\partial S/\partial X_i)L}{AE} = 0$$

Thus, for a statically indeterminate truss with $n$ redundant elements, we have a set of $n$ simultaneous compatibility equations available for their solution, namely,

$$\begin{Bmatrix} \displaystyle\sum \frac{S(\partial S/\partial X_1)L}{AE} \\ \displaystyle\sum \frac{S(\partial S/\partial X_2)L}{AE} \\ \cdot \\ \cdot \\ \cdot \\ \displaystyle\sum \frac{S(\partial S/\partial X_n)L}{AE} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{6-27}$$

**Example 6-11**

For the fixed-end beam under general loading shown in Fig. 6-14(a), derive a working formula for solving the end reactions at $A$.

We select the left-end reaction components $M_A$ and $V_A$ as redundants, as shown in Fig. 6-14(b). The primary structure is a cantilever subjected to the original loads on the span together with the redundant forces $M_A$ and $V_A$ at the left end. Aplying the method of least work, we obtain

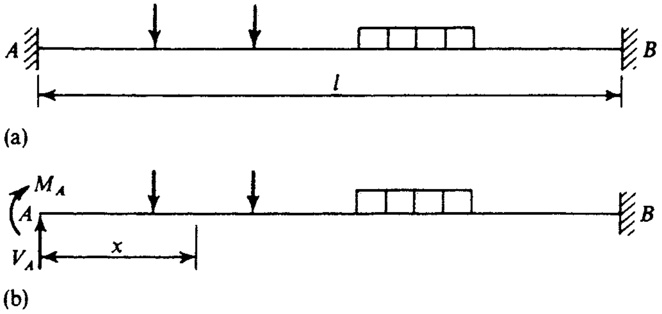


Fig. 6-14

$$\frac{\partial W}{\partial M_A} = \int_0^l \frac{M(\partial M/\partial M_A)\,dx}{EI} = 0 \tag{6-28}$$

$$\frac{\partial W}{\partial V_A} = \int_0^l \frac{M(\partial M/\partial V_A)\,dx}{EI} = 0 \tag{6-29}$$

Since the bending moment at any section of the primary structure is given by

$$M = M' + M_A + V_A x$$

where $M'$ indicates the bending moment at the same section of the primary structure resulting from the original loads on the span, we have

$$\frac{\partial M}{\partial M_A} = 1 \qquad \text{and} \qquad \frac{\partial M}{\partial V_A} = x$$

Substituting these in Eqs. 6-28 and 6-29 results in the following two equations

$$\int_0^l \frac{M\,dx}{EI} = 0 \tag{6-30}$$

$$\int_0^l \frac{Mx\,dx}{EI} = 0 \tag{6-31}$$

to solve for redundants $M_A$ and $V_A$.

For a beam of uniform section with constant $EI$, Eqs. 6-30 and 6-31 reduce to

$$\int_0^l M\,dx = 0 \tag{6-32}$$

$$\int_0^l Mx\,dx = 0 \tag{6-33}$$

As an illustration, let us find the fixed-end moments of the beam shown in Fig. 6-15.

Taking the origin at $A$, we note that

$$M = M_A + V_A x \qquad 0 \leqslant x \leqslant a$$

$$M = M_A + V_A x - P(x - a) \qquad a \leqslant x \leqslant l$$

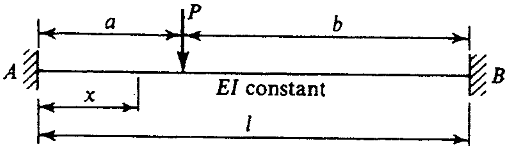


Fig. 6-15

Applying Eqs. 6-32 and 6-33 gives

$$\int_0^l M\,dx = \int_0^a (M_A + V_A x)\,dx + \int_a^l [M_A + V_A x - P(x - a)]\,dx$$

$$= \int_0^l (M_A + V_A x)\,dx + \int_a^l [-P(x - a)]dx = 0$$

or

$$M_A l + \frac{V_A l^2}{2} - \frac{Pb^2}{2} = 0 \qquad (6\text{-}34)$$

and

$$\int_0^l Mx\,dx = \int_0^a (M_A + V_A x)x\,dx + \int_a^l [M_A + V_A x - P(x - a)]x\,dx$$

$$= \int_0^l (M_A + V_A x)x\,dx + \int_a^l [-P(x - a)]x\,dx = 0$$

or

$$\frac{M_A l^2}{2} + \frac{V_A l^3}{3} - \frac{Pb^2(a + 2l)}{6} = 0 \qquad (6\text{-}35)$$

Solving Eqs. 6-34 and 6-35 simultaneously, we obtain

$$M_A = -\frac{Pab^2}{l^2} \qquad V_A = \frac{Pb^2(l + 2a)}{l^3}$$

Similarly,

$$M_B = -\frac{Pa^2b}{l^2} \qquad V_B = \frac{Pa^2(l + 2b)}{l^3}$$

**Example 6-12**

Analyze the frame shown in Fig. 6-16(a) by taking the internal shear, thrust, and moment in the midspan section of the beam as redundants.

Because of symmetry, the shear must be zero in the midspan section $e$ of the beam, and only thrust and bending moment are left as redundants, as shown in Fig. 6-16(a). The solution can be simplified by considering only half of the frame, as shown in Fig. 6-16(b) and Table 6-5.

Applying

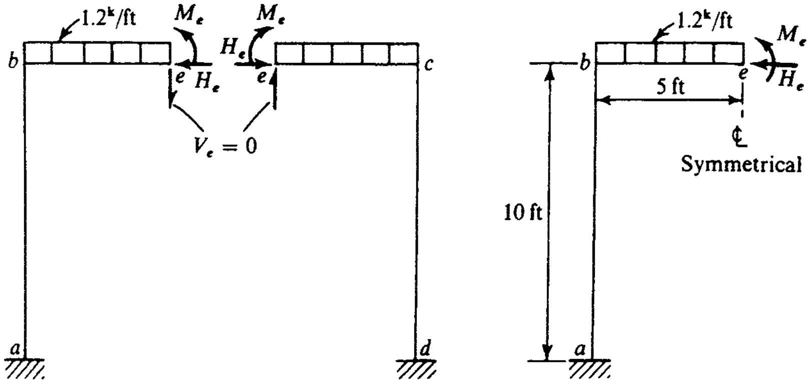
$$\frac{\partial W}{\partial M_e} = 0 \qquad \text{or} \qquad \int_F \frac{M(\partial M/\partial M_e)\,dx}{EI} = 0$$

we have

$$\frac{2}{EI}\left[\int_0^5 \left(M_e - \frac{(1.2)x^2}{2}\right)(1)\,dx + \int_0^{10} (M_e + H_e x - 15)(1)\,dx\right] = 0$$

or

$$3M_e + 10H_e - 35 = 0 \qquad (6\text{-}36)$$



(a) (b)

Fig. 6-16

**TABLE 6-5**

| Member | Origin | Limit (ft) | $M$ (ft-kips) | $\dfrac{\partial M}{\partial M_e}$ | $\dfrac{\partial M}{\partial H_e}$ (ft) |
|---|---|---|---|---|---|
| *eb* | *e* | 0 to 5 | $M_e - \dfrac{(1.2)x^2}{2}$ | 1 | 0 |
| *ba* | *b* | 0 to 10 | $M_e + H_e x - 15$ | 1 | $x$ |

Applying

$$\frac{\partial W}{\partial H_e} = 0 \qquad \text{or} \qquad \int_F \frac{M(\partial M/\partial H_e)\,dx}{EI} = 0$$

we have

$$\frac{2}{EI}\left[\int_0^{10} (M_e + H_e x - 15)x\,dx\right] = 0$$

or

$$3M_e + 20H_e - 45 = 0 \tag{6-37}$$

Solving Eqs. 6-36 and 6-37 simultaneously gives

$$H_e = 1.0\text{ kip} \qquad M_e = 8.33\text{ ft-kips}$$

from which we obtain

$$H_a = 1.0\text{ kip} \qquad M_a = 3.33\text{ ft-kips}$$

as previously found.

For a highly indeterminate rigid frame, such as the one shown in Fig. 6-17(a), the procedure of the analysis remains the same. The frame is statically indeterminate to the 24th degree. We may cut it back to three determinate structures and substitute the redundants $X_1$, $X_2$, . . . , $X_{24}$ at the cut sections as shown in Fig. 6-17(b).

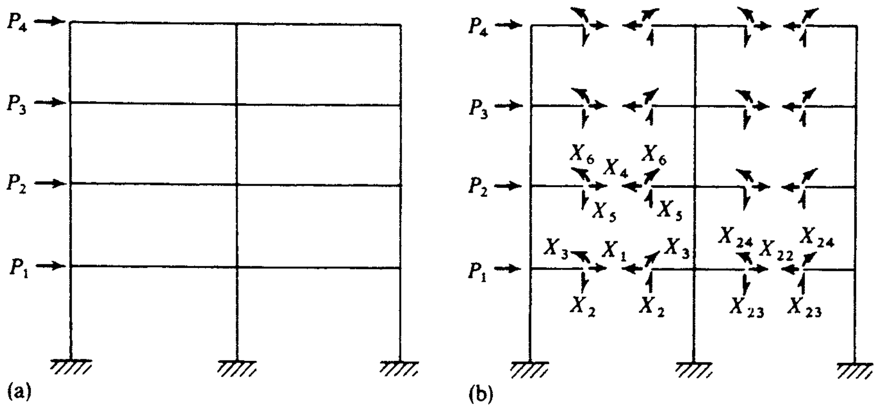


Fig. 6-17

From least work, we have 24 equations to solve for all the redundants simultaneously, namely,

$$\begin{Bmatrix} \dfrac{\partial W}{\partial X_1} \\ \dfrac{\partial W}{\partial X_2} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial W}{\partial X_{24}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix}$$

where $W$ is the total strain energy of the frame due to the external loads and redundant forces. The principle is neat and elegant, whereas the numerical calculations involved in the equations above are so cumbersome that it is almost impossible for a structural engineer to reach an exact solution for the system with only a slide rule or desk calculator. To handle a practical problem like this, a grossly simplified model of the actual structure was often used. However, with the advent of the digital computer, the solving of simultaneous equations can now be performed in a matter of minutes.

**Example 6-13**

Analyze the truss in Fig. 6-18(a). Assume that $E = 30{,}000$ kips/in.$^2$ and $L(\text{ft})/A(\text{in.}^2) = 1$ for all members.

The truss is statically indeterminate to the second degree. We may take bars $bC$ and $Cd$ as redundant members. As shown in Fig. 6-18(b), these bars are cut and replaced by redundant axial forces $X_1$ and $X_2$, respectively. The internal force for each bar is then computed in terms of the external load and redundant forces as indicated. The unknowns $X_1$ and $X_2$ are then solved by the simultaneous equations

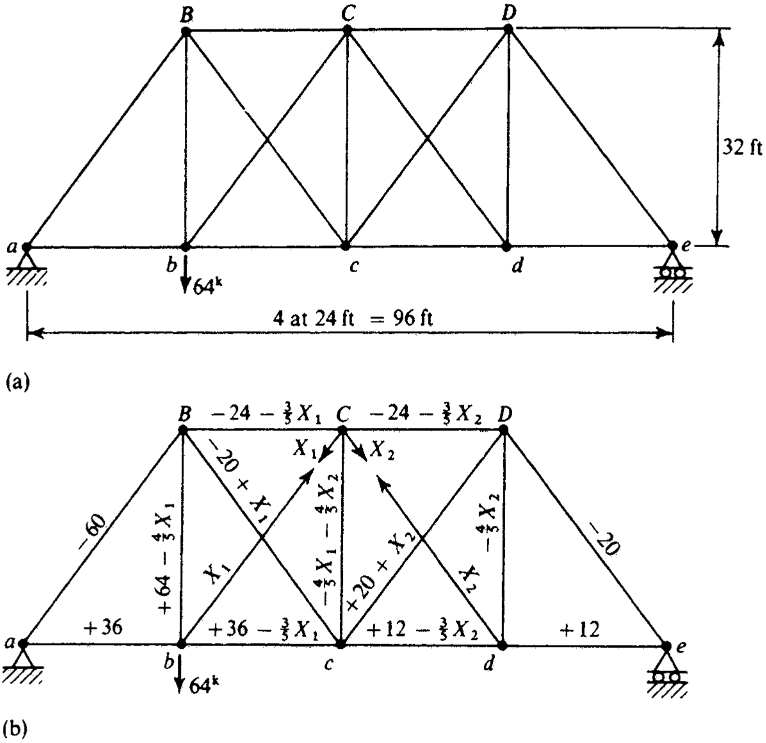


**Fig. 6-18**

$$\sum \frac{S(\partial S/\partial X_1)L}{AE} = 0 \qquad \text{and} \qquad \sum \frac{S(\partial S/\partial X_2)L}{AE} = 0$$

as prepared in Table 6-6. Setting

$$-78.4 + 4X_1 + 0.64X_2 = 0 \tag{6-38}$$

$$27.2 + 0.64X_1 + 4X_2 = 0 \tag{6-39}$$

and solving Eqs. 6-38 and 6-39 simultaneously, we obtain

$$X_1 = +21.2 \text{ kips} \qquad X_2 = -10.2 \text{ kips}$$

The answer for each of the bar forces is given in the last column of Table 6-6. Note that this procedure can be extended to trusses with many redundants.

Structures made up of some members which are two-force members carrying only axial forces and others which are not are called composite structures. They are conveniently analyzed by the method of least work, as illustrated in the following example.

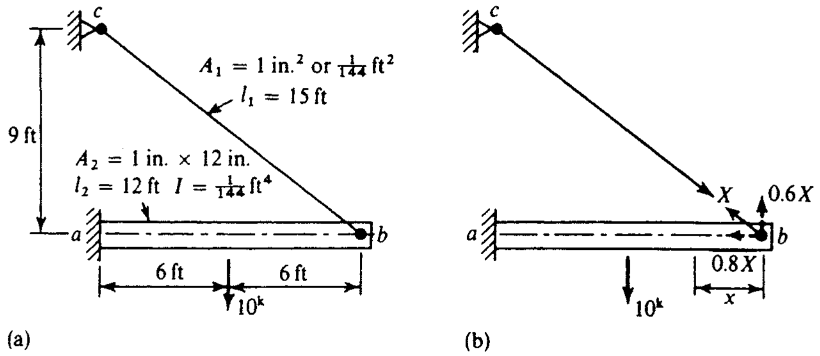
**Example 6-14**

Figure 6-19(a) shows a cantilever beam whose other end is supported by a rod. Find the force in the rod. $E = 30{,}000$ kips/in.$^2$.

The structure is statically indeterminate to the first degree. Select the force

**TABLE 6-6**

| Member | $\frac{L}{A}$ (ft/in.$^2$) | $S$ (kips) | $\frac{\partial S}{\partial X_1}$ | $\frac{\partial S}{\partial X_2}$ | $\frac{S(\partial S/\partial X_1)L}{A}$ (ft-kips/in.$^2$) | $\frac{S(\partial S/\partial X_2)L}{A}$ (ft-kips/in.$^2$) | Answer (kips) |
|---|---|---|---|---|---|---|---|
| *ab* | 1 | $36 + 0 + 0$ | 0 | 0 | 0 | 0 | +36 |
| *bc* | 1 | $36 - (\frac{3}{5})X_1 + 0$ | $-\frac{3}{5}$ | 0 | $-21.6 + (\frac{9}{25})X_1 + 0$ | 0 | +23.3 |
| *cd* | 1 | $12 + 0 - (\frac{3}{5})X_2$ | 0 | $-\frac{3}{5}$ | 0 | $-7.2 + 0 + (\frac{9}{25})X_2$ | +18.2 |
| *de* | 1 | $12 + 0 + 0$ | 0 | 0 | 0 | 0 | +12 |
| *BC* | 1 | $-24 - (\frac{3}{5})X_1 + 0$ | $-\frac{3}{5}$ | 0 | $14.4 + (\frac{9}{25})X_1 + 0$ | 0 | −36.7 |
| *CD* | 1 | $-24 + 0 - (\frac{3}{5})X_2$ | 0 | $-\frac{3}{5}$ | 0 | $14.4 + 0 + (\frac{9}{25})X_2$ | −17.8 |
| *aB* | 1 | $-60 + 0 + 0$ | 0 | 0 | 0 | 0 | −60 |
| *Bb* | 1 | $64 - (\frac{4}{5})X_1 + 0$ | $-\frac{4}{5}$ | 0 | $-51.2 + (\frac{16}{25})X_1 + 0$ | 0 | +47 |
| *Bc* | 1 | $-20 + X_1 + 0$ | +1 | 0 | $-20 + X_1 + 0$ | 0 | + 1.2 |
| *bC* | 1 | $0 + X_1 + 0$ | +1 | 0 | $0 + X_1 + 0$ | 0 | +21.2 |
| *Cc* | 1 | $0 - (\frac{4}{5})X_1 - (\frac{4}{5})X_2$ | $-\frac{4}{5}$ | $-\frac{4}{5}$ | $0 + (\frac{16}{25})X_1 + (\frac{16}{25})X_2$ | $0 + (\frac{16}{25})X_1 + (\frac{16}{25})X_2$ | − 8.8 |
| *Cd* | 1 | $0 + 0 + X_2$ | 0 | +1 | 0 | $0 + 0 + X_2$ | −10.2 |
| *cD* | 1 | $20 + 0 + X_2$ | 0 | +1 | 0 | $20 + 0 + X_2$ | + 9.8 |
| *Dd* | 1 | $0 + 0 - (\frac{4}{5})X_2$ | 0 | $-\frac{4}{5}$ | 0 | $0 + 0 + (\frac{16}{25})X_2$ | + 8.2 |
| *De* | 1 | $-20 + 0 + 0$ | 0 | 0 | 0 | 0 | −20 |
| | | | | $\Sigma$ | $-78.4 + 4X_1 + 0.64X_2$ | $27.2 + 0.64X_1 + 4X_2$ | |



(a) (b)

Fig. 6-19

in the tie rod as the redundant $X$ as shown in Fig. 6-19(b). Then the internal work in the rod is

$$\frac{X^2 l_1}{2A_1 E}$$

and the internal work in the beam is equal to

$$\int_0^6 \frac{(0.6Xx)^2\,dx}{2EI} + \int_6^{12} \frac{[0.6Xx - 10(x-6)]^2\,dx}{2EI} + \frac{(-0.8X)^2 l_2}{2A_2 E}$$

Applying $\partial W/\partial X = 0$ gives

$$\frac{Xl_1}{A_1 E} + \int_0^6 \frac{(0.6Xx)(0.6x)\,dx}{EI} + \int_6^{12} \frac{[0.6Xx - 10(x-6)][0.6x]\,dx}{EI} + \frac{(-0.8X)(-0.8)l_2}{A_2 E} = 0$$

or

$$\frac{15X}{1/144} + \frac{207.4X - 3{,}024 + 1{,}944}{1/144} + \frac{(0.64X)(12)}{12/144} = 0$$

After simplifying, we find that

$$15X + 207.4X - 1{,}080 + 0.64X = 0$$

which yields

$$X = 4.84 \text{ kips} \qquad \text{(tension)}$$

The effect of the axial force of beam on the strain energy is small and can be neglected.

## 6-6 INFLUENCE LINES FOR STATICALLY INDETERMINATE STRUCTURES: THE MÜLLER-BRESLAU PRINCIPLE

We recall that the Müller-Breslau principle was used to construct the influence lines for statically determinate structures (see Sec. 4-3). We shall demonstrate that the principle is equally applicable to obtaining the influence lines for statically indeterminate linear structures.

Suppose that we want the influence line for the reaction at support $b$ of the indeterminate beam $abc$ shown in Fig. 6-20(a). The influence ordinate at any point $i$ a distance $x$ from the left end is obtained by placing a unit load at that point and computing the reaction at support $b$. The procedure for finding this reaction is as follows:

1. Remove the support at $b$ and introduce in its place a redundant reaction called $R_b$.
2. Consider beam $ac$ as the primary structure subjected to the combined effects of the unit force at $i$ and $R_b$ [see Fig. 6-20(b)].
3. Use the condition of compatibility that the total deflection at point $b$ must be zero,

$$\Delta_b = R_b\delta_{bb} - \delta_{bi} = 0$$

See Fig. 6-20(c) for $\delta_{bi}$ and Fig. 6-20(d) for $\delta_{bb}$.

$$R_b = \frac{\delta_{bi}}{\delta_{bb}}$$

4. Use the reciprocity

$$\delta_{bi} = \delta_{ib}$$

to obtain

$$R_b = \frac{\delta_{ib}}{\delta_{bb}} \tag{6-40}$$

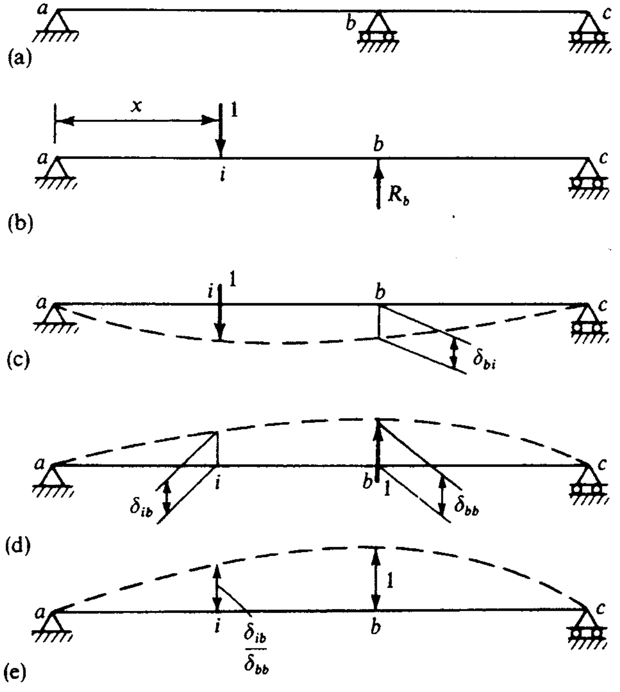


Fig. 6-20

Note that in Eq. 6-40 the numerator $\delta_{ib}$ represents the ordinate of the deflection curve of the primary beam *ac* caused by applying a unit force at *b*. The denominator $\delta_{bb}$ is only a special case of $\delta_{ib}$ (i.e., $\delta_{bb} = \delta_{ib}$ if $i = b$), as shown in Fig. 6-20(d). Each ordinate of the curve of Fig. 6-20(d) divided by $\delta_{bb}$ will give the corresponding influence ordinate for $R_b$ (see Eq. 6-40), as shown in Fig. 6-20(e).

Referring to Fig. 6-20(e), we note that at *b*, $\delta_{bb}/\delta_{bb} = 1$. Hence, the influence line for $R_b$ is nothing more than the deflected structure resulting from removal of the support at *b* and introduction in its place of a unit deflection along the line of reaction.

We have hitherto used the reaction of a support as illustration. This, then, is the Müller-Breslau principle and may be stated as follows: *The ordinates of the influence line for any element* (reaction, axial force, shear, or moment) *of a structure equal those of the deflection curve obtained by removing the restraint corresponding to that element from the structure and introducing in its place a unit load divided by the deflection at the point of application of the unit load.*

This may be rephrased as: *The deflected structure resulting from a unit displacement corresponding to the action for which the influence line is desired gives the influence line for that action.*

The Müller-Breslau principle provides a very convenient method for sketching qualitative influence lines for indeterminate structures and is the basis for certain experimental model analyses.

1. In the simplest case the influence line for a reaction component can be sketched by removing the restraint and allowing the reaction to move through a *unit displacement*. The deflected structure will then be the influence line for the reaction.

Thus, the dashed line in Fig. 6-21(a) shows the influence line for the vertical reaction at support *a* of the three-span continuous beam. In Fig. 6-21(b) the dashed line indicates the influence line for the fixed-end moment at support *a* of the fixed-end beam *ab*. The curve is obtained by replacing the fixed support at *a* with a hinge support and by introducing a unit rotation. Figure 6-21(c) shows the construction of the influence line for the horizontal reaction component at the fixed support *d* of a portal frame. Note that the fixed support at *d* is replaced by a roller and slide acted on by a horizontal force so as to produce a unit horizontal displacement.

Note the following:

a. The vertical deflections of the structure will be influence line ordinates for the vertical loads on the structure.

b. The horizontal deflections of the structure will be influence line ordinates for the horizontal loads on the structure.

c. The rotation of the tangents of the structure will be influence line ordinates for the moment load on the structure.

2. The moment influence line for a section of a beam or rigid frame can be

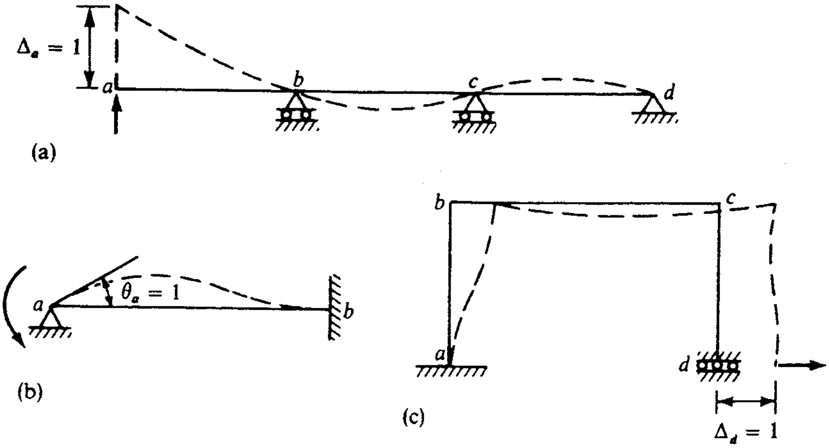


Fig. 6-21

drawn by cutting the section and allowing a pair of equal and opposite moments to produce a *unit relative rotation* (but no relative translation) for the two sides of the section considered. The deflected structure will then be the influence line for the moment. Thus, the influence line for the moment at the midspan section of a three-span continuous beam is the dashed line in Fig. 6-22.

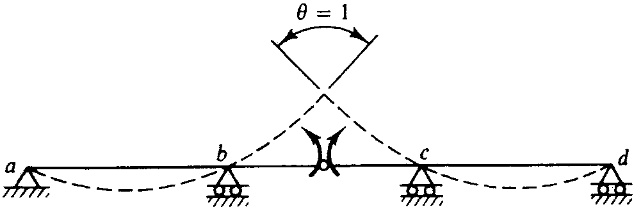


Fig. 6-22

3. The shear influence line for a section of a beam or rigid frame can be drawn by cutting the section and applying a pair of equal and opposite shearing forces to produce a *unit relative transverse displacement* (but no relative rotation) for the two sides of the section considered. The deflected structure will then be the influence line for the shear. The influence line for the shear at the midspan section of the fixed-end beam *ab* is shown in Fig. 6-23 by the dashed lines.

4. To obtain the influence line for the axial force in a bar, we cut the bar and apply a pair of equal and opposite axial forces so as to cause a *unit relative axial displacement* for the two cut ends. The deflected structure will give the desired influence line.

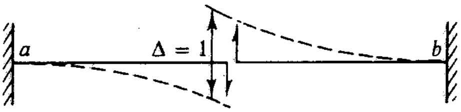


Fig. 6-23

Figure 6-24 serves as a simple illustration of obtaining the influence line for the bar force in *bC* of the indeterminate truss. The vertical components of the panel point deflections are thus the influence ordinates for the vertical panel loads.

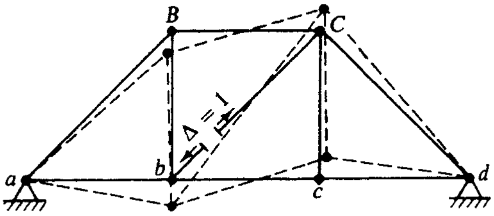


Fig. 6-24

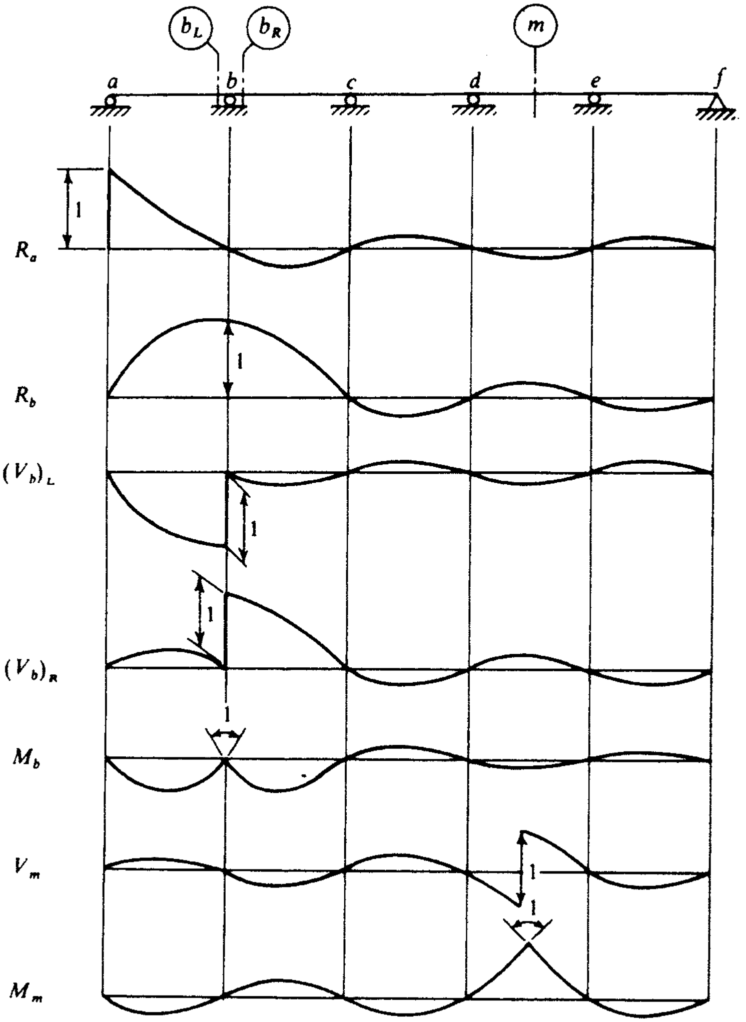


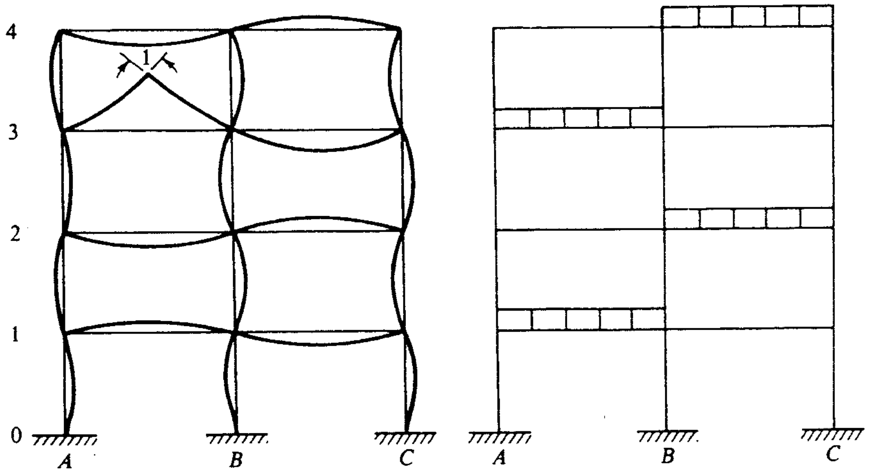
Fig. 6-25

For highly indeterminate continuous beams or rigid frames the technique of sketching qualitative influence lines, based on the Müller-Breslau principle, is extremely useful in determining the loading patterns for design. Figure 6-25 shows typical influence lines for a five-span continuous beam.

The sketches in Fig. 6-25 indicate that if a maximum $R_a$ is desired, then spans *ab*, *cd*, and *ef* should be loaded; if the maximum values of reaction, of shear, and of bending moment at *b* are desired, then spans *ab*, *bc*, and *de* should be loaded.

Figure 6-26(a) shows the influence line for the positive moment at the midspan section of *A*3-*B*3 of the frame shown. The uniform loading pattern for obtaining the maximum positive moment of this section is shown in Fig. 6-26(b).

Numerical examples for influence lines of statically indeterminate structures follow.



(a) (b)

Fig. 6-26

**Example 6-15**

Using the Müller-Breslau principle, construct the influence line for the reaction at support *b* of Fig. 6-27(a).

We begin by removing the support at *b* and placing a unit load along the line of reaction, as shown in Fig. 6-27(b). The ordinates of the resulting deflection curve, in Fig. 6-27(b), divided by $\delta_{bb}$ give the corresponding influence ordinates for the reaction at *b*, called $R_b$.

Probably the easiest method for computing the ordinate of the curve in Fig. 6-27(b) at any point *i* a distance *x* from the left end is the conjugate-beam method, shown in Fig. 6-27(c).

Thus,

$$\delta_{ib} = \left(\frac{l^2}{4EI}\right)(x) - \left(\frac{x}{2EI}\right)\left(\frac{x}{2}\right)\left(\frac{x}{3}\right) = \frac{3l^2x - x^3}{12EI} \qquad (0 \leq x \leq l)$$

Substituting $x = l$ in the expression above gives

$$\delta_{bb} = \frac{l^3}{6EI}$$

The influence ordinate for any point $i$ $(0 \leqslant x \leqslant l)$ is governed by the equation

$$R_b = \frac{\delta_{ib}}{\delta_{bb}} = \frac{3l^2x - x^3}{2l^3}$$

as shown in Fig. 6-27(d). Because of symmetry about the middle support, we can accomplish the other half, as shown by the dashed line in Fig. 6-27(d).

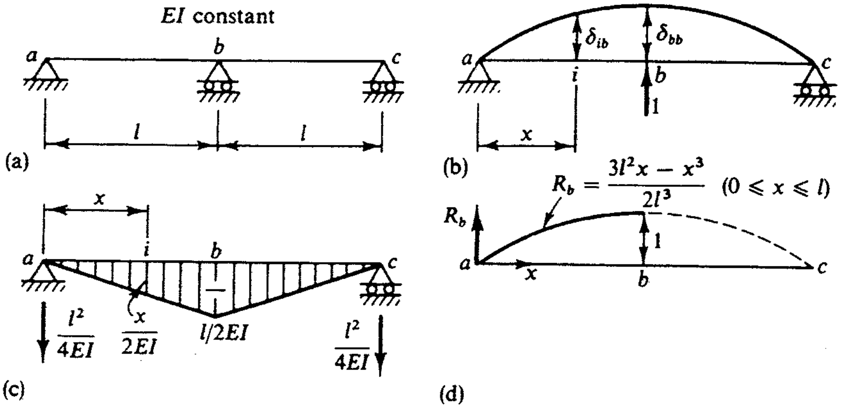


Fig. 6-27

**Example 6-16**

Compute the ordinates at 2-ft intervals of the influence line for the moment at the midspan section $d$ of $ab$ for the beam shown in Fig. 6-28(a).

The ability to resist moment at section $d$ is first destroyed by inserting a pin. Unit couples are applied on each side of the pin to produce certain relative rotation, denoted by $\theta$, between the two sides, as shown in Fig. 6-28(b). The conjugate beam and elastic load are then obtained, as shown in Fig. 6-28(c). Note that we assume $EI = 1$ ($EI$ will be canceled out in the final stage of calculation); hence, the elastic load of Fig. 6-28(c) is the moment diagram based on Fig. 6-28(b).

Referring to Fig. 6-28(c), we may solve reactions $R_a$, $R_c$, and $R_d$ by the equilibrium equations and the condition equation $M_b = 0$. Thus,

$$\sum M_a = 0 \qquad 20R_c + 5R_d - \frac{(2)(20)}{2}(10) = 0 \tag{6-41}$$

$$\sum M_c = 0 \qquad 15R_d - 20R_a - \frac{(2)(20)}{2}(10) = 0 \tag{6-42}$$

$$M_b = 0 \qquad 10R_c - \frac{(2)(10)}{2}\left(\frac{10}{3}\right) = 0 \tag{6-43}$$

Solving Eqs. 6-41, 6-42, and 6-43 simultaneously, we obtain

$$R_a = 10 \qquad R_c = 3.33 \qquad R_d = 26.67$$

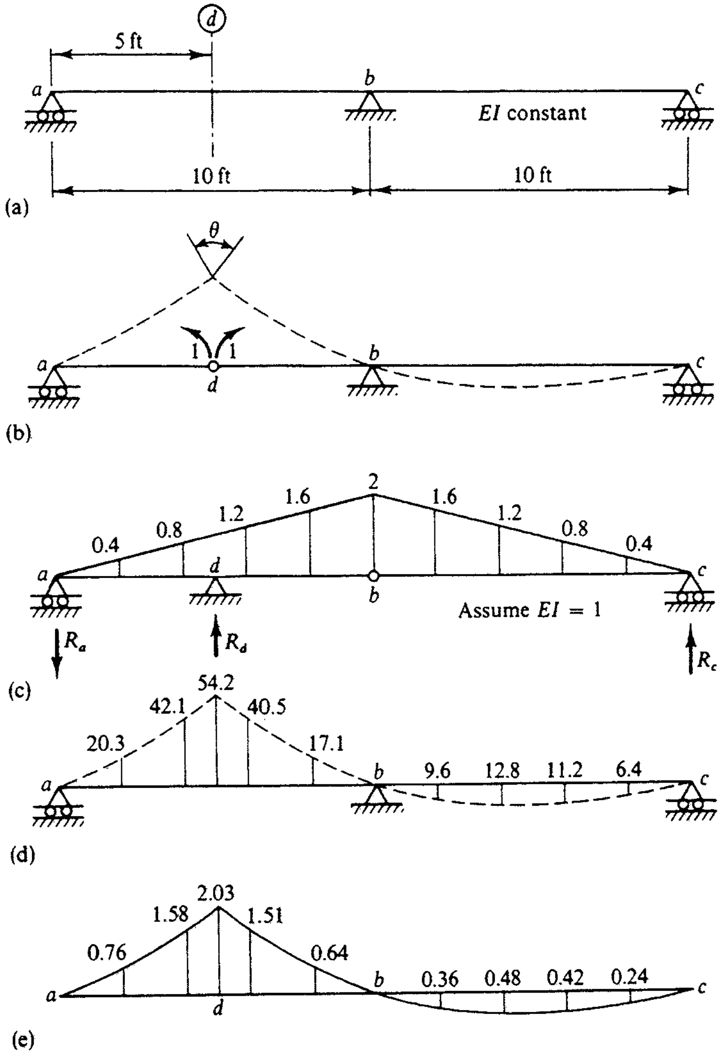


Fig. 6-28

Note that $R_d$ is the shear difference between the left and right sides at section $d$ of the conjugate beam and thus equals the relative rotation $\theta$ between corresponding portions of the distorted beam in Fig. 6-28(b). The various moments of the conjugate beam at 2-ft intervals and at point $d$ which correspond to the deflections of the distorted beam in Fig. 6-28(b) are computed as shown in Fig. 6-28(d).

These values divided by 26.67 (so as to make $\theta = 1$) will give the ordinates of the influence line for the moment at section $d$, as shown in Fig. 6-28(e).

**Example 6-17**

Compute the influence ordinates at 2-ft intervals for the shear at the midspan section $d$ of $ab$ for the beam shown in Fig. 6-29(a).

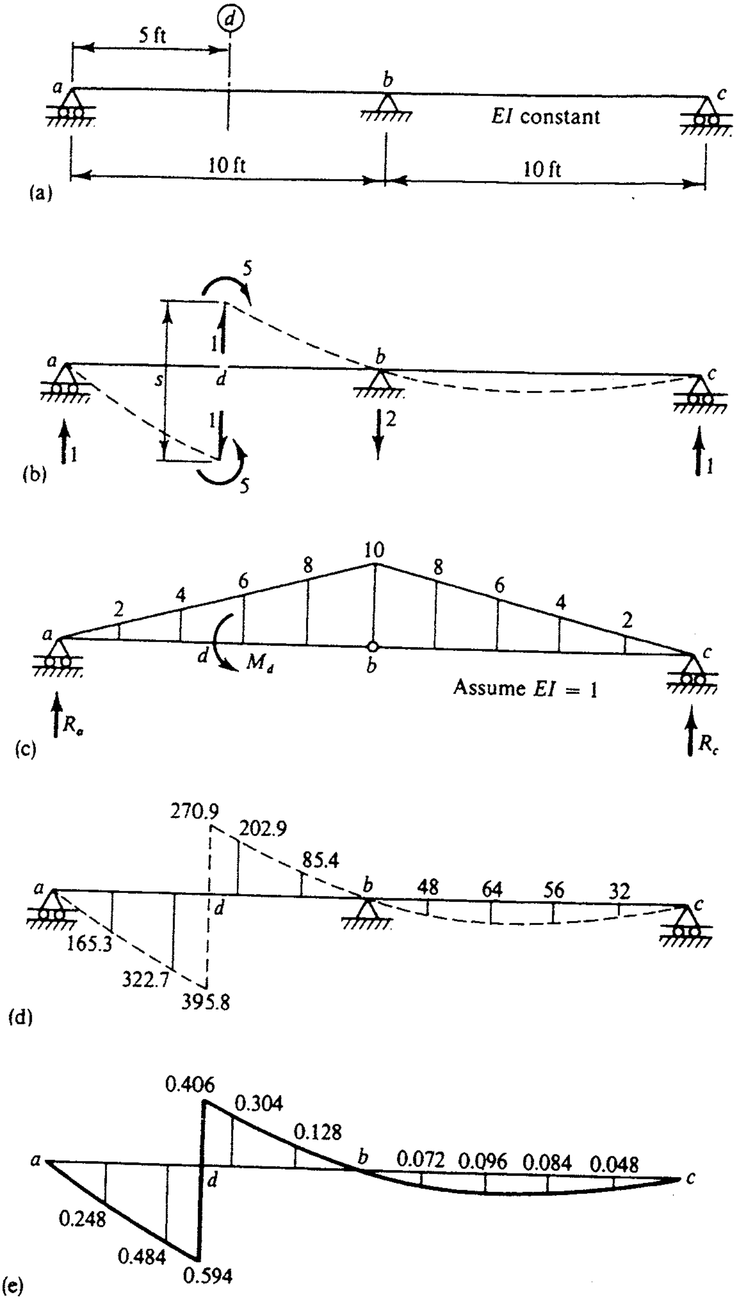

(a)
d
5 ft
a
b
c
EI constant
10 ft
10 ft
(b)
5
1
a
s
d
b
c
1
2
1
1
5
(c)
10
8
8
6
6
4
4
2
2
a
d
$M_d$
b
c
Assume $EI = 1$
$R_a$
$R_c$
(d)
270.9
202.9
85.4
a
d
b
48
64
56
32
c
165.3
322.7
395.8
(e)
0.406
0.304
0.128
a
d
b
0.072
0.096
0.084
0.048
c
0.248
0.484
0.594

Fig. 6-29

We start by removing the shearing resistance at section *d* without impairing the capacity for resistance to moment. This can be accomplished by cutting the beam and inserting a slide device. Next, we apply a pair of equal and opposite unit forces to produce certain relative vertical displacement, denoted by *s*, between the two cut ends without causing relative rotation, as shown in Fig. 6-29(b). Also indicated in Fig. 6-29(b) are the induced reactions and moments at the cut ends required by equilibrium.

The conjugate beam together with the elastic loads is shown in Fig. 6-29(c). Attention should be paid to the elastic load $M_d$ acting at *d*. This is necessary since the relative vertical displacement (without relative rotation) between the two cut ends at section *d* of the distorted beam in Fig. 6-29(b) requires a moment difference (without a shear difference) for the two sides at *d* of the conjugate beam shown in Fig. 6-29(c). This can be fulfilled only by applying a moment at *d* for the conjugate beam. Referring to Fig. 6-29(c), we solve reactions $R_a$, $R_c$, and $M_d$ by the equilibrium equations and the condition equation $M_b = 0$. Thus,

$$\sum M_a = 0 \qquad 20R_c + M_d - \frac{(10)(20)}{2}(10) = 0 \tag{6-44}$$

$$\sum M_c = 0 \qquad 20R_a - M_d - \frac{(10)(20)}{2}(10) = 0 \tag{6-45}$$

$$M_b = 0 \qquad 10R_c - \frac{(10)(10)}{2}\left(\frac{10}{3}\right) = 0 \tag{6-46}$$

Solving Eqs. 6-44, 6-45, and 6-46 simultaneously, we obtain

$$R_a = 83.33 \qquad R_c = 16.67 \qquad M_d = 666.67$$

Note that $M_d$ is the moment difference between the left and right sides at section *d* of the conjugate beam and therefore equals the relative deflection *s* between the corresponding portions of the distorted beam shown in Fig. 6-29(b). The various moments of the conjugate beam at 2-ft intervals and at *d* are then computed, as shown in Fig. 6-29(d).

These values divided by $M_d = 666.67$ (so as to make $s = 1$) will give the ordinates of the shear influence line for section *d*, as shown in Fig. 6-29(e).

For highly indeterminate structures the influence ordinates for various functions may be found by using a computer for a number of equations based on consistent deformations. Let us consider the four-span continuous beam shown in Fig. 6-30(a). Find the influence line for the reaction at support *b*, called $R_b$.

We start by removing support *b* and applying to it a force $X_b$ along the line of reaction so as to produce a unit displacement at *b*. Then, by the Müller-Breslau principle the elastic line of the distorted beam, shown by the dashed line in Fig. 6-30(b), will be the influence line for $R_b$. Also indicated in Fig. 6-30(b) are the induced reactions at supports *c* and *d*, called $X_c$ and $X_d$, respectively, and the ordinate of the curve at any point *i*, called $\Delta_i$, at distance *x* from the left end.

To obtain the value of $\Delta_i$ by the method of consistent deformations, we regard the indeterminate beam in Fig. 6-30(b) as a simple beam *ae* (primary

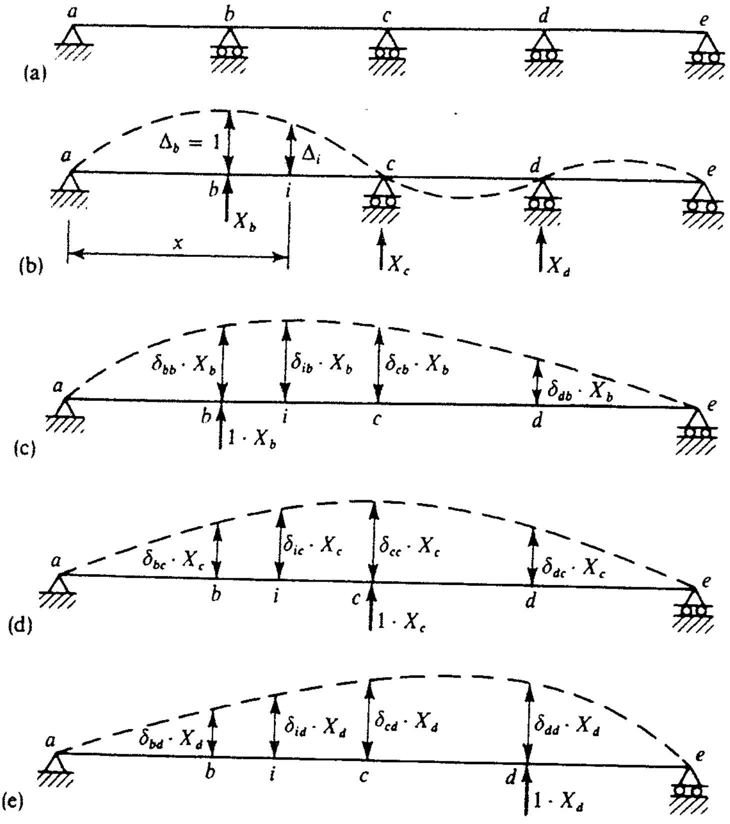


**Fig. 6-30**

structure) subjected to forces $X_b$, $X_c$, and $X_d$, the effects of which can be separated by the principle of superposition into three basic cases, as shown in Fig. 6-30(c), (d), and (e), respectively. Thus,

$$\Delta_i = \delta_{ib}X_b + \delta_{ic}X_c + \delta_{id}X_d$$

in which the unknowns $X_b$, $X_c$, and $X_d$ can be solved by the compatibility condition

$$\begin{Bmatrix} \Delta_b \\ \Delta_c \\ \Delta_d \end{Bmatrix} = \begin{bmatrix} \delta_{bb} & \delta_{bc} & \delta_{bd} \\ \delta_{cb} & \delta_{cc} & \delta_{cd} \\ \delta_{db} & \delta_{dc} & \delta_{dd} \end{bmatrix} \begin{Bmatrix} X_b \\ X_c \\ X_d \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{6-47}$$

In a similar manner, we can find the influence line for the reaction at each of the other supports. This done, the influence lines for the moment and shear at various points can be deduced from them by simple statics.

## PROBLEMS

**6-1.** Analyze the beam in Fig. 6-31 by the method of consistent deformations. (a) Use the reaction at center support *b* as redundant. (b) Use the internal moment at *b* as redundant. Assume constant *EI*.

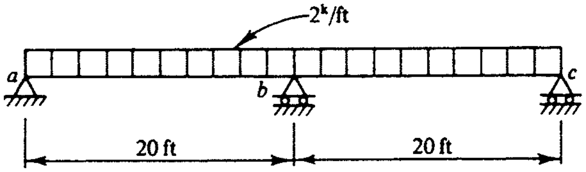


Fig. 6-31

**6-2.** Determine the reaction at *b* in Fig. 6-32 by the method of consistent deformations. Assume constant *EI*.

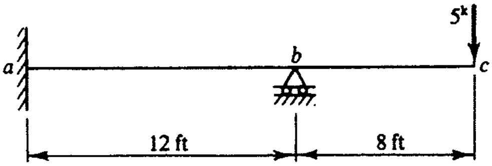


Fig. 6-32

**6-3.** Find the reaction at *b* in Fig. 6-33 by the method of consistent deformations. Assume constant *E*.

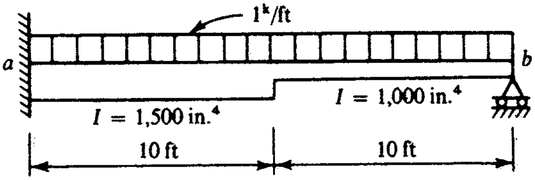


Fig. 6-33

**6-4.** For the system shown in Fig. 6-34 determine, by the method of consistent deformations, the reaction at support *e*. The flexibility of the spring $f$ = 0.2 cm/kN of force; the bending rigidity of the beam $EI$ = 30,000 kN·m$^2$.

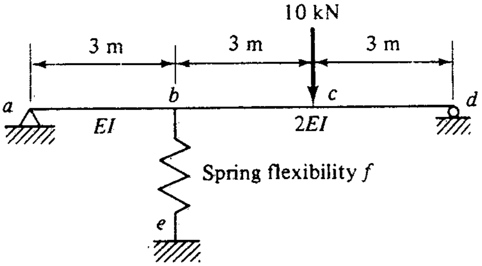


Fig. 6-34

**6-5.** Find the fixed-end moments for the beams in Fig. 6-35 by the method of consistent deformations. Assume constant *EI*.

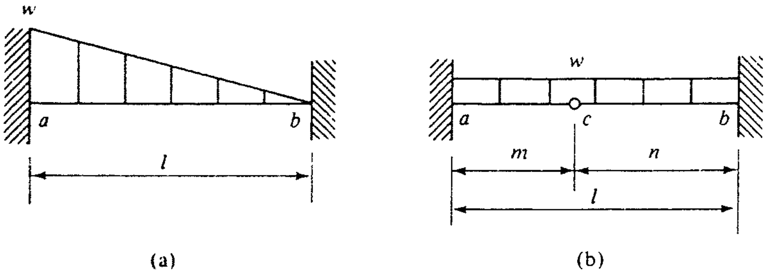


**Fig. 6-35**

**6-6.** Use the method of consistent deformations to determine the horizontal reaction at support *c* of the rigid frame shown in Fig. 6-36. Assume that $E = 20{,}000 \text{ kN/cm}^2$ and $I = 20{,}000 \text{ cm}^4$.

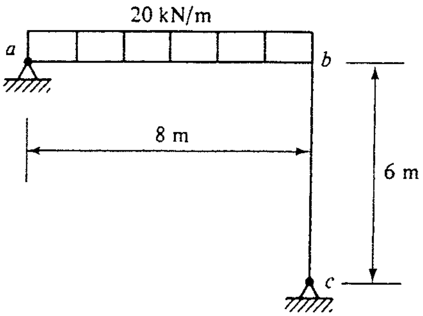


**Fig. 6-36**

**6-7.** Analyze the rigid frame shown in Fig. 6-37 by the method of consistent deformations. Use the reaction components at support *a* as the redundants. All members have the same value of *EI*.

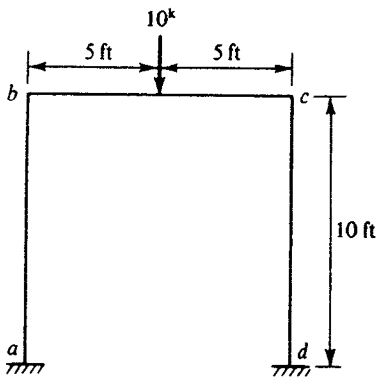


**Fig. 6-37**

**6-8.** Repeat Prob. 6-7 using the moment forces at joints *b*, *c*, and *d* as the redundants.

**6-9.** Analyze the truss in Fig. 6-38 by the method of consistent deformations. Assume that $E = 20{,}000$ kN/cm$^2$ and $A = 25$ cm$^2$ for all members.

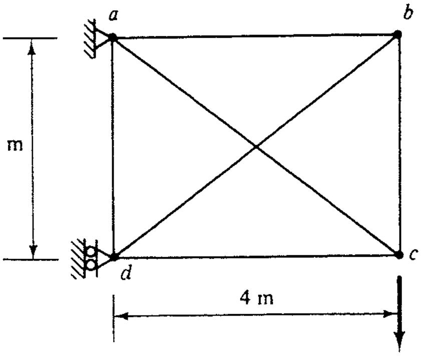


**Fig. 6-38**

**6-10.** Analyze each of the trusses in Fig. 6-39 by the method of consistent deformations. Assume that $E = 30{,}000$ kips/in.$^2$ and $L(\text{ft})/A(\text{in.}^2) = 2$ for all members.

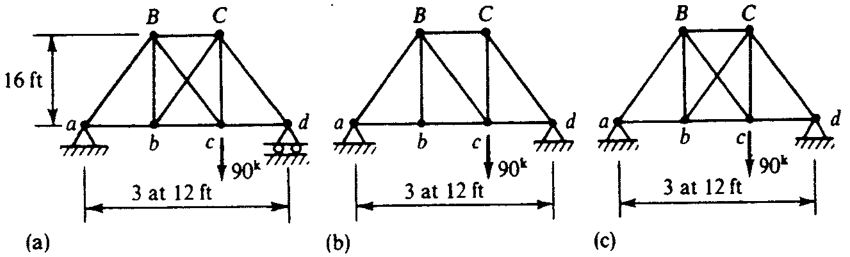


**Fig. 6-39**

**6-11.** Analyze the truss in Fig. 6-39(a) (without the external load) subject to a rise in temperature of 50°F for member *BC*. Assume that $\alpha = 0.0000065$ in./in./1°F.

**6-12.** Repeat Prob. 6-1 using Castigliano's compatibility equation.

**6-13.** Repeat Prob. 6-5 using Castigliano's compatibility equation.

**6-14.** Repeat Prob. 6-6 using Castigliano's compatibility equation.

**6-15.** Repeat Prob. 6-9 using Castigliano's compatibility equation.

**6-16.** Find the internal force for the tie rod *ac* of the composite structure shown in Fig. 6-40, and sketch the moment diagram for member *ab*. $E = 30{,}000$ kips/in.$^2$.

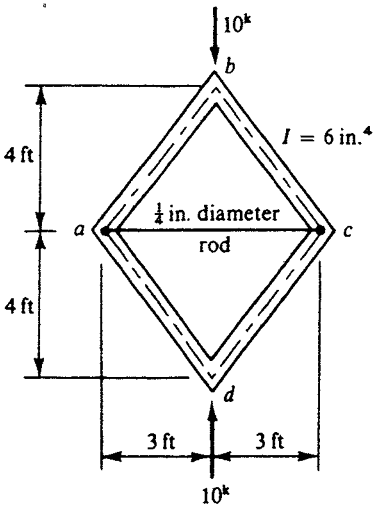


Fig. 6-40

**6-17.** Find the internal forces for all rods of the composite structure shown in Fig. 6-41 and sketch the moment diagram for beam *ab*. $E = 30{,}000$ kips/in.$^2$.

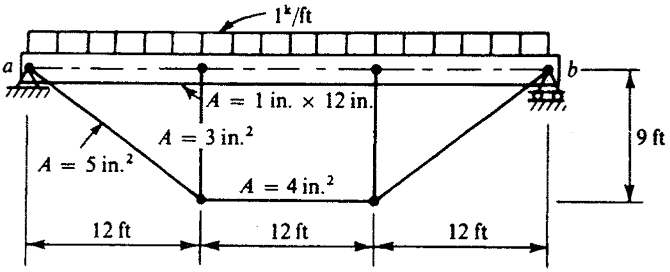


Fig. 6-41

**6-18.** Compute the ordinates at 2-m intervals of the influence line for the reaction at *a* of the beam shown in Fig. 6-42.

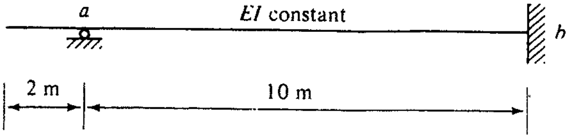


Fig. 6-42

**6-19.** Compute the influence ordinates at 2-m intervals for the end moment at $a$ of the beam shown in Fig. 6-43.

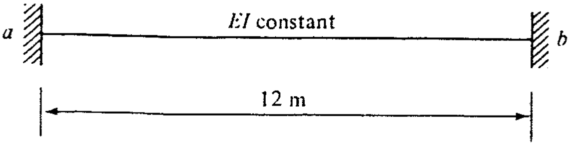


**Fig. 6-43**

**6-20.** Refer to the beam in Fig. 6-44. Compute the influence ordinates at 2-m intervals for the following elements: (a) the reaction at support $a$, (b) the moment at $b$, and (c) the shear at the midspan section of $ab$.

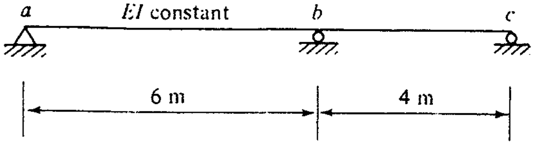


**Fig. 6-44**

**6-21.** Sketch the influence lines for $R_a$, $R_c$, $M_c$, $V_c$ (left), $V_c$ (right), and $M$ and $V$ for the midspan section of $bc$ of the beam shown in Fig. 6-45.

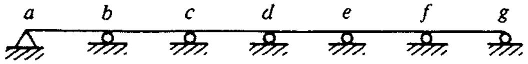


**Fig. 6-45**

**6-22.** Sketch the influence lines for the shear and moment in the midspan section of member $ab$ of the rigid frame shown in Fig. 6-46.

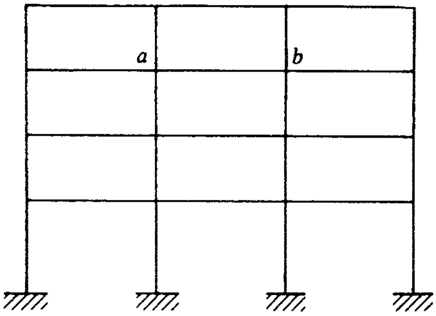


**Fig. 6-46**

# 7

# *Slope-Deflection Method*

## 7-1 GENERAL

Throughout the preceding chapter the methods of analyzing statically indeterminate structures have used forces (reactions or internal forces) as the basic unknowns. These are often referred to as the force or action methods. Displacements, however, may be used equally well as unknowns. Methods using displacements as the basic unknowns are called displacement methods. One of the important displacement methods is the *method of slope deflection*, based on determining the rotations and deflections of various joints from which the end moments for each member are found.

The slope-deflection method may be used in analyzing all types of statically indeterminate beams and rigid frames composed of prismatic or nonprismatic members. However, in this chapter we discuss exclusively beams and rigid frames made of prismatic members.

## 7-2 BASIC SLOPE-DEFLECTION EQUATIONS

The basis of the slope-deflection method lies in the *slope-deflection equations*, which express the end moments of each member in terms of the end distortions of that member.

Consider member *ab* shown in Fig. 7-1, which is isolated from a loaded statically indeterminate beam or rigid frame (not shown). The member is deformed (see the dashed line) with end rotations $\theta_a$ and $\theta_b$ and relative deflection $\Delta$ between the ends. Obviously, the induced end moments at *a* and *b*, called $M_{ab}$ and $M_{ba}$,

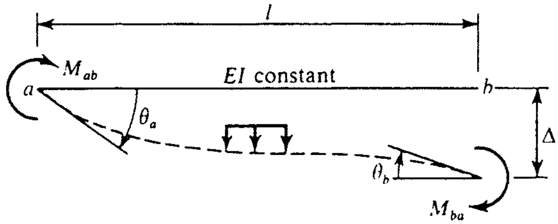


**Fig. 7-1**

are related to the elastic distortions at both ends as well as to the load on span *ab*, if any. Thus,

$$M_{ab} = f(\theta_a, \theta_b, \Delta, \text{load on span}) \tag{7-1}$$

$$M_{ba} = g(\theta_a, \theta_b, \Delta, \text{load on span}) \tag{7-2}$$

where *f* and *g* are symbols for functions.

To find the expressions of Eqs. 7-1 and 7-2, let us first establish the following sign convention for slope deflection:

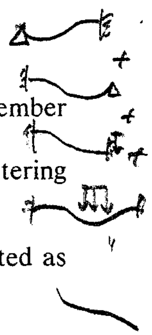
1. The moment acting on the end of a member (not a joint) is positive when clockwise.
2. The rotation at the end of a member is positive when the tangent to the deformed curve at the end rotates clockwise from its original position.
3. The relative deflection between ends of a member is positive when it corresponds to a clockwise rotation of the member (the straight line joining the ends of the elastic curve).

[illegible] clockwise is +ve.

All signs of end distortions and moments shown in Fig. 7-1 are positive. The sign conventions established here are purely arbitrary and could be replaced by any other convenient system; but once these conventions have been adopted, we will restrict ourselves to this system.

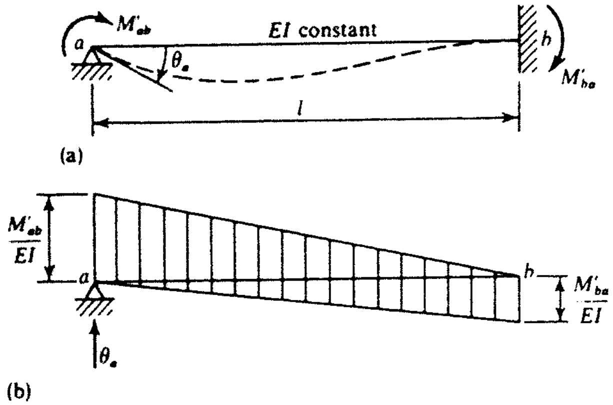
Next, let us refer to Fig. 7-1 and observe that the end moments $M_{ab}$ and $M_{ba}$ may be considered as the algebraic sum of four separate effects:

1. The moment due to end rotation $\theta_a$ while the other end *b* is fixed
2. The moment due to end rotation $\theta_b$ while end *a* is fixed
3. The moment due to a relative deflection $\Delta$ between the ends of the member without altering the existing slopes of tangents at the ends
4. The moment caused by placing the actual loads on the span without altering the existing end distortions



In each of the cases the corresponding end moments can be evaluated as follows:

1. Consider member *ab* supported as shown in Fig. 7-2(a). The dashed lines indicate the deformed shape. Note that end *a* is rotated through an angle $\theta_a$, whereas end *b* is fixed ($\theta_b = 0$); there is no relative end displacement between *a* and *b* ($\Delta = 0$). The corresponding end moments at *a* and *b*, denoted by



(1)

Fig. 7-2

$M'_{ab}$ and $M'_{ba}$, respectively, can best be found by the method of conjugate beam, which is shown in Fig. 7-2(b) with the moment diagram divided by $EI$ as its elastic loads and $\theta_a$ as its reaction so that the positive shear in the conjugate beam gives the desired positive slope in the actual beam. From equilibrium conditions:

$\frac{1}{2} \frac{M_{ab}}{EI} \times l$ = area of

moment area method

area = $\frac{M}{EI}$

1/3, 2/3 etc -> distance of centroid to pt

$$\sum M_a = 0 \qquad \underbrace{\left(\frac{M'_{ab}l}{2EI}\right)}\left(\frac{l}{3}\right) - \left(\frac{M'_{ba}l}{2EI}\right)\left(\frac{2l}{3}\right) = 0 \tag{7-3}$$

$$\sum M_b = 0 \qquad \underbrace{(\theta_a l)}_{?} - \left(\frac{M'_{ab}l}{2EI}\right)\left(\frac{2l}{3}\right) + \left(\frac{M'_{ba}l}{2EI}\right)\left(\frac{l}{3}\right) = 0 \tag{7-4}$$

From Eq. 7-3,

$$M'_{ba} = \tfrac{1}{2} M'_{ab} \tag{7-5}$$

Substituting Eq. 7-5 in Eq. 7-4 gives

$$M'_{ab} = \frac{4EI\theta_a}{l} \qquad M = f(\theta) \tag{7-6}$$

Thus,

$$M'_{ba} = \frac{2EI\theta_a}{l} \tag{7-7}$$

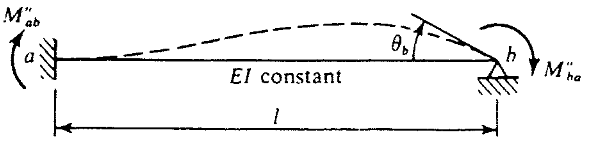
2. Consider member $ab$ supported and deformed as shown in Fig. 7-3, where end $b$ is rotated through an angle $\theta_b$ and end $a$ is fixed. The corresponding end moment at $b$, called $M''_{ba}$, and the moment at $a$, called $M''_{ab}$, are obtained similarly:

$$M''_{ab} = \tfrac{1}{2} M''_{ba} \tag{7-8}$$

$$M''_{ab} = \frac{2EI\theta_b}{l} \tag{7-9}$$

$$M''_{ba} = \frac{4EI\theta_b}{l} \tag{7-10}$$

3. To find the moments developed at the ends as the result of a pure relative



②

Fig. 7-3

deflection Δ between the two ends without causing end rotations, let us consider the fixed-end beam in Fig. 7-4(a). Because of the symmetry of the deformation with reference to the midpoint of the member [see the dashed lines in Fig. 7-4(a)], the two end moments must be equal. Thus, if we let $M'''_{ab}$ and $M'''_{ba}$ be the end moments at $a$ and $b$, respectively, we have

$$M'''_{ab} = M'''_{ba} = -M$$

The negative sign indicates that $M'''_{ab}$ and $M'''_{ba}$ are counterclockwise. The value of $M$ may be found by the method of conjugate beam, as shown in Fig. 7-4(b). Note that, besides the distributed elastic loads of the $M/EI$ diagram, a couple acts at end $b$ equal to Δ corresponding to the deflection at $b$ of the base structure. From $\Sigma M = 0$, we have

$$\left(\frac{Ml}{2EI}\right)\left(\frac{l}{3}\right) - \Delta = 0$$

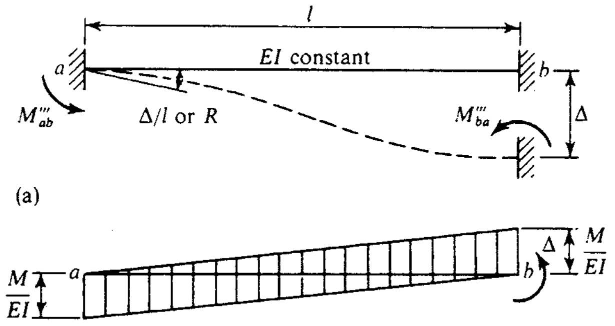
or

$$M = \frac{6EI\Delta}{l^2}$$

Thus, the moments developed at the ends of a member due to a pure relative end displacement are given by

$$M'''_{ab} = M'''_{ba} = -\frac{6EI\Delta}{l^2} \tag{7-11}$$

4. Finally, the moments induced at the ends of a member without causing end distortions when the external loads are placed on the span are nothing more than the fixed-end moments, usually denoted by $M^F_{ab}$ and $M^F_{ba}$.



③

(a)

(b)

Fig. 7-4

Summing up the four elements listed above, we have

$$M_{ab} = M'_{ab} + M''_{ab} + M'''_{ab} \pm M^F_{ab}$$

$$M_{ba} = M'_{ba} + M''_{ba} + M'''_{ba} \pm M^F_{ba}$$

Using Eqs. 7-6, 7-7, and 7-9 to 7-11, we find that

$$M_{ab} = \frac{4EI\theta_a}{l} + \frac{2EI\theta_b}{l} - \frac{6EI\Delta}{l^2} \pm M^F_{ab}$$

$$M_{ba} = \frac{2EI\theta_a}{l} + \frac{4EI\theta_b}{l} - \frac{6EI\Delta}{l^2} \pm M^F_{ba}$$

Rearranging gives

$$M_{ab} = 2E\frac{I}{l}\left(2\theta_a + \theta_b - 3\frac{\Delta}{l}\right) \pm M^F_{ab} \tag{7-12}$$

$$M_{ba} = 2E\frac{I}{l}\left(2\theta_b + \theta_a - 3\frac{\Delta}{l}\right) \pm M^F_{ba} \tag{7-13}$$

which are the basic equations of slope deflection for a general deformed member of uniform cross section. The equations express end moment $M_{ab}$ and $M_{ba}$ in terms of the end slopes ($\theta_a$, $\theta_b$), the relative deflection between the two ends ($\Delta$), and the loading on the span *ab*.

If we let

$$\frac{I}{l} = K \qquad \frac{\Delta}{l} = R$$

$K$ being the *stiffness factor of the member* and $R$ the rigid-body *rotation of the member* [see Fig. 7-4(a)], the equations become

$$M_{ab} = 2EK(2\theta_a + \theta_b - 3R) \pm M^F_{ab} \tag{7-14}$$

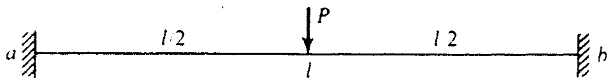
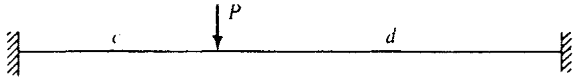
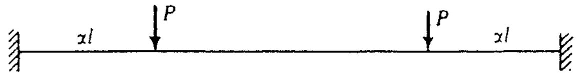
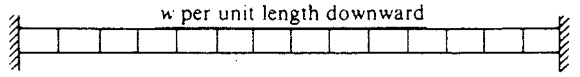
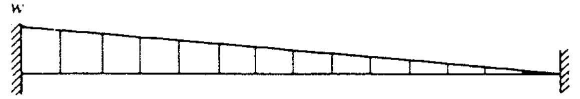
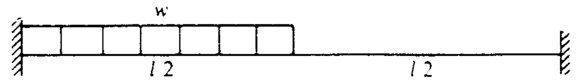
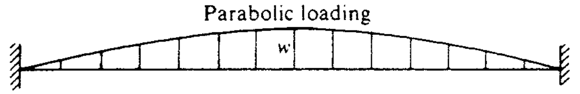
$$M_{ba} = 2EK(2\theta_b + \theta_a - 3R) \pm M^F_{ba} \tag{7-15}$$

The signs and values of $M^F_{ab}$ and $M^F_{ba}$ depend on the loading condition on span *ab*. If the member *ab* carries no load itself, then $M^F_{ab} = M^F_{ba} = 0$. See Secs. 6-2 and 6-5 for the evaluation of the fixed-end moments. For reference, the fixed-end moments for a straight member of constant $EI$ due to the common types of loading are given in Table 7-1.

## 7-3 PROCEDURE OF ANALYSIS BY THE SLOPE-DEFLECTION METHOD

The slope-deflection method consists of writing a series of slope-deflection equations expressing the end moments for all members in terms of the slope (rotation) and the deflection (relative translation) of various joints, or of quantities proportional to them, and of solving these unknown displacements by a number of equilibrium equations which these end moments must satisfy. Once the displacements have been determined, the end moments for each member may be figured. The solution thus obtained is unique, since it satisfies the equilibrium equations and end conditions (compatibility conditions) embedded in the slope-deflection equations.

## * TABLE 7-1

| $M^F_{ab}$ | Loading Case | $M^F_{ba}$ |
|---|---|---|
| $-\frac{Pl}{8}$ |  | $+\frac{Pl}{8}$ |
| $-\frac{Pcd^2}{l^2}$ |  | $+\frac{Pc^2d}{l^2}$ |
| $-\alpha(1-\alpha)Pl$ |  | $+\alpha(1-\alpha)Pl$ |
| $-\frac{wl^2}{12}$ |  | $+\frac{wl^2}{12}$ |
| $-\frac{wl^2}{20}$ |  | $+\frac{wl^2}{30}$ |
| $-\frac{11wl^2}{192}$ |  | $+\frac{5wl^2}{192}$ |
| $-\frac{wl^2}{15}$ |  | $+\frac{wl^2}{15}$ |
| $+\frac{Md(2c-d)}{l^2}$ |  | $+\frac{Mc(2d-c)}{l^2}$ |

To illustrate, let us consider the frame in Fig. 7-5(a). First we draw the free-body diagrams for all members, as shown in Fig. 7-5(b), where the unknown end moments for each member are assumed positive (i.e., acting clockwise according to our sign convention).

Next, we observe that ends *a* and *d* of the frame are fixed and will undergo no rotation ($\theta_a = \theta_d = 0$) or linear displacement. Joint *b*, owing to the restriction of length *ab* (we neglect the small change in length in *ab* due to the axial forces) and the support *a*, cannot move otherwise but rotates about *a*. However, since the deformations of the frame are extremely small as compared to the length, we may replace arc length with tangent length without appreciable error. With

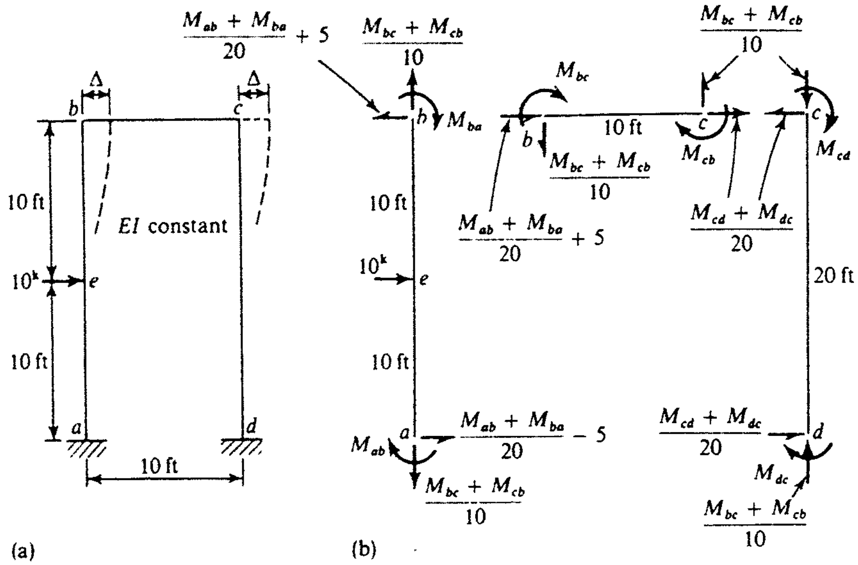


Fig. 7-5

joint $b$, and therefore $c$, moving a horizontal distance Δ to the right [see Fig. 7-5(a)], there is a relative deflection Δ between joints $a$ and $b$ and also joints $c$ and $d$. There is no relative deflection between joints $b$ and $c$ if the small lengthening or shortening in $ab$ or $cd$, caused primarily by axial forces, is neglected. There are some joint rotations at $b$ and $c$. Attention should be paid to the fact that when a rigid frame is deformed, each rigid joint is considered to rotate as a whole. For instance, members $ba$ and $bc$ rotate the same angle $\theta_b$ at joint $b$. Similarly, members $cb$ and $cd$ rotate the same angle $\theta_c$ at joint $c$.

To determine the end moments for each member shown in Fig. 7-5(b), we write a series of slope-deflection equations as follows:

$$
\begin{aligned}
M_{ab} &= 2EK_{ab}(\theta_b - 3R) - M_{ab}^F = 2E\left(\frac{I}{20}\right)(\theta_b - 3R) - 25 \\
M_{ba} &= 2EK_{ab}(2\theta_b - 3R) + M_{ba}^F = 2E\left(\frac{I}{20}\right)(2\theta_b - 3R) + 25 \\
M_{bc} &= 2EK_{bc}(2\theta_b + \theta_c) = 2E\left(\frac{I}{10}\right)(2\theta_b + \theta_c) \\
M_{cb} &= 2EK_{bc}(2\theta_c + \theta_b) = 2E\left(\frac{I}{10}\right)(2\theta_c + \theta_b) \\
M_{cd} &= 2EK_{cd}(2\theta_c - 3R) = 2E\left(\frac{I}{20}\right)(2\theta_c - 3R) \\
M_{dc} &= 2EK_{cd}(\theta_c - 3R) = 2E\left(\frac{I}{20}\right)(\theta_c - 3R)
\end{aligned}
\qquad (7\text{-}16)
$$

Involved in these expressions are three unknowns: $\theta_b$, $\theta_c$, and $R$ (or $\Delta/20$). These can be solved by the three equations of statics the end moments must satisfy.

By taking joints $b$ and $c$ as free bodies, we immediately obtain two equilibrium equations:

$$\sum M_{\text{joint } b} = 0 \qquad \text{or} \qquad M_{ba} + M_{bc} = 0 \tag{7-17}$$

$$\sum M_{\text{joint } c} = 0 \qquad \text{or} \qquad M_{cb} + M_{cd} = 0 \tag{7-18}$$

Usually, we have as many joint equilibrium equations as the number of joint displacement unknowns involved. However, with the member rotation $R$ unknown, a third equation may be more conveniently secured from the equilibrium of the structure. Referring to Fig. 7-5, by taking the whole frame as free body, we see that the horizontal shear in ends $a$ and $d$ must balance the horizontal external force acting on the frame. Thus,

$$10 + \left(\frac{M_{ab} + M_{ba}}{20} - 5\right) + \left(\frac{M_{cd} + M_{dc}}{20}\right) = 0$$

or

$$M_{ab} + M_{ba} + M_{cd} + M_{dc} + 100 = 0 \tag{7-19}$$

Before we try to substitute the expressions of Eq. 7-16 in Eqs. 7-17, 7-18, and 7-19, we should note that if our purpose is to determine the end moments but not to obtain the exact values of the slope and deflection of each joint, then we may substitute the relative values for the coefficients $2EI/l$, usually some simple integers, in the slope-deflection equations in order to facilitate the calculation. This can be done because such a substitution will only magnify the values of $\theta$ and $\Delta$ but will not affect the final result of the end moments. Thus, if we set

$$2E\frac{I}{20} = 1$$

accordingly,

$$2E\left(\frac{I}{10}\right) = 2$$

and the moment expressions of Eq. 7-16 become

$$\begin{aligned} M_{ab} &= \theta_b - 3R - 25 \\ M_{ba} &= 2\theta_b - 3R + 25 \\ M_{bc} &= 2(2\theta_b + \theta_c) \\ M_{cb} &= 2(2\theta_c + \theta_b) \\ M_{cd} &= 2\theta_c - 3R \\ M_{dc} &= \theta_c - 3R \end{aligned} \tag{7-20}$$

Substituting these in Eqs. 7-17, 7-18, and 7-19 yields

$$6\theta_b + 2\theta_c - 3R + 25 = 0 \tag{7-21}$$

$$2\theta_b + 6\theta_c - 3R = 0 \tag{7-22}$$

$$3\theta_b + 3\theta_c - 12R + 100 = 0 \tag{7-23}$$

Solving Eqs. 7-21, 7-22, and 7-23 simultaneously, we obtain

$$\theta_b = -1.20 \qquad \theta_c = 5.05 \qquad R = 9.30$$

Note that the values thus obtained are only the relative values of the slope and the deflection for the various joints. They must be divided by the factor $2EI/20$ to give the absolute values of the slope and deflection.

To determine the end moments for each member of the frame, we substitute $\theta_b = -1.20$, $\theta_c = 5.05$, $R = 9.30$ in Eq. 7-20 and obtain

$$M_{ab} = -54.10 \text{ ft-kips} \qquad \text{(counterclockwise)}$$
$$M_{ba} = -5.30 \text{ ft-kips} \qquad \text{(counterclockwise)}$$
$$M_{bc} = +5.30 \text{ ft-kips} \qquad \text{(clockwise)}$$
$$M_{cb} = +17.80 \text{ ft-kips} \qquad \text{(clockwise)}$$
$$M_{cd} = -17.80 \text{ ft-kips} \qquad \text{(counterclockwise)}$$
$$M_{dc} = -22.80 \text{ ft-kips} \qquad \text{(counterclockwise)}$$

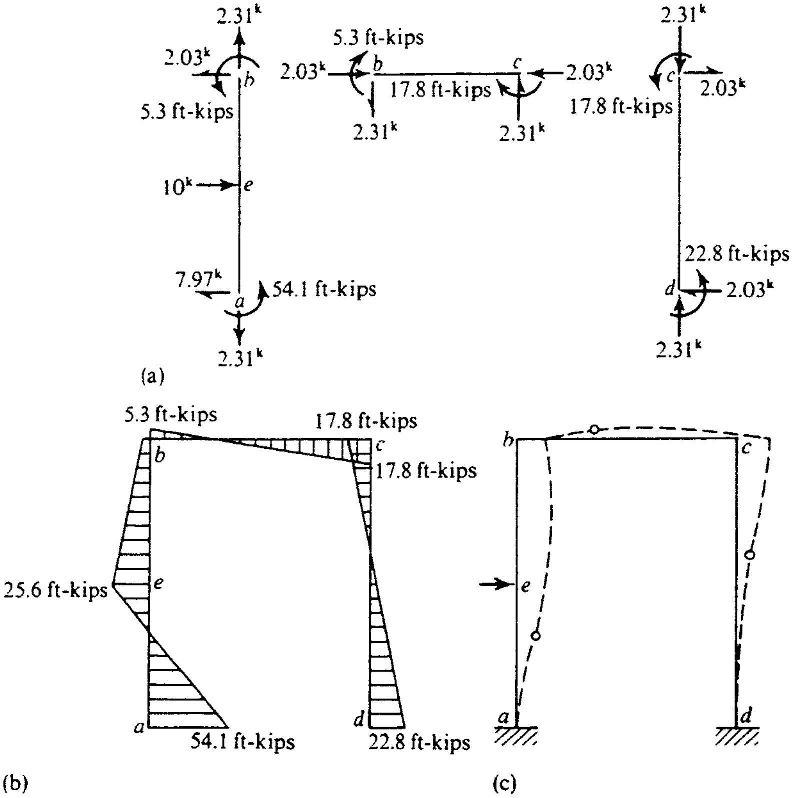


Fig. 7-6

The answer diagram for the end actions for each member of the frame is shown in Fig. 7-6(a), which is based on Fig. 7-5(b).

The moment diagram for the frame is drawn as in Fig. 7-6(b). The moment is plotted on the compressive side of each member. In this particular case each member has one point of inflection corresponding to the point of zero moment.

Finally, we can sketch the elastic curve of the deformed structure, as shown in Fig. 7-6(c), by using the values (or relative values) of the joint rotations and deflections together with the bending moment diagram. Note particularly the following:

1. The elastic curve of the deformed frame bends according to the bending moment diagram.
2. Both joints *b* and *c* deflect to the right the same horizontal distance.
3. Joint *b* rotates counterclockwise while joint *c* rotates clockwise.
4. Since joints *b* and *c* are rigid, the tangents to the elastic curves *ba* and *bc* at *b* and the tangents to the elastic curves *cb* and *cd* at *c* should be perpendicular to each other so as to maintain the original formation at the joints of the unloaded frame.

## 7-4 ANALYSIS OF STATICALLY INDETERMINATE BEAMS BY THE SLOPE-DEFLECTION METHOD

The application of the slope-deflection method in solving statically indeterminate beams will be illustrated in the following examples.

**Example 7-1**

Figure 7-7(a) shows a continuous, two-section beam, all the supports of which are immovable. We wish to draw the shear and bending moment diagrams for the beam. We solve for the end moments and shears as follows:

1. Since $2EK_{ab} = 2E(2I)/16 = EI/4$ and $2EK_{bc} = 2EI/12 = EI/6$, if we let $2EK_{ab} = 3$, then $2EK_{bc} = 2$ relatively. The relative values of $2EK$ are shown circled in Fig. 7-7(a).
2. By inspection $\theta_a = 0$ (fixed end at *a*) and $R_{ab} = R_{bc} = 0$ (immovable supports at *a*, *b*, *c*).
3. We calculate the fixed-end moments as follows:

$$M^F_{ab} = -\frac{(7.5)(16)}{8} = -15 \text{ ft-kips}$$

$$M^F_{ba} = +\frac{(7.5)(16)}{8} = +15 \text{ ft-kips}$$

$$M^F_{bc} = -\frac{(1)(12)^2}{12} = -12 \text{ ft-kips}$$

$$M^F_{cb} = +\frac{(1)(12)^2}{12} = +12 \text{ ft-kips}$$

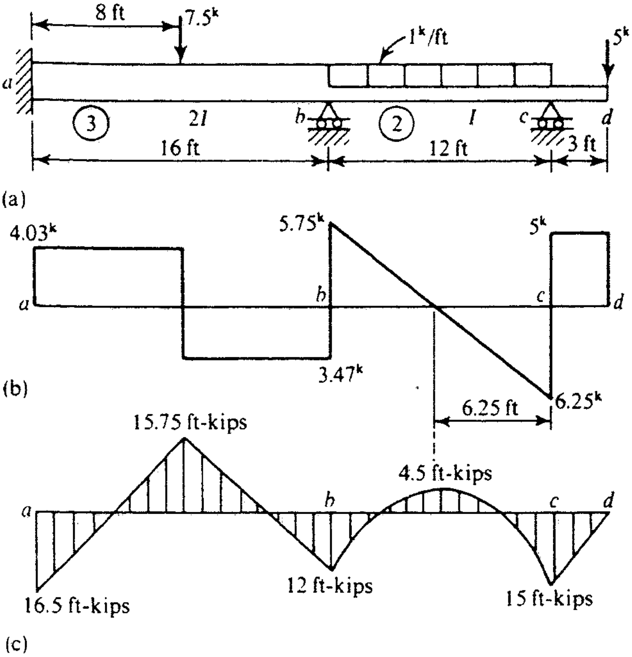


Fig. 7-7

4. We write the slope-deflection equations using the relative $2EK$ values.

$$M_{ab} = (3)(\theta_b) - 15 \tag{7-24}$$

$$M_{ba} = (3)(2\theta_b) + 15 = 6\theta_b + 15 \tag{7-25}$$

$$M_{bc} = (2)(2\theta_b + \theta_c) - 12 = 4\theta_b + 2\theta_c - 12 \tag{7-26}$$

$$M_{cb} = (2)(2\theta_c + \theta_b) + 12 = 2\theta_b + 4\theta_c + 12 \tag{7-27}$$

5. Involved in the equations above are two unknowns, $\theta_b$ and $\theta_c$, that can be solved by two joint conditions:

$$\sum M_{\text{joint } b} = M_{ba} + M_{bc} = 0 \tag{7-28}$$

$$M_{cb} = (5)(3) = 15 \tag{7-29}$$

Substituting Eqs. 7-25 and 7-26 in Eq. 7-28, and Eq. 7-27 in Eq. 7-29 yields

$$10\theta_b + 2\theta_c + 3 = 0 \tag{7-30}$$

and

$$2\theta_b + 4\theta_c - 3 = 0 \tag{7-31}$$

Solving yields $\theta_b = -\frac{1}{2} \qquad \theta_c = 1$

6. Substituting the above values in step 4, we obtain

$$M_{ab} = -16.5 \text{ ft-kips}$$

$$M_{ba} = +12.0 \text{ ft-kips}$$

$$M_{bc} = -12.0 \text{ ft-kips}$$

$$M_{cb} = +15.0 \text{ ft-kips}$$

7. Having determined the end moments for each member, we can find the end shears and, therefore, the reactions.

$$R_a = V_{ab} = \frac{7.5}{2} + \frac{16.5 - 12}{16} = 4.03 \text{ kips} \qquad \text{(up)}$$

$$V_{ba} = \frac{-7.5}{2} + \frac{16.5 - 12}{16} = -3.47 \text{ kips} \qquad \text{(up)}$$

$$V_{bc} = \frac{12}{2} - \frac{15 - 12}{12} = 5.75 \text{ kips} \qquad \text{(up)}$$

$$R_b = 3.47 + 5.75 = 9.22 \text{ kips} \qquad \text{(up)}$$

$$V_{cb} = -\frac{12}{2} - \frac{15 - 12}{12} = -6.25 \text{ kips} \qquad \text{(up)}$$

$$V_{cd} = 5 \text{ kips} \qquad \text{(up)}$$

$$R_c = 6.25 + 5 = 11.25 \text{ kips} \qquad \text{(up)}$$

8. We now draw the shear and moment diagrams, as shown in Fig. 7-7(b) and (c), respectively.

**Example 7-2**

For the system and load shown in Fig. 7-8, find a general expression for the spring force. Given $E = 20{,}000$ kN/cm$^2$, $I = 5{,}000$ cm$^4$, $k = 5$ kN/cm, $w = 4$ kN/m, and $l = 3$ m, determine the value of the spring force.

Assuming that the contraction of spring at $b$ is $\Delta$, we write the moment equations as

$$M_{ab} = 2E\frac{I}{l}\left(\theta_b - 3\frac{\Delta}{l}\right) - \frac{wl^2}{12} \tag{7-32}$$

$$M_{ba} = 2E\frac{I}{l}\left(2\theta_b - 3\frac{\Delta}{l}\right) + \frac{wl^2}{12} \tag{7-33}$$

Using $M_{ba} = 0$, we have

$$\theta_b = \frac{3\Delta}{2l} - \frac{wl^3}{48EI} \tag{7-34}$$

Substituting Eq. 7-34 in Eq. 7-32 gives

$$M_{ab} = -\left(\frac{3EI\Delta}{l^2} + \frac{wl^2}{8}\right) \tag{7-35}$$

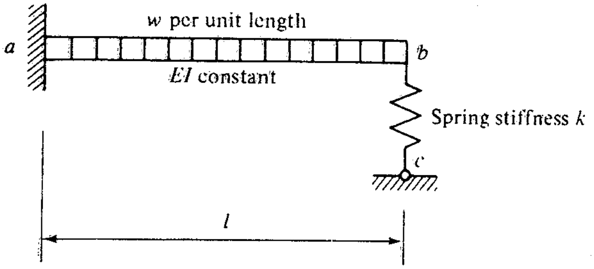


Fig. 7-8

Let the spring force be $X$; hence, $X = k\Delta$ or $\Delta = X/k$. Thus,

$$M_{ab} = -\left(\frac{3EIX}{kl^2} + \frac{wl^2}{8}\right) \tag{7-36}$$

Since the compressive force in the spring is equal to the end shear at $b$,

$$X = \frac{wl}{2} + \frac{M_{ab}}{l}$$

Using Eq. 7-36, we obtain

$$X = \frac{3}{8} wl \left(\frac{1}{1 + \dfrac{3EI}{kl^3}}\right) \tag{7-37}$$

A substitution of $E = 20{,}000$ kN/cm$^2$, $I = 5{,}000$ cm$^4$, $k = 5$ kN/cm, $l = 3$ m $= 300$ cm, and $wl = (4)(3) = 12$ kN in Eq. 7-37 yields

$$X = \left(\frac{3}{8}\right)(12)\left(\frac{1}{1 + \dfrac{(3)(10)^8}{(5)(3)^3(10)^6}}\right) = 1.2 \text{ kN}$$

Some special cases can be derived by specifying the value of spring stiffness. They are given in Fig. 7-9. For instance, $k = 0$ means no axial resistance (free end) (i.e., $X = 0$); $k = \infty$ means an unyielding support, $X = \frac{3}{8} wl$. If the spring is replaced by an elastic link with length $L$, cross-sectional area $A$, and modulus of elasticity $E$, then we use $k = AE/L$ in Eq. 7-37.

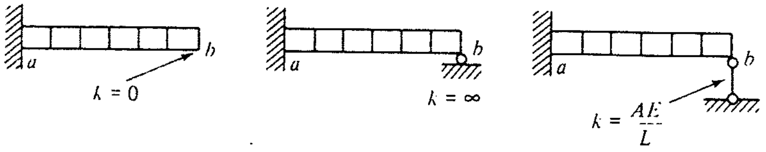


Fig. 7-9

## 7-5 ANALYSIS OF STATICALLY INDETERMINATE RIGID FRAMES WITHOUT JOINT TRANSLATION BY THE SLOPE-DEFLECTION METHOD

Some rigid frames, such as those shown in Fig. 7-10(a)–(d), are so constructed that translations of joints are prevented. Others, although capable of joint translation in construction, will undergo no joint translation because of the symmetry of the structure and the loading about a certain axis, such as those shown in Fig. 7-10(e)–(g).

In both cases

$$R = 0$$

in the equations of slope deflection, so the analysis is considerably simplified, as we shall see in the following examples.

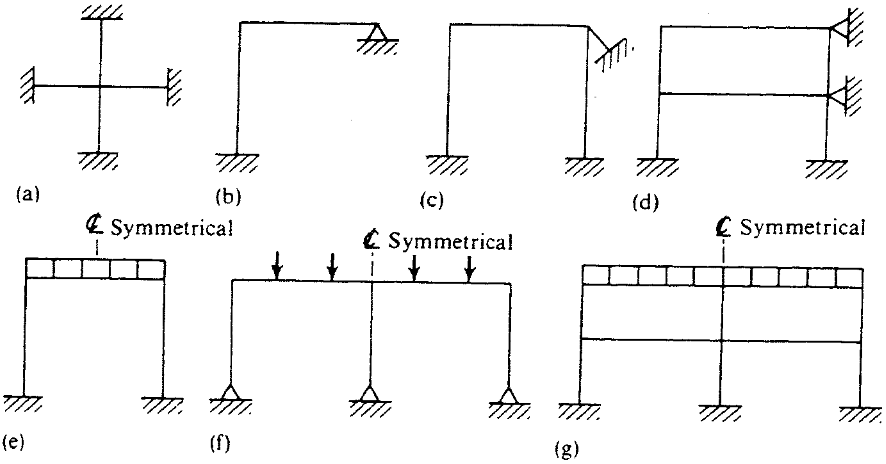


Fig. 7-10

**Example 7-3**

The end moments for the frame shown in Fig. 7-11 were solved by the force method (Examples 6-7 and 6-12) and will be re-solved by slope deflection.

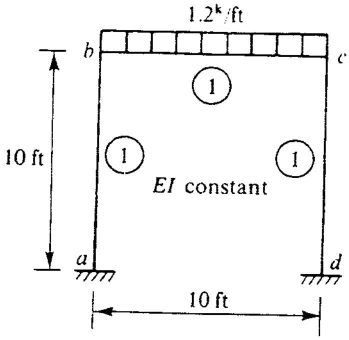


Fig. 7-11

The analysis is as follows:

1. The relative $2EK$ values for all members are shown encircled.
2. $\theta_a = \theta_d = R = 0$, and because of symmetry, $\theta_c = -\theta_b$.
3. $M^F_{bc} = -M^F_{cb} = -(1.2)(10)^2/12 = -10$ ft-kips.
4. Equations of slope deflection are then given by

$$M_{ab} = -M_{dc} = (1)(\theta_b) \tag{7-38}$$

$$M_{ba} = -M_{cd} = (1)(2\theta_b) \tag{7-39}$$

$$M_{bc} = -M_{cb} = (1)(2\theta_b - \theta_b) - 10 \tag{7-40}$$

5. There is only one unknown, $\theta_b$, involved in this analysis, and it can be solved by

$$\sum M_{\text{joint } b} = M_{ba} + M_{bc} = 0 \tag{7-41}$$

Substituting Eqs. 7-39 and 7-40 in Eq. 7-41 gives

$$3\theta_b - 10 = 0$$

from which

$$\theta_b = 3.33$$

6. Going back to step 4, we obtain

$$M_{ab} = -M_{dc} = 3.33 \text{ ft-kips}$$
$$M_{ba} = -M_{cd} = 6.67 \text{ ft-kips}$$
$$M_{bc} = -M_{cb} = -6.67 \text{ ft-kips}$$

**Example 7-4**

Analyze the frame in Fig. 7-12 if the support at *a* yields 0.0016 rad clockwise. Assume that $EI = 10{,}000$ kips-ft$^2$.

$$M_{ab} = \frac{2EI}{10}(2\theta_a + \theta_b) = \frac{EI}{5}[(2)(0.0016) + \theta_b]$$

$$M_{ba} = \frac{2EI}{10}(2\theta_b + \theta_a) = \frac{EI}{5}(2\theta_b + 0.0016)$$

$$M_{bc} = \frac{2E(2I)}{20}(2\theta_b + \theta_c) = \frac{EI}{5}(2\theta_b) \qquad (\theta_c = 0)$$

$$M_{cb} = \frac{2E(2I)}{20}(2\theta_c + \theta_b) = \frac{EI}{5}(\theta_b)$$

The unknown $\theta_b$ is solved by

$$\sum M_{\text{joint } b} = M_{ba} + M_{bc} = 0$$

or

$$4\theta_b + 0.0016 = 0$$

from which

$$\theta_b = -0.0004$$

With $\theta_b$ determined, all end moments can be figured as

$$M_{ab} = \frac{10{,}000}{5}(0.0032 - 0.0004) = 5.6 \text{ ft-kips}$$

$$M_{ba} = \frac{10{,}000}{5}(-0.0008 + 0.0016) = 1.6 \text{ ft-kips}$$

$$M_{bc} = \frac{10{,}000}{5}(-0.0008) = -1.6 \text{ ft-kips}$$

$$M_{cb} = \frac{10{,}000}{5}(-0.0004) = -0.8 \text{ ft-kips}$$

The deformed structure is indicated by the dashed lines in Fig. 7-12.

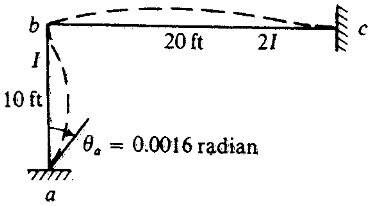


Fig. 7-12

## 7-6 ANALYSIS OF STATICALLY INDETERMINATE RIGID FRAMES WITH ONE DEGREE OF FREEDOM OF JOINT TRANSLATION BY THE SLOPE-DEFLECTION METHOD

Figure 7-13 shows several examples of rigid frames with one degree of freedom of joint translation. In each of these examples, if the translation of one joint is given or assumed, the translation of all other joints can be deduced from it.

For instance, suppose that, in the frame in Fig. 7-13(a), joint *a* moves to *a'* a distance Δ. Since we neglect any slight change in the length of a member due to axial forces, and since the rotations of members are small, joint *a* moves essentially perpendicular to member *Aa* and, in this case, horizontally. Similarly, joint *b* at the top of column *Bb* must move horizontally. Furthermore, as *b* is the end of member *ab* and the change of axial length of *ab* is neglected, the horizontal movement of *b* [see *bb'* in Fig. 7-13(a)] must also be Δ (i.e., *aa'* = *bb'* = Δ).

In the same manner, we reason that if joint *a* in Fig. 7-13(b) moves a horizontal distance Δ to *a'*, the tops of all the other columns must have the same horizontal displacement.

Let us now consider the case of Fig. 7-13(c) in which the frame is acted on by a lateral force at the top of column *Aa*. Joint *a* cannot move other than

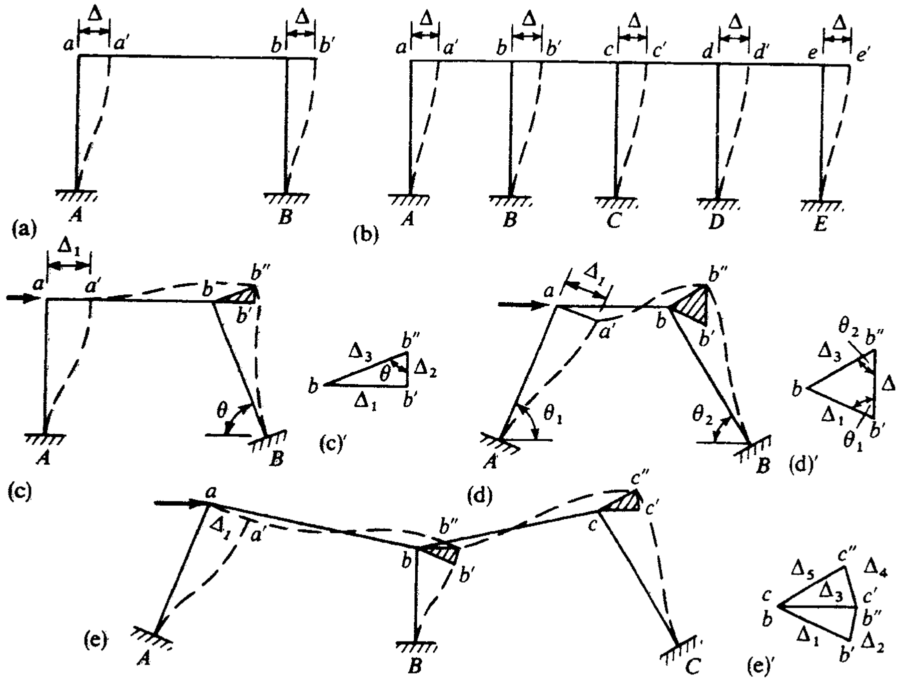


Fig. 7-13

horizontally, say a distance $\Delta_1$ to $a'$. To find the final location of joint $b$, we imagine that the frame is temporarily disconnected at $b$. Point $b$, being the end of member $ab$, will move to $b'$ the displacement $\Delta_1$, if free of other effects except that due to the movement of $a$. However, $b$ is also the end of member $Bb$; the final position of $b$, called $b''$, must be determined by two arcs that restrict the motion of $b'$—one from $a'$ with a radius equal to $ab$ (or $a'b'$) and the other from $B$ with a radius equal to $Bb$. Since the deformations of the frame are very small in proportion to the length, it is permissible to substitute the tangents for the arcs, as shown in Fig. 7-13(c). The displacement diagram is shown separately in Fig. 7-13(c)′, for which we note that $\Delta_1$ is the relative displacement between the ends of member $Aa$, $\Delta_2$ that between the ends of $ab$, and $\Delta_3$ that between the ends of $Bb$. The relationship between $\Delta_1$, $\Delta_2$, and $\Delta_3$ can be expressed by the sine law:

$$\frac{\Delta_1}{\sin\theta} = \frac{\Delta_2}{\sin(90° - \theta)} = \frac{\Delta_3}{\sin 90°}$$

from which
$$\Delta_1 = \Delta_2 \tan\theta = \Delta_3 \sin\theta$$

The joint displacements of Fig. 7-13(d) are similar to that described for Fig. 7-13(c), noting that joint $a$ should move perpendicularly to member $Aa$. From the displacement diagram shown in Fig. 7-13(d)′ we see that

$$\frac{\Delta_1}{\sin\theta_2} = \frac{\Delta_2}{\sin(\theta_1 + \theta_2)} = \frac{\Delta_3}{\sin\theta_1}$$

The procedure described above is now extended to a two-span frame such as the one shown in Fig. 7-13(e) together with the joint-displacement diagram in Fig. 7-13(e)′. It may be extended to any number of spans. With the relative deflection between the ends for each member clarified, it becomes a simple matter to apply the slope-deflection equations.

**Example 7-5**

Find the end moments for each member of the portal frame shown in Fig. 7-14(a) resulting from a lateral force $P$ acting on the top of the column. Assume constant $EI$ throughout the entire frame.

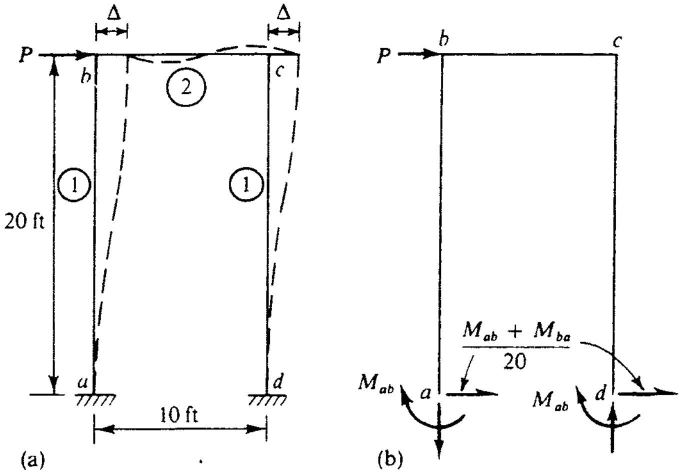
Because of the lateral force $P$ acting at $b$, the frame deflects to the right. Both joints $b$ and $c$ move the same horizontal distance $\Delta$, as indicated. There are also rotations $\theta_b$ at joint $b$ and $\theta_c$ at joint $c$. Now, since an equal and opposite force $P$ acting at $c$ would completely balance the original force at $b$, and would thus return the structure to the original position (except for some small change of length in $bc$), $\theta_b$ must be equal to $\theta_c$. Thus,

$$M_{ab} = M_{dc} = (1)(\theta_b - 3R) \qquad (\theta_a = \theta_d = 0)$$

$$M_{ba} = M_{cd} = (1)(2\theta_b - 3R)$$

$$M_{bc} = M_{cb} = (2)(2\theta_b + \theta_b) = 6\theta_b$$

This special case in which the end moments and joint rotations of one side of the center-line axis of the structure are the same as those of other side is termed *antisymmetry*, in contrast to the case of *symmetry*, in which the values of the end



(a) (b)

Fig. 7-14

moments and joint rotations of one side of the center-line axis of the structure are equal but opposite to those of the other side, according to the sign convention of slope deflection.

The unknowns, $\theta_b$ and $R$, are then solved by two equilibrium equations, one for joint $b$ and the other for the entire frame:

$$\sum M_{\text{joint } b} = M_{ba} + M_{bc} = 0$$

or

$$8\theta_b - 3R = 0 \tag{7-42}$$

By isolating the frame from the supports [Fig. 7-14(b)],

$$\sum F_x = (2)\frac{M_{ab} + M_{ba}}{20} + P = 0$$

or

$$3\theta_b - 6R + 10P = 0 \tag{7-43}$$

Solving Eqs. 7-42 and 7-43 simultaneously, we obtain

$$\theta_b = \frac{10}{13}P \qquad 3R = \frac{80}{13}P$$

Substituting these in equations for end moments, we obtain

$$M_{ab} = M_{dc} = -\frac{70}{13}P$$

$$M_{ba} = M_{cd} = -\frac{60}{13}P$$

$$M_{bc} = M_{cb} = +\frac{60}{13}P$$

**Example 7-6**

Draw the bending moment diagram for the frame shown in Fig. 7-15(a). The relative values of $2EK$ are circled.

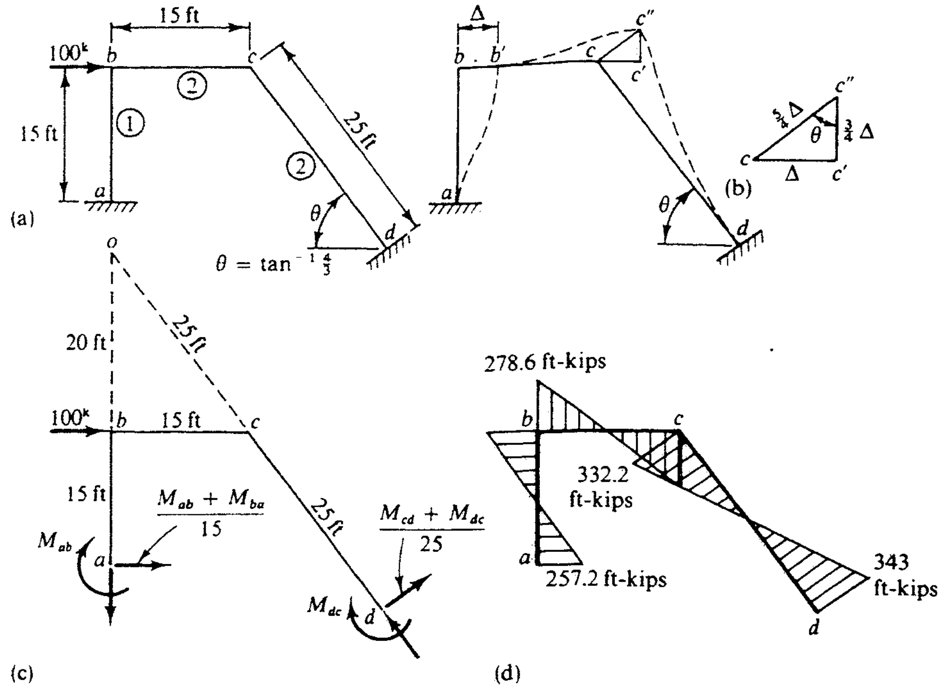


**Fig. 7-15**

To find the end moments, we begin by sketching the relative displacement diagram, as shown in Fig. 7-15(b). Thus,

$$R_{ab} = \frac{\Delta}{l_{ab}} = \frac{\Delta}{15}$$

$$R_{bc} = -\frac{\frac{3}{4}\Delta}{l_{bc}} = -\frac{\frac{3}{4}\Delta}{15} = -\frac{\Delta}{20}$$

$$R_{cd} = \frac{\frac{5}{4}\Delta}{l_{cd}} = \frac{\frac{5}{4}\Delta}{25} = \frac{\Delta}{20}$$

If we let $R_{ab} = R$, then $R_{bc} = -3R/4$ and $R_{cd} = 3R/4$. The expressions for the various end moments are now written

$$M_{ab} = (1)(\theta_b - 3R) \qquad (\theta_a = 0)$$

$$M_{ba} = (1)(2\theta_b - 3R)$$

$$M_{bc} = (2)[2\theta_b + \theta_c + (3)(\tfrac{3}{4}R)]$$

$$M_{cb} = (2)[2\theta_c + \theta_b + (3)(\tfrac{3}{4}R)]$$

$$M_{cd} = (2)[2\theta_c - (3)(\tfrac{3}{4}R)] \qquad (\theta_d = 0)$$

$$M_{dc} = (2)[\theta_c - (3)(\tfrac{3}{4}R)]$$

Two of the three condition equations required to evaluate the three independent unknowns $\theta_b$, $\theta_c$, and $R$ are from $\Sigma M = 0$ for joints $b$ and $c$. Thus,

$$M_{ba} + M_{bc} = 0$$

$$6\theta_b + 2\theta_c + 1.5R = 0 \tag{7-44}$$

$$M_{cb} + M_{cd} = 0$$

$$\theta_b + 4\theta_c = 0 \tag{7-45}$$

The third condition equation can best be found by expressing $\Sigma M_o = 0$ for the entire frame, $o$ being the center of moment chosen at the intersection of the two legs [see Fig. 7-15(c)], since this eliminates the axial forces from the equation. Thus,

$$M_{ab} + M_{dc} - (100)(20) - \left(\frac{M_{ab} + M_{ba}}{15}\right)(35) - \left(\frac{M_{cd} + M_{dc}}{25}\right)(50) = 0$$

$$-6\theta_b - 10\theta_c + 24.5R - 2{,}000 = 0 \tag{7-46}$$

Solving Eqs. 7-44, 7-45, and 7-46 simultaneously, we obtain

$$\theta_b = -21.4 \qquad \theta_c = 5.36 \qquad R = 78.6$$

Substitution of these values in the moment expressions yields

$$M_{ab} = -257.2 \text{ ft-kips} \qquad M_{ba} = -278.6 \text{ ft-kips}$$

$$M_{bc} = 278.6 \text{ ft-kips} \qquad M_{cb} = 332.2 \text{ ft-kips}$$

$$M_{cd} = -\ 332.2 \text{ ft-kips} \qquad M_{dc} = -343.0 \text{ ft-kips}$$

as shown in Fig. 7-15(d).

## 7-7 ANALYSIS OF STATICALLY INDETERMINATE RIGID FRAMES WITH TWO DEGREES OF FREEDOM OF JOINT TRANSLATION BY THE SLOPE-DEFLECTION METHOD

The number of degrees of freedom for joint translation in a rigid frame equals the number of independent joint translations that can be given to the frame. Figure 7-16 shows several examples of rigid frames with two degrees of freedom of joint translation.

Figure 7-16(a) shows an unsymmetrical gable bent subjected to a vertical load at the top. Under this pressure joint $b$ will move a distance $\Delta_1$ to $b'$ and joint $d$ a distance $\Delta_2$ to $d'$, as indicated. To locate the position of $c$, let us imagine that joint $c$ is temporarily disconnected. Joint $c$, being the end of member $bc$, will move to $c'$ ($cc' = bb' = \Delta_1$). Joint $c$, being the end of member $cd$, will move to $c''$ ($cc'' = dd' = \Delta_2$). The final position of $c$ is $c'''$, which is the intersection of the line perpendicular to $bc$ drawn at $c'$ and the line perpendicular to $cd$ drawn at $c''$, as indicated in Fig. 7-16(a). From the displacement diagram $\Delta_3$ (the relative displacement between ends $b$ and $c$) and $\Delta_4$ (that between $c$ and $d$) is related to $\Delta_1$ (that between $a$ and $b$) and $\Delta_2$ (that between $d$ and $e$) by the

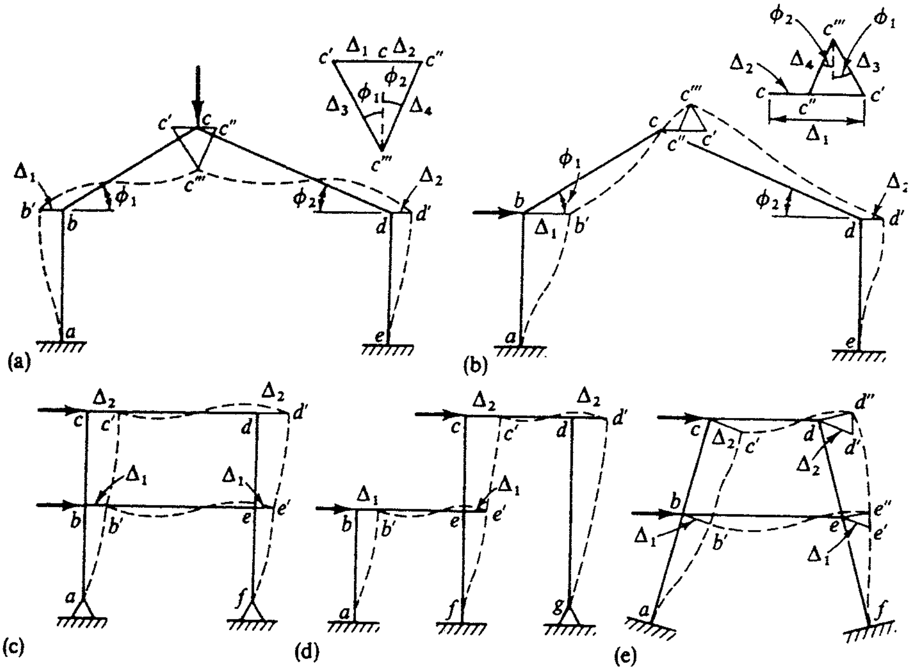


Fig. 7-16

sine law,

$$\frac{\Delta_1 + \Delta_2}{\sin(\phi_1 + \phi_2)} = \frac{\Delta_3}{\sin (90° - \phi_2)} = \frac{\Delta_4}{\sin (90° - \phi_1)}$$

or

$$\frac{\Delta_1 + \Delta_2}{\sin (\phi_1 + \phi_2)} = \frac{\Delta_3}{\cos \phi_2} = \frac{\Delta_4}{\cos \phi_1}$$

so that $\Delta_3$ and $\Delta_4$ can be deduced from $\Delta_1$ and $\Delta_2$.

The case shown in Fig. 7-16(b) is similar to that of Fig. 7-16(a) except that we assume the tops of the two legs move in the same direction as the result of a lateral force applied at *b*. The relationships of joint displacements are expressed by

$$\frac{\Delta_1 - \Delta_2}{\sin (\phi_1 + \phi_2)} = \frac{\Delta_3}{\sin (90° - \phi_2)} = \frac{\Delta_4}{\sin (90° - \phi_1)}$$

or

$$\frac{\Delta_1 - \Delta_2}{\sin (\phi_1 + \phi_2)} = \frac{\Delta_3}{\cos \phi_2} = \frac{\Delta_4}{\cos \phi_1}$$

Figure 7-16(c) and (e) show the joint displacements in two-story frames, and Fig. 7-16(d) shows the joint displacements in a two-stage frame. In each of these cases, the arrangement of the structure is such that the joint translations of the first floor are not required to be in any fixed relationship to those of the second floor (i.e., the translations of joints *b* and *e* are not in a fixed ratio to the translations of joints *c* and *d*). The procedure for finding joint displacements on each of the floors is the same as that discussed in Sec. 7-6 for one-story frames. With the relative end displacement for each member consistently determined, it becomes rather easy to analyze the frame by the method of slope deflection.

**Example 7-7**

Find all the end moments for the gable bent shown in Fig. 7-17(a). Assume constant *EI* throughout the entire frame so that the relative $2EK$ values are the circled numbers.

We start by sketching the joint displacements of the frame as shown in Fig. 7-17(b), for which we note that

$$\Delta_3 = \frac{(\Delta_1 + \Delta_2)\sin(90^\circ - \phi_2)}{\sin(\phi_1 + \phi_2)} = \frac{(\Delta_1 + \Delta_2)\cos\phi_2}{\sin[180^\circ - (\phi_1 + \phi_2)]} = \frac{4}{5}(\Delta_1 + \Delta_2)$$

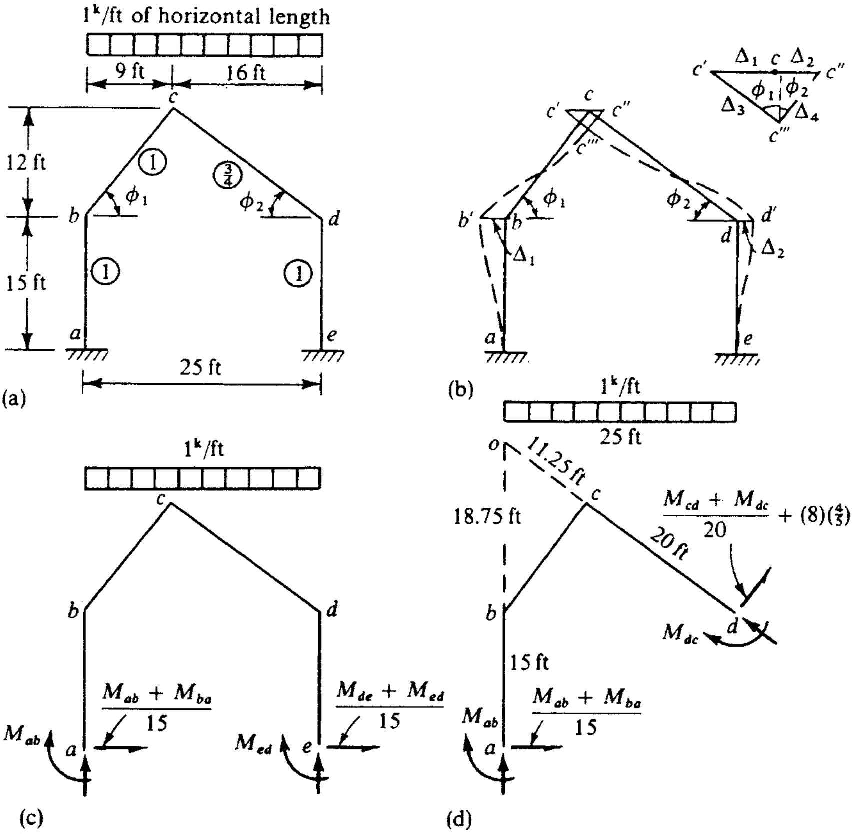


Fig. 7-17

since, if we refer to Fig. 7-17(a), $\cos\phi_2 = \frac{4}{5}$ and $180° - (\phi_1 + \phi_2) = \sphericalangle c = 90°$ in this case. Similarly,

$$\Delta_4 = \frac{(\Delta_1 + \Delta_2)\cos\phi_1}{\sin[180° - (\phi_1 + \phi_2)]} = \frac{3}{5}(\Delta_1 + \Delta_2)$$

Thus, the rotation of each member can be expressed in terms of $\Delta_1$ and $\Delta_2$.

$$R_{ab} = -\frac{\Delta_1}{15} \qquad R_{bc} = \left(\frac{1}{15}\right)\left(\frac{4}{5}\right)(\Delta_1 + \Delta_2)$$

$$R_{cd} = -\left(\frac{1}{20}\right)\left(\frac{3}{5}\right)(\Delta_1 + \Delta_2) \qquad R_{de} = \frac{\Delta_2}{15}$$

If we let $\Delta_1/15 = R_1$ and $\Delta_2/15 = R_2$, we have

$$R_{ab} = -R_1 \qquad R_{bc} = 0.8(R_1 + R_2)$$

$$R_{cd} = -0.45(R_1 + R_2) \qquad R_{de} = R_2$$

The fixed-end moments are found to be

$$M_{bc}^F = -M_{cb}^F = -\frac{(1)(9)^2}{12} = -6.750 \text{ ft-kips}$$

$$M_{cd}^F = -M_{dc}^F = -\frac{(1)(16)^2}{12} = -21.333 \text{ ft-kips}$$

The expressions for various end moments are then written

$$\begin{aligned}
M_{ab} &= (1)(\theta_b + 3R_1) \qquad (\theta_a = 0)\\
M_{ba} &= (1)(2\theta_b + 3R_1)\\
M_{bc} &= (1)[2\theta_b + \theta_c - 2.4(R_1 + R_2)] - 6.750\\
M_{cb} &= (1)[2\theta_c + \theta_b - 2.4(R_1 + R_2)] + 6.750\\
M_{cd} &= (\tfrac{3}{4})[2\theta_c + \theta_d + 1.35(R_1 + R_2)] - 21.333\\
M_{dc} &= (\tfrac{3}{4})[2\theta_d + \theta_c + 1.35(R_1 + R_2)] + 21.333\\
M_{de} &= (1)(2\theta_d - 3R_2) \qquad (\theta_e = 0)\\
M_{ed} &= (1)(\theta_d - 3R_2)
\end{aligned}$$

Five statical equations are needed to solve the independent unknowns $\theta_b$, $\theta_c$, $\theta_d$, $R_1$, and $R_2$—three from $\sum M = 0$ for joints $b$, $c$, and $d$; one from shear balance ($\sum F_x = 0$) for the entire frame [see Fig. 7-17(c)]; and one from $\sum M_o = 0$ for the portion of the frame shown in Fig. 7-17(d). Thus,

$$\sum M_{\text{joint } b} = M_{ba} + M_{bc} = 0$$

$$4\theta_b + \theta_c + 0.6R_1 - 2.4R_2 - 6.750 = 0 \tag{7-47}$$

$$\sum M_{\text{joint } c} = M_{cb} + M_{cd} = 0$$

$$\theta_b + 3.5\theta_c + 0.75\theta_d - 1.387(R_1 + R_2) - 14.583 = 0 \tag{7-48}$$

$$\sum M_{\text{joint } d} = M_{dc} + M_{de} = 0$$

$$0.75\theta_c + 3.5\theta_d + 1.013R_1 - 1.987R_2 + 21.333 = 0 \tag{7-49}$$

$\Sigma F_x = 0$ for the frame of Fig. 7-17(c):

$$\frac{M_{ab} + M_{ba}}{15} + \frac{M_{de} + M_{ed}}{15} = 0$$

$$\theta_b + \theta_d + 2R_1 - 2R_2 = 0 \tag{7-50}$$

$\Sigma M_o = 0$ for the portion of the frame [Fig. 7-17(d)]:

$$M_{ab} - \left(\frac{M_{ab} + M_{ba}}{15}\right)(33.75) - \left(\frac{M_{cd} + M_{dc}}{20}\right)(31.25) - (8)\left(\frac{4}{5}\right)(31.25)$$

$$+ M_{dc} + \frac{(1)(25)^2}{2} = 0$$

$$(\theta_b + 3R_1) - 2.25(3\theta_b + 6R_1) - 1.5625[2.25\theta_c + 2.25\theta_d + 2.025(R_1 + R_2)]$$

$$- 200 + [0.75\theta_c + 1.5\theta_d + 1.013(R_1 + R_2) + 21.333] + 312.5 = 0$$

or

$$5.75\theta_b + 2.766\theta_c + 2.016\theta_d + 12.651R_1 + 2.151R_2 - 133.833 = 0 \tag{7-51}$$

Solving Eqs. 7-47, 7-48, 7-49, 7-50, and 7-51 simultaneously, we obtain

$$\theta_b = 0.341 \qquad \theta_c = 11.039 \qquad \theta_d = -8.342$$

$$R_1 = 8.543 \qquad R_2 = 4.561$$

Substituting these in the expressions for the various end moments, we obtain

$$M_{ab} = 26 \text{ ft-kips}$$

$$M_{ba} = -M_{bc} = 26.3 \text{ ft-kips}$$

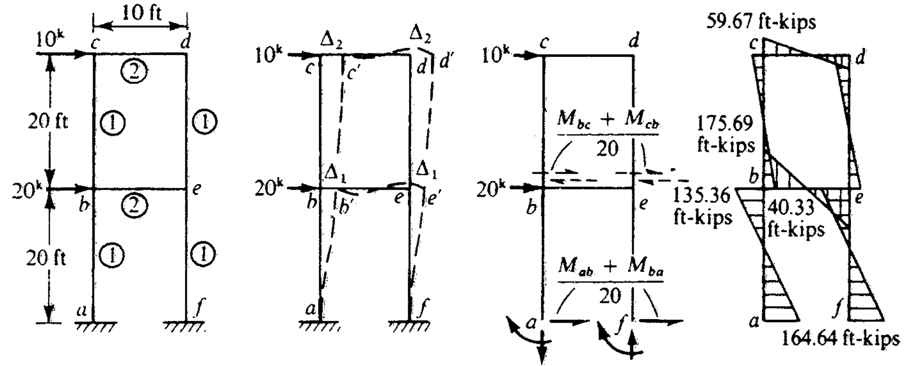
$$M_{cb} = -M_{cd} = -2.3 \text{ ft-kips}$$

$$M_{dc} = -M_{de} = 30.3 \text{ ft-kips}$$

$$M_{ed} = -22 \text{ ft-kips}$$

**Example 7-8**

Analyze the frame in Fig. 7-18(a). Assume that all members are of uniform cross section, so the relative values of $2EK$ are the circled numbers. Because of the



(a) (b) (c) (d)

Fig. 7-18

action of the lateral forces, the frame will deflect to the right. Assume that joints $b$ and $e$, on the first-floor level, move a horizontal distance $\Delta_1$; joints $c$ and $d$, on the second-floor level, move a horizontal distance $\Delta_2$, as shown in Fig. 7-18(b). Antisymmetry exists in this case. Thus,

$$\theta_b = \theta_e \qquad \theta_c = \theta_d \qquad R_{ab} = R_{ef} = \frac{\Delta_1}{20} \qquad R_{bc} = R_{de} = \frac{\Delta_2 - \Delta_1}{20}$$

Now since $\theta_a = \theta_f = R_{be} = R_{cd} = 0$

the problem involves a total of four unknowns, $\theta_b$, $\theta_c$, $R_{ab}$, and $R_{bc}$, which are to be solved by four equations of statics, two from $\Sigma M = 0$ for joints $b$ and $c$ and the other two from shear balance for the frame.

The equations expressing end moments are then written:

$$\begin{aligned}
M_{ab} &= M_{fe} = (1)(\theta_b - 3R_{ab}) && (\theta_a = 0)\\
M_{ba} &= M_{ef} = (1)(2\theta_b - 3R_{ab}) &&\\
M_{bc} &= M_{ed} = (1)(2\theta_b + \theta_c - 3R_{bc}) &&\\
M_{be} &= M_{eb} = (2)(2\theta_b + \theta_e) = 6\theta_b && (\theta_b = \theta_e)\\
M_{cb} &= M_{de} = (1)(2\theta_c + \theta_b - 3R_{bc}) &&\\
M_{cd} &= M_{dc} = (2)(2\theta_c + \theta_d) = 6\theta_c && (\theta_c = \theta_d)
\end{aligned}$$

The equations from joint equilibrium in moment are then established:

$$M_{ba} + M_{bc} + M_{be} = 0$$

$$10\theta_b + \theta_c - 3R_{ab} - 3R_{bc} = 0 \tag{7-52}$$

$$M_{cb} + M_{cd} = 0$$

$$\theta_b + 8\theta_c - 3R_{bc} = 0 \tag{7-53}$$

The third equation is from shear balance for the entire frame isolated from the supports [see Fig. 7-18(c)]:

$$(2)\left(\frac{M_{ab} + M_{ba}}{20}\right) + 20 + 10 = 0$$

$$\theta_b - 2R_{ab} + 100 = 0 \tag{7-54}$$

The fourth equation results from considering the free body cut out by a horizontal section just above $be$; the shear in the two legs [see the dashed lines in Fig. 7-18(c)] must balance the lateral force of 10 kips. Thus,

$$(2)\left(\frac{M_{bc} + M_{cb}}{20}\right) + 10 = 0$$

$$3\theta_b + 3\theta_c - 6R_{bc} + 100 = 0 \tag{7-55}$$

Solving Eqs. 7-52, 7-53, 7-54, and 7-55 simultaneously, we obtain

$$\theta_b = 29.282 \qquad \theta_c = 9.945$$

$$R_{ab} = 64.641 \qquad R_{cd} = 36.282$$

Substituting these in the moment equations, we arrive at

$$M_{ab} = M_{fe} = -164.64 \text{ ft-kips}$$
$$M_{ba} = M_{ef} = -135.36 \text{ ft-kips}$$
$$M_{bc} = M_{ed} = -40.33 \text{ ft-kips}$$
$$M_{be} = M_{eb} = +175.69 \text{ ft-kips}$$
$$M_{cb} = M_{de} = -59.67 \text{ ft-kips}$$
$$M_{cd} = M_{dc} = +59.67 \text{ ft-kips}$$

as plotted in Fig. 7-18(d).

## 7-8 ANALYSIS OF STATICALLY INDETERMINATE RIGID FRAMES WITH SEVERAL DEGREES OF FREEDOM OF JOINT TRANSLATION BY THE SLOPE-DEFLECTION METHOD

The slope-deflection procedure described in the preceding sections may be extended to the analysis of frames with more than two degrees of freedom of joint translation. Solving simultaneous slope-deflection equations in terms of joint rotations and translations, although time consuming, is frequently simpler and more easily applied than the force method previously discussed. Consider the four-story, two-bay building frame shown in Fig. 7-19. It would require the solution of 24 simultaneous equations in terms of total force redundants, compared with 16 equations needed by the method of slope deflection: 12 equations expressing $\Sigma M = 0$ for each of the 12 joints having rotations and four equations expressing the shear balance for each floor level of the four stories.

The term *kinematic indeterminacy* is sometimes used to describe a structure

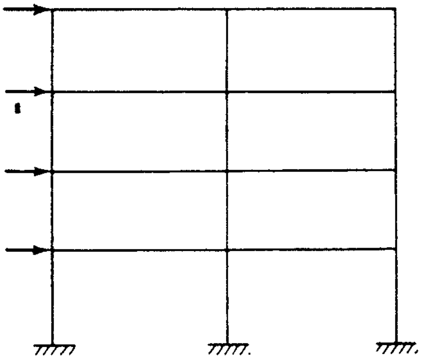


Fig. 7-19

with freedom of joint displacements. A structure having *n* degrees of freedom of joint displacements (translations and rotations) is kinematically indeterminate to the *n*th degree and requires the solution of *n* independent kinematic redundants from the equal number of equilibrium equations. Therefore, the frame shown in Fig. 7-19 is statically indeterminate to the 24th degree but kinematically indeterminate to the 16th degree. Obviously a structure with all its joints restrained against any displacement is said to be kinematically determinate.

## 7-9 MATRIX FORMULATION OF SLOPE-DEFLECTION PROCEDURE

In the slope-deflection method, the joint displacements appear as the basic unknown in the simultaneous equations which express the joint equilibrium of forces in terms of joint rotations and translations. The matrix formulation of these equilibrium equations will readily throw light on the analysis, revealing the essence of the matrix stiffness procedure. This can best be illustrated by going over the problem in Fig. 7-5.

We start by Eq. 7-16 and, with some modifications, rewrite the slope-deflection equations as

$$\begin{aligned}
M_{ab} &= 2E\frac{I_{ab}}{l_{ab}}\left(\theta_b - \frac{3\Delta}{l_{ab}}\right) - 25 \\
M_{ba} &= 2E\frac{I_{ab}}{l_{ab}}\left(2\theta_b - \frac{3\Delta}{l_{ab}}\right) + 25 \\
M_{bc} &= 2E\frac{I_{bc}}{l_{bc}}(2\theta_b + \theta_c) \\
M_{cb} &= 2E\frac{I_{bc}}{l_{bc}}(2\theta_c + \theta_b) \\
M_{cd} &= 2E\frac{I_{cd}}{l_{cd}}\left(2\theta_c - \frac{3\Delta}{l_{cd}}\right) \\
M_{dc} &= 2E\frac{I_{cd}}{l_{cd}}\left(\theta_c - \frac{3\Delta}{l_{cd}}\right)
\end{aligned} \tag{7-56}$$

We also rewrite the joint equilibrium equations as

$$\begin{aligned}
M_{ba} + M_{bc} &= 0 \\
M_{cb} + M_{cd} &= 0 \\
\frac{M_{ab} + M_{ba}}{l_{ab}} + \frac{M_{cd} + M_{dc}}{l_{cd}} + 5 &= 0
\end{aligned} \tag{7-57}$$

Substituting the various expressions of Eq. 7-56 in Eq. 7-57, we have

$$\begin{aligned}
\left(\frac{4EI_{ab}}{l_{ab}} + \frac{4EI_{bc}}{l_{bc}}\right)\theta_b + \frac{2EI_{bc}}{l_{bc}}\theta_c - \frac{6EI_{ab}}{l_{ab}^2}\Delta + 25 &= 0 \\
\frac{2EI_{bc}}{l_{bc}}\theta_b + \left(\frac{4EI_{bc}}{l_{bc}} + \frac{4EI_{cd}}{l_{cd}}\right)\theta_c - \frac{6EI_{cd}}{l_{cd}^2}\Delta &= 0 \\
\frac{6EI_{ab}}{l_{ab}^2}\theta_b + \frac{6EI_{cd}}{l_{cd}^2}\theta_c - \left(\frac{12EI_{ab}}{l_{ab}^3} + \frac{12EI_{cd}}{l_{cd}^3}\right)\Delta + 5 &= 0
\end{aligned} \tag{7-58}$$

Equation 7-58 in matrix form is

$$\begin{array}{ccccc}
\text{FJA} & & \begin{array}{ccc} \theta_b = 1 & \qquad \theta_c = 1 & \qquad \Delta = 1 \end{array} & & \text{JL} \\
\left\{\begin{matrix} 25 \\ 0 + \\ -5 \end{matrix}\right\} & + & \begin{bmatrix}
\dfrac{4EI_{ab}}{l_{ab}} + \dfrac{4EI_{bc}}{l_{bc}} & \dfrac{2EI_{bc}}{l_{bc}} & -\dfrac{6EI_{ab}}{l_{ab}^2} \\
\dfrac{2EI_{bc}}{l_{bc}} & \dfrac{4EI_{bc}}{l_{bc}} + \dfrac{4EI_{cd}}{l_{cd}} & -\dfrac{6EI_{cd}}{l_{cd}^2} \\
-\dfrac{6EI_{ab}}{l_{ab}^2} & -\dfrac{6EI_{cd}}{l_{cd}^2} & \dfrac{12EI_{ab}}{l_{ab}^3} + \dfrac{12EI_{cd}}{l_{cd}^3}
\end{bmatrix} \left\{\begin{matrix} \theta_b \\ \theta_c \\ \Delta \end{matrix}\right\} & = & \left\{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\}
\end{array} \tag{7-59}$$

Equation 7-59 can be put into generalized form

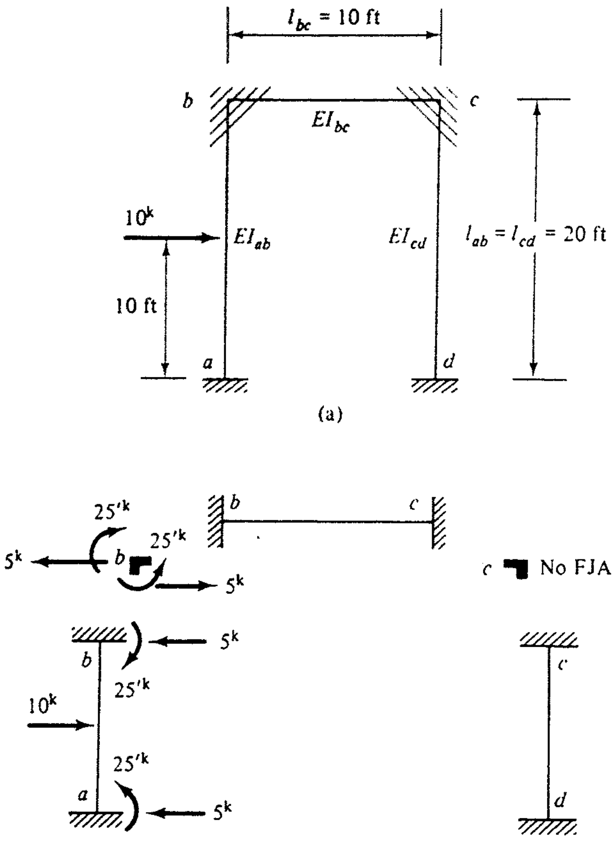
$$F' + SD = F \tag{7-60}$$

We notice that the column matrix $F'$ on the left-hand side of Eq. 7-59 or 7-60 represents the fixed-joint action (FJA) under applied member loads, while the square matrix $S$ is a stiffness matrix, each column of which gives the various joint actions required to produce a certain unit joint displacement. Note that the stiffness matrix is symmetrical. The column matrix $D$ represents joint displacements. The column matrix $F$ on the right-hand side of the same equation contains the actual joint loads (JL) corresponding to the joint displacements. In our case there are no loads actually acting on joints $b$ and $c$, so the elements in the JL matrix are zeros.

It may be interesting to compare Eq. 7-60 with Eq. 6-11 and find out the duality between the displacement method (stiffness method) and the force method (flexibility method).

The analysis therefore can be performed as follows:

1. Analyze a restrained structure with all joints fixed (kinematically determinate) and subjected to member loads only [Fig. 7-20(a)]. With reference to Fig. 7-20(b), we see that locking the joint $b$ against rotation requires an artificial moment force of 25 ft-kips acting clockwise in order to balance the internal



(a)

(b)

**Fig. 7-20**

moment exerted by member *ab* at that joint. Likewise, preventing the lateral translation of joint *b* requires a lateral force of 5 kips acting to the left to balance the internal shear exerted by member *ab* at joint *b*. Simply, the fixed-joint action is obtained from the algebraic sum of fixed-end actions of the related members.

2. Apply joint displacements individually and successively to joints *b* and *c* (free joints) so that the altered structure is restored to the actual displaced configuration. Figure 7-21 shows the separate cases of unit joint displacement from which we obtained the joint forces corresponding to these displacements. For instance, turning a unit rotation of joint *b* would require a moment force of $(4EI_{ab}/l_{ab} + 4EI_{bc}/l_{bc})$ at joint *b*, a moment force of $2EI_{bc}/l_{bc}$ at joint *c*, and a lateral force of $-6EI_{ab}/l_{ab}^2$ (acting to the left) at *b*. The rest can similarly be explained. These are stiffness coefficients which construct the stiffness matrix of Eq. 7-59.

3. Steps 1 and 2 accomplish the geometric configuration (compatibility). It remains for us to say that the sum of joint forces thus obtained must be equal to the actual joint loads (equilibrium). This completes our formulation.

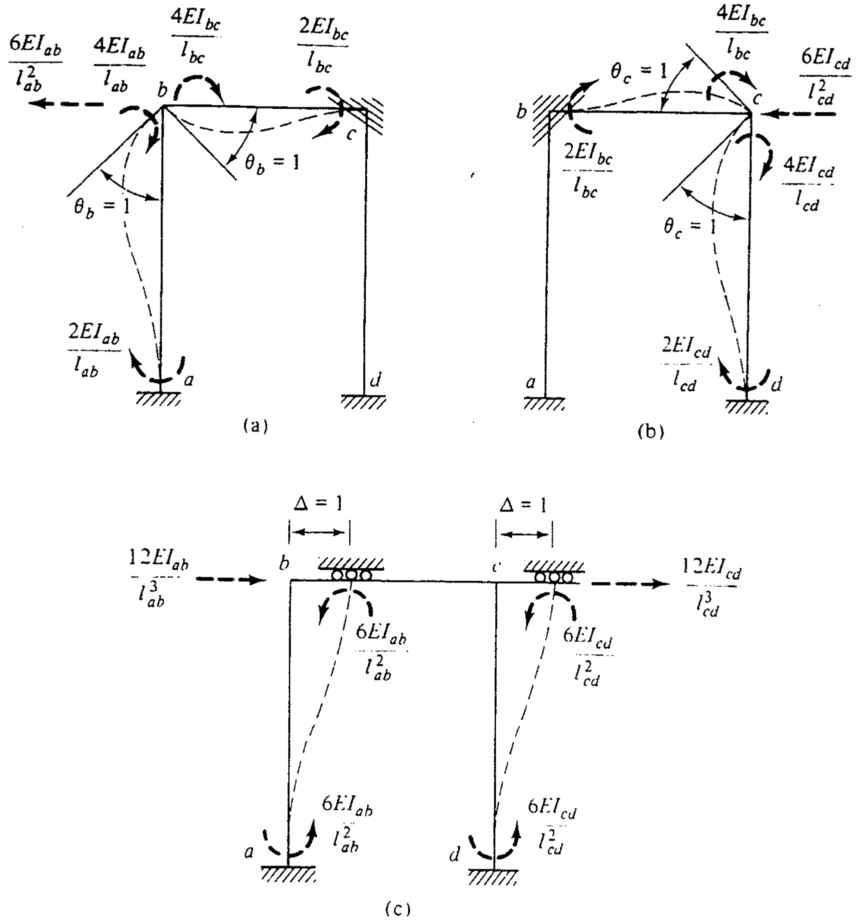


**Fig. 7-21**

Using $2EI_{ab}/l_{ab} = 2EI_{cd}/l_{cd} = 1$, $2EI_{bc}/l_{bc} = 2$, and $l_{ab} = l_{cd} = 2l_{bc} = 20$, we reduce Eq. 7-59 to

$$\begin{Bmatrix} 25 \\ 0 \\ -5 \end{Bmatrix} + \begin{bmatrix} 6 & 2 & -\frac{3}{20} \\ 2 & 6 & -\frac{3}{20} \\ -\frac{3}{20} & -\frac{3}{20} & \frac{12}{400} \end{bmatrix} \begin{Bmatrix} \theta_b \\ \theta_c \\ \Delta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7-61}$$

Replacing $\Delta$ with $R$ ($R = \Delta/20$) in Eq. 7-61 gives

$$\begin{Bmatrix} 25 \\ 0 \\ -5 \end{Bmatrix} + \begin{bmatrix} 6 & 2 & -3 \\ 2 & 6 & -3 \\ -\frac{3}{20} & -\frac{3}{20} & \frac{12}{20} \end{bmatrix} \begin{Bmatrix} \theta_b \\ \theta_c \\ R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7-62}$$

which is identical to the set of simultaneous equations given in Eqs. 7-21 to 7-23.

## PROBLEMS

7-1. Analyze the beam shown in Fig. 7-22 by slope deflection. Draw the shear and moment diagrams.

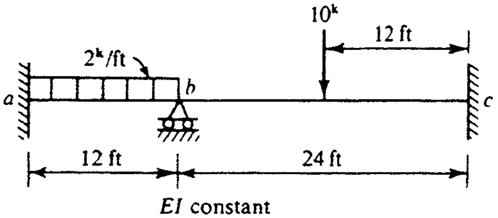


Fig. 7-22

7-2. Figure 7-23 shows a frame of uniform cross section. Find all the end moments by slope deflection, and sketch the deformed structure.

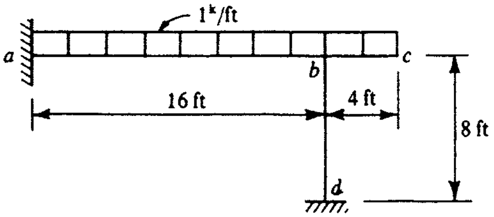


Fig. 7-23

7-3. Analyze the beam shown in Fig. 7-24 by slope deflection. Draw the shear and moment diagrams.

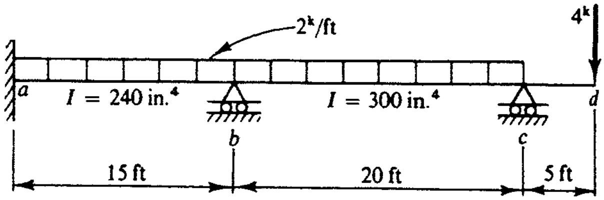


Fig. 7-24

7-4. Find all the end moments by slope deflection for the rigid frame shown in Fig. 7-25. Draw the moment diagram, and sketch the deformed structure.

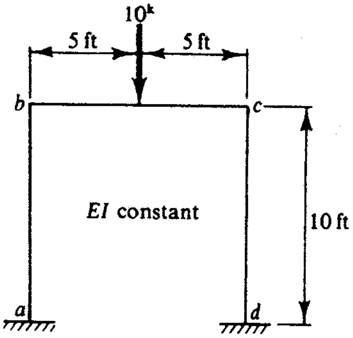


Fig. 7-25

**7-5.** In Fig. 7-24 remove all the loads, and assume that the support *b* settles vertically 0.5 in. Find all the end moments by slope deflection. $E = 30{,}000$ kips/in.$^2$.

**7-6.** In Fig. 7-25 remove the load, and assume that a rotational yield of 0.002 radian clockwise and a linear yield downward of 0.1 in. occur at support *a*. Find the moment diagram. $EI = 10{,}000$ kips-ft$^2$.

**7-7.** For the system and load shown in Fig. 7-26, use the method of slope deflection to find the spring force if $EI = 2 \times 10^8$ kN-cm$^2$ and the axial stiffness of the spring $k = 8$ kN/cm.

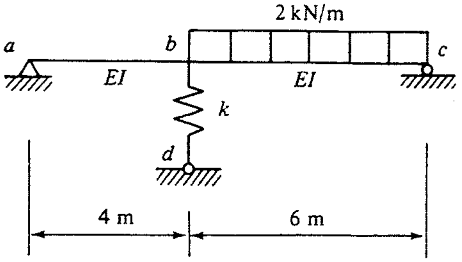


Fig. 7-26

**7-8.** For the load and frame shown in Fig. 7-27, use the method of slope deflection to find the reaction at supports *a* and *c*, and sketch the deformed structure.

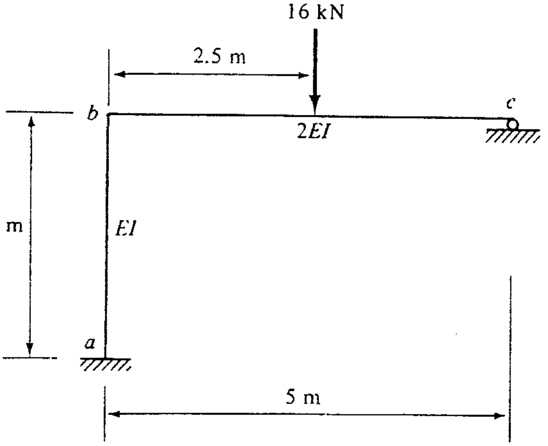


Fig. 7-27

**7-9.** Find all the end moments by slope deflection for the rigid frame shown in Fig. 7-28. Draw the moment diagram, and sketch the deformed structure. [*Hint:* There are two slopes at $c$ (i.e., $\theta_{cb}$ and $\theta_{cd}$). Use the condition $M_{cb} = M_{cd} = 0$ to evaluate the two unknowns.]

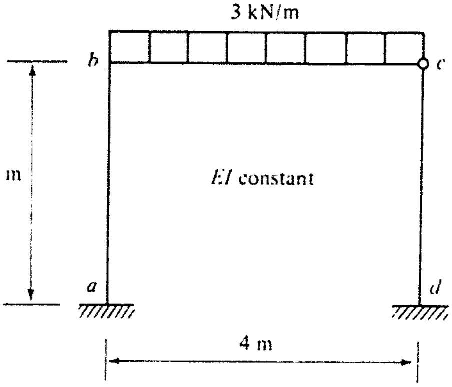


Fig. 7-28

**7-10.** For the system shown in Fig. 7-29, use the method of slope deflection to find the reaction at support $d$ if $E = 20{,}000\ \text{kN/cm}^2$, $I = 40{,}000\ \text{cm}^4$, and the axial stiffness of the spring $k = 5\ \text{kN/cm}$.

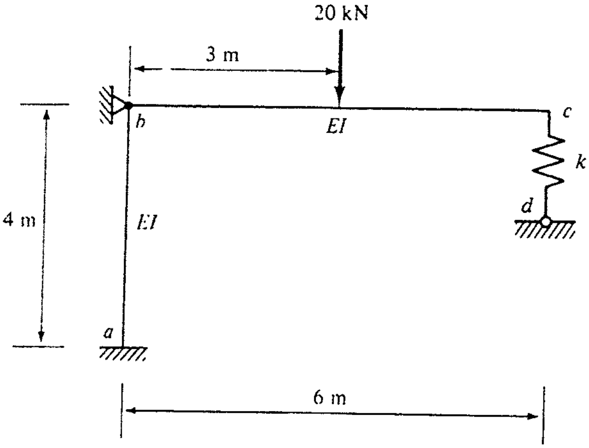


Fig. 7-29

**7-11.** Analyze each of the frames shown in Fig. 7-30 by slope deflection, and draw the moment diagram. Assume constant *EI*.

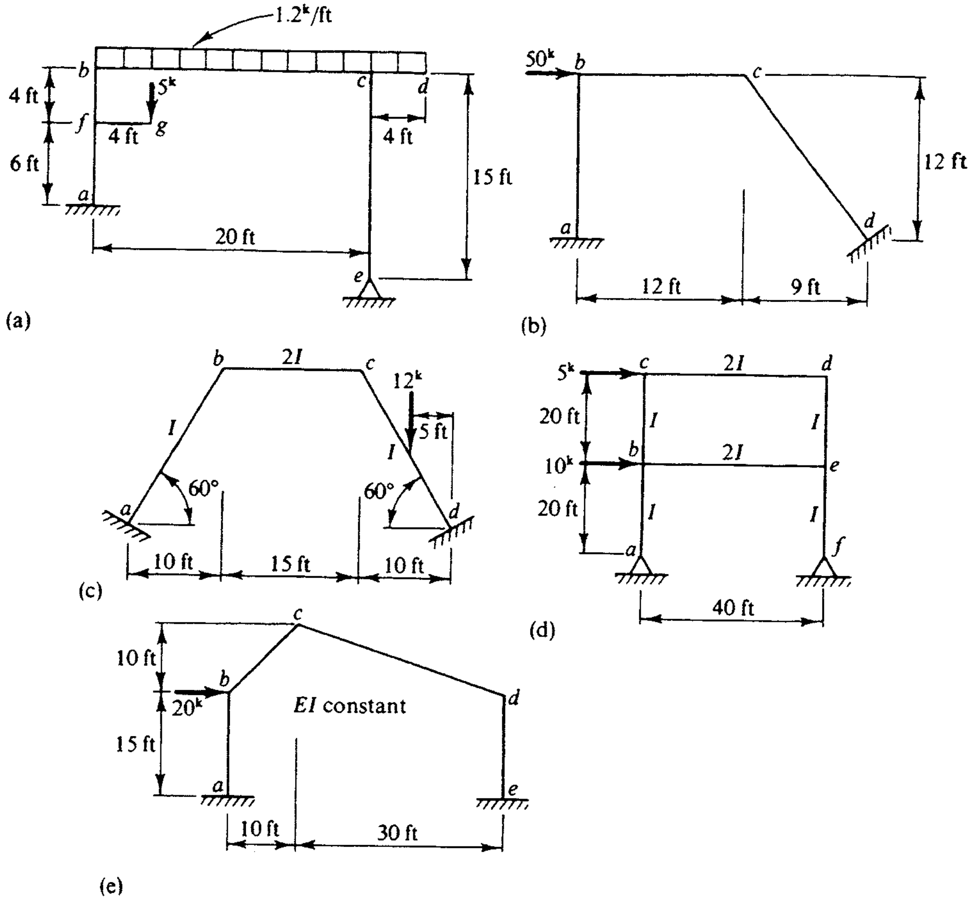


Fig. 7-30

**7-12.** Find the fixed-end moments for the beams shown in Fig. 7-31 by slope deflection. Assume that $w = 3$ kN/m, $l = 4$ m.

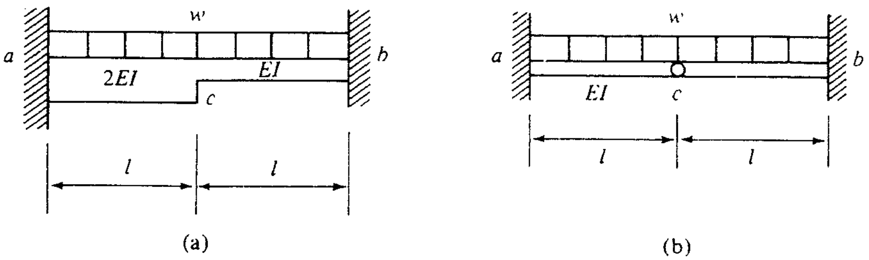


Fig. 7-31

**7-13.** Repeat Prob. 7-3 by the matrix stiffness procedure.

**7-14.** Repeat Prob. 7-11 shown in Fig. 7-30(b) by the matrix stiffness procedure.

**7-15.** Use the matrix stiffness method to analyze the frame shown in Fig. 7-32.

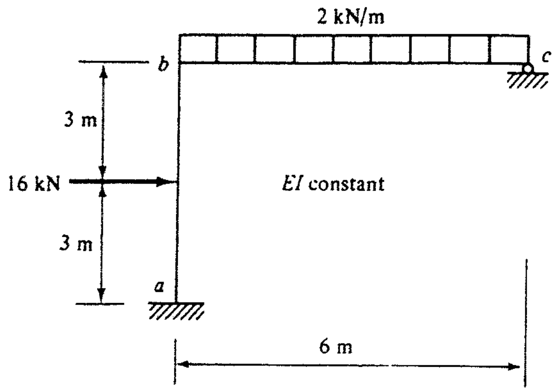


**Fig. 7-32**

# 8

# *Moment-Distribution Method*

## 8-1 GENERAL

The method of *moment distribution* was originated by Hardy Cross in 1930 in a paper entitled "Analysis of Continuous Frames by Distributing Fixed-End Moments." It is the method normally used to analyze all types of statically indeterminate beams and rigid frames in which the members are primarily subjected to bending. The process of moment distribution is initiated by the basic slope-deflection equation (see Sec. 7-2); the moment acting on the end of a member is the algebraic sum of four effects:

1. The moment due to the loads on the member if the member is considered as a fixed-end beam (the fixed-end moment).
2. The moment due to the rotation of the near end (this end) while the far end (the other end) is fixed.
3. The moment due to the rotation of the far end, the near end being fixed.
4. The moment due to the relative translation between the two ends of the member.

This suggests that one line of attack might be to allow these effects to take place separately through a series of steps, first *locking* the joints and then *unlocking* them. For instance, in a rigid-joint structure without joint translation, once the joints are locked (held against rotation), each member is in the state of a fixed-end beam. By unlocking (releasing) a joint, we find that resisting moments will be developed or distributed at the near ends of the members meeting at the joint in proportion to their stiffnesses or according to their distribution factors. At the same time moments will be induced or carried over to the far ends of these members according to their carry-over factors. Joints may be successively released and reheld, one by one, as many times as necessary until each joint will have rotated into its actual, or nearly actual, position. Thus, the process is essentially one of successive approximations which can be carried to any degree of accuracy desired. We shall define the terms *stiffness, distribution factor*, and *carry-over factor* and explain them step by step in the following sections.

It must be noted that the method of moment distribution, although it depends on solving slope-deflection equations, is nevertheless a new approach in structural analysis. For instance, determining the end moments of a highly indeterminate rigid frame with joint translations prevented does not require solving any simultaneous equations, in contrast to the previously discussed methods. Even in analyzing rigid frames having joint translations, the method of moment distribution usually does not involve as many simultaneous equations as are required by any of the methods already discussed. Furthermore, this method is adaptable to computer programming, since it is cyclic. As far as hand calculations are concerned, it is regarded as the most ingenious and convenient method contributed to structural engineering.

The method of moment distribution can be applied to structures composed of prismatic members or nonprismatic members. The present chapter is confined to beams and frames of prismatic members.

The sign convention adopted is the same as that previously suggested for the slope-deflection method: clockwise end moment and rotation of a member are considered positive.

## 8-2 FIXED-END MOMENT

The application of the method of moment distribution requires knowledge of the moments developed at the ends of loaded beams with both ends built in. These moments are called fixed-end moments, often denoted by the symbol $M^F$ or F.E.M. in tables and illustrations.

The determination of fixed-end moments was discussed in Sec. 6-2 and 6-5. For a straight prismatic member the fixed-end moments due to common types of loading were given in Table 7-1.

## 8-3 STIFFNESS, DISTRIBUTION FACTOR, AND DISTRIBUTION OF EXTERNAL MOMENT APPLIED TO A JOINT

For a member of uniform section (constant $EI$), the *stiffness* (or more specifically the *rotational stiffness*) is defined as the end moment required to produce a unit rotation at one end of the member while the other end is fixed.

Consider member *ab* in Fig. 8-1 with a constant section. End *b* is fixed,

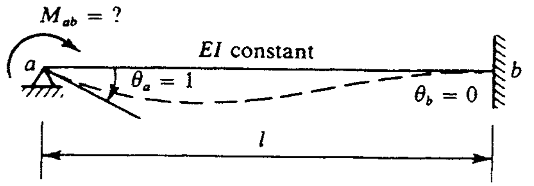


Fig. 8-1

and end $a$ is allowed to rotate. The end moment required at end $a$ to rotate $\theta_a = 1$, whereas $\theta_b = 0$ is given by

$$M_{ab} = 2E\frac{I}{l}(2\theta_a + \theta_b) = 2E\frac{I}{l}(2 + 0) = 4E\frac{I}{l}$$
$$= 4EK$$

This moment is defined as *stiffness* and is denoted by $S$. Thus,

$$S = 4E\frac{I}{l} \qquad (8\text{-}1)$$
$$= 4EK$$

$I/l$ or $K$ being the *stiffness factor*.

Let us turn to Fig. 8-2(a), which shows a frame composed of four members,

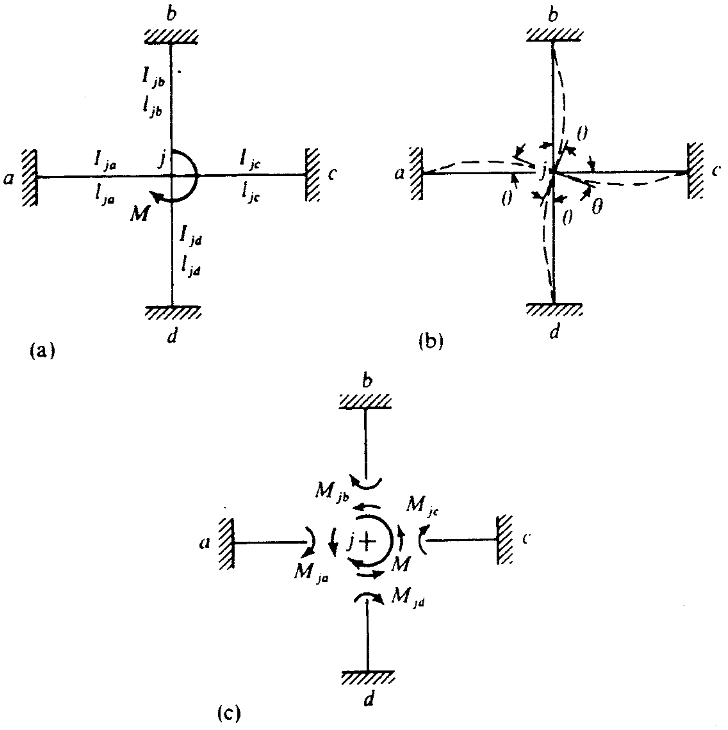


Fig. 8-2

each with one end fixed and the other end rigidly connected at joint $j$ whose translation is prevented. If a clockwise moment $M$ is applied to the joint, it will cause the joint to rotate clockwise through an angular deformation $\theta$, as shown in Fig. 8-2(b). Since $j$ is a rigid joint, each tangent to the elastic curve of the connected end rotates the same angle $\theta$. The applied moment $M$ is resisted by the four members meeting at the joint. The resisting moments $M_{ja}$, $M_{jb}$, $M_{jc}$, and $M_{jd}$ will be induced at the ends of the four members to balance the effect of the external moment $M$, as shown in Fig. 8-2(c).

Equilibrium of the joint requires that

$$M_{ja} + M_{jb} + M_{jc} + M_{jd} = M \tag{8-2}$$

and the slope-deflection equations for the four members give

$$\begin{aligned} M_{ja} &= 4E\frac{I_{ja}}{l_{ja}}\theta = 4EK_{ja}\theta = S_{ja}\theta \\ M_{jb} &= 4E\frac{I_{jb}}{l_{jb}}\theta = 4EK_{jb}\theta = S_{jb}\theta \\ M_{jc} &= 4E\frac{I_{jc}}{l_{jc}}\theta = 4EK_{jc}\theta = S_{jc}\theta \\ M_{jd} &= 4E\frac{I_{jd}}{l_{jd}}\theta = 4EK_{jd}\theta = S_{jd}\theta \end{aligned} \tag{8-3}$$

Equation 8-3 shows that when an external moment is applied to a joint, the resisting moments developed at the near ends of the members meeting at the joint, while the other ends are all fixed, are in direct proportion to the rotational stiffnesses.

Substituting Eq. 8-3 in Eq. 8-2, we obtain

$$(S_{ja} + S_{jb} + S_{jc} + S_{jd})\theta = M$$

or

$$4E(K_{ja} + K_{jb} + K_{jc} + K_{jd})\theta = M$$

Thus,

$$\theta = \frac{M}{4E\,\Sigma K} \tag{8-4}$$

where $\Sigma K = K_{ja} + K_{jb} + K_{jc} + K_{jd}$.

From Eqs. 8-3 and 8-4, we see that

$$\begin{aligned} M_{ja} &= \frac{K_{ja}}{\Sigma K}M = D_{ja}M \\ M_{jb} &= \frac{K_{jb}}{\Sigma K}M = D_{jb}M \\ M_{jc} &= \frac{K_{jc}}{\Sigma K}M = D_{jc}M \\ M_{jd} &= \frac{K_{jd}}{\Sigma K}M = D_{jd}M \end{aligned} \tag{8-5}$$

in which the ratio $K_{ji}/\Sigma K$ or $D_{ji}$ ($i = a, b, c, d$) is defined as the *distribution factor*. Thus, a moment resisted by a joint will be distributed among the connecting members in proportion to their distribution factors. In determining the distribution factors, only the relative $K$ values for connected members are needed. Thus, in most cases we are concerned with the *relative stiffness* rather than the absolute stiffness (Eq. 8-1).

## 8-4 CARRY-OVER FACTOR AND CARRY-OVER MOMENT

Referring to Fig. 8-2, we evaluate the moments at the far ends (fixed ends) of the four members by the slope-deflection method as

$$\begin{aligned} M_{aj} &= 2E\frac{I_{ja}}{l_{ja}}\theta = \left(\frac{1}{2}\right)M_{ja} \\ M_{bj} &= 2E\frac{I_{jb}}{l_{jb}}\theta = \left(\frac{1}{2}\right)M_{jb} \\ M_{cj} &= 2E\frac{I_{jc}}{l_{jc}}\theta = \left(\frac{1}{2}\right)M_{jc} \\ M_{dj} &= 2E\frac{I_{jd}}{l_{jd}}\theta = \left(\frac{1}{2}\right)M_{jd} \end{aligned} \tag{8-6}$$

Equation 8-6 indicates that the moment induced at the far end (fixed) of a prismatic member equals *one-half* the distributed moment at the near end. The ratio ($\frac{1}{2}$) is called the *carry-over factor*, and the induced moments $M_{aj}$, $M_{bj}$, $M_{cj}$, and $M_{dj}$ are called the *carry-over moments*.

In general, the *carry-over factor* may be defined as the ratio of the induced moment at the far end, which is fixed, to the applied moment at the near end, which is prevented from translation but is allowed to rotate. Consider Fig. 8-3. If an end moment $M_{ab}$ is applied at the near end $a$, then the moment induced at the far end $b$, called $M_{ba}$, is given by

$$M_{ba} = C_{ab}M_{ab} \tag{8-7}$$

where $C_{ab}$ is the carry-over factor from $a$ to $b$. For a member of uniform $EI$,

$$M_{ab} = 2E\frac{I}{l}(2\theta_a)$$

$$M_{ba} = 2E\frac{I}{l}(\theta_a) = \frac{1}{2}M_{ab}$$

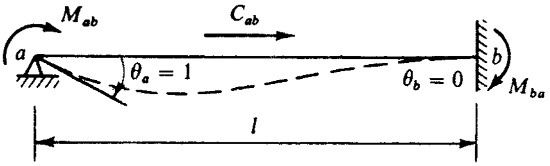


Fig. 8-3

Therefore, $C_{ab} = \frac{1}{2}$

If we consider end *a* as the far end (fixed), and end *b* as the near end (allowed to rotate), we can, in a like manner, prove that

$$C_{ba} = \tfrac{1}{2}$$

Thus for a member of uniform section,

$$C_{ab} = C_{ba} = \tfrac{1}{2} \tag{8-8}$$

We recapitulate some of the main points of Secs. 8-3 and 8-4 as follows. When an external moment is applied to a joint whose translation is prevented, the joint rotates; but the rotation is checked by members meeting at the joint. The resisting moment is then distributed to the near ends of the connected members according to their distribution factors, provided that all the far ends of these members are fixed. The distributed moment to the near end for each member based on the free body of the member (not the joint) equals the applied moment times the distribution factor bearing the same sign as that of the applied moment. Meanwhile, moment is carried over to the far end of each member, which equals one-half the distributed moment to the near end and bears the same sign.

In subsequent illustrations and tables we often use the symbols D.M. to denote the distributed moment; D.F., the distribution factor; C.O.M., the carry-over moment; and C.O.F., the carry-over factor.

**Example 8-1**

For the loaded frame shown in Fig. 8-4(a), find the end moments at *a* and *c*.

We begin by putting the portion *abc* of Fig. 8-4(a) into its equivalent [Fig. 8-4(b)] and obtaining the relative *K* values for members *ab* and *bc* (circled). It is then readily seen that the end moments at *a* and *c* are the carry-over moments due to the external moment 60 kN · m applied to joint *b*. Thus,

$$M_{ab} = \frac{1}{2} M_{ba} = \left(\frac{1}{2}\right)(-60)\left(\frac{1}{3+1}\right) = -7.5 \text{ kN} \cdot \text{m}$$

$$M_{cb} = \frac{1}{2} M_{bc} = \left(\frac{1}{2}\right)(-60)\left(\frac{3}{3+1}\right) = -22.5 \text{ kN} \cdot \text{m}$$

The negative signs indicate counterclockwise moments.

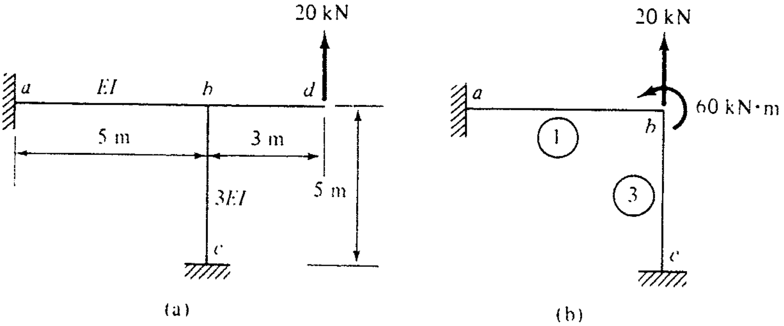


Fig. 8-4

## 8-5 THE PROCESS OF LOCKING AND UNLOCKING: ONE JOINT

The essence of moment distribution lies in locking and unlocking the joints based on the principle of superposition; that is, the effect of an artificial moment applied to a rigid joint of the frame and then eliminated is the same as no effect on the actual structure, since the two actions are neutralized. In this section our attention is confined to those frames that have all joints (including supported ends) fixed except one, which is allowed to rotate. Figure 8-5(a) shows the case which may then be considered as the superposition of effects of Fig. 8-5(b) and (c).

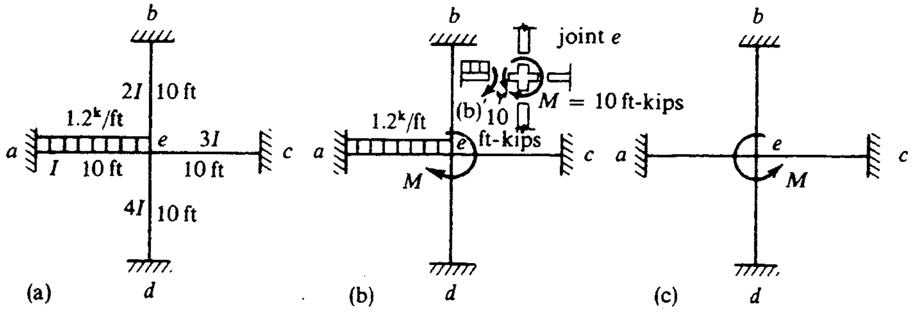


Fig. 8-5

In connection with the setup in Fig. 8-5, we note the following:

1. Suppose that the artificial moment $M$ imposed on joint $e$ [see Fig. 8-5(b)] is so chosen as to just lock the joint against rotation (keeping $\theta_e = 0$) under the original loading (in the present case, the uniform load over span $ae$). Each member of the frame is then in the state of a fixed-end beam; consequently, fixed-end moments will be developed at the ends of member $ae$:

$$M^F_{ea} = -M^F_{ae} = \frac{(1.2)(10)^2}{12} = 10 \text{ ft-kips}$$

all other member ends being subjected to no moment. The results are shown in row 1 of Table 8-1.

**TABLE 8-1**

| | | End Moment (ft-kips) | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | *ae* | | *be* | | *ce* | | *de* | |
| Row | Step | $M_{ae}$ | $M_{ea}$ | $M_{be}$ | $M_{eb}$ | $M_{ce}$ | $M_{ec}$ | $M_{de}$ | $M_{ed}$ |
| 1 | F.E.M. | −10 | +10 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | D.M. | 0 | −1 | 0 | −2 | 0 | −3 | 0 | −4 |
| 3 | C.O.M. | −0.5 | 0 | −1 | 0 | −1.5 | 0 | −2 | 0 |
| 4 | Σ | −10.5 | +9 | −1 | −2 | −1.5 | −3 | −2 | −4 |

Referring to Fig. 8-5(b)', we find that the equilibrium condition

$$\sum M_{\text{joint } e} = 0$$

requires that the locking moment

$$M = 10 \text{ ft-kips}$$

act clockwise on joint $e$.

2. Next, let us release the joint $e$ from the artificial restraint, that is, apply to it an unlocking moment equal and opposite to the locking moment. Referring to Fig. 8-5(c), we see a counterclockwise $M$ equal to the value of 10 ft-kips:

$$M = -10 \text{ ft-kips}$$

is thus applied to joint $e$.

As a result of this unlocking moment, the resisting moment will be distributed at the near ends and carried over to the far ends, as described in preceding sections. Thus,

$$M_{ea} = (-10)\left(\frac{1}{10}\right) = -1.0 \text{ ft-kip} \qquad M_{ae} = (-1)\left(\frac{1}{2}\right) = -0.5 \text{ ft-kip}$$

$$M_{eb} = (-10)\left(\frac{2}{10}\right) = -2.0 \text{ ft-kips} \qquad M_{be} = (-2)\left(\frac{1}{2}\right) = -1.0 \text{ ft-kip}$$

$$M_{ec} = (-10)\left(\frac{3}{10}\right) = -3.0 \text{ ft-kips} \qquad M_{ce} = (-3)\left(\frac{1}{2}\right) = -1.5 \text{ ft-kips}$$

$$M_{ed} = (-10)\left(\frac{4}{10}\right) = -4.0 \text{ ft-kips} \qquad M_{de} = (-4)\left(\frac{1}{2}\right) = -2.0 \text{ ft-kips}$$

They are shown in rows 2 and 3 of Table 8-1, respectively.

3. The sum of the results from steps 1 and 2 gives the solution, as shown in row 4 of Table 8-1.

In analyzing problems like this, with all ends fixed except one rigid joint allowed to rotate, the method of moment distribution provides a very rapid tool, since it involves only one round of locking and unlocking. More examples to illustrate this process follow.

**Example 8-2**

The end moments at $a$, $b$, and $c$ for the beam and loading shown in Fig. 8-6(a) may be obtained by locking joint $b$ and then unlocking it, as indicated in Fig. 8-6(b) and (c). The complete analysis is shown in Fig. 8-6(d), which contains the following steps:

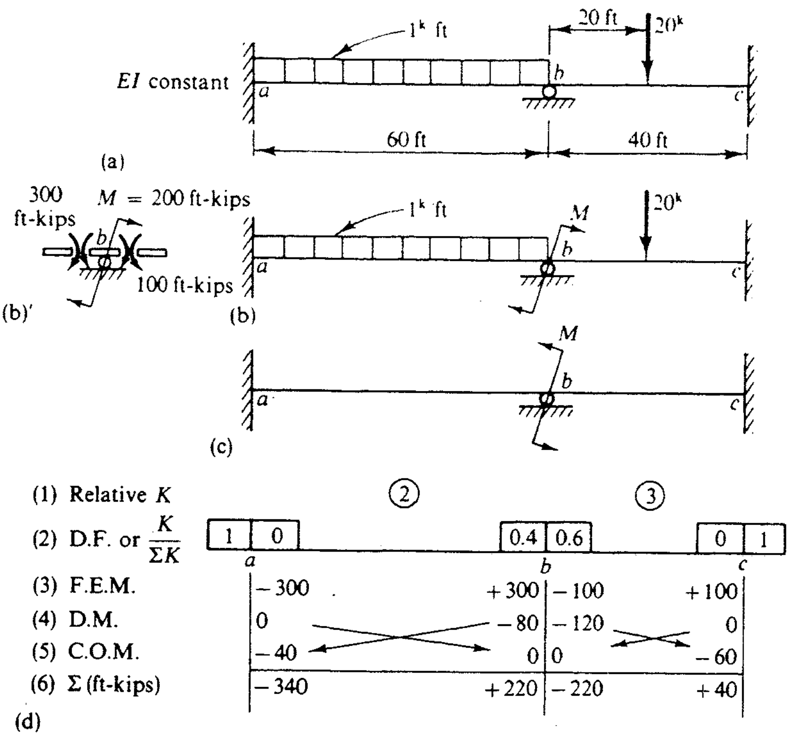
1. The values of stiffness for members $ab$ and $bc$ are found to be

$$K_{ab} = \frac{4EI}{60} \qquad K_{bc} = \frac{4EI}{40}$$

Multiplying each by $30/EI$ gives the relative $K$ values (circled).

2. The distribution factors are computed according to $K/\Sigma K$. Thus,

$$D_{ba} = \frac{2}{2+3} = 0.4 \qquad D_{bc} = \frac{3}{2+3} = 0.6$$



| | | a | b | b | c | |
|---|---|---|---|---|---|---|
| (1) Relative $K$ | | ② | | | ③ | |
| (2) D.F. or $\frac{K}{\Sigma K}$ | 1 | 0 | 0.4 | 0.6 | 0 | 1 |
| (3) F.E.M. | | −300 | +300 | −100 | +100 | |
| (4) D.M. | | 0 | −80 | −120 | 0 | |
| (5) C.O.M. | | −40 | 0 | 0 | −60 | |
| (6) Σ (ft-kips) | | −340 | +220 | −220 | +40 | |

**Fig. 8-6**

We consider the immovable supports at $a$ and $c$ with infinite stiffness so that

$$D_{ab} = D_{cb} = 0$$

The distribution factors are indicated in the attached box at each of the joints.

3. The locking joint $b$ artificially puts members $ab$ and $bc$ in the state of fixed-end beams. We write the fixed-end moments as

$$M_{ab}^F = -M_{ba}^F = -\frac{(1)(60)^2}{12} = -300 \text{ ft-kips}$$

$$M_{bc}^F = -M_{cb}^F = -\frac{(20)(40)}{8} = -100 \text{ ft-kips}$$

Note that locking joint $b$ means applying an external clockwise moment equal to 200 ft-kips, required by the equilibrium of moments for that joint [see Fig. 8-6(b)'].

4. Unlocking joint $b$ (i.e., eliminating the artificial restraint acting on joint $b$) means applying an external counterclockwise moment equal to 200 ft-kips. We write the distributed moment for each of the near ends according to the distribution factor.

5. Write down the carry-over moment for each of the far ends equal to one-half the distributed moment of the near end.

6. The sum of the results from steps 3, 4, and 5 gives the solution.

**Example 8-3**

Find the end moments for the frame shown in Fig. 8-7(a) resulting from the rotational yield of support *a* 0.0016 rad clockwise. $EI = 10{,}000$ kips-ft$^2$. Note that this problem was solved by slope deflection in Example 7-4.

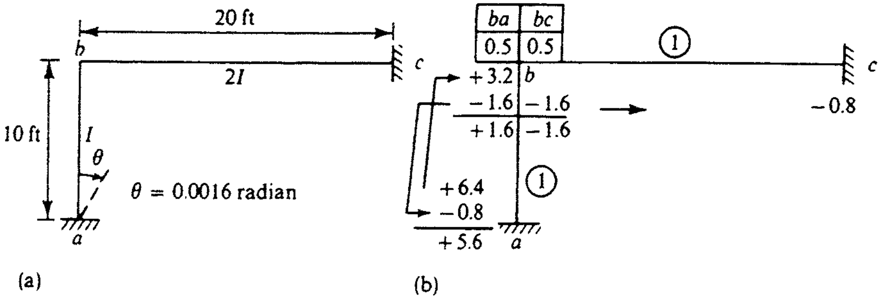


Fig. 8-7

The moment required to produce a rotation of 0.0016 rad at *a* is given by

$$M = \frac{4EI_{ab}\theta_a}{l_{ab}} = \frac{(4)(10{,}000)(0.0016)}{10} = 6.4 \text{ ft-kips}$$

if joint *b* is temporarily fixed. Half the amount of this moment will be carried over to end *b* of member *ab*. By releasing joint *b*, a process of distribution and carry-over takes place, as recorded in Fig. 8-7(b). This gives

$$M_{ab} = 5.6 \text{ ft-kips} \qquad M_{ba} = -M_{bc} = 1.6 \text{ ft-kips} \qquad M_{cb} = -0.8 \text{ ft-kip}$$

## 8-6 THE PROCESS OF LOCKING AND UNLOCKING: TWO OR MORE JOINTS

For a rigid frame or continuous beam having no joint translation but involving more than one joint permitted to rotate, the process of moment distribution consists of the repeated application of the principle of superposition, as stated briefly in the following steps:

1. The joints are first locked; all members, accordingly, are fixed-end. Write the fixed-end moments for all members.
2. The joints are then unlocked. Only one joint at a time is selected to be unlocked. While one joint is unlocked, the rest of joints are assumed to be held against rotation.

Calculate the unlocking moment at this joint, and write distributed moments for the near ends of the members meeting at this joint.

3. Also write down the carry-over moments at the far ends of these members. Note that the carry-over moments constitute a new set of fixed-end moments for the far ends.

4. Relock the joint, and select the next joint to be unlocked. Repeat steps 2 and 3.

Note that after a joint is unlocked and the moments at a joint distributed, the joint is in balance, or in equilibrium, since the artificial restraint is removed. However, there are other joints still locked by external means; hence, the next step is to relock the joint and then proceed to unlock the next joint. The process of locking and unlocking each joint only once constitutes *one cycle* of moment distribution.

5. Joints are unlocked and relocked one by one; therefore, steps 2 and 3 are repeated several times. The process can be halted as soon as the carry-over moments are so small that we are willing to neglect them.
6. Sum up the moments to obtain the final result.

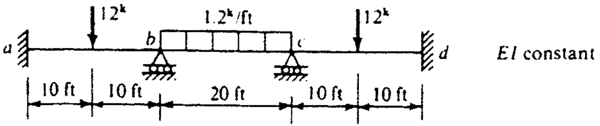
We see that the analysis starts from an alteration of the original structure by locking all joints against rotation. This means that artificial restraints are actually applied to the original structure. The altered structure, consisting of a number of fixed-end members, is then modified by unlocking and relocking joints one by one until all artificial restraints are removed or diminished to a sufficiently small amount. Thus, moment distribution is a method of successive approximations by which the exact results can be approached with the desired degree of precision.

The complete analysis of a loaded three-span continuous beam by moment distribution, shown in Fig. 8-8, will serve to illustrate the foregoing procedure.

The presentation of moment distribution for the preceding illustration may be rearranged as shown in Fig. 8-9. At first glance it seems as if joints $b$ and $c$ were locked and then unlocked simultaneously. However, the performance can still be considered under the restriction of unlocking one joint at a time. For the loads given, the fixed-end moments are recorded in step 1 (see Fig. 8-9). Next, we may consider joint $c$ as being held against rotation and joint $b$ as being unlocked first. The unlocking moment $+10$ at $b$ is then equally distributed to the near ends of members $ba$ and $bc$ as indicated in step 2, and one-half of the amount is carried over to the far ends of these members as indicated in step 3. Next, we consider joint $b$ as locked and joint $c$ as released, but only partially, by applying an unlocking moment of $-(40 - 30)$ or $-10$ to it, since the complete releasing of joint $c$ would require an unlocking moment of $-(40 - 30 + 2.5)$ or $-12.5$ for the time being. This unlocking moment $-10$ is equally distributed to the near ends of members $cb$ and $cd$, and one-half of the amount is carried over to the far ends of these members as indicated in steps 2 and 3, leaving the just-received carry-over moment $+2.5$ to be handled later.

Referring to step 3 of Fig. 8-9, we find the carry-over moments form a new set of fixed-end moments for the beam, and a process of unlocking joints can be carried out in a similar manner. The process will thus be repeated in a cyclic fashion until the carry-over moments are neglected.

In fact, if the carry-over moments are neglected, the joints, after being unlocked, are in balance (i.e., no external constraint exists). This gives the



| | | | | | | |
|---|---|---|---|---|---|---|
| K | ① | | ① | | ① | |
| K/ΣK | 0 | 0.5 | 0.5 | 0.5 | 0.5 | 0 |
| Lock all joints F.E.M. | −30 | +30 | −40 | +40 | −30 | +30 |
| Unlock joint b D.M. | | +5 | +5 | | | |
| Unlock joint b C.O.M. | +2.5 | | | +2.5 | | |
| Unlock joint c D.M. | | | | −6.25 | −6.25 | |
| Unlock joint c C.O.M. | | | −3.13 | | | −3.13 |
| Unlock joint b D.M. | | +1.57 | +1.56 | | | |
| Unlock joint b C.O.M. | +0.78 | | | +0.78 | | |
| Unlock joint c D.M. | | | | −0.39 | −0.39 | |
| Unlock joint c C.O.M. | | | −0.20 | | | −0.20 |
| Unlock joint b D.M. | | +0.10 | +0.10 | | | |
| Unlock joint b C.O.M. | +0.05 | | | +0.05 | | |
| Unlock joint c D.M. | | | | −0.02 | −0.03 | |
| Σ (ft-kips) | −26.67 | +36.67 | −36.67 | +36.67 | −36.67 | +26.67 |

Locking joint *b* requires moment of − 10 as enclosed by dashed line.

To eliminate this artificial moment we apply + 10 to joint *b*. After the process of distribution, joint *b* is in balanced state.

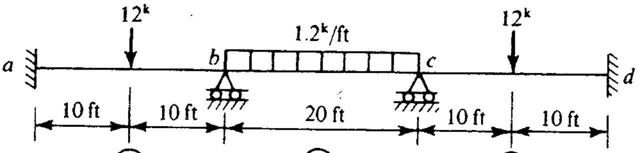
The C.O.M. constitutes additional F.E.M. to the far end.

Now joint *c* is subjected to the total artificial moment of + 12.5 as enclosed by dashed line. After distributing the moment of − 12.5 to each side and carry-over, joint *c* is balanced, while joint *b* is again unbalanced because of the C.O.M. of − 3.13.

⋮

The process of unlocking and relocking joints *b* and *c* alternatively is continued until the C.O.M. is small enough to be neglected.

**Fig. 8-8**



| | | | | | | | |
|---|---|---|---|---|---|---|---|
| K | ① | | ① | | ① | | |
| K/ΣK | 0 | 0.5 | 0.5 | 0.5 | 0.5 | 0 | |
| (1) F.E.M. | −30 | +30 | −40 | +40 | −30 | +30 | |
| (2) D.M. | | +5 | +5 | −5 | −5 | | |
| (3) C.O.M. | +2.5 | | −2.5 | +2.5 | | −2.5 | First cycle |
| (4) D.M. | | +1.25 | +1.25 | −1.25 | −1.25 | | |
| (5) C.O.M. | +0.63 | | −0.63 | +0.63 | | −0.63 | Second cycle |
| (6) D.M. | | +0.32 | +0.31 | −0.31 | −0.32 | | |
| (7) C.O.M. | +0.16 | | −0.16 | +0.16 | | −0.16 | Third cycle |
| (8) D.M. | | +0.08 | +0.08 | −0.08 | −0.08 | | |
| (9) C.O.M. | +0.04 | | −0.04 | +0.04 | | −0.04 | Fourth cycle |
| (10) D.M. | | +0.02 | +0.02 | −0.02 | −0.02 | | |
| Σ (ft-kips) | −26.67 | +36.67 | −36.67 | +36.67 | −36.67 | +26.67 | Fifth cycle |

**Fig. 8-9**

approximate solution for the analysis. For instance, the first approximation may be obtained from the sum of steps 1 and 2, the result of the first cycle; the second approximation may be obtained from the sum of steps 1 to 4, the result up to the second cycle; and so on. It is interesting to note that, in this particular problem, even the first cycle yields a good approximation of the exact solution. After two or three cycles, the carry-over values become negligible.

Note that the beam and the loading shown in Fig. 8-9 are symmetrical, the data presented on each side of the line of symmetry are equal in magnitude but opposite in sign. This special display suggests that some modification could be made in order to facilitate the process of moment distribution by working with only half the structure. See Sec. 8-7 for *modified stiffness*. This problem will be re-solved in Example 8-6 by using modified stiffness.

More examples are given to illustrate the cyclic process.

**Example 8-4**

Analyze the frame in Fig. 8-10 by moment distribution. The relative stiffnesses for the frame members are computed first as

$$K_{\underset{(cd)}{ab}} = \frac{2I}{30} \qquad \text{say} \qquad 4$$

$$K_{bc} = \frac{2I}{40} \qquad \text{say} \qquad 3$$

$$K_{\underset{(cf)}{be}} = \frac{I}{20} \qquad \text{say} \qquad 3$$

Next, the fixed-end moments for the loaded member $bc$ are found to be

$$M_{bc}^{F} = -\frac{(60)(10)(30)^2}{(40)^2} + \frac{(60)(30)(10)^2}{(40)^2} = -225 \text{ ft-kips}$$

$$M_{cb}^{F} = +\frac{(60)(30)(10)^2}{(40)^2} - \frac{(60)(10)(30)^2}{(40)^2} = -225 \text{ ft-kips}$$

The complete analysis is shown in Fig. 8-11. Note that the intermediate values of the moments at fixed ends $e$ and $f$ are not shown. Since members $be$ and $cf$ are not loaded, it is evident that

$$M_{eb} = \tfrac{1}{2}M_{be} = 32.2 \text{ ft-kips} \qquad M_{fc} = \tfrac{1}{2}M_{cf} = 32.2 \text{ ft-kips}$$

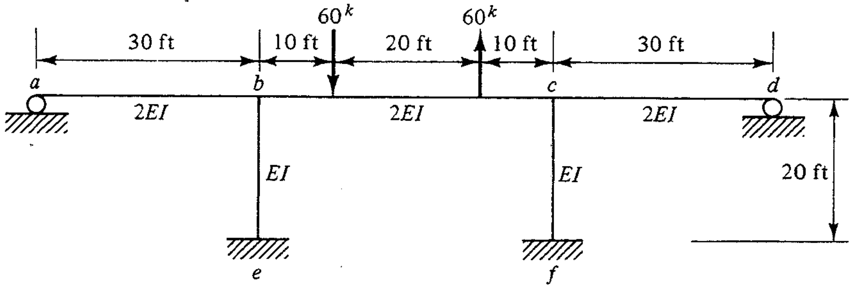


Fig. 8-10

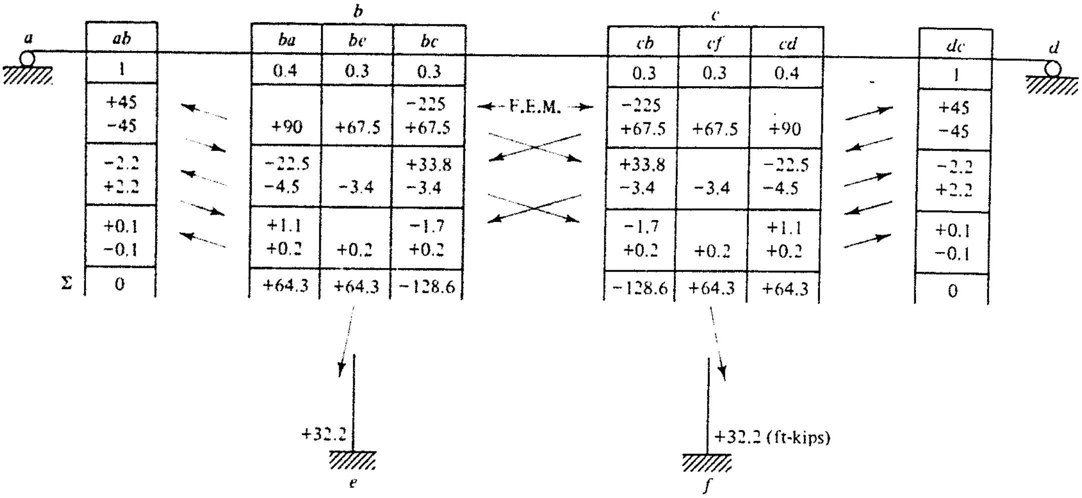


Fig. 8-11

From Fig. 8-11 the values of the moments obtained on the left side of the center line of the structure are exactly the same as those on the right. Such a special display is referred to as *antisymmetry*, which yields

$$\theta_b = \theta_c$$

An adjustment can be made to the stiffness of the center beam *bc* that will permit the analyst to work with only half the structure. We also find that the final result $M_{ab} = M_{dc} = 0$ is known beforehand. The convergence of moment distribution may be improved by using modified stiffness in beams *ab* and *dc*. See Sec. 8-7 for modified stiffness. This problem is re-solved in Example 8-7.

## 8-7 MODIFIED STIFFNESSES

The examples given in Sec. 8-6 have illustrated three special cases that suggest that some modifications for simplifying the moment-distribution process might be found by recognizing certain known conditions.

Consider the frame subjected to a clockwise external moment $M$ applied to the connecting joint shown in Fig. 8-12(a), for which we note the following:

1. $\theta_a = 0$; that is, the member is fixed at end $a$.
2. $M_b = 0$; that is, the member is simply supported at end $b$.
3. $\theta_c = -\theta$; that is, the member rotates through an equal but opposite angle at the other end $c$ as in the case of symmetry.
4. $\theta_d = \theta$; that is, the member rotates through the same angle at the other end $d$ as in the case of antisymmetry.

Referring to Fig. 8-12(b), we find the equilibrium of joint $j$ requires that

$$M_{ja} + M_{jb} + M_{jc} + M_{jd} = M \tag{8-9}$$

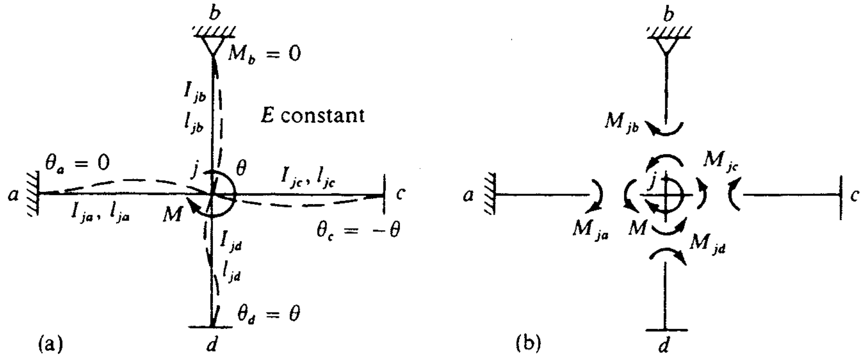


**Fig. 8-12**

From the slope-deflection equations we obtain

$$M_{ja} = 4EK_{ja}\theta \tag{8-10}$$

and
$$M_{jb} = 2EK_{jb}(2\theta + \theta_b)$$

but
$$M_{bj} = 2EK_{jb}(2\theta_b + \theta) = 0$$

or
$$\theta_b = -\frac{\theta}{2}$$

Hence,
$$M_{jb} = 2EK_{jb}\left(2\theta - \frac{\theta}{2}\right) = 4E\left(\frac{3}{4}K_{jb}\right)\theta$$
$$= 4EK'_{jb}\theta \tag{8-11}$$

where we let

$$K'_{jb} = \tfrac{3}{4}K_{jb} \tag{8-12}$$

$K'_{jb}$ being called the *modified stiffness factor* for member *jb*. Similarly,

$$M_{jc} = 2EK_{jc}(2\theta - \theta) = 4E(\tfrac{1}{2}K_{jc})\theta$$
$$= 4EK'_{jc}\theta \tag{8-13}$$

where
$$K'_{jc} = \tfrac{1}{2}K_{jc} \tag{8-14}$$

$$M_{jd} = 2EK_{jd}(2\theta + \theta) = 4E(\tfrac{3}{2}K_{jd})\theta$$
$$= 4EK'_{jd}\theta \tag{8-15}$$

where
$$K'_{jd} = \tfrac{3}{2}K_{jd} \tag{8-16}$$

Substituting Eqs. 8-10, 8-11, 8-13, and 8-15 in Eq. 8-9 yields

$$4E(K_{ja} + K'_{jb} + K'_{jc} + K'_{jd})\theta = M$$

or
$$\theta = \frac{M}{4E\sum K'} \tag{8-17}$$

where
$$\sum K' = K_{ja} + K'_{jb} + K'_{jc} + K'_{jd}$$

Substituting Eq. 8-17 in Eqs. 8-10, 8-11, 8-13, and 8-15 yields

$$\begin{aligned} M_{ja} &= \frac{K_{ja}}{\sum K'}M \\ M_{jb} &= \frac{K'_{jb}}{\sum K'}M \\ M_{jc} &= \frac{K'_{jc}}{\sum K'}M \\ M_{jd} &= \frac{K'_{jd}}{\sum K'}M \end{aligned} \tag{8-18}$$

Thus, it is seen that when an external moment is applied to joint $j$, the distributed moments to the near ends of the members meeting at the joint are in direct proportion to their modified stiffness factors if the conditions of the far ends are known. By using the modified stiffness factor for one end, we actually eliminate the carry-over to the other end except for writing down the final result for that part.

Let us recapitulate the modified stiffness factors for various end conditions:

1. If the one end is simply supported, the modified stiffness factor for the other end is given by

$$K' = \tfrac{3}{4}K \tag{8-19}$$

2. If the one end is symmetrical to the other end, then

$$K' = \tfrac{1}{2}K \tag{8-20}$$

3. If the one end is antisymmetrical to the other end, then

$$K' = \tfrac{3}{2}K \tag{8-21}$$

$K'$ denotes the modified stiffness factor. In parallel with Eq. 8-1, we may have

$$S' = 4EK' \tag{8-22}$$

$S'$ being called the *modified stiffness*. A general definition for *modified stiffness* is the end moment required to produce a unit rotation at this end (simple end), while the other end remains in the actual conditions.

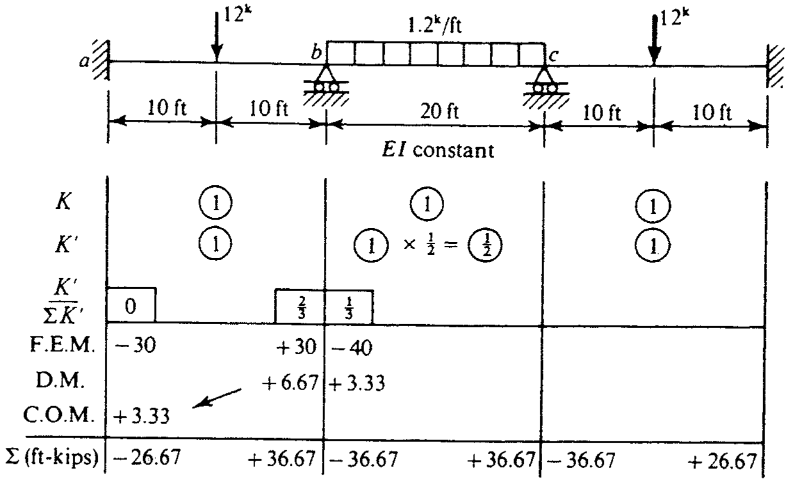
The following examples illustrate the process of moment distribution by using the modified $K$ values.

**Example 8-5**

Analyze the symmetrical beam in Fig. 8-13 by using the modified $K$ value in the center span.

**Example 8-6**

Analyze the symmetrical frame in Fig. 8-14, which was solved by the methods of consistent deformations, least work, and slope deflection. Once again the method



| | | | | | | |
|---|---|---|---|---|---|---|
| K | ① | | ① | | ① | |
| K' | ① | | ① × ½ = (½) | | ① | |
| K'/ΣK' | 0 | ⅔ | ⅓ | | | |
| F.E.M. | −30 | +30 | −40 | | | |
| D.M. | | +6.67 | +3.33 | | | |
| C.O.M. | +3.33 | | | | | |
| Σ (ft-kips) | −26.67 | +36.67 | −36.67 | +36.67 | −36.67 | +26.67 |

Fig. 8-13

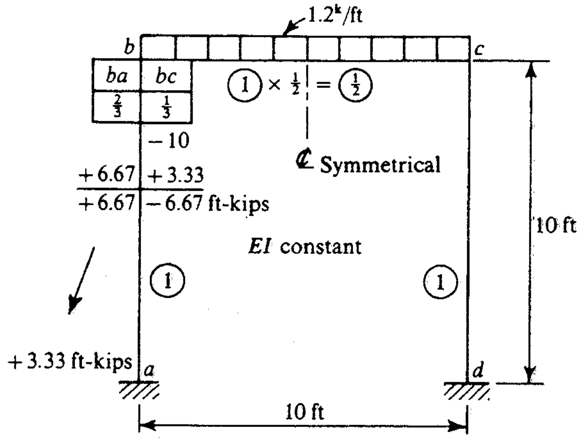


Fig. 8-14

of moment distribution demonstrates its superiority over any other method previously discussed.

**Example 8-7**

Analyze the frame in Fig. 8-10 by using the modified stiffness in beams *ab*, *bc*, and *cd*. The solution is given in Fig. 8-15, which agrees with the result of Example 8-4.

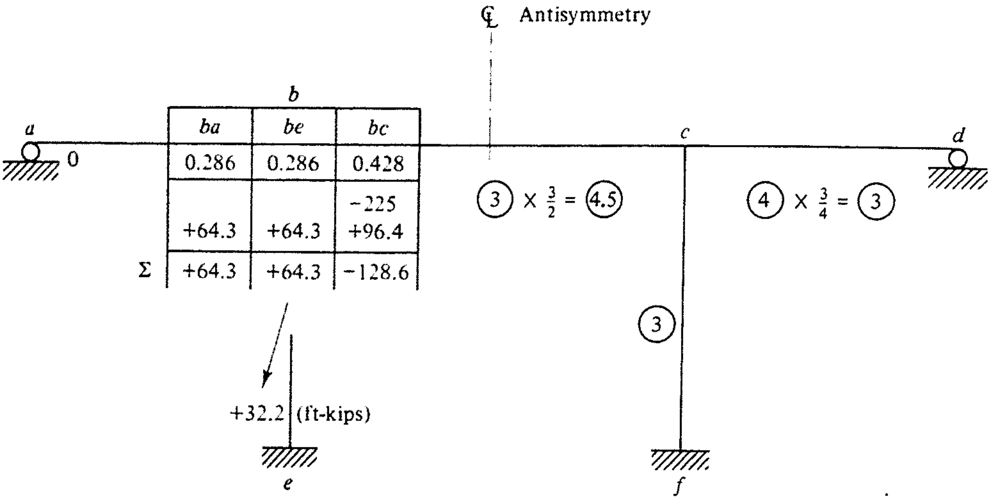
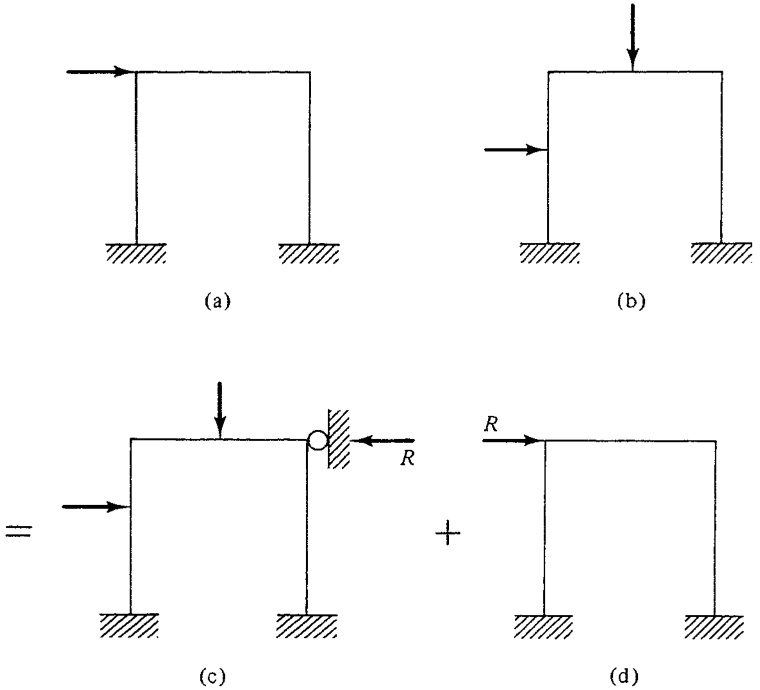


Fig. 8-15

## 8-8 THE TREATMENT OF JOINT TRANSLATIONS

The procedure of moment distribution discussed thus far is based on the restriction that the joints of the structure do not move. However, many frames encountered in practice undergo joint translations. There are, in general, two types of loaded frames in which joint translations are involved. The first is a frame under the action of a lateral force applied at a joint, such as the one shown in Fig. 8-16(a); the second is a frame that, together with loads acting on its members, forms an unsymmetrical system, such as the one shown in Fig. 8-16(b). To handle the latter type of frame, we may resort to the principle of superposition. An artificial holding force to prevent joint translation is imposed on the structure and is subsequently eliminated. Thus, the frame in Fig. 8-16(b) can be considered as the superposed effect of the two separate systems indicated in Fig. 8-16(c) and (d). In the first place [Fig. 8-16(c)], the translation of joints is prevented by providing an artificial support at the top of the column so that moment distribution can be carried out in the usual manner. The required holding force $R$ is then obtained by statics. The next step [Fig. 8-16(d)] is to eliminate the artificial restraint by applying to the top of column a lateral force equal to $R$ but opposite in direction. The resulting configuration is the same as that shown in Fig. 8-16(a). Therefore, the problem now reduces to dealing with a frame under lateral forces applied at the joints.

To handle this type of frame, we consider the frame in Fig. 8-17(a) in which joints $a$ and $d$ are fixed, whereas joints $b$ and $c$ undergo both translation and rotation because of the lateral force $P$ applied at the top of column. It is obvious that if the joint translation of $b$ is specified as $\Delta$, then the joint translation of $c$ is determined and, in this case, is also $\Delta$. Now the distortion imposed on joints



(a) (b)

(c) (d)

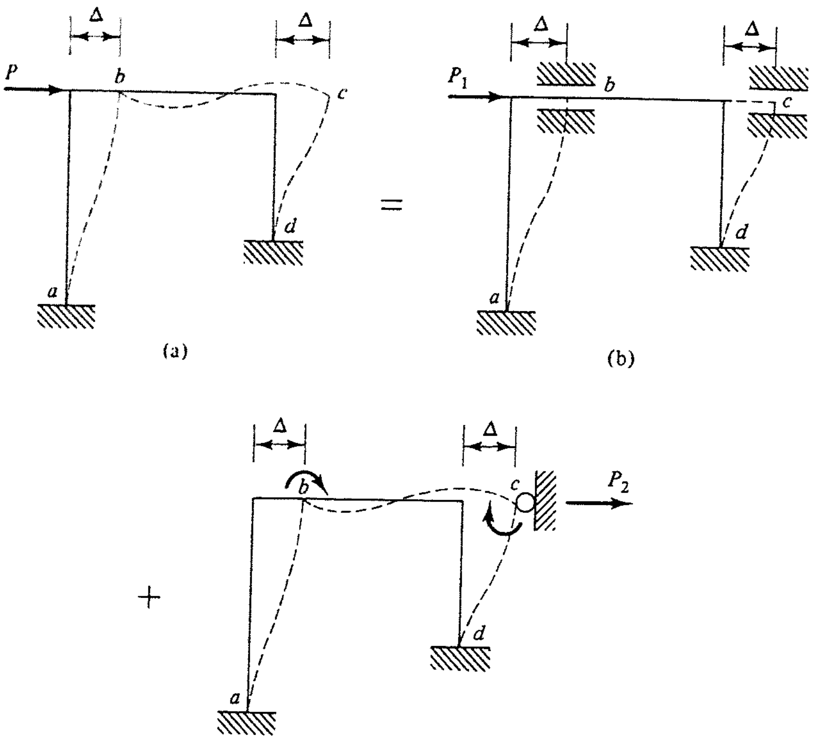
Fig. 8-16

*b* and *c* may be regarded as the superposed effect of the two following separate steps:

1. Translation without rotation [see Fig. 8-17(b)]
2. Rotation without translation [see Fig. 8-17(c)]

In step 1 joints *b* and *c* are locked against rotation ($\theta_b = \theta_c = 0$), and the joint translation $\Delta$ is produced by applying a lateral force $P_1$. It is clear that some external restraints (i.e., locking moments) are required at joints *b* and *c* in order to hold both joints against rotation. Also, end moments will be induced in the members having relative joint translations according to Eq. 7-11, that is, the respective value of $-6EI\Delta/l^2$ for each column end in the present case. We call them the F.E.M. due to joint translations, from which the lateral force $P_1$ can be figured.

In step 2 further joint translations are checked by providing an artificial support at the top of the column. Joints *b* and *c* are then unlocked so that they finally rotate to their actual positions. Recall that to unlock a joint means to apply a moment equal but opposite to the locking moment at the joint. From the resulting end moments, the holding force $P_2$ at the artificial support can be figured.



(a) (b) (c)

Fig. 8-17

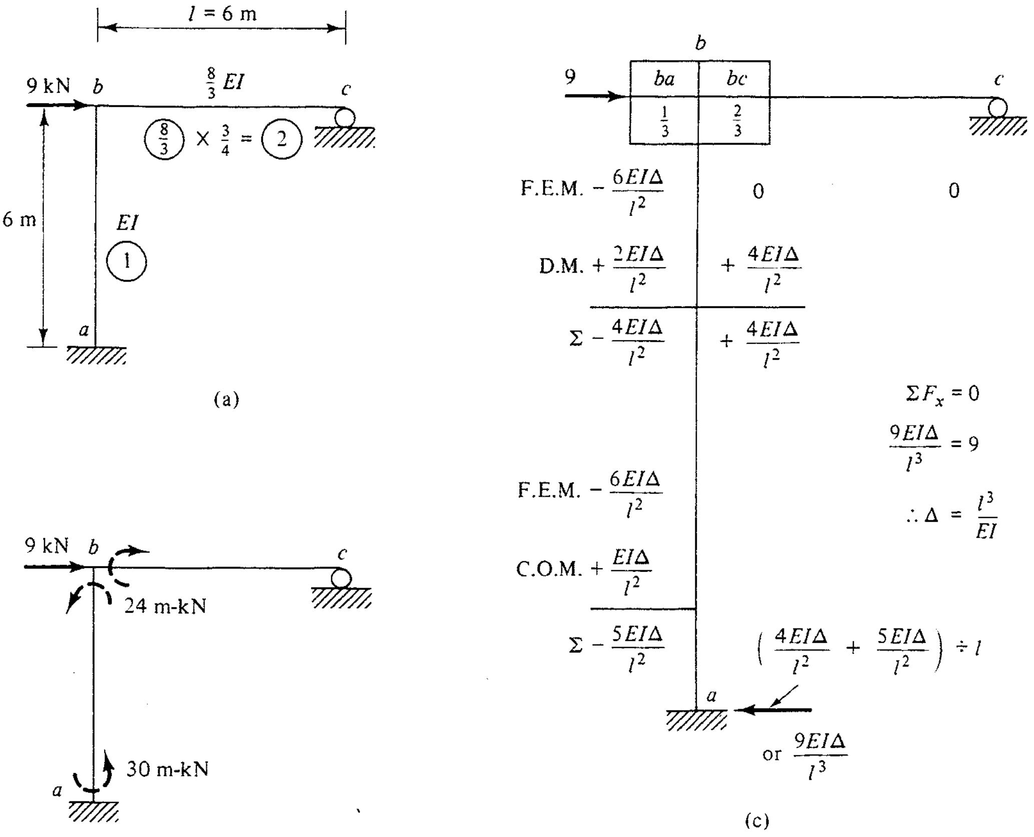
The preceding two steps complete the procedure of moment distribution for frames having joint translations under joint loads. The procedure differs from that for frames without joint translation under member loads, only in the source of fixed-end moments. In the former case, the fixed-end moments arise from pure joint translations, whereas in the latter case, the fixed-end moments are due to loads acting on the fixed beams as described in Sec. 8-2.

Referring to Fig. 8-17, we notice that the lateral force applied at the joints, that is, the sum of $P_1$ and $P_2$, is found to be a function of Δ, a value that we usually do not know at the outset. However, Δ can be solved by the force condition

$$P_1 + P_2 = P$$

After the value of Δ is found, the resulting end moments, which are also expressed in terms of Δ, can readily be determined.

As a simple illustration, let us analyze the frame shown in Fig. 8-18(a) by the method of moment distribution. Because of the lateral force acting on the column top, joint *b* and also joint *c* will move to the right a distance Δ, causing



(a)

(b)

(c)

**Fig. 8-18**

a relative deflection between joints $a$ and $b$. We thus write the fixed-end moments due to joint translation equal to $-6EI\Delta/l^2$ at column ends $a$ and $b$. Since $M_c = 0$ is known beforehand, we use modified stiffness for member $bc$ to simplify the calculation. Next, we perform the process of distribution and carry-over. The complete analysis is given in Fig. 8-18(b). The resulting end moments are found to be consistent with a horizontal reaction $9EI\Delta/l^3$ at support $a$, which should be equal to the applied lateral force of 9 kN from $\Sigma F_X = 0$. Thus we have

$$\frac{9EI\Delta}{l^3} = 9$$

or

$$\Delta = \frac{l^3}{EI}$$

Substituting this value in the result shown in Fig. 8-18(b), we obtain the answer diagram given in Fig. 8-18(c).

Practically, we often start with a convenient value for $\Delta$, or F.E.M., to carry out the moment-distribution process and then correct the result thus obtained by a constant of proportionality. The problem is re-solved in Fig. 8-19.

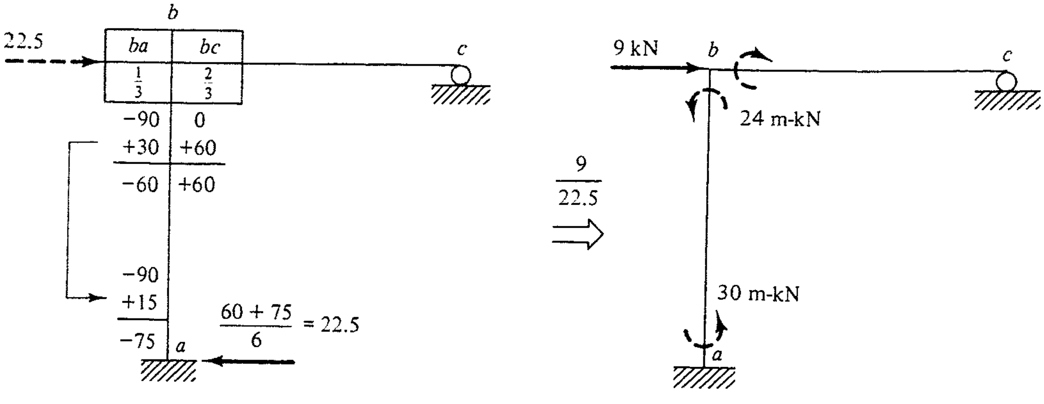


Fig. 8-19

Note that we choose F.E.M. = −90 as a start. The final result of end moments is associated with a lateral force equal to 22.5 kN. Multiplying the obtained result by a correction ratio of 9/22.5 will give the answer.

## 8-9 ANALYSIS OF STATICALLY INDETERMINATE RIGID FRAMES WITH ONE DEGREE OF FREEDOM OF JOINT TRANSLATION BY MOMENT DISTRIBUTION

**Example 8-8**

Figure 8-20(a) shows a loaded one-story bent with an inclined leg. The relative $K$ value for each member is circled. The end moments were solved by slope deflection in Example 7-6. Let us now re-solve them by moment distribution.

We begin by finding the relative end displacement for each member as shown in Fig. 8-20(b). Next, we assume fixed-end moments, due to consistent joint translations, in proportion to the value of $-6EK\Delta/l$:

$$M^F_{ab} = M^F_{ba} = -\frac{6E(1)(\Delta)}{15} \qquad \text{say} \qquad -100$$

$$M^F_{bc} = M^F_{cb} = +\frac{6E(2)[(3/4)\Delta]}{15} \qquad \text{say} \qquad +150$$

$$M^F_{cd} = M^F_{dc} = -\frac{6E(2)[(5/4)\Delta]}{25} \qquad \text{say} \qquad -150$$

The process of moment distribution for this particular setup is shown in Fig. 8-20(c). Referring to Fig. 8-20(d), by applying $\Sigma M_o = 0$ for the entire frame, we

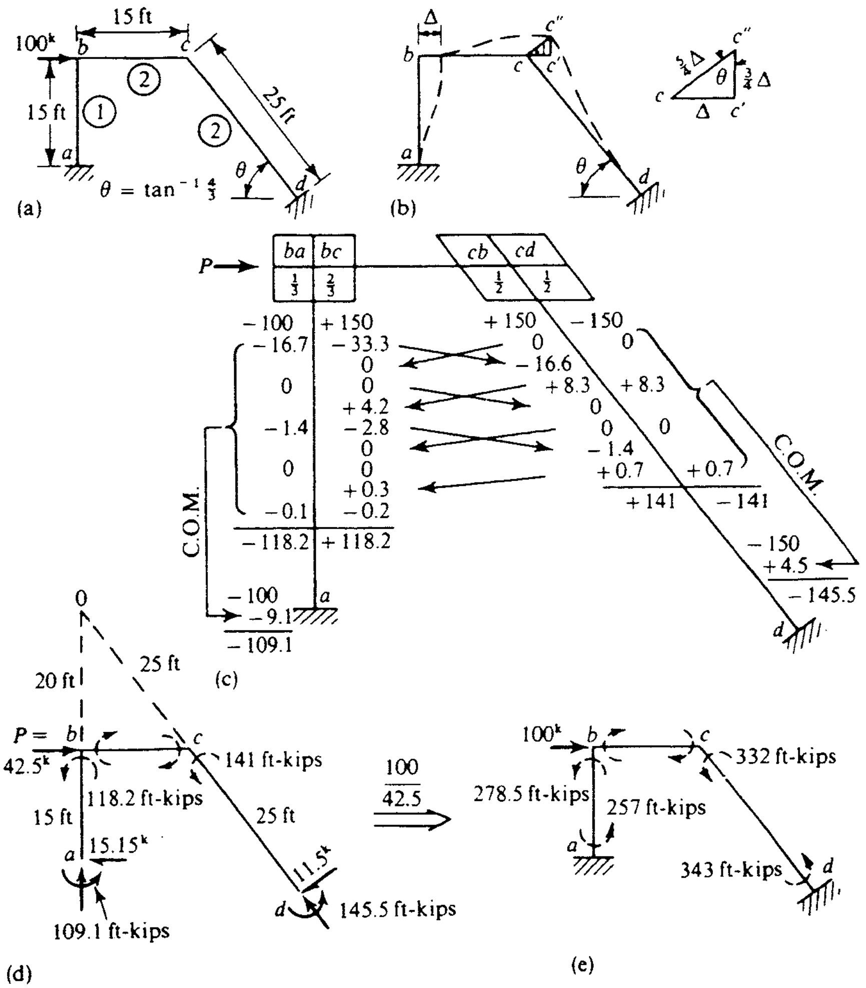


**Fig. 8-20**

find the resulting moments are consistent with a horizontal force $P = 42.5$ kips acting at $b$:

$$(15.15)(35) + (11.5)(50) - 109.1 - 145.5 - 20P = 0$$

$$P = 42.5 \text{ kips}$$

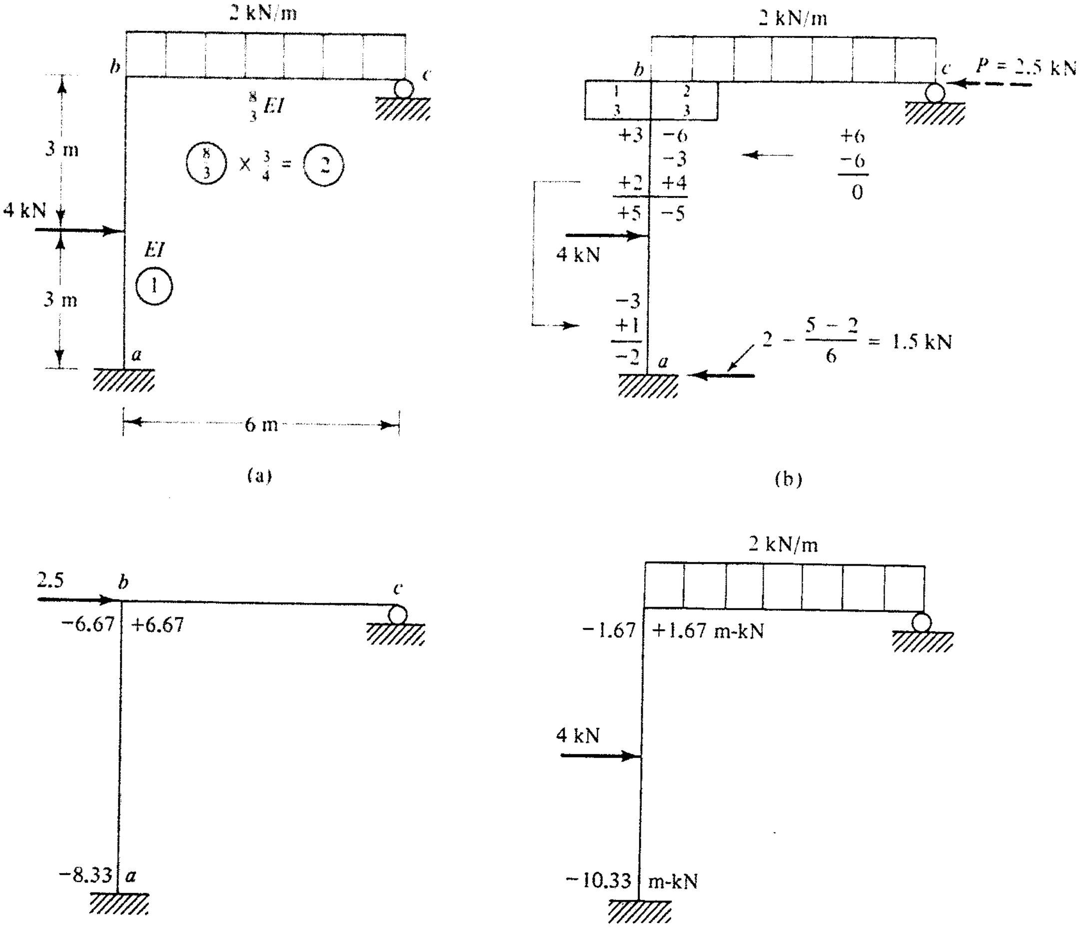
The final result, which is obtained by multiplying all the moments in Fig. 8-20(d) by the ratio 100/42.5, is shown in Fig. 8-20(e).

### Example 8-9

Determine all the end moments of the loaded frame in Fig. 8-21(a).

The complete analysis is as follows:

1. Hold the loaded frame at the top of the column, say at joint $c$, against sidesway,



(a) (b)

(c) (d)

Fig. 8-21

and obtain the end moments by the usual moment-distribution procedures [Fig. 8-21(b)].

2. From $\Sigma F_x = 0$ for the entire frame, calculate the holding force needed to prevent sidesway. In this case, it is

$$4 - 1.5 = 2.5 \text{ kN}$$

acting to the left, as indicated in Fig. 8-21(b).

3. Remove the artificial holding force by the application of an equal and opposite force at the top of the column, and find the resulting end moments [see Fig. 8-21(c)].

4. Add the end moments from steps 1 and 3 to obtain the final solution [see Fig. 8-21(d)].

Note that the analysis for step 3 [Fig. 8-21(c)] can easily be accomplished by taking advantage of the result shown in Fig. 8-19.

The technique employed in the preceding examples can be used in analyzing any frame having one degree of freedom of joint translation.

## 8-10 ANALYSIS OF STATICALLY INDETERMINATE RIGID FRAMES WITH TWO DEGREES OF FREEDOM OF JOINT TRANSLATION BY MOMENT DISTRIBUTION

A rigid frame having two degrees of freedom of joint translation can be analyzed by breaking it down into two independent cases in each of which only one degree of freedom of joint translation is allowed to occur.

Consider the two-story frame in Fig. 8-22(a). To handle it, let us refer to two separate cases, as shown in Fig. 8-22(b) and (c), each involving only one degree of freedom of joint translation. Each of these cases can be analyzed by the method of moment distribution previously described. For example, in the case in Fig. 8-22(b), joints $c$ and $d$ are held from translation by providing a lateral support at $d$, whereas joints $b$ and $e$ are displaced horizontally because of a force $X_1$ applied laterally at $b$. A moment-distribution solution can then be obtained, and all the end moments, shearing forces, and reactions are in terms of $X_1$. Let $mX_1$ denote the corresponding reaction of the lateral support at $d$, $m$ being a constant of proportionality. Note that the deflected shape (dashed line) indicates the initial position of the frame where the joint displacement of $b$ and $e$ has been introduced with all joints held against rotation. The final elastic curve after the joints have been released is not shown.

A similar solution can be carried out for the case in Fig. 8-22(c), where the pushing force is $X_2$ and the reaction of the lateral support at $e$ is $nX_2$, $n$ being a constant of proportionality.

The superposition of effects in the cases shown in Fig. 8-22(b) and (c) results in the case shown in Fig. 8-22(d), in which all the internal forces and reactions can be found as the linear combination of $X_1$ and $X_2$. Comparing Fig.

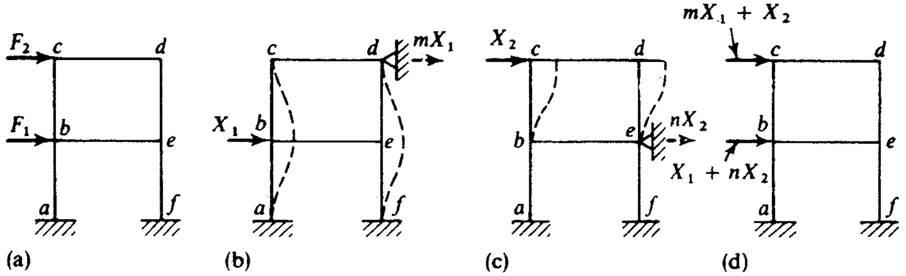


Fig. 8-22

8-22(d) with Fig. 8-22(a), we see that (d) will be the solution of (a) by solving $X_1$ and $X_2$ from

$$X_1 + nX_2 = F_1 \tag{8-23}$$

$$mX_1 + X_2 = F_2 \tag{8-24}$$

A similar procedure can be used to analyze the two-stage bent shown in Fig. 8-23(a) or the gable bent shown in Fig. 8-23(b).

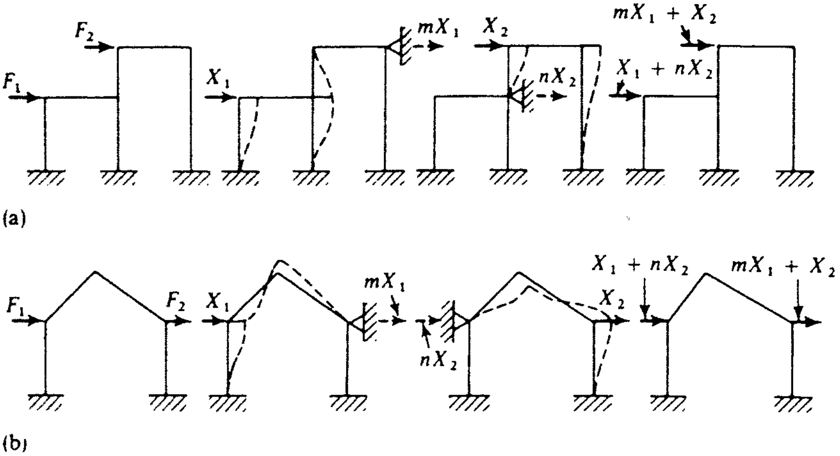


Fig. 8-23

**Example 8-10**

Find all the end moments for the frame in Fig. 8-24. Assume the same *EI* for all members. This is the same problem as Example 7-8, which was solved by the method of slope deflection.

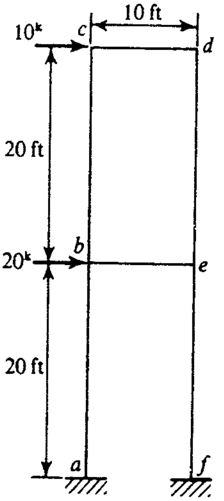


Fig. 8-24

To determine all the end moments by the method of moment distribution, we begin with the analysis of the frame in Fig. 8-25(a). The antisymmetry of the system permits us to use modified stiffnesses for members *be* and *cd* and to work with only half the structure while balancing moments, as shown in Fig. 8-25(b). In this case joints *c* and *d* are held from translation by the lateral support at *d*, and joints *b* and *e* are displaced horizontally an equal distance because of the lateral action applied at *b*. Consistent with the joint translation and the property of structure, we may assume the fixed-end moments for the columns as follows:

$$M^F_{ab} = M^F_{ba} = M^F_{ef} = M^F_{fe} = -100$$

$$M^F_{bc} = M^F_{cb} = M^F_{de} = M^F_{ed} = +100$$

After releasing the joints and balancing the moments, we obtain a set of end moments [see Fig. 8-25(b)] consistent with a pushing force of 36.13 applied at *b* and a lateral reaction of 16.5 acting to the left at *d* [see Fig. 8-25(c)]. Note that the reaction at *d* is first obtained from the balance of shear of the upper story (i.e., from $\Sigma F_x = 0$ for the portion of frame just above the level *be*). The pushing force at *b* is then determined by applying $\Sigma F_x = 0$ for the entire frame. Multiplying the result of Fig. 8-25(c) by the ratio $X_1/36.13$ gives the solution of Fig. 8-25(a), as shown in Fig. 8-25(d).

Next, let us follow the same procedure and analyze the frame in Fig. 8-26(a), in which joints *b* and *e* are held from translation by the lateral support at *e*, whereas joints *c* and *d* are displaced horizontally an equal distance due to a pushing force applied at *c*. Consistent with the sidesway and the property of structure, the fixed-end moments for the columns *bc* and *de* may be assumed to be

$$M^F_{bc} = M^F_{cb} = M^F_{de} = M^F_{ed} = -100$$

Taking advantage of antisymmetry by using modified stiffnesses in members *be* and *cd* and working with only half the structure, we complete the process of moment distribution as shown in Fig. 8-26(b). The resulting end moments are found to be consistent with a pushing force of 13.75 applied at *c* and a lateral reaction of 16.38 acting to the left at *e* [see Fig. 8-26(c)]. In this case the force at *c* is first obtained from the balance of shear of the upper story, and the reaction at *e* is then determined from $\Sigma F_x = 0$ for the entire frame.

Multiplying the result of Fig. 8-26(c) by the ratio $X_2/13.75$ gives the solution for the frame in Fig. 8-26(a), as shown in Fig. 8-26(d).

Let us now sum up the findings in the preceding two steps [Figs. 8-25(d) and 8-26(d)], as shown in Fig. 8-27(a). Moments and forces are taken positive following the indicated directions.

For solving the frame in Fig. 8-24, we set

$$X_1 - 1.19X_2 = 20 \tag{8-25}$$

$$X_2 - 0.455X_1 = 10 \tag{8-26}$$

from which

$$X_1 = 69.55$$

$$X_2 = 41.64$$

Substituting these in Fig. 8-27(a) yields the desired solution, as shown in Fig. 8-27(b).

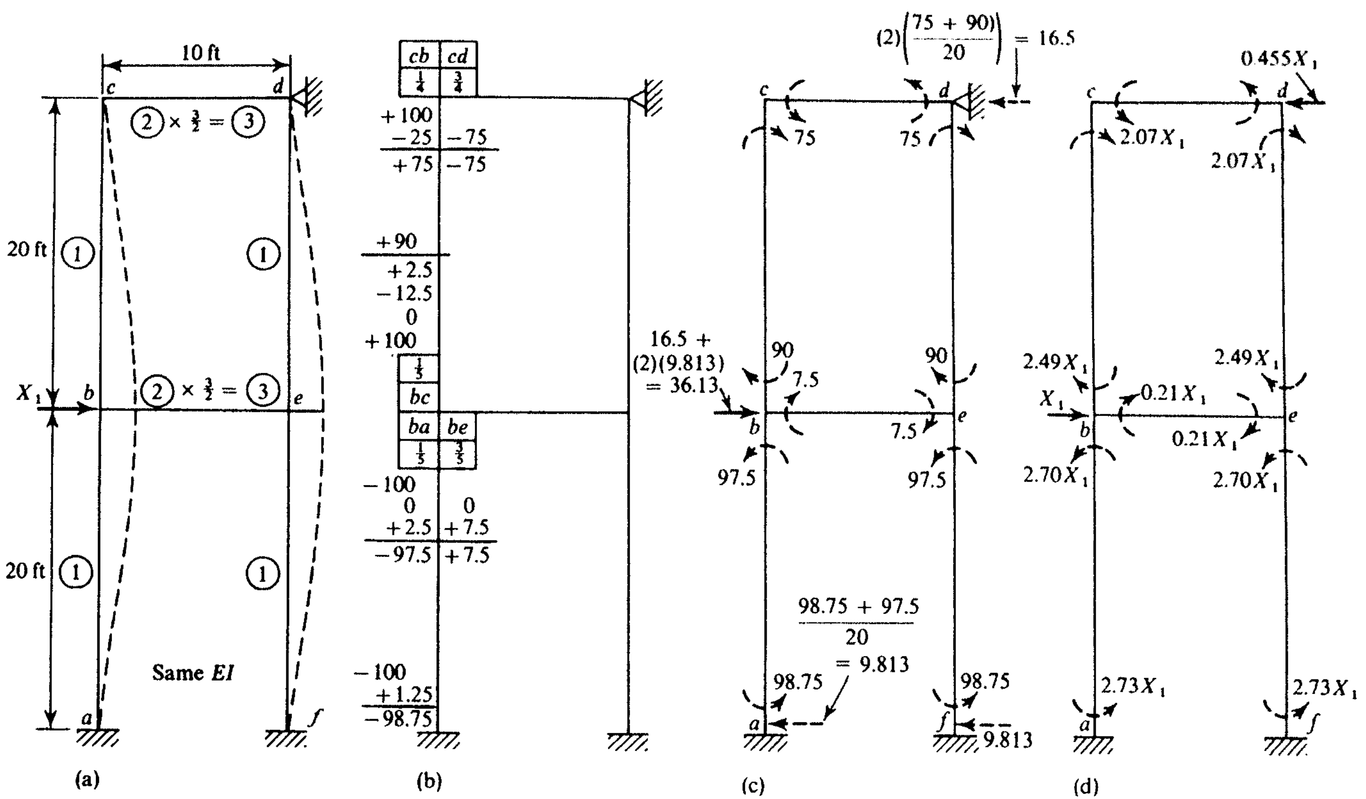

10 ft
c
d
② × 3/2 = ③
20 ft
①
①
X₁
b
② × 3/2 = ③
e
20 ft
①
①
Same EI
a
f
(a)
cb
cd
1/4
3/4
+100
−25
+75
−75
−75
+90
+2.5
−12.5
0
+100
1/5
bc
ba
be
1/5
3/5
−100
0
+2.5
−97.5
0
+7.5
+7.5
−100
+1.25
−98.75
(b)
(2)((75 + 90)/20) = 16.5
c
d
75
75
16.5 + (2)(9.813) = 36.13
90
7.5
90
7.5
b
e
97.5
97.5
(98.75 + 97.5)/20 = 9.813
98.75
98.75
a
f
9.813
(c)
0.455X₁
c
d
2.07X₁
2.07X₁
2.49X₁
2.49X₁
X₁
0.21X₁
0.21X₁
b
e
2.70X₁
2.70X₁
2.73X₁
2.73X₁
a
f
(d)

Fig. 8-25

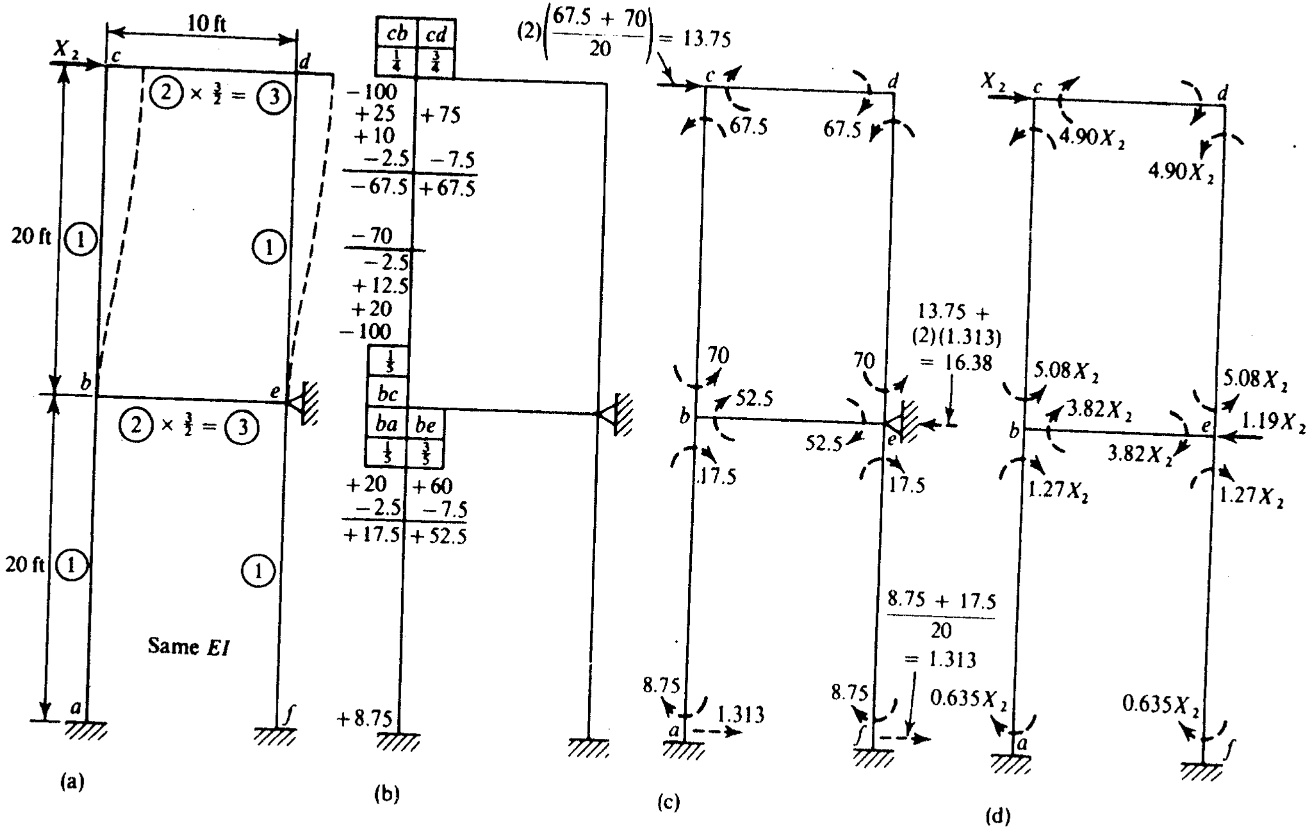

10 ft
20 ft
20 ft
$X_2$
a
b
c
d
e
f
② × 3/2 = ③
①
Same EI
cb
cd
1/4
3/4
−100
+25
+10
−2.5
−67.5
+75
−7.5
+67.5
−70
−2.5
+12.5
+20
−100
1/3
bc
ba
be
+20
−2.5
+17.5
+60
−7.5
+52.5
+8.75
(2)((67.5 + 70)/20) = 13.75
67.5
70
52.5
17.5
8.75
1.313
13.75 + (2)(1.313) = 16.38
(8.75 + 17.5)/20 = 1.313
$4.90X_2$
$5.08X_2$
$3.82X_2$
$1.19X_2$
$1.27X_2$
$0.635X_2$
(a)
(b)
(c)
(d)

Fig. 8-26

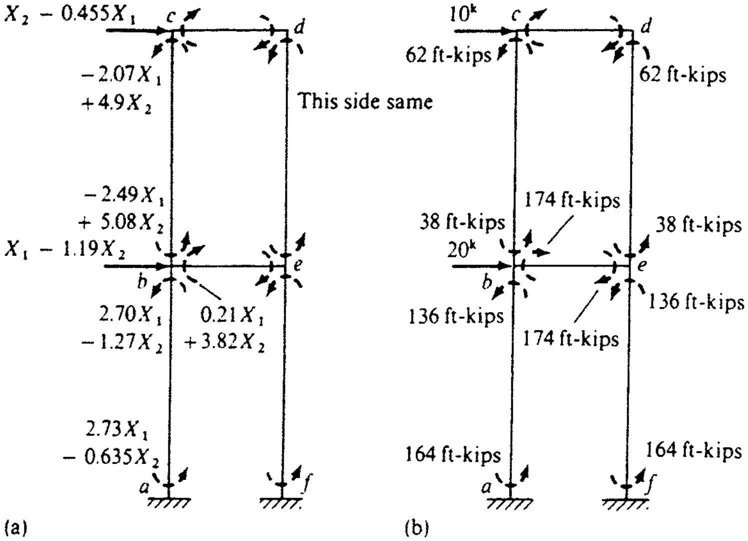


Fig. 8-27

The small discrepancy between the foregoing result and the more exact solution obtained by the slope-deflection method (see Example 7-8) is due to the fact that we complete the moment distribution in each of the foregoing cases with only two cycles. However, two cycles have provided sufficient accuracy for practical purposes.

**Example 8-11**

Find all the end moments for the frame in Fig. 8-28(a). Assume the same *EI* for all members.

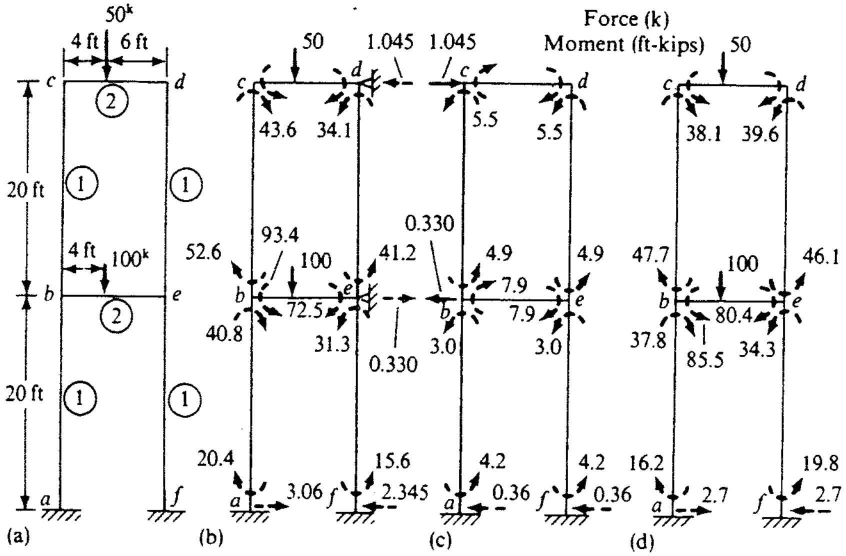


Fig. 8-28

The analysis is as follows:

1. The frame is completely prevented from sidesway by introducing lateral supports at joints *d* and *e*. Following the usual procedures for moment distribution, we determine all the end moments and find that they are consistent with a lateral holding force of 1.045 kips acting to the left at *d* and a lateral holding force of 0.330 kip acting to the right at *e*. The results are recorded, as in Fig. 8-28(b), by the dashed symbols. (See also Fig. 8-29 for the process of moment distribution in this step.)

2. Eliminate these artificial restraints by applying a set of equal and opposite forces to the corresponding joints, as shown in Fig. 8-28(c). To analyze it, we take advantage of the results of the preceding example [see Fig. 8-27(a)] and set

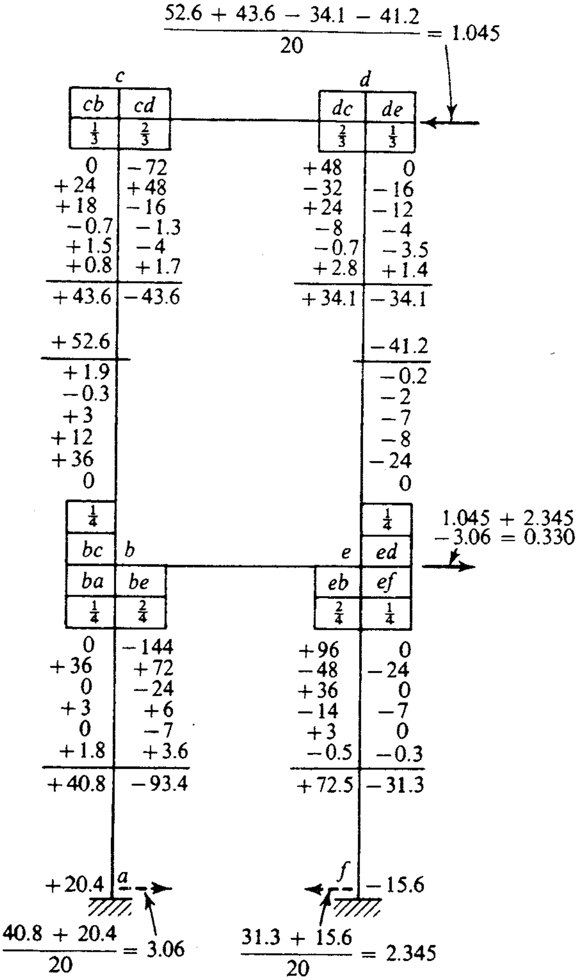


Fig. 8-29

$$X_1 - 1.190X_2 = -0.330 \tag{8-27}$$

$$-0.455X_1 + X_2 = 1.045 \tag{8-28}$$

and solve for $X_1 = 1.99 \qquad X_2 = 1.95$

Substituting these in Fig. 8-27(a) yields the result recorded in Fig. 8-28(c) by the dashed symbols.

3. The superposition of steps 1 and 2 will give the final solution of the moments, as in Fig. 8-28(d).

## 8-11 ANALYSIS OF STATICALLY INDETERMINATE RIGID FRAMES WITH SEVERAL DEGREES OF FREEDOM OF JOINT TRANSLATION BY MOMENT DISTRIBUTION

The procedures of moment distribution used in the analysis of frames having two degrees of joint translation can be extended to the analysis of frames having multiple degrees of freedom with respect to sidesway. For a frame with $n$ degrees of freedom of joint translation, the moment-distribution solution may be broken down into $(n + 1)$ separate cases including:

1. A case in which the frame is held from joint translation by introducing $n$ artificial supports
2. $n$ independent cases in each of which only one degree of freedom of joint translation is allowed to occur by maintaining $(n - 1)$ holding forces at other points where sidesway would take place

The superposition of these $(n + 1)$ cases will give the final solution provided that the artificial holding forces in step 1 are all eliminated by applying the results obtained in step 2. This necessitates the solution of $n$ simultaneous equations.

To illustrate this procedure, consider the four-story, two-bay building bent, which has four degrees of freedom of joint translation, shown in Fig. 8-30(a). The first step is to introduce artificial lateral supports at all the floors, as in Fig.

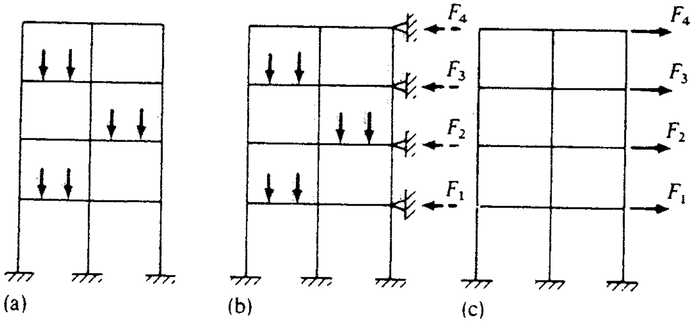


**Fig. 8-30**

8-30(b), to prevent the frame from sidesway. All the loads are then applied, and a regular moment distribution without joint translation is performed to obtain all the end moments from which the reactions (the holding forces) at these supports can be found. We let $F_1$, $F_2$, $F_3$, and $F_4$ denote these forces.

The next step is to eliminate all these artificial forces by applying a set of equal and opposite forces at the corresponding points, as shown in Fig. 8-30(c). The sum of the two steps of Fig. 8-30(b) and (c) will give the final solution.

However, to obtain the solution for the frame in Fig. 8-30(c) would require the complete analysis of four independent cases, as shown in Fig. 8-31. In each of the four cases, only one degree of freedom of sidesway is involved. We have, therefore, no difficulty in determining all the end moments and the reactions of the frame in terms of the lateral force applied to the floor level where the sidesway is not inhibited. For instance, in Fig. 8-31(a) a lateral force $X_1$ is applied to the first floor level which deflects sidewise. Note that the deflected shape (dashed line) indicates only the initial position of the frame before the balancing moments. Following the procedure described in Sec. 8-9, we find the reactions of the lateral supports at the second, third, and fourth floor levels to be $r_{21}X_1$, $r_{31}X_1$, and $r_{41}X_1$, respectively, $r_{21}$, $r_{31}$, $r_{41}$ being constants of proportionality. The other cases shown in Fig. 8-31 are similarly treated. It now remains to find $X_1$, $X_2$, $X_3$, and $X_4$ by solving the four simultaneous equations,

$$X_1 + r_{12}X_2 + r_{13}X_3 + r_{14}X_4 = F_1 \tag{8-29}$$

$$r_{21}X_1 + X_2 + r_{23}X_3 + r_{24}X_4 = F_2 \tag{8-30}$$

$$r_{31}X_1 + r_{32}X_2 + X_3 + r_{34}X_4 = F_3 \tag{8-31}$$

$$r_{41}X_1 + r_{42}X_2 + r_{43}X_3 + X_4 = F_4 \tag{8-32}$$

With $X_1$, $X_2$, $X_3$, and $X_4$ determined and substituted in the cases of Fig. 8-31(a), (b), (c), and (d), respectively, we superpose the results to reach the solution for the frame in Fig. 8-30(c).

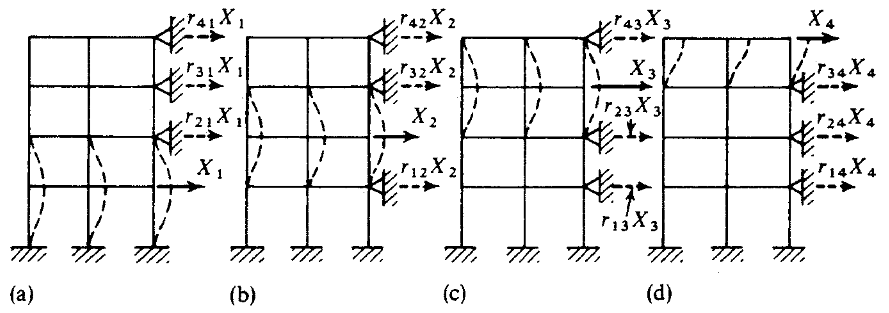


**Fig. 8-31**

## 8-12 MATRIX FORMULATION OF THE MOMENT-DISTRIBUTION PROCEDURE

The moment distribution is a cyclic process and tends to be convergent to an exact solution. This can best be illustrated by the matrix formulation for the moment-distribution process listed in Fig. 8-9.

We begin with a column matrix $A$ representing the fixed-end moments. In this case,

$$A = \left\{ \begin{matrix} M_{ab}^F \\ M_{ba}^F \\ M_{bc}^F \\ M_{cb}^F \\ M_{cd}^F \\ M_{dc}^F \end{matrix} \right\} = \left\{ \begin{matrix} -30 \\ 30 \\ -40 \\ 40 \\ -30 \\ 30 \end{matrix} \right\} \tag{8-33}$$

See the first row of Fig. 8-9.

Next, we form a distribution matrix $B$ with distribution factors as its constituents in such a way that the matrix product $BA$ (a column matrix) will give the respective distributed moments. Thus,

$$BA = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \left\{ \begin{matrix} -30 \\ 30 \\ -40 \\ 40 \\ -30 \\ 30 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ 5 \\ 5 \\ -5 \\ -5 \\ 0 \end{matrix} \right\} \tag{8-34}$$

See the second row of Fig. 8-9. Apparently, the resulting moment of the first cycle is

$$M_1 = A + BA = (I + B)A \tag{8-35}$$

in which $I$ is a unit matrix.

Next, let the matrix $C$ perform as a carry-over operator so that the matrix product $CBA$, a column matrix, will represent the respective carry-over moments. Using the result of Eq. 8-34, we have in this case

$$CBA = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \left\{ \begin{matrix} 0 \\ 5 \\ 5 \\ -5 \\ -5 \\ 0 \end{matrix} \right\} = \left\{ \begin{matrix} 2.5 \\ 0 \\ -2.5 \\ 2.5 \\ 0 \\ -2.5 \end{matrix} \right\} \tag{8-36}$$

See the third row of Fig. 8-9. The carry-over moments constitute a new set of fixed-end moments. We repeat the same process by replacing $A$ with $CBA$ in Eq. 8-35 to obtain the moments of the second cycle. Thus,

$$M_2 = (I + B)CBA \tag{8-37}$$

For simplicity, we introduce $D = CB$, and Eq. 8-37 becomes

$$M_2 = (I + B)DA \tag{8-38}$$

Similarly, we can have

$$\begin{aligned} M_3 &= (I + B)(D)^2A \\ M_4 &= (I + B)(D)^3A \qquad (8\text{-}39) \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

in which $(D)^2 = (D)(D)$, $(D)^3 = (D)(D)(D)$, and so on. (Note that at the end of first cycle, the total moment is $M_1$; at the end of second cycle, the total moment is $M_1 + M_2$, and so on.) Summing up $M_1, M_2, \ldots,$ we obtain the final resulting moment:

$$\begin{aligned} M &= (I + B)[A + DA + (D)^2A + (D)^3A + \cdots] \\ &= (I + B)\left(\frac{1}{I - D}\right)(A) \\ &= (I + B)(I - D)^{-1}(A) \qquad (8\text{-}40) \end{aligned}$$

In our case

$$I + B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8-41}$$

$$D = CB = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \end{bmatrix}$$

$$(I - D)^{-1} = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & -\frac{4}{15} & -\frac{4}{15} & \frac{1}{15} & \frac{1}{15} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{15} & \frac{16}{15} & -\frac{4}{15} & -\frac{4}{15} & 0 \\ 0 & -\frac{4}{15} & -\frac{4}{15} & \frac{16}{15} & \frac{1}{15} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{15} & \frac{1}{15} & -\frac{4}{15} & -\frac{4}{15} & 1 \end{bmatrix} \qquad (8\text{-}42)$$

Substituting Eqs. 8-33, 8-41, and 8-42 in Eq. 8-40 gives

$$M = \begin{Bmatrix} M_{ab} \\ M_{ba} \\ M_{bc} \\ M_{cb} \\ M_{cd} \\ M_{dc} \end{Bmatrix} = \begin{Bmatrix} -26.67 \\ 36.67 \\ -36.67 \\ 36.67 \\ -36.67 \\ 26.67 \end{Bmatrix} \qquad (8\text{-}43)$$

which can be checked with the result shown in the last row of Fig. 8-9.

## PROBLEMS

**8-1.** Solve the end moments for Prob. 7-1 by moment distribution.

**8-2.** Solve the end moments for Prob. 7-2 by moment distribution.

**8-3.** Solve the end moments for Prob. 7-3 by moment distribution.

**8-4.** Solve the end moments for Prob. 7-4 by moment distribution: (a) not using modified stiffness; (b) using a modified $K$ value in the center span.

**8-5.** Analyze the box shown in Fig. 8-32 by moment distribution. Assume constant $EI$.

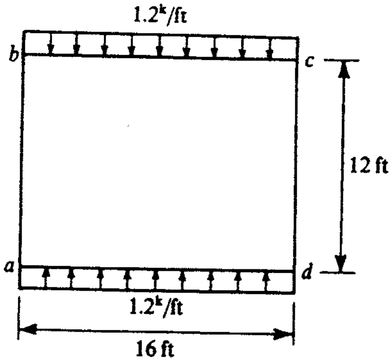


Fig. 8-32

**8-6.** Analyze the continuous beam in Fig. 8-33 by moment distribution. Assume constant *EI*.

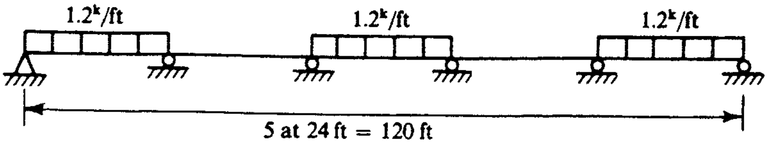


Fig. 8-33

**8-7.** Analyze the continuous beam in Fig. 8-34 by moment distribution. Take advantage of modified stiffnesses, replacing the unsymmetrical loading system with a symmetrical and an antisymmetrical system. Assume constant *EI*.

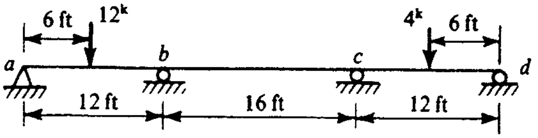


Fig. 8-34

**8-8.** Analyze the frame in Fig. 8-35 by moment distribution and find the reaction at support *c*.

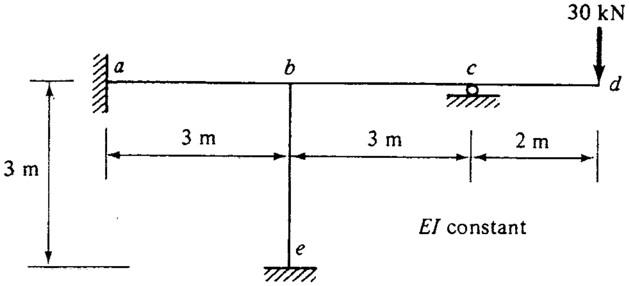


Fig. 8-35

**8-9.** With reference to Fig. 8-36, find the stiffness for end *a* of member *ab* if a hinge connection is inserted in the member at *c* as shown.

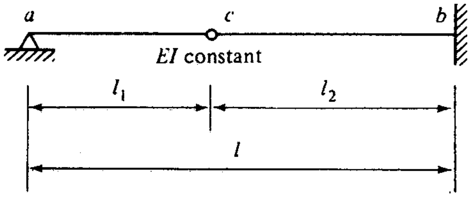


Fig. 8-36

**8-10.** Use the method of moment distribution to find the fixed-end moment and the spring force for the beam shown in Fig. 8-37.

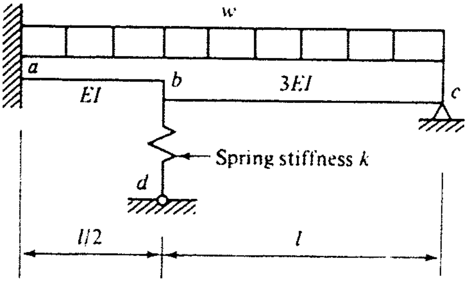
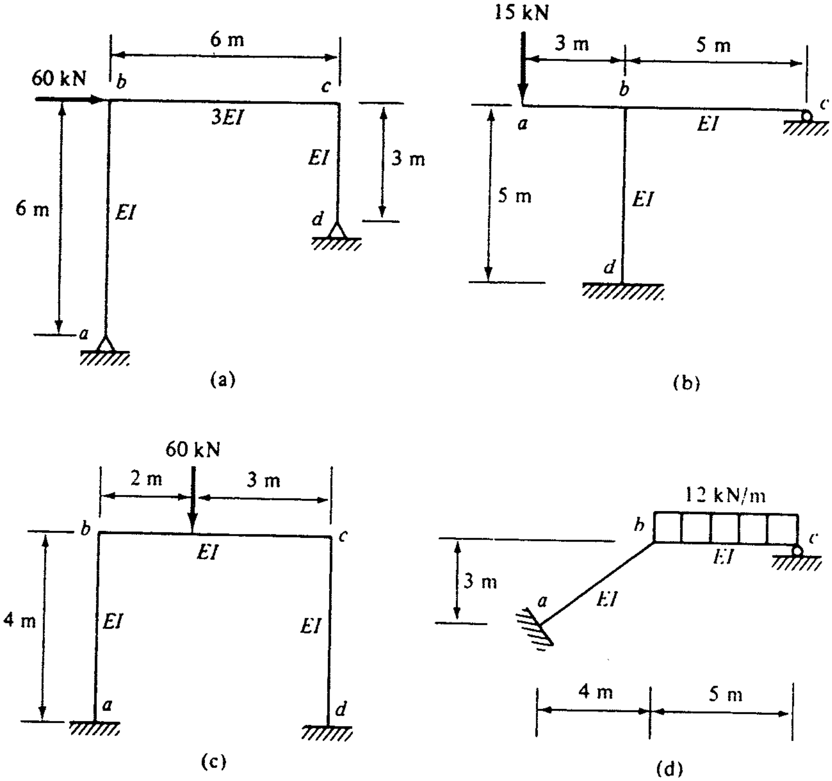


**Fig. 8-37**

**8-11.** Analyze each of the frames in Fig. 8-38 by moment distribution.



(a) (b) (c) (d)

**Fig. 8-38**

**8-12.** Use the method of moment distribution to find the fixed-end moments for each of the beams of Prob. 7-12.

**8-13.** Solve the frame in Fig. 8-39 by moment distribution.

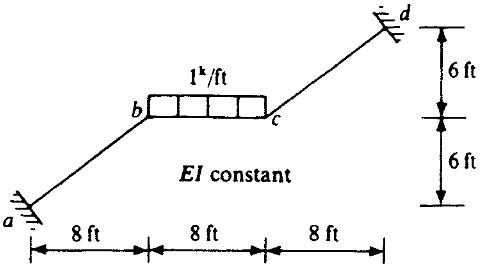


**Fig. 8-39**

**8-14.** Solve each frame of Prob. 7-11 by moment distribution.

**8-15.** Analyze the frame in Fig. 8-40 by moment distribution.

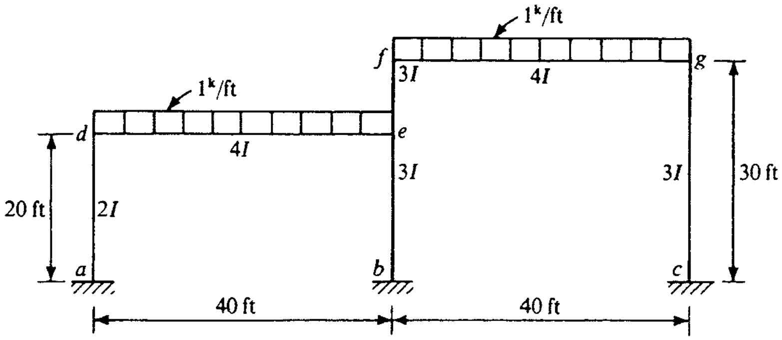


**Fig. 8-40**

**8-16.** Use the matrix formulation of moment distribution to solve Probs. 8-1 to 8-3.

# 9

# *Matrix Force Method*

## 9-1 GENERAL

During the past decades, the rapid development of computers and the growing demand for better methods of analysis for complex and lightweight structures led to the development of methods for matrix analysis of structures. The use of matrix notation in expressing structural theory is in itself simple and elegant, but the practical value of matrix analysis would not have become clear without the invention of the high-speed digital computer.

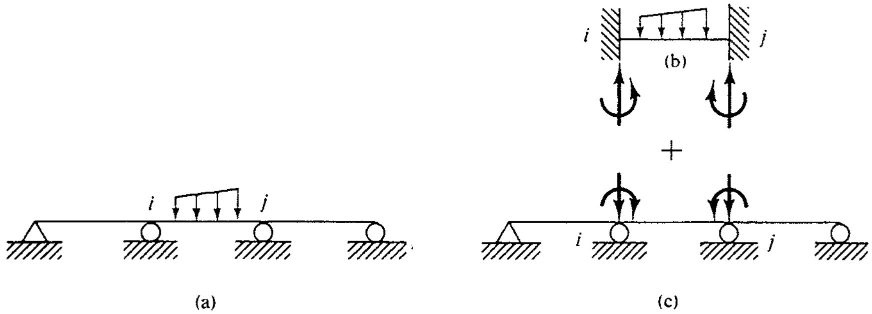
It is true that the classical methods of structural analysis, such as the method of consistent deformations and the slope-deflection method, which had only limited use in the past because of operational difficulties in solving a great number of simultaneous equations, have now regained their strength because of the advent of the digital computer. It is also true that these methods can be conveniently expressed in matrix forms. However, the matrix method, to be discussed in this and the subsequent chapter, has its unique theoretical basis and particular procedures. The idea is based on the finite-element concept that enables the step-by-step buildup of the force-displacement relationship of a structure from the basic elements of which the structure is composed.

The matrix analysis of structures commonly falls into two categories: the force method (flexibility method) and the displacement method (stiffness method). The force method treats the member forces as the basic unknowns and relates the forces to the corresponding displacements by flexibility matrices, whereas the displacement method regards the nodal displacements as the basic unknowns and relates the displacements to the corresponding forces by stiffness matrices. As will be seen, a duality exists between the two approaches. We discuss the force method in this chapter and the displacement method in Chapter 10.

## 9-2 BASIC CONCEPTS OF STRUCTURES

Structures such as trusses, beams, and rigid frames are defined as assemblages of structural elements jointed together at a finite number of discrete points, called *nodes* or *nodal points,* and loaded only at these points. The term node is used instead of joint because often the application point of a concentrated load, not the conventional joint, is taken as a node.

Distributed loads or any other type of member loads acting between the nodes can be replaced by equivalent loads at the nodes. To illustrate, let us consider the three-span continuous beam shown in Fig. 9-1(a), whose center span is subjected to a distributed load. Let us first fix artificially the joints *i* and *j* of the center span [Fig. 9-1(b)] and then release them [Fig. 9-1(c)]. Since the application and removal of the artificial forces are neutralized, the original configuration of Fig. 9-1(a) is therefore statically equivalent to the combined effects of Fig. 9-1(b) and (c). The equivalent nodal loads at *i* and *j* in Fig. 9-1(c) are the reverse of fixed-end actions (moments and shears) in Fig. 9-1(b). Note that the final forces and displacements in the loaded member *i–j* must be obtained by superposing the effects of the fixed-end beam and those resulting from the nodal-force analysis of the original structure.



(a) (c)

Fig. 9-1

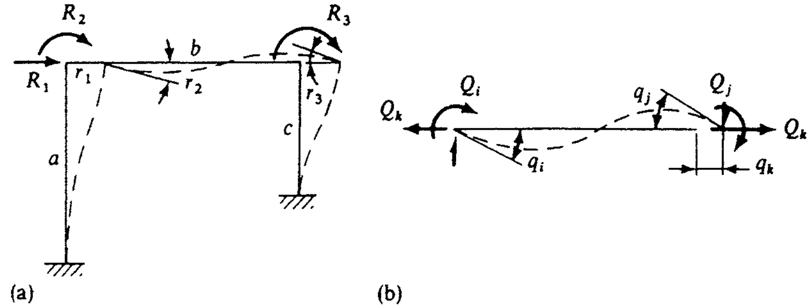
Any complicated structure can be cut into simpler components. For instance, a truss may be considered as composed of many two-force members pin-connected at their ends. A rigid frame may be taken as a composition of a number of three-force members or other convenient units. The behavior of the subdivided elements, as well as the whole structure, must satisfy the following basic conditions:

1. Equilibrium of forces
2. Compatibility of displacements
3. Force-displacement relationship specified by the geometric and elastic properties of the elements

These conditions are generally required by a linear structure no matter what method is used.

In the systematic analysis of structures, it is essential to introduce the notation used for force and displacement in this and Chapter 10. Consider the frame in Fig. 9-2(a), which is composed of three elements (members *a*, *b*, and *c*) subjected to external loads (nodal forces) denoted by $R_1$, $R_2$, and $R_3$ with the corresponding nodal displacements $r_1$, $r_2$, and $r_3$, respectively.

A typical member element is shown in Fig. 9-2(b) and is generally subjected to the internal forces of the end moments, denoted by $Q_i$ and $Q_j$, and the axial forces, denoted by $Q_k$. The member is also subjected to end shears; however, the end shears can be expressed in terms of the end moments and therefore are not considered independent forces. Associated with the end moments and axial forces are the end rotations $q_i$ and $q_j$ and the axial elongation $q_k$. The signs shown in Fig. 9-2 are considered as positive. A superscript is used for these quantities to identify their belonging to a particular member. For instance, $Q_i^a$, $Q_j^a$, and $Q_k^a$ indicate the internal forces for member *a*, and $q_i^a$, $q_j^a$, and $q_k^a$ their corresponding deformations.



(a) (b)

**Fig. 9-2**

In matrix representation, we use a column matrix $R$,

$$R = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

to denote all the nodal forces, and a column matrix $r$,

$$r = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

to denote the corresponding nodal displacements.

Also, we use a column matrix $Q$,

$$Q = \begin{Bmatrix} Q_i^a \\ Q_j^a \\ \cdot \\ \cdot \\ \cdot \\ Q_i^b \\ Q_j^b \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

to denote all the internal end forces for members $a$, $b$, . . . , and a column matrix $q$,

$$q = \begin{Bmatrix} q_i^a \\ q_j^a \\ \cdot \\ \cdot \\ \cdot \\ q_i^b \\ q_j^b \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

to denote the corresponding internal displacements.

Frequently involved in the subsequent discussion is the principle of virtual work, which serves in many cases as an effective substitute for the equations of equilibrium or compatibility. The principle simply states that for an elastic structure in equilibirum, the external virtual work is equal to the internal virtual work (virtual strain energy). The virtual work may be the result of either a virtual displacement or a virtual force. Thus, if virtual displacements are used, we have

$$\delta r^T R = \delta q^T Q \tag{9-1}$$

On the other hand, if virtual forces are used,

$$\delta R^T r = \delta Q^T q \tag{9-2}$$

## 9-3 EQUILIBRIUM, FORCE TRANSFORMATION MATRIX

For a statically determinate structure, each of the member forces may be expressed in terms of the external nodal loads by using the equilibrium conditions of the system alone. Thus,

$$Q_1 = b_{11}R_1 + b_{12}R_2 + \dots + b_{1n}R_n$$

$$
\begin{aligned}
Q_2 &= b_{21}R_1 + b_{22}R_2 + \dots + b_{2n}R_n \\
&\ \ \vdots \\
Q_m &= b_{m1}R_1 + b_{m2}R_2 + \dots + b_{mn}R_n
\end{aligned}
\tag{9-3}
$$

in which $Q_1 = Q_i^a$, $Q_2 = Q_j^a$, . . . . Observe that $R_1, R_2, \dots, R_n$ represent the total set of applied loads and $Q_1, Q_2, \dots, Q_m$ the total set of member forces. No connection between the subscripts on $R$ and $Q$ is implied.

The matrix form for Eq. 9-3 is

$$Q = bR \tag{9-4}$$

where

$$
b = \begin{bmatrix}
b_{11} & b_{12} & \dots & b_{1n} \\
b_{21} & b_{22} & \dots & b_{2n} \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
b_{m1} & b_{m2} & \dots & b_{mn}
\end{bmatrix}
\tag{9-5}
$$

is called the *force transformation matrix*, which relates the internal forces to the external forces. Matrix $b$ is usually a rectangular matrix in which the typical element $b_{ij}$ is the value of the internal force component $Q_i$ caused by a unit value of external load $R_j$. Note that $b$ is merely an expression of equilibrium for the system.

As for a statically indeterminate structure, the internal member forces cannot be expressed in terms of the external loads by equilibrium alone. However, as previously stated (see Sec. 6-1), a statically indeterminate structure can be made determinate by removing the redundant elements. The statically determinate and stable structure that remains after the removal of the extra restraints is called a *primary structure*. We then consider the original structure as equivalent to the primary structure subjected to the combined influences of the applied loads and the unknown redundant forces, thereby treating the redundants as a part of the external loads of unknown magnitude. In this way, we can express member forces in terms of the original applied loads $R$ and the redundant forces $X$ as

$$Q = \underset{\text{original}}{b_R R} + \underset{\text{redundant forces}}{b_X X} \tag{9-6}$$

or

$$Q = [b_R | b_X] \begin{Bmatrix} R \\ \hline X \end{Bmatrix} \tag{9-7}$$

where $b_R$ and $b_X$ are force transformation matrices representing the separate influences of the known applied loads $R$ and the unknown redundants $X$ on the member forces. They are geneally rectangular matrices.

## 9-4 COMPATIBILITY

Compatibility is a continuity condition on the displacements of the structure after the external loads are applied to the structure. Compatibility must be brought into the analysis of statically indeterminate structures since the equilibrium equations alone do not suffice to solve the problem.

If we let $r_X$ denote the prescribed displacement matrix corresponding to the redundant force matrix $X$, the compatibility conditions used in the force method for solving a static structure are that the displacements at all the cuts of redundant points caused by the original applied loads, and the redundant forces must be made to be equal to $r_X$ in order that the continuity of the structure can be maintained. For a loaded structure mounted on rigid supports, the gap in the displacements at redundant points resulting from applied loads is precisely removed by the redundant forces. Therefore, the compatibility condition is

$$r_X = 0 \tag{9-8}$$

## 9-5 FORCE-DISPLACEMENT RELATIONSHIP, FLEXIBILITY COEFFICIENT, FLEXIBILITY MATRIX

A *flexibility coefficient* $f_{ij}$ is the displacement at point $i$ due to a unit action at point $j$, all other points being unloaded. Apparently, the flexibility coefficient constitutes a relationship between deformation and force. Applying the principle of superposition, we may express the deformation at any point of a system caused by a set of forces in terms of the flexibility coefficients.

Our intention is, first of all, to establish the relationship between the member displacements and the member forces of a structure. Consider a typical member $a$ taken from a plane structure as shown in Fig. 9-3. As before, the member forces are represented by a column matrix $Q^a$,

$$Q^a = \begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_k^a \end{Bmatrix}$$

and the corresponding member deformations are represented by a column matrix $q^a$,

$$q^a = \begin{Bmatrix} q_i^a \\ q_j^a \\ q_k^a \end{Bmatrix}$$

Fig. 9-3

Note that the clockwise end moments and rotations and the tensile axial forces and elongation are considered as positive.

Using the flexibility coefficient $f^a_{ij}$, we may express each of the member deformations in terms of the separate influences of the whole set of member forces:

$$\begin{aligned} q^a_i &= f^a_{ii}Q^a_i + f^a_{ij}Q^a_j + f^a_{ik}Q^a_k \\ q^a_j &= f^a_{ji}Q^a_i + f^a_{jj}Q^a_j + f^a_{jk}Q^a_k \\ q^a_k &= f^a_{ki}Q^a_i + f^a_{kj}Q^a_j + f^a_{kk}Q^a_k \end{aligned} \tag{9-9}$$

or in matrix form,

$$q^a = f^aQ^a \tag{9-10}$$

in which

$$f^a = \begin{bmatrix} f^a_{ii} & f^a_{ij} & f^a_{ik} \\ f^a_{ji} & f^a_{jj} & f^a_{jk} \\ f^a_{ki} & f^a_{kj} & f^a_{kk} \end{bmatrix} \tag{9-11}$$

is defined as the *element flexibility matrix*. Clearly, the coefficient, for instance $f^a_{ii}$, is given by

$$f^a_{ii} = q^a_i \quad \text{as} \quad Q^a_i = 1 \qquad Q^a_j = Q^a_k = 0$$

The rest can similarly be defined.

The descriptions above refer to an individual element. For a structure consisting of $a$, $b$, . . . elements, we have

$$\begin{aligned} q^a &= f^aQ^a \\ q^b &= f^bQ^b \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

Let

$$q = \begin{Bmatrix} q^a \\ q^b \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \quad \text{and} \quad Q = \begin{Bmatrix} Q^a \\ Q^b \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

These equations can be put in the matrix form

$$q = fQ \tag{9-12}$$

where

$$f = \begin{bmatrix} f^a & & & & \\ & f^b & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \tag{9-13}$$

which is a diagonal matrix with element flexibility matrices as its constituents.

Since the flexibility coefficients of Eq. 9-11 serve to relate the member deformations to the member forces, they are certainly governed by the geometric and material properties of the member. Suppose that the member is prismatic with length $L$, cross-sectional area $A$, moment of inertia $I$, and modulus of elasticity $E$ and regarded as simply supported. The elements in the first column of $f^a$ are, by definition, the member deformation resulting from $Q_i^a = 1$. These are found to be

$$f_{ii}^a = \text{rotation of the left end} = \frac{L}{3EI}$$

$$f_{ji}^a = \text{rotation at the right end} = -\frac{L}{6EI}$$

$$f_{ki}^a = \text{elongation of the member} = 0$$

Note that $f_{ii}^a$ and $f_{ji}^a$ can easily be determined by the conjugate-beam method and that $f_{ki}^a = 0$ is apparent. All the other elements can be obtained similarly. Thus, the member flexibility matrix is given by

$$f^a = \begin{bmatrix} \frac{L}{3EI} & -\frac{L}{6EI} & 0 \\ -\frac{L}{6EI} & \frac{L}{3EI} & 0 \\ 0 & 0 & \frac{L}{AE} \end{bmatrix} \tag{9-14}$$

Note that the member flexibility matrix is symmetric because of reciprocity.

If the effect of axial forces in the member is neglected, as is usually done in rigid-frame analysis, then

$$f^a = \frac{L}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{9-15}$$

For a truss member subjected to axial forces only,

$$f^a = \left[\frac{L}{AE}\right] \tag{9-16}$$

We have already established the relationship between the member deformations and the member forces (the internal relationship). Next we must establish the relationship between the nodal displacements $r$ and the nodal forces $R$ (the external relationship). This may be accomplished by using the technique of virtual work. We start by Eq. 9-2:

$$\delta R^T r = \delta Q^T q \tag{9-17}$$

Based on Eqs. 9-4 and 9-12, namely,

$$Q = bR \qquad \text{and} \qquad q = fQ$$

we have

$$\delta Q = b\,\delta R$$

or

$$\delta Q^T = \delta R^T b^T \tag{9-18}$$

and
$$q = fbR \tag{9-19}$$

Substituting Eqs. 9-18 and 9-19 in Eq. 9-17 yields

$$\delta R^T r = \delta R^T b^T fbR$$

from which
$$r = b^T fbR \tag{9-20}$$

If we let

$$F = b^T fb \tag{9-21}$$

$F$ being called the *total flexibility matrix* or *flexibility matrix of the structure*, then

$$r = FR \tag{9-22}$$

For a statically determinate structure Eq. 9-22 gives a direct solution of all the nodal displacements in terms of the external nodal forces.

If the structure is statically indeterminate, then the external work must include the work caused by the redundant forces; also, Eq. 9-6 must be used instead of Eq. 9-4. To relate the displacements to the corresponding forces, we begin the derivation by

$$\delta R^T r_R + \delta X^T r_X = \delta Q^T q \tag{9-23}$$

where $r_R$ and $r_X$ are displacements corresponding to nodal forces $R$ and redundant forces $X$, respectively. Using equilibrium Eq. 9-6 and virtual force, we have

$$\delta Q = b_R \delta R + b_X \delta X$$

so that
$$\delta Q^T = \delta R^T b_R^T + \delta X^T b_X^T \tag{9-24}$$

Also, because of Eqs. 9-12 and 9-6,

$$q = fQ = fb_R R + fb_X X \tag{9-25}$$

Substituting Eqs. 9-24 and 9-25 in Eq. 9-23 yields

$$\delta R^T r_R + \delta X^T r_X = (\delta R^T b_R^T + \delta X^T b_X^T)(fb_R R + fb_X X)$$

or
$$\delta R^T r_R + \delta X^T r_X = \delta R^T (b_R^T fb_R R + b_R^T fb_X X) + \delta X^T (b_X^T fb_R R + b_X^T fb_X X)$$

Comparing the virtual forces on the left and the right sides of the preceding equation, we have

$$r_R = b_R^T fb_R R + b_R^T fb_X X$$
$$r_X = b_X^T fb_R R + b_X^T fb_X X$$

These may be arranged as

$$r_R = F_{RR} R + F_{RX} X \tag{9-26}$$
$$r_X = F_{XR} R + F_{XX} X \tag{9-27}$$

if we let
$$\begin{aligned} F_{RR} &= b_R^T fb_R \qquad F_{RX} = b_R^T fb_X \\ F_{XR} &= b_X^T fb_R \qquad F_{XX} = b_X^T fb_X \end{aligned} \tag{9-28}$$

Equations 9-26 and 9-27 may be put in matrix form as

$$\begin{Bmatrix} r_R \\ \hdashline r_X \end{Bmatrix} = \left[\begin{array}{c:c} F_{RR} & F_{RX} \\ \hdashline F_{XR} & F_{XX} \end{array}\right] \begin{Bmatrix} R \\ \hdashline X \end{Bmatrix} \tag{9-29}$$

For structures on rigid supports Eq. 9-29 becomes

$$\begin{Bmatrix} r_R \\ \hdashline r_X = 0 \end{Bmatrix} = \left[\begin{array}{c:c} F_{RR} & F_{RX} \\ \hdashline F_{XR} & F_{XX} \end{array}\right] \begin{Bmatrix} R \\ \hdashline X \end{Bmatrix} \tag{9-30}$$

The compatibility condition is, therefore,

$$F_{XR}R + F_{XX}X = 0 \tag{9-31}$$

from which

$$X = -F_{XX}^{-1}F_{XR}R \tag{9-32}$$

Equation 9-32 expresses the solution for the redundants.

Substituting Eq. 9-32 in Eq. 9-26, we finally relate the unknown nodal displacements to the corresponding applied nodal forces, in a statically indeterminate structure, covering the effects of the redundant forces.

$$r_R = (F_{RR} - F_{RX}F_{XX}^{-1}F_{XR})R$$

or simply

$$r_R = F'R \tag{9-33}$$

if we let

$$F' = F_{RR} - F_{RX}F_{XX}^{-1}F_{XR} \tag{9-34}$$

$F'$ being the *flexibility matrix of the indeterminate structure.*

With the redundants $X$ found to be $-F_{XX}^{-1}F_{XR}R$, the member forces are then solved by equilibrium:

$$Q = b_R R + b_X X = (b_R - b_X F_{XX}^{-1}F_{XR})R$$

That is,

$$Q = b'R \tag{9-35}$$

if we let

$$b' = b_R - b_X F_{XX}^{-1}F_{XR} \tag{9-36}$$

$b'$ being the *force transformation matrix of the indeterminate structure,* relating directly the member forces to the applied nodal loads covering the effects of the redundants.

An alternative from of $F'$ may be obtained in terms of $b'$:

$$F' = b_R^T f b' \tag{9-37}$$

since

$$F' = F_{RR} - F_{RX}F_{XX}^{-1}F_{XR}$$
$$= b_R^T f b_R - b_R^T f b_X F_{XX}^{-1}F_{XR} = b_R^T f(b_R - b_X F_{XX}^{-1}F_{XR})$$

which yields Eq. 9-37. Note that Eq. 9-34 for finding $F'$ is quite general but that the alternative form given by Eq. 9-37 is more convenient if $b'$ is first determined.

The following identity is useful for checking results:

$$b_X^T f b' = 0 \tag{9-38}$$

This can easily be proved as follows:

$$b_X^T f b' = b_X^T f (b_R - b_X F_{XX}^{-1} F_{XR}) = F_{XR} - F_{XX} F_{XX}^{-1} F_{XR}$$
$$= F_{XR} - F_{XR} = 0$$

## 9-6 ANALYSIS OF STATICALLY DETERMINATE STRUCTURES BY THE MATRIX FORCE METHOD

As developed in Sec. 9-3, for a statically determinate structure, the internal forces $Q$ can be solved by equilibrium alone:

$$Q = bR$$

See Eq. 9-4.

Also the nodal displacements $r$ can be solved by

$$r = b^T f b R$$

See Eq. 9-20.

Assume that the purely statical task of evaluating the force transformation matrix $b$ is not difficult, although this phase of analysis may cost a considerable amount of labor in complicated problems.

The procedure for analyzing a statically determinate structure by the force method is as follows:

1. Define the external nodal loads $R$.
2. Define the internal member forces $Q$.
3. Determine the force transformation matrix $b$.

Consider the elements of the first column of $b$. If we let

$$R_1 = 1 \qquad R_2 = R_3 = \cdots = R_n = 0$$

it is readily seen from Eq. 9-3 that $Q_1, Q_2, \ldots, Q_m$ are the elements of the first column. The rest can be obtained similarly.

4. The internal member forces $Q$ are then solved by

$$Q = bR$$

5. Determine individual element flexibility matrices $f^a, f^b, \ldots$ according to Eq. 9-15 or 9-16, and assemble them as a diagonal matrix,

$$f = \begin{bmatrix} f^a & & & \\ & f^b & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{bmatrix}$$

6. Compute the flexibility matrix of the structure,

$$F = b^T f b$$

7. Find the nodal displacements $r$,

$$r = FR$$

**Example 9-1**

Find the bar forces of the truss shown in Fig. 9-4. Find also the deflections corresponding to the applied loads $R_1$ and $R_2$. Assume that $L/A = 1$ for all members.

The load matrix is

$$R = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

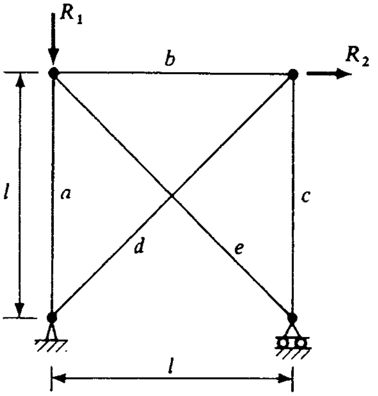


Fig. 9-4

The truss has five bars designated by $a$, $b$, $c$, $d$, and $e$. The member-force matrix is

$$Q = \begin{Bmatrix} Q^a \\ Q^b \\ Q^c \\ Q^d \\ Q^e \end{Bmatrix}$$

The force transformation matrix $b$ is given by

$$b = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

in which the first column contains the bar forces of the truss in Fig. 9-4 in the order $a$, $b$, $c$, $d$, $e$, resulting from $R_1 = 1$, $R_2 = 0$. The second column contains the corresponding bar forces resulting from $R_2 = 1$, $R_1 = 0$. From equilibrium

$$\begin{Bmatrix} Q^a \\ Q^b \\ Q^c \\ Q^d \\ Q^e \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

For individual members the flexibility matrices are found to be

$$f^a = f^b = f^c = f^d = f^e = \frac{1}{E}$$

since $L/A = 1$ for all members. Thus, the diagonal matrix is

$$f = \begin{bmatrix} f^a & & & & \\ & f^b & & & \\ & & f^c & & \\ & & & f^d & \\ & & & & f^e \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

The total flexibility matrix is then determined:

$F = b^Tfb$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & \sqrt{2} & 0 \end{bmatrix} \frac{1}{E}\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{E}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

The nodal displacements $r$ are solved by

$$r = FR$$

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

or

$$r_1 = \frac{R_1}{E} \qquad r_2 = \frac{3R_2}{E}$$

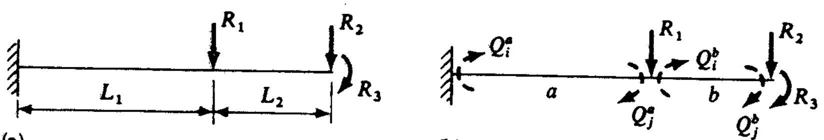
**Example 9-2**

Find the deflections corresponding to the applied loads for the cantilever beam shown in Fig. 9-5(a). Assume constant $EI$.

Since the loaded point of $R_1$ must be considered as a nodal point, it divides the beam into two segments, designated as member $a$ and member $b$ in Fig. 9-5(b). The internal member forces are shown by dashed lines. From equilibrium

$$R_1 = 1 \; R_2 = 1 \; R_3 = 1$$

$$Q = bR \qquad \begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \end{Bmatrix} = \begin{bmatrix} -L_1 & -(L_1 + L_2) & -1 \\ 0 & L_2 & 1 \\ 0 & -L_2 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix}$$



(a) (b)

Fig. 9-5

Note that the elements of the first column of matrix $b$ are the member forces caused by $R_1 = 1$, $R_2 = R_3 = 0$ for the beam shown in Fig. 9-5(b). This gives

$$Q_i^a = -L_1 \qquad Q_j^a = Q_i^b = Q_j^b = 0$$

The second column of matrix $b$ contains the member forces resulting from $R_2 = 1$, $R_1 = R_3 = 0$. Thus,

$$Q_i^a = -(L_1 + L_2) \qquad Q_j^a = L_2 \qquad Q_i^b = -L_2 \qquad Q_j^b = 0$$

And the third column of matrix $b$ contains the member forces due to a unit couple applied only at the free end of the beam (i.e., $R_3 = 1$, $R_1 = R_2 = 0$). This gives

$$Q_i^a = -1 \qquad Q_j^a = 1 \qquad Q_i^b = -1 \qquad Q_j^b = 1$$

The individual member flexibility matrices are

$$f^a = \frac{1}{6EI}\begin{bmatrix} 2L_1 & -L_1 \\ -L_1 & 2L_1 \end{bmatrix} \qquad f^b = \frac{1}{6EI}\begin{bmatrix} 2L_2 & -L_2 \\ -L_2 & 2L_2 \end{bmatrix}$$

from which

$$f = \frac{1}{6EI}\begin{bmatrix} 2L_1 & -L_1 & 0 & 0 \\ -L_1 & 2L_1 & 0 & 0 \\ 0 & 0 & 2L_2 & -L_2 \\ 0 & 0 & -L_2 & 2L_2 \end{bmatrix}$$

The total flexibility matrix $F$ is obtained from

$$F = b^Tfb = \begin{bmatrix} -L_1 & 0 & 0 & 0 \\ -(L_1 + L_2) & L_2 & -L_2 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \left(\frac{1}{6EI}\right)$$

$$\cdot \begin{bmatrix} 2L_1 & -L_1 & 0 & 0 \\ -L_1 & 2L_1 & 0 & 0 \\ 0 & 0 & 2L_2 & -L_2 \\ 0 & 0 & -L_2 & 2L_2 \end{bmatrix}\begin{bmatrix} -L_1 & -(L_1 + L_2) & -1 \\ 0 & L_2 & 1 \\ 0 & -L_2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{L_1^3}{3EI} & \dfrac{2L_1^3 + 3L_1^2L_2}{6EI} & \dfrac{L_1^2}{2EI} \\ \dfrac{2L_1^3 + 3L_1^2L_2}{6EI} & \dfrac{(L_1 + L_2)^3}{3EI} & \dfrac{(L_1 + L_2)^2}{2EI} \\ \dfrac{L_1^2}{2EI} & \dfrac{(L_1 + L_2)^2}{2EI} & \dfrac{L_1 + L_2}{EI} \end{bmatrix}$$

Thus,

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{bmatrix} \dfrac{L_1^3}{3EI} & \dfrac{2L_1^3 + 3L_1^2L_2}{6EI} & \dfrac{L_1^2}{2EI} \\ \dfrac{2L_1^3 + 3L_1^2L_2}{6EI} & \dfrac{(L_1 + L_2)^3}{3EI} & \dfrac{(L_1 + L_2)^2}{2EI} \\ \dfrac{L_1^2}{2EI} & \dfrac{(L_1 + L_2)^2}{2EI} & \dfrac{L_1 + L_2}{EI} \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix}$$

or

$$r_1 = \frac{R_1L_1^3}{3EI} + \frac{R_2(2L_1^3 + 3L_1^2L_2)}{6EI} + \frac{R_3L_1^2}{2EI} \qquad (9\text{-}39)$$

$$r_2 = \frac{R_1(2L_1^3 + 3L_1^2L_2)}{6EI} + \frac{R_2(L_1 + L_2)^3}{3EI} + \frac{R_3(L_1 + L_2)^2}{2EI} \tag{9-40}$$

$$r_3 = \frac{R_1L_1^2}{2EI} + \frac{R_2(L_1 + L_2)^2}{2EI} + \frac{R_3(L_1 + L_2)}{EI} \tag{9-41}$$

As a particular problem, find the vertical deflection and the rotation at the free end of the loaded cantilever beam shown in Fig. 9-6.

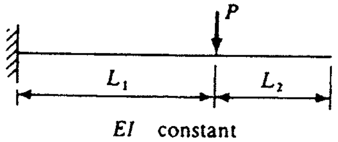


Fig. 9-6

To do this, we set $R_1 = P$, $R_2 = R_3 = 0$ in Eq. 9-40 to obtain

$$r_2 = \frac{P(2L_1^3 + 3L_1^2L_2)}{6EI}$$

which is the resulting vertical deflection of the end of the beam, and we set $R_1 = P$, $R_2 = R_3 = 0$ in Eq. 9-41 to obtain

$$r_3 = \frac{PL_1^2}{2EI}$$

which is the resulting rotation of the end of the beam.

## 9-7 ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES BY THE MATRIX FORCE METHOD

As developed in Secs. 9-3 to 9-5, the procedures for analyzing a statically indeterminate structure by the force method are given as follows:

1. Define the external loads $R$.
2. Define the internal member forces $Q$, and specify the redundants $X$.
3. Calculate the force transformation matrices $b_R$ and $b_X$ from equilibrium:

$$Q = [b_R | b_X] \left\{ \frac{R}{X} \right\}$$

4. Determine the individual element flexibility matrices $f^a$, $f^b$, . . . , and assemble them to obtain $f$:

$$f = \begin{bmatrix} f^a & & & \\ & f^b & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{bmatrix}$$

5. Calculate $F_{XR}$:

$$F_{XR} = b_X^T f b_R$$

6. Calculate $F_{XX}$:

$$F_{XX} = b_X^T f b_X$$

7. Find the inverse of $F_{XX}$.
8. Solve the redundants $X$ by

$$X = -F_{XX}^{-1} F_{XR} R$$

and substitute $X$ in the equilibrium equation to obtain the member forces $Q$.

9. Alternatively, we may find $b'$ by

$$b' = b_R - b_X F_{XX}^{-1} F_{XR}$$

and obtain the member forces $Q$ by

$$Q = b'R$$

10. If the nodal displacements are desired, calculate $F'$ by

$$F' = b_R^T f b'$$

and find $r_R$ by

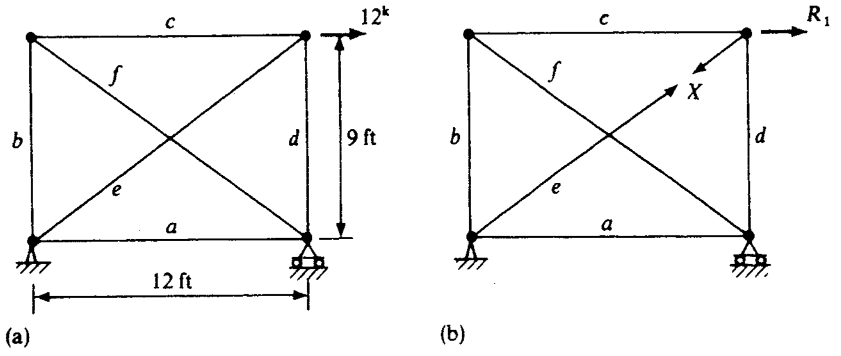
$$r_R = F'R$$

As seen in the latter part of Example 9-2, if the points where the displacements are desired are not actually loaded, then we must apply fictitious loads of zero value at these points in order to carry out the procedures listed above.

**Example 9-3**

Find the bar forces of the truss in Fig. 9-7(a) by the force method. Also find the nodal displacement corresponding to the applied load. Assume that $E = 30{,}000$ kips/in.$^2$ and $L(\text{ft})/A(\text{in.}^2) = 1$ for all members.

The truss shown in Fig. 9-7(a) is statically indeterminate to the first degree. Let us select bar $e$ as the redundant and denote the external load of 12 kips by $R_1$, as shown in Fig. 9-7(b). The bar forces are denoted by $Q^a$, $Q^b$, . . . , $Q^f$. From equilibrium based on the primary structure of Fig. 9-7(b),



(a) (b)

Fig. 9-7

$$\begin{Bmatrix} Q^a \\ Q^b \\ Q^c \\ Q^d \\ Q^e \\ Q^f \end{Bmatrix} = \begin{array}{c} \begin{array}{cc} R_1 = 1 & X = 1 \end{array} \\ \left[\begin{array}{c|c} 1 & -\frac{4}{5} \\ \frac{3}{4} & -\frac{3}{5} \\ 1 & -\frac{4}{5} \\ 0 & -\frac{3}{5} \\ 0 & 1 \\ -\frac{5}{4} & 1 \end{array}\right] \\ \begin{array}{cc} b_R & b_X \end{array} \end{array} \begin{Bmatrix} R_1 \\ X \end{Bmatrix}$$

since $L/A = 1$ for all members

$$f = \frac{1}{E}\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Thus,

$$F_{XR} = b_X^T f b_R$$

$$= [-\tfrac{4}{5}\ -\tfrac{3}{5}\ -\tfrac{4}{5}\ -\tfrac{3}{5}\ 1\ 1]\left(\frac{1}{E}\right)\begin{Bmatrix} 1 \\ \frac{3}{4} \\ 1 \\ 0 \\ 0 \\ -\frac{5}{4} \end{Bmatrix} = -\frac{3.3}{E}$$

$$F_{XX} = b_X^T f b_X$$

$$= [-\tfrac{4}{5}\ -\tfrac{3}{5}\ -\tfrac{4}{5}\ -\tfrac{3}{5}\ 1\ 1]\left(\frac{1}{E}\right)\begin{Bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \\ -\frac{4}{5} \\ -\frac{3}{5} \\ 1 \\ 1 \end{Bmatrix} = \frac{4}{E}$$

$$F_{XX}^{-1} = \frac{E}{4}$$

The redundant force $X$ is then solved by

$$X = -F_{XX}^{-1}F_{XR}R$$

$$= -\left(\frac{E}{4}\right)\left(-\frac{3.3}{E}\right)(12) = 9.9 \text{ kips}$$

Substituting in the equilibrium equation, we obtain

$$\begin{Bmatrix} Q^a \\ Q^b \\ Q^c \\ Q^d \\ Q^e \\ Q^f \end{Bmatrix} = \begin{bmatrix} 1 & -\frac{4}{5} \\ \frac{3}{4} & -\frac{3}{5} \\ 1 & -\frac{4}{5} \\ 0 & -\frac{3}{5} \\ 0 & 1 \\ -\frac{5}{4} & 1 \end{bmatrix}\begin{Bmatrix} 12 \\ 9.9 \end{Bmatrix} = \begin{Bmatrix} 4.08 \\ 3.06 \\ 4.08 \\ -5.94 \\ 9.90 \\ -5.10 \end{Bmatrix} \text{kips}$$

Alternatively, we find

$$b' = b_R - b_X F_{XX}^{-1} F_{XR}$$

$$= \begin{Bmatrix} 1 \\ \frac{3}{4} \\ 1 \\ 0 \\ 0 \\ -\frac{5}{4} \end{Bmatrix} - \begin{Bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \\ -\frac{4}{5} \\ -\frac{3}{5} \\ 1 \\ 1 \end{Bmatrix} \left(\frac{E}{4}\right)\left(-\frac{3.3}{E}\right) = \begin{Bmatrix} 0.340 \\ 0.255 \\ 0.340 \\ -0.495 \\ 0.825 \\ -0.425 \end{Bmatrix}$$

and obtain $Q$ by $Q = b'R$:

$$\begin{Bmatrix} Q^a \\ Q^b \\ Q^c \\ Q^d \\ Q^e \\ Q^f \end{Bmatrix} = \begin{Bmatrix} 0.340 \\ 0.255 \\ 0.340 \\ -0.495 \\ 0.825 \\ -0.425 \end{Bmatrix} (12) = \begin{Bmatrix} 4.08 \\ 3.06 \\ 4.08 \\ -5.94 \\ 9.90 \\ -5.10 \end{Bmatrix} \text{kips}$$

To find $r_1$, we first calculate the flexibility matrix of structure $F'$:

$$F' = b_R^T f b'$$

$$= [1 \ \tfrac{3}{4} \ 1 \ 0 \ 0 \ -\tfrac{5}{4}] \left(\frac{1}{E}\right) \begin{Bmatrix} 0.340 \\ 0.255 \\ 0.340 \\ -0.495 \\ 0.825 \\ -0.425 \end{Bmatrix} = \frac{1.4}{E}$$

The displacement $r_1$ is then solved:

$$r_1 = F'R_1$$

$$= \left(\frac{1.4}{E}\right)(12) = \frac{(1.4)(12)}{30{,}000} = 0.00056 \text{ ft}$$

in the direction of the applied load.

**Example 9-4**

Find the member forces (end moments) of the rigid frame in Fig. 9-8(a) by the force method. $E$ is constant.

The frame shown in Fig. 9-8(a) is statically indeterminate to the second degree. It may be made determinate by inserting two pins as in Fig. 9-8(b). Then the structure is subjected to the original applied loads denoted by $R_1$ and $R_2$ together with the redundant couples $X_1$ and $X_2$. The member forces (end moments) in Fig. 9-8(b), $Q_i^a$, $Q_j^a$, . . . , are shown by dashed lines.

The force transformation matrix is obtained by considering the influences of $R_1 = 1$, $R_2 = 1$, $X_1 = 1$, and $X_2 = 1$ successively and separately, as shown in Fig. 9-9.

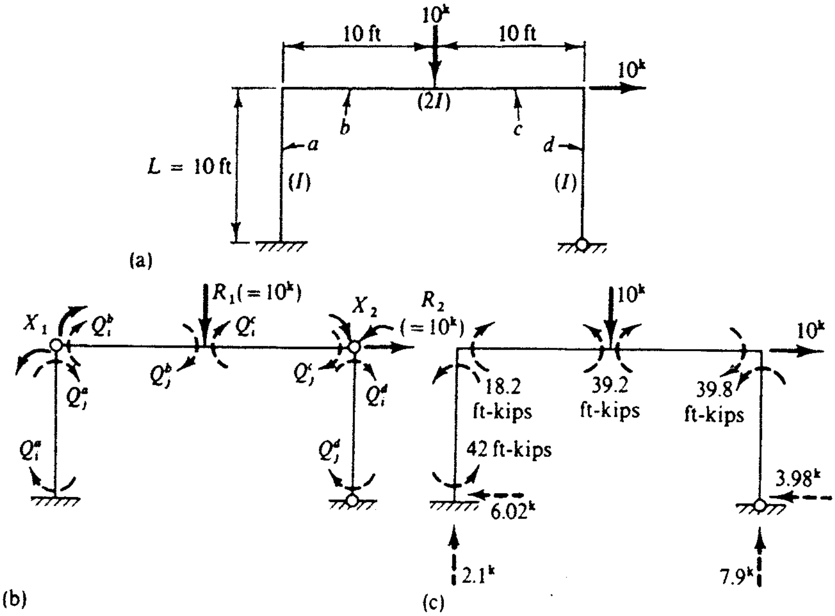


Fig. 9-8

$$
\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \\ Q_i^c \\ Q_j^c \\ Q_i^d \\ Q_j^d \end{Bmatrix}
=
\begin{array}{cc|cc}
R_1=1 & R_2=1 & X_1=1 & X_2=1 \\
\hline
0 & -L & 1 & 1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 \\
-\dfrac{L}{2} & 0 & -\dfrac{1}{2} & \dfrac{1}{2} \\
\dfrac{L}{2} & 0 & \dfrac{1}{2} & -\dfrac{1}{2} \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
\hline
\multicolumn{2}{c|}{b_R} & \multicolumn{2}{c}{b_X}
\end{array}
\begin{Bmatrix} R_1 \\ R_2 \\ \hline X_1 \\ X_2 \end{Bmatrix}
$$

From individual member flexibility matrices, we form

$$
f = \frac{L}{6EI}\begin{bmatrix}
2 & -1 & & & & & & \\
-1 & 2 & & & & & & \\
 & & 1 & -\frac{1}{2} & & & & \\
 & & -\frac{1}{2} & 1 & & & & \\
 & & & & 1 & -\frac{1}{2} & & \\
 & & & & -\frac{1}{2} & 1 & & \\
 & & & & & & 2 & -1 \\
 & & & & & & -1 & 2
\end{bmatrix}
$$

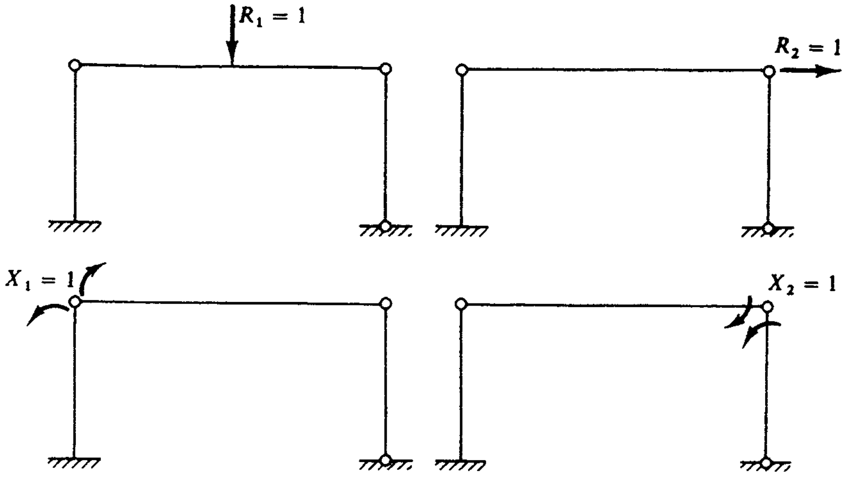


Fig. 9-9

Using $b_R$, $b_X$, and $f$ found previously, we obtain

$$F_{XR} = b_X^T f b_R = \frac{L^2}{6EI}\begin{bmatrix} \frac{3}{4} & -3 \\ -\frac{3}{4} & -2 \end{bmatrix}$$

$$F_{XX} = b_X^T f b_X = \frac{L}{6EI}\begin{bmatrix} 8 & 2 \\ 2 & 6 \end{bmatrix}$$

and

$$F_{XX}^{-1} = \frac{6EI}{L}\frac{\begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}^T}{\begin{vmatrix} 8 & 2 \\ 2 & 6 \end{vmatrix}} = \left(\frac{6EI}{L}\right)\left(\frac{1}{44}\right)\begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$$

The force transformation matrix of the indeterminate structure is

$$b' = b_R - b_X F_{XX}^{-1} F_{XR}$$

$$= \begin{bmatrix} 0 & -L \\ 0 & 0 \\ 0 & 0 \\ -\frac{L}{2} & 0 \\ \frac{L}{2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\cdot \left(\frac{6EI}{L}\right)\left(\frac{1}{44}\right)\begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}\left(\frac{L^2}{6EI}\right)\begin{bmatrix} \frac{3}{4} & -3 \\ -\frac{3}{4} & -2 \end{bmatrix}$$

$$= \frac{L}{88}\begin{bmatrix} 0 & -88 \\ 0 & 0 \\ 0 & 0 \\ -44 & 0 \\ 44 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} - \frac{L}{88}\begin{bmatrix} -3 & -48 \\ -12 & 28 \\ 12 & -28 \\ -13.5 & 4 \\ 13.5 & -4 \\ -15 & -20 \\ 15 & 20 \\ 0 & 0 \end{bmatrix} = \frac{L}{88}\begin{bmatrix} 3 & -40 \\ 12 & -28 \\ -12 & 28 \\ -30.5 & -4 \\ 30.5 & 4 \\ 15 & 20 \\ -15 & -20 \\ 0 & 0 \end{bmatrix}$$

The end moments are then solved by $Q = b'R$:

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \\ Q_i^c \\ Q_j^c \\ Q_i^d \\ Q_j^d \end{Bmatrix} = \frac{L}{88}\begin{bmatrix} 3 & -40 \\ 12 & -28 \\ -12 & 28 \\ -30.5 & -4 \\ 30.5 & 4 \\ 15 & 20 \\ -15 & -20 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Using $L = 10$ ft and $R_1 = R_2 = 10$ kips, we obtain

$$Q_i^a = -42 \text{ ft-kips}$$

$$Q_j^a = -Q_i^b = -18.2 \text{ ft-kips}$$

$$Q_j^b = -Q_i^c = -39.2 \text{ ft-kips}$$

$$Q_j^c = -Q_i^d = 39.8 \text{ ft-kips}$$

$$Q_j^d = 0$$

To check, we find that the identity

$$b_X f b' = 0$$

is satisfied by substituting in the values of $b_X$, $f$, and $b'$ previously found. The answer diagram for the end moments together with the reactions at the supports found by statics is shown by the dashed line in Fig. 9-8(c).

**Example 9-5**

Find the end moments for the rigid frame shown in Fig. 9-10(a) by the force method. Assume constant *EI*.

The equivalent form of the given loaded frame is shown in Fig. 9-10(b). Because of symmetry, the vertical reaction at each support of the frame is known to be 6 kips acting upward, as indicated. If only flexural deformation is considered, then the nodal axial forces, shown in the frame in Fig. 9-10(b), only increase the compression in the two columns but cause no effect on the end moments of the frame and can therefore be neglected in the nodal-force analysis for obtaining end moments. The primary structure may be chosen as the one shown in Fig. 9-11, subjected to nodal moments $R_1$ and $R_2$ and redundant reaction components of the left support, denoted by $X_1$ and $X_2$. Those shown by dashed lines are member end moments $Q_i^a$, $Q_j^a$, . . . . They can be expressed in terms of $R$ and $X$ as

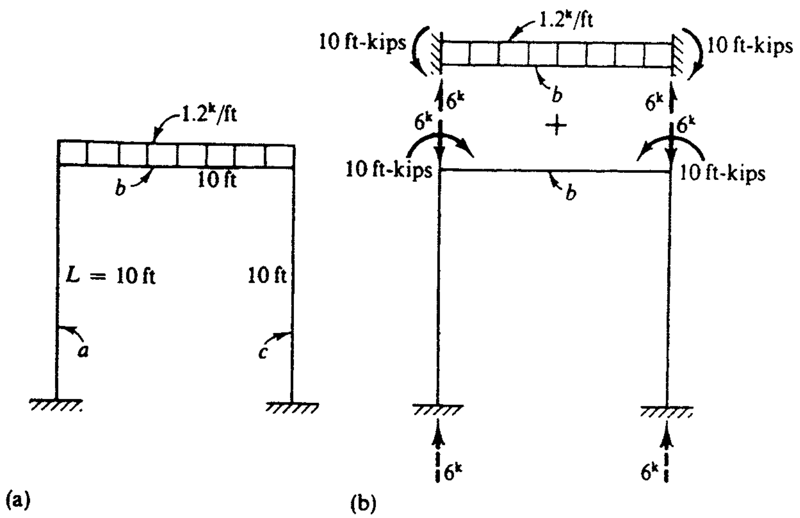


Fig. 9-10

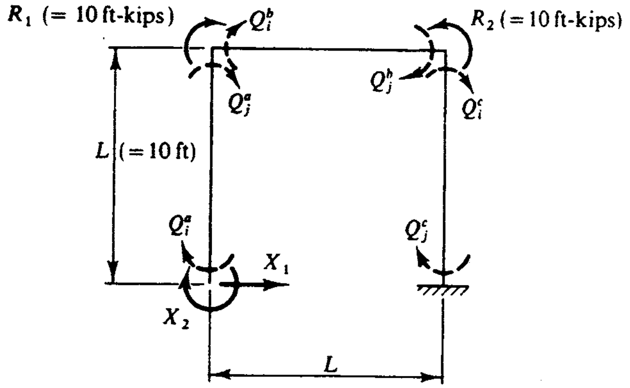


Fig. 9-11

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \\ Q_i^c \\ Q_j^c \end{Bmatrix} = \begin{array}{c} \begin{array}{cccc} R_1 = 1 & R_2 = 1 & x_1 = 1 & x_2 = 1 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & L & -1 \\ 1 & 0 & -L & 1 \\ -1 & 0 & L & -1 \\ 1 & -1 & -L & 1 \\ -1 & 1 & 0 & -1 \end{array}\right] \\ \begin{array}{cc} b_R & b_X \end{array} \end{array} \begin{Bmatrix} R_1 \\ R_2 \\ \hline X_1 \\ X_2 \end{Bmatrix}$$

From the member flexibility matrices, we form

$$f = \frac{L}{6EI}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & & & & \\ & & 2 & -1 & & \\ & & -1 & 2 & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

Thus,

$$F_{XR} = b_X^T f b_R$$

$$= \begin{bmatrix} 0 & L & -L & L & -L & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}\left(\frac{L}{6EI}\right)$$

$$\cdot\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & & & & \\ & & 2 & -1 & & \\ & & -1 & 2 & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{L}{6EI}\begin{bmatrix} -9L & 3L \\ 12 & -6 \end{bmatrix}$$

Similarly,

$$F_{XX} = b_X^T f b_X$$

$$= \frac{L}{6EI}\begin{bmatrix} 10L^2 & -12L \\ -12L & 18 \end{bmatrix}$$

from which

$$F_{XX}^{-1} = \frac{6EI}{L}\frac{\begin{bmatrix} 18 & 12L \\ 12L & 10L^2 \end{bmatrix}^T}{(180L^2 - 144L^2)} = \frac{6EI}{L}\frac{\begin{bmatrix} 18 & 12L \\ 12L & 10L^2 \end{bmatrix}}{36L^2}$$

The force transformation matrix of the indeterminate structure is

$$b' = b_R - b_X F_{XX}^{-1} F_{XR}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ L & -1 \\ -L & 1 \\ L & -1 \\ -L & 1 \\ 0 & -1 \end{bmatrix}\left(\frac{1}{36L^2}\right)\begin{bmatrix} 18 & 12L \\ 12L & 10L^2 \end{bmatrix}\begin{bmatrix} -9L & 3L \\ 12 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ -\frac{5}{6} & \frac{1}{6} \\ \frac{5}{6} & -\frac{1}{6} \\ -\frac{5}{6} & \frac{1}{6} \\ \frac{5}{6} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{5}{6} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

The end moments based on the nodal-force analysis are then solved by $Q = b'R$:

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \\ Q_i^c \\ Q_j^c \end{Bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{5}{6} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{Bmatrix} 10 \\ 10 \end{Bmatrix} = \begin{Bmatrix} \frac{10}{3} \\ \frac{20}{3} \\ \frac{10}{3} \\ -\frac{10}{3} \\ -\frac{20}{3} \\ -\frac{10}{3} \end{Bmatrix} \text{ft-kips}$$

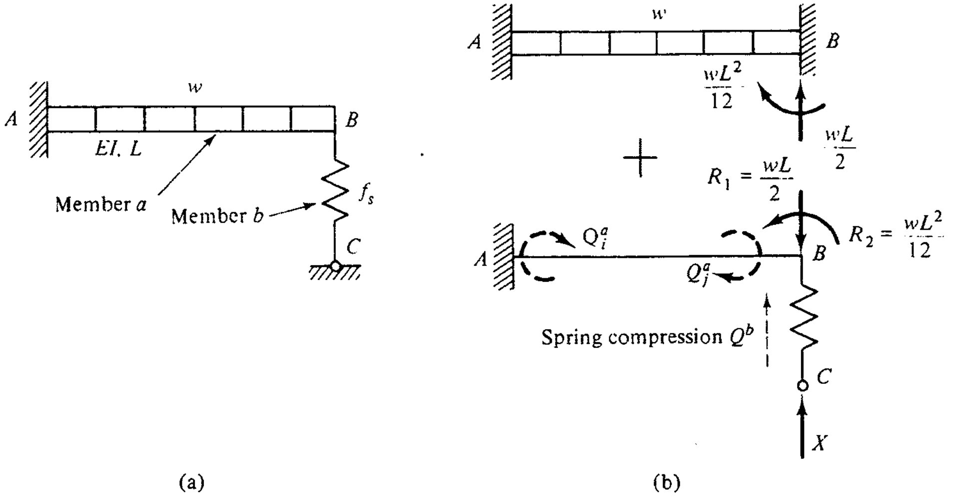
The final result is obtained by adding the fixed-end moments [see upper part of Fig. 9-10(b)] to the end moments of member $b$. Thus,

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \\ Q_i^c \\ Q_j^c \end{Bmatrix} = \begin{Bmatrix} \frac{10}{3} \\ \frac{20}{3} \\ \frac{10}{3} - 10 \\ -\frac{10}{3} + 10 \\ -\frac{20}{3} \\ -\frac{10}{3} \end{Bmatrix} = \begin{Bmatrix} \frac{10}{3} \\ \frac{20}{3} \\ -\frac{20}{3} \\ \frac{20}{3} \\ -\frac{20}{3} \\ -\frac{10}{3} \end{Bmatrix} \text{ft-kips}$$

**Example 9-6**

Use the matrix force method to find the reaction at support $C$, and the deflection and slope at $B$, for the beam shown in Fig. 9-12(a). Assume that the spring flexibility is $f_s$.

We consider the spring as a member and therefore the system is composed of the beam portion, denoted as member $a$, and the spring, denoted as member $b$. The whole can be separated into two parts: the fixed-end beam under a uniform load and the system subjected to nodal forces $R_1$, $R_2$ at $B$ and redundant reaction $X$ at $C$ as shown in Fig. 9-12(b). To obtain the nodal displacements at $B$ and reaction at $C$, it is necessary only to analyze the nodal-load system of Fig. 9-12(b).



(a) (b)

Fig. 9-12

We first relate the end moments $Q_i^a$, $Q_j^a$ of member $a$, and the spring force $Q^b$ to the external loads $R_1$, $R_2$, and $X$ as

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q^b \end{Bmatrix} = \begin{array}{c} \left[\begin{array}{cc:c} -L & 1 & L \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right] \\ \begin{array}{cc} b_R \qquad & \qquad b_X \end{array} \end{array} \begin{Bmatrix} R_1 \\ R_2 \\ \hdashline X \end{Bmatrix}$$

The flexibility matrix of member $a$ is

$$f^a = \begin{bmatrix} \dfrac{L}{3EI} & -\dfrac{L}{6EI} \\ -\dfrac{L}{6EI} & \dfrac{L}{3EI} \end{bmatrix}$$

The flexibility of member $b$ (spring) is $f_s$. Therefore, the assembled flexibility is

$$f = \begin{bmatrix} \dfrac{L}{3EI} & -\dfrac{L}{6EI} & \\ -\dfrac{L}{6EI} & \dfrac{L}{3EI} & \\ & & f_s \end{bmatrix}$$

With $b_R$, $b_X$, and $f$ obtained, we have

$$F_{XR} = b_X^T f b_R = \begin{bmatrix} -\dfrac{L^3}{3EI} & \dfrac{L^2}{2EI} \end{bmatrix}$$

$$F_{XX} = b_X^T f b_X = \frac{L^3}{3EI} + f_s$$

$$F_{XX}^{-1} = \frac{1}{L^3/3EI + f_s}$$

The redundant force $X$, which is equal to the spring force $Q^b$, is determined by

$$\begin{aligned} X &= -F_{XX}^{-1} F_{XR} R \\ &= -\left(\frac{1}{L^3/3EI + f_s}\right) \begin{bmatrix} -\dfrac{L^3}{3EI} & \dfrac{L^2}{2EI} \end{bmatrix} \begin{Bmatrix} \dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} \\ &= \frac{3}{8} wL \left(\frac{1}{1 + 3EIf_s/L^3}\right) \end{aligned}$$

Apparently, if $f_s = 0$,

$$X = \tfrac{3}{8} wL$$

To find the deflection and slope at $B$, we use

$$r_R = F_{RR} R + F_{RX} X$$

Now since

$$F_{RR} = b_R^T f b_R = \begin{bmatrix} \dfrac{L^3}{3EI} & -\dfrac{L^2}{2EI} \\ -\dfrac{L^2}{2EI} & \dfrac{L}{EI} \end{bmatrix}$$

$$F_{RX} = b_R^T f b_X = \begin{bmatrix} -\dfrac{L^3}{3EI} \\ \dfrac{L^2}{2EI} \end{bmatrix}$$

we reach

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{bmatrix} \dfrac{L^3}{3EI} & -\dfrac{L^2}{2EI} \\ -\dfrac{L^2}{2EI} & \dfrac{L}{EI} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} + \begin{bmatrix} -\dfrac{L^3}{3EI} \\ \dfrac{L^2}{2EI} \end{bmatrix} (X)$$

Using

$$R_1 = \frac{wL}{2} \qquad R_2 = \frac{wL^2}{12} \qquad X = \frac{3}{8} wL \left( \frac{1}{1 + 3EIf_s/L^3} \right)$$

we obtain

$$\begin{aligned} r_1 \text{ (deflection)} &= \frac{wL^4}{8EI} - \frac{L^3}{3EI} \left( \frac{\frac{3}{8}wL}{1 + 3EIf_s/L^3} \right) \\ &= \frac{3}{8} wL \left( \frac{1}{3EI/L^3 + 1/f_s} \right) \\ r_2 \text{ (slope)} &= -\frac{wL^3}{6EI} + \frac{L^2}{2EI} \left( \frac{\frac{3}{8}wL}{1 + 3EIf_s/L^3} \right) \\ &= \left( -\frac{w}{2} + \frac{wL^3}{48EIf_s} \right) \left( \frac{1}{3EI/L^3 + 1/f_s} \right) \end{aligned}$$

As a check, if $f_s = 0$ (rigid support), we have

$$r_1 = 0 \qquad r_2 = \frac{wL^3}{48EI} \curvearrowright$$

If $f_s = \infty$ (free end),

$$r_1 = \frac{wL^4}{8EI} \downarrow \qquad r_2 = -\frac{wL^3}{6EI} \curvearrowright$$

The foregoing procedure for fixing a loaded beam is not limited to the case of distributed loads. The procedure can also be applied to members subjected to a set of concentrated loads, if reducing the number of nodes is desirable.

## 9-8 ON THE NOTION OF PRIMARY STRUCTURE

The procedures for the analysis of statically indeterminate structures by the force method already discussed are based on the concept of *primary structure* previously developed in the method of consistent deformations. The notion of primary structure serves a convenient means of setting up an equilibrium equation. However, if we, without considering the notion of primary structure, examine the basic equation

$$Q = b_R R + b_X X$$

we observe that it merely states that $Q$ is linearly related to a set of applied forces $R$ and a set of unknown forces $X$. The equation itself does not necessarily suggest a primary structure. As a result, we may separate these two sets of influences from the two independent force systems imposed on the original structure. Doing so does not violate the truth of the preceding equation but certainly broadens our view of handling the problem.

Now $b_R$ represents an array of member forces in equilibrium with unit applied loads based on the original structure. More specifically, each column of $b_R$ represents member forces in equilibrium with a certain unit load applied to the original structure. Since the original structure is statically indeterminate, many equilibrating systems may be chosen from to establish each column of $b_R$.

Likewise, each column of $b_X$ can be thought of as an independent self-equilibrating internal force system for the original structure. For a structure indeterminate to the $n$th degree, $b_X$ will represent any group of $n$ independent self-equilibrating member force systems, one for each redundant.

If it is convenient, these member forces may be determined by introducing a primary structure. However, in a larger sense, the traditional notion of primary structure is not essential to the analysis of a statically indeterminate structure; rather, it introduces unnecessary restrictions to the analysis.

**Example 9-7**

Solve the bar forces of the truss in Fig. 9-7(a) (Example 9-3) by the preceding generalized procedures.

**Solution 1** Disregarding the notion of a primary structure, we may choose a set of member forces in equilibrium with external load $R_1 = 1$, as shown in Fig. 9-13(a), and a set of self-equilibrating internal forces, as shown in Fig. 9-13(b). Thus,

$$b_R = \begin{Bmatrix} 5 \\ \frac{15}{4} \\ 5 \\ 3 \\ -5 \\ -\frac{25}{4} \end{Bmatrix} \qquad b_X = \begin{Bmatrix} -4 \\ -3 \\ -4 \\ -3 \\ 5 \\ 5 \end{Bmatrix}$$

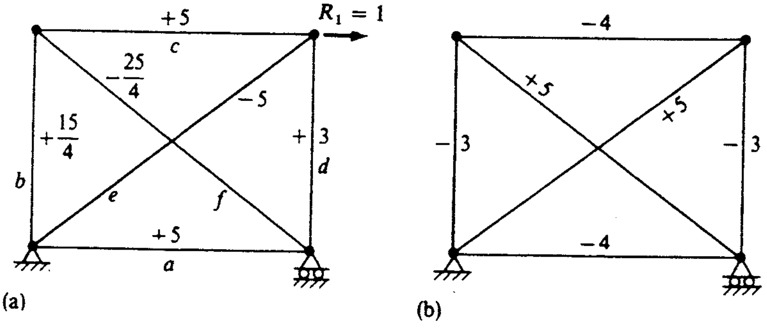


Fig. 9-13

There are, of course, many other choices that might be made.

$$F_{XR} = b_X^T f b_R$$

$$= [-4 \quad -3 \quad -4 \quad -3 \quad 5 \quad 5]\left(\frac{1}{E}\right)\begin{Bmatrix} 5 \\ \frac{15}{4} \\ 5 \\ 3 \\ -5 \\ -\frac{25}{4} \end{Bmatrix} = -\frac{116.5}{E}$$

$$F_{XX} = b_X^T f b_X$$

$$= [-4 \quad -3 \quad -4 \quad -3 \quad 5 \quad 5]\left(\frac{1}{E}\right)\begin{Bmatrix} -4 \\ -3 \\ -4 \\ -3 \\ 5 \\ 5 \end{Bmatrix} = \frac{100}{E}$$

$$F_{XX}^{-1} = \frac{E}{100}$$

The force transformation matrix is then determined:

$$b' = b_R - b_X F_{XX}^{-1} F_{XR}$$

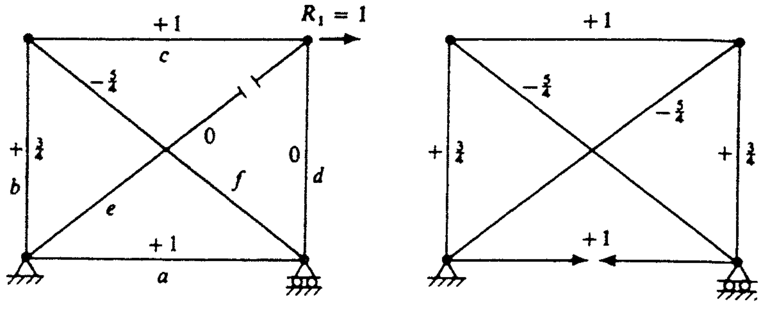
$$= \begin{Bmatrix} 5 \\ \frac{15}{4} \\ 5 \\ 3 \\ -5 \\ -\frac{25}{4} \end{Bmatrix} - \begin{Bmatrix} -4 \\ -3 \\ -4 \\ -3 \\ 5 \\ 5 \end{Bmatrix}\left(\frac{E}{100}\right)\left(\frac{-116.5}{E}\right) = \begin{Bmatrix} 5 \\ 3.75 \\ 5 \\ 3 \\ -5 \\ -6.25 \end{Bmatrix} - \begin{Bmatrix} 4.660 \\ 3.495 \\ 4.660 \\ 3.495 \\ -5.825 \\ -5.825 \end{Bmatrix} = \begin{Bmatrix} 0.340 \\ 0.255 \\ 0.340 \\ -0.495 \\ 0.825 \\ -0.425 \end{Bmatrix}$$

This is the same $b'$ obtained in Example 9-3 and will lead to the same final results for the bar forces.

**Solution 2** It may be interesting to point out that when primary structure is used in analyzing an indeterminate structure, the same final results will be obtained if different primary structures are chosen in developing $b_R$ and $b_X$.

To illustrate, let us first take member $e$ as the redundant. The bar forces associated with the given primary structure due to external load $R_1 = 1$ are elements of $b_R$, as indicated in Fig. 9-14(a). Next, let member $a$ be chosen as the redundant. Setting the redundant force equal to unity, we obtain a set of internal forces in equilibrium [Fig. 9-14(b)], which forms $b_X$.

$$b_R = \begin{Bmatrix} 1 \\ \frac{3}{4} \\ 1 \\ 0 \\ 0 \\ -\frac{5}{4} \end{Bmatrix} \qquad b_X = \begin{Bmatrix} 1 \\ \frac{3}{4} \\ 1 \\ \frac{3}{4} \\ -\frac{5}{4} \\ -\frac{5}{4} \end{Bmatrix}$$



(a) (b)

Fig. 9-14

$$F_{XR} = b_X^T f b_R$$

$$= [1 \quad \tfrac{3}{4} \quad 1 \quad \tfrac{3}{4} \quad -\tfrac{5}{4} \quad -\tfrac{5}{4}]\left(\frac{1}{E}\right)\begin{Bmatrix} 1 \\ \frac{3}{4} \\ 1 \\ 0 \\ 0 \\ -\frac{5}{4} \end{Bmatrix} = \frac{66}{16E}$$

$$F_{XX} = b_X^T f b_X$$

$$= [1 \quad \tfrac{3}{4} \quad 1 \quad \tfrac{3}{4} \quad -\tfrac{5}{4} \quad -\tfrac{5}{4}]\left(\frac{1}{E}\right)\begin{Bmatrix} 1 \\ \frac{3}{4} \\ 1 \\ \frac{3}{4} \\ -\frac{5}{4} \\ -\frac{5}{4} \end{Bmatrix} = \frac{100}{16E}$$

$$F_{XX}^{-1} = \frac{16E}{100}$$

The force transformation matrix $b'$ is found to be

$$b' = b_R - b_X F_{XX}^{-1} F_{XR} = \begin{Bmatrix} 1 \\ \frac{3}{4} \\ 1 \\ 0 \\ 0 \\ -\frac{5}{4} \end{Bmatrix} - \begin{Bmatrix} 1 \\ \frac{3}{4} \\ 1 \\ \frac{3}{4} \\ -\frac{5}{4} \\ -\frac{5}{4} \end{Bmatrix}\left(\frac{16E}{100}\right)\left(\frac{66}{16E}\right) = \begin{Bmatrix} 0.340 \\ 0.255 \\ 0.340 \\ -0.495 \\ 0.825 \\ -0.425 \end{Bmatrix}$$

the same as previously obtained.

## PROBLEMS

**9-1.** Use the force method to find the vertical deflection at each of the loaded points of the beam shown in Fig. 9-15. Assume constant $EI$.

**9-2.** Use the force method to find the slope and deflection at the loaded point of the beam shown in Fig. 9-16.

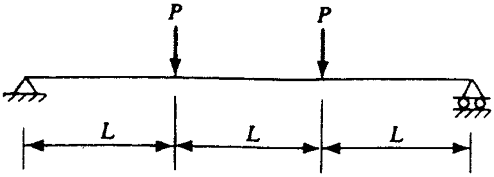


Fig. 9-15

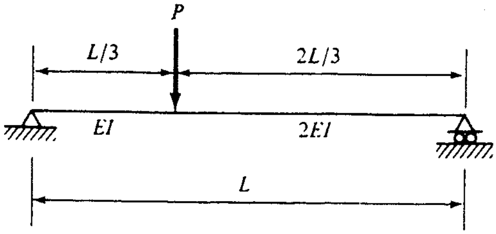


Fig. 9-16

9-3. Find, by the force method, all the bar forces and the vertical deflection at each of the loaded joints of the truss shown in Fig. 9-17. Assume that $A = 10 \text{ in.}^2$, $E = 30{,}000 \text{ kips/in.}^2$ for all members.

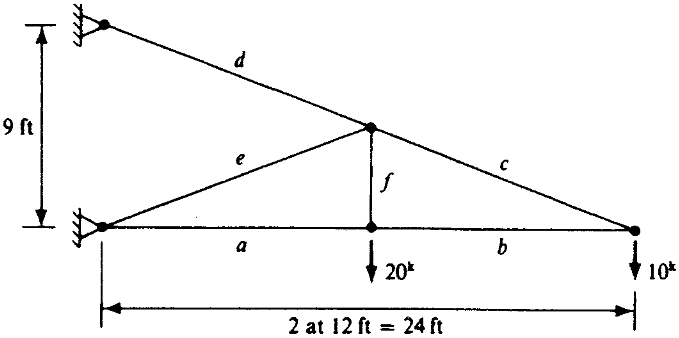


Fig. 9-17

9-4. Find, by the force method, all the member forces (end moments) and the nodal displacements corresponding to the applied loads for the frame in Fig. 9-18. Assume constant $EI$.

9-5. Find, by the force method, the slope and deflection at the loaded end of the beam shown in Fig. 9-19. Assume constant $EI$.

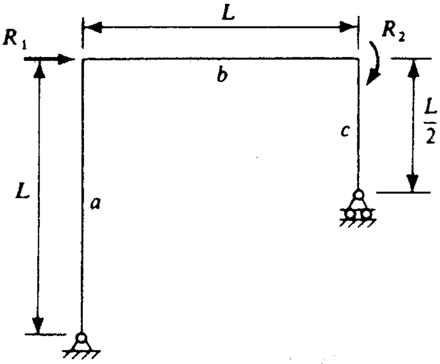


Fig. 9-18

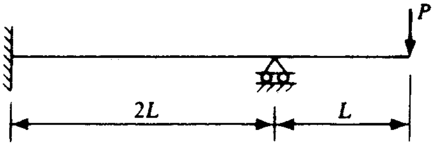


Fig. 9-19

9-6. Find, by the force method, the bar forces and the deflection components at the loaded point of the truss in Fig. 9-20. Assume that $A = 10$ in.$^2$ and $E = 30{,}000$ kips/in.$^2$ for all members.

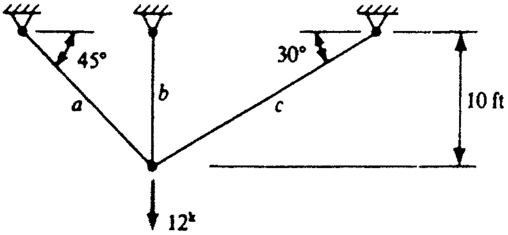


Fig. 9-20

9-7. The truss shown in Fig. 9-21 is statically indeterminate to the first degree. Choose the axial force in bar $c$ as redundant. Use the force method to find the bar forces and the deflection components corresponding to the applied loads. Assume that $A = 50$ cm$^2$ and $E = 20{,}000$ kN/cm$^2$ for all bars.

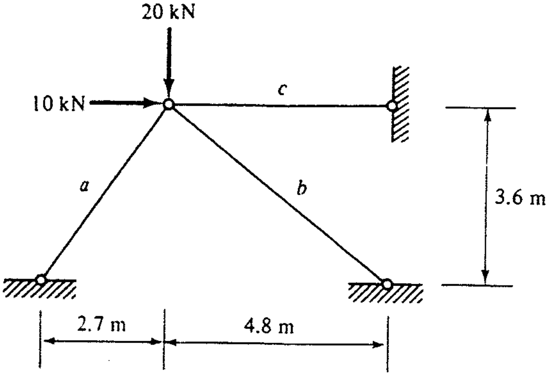


Fig. 9-21

9-8. Use the force method to obtain the member end moments for the frame shown in Fig. 9-22. Assume constant $EI$.

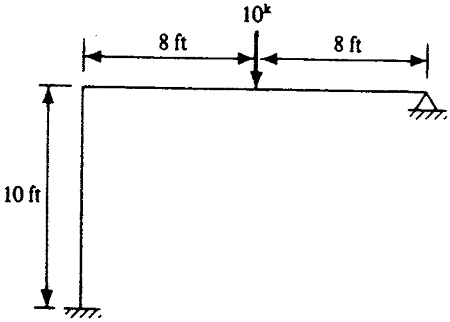


Fig. 9-22

**9-9.** Use the force method to find the member end moments for the frame shown in Fig. 9-23. Assume constant $EI$.

**9-10.** Refer to Fig. 9-24. Find, by the force method, the slope and deflection of the beam at the point $B$.

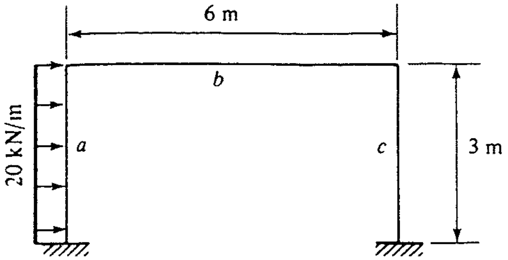


**Fig. 9-23**

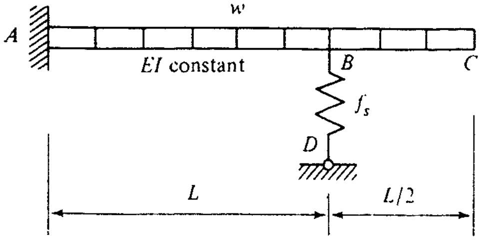


**Fig. 9-24**

**9-11.** Use the force method to find the member end moments for the gable bent in Fig. 9-25. Assume constant $EI$.

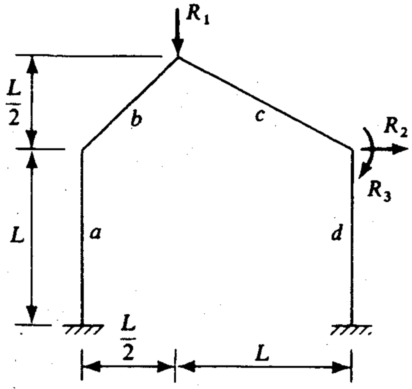


**Fig. 9-25**

**9-12.** Solve Prob. 9-6 disregarding the notion of primary structure.

**9-13.** Solve Prob. 9-7 disregarding the notion of primary structure.

**9-14.** Solve Prob. 9-8 by using different primary structures for developing the force transformation matrices $b_R$ and $b_X$.

# 10

# Matrix Displacement Method

## 10-1 GENERAL

As previously pointed out, the force method and the displacement method represent two different approaches to analyzing structures. The basic concepts of the structure remain the same (see Sec. 9-2). The fundamental difference between these two methods is that the force method chooses the member forces as the basic unknowns, whereas the displacement method chooses the nodal displacements as the basic unknowns. Like the force method, the basic equations of the displacement method are derived from

1. The equilibrium of forces
2. The compatibility of displacements
3. The force-displacement relationship

The compatibility condition is first satisfied by correlating the external nodal displacements to the end deformations of the members. The force-displacement relationship is then established between the member end forces and deformations and between the nodal forces and nodal displacements. Finally, using nodal equilibrium equations, we solve for the unknown nodal displacements and, therefore, for the member forces and deformations of the structure.

## 10-2 COMPATIBILITY, DISPLACEMENT TRANSFORMATION MATRIX

The compatibility used in the displacement method is that the geometry of deformation must be such that the elements of structure fit together at the nodal points; that is, the member deformations $q$ should be consistently related to the nodal displacements $r$. Let $a_{ij}$ represent the value of member deformation $q_i$ caused by a unit nodal displacement $r_j$. The total value of each member deformation caused by all the nodal displacements may be written as

$$\begin{aligned}
q_1 &= a_{11}r_1 + a_{12}r_2 + \cdots + a_{1n}r_n \\
q_2 &= a_{21}r_1 + a_{22}r_2 + \cdots + a_{2n}r_n \\
&\vdots \\
q_m &= a_{m1}r_1 + a_{m2}r_2 + \cdots + a_{mn}r_n
\end{aligned} \tag{10-1}$$

in which $q_1 = q_i^a$, $q_2 = q_j^a$, . . . represent the total set of member deformations and $r_1, r_2, \ldots, r_n$ the total set of nodal displacements. Note that no connection between the subscripts on $q$ and $r$ is implied. In matrix form,

$$\begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_m \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot \quad \cdot & \cdot \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{Bmatrix}$$

That is,

$$q = ar \tag{10-2}$$

where

$$a = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot \quad \cdot & \cdot \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix} \tag{10-2a}$$

called the *displacement transformation matrix,* which relates the internal member deformations to the external nodal displacements. Matrix $a$ is usually a rectangular matrix. It is simply a geometric transformation of coordinates representing the compatibility of the displacements of a system.

For example, let us consider the truss in Fig. 10-1. The possible nodal displacements are one linear displacement at the roller support, denoted by $r_1$, and two linear displacement components at the top joint, denoted by $r_2$ and $r_3$. Also indicated in Fig. 10-1 are the member deformations of bar $c$ as the result of the separate influences of $r_1$, $r_2$, and $r_3$, which gives the total $q^c$ as

$$q^c = 0.6r_1 - 0.6r_2 + 0.8r_3$$

$q^a$ and $q^b$ can be similarly obtained. The result, put in matrix form, is

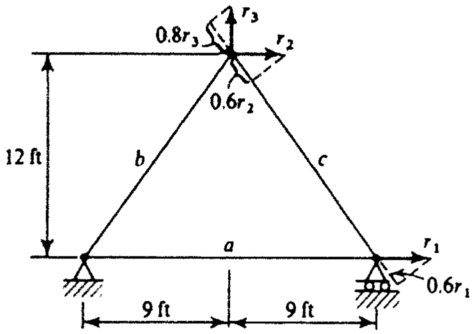


Fig. 10-1

$$\begin{Bmatrix} q^a \\ q^b \\ q^c \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0.6 & -0.6 & 0.8 \end{bmatrix}}_{a} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

Note that the displacement transformation matrix in this case happens to be a square matrix.

If the same forces and displacements are involved, the relationship between the displacement transformation matrix and the force transformation matrix can be obtained by virtual work as follows:

$$\delta R^T r = \delta Q^T q$$

Because of Eq. 9-4, $Q = bR$

or $\delta Q^T = \delta R^T b^T$

and Eq. 10-2, $q = ar$

we have $\delta R^T r = \delta R^T b^T a r$

from which

$$b^T a = I \tag{10-3}$$

Similarly, we can prove

$$a^T b = I \tag{10-4}$$

## 10-3 FORCE-DISPLACEMENT RELATIONSHIP, STIFFNESS COEFFICIENT, STIFFNESS MATRIX

A *stiffness coefficent* $k_{ij}$ is defined as the force developed at point $i$ due to a unit displacement at point $j$, all other points (nodes) being fixed. Like the flexibility coefficient, the stiffness coefficient constitutes a relationship between force and displacement. Applying the principle of superposition, we may express the force component at any point of a system in terms of a set of nodal displacements.

The first step in this analysis is to express the end forces in terms of the end deformations of an individual member. Using the stiffness coefficients and the notation defined in Fig. 9-2, we have

$$\begin{aligned} Q_i^a &= k_{ii}^a q_i^a + k_{ij}^a q_j^a + k_{ik}^a q_k^a \\ Q_j^a &= k_{ji}^a q_i^a + k_{jj}^a q_j^a + k_{jk}^a q_k^a \\ Q_k^a &= k_{ki}^a q_i^a + k_{kj}^a q_j^a + k_{kk}^a q_k^a \end{aligned} \tag{10-5}$$

in which $q_i^a$, $q_j^a$, and $q_k^a$ are the end deformations of a particular member $a$ and $Q_i^a$, $Q_j^a$, and $Q_k^a$ are the corresponding member forces. It is clear that the stiffness coefficient, say $k_{ii}^a$, is defined as

$$k_{ii}^a = Q_i^a \quad \text{as} \quad q_i^a = 1 \qquad q_j^a = q_k^a = 0$$

The rest can be defined similarly.

Equation 10-5 in matrix form is

$$Q^a = k^a q^a \tag{10-6}$$

in which

$$k^a = \begin{bmatrix} k_{ii}^a & k_{ij}^a & k_{ik}^a \\ k_{ji}^a & k_{jj}^a & k_{jk}^a \\ k_{ki}^a & k_{kj}^a & k_{kk}^a \end{bmatrix} \tag{10-6a}$$

is defined as the *element stiffness matrix.*

If Eq. 10-6 is premultiplied by $(k^a)^{-1}$,

$$(k^a)^{-1}Q^a = (k^a)^{-1}k^a q^a$$

or

$$q^a = (k^a)^{-1}Q^a \tag{10-7}$$

Comparing Eq. 10-7 with Eq. 9-10,

$$q^a = f^a Q^a$$

we see that

$$f^a = (k^a)^{-1} \tag{10-8}$$

Thus, *the element flexibility matrix is the inverse of the element stiffness matrix, and vice versa.*

The descriptions above refer to an individual element. For the entire assemblage composed of $a$, $b$, . . . elements, we have

$$\begin{aligned} Q^a &= k^a q^a \\ Q^b &= k^b q^b \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

Since

$$Q = \begin{Bmatrix} Q^a \\ Q^b \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \qquad q = \begin{Bmatrix} q^a \\ q^b \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

the preceding equations can be assembled as

$$Q = kq \tag{10-9}$$

where

$$k = \begin{bmatrix} k^a & & & & \\ & k^b & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \tag{10-9a}$$

which is a diagonal matrix with individual element stiffness matrices as its constituents.

Refer to Eqs. 10-5 and 10-6a, and consider a prismatic member with length $L$, cross-sectional area $A$, moment of inertia $I$, and modulus of elasticity $E$. The elements in the first column of $k^a$ are, by definition, the member forces resulting from $q_i^a = 1$, that is, a unit rotation at the left end of the member (see Fig. 9-3). Thus,

$$k_{ii}^a = \text{moment at the left end} = \frac{4EI}{L}$$

$$k_{ji}^a = \text{moment at the right end} = \frac{2EI}{L}$$

$$k_{ki}^a = \text{axial force of the member} = 0$$

Note that $k_{ii}^a$ and $k_{ji}^a$ can easily be obtained by the slope-deflection method; $k_{ki}^a = 0$ is apparent. All the other elements of $k^a$ are similarly determined. Thus,

$$k^a = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & 0 \\ \frac{2EI}{L} & \frac{4EI}{L} & 0 \\ 0 & 0 & \frac{AE}{L} \end{bmatrix} \tag{10-10}$$

which is a symmetric matrix. As a check, we note that

$$f^a k^a = \begin{bmatrix} \frac{L}{3EI} & -\frac{L}{6EI} & 0 \\ -\frac{L}{6EI} & \frac{L}{3EI} & 0 \\ 0 & 0 & \frac{L}{AE} \end{bmatrix} \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & 0 \\ \frac{2EI}{L} & \frac{4EI}{L} & 0 \\ 0 & 0 & \frac{AE}{L} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When the effect of axial forces is disregarded, as is usually done in rigid frame analysis,

$$k^a = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \tag{10-11}$$

For a pin-connected truss,

$$k^a = \left[\frac{AE}{L}\right] \tag{10-12}$$

The second step in the analysis is to express the nodal forces in terms of the corresponding nodal displacements. This can be accomplished easily by the method of vitual work. Let us start by Eq. 9-1:

$$\delta r^T R = \delta q^T Q \tag{10-13}$$

From Eqs. 10-2 and 10-9,

$$q = ar \qquad \text{and} \qquad Q = kq$$

we have

$$\delta q = a\delta r$$

or

$$\delta q^T = \delta r^T a^T \tag{10-14}$$

and

$$Q = kar \tag{10-15}$$

Substituting Eqs. 10-14 and 10-15 in Eq. 10-13 gives

$$\delta r^T R = \delta r^T a^T kar$$

from which

$$R = a^T kar \tag{10-16}$$

or

$$R = Kr \tag{10-17}$$

if we make

$$K = a^T ka \tag{10-18}$$

$K$ being called the *total stiffness matrix,* or *the stiffness matrix of structure,* which directly relates the nodal forces to the nodal displacements of a structure.

If we premultiply Eq. 10-17 with $K^{-1}$ on both sides, we have

$$K^{-1}R = K^{-1}Kr = r$$

or

$$r = K^{-1}R \tag{10-19}$$

Comparing Eq. 10-19 with Eq. 9-22,

$$r = FR$$

we have

$$F = K^{-1} \tag{10-20}$$

It is thus seen that *the total flexibility matrix is the inverse of the total stiffness matrix, and vice versa if the same forces and displacements are involved.*

## 10-4 EQUILIBRIUM

Refer to Eq. 10-17:

$$R = Kr$$

If $r$ denotes the elements of all possible unknown nodal displacements (not including the known support or boundary conditions), then $R$ must denote all the corresponding nodal forces. The equilibrium of each node requires that the possible nodal forces, expressed in terms of unknown nodal displacements, must be equal to the applied loads. Thus, if these nodal loads are given, we can solve for the unknown nodal displacements by Eq. 10-19,

$$r = K^{-1}R$$

and for the member forces by Eqs. 10-2 and 10-9,

$$Q = kq = kar$$

## 10-5 ANALYSIS OF STRUCTURES BY THE MATRIX DISPLACEMENT METHOD

It is interesting that in the discussion of the displacement method the question of statical redundancy did not arise. The displacement method can apply with equal ease to statically determinate structures and statically indeterminate structures. The procedures of analysis by the displacement method are contained in the following steps:

1. Define all the possible unknown nodal displacements $r$.

Generally, a pin-connected node has two linear displacement components, with the rotation of the pin considered free of the connected members. A rigidly connected node has three displacement components, two linear and one rotational, but the linear displacements may be excepted if the axial deformations of the connected members are neglected and sidesway prevented. No displacements are assigned to the nodes that cannot move. Thus, in a pin-connected truss the hinged support is considered completely restrained; the roller support has one liner movement. In a rigid frame the built-in support undergoes no displacement; a hinged support can have only an angular displacement, whereas a roller support has one angular displacement and one linear displacement.

2. Determine the displacement transformation matrix $a$ from geometric configuration.

3. Determine the individual element stiffness matrices $k^a$, $k^b$, . . . according to Eq. 10-11 or 10-12, and assemble them in a diagonal matrix:

$$k = \begin{bmatrix} k^a & & & & \\ & k^b & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}$$

4. Compute the stiffness matrix of structure $K$:

$$K = a^T ka$$

5. Obtain the inverse of $K$.

6. Compute nodal displacements $r$ by

$$r = K^{-1}R$$

Note that $R$ is in one-to-one correspondence with $r$. Some of the $R$ are the actual loads; others are zero if no load is applied there. All the $R$ are known.

7. Compute the member forces $Q$ by

$$Q = kar$$

8. Distributed loads are handled indirectly by the procedure outlined in Sec. 9-2.

**Example 10-1**

Compute the nodal displacements and bar forces for the truss shown in Fig. 10-2(a). Assume that $E = 30{,}000$ kips/in.$^2$ and $L$(ft)/$A$(in.$^2$) = 1 for all members.

The roller support has a possible displacement $r_1$, and the top joint has possible displacement components $r_2$ and $r_3$, as indicated in Fig. 10-2(b). The nodal forces $R_1$, $R_2$, and $R_3$ correspond to the nodal displacements. Note that $R_1 = R_2 = 0$ and $R_3 = -8$ kips in this problem.

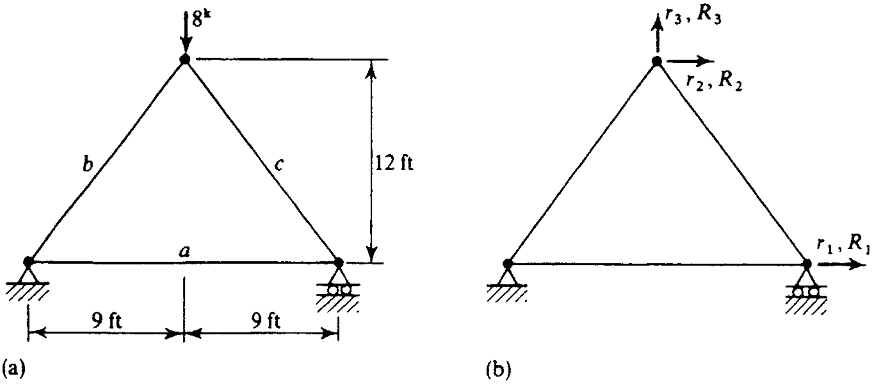


Fig. 10-2

Using the results of the example in Sec. 10-2, the displacement transformation matrix is

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0.6 & -0.6 & 0.8 \end{bmatrix}$$

The individual member stiffness matrix is determined by Eq. 10-12,

$$\left[\frac{AE}{L}\right]$$

Since $A/L = 1$ for all members,

$$k^a = k^b = k^c = E$$

from which we form the diagonal matrix

$$k = E\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Thus, the total stiffness matrix is

$$K = a^t k a = \begin{bmatrix} 1 & 0 & 0.6 \\ 0 & 0.6 & -0.6 \\ 0 & 0.8 & 0.8 \end{bmatrix} (E) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0.6 & -0.6 & 0.8 \end{bmatrix}$$

$$= E \begin{bmatrix} 1.36 & -0.36 & 0.48 \\ -0.36 & 0.72 & 0 \\ 0.48 & 0 & 1.28 \end{bmatrix}$$

$$K^{-1} = \frac{1}{E} \begin{bmatrix} 1 & 0.5 & -0.375 \\ 0.5 & 1.639 & -0.188 \\ -0.375 & -0.188 & 0.922 \end{bmatrix}$$

The nodal displacements are then determined by $r = K^{-1} R$:

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & 0.5 & -0.375 \\ 0.5 & 1.639 & -0.188 \\ -0.375 & -0.188 & 0.922 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -8 \end{Bmatrix} = \frac{1}{E} \begin{Bmatrix} 3 \\ 1.5 \\ -7.38 \end{Bmatrix}$$

Using $E = 30{,}000$ kips/in.$^2$, we obtain

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} 0.0001 \\ 0.00005 \\ -0.00027 \end{Bmatrix} \text{ft}$$

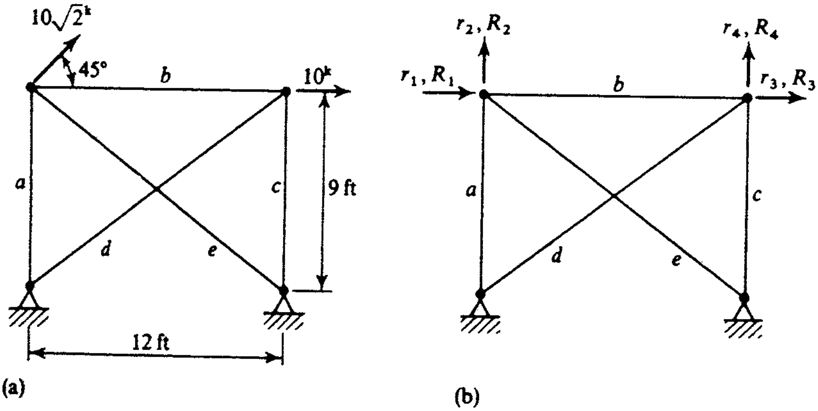
Finally, the bar forces are solved by $Q = kar$:

$$\begin{Bmatrix} Q^a \\ Q^b \\ Q^c \end{Bmatrix} = (E) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0.6 & -0.6 & 0.8 \end{bmatrix} \left(\frac{1}{E}\right) \begin{Bmatrix} 3 \\ 1.5 \\ -7.38 \end{Bmatrix} = \begin{Bmatrix} 3 \\ -5 \\ -5 \end{Bmatrix} \text{ kips}$$

**Example 10-2**

Solve the bar forces for the truss in Fig. 10-3(a). Assume constant $E$ and $L(\text{ft})/A(\text{in.}^2) = 1$ for all members.

The assigned nodal displacements and the corresponding nodal forces are shown in Fig. 10-3(b). In this problem $R_1 = R_2 = R_3 = 10$ kips, $R_4 = 0$.



(a)

(b)

Fig. 10-3

From compatibility,

$$\begin{Bmatrix} q^a \\ q^b \\ q^c \\ q^d \\ q^e \end{Bmatrix} = \underset{a}{\begin{array}{c} \begin{array}{cccc} r_1 = 1 & r_2 = 1 & r_3 = 1 & r_4 = 1 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.8 & 0.6 \\ -0.8 & 0.6 & 0 & 0 \end{bmatrix} \end{array}} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix}$$

Since $L/A = 1$ for all members, the individual member stiffness matrices are found to be

$$k^a = k^b = k^c = k^d = k^e = E$$

from which

$$k = E\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

Thus, we have the total stiffness matrix:

$$K = a^T k a$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 & -0.8 \\ 1 & 0 & 0 & 0 & 0.6 \\ 0 & 1 & 0 & 0.8 & 0 \\ 0 & 0 & 1 & 0.6 & 0 \end{bmatrix} (E) \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.8 & 0.6 \\ -0.8 & 0.6 & 0 & 0 \end{bmatrix}$$

$$= E\begin{bmatrix} 1.64 & -0.48 & -1 & 0 \\ -0.48 & 1.36 & 0 & 0 \\ -1 & 0 & 1.64 & 0.48 \\ 0 & 0 & 0.48 & 1.36 \end{bmatrix}$$

$$K^{-1} = \frac{1}{2.150E}\begin{bmatrix} 2.721 & 0.960 & 1.850 & -0.653 \\ 0.960 & 1.920 & 0.653 & -0.230 \\ 1.850 & 0.653 & 2.721 & -0.960 \\ -0.653 & -0.230 & -0.960 & 1.296 \end{bmatrix}$$

The nodal displacements are given by $r = K^{-1} R$:

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \frac{1}{2.150E}\begin{bmatrix} 2.721 & 0.960 & 1.850 & -0.653 \\ 0.960 & 1.920 & 0.653 & -0.230 \\ 1.850 & 0.653 & 2.721 & -0.960 \\ -0.653 & -0.230 & -0.960 & 1.296 \end{bmatrix}\begin{Bmatrix} 10 \\ 10 \\ 10 \\ 0 \end{Bmatrix} = \frac{1}{E}\begin{Bmatrix} 25.7 \\ 16.4 \\ 24.3 \\ -8.57 \end{Bmatrix}$$

The bar forces are then obtained from $Q = kar$.

$$\begin{Bmatrix} Q^a \\ Q^b \\ Q^c \\ Q^d \\ Q^e \end{Bmatrix} = (E)\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.8 & 0.6 \\ -0.8 & 0.6 & 0 & 0 \end{bmatrix}\left(\frac{1}{E}\right)\begin{Bmatrix} 25.7 \\ 16.4 \\ 24.3 \\ -\ 8.57 \end{Bmatrix} = \begin{Bmatrix} 16.40 \\ -\ 1.40 \\ -\ 8.57 \\ 14.30 \\ -10.72 \end{Bmatrix} \text{ kips}$$

**Example 10-3**

Find all the end moments for the frame shown in Fig. 10-4(a). Assume constant $EI$.

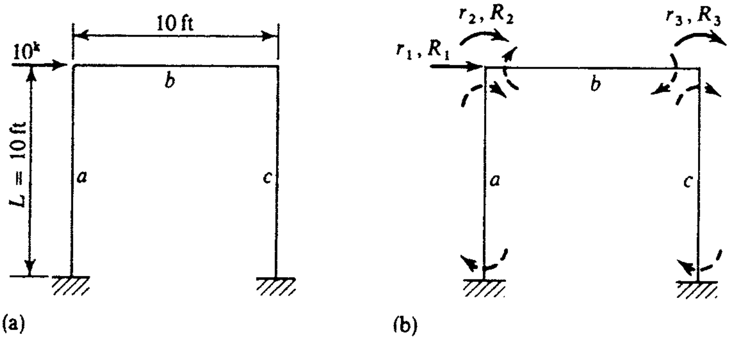
We assume unknown nodal displacements $r_1$, $r_2$, and $r_3$ and their corresponding forces $R_1$, $R_2$, and $R_3$, as shown in Fig. 10-4(b). Note that $r_1$ represents the sidesway of the frame and that $r_2$ and $r_3$ represent the rotations at the joints. In the present case $R_1 = 10$ kips, $R_2 = R_3 = 0$. The dashed lines in Fig. 10-4(b) indicate the assumed directions for the member moments, $Q_i^a$, $Q_j^a$, . . . , and their corresponding member rotations, $q_i^a$, $q_j^a$, . . . . From compatibility

$$\begin{Bmatrix} q_i^a \\ q_j^a \\ q_i^b \\ q_j^b \\ q_i^c \\ q_j^c \end{Bmatrix} = \underset{a}{\begin{array}{c} \begin{array}{ccc} r_1 = 1 & r_2 = 1 & r_3 = 1 \end{array} \\ \begin{bmatrix} -\frac{1}{10} & 0 & 0 \\ -\frac{1}{10} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{10} & 0 & 1 \\ -\frac{1}{10} & 0 & 0 \end{bmatrix} \end{array}} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

Since the members are identical, the individual member stiffness matrices are the same:

$$k^a = k^b = k^c = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

See Eq. 10-11.



(a) (b)

Fig. 10-4

Using Eq. 10-9a, we have

$$k = \frac{EI}{L}\begin{bmatrix} 4 & 2 & & & & \\ 2 & 4 & & & & \\ & & 4 & 2 & & \\ & & 2 & 4 & & \\ & & & & 4 & 2 \\ & & & & 2 & 4 \end{bmatrix}$$

Thus, the total stiffness matrix is

$$K = a^T ka$$

$$= \begin{bmatrix} -\frac{1}{10} & -\frac{1}{10} & 0 & 0 & -\frac{1}{10} & -\frac{1}{10} \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\cdot\left(\frac{EI}{L}\right)\begin{bmatrix} 4 & 2 & & & & \\ 2 & 4 & & & & \\ & & 4 & 2 & & \\ & & 2 & 4 & & \\ & & & & 4 & 2 \\ & & & & 2 & 4 \end{bmatrix}\begin{bmatrix} -\frac{1}{10} & 0 & 0 \\ -\frac{1}{10} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{10} & 0 & 1 \\ -\frac{1}{10} & 0 & 0 \end{bmatrix}$$

$$= \frac{EI}{L}\begin{bmatrix} 0.24 & -0.6 & -0.6 \\ -0.6 & 8 & 2 \\ -0.6 & 2 & 8 \end{bmatrix}$$

$$K^{-1} = \frac{L}{EI}\left(\frac{1}{10.08}\right)\begin{bmatrix} 60 & 3.6 & 3.6 \\ 3.6 & 1.56 & -0.12 \\ 3.6 & -0.12 & 1.56 \end{bmatrix}$$

The nodal displacements are then obtained from $r = K^{-1}R$:

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \frac{L}{EI}\left(\frac{1}{10.08}\right)\begin{bmatrix} 60 & 3.6 & 3.6 \\ 3.6 & 1.56 & -0.12 \\ 3.6 & -0.12 & 1.56 \end{bmatrix}\begin{Bmatrix} 10 \\ 0 \\ 0 \end{Bmatrix} = \frac{L}{EI}\left(\frac{1}{10.08}\right)\begin{Bmatrix} 600 \\ 36 \\ 36 \end{Bmatrix}$$

Finally, the end moments are determined by $Q = kar$. Using the values of $k$, $a$, and $r$ previously obtained gives

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \\ Q_i^c \\ Q_j^c \end{Bmatrix} = \begin{Bmatrix} -28.6 \\ -21.4 \\ 21.4 \\ 21.4 \\ -21.4 \\ -28.6 \end{Bmatrix} \text{ ft-kips}$$

**Example 10-4**

The end moments of the rigid frame shown in Fig. 10-5 were solved by the force method in Example 9-5 and will now be re-solved by the displacement method.

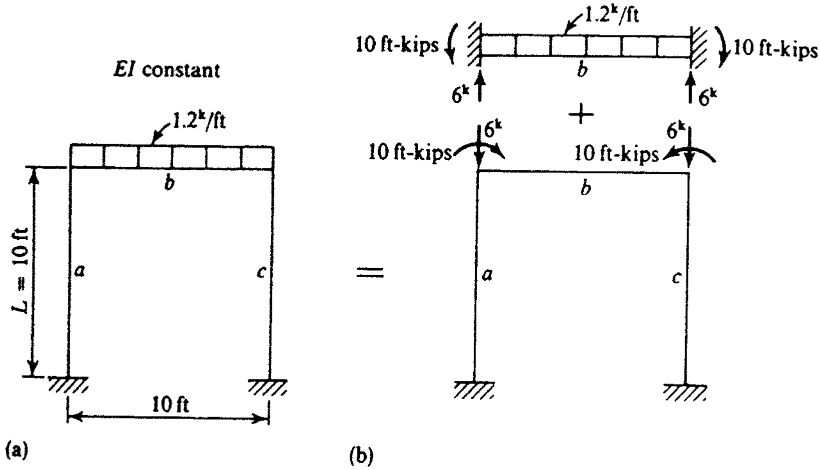


Fig. 10-5

Neglecting the effect of axial deformations, the frame prepared for nodal-force analysis is shown in Fig. 10-6, where $r_1$ and $r_2$ denote the joint rotations and $R_1$ and $R_2$ denote the corresponding moments. In this problem $R_1 = -R_2 = 10$ ft-kips. The dashed lines indicate assumed directions for member end moments, $Q_i^a, Q_j^a, \ldots$, and their corresponding member end rotations, $q_i^a, q_j^a, \ldots$. The

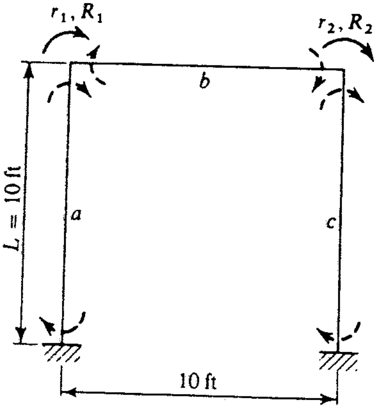


Fig. 10-6

latter can be expressed in terms of $r_1$ and $r_2$ as

$$\begin{Bmatrix} q_i^a \\ q_j^a \\ q_i^b \\ q_j^b \\ q_i^c \\ q_j^c \end{Bmatrix} = \underbrace{\begin{array}{cc} r_1 = 1 & r_2 = 1 \\ \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right. & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{array}\right] \end{array}}_{a} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

The diagonal matrix $k$ is the same as that obtained in the preceding example. The stiffness matrix of structure is obtained by

$$K = a^T k a$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \left(\frac{EI}{L}\right) \begin{bmatrix} 4 & 2 & & & & \\ 2 & 4 & & & & \\ & & 4 & 2 & & \\ & & 2 & 4 & & \\ & & & & 4 & 2 \\ & & & & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{EI}{L} \begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix}$$

Thus,

$$K^{-1} = \frac{L}{EI} \frac{\begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix}^T}{60} = \frac{L}{30EI} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$$

and

$$r = K^{-1} R$$

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \frac{L}{30EI} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} 10 \\ -10 \end{Bmatrix} = \frac{L}{30EI} \begin{Bmatrix} 50 \\ -50 \end{Bmatrix}$$

Then the member end moments based on the nodal-force analysis (Fig. 10-6) are determined by $Q = kar$. Using the values of $k$, $a$, and $r$ previously found gives

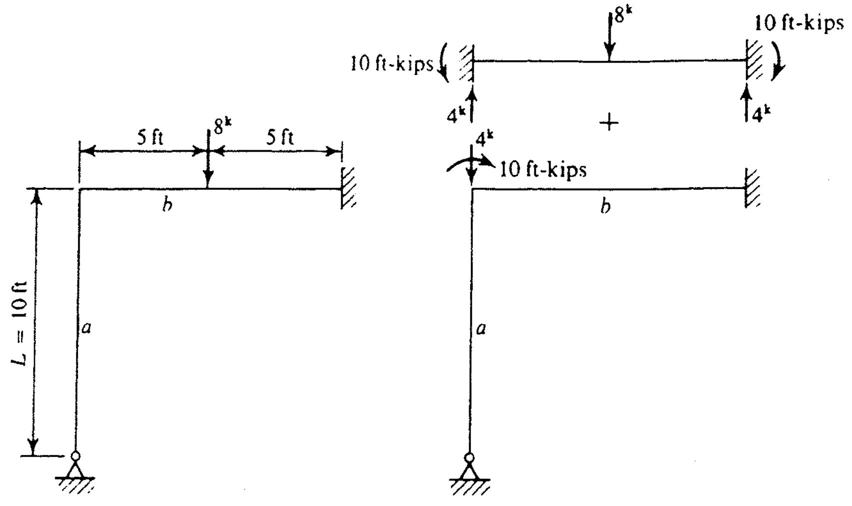
$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \\ Q_i^c \\ Q_j^c \end{Bmatrix} = \begin{Bmatrix} 3.33 \\ 6.67 \\ 3.33 \\ -3.33 \\ -6.67 \\ -3.33 \end{Bmatrix} \text{ ft-kips}$$

The end moments of member $b$, $Q_i^b$ and $Q_j^b$, must be corrected by adding the fixed-end moments shown in the upper part of Fig. 10-5(b). The final result is

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \\ Q_i^c \\ Q_j^c \end{Bmatrix} = \begin{Bmatrix} 3.33 \\ 6.67 \\ -6.67 \\ 6.67 \\ -6.67 \\ -3.33 \end{Bmatrix} \text{ ft-kips}$$

**Example 10-5**

Obtain the end moments for the frame shown in Fig. 10-7(a). Use the equivalent form in Fig. 10-7(b) for the analysis so that the size of the matrices will be reduced. Assume constant *EI*.



(a) (b)

**Fig. 10-7**

If the effect of axial deformations is neglected, the frame prepared for nodal-force analysis is shown in Fig. 10-8. The frame is subjected to the joint rotations $r_1$ and $r_2$ and the corresponding nodal moments $R_1$ and $R_2$. In this case, $R_1 = 0$ and $R_2 = 10$ ft-kips.

As before, the dashed lines indicate the assumed directions for the member end moments $Q_i^a$, $Q_j^a$, $Q_i^b$, and $Q_j^b$ and their corresponding end rotations $q_i^a$, $q_j^a$, $q_i^b$, and $q_j^b$. The latter can be expressed in terms of $r_1$ and $r_2$ from compatibility as

$$\begin{Bmatrix} q_i^a \\ q_j^a \\ q_i^b \\ q_j^b \end{Bmatrix} = \begin{array}{c} \begin{array}{cc} r_1 = 1 & r_2 = 1 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ a \end{array} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

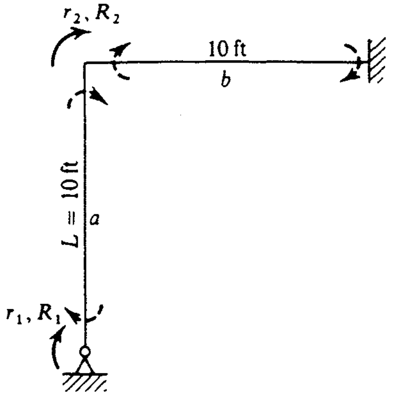


Fig. 10-8

Since the members are identical,

$$k = \frac{EI}{L}\begin{bmatrix} 4 & 2 & & \\ 2 & 4 & & \\ & & 4 & 2 \\ & & 2 & 4 \end{bmatrix}$$

The total stiffness matrix is then determined.

$$K = a^T k a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}\left(\frac{EI}{L}\right)\begin{bmatrix} 4 & 2 & & \\ 2 & 4 & & \\ & & 4 & 2 \\ & & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix}$$

from which

$$K^{-1} = \frac{L}{EI}\,\frac{\begin{bmatrix} 8 & -2 \\ -2 & 4 \end{bmatrix}^T}{28} = \frac{L}{14EI}\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

The nodal displacements are expressed by

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = K^{-1}R = \frac{L}{14EI}\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \frac{L}{14EI}\begin{Bmatrix} -10 \\ 20 \end{Bmatrix}$$

Thus, the end moments from nodal-force analysis, based on Fig. 10-8, are found to be

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \end{Bmatrix} = kar = \frac{EI}{L}\begin{bmatrix} 4 & 2 & & \\ 2 & 4 & & \\ & & 4 & 2 \\ & & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\left(\frac{L}{14EI}\right)\begin{Bmatrix} -10 \\ 20 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4.28 \\ 5.72 \\ 2.86 \end{Bmatrix} \text{ ft-kips}$$

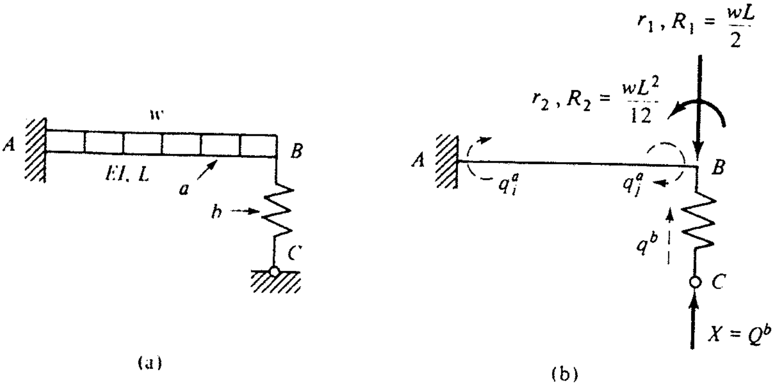
After adding the fixed-end moments [see the upper part of Fig. 10-7(b)] to $Q_i^b$ and $Q_j^b$, we obtain the final solution as

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \\ Q_j^b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4.28 \\ -4.28 \\ 12.86 \end{Bmatrix} \text{ ft-kips}$$

**Example 10-6**

The deflection and slope at $B$ for the system shown in Fig. 10-9(a) were solved by the force method in Example 9-6, and will be resolved by the displacement method. Note that the spring stiffness is $k_s$ or $1/f_s$.

As in Example 9-6, to obtain the nodal displacements at $B$, it is necessary only to analyze the system under nodal loads shown in Fig. 10-9(b). Also shown in Fig. 10-9(b) by dashed lines are the member deformations $q_i^a$, $q_j^a$, and $q^b$ (spring contraction).



(a) (b)

**Fig. 10-9**

We first relate $q_i^a$, $q_j^a$, and $q^b$ to the external nodal displacements $r_1$ and $r_2$ as

$$\begin{Bmatrix} q_i^a \\ q_j^a \\ q^b \end{Bmatrix} = \underset{a}{\begin{bmatrix} -\frac{1}{L} & 0 \\ -\frac{1}{L} & -1 \\ 1 & 0 \end{bmatrix}} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

From the beam stiffness

$$k^a = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

and the spring stiffness $k^b = k_s$, we form

$$k = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} & \\ \frac{2EI}{L} & \frac{4EI}{L} & \\ & & k_s \end{bmatrix}$$

With matrices $a$ and $k$ determined, we obtain

$$K = a^T k a = \begin{bmatrix} \dfrac{12EI}{L^3} + k_s & \dfrac{6EI}{L^2} \\ \dfrac{6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix}$$

$$K^{-1} = \frac{\begin{bmatrix} \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} \\ -\dfrac{6EI}{L^2} & \dfrac{12EI}{L^3} + k_s \end{bmatrix}}{\dfrac{4EI}{L}\left(\dfrac{3EI}{L^3} + k_s\right)}$$

Using

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = [K]^{-1} \begin{Bmatrix} R_1 = \dfrac{wL}{2} \\ R_2 = \dfrac{wL^2}{12} \end{Bmatrix}$$

we solve for

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \frac{1}{\dfrac{3EI}{L^3} + k_s} \begin{Bmatrix} \dfrac{3}{8}wL \\ -\dfrac{1}{2}w + \dfrac{k_s wL^3}{48EI} \end{Bmatrix}$$

which can be checked with the result found in Example 9-6.

## 10-6 USE OF THE MODIFIED MEMBER STIFFNESS MATRIX

Referring to Eq. 10-11, we find the stiffness matrix for a uniform member in frame analysis is given by

$$k^a = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

Note that the elements in the first column of the matrix are the member end moments obtained by producing in this end ($i$ end) a unit rotation, the other end ($j$ end) being fixed. The procedure described for obtaining these values is exactly the same as that for finding the stiffness of the $i$ end and its carry-over value to the $j$ end in the method of moment distribution, as illustrated in Fig. 10-10. Recall, in the moment-distribution procedures, that if the actual condition of the other end is known, then the computation can be simplified by using the modified stiffness of this end and omitting the presentation of the other portion of structure in the analysis. This technique can also be applied to the displacement method by introducing the *modified member stiffness matrix*. We note the following special cases:

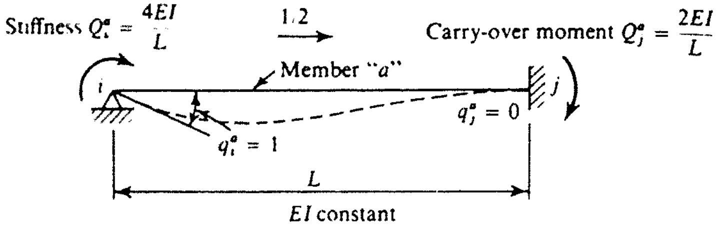


**Fig. 10-10**

1. When the other end is simply supported, then the moment needed to produce a unit rotation in this end is $3EI/L$; that is, $Q_i^a = 3EI/L$ for $q_i^a = 1$, as indicated in Fig. 10-11.

Modified stiffness

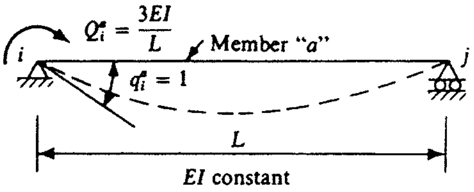


**Fig. 10-11**

The modified member stiffness matrix, if we disregard the factors of $Q_j^a$ and $q_j^a$, is then given by

$$(k^a)' = \left[\frac{3EI}{L}\right] \tag{10-21}$$

where $(k^a)'$ denotes the modified member stiffness matrix for member $a$.

2. When the other end rotates an equal but opposite angle to that of this end (the case of symmetry),

$$(k^a)' = \left[\frac{2EI}{L}\right] \tag{10-22}$$

3. When the other end rotates the same angle as that of this end (the case of antisymmetry),

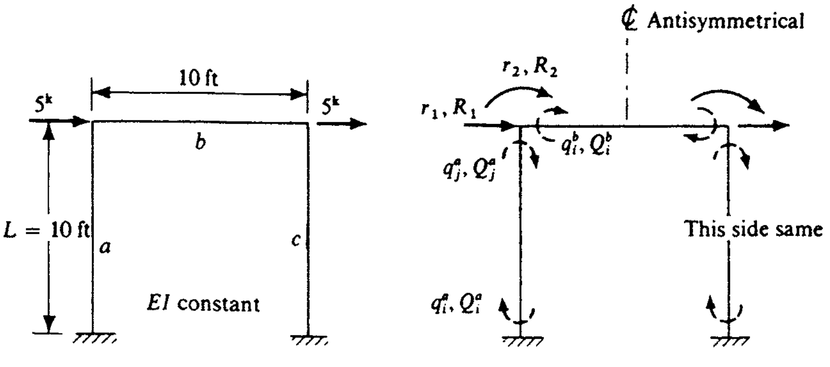
$$(k^a)' = \left[\frac{6EI}{L}\right] \tag{10-23}$$

Application of the procedures just described is illustrated by re-solving Examples 10-3 to 10-5 of Sec. 10-5 as follows:

**Example 10-7**

Re-solve Example 10-3 by using the modified stiffness matrix.

The frame in Fig. 10-4(a) may be put in the form of Fig. 10-12(a). The structure and its loading represent a case of antisymmetry for which we may assume the nodal displacements and their corresponding nodal forces as shown in Fig. 10-12(b). Observe that $R_1 = 5$ kips and $R_2 = 0$.



(a) (b)

**Fig. 10-12**

The analysis can be simplified by working with only half the structure if the modified stiffness matrix for member $b$ is used:

$$(k^b)' = \left[\frac{6EI}{L}\right]$$

Hence,

$$k = \begin{bmatrix} k^a & \\ & (k^b)' \end{bmatrix} = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

From compatibility

$$\begin{Bmatrix} q_i^a \\ q_j^a \\ q_i^b \end{Bmatrix} = \underbrace{\begin{array}{cc} r_1 = 1 & r_2 = 1 \\ \left[\begin{matrix} -\frac{1}{10} \\ -\frac{1}{10} \\ 0 \end{matrix}\right. & \left.\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right] \end{array}}_{a} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

Using the values already found for $a$ and $k$, we have

$$K = a^Tka = \frac{EI}{L}\begin{bmatrix} 0.12 & -0.6 \\ -0.6 & 10 \end{bmatrix}$$

$$K^{-1} = \frac{L}{0.84EI}\begin{bmatrix} 10 & 0.6 \\ 0.6 & 0.12 \end{bmatrix}$$

$$r = K^{-1}R = \frac{L}{0.84EI}\begin{bmatrix} 10 & 0.6 \\ 0.6 & 0.12 \end{bmatrix}\begin{Bmatrix} 5 \\ 0 \end{Bmatrix} = \frac{L}{0.84EI}\begin{Bmatrix} 50 \\ 3 \end{Bmatrix}$$

Thus,

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \end{Bmatrix} = kar = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{10} & 0 \\ -\frac{1}{10} & 1 \\ 0 & 1 \end{bmatrix}\left(\frac{L}{0.84EI}\right)\begin{Bmatrix} 50 \\ 3 \end{Bmatrix}$$

$$= \begin{Bmatrix} -28.6 \\ -21.4 \\ 21.4 \end{Bmatrix} \text{ ft-kips}$$

The results, $Q_i^a = Q_j^c = -28.6$ ft-kips, $Q_j^a = Q_i^c = -21.4$ ft-kips, and $Q_i^b = Q_j^b = 21.4$ ft-kips, are the same as previously found in Example 10-3.

**Example 10-8**

Re-solve Example 10-4 by using the modified stiffness matrix.

The portal frame shown in Fig. 10-5(a) is symmetrical about the center line of the beam. The frame assumed for nodal-force analysis may be given as in Fig. 10-13. Referring to Fig. 10-5(b), we note that $R_1 = 10$ ft-kips. The analysis can be simplified by working with only half the frame and using the modified stiffness matrix for member $b$:

$$(k^b)' = \left[\frac{2EI}{L}\right]$$

Thus,

$$k = \begin{bmatrix} k^a & \\ & (k^b)' \end{bmatrix} = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

From compatibility,

$$\begin{Bmatrix} q_i^a \\ q_j^a \\ q_i^b \end{Bmatrix} = \underset{a}{\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}} [r_1]$$

With the values of $a$ and $k$ found, we obtain

$$K = a^T k a = \left[\frac{6EI}{L}\right]$$

It follows that

$$K^{-1} = \left[\frac{L}{6EI}\right]$$

$$r_1 = K^{-1}R_1 = \left[\frac{L}{6EI}\right][10] = \left[\frac{10L}{6EI}\right]$$

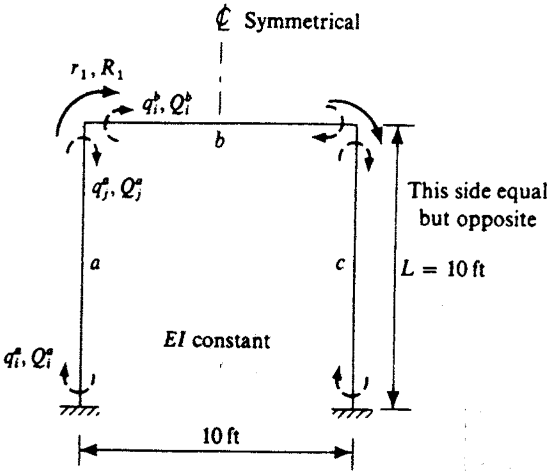


Fig. 10-13

With the values of $a$, $k$, and $r_1$ found, we obtain the end moments from nodal-force analysis as

$$\begin{Bmatrix} Q_i^a \\ Q_j^a \\ Q_i^b \end{Bmatrix} = kar_1 = \begin{Bmatrix} 3.33 \\ 6.67 \\ 3.33 \end{Bmatrix} \text{ ft-kips}$$

Adding the fixed-end moment of $-10$ ft-kips [see the upper part of Fig. 10-5(b)] to $Q_i^b$ in the result above and using symmetry, we obtain the final solution as

$$Q_i^a = -Q_j^c = 3.33 \text{ ft-kips}$$
$$Q_j^a = -Q_i^c = 6.67 \text{ ft-kips}$$
$$Q_i^b = -Q_j^b = -6.67 \text{ ft-kips}$$

These are the same as previously obtained in Example 10-4.

**Example 10-9**

Re-solve Example 10-5 by using modified member stiffness.

Refer to Fig. 10-7. Since the moment at the hinged end is zero, we may simplify the computation by using the modified stiffness matrix $[3EI/L]$ for member $a$ and by assuming the frame for nodal-force analysis as in Fig. 10-14. Note that in present case $R_1 = 10$ ft-kips.

$$k = \begin{bmatrix} (k^a)' & \\ & k^b \end{bmatrix} = \frac{EI}{L}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

From compatibility

$$\begin{Bmatrix} q_j^a \\ q_i^b \\ q_j^b \end{Bmatrix} = \underset{a}{\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}} [r_1]$$

With $k$ and $a$ determined, we have

$$K = a^Tka = \frac{EI}{L}[7]$$

$$r_1 = K^{-1}R_1 = \frac{L}{EI}\left[\frac{10}{7}\right]$$

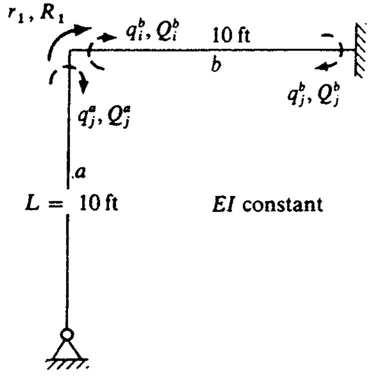


Fig. 10-14

Thus, the end moments from nodal-force analysis are given by

$$\begin{Bmatrix} Q_j^a \\ Q_i^b \\ Q_j^b \end{Bmatrix} = kar_1 = \frac{EI}{L}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}\left(\frac{L}{EI}\right)\left[\frac{10}{7}\right] = \begin{Bmatrix} 4.28 \\ 5.72 \\ 2.86 \end{Bmatrix} \quad \text{ft-kips}$$

After the values of $Q_i^b$ and $Q_j^b$ are corrected by adding the fixed-end moment [see the upper part of Fig. 10-7(b)], we obtain the final solution as previously found in Example 10-5:

$$\begin{Bmatrix} Q_j^a \\ Q_i^b \\ Q_j^b \end{Bmatrix} = \begin{Bmatrix} 4.28 \\ -4.28 \\ 12.86 \end{Bmatrix} \quad \text{ft-kips}$$

## 10-7 THE GENERAL FORMULATION OF THE MATRIX DISPLACEMENT METHOD

It may be interesting to point out that in a frame structure, there exist generally two kinds of node: those with unknown displacements associated with known forces (unprescribed node or free node) and those with known displacements associated with unknown forces (prescribed node). Throughout the previous sections of this chapter, we dealt only with the former kind of node. In a more general case, however, the formulation of displacement procedure should include also the unknown forces (e.g., the support reactions) and the corresponding known displacements at the prescribed nodes. As will be seen, the matrix displacement formulation in this manner leads to a direct analogy with the matrix force presentation as developed in Sec. 9-5. The procedure can be summarized as follows:

1. Separate the nodal displacements into two categories: the unprescribed displacements $r_R$ corresponding to the applied nodal loads $R$ and the prescribed displacements $r_X$ corresponding to the forces $X$ yet to be determined.

2. Establish the displacement transformation matrix to relate the element end displacements $q$ with the nodal displacements $r_R$ and $r_X$:

$$q = \underset{a}{[a_R \mid a_X]}\begin{Bmatrix} r_R \\ \hdashline r_X \end{Bmatrix} \tag{10-24}$$

where the transformation matrix $a$ is partitioned into $a_R$ and $a_X$, indicating the separate influences from $r_R$ and $r_X$.

3. Assemble the element stiffness matrices $k^a$, $k^b$, . . . to obtain

$$k = \begin{bmatrix} k^a & & & & \\ & k^a & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix} \tag{10-25}$$

4. Construct the total stiffness matrix by the process

$$K = a^T k a$$

$$= \begin{Bmatrix} a_R^T \\ \cdots \\ a_X^T \end{Bmatrix} [k][a_R \;\vdots\; a_X] \tag{10-26}$$

$$= \left[\begin{array}{c:c} a_R^T k a_R & a_R^T k a_X \\ \hdashline a_X^T k a_R & a_X^T k a_X \end{array}\right]$$

5. The overall stiffness expressed in Eq. 10-26 relates the total nodal forces $R$ and $X$ to the corresponding displacements $r_R$ and $r_X$. The general formulation can be written as

$$\begin{Bmatrix} R \\ \cdots \\ X \end{Bmatrix} = \left[\begin{array}{c:c} K_{RR} & K_{RX} \\ \hdashline K_{XR} & K_{XX} \end{array}\right] \begin{Bmatrix} r_R \\ \cdots \\ r_X \end{Bmatrix} \tag{10-27}$$

where

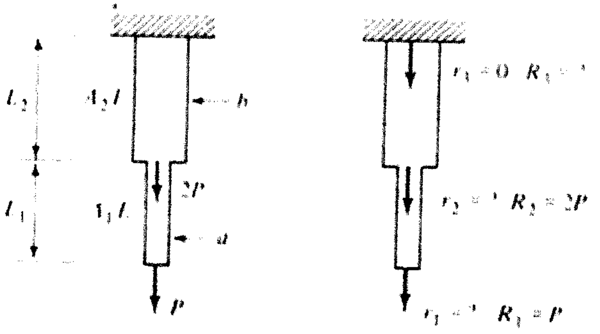
$$K_{RR} = a_R^T k a_R \qquad K_{RX} = a_R^T k a_X$$
$$K_{XR} = a_X^T k a_R \qquad K_{XX} = a_X^T k a_X \tag{10-28}$$

The frequently encountered case is that the prescribed displacements are null (i.e., $r_X = 0$). Then Eq. 10-27 reduces to

$$\begin{Bmatrix} R \\ \cdots \\ X \end{Bmatrix} = \left[\begin{array}{c:c} K_{RR} & K_{RX} \\ \hdashline K_{XR} & K_{XX} \end{array}\right] \begin{Bmatrix} r_R \\ \cdots \\ 0 \end{Bmatrix} \tag{10-29}$$

**Example 10-10**

Consider the hanger made up by a two-section bar as shown in Fig. 10-15(a). The bar is subjected to load of $P$ at the free end, and load of $2P$ at the point where the change of section takes place. Find the elongations measured at the loaded points, and the reaction at the fixed support [see Fig. 10-15(b)]. Assume that the bar weight is negligible.



(a) (b)

**Fig. 10-15**

The bar elongations $q^a$ and $q^b$ are related to the nodal displacements $r_1$, $r_2$, and $r_3$ ($r_3 = 0$) as

$$\begin{Bmatrix} q^a \\ q^b \end{Bmatrix} = \begin{array}{c} \left[\begin{array}{cc:c} 1 & -1 & 0 \\ 0 & 1 & -1 \end{array}\right] \\ \begin{array}{cc} \quad a_R & \qquad a_X \end{array} \end{array} \left\{\begin{array}{c} r_1 \\ r_2 \\ \hdashline r_3 \end{array}\right\}$$

From the element stiffness matrices $k^a = A_1E/L_1$ and $k^b = A_2E/L_2$, we obtain

$$k = \begin{bmatrix} \dfrac{A_1E}{L_1} & \\ & \dfrac{A_2E}{L_2} \end{bmatrix}$$

With matrices $a_R$, $a_X$, and $k$ determined, the partitioned matrices for the total stiffness matrix are then calculated:

$$K_{RR} = a_R^T k a_R = \begin{bmatrix} \dfrac{A_1E}{L_1} & -\dfrac{A_1E}{L_1} \\ -\dfrac{A_1E}{L_1} & \dfrac{A_1E}{L_1} + \dfrac{A_2E}{L_2} \end{bmatrix}$$

$$K_{RX} = a_R^T k a_X = \begin{bmatrix} 0 \\ -\dfrac{A_2E}{L_2} \end{bmatrix}$$

$$K_{XR} = a_X^T k a_R = \begin{bmatrix} 0 & -\dfrac{A_2E}{L_2} \end{bmatrix}$$

$$K_{XX} = a_X^T k a_X = \begin{bmatrix} \dfrac{A_2E}{L_2} \end{bmatrix}$$

Hence, the force-displacement relationship is

$$\left\{\begin{array}{c} R_1 = P \\ R_2 = 2P \\ \hdashline R_3 = ? \end{array}\right\} = \left[\begin{array}{cc:c} \dfrac{A_1E}{L_1} & -\dfrac{A_1E}{L_1} & 0 \\ -\dfrac{A_1E}{L_1} & \dfrac{A_1E}{L_1} + \dfrac{A_2E}{L_2} & -\dfrac{A_2E}{L_2} \\ \hdashline 0 & -\dfrac{A_2E}{L_2} & \dfrac{A_2E}{L_2} \end{array}\right] \left\{\begin{array}{c} r_1 = ? \\ r_2 = ? \\ \hdashline r_3 = 0 \end{array}\right\}$$

We solve the unknown displacements $r_1$ and $r_2$ by

$$\begin{aligned} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} &= [K_{RR}]^{-1} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} \\ &= \begin{bmatrix} \dfrac{A_1E}{L_1} & -\dfrac{A_1E}{L_1} \\ -\dfrac{A_1E}{L_1} & \dfrac{A_1E}{L_1} + \dfrac{A_2E}{L_2} \end{bmatrix}^{-1} \begin{Bmatrix} P \\ 2P \end{Bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \frac{L_1}{A_1E} + \frac{L_2}{A_2E} & \frac{L_2}{A_2E} \\ \frac{L_2}{A_2E} & \frac{L_2}{A_2E} \end{bmatrix} \begin{Bmatrix} P \\ 2P \end{Bmatrix}$$

Therefore,

$$r_1 = \frac{PL_1}{A_1E} + \frac{3PL_2}{A_2E}$$

$$r_2 = \frac{3PL_2}{A_2E}$$

Finally, we obtain the end reaction $R_3$ by

$$R_3 = [K_{XR}] \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{A_2E}{L_2} \end{bmatrix} \begin{Bmatrix} \frac{PL_1}{A_1E} + \frac{3PL_2}{A_2E} \\ \frac{3PL_2}{A_2E} \end{Bmatrix} = -3P$$

These results can easily be checked.

## 10-8 COMPARISON OF THE FORCE METHOD AND THE DISPLACEMENT METHOD

The force method and the displacement method represent two parallel ways of analyzing structures. The basic procedures for the two methods may be briefly recapitulated as in Table 10-1. The duality between the two methods is apparent. The choice of the methods mainly lies in the accuracy of the solution and the ease of computation which in turn would depend upon the idealization of the structure, the rounding off of error, and the type of and formulation of the problem. Generally, except for structures that involve many joint displacements but few force redundants, the displacement method is often preferred. Some of the reasons are as follows:

**TABLE 10-1**

| Force Method | Displacement Method |
|---|---|
| (1) Select member forces as basic unknowns. | (1) Select nodal displacements as basic unknowns. |
| (2) Establish the force transformation matrix. | (2) Establish the displacemen transformation matrix. |
| (3) Evaluate member flexibility matrices. | (3) Evaluate member stiffness matrices. |
| (4) Obtain the total flexibility matrix. | (4) Obtain the total stiffness matrix. |
| (5) Express the nodal displacements in terms of the nodal forces. | (5) Express the nodal forces in terms of the nodal displacements. |

1. In the displacement method the irrelevancy of force redundancy enables the use of the same procedures for analyzing statically determinate structures and statically indeterminate structures.

2. It is much easier to form the displacement transformation matrix than the force transformation matrix, since the effects of displacements are often localized.

3. It is found that the displacement method usually produces a well-conditioned stiffness matrix of structure; whereas in the force method, a well-conditioned flexibility matrix of the structure depends upon a good choice of force redundants.

Note that a *well-conditioned matrix* is the one for which the largest terms lie on the main diagonal and is, thus, most suitable for computer operation.

## PROBLEMS

**10-1.** Find, by the force method, the deflection and slope at the free end for the beam shown in Fig. 10-16. Repeat it by the displacement method and compare.

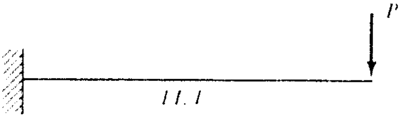


**Fig. 10-16**

**10-2.** Find, by the force method, the bar forces and the nodal displacements for the truss shown in Fig. 10-17. Assume that $L/AE = 1$ for all members. Repeat it by the displacement method and compare.

**10-3.** Solve Prob. 9-6 by the displacement method.

**10-4.** Solve Prob. 9-10 by the displacement method.

**10-5.** Analyze the frame shown in Fig. 10-18 by the displacement method. Assume that $EI/L = 1$ for both members.

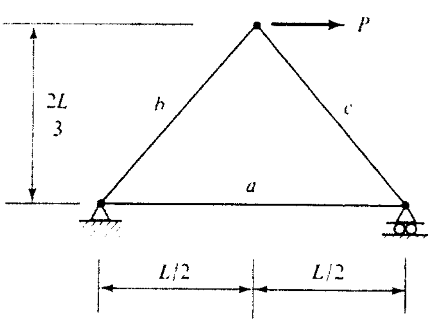


**Fig. 10-17**

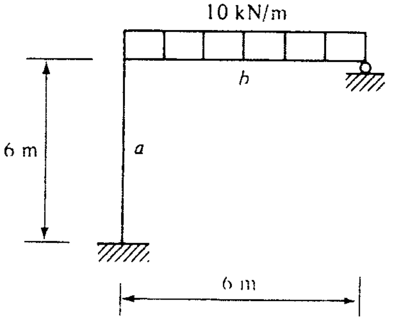


**Fig. 10-18**

**10-6.** Find, by the displacement method, the slope and deflection at midspan section $C$ for the simple beam under a uniform load shown in Fig. 10-19. Use modified member stiffness to simplify calculation.

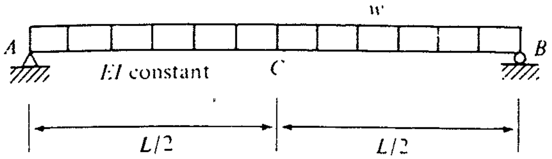


Fig. 10-19

**10-7.** Find, by the displacement method, the moment at support $B$ of the beam shown in Fig. 10-20.

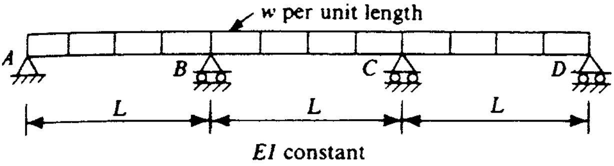


Fig. 10-20

**10-8.** Obtain, by the displacement method, all the end moments for the rigid frame shown in Fig. 10-21.

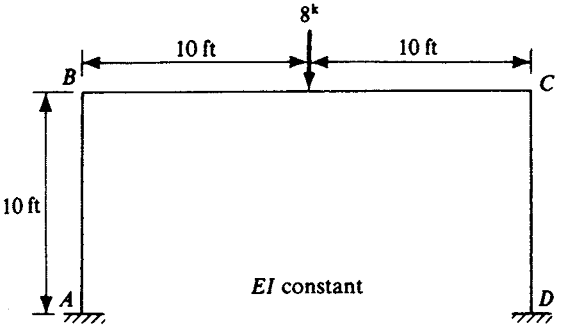


Fig. 10-21

**10-9.** Obtain, by the displacement method, all the end moments for the frame of Fig. 10-22, using the modified member stiffness matrix for the center beam due to antisymmetry.

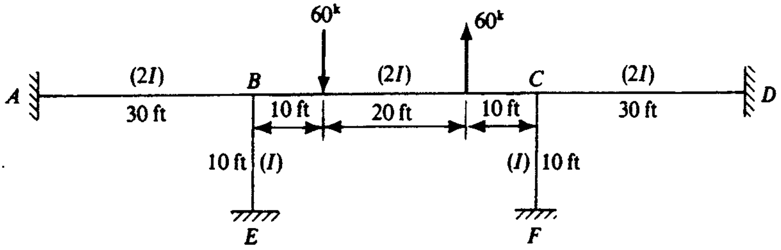


Fig. 10-22

**10-10.** For the composite system in Fig. 10-23, construct the displacement transformation matrix consistent with the nodal displacements shown.

**10-11.** Use the displacement method to obtain the end moments for the frame shown in Fig. 10-24. Assume constant $EI$.

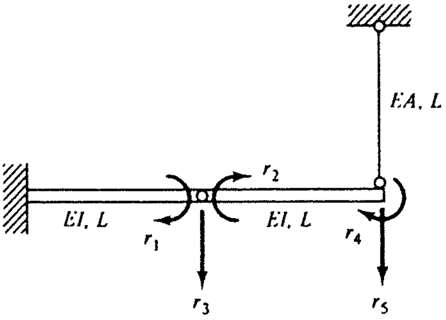


Fig. 10-23

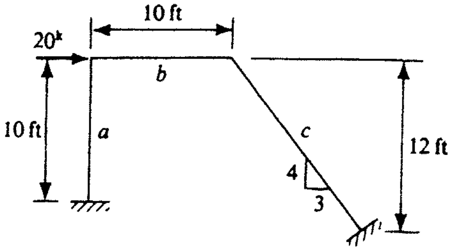


Fig. 10-24

**10-12.** Use the general formulation of displacement method to solve the frame in Fig. 10-18.

# 11

# *Direct Stiffness Method*

## 11-1 GENERAL

In the preceding chapter, we discussed the matrix displacement method (stiffness method) for analyzing simple structures. The structure stiffness matrix was established by

$$K = a^T ka$$

However, if the structure to be analyzed is large and complicated and if member end actions include more force components, then not only must the size of element stiffness matrices be increased to accommodate these requirements but the establishment of the displacement transformation matrix $a$ also becomes involved. Therefore, the preceding procedure in forming the total stiffness matrix $K$ would appear inefficient.

Alternatively, we may treat each individual member or element as a structure (member-structure) and obtain the stiffness matrix for the member-structure by using the smaller matrices $a$, $a^T$, and $k$ of the member. The total stiffness of the entire structure is then constructed by superimposing the stiffness matrices of the individual structural members. This method, referred to as the *direct stiffness method,* casts the analysis into a more formalized format, which may be readily programmed on a digital computer.

Both the matrix displacement method and the direct stiffness method are intended to establish the same total structure stiffness matrix $K$ to relate the nodal forces and displacements, as given by the general expression of Eq.

10-27:

$$\left\{\begin{array}{c} R \\ \hline X \end{array}\right\} = \underbrace{\left[\begin{array}{c|c} K_{RR} & K_{RX} \\ \hline K_{XR} & K_{XX} \end{array}\right]}_{K} \left\{\begin{array}{c} r_R \\ \hline r_X \end{array}\right\}$$

which separates the influences from unprescribed (unknown) nodal displacements $r_R$ and prescribed nodal displacements $r_X$ on $K$. The direct stiffness method differs from the matrix displacement method only in the manner by which the total stiffness is constructed. It involves the transforming of the element stiffness matrices in local coordinates to global coordinates and superposing them through numbering identification to obtain the structure stiffness matrix. For complex structure, this difference is important.

## 11-2 ELEMENT STIFFNESS MATRIX IN LOCAL COORDINATES

Consider a straight beam element of uniform cross section. Associated with the element are the generalized end displacements $q$, including rotations, normal translations, and axial deformations, and the corresponding generalized forces $Q$, including moments, shears, and axial forces. With the nodal coordinates numbered as shown in Fig. 11-1 for the element $i$, with its two ends $j$ and $k$, the force-displacement characteristics of the element can be given by means of the matrix equations

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & 0 & 0 \\ k_{21} & k_{22} & k_{23} & k_{24} & 0 & 0 \\ k_{31} & k_{32} & k_{33} & k_{34} & 0 & 0 \\ k_{41} & k_{42} & k_{43} & k_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{55} & k_{56} \\ 0 & 0 & 0 & 0 & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}$$

or

$$Q = kq \tag{11-1}$$

where $k$ is the element stiffness matrix with reference to a set of local orthogonal coordinate axes $x$ and $y$, $x$ being taken along the centroidal axis of the member. Note that $k$ contains stiffness coefficients $k_{\alpha\beta}$; $k_{\alpha\beta}$, as already defined, is the force $Q$ induced at coordinate $\alpha$ due to a unit displacement $q$ at coordinate $\beta$ while all other $q$'s are zero. We assume the change in length of the element due to flexural deformations is negligible. It follows that axial end forces are not required to maintain static equilibrium of the restrained beam element when it is subjected to a unit value of end rotation or normal translation. Reciprocally, no end moments or shear forces are required for the restrained beam element when it is subjected to a unit value of axial deformation.

The element stiffness matrix is generated by applying a unit value of each end displacement in turn, and the corresponding column of the matrix gives the

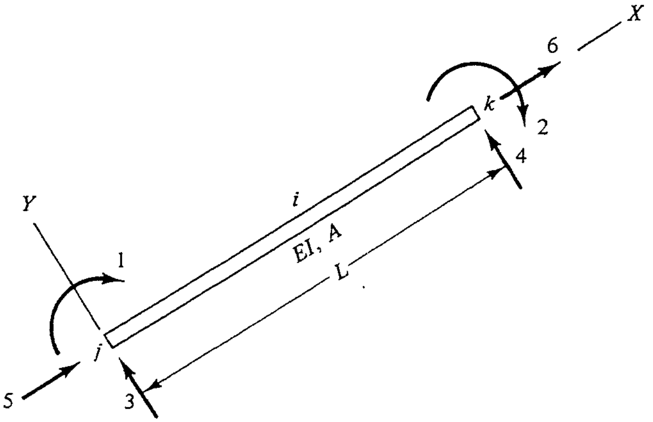


Figure 11-1

various end forces induced in the restrained element. Using the information developed in Sec. 7-2 and Eq. 10-10, we write the element stiffness matrix of Eq. 11-1 for a prismatic member as:

$$
\begin{array}{c}
\begin{array}{cccccc} \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \end{array} \\
k = \left[\begin{array}{cccccc}
\frac{4EI}{L} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & 0 & 0 \\
\frac{2EI}{L} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{6EI}{L^2} & 0 & 0 \\
-\frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{12EI}{L^3} & 0 & 0 \\
\frac{6EI}{L^2} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{12EI}{L^3} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{AE}{L} & -\frac{AE}{L} \\
0 & 0 & 0 & 0 & -\frac{AE}{L} & \frac{AE}{L}
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
\qquad (11\text{-}2)
$$

## 11-3 ROTATIONAL TRANSFORMATION OF A COORDINATE SYSTEM

The stiffness matrix given in Eq. 11-2 is determined with respect to the local, or member, coordinate system. When several members from different directions meet at a joint, each of the local stiffness matrices of connected members must be first transformed into a stiffness matrix based on a common coordinate system before their influences can be added to obtain the joint stiffness coefficients. It is, therefore, necessary to establish a global, or structure, coordinate system, in terms of which all element stiffness matrices must be written before assembly.

Consider a plane vector **V** (force or displacement), as in Fig. 11-2, represented by its components $V_x$, $V_y$ in the set of local orthogonal $XY$ system or $\overline{V}_x$, $\overline{V}_y$ in the set of global orthogonal $\overline{X}\overline{Y}$ system. Let the angle between $X$ and $\overline{X}$ be $\theta$, as shown. We may express $V_x$ and $V_y$ in terms of $\overline{V}_x$ and $\overline{V}_y$ as

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \overline{V}_x \\ \overline{V}_y \end{Bmatrix}$$

or

$$\mathbf{V} = \theta\overline{\mathbf{V}} \tag{11-3}$$

in which $\theta$ represents a rotation matrix from the $\overline{XY}$ system to the $XY$ system. Inversely,

$$\overline{\mathbf{V}} = \theta^{-1}\mathbf{V} \tag{11-4}$$

That is,

$$\begin{Bmatrix} \overline{V}_x \\ \overline{V}_y \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} V_x \\ V_y \end{Bmatrix} \tag{11-5}$$

We see that

$$\theta^{-1} = \theta^T \tag{11-6}$$

Hence the rotation matrix $\theta$ is said to be *orthogonal*. Thus

$$\overline{\mathbf{V}} = \theta^T\mathbf{V} \tag{11-7}$$

Let us consider the beam element of Fig. 11-1, now shown in Fig. 11-3(a). Following Eq. 11-3, we can express the forces or displacements at both ends in terms of those in the global $\overline{XY}$ system [Fig. 11-3(b)]. Thus,

$$\begin{Bmatrix} Q_5 \\ Q_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \overline{Q}_5 \\ \overline{Q}_3 \end{Bmatrix} \tag{11-8}$$

and

$$\begin{Bmatrix} Q_6 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \overline{Q}_6 \\ \overline{Q}_4 \end{Bmatrix} \tag{11-9}$$

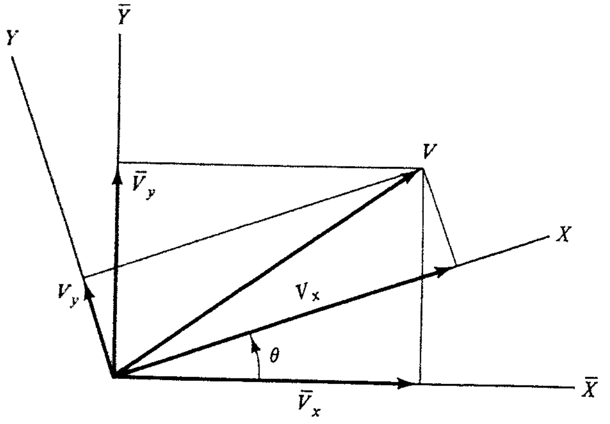
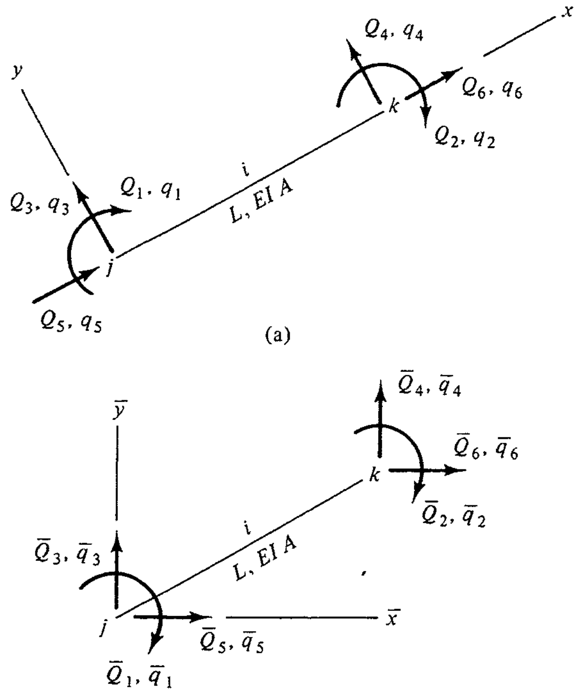


Figure 11-2



(a)

(b)

**Figure 11-3**

The end moments are not affected by the transformation of coordinate system. Thus

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \overline{Q}_1 \\ \overline{Q}_2 \end{Bmatrix} \tag{11-10}$$

Collecting Eqs. 11-8 through 11-10 and rearranging gives:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 0 & 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \end{Bmatrix}$$

local global

or

$$Q = a\overline{Q} \tag{11-11}$$

where $a$ is a square matrix called the *transformation matrix*, which transforms the member end forces from local coordinates to global coordinates. Inversely, we can establish

$$\overline{Q} = a^{-1}Q \tag{11-12}$$

We see that

$$a^{-1} = a^{T} \tag{11-13}$$

that is, the transformation matrix $a$ is orthogonal. Therefore,

$$\overline{Q} = a^{T}Q \tag{11-14}$$

The foregoing transformation process described for end forces is equally valid for the end displacements.

Refer to Fig. 11-4 for a member $i$. Given the position coordinates for both ends as $(\overline{x}_j, \overline{y}_j)$ and $(\overline{x}_k, \overline{y}_k)$, we can compute

$$\cos\theta = \frac{\overline{x}_k - \overline{x}_j}{L} \tag{11-15}$$

and

$$\sin\theta = \frac{\overline{y}_k - \overline{y}_j}{L} \tag{11-16}$$

where

$$L = \sqrt{(\overline{x}_k - \overline{x}_j)^2 + (\overline{y}_k - \overline{y}_j)^2} \tag{11-17}$$

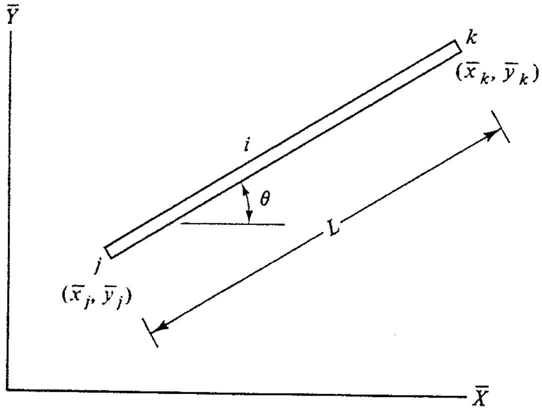


**Figure 11-4**

## 11-4 ELEMENT STIFFNESS MATRIX IN GLOBAL COORDINATES

Consider the inclined member of Fig. 11-3(a), in which the end forces and displacements are situated in a local system. An application of Eqs. 11-1 and 11-2 gives

$$\left\{\begin{matrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{matrix}\right\} = \begin{bmatrix} \frac{4EI}{L} & & & & & \\ \frac{2EI}{L} & \frac{4EI}{L} & & & \text{sym.} & \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & & & \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{12EI}{L^3} & & \\ 0 & 0 & 0 & 0 & \frac{AE}{L} & \\ 0 & 0 & 0 & 0 & -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \left\{\begin{matrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{matrix}\right\}$$

or

$$Q = kq \tag{11-18}$$

in which $k$ is the element stiffness matrix in local coordinates.

By vector transformation 11-11,

$$Q = a\overline{Q} \tag{11-19}$$

in which $\overline{Q}$ represents the end forces expressed in the global system shown in Fig. 11-3(b). Similarly, for the corresponding displacement vectors,

$$q = a\overline{q} \tag{11-20}$$

Substituting Eqs. 11-19 and 11-20 into Eq. 11-18, we have

$$a\overline{Q} = ka\overline{q}$$

or

$$\overline{Q} = a^{-1}ka\overline{q} \tag{11-21}$$

In view of the orthogonality of $a$, that is, $a^{-1} = a^T$, Eq. 11-21 becomes

$$\overline{Q} = a^Tka\overline{q} \tag{11-22}$$

We now define

$$\overline{k} = a^Tka \tag{11-23}$$

as the element stiffness matrix in global coordinates and therefore write the force-displacement relation in the global system as

$$\overline{Q} = \overline{k}\overline{q} \tag{11-24}$$

With $a$ defined in Eq. 11-11 and $k$ as in Eq. 11-18, we perform the matrix operation indicated in Eq. 11-23 and obtain the element stiffness matrix $\overline{k}$ in global coordinates, as in Eq. 11-25.

$$\bar{k} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline \frac{4EI}{L} & & & & & & 1 \\ \frac{2EI}{L} & \frac{4EI}{L} & & \text{sym.} & & & 2 \\ -\frac{6EI}{L^2}C_x & -\frac{6EI}{L^2}C_x & \frac{12EI}{L^3}C_x^2 + \frac{AE}{L}C_y^2 & & & & 3 \\ \frac{6EI}{L^2}C_x & \frac{6EI}{L^2}C_x & -\frac{12EI}{L^3}C_x^2 - \frac{AE}{L}C_y^2 & \frac{12EI}{L^3}C_x^2 + \frac{AE}{L}C_y^2 & & & 4 \\ \frac{6EI}{L^2}C_y & \frac{6EI}{L^2}C_y & \left(-\frac{12EI}{L^3} + \frac{AE}{L}\right)C_xC_y & \left(\frac{12EI}{L^3} - \frac{AE}{L}\right)C_xC_y & \frac{12EI}{L^3}C_y^2 + \frac{AE}{L}C_x^2 & & 5 \\ -\frac{6EI}{L^2}C_y & -\frac{6EI}{L^2}C_y & \left(\frac{12EI}{L^3} - \frac{AE}{L}\right)C_xC_y & \left(-\frac{12EI}{L^3} + \frac{AE}{L}\right)C_xC_y & -\frac{12EI}{L^3}C_y^2 - \frac{AE}{L}C_x^2 & \frac{12EI}{L^3}C_y^2 + \frac{AE}{L}C_x^2 & 6 \end{array} \tag{11-25}$$

in which $C_x = \cos\theta$ and $C_y = \sin\theta$. By Eq. 11-25 all element stiffness matrices in local coordinates are transformed with respect to the single set of orthogonal common axes $\bar{X}$ and $\bar{Y}$; the overall structure stiffness matrix can then be developed with respect to this system of axes.

## 11-5 A SPECIAL CASE: ELEMENT STIFFNESS MATRIX FOR A TRUSS MEMBER

For a truss member pin-connected at its ends, since the end moments and rotations are irrelevant, nodal quantities at each end are identified as in Figs. 11-5(a) and (b) with respect to both local and global coordinate systems. The member stiffness matrix in local coordinates is therefore given by

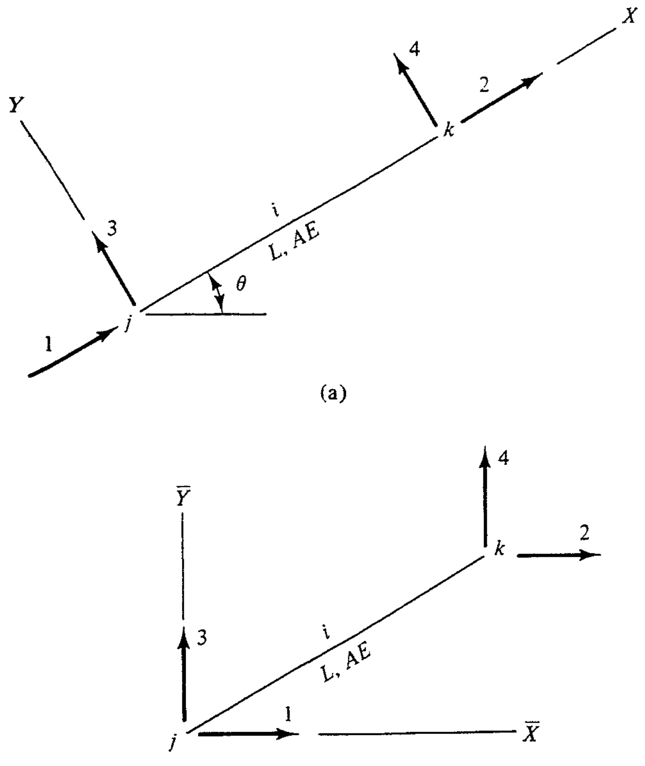
$$k = \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \hline \frac{AE}{L} & -\frac{AE}{L} & 0 & 0 & 1 \\ -\frac{AE}{L} & \frac{AE}{L} & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{array} \tag{11-26}$$

Since

$$\begin{Bmatrix} Q_1 \\ Q_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_3 \end{Bmatrix} \quad \begin{Bmatrix} Q_2 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \bar{Q}_2 \\ \bar{Q}_4 \end{Bmatrix}$$

we establish

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \\ \bar{Q}_3 \\ \bar{Q}_4 \end{Bmatrix}$$



(a)

(b)

**Figure 11-5**

or

$$Q = a\overline{Q} \tag{11-27}$$

in which $a$ is a transformation matrix. Using Eq. 11-23, that is,

$$\overline{k} = a^T k a$$

and performing the matrix multiplications, we obtain the element stiffness matrix for a truss member in global coordinates as

$$\overline{k} = \frac{AE}{L}\begin{bmatrix} C_x^2 & & \text{sym.} & \\ -C_x^2 & C_x^2 & & \\ C_xC_y & -C_xC_y & C_y^2 & \\ -C_xC_y & C_xC_y & -C_y^2 & C_y^2 \end{bmatrix} \tag{11-28}$$

in which $C_x = \cos\theta$ and $C_y = \sin\theta$, as previously defined.

A helical spring is similar to a truss member in that it provides only axial resistance. If the spring has a stiffness $S$, we may replace $AE/L$ with $S$ in Eq. 11-28 to obtain the element stiffness matrix for a helical spring in global coordinates.

## 11-6 STRUCTURE STIFFNESS MATRIX

The structure stiffness matrix is to relate the nodal forces and corresponding displacements of a structure. Consider the structure in Fig. 11-6(a) with 1, 2, . . . , $i$, . . . , $n$ nodal coordinates. The nodal forces $R$ can be linearly related to the corresponding displacements $r$ in the form

$$\begin{Bmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_i \\ \cdot \\ \cdot \\ \cdot \\ R_n \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1i} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2i} & \cdots & K_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{i1} & K_{i2} & \cdots & K_{ii} & \cdots & K_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{n1} & K_{n2} & \cdots & K_{ni} & \cdots & K_{nn} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_i \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{Bmatrix}$$

or, simply,

$$R = Kr \tag{11-29}$$

in which $K$ is called the *structure stiffness matrix* or *total stiffness matrix*. A typical stiffness coefficient $K_{\alpha\beta}$ is the force induced at coordinate $\alpha$ due to a unit displacement at coordinate $\beta$, all other coordinate displacements being zero. A stiffness coefficient of any nodal coordinate is obtained by summing the element stiffness coefficients of the same subscripts from the members that frame into that node. To illustrate this procedure, let us try to find $K_{ii}$. Consider the coordinate $i$ of Fig. 11-6(a). The equilibrium at coordinate $i$ shown in Fig. 11-6(b) requires the nodal force $R_i$ equal to the sum of internal forces acting on the respective elements $a$, $b$, and $c$ common to coordinate $i$. We assume that coordinate transformation has already been applied, so that the element forces and displacements refer to the global coordinate system. Thus,

$$R_i = \overline{Q}_i^a + \overline{Q}_i^b + \overline{Q}_i^c \tag{11-30}$$

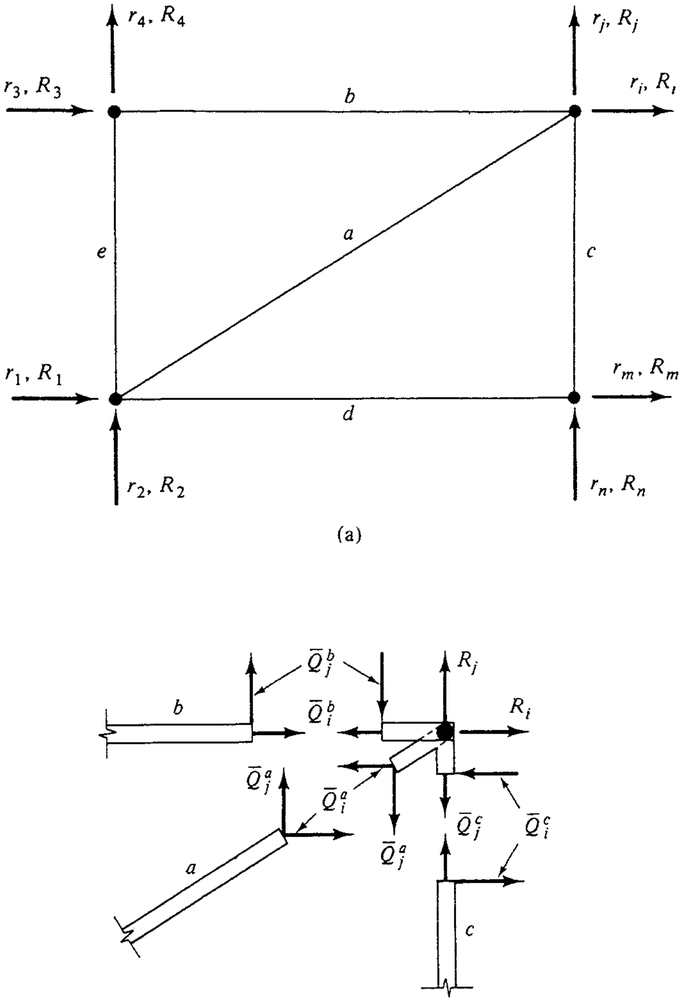
where $Q_i^a$ is the force in the $i$-direction on element $a$. The rest are similarly defined.

By definition, $K_{ii}$ is the value of $R_i$ caused only by $r_i = 1$, all other $r$ being zero. That is

$$R_i = K_{ii}r_i = K_{ii} \tag{11-31}$$

Also, with $r_i = 1$, the compatibility requires that all member ends meeting at $i$ move the same displacement. That is,

$$\overline{q}_i^a = \overline{q}_i^b = \overline{q}_i^c = r_i = 1 \tag{11-32}$$



(a)

(b)

**Figure 11-6**

Therefore,

$$\overline{Q}_i^a = \overline{k}_{ii}^a \overline{q}_i^a = \overline{k}_{ii}^a$$
$$\overline{Q}_i^b = \overline{k}_{ii}^b \overline{q}_i^b = \overline{k}_{ii}^b \qquad (11\text{-}33)$$
$$\overline{Q}_i^c = \overline{k}_{ii}^c \overline{q}_i^c = \overline{k}_{ii}^c$$

Substituting Eqs. 11-31 and 11-33 into Eq. 11-30 gives

$$K_{ii} = \overline{k}_{ii}^a + \overline{k}_{ii}^b + \overline{k}_{ii}^c \qquad (11\text{-}34)$$

In general, we can write the structure stiffness coefficients at node $i$ as

$$K_{ik} = \bar{k}^{a}_{ik} + \bar{k}^{b}_{ik} + \bar{k}^{c}_{ik} \qquad (k = 1, 2, \ldots, i, \ldots, n) \tag{11-35}$$

This demonstrates that the structure stiffness coefficients $K_{ik}$ can be formed by the superposition of the element stiffness coefficients with common subscripts from the members meeting at coordinate $i$, and this summation process is based on the nodal equilibrium of forces expressed in terms of displacements.

## 11-7 THE PROCEDURE OF THE DIRECT STIFFNESS METHOD IN ANALYZING FRAMED STRUCTURES

In view of the foregoing reasoning, the general procedure for the direct stiffness analysis of framed structures is outlined by the following steps:

1. Label the separate elements of the structure. The interconnection points and the supporting points are considered as structure nodes.
2. Identify all nodal coordinates by numbers (nodes of unknown displacement first) with reference to a set of global axes.
3. For each element, identify member end coordinates and establish the element stiffness matrix with reference to the local axes.
4. Transform each of the element stiffness matrices in local coordinates to global coordinates.
5. Superimpose the element stiffness matrices in global coordinates to obtain the structure stiffness matrix. Be sure that the nodal numbering of each element corresponds to that of the structure so that the structure stiffness coefficients $K_{ik}$ are the collection of all the element stiffness coefficients bearing the same subscripts $ik$; that is,

$$K_{ik} = \sum_{m} \bar{k}^{m}_{ik} \tag{11-36}$$

where the summation extends over all $m$ elements meeting at node $i$.

6. The structure stiffness matrix relates the total nodal forces to the corresponding total nodal displacements in the form

$$\left\{\begin{array}{c} R \\ \hline X \end{array}\right\} = \left[\begin{array}{c|c} K_{RR} & K_{RX} \\ \hline K_{XR} & K_{XX} \end{array}\right] \left\{\begin{array}{c} r_R \\ \hline r_X \end{array}\right\} \tag{11-37}$$

We divide the nodal displacements into two groups: one is the unknown nodal displacement $r_R$ corresponding to known nodal forces $R$; the other is the known nodal displacements $r_X$ corresponding to unknown nodal forces $X$.

The solutions are obtained in two steps: first for $r_R$ and then for $X$.

7. Member end forces obtained in global coordinates as linear functions of the nodal displacements in global coordinates are finally transformed back to local coordinates in a form of moment, shear, and axial force.

Note that the distributed or other loads applied on the member span are converted into equivalent nodal loads by a change of sign of fixed-joint actions;

the equivalent nodal loads must be expressed in global coordinates. When calculating member forces, the fixed-joint actions in global coordinates are transformed into local coordinates according to Eq. 11-11.

## 11-8 ILLUSTRATIVE EXAMPLES

Although the direct stiffness procedure is most suitable for analyzing complicated problems carried out by the computer, the matrix examples given in this section, in general, are short enough to be performed and checked by hand calculation. They serve to illustrate the basic procedure for finding the unknown displacements and reactions.

**Example 11-1**

Analyze the truss shown in Fig. 11-7(a) by the direct stiffness method. Assume $L/AE = 1$ for all members.

The first step in developing the analysis is to identify the elements and number the structure nodes (nodes with unknown displacements first). This has been done in Fig. 11-7(b). The complete nodal displacements and corresponding nodal forces are written as

$$\left\{\begin{array}{c} r_R \\ \hdashline r_X \end{array}\right\} = \left\{\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ \hdashline r_4 \\ r_5 \\ r_6 \end{array}\right\} \qquad \left\{\begin{array}{c} R \\ \hdashline X \end{array}\right\} = \left\{\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ \hdashline R_4 \\ R_5 \\ R_6 \end{array}\right\} \tag{11-38}$$

which are partitioned with respect to unknown displacements $\{r_R\} = \{r_1\ r_2\ r_3\}$ corresponding to known nodal forces $\{R\} = \{R_1\ R_2\ R_3\}$ and known nodal displacements $\{r_X\} = \{r_4\ r_5\ r_6\}$ corresponding to unknown reactions $\{X\} = \{R_4\ R_5\ R_6\}$. In the present case, we have known values:

$$r_4 = r_5 = r_6 = R_1 = R_3 = 0 \quad \text{and} \quad R_2 = P$$

We wish to solve the unknown joint displacements $r_1$, $r_2$, and $r_3$ and the unknown support reactions $R_4$, $R_5$, and $R_6$.

Consistent with the nodal numbering of the structure in Fig. 11-7(b), we next label the nodes (end coordinates) of each element, as shown in Fig. 11-7(c).

The stiffness matrix for each element in global coordinates is then computed according to Eq. 11-28. Each relates the nodal forces and displacements of the element considered.

*Element a* $\qquad \dfrac{AE}{L} = 1 \qquad C_x = \cos\theta = 1 \qquad C_y = \sin\theta = 0$

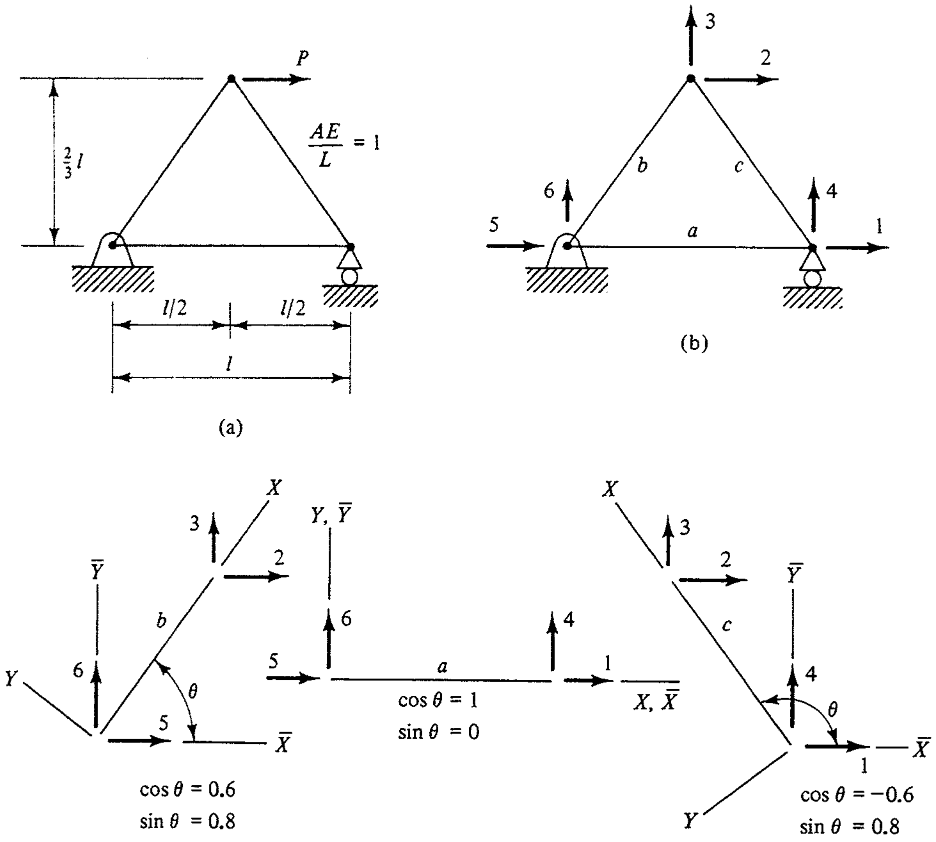
$$\bar{k}^a = \begin{array}{c} \begin{array}{cccc} 5 & 1 & 6 & 4 \end{array} \\ \left[\begin{array}{cccc} 1 & & & \\ -1 & 1 & \text{sym.} & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 1 \\ 6 \\ 4 \end{array} \tag{11-39}$$

*Element b* $\quad \dfrac{AE}{L} = 1 \qquad C_x = \cos\theta = 0.6 \qquad C_y = \sin\theta = 0.8$

$$\bar{k}^b = \begin{array}{c} \begin{array}{cccc} 5 & 2 & 6 & 3 \end{array} \\ \left[\begin{array}{cccc} 0.36 & & \text{sym.} & \\ -0.36 & 0.36 & & \\ 0.48 & -0.48 & 0.64 & \\ -0.48 & 0.48 & -0.64 & 0.64 \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 2 \\ 6 \\ 3 \end{array} \tag{11-40}$$

*Element c* $\quad \dfrac{AE}{L} = 1 \qquad C_x = \cos\theta = -0.6 \qquad C_y = \sin\theta = 0.8$

$$\bar{k}^c = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 4 & 3 \end{array} \\ \left[\begin{array}{cccc} 0.36 & & \text{sym.} & \\ -0.36 & 0.36 & & \\ -0.48 & 0.48 & 0.64 & \\ 0.48 & -0.48 & -0.64 & 0.64 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 3 \end{array} \tag{11-41}$$



(a)

(b)

(c)

**Figure 11-7**

Superimposing element stiffness matrices given in Eqs. 11-39 to 11-41, we establish the structure stiffness matrix as

$$K = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1.36 & & & & \text{sym.} & \\ -0.36 & 0.72 & & & & \\ 0.48 & 0 & 1.28 & & & \\ -0.48 & 0.48 & -0.64 & 0.64 & & \\ -1 & -0.36 & -0.48 & 0 & 1.36 & \\ 0 & -0.48 & -0.64 & 0 & 0.48 & 0.64 \end{array}\right] \end{array} \begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array} \tag{11-42}$$

which relates the total set of nodal forces and displacements; that is,

$$\begin{Bmatrix} R_1\\R_2\\R_3\\\hline R_4\\R_5\\R_6 \end{Bmatrix} = \begin{array}{c} \begin{array}{cc} K_{RR} & \qquad\qquad K_{RX} \end{array} \\ \left[\begin{array}{ccc|ccc} 1.36 & -0.36 & 0.48 & -0.48 & -1 & 0 \\ -0.36 & 0.72 & 0 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0 & 1.28 & -0.64 & -0.48 & -0.64 \\ \hline -0.48 & 0.48 & -0.64 & 0.64 & 0 & 0 \\ -1 & -0.36 & -0.48 & 0 & 1.36 & 0.48 \\ 0 & -0.48 & -0.64 & 0 & 0.48 & 0.64 \end{array}\right] \\ \begin{array}{cc} K_{XR} & \qquad\qquad K_{XX} \end{array} \end{array} \begin{Bmatrix} r_1\\r_2\\r_3\\\hline r_4\\r_5\\r_6 \end{Bmatrix} \tag{11-43}$$

Applying the compatibility condition $r_X = 0$, we solve the unknown displacement $r_R$ in terms of known forces $R$ by

$$r_R = K_{RR}^{-1}R \tag{11-44}$$

or

$$\begin{aligned} r_R = \begin{Bmatrix} r_1\\r_2\\r_3 \end{Bmatrix} &= \begin{bmatrix} 1.36 & -0.36 & 0.48 \\ -0.36 & 0.72 & 0 \\ 0.48 & 0 & 1.28 \end{bmatrix}^{-1} \begin{Bmatrix} R_1\\R_2\\R_3 \end{Bmatrix} \\ &= \begin{bmatrix} 1 & 0.5 & 0.375 \\ 0.5 & 1.639 & -0.188 \\ 0.375 & -0.188 & 0.922 \end{bmatrix} \begin{Bmatrix} 0\\P\\0 \end{Bmatrix} \\ &= \begin{Bmatrix} 0.5\\1.639\\-0.188 \end{Bmatrix} \frac{PL}{AE} \qquad \left(\frac{L}{AE} = 1\right) \end{aligned} \tag{11-45}$$

The support reactions are then solved by

$$X = K_{XR}r_R \tag{11-46}$$

or

$$X = \begin{Bmatrix} R_4\\R_5\\R_6 \end{Bmatrix} = \begin{bmatrix} -0.48 & 0.48 & -0.64 \\ -1 & -0.36 & -0.48 \\ 0 & -0.48 & -0.64 \end{bmatrix} \begin{Bmatrix} 0.5\\1.639\\-0.188 \end{Bmatrix} = \begin{Bmatrix} 0.667\\-1\\-0.667 \end{Bmatrix} P \tag{11-47}$$

With all the nodal displacements found, the member end forces in global coordinates are obtained by using the element stiffness matrix previously established.

Thus,
*Element a*

$$\begin{Bmatrix} \overline{Q}_5^a \\ \overline{Q}_1^a \\ \overline{Q}_6^a \\ \overline{Q}_4^a \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \overline{q}_5^a \\ \overline{q}_1^a \\ \overline{q}_6^a \\ \overline{q}_4^a \end{Bmatrix} = \begin{Bmatrix} -0.5 \\ 0.5 \\ 0 \\ 0 \end{Bmatrix} P \tag{11-48}$$

Note that $\overline{q}_5^a = r_5 = 0$, $\overline{q}_1^a = r_1 = 0.5$, $\overline{q}_6^a = r_6 = 0$, and $\overline{q}_4^a = r_4 = 0$.
*Element b*

$$\begin{Bmatrix} \overline{Q}_5^b \\ \overline{Q}_2^b \\ \overline{Q}_6^b \\ \overline{Q}_3^b \end{Bmatrix} = \begin{bmatrix} 0.36 & -0.36 & 0.48 & -0.48 \\ -0.36 & 0.36 & -0.48 & 0.48 \\ 0.48 & -0.48 & 0.64 & -0.64 \\ -0.48 & 0.48 & -0.64 & 0.64 \end{bmatrix} \begin{Bmatrix} \overline{q}_5^b \\ \overline{q}_2^b \\ \overline{q}_6^b \\ \overline{q}_3^b \end{Bmatrix} = \begin{Bmatrix} -0.500 \\ 0.500 \\ -0.667 \\ 0.667 \end{Bmatrix} P \tag{11-49}$$

Note that $\overline{q}_5^b = r_5 = 0$, $\overline{q}_2^b = r_2 = 1.639$, $\overline{q}_6^b = r_6 = 0$, and $\overline{q}_3^b = r_3 = -0.118$.
*Element c*

$$\begin{Bmatrix} \overline{Q}_1^c \\ \overline{Q}_2^c \\ \overline{Q}_4^c \\ \overline{Q}_3^c \end{Bmatrix} = \begin{bmatrix} 0.36 & -0.36 & -0.48 & 0.48 \\ -0.36 & 0.36 & 0.48 & -0.48 \\ -0.48 & 0.48 & 0.64 & -0.64 \\ 0.48 & -0.48 & -0.64 & 0.64 \end{bmatrix} \begin{Bmatrix} \overline{q}_1^c \\ \overline{q}_2^c \\ \overline{q}_4^c \\ \overline{q}_3^c \end{Bmatrix} = \begin{Bmatrix} -0.500 \\ 0.500 \\ 0.667 \\ -0.667 \end{Bmatrix} P \tag{11-50}$$

Note that $\overline{q}_1^c = r_1 = 0.5$, $\overline{q}_2^c = r_2 = 1.639$, $\overline{q}_4^c = r_4 = 0$, and $\overline{q}_3^c = r_3 = -0.188$.

Member forces $Q$ in local coordinates are obtained by transforming $\overline{Q}$ in global coordinates through the equation

$$Q = a\overline{Q} \tag{11-51}$$

where $a$ is the transformation matrix defined in Eq. 11-27, that is,

$$a = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{11-52}$$

Thus, we have the following
*Element a*

$$\begin{Bmatrix} Q_5^a \\ Q_1^a \\ Q_6^a \\ Q_4^a \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -0.5 \\ 0.5 \\ 0 \\ 0 \end{Bmatrix} P = \begin{Bmatrix} -0.5 \\ 0.5 \\ 0 \\ 0 \end{Bmatrix} P \tag{11-53}$$

that is, the member $a$ is subjected to an axial tension of $P/2$.
*Element b*

$$\begin{Bmatrix} Q_5^b \\ Q_2^b \\ Q_6^b \\ Q_3^b \end{Bmatrix} = \begin{bmatrix} 0.6 & 0 & 0.8 & 0 \\ 0 & 0.6 & 0 & 0.8 \\ -0.8 & 0 & 0.6 & 0 \\ 0 & -0.8 & 0 & 0.6 \end{bmatrix} \begin{Bmatrix} -0.500 \\ 0.500 \\ -0.667 \\ 0.667 \end{Bmatrix} P = \begin{Bmatrix} -0.833 \\ 0.833 \\ 0 \\ 0 \end{Bmatrix} P \tag{11-54}$$

that is, the member $b$ is subjected to an axial tension of $5P/6$.

*Element c*

$$\begin{Bmatrix} Q_1^c \\ Q_2^c \\ Q_4^c \\ Q_3^c \end{Bmatrix} = \begin{bmatrix} -0.6 & 0 & 0.8 & 0 \\ 0 & -0.6 & 0 & 0.8 \\ -0.8 & 0 & -0.6 & 0 \\ 0 & -0.8 & 0 & -0.6 \end{bmatrix} \begin{Bmatrix} -0.500 \\ 0.500 \\ 0.667 \\ -0.667 \end{Bmatrix} P = \begin{Bmatrix} 0.833 \\ -0.833 \\ 0 \\ 0 \end{Bmatrix} P \qquad (11\text{-}55)$$

Therefore, member $c$ is subjected to an axial compression of $5P/6$.

**Example 11-2**

Analyze the rigid frame shown in Fig. 11-8(a) by the direct stiffness method. The frame is composed of two identical members denoted by $a$ and $b$ and is subjected to a uniform load over the horizontal member $b$ only.

The equivalent form of the given loaded frame is shown in Fig. 11-8(b), which consists of two parts: the original frame under the equivalent nodal loads and a fixed-end beam $b$. Arbitrary numbering of nodal coordinates are given to the frame as in Fig. 11-8(c). Thus we write the complete nodal displacements and forces as

$$\begin{Bmatrix} r_R \\ \text{----} \\ r_X \end{Bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ \text{----} \\ r_6 = 0 \\ r_7 = 0 \\ r_8 = 0 \\ r_9 = 0 \end{Bmatrix} \qquad \begin{Bmatrix} R \\ \text{----} \\ X \end{Bmatrix} = \begin{Bmatrix} R_1 = \dfrac{wL^2}{12} \\ R_2 = 0 \\ R_3 = \dfrac{-wL}{2} \\ R_4 = \dfrac{-wL^2}{12} \\ R_5 = 0 \\ \text{----------------} \\ R_6 - \dfrac{wL}{2} \\ R_7 \\ R_8 \\ R_9 \end{Bmatrix} \qquad (11\text{-}56)$$

Note that $r_1$, $r_2$, $r_3$, $r_4$, and $r_5$ are unknown nodal displacements and $R_6$, $R_7$, $R_8$, and $R_9$ represent support reactions. We wish to determine these unknown quantities and obtain also the member end forces.

Let us first use Eq. 11-25 to establish the element stiffness matrices in global coordinates for elements $a$ and $b$ as follows:

*Element a* $\qquad C_x = \cos\theta = 0.8 \qquad C_y = \sin\theta = 0.6$

$$\bar{k}^a = \begin{array}{c} \begin{array}{cccccc} 7 & 1 & 9 & 3 & 8 & 2 \end{array} \\ \left[\begin{array}{cccccc} \frac{4EI}{L} & & & & \text{sym.} & \\ \frac{2EI}{L} & \frac{4EI}{L} & & & & \\ -\frac{4.8EI}{L^2} & -\frac{4.8EI}{L^2} & \frac{7.68EI}{L^3}+\frac{0.36AE}{L} & & & \\ \frac{4.8EI}{L^2} & \frac{4.8EI}{L^2} & -\frac{7.68EI}{L^3}-\frac{0.36AE}{L} & \frac{7.68EI}{L^3}+\frac{0.36AE}{L} & & \\ \frac{3.6EI}{L^2} & \frac{3.6EI}{L^2} & -\frac{5.76EI}{L^3}+\frac{0.48AE}{L} & \frac{5.76EI}{L^3}-\frac{0.48AE}{L} & \frac{4.32EI}{L^3}+\frac{0.64AE}{L} & \\ -\frac{3.6EI}{L^2} & -\frac{3.6EI}{L^2} & \frac{5.76EI}{L^3}-\frac{0.48AE}{L} & -\frac{5.76EI}{L^3}+\frac{0.48AE}{L} & -\frac{4.32EI}{L^3}-\frac{0.64AE}{L} & \frac{4.32EI}{L^3}+\frac{0.64AE}{L} \end{array}\right] \begin{array}{c} 7 \\ 1 \\ 9 \\ 3 \\ 8 \\ 2 \end{array} \end{array} \qquad (11\text{-}57)$$

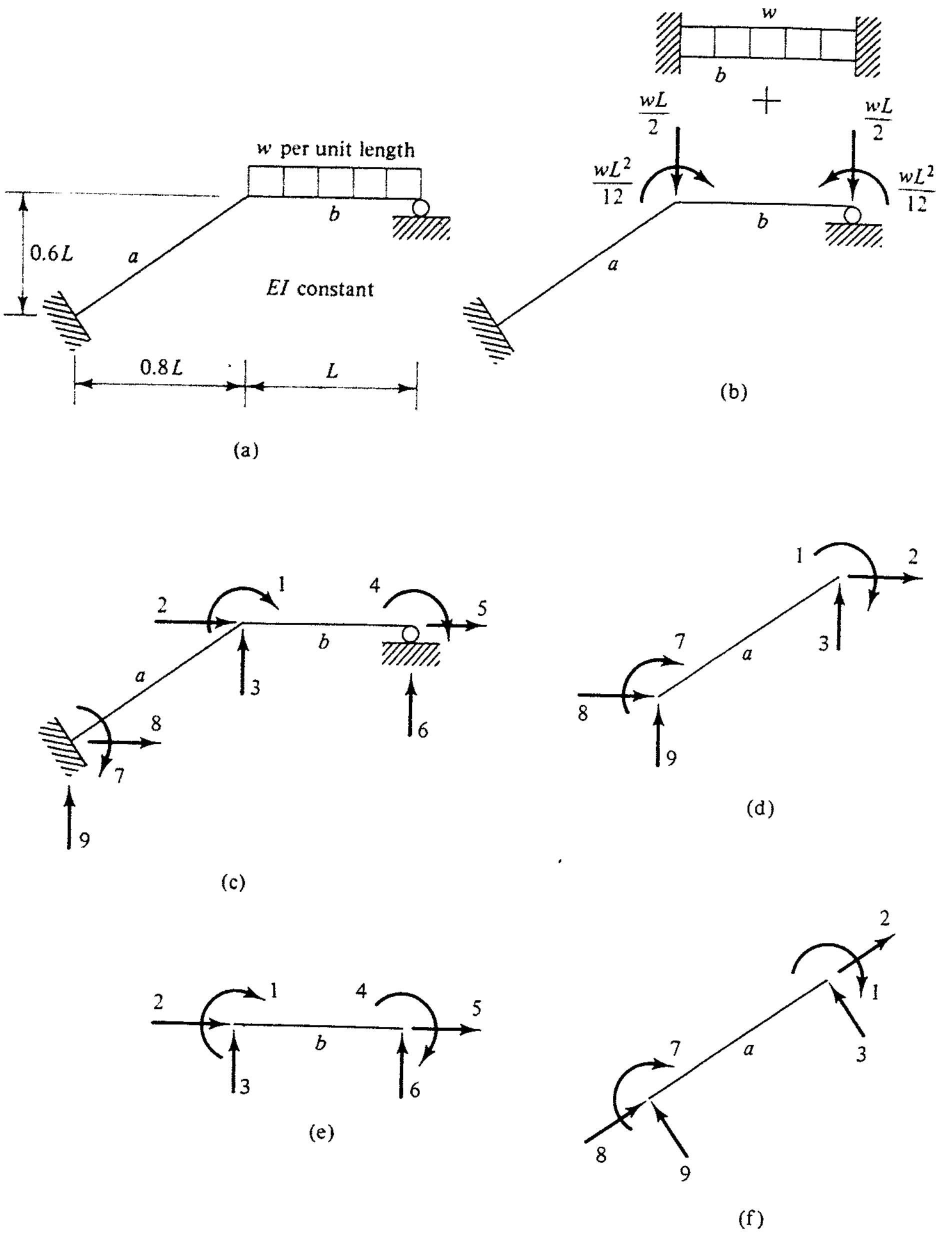

w per unit length
b
0.6L
a
EI constant
0.8L
L
(a)
w
b
+
wL/2
wL/2
wL²/12
wL²/12
b
a
(b)
1
2
3
4
5
6
7
8
9
a
b
(c)
1
2
3
7
8
9
a
(d)
1
2
3
4
5
6
b
(e)
1
2
3
7
8
9
a
(f)

Figure 11-8

*Element b* $\quad C_x = \cos\theta = 1 \qquad C_y = \sin\theta = 0$

$$\bar{k}^b = \begin{array}{c} \begin{array}{cccccc} 1 & 4 & 3 & 6 & 2 & 5 \end{array} \\ \left[\begin{array}{cccccc} \frac{4EI}{L} & & & & & \\ \frac{2EI}{L} & \frac{4EI}{L} & & \text{sym.} & & \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & & & \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{12EI}{L^3} & & \\ 0 & 0 & 0 & 0 & \frac{AE}{L} & \\ 0 & 0 & 0 & 0 & -\frac{AE}{L} & \frac{AE}{L} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 4 \\ 3 \\ 6 \\ 2 \\ 5 \end{array} \tag{11-58}$$

Superimposing element stiffness matrices given in Eqs. 11-57 and 11-58 then gives the structure stiffness matrix, which relates all the nodal forces to the corresponding displacements; that is,

$$\left\{\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ \hline R_6 - \frac{wL}{2} \\ R_7 \\ R_8 \\ R_9 \end{array}\right\} = \overset{K_{RR} \qquad\qquad\qquad\qquad K_{RX}}{\underset{K_{XR} \qquad\qquad\qquad\qquad K_{XX}}{\left[\begin{array}{ccccc|cccc} \frac{8EI}{L} & & & & & & & & \\ -\frac{3.6EI}{L^2} & \frac{4.32EI}{L^3} + \frac{1.64AE}{L} & & & & & & & \\ -\frac{1.2EI}{L^2} & -\frac{5.76EI}{L^3} + \frac{0.48AE}{L} & \frac{19.68EI}{L^3} + \frac{0.36AE}{L} & & & & \text{Sym.} & & \\ \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} & & & & & \\ 0 & -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & & & & \\ \hline \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & & & \\ \frac{2EI}{L} & -\frac{3.6EI}{L^2} & \frac{4.8EI}{L^2} & 0 & 0 & 0 & \frac{4EI}{L} & & \\ \frac{3.6EI}{L^2} & -\frac{4.32EI}{L^3} - \frac{0.64AE}{L} & \frac{5.76EI}{L^3} - \frac{0.48AE}{L} & 0 & 0 & 0 & \frac{3.6EI}{L^2} & \frac{4.32EI}{L^3} + \frac{0.64AE}{L} & \\ -\frac{4.8EI}{L^2} & \frac{5.76EI}{L^3} - \frac{0.48AE}{L} & -\frac{7.68EI}{L^3} - \frac{0.36AE}{L} & 0 & 0 & 0 & -\frac{4.8EI}{L^2} & -\frac{5.76EI}{L^3} + \frac{0.48AE}{L} & \frac{7.68EI}{L^3} + \frac{0.36AE}{L} \end{array}\right]}} \left\{\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ \hline r_6 \\ r_7 \\ r_8 \\ r_9 \end{array}\right\} \tag{11-59}$$

The inverse of stiffness matrix cannot be obtained conveniently in literal form. This solution must be continued by substituting numerical data or relative values for the geometric properties. If we select $AL^2/I = 2{,}000$ and use the data of Eq. 11-56, we can write Eq. 11-59 as

$$\left\{\begin{array}{c} \dfrac{wL^2}{12} \\ 0 \\ -\dfrac{wL^2}{2} \\ -\dfrac{wL^2}{12} \\ 0 \\ \hdashline R_6L - \dfrac{wL^2}{2} \\ R_7 \\ R_8L \\ R_9L \end{array}\right\} = \frac{EI}{L}\left[\begin{array}{ccccc:cccc} 8 & & & & & & & & \\ -3.6 & 3284.32 & & & & & \text{sym.} & & \\ -1.2 & 954.24 & 739.68 & & & & & & \\ 2 & 0 & -6 & 4 & & & & & \\ 0 & -2000 & 0 & 0 & 2000 & & & & \\ \hdashline 6 & 0 & -12 & 6 & 0 & 12 & & & \\ 2 & -3.6 & 4.8 & 0 & 0 & 0 & 4 & & \\ 3.6 & -1284.32 & -954.24 & 0 & 0 & 0 & 3.6 & 1284.32 & \\ -4.8 & -954.24 & -727.68 & 0 & 0 & 0 & -4.8 & 954.24 & 727.68 \end{array}\right]\left\{\begin{array}{c} r_1 \\ r_2/L \\ r_3/L \\ r_4 \\ r_5/L \\ \hdashline r_6/L = 0 \\ r_7 = 0 \\ r_8/L = 0 \\ r_9/L = 0 \end{array}\right\} \tag{11-60}$$

in which the coefficients of the stiffness matrix appear to be *dimensionless*.

Solving for the unknown nodal displacements, we have

$$\left\{\begin{array}{c} r_1 \\ \dfrac{r_2}{L} \\ \dfrac{r_3}{L} \\ r_4 \\ \dfrac{r_5}{L} \end{array}\right\} = \left[\begin{array}{ccccc} 8 & & \text{sym.} & & \\ -3.6 & 3284.32 & & & \\ -1.2 & 954.24 & 739.68 & & \\ 2 & 0 & -6 & 4 & \\ 0 & -2000 & 0 & 0 & 2000 \end{array}\right]^{-1}\left\{\begin{array}{c} \dfrac{1}{12} \\ 0 \\ -\dfrac{1}{2} \\ -\dfrac{1}{12} \\ 0 \end{array}\right\}\frac{wL^3}{EI}$$

$$= \left[\begin{array}{ccccc} 0.164836 & & & & \\ 0.0257319 & 0.0302499 & & \text{sym.} & \\ -0.0340109 & -0.0395686 & 0.0531275 & & \\ -0.133434 & -0.0722188 & 0.0966966 & 0.461762 & \\ 0.0257319 & 0.0302499 & -0.0395686 & -0.0722188 & 0.0307499 \end{array}\right]\left\{\begin{array}{c} \dfrac{1}{12} \\ 0 \\ -\dfrac{1}{2} \\ -\dfrac{1}{12} \\ 0 \end{array}\right\}\frac{wL^3}{EI}$$

$$= \left\{\begin{array}{r} 0.0418613 \\ 0.0279469 \\ -0.0374560 \\ -0.0979480 \\ 0.0279469 \end{array}\right\}\frac{wL^3}{EI} \tag{11-61}$$

With the unknown nodal displacements determined, we solve the unknown reactions:

$$\left\{\begin{array}{l} R_6L - \dfrac{wL^2}{2} \\ R_7 \\ R_8L \\ R_9L \end{array}\right\} = \left[\begin{array}{ccccc} 6 & 0 & -12 & 6 & 0 \\ 2 & -3.6 & 4.8 & 0 & 0 \\ 3.6 & -1284.32 & -954.24 & 0 & 0 \\ -4.8 & -954.24 & -727.68 & 0 & 0 \end{array}\right]\left\{\begin{array}{r} 0.0418613 \\ 0.0279469 \\ -0.0374560 \\ -0.0979480 \\ 0.0279469 \end{array}\right\} wL^2 \tag{11-62}$$

$$= \left\{\begin{array}{r} 0.112953 \\ -0.196675 \\ 0 \\ 0.387047 \end{array}\right\} wL^2$$

which gives

$$R_6 = 0.613wL \qquad R_7 = -0.197wL^2 \qquad R_8 = 0 \qquad R_9 = 0.387wL$$

Member end forces in global coordinates are obtained by the matrix multiplication of the respective element stiffness matrices and nodal displacements in global coordinates.

*Element a* Fig 11-8(d) shows the end coordinates for member $a$ in global coordinate system. Using the element stiffness matrix of Eq. 11-57, we obtain:

$$\begin{Bmatrix} \overline{Q}_7^a \\ \overline{Q}_1^a \\ \overline{Q}_9^a L \\ \overline{Q}_3^a L \\ \overline{Q}_8^a L \\ \overline{Q}_2^a L \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} 4 & & & \text{sym.} & & \\ 2 & 4 & & & & \\ -4.8 & -4.8 & 727.68 & & & \\ 4.8 & 4.8 & -727.68 & 727.68 & & \\ 3.6 & 3.6 & 954.24 & -954.24 & 1284.32 & \\ -3.6 & -3.6 & -954.24 & 954.24 & -1284.32 & 1284.32 \end{bmatrix}$$

$$\cdot \begin{Bmatrix} 0 \\ 0.0418613 \\ 0 \\ -0.0374560 \\ 0 \\ 0.0279469 \end{Bmatrix} \frac{wL^3}{EI} \tag{11-63}$$

$$= \begin{Bmatrix} -0.196675 \\ -0.112953 \\ 0.386999 \\ -0.386999 \\ 0 \\ 0 \end{Bmatrix} wL^2$$

or end moments $\overline{Q}_1^a = -0.113wL^2$ and $\overline{Q}_7^a = -0.197wL^2$, horizontal forces $\overline{Q}_2^a = \overline{Q}_8^a = 0$, and vertical forces $\overline{Q}_3^a = -\overline{Q}_9^a = -0.387wL$.

*Element b* Refer to Fig. 11-8(e) for end coordinates, which are the same in the global or local coordinate system, since member $b$ is horizontal. Using the element stiffness matrix of Eq. 11-58, we can determine the end forces for member $b$ in a nodal-force analysis. The final forces in $b$ must include the fixed-end forces in Fig. 11-8(b). Thus,

$$\begin{Bmatrix} \overline{Q}_1^b \\ \overline{Q}_4^b \\ \overline{Q}_3^b L \\ \overline{Q}_6^b L \\ \overline{Q}_2^b L \\ \overline{Q}_5^b L \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} 4 & & & \text{sym.} & & \\ 2 & 4 & & & & \\ -6 & -6 & 12 & & & \\ 6 & 6 & -12 & 12 & & \\ 0 & 0 & 0 & 0 & 2{,}000 & 2{,}000 \\ 0 & 0 & 0 & 0 & -2{,}000 & \\ & & & & & \end{bmatrix} \begin{Bmatrix} 0.0418613 \\ -0.0979480 \\ -0.0374560 \\ 0 \\ 0.0279469 \\ 0.0279469 \end{Bmatrix} \frac{wL^3}{EI} + \begin{Bmatrix} -1/12 \\ 1/12 \\ 1/2 \\ 1/2 \\ 0 \\ 0 \end{Bmatrix} wL^2$$

$$= \begin{Bmatrix} 0.112955 \\ 0 \\ 0.387047 \\ 0.612953 \\ 0 \\ 0 \end{Bmatrix} wL^2 \tag{11-64}$$

or end moments $\overline{Q}_1^b = 0.113wL^2$ and $\overline{Q}_4^b = 0$, axial forces $\overline{Q}_2^b = \overline{Q}_5^b = 0$, and end shears $\overline{Q}_3^b = 0.387wL$ and $\overline{Q}_6^b = 0.613wL$.

The end forces for member *a* of Eq. 11-63 can be expressed in local coordinates [Fig. 11-8(e)] by applying the transformation matrix of Eq. 11-11. Thus,

$$\begin{Bmatrix} Q_7^a \\ Q_1^a \\ Q_9^a L \\ Q_3^a L \\ Q_8^a L \\ Q_2^a L \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & -0.6 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & -0.6 \\ 0 & 0 & 0.6 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.8 \end{bmatrix} \begin{Bmatrix} -0.196675 \\ -0.112953 \\ 0.386999 \\ -0.386999 \\ 0 \\ 0 \end{Bmatrix} wL^2 \tag{11-65}$$

$$= \begin{Bmatrix} -0.196675 \\ -0.112953 \\ 0.309599 \\ -0.309599 \\ 0.232199 \\ -0.232199 \end{Bmatrix} wL^2$$

or end moments $Q_1^a = -0.113wL^2$ and $Q_7^a = -0.197wL^2$, end shears $Q_3^a = -Q_9^a = -0.31wL$, and axial forces $Q_2^a = -Q_8^a = -0.232wL$.

## 11-9 COMPUTER PROGRAMS FOR FRAMED STRUCTURES

With the advent of the digital computer, structural problems can readily be solved by regimented computer procedures. Illustrative programs can be found in numerous texts. The general outline of programs for stiffness method is contained in the following steps:

1. Input of structural data including structure parameters and elastic moduli, joint information (number and coordinates), member information (designations, properties, and orientation), and joint restraint list.
2. Formulation of stiffness matrices, including the generation of a structure stiffness matrix from member stiffness matrices and the inversion of the structure matrix.
3. Input of load data indicating number of elements, joints, and loading conditions, which are further classified as equivalent joint loads and combined joint loads.
4. Calculation and output of results, giving joint displacements, support reactions, and member end actions.

The further development of person-machine communication leads to programming systems that are problem-oriented, enabling the solution of an entire group of related problems rather than one specific isolated problem.

## PROBLEMS

**11-1.** Determine, by the direct stiffness method, the support reactions for the beam shown in Fig. 11-9.

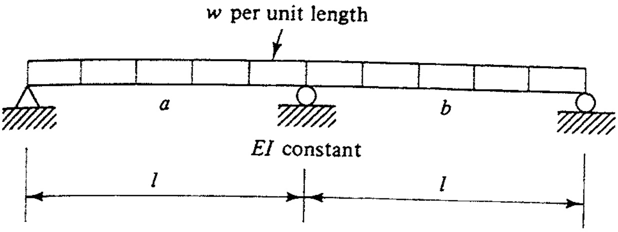


Figure 11-9

**11-2.** Determine, by the direct stiffness method, the member end forces and support reactions for the beam shown in Fig. 11-10.

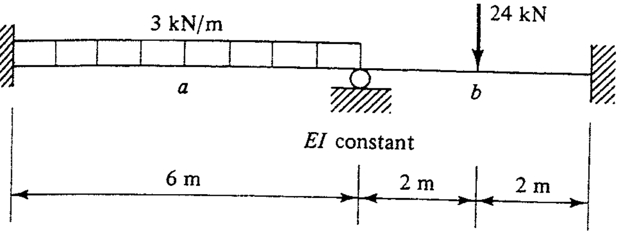


Figure 11-10

**11-3.** Determine, by the direct stiffness method, the bar forces, support reactions, and nodal displacements for the truss shown in Fig. 11-11. Assume constant axial rigidity $EA$ for all members.

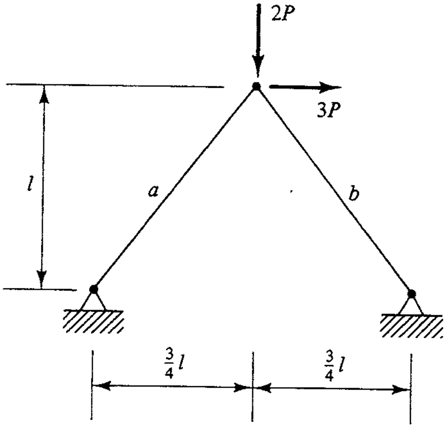


Figure 11-11

**11-4.** Determine, by the direct stiffness method, the bar forces and support reactions for the truss shown in Fig. 11-12. Assume $A/L = 1$ for all members.

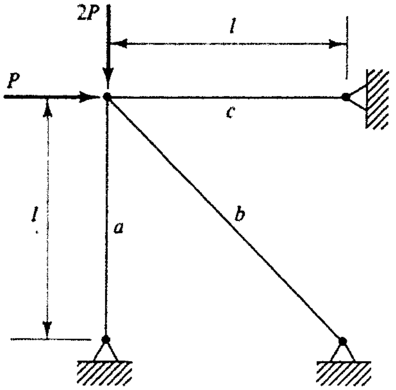


Figure 11-12

11-5. Find, by the direct stiffness method, the member end forces and support reactions for the rigid frame shown in Fig. 11-13.

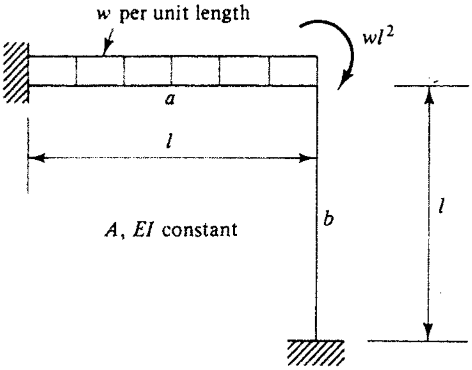


Figure 11-13

11-6. Find, by the direct stiffness method, the member end forces and support reactions for the rigid frame shown in Fig. 11-14.

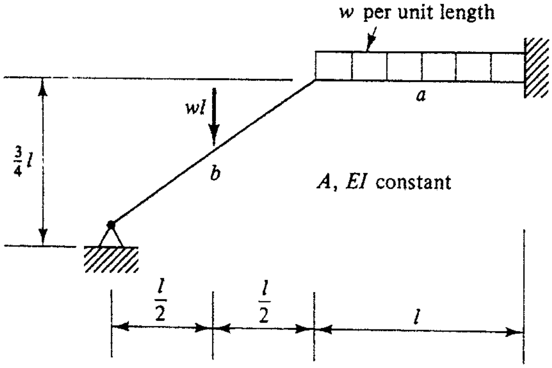


Figure 11-14

**11-7.** Use the direct stiffness method to construct the total stiffness matrix for the frame shown in Fig. 11-15. Obtain the nodal displacements.

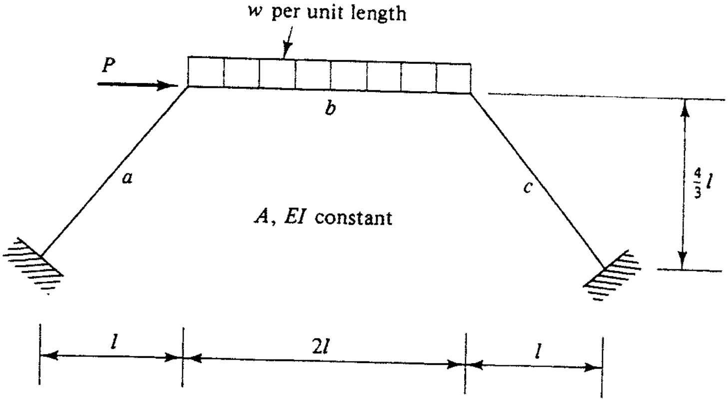


Figure 11-15

# 12

# The Treatment of Nonprismatic Members

## 12-1 GENERAL

In the previous discussion of framed-structure analysis, we were concerned only with structures composed of uniform elements. In many instances, however, the members forming a structure are nonprismatic, or in the more general case, have variable rigidities such as those shown in Fig. 12-1. The fundamental concepts of analysis remain the same as if the structure were built up by prismatic members. However, the expression for member constants, including the fixed-end actions, the flexibility and stiffness coefficients, and the stiffness and carry-over factor necessary for a moment distribution, derived specifically for prismatic members, are no longer valid for nonprismatic members. These constants applied for nonprismatic members must be first determined so that the analysis of structure either by slope-deflection equations, or by moment distribution, or by matrix procedure can be carried out in the usual manner.

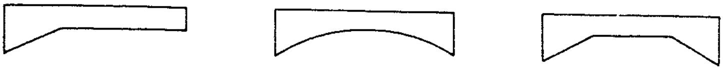
In this chapter we develop various integral formulas expressing these constants and demonstrate how to employ a numerical approach to approximate an integration.

## 12-2 FIXED-END ACTIONS

Consider a member of varying fluxural rigidity with both ends fixed and subjected to the bending action caused by member loads, as shown in Fig. 12-2. The general expression for the fixed-end moments may be found by the method of

Tapered member

Stepped member



Haunched member

Figure 12-1

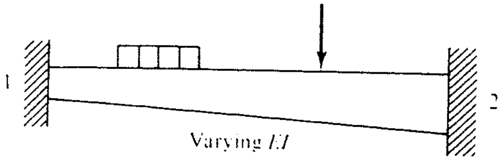


Figure 12-2

least work, using Eqs. 6-30 and 6-31:

$$\int_0^l \frac{M\,dx}{EI} = 0 \tag{12-1}$$

$$\int_0^l \frac{Mx\,dx}{EI} = 0 \tag{12-2}$$

If the member is made of the same material, we can assume that $E$ is a constant. The preceding expressions thus become

$$\int_0^l \frac{M\,dx}{I} = 0 \tag{12-3}$$

$$\int_0^l \frac{Mx\,dx}{I} = 0 \tag{12-4}$$

where both $M$ and $I$ are functions of $x$.

As an illustration, let us find $M_1$ and $V_1$ for the fixed beam shown in Fig. 12-3 due to a uniform load over the entire span.

The moment at any section distance $x$ from the left end is

$$M = M_1 + V_1 x - \frac{wx^2}{2}$$

Substituting $M$ in Eqs. 12-3 and 12-4 gives

$$M_1 \int_0^l \frac{dx}{I} + V_1 \int_0^l \frac{x\,dx}{I} - \frac{w}{2} \int_0^l \frac{x^2\,dx}{I} = 0 \tag{12-5}$$

$$M_1 \int_0^l \frac{x\,dx}{I} + V_1 \int_0^l \frac{x^2\,dx}{I} - \frac{w}{2} \int_0^l \frac{x^3\,dx}{I} = 0 \tag{12-6}$$

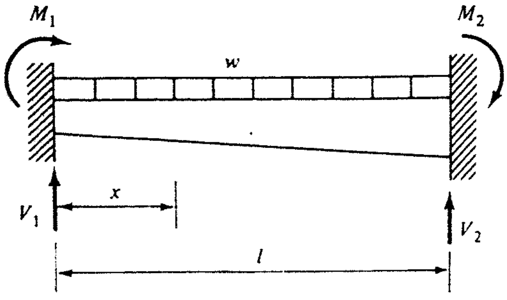


Figure 12-3

In matrix form,

$$\begin{bmatrix} \int_0^l \frac{dx}{I} & \int_0^l \frac{x\,dx}{I} \\ \int_0^l \frac{x\,dx}{I} & \int_0^l \frac{x^2\,dx}{I} \end{bmatrix} \begin{Bmatrix} M_1 \\ V_1 \end{Bmatrix} = \frac{w}{2} \begin{Bmatrix} \int_0^l \frac{x^2\,dx}{I} \\ \int_0^l \frac{x^3\,dx}{I} \end{Bmatrix} \tag{12-7}$$

Eliminating $V_1$ from Eqs. 12-5 and 12-6 yields the fixed-end moment at 1:

$$M_1 = \left(\frac{w}{2}\right) \frac{\left(\int_0^l (x^2\,dx)/I\right)^2 - \int_0^l (x\,dx)/I \int_0^l (x^3\,dx)/I}{\int_0^l dx/I \int_0^l (x^2\,dx)/I - \left(\int_0^l (x\,dx)/I\right)^2} \tag{12-8}$$

The fixed-end moment at 2 can also be obtained from Eq. 12-8 by taking the integral origin at 2 and using reverse sign. For a member of varying $I$, $M_1$ and $M_2$ are not equal except in a symmetrical system. For a member of uniform section, Eq. 12-8 reduces to

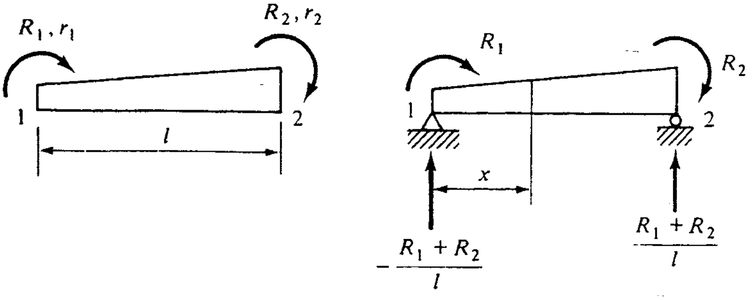
$$M_1 = -M_2 = \left(\frac{w}{2}\right) \frac{\left(\int_0^l x^2\,dx\right)^2 - \int_0^l x\,dx \int_0^l x^3\,dx}{\int_0^l dx \int_0^l x^2\,dx - \left(\int_0^l x\,dx\right)^2} = -\frac{wl^2}{12} \tag{12-9}$$

With $M_1$ and $M_2$ obtained, we can find $V_1$ and $V_2$ by simple statics.

## 12-3 THE ROTATIONAL FLEXIBILITY MATRIX OF A BEAM ELEMENT

Consider the beam element with a variable cross section subjected to end moments $R_1$ and $R_2$ with the corresponding rotations $r_1$ and $r_2$, as in Fig. 12-4(a). In order to find flexibility coefficients, we must have the element properly supported. If we regard the element as a simple beam, as in Fig. 12-4(b), then the bending moment at any section distance $x$ from the left end is

$$M = R_1 - \frac{(R_1 + R_2)x}{l} \tag{12-10}$$



(a) (b)

Figure 12-4

By the method of least work, we obtain the end rotations $r_1$ and $r_2$ as

$$\begin{aligned} r_1 = \frac{\partial W}{\partial R_1} &= \int_0^l \frac{M(\partial M/\partial R_1)\,dx}{EI} \\ &= \int_0^l \frac{[R_1 - (R_1 + R_2)(x/l)][1 - (x/l)]\,dx}{EI} \end{aligned} \tag{12-11}$$

$$\begin{aligned} r_2 = \frac{\partial W}{\partial R_2} &= \int_0^l \frac{M(\partial M/\partial R_2)\,dx}{EI} \\ &= \int_0^l \frac{[R_1 - (R_1 + R_2)(x/l)][-x/l]\,dx}{EI} \end{aligned} \tag{12-12}$$

Setting $R_1 = 1$ and $R_2 = 0$ in Eqs. 12-11 and 12-12, respectively, leads to the rotational flexibility coefficients $f_{11}$ and $f_{21}$:

$$f_{11} = \int_0^l \frac{(l - x)^2\,dx}{EIl^2} \tag{12-13}$$

$$f_{21} = -\int_0^l \frac{(l - x)(x)\,dx}{EIl^2} \tag{12-14}$$

Similarly, setting $R_1 = 0$ and $R_2 = 1$ in Eqs. 12-11 and 12-12, respectively, yields

$$f_{12} = -\int_0^l \frac{(l - x)(x)\,dx}{EIl^2} \tag{12-15}$$

$$f_{22} = \int_0^l \frac{x^2\,dx}{EIl^2} \tag{12-16}$$

Collecting Eqs. 12-13 to 12-16 and assuming a constant $E$, we form the rotational flexibility matrix:

$$[f] = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

$$= \frac{1}{EI^2}\begin{bmatrix} \displaystyle\int_0^l \frac{(l-x)^2\,dx}{I} & \displaystyle -\int_0^l \frac{(l-x)(x)\,dx}{I} \\ \displaystyle -\int_0^l \frac{(l-x)(x)\,dx}{I} & \displaystyle\int_0^l \frac{x^2\,dx}{I} \end{bmatrix} \qquad (12\text{-}17)$$

For a member of uniform cross section, this expression reduces to

$$[f] = \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad (12\text{-}18)$$

## 12-4 THE ROTATIONAL STIFFNESS MATRIX OF A BEAM ELEMENT

Consider the same beam element in Fig. 12-4(a). By definition, the rotational stiffness coefficients $k_{11}$ and $k_{21}$ are the respective moments at coordinates 1 and 2 due to a unit rotation of coordinate 1 ($r_1 = 1$). Similarly, the rotational stiffness coefficients $k_{12}$ and $k_{22}$ are the respective moments at coordinates 1 and 2 due to a unit rotation at coordinate 2 ($r_2 = 1$). They are illustrated in Fig. 12-5(a) and (b).

To find the integral expression for the rotational stiffness matrix, we need only perform the inversion for the rotational flexibility matrix given by Eq. 12-17. Thus,

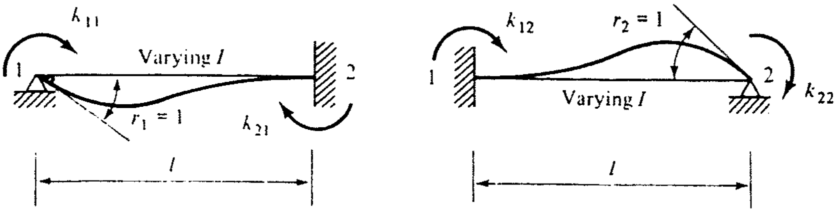
$$[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \displaystyle\frac{1}{EI^2}\int_0^l \frac{(l-x)^2\,dx}{I} & \displaystyle -\frac{1}{EI^2}\int_0^l \frac{(l-x)(x)\,dx}{I} \\ \displaystyle -\frac{1}{EI^2}\int_0^l \frac{(l-x)(x)\,dx}{I} & \displaystyle\frac{1}{EI^2}\int_0^l \frac{x^2\,dx}{I} \end{bmatrix}^{-1}$$

$$= \frac{E}{\displaystyle\int_0^l \frac{dx}{I}\int_0^l \frac{x^2\,dx}{I} - \left(\int_0^l \frac{x\,dx}{I}\right)^2}\begin{bmatrix} \displaystyle\int_0^l \frac{x^2\,dx}{I} & \displaystyle\int_0^l \frac{(l-x)(x)\,dx}{I} \\ \displaystyle\int_0^l \frac{(l-x)(x)\,dx}{I} & \displaystyle\int_0^l \frac{(l-x)^2\,dx}{I} \end{bmatrix} \qquad (12\text{-}19)$$

Equation 12-19 shows that the stiffness matrix is symmetrical (i.e., $k_{12} = k_{21}$). For a member of uniform cross section, Eq. 12-19 reduces to

$$[k] = \frac{2EI}{l}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (12\text{-}20)$$



(a) (b)

Figure 12-5

## 12-5 THE GENERALIZED SLOPE-DEFLECTION EQUATIONS

The slope-deflection equations derived for a typical uniform member 1–2 (see Sec. 7-2) are

$$M_{12} = 2E\frac{I}{l}\left[2\theta_1 + \theta_2 - 3\left(\frac{\Delta}{l}\right)\right] + M_{12}^F \tag{12-21}$$

$$M_{21} = 2E\frac{I}{l}\left[2\theta_2 + \theta_1 - 3\left(\frac{\Delta}{l}\right)\right] + M_{21}^F \tag{12-22}$$

In cases involving a variable moment of inertia, these equations are not valid and some generalized slope-deflection equations must be formed. To do this, we recall that the basic slope-deflection equations are derived from the sum of four separate effects:

1. The rigid body translation $\Delta$ between two ends. This is equivalent to a rigid body rotation $\Delta/l$ or $R$ of the whole member.
2. The rotation $(\theta_1 - \Delta/l)$ at end 1 only.
3. The rotation $(\theta_2 - \Delta/l)$ at end 2 only.
4. The application of member loads with end displacement prevented.

These steps are recaptured as in Fig. 12-6 and are also applicable to the instances of nonuniform member.

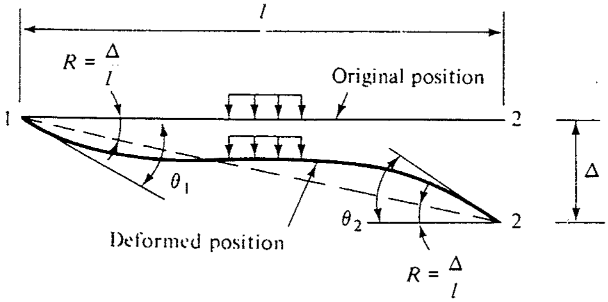


Figure 12-6

Now the rigid body motion of step 1 causes no moments. Step 2 produces

$$M_{12} = k_{11}\left(\theta_1 - \frac{\Delta}{l}\right) \tag{12-23}$$

$$M_{21} = k_{21}\left(\theta_1 - \frac{\Delta}{l}\right) \tag{12-24}$$

Similarly, step 3 gives

$$M_{12} = k_{12}\left(\theta_2 - \frac{\Delta}{l}\right) \tag{12-25}$$

$$M_{21} = k_{22}\left(\theta_2 - \frac{\Delta}{l}\right) \tag{12-26}$$

Note that $k_{11}$, $k_{12}$, $k_{21}$, and $k_{22}$ are the rotational stiffness coefficients defined in Eq. 12-19.

The end moments corresponding to step 4 are the fixed end moments discussed in Sec. 12-2. They are usually denoted by $M_{12}^F$ and $M_{21}^F$ in slope-deflection equations.

Collecting these effects, we arrive at the generalized slope-deflection equations applicable to cases involving variable moment of inertia:

$$M_{12} = k_{11}\theta_1 + k_{12}\theta_2 - (k_{11} + k_{12})\frac{\Delta}{l} + M_{12}^F \tag{12-27}$$

$$M_{21} = k_{21}\theta_1 + k_{22}\theta_2 - (k_{21} + k_{22})\frac{\Delta}{l} + M_{21}^F \tag{12-28}$$

Assembling Eqs. 12-27 and 12-28 in a matrix and using $R = \Delta/l$, we have

$$\begin{Bmatrix} M_{12} - M_{12}^F \\ M_{21} - M_{21}^F \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \theta_1 - R \\ \theta_2 - R \end{Bmatrix} \tag{12-29}$$

which gives the relationship between member end forces and displacements. For a member of uniform section,

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \frac{2EI}{l}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{12-30}$$

and Eqs. 12-27 and 12-28 reduce to Eqs. 12-21 and 12-22.

With the fixed-end moments and rotational stiffness coefficients for each member determined, application of generalized slope-deflection equations 12-27 and 12-28 in analyzing frames composed of nonuniform members can be carried out by the procedure given in Chapter 7.

## 12-6 THE STIFFNESS AND CARRY-OVER FACTOR FOR MOMENT DISTRIBUTION

To develop an expression for stiffness and carry-over factor for a member of varying $I$ in moment distribution, let us recall the definition of rotational stiffness for an end of a member as the end moment required to produce a unit rotation at this end (simple end) while the other end is fixed; and the definition of carry-over factor as the ratio of induced moment at the other end (fixed) to the applied moment at this end. See Fig. 12-7, in which we use the conventional notation $S_{12}$ in moment distribution as the rotational stiffness of end 1 (this end) of member 1–2; $C_{12}$ as the carry-over factor from end 1 to end 2. By definition, $S_{12}$ is the same stiffness coefficient $k_{11}$ given in Eq. 12-19. Thus,

$$S_{12} = \frac{E\int_0^l (x^2\,dx)/I}{\int_0^l dx/I \int_0^l (x^2\,dx)/I - \left(\int_0^l (x\,dx)/I\right)^2} \tag{12-31}$$

The rotational stiffness of end 2 of member 1–2, denoted by $S_{21}$, can similarly be explained and is equal to $k_{22}$ given in Eq. 12-19. Thus,

$$S_{21} = \frac{E\int_0^l (l - x)^2\,dx/I}{\int_0^l dx/I \int_0^l (x^2\,dx)/I - \left(\int_0^l (x\,dx)/I\right)^2} \tag{12-32}$$

Note that for a member of nonuniform cross section, $S_{21}$ is usually not equal to $S_{12}$. For a member of constant cross section,

$$S_{12} = S_{21} = \frac{4EI}{l} \tag{12-33}$$

Again, refer to the setup in Fig. 12-7. The induced moment $M_{21}$ at end 2 is the carry-over moment from end 1 and is therefore equal to $C_{12}S_{12}$ or $C_{12}k_{11}$. But it can also be interpreted as the moment restraint at end 2 due to a unit rotation only at end 1 and therefore is equal to the stiffness coefficient $k_{21}$ in Eq. 12-19. Thus,

$$C_{12} = \frac{k_{21}}{k_{11}} \tag{12-34}$$

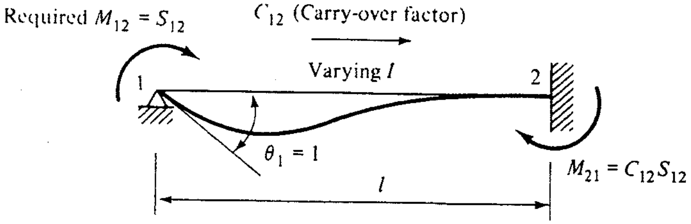


Figure 12-7

Using the expressions of $k_{11}$ and $k_{21}$ in Eq. 12-19, we obtain

$$C_{12} = \frac{\int_0^l (l - x)x\, dx/I}{\int_0^l x^2\, dx/I} \tag{12-35}$$

By the same reasoning,

$$C_{21} = \frac{k_{12}}{k_{22}} \tag{12-36}$$

Using the expressions of $k_{12}$ and $k_{22}$ in Eq. 12-19 gives

$$C_{21} = \frac{\int_0^l (l - x)x\, dx/I}{\int_0^l (l - x)^2\, dx/I} \tag{12-37}$$

Note that for a member of nonuniform cross section, $C_{12}$ and $C_{21}$ are usually not equal. For a member of constant cross section,

$$C_{12} = C_{21} = \tfrac{1}{2} \tag{12-38}$$

Since $C_{12}S_{12} = k_{21}$, $C_{21}S_{21} = k_{12}$, and $k_{12} = k_{21}$, we reach

$$C_{12}S_{12} = C_{21}S_{21} \tag{12-39}$$

This relationship provides a check on separately computed value of the stiffness and carry-over factors.

The determination of fixed-end moments, stiffnesses, and carry-over factors prerequisite to a moment distribution procedure usually involves a large amount of computation. Fortunately, the values of a considerable number of these factors for the more common types of nonprismatic members have been published for the convenience of structural engineers. One such source is the *Handbook of Frame Constants* published by the Portland Cement Association. The member stiffness matrix and, therefore, the flexibility matrix required for a matrix analysis of frames can be deduced from these data by using the relationships

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} S_{12} & C_{21}S_{21} \\ C_{12}S_{12} & S_{21} \end{bmatrix} = \begin{bmatrix} S_{12} & C_{12}S_{12} \\ C_{12}S_{12} & S_{21} \end{bmatrix} \tag{12-40}$$

## 12-7 FIXED-END MOMENT DUE TO JOINT TRANSLATION

When frame sidesway is involved, it is necessary to determine the fixed-end moments due to joint translation for the relevant members before we can carry out a moment distribution. We recall that the fixed-end moment developed at either end of a prismatic member because of relative end displacement equals $-6EI\Delta/l^2$. This is no longer valid for a member of nonuniform cross section.

Consider member 1–2 having varying $I$ subjected to a pure relative end translation Δ, as shown in Fig. 12-8. The moment restraints at end 1 and 2 can readily be obtained by setting $\theta_1 = \theta_2 = M_{12}^F = M_{21}^F = 0$ in the generalized slope-deflection equations 12-27 and 12-28. Thus,

$$M_{12} = -(k_{11} + k_{12})\frac{\Delta}{l} \tag{12-41}$$

$$M_{21} = -(k_{21} + k_{22})\frac{\Delta}{l} \tag{12-42}$$

in which $k_{11}$, $k_{12}$, $k_{21}$, and $k_{22}$ are stiffness coefficients defined by Eq. 12-19. Equations 12-41 and 12-42 in terms of the conventional notation of moment distribution are given as

$$M_{12} = -(S_{12} + C_{21}S_{21})\frac{\Delta}{l} \tag{12-43}$$

$$= -S_{12}(1 + C_{12})\frac{\Delta}{l}$$

$$M_{21} = -(S_{21} + C_{12}S_{12})\frac{\Delta}{l} \tag{12-44}$$

$$= -S_{21}(1 + C_{21})\frac{\Delta}{l}$$

by using the relationship $C_{12}S_{12} = C_{21}S_{21}$. For a member of uniform cross section, $S_{12} = S_{21} = 4EI/l$, $C_{12} = C_{21} = \frac{1}{2}$, and the expressions above reduce to

$$M_{12} = M_{21} = -\frac{6EI\Delta}{l^2} \tag{12-45}$$

If end 2 is hinged, the modified fixed-end moment at 1, called $M_{12}'$, resulting from the relative end translation Δ can be found by first assuming both ends fixed and subsequently restoring end 2 to its original hinged condition. Thus,

$$M_{12}' = M_{12} - C_{21}M_{21}$$

Using Eqs. 12-43 and 12-44, we obtain

$$M_{12}' = -S_{12}(1 - C_{12}C_{21})\frac{\Delta}{l} \tag{12-46}$$

For a member of uniform cross section, this expression reduces to

$$M_{12}' = -\frac{3EI\Delta}{l^2} \tag{12-47}$$

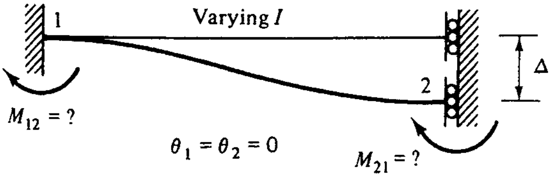


Figure 12-8

## 12-8 MODIFIED STIFFNESS FOR MOMENT DISTRIBUTION

The modified stiffness of this end of a member may be defined as the end moment required to produce a unit rotation at this end (simple end) while the other end remains in the actual condition other than being fixed.

*Case* 1 The modified stiffness for end 1 of member 1–2, if end 2 is simply supported, is given by

$$S'_{12} = S_{12}(1 - C_{12}C_{21}) \tag{12-48}$$

in which $S'_{12}$ denotes the modified stiffness and $S_{12}$ is the stiffness found by the usual manner. The equation above can be proved as follows.

By definition, the setup in Fig. 12-9(a) gives the configuration for finding the modified stiffness for end 1 of member 1–2, the other end being simply supported. To accomplish this, we may break member 1–2 down into two separate steps, as shown in Fig. 12-9(b) and (c). In Fig. 12-9(b) we temporarily lock end 2 against rotation ($\theta_2 = 0$). A moment $S_{12}$ applied at end 1 will produce a unit rotation at 1 and induce a carry-over moment of $C_{12}S_{12}$ at 2. In Fig. 12-9(c) we release end 2 to its actual condition of zero moment and at the same time lock end 1 against further rotation. A moment of $-C_{12}S_{12}$ must be developed at end 2 and, consequently, $-C_{21}C_{12}S_{12}$ will be carried over to end 1. The sum of the foregoing two steps for end 1 gives

$$\begin{aligned} S'_{12} &= S_{12} - C_{21}C_{12}S_{12} \\ &= S_{12}(1 - C_{12}C_{21}) \end{aligned}$$

as asserted.

For a prismatic member $C_{12} = C_{21} = \frac{1}{2}$; therefore,

$$S'_{12} = \tfrac{3}{4}S_{12} \tag{12-49}$$

*Case* 2 The modified stiffness for end 1 of member 1–2, if end 2 rotates an equal but opposite angle to that of end 1 as in the case of symmetry, is given

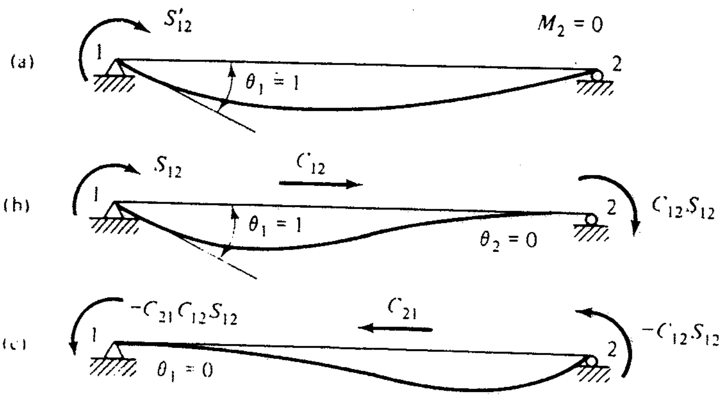


Figure 12-9

by

$$S'_{12} = S_{12}(1 - C_{12}) \tag{12-50}$$

To prove this, we refer to Fig. 12-10(a), which is postulated according to the definition of modified stiffness for the present case.

As before, we break this into two separate steps, as shown in Fig. 12-10(b) and (c). Figure 12-10(b) shows the usual way of determining $S_{12}$. In Fig. 12-10(c) a moment of $-S_{21}$ is applied at end 2 necessary to bring it back to its actual position. Consequently, a moment of $-C_{21}S_{21}$ is carried over to end 1. The sum of the results of these two steps for end 1 gives

$$S'_{12} = S_{12} - C_{21}S_{21}$$

On substituting $C_{12}S_{12}$ for $C_{21}S_{21}$ in the expression above, we obtain

$$S'_{12} = S_{12}(1 - C_{12})$$

as asserted.

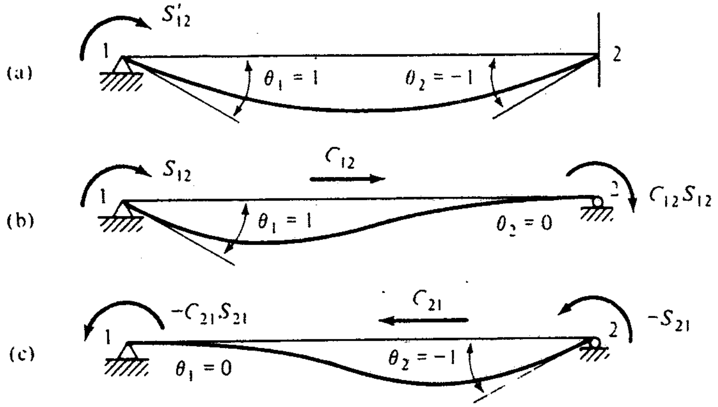


**Figure 12-10**

For a prismatic member $C_{12} = \frac{1}{2}$; therefore,

$$S'_{12} = \tfrac{1}{2}S_{12} \tag{12-51}$$

*Case* 3 The modified stiffness of end 1 of member 1–2, if end 2 rotates an angle equal to that of end 1 as in the case of antisymmetry, is given by

$$S'_{12} = S_{12}(1 + C_{12}) \tag{12-52}$$

To prove this, we refer to Fig. 12-11(a) for the setup of $S'_{12}$. As before, this may be considered as the superposition of two separate cases, as shown in Fig. 12-11(b) and (c). Consequently,

$$S'_{12} = S_{12} + C_{21}S_{21}$$

On substituting $C_{12}S_{12}$ for $C_{21}S_{21}$, we obtain

$$S'_{12} = S_{12}(1 + C_{12})$$

as asserted.

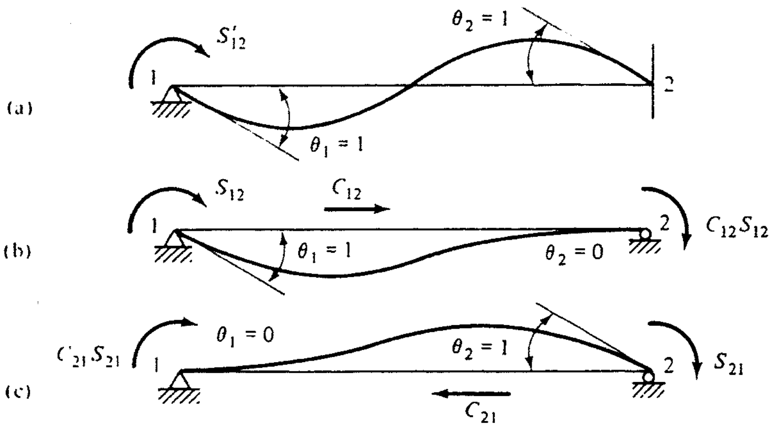


Figure 12-11

For a prismatic member $C_{12} = \frac{1}{2}$; therefore,

$$S'_{12} = \tfrac{3}{2}S_{12} \tag{12-53}$$

## 12-9 A NUMERICAL SOLUTION

One of the most frequently used methods of approximate integration is the Simpson's one-third rule. Consider an integral $\int f(x)\,dx$ between the limit $a$ and $b$. If the integral from $x = a$ to $x = b$ is divided into $n$ equal parts, where $n$ is an even number, and if $y_0, y_1, \ldots, y_{n-1}, y_n$ are the ordinates of the curve $y = f(x)$ at these points of subdivision (Fig. 12-12), then according to Simpson's one-third rule,

$$\int_a^b f(x)\,dx \approx \frac{\Delta}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \tag{12-54}$$

where $\Delta = (b - a)/n$. Simpson's rule gives an accurate result if $f(x)$ is a linear, quadratic, or cubic function.

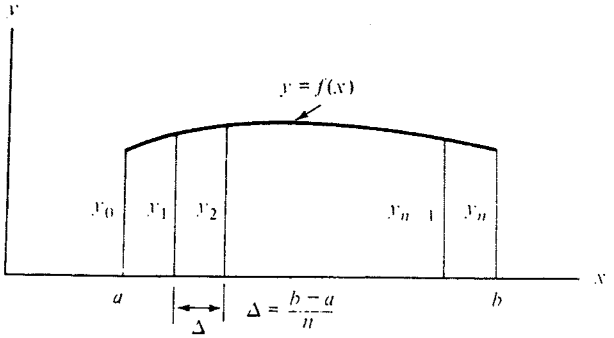


Figure 12-12

To illustrate, let us solve the rotational flexibility matrix of a beam element expressed in Eq. 12-17. Consider the coefficient $f_{22}$,

$$\frac{1}{El^2}\int_0^l \frac{x^2\,dx}{I} \qquad [I = I(x)]$$

We divide the length $l$ into 10 equal segments, each with $0.1l$. An application of Eq. 12-54 yields

$$\begin{aligned} f_{22} &= \frac{1}{El^2}\int_0^l \frac{x^2\,dx}{I} \\ &= \frac{0.1l}{3E}\left[\frac{(1)(0)^2}{I_0} + \frac{(4)(0.1)^2}{I_1} + \frac{(2)(0.2)^2}{I_2} + \frac{(4)(0.3)^2}{I_3} + \frac{(2)(0.4)^2}{I_4} + \frac{(4.0)(0.5)^2}{I_5}\right. \\ &\quad \left. + \frac{(2)(0.6)^2}{I_6} + \frac{(4)(0.7)^2}{I_7} + \frac{(2)(0.8)^2}{I_8} + \frac{(4)(0.9)^2}{I_9} + \frac{(1)(1)^2}{I_{10}}\right] \end{aligned} \tag{12-55}$$

with $I_0 = I(0)$, $I_1 = I(0.1l)$, $I_2 = I(0.2l)$, and so on. In matrix form,

$$f_{22} = \frac{0.1l}{3E}[I_0^{-1} \quad I_1^{-1} \quad I_2^{-1} \quad I_3^{-1} \quad I_4^{-1} \quad I_5^{-1} \quad I_6^{-1} \quad I_7^{-1} \quad I_8^{-1} \quad I_9^{-1} \quad I_{10}^{-1}] \begin{Bmatrix} 0 \\ 0.04 \\ 0.08 \\ 0.36 \\ 0.32 \\ 1 \\ 0.72 \\ 1.96 \\ 1.28 \\ 3.24 \\ 1 \end{Bmatrix} \tag{12-56}$$

In the same manner, we obtain

$$\begin{aligned} f_{11} &= \frac{1}{El^2}\int_0^l \frac{(l-x)^2\,dx}{I} \\ &= \frac{0.1l}{3E}\left[\frac{(1)(1)^2}{I_0} + \frac{(4)(0.9)^2}{I_1} + \frac{(2)(0.8)^2}{I_2} + \frac{(4)(0.7)^2}{I_3} + \frac{(2)(0.6)^2}{I_4} + \frac{(4)(0.5)^2}{I_5}\right. \\ &\quad \left. + \frac{(2)(0.4)^2}{I_6} + \frac{(4)(0.3)^2}{I_7} + \frac{(2)(0.2)^2}{I_8} + \frac{(4)(0.1)^2}{I_9} + \frac{(1)(0)^2}{I_{10}}\right] \end{aligned}$$

$$= \frac{0.1l}{3E}[I_0^{-1} \quad I_1^{-1} \quad I_2^{-1} \quad I_3^{-1} \quad I_4^{-1} \quad I_5^{-1} \quad I_6^{-1} \quad I_7^{-1} \quad I_8^{-1} \quad I_9^{-1} \quad I_{10}^{-1}] \cdot \begin{Bmatrix} 1 \\ 3.24 \\ 1.28 \\ 1.96 \\ 0.72 \\ 1 \\ 0.32 \\ 0.36 \\ 0.08 \\ 0.04 \\ 0 \end{Bmatrix} \tag{12-57}$$

$$f_{12} = f_{21} = -\frac{1}{El^2}\int_0^l \frac{(l - x)(x)dx}{I}$$

$$= -\frac{0.1l}{3E}\left[\frac{(1)(1)(0)}{I_0} + \frac{(4)(0.9)(0.1)}{I_1} + \frac{(2)(0.8)(0.2)}{I_2} + \frac{(4)(0.7)(0.3)}{I_3}\right.$$

$$+ \frac{(2)(0.6)(0.4)}{I_4} + \frac{(4)(0.5)^2}{I_5} + \frac{(2)(0.4)(0.6)}{I_6} + \frac{(4)(0.3)(0.7)}{I_7}$$

$$\left.+ \frac{(2)(0.2)(0.8)}{I_8} + \frac{(4)(0.1)(0.9)}{I_9} + \frac{(1)(0)(1)}{I_{10}}\right]$$

$$= \frac{0.1l}{3E}[I_0^{-1} \quad I_1^{-1} \quad I_2^{-1} \quad I_3^{-1} \quad I_4^{-1} \quad I_5^{-1} \quad I_6^{-1} \quad I_7^{-1} \quad I_8^{-1} \quad I_9^{-1} \quad I_{10}^{-1}] \cdot \begin{Bmatrix} 0 \\ -0.36 \\ -0.32 \\ -0.84 \\ -0.48 \\ -1 \\ -0.48 \\ -0.84 \\ -0.32 \\ -0.36 \\ 0 \end{Bmatrix} \tag{12-58}$$

If we let $c_{11} = f_{11}$, $c_{12} = f_{12}$, $c_{21} = f_{22}$, and $c_{22} = f_{21}$, we have

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \frac{0.1l}{3E}\begin{bmatrix} I_0^{-1} & I_1^{-1} & I_2^{-1} & I_3^{-1} & I_4^{-1} & I_5^{-1} & I_6^{-1} & I_7^{-1} & I_8^{-1} & I_9^{-1} & I_{10}^{-1} \\ I_{10}^{-1} & I_9^{-1} & I_8^{-1} & I_7^{-1} & I_6^{-1} & I_5^{-1} & I_4^{-1} & I_3^{-1} & I_2^{-1} & I_1^{-1} & I_0^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3.24 & -0.36 \\ 1.28 & -0.32 \\ 1.96 & -0.84 \\ 0.72 & -0.48 \\ 1 & -1 \\ 0.32 & -0.48 \\ 0.36 & -0.84 \\ 0.08 & -0.32 \\ 0.04 & -0.36 \\ 0 & 0 \end{bmatrix}$$

or simply

$$[C] = [A][B] \tag{12-59}$$

For a prismatic member, the $I$-value for each segment is constant. Therefore,

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \tag{12-60}$$

That is, the rotational flexibility matrix becomes

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

as expected.

With element rotational flexibility coefficients determined, the element rotational stiffness coefficients as well as other constants can readily be derived.

## PROBLEMS

**12-1.** Figure 12-13 shows a beam of varying cross section with both ends fixed and subjected to the bending action of a concentrated load $P$. Find the integral expressions for the fixed-end moments.

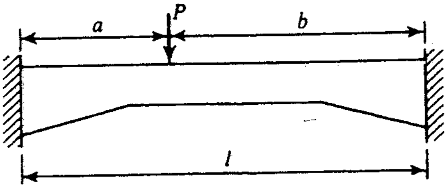


**Figure 12-13**

**12-2.** For the beam element of constant width shown in Fig. 12-14, find, by the numerical method, the rotational flexibility matrix. Assume that the element is simply supported.

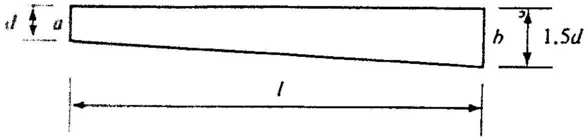


**Figure 12-14**

**12-3.** Use the result of Prob. 12-2 to find the rotational stiffness matrix for the same element.

**12-4.** For the haunched member of constant width shown in Fig. 12-15, find the stiffnesses and carry-over factors by the numerical method or from the table of frame constants if available.

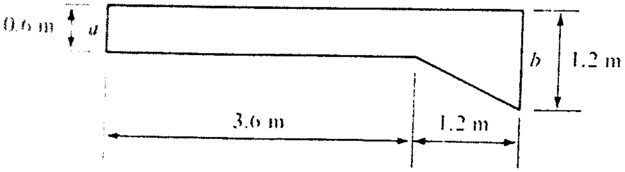


**Figure 12-15**

**12-5.** Using the result of Prob. 12-4, give the rotational stiffness and flexibility matrices for the element shown in Fig. 12-15.

**12-6.** Use the numerical method or table of frame constants to determine the fixed-end moments for the haunched beam of Fig. 12-15 due to a uniform load of 5 kN/m over the entire span.

**12-7.** Use the relevant calculation from Probs. 12-4 to 12-6 to find, by moment distribution, the end moments for the beam in Fig. 12-16 subjected to a uniform load of 5 kN/m over the entire span.

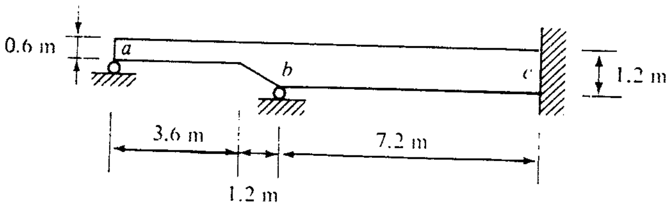


**Figure 12-16**

**12-8.** Repeat Prob. 12-7 by the method of slope deflection.

**12-9.** Repeat Prob. 12-7 by the matrix force method.

**12-10.** Repeat Prob. 12-7 by the matrix displacement method.

# 13

# *Matrix Analysis of Elastic Stability*

## 13-1 GENERAL

In Chap. 2 we discussed in detail the statical stability and the geometric stability of simple framed structures. They are judged by the number and arrangement of elements and supports of which the structures are composed and are independent of the externally applied forces. However, the stability with which we are now concerned is a different kind of stability, which may be called the *stability of equilibrium* for an elastic structural system. According to Kirchhoff's *uniqueness theorem*, an elastic body can assume one and only one equilibrium configuration under a given external loading. The theorem holds true as long as the structure is linear. When the uniqueness of solution is established, we need to find only the solution of a given boundary-value problem; that solution is the solution. However, there are cases that show the violation of the uniqueness of the solution in one way or another and are characterized by the disproportionality between the loads and displacements (nonlinear). For instance, elastic columns and plates can buckle, narrow beams can collapse laterally, and framed structures can become unstable under certain specified loads. These problems are all connected with the loss of uniqueness of solution. Under certain circumstances two or more solutions may become possible, and the circumstances are said to cause the instability.

In a classical scientific sense, the stability of an elastic system can be investigated by subjecting it to an infinitesimal disturbance from its equilibrium position. If the system returns to its original position upon removal of the disturbance, it is considered to be *stable*. If it does not return, it is *unstable*. The borderline between stable and unstable equilibrium is referred to as *neutral* *equilibrium*. Our problem is to examine if there can exist an alternative state (or states) of equilibrium in a displayed position from which critical loads can be evaluated.

In the sense of modern matrix analysis of a linear structure, instability of a structure means the loss of structure stiffness. Consider a framed structure with nodal forces $\{R\}$ and the corresponding displacements $\{r\}$, which can be generally related by a stiffness matrix $[K]$:

$$[K]\{r\} = \{R\} \tag{13-1}$$

When the effects of axial forces are included, it is shown (Sec. 13-2) that the elements of $[K]$ are no longer constants but are a function of axial forces. As the axial effects become critical, it is possible to have the structure acquire some small displacements $\{\delta r\}$ without increase in $\{R\}$. Thus,

$$[K]\{r + \delta r\} = \{R\} \tag{13-2}$$

From Eqs. 13-1 and 13-2, we reach the solution

$$[K]\{\delta r\} = \{0\} \tag{13-3}$$

For a nontrivial solution (i.e., other than $\{\delta r\} = \{0\}$) to exist, $[K]$ must be singular, that is,

$$|K| = 0 \tag{13-4}$$

Equation 13-4 implies that the inverse of the stiffness matrix is infinitely large or that the total stiffness matrix is vanishingly small, thus serving a criterion for evaluating the critical loads.

As an illustration, consider a vertical bar *ab* hinged at the bottom and supported by a spring *bc* at the top carrying an axial load $P$, as shown in Fig. 13-1(a). Assume that the bar is infinitely rigid. It is intuitively correct that if the load $P$ acts truly along *ab*, the bar is in equilibrium for any magnitude of

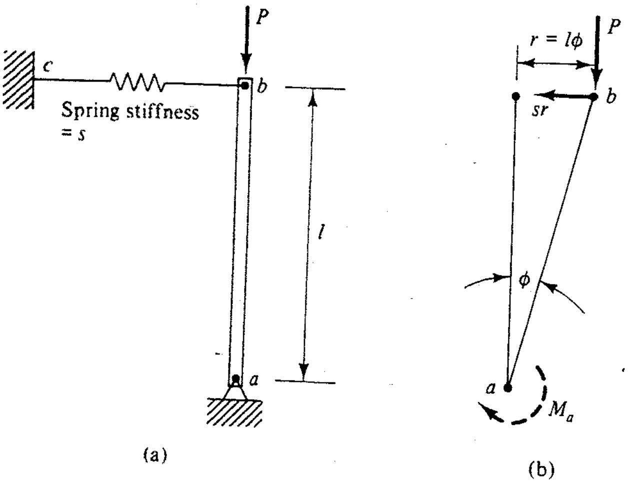


Figure 13-1

$P$. However, it is interesting to investigate the stability of the equilibrium for the system.

The system has a single degree of freedom, that is, the rotation of the bar about hinge $a$, or its equivalent, the lateral displacement $r$ at top $b$, since for small rotation $\phi$, $r = l\phi$. The problem may first be investigated in the following classical approach.

Suppose the bar is turned a small rotation $\phi$ (clockwise) about $a$, corresponding to a small disturbing moment $M_a$ (shown as a dashed line). The bar is brought to rest in this position; the balanced force system is shown in Fig. 13-1(b). The force in the spring is given by $sr$, $s$ being the spring stiffness. Thus,

$$M_a = srl \cos \phi - Pr$$

For small $\phi$, it becomes

$$M_a = srl - Pr \tag{13-5}$$

After the cause of disturbance is removed (setting $M_a = 0$), we notice that if $srl > Pr$, or

$$P < sl \tag{13-6}$$

the bar tends to rotate in the counterclockwise direction, that is, to turn back to the original stable position. However, if

$$P > sl \tag{13-7}$$

the bar tends to accelerate clockwise, and the original configuration of equilibrium becomes unstable. If

$$P = sl \tag{13-8}$$

then the system is in neutral equilibrium and the load $P = sl$ is called the *critical load*.

We see that the criterion for stability of equilibrium depends on the relation between the magnitude of the axial load $P$, spring stiffness $s$, and structural geometry $l$ and is independent of the magnitude of the displacement $r$ or $\phi$, as long as the displacement is small.

The problem may also be investigated by the loss of structure stiffness. Refer to Fig. 13-1(a). For a small rotation at $a$, we have—from Eq. 13-5—the corresponding disturbing moment at $a$ given by

$$M_a = srl - Pr$$

This is equivalent to applying a disturbing force $R = M_a/l$ to cause a small lateral displacement $r$ at the top $b$ (Fig. 13-2). Thus,

$$R = \frac{M_a}{l} = sr - \frac{Pr}{l} = (s - \frac{P}{l})(r) \tag{13-9}$$

The stiffness is therefore given by

$$K = s - \frac{P}{l} \tag{13-10}$$

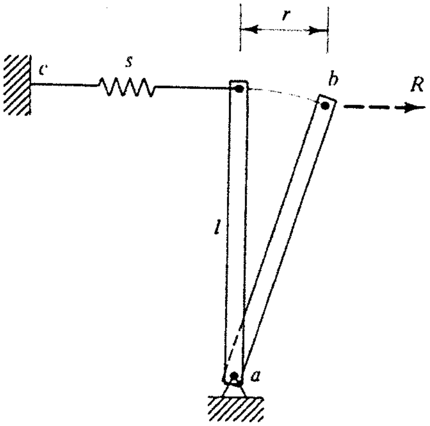


Figure 13-2

When the stiffness vanishes ($K = 0$), the system becomes unstable. Thus,

$$P_{cr} = sl \tag{13-11}$$

as previously found.

Note that the structure stiffness in general consists of two parts: an elastic stiffness $s$ and a geometric stiffness $P/l$, which depends on the geometry of the structure and the axial action.

## 13-2 STIFFNESS MATRIX FOR A BEAM ELEMENT SUBJECT TO AXIAL FORCE

Consider the element of Fig. 13-3(a) subjected to an axial force $P$ and a set of end actions $\{Q\}$ in which $Q_1$ and $Q_2$ are the end moments, whereas $Q_3$ and $Q_4$ are the end shears. The corresponding end displacements $\{q\}$ are shown in Fig. 13-3(b). All signs of forces and displacements depicted in Fig. 13-3 are taken as positive. We are to establish the element stiffness matrix in the presence of the axial forces, using the principle of conservation of energy.

We assume that the application of axial load $P$ takes place first; finally $\{Q\}$ with $\{q\}$ are set in, with $P$ kept constant. The system is in equilibrium not only in the original configuration but also in the later displaced position. During the end of transition, the external work done is given by

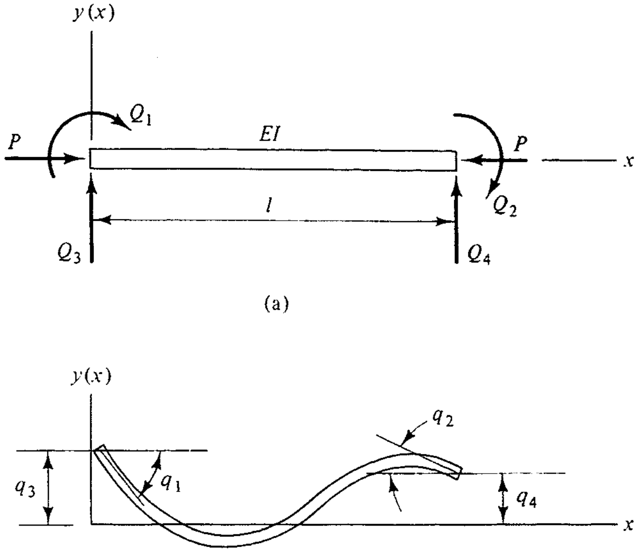
$$W_E = \frac{1}{2}\{q\}^T\{Q\} + \frac{P}{2}\int_0^l (y')^2\,dx \tag{13-12}$$

including the work of $\{Q\}$ and that due to $P$.

The term

$$\frac{1}{2}\int_0^l (y')^2\,dx \tag{13-13}$$

represents the axial displacement of one end relative to the other end resulting from bending. This can be proved as follows.



(a)

(b)

Figure 13-3

We assume that the axial deformation in the beam has taken place before the axial force $P$ reaches its full value. Therefore, the total beam length $l$ would not change when the end actions $\{Q\}$ are applied in the presence of axial force $P$. Consider an element $mn$ of the beam with the length $dx$ before bending (Fig. 13-4). This element has already been under compression. After bending, $mn$ displaces to the position $m'n'$. Since there is no change of length before and after bending, the arc length $m'n'$ is equal to $dx$. The shortening of $mn$ in the $x$ direction is, therefore,

$$\begin{aligned} mn - m'n'' &= dx - \left[(dx)^2 - \left(\frac{dy}{dx}dx\right)^2\right]^{1/2} \\ &= dx - dx\left[1 - \left(\frac{dy}{dx}\right)^2\right]^{1/2} \\ &= dx - dx\left[1 - \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + \cdots\right] \\ &\approx \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx \end{aligned}$$

where the higher-order terms are neglected in the last step. The approach of the ends of the column is therefore given by Eq. (13-13).

The internal work, that is, the bending strain energy stored in the member,

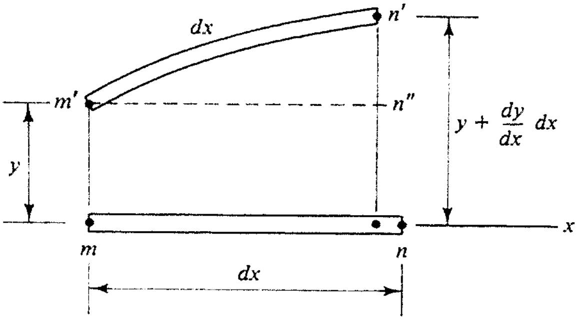


Figure 13-4

is given by

$$W_I = \frac{1}{2}\int_0^l \frac{M^2\,dx}{EI} = \frac{EI}{2}\int_0^l (y'')^2\,dx \tag{13-14}$$

Setting $W_E = W_I$ gives

$$\frac{1}{2}\{q\}^T\{Q\} + \frac{P}{2}\int_0^l (y')^2\,dx = \frac{EI}{2}\int_0^l (y'')^2\,dx \tag{13-15}$$

Using the relationship $\{Q\} = [k]\{q\}$ in Eq. 13-15, in which $[k]$ is the element stiffness matrix, we obtain

$$\{q\}^T[k]\{q\} = EI\int_0^l (y'')^2\,dx - P\int_0^l (y')^2\,dx \tag{13-16}$$

In order to evaluate the stiffness matrix $[k]$, it is necessary to express the transverse displacement $y(x)$ in terms of the end displacements $\{q\}$. This can be accomplished by assuming a displacement function in a polynomial series

$$y(x) = a_1 + a_2x + a_3x^2 + a_4x^3 \tag{13-17}$$

and applying the boundary conditions,

$$\begin{aligned} y &= q_3, & y' &= -q_1 & &\text{at } x = 0 \\ y &= q_4, & y' &= -q_2 & &\text{at } x = l \end{aligned} \tag{13-18}$$

to determine the four coefficients. A cubic series is used since the third derivative (the shear) is constant and the second derivative (the moment) is linear, and these are consistent with the nodal-force pattern assumed for the beam element.

The coefficients are found to be

$$\begin{aligned} a_1 &= q_3 \\ a_2 &= -q_1 \\ a_3 &= \frac{3(q_4 - q_3) + (2q_1 + q_2)l}{l^2} \\ a_4 &= \frac{2(q_3 - q_4) - (q_1 + q_2)l}{l^3} \end{aligned} \tag{13-19}$$

Substituting these into Eq. 13-17 and rearranging, we express $y$ in terms of the end distortions $\{q\}$ in matrix form as

$$y = \left[\left(-x + \frac{2x^2}{l} - \frac{x^3}{l^2}\right)\left(\frac{x^2}{l} - \frac{x^3}{l^2}\right)\left(1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right)\left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right)\right]\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

or simply

$$y = [\phi]\{q\} \tag{13-20}$$

in which $[\phi] = [\phi_1(x)\ \phi_2(x)\ \phi_3(x)\ \phi_4(x)]$ with $\phi_1(x) = -x + 2x^2/l - x^3/l^2$, and so on. Differentiating Eq. 13-20, we write

$$y' = [\phi']\{q\} \tag{13-21}$$

and

$$y'' = [\phi'']\{q\} \tag{13-22}$$

where

$$[\phi'] = [\phi_1'(x)\ \phi_2'(x)\ \phi_3'(x)\ \phi_4'(x)]$$

$$= \left[\left(-1 + \frac{4x}{l} - \frac{3x^2}{l^2}\right)\left(\frac{2x}{l} - \frac{3x^2}{l^2}\right)\left(-\frac{6x}{l^2} + \frac{6x^2}{l^3}\right)\left(\frac{6x}{l^2} - \frac{6x^2}{l^3}\right)\right] \tag{13-23}$$

and

$$[\phi''] = [\phi_1''(x)\ \phi_2''(x)\ \phi_3''(x)\ \phi_4''(x)] \tag{13-24}$$

$$= \left[\left(\frac{4}{l} - \frac{6x}{l^2}\right)\left(\frac{2}{l} - \frac{6x}{l^2}\right)\left(-\frac{6}{l^2} + \frac{12x}{l^3}\right)\left(\frac{6}{l^2} - \frac{12x}{l^3}\right)\right]$$

Since $(y')^2 = (y')^T(y')$ and $(y'')^2 = (y'')^T(y'')$, using Eqs. 13-21 and 13-22, we may write

$$(y')^2 = \{q\}^T[\phi']^T[\phi']\{q\} = \{q\}^T[\phi_i'\ \phi_j']\{q\} \tag{13-25}$$

in which

$$[\phi_i'\phi_j'] = [\phi']^T[\phi'] = \begin{bmatrix} \phi_1'\phi_1' & & \text{sym.} & \\ \phi_2'\phi_1' & \phi_2'\phi_2' & & \\ \phi_3'\phi_1' & \phi_3'\phi_2' & \phi_3'\phi_3' & \\ \phi_4'\phi_1' & \phi_4'\phi_2' & \phi_4'\phi_3' & \phi_4'\phi_4' \end{bmatrix} \tag{13-26}$$

and

$$(y'')^2 = \{q\}^T[\phi'']^T[\phi'']\{q\} = \{q\}^T[\phi_i''\phi_j'']\{q\} \tag{13-27}$$

in which

$$[\phi_i''\phi_j''] = [\phi'']^T[\phi''] = \begin{bmatrix} \phi_1''\phi_1'' & & \text{sym.} & \\ \phi_2''\phi_1'' & \phi_2''\phi_2'' & & \\ \phi_3''\phi_1'' & \phi_3''\phi_2'' & \phi_3''\phi_3'' & \\ \phi_4''\phi_1'' & \phi_4''\phi_2'' & \phi_4''\phi_3'' & \phi_4''\phi_4'' \end{bmatrix} \tag{13-28}$$

Substituting Eqs. 13-25 and 13-27 into Eq. 13-16 gives

$$\{q\}^T[k]\{q\} = \{q\}^T\left[EI\int_0^l [\phi_i''\phi_j'']\,dx - P\int_0^l [\phi_i'\phi_j']\,dx\right]\{q\} \tag{13-29}$$

from which we obtain the stiffness matrix $[k]$ of the beam-column as

$$[k] = \left[ EI \int_0^l [\phi_i''\phi_j''] \, dx \right] - \left[ P \int_0^l [\phi_i'\phi_j'] \, dx \right] \tag{13-30}$$

Using the expressions given in Eqs. 13-23 and 13-24 for $[\phi']$ and $[\phi'']$ and Eqs. 13-26 and 13-28 for $[\phi_i'\phi_j']$ and $[\phi_i''\phi_j'']$, we obtain

$$[k] = \begin{bmatrix} \frac{4EI}{l} & & \text{sym.} & \\ \frac{2EI}{l} & \frac{4EI}{l} & & \\ -\frac{6EI}{l^2} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} & \\ \frac{6EI}{l^2} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & \frac{12EI}{l^3} \end{bmatrix} - P \begin{bmatrix} \frac{2l}{15} & & \text{sym.} & \\ -\frac{l}{30} & \frac{2l}{15} & & \\ -\frac{1}{10} & -\frac{1}{10} & \frac{6}{5l} & \\ \frac{1}{10} & \frac{1}{10} & -\frac{6}{5l} & \frac{6}{5l} \end{bmatrix} \tag{13-31}$$

We recognize that the stiffness matrix is a function of the axial force $P$ and is made up of two parts: The first is the elastic stiffness matrix of an ordinary beam element, and the second may be called the *geometrical matrix* (or *stability matrix*) representing the axial effect with force $P$ as a parameter of influence.

The matrix of Eq. 13-31 is derived for an axial compression, but it serves equally well for a member subjected to tension by changing only the minus sign before $P$ to plus.

## 13-3 ELASTIC STABILITY OF A PRISMATIC COLUMN

The stiffness matrix given in Eq. 13-31 can be used to obtain the first approximation of the critical loads for an axially loaded straight column of uniform $EI$ with various end conditions. In some instances, the results are strikingly good.

Consider the column shown in Fig. 13-5(a). The general nodal forces and displacements are related by the stiffness matrix as

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{l} - \frac{2Pl}{15} & & \text{sym.} & \\ \frac{2EI}{l} + \frac{Pl}{30} & \frac{4EI}{l} - \frac{2Pl}{15} & & \\ -\frac{6EI}{l^2} + \frac{P}{10} & -\frac{6EI}{l^2} + \frac{P}{10} & \frac{12EI}{l^3} - \frac{6P}{5l} & \\ \frac{6EI}{l^2} - \frac{P}{10} & \frac{6EI}{l^2} - \frac{P}{10} & -\frac{12EI}{l^3} + \frac{6P}{5l} & \frac{12EI}{l^3} - \frac{6P}{5l} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} \tag{13-32}$$

The simplest case is a fixed-slide column [Fig. 13-5(b)], in which only

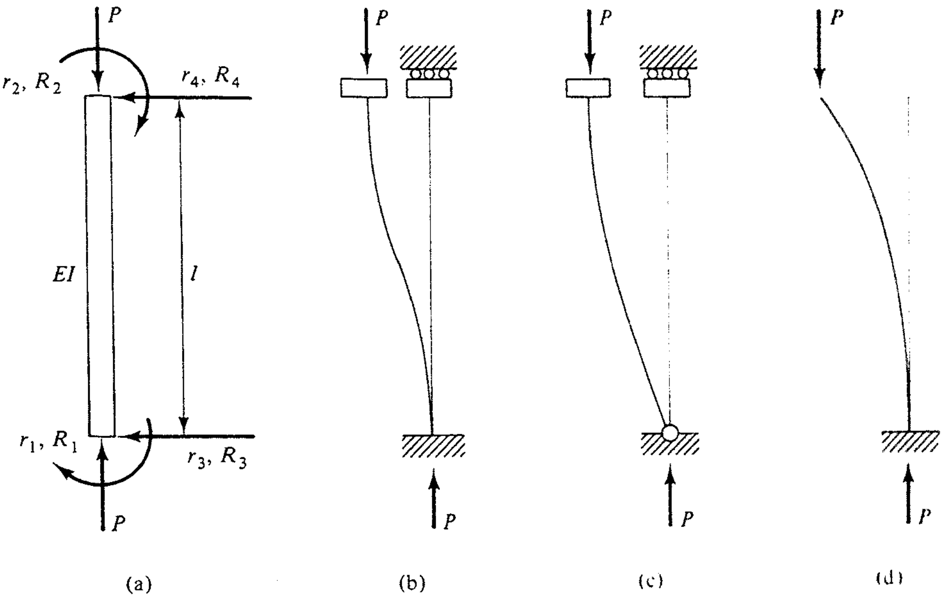


**Figure 13-5**

displacement $r_4$ can occur while $r_1 = r_2 = r_3 = 0$. The force-displacement relationship is therefore given by

$$R_4 = \underbrace{\left[\frac{12EI}{l^3} - \frac{6P}{5l}\right]}_{K} r_4 \tag{13-33}$$

The critical load is obtained by setting $|K| = 0$, that is,

$$\frac{12EI}{l^3} - \frac{6P}{5l} = 0 \tag{13-34}$$

from which

$$P_{cr} = \frac{10EI}{l^2} \tag{13-35}$$

Compared with the theoretical value $P_{cr} = \pi^2 EI/l^2$, the error is only 1.3% on the high side.

If the fixed end is changed to a hinged end as in Fig. 13-5(c), then the equation of equilibrium corresponding to possible displacements $r_1$ and $r_4$ becomes

$$\begin{Bmatrix} R_1 \\ R_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{4EI}{l} - \frac{2Pl}{15} & \frac{6EI}{l^2} - \frac{P}{10} \\ \frac{6EI}{l^2} - \frac{P}{10} & \frac{12EI}{l^3} - \frac{6P}{5l} \end{bmatrix}}_{K} \begin{Bmatrix} r_1 \\ r_4 \end{Bmatrix} \tag{13-36}$$

Setting $|K| = 0$ yields

$$\frac{12(EI)^2}{l^4} - 5.2\frac{PEI}{l^2} + 0.15\,P^2 = 0 \tag{13-37}$$

Let $\lambda = EI/l^2$. Equation 13-37 becomes

$$0.15P^2 - 5.2\lambda P + 12\lambda^2 = 0 \tag{13-38}$$

Solving for $P$ gives the smaller value, equal to $2.5\lambda$. Thus

$$P_{cr} = \frac{2.5EI}{l^2} \tag{13-39}$$

which is also very close to the exact value $(\pi^2 EI/4l^2)$, with an error 1.3% on the high side.

For a fixed-free column, as shown in 13-5($d$), $r_1 = r_3 = 0$. We have the force-displacement relationship corresponding to $r_2$ and $r_4$ as

$$\begin{Bmatrix} R_2 \\ R_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{4EI}{l} - \dfrac{2Pl}{15} & \dfrac{6EI}{l^2} - \dfrac{P}{10} \\ \dfrac{6EI}{l^2} - \dfrac{P}{10} & \dfrac{12EI}{l^3} - \dfrac{6P}{5l} \end{bmatrix}}_{K} \begin{Bmatrix} r_2 \\ r_4 \end{Bmatrix} \tag{13-40}$$

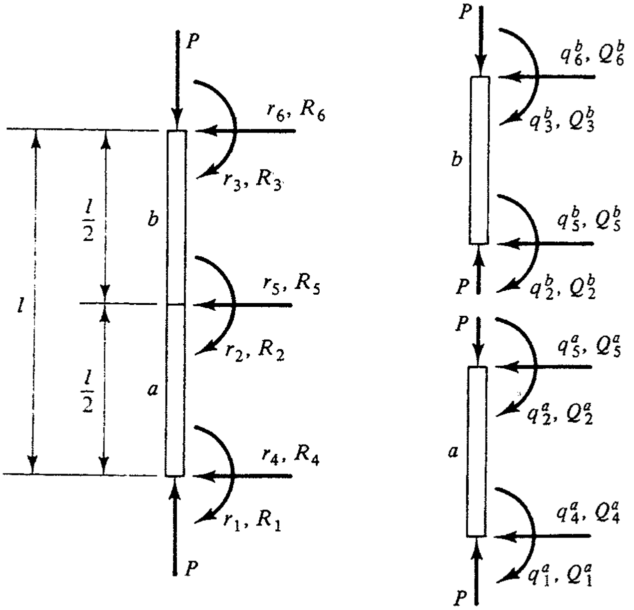
Setting $|K| = 0$ gives the same equation as 13-37. Therefore,

$$P_{cr} = \frac{2.5EI}{l^2} \tag{13-41}$$

The theoretical $P_{cr}$ value in this case is also $\pi^2 EI/4l^2$. The error is 1.3% on the high side.

The result is not satisfactory when applied to the case of hinged-hinged column or hinged-fixed column; besides, the procedure is not directly applicable to the fixed-fixed column. The way to handle this situation is to idealize the column into two identical elements and to construct the column stiffness matrix through the superimposing of the element stiffness matrices. The numbering scheme for external nodal coordinates of displacements and forces are shown in Fig. 13-6(a). Corresponding internal element end coordinates of displacements and forces are shown in Fig. 13-6(b). Since the element coordinates (local) coincide with the structure coordinates (global), it is convenient to use the direct stiffness method to establish the column stiffness matrix.

In view of Eq. 13-31 and Fig. 13-6(b), we establish the stiffness matrices



(a) (b) **Figure 13-6**

for elements $a$ and $b$ as

$$[k^a] = \begin{array}{c} \begin{array}{cccc} 1 \qquad\qquad & 2 \qquad\qquad & 4 \qquad\qquad & 5 \end{array} \\ \left[\begin{array}{cccc} \frac{8EI}{l} - \frac{Pl}{15} & & \text{sym.} & \\ \frac{4EI}{l} + \frac{Pl}{60} & \frac{8EI}{l} - \frac{Pl}{15} & & \\ -\frac{24EI}{l^2} + \frac{P}{10} & -\frac{24EI}{l^2} + \frac{P}{10} & \frac{96EI}{l^3} - \frac{12P}{5l} & \\ \frac{24EI}{l^2} - \frac{P}{10} & \frac{24EI}{l^2} - \frac{P}{10} & -\frac{96EI}{l^3} + \frac{12P}{5l} & \frac{96EI}{l^3} - \frac{12P}{5l} \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 4 \\ 5 \end{array} \end{array} \qquad (13\text{-}42)$$

$$[k_b] = \begin{array}{c} \\ \left[\begin{array}{cccc} 2 & 3 & 5 & 6 \\ \frac{8EI}{l} - \frac{Pl}{15} & & \text{sym.} & \\ \frac{4EI}{l} + \frac{Pl}{60} & \frac{8EI}{l} - \frac{Pl}{15} & & \\ -\frac{24EI}{l^2} + \frac{P}{10} & -\frac{24EI}{l^2} + \frac{P}{10} & \frac{96EI}{l^3} - \frac{12P}{5l} & \\ \frac{24EI}{l^2} - \frac{P}{10} & \frac{24EI}{l^2} - \frac{P}{10} & -\frac{96EI}{l^3} + \frac{12P}{5l} & \frac{96EI}{l^3} - \frac{12P}{5l} \end{array}\right] \end{array} \begin{array}{c} \\ 2 \\ 3 \\ 5 \\ 6 \end{array} \qquad (13\text{-}43)$$

From these, we construct the column stiffness matrix to relate the nodal forces $\{R\}$ and displacements $\{r\}$, with each stiffness coefficient collected from element stiffness coefficients bearing the same subscripts. Thus,

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{8EI}{l} - \frac{Pl}{15} & & & \text{sym.} & & \\ \frac{4EI}{l} + \frac{Pl}{60} & \frac{16EI}{l} - \frac{2Pl}{15} & & & & \\ 0 & \frac{4EI}{l} + \frac{Pl}{60} & \frac{8EI}{l} - \frac{Pl}{15} & & & \\ -\frac{24EI}{l^2} + \frac{P}{10} & -\frac{24EI}{l^2} + \frac{P}{10} & 0 & \frac{96EI}{l^3} - \frac{12P}{5l} & & \\ \frac{24EI}{l^2} - \frac{P}{10} & 0 & -\frac{24EI}{l^2} + \frac{P}{10} & -\frac{96EI}{l^3} + \frac{12P}{5l} & \frac{192EI}{l^3} - \frac{24P}{5l} & \\ 0 & \frac{24EI}{l^2} - \frac{P}{10} & \frac{24EI}{l^2} - \frac{P}{10} & 0 & -\frac{96EI}{l^3} + \frac{12P}{5l} & \frac{96EI}{l^3} - \frac{12P}{5l} \end{bmatrix}}_{K} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{Bmatrix}$$

(13-44)

We shall use Eq. 13-44 to determine the critical load for columns with various end conditions, as shown in Fig. 13-7.

1. Fixed-fixed column

For a column with both ends built in [Fig. 13-7(a)], all nodal displacements except lateral displacement $r_5$ vanish [see coordinates in Fig. 13-6(a)]. The force-displacement equation reduces to

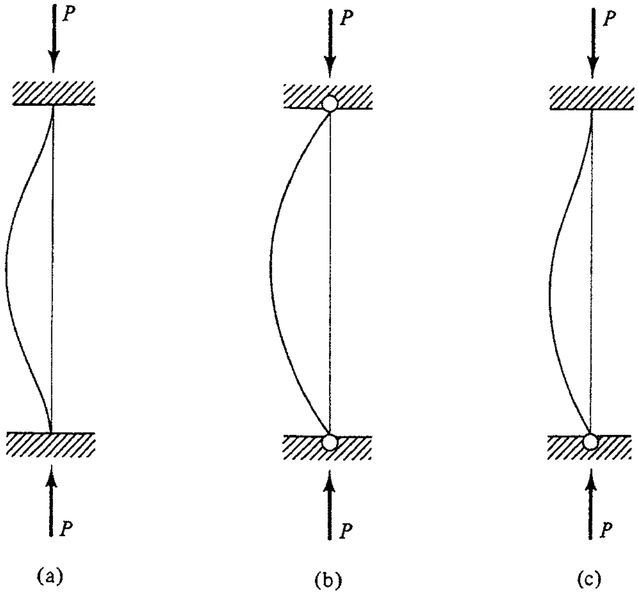
$$R_5 = \underbrace{\left[\frac{192EI}{l^3} - \frac{24P}{5l}\right]}_{K} r_5 \qquad (13\text{-}45)$$

Setting $|K| = 0$, or

$$\frac{192EI}{l^3} - \frac{24P}{5l} = 0$$

we obtain

$$P_{cr} = \frac{40EI}{l^2} \qquad (13\text{-}46)$$



(a) (b) (c) **Figure 13-7**

Comparing this result with the theoretical value

$$P_{cr} = \frac{4\pi^2 EI}{l^2} = \frac{39.48EI}{l^2}$$

we find that the error is 1.3% on the high side.

2. Hinged-hinged column

For a hinged-hinged column [Fig. 13-7(b)], only $r_1$, $r_3$, and $r_5$ are free to occur. The force-displacement relationship is therefore

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{8EI}{l} - \dfrac{Pl}{15} & 0 & \dfrac{24EI}{l^2} - \dfrac{P}{10} \\ 0 & \dfrac{8EI}{l} - \dfrac{Pl}{15} & -\dfrac{24EI}{l^2} + \dfrac{P}{10} \\ \dfrac{24EI}{l^2} - \dfrac{P}{10} & -\dfrac{24EI}{l^2} + \dfrac{P}{10} & \dfrac{192EI}{l^3} - \dfrac{24P}{5l} \end{bmatrix}}_{K} \begin{Bmatrix} r_1 \\ r_3 \\ r_5 \end{Bmatrix} \tag{13-47}$$

Because of symmetrical displacement, $r_3 = -r_1$. This enables us to work with half the column, and the force-displacement relationship reduces to

$$\begin{Bmatrix} R_1 \\ R_5 \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{8EI}{l} - \dfrac{Pl}{15} & \dfrac{24EI}{l^2} - \dfrac{P}{10} \\ 2\left(\dfrac{24EI}{l^2} - \dfrac{P}{10}\right) & \dfrac{192EI}{l^3} - \dfrac{24P}{5l.} \end{bmatrix}}_{K'} \begin{Bmatrix} r_1 \\ r_5 \end{Bmatrix} \tag{13-48}$$

The critical load is obtained by setting the determinant of the modified stiffness to zero, that is, $|K'| = 0$, and solving for the smaller value of $P$. To simplify calculations we introduce

$$P = \frac{\lambda EI}{l^2}$$

and divide each term by $EI/l$. The condition $|K'| = 0$ then takes the form

$$\begin{vmatrix} 8 - \dfrac{\lambda}{15} & \dfrac{1}{l}\left(24 - \dfrac{\lambda}{10}\right) \\ \dfrac{2}{l}\left(24 - \dfrac{\lambda}{10}\right) & \dfrac{1}{l^2}\left(192 - \dfrac{24}{5}\lambda\right) \end{vmatrix} = 0 \qquad (13\text{-}49)$$

which yields

$$3\lambda^2 - 416\lambda + 3{,}840 = 0 \qquad (13\text{-}50)$$

The smaller root of Eq. 13-50 is $\lambda = 9.95$. Therefore,

$$P_{cr} = \frac{9.95EI}{l^2} \qquad (13\text{-}51)$$

The theoretical value is $\pi^2 EI/l^2$, or $9.87EI/l^2$. We find that the error is less than 1% on the high side.

3. Hinged-fixed column

For a hinged-fixed column, as shown in Fig. 13-7(c), the nodal displacements allowed to take place are $r_1$, $r_2$, and $r_5$. Consequently the corresponding force-displacement relationship is

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_5 \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{8EI}{l} - \dfrac{Pl}{15} & \dfrac{4EI}{l} + \dfrac{Pl}{60} & \dfrac{24EI}{l^2} - \dfrac{P}{10} \\ \dfrac{4EI}{l} + \dfrac{Pl}{60} & \dfrac{16EI}{l} - \dfrac{2Pl}{15} & 0 \\ \dfrac{24EI}{l^2} - \dfrac{P}{10} & 0 & \dfrac{192EI}{l^3} - \dfrac{24P}{5l} \end{bmatrix}}_{K} \begin{Bmatrix} r_1 \\ r_2 \\ r_5 \end{Bmatrix} \qquad (13\text{-}52)$$

The critical load is obtained by setting $|K| = 0$ and solving for the smallest value of $P$. As before, we let

$$P = \frac{\lambda EI}{l^2}$$

and divide each term by $EI/l$. The condition $|K| = 0$ then becomes

$$\begin{vmatrix} 8 - \dfrac{\lambda}{15} & 4 + \dfrac{\lambda}{60} & \dfrac{1}{l}\left(24 - \dfrac{\lambda}{10}\right) \\ 4 + \dfrac{\lambda}{60} & 16 - \dfrac{2}{15}\lambda & 0 \\ \dfrac{1}{l}\left(24 - \dfrac{\lambda}{10}\right) & 0 & \dfrac{1}{l^2}\left(192 - \dfrac{24}{5}\lambda\right) \end{vmatrix} = 0 \qquad (13\text{-}53)$$

The smallest root of $\lambda$ that satisfies the preceding equation is 20.71, giving

$$P_{cr} = \frac{20.71EI}{l^2} \tag{13-54}$$

The theoretical value is $20.19EI/l^2$. The error is 2.5% on the high side.

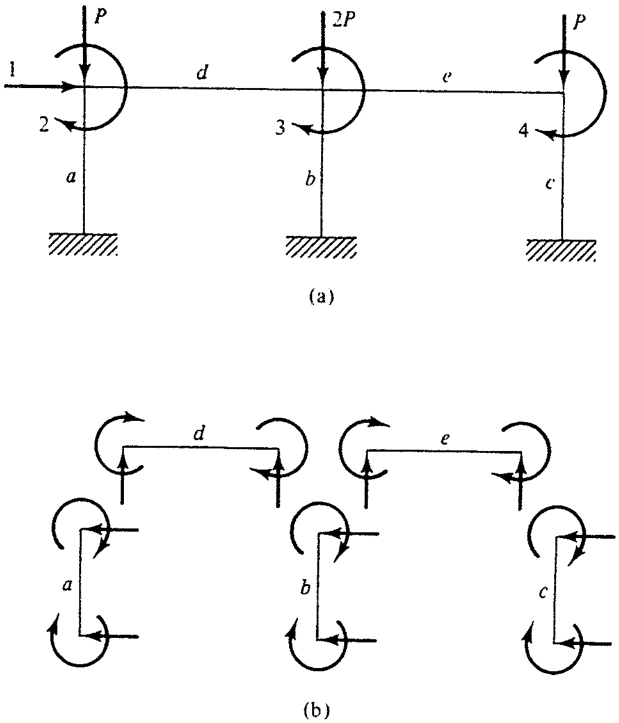
From the preceding examples, we see that the critical loads obtained by the stiffness method always err on the high side. This is due to the fact that an approximate function (a cubic polynomial instead of a trigonometric series) is used to describe the displaced configuration of the member and any deviation from the true deflected shape amounts to introducing additional constraints to the member. These constraints cause the member to be stiffer than the original one and therefore result in a higher buckling load than the actual one. Note that this fact generally provides an upper bound to the stiffness matrix.

## 13-4 ELASTIC STABILITY OF A RIGID FRAME

In the analysis of a rigid frame, if the axial effect on the bending stiffness is not ignored, then we must use Eq. 13-31 to calculate the stiffness for each element. However, the axial forces in the members of a statically indeterminate rigid frame are generally not known exactly at the outset. We therefore must begin with estimated axial forces and determine the stiffness for each member accordingly. The frame is then analyzed as a linear structure, following the usual procedure of stiffness method. If the final results of analysis indicate that the obtained axial forces differ considerably from the estimated values, then revision to the analysis must be made.

In handling the stability of a rigid frame (we limit the discussion to the elastic buckling of rigid frames within their own plane), the problem is considerably simplified by assuming that the members of the frame are subjected to axial loads only (known external forces). Before buckling sets in, no bending moments, and hence no shears, of any kind are induced. In practical instances, the loads usually act at intermediate locations on the members. However, such loads can be replaced by statically equivalent loads at the joints. It can be shown that the critical load obtained in this manner does differ not much from a more exact analysis in which the effect of bending moment produced by intermediate loads is included.

Consider now a structural system composed of five members designated by *a*, *b*, *c*, *d*, and *e* [Fig. 13-8(a)]. Four system coordinates are assigned to identify external nodal displacements $r_i$ ($i = 1, 2, 3, 4$) and the corresponding nodal forces $R_i$. Each member has four coordinates to define its end forces or displacements, as shown in Fig. 13-8(b). The element stiffnesses $[k^a]$, $[k^b]$, . . . , $[k^e]$ are first determined by applying Eq. 13-31 corresponding to nodal movement. In this particular case we note that no effect of $P$ is included in the stiffness of member *d* or *e*, since initial axial loads are not present. The overall



(a)

(b)

Figure 13-8

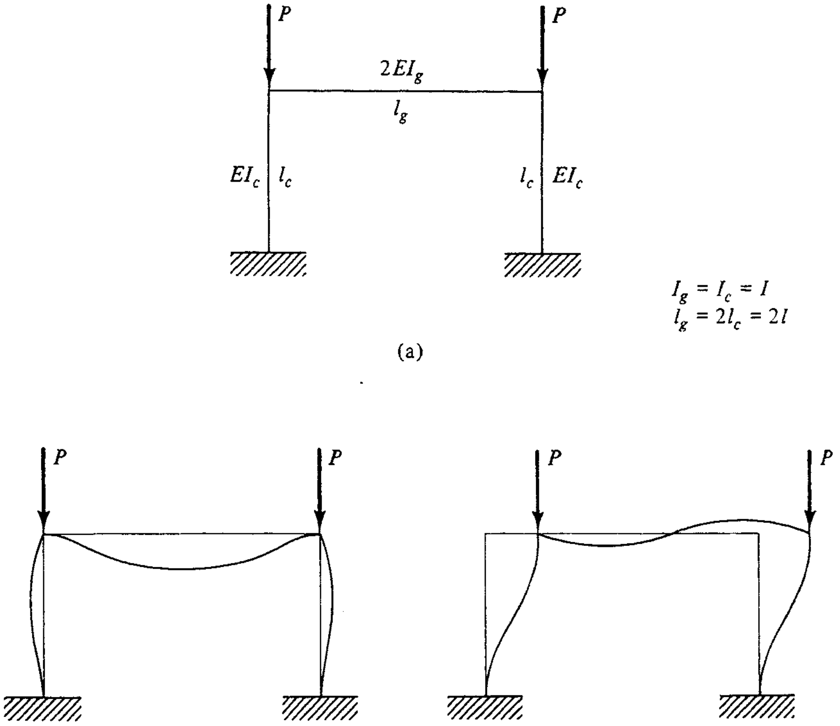
force-displacement relationship of the system is then established in the form

$$\{R\} = [K]\{r\}$$

in which the structure stiffness $[K]$ can be achieved by the direct stiffness method.

As axial loads increase continuously (we assume all loads increase in the same proportion), the structure may become unstable. The condition that at the critical stage the total stiffness $[K]$ must vanish, or $|K| = 0$, will give the critical buckling loads. A frame may have several critical loads and associated modes of buckling. The lowest of these loads is called the *first critical load,* and the associated mode is called the *fundamental mode.*

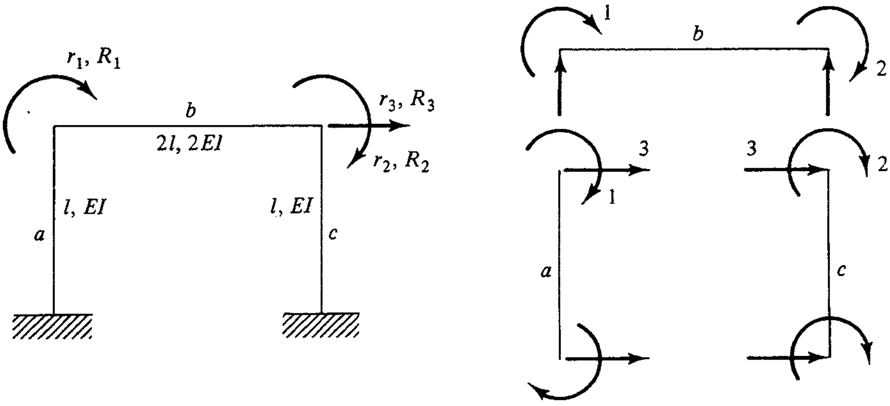
As a numerical example, let us discuss the stability of the simple portal frame shown in Fig. 13-9(a). This frame may undergo two possible modes of buckling: One is the symmetrical mode [Fig. 13-9(b)], and the other is the antisymmetrical mode [Fig. 13-9(c)]. The first type of buckling can occur only when the frame is braced against a joint translation. For the purpose of generality, we work with the frame corresponding to the sidesway mode of buckling. Structure (external) and member (internal) coordinates corresponding to possible nodal displacements are shown by numbers in Fig. 13-10(a) and (b), respectively.



(a)

(b) (c)

**Figure 13-9**



(a) (b)

**Figure 13-10**

In accordance with Eq. 13-31, we have the stiffness matrix for element $a$,

$$[k^a] = \begin{array}{c c c} 1 & 3 & \\ \left[\begin{array}{c} \dfrac{4EI}{l} - \dfrac{2Pl}{15} \\ -\dfrac{6EI}{l^2} + \dfrac{P}{10} \end{array}\right. & \left.\begin{array}{c} -\dfrac{6EI}{l^2} + \dfrac{P}{10} \\ \dfrac{12EI}{l^3} - \dfrac{6P}{5l} \end{array}\right] & \begin{array}{c} 1 \\ 3 \end{array} \end{array} \tag{13-55}$$

Similarly, for element $c$,

$$[k^c] = \begin{array}{c c c} 2 & 3 & \\ \left[\begin{array}{c} \dfrac{4EI}{l} - \dfrac{2Pl}{15} \\ -\dfrac{6EI}{l^2} + \dfrac{P}{10} \end{array}\right. & \left.\begin{array}{c} -\dfrac{6EI}{l^2} + \dfrac{P}{10} \\ \dfrac{12EI}{l^3} - \dfrac{6P}{5l} \end{array}\right] & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \tag{13-56}$$

The horizontal girder (element $b$) is not subjected to initial axial load. Thus,

$$[k^b] = \begin{array}{c c c} 1 & 2 & \\ \left[\begin{array}{c} \dfrac{4EI}{l} \\ \dfrac{2EI}{l} \end{array}\right. & \left.\begin{array}{c} \dfrac{2EI}{l} \\ \dfrac{4EI}{l} \end{array}\right] & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \tag{13-57}$$

Collecting the element stiffnesses of Eqs. 13-55 through 13-57 by the direct stiffness approach, we establish the structure stiffness matrix $[K]$ to relate nodal forces $\{R\}$ and the corresponding displacements $\{r\}$.

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \underset{K}{\begin{bmatrix} \dfrac{8EI}{l} - \dfrac{2Pl}{15} & \dfrac{2EI}{l} & -\dfrac{6EI}{l^2} + \dfrac{P}{10} \\ \dfrac{2EI}{l} & \dfrac{8EI}{l} - \dfrac{2Pl}{15} & -\dfrac{6EI}{l^2} + \dfrac{P}{10} \\ -\dfrac{6EI}{l^2} + \dfrac{P}{10} & -\dfrac{6EI}{l^2} + \dfrac{P}{10} & \dfrac{24EI}{l^3} - \dfrac{12P}{5l} \end{bmatrix}} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} \tag{13-58}$$

At the critical load, $|K| = 0$. To simplify presentation, we may set

$$P = \frac{\lambda EI}{l^2}$$

and divide each term by $EI/l$. Thus, we have

$$|K| = \begin{vmatrix} 8 - \dfrac{2}{15}\lambda & 2 & \dfrac{1}{l}\left(-6 + \dfrac{\lambda}{10}\right) \\ 2 & 8 - \dfrac{2}{15}\lambda & \dfrac{1}{l}\left(-6 + \dfrac{\lambda}{10}\right) \\ \dfrac{1}{l}\left(-6 + \dfrac{\lambda}{10}\right) & \dfrac{1}{l}\left(-6 + \dfrac{\lambda}{10}\right) & \dfrac{1}{l^2}\left(24 - \dfrac{12}{5}\lambda\right) \end{vmatrix} = 0 \tag{13-59}$$

which gives the smallest root of λ equal to 7.5. Therefore,

$$P_{cr} = \frac{7.5EI}{l^2} \tag{13-60}$$

In the case of the symmetrical buckling mode, where the sidesway is prevented, we have $r_3 = 0$. The stiffness matrix is obtained by erasing the third row and column of $[K]$ in Eq. 13-58; that is,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{8EI}{l} - \frac{2Pl}{15} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{8EI}{l} - \frac{2Pl}{15} \end{bmatrix}}_{K} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} \tag{13-61}$$

Using $P = \lambda EI/l^2$ and dividing each term by $EI/l$, we have the condition $|K| = 0$ as

$$\begin{vmatrix} 8 - \frac{2}{15}\lambda & 2 \\ 2 & 8 - \frac{2}{15}\lambda \end{vmatrix} = 0 \tag{13-62}$$

which gives the smaller root of λ equal to 45. Thus the first approximation for the buckling load is

$$P_{cr} = \frac{45EI}{l^2} \tag{13-63}$$

The $P_{cr}$ obtained in Eq. 13-63 is obviously too high, for it is higher than $4\pi^2EI/l^2$, the critical load for a fixed-end column. To improve the result, we may divide each of the frame members into two elements and analyze the system accordingly. This is left as an exercise.

The buckling load for the frame is usually expressed in the form

$$P_{cr} = \frac{\pi^2EI_c}{(\alpha l_c)^2} \tag{13-64}$$

in which $\alpha l_c$ is called the *effective length* of the column and $\alpha$ is called the *effective-length factor*. Generally speaking, in the symmetrical buckling mode, the effective length of the column would be less than the actual length (i.e., $\alpha < 1$) because of the restraint provided by the adjacent girder, whereas in the antisymmetrical mode, the effective length of the column will exceed its actual length (i.e., $\alpha > 1$) because of the sidesway effect. The $\alpha$-value depends on a particular buckling configuration and can be determined from the ratio of the stiffness factor $I/l$ of column section to that of girder section. If we let this ratio be

$$G = \frac{I_c/l_c}{I_g/l_g} \tag{13-65}$$

we can plot $\alpha$-values against $G$ to give curves for the symmetrical and antisymmetrical modes (Fig. 13-11). Note that $G = 0$ and $G = \infty$, as indicated in each curve of Fig. 13-11, represent two extreme conditions of end restraint. The condition $G = 0$ implies that $I_g/l_g = \infty$, meaning that the top of column is infinitely rigid. This gives a fixed-fixed column with $\alpha = \frac{1}{2}$ in the symmetric mode and a fixed-slide column with $\alpha = 1$ in the antisymmetrical mode. On the other hand, the condition $G = \infty$ implies that $I_g/l_g = 0$, meaning that the top of column is completely flexible, that is, pin-ended. This gives a fixed-hinged column with $\alpha = 0.7$ in the symmetrical mode and a fixed-free column with $\alpha = 2$ in the antisymmetrical mode.

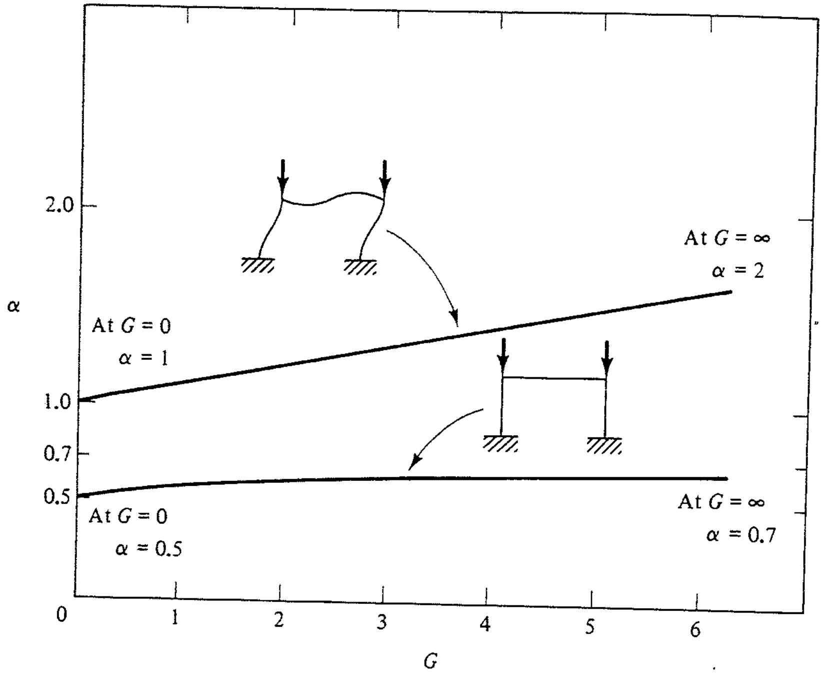


Figure 13-11

## PROBLEMS

13-1. Refer to Fig. 13-12. The system consists of two pin-connected rigid bars, each of length $l$, and a lateral spring support with stiffness $s$. The bars are subjected to a load $P$ acting vertically downward. Determine the critical $P$.

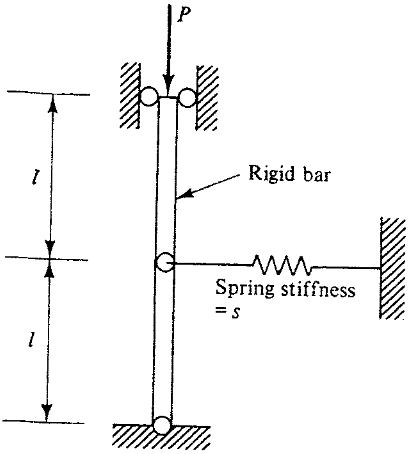


Figure 13-12

**13-2.** Discuss the stability of the system shown in Fig. 13-13, which is the same structure of Prob. 13-1 except that the top end is replaced by a lateral spring support.

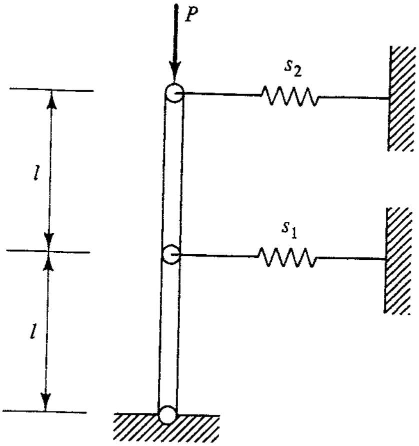


Figure 13-13

**13-3.** Determine, by the direct stiffness method, the critical load for the column shown in Fig. 13-14.

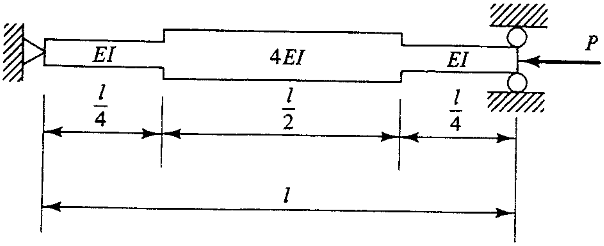


Figure 13-14

**13-4.** Repeat Prob. 13-3 if the end supports are fixed.

**13-5.** Determine, by the direct stiffness method, the critical load for the beam shown in Fig. 13-15.

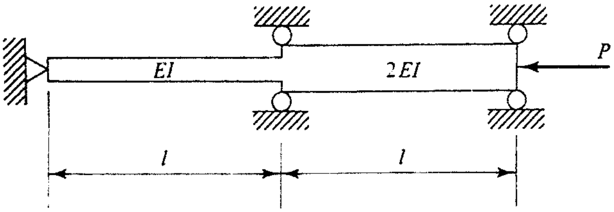


Figure 13-15

**13-6.** Figure 13-16 shows the same portal frame discussed in Sec. 13-4. Assuming a symmetrical mode of buckling and using two elements for each column, obtain, by the direct stiffness method, the critical load.

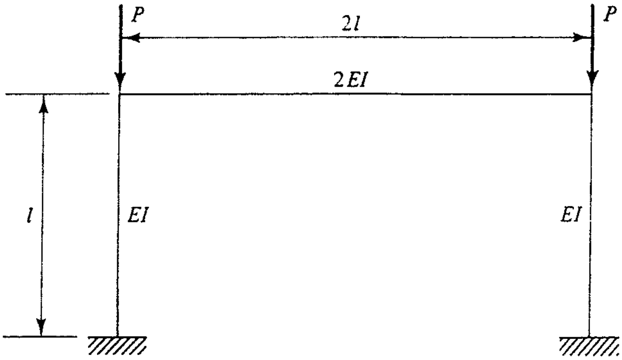


Figure 13-16

**13-7.** Find, by the direct stiffness method, the critical *P* for the rigid frame shown in Fig. 13-17.

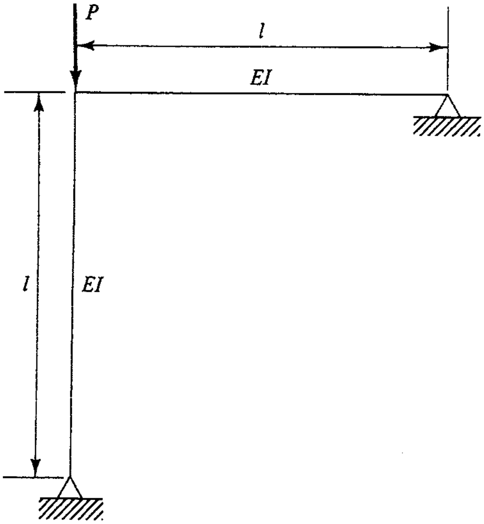


Figure 13-17

**13-8.** Discuss the stability for the portal frame with hinged supports, as shown in Fig. 13-18.

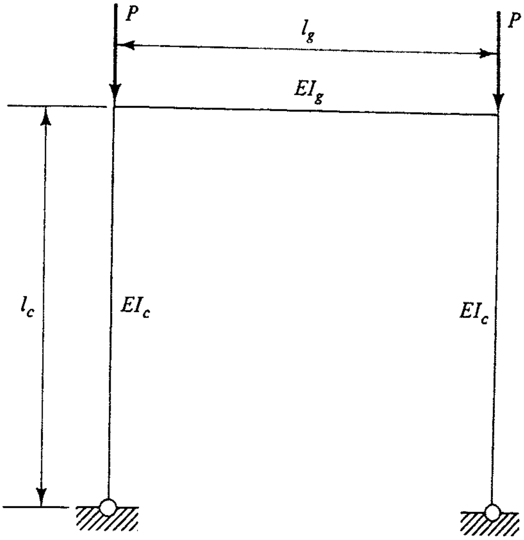


Figure 13-18

# 14

# Structural Dynamics

## 14-1 GENERAL

Throughout the previous chapters we have concerned ourselves with structures subjected only to statical loads producing displacements, which are independent of time. We shall now deal with structural analysis under dynamic loads. In one sense, the term *dynamic* may be defined as time-varying; thus a dynamic load is the load of which the magnitude, direction, or position varies with time. Likewise, the structural response (for example, displacements, internal forces, stresses, strains) to a dynamic load is also time-varying. A further distinction between static and dynamic analysis is that in a dynamic analysis it is necessary to take into account the forces produced by the inertia of the accelerating masses, whereas in static analysis the inertia forces are generally neglected. However, by the well-known d'Alembert's principle, the dynamic problem can be reduced to a static one provided forces equal to the product of masses and accelerations (with a negative sign) are introduced.

For evaluating structural response to dynamic loads, we have two basically different approaches: the deterministic and the nondeterministic. If the time-variation of loading is fully defined, the analysis of structural response is said to be deterministic. On the other hand, if the time-variation of loading is random and can only be defined in a statistical sense, then the analysis is nondeterministic. In this chapter we consider only the deterministic dynamic analysis.

Any structure possessing mass and elasticity is capable of vibration when disturbed from its equilibrium configuration. In general, the oscillatory motions of the disturbed structure are periodic, that is, they repeat themselves in equal intervals of time, called *periods*. The number of complete cycles of motion in a unit of time is referred to as the *frequency* of vibration. There are free and forced vibrations. *Free vibration* takes place when an elastic system vibrates under the action inherent in the system itself without being impressed by external forces. The system under free vibration vibrates at one or more of its *natural frequencies*, which are properties of the elastic system. *Forced vibration* takes place under the excitation of external force and at the frequency of the exciting force, which is independent of the natural frequencies of the system. When the frequency of the exciting force coincides with one of the natural frequencies of the system, *resonance* occurs and dangerously large amplitudes may result. Therefore, the calculation of natural frequency is of practical importance. In reality, there always exist forces against the vibration, such as friction, air resistance, or imperfect elasticity. Since no active forces are supplied in free vibration, these resisting forces cause the amplitude to diminish gradually until the motion ceases. This type of motion is called *damped free vibration*. On the other hand, forced vibration may be maintained at constant amplitude by externally supplied forces.

The *degrees of freedom* refer to the number of independent coordinates necessary to describe the motion of a system. Since an infinite number of coordinates are needed to describe the motion of the distributed mass of an elastic structure (e.g., a beam), the structure therefore has an infinite number of degrees of freedom. However, in many cases, we can use simple procedures to limit the infinite number of degrees of freedom to finite degrees of freedom, so that it is practical to carry out the analysis.

An elastic system with more than one degree of freedom usually vibrates without definite pattern. Only under certain specified conditions will some simple and orderly motions called *natural modes* (principal modes) of vibration take place. In natural modes of vibration, each point in the system follows a definite pattern of equal frequency (natural frequency). Moreover, the more general type of motion can always be represented by the superposition of natural modes of vibration.

In general, the structural response to dynamic loading is expressed basically in terms of the displacements of structure, other responses—such as internal forces, stresses, and strains—being treated as a secondary phase of the analysis, which can be derived from displacement patterns.

## 14-2 LUMPED MASSES

In the dynamic system, the analysis is complicated by taking into consideration the inertia forces, which are related to the time-displacement history and mass pattern of the structure. For a structural system having distributed mass and distributed elastic properties, an infinite number of coordinates are necessary to define the motion. In practice, we select only a finite number of coordinates to analyze a distributed-mass system, using either lumped masses or consistent masses. In this chapter we limit our discussion to the lumped mass system.

Consider the beam of Fig. 14-1(a). Assume that the beam is divided into segments and the mass of each segment is lumped (concentrated) into rigid bodies at discrete nodes, as shown in Fig. 14-1(b), the distribution of the segment mass to these nodes being determined by simple statics. These lumped masses are connected by elastic massless springs (treating the member segments as springs), which possess the original force-displacement properties of the member. Such an idealized representation of a structure is called a *model.* The number of coordinates necessary to describe the motion of each mass depends upon the type of motion. In a three-dimensional space, six coordinates are required to represent the degrees of freedom of the motion for each mass. However, in linear or plane motion, the number of coordinates can be considerably reduced.

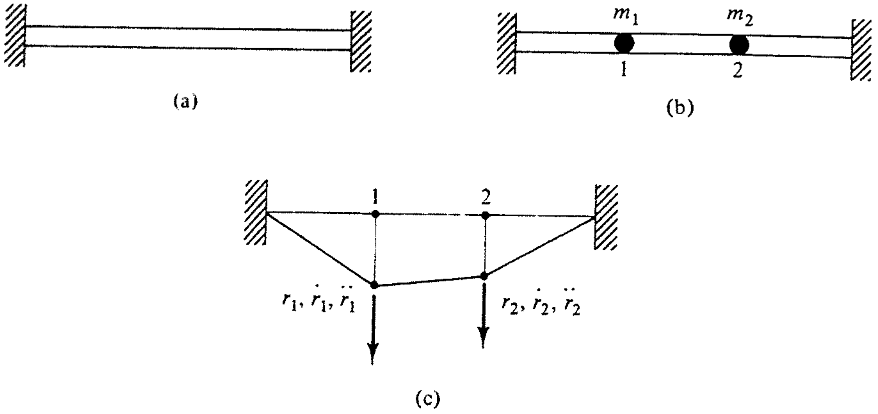


**Figure 14-1**

For a system in which only one translational degree of freedom is specified at any node, the lumped-mass matrix has a diagonal form. To illustrate, refer to Fig. 14-1(c). The state of motion of two masses $m_1$ and $m_2$ is described by the displacements $r_1$ and $r_2$, the velocities $\dot{r}_1$ and $\dot{r}_2$, and the acceleration $\ddot{r}_1$ and $\ddot{r}_2$, as shown. Writing Newton's second law for the independent force equations, we have

$$\begin{aligned} R_1 &= m_1\ddot{r}_1 \\ R_2 &= m_2\ddot{r}_2 \end{aligned} \tag{14-1}$$

In matrix form

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{Bmatrix} \tag{14-2}$$

For $n$ masses, each associated with one independent translational coordinate, we can write

$$\begin{Bmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_n \end{Bmatrix} = \begin{bmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & m_n \end{bmatrix} \begin{Bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{r}_n \end{Bmatrix}$$

or simply

$$\{R\} = [m]\{\ddot{r}\} \tag{14-3}$$

in which $[m]$ is a lumped mass matrix containing the lumped masses $m_1$, $m_2, \ldots, m_n$ along its diagonal.

## 14-3 FORMULATION OF THE EQUATION OF MOTION

The basic elements involved in a linearly elastic structural dynamics system include the mass, the energy-loss mechanism (damping), the elastic properties (flexibility or stiffness), and the external source of excitation (loading). In its simplest model is a single-degree-of-freedom vibrating system shown in Fig. 14-2(a), in which the entire mass is concentrated in a rigid block under a time-varying load $F(t)$. Movement of the block is constrained by the roller supports, so that only horizontal translation ($x$ direction) is permitted. The position of the block at any time is described by $x$. The elastic resistance to displacement is provided by the weightless spring of stiffness $s$; and the energy-loss mechanism is represented by the damper with damping coefficient $c$ (a viscous damping is assumed). By d'Alembert's principle, equilibrium is maintained [see Fig. 14-2(b)] between the external-loading mechanism $F(t)$ and the resistance of the system, including the mass (the inertia force), the damping, and the spring. Thus

$$F_M + F_D + F_S = F(t) \tag{14-4}$$

or

$$m\ddot{x} + c\dot{x} + sx = F(t) \tag{14-5}$$

in which the viscous damping force $F_D$ is taken equal to the product of damping coefficient $c$ and velocity $\dot{x}$.

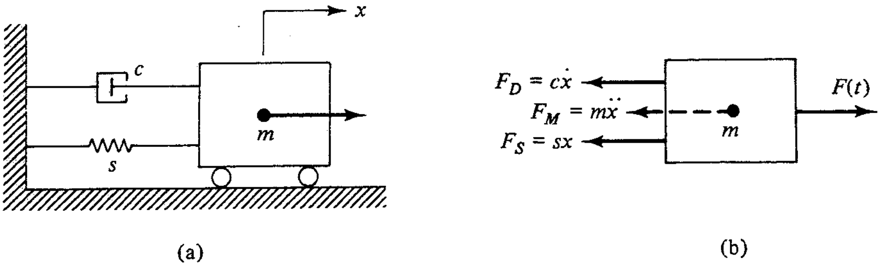


Figure 14-2

One class of motions is called *free vibrations*, which take place by setting the applied force $F(t)$ equal to zero in Eq. 14-5. Thus,

$$m\ddot{x} + c\dot{x} + sx = 0 \qquad (14\text{-}6)$$

The situation can further be simplified by removing the damping force (i.e., $c = 0$); then the equation of motion reduces to

$$m\ddot{x} + sx = 0 \qquad (14\text{-}7)$$

which represents *undamped free vibrations*.

In general, the equation of motion of any single-degree-of-freedom complex system can be presented in the form of Eq. 14-5 with some modifications, and the dynamic response of linearly elastic multi-degree-of-freedom systems is practically evaluated by utilizing the basic ideas developed for the single-degree-of-freedom system, as is seen in various sections to follow.

## 14-4 UNDAMPED FREE VIBRATION OF LUMPED SINGLE-DEGREE-OF-FREEDOM SYSTEMS

To introduce elastic vibrations, let us first consider the undamped free vibration for a lumped single-degree-of-freedom system, shown in Fig. 14-3. Let us give the mass an initial displacement and suddenly release it at time $t = 0$. We can describe the motion of the body by Eq. 14-7, namely,

$$m\ddot{x} + sx = 0$$ — no damping (c

For convenience, we write the equation as

$$\ddot{x} + \omega^2 x = 0 \qquad (14\text{-}8)$$

with

$$\omega = \sqrt{\frac{s}{m}} \qquad (14\text{-}9)$$

The general solution of Eq. (14-8) is

$$x(t) = A_1 \sin \omega t + A_2 \cos \omega t \qquad (14\text{-}10)$$

Equation 14-10 gives the displacement of the mass at any time $t$. Now, since

$$A_1 \sin \omega t + A_2 \cos \omega t = A_1 \sin (\omega t + 2\pi) + A_2 \cos (\omega t + 2\pi)$$

$$= A_1 \sin \omega\left(t + \frac{2\pi}{\omega}\right) + A_2 \cos \omega\left(t + \frac{2\pi}{\omega}\right)$$

= [illegible]

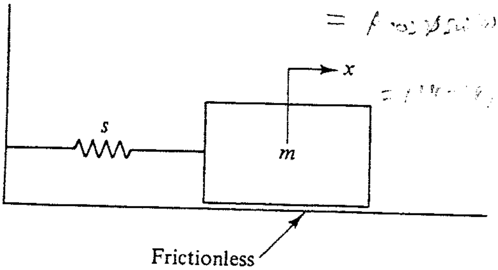


Figure 14-3

the displacement $x$ of Eq. 14-10 has the same value at time $t$ as at time $t + 2\pi/\omega$. We say that the equation is periodic. The *period*, commonly given in seconds, is denoted by $T$. Thus

$$T = \frac{2\pi}{\omega} \tag{14-11}$$

which is referred to as the *natural period of vibration*. The reciprocal of $T$ is the *natural frequency*, usually denoted by $f$. Thus

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{14-12}$$

The unit of $f$ is cycles per second. The quantity $\omega$ is called the *natural angular frequency* of the motion.

The two constants $A_1$ and $A_2$ in the general solution of Eq. 14-10 can be determined from given initial conditions of motion. For example, the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ will give $A_2 = x_0$ and $A_1 = \dot{x}_0/\omega$. The general solution of Eq. 14-10 can be written in an alternative form if we use the identity

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$$

→ trigo. identity

and introduce two new constants $A$ and $\psi$ such that

$$\begin{aligned} A_1 &= A\cos\psi \\ A_2 &= A\sin\psi \end{aligned} \tag{14-13}$$

The result of substituting Eq. 14-13 into Eq. 14-10 is

$$x(t) = A\sin(\omega t + \psi) \tag{14-14}$$

amplitude phase angle

where the constants

$$\begin{aligned} A &= \sqrt{A_1^2 + A_2^2} \\ \psi &= \tan^{-1}\frac{A_2}{A_1} \end{aligned} \tag{14-15}$$

are evaluated with the initial conditions. The constant $\psi$ in Eq. 14-14 is referred to as the *phase angle*, and $A$ is called the *amplitude* of the motion.

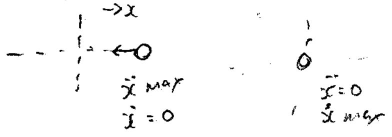
A graph of $x(t)$ against $t$ for Eq. 14-14 or Eq. 14-10 is shown in Fig. 14-4. Such a motion may be represented by the projection of a rotating vector **A** on a vertical diameter as it moves around a circle with constant angular velocity $\omega$, as shown. It is readily seen that the motion is harmonic. Note that all harmonic motion is periodic, but not all periodic motion is harmonic.

The velocity and acceleration are obtained by time derivatives of Eq. 14-14. Thus

$$\dot{x}(t) = \omega A\cos(\omega t + \psi) = \omega A\sin\left[(\omega t + \psi) + \frac{\pi}{2}\right] \tag{14-16}$$

$$\ddot{x}(t) = -\omega^2 A\sin(\omega t + \psi) = \omega^2 A\sin[(\omega t + \psi) + \pi] \tag{14-17}$$

These equations indicate that the velocity and acceleration are also harmonic



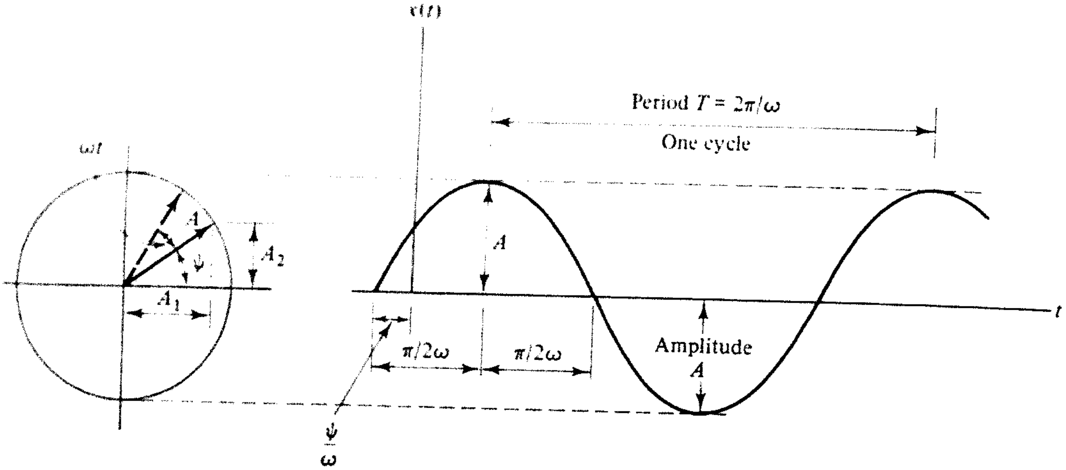


**Figure 14-4**

and can be represented by the vectors $\omega\mathbf{A}$ and $\omega^2\mathbf{A}$, respectively, rotating with the same speed as the displacement vector but in the respective positions of 90° and 180° anead of it.

Let us consider next the lumped-mass model for a one-story frame performing a single-degree-of-freedom free vibration. In the idealization [Fig. 14-5(a)], we assume that the beam is completely rigid and supported by elastic massless columns. The masses are concentrated on the beam level. Axial deformation of members is neglected. With these assumptions, the mass can have only horizontal motion, which is resisted by the lateral stiffnesses of columns. The behavior of the frame is therefore analogous to the spring-supported mass shown in Fig. 14-5(b). The resistance to the horizontal displacement can be represented by an equivalent spring with a stiffness $s$ equal to the sum of lateral stiffness

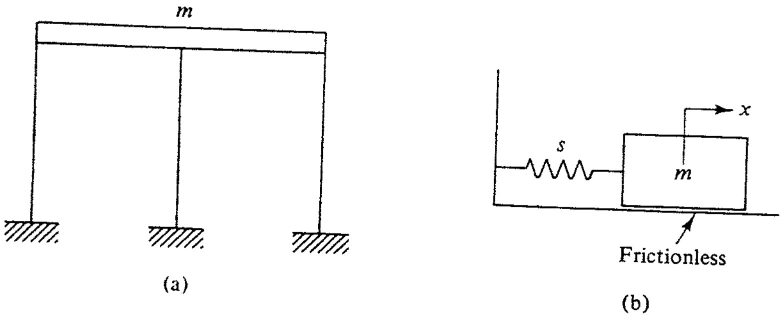


**Figure 14-5**

(the shear force corresponding to a unit lateral displacement) at top of the columns. In the present case

$$s = \sum \frac{12EI}{L^3} \tag{14-18}$$

for a fixed base of columns, assuming that the effect of girder flexibility is negligible.

**Example 14-1**

Consider the frame shown in Fig. 14-6(a). Assume the beam weight plus loading is $8W$ and the weight of two columns are $0.8W$ and $1.2W$, respectively. We wish to determine the approximate value of the natural frequency and natural period of vibration.

The total mass $M$ lumped is $9W/g$ ($g$ is gravitational acceleration) if we include one-half of the column masses at the beam level [see Fig. 14-6(b)]. The equivalent spring stiffness is

$$s = \frac{12EI}{L^3} + \frac{12(2EI)}{L^3} = \frac{36EI}{L^3}$$

It follows from Eq. 14-9 that the natural angular frequency of vibration is given by

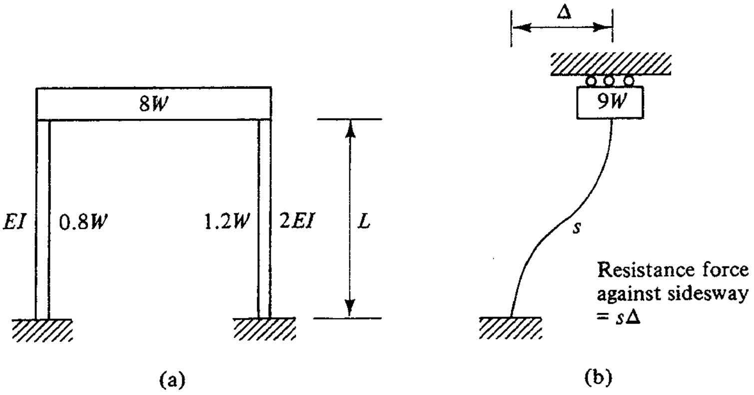
$$\omega = \sqrt{\frac{s}{m}} = \sqrt{\frac{36EI/L^3}{9W/g}} = 2\sqrt{\frac{EIg}{WL^3}}$$

Thus the natural frequency of vibration is

$$f = \frac{\omega}{2\pi} = \frac{1}{\pi}\sqrt{\frac{EIg}{WL^3}}$$

and the natural period of vibration is

$$T = \frac{1}{f} = \pi\sqrt{\frac{WL^3}{EIg}}$$

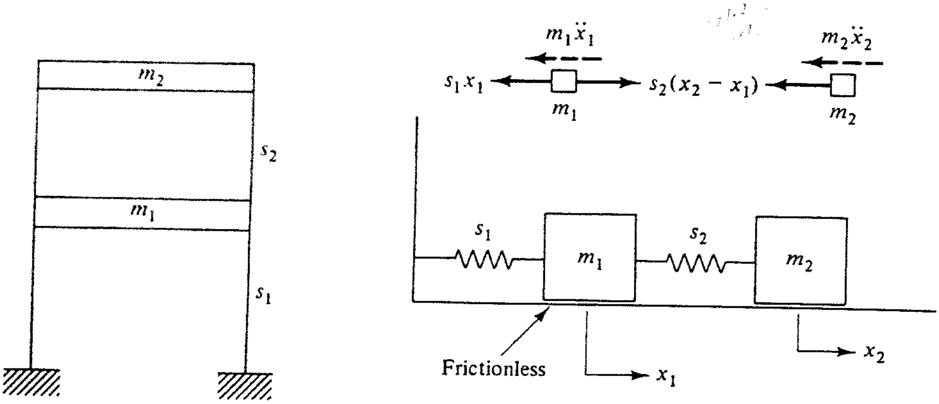


(a) (b)

Figure 14-6

## 14-5 UNDAMPED FREE VIBRATION OF LUMPED MULTI-DEGREE-OF-FREEDOM SYSTEMS

The undamped free vibration of structural systems with more than one degree of freedom can be investigated by a procedure similar to the lumped mass model developed in the last section. As an illustration, let us refer to Fig. 14-7(a), which represents the lumped mass system for a two-story shear building with the masses concentrated at the floor levels. We assume that the beam stiffnesses are infinite in comparison with the column stiffnesses. Axial member deformations are neglected. It follows from these assumptions that the system has two degrees of freedom, represented by the independent lateral translations $x_1$ and $x_2$ at two floor levels, where masses $m_1$ and $m_2$ are lumped. The frame thus behaves like the spring-supported system of two masses shown in Fig. 14-7(b). The stiffnesses $s_1$ and $s_2$ of equivalent springs for the frame are evaluated, respectively, by the combined lateral stiffness of columns in each story, as illustrated in Example 14-1.



(a) (b)

Figure 14-7

For the system shown in Fig. 14-7, the equations of motion are

$$\begin{aligned} m_1\ddot{x}_1 + (s_1 + s_2)x_1 - s_2x_2 &= 0 \\ m_2\ddot{x}_2 - s_2x_1 + s_2x_2 &= 0 \end{aligned} \tag{14-19}$$

In matrix form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} s_1 + s_2 & -s_2 \\ -s_2 & s_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14-20}$$

with $s_{11} = s_1 + s_2$, $s_{12} = s_{21} = -s_2$, and $s_{22} = s_2$.

The above reasoning can be extended to an $n$-degree-of-freedom system with $n$ lumped masses. The general equation of motion can be expressed as

$$[m]\{\ddot{x}\} + [s]\{x\} = \{0\} \tag{14-21}$$

where $[m]$ is a diagonal lumped mass matrix,

$$[m] = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{bmatrix} \tag{14-22}$$

and $[s]$ is a stiffness matrix,

$$[s] = \begin{bmatrix} s_{11} & s_{12} & \cdot & \cdot & \cdot & s_{1n} \\ s_{21} & s_{22} & \cdot & \cdot & \cdot & s_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ s_{n1} & s_{n2} & \cdot & \cdot & \cdot & s_{nn} \end{bmatrix} \tag{14-23}$$

Premultiplying by $[m]^{-1}$ reduces Eq. 14-21 to

$$\{\ddot{x}\} + [m]^{-1}[s]\{x\} = \{0\} \tag{14-24}$$

or

$$\{\ddot{x}\} + [D]\{x\} = \{0\} \tag{14-25}$$

with

$$[D] = [m]^{-1}[s] \tag{14-26}$$

referred to as the *dynamic matrix*. The solution of Eq. 14-25 is a set of harmonic motion expressed in the form

$$x_i = A_i \sin(\omega t + \psi) \qquad (i = 1, 2, \ldots, n) \tag{14-27}$$

or, in matrix form,

$$\{x\} = \sin(\omega t + \psi)\{A\} \tag{14-28}$$

in which the column matrix $\{A\}$ represents the amplitudes of the deflected shape at the $n$ coordinates. Noting that from 14-17

$$\{\ddot{x}\} = -\omega^2 \sin(\omega t + \psi)\{A\} = -\omega^2\{x\} \tag{14-29}$$

we find after substitution in Eq. 14-25 that

$$-\omega^2\{x\} + [D]\{x\} = \{0\} \tag{14-30}$$

[illegible]

Substituting Eq. 14-28 into Eq. 14-30 yields

$$([D] - \omega^2[\mathrm{I}])\{A\} = \{0\} \tag{14-31}$$

Equation 14-31 represents an eigenvalue (characteristic value) problem. For a nontrivial solution of $\{A\}$ to exist, we must have

$$|[D] - \omega^2[\mathrm{I}]| = 0 \tag{14-32}$$

determinant = 0

Upon expending this determinant, we obtain an $n$th-degree polynomial in $\omega^2$. The solution of the polynomial results in $n$ eigenvalues of $\omega^2$. The smallest eigenvalue corresponds to the smallest natural angular frequency, which is called the *first mode frequency*.

For each eigenvalue of $\omega^2$, a set of Eq. 14-31 can be established. However, unique values for amplitudes $\{A\}$ or displacements $\{x\}$ cannot be determined; only the relative amplitudes or the displacement shapes can be found. In other words, for each eigenvalue of $\omega^2$, we obtain only a mode configuration called a *natural mode*, which is defined by a *modal column* (eigenvector or characteristic vector) $\{A\}$. If one of the amplitudes in natural mode is set to unity, the rest of the values $\{A\}$ can be determined relative to this reference value. This process of normalization gives the *normal mode characteristic shape*.

**Example 14-2**

Consider the lumped mass model for a two-story shear building shown in Fig. 14-8. Assuming identical columns and the same amount of mass concentrated at each floor level (including the contribution of column mass), we wish to find the natural angular frequencies and the corresponding normal mode characteristic shapes.

The mass matrix is

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

The stiffness matrix is determined by Eq. 14-20:

$$[s] = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} s_1 + s_2 & -s_2 \\ -s_2 & s_2 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 48 & -24 \\ -24 & 24 \end{bmatrix}$$

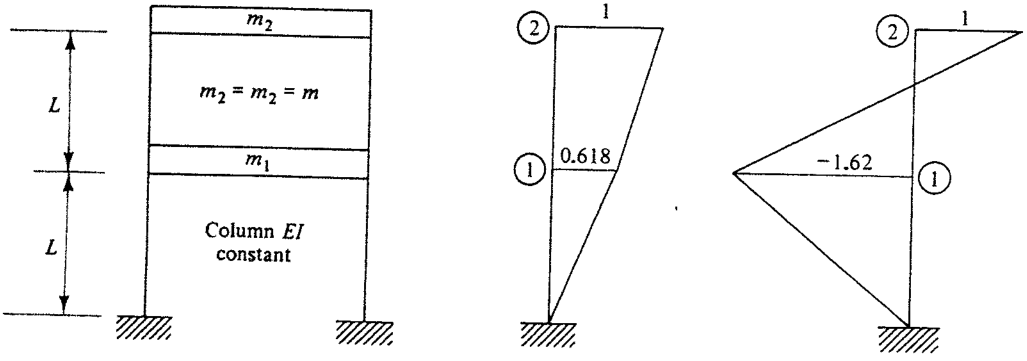
Consequently, we have the dynamic matrix

$$[D] = [m]^{-1}[s] = \frac{EI}{mL^3}\begin{bmatrix} 48 & -24 \\ -24 & 24 \end{bmatrix}$$

The expression of Eq. 14-32 for obtaining the natural angular frequency is, therefore,

$$\left| \frac{EI}{mL^3}\begin{bmatrix} 48 & -24 \\ -24 & 24 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

For simplicity of presentation, we let $EI/mL^3 = 10\,\mathrm{s}^{-2}$. The last expression becomes



(a) (b) (c)

Figure 14-8

$$\begin{vmatrix} 480 - \omega^2 & -240 \\ -240 & 240 - \omega^2 \end{vmatrix} = 0$$

Using $\lambda = \omega^2$ and expanding this determinant leads to a quadratic equation,

$$\lambda^2 - 720\,\lambda + 57{,}600 = 0$$

which is found to have roots $\lambda_1 = \omega_1^2 = 91.7$ and $\lambda_2 = \omega_2^2 = 628.3$, or

$$\omega_1 = 9.6 \text{ rad/s} \qquad \omega_2 = 25.1 \text{ rad/s}$$

The governing expression for determining the ratio between amplitudes $A_1$ and $A_2$ is given by Eq. 14-31. Thus,

$$\frac{EI}{mL^3}\begin{bmatrix} 48 & -24 \\ -24 & 24 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} - \omega^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assuming $EI/mL^3 = 10$ as before, we have

$$\begin{bmatrix} 480 & -240 \\ -240 & 240 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} - \omega^2\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Substituting $\omega_1^2 = 91.7$, we find the ratio $A_2/A_1$ is $1/0.618$. The modal column (eigenvector) of the first normal mode is, therefore,

$$\{A\}_1 = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix}$$

Similarly, with $\omega_2^2 = 628.3$, we obtain $A_2/A_1 = -1/1.62$ and the modal column of the second normal mode,

$$\{A\}_2 = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}_2 = \begin{Bmatrix} -1.62 \\ 1 \end{Bmatrix}$$

The two normal mode characteristic shapes based on $A_2 = 1$ are shown in Fig. 14-8(b) and (c).

---

There exists an important property called *orthogonality* between two modal columns (natural modes or normal modes) with respect to mass and stiffness matrices. These relationships are expressed by

$$\{A\}_1^T[m]\{A\}_2 = 0 \tag{14-33}$$

and

$$\{A\}_1^T[s]\{A\}_2 = 0 \tag{14-34}$$

Equations 14-33 and 14-34 are satisfied by a direct inserting of the matrices

$$\{A\}_1 = \{0.618 \quad 1\}$$

$$\{A\}_2 = \{-1.62 \quad 1\}$$

$$[m] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

and

$$[s] = \frac{EI}{L^3}\begin{bmatrix} 48 & -24 \\ -24 & 24 \end{bmatrix}$$

into the two expressions.

In the general case of an $n$-degree-of-freedom system, there are $n$ eigenvalues $\omega_i^2$ and $n$ corresponding eigenvectors $\{A\}_i$ ($i = 1, 2, \ldots, n$). The orthogonality properties between two distinct modes $j$ and $k$ are given as

$$\{A\}_j^T[m]\{A\}_k = 0 \tag{14-35}$$

$$\{A\}_j^T[s]\{A\}_k = 0 \tag{14-36}$$

which we prove as follows.

Consider two eigenvectors $\{A\}_j$ and $\{A\}_k$ corresponding, respectively, to eigenvalues $\omega_j^2$ and $\omega_k^2$. Since each eigenvector and its eigenvalue satisfy the equation of motion of Eq. 14-31, that is,

$$([D] - \omega^2[\mathrm{I}])\{A\} = \{0\} \qquad ([D] = [m]^{-1}[s])$$

or

$$[s]\{A\} = \omega^2[m]\{A\} \tag{14-37}$$

we have

$$[s]\{A\}_j = \omega_j^2[m]\{A\}_j \tag{14-38}$$

and

$$[s]\{A\}_k = \omega_k^2[m]\{A\}_k \tag{14-39}$$

Premultiplying Eq. 14-38 by $\{A\}_k^T$ gives

$$\{A\}_k^T[s]\{A\}_j = \omega_j^2\{A\}_k^T[m]\{A\}_j \tag{14-40}$$

Taking the transpose of Eq. 14-40 and noting that $[m]$ and $[s]$ are both symmetric matrices, we obtain

$$\{A\}_j^T[s]\{A\}_k = \omega_j^2\{A\}_j^T[m]\{A\}_k \tag{14-41}$$

Premultiplying Eq. 14-39 by $\{A\}_j^T$ gives

$$\{A\}_j^T[s]\{A\}_k = \omega_k^2\{A\}_j^T[m]\{A\}_k \tag{14-42}$$

Subtracting Eq. 14-41 from Eq. 14-42, we obtain

$$(\omega_k^2 - \omega_j^2)\{A\}_j^T[m]\{A\}_k = 0 \tag{14-43}$$

In case the frequencies are distinct, that is, $\omega_j \neq \omega_k$, the product of the matrices in Eq. 14-43 vanishes. Thus

$$\{A\}_j^T[m]\{A\}_k = 0 \qquad \text{for } j \neq k \tag{14-44}$$

In view of Eq. 14-41 or Eq. 14-42,

$$\{A\}_j^T[s]\{A\}_k = 0 \qquad \text{for } j \neq k \tag{14-45}$$

This completes our statement that the natural modes, or normal modes, are orthogonal with respect to the mass and stiffness matrices.

## 14-6 DAMPED FREE VIBRATION

Damping is associated with energy dissipation. Viscous damping is encountered by bodies moving through a fluid or structures subjected to motion relative to a base in an earthquake. For the free vibration of a spring and lumped mass system with viscous damping, as shown in Fig. 14-2, the equation of motion is given in 14-6; that is,

$$m\ddot{x} + c\dot{x} + sx = 0 \tag{14-46}$$

The equation has a solution of the form $x = e^{\alpha t}$, which, on substitution, yields

$$(m\alpha^2 + c\alpha + s)e^{\alpha t} = 0$$

or

$$\alpha^2 + \frac{c}{m}\alpha + \frac{s}{m} = 0 \tag{14-47}$$

Hence

$$\alpha_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}} \tag{14-48}$$

The general solution may be written as

$$x = B_1e^{\alpha_1 t} + B_2e^{\alpha_2 t} \tag{14-49}$$

where $B_1$ and $B_2$ are constants depending on the initial conditions of motion.

The behavior of the damped system depends upon whether the radical of Eq. 14-48 is real, zero, or imaginary. If the radical is zero, that is,

$$\left(\frac{c}{2m}\right)^2 = \frac{s}{m} \tag{14-50}$$

we call the system critically damped, and we say the corresponding damping coefficient is the *critical damping coefficient,* denoted by $c_{cr}$. Thus, from Eq. 14-50 we have

$$c_{cr} = 2\sqrt{sm} \tag{14-51}$$

It is convenient to introduce the nondimensional ratio

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{sm}} \tag{14-52}$$

referred to as the *damping ratio*. Then we can write the equation of motion, Eq. 14-46, in terms of the damping ratio and natural frequency as

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2x = 0 \tag{14-53}$$

in which $\omega = \sqrt{s/m}$, as previously defined. The roots of Eq. 14-48 become

$$\alpha_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega \tag{14-54}$$

There are three forms of damping, depending upon whether $\zeta$ is greater than, less than, or equal to 1.

*Case* 1 $\zeta > 1$ (Overdamping). The radical $\sqrt{\zeta^2 - 1}$ is real and less than $\zeta$, and hence $\alpha_1$ and $\alpha_2$ are negative. The general solution is

$$x = B_1e^{(-\zeta+\sqrt{\zeta^2-1})\omega t} + B_2e^{(-\zeta-\sqrt{\zeta^2-1})\omega t} \tag{14-55}$$

Equation 14-55 represents an exponentially decreasing function. As $t$ increases, the displacement tends to be zero. The motion is not periodic, and the damping is large enough to prevent oscillation.

*Case* 2 $\zeta < 1$ (Underdamping). In this case the radical $\sqrt{\zeta^2 - 1}$ is imaginary, and the roots of Eq. 14-54 can be written as

$$\alpha_{1,2} = (-\zeta \pm i\sqrt{1 - \zeta^2})\omega \tag{14-56}$$

The general solution becomes

$$x = e^{-\zeta\omega t}(B_1 \sin\sqrt{1 - \zeta^2}\omega t + B_2 \cos\sqrt{1 - \zeta^2}\omega t) \tag{14-57}$$

or, equivalently,

$$x = e^{-\zeta\omega t}B \sin(\sqrt{1 - \zeta^2}\omega t + \psi) \tag{14-58}$$

with

$$B = \sqrt{B_1^2 + B_2^2}$$

$$\psi = \tan^{-1}\frac{B_2}{B_1} \tag{14-59}$$

Either Eq. 14-57 or Eq. 14-58 represents a case of oscillatory motion with decaying amplitude. The rate of decay of the amplitude of oscillation is determined by the damping factor $e^{-\zeta\omega t}$. The natural frequency of oscillation is

$$f = \frac{\sqrt{1 - \zeta^2}}{2\pi}\omega \tag{14-60}$$

When $\zeta = 0$ (i.e., $c = 0$), the expression for $f$ reduces to that of undamped motion. The damping decreases the natural frequency of oscillation by the factor $\sqrt{1 - \zeta^2}$.

*Case* 3 $\zeta = 1$ (Critical Damping). Since the radical in Eq. 14-54 is zero for critical damping, the roots $\alpha_1$ and $\alpha_2$ are identical and equal to $-\omega$. Equation 14-49 becomes

$$x = (B_1 + B_2)e^{-\omega t} = Be^{-\omega t} \tag{14-61}$$

Equation 14-61 contains only one constant $B$, which is insufficient to represent the general solution. In this case, we introduce a function of the form $te^{-\omega t}$ that satisfies the equation of motion and write the general solution as

$$x = (B + Ct)e^{-\omega t} \tag{14-62}$$

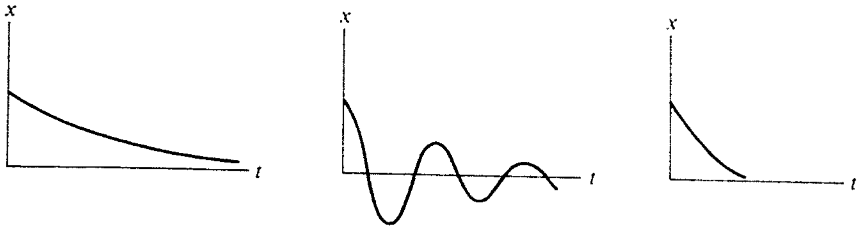
The system of critical damping represents the transition between the nonoscillatory and oscillatory motions. The motion is not periodic as in the case of overdamping, but it has the smallest damping possible for a periodic motion.

All three motions are illustrated schematically in Fig. 14-9.

For an $n$-degree-of-freedom system with lumped masses, the equation of motion becomes

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [s]\{x\} = \{0\} \tag{14-63}$$

where $[m]$ and $[s]$ are lumped mass and stiffness matrices previously defined by Eqs. 14-22 and 14-23, respectively, and $[c]$ is the *damping matrix*, given by



Overdamping Underdamping Critical damping

Figure 14-9

$$[c] = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \qquad (14\text{-}64)$$

which relates the coordinate damping forces to the corresponding velocities. A typical damping coefficient $c_{ij}$ is given by

$$c_{ij} = \text{force corresponding to coordinate } i \text{ caused by}$$
$$\text{a unit velocity of coordinate } j \text{ only}$$

In practice the damping matrix is conveniently assumed to be proportional to the mass matrix or stiffness matrix or both. Thus,

$$[c] = c_1[m] + c_2[s]$$

where $c_1$ and $c_2$ are constants of proportionality. This enables making use of coordinate transformation to uncouple the system so that the solution may proceed by using the single-degree-of-freedom lumped block. See Sec. 14-8 for normal coordinates.

Note that the evaluation of a damping property such as $c(x)$ is practically difficult. Therefore, the damping is usually expressed in terms of the damping ratio found from experiments on similar structures rather than by an explicit damping matrix.

## 14-7 FORCED VIBRATION: STEADY-STATE SOLUTION

If the spring-mass system considered in the preceding section is excited by a time-dependent force that varies harmonically, then we may write the equation of motion as

$$m\ddot{x} + c\dot{x} + sx = F_0 \sin \overline{\omega} t \qquad (14\text{-}65)$$

The solution of Eq. 14-65 is given by the combination of the *complementary* *solution* (transient solution) representing the free viscously damped vibration $(F_0 = 0)$ and a *particular solution* representing the forced oscillations. Note that the damped motion will soon disappear and only the steady-state forced oscillation remains.

Let us assume a particular solution of the form

$$x = A \sin (\bar{\omega} t - \psi) \tag{14-66}$$

where $A$ is the amplitude and $\psi$ is the phase angle by which the motion lags the impressed force. $A$ and $\psi$ are determined by substituting Eq. 14-66 and its time derivatives:

$$\dot{x} = \bar{\omega} A \cos (\bar{\omega} t - \psi) = \bar{\omega} A \sin \left( \bar{\omega} t - \psi + \frac{\pi}{2} \right)$$

$$\ddot{x} = -\bar{\omega}^2 A \sin (\bar{\omega} t - \psi)$$

into Eq. 14-65. Thus,

$$-m\bar{\omega}^2 A \sin (\bar{\omega} t - \psi) + c\bar{\omega} A \sin \left( \bar{\omega} t - \psi + \frac{\pi}{2} \right) + sA \sin (\bar{\omega} t - \psi) = F_0 \sin \bar{\omega} t$$

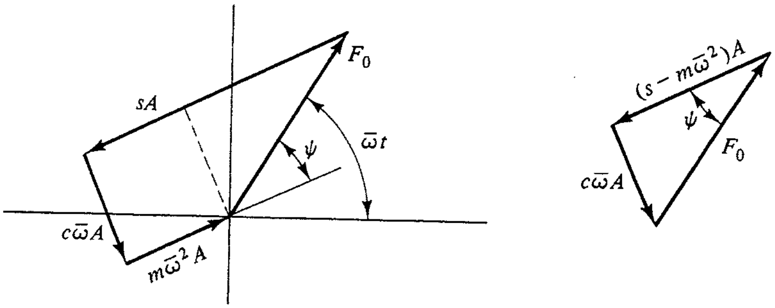
or

$$\begin{aligned} m\bar{\omega}^2 A \sin (\bar{\omega} t - \psi) - c\bar{\omega} A \sin \left( \bar{\omega} t - \psi + \frac{\pi}{2} \right) \\ - sA \sin (\bar{\omega} t - \psi) + F_0 \sin \bar{\omega} t = 0 \end{aligned} \tag{14-67}$$

Equation 14-67 represents the equilibrium between the inertia force, damping force, elastic force, and impressed force and can be shown graphically in Fig. 14-10, from which we determine

$$A = \frac{F_0}{\sqrt{(s - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}} \tag{14-68}$$

$$\psi = \tan^{-1} \frac{c\bar{\omega}}{s - m\bar{\omega}^2} \tag{14-69}$$



(a) (b)

Figure 14-10

By dividing the numerator and denominator of Eqs. 14-68 and 14-69 by $s$ and substituting the relations $\omega^2 = s/m$ and $\zeta = c/2\sqrt{sm}$, we obtain

$$A = \frac{F_0/s}{\sqrt{[1 - (\overline{\omega}/\omega)^2]^2 + [2\zeta(\overline{\omega}/\omega)]^2}} \tag{14-70}$$

$$\psi = \tan^{-1} \frac{2\zeta(\overline{\omega}/\omega)}{1 - (\overline{\omega}/\omega)^2} \tag{14-71}$$

The factor $F_0/s$ in Eq. 14-70 can be regarded as the displacement that a static force of magnitude $F_0$ would produce. Then the ratio

$$\frac{1}{\sqrt{[1 - (\overline{\omega}/\omega)^2]^2 + [2\zeta(\overline{\omega}/\omega)]^2}} \tag{14-72}$$

represents the *magnification factor* between the amplitude of oscillations produced dynamically by a force $F_0$ and the displacement produced by the same force considered as a static load. We observe that for small values of the damping factor $\zeta$, the amplitude can become very large when $\overline{\omega} = \omega$. This phenomenon is called *resonance*. A major application of the theory of vibrations is devoted to the avoidance of resonance in the design of structures that are subjected to dynamic forces.

If damping is neglected in Eq. 14-65, then the particular solution takes the form

$$x = A \sin \overline{\omega} t \tag{14-73}$$

with

$$A = \frac{F_0/s}{1 - (\overline{\omega}/\omega)^2} \tag{17-74}$$

## 14-8 NORMAL COORDINATES

The deflected shape of a linear structure can usually be expressed as the sum of a series of specified displacement patterns. For example, the deflection of a simple beam of length $L$ under transverse loading can be represented by the Fourier series of the form

$$y = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{2\pi x}{L} + \cdots + a_n \sin \frac{n\pi x}{L} \tag{14-75}$$

The undetermined $a$'s (amplitudes) of the sine-wave shapes may be considered as the coordinates of the system. In general, any deflection configuration represented by coordinates $\{r\}$ can be described by an arbitrary set of compatible mode shapes $[\phi]$ with undeterminated amplitude $\{u\}$; that is,

$$\{r\} = [\phi]\{u\} \tag{14-76}$$

in which $\{u\}$ are referred to as *generalized coordinates*. The number of assumed shape patterns represents the degrees of freedom.

To illustrate, let us consider the cantilever column shown in Fig. 14-11(a), for which the lateral deflection is defined by coordinates $\{r_1 \quad r_2 \quad r_3\}$ at three levels. Three arbitrary mode shapes of deflection compatible with the constraints are assumed as in Figs. 14-11(b), (c), and (d). The lateral deflection described by original independent coordinates $\{r_1 \quad r_2 \quad r_3\}$ can be expressed as the linear combination of the three mode shapes $\{\phi\}_i$, each modified by an amplitude $u_i$ $(i = 1, 2, 3)$. Thus

$$\begin{aligned}\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} &= \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} u_1 + \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} u_2 + \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} u_3 \\ &= \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}\end{aligned} \tag{14-77}$$

It is apparent that matrix $[\phi]$ serves to transform from generalized coordinates $\{u\}$ to the geometric coordinates $\{r\}$. As $\{u\}$ are independent coordinates, matrix $[\phi]$ is nonsingular and can be inverted. Thus it is always possible to determine $\{u\}$ for a given configuration $\{r\}$. In the present instance,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}^{-1} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} \tag{14-78}$$

If the assumed displacement patterns are the $n$ natural modes or normal modes of undamped free vibration such that the modal matrix $[\phi]$ in Eq. 14-76 is formed by the eigenvectors $\{A\}$, that is,

$$[\phi] = [\{A\}_1 \, \{A\}_2 \cdots \{A\}_n] \tag{14-79}$$

then we can make use of the orthogonality property of $\{A\}$ in the evaluation of

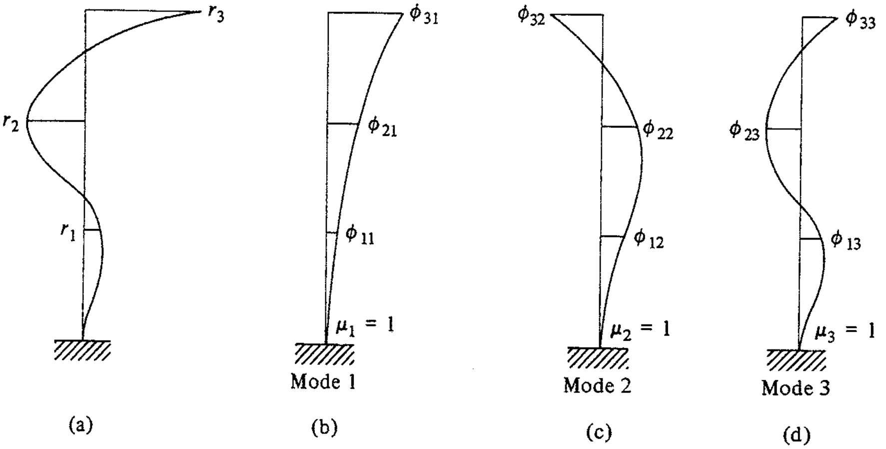


Figure 14-11

$[u]$ to avoid solving simultaneous equations, as in Eq. 14-78. To do this, we may premultiply each side of Eq. 14-76 with the product of $[\phi]^T$ and the mass matrix $[m]$; thus,

$$[\phi]^T[m]\{r\} = [\phi]^T[m][\phi]\{u\} \tag{14-80}$$

In view of Eq. 14-79 and because of the orthogonality property with respect to mass, the term $[\phi]^T[m]\{\phi\}$ on the right-hand side of Eq. 14-80 can be expanded as

$$\begin{aligned}[\phi]^T[m][\phi] &= \begin{bmatrix} \{A\}_1^T[m]\{A\}_1 & \{A\}_1^T[m]\{A\}_2 & \cdots & \{A\}_1^T[m]\{A\}_n \\ \{A\}_2^T[m]\{A\}_1 & \{A\}_2^T[m]\{A\}_2 & \cdots & \{A\}_2^T[m]\{A\}_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \{A\}_n^T[m]\{A\}_1 & \{A\}_n^T[m]\{A\}_2 & \cdots & \{A\}_n^T[m]\{A\}_n \end{bmatrix} \\ &= \begin{bmatrix} \{A\}_1^T[m]\{A\}_1 & & & & \\ & \{A\}_2^T[m]\{A\}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \{A\}_n^T[m]\{A\}_n \end{bmatrix}\end{aligned} \tag{14-81}$$

which is a diagonal matrix. If we let $\{A\}_1^T[m]\{A\}_1 = \overline{m}_1$, $\{A\}_2^T[m]\{A\}_2 = \overline{m}_2, \ldots,$ Eq. 14-81 will take the form

$$[\phi]^T[m][\phi] = \begin{bmatrix} \overline{m}_1 & & & & \\ & \overline{m}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \overline{m}_n \end{bmatrix} = [\overline{m}] \tag{14-82}$$

Substituting Eq. 14-82 into Eq. 14-80 and using matrix multiplication, we obtain a set of $n$ uncoupled equations,

$$\begin{aligned} \{A\}_1^T[m]\{r\} &= \overline{m}_1 u_1 \\ \{A\}_2^T[m]\{r\} &= \overline{m}_2 u_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ \{A\}_n^T[m]\{r\} &= \overline{m}_n u_n \end{aligned} \tag{14-83}$$

which can be used to solve for coordinates $u_1, u_2, \ldots, u_n$. These mode-amplitude generalized coordinates are referred to as *normal coordinates*. From Eq. 14-83 any normal coordinate $u_i$ is given by

$$u_i = \frac{\{A\}_i^T[m]\{r\}}{\overline{m}_i} \qquad (i = 1, 2, \ldots, n) \tag{14-84}$$

The normal coordinates are important for dynamic response analysis of a linear system in that such analysis yields uncoupled equations of motion, as is discussed in the next section.

## 14-9 RESPONSE TO DYNAMIC FORCES: UNCOUPLED EQUATIONS OF MOTION

On the assumption that damping can be neglected, the equation of motion for an $n$-degree-of-freedom system is

$$[m]\{\ddot{x}\} + [s]\{x\} = \{F\} \tag{14-85}$$

The stiffness matrix $[s]$ in Eq. 14-85 is generally not a diagonal matrix. For a distributed-mass system, the mass matrix is also not diagonal. The orthogonality properties of the natural modes or normal modes with respect to $[m]$ and $[s]$ now may be used to uncouple Eq. 14-85.

Let

$$\{x\} = [\phi]\{u\} \tag{14-86}$$

in which the transformation matrix $[\phi]$ is formed by modal columns $\{A\}$ as defined by Eq. 14-79 and $\{u\}$ represents the normal coordinates. It follows that

$$\{\ddot{x}\} = [\phi]\{\ddot{u}\} \tag{14-87}$$

Substituting Eqs. 14-86 and 14-87 into 14-85 leads to

$$[m][\phi]\{\ddot{u}\} + [s][\phi]\{u\} = \{F\} \tag{14-88}$$

Premultiplying each side of Eq. 14-88 by $[\phi]^T$ gives

$$[\phi]^T[m][\phi]\{\ddot{u}\} + [\phi]^T[s][\phi]\{u\} = [\phi]^T\{F\} \tag{14-89}$$

Because of the orthogonality with respect to the mass matrix, we have, from Eq. 14-81 or Eq. 14-82,

$$[\phi]^T[m][\phi] = [\overline{m}] \tag{14-90}$$

where $[\overline{m}]$ is a diagonal matrix given by

$$[\overline{m}] = \begin{bmatrix} \overline{m}_1 & & & & \\ & \overline{m}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \overline{m}_n \end{bmatrix}$$

$$= \begin{bmatrix} \{A\}_1^T[m]\{A\}_1 & & & & \\ & \{A\}_2^T[m]\{A\}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \{A\}_n^T[m]\{A\}_n \end{bmatrix} \tag{14-91}$$

Following the orthogonality with respect to the stiffness matrix, we can similarly obtain

$$[\phi]^T[s][\phi] = [\bar{s}] \tag{14-92}$$

with

$$[\bar{s}] = \begin{bmatrix} \bar{s}_1 & & & & \\ & \bar{s}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \bar{s}_n \end{bmatrix} = \begin{bmatrix} \{A\}_1^T[s]\{A\}_1 & & & & \\ & \{A\}_2^T[s]\{A\}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \{A\}_n^T[s]\{A\}_n \end{bmatrix} \tag{14-93}$$

Substituting Eqs. 14-90 and 14-92 into Eq. 14-89 and using the notion

$$[\phi]^T\{F\} = \{\bar{F}\} \tag{14-94}$$

we have

$$[\bar{m}]\{\ddot{u}\} + [\bar{s}]\{u\} = \{\bar{F}\} \tag{14-95}$$

which represents the equation of motion in normal coordinates for forced undamped vibration. The uncoupled form of Eq. 14-95 enables us to determine the response in each normal mode of vibration separately as an independent single-degree-of-freedom system in normal coordinates and then to superimpose these by the transformation of Eq. 14-86 to obtain the response in the original coordinates. This procedure is called the *mode-superposition method.*

If we use the relationship of Eq. 14-38 for $[s]$ and $[m]$, that is,

$$[s]\{A\}_j = \omega_j^2[m]\{A\}_j$$

and premultiply this equation with $\{A\}_j^T$, we have generally

$$\{A\}_j^T[s]\{A\}_j = \omega_j^2\{A\}_j^T[m]\{A\}_j \qquad (j = 1, 2, \ldots, n) \tag{14-96}$$

Substituting Eq. 14-96 into Eq. 14-93, we obtain

$$\begin{bmatrix} \bar{s}_1 & & & & \\ & \bar{s}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \bar{s}_n \end{bmatrix}$$

$$= \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_n^2 \end{bmatrix} \begin{bmatrix} \{A\}_1^T[m]\{A\}_1 & & & \\ & \{A\}_2^T[m]\{A\}_2 & & \\ & & \ddots & \\ & & & \{A\}_n^T[m]\{A\}_n \end{bmatrix}$$

or

$$[\bar{s}] = [\Omega][\bar{m}] \tag{14-97}$$

Note that $[\Omega][\bar{m}] = [\bar{m}][\Omega]$.

Equation 14-97 relates the generalized stiffness to the generalized mass by the frequency of vibration. Thus for any mode $j$, we have

$$\bar{s}_j = \omega_j^2 \bar{m}_j \qquad (j = 1, 2, \ldots, n) \tag{14-98}$$

Using Eq. 14-97, we can write Eq. 14-95 in the form

$$[\bar{m}]\{\ddot{u}\} + [\bar{m}][\Omega]\{u\} = \{\bar{F}\} \tag{14-99}$$

Equation 14-99 represents a set of $n$ uncoupled differential equations; that is,

$$\begin{aligned} \bar{m}_1\ddot{u}_1 + \bar{m}_1\omega_1^2 u_1 &= \bar{F}_1 \\ \bar{m}_2\ddot{u}_2 + \bar{m}_2\omega_2^2 u_2 &= \bar{F}_2 \\ &\vdots \\ \bar{m}_n\ddot{u}_n + \bar{m}_n\omega_n^2 u_n &= \bar{F}_n \end{aligned} \tag{14-100}$$

or

$$\ddot{u}_j + \omega_j^2 u_j = \frac{\bar{F}_j}{\bar{m}_j} \qquad (j = 1, 2, \ldots, n) \tag{14-101}$$

in which

$$\bar{F}_j = \{A\}_j^T\{F\} \tag{14-102}$$

**Example 14-3**

Determine the response of the system shown in Fig. 14-8 to a set of harmonic forces

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \sin \bar{\omega} t \\ 2F_0 \sin \bar{\omega} t \end{Bmatrix}$$

at levels 1 and 2, respectively.

The governing differential equation for determining the response is expressed by Eq. 14-99; that is,

$$[\bar{m}]\{\ddot{u}\} + [\bar{m}][\Omega]\{u\} = \{\bar{F}\}$$

We first use the modal columns (see Example 14-2)

$$\{A\}_1 = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix} \qquad \{A\}_2 = \begin{Bmatrix} -1.620 \\ 1 \end{Bmatrix}$$

to form the transformation matrix $[\phi]$. Thus

$$[\phi] = \begin{bmatrix} 0.618 & -1.620 \\ 1 & 1 \end{bmatrix} \tag{14-103}$$

The generalized mass matrix $[\overline{m}]$ is then given by Eq. 14-90; that is,

$$\begin{aligned}[\overline{m}] &= [\phi]^T[m][\phi] \\ &= \begin{bmatrix} 0.618 & 1 \\ -1.620 & 1 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 0.618 & -1.620 \\ 1 & 1 \end{bmatrix} \\ &= m \begin{bmatrix} 1.382 & 0 \\ 0 & 3.624 \end{bmatrix}\end{aligned} \tag{14-104}$$

Substituting the natural angular frequencies $\omega_1^2 = 91.7$ and $\omega_2^2 = 628.3$ from Example 14-2 into Eq. 14-97 gives

$$[\Omega] = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} 91.7 & 0 \\ 0 & 628.3 \end{bmatrix} \tag{14-105}$$

Note that we have used $EI/mL^3 = 10$ in developing $\omega_1^2$ and $\omega_2^2$.

The generalized force matrix is calculated by Eq. 14-94

$$\begin{aligned}\{\overline{F}\} &= [\phi]^T\{F\} \\ &= \begin{bmatrix} 0.618 & 1 \\ -1.620 & 1 \end{bmatrix} \begin{Bmatrix} F_0 \sin \overline{\omega} t \\ 2F_0 \sin \overline{\omega} t \end{Bmatrix} = \begin{Bmatrix} 2.618F_0 \sin \overline{\omega} t \\ 0.380F_0 \sin \overline{\omega} t \end{Bmatrix}\end{aligned} \tag{14-106}$$

Substituting matrices $[\overline{m}]$, $[\Omega]$, and $\{\overline{F}\}$ into the equation of motion of 14-99 yields

$$\begin{aligned} m\begin{bmatrix} 1.382 & 0 \\ 0 & 3.624 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + m \begin{bmatrix} 1.382 & 0 \\ 0 & 3.624 \end{bmatrix} \begin{bmatrix} 91.7 & 0 \\ 0 & 628.3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ = \begin{Bmatrix} 2.618F_0 \sin \overline{\omega} t \\ 0.380F_0 \sin \overline{\omega} t \end{Bmatrix}\end{aligned} \tag{14-107}$$

Equation 14-107 represents two uncoupled equations of motion, each with one degree of freedom:

$$(1.382m)\ddot{u}_1 + (126.73m)u_1 = 2.618F_0 \sin \overline{\omega} t \tag{14-108}$$

$$(3.624m)\ddot{u}_2 + (2276.96m)u_2 = 0.380F_0 \sin \overline{\omega} t \tag{14-109}$$

The particular solution for these steady-state undamped, forced vibrations is given by Eqs. 14-73 and 14-74. Consequently,

$$u_1 = \frac{2.618F_0 \sin \overline{\omega} t}{(126.73m)(1 - \overline{\omega}^2/91.7)} = \frac{1.894F_0}{m(91.7 - \overline{\omega}^2)} \sin \overline{\omega} t \tag{14-110}$$

$$u_2 = \frac{0.380F_0 \sin \overline{\omega} t}{(2276.96m)(1 - \overline{\omega}^2/628.3)} = \frac{0.105F_0}{m(628.3 - \overline{\omega}^2)} \sin \overline{\omega} t \tag{14-111}$$

Finally, the displacements $\{u\}$ are transformed by $\{x\} = [\phi]\{u\}$ to obtain the total displacements $\{x\}$. Thus,

$$\begin{aligned}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} &= \begin{bmatrix} 0.618 & -1.620 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{1.894}{91.7 - \overline{\omega}^2} \\ \dfrac{0.105}{628.3 - \overline{\omega}^2} \end{Bmatrix} \left( \frac{F_0 \sin \overline{\omega} t}{m} \right) \\ &= \begin{Bmatrix} \dfrac{1.170}{91.7 - \overline{\omega}^2} - \dfrac{0.170}{628.3 - \overline{\omega}^2} \\ \dfrac{1.894}{91.7 - \overline{\omega}^2} + \dfrac{0.105}{628.3 - \overline{\omega}^2} \end{Bmatrix} \left( \frac{F_0 \sin \overline{\omega} t}{m} \right)\end{aligned} \tag{14-112}$$

If we desire to include the damping effect, then the equation of motion for an $n$-degree-of-freedom system is given by

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [s]\{x\} = \{F\} \tag{14-113}$$

Following procedures similar to those in changing from Eq. 14-85 to Eq. 14-89, we transform Eq. 14-113 to normal coordinates $\{u\}$:

$$[\phi]^T[m][\phi]\{\ddot{u}\} + [\phi]^T[c][\phi]\{\dot{u}\} + [\phi]^T[s][\phi]\{u\} = [\phi]^T\{F\} \tag{14-114}$$

As pointed out in Sec. 14-6, we may, for the sake of convenience, express the damping matrix $[c]$ as a linear combination of the mass matrix $[m]$ and the stiffness matrix $[s]$. If such is the case, the orthogonality condition can apply to the damping matrix also, so that the term $[\phi]^T[c][\phi]$ in Eq. 14-114 is a diagonal matrix, denoted by $[\overline{c}]$. Equation 14-114 then reduces to

$$[\overline{m}]\{\ddot{u}\} + [\overline{c}]\{\dot{u}\} + [\overline{s}]\{u\} = \{\overline{F}\} \tag{14-115}$$

which gives a set of $n$ uncoupled equations in normal coordinates, each representing an independent system with a single degree of freedom. The displacements $\{u\}$ are finally transformed to the original $\{x\}$ by

$$\{x\} = [\phi]\{u\}$$

thereby superimposing $n$ separate modes in normal coordinates to reach the total displacements.

## 14-10 A LITTLE BIT OF EARTHQUAKE RESPONSE

In this section, as the title suggests, we intend to discuss the earthquake response only in its simplest form and within the scope of deterministic-response analysis for a linear lumped mass system.

Refer to Fig. 14-12(a) for a single-degree-of-freedom system with a lumped mass $m$ (representing the concentrated mass of a column-beam element) supported by an elastic massless column with lateral stiffness $s$ and connected to a damper with viscous damping coefficient $c$. Let the system undergo a vibration because of base motion $x_g$ (assumed to be in a horizontal direction), which is a time-varying displacement of the base from its original position. In the absence of an external exciting force, the only forces acting on the mass are an inertia force $m(\ddot{x}_g + \ddot{x})$, a viscous damping force $c\dot{x}$, and an elastic force $sx$, $x$ being the

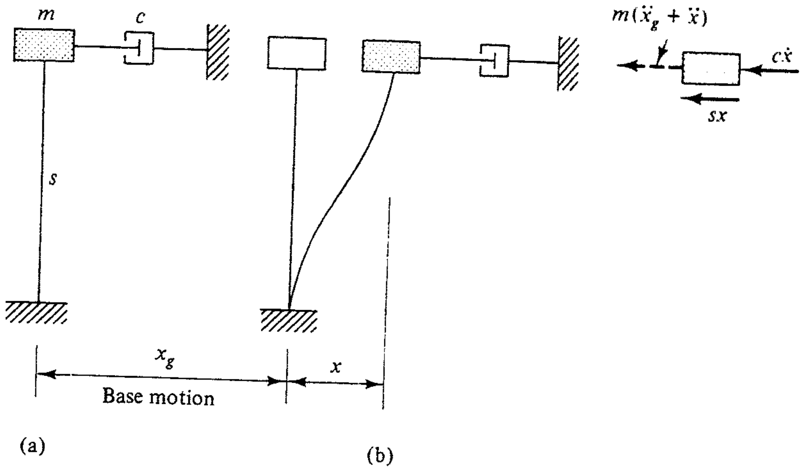


**Figure 14-12**

relative displacement of the top from the base. See Fig. 14-12(b). Note that the inertia force depends upon the total motion, whereas the damping and elastic forces depend upon the motion of mass relative to the base. Therefore, the equation of motion is

$$m(\ddot{x}_g + \ddot{x}) + c\dot{x} + sx = 0 \tag{14-116}$$

or

$$m\ddot{x} + c\dot{x} + sx = -m\ddot{x}_g \tag{14-117}$$

The alternative form of Eq. 14-117 may be given as

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = -\ddot{x}_g \tag{14-118}$$

where $\omega = \sqrt{s/m}$ (natural frequency) and $\zeta = c/2\sqrt{sm}$ (damping ratio), as previously defined in Eqs. 14-9 and 14-52, respectively. Comparing Eq. 14-117 with Eq. 14-5, we see that the effect of base motion is equivalent to an externally applied force $(-m\ddot{x}_g)$. The determination of $\ddot{x}_g$ is based on the analysis of response spectra retained from previous earthquakes.

With respect to a lumped multi-degree-of-freedom system subjected to base motion, the earthquake-response analysis can be carried out in a manner similar to that for a single-degree-of-freedom system. By analogy with Eq. 14-116, we can write the equation of motion in matrix notation as

$$[m]\{\ddot{x}_g\{\mathrm{I}\} + \{\ddot{x}\}\} + [c]\{\dot{x}\} + [s]\{x\} = \{0\} \tag{14-119}$$

in which {I} is a unit column matrix and the term $\ddot{x}_g\{\mathrm{I}\}$ expresses the fact that each mass of the system experiences the same acceleration due to base motion. Rearranging Eq. 14-119 gives

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [s]\{x\} = -[m]\{\mathrm{I}\}\ddot{x}_g \tag{14-120}$$

where the term on the right-hand side of the equation represents the effective earthquake forces. The solution of the earthquake response of Eq. 14-120 can be achieved by following the mode-superposition method discussed in the preceding section, that is, by first transforming the earthquake response in the original coordinates to that in normal coordinates to obtain a set of uncoupled equations, each representing the response of a single-degree-of-freedom lumped block, thus enabling us to determine the responses in each normal mode separately, and then superimposing these by the transformation matrix to obtain the response in the original coordinates.

In practice, since ground motion tends to excite strongly only the lowest modes of vibration, the earthquake response for a system of many degrees of freedom can often be approximated by carrying out the analysis based on a few normal coordinates.

## PROBLEMS

**14-1.** Determine the natural frequency and period of vibration of a mass $m$ attached to the free end of a light cantilever beam of length $L$ and flexural rigidity $EI$.

**14-2.** Determine the natural angular frequencies of vibration for the frame shown in Fig. 14-13. Assume that the beams are infinitely rigid in comparison with column stiffness and that the masses are lumped at two floor levels.

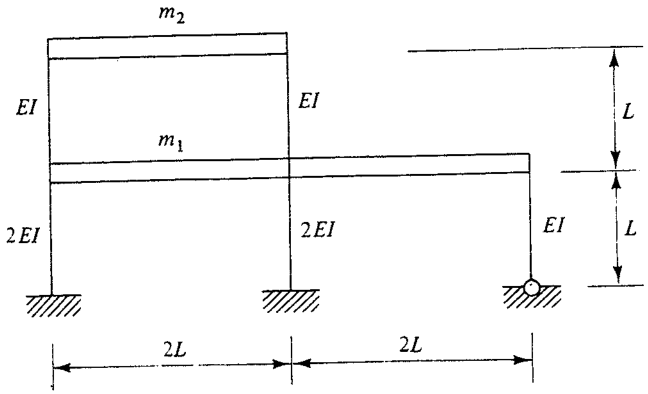


**Figure 14-13**

**14-3.** Find the natural angular frequencies and characteristic shapes for the two-degree-of-freedom systems shown in Fig. 14-14.

**14-4.** Consider the lumped-mass system shown in Fig. 14-15, which has three degrees of freedom represented by the lateral translations $x_1$, $x_2$, and $x_3$ at three floor levels. Give the equation of motion of undamped free vibration and obtain the natural angular frequencies.

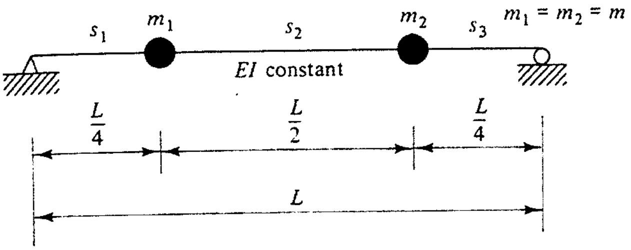


Figure 14-14

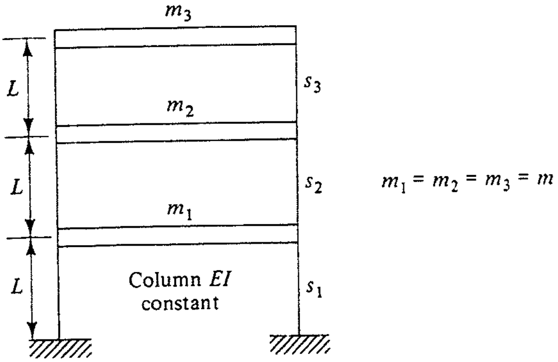


Figure 14-15

**14-5.** For the free vibration of Prob. 14-4, obtain characteristic shapes. Check your result by the orthogonality property.

**14-6.** Determine the response of the system of Prob. 14-4 to a set of harmonic forces

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} F_0 \sin \overline{\omega} t \\ 2F_0 \sin \overline{\omega} t \\ F_0 \sin \overline{\omega} t \end{Bmatrix}$$

# *Answers to Selected Problems*

## CHAPTER 2

**2-1.** **(a)** Stable and indeterminate to the fifth degree
**(b)** Unstable
**(c)** Stable and determinate
**(d)** Stable and indeterminate to the second degree

**2-2.** **(b)** Stable and indeterminate to the third degree
**(c)** Unstable externally
**(d)** Unstable internally

**2-3.** **(b)** Stable and indeterminate to the fifth degree
**(e)** Stable and indeterminate to the fourth degree
**(f)** Unstable externally
**(g)** Stable and indeterminate to the 102nd degree

## CHAPTER 3

**3-3.** **(b)** $S_{ab} = S_{bc} = +14.6$ kN $\quad S_{aB} = +9.1$ kN
$S_{Bb} = +17.5$ kN $\quad S_{Bc} = -18.6$ kN
**(d)** $S_{bc} = S_{cd} = S_{de} = S_{ef} = 0$
$S_{ag} = -S_{bg} = 26.9$ kN
**(e)** $S_{ab} = S_{bc} = 0 \quad S_{aB} = -11.3$ kN
$S_{Bc} = -14.4$ kN

**3-4.** **(a)** $S_a = -32$ kips $\quad V_b = +12$ kips $\quad S_c = +24$ kips
**(b)** $H_a = +18.75$ kips $\quad V_b = -15$ kips $\quad S_c = -6.25$ kips
**(c)** $V_a = +11.6$ kips $\quad H_b = +166.7$ kips $\quad S_c = -20$ kips

(d) $S_a = +20$ kips $V_b = +100$ kips $V_c = -45$ kips
(e) $S_a = -50$ kips

3-5. $V_a = -20$ kips $V_b = +10$ kips $V_c = -20$ kips $S_d = 0$

3-6. $S_{AB} = S_{BC} = S_{CD} = S_{BF} = S_{CF} = 0$ $S_{AD} = +15.4$ kips
$S_{AE} = +11.2$ kips $S_{DE} = -13.6$ kips $S_{EF} = -10$ kips

3-7. $S_{ad} = +7.85$ kN

3-8. (d) $M_c = +108$ ft-kips
(e) $M_b = +31.25$ ft-kips
(f) $M_b = -144$ ft-kips

3-9. (a) $M_c = +33.3$ kN • m
(b) $M_b = +71.4$ kN • m

## CHAPTER 4

4-1. $V = 49.33$ kips $M = 458.67$ ft-kips

4-4. $R_A = 45$ kN $R_C = 175$ kN $R_E = 30$ kN
$V_B = -55$ kN $M_B = -90$ kN • m $M_C = -180$ kN • m

4-8. (a) $R = 22.78$ kips
(b) $V = 14.44$ kips $M = 216.67$ ft-kips

4-9. $V_a = +29.68$ kips (maximum tension)
$V_a = -4.4$ kips (maximum compression)
$S_b = +52.75$ kips

4-10. $R_C = 156$ kN $M_D = 111$ kN • m

4-11. $M_E = 135$ kN • m $R_E = 55$ kN

## CHAPTER 5

5-1. $\Delta_c = 5wl^4/384EI$ (down) $\theta_a = wl^3/24EI$ (clockwise)

5-2. $\Delta_p = 0.0147Pl^3/EI$ (down) $\Delta$(at midspan) $= 0.0236Pl^3/EI$ (down)

5-3. $\Delta = 0.54$ in. (down) $\theta = 0.006$ rad (counterclockwise)

5-4. $\Delta_h = 0.43$ in. (right) $\Delta_v = 0.696$ in. (down)
$\Delta_r = 0.006$ rad (counterclockwise)

5-5. $\theta_b = 0.0023$ radian (clockwise) $\Delta_b = 0.375$ cm (down)
$\theta_c = 0.0038$ radian (clockwise) $\Delta_c = 1.35$ cm (down)

5-6. $\Delta_1 = 1.15$ cm (right) $\Delta_2 = 0.59$ cm (down)
$\Delta_3 = 0.0039$ radian (counterclockwise)

5-7. $\Delta_1 = 0.1$ cm (right) $\Delta_2 = 0.12$ cm (down)

5-8. (a) $\Delta_B = 0.00746$ ft (down)
(b) $\Delta_C = 0.00278$ ft (right)
(c) $\Delta_{bC} = 0.0002$ ft (toward each other)
(d) $\theta = 0.000434$ rad (clockwise)

**5-20.** $\theta_c = 0.003$ radian (counterclockwise, at the left side)
$\theta_c = 0.0017$ radian (clockwise, at the right side)
$\Delta_b = 0.267$ cm (down)

## CHAPTER 6

**6-1.** **(a)** $R_b = 50$ kips (up)
**(b)** $M_b = -100$ ft-kips

**6-2.** $R_b = 10$ kips (up)

**6-3.** $R_b = 7.28$ kips (up)

**6-4.** $R_e = 0.916$ kN (up)

**6-5.** **(b)** $M_a = \text{wm}\ (\text{m}^4 + 4\text{mn}^3 + 3\text{n}^4)/8(\text{m}^3 + \text{n}^3)$

**6-6.** $H_c = 15.24$ kN (left)

**6-7.** $H_a = 1.25$ kips (right) $V_a = 5$ kips (up) $M_a = 4.16$ ft-kips (clockwise)

**6-9.** $S_{bc} = 5.25$ kN

**6-10.** **(a)** $S_{bC} = -0.75$ kip
**(b)** $H_d = 30$ kips (left)
**(c)** $S_{bC} = -5.4$ kips $H_d = 31.1$ kips (left)

**6-11.** $S_{Bc} = S_{bC} = +8.75$ kips

**6-16.** $S_{ac} = +4.25$ kips $M_a = -3.25$ ft-kips
The effect of axial force in beam is neglected.

**6-17.** 21.75 kips, 17.4 kips, and 13.05 kips

**6-18.** The influence ordinates are 1.3, 1, 0.704, 0.432, 0.208, 0.056, 0

**6-19.** The influence ordinates are 0, 1.39, 1.78, 1.50, 0.89, 0.28, 0

**6-20.** **(a)** The influence ordinates are 1, 0.578, 0.222, 0, −0.05, 0
**(b)** The influence ordinates are 0, −0.532, −0.668, 0, −0.30, 0
**(c)** The influence ordinates are 0, −0.422, 0.222, 0, −0.05, 0

## CHAPTER 7

**7-1.** $M_{ab} = -22$ ft-kips $M_{ba} = -M_{bc} = 28$ ft-kips $M_{cb} = 31$ ft-kips

**7-2.** $M_{ab} = -23.55$ ft-kips $M_{ba} = 16.89$ ft-kips
$M_{bd} = -8.89$ ft-kips $M_{db} = -4.44$ ft-kips

**7-3.** $M_{ab} = -22.1$ ft-kips $M_{ba} = -M_{bc} = 68.3$ ft-kips
$M_{cb} = 20$ ft-kips

**7-5.** $M_{ab} = -44.8$ ft-kips $M_{ba} = -M_{bc} = -34.4$ ft-kips

**7-6.** $M_{ab} = 4.9$ ft-kips $M_{ba} = -M_{bc} = -0.9$ ft-kip
$M_{cb} = -M_{cd} = 2.24$ ft-kips $M_{dc} = -1.76$ ft-kips

**7-7.** Spring force = 3.58 kN (compression)

**7-8.** $R_c = 7.35$ kN $\quad M_{ab} = -3.26$ kN • m

**7-9.** $M_{ab} = -0.63$ kN • m $\quad M_{ba} = -M_{bc} = 2.49$ kN • m
$M_{dc} = -1.87$ kN • m

**7-10.** $R_d = 3.03$ kN

**7-11.** (a) $M_{ab} = -14.9$ ft-kips $\quad M_{ba} = -M_{bc} = 14.9$ ft-kips
$M_{cb} = 39.4$ ft-kips $\quad M_{ce} = -29.8$ ft-kips
(b) $M_{ab} = -120.8$ ft-kips $\quad M_{ba} = -M_{bc} = -112.6$ ft-kips
$M_{cb} = -M_{cd} = 104$ ft-kips $\quad M_{dc} = -103.6$ ft-kips
(c) $M_{ab} = 8.52$ ft-kips $\quad M_{ba} = -M_{bc} = 9.58$ ft-kips
$M_{cb} = -M_{cd} = -1$ ft-kip $\quad M_{dc} = 26.74$ ft-kips
(d) $M_{ba} = M_{ef} = -150$ ft-kips $\quad M_{be} = M_{eb} = 156.25$ ft-kips
$M_{bc} = M_{ed} = -6.25$ ft-kips $\quad M_{cb} = M_{de} = -43.75$ ft-kips
$M_{cd} = M_{dc} = 43.75$ ft-kips
(e) $M_{ab} = -118.1$ ft-kips $\quad M_{ba} = -M_{bc} = -82$ ft-kips
$M_{cb} = -M_{cd} = 11.5$ ft-kips $\quad M_{dc} = -M_{de} = 25.8$ ft-kips
$M_{ed} = -74.1$ ft-kips

**7-12.** (a) $M_{ac} = 18.5$ kN • m $\quad M_{bc} = 14.2$ kN • m
(b) $M_{ac} = -M_{bc} = -24$ kN • m

**7-15.** $M_{ab} = -41.25$ kN • m $\quad M_{ba} = -M_{bc} = -6.75$ kN • m

## CHAPTER 8

**8-5.** End moment = 14.6 ft-kips

**8-6.** Moments at interior supports are 36.5 ft-kips and 27.3 ft-kips

**8-7.** $M_{ba} = -M_{bc} = 11.4$ ft-kips $\quad M_{cb} = -M_{cd} = 0.6$ ft-kip

**8-8.** $R_c = 57.27$ kN

**8-9.** $K_{ab} = 3EIl_1^2/(l_1^3 + l_2^3)$

**8-10.** $M_{ba} = -M_{bc} = \dfrac{57wl^2}{816} - \dfrac{288EI\Delta}{17l^2}$

**8-11.** (a) $M_{bc} = 61.2$ kN • m
(b) $M_{bc} = -33.8$ kN • m
(c) $M_{bc} = -26.7$ kN • m
(d) $M_{bc} = 33.9$ kN • m

**8-13.** End moment = 22.10 ft-kips

**8-15.** $M_{ad} = 50.9$ ft-kips $\quad M_{be} = -12.2$ ft-kips
$M_{cg} = -35.6$ ft-kips $\quad M_{da} = -M_{de} = 85.8$ ft-kips
$M_{ed} = 119.8$ ft-kips $\quad M_{eb} = -48.5$ ft-kips
$M_{ef} = -71.3$ ft-kips $\quad M_{fe} = -M_{fg} = 108.8$ ft-kips
$M_{gf} = -M_{gc} = 78.1$ ft-kips

## CHAPTER 9

**9-1.** $r_p = 5PL^3/6EI$ (down)

**9-3.** $r$ (under 10 kips load) $= 0.01608$ ft (down)
$r$ (under 20 kips load) $= 0.00548$ ft (down)

**9-4.** $r_1 = (4L^3R_1 - L^2R_2)/6EI \qquad r_2 = (-L^2R_1 + 2LR_2)/6EI$

**9-5.** Deflection $= 5PL^3/6EI$ Slope $= PL^2/EI$

**9-6.** $\begin{Bmatrix} Q^a \\ Q^b \\ Q^c \end{Bmatrix} = \begin{Bmatrix} 3.36 \\ 8.28 \\ 2.76 \end{Bmatrix}$ kips $\qquad \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \dfrac{1}{AE}\begin{Bmatrix} 83.2 \\ -15.72 \end{Bmatrix}$ ft

**9-7.** $S = -6.725$ kN

**9-9.** $Q_i^a = -44 \text{ kN} \cdot \text{m} \qquad Q_j^a = -10 \text{ kN} \cdot \text{m} \qquad Q_j^c = -23.3 \text{ kN} \cdot \text{m}$

**9-10.** $\text{slope} = \dfrac{1}{96EI}\left(wL^3 + \dfrac{51\,wL^3}{1 + L^3/3EIf_s}\right)$

$\text{deflection} = \dfrac{17\,wL^4}{48EI(1 + L^3/3EIf_s)}$

**9-11.** $Q_i^a = 0.080LR_1 - 0.236LR_2 - 0.274R_3$
$Q_j^d = -0.094LR_1 - 0.392LR_2 + 0.030R_3$

## CHAPTER 10

**10-5.** The moment at the fixed support $= -11.25$ kN • m

**10-7.** $M_B = wL^2/10$

**10-8.** $M_{AB} = 8$ ft-kips $\qquad M_{BA} = 16$ ft-kips

**10-9.** $M_{AB} = 31$ ft-kips $\qquad M_{BA} = 62$ ft-kips
$M_{BE} = 93.2$ ft-kips $\qquad M_{EB} = 46.6$ ft-kips
$M_{BC} = -155.2$ ft-kips

**10-10.**

$$a = \begin{bmatrix} \dfrac{2EI}{L} & 0 & -\dfrac{6EI}{L^2} & 0 & 0 \\ \dfrac{4EI}{L} & 0 & -\dfrac{6EI}{L^2} & 0 & 0 \\ 0 & \dfrac{4EI}{L} & \dfrac{6EI}{L^2} & \dfrac{2EI}{L} & -\dfrac{6EI}{L^2} \\ 0 & \dfrac{2EI}{L} & \dfrac{6EI}{L^2} & \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} \\ 0 & 0 & 0 & 0 & \dfrac{AE}{L} \end{bmatrix}$$

**10-11.** $Q_i^a = -46.48$ ft-kips $\qquad Q_j^a = -Q_i^b = -42.25$ ft-kips
$Q_j^b = -Q_i^c = 33.80$ ft-kips $\qquad Q_j^c = -30.98$ ft-kips

## CHAPTER 11

**11-2.** Fixed-end moments are $-8.4$ kN • m (left) and 12.9 kN • m (right). The rotation at the interior support is $1.8/EI$ rad.

**11-3.** $Q^a = 1.25P \qquad Q^b = -3.75P$
Displacements of the top joint are $5.208Pl/AE$ (right) and $1.953Pl/AE$ (down).

**11-4.** $Q^a = -1.25P \qquad Q^b = -1.06P \qquad Q^c = -0.25P$
Displacements of the joint are $0.25Pl/AE$ (right) and $1.25\ Pl/AE$ (down).

**11-5.** End moments of the beam are $0.13\ wl^2$ and $0.54\ wl^2$. End moments of the column are $0.46\ wl^2$ and $0.22\ wl^2$.

**11-6.** Fixed-end moment is $0.08\ wl^2$ including the axial effect.

## CHAPTER 12

**12-2.** $f = \dfrac{l}{3EI}\begin{bmatrix} 0.73 & -0.27 \\ -0.27 & 0.40 \end{bmatrix}$

**12-3.** $k = \dfrac{EI}{l}\begin{bmatrix} 5.48 & 3.70 \\ 3.70 & 10 \end{bmatrix}$

**12-4.** $C_{ab} = 0.743 \qquad C_{ba} = 0.462$
$S_{ab} = 4.62\ EI_a/l \qquad S_{ba} = 7.43\ EI_a/l$

**12-5.** $k = \dfrac{EI_a}{l}\begin{bmatrix} 4.62 & 3.43 \\ 3.43 & 7.43 \end{bmatrix}$

$f = \dfrac{l}{EI_a}\begin{bmatrix} 0.329 & -0.152 \\ -0.152 & 0.205 \end{bmatrix}$

**12-6.** $M^F_{ab} = -7.5$ kN • m $\qquad M^F_{ba} = 14.4$ kN • m

**12-7.** $M_{ba} = 20.3$ kN • m $\qquad M_{cb} = 22.3$ kN • m

**12-8.** The moment at the fixed support = 12.85 kN • m

## CHAPTER 13

**13-1.** $P_{cr} = \dfrac{sl}{2}$

**13-2.** If we let $s_1 = 1$ and $s_2 = 2$, then the first-mode $P_{cr} = 0.439l$ and the second-mode $P_{cr} = 4.561l$

**13-3.** The theoretical $P_{cr} = 26EI/l^2$

**13-6.** The theoretical $P_{cr} = 25.2EI/l^2$

**13-7.** The theoretical $P_{cr} = 28.3EI/l^2$

## CHAPTER 14

**14-1.** $f = \dfrac{1}{2\pi}\sqrt{\dfrac{3EI}{mL^3}}$ cycle/s

**14-2.** $\omega_1 = 9.19 \sqrt{\dfrac{EI}{mL^3}}$ rad/s

$\omega_2 = 3.81 \sqrt{\dfrac{EI}{mL^3}}$ rad/s

**14-3.** $\omega_1 = 13.86 \sqrt{\dfrac{EI}{mL^3}}$ rad/s

$\omega_2 = 19.60 \sqrt{\dfrac{EI}{mL^3}}$ rad/s

Two modes of the characteristic shape: The symmetrical and the antisymmetrical

**14-4.** $\omega_1 = 2.180 \sqrt{\dfrac{EI}{mL^3}}$ rad/s

$\omega_2 = 6.109 \sqrt{\dfrac{EI}{mL^3}}$ rad/s

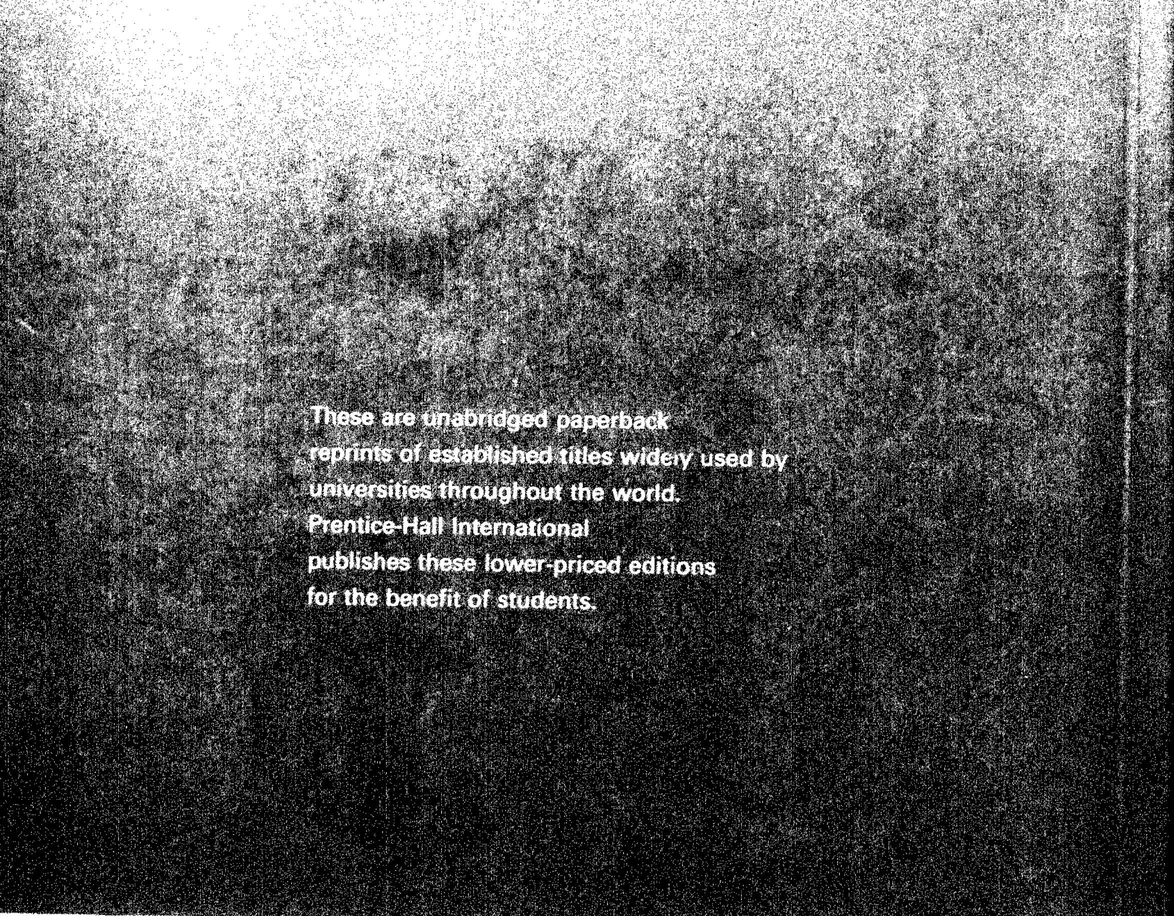
$\omega_3 = 8.828 \sqrt{\dfrac{EI}{mL^3}}$ rad/s

**14-5.** $\{A\}_1 = \begin{Bmatrix} 0.445 \\ 0.802 \\ 1 \end{Bmatrix}$

$\{A\}_2 = \begin{Bmatrix} -1.247 \\ -0.555 \\ 1 \end{Bmatrix}$

$\{A\}_3 = \begin{Bmatrix} 1.802 \\ -2.247 \\ 1 \end{Bmatrix}$

# Index



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