ENGINEERING MECHANICS

Volume 2

Stresses, Strains, Displacements

C. Hartsuijker and J.W. Welleman



Springer

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Volume 2: Stresses, Strains, Displacements

by

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Springer

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-1-4020-4123-5 (HB)
ISBN 978-1-4020-5763-2 (e-book)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

www.springer.com

This is a translation of the original Dutch work "Toegepaste Mechanica, Deel 2: Spanningen, Vervormingen, Verplaatsingen", 2001,
Academic Service, The Hague, The Netherlands.

Printed on acid-free paper

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Preface

This Volume is the second of a series of two:

- Volume 1: Equilibrium
- Volume 2: Stresses, deformations and displacements

These volumes introduce the fundamentals of structural and continuum mechanics in a comprehensive and consistent way. All theoretical developments are presented in text and by means of an extensive set of figures. Numerous examples support the theory and make the link to engineering practice. Combined with the problems in each chapter, students are given ample opportunities to exercise.

The book consists of distinct modules, each divided into sections which are conveniently sized to be used as lectures. Both formal and intuitive (engineering) arguments are used in parallel to derive the important principles. The necessary mathematics is kept to a minimum however in some parts basic knowledge of solving differential equations is required.

The modular content of the book shows a clear order of topics concerning stresses and deformations in structures subject to bending and extension. Chapter 1 deals with the fundamentals of material behaviour and the introduction of basic material and deformation quantities. In Chapter 2 the fibre model is introduced to describe the behaviour of line elements subject to extension (tensile or compressive axial forces). A formal approach is followed in which the three basic relationships (the kinematic, constitutive and static relationships) are used to describe the displacement field with a second order differential equation. Numerous examples show the influence of the boundary conditions and loading conditions on the solution of the displacement field. In Chapter 3 the cross-sectional quantities such as centre of mass or centre of gravity, centroid, normal (force) centre, first moments of area or static moments, and second moments of area or moments of inertia are introduced as well as the polar moment of inertia. The influence of the translation of the coordinate system on these quantities is also investigated, resulting in the parallel axis theorem or *Steiner's rule* for the static moments and moments of inertia. With the definitions of Chapters 1 to 3 the complete theory for bending and extension is combined in Chapter 4 which describes the fibre model subject to extension and bending (Euler–Bernoulli theory). The same framework is used as in Chapter 2 by defining the kinematic, constitutive and static relationships, in order to obtain the set of differential equations to describe the combined behaviour of extension and bending. By choosing a specific location of the coordinate system through the normal (force) centre, we introduce the uncoupled description of extension and bending. The strain and stress distribution in a cross-section are introduced and engineering expressions are resolved for cross-sections with at least one axis of symmetry. In this chapter also some special topics are covered like the core of a cross-section, and the influence of temperature effects.

For non-constant bending moment distributions, beams have to transfer shear forces which will lead to shear stresses in longitudinal and transversal section planes. Based on the equilibrium conditions only, expressions for the shear flow and the shear stresses will be derived. Field of applications are (glued or dowelled) interfaces between different materials in a composite cross-section and the stresses in welds. Special attention is also given to thin-walled sections and the definition of the shear (force) centre for thin-walled sections. This chapter focuses on homogeneous cross-sections with at least one axis of symmetry. Shear deformation is not considered.

Chapter 6 deals with torsion, which is treated according to the same concept as in the previous chapters; linear elasticity is assumed. The elementary theory is used on thin-walled tubular sections. Apart from the deformations also shear stress distributions are obtained. Special cases like solid circular sections and open thin-walled sections are also treated.

Structural behaviour due to extension and or bending is treated in Chapters 7 and 8. Based on the elementary behaviour described in Chapters 2 and 4 the structural behaviour of trusses is treated in Chapter 7 and of beams in Chapter 8. The deformation of trusses is treated both in a formal (analytical) way and in a practical (graphical) way with aid of a relative displacement graph or so-called Williot diagram. The deflection theory for beams is elaborated in Chapter 8 by solving the differential equations and the introduction of (practical) engineering methods to obtain the displacements and deformations based on the moment distribution. With these engineering formulae, forget-me-nots and moment-area theorems, numerous examples are treated. Some special cases like temperature effects are also treated in this chapter.

Chapter 9 shows a comprehensive description of the fibre model on unsymmetrical and or inhomogeneous cross-sections. Much of the earlier presented derivations are now covered by a complete description using a two letter symbol approach. This formal approach is quite unique and offers a fast and clear method to obtain the strain and stress distribution in arbitrary cross-sections by using an initially given coordinate system with its origin located at the normal centre of the cross-section. Although a complete description in the principal coordinate system is also presented, it will become clear that a description in the initial coordinate system is to be preferred. Centres of force and core are also treated in this comprehensive theory, as well as the full description of the shear flow in an arbitrary cross-section. The last part of this chapter shows the application of this theory on numerous examples of both inhomogeneous and unsymmetrical cross-sections. Special attention is also given to thin-walled sections as well as the shear (force) centre of unsymmetrical thin-walled sections which is of particular interest in steel structures design.

This latter chapter is not necessarily regarded as part of a first introduction into stresses and deformations but would be more suitable for a second or third course in Engineering Mechanics. However, since this chapter offers the complete and comprehensive description of the theory, it is an essential part of this volume.

We do realise, however, that finding the right balance between abstract fundamentals and practical applications is the prerogative of the lecturer. He or she should therefore decide on the focus and selection of the topics treated in this volume to suit the goals of the course in question.

The authors want to thank especially the reviewer Professor Graham M.L. Gladwell from the University of Waterloo (Canada) for his tedious job to improve the Dutch-English styled manuscript into readable English. We also thank Jolanda Karada for her excellent job in putting it all together and our publisher Nathalie Jacobs who showed enormous enthusiasm and patience to see this series of books completed and to have them published by Springer.

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Delft, The Netherlands
July 2007

Foreword

Structural or Engineering Mechanics is one of the core courses for new students in engineering studies. At Delft University of Technology a joint educational program for Statics and Strength of Materials has been developed by the Koiter Institute, and has subsequently been incorporated in the curricula of faculties like Civil Engineering, Aeronautical Engineering, Architectural Engineering, Mechanical Engineering, Maritime Engineering and Industrial Design.

In order for foreign students also to be able to benefit from this program an English version of the Dutch textbook series written by Coenraad Hartsuijker, which were already used in most faculties, appeared to be necessary. It is fortunate that in good cooperation between the writers, Springer and the Koiter Institute Delft, an English version of two text books could be realized, and it is believed that this series of books will greatly help the student to find his or her way into Engineering or Structural Mechanics.

Indeed, the volumes of this series offer some advantages not found elsewhere, at least not to this extent. Both formal and intuitive approaches are used, which is more important than ever. The books are modular and can also be used for self-study. Therefore, they can be used in a flexible manner and will fit almost any educational system. And finally, the SI system is used consistently. For these reasons it is believed that the books form a very valuable addition to the literature.

René de Borst
Scientific Director, Koiter Institute Delft

1 Material Behaviour

To calculate the stresses and deformations in structures, we have to know the *material behaviour*, which can be obtained only by experiments.

Through standardised tests, the material properties are laid down in a number of specific quantities. One of these tests is the *tensile test*, described in Section 1.1, resulting in a so-called *stress-strain diagram*.

Section 1.2 looks at stress-strain diagrams for a number of materials.

This book addresses mainly materials with a linear-elastic behaviour, which obey *Hooke's Law*. Section 1.3 devotes attention to the linear behaviour of materials, such as steel, aluminium, concrete and wood.

1.1 Tensile test

Strength, stiffness and ductility are important material properties and can be described as follows:

- *strength* – the resistance that has to be overcome to break the cohesion of the material;
- *stiffness* – the resistance against deformation;
- *ductility* – the capacity to undergo large strains before fracture occurs.

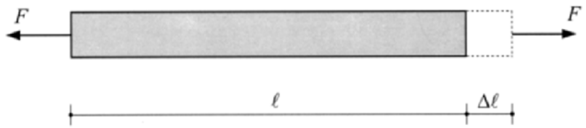

Figure 1.1 Prismatic bar subject to tension.

Figure 1.2 Test bar.

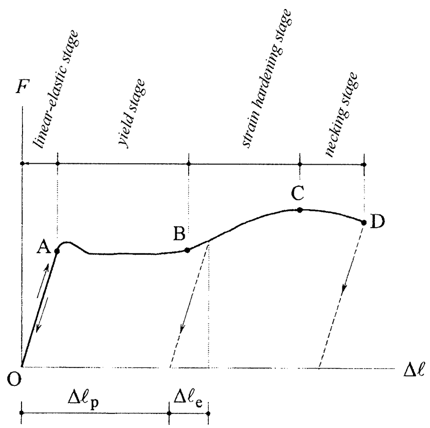

Figure 1.3 Force-elongation diagram (F-$\Delta\ell$ diagram).

An important and often-used test for determining the *strength*, *stiffness* and *ductility* of a material is the tensile test. In tensile tests, a specimen of the material in the form of a bar is placed in a so-called *tensile testing machine*. The test bar is slowly stretched until fracture occurs. For each applied elongation $\Delta\ell$ the required strength F[1] is measured, and both values are plotted in a so-called *F-$\Delta\ell$ diagram* or *force-elongation diagram*.

Figure 1.1 shows a prismatic bar; prismatic means that the bar has a uniform cross-section. To prevent fracture of the bar near the ends a test bar is shaped at the ends as in Figure 1.2. In that case, $\Delta\ell$ is the elongation of the distance ℓ between two measuring points on the prismatic part of the bar.

Figure 1.3 shows the *force-elongation diagram* or F-$\Delta\ell$ diagram (not to scale) for hot rolled steel (mild steel) in tension.

There are four stages in this F-$\Delta\ell$ diagram.

- *Linear-elastic stage* – path OA
  This part of the diagram is practically straight. Up to point A there seems to be a proportionality (*linear relationship*) between the force F and elongation $\Delta\ell$. If the load in A is removed, the same path is followed in opposite direction until point O is again reached. In other words, if the force is removed, the bar springs back to its original length. This type of behaviour is known as *elastic*.
- *Yield stage or plastic stage* – path AB
  Path AB of the diagram generally includes a number of "bumps" but is otherwise virtually horizontal. This means that the elongation of the

[1] If a change in length is applied and the required force is measured, the test is referred to as being *deformation-driven*. If, however, a load is applied and the associated change in length is measured, the test is said to be *load-driven*. In general, deformation-driven tests and load-driven tests give different results.

bar increases with a nearly constant load. This phenomenon is known as *yielding* or *plastic flow* of the material.

- *Strain hardening stage* – path BC
  When the deformation becomes larger, the material may offer additional resistance. The required force to obtain the elongation increases. This is called *strain hardening*.
- *Necking stage* – path CD
  Beyond point C, the load decreases with increasing elongation. Locally the bar produces a pronounced *necking* (see Figure 1.4) that increases until fracture occurs at D. At fracture the load falls away and both parts of the bar spring back a little elastically.

Figure 1.4 Local necking of a test bar.

If somewhere between point A (the *limit of proportionality*) and point D at which *fracture* occurs) the load is removed, the test bar reverts a little elastically. The return path (unloading path) is a nearly straight line parallel to OA. In Figure 1.3 this is shown by means of the dashed line. Once the load has been released to zero the bar demonstrates a *permanent set* or *plastic elongation* $\Delta\ell_p$; the *elastic elongation* was $\Delta\ell_e$.

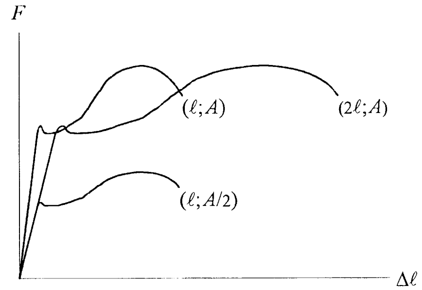

Figure 1.5 F-$\Delta\ell$ diagrams for bars with various dimensions.

The F-$\Delta\ell$ diagram depends not only on the material, but also on the dimensions of the test bar, namely the length ℓ between the measuring points on the prismatic part of the bar, and the area A of the cross-section. Figure 1.5 shows the F-$\Delta\ell$ diagrams for three bars made of the same material but with different dimensions.

If the (prismatic) bar is chosen twice as long without changing the load F, then the elongation is twice as large. We can see this by looking at the behaviour of the two bars in Figure 1.6, attached one behind the other. The total elongation is the sum of the elongations of each of the bars. The elongation $\Delta\ell$ is therefore proportional to the length ℓ of the bar.

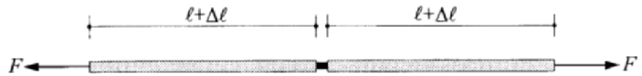

Figure 1.6 Due to the same force F, a member that is twice as long has an elongation that is twice as large.

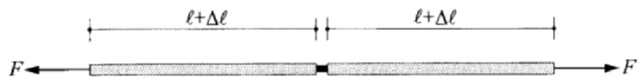

Figure 1.6 Due to the same force F, a member that is twice as long has an elongation that is twice as large.

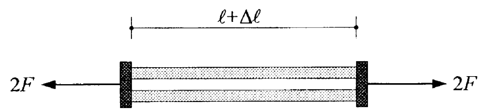

Figure 1.7 For a cross-section that is twice as large, the required force to get the same elongation $\Delta\ell$ is twice as large.

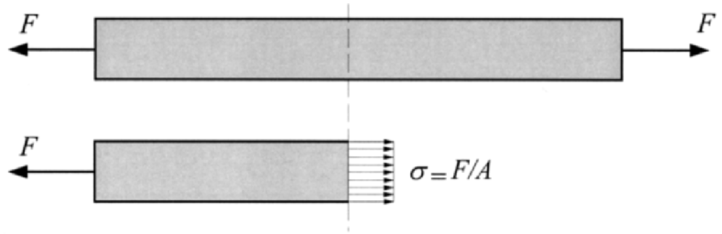

Figure 1.8 The normal stress $\sigma = N/A$ due to extension is constant in a homogeneous cross-section.

To eliminate the influence of the length of the test bar, we plot $\varepsilon = \Delta\ell/\ell$ on the horizontal axis instead of $\Delta\ell$. The (dimensionless) deformation quantity

$$\varepsilon = \frac{\Delta\ell}{\ell} = \frac{\text{elongation}}{\text{original length}}$$

is referred to as the *strain* of the bar.

If the cross-section A of the bar is chosen twice as large, a doubled load F is required to get the same elongation $\Delta\ell$. Refer to the behaviour of the two parallel bars in Figure 1.7. To get an elongation $\Delta\ell$ each bar has to be subjected to a normal force F, and the total load on the system of two bars is $2F$. Therefore the force F is proportional to the area A of the cross-section of the bar.

To eliminate the influence of the area of the cross-section, we plot the quantity

$$\sigma = \frac{F}{A}$$

along the vertical axis instead of F; σ is the *normal stress* in the cross-section.

In general, the normal stress varies across the cross-section and $\sigma = F/A$ should be seen as the "*average*" normal stress in the cross-section. If the cross-section is homogeneous (the cross-section consists of the same material everywhere) and the cross-section in question is far enough away from the ends of the bar where the loads are applied (these are *disruption zones*), then the normal stress due to the tensile force is roughly constant over the cross-section (see Figure 1.8).

By converting the force-elongation diagram (F-$\Delta\ell$ diagram) into a *stress-strain diagram* (σ-ε diagram) we eliminate the influence of the bar dimensions on the result of the tension test. So test bars of various dimensions lead to almost the same σ-ε diagrams.[1]

The values found by experiments are of course subject to dispersion. In addition, they depend on the way the experiments are performed, such as the speed at which the load is increased. For all materials the test results are influenced by temperature, and for wood and concrete, for example, humidity also plays a role.

1.2 Stress-strain diagrams

Figure 1.9 shows a σ-ε diagram with a distinct yield range. The specific quantities by which the shape of the stress-strain diagram is more or less determined are

- f_y – the yield point;[2]
- f_t – the tensile strength;[3]
- ε_y – the yield strain, that is the strain at the start of the yield stage;
- ε_{pl} – the strain at the end of the yield stage;

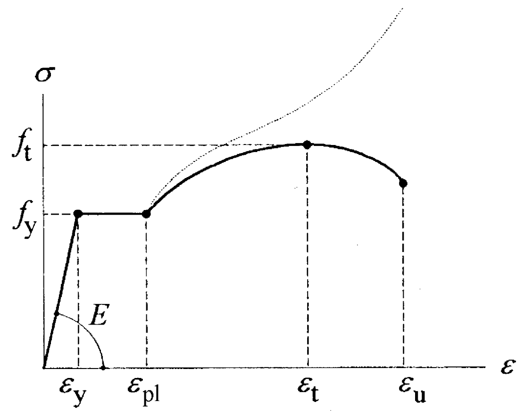

Figure 1.9 σ-ε diagram with a distinct yield region.

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1 Due to the local character of necking, the strain at fracture may differ per test bar.

2 Also referred to as *yield stress* or *yield strength*. Strength quantities in the σ-ε diagram are indicated by the kernel symbol f instead of σ.

3 Also referred to as *ultimate* (*tensile*) *stress*. To calculate the stress σ, the force may be divided by the original area A of the cross-section, or by the actual cross-section A' which will have decreased from A through transverse contradiction, and necking. Since A' is less than A, the 'true' stress F/A' is larger than the 'nominal' stress F/A. In building practice, attention is restricted to the nominal stress.

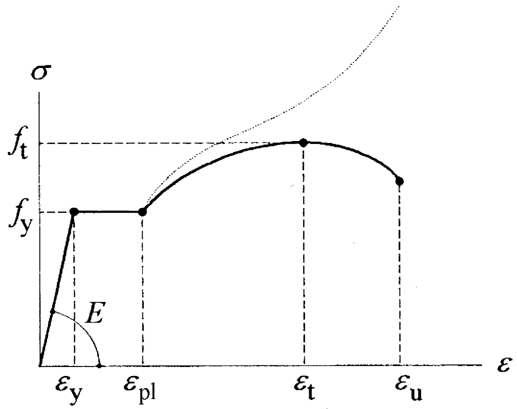

Figure 1.9 σ-ε diagram with a distinct yield region.

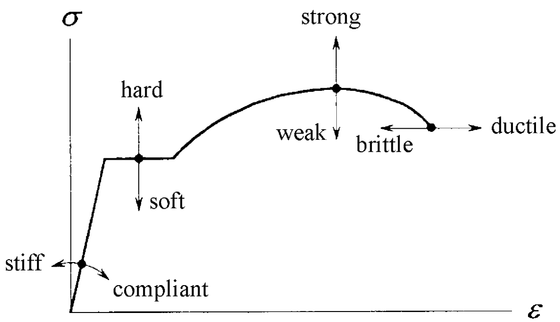

Figure 1.10 Material properties.

ε_t – the strain associated with the tensile strength f_t;
ε_u – the strain at fracture.

In the elastic range, there is a linear relationship between the stress σ and strain ε:

$$\sigma = E\varepsilon.$$

The proportionality factor E is a *material constant* and is known as the *modulus of elasticity* or *Young's modulus*. The modulus of elasticity characterises the resistance (*stiffness*) of the material with respect to *deformations due to a change in length*. In the σ-ε diagram, the modulus of elasticity is equal to the slope $E = \sigma/\varepsilon$ of the path in the linear-elastic stage.

Since the strain ε is dimensionless, the modulus of elasticity E has the dimension of a stress (force/area).

In Figure 1.10, the concepts stiffness, strength, etc., are shown in the σ-ε diagram.

- a *stiff* material has a larger modulus of elasticity E than a *compliant* material;
- a *hard* material has a larger yield point f_y than a *soft* material;
- a *strong* material has a higher tensile strength f_t than a *weak* material;
- a *ductile* material has a larger strain ε_u at fracture than a *brittle* material.

Ductile materials include most metals, such as steel, aluminium, etc. For metals, the σ-ε diagrams for tension and compression are generally equal, so the compressive strength[1] f'_c is equal to the tensile strength f_t.

Materials in which fracture occurs with minor strain are known as brittle materials. Examples include concrete, stone, cast iron, glass. With stone-like materials, the diagrams for tension and compression generally differ and the compressive strength is generally larger than the tensile strength (see Figure 1.11).

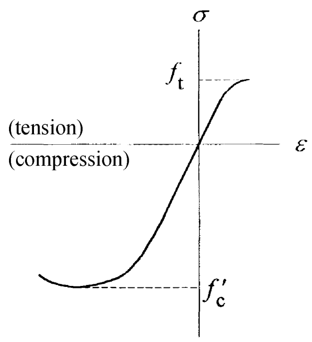

Figure 1.11 σ-ε diagram for a brittle material.

The σ-ε diagrams for a number of materials are shown below.

• *Steel*
Figure 1.12 shows the σ-ε diagram for steel Fe 360. The diagram is not drawn to scale. For the tensile strength f_t and the yield point f_y we use

$$f_t = 360\ \text{N/mm}^2 \quad \text{and} \quad f_y = 235\ \text{N/mm}^2.$$

The modulus of elasticity is

$$E = 210\ \text{GPa}.$$

The stress at which yielding starts is easy to determine:

$$\varepsilon_y = \frac{\sigma_y}{E} = \frac{235\ \text{N/mm}^2}{210 \times 10^3\ \text{N/mm}^2} = 0.00112.$$

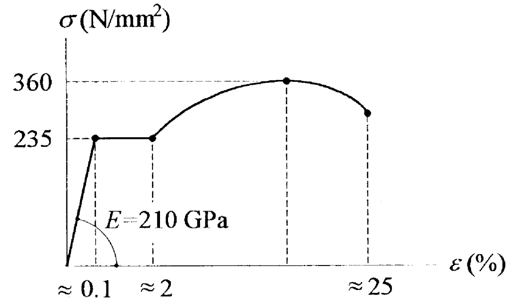

Figure 1.12 σ-ε diagram for steel Fe 360.

[1] In mechanics, it is the convention to call normal stresses positive if they are tensile. If one is primarily dealing with compressive stresses, it may be convenient to call compressive stresses positive. In that case, the prime is used for the change in sign. See also Volume 1, Section 6.5.

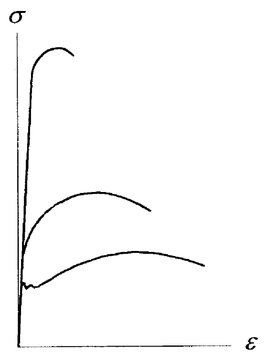

Figure 1.13 σ-ε diagrams for various grades of steel.

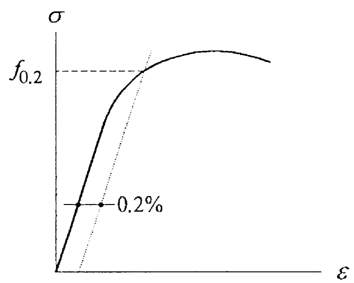

Figure 1.14 The 0.2% offset yield strength $f_{0.2}$.

If the yield strain, a dimensionless quantity, is expressed in percentages, then

$$\varepsilon_y = 0.112\%.$$

The strain hardening starts at the strain ε_{pl} that is some 20 times as large as ε_y:

$$\varepsilon_{pl} \approx 2\%.$$

The strain at fracture ε_u is another 10 to 15 times larger:

$$\varepsilon_u \approx 25\%.$$

Structural steel Fe 360 is a ductile material with a distinct yield point and a relatively low tensile strength. Adding small amounts of alloying elements during steel preparation or the *cold working*[1] of steel results in grades of steel with considerably higher tensile strengths. Figure 1.13 shows (more or less to scale) σ-ε diagrams for different grades of steel. It is clear that:

- All grades of steel have more or less the same modulus of elasticity E.
- The higher the tensile strength f_t, the smaller the fracture strain ε_u. In other words, the ductility decreases with increasing strength.
- For grades of steel with a high tensile strength (Fe 600 and above) there is a gradual transition from the linear-elastic stage to the strain-hardening stage. There is no yield stage.

For steel without yield stage, the yield point f_y is chosen by the so-called *offset* method. This is illustrated in Figure 1.14, where a line offset an (ar-

[1] That is drawing and rolling the steel to its finished dimensions at room temperature.

bitrary) amount of 0.2% is drawn parallel to the initial σ-ε diagram. The yield point f_y is now replaced by the 0.2% *offset yield strength*, indicated as $f_{0.2}$:

$$f_y = f_{0.2}.$$

• *Aluminium*
Aluminium is a ductile material: it can undergo large deformations before fracture occurs. But there is no clear yield point (see Figure 1.15). Also here the 0.2% *offset yield strength* is used.

The modulus of elasticity is:

$$E = 70 \text{ GPa}.$$

The modulus of elasticity of aluminium is about a third that of steel. Aluminium is therefore approximately three times as compliant as steel. This means that in the elastic stage the deformations of an aluminium structure are about three times as large as the deformations of the same structure constructed in steel.

As with steel, the properties of aluminium depend strongly on the alloying elements, the method of preparation and the after-treatment.

• *Rubber*
For rubber, there is a linear-elastic relationship between stress and strain up to very high strains (10 to 20%). Beyond the linear-elastic stage, the properties depend on the type of rubber (see Figure 1.16). In the non-linear area, rubber may still behave elastically for a long time. In that case the same path for loading and unloading is followed in the σ-ε diagram (*non-linear elasticity*). Some soft types of rubber are capable of huge elongations. The strain at fracture may be 800%. Just before fracture, there is generally a

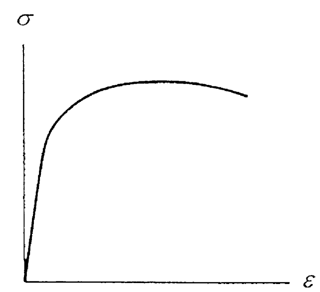

Figure 1.15 Example of a σ-ε diagram for aluminium.

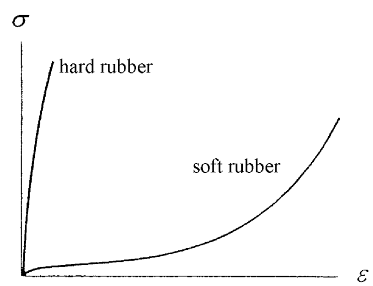

Figure 1.16 Two σ-ε diagrams for rubber.

Figure 1.17 σ-ε diagram for glass.

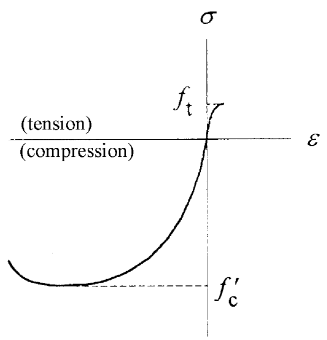

Figure 1.18 Example of a σ-ε diagram for concrete.

clear increase in stiffness. This behaviour can be verified by stretching a common elastic band.

• *Glass*
Glass behaves linearly until it breaks (see Figure 1.17). Glass is an ideal brittle material. The modulus of elasticity and tensile strength depend on the type of glass. The tensile strength of glass fibres may be up to 100 times as large as that of plate glass.

• *Concrete*
Concrete is a stone-like material with a small tensile strength and a large compressive strength (see Figure 1.18). For strength calculation, one uses extensively idealised diagrams. For deformation calculation, a linear-elastic material behaviour is assumed with a modulus of elasticity in which all time-dependent effects have been taken into account. Concrete is some 6 to 8 times as compliant as steel.

• *Wood*
Wood is an *anisotropic material*: due to its fibre structure, the material properties are not the same in all directions.[1] The σ-ε diagram for wood is therefore less explicit. It depends on many factors: in addition to the direction of the fibre there is the humidity and speed of loading. Moreover, the behaviour under tension and compression differ. With respect to tension, the behaviour of wood is brittle and fracture occurs suddenly. When subject to compression wood seems relatively ductile; the fibres fold but continue to offer resistance.

---

1 In *isotropic* materials the material properties are the same in all directions. In *anisotropic* materials the material properties depend on the direction.

1.3 Hooke's Law

For materials with a sufficiently long yield stage (ductile materials such as steel Fe 360), the σ-ε diagram is often simplified to that in Figure 1.19 for an *elastic-plastic* material.

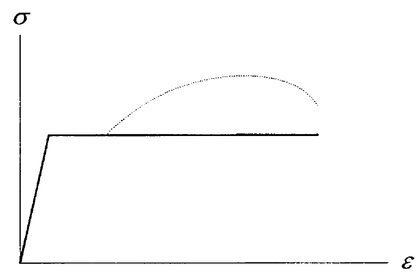

Figure 1.19 σ-ε diagram for an elastic-plastic material.

In building practice, we are mainly interested in the situation in which a structure or part of the structure reaches a so-called limit state.[1] Here, we distinguish between *ultimate limit states* and *serviceability limit states*:

- *Ultimate limit states* are states at which the structure or part of it collapses. This may be due to a *loss of equilibrium* (e.g. through turning over, sliding, floating or instability) or to a *loss of carrying capacity* (because the structure is not strong enough in one or more of its parts to transfer the forces to which they are subjected).
- *Serviceability limit states* are states in which the structure or part of it no longer functions appropriately (e.g. due to excessive deformations, vibrations, cracking, etc.), often long before the structure collapses.

When in an *ultimate limit state* the structure collapses because one or more structural parts are no longer strong enough to transfer the forces, the material will be loaded to its ultimate in these parts, and ductile materials will be loaded far into the plastic region. The associated ultimate load (*yield load*) for ductile materials is determined by the *theory of plasticity*.[2] Since the linear-elastic stage is of minor importance here, the σ-ε diagram is often simplified to that of a *rigid-plastic material* (see Figure 1.20).

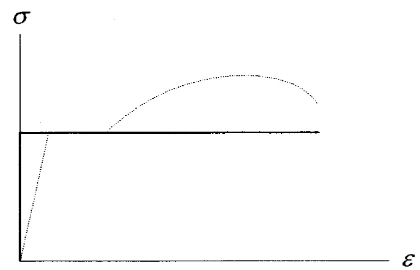

Figure 1.20 σ-ε diagram for a rigid-plastic material.

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1 See also Volume 1, Section 6.2.4.

2 Also referred to as the *theory of plastic design*, *ultimate-load design* or *limit design*.

In a *serviceability limit state* the deformations are generally so small that they are on the linear-elastic path of the σ-ε diagram, sufficiently far from the yield point. Calculations relating to the *serviceability limit states* are therefore performed according to the *linear theory of elasticity*, based on the proportionality between stress σ and strain ε:

$$\sigma = E\varepsilon.$$

The proportionality between stress and strain was found by Robert Hooke (1635–1703) and is known as *Hooke's Law*. Hooke formulated the law as “ut tensio sic vis” (as is the tension so is the force), and published it in 1678 as the anagram “ceiiinosssttuv”.

$\sigma = E\varepsilon$ is Hooke's law in its simplest form.[1]

Note that the use of the word “law” can be somewhat misleading. The character of this law is somewhat different to those of other generally applicable laws such as those of Newton. Hooke's law is no more than a good representation of certain results found by experiments. The approach is very good for the elastic stage in metals.

For wooden beams subject to moderate forces the approach is reasonable; time-dependent influences are corrected by a creep factor.

For concrete, the approximation is not so good. Under compression, the relationship between stress and strain is barely linear. Time-dependent influences (shrinkage and creep) are other complicating factors. However,

[1] In Chapter 6, where the shear stresses due to torsion is covered, Hooke's law appears in an entirely different guise. Hooke's law is covered from a general perspective in Volume 4.

in serviceability limit states, a linear-elastic material behaviour is also assumed for concrete. The time-dependent effects are taken into account in the modulus of elasticity.

Figure 1.21 shows the first part of the linear-elastic stage for different materials in one σ-ε diagram. The slope of each path represents the modulus elasticity $E = \sigma/\varepsilon$, the material property that characterises the *stiffness* of the material against deformation through a change in length. The figure provides an idea of the differences in stiffness between the various materials in the elastic stage.

In the next chapters it is assumed that the stresses and strains remain within the linear-elastic stage and follow Hooke's law.

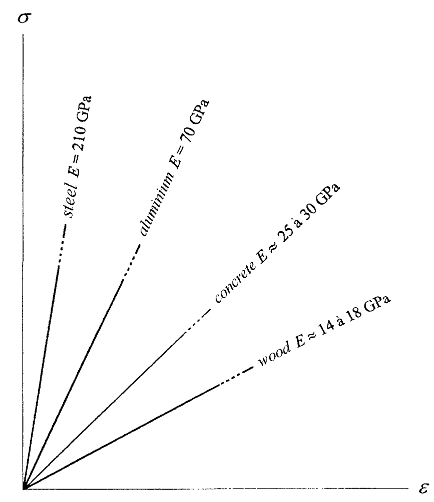

Figure 1.21 The linear-elastic paths for various materials in one σ-ε diagram.

2

Bar Subject to Extension

A bar is a body of which the two cross-sectional dimensions are considerably smaller than the third dimension, the length. A bar is one of the most frequently used types of structural members. To understand something about the behaviour of bar type structures, it is first necessary to understand the behaviour of a single bar.

This chapter addresses the case of a bar subject to extension. We talk of *extension* when the (straight) bar remains straight after deformation and does not bend.[1]

Section 2.1 addresses the assumptions that are the basis of the *fibre model*, a physical model with which it is easier to imagine the behaviour of a bar. It is also assumed that the cross-section of the bar is *homogeneous* and that the material behaves *linear elastically*.

Three basic relationships can be distinguished when describing the behaviour of a bar, namely the *kinematic relationships*, the *constitutive relationships* and the *static relationships* or *equilibrium relationships*. They are derived for extension in Section 2.2.

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[1] Chapter 4 addresses combined bending and extension.

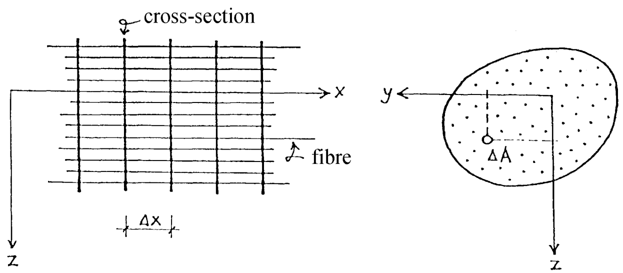

Figure 2.1 The fibre model for a bar. The behaviour of the model is described in a xyz coordinate system with the x axis parallel to the fibres and the yz plane perpendicular to the fibres and parallel to the cross-sections.

Section 2.3 addresses the strain and stress distribution in a cross-section due to extension.

The point of application of the resultant N of all normal stresses in the cross-section, due to extension, is known as the *normal force centre* or *normal centre* NC. Section 2.4 addresses the location of the normal centre, which plays an important role for bending with extension.[1]

A mathematical description of the extension problem is given in Section 2.5, where the three basic relationships from Section 2.2 are put together to give the *differential equation for extension*.

Next follows a number of examples: calculating the changes in length and displacements in Section 2.6 and working with the differential equation in Section 2.7.

In Section 2.8 some remarks are made on the difference that may be noticed between the formal approach used in this book and engineering practice. This chapter ends with a number of problems in Section 2.9.

2.1 The fibre model

In order to imagine the behaviour of a bar, we create a *physical model*. A condition is that the results of the model have to give a sufficiently accurate picture of reality. It is always the experiment that must confirm the correctness of the chosen model and the associated assumptions.

A model that seems to function effectively is the so-called *fibre model* (see Figure 2.1). This model is based on the following assumptions:

---

[1] See Chapter 4.

- Inspired by the structure of wood, the member is considered to consist of a very large number of parallel *fibres* in the longitudinal direction. Later we will look at the limiting case in which the number of fibres is so large that the area ΔA of a single fibre approaches zero.
- The fibres are kept together by a very large number of absolutely rigid planes perpendicular to the direction of the fibres. These rigid planes are known as *cross-sections*. Later we will look at the limiting case in which the number of cross-sections is so large that the distance Δx between two consecutive cross-sections approaches zero.
- The plane *cross-sections remain plane and normal to the longitudinal fibres* of the beam, even after deformation. This assumption is known as *Bernoulli's hypothesis*.[1]

To describe the behaviour of the model, we use an xyz coordinate system with the x axis parallel to the fibres and the yz plane parallel to the cross-sections, perpendicular to the direction of the fibres.

The location of a cross-section is defined by its x coordinate; the location of a fibre is defined by its y and z coordinates.

Later we will see that the behaviour of the bar is most easily described when the x axis is selected along a particular preferred fibre through the *normal centre* NC. This fibre is known as the *bar axis*. As long as the location of the normal centre and bar axis are not yet known, the x axis is defined along an arbitrary fibre that may even lie outside the cross-section.

The following assumptions are made with respect to the *material behaviour*:

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[1] Named after the Swiss Jacob Bernoulli (1654–1705), from a famous family of mathematicians and physicists.

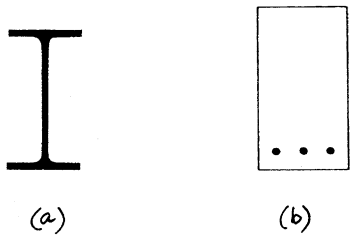


Figure 2.2 The cross-section is (a) homogeneous for steel beam, and (b) inhomogeneous for a reinforced concrete beam.

- All the fibres consist of the same material and therefore have the same material properties. In this case, the cross-section of the bar is said to be *homogeneous*.
- The material behaves *linear-elastically* and follows *Hooke's law*, with a linear relationship between stress σ and strain ε:

$$\sigma = E\varepsilon.$$

Note that in a homogeneous cross-section all the fibres have the same modulus of elasticity E (see Figure 2.2a).

If the fibres do not all have the same modulus of elasticity, because they consist of different materials, the cross-section is said to be *inhomogeneous*.[1] In this way, a reinforced concrete beam has an inhomogeneous cross-section, because the "concrete fibres" and "steel fibres" have different moduli of elasticity (see Figure 2.2b).

2.2 The three basic relationships

When investigating the behaviour of a bar, we distinguish three different basic relationships:
- Static or equilibrium relationships.
- Constitutive relationships.
- Kinematic relationships.

Static or equilibrium relationships
The static relationships link the load (due to external forces) and the section forces. They follow from the equilibrium.

---

[1] Inhomogeneous cross-sections are covered in Chapter 9.

Constitutive relationships
The constitutive relationships link the section forces and the associated deformations. They follow from the behaviour of the material (linear-elastic in this case).

Kinematic relationships
The kinematic relationships link the deformations and the displacements. They are the result of a permanent cohesion within the bar – holes do not suddenly appear. The kinematic relationships are independent of the material behaviour.

The three basic relationships allow us to link the load (due to external forces) and the associated displacements. In Figure 2.3 this is schematically shown for a bar subject to extension.

Below the three basic relationships are discussed in a reversed order.

2.2.1 The kinematic relationship

In this section we look for the relationship between the deformation and displacement for a bar subject to extension.

In Section 1.1, the strain ε was introduced as a deformation quantity. For the bar in a tensile test it was defined as

$$\varepsilon = \frac{\Delta \ell}{\ell} = \frac{\text{elongation}}{\text{original length}} .$$

Below, this definition is used also for the strain of the individual fibres.

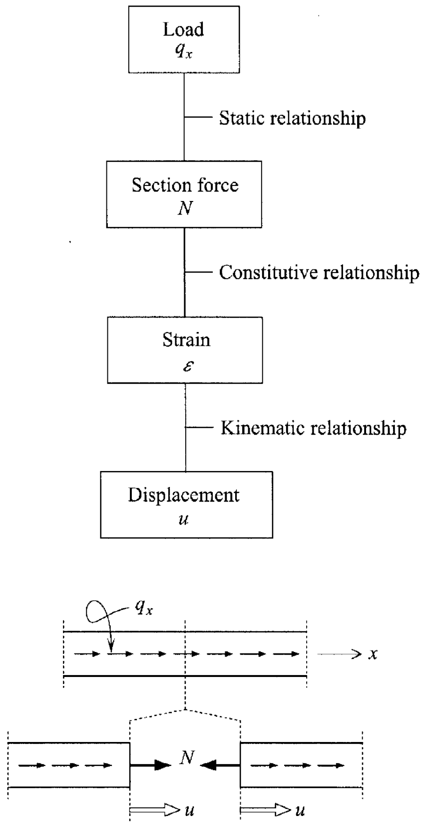


Figure 2.3 Schematic representation of the link between load and displacement for a bar subject to extension. To obtain this link, we have to use all three basic relationships: kinematic, constitutive, and static. The deformation quantity ε represents the *strain* of the bar.

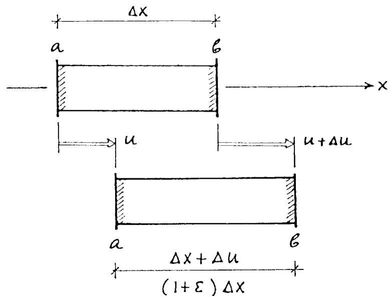


Figure 2.4 A small bar segment with length Δx, before and after the deformation by extension.

Figure 2.4 shows a small segment of a bar subject to extension. The segment has a length Δx, and is bounded by the end-sections a and b.

If the bar changes length due to tension or compression, the cross-sections will move with respect to one another.[1] Assume end-section a moves in the x direction by a distance u and end-section b moves by a distance $u + \Delta u$.

All longitudinal fibres between the end-sections a and b have the same original length "ℓ". This length is equal to the distance Δx between both end-sections. The elongation "$\Delta\ell$" of the fibres is equal to the difference in displacement Δu between the end-sections b and a.

For extension, all fibres undergo the same strain ε:

$$\varepsilon = \frac{\text{“}\Delta\ell\text{”}}{\text{“}\ell\text{”}} = \frac{\text{elongation}}{\text{original length}} = \frac{\Delta u}{\Delta x}.$$

The limit of $\Delta u/\Delta x$ as Δx tends to zero is known as the derivative of u with respect to x:

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\mathrm{d}u}{\mathrm{d}x}\,.$$

The strain of the fibres is therefore

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}\,.$$

This is the *kinematic relationship* for extension; it provides a link between

---

[1] Remember that the bar will not bend (curve) if there is no bending, but only extension.

the deformation quantity ε (the strain of the fibres in the bar) and the displacement u (of a cross-section in x direction).

For a bar segment the change in length "$\Delta\ell$" is equal to the difference in displacement between the end-sections:

$$\text{"}\Delta\ell\text{"} = \Delta u = \varepsilon\Delta x.$$

The total change in length of the bar is found by summing all contributions $\varepsilon\Delta x$ of the individual segments over the entire length of the bar:

$$\Delta\ell = \int_{\ell} \varepsilon \, \mathrm{d}x.$$

This relationship is the basis for the formulae for calculating the change in length of a bar. Examples are given in Section 2.6.

2.2.2 The constitutive relationship

This section looks at the relationship between deformation and section force for a bar subject to extension. This relationship is dependent on the behaviour of the material, i.e. the modulus of elasticity E.

The resultant of the normal stress σ in fibre (y, z) with area ΔA is a small force ΔN (see Figure 2.5):

$$\Delta N = \sigma\Delta A.$$

In a linear-elastic material, the fibres follow Hooke's law:

$$\sigma = E\varepsilon$$

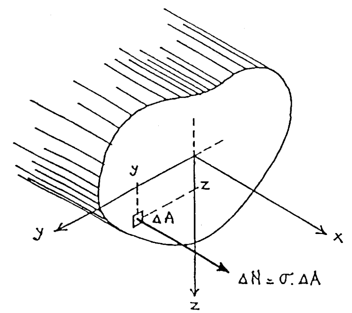


Figure 2.5 The resultant of the normal stress σ in fibre (y, z) with area ΔA is a small force $\Delta N = \sigma\Delta A$.

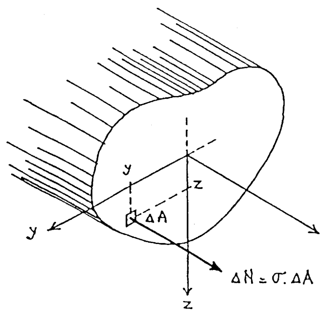


Figure 2.5 The resultant of the normal stress σ in fibre (y, z) with area ΔA is a small force $\Delta N = \sigma \Delta A$.

so that

$$\Delta N = \sigma \Delta A = E\varepsilon \Delta A.$$

The total normal force N is found by summing the contributions of all the fibres or, in other words, integrating all the forces ΔN with respect to the cross-section A:

$$N = \int_A \sigma \, \mathrm{d}A = \int_A E\varepsilon \, \mathrm{d}A.$$

For extension, all the fibres undergo the same elongation, and ε can be placed outside the integral. If the cross-section is homogeneous, all fibres have the same modulus of elasticity and E can also be placed outside the integral. Hence

$$N = E\varepsilon \int_A \mathrm{d}A$$

or

$$N = EA\varepsilon.$$

This is the *constitutive relationship* for extension. It links the normal force N (a section force) and the strain ε (a deformation quantity). The constitutive relationship depends on the behaviour (constitution) of the material as it includes the modulus of elasticity E.

EA is known as the *axial stiffness* of the bar. The axial stiffness is a measure of the resistance of the bar to axial deformation, and depends on both the modulus of elasticity E of the material and the area A of the cross-section.

2.2.3 The static relationship

Static or equilibrium relationships link load and section forces. They follow from the equilibrium of a small member segment, and were derived in Volume 1, Section 11.1. There, we found that extension (only normal forces) and bending (bending moments and shear forces) can be treated separately.

We recapitulate the derivation of the static relationship for extension.

In Figure 2.6, a small segment with length Δx has been isolated from a bar. The bar segment is subject to the distributed loads q_x and q_z. The loads act on the bar axis (for clarity this is not drawn as such for q_z). When the length Δx of the bar segment is sufficiently small, the distributed loads q_x and q_z can be considered uniformly distributed.

The (unknown) section forces on the right-hand and left-hand section planes are shown in accordance with their positive directions. The section forces are functions of x, and are generally different in the two section planes. Assume that the forces on the left-hand section plane are N, V and M. Also assume that these forces increase over distance Δx by ΔN, ΔV and ΔM respectively. The forces on the right-hand section plane are then $(N + \Delta N)$, $(V + \Delta V)$ and $(M + \Delta M)$.

The force equilibrium of the bar segment in the x direction gives

$$\sum F_x = -N + (N + \Delta N) + q_x \, \Delta x = 0$$

or

$$\Delta N + q_x \, \Delta x = 0.$$

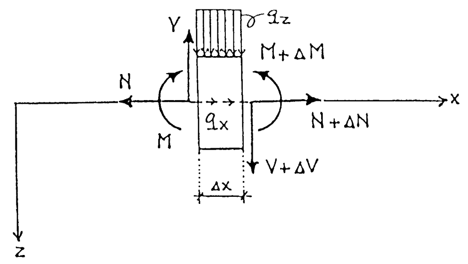


Figure 2.6 The section forces on a bar segment with small length Δx ($\Delta x \to 0$).

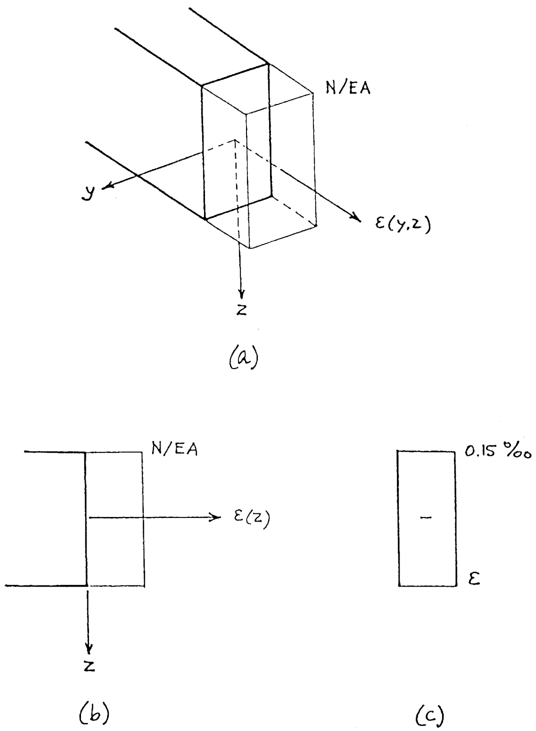


Figure 2.7 Strain diagram due to extension: the strain is uniformly distributed over the cross-section. (a) Spatial representation; (b) and (c) two-dimensional representations.

After dividing by Δx we find

$$\frac{\Delta N}{\Delta x} + q_x = 0.$$

In the limit $\Delta x \to 0$, the equation for the force equilibrium for an elementary bar segment changes into the first-order differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0.$$

This is the *static relationship* for extension.

Comment: The derivation is invalid when a concentrated force F_x is acting on the bar segment. In that case, there is a step change in the variation of the normal force N. As a function of x, N is then no longer continuous and differentiable.

2.3 Strain diagram and normal stress diagram

In a bar subject to extension, all fibres undergo the same elongation, regardless of the material behaviour (see Section 2.2.1).

Using the constitutive relationship

$$N = EA\varepsilon,$$

we find a uniform strain over the cross-section:

$$\varepsilon = \frac{N}{EA}.$$

Figure 2.7a shows the uniform strain distribution over a rectangular cross-section in a *strain diagram*. Here, along each fibre (y, z) the value of the associated strain $\varepsilon(y, z)$ is plotted. It is the convention to plot the positive values in the positive x direction and the negative values in the negative x direction.

In principle, the strain diagram is a spatial figure. If the strain is independent of the y coordinate, as in this case, then the figure can be simplified into a plane diagram (see Figure 2.7b).

One often leaves out the axes, and the sign associated with the strain is placed within the diagram. So we can see in Figure 2.7c that the strain is constant over the cross-section, that it is negative, and has the value 0.15×10^{-3} ($= 0.15$‰).

In a bar with homogeneous cross-section, all fibres have the same modulus of elasticity E. If such a bar is subject to extension, the fibres are not only subject to the same strain, but also to the same normal stress:

$$\sigma = E\varepsilon = E\frac{N}{EA} = \frac{N}{A}\,.$$

Figure 2.8a shows the uniform distribution of the normal stresses in a *normal stress diagram*. Here, in the same way as in the strain diagram, the value of the normal stress $\sigma(y, z)$ in each fibre (y, z) is plotted along that fibre.

Like the strain diagram, the stress diagram is a spatial figure. If the stresses are independent of the y coordinate, it can be simplified into a plane diagram (see Figure 2.8b).

Here too the axes are generally omitted and the sign of the stress is placed within the diagram. Figure 2.8c shows that the normal stress is constant

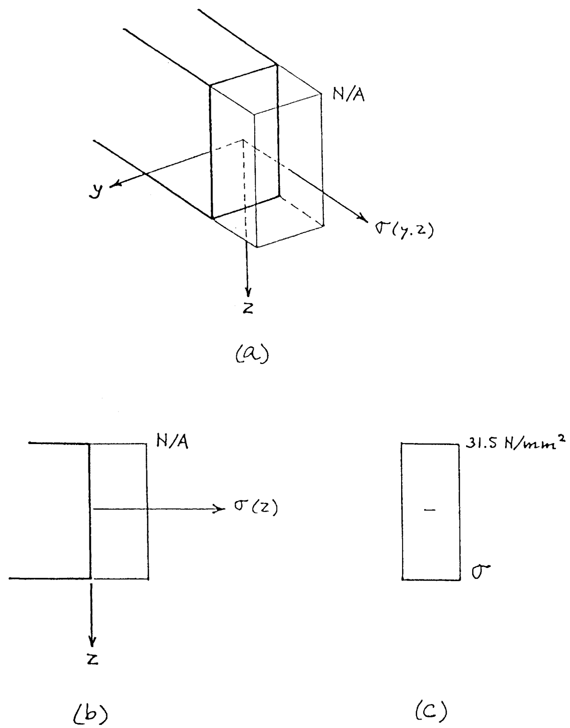


Figure 2.8 Normal stress diagram due to extension: in a homogeneous cross-section the normal stress is uniformly distributed. (a) Spatial representation; (b) and (c) two-dimensional representations.

over the cross-section, and that it is a compressive stress of 31.5 N/mm^2.

Comment: In bars subject to extension, all fibres undergo the same strain ε, irrespective whether the cross-section is homogeneous or inhomogeneous. On the other hand, in a bar subject to extension all fibres have the same normal stress σ if and only if the cross-section is homogeneous: in an inhomogeneous cross-section, the normal stresses due to extension are no longer uniformly distributed.

2.4 Normal centre and bar axis

This section addresses the location of the normal centre NC of a cross-section, and by consequence the location of the bar axis. To locate NC we must consider bending moments for which we follow a formal approach that can differ from engineering practice. In Section 2.7 the difference between the formal approach and the approach used in engineering practice is described.

The resultant of all normal stresses due to extension is the normal force N. For a homogeneous cross-section,

$$N = \int_A \sigma \, \mathrm{d}A = \sigma A.$$

The point of application of the normal force N is defined as the *normal force centre* or *normal centre* of the cross-section, indicated by NC. The fibre through the normal centre NC is defined as the *bar axis*.

Later we shall see that the behaviour of a bar is most easily described in a coordinate system with the x axis along the bar axis. It is therefore important to know the location of the normal centre NC. This problem is solved in two ways, given below:

a. in a xyz coordinate system with the x axis through the normal centre NC of the cross-section, and along the bar axis;
b. in a $\overline{xyz}$ coordinate system with the $\bar{x}$ axis along an arbitrary fibre.

Solution a:
Assume the x axis passes through the normal centre NC of the cross-section, the point of application of the resultant of all normal stresses due to extension.

The resultant of the normal stress σ in fibre (y, z) with area ΔA is a small force ΔN:

$$\Delta N = \sigma A.$$

This small force at fibre (y, z) is statically equivalent to an equal small force ΔN_x[1] at the normal centre NC (the origin of the yz coordinate system), together with two small bending moments ΔM_y and ΔM_z, acting in the xy plane and xz plane respectively (see Figure 2.9):

$$\Delta M_y = y \Delta N = y \sigma \Delta A,$$

$$\Delta M_z = z \Delta N = z \sigma \Delta A.$$

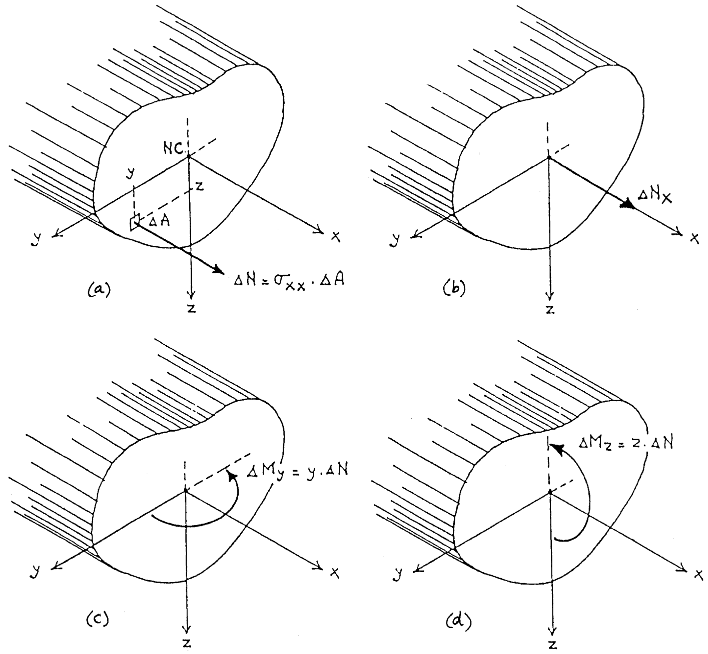


Figure 2.9 By moving the small normal force ΔN from fibre (y, z) towards the x axis, we generate small bending moments ΔM_y and ΔM_z.

[1] In the notation "N_x" the index x indicates that the normal force N acts along the x axis. Since it is the convention to let the normal force apply at the bar axis and to select the x axis there, the index is generally omitted. In this section we are also using a coordinate system for which the x axis does not coincide with the member axis. Therefore the index x is temporarily used.

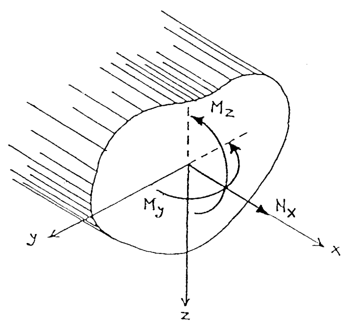


Figure 2.10 The section forces N_x, M_y and M_z due to the normal stresses in the cross-section.

If we sum the contributions of all small forces ΔN over the entire cross-section, we find the normal force

$$N_x = \int_A \sigma \, \mathrm{d}A = \sigma \int_A \mathrm{d}A = \sigma A,$$

and the bending moments

$$M_y = \int_A \sigma y \, \mathrm{d}A = \sigma \int_A y \, \mathrm{d}A \quad \text{(bending moment acting in the } xy \text{ plane)},$$

$$M_z = \int_A \sigma z \, \mathrm{d}A = \sigma \int_A z \, \mathrm{d}A \quad \text{(bending moment acting in the } xz \text{ plane)}.$$

Since in a homogeneous cross-section the normal stress σ due to extension is uniformly distributed (i.e. independent of the coordinates of the fibre with small area dA), σ can be placed outside the integrals.

The section forces N_x, M_y and M_z due to the normal stresses in the cross-section are shown in Figure 2.10.

If the resultant of all normal stresses has its line of action through the normal centre NC, M_y and M_z have to be zero. The location of the normal centre (the bar axis) in a homogeneous cross-section apparently follows from the condition:

$$\int_A y \, \mathrm{d}A = 0 \quad \text{and} \quad \int_A z \, \mathrm{d}A = 0.$$

This implies that in a *homogeneous cross-section* the location of the normal centre NC is determined exclusively by the geometry (shape) of the cross-section. The normal centre then coincides with the *centroid* of the cross-section,[1] as we shall see in Chapter 3.

Solution b:
We can also work in a $\overline{xyz}$ coordinate system with the $\bar{x}$ axis chosen along an arbitrary fibre that need not coincide with the bar axis.

The small force $\Delta N = \sigma \Delta A$ at fibre $(\bar{y}, \bar{z})$ is statically equivalent to an equal small force $\Delta N_{\bar{x}}$ at the $\bar{x}$ axis, together with two small moments $\Delta M_{\bar{y}}$ and $\Delta M_{\bar{z}}$ acting in the $\overline{xy}$ plane and $\overline{xz}$ plane respectively (see Figure 2.11):

$$\Delta M_{\bar{y}} = \bar{y}\Delta N = \bar{y}\sigma \Delta A,$$

$$\Delta M_{\bar{z}} = \bar{z}\Delta N = \bar{z}\sigma \Delta A.$$

Summing the contributions of all small forces ΔN over the entire cross-section A, leads to

$$N_{\bar{x}} = \int_A \sigma \, \mathrm{d}A = \sigma \int_A \mathrm{d}A = \sigma A,$$

$$M_{\bar{y}} = \int_A \sigma \bar{y} \, \mathrm{d}A = \sigma \int_A \bar{y} \, \mathrm{d}A \tag{2.1a}$$

$= $ bending moment acting in the $\overline{xy}$ plane,

$$M_{\bar{z}} = \int_A \sigma \bar{z} \, \mathrm{d}A = \sigma \int_A \bar{z} \, \mathrm{d}A \tag{2.1b}$$

$= $ bending moment acting in the $\overline{xz}$ plane.

[1] Chapter 3 addresses the location of the centroid in further detail.

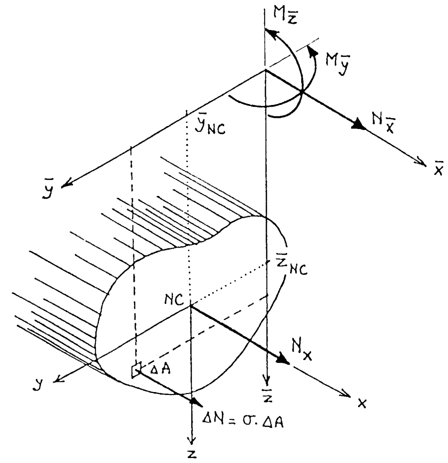


Figure 2.11 The bending moments $M_{\overline{y}}$ and $M_{\overline{z}}$ when the $\overline{x}$ axis is not chosen along the bar axis, and the point of application of $N_{\overline{x}}$ does not coincide with the normal centre NC of the cross-section.

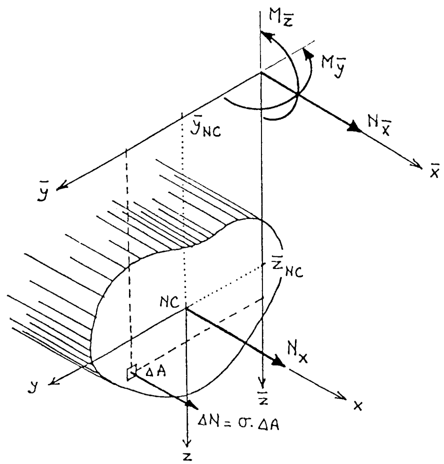


Figure 2.11 The bending moments $M_{\overline{y}}$ and $M_{\overline{z}}$ when the $\overline{x}$ axis is not chosen along the bar axis, and the point of application of $N_{\overline{x}}$ does not coincide with the normal centre NC of the cross-section.

If the line of action of the resultant of all normal stresses due to extension passes through the normal centre NC, with coordinates $(\bar{y}_{\text{NC}}, \bar{z}_{\text{NC}})$, then

$$M_{\bar{y}} = N_x \cdot \bar{y}_{\text{NC}} = \sigma A \cdot \bar{y}_{\text{NC}}, \tag{2.2a}$$

$$M_{\bar{z}} = N_x \cdot \bar{z}_{\text{NC}} = \sigma A \cdot \bar{z}_{\text{NC}}. \tag{2.2b}$$

When Equation (2.2) is equated to Equation (2.1), we find the coordinates $(\bar{y}_{\text{NC}}, \bar{z}_{\text{NC}})$ of the normal centre NC:

$$\bar{y}_{\text{NC}} = \frac{\int_A \bar{y}\, \mathrm{d}A}{A} \quad \text{and} \quad \bar{z}_{\text{NC}} = \frac{\int_A \bar{z}\, \mathrm{d}A}{A}\ .$$

These are the coordinates of the centroid of the cross-section.

Conclusion: *In a homogeneous cross-section the normal centre NC coincides with the centroid of the cross-section.*

Comment: The *bar axis* was defined as the fibre through the normal centre NC. It is often said that the bar axis is in the centroid of the cross-section. This is true only for homogeneous cross-sections. For inhomogeneous cross-sections it is untrue. Therefore *it is preferable to define the bar axis as the fibre through the normal centre* NC, the point of application of the resultant of all normal stresses due to extension.

2.5 Mathematical description of the extension problem

In Section 2.5.1, the three basic equations from Section 2.2 are combined to form a single, second-order differential equation in the displacement u. This differential equation for extension can be solved by repeated integration.

The general solution contains two (still unknown) integration constants. Section 2.5.2 describes how these integration constants follow from the boundary conditions (joining and/or end conditions).

Numerical examples, in which the differential equation is used, are given in Section 2.7.

2.5.1 The differential equation for extension

The three basic equations for extension are (see Section 2.2)

Kinematic relationship: $$\varepsilon = u', \tag{2.3}$$

Constitutive relationship: $$N = EA\varepsilon, \tag{2.4}$$

Static relationship: $$N' + q_x = 0. \tag{2.5}$$

By substituting the strain ε from Equation (2.3) into Equation (2.4) we find

$$N = EAu'.$$

Substituting this expression for N in (2.5) gives

$$(EAu')' + q_x = 0. \tag{2.6}$$

This is a second-order[1] differential equation for the axial displacement u. When the axial stiffness EA is constant (independent of x), the bar is known as *prismatic* and the differential equation can be simplified to

[1] The order of the differential equation is determined by the highest derivative.

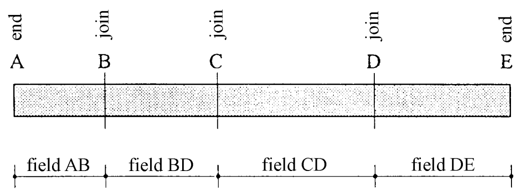


Figure 2.12 A bar divided into fields. The boundary conditions are found at the bar ends and at the joins of two adjacent fields.

$$EAu'' + q_x = 0.$$

or

$$EAu'' = -q_x. \tag{2.7}$$

For a prismatic bar, the extension problem reduces to an equation involving just the second derivative in the axial displacement u.

2.5.2 Boundary conditions: joining and/or end conditions

The differential equation for extension can be solved through repeated integration whether or not EA is constant and the bar is prismatic. Each integration gives an (still unknown) integration constant. The total number of integration constants in the general solution is two; this number is equal to the order of the differential equation.

For a prismatic bar differential equation (2.7) applies only for regions in which the displacement u and the normal force $N = EAu'$ are continuous and/or continuous differentiable, and in which the axial stiffness EA is constant. Such a region is known as a *field* (see Figure 2.12).

Each field has its own solution with its own two integration constants.

The integration constants follow from the *boundary conditions* on the boundaries of a field. At the join of two adjacent fields the boundary condition is called a *joining condition*. At the end of a field without an adjacent field, the boundary condition is called an *end condition*.

Joining conditions:
There are always two conditions per join: one relates to the displacement u, the other relates to the normal force $N = EAu'$.

Joining condition related to the displacement u

At a join of two fields, the displacement u must be continuous – holes cannot suddenly appear in the bar! Therefore the following applies at join B, where the fields AB and BC are connected[1] (see Figure 2.13a):

$$u_{\rm B}^{\rm AB} = u_{\rm B}^{\rm BC}.$$

Joining condition related to the normal force $N = EAu'$

If a concentrated force $F_{x;\rm B}$ is acting at join B, the joining condition follows from the force equilibrium in x direction of a small bar segment at the join. With a length Δx of the segment, and $\Delta x \to 0$ we find (see Figure 2.13b)

$$-N_{\rm B}^{\rm AB} + N_{\rm B}^{\rm BC} + F_{x,\rm B} = 0$$

or

$$N_{\rm B}^{\rm AB} = N_{\rm B}^{\rm BC} + F_{x,\rm B}.$$

If there is a distributed load q_x it is not included in the equilibrium equation, because the influence of $q_x \Delta x$ is negligibly small with respect to the other terms in the equilibrium equation when $\Delta x \to 0$.

With normal force N expressed in terms of the displacement u, the joining condition at B is

$$(EAu')_{\rm B}^{\rm AB} = (EAu')_{\rm B}^{\rm BC} + F_{x,\rm B}.$$

If join B is unloaded ($F_{x,\rm B} = 0$), then the normal force N is continuous:

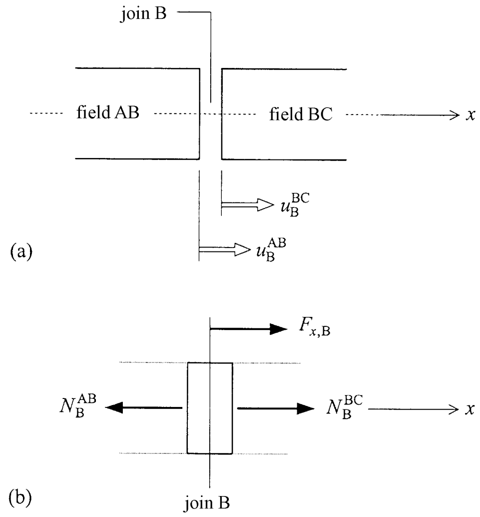


Figure 2.13 The joining conditions related to u and N. (a) At a join of two adjacent fields, the displacement u must be continuous: $u_{\rm B}^{\rm AB} = u_{\rm B}^{\rm BC}$. (b) The force equilibrium of a small bar segment at a join of two fields gives $N_{\rm B}^{\rm AB} = N_{\rm B}^{\rm BC} + F_{x,\rm B}$.

[1] Field boundaries (locations) are indicated by a sub-index and the fields (regions) are indicated by an upper index.

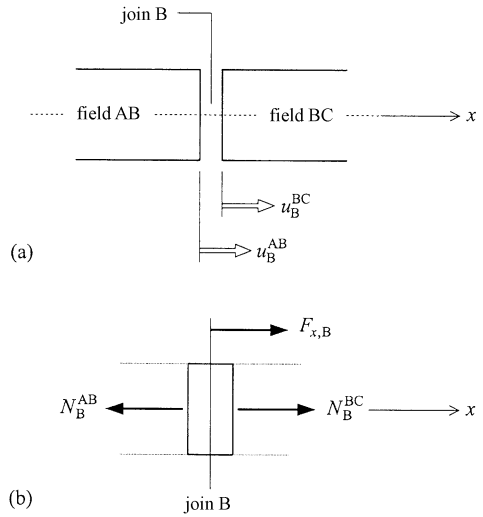


Figure 2.13 The joining conditions related to u and N. (a) At a join of two adjacent fields, the displacement u must be continuous: $u_B^{AB} = u_B^{BC}$. (b) The force equilibrium of a small bar segment at a join of two fields gives $N_B^{AB} = N_B^{BC} + F_{x,B}$.

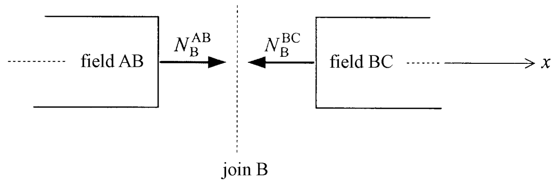


Figure 2.14 Principle of action and reaction: N is continuous in an unloaded join of two fields.

$$N_B^{AB} = N_B^{BC}.$$

This is in line with the principle of action and reaction (see Figure 2.14).

- *End conditions*

At a bar end there is always one boundary condition: an end condition. At the bar end, either the displacement u, or the normal force $N = EAu'$ is prescribed. If u is prescribed (this happens at a support[1]), then N is unknown; and *vice versa*, if N is prescribed (this occurs at a non-supported end), then u is unknown. The prescribed value of N follows from the equilibrium of a small end segment with length Δx for which $\Delta x \to 0$.

Examples are given in Section 2.7.

2.6 Examples relating to changes in length and displacements

In this section, only the kinematic and constitutive equations are used. Section 2.6.1 includes a brief summary of the common formulae for calculating changes in length for members subject to extension. Sections 2.6.2 to 2.6.5 include a number of examples in which the formulae are used to calculate a change in length or a displacement.

2.6.1 Summary of the formulae for a change in length

For the change in length of a member we have

---

1 See also Volume 1, Section 4.3.1.

$$\Delta\ell = \int_\ell \varepsilon \, \mathrm{d}x .$$

With $\varepsilon = N/EA$ (the constitutive relationship) we find

$$\Delta\ell = \int_\ell \frac{N}{EA} \, \mathrm{d}x .$$

This formulation is particularly useful when the variation of the normal force in the member is known. We look at three cases in more detail.

Note: If we plot the strain $\varepsilon = N/EA$ as a function of x in a so-called ε *diagram* then the change in length of the member is equal to the area of that ε diagram.

- *A prismatic member*

For a prismatic member, the axial stiffness EA is constant (independent of x), and can be placed outside the integral:

$$\Delta\ell = \frac{1}{EA} \int_\ell N \, \mathrm{d}x .$$

The change in length of the member is equal to the area of the N diagram, divided by EA.

- *A prismatic member with constant normal force*

For prismatic members with constant normal force, both EA and N can be placed outside the integral:

$$\Delta\ell = \frac{N}{EA} \int_\ell \mathrm{d}x = \frac{N\ell}{EA} .$$

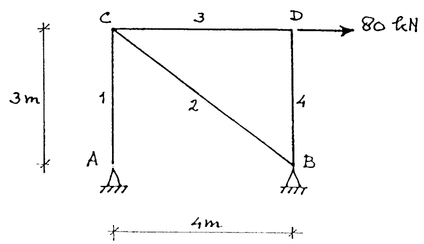


Figure 2.15 A truss.

Table 2.1

Member i	N^i (kN)	ℓ^i (m)	$(EA)^i$ (kN)	$\Delta\ell^i = \left(\frac{N\ell}{EA}\right)^i$ (m)
1	+60	3	200×10^3	$+0.9 \times 10^{-3}$
2	–100	5	250×10^3	-2.0×10^{-3}
3	+80	3	200×10^3	$+1.6 \times 10^{-3}$
4	0	4	200×10^3	0

• *A non-prismatic member with constant normal force*
The normal force N can be placed outside the integral, but the axial stiffness EA (as a function of x) must remain in it:

$$\Delta\ell = N \int_\ell \frac{1}{EA}\,\mathrm{d}x.$$

2.6.2 Change in length of a truss members

For the truss in Figure 2.15, the truss members are prismatic (as is usual). Members 1, 3 and 4 have an axial stiffness of 200 MN; diagonal member 2 has a different axial stiffness of 250 MN.

Question:
Determine the change in length of each of the members.

Solution:
The normal force in a truss member is constant. For a prismatic member with a constant normal force,

$$\Delta\ell = \frac{N\ell}{EA}.$$

The length ℓ and axial stiffness EA are known for each member. To determine the changes in length we have to know the member forces also. These have to be calculated first. The results are shown in Table 2.1.[1]

[1] Remember that the upper index is used to indicate the members.

Comment: With these calculations, the units have to be taken into account carefully. It is best to indicate the units, e.g. m and kN, at the start. In that case, the axial stiffness has to be converted from MN into kN. The signs for tension/elongation and compression/shortening must also be watched carefully: an error in the sign of the normal force gives an elongation instead of a shortening, and *vice versa*.

From the calculation, we see that member 3 undergoes the largest elongation with 1.6 mm. The largest shortening is 2 mm, and occurs in diagonal member 2.

Due to the change in length of the members, the (free) joints C and D will move. The calculation of the displacements of these joints is covered in Chapter 7.

2.6.3 Column from a three-storey building

Figure 2.16a shows column ABCD from a three-storey building. Dimensions, loads and axial stiffnesses are shown in the figure. The storey levels are numbered from (1) to (3).

Questions:

a. Determine the variation of the strain along the height of column ABCD (ε diagram).
b. Determine the change in length of column ABCD.
c. Determine the vertical displacement of B, C and D respectively.

Solution:
Column ABCD is not prismatic, nor is the normal force constant, see the N diagram in Figure 2.16b. Per storey, the column is prismatic and the normal

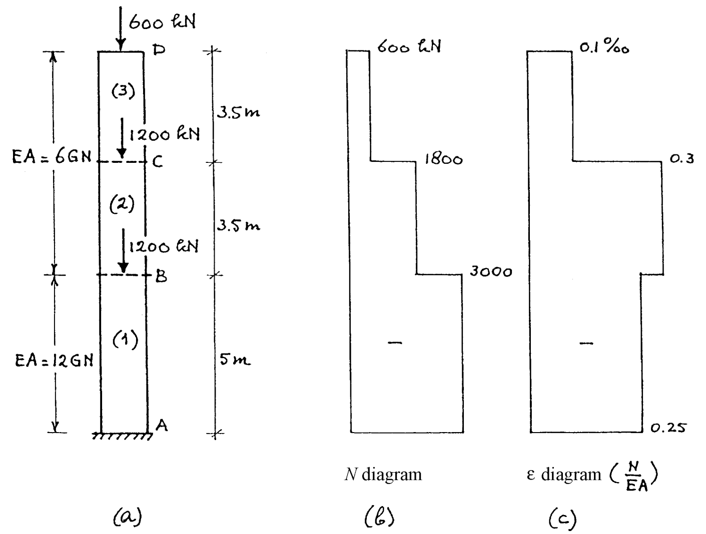


Figure 2.16 (a) Column from a three-storey building with (b) the N diagram and (c) the ε diagram.

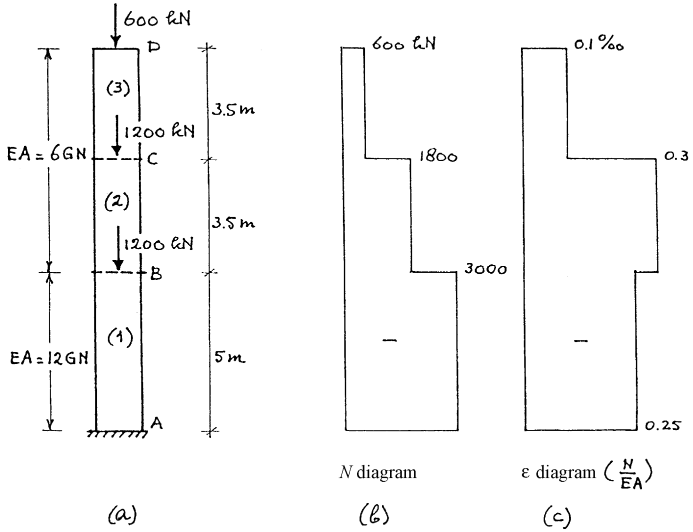


Figure 2.16 (a) Column from a three-storey building with (b) the N diagram and (c) the ε diagram.

force is constant. The column can therefore be seen as a stack of prismatic members with constant normal forces.

Per storey level i we can therefore use

$$\Delta\ell^i = \frac{N^i \ell^i}{(EA)^i}.$$

When calculating the strains, changes in length and displacements, we will hereafter use the units mm and N.

a. The variation of the strain ε in column ABCD (the ε diagram for ABCD). Part AB (1st storey):

$$\varepsilon^{(1)} = \frac{N^{(1)}}{EA^{(1)}} = \frac{-3000 \times 10^3 \text{ N}}{12 \times 10^9 \text{ N}} = -0.25 \times 10^{-3} = -0.25‰.$$

Part BC (2nd storey):

$$\varepsilon^{(2)} = \frac{N^{(2)}}{EA^{(2)}} = \frac{-1800 \times 10^3 \text{ N}}{6 \times 10^9 \text{ N}} = -0.3 \times 10^{-3} = -0.3‰.$$

Part CD (3rd storey):

$$\varepsilon^{(3)} = \frac{N^{(3)}}{EA^{(3)}} = \frac{-600 \times 10^3 \text{ N}}{6 \times 10^9 \text{ N}} = -0.1 \times 10^{-3} = -0.1‰.$$

Figure 2.16c shows the variation of the strain $\varepsilon = N/EA$ in a ε diagram over the total height of the column.

The strain is a maximum in field BC, the second storey, even though the normal force is not the largest there.

b. The change in length of column ABCD.
First the change in length per storey (field) is determined:

AB (1st storey):

$$\Delta\ell^{(1)} = \frac{N^{(1)}\ell^{(1)}}{EA^{(1)}} = \frac{(-3000 \times 10^3 \text{ N})(5 \times 10^3 \text{ mm})}{12 \times 10^9 \text{ N}} = -1.25 \text{ mm}.$$

BC (2nd storey):

$$\Delta\ell^{(2)} = \frac{N^{(2)}\ell^{(2)}}{EA^{(2)}} = \frac{(-1800 \times 10^3 \text{ N})(3.5 \times 10^3 \text{ mm})}{6 \times 10^9 \text{ N}} = -1.05 \text{ mm}.$$

CD (3rd storey):

$$\Delta\ell^{(3)} = \frac{N^{(3)}\ell^{(3)}}{EA^{(3)}} = \frac{(-600 \times 10^3 \text{ N})(3.5 \times 10^3 \text{ mm})}{6 \times 10^9 \text{ N}} = -0.35 \text{ mm}.$$

The change in length of column ABCD is therefore

$$\Delta\ell = \Delta\ell^{(1)} + \Delta\ell^{(2)} + \Delta\ell^{(3)} = -2.65 \text{ mm}.$$

The column shortens by 2.65 mm.

The change in length of the column can also be found from the area of the ε diagram (N/EA diagram):

$$\Delta\ell = \int_\ell \varepsilon \, \mathrm{d}x = \int_\ell \frac{N}{EA} \, \mathrm{d}x$$

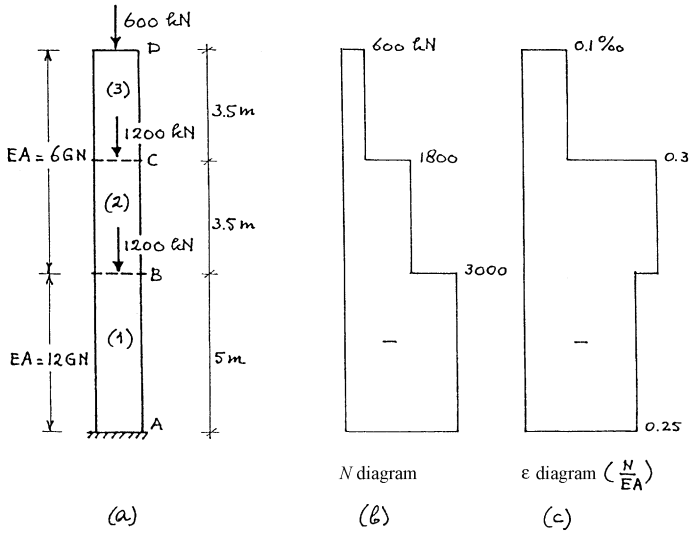


Figure 2.16 (a) Column from a three-storey building with (b) the N diagram and (c) the ε diagram.

$$= (-0.25 \times 10^{-3})(5 \times 10^3 \text{ mm})$$
$$+ (-0.3 \times 10^{-3})(3.5 \times 10^3 \text{ mm})$$
$$+ (-0.1 \times 10^{-3})(3.5 \times 10^3 \text{ mm}$$
$$= -1.25 - 1.05 - 0.35 = -2.65 \text{ mm}$$

c. The displacements at B, C and D.
Column AB shortens by 1.25 mm, therefore B drops by 1.25 mm. Column BC shortens by 1.05 mm; C drops by $1.25 + 1.05 = 2.30$ mm. Column CD shortens by 0.35 mm; D drops by $1.25 + 1.05 + 0.35 = 2.65$ mm, the same amount as the shortening of the entire column.

2.6.4 Prismatic column subject to its dead weight

Figure 2.17a shows a prismatic column, with length ℓ, cross-section A and a total dead weight G. The modulus of elasticity is E.

Questions:
Due to the dead weight determine:

a. The variation of N and ε as functions of x.
b. The variation of the displacement u as function of x.
c. The vertical displacement at the top of the column.

Solution:
a. The variation of N and ε as function of x.
The dead weight can be seen as a uniformly distributed axial load q_x along the bar axis:

$$q_x = -\frac{G}{\ell}\,.$$

Since the distributed load is acting opposite to the x direction, q_x is negative. The equilibrium of the isolated part of the column in Figure 2.17b gives

$$N(x) = \frac{G}{\ell}x - G = -G\left(1 - \frac{x}{\ell}\right)$$

so that

$$\varepsilon(x) = \frac{N(x)}{EA} = -\frac{G}{EA}\left(1 - \frac{x}{\ell}\right).$$

The normal force N and strain ε are linear over the height of the column. Figure 2.17b shows the N diagram and ε diagram.

b. The variation of the displacement u as a function of x.
The displacement u (positive in the positive x direction) follows from

$$\Delta u = u(x) - u(0) = \int_0^x \varepsilon \, dx$$

so that

$$u(x) = u(0) + \int_0^x \varepsilon \, dx.$$

At the support $x = 0$ the displacement is zero: $u(0) = 0$. With this end condition we find

$$u(x) = \int_0^x \varepsilon \, dx = -\frac{G}{EA}\int_0^x \left(1 - \frac{x}{\ell}\right) dx = -\frac{G\ell}{EA}\left(\frac{x}{\ell} - \frac{1}{2}\frac{x^2}{\ell^2}\right).$$

The displacement is parabolic (quadratic in x) and is negative everywhere; this means that the displacement is directed downwards.

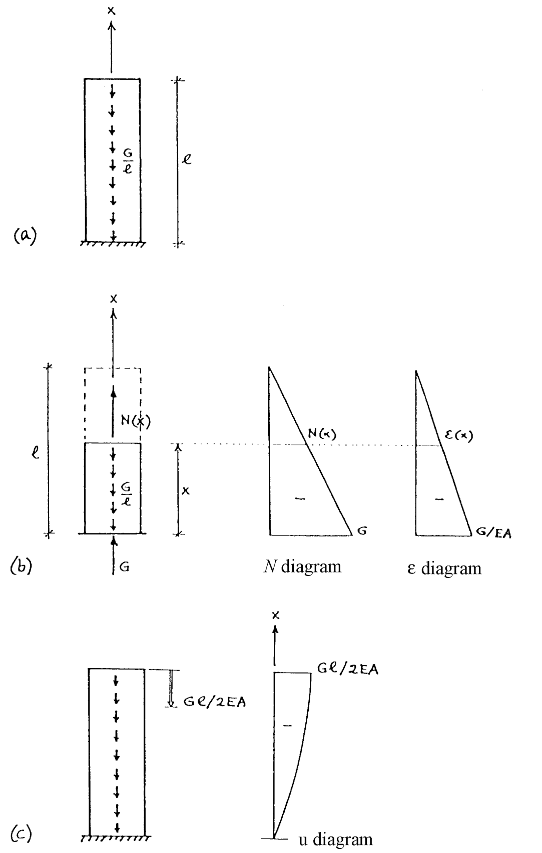


Figure 2.17 (a) A prismatic column, fixed at its base and free at the top, subject to its dead weight. The associated (b) N and ε diagrams, and (c) the displacement diagram or u diagram.

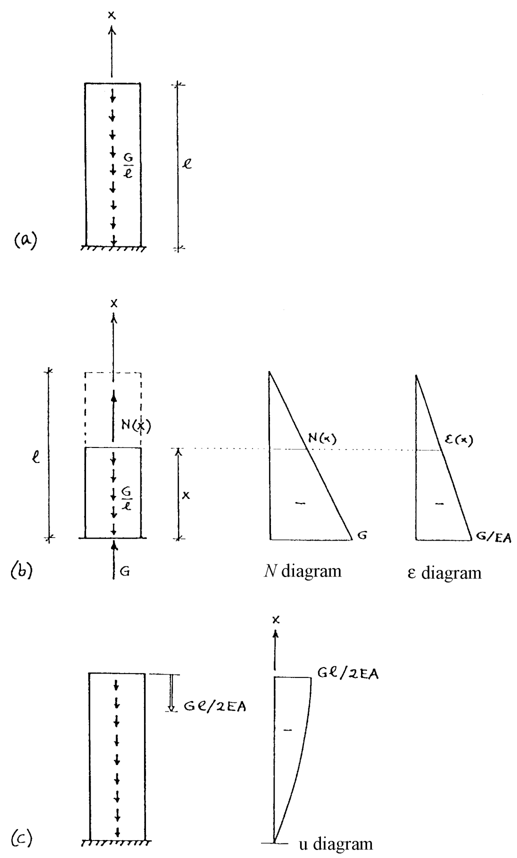


Figure 2.17 (a) A prismatic column, fixed at its base and free at the top, subject to its dead weight. The associated (b) N and ε diagrams, and (c) the displacement diagram or u diagram.

The variation of the displacement u along the x axis is shown in a *displacement diagram* or *u diagram* (see Figure 2.17c).

c. The vertical displacement at the top of the column.
The displacement at the top $x = \ell$ can be found from the expression for $u(x)$:

$$u(\ell) = -\frac{G\ell}{2EA}.$$

The minus sign means that the column shortens.

Note that for the displacement at the top also applies

$$u(\ell) = \int_0^\ell \varepsilon \, dx.$$

This integral can be interpreted as the area of the ε diagram. That is the area of the triangle in Figure 2.17b, which is easy to determine:

$$u(\ell) = \tfrac{1}{2} \cdot \left(-\frac{G}{EA}\right) \cdot \ell = -\frac{G\ell}{2EA}.$$

2.6.5 Non-prismatic column with constant normal force

The column in Figure 2.18a has a length ℓ and a square cross-section of which the side varies linearly from a at the ends to $2a$ at the middle. The modulus of elasticity is E.

Question:
Determine the change in length of the column due to the compressive force F.

Comment: This problem requires some mathematical skill, such as the substitution of variables.

Solution:
For the change in length of the non-prismatic column with constant normal force we have

$$\Delta\ell = N \int_{\ell} \frac{1}{EA}\,\mathrm{d}x = \frac{N}{E} \int_{\ell} \frac{1}{A}\,\mathrm{d}x.$$

Since the area $A = A(x)$ of the cross-section is a function of x, it has to remain inside the integral.

Symmetry considerations make it possible to consider half the column (see Figure 2.18b). If $\Delta\ell$ is the change in length of the total column, then $\Delta\ell/2$ is the change in length of half the column:

$$\tfrac{1}{2}\Delta\ell = \frac{N}{E} \int_0^{\ell/2} \frac{1}{A(x)}\,\mathrm{d}x,$$

which, with $N = -F$, implies

$$\Delta\ell = -\frac{2F}{E} \int_0^{\ell/2} \frac{1}{A(x)}\,\mathrm{d}x.$$

We now have to find the area of the cross-section as a function of x. For the width $b(x)$ at height x we have

$$b(x) = 2a\left(1 - \frac{x}{\ell}\right),$$

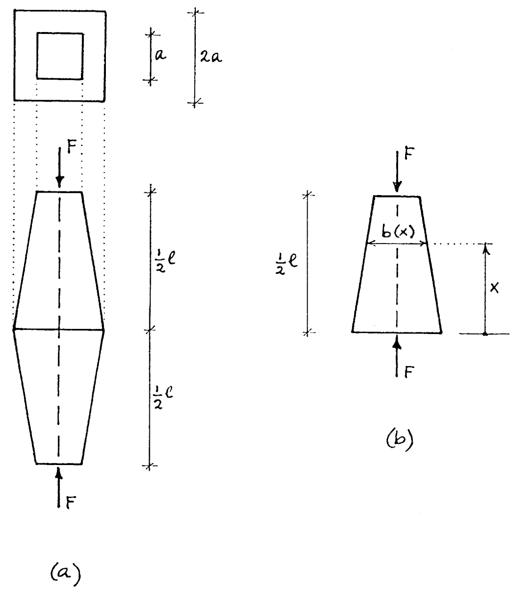


Figure 2.18 (a) A non-prismatic column with constant normal force. (b) For calculations, half the column can be considered on basis of symmetry.

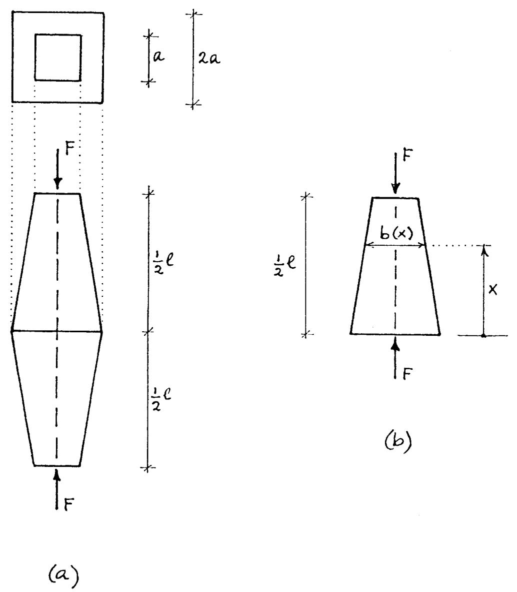


Figure 2.18 (a) A non-prismatic column with constant normal force. (b) For calculations, half the column can be considered on basis of symmetry.

and for the area $A(x)$ of the cross-section there

$$A(x) = \{b(x)\}^2 = 4a^2 \left(1 - \frac{x}{\ell}\right)^2 .$$

Solving the integral gives the following:

$$\int_0^{\ell/2} \frac{1}{A(x)}\, \mathrm{d}x = \int_0^{\ell/2} \frac{1}{4a^2 \left(1 - \frac{x}{\ell}\right)^2}\, \mathrm{d}x$$

$$= \frac{1}{4a^2} \int_0^{\ell/2} \frac{1}{\left(1 - \frac{x}{\ell}\right)^2} \frac{\mathrm{d}\left(1 - \frac{x}{\ell}\right)}{\left(-\frac{1}{\ell}\right)} .$$

Select a new variable $\tilde{x}$:

$$\tilde{x} = 1 - \frac{x}{\ell} .$$

With this new variable the integration limits $x = 0$ and $x = \ell/2$ change to $\tilde{x} = 1$ and $\tilde{x} = 1/2$ respectively, so that

$$\int_0^{\ell/2} \frac{1}{A(x)}\, \mathrm{d}x = -\frac{\ell}{4a^2} \int_1^{1/2} \frac{1}{\tilde{x}^2}\, \mathrm{d}\tilde{x} = -\frac{\ell}{4a^2} \left(-\frac{1}{\tilde{x}}\right)\bigg|_1^{1/2} = \frac{\ell}{4a^2} .$$

For the change in length of the column we now find

$$\Delta\ell = -\frac{2F}{E} \int_0^{\ell/2} \frac{1}{A(x)}\, \mathrm{d}x = -\frac{2F}{E} \frac{\ell}{4a^2} = -\frac{F\ell}{2Ea^2} .$$

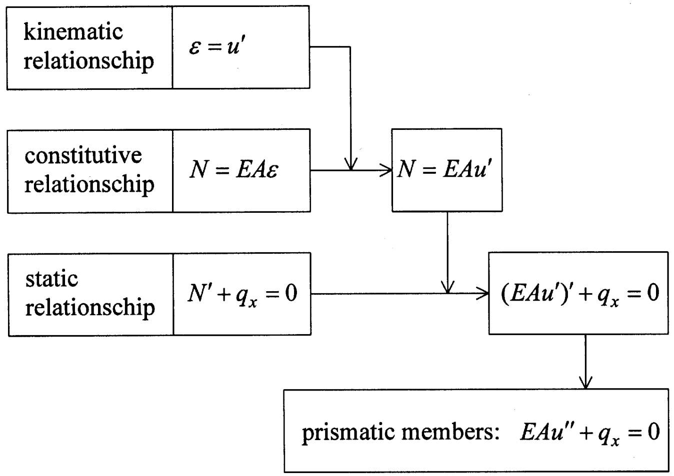
The minus sign indicates that the column shortens.

2.7 Examples relating to the differential equation for extension

In Section 2.6 we used just the kinematic and constitutive relationships. In this section we now involve the static relationship (equilibrium equations). First a summary is given of the various formulae that form the basis for the differential equation for extension, and that are required to satisfy the boundary conditions (Section 2.7.1). Next two examples are given. The examples relate to a column subject to extension, that in the first case is statically determinate (Section 2.7.2), and in the second case is statically indeterminate (Section 2.7.3).

2.7.1 Summary of the formulae for extension

Before dealing with the two examples in which the problem of extension is solved with the help of the differential equation, we first provide a summary of the various formulae related to extension in the scheme alongside.



For a prismatic member, starting from the second-order differential equation for extension, we find the normal force N by integrating once:

$$N = EAu' = -\int q_x \, dx,$$

and after integrating again we find the displacement u:

$$EAu = -\int \left(\int q_x \, dx \right) dx.$$

With each integration there appears an unknown integration constant. This means that the expression for the normal force N contains one integration constant and that for the displacement u contains two integration constants.

The unknown integration constants follow from the boundary conditions (end and/or joining conditions). These conditions relate to the magnitude of N and/or u on a field boundary. A member end always gives one boundary condition (an end condition); a field join always gives two boundary conditions (joining conditions).

2.7.2 Prismatic column fixed at one side, and subject to its dead weight

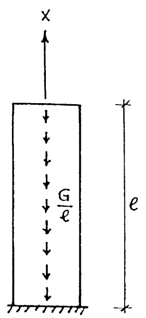


Figure 2.19 A prismatic column, fixed at the base and free at the top, subject to its dead weight.

Figure 2.19 shows a prismatic column, fixed at the base and free at the top, with length ℓ, cross-section A and a total dead weight G. The modulus of elasticity is E.

The same column was discussed in Section 2.6.4, but in a different way.

Question:
Use the differential equation for extension to determine the variation of the displacement u and normal force N due to the dead weight of the column.

Solution:
The dead weight can be seen as a uniformly distributed axial load q_x along the column axis:

$$q_x = -\frac{G}{\ell}\,.$$

The differential equation for extension is now (pay attention to the signs!)

$$EAu'' = -q_x = +\frac{G}{\ell}\,.$$

By repeated integration we find

$$N = EAu' = \frac{G}{\ell}x + C_1,$$

$$EAu = \tfrac{1}{2}\,\frac{G}{\ell}x^2 + C_1 x + C_2.$$

The constants C_1 and C_2 follow from the boundary conditions, in this case end conditions. At an end, either the magnitude of the normal force N is known or the magnitude of the displacement u.

The column is fixed at the base, at $x = 0$, and cannot move here.[1] This gives the first boundary condition:

$$x = 0;\ u = 0.$$

Furthermore, the column is unloaded at the top at $x = \ell$. There the normal force is zero.[2] This gives the second boundary condition:

$$x = \ell;\ N = EAu' = 0.$$

The first boundary condition leads to

$$C_2 = 0,$$

and the second to

1 The normal force N is initially unknown here.
2 The displacement u is initially unknown here.

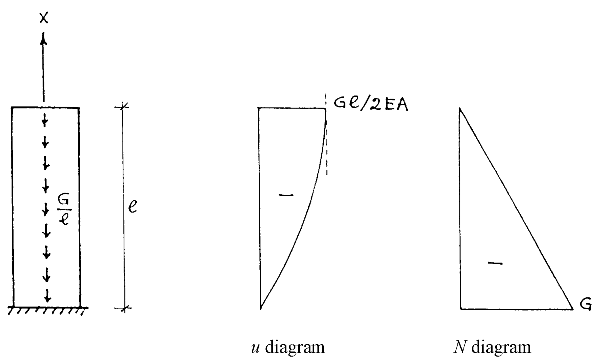


Figure 2.20 The column with associated u and N diagrams.

$$C_1 = -G.$$

For the variation of the displacement u we find

$$EAu = \tfrac{1}{2}\,\frac{G}{\ell}x^2 - Gx$$

or, rewritten,

$$u = \frac{G\ell}{EA}\left(\tfrac{1}{2}\,\frac{x^2}{\ell^2} - \frac{x}{\ell}\right).$$

The variation of the normal force N is

$$N = EAu' = G\left(\frac{x}{\ell} - 1\right).$$

Note that the normal force N is proportional to the slope of the u diagram. Note in particular that $u = -G\ell/(2EA)$ at the top, and $N = -G$ at the base.

Figure 2.20 shows the u diagram and N diagram. The same results were found in Section 2.6.4. Since the structure is statically determinate, the variation of the normal force could be derived directly from the equilibrium there, without using the strain-displacement (kinematic) and stress-strain (constitutive) equations.

2.7.3 Prismatic column fixed at two sides, and subject to its dead weight

Figure 2.21 shows a prismatic column fixed at both ends, with length ℓ, cross-section A and a total dead weight G. The modulus of elasticity is E.

Question:
Use the differential equation for extension to determine the variation of the displacement u and normal force N due to the dead weight of the column.

Solution:
The difference between this and the previous example is that the column is now fixed at both ends. As a consequence the force flow is indeterminate. This means that the support reactions and the variation of the normal force can no longer be derived directly from the equilibrium equations. There are an infinite number of force flows that satisfy the equilibrium equations.

The correct force flow satisfies not only the equilibrium equations, but also the condition that the deformed column has to fit exactly between both fixed supports. Therefore, the actual force flow satisfies not only the equilibrium equations but also the stress-strain (constitutive) equations and the strain-displacement (kinematic) equations.

When performing calculations for a statically indeterminate structure we therefore need all three basic relationships: kinematic, constitutive and static, as shown below.

With

$$q_x = -\frac{G}{\ell}$$

the differential equation for extension is

$$EAu'' = -q_x = \frac{G}{\ell}\,.$$

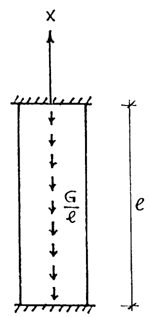


Figure 2.21 A prismatic column, at both ends fixed, subject to its dead weight.

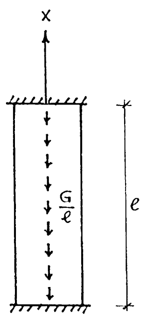


Figure 2.21 A prismatic column, at both ends fixed, subject to its dead weight.

After repeated integration we find

$$N = EAu' = \frac{G}{\ell}x + C_1,$$

$$EAu = \tfrac{1}{2}\,\frac{G}{\ell}x^2 + C_1 x + C_2.$$

So far, the results are exactly the same as those for the statically determinate column in Section 2.7.2. The difference occurs in the boundary conditions (end conditions).

The column is fixed at its base ($x = 0$) and at its top ($x = \ell$). This leads to the following two boundary conditions:

$$x = 0;\ u = 0,$$

$$x = \ell;\ u = 0.$$

The first boundary condition gives

$$C_2 = 0$$

and the second

$$C_1 = -\tfrac{1}{2}\,G.$$

For the displacement u as a function of x we find

$$EAu = \tfrac{1}{2}\,\frac{G}{\ell}x^2 - \tfrac{1}{2}\,Gx$$

or, rewritten,

$$u = \frac{G\ell}{2EA}\left(\frac{x^2}{\ell^2} - \frac{x}{\ell}\right) \cdot N = EAu' = G\left(\frac{x}{\ell} - \tfrac{1}{2}\right).$$

For the normal force N we find

$$N = EAu' = G\left(\frac{x}{\ell} - \tfrac{1}{2}\right).$$

The distribution of the displacement u and normal force N along the column axis is shown in Figure 2.22, in a u and N diagram respectively.

The displacement u is parabolic. The maximum displacement occurs at the middle $(x - \ell/2)$:

$$u\left(x = \tfrac{1}{2}\ell\right) = \tfrac{1}{8}\,\frac{G\ell}{EA}.$$

The normal force N is linear. At the base $(x = 0)$ there is a compressive force of $G/2$ and at the top $(x = \ell)$ there is a tensile force of $G/2$. At the middle $(x = \ell/2)$ the normal force is zero. Note that the normal force is proportional to the slope of the u diagram.

The support reactions in Figure 2.22 are derived from the N diagram.

Check: In Section 2.6.1 it was noted that the change in length $\Delta\ell$ for a prismatic member is equal to the area of the N diagram, divided by EA. For the column in this example, the total area of the N diagram is zero. Therefore, the total change in length of the column is zero and the deformed column indeed fits between both fixed supports.

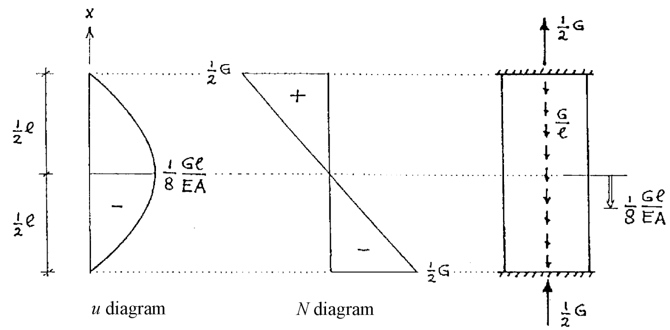


Figure 2.22 The column with associated u and N diagrams. The support reactions are derived from the N diagram.

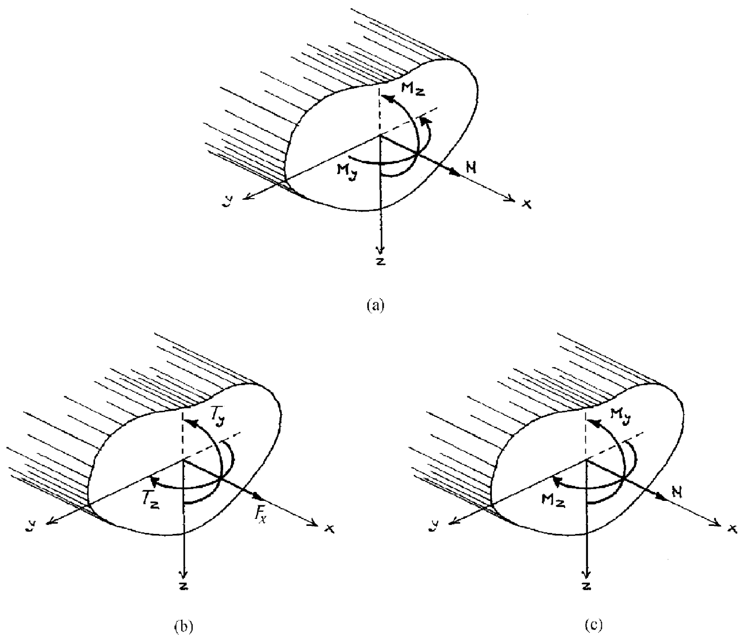


Figure 2.23 (a) The bending moments (section forces) M_y and M_z in the formal approach. The positive directions of M_y and M_z, acting in the xy and xz plane respectively, follow from their definitions: $M_y = \int_A y\sigma \, \mathrm{d}A$ and $M_z = \int_A z\sigma \, \mathrm{d}A$. (b) The moments T_y and T_z about the y and z axes respectively. By definition their positive directions are found from the right-hand or corkscrew rule. Note that for a positive cross-sectional plane, $M_y = -T_z$ and $M_z = +T_y$. (c) In engineering practice the bending moments (section forces) M_y and M_z on a positive cross-sectional plane are often defined in the same way as T_y and T_z. On a negative cross-sectional plane M_y and M_z are then opposite to T_y and T_z.

2.8 Formal approach and engineering practice

This book uses a formal definition of bending moments which is consistent with the stress definitions used in continuum mechanics[1] (see Volume 1, Section 10.1.3). However it is sad to see that in engineering practice a different notation is often used, mainly based on historical and pragmatic grounds.

The formal definitions are shown in Figure 2.23a. Here, the bending moments M_y and M_z are defined as *moments acting in the* xy *and* xz *plane* respectively.

In Volume 1, Sections 3.1.4, 3.1.5, and 3.3 the definitions are given for the moments of a force or couple about a point or about a line. To distinguish these moments from the bending moments M (section forces), we used a different symbol T. So T_y and T_z are the *moments about the* y *and* z *axis* respectively, as shown in Figure 2.23b. By definition the positive directions of T_y and T_z are found from the *right-hand rule*[2] or *corkscrew rule*.[3] Note that for a positive cross-sectional plane (formal approach)

$$M_y = -T_z,$$
$$M_z = +T_y.$$

However, in engineering practice the bending moments (section forces) M_y and M_z for a positive cross-sectional plane are usually defined in the same

[1] The formal definition also leads to a consistent tensor notation, which is out of the scope of this volume.
[2] See Volume 1, Section 1.3.2.
[3] See Volume 1, Section 3.3.1.

way as T_y and T_z. Hence, the bending moments M_y and M_z are *moments about the* y *and* z *axis* respectively, as shown in Figure 2.23c. The bending moments on the negative cross-sectional plane are equal and opposite.

Note from the above that the definitions of the bending moments M_y and M_z according to the formal approach are not the same as those often used in engineering practice, and their positive directions may be opposite. The consequences will be discussed later.

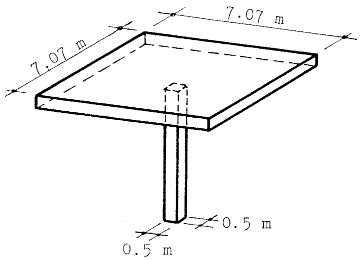
Students have to be aware that different books may use different definitions. Older books even use coordinate systems in which the beam axis is the z axis.

Once the definitions are clear, the actual coordinate system used is for *engineering purposes* of no concern. However the authors explicitly use the formal definitions for *educational purposes* in order to obtain a consistent and convenient set of equations. In some sections in this book, pronounced differences between this formal approach and the engineering approach will be discussed.

2.9 Problems

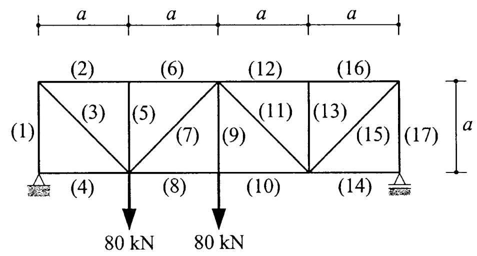
Mixed problems on stress, strain and change in length due to extension (Sections 2.1 to 2.6).

2.1 A column supports a "mushroom floor" of 50 m^2. The weight of floor and inventory is 12.5 kN/m^2. The column has a rectangular cross-section of 500 × 500 mm^2.



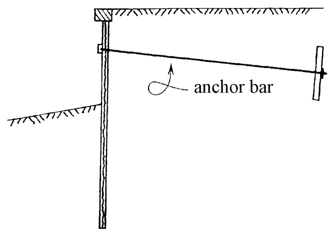
Question:
Determine the compressive stress in the column.

2.2 In the truss shown, all diagonal members have a cross-sectional area of 1400 mm^2. The other members have a cross-sectional area of 800 mm^2. The truss is loaded by two forces of 80 kN.



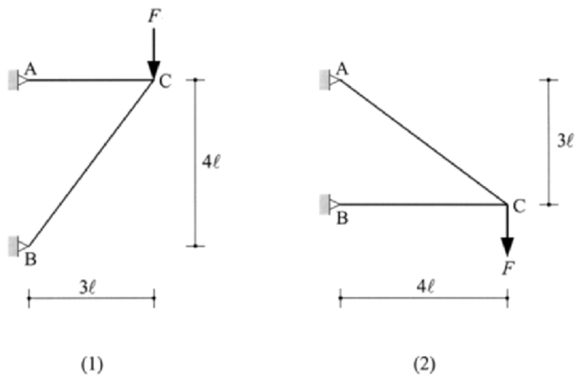
Questions:
a. Determine the stresses in the top chord members.
b. Determine the stresses in the bottom chord members.
c. Determine the stresses in the verticals.
d. Determine the stresses in the diagonals.

2.3 An anchor bar of a bank protection has a circular cross-section with a diameter of 30 mm. The (design value of the) strength is 100 N/mm.



Question:
Determine the admissible tensile force in the anchor bar.

2.4: 1–2 In both trusses, the members AC and BC have the same cross-sectional area $A = 800\ \text{mm}^2$. The tensile stresses in the truss may not exceed $140\ \text{N/mm}^2$, and the compressive stresses may not exceed $80\ \text{N/mm}^2$.

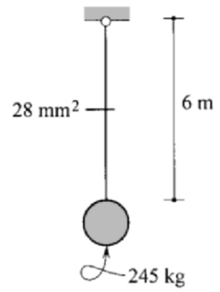


Question:
Determine the maximum vertical force F at C that may be exerted on the truss. Which member is critical?

2.5 A steel wire with cross-sectional area $A = 150\ \text{mm}^2$ is subject to tension by a force F. The modulus of elasticity is $E = 210 \times 10^3\ \text{N/mm}^2$. The yield stress is $f_y = 235\ \text{N/mm}^2$.

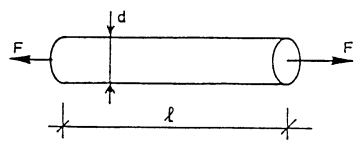
Questions:
a. Determine the strain per mil if $F = 25.2$ kN.
b. Determine the force F for which the yield stress is reached.

2.6 A mass of 245 kg is suspended from a steel wire with a cross-sectional area of 28 mm^2 and a length of 6 m. The modulus of elasticity is $E = 210$ GPa. The dead weight of the wire is neglected. Assume the gravitational field strength is $g = 10$ N/kg.



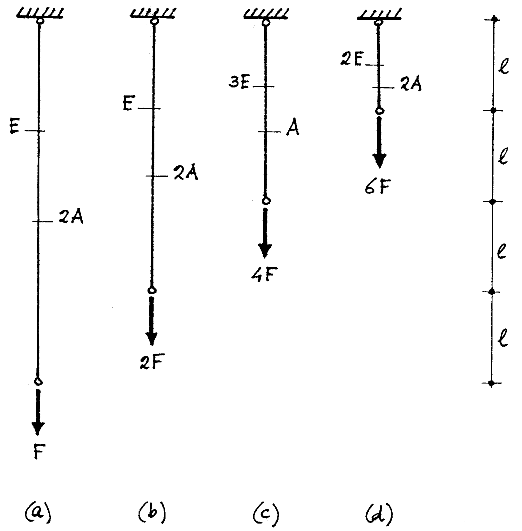
Questions:
a. Determine the stress in the wire.
b. Determine the strain of the wire.

2.7 A prismatic bar of length ℓ has a circular cross-section with diameter d. The bar is loaded by a tensile force F. At the bar a strain ε is measured. The modulus of elasticity of the material is E. In the calculation use $\ell = 0.85$ m, $d = 20$ mm, $\varepsilon = 0.47‰$ and $E = 210$ GPa.



Questions:
a. Determine the normal stress in the cross-section in N/mm^2.
b. Determine the axial stiffness of the bar in MN.
c. Determine the magnitude of the tensile force F in kN.
d. Determine the elongation of the bar in mm.

2.8 Four different wires are loaded by four different forces. All requisite information can be found from the figure.



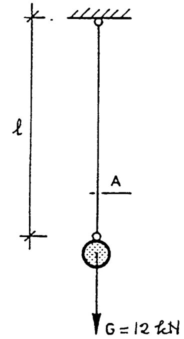
Questions:

a. Which wire has the largest normal stress?
b. Which wire has the largest strain?
c. Which wire has the largest elongation?

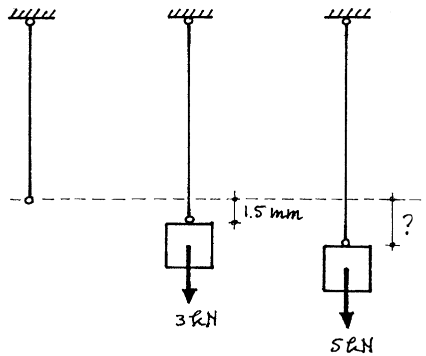
2.9 A block with weight $G = 12$ kN is suspended from a steel wire with length ℓ and cross-sectional area A. The modulus of elasticity is $E = 210 \times 10^3$ MPa. Due to the weight G the wire may not lengthen by more than 2 mm and the stress may not exceed 240 N/mm^2.

Questions:

a. Determine the minimum cross-sectional area A for $\ell = 1.5$ m.
b. Determine the minimum cross-sectional area A for $\ell = 2.1$ m.



2.10 When a weight of 3 kN is suspended from a wire, the wire lengthens by 1.5 mm.



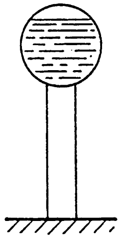
Question:

Determine the elongation if the weight is 5 kN.

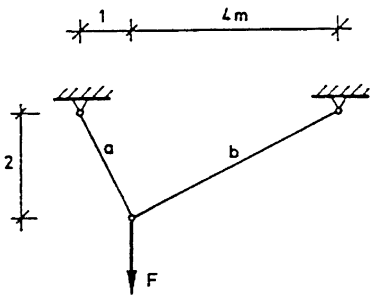
2.11 A water tower consists of a prismatic steel column that supports a spherical water reservoir. The reservoir has a dead weight of 200 kN and a volume of 100 m^3. The steel column shortens 36 mm when the reservoir is entirely filled. The dead weight of the column is ignored. Assume the specific weight of water is 10 kN/m^3.

Questions:

a. Determine the shortening of the steel column when the reservoir is 60% full.
b. Determine the shortening of the steel column when the reservoir is empty.

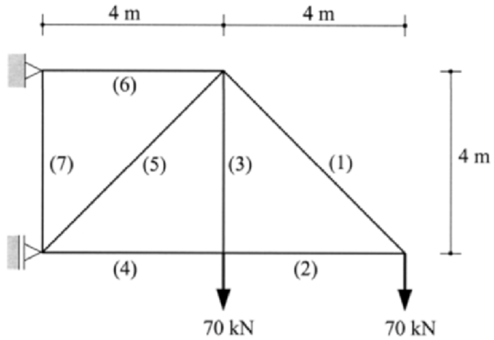


2.12 In the structure shown, the wires a and b are of the same material and have the same cross-section. The structure is loaded by the force F.



Question:
Determine the ratio $\Delta\ell^{(a)}/\Delta\ell^{(b)}$ if $\Delta\ell^{(a)}$ and $\Delta\ell^{(b)}$ are the elongations of the wires a and b respectively.

2.13 All members in the truss have the same axial stiffness of 280 MN. The truss is loaded by two forces of 70 kN.

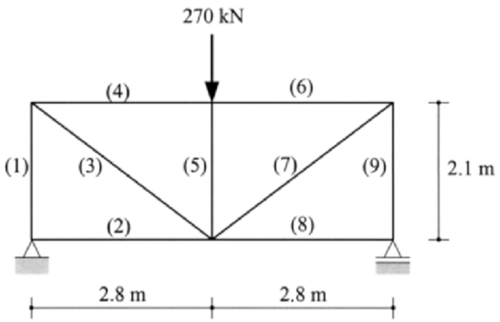


Question:
Determine the changes in length of the members.

2.14 All members in the truss have the same cross-sectional area $A = 1500$ mm^2. The modulus of elasticity is $E = 70$ GPa. The truss is loaded by a force of 270 kN.

Questions:

a. Determine the stresses in the members.
b. Determine the strains in the members, in per mille.
c. Determine the changes in length of the members.



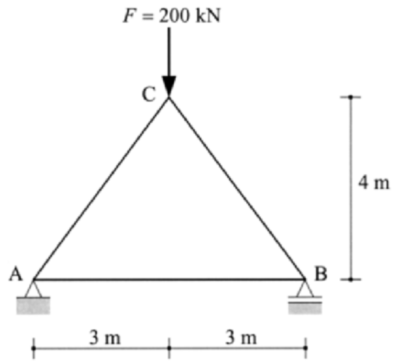
2.15 See the truss in problem 2.14, but now all members loaded by tension have a cross-sectional area $A = 1500$ mm^2, and all members loaded by compression have a cross-sectional area $A = 2000$ mm^2. The modulus of elasticity is $E = 70$ GPa.

Questions:

a. Determine the stresses in the members.
b. Determine the strains in the members, in ‰.
c. Determine the changes in length of the members.

2.16 All members in the truss have the same axial stiffness $EA = 150$ MN. The truss is loaded at C by a vertical force $F = 200$ kN.

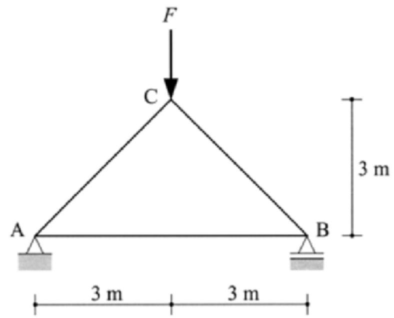
Question:
Determine the displacement of the roller at B.



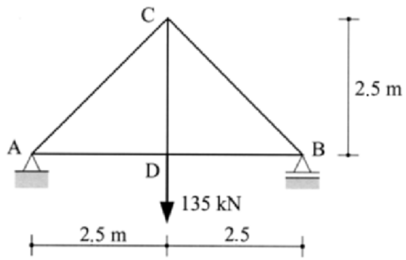
2.17 All members in the truss have the same axial stiffness $EA = 150$ MN. The truss is loaded at C by a vertical force F. In consequence of this, the roller at B moves a distance u.

Questions:

a. Determine F if $u = 4$ mm.
b. Find u if $F = 175$ kN.



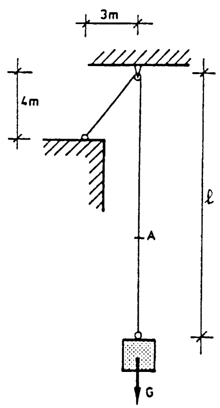
2.18 All members in the truss have the same axial stiffness $EA = 75$ MN. The truss is loaded at D by a vertical force of 135 kN.



Questions:

a. Determine the change in length of member CD.
b. Determine the displacement of the roller at B.

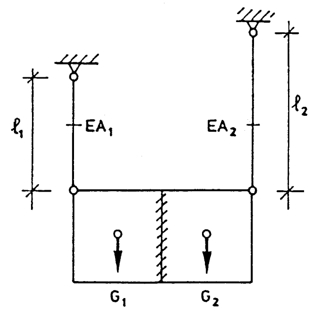
2.19 A block with weight $G = 48$ kN is suspended from a wire that runs over a pulley without friction. See the figure for the dimensions. The cross-sectional area of the wire is $A = 38$ mm^2; the length is $\ell = 14$ m. The modulus of elasticity is $E = 200$ GPa. The dead weight of the wire is neglected.



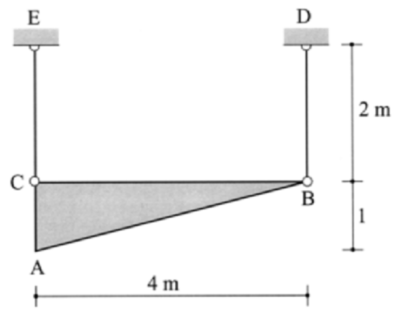
Question:
How far does the block drop due to the elongation of the wire?

2.20 Two square blocks with different weights G_1 and G_2 are glued together, and suspended from two wires. The wires are of different lengths ℓ_1 and ℓ_2 and have different strain rigidities EA_1 and EA_2. In the calculation use $\ell = 1.5$ m, $\ell_2 = 2.0$ m, $G_1 = 18$ kN and $G_2 = 6$ kN.

Question:
Determine the ratio EA_1/EA_2 for which the blocks, subject to their dead weight, drop without rotating.



2.21 A triangular homogeneous slab of constant thickness is suspended from two steel wires BD and CE of equal length. The cross-sectional area A of the wires is different so that B and C drop the same amount.

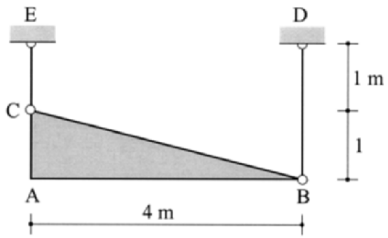


Question:
Determine the ratio $A^{(CE)}/A^{(BD)}$.

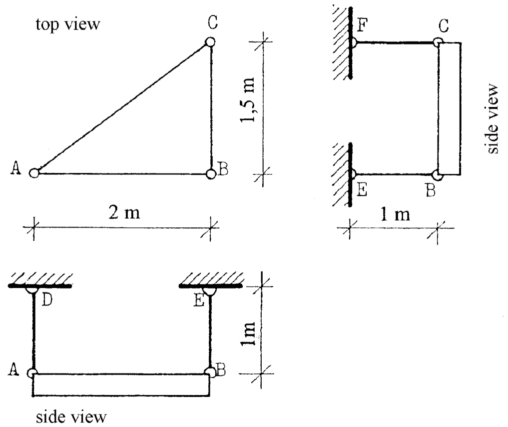
2.22 A triangular steel plate ABC of constant thickness is suspended from two steel wires BD and CE.

Questions:

a. Determine the ratio $\Delta\ell^{(BD)}/\Delta\ell^{(CE)}$ if the cross-sectional area of the steel wires has been selected so that the same stress occurs in both wires.
b. Determine the ratio $\Delta\ell^{(BD)}/\Delta\ell^{(CE)}$ if both wires have the same cross-sectional area.



2.23 A triangular homogeneous slab ABC of constant thickness is suspended at its corners with vertical wires from the ceiling. All wires have the same length, cross-sectional area and modulus of elasticity.



Questions:

Which statement is correct after the suspension?

a. A will hang lower than B and C.
b. B will hang lower than A and C.
c. C will hang lower than A and B.
d. A, B and C will be at the same height.

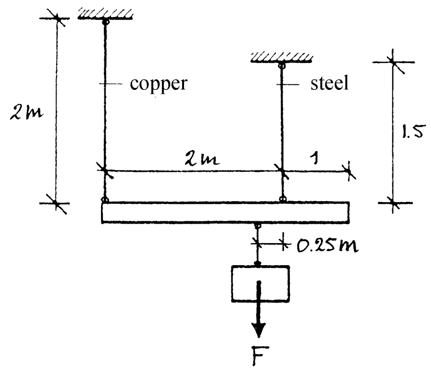
First use your intuition and then check your answer by analysis.

2.24 A rigid beam with a dead weight of 3 kN/m is suspended from two vertical bars, one made of copper, the other of steel. Before connecting the beam to the bars, the lower ends of the bars are at the same level. A load F is also suspended from the beam as indicated.

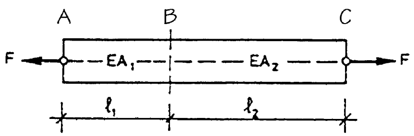
For the copper bar: $E_c = 120$ GPa and $A_c = 100$ mm^2.
For the steel bar: $E_s = 210$ GPa and $A_s = 200$ mm^2.

Questions:

a. Determine the magnitude of the load F for which the beam remains horizontal.
b. Determine the associated drop of the beam.
c. Determine the associated stress in the steel bar.
d. Determine the associated stress in the copper bar.

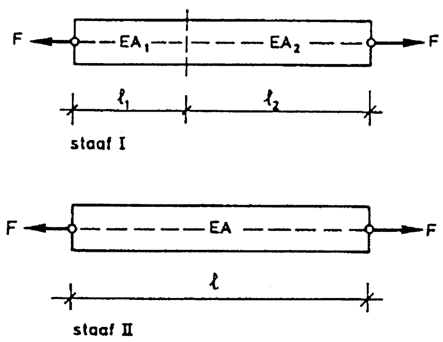


2.25 Bar ABC consists of the two parts AB and BC with different length and different axial stiffnesses (see the figure). The bar is loaded by a tensile force $F = 30$ kN. In the calculation use
$\ell_1 = 0.6$ m, $\ell_2 = 1.2$ m, $EA_1 = 6$ MN and $EA_2 = 8$ MN.



Questions:
a. Determine the strain of AB in ‰.
b. Determine the elongation of BC in mm.
c. Determine the total elongation of ABC in mm.

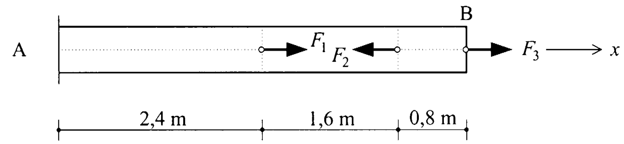
2.26 Compound bar (I), with length $\ell = \ell_1 + \ell_2$, has an axial stiffness EA_1 over length ℓ_1 and an axial stiffness EA_2 over length ℓ_2. Due to a tensile force F, an equally long prismatic bar (II) with axial stiffness EA undergoes the same elongation $\Delta\ell$ as compound bar (I). In the calculation use $\ell_1 = 1.8$ m, $\ell_2 = 2.4$ m, $EA_1 = 30$ MN and $EA_2 = 40$ MN.



Questions:
a. Determine the axial stiffness EA of bar (II) in MN.
b. Determine the elongation of both bars if $F = 70$ kN.

2.27: 1–3 A prismatic bar AB with cross-sectional area $A = 240$ mm^2 is fixed at A and free at B, and is loaded by three axial forces F_1, F_2 and F_3. The modulus of elasticity is $E = 200$ GPa. There are three different cases of loading:

(1) $F_1 = 25$ kN, $F_2 = 15$ kN and $F_3 = 30$ kN.
(2) $F_1 = 25$ kN, $F_2 = 45$ kN and $F_3 = 25$ kN.
(3) $F_1 = 16$ kN, $F_2 = 60$ kN and $F_3 = 12$ kN.

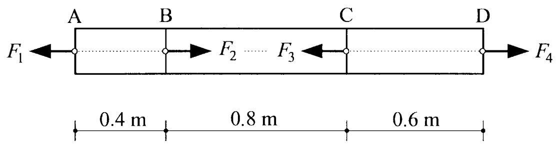


Questions:
a. Determine the N diagram.
b. Determine the variation of the strain along the bar (the ε diagram).
c. Determine the displacement $u_{x;\mathrm{B}}$.

2.28: 1–2 Bar ABCD has a uniform cross-sectional area $A = 400\ \text{mm}^2$ and consists of three materials with different modulus of elasticity:

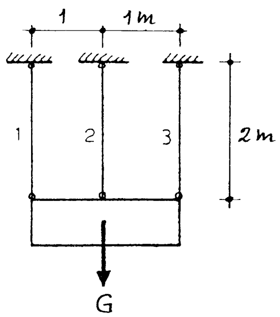
$$E^{(AB)} = 25\ \text{GPa},\ E^{(BC)} = 80\ \text{GPa},\ \text{ and } E^{(CD)} = 200\ \text{GPa}.$$

There are two cases of loading:
(1) $F_1 = 50$ kN, $F_2 = 10$ kN, $F_3 = 20$ kN and $F_4 = 60$ kN.
(2) $F_1 = 25$ kN, $F_2 = 45$ kN, $F_3 = 60$ kN and $F_4 = 40$ kN.



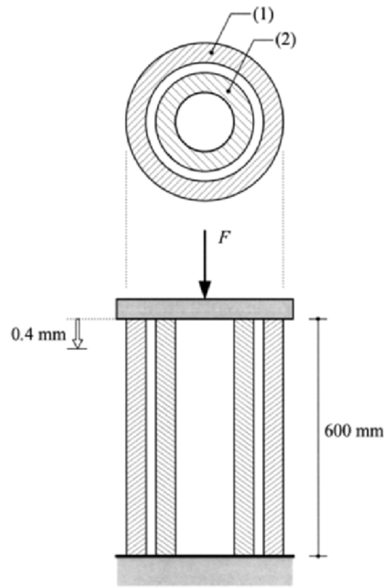
Questions:
a. Determine the N diagram.
b. Determine the variation of the strain along the bar (the ε diagram).
c. Determine the change in length of bar segments AB, BC and CD.
d. Determine the change in length of the total bar.

2.29 A rigid block with weight G is suspended from three vertical bars of equal cross-sectional area $A = 250\ \text{mm}^2$ and equal length $\ell = 2$ m. The outer bars (1) and (3) are made of copper and the centre bar (2) is made of steel. Due to the weight G all bars lengthen by 0.96 mm. The modulus of elasticity of copper is $E_c = 125$ GPa and that of steel is $E_s = 200$ GPa.



Questions:
a. Determine the strains and stresses in the bars.
b. Determine the forces in the bars.
c. Determine weight G of the block.

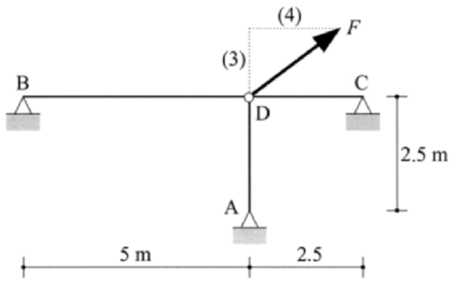
2.30 Two interlocking tubes (1) and (2) with a length of 600 mm are loaded by means of a rigid cover plate by a compressive force F. As a result, both tubes shorten by 0.4 mm. In the calculation use
$A^{(1)} = 3000\ \text{mm}^2$, $A^{(2)} = 1500\ \text{mm}^2$,
$E^{(1)} = 100$ GPa and $E^{(2)} = 70$ GPa.



Questions:
a. Determine the normal force in the outer tube (1).
b. Determine the normal force in the inner tube (2).
c. Determine the magnitude of force F.
d. Determine the amount of shortening if $F = 420$ kN.

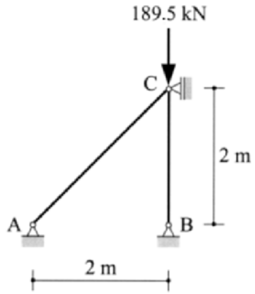
2.31 As problem 2.30, but now $E^{(1)} = 70$ GPa and $E^{(2)} = 100$ GPa.

2.32 The members AD, BD and CD are connected by a hinge at joint D. All members have the same axial stiffness $EA = 125$ MN. Due to the force F the horizontal (component of the) displacement of joint D is 1.6 mm.



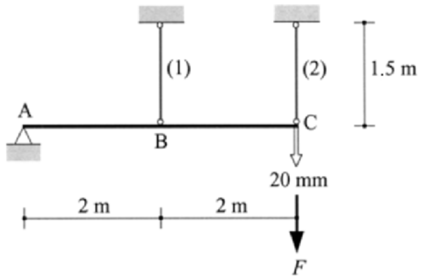
Questions:
a. Determine the forces in members BD and CD.
b. Determine the magnitude of force F from the equilibrium of joint D.
c. Determine the vertical (component of the) displacement of joint D.

2.33 In the structure shown, members AB and BC have the same axial stiffness $EA = 50$ MN. Due to the vertical force of 189.5 kN at C, AC shortens by 3.96 mm.



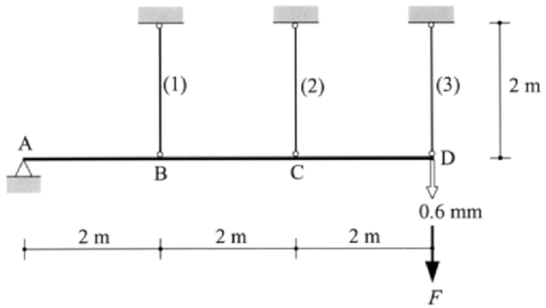
Questions:
a. Determine the force in member AC.
b. Determine the force in member BC from the force equilibrium of joint C.
c. Determine the drop of joint C.
d. Determine the support reactions at A, B and C.

2.34 An entirely rigid and weightless beam ABC is supported by a hinge at A and suspended from two vertical bars at B and C. In unloaded condition, the beam is horizontal. C drops by 20 mm under influence of force F at C. In the calculation, use for the axial stiffnesses of the bars $EA^{(1)} = 1500$ kN and $EA^{(2)} = 3000$ kN.



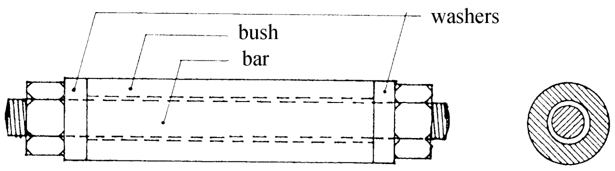
Questions:
a. Determine the strains in the bars.
b. Determine the forces in the bars.
c. Determine the magnitude of force F.
d. Determine the magnitude and direction of the support reaction at A.

2.35 An entirely rigid and weightless beam ABCD is supported by a hinge at A and is suspended from the vertical bars (1) to (3) at B, C and D. In unloaded condition, the beam is horizontal. As a result of force F at D, D drops by 0.6 mm. In the calculation use $A^{(1)} = A^{(2)} = A^{(3)} = 1000 \text{ mm}^2$ and $E^{(1)} = 200$ GPa, $E^{(2)} = 100$ GPa, $E^{(3)} = 300$ GPa.



Questions:
a. Determine the strains and stresses in the bars.
b. Determine the forces in the bars.
c. Determine the magnitude of force F.
d. Determine the magnitude and direction of the support reaction at A.

2.36 A steel bar, with screw thread at each end, is enclosed in a cylindrical steel bush with a length of 300 mm.

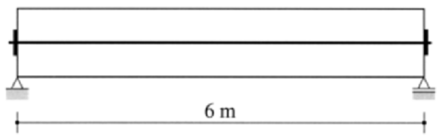


side view cross-section

It is assumed that the washers at the ends of the bush are non-deformable and have negligible thickness. One of the bolts is turned until there is a tensile force of 56 kN in the bar. The cross-sectional area of the bar is $A^{\text{bar}} = 500 \text{ mm}^2$, and of the bush $A^{\text{bush}} = 1000 \text{ mm}^2$. The modulus of elasticity is $E = 210$ GPa.

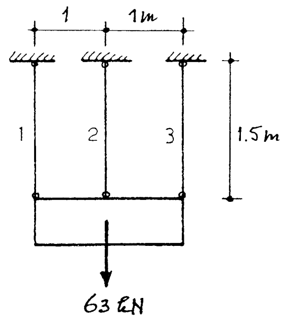
Questions:
a. Determine the shortening of the bush.
b. Determine the elongation of the bar.
c. Determine the length by which the bolt must be turned to achieve a tensile force of 56 kN in the bar.

2.37 In a centrically prestressed concrete beam with a length of 6 m there is the prestressing force $F_p = 1100$ kN. The prestressing tendon is stressed using a jack until the required prestressing force has been achieved. After that, the tendon is locked in position with end anchorage devices.
For the concrete beam, $A_c = 63.2 \times 10^3 \text{ mm}^2$ and $E_c = 30$ GPa. For the prestressing tendon, $A_p = 900 \text{ mm}^2$ and $E_p = 210$ GPa.



Questions:
a. Determine the shortening of the concrete beam due to the prestressing force.
b. Determine elongation of the tendon.
c. Determine the stroke of the jack (the length by which the jack has to draw out the tendon) in order to achieve the prestressing force of 1100 kN.

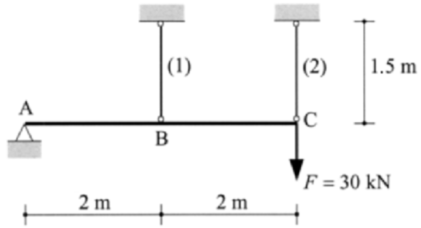
2.38 A rigid block with a weight of 63 kN is suspended from three vertical bars of equal cross-sectional area $A = 250\,\text{mm}^2$ and equal length $\ell = 1.5$ m. The outer bars (1) and (3) are made of steel and the centre bar (2) is made of copper. The modulus of elasticity of copper is $E_c = 125$ GPa and that of steel is $E_s = 200$ GPa.



Questions:
a. Determine the vertical displacement of the block.
b. Determine the forces in the bars.
c. Determine the stresses in the bars.

2.39 As problem 2.38, but now the outer bars (1) and (3) are made of copper and centre bar (2) is made of steel.

2.40 An entirely rigid and weightless beam ABC is supported by a hinge at A and suspended from two vertical bars at B and C. In the unloaded state, the beam is horizontal. The structure is loaded at C by the force $F = 30$ kN. For the axial stiffnesses of the bars use $EA^{(1)} = 3000$ kN and $EA^{(2)} = 1500$ kN.

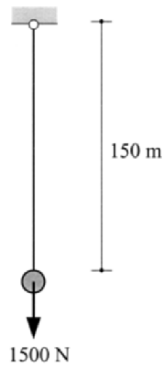


Questions:
a. How much does C drop?
b. How large are the strains and forces in the bars?
c. Determine the magnitude and direction of the support reaction at A.

2.41 A cable is wound stress-free around a drum. From the drum 633.5 m of cable is released into a deep mine shaft. The cable bears only its dead weight. The mass density of the cable is 7.85×10^3 kg/m^3. The modulus of elasticity of the cable is $E = 90 \times 10^3$ N/mm^2. Assume the gravitational field strength is 10 N/kg.

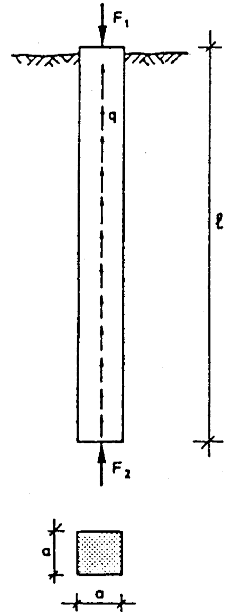
Questions:
a. Determine the elongation of the free-hanging cable.
b. Determine the length for which a maximum stress of 130 N/mm^2 is achieved in the cable.

2.42 A load of 1500 N is suspended from a 150-metre steel wire with circular cross-section and diameter of 6 mm. The mass density of steel is $\gamma = 78.5$ kN/m^3 and the modulus of elasticity is $E = 210$ GPa. Assume the gravitational field strength is $g = 10$ N/kg.



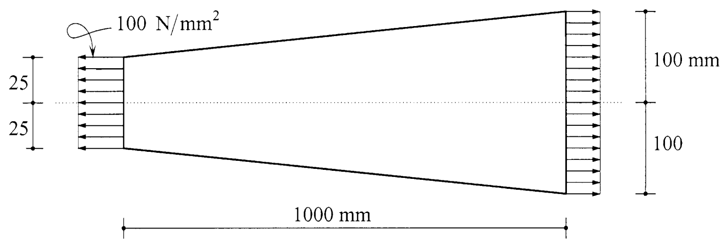
Questions:
a. Determine the elongation of the steel wire due to the load.
b. Determine the elongation of the steel wire due to its dead weight.
c. Determine the total elongation of the steel wire.
d. Determine the maximum normal stress in the steel wire.

2.43 A pile in the ground is loaded by a force F_1. The pile bears this load in a part F_2 at the end, and the rest on friction. The friction forces are modelled as a uniformly distributed axial line load q. The pile has a length ℓ and a square cross-sectional area $a \times a$ (see the figure). The modulus of elasticity is set at 25 GPa. In the calculation use $\ell = 24$ m, $a = 300$ mm, $F_1 = 2.55$ MN and $F_2 = 1.35$ MN.



Questions:
a. Determine the uniformly distributed load q due to friction.
b. Determine the shortening of the pile.

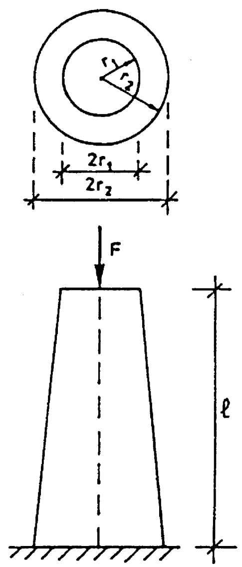
2.44 A 12 mm-thick tapered steel plate is loaded on two opposite sides by uniformly distributed tensile stresses. The stress on the left-hand side is 100 N/m^2. The modulus of elasticity is $E = 210$ GPa.



Questions:

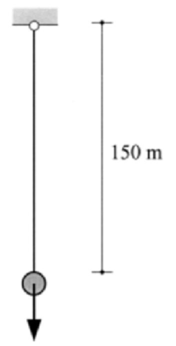
a. Determine the tensile stress on the right-hand side.
b. Determine the normal force in the plate modelled as a line element.
c. Provide a rough estimate of the change in length of the plate , without extensive calculation.
d. Determine the change in length of the plate modelled as a line element accurately.

2.45 A concrete column in the shape of a truncated cone is loaded by a compressive force F (see the figure). The modulus of elasticity is 25 GPa. In the calculation use $F = 4$ MN, $\ell = 2.8$ m, $r_1 = 150$ mm and $r_2 = 250$ mm.



Question:
Determine the shortening of the column.

2.46 A load of 1500 N is suspended from a 150-metre steel wire with a diameter of 6 mm. The mass density of steel is $\gamma = 78.5\ \text{kN/m}^3$ and the modulus of elasticity is $E = 210\ \text{GPa}$. Assume the gravitational field strength is $g = 10$ N/kg.

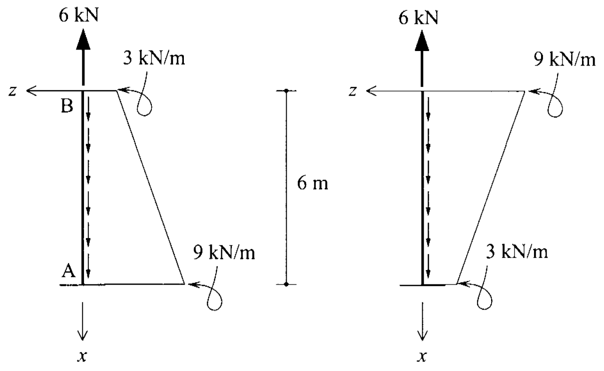


Questions:

a. Determine the vertical displacement u as a function of the distance x (in mm) from the point of suspension. Draw the displacement diagram.
b. Determine the displacement at the free end.

The differential equation for extension (Sections 2.5 to 2.7)

2.47: 1–2 The prismatic column AB, with a fixed support at A and a free end at B, is 6 metres high and has an axial stiffness $EA = 9$ MN. The column is subject to extension in two ways.

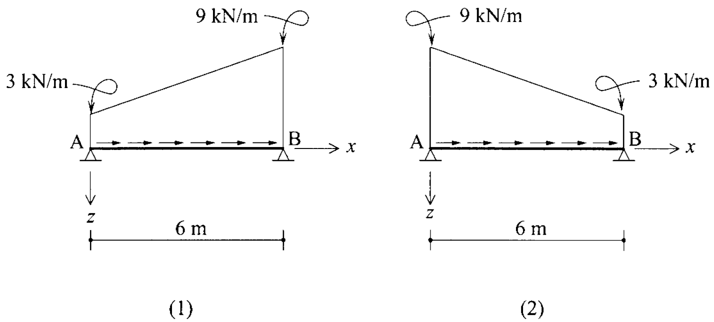


(1) (2)

Questions:

a. Write down the distributed load as a function of x.
b. Using the differential equation for extension, determine the normal force N and displacement u as functions of x.
c. Sketch the N diagram and u diagram.
d. Determine the support reactions; draw them in the directions in which they act on the column.
e. Determine the displacement of column end B.

2.48: 1–2 The prismatic bar AB, supported by hinges at both ends, is axially loaded in two different ways. The length of the bar is 6 metres; the axial stiffness is $EA = 9$ MN.



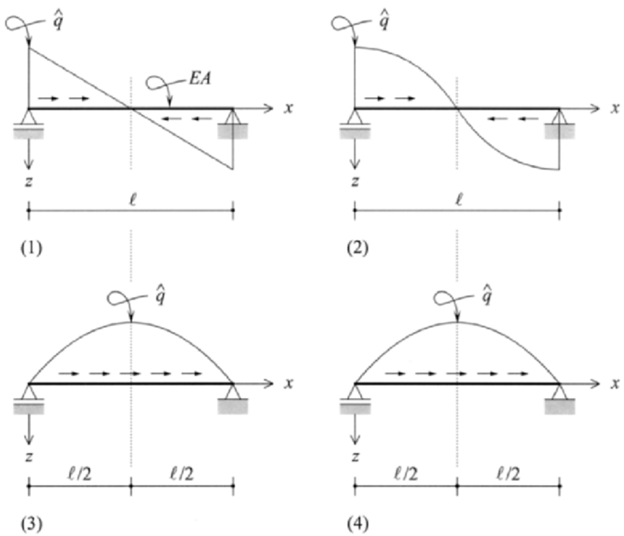
a. Write down the distributed load as a function of x.
b. Using the differential equation for extension, determine both the normal force N and displacement u as functions of x.
c. Sketch the N diagram and u diagram.
d. Determine the support reactions; draw them in the directions in which they act on the bar.

2.49: 1–4 A simply supported prismatic member with length ℓ and axial stiffness EA is subject to extension by four different distributed axial loads $q(x)$ with top value $\hat{q}$:

$$(1)\ q(x) = \hat{q} \cdot \left(1 - 2\frac{x}{\ell}\right), \qquad (2)\ q(x) = \hat{q}\cos\frac{\pi x}{\ell},$$

$$(3)\ q(x) = 4\,\hat{q} \cdot \left(\frac{x}{\ell} - \frac{x^2}{\ell^2}\right), \quad (4)\ q(x) = \hat{q}\sin\frac{\pi x}{\ell}.$$

In the calculation use $\ell = 5$ mm, $\hat{q} = 2.4$ kN/m and $EA = 2$ MN.



Questions:
a. Using the differential equation for extension, determine both the normal force N and displacement u as functions of x.
b. Sketch the N diagram and u diagram.
c. Determine the support reactions; draw them as they act on the member.
d. Determine the displacement of the member end at the roller support.

2.50: 1–4 A prismatic member of length ℓ is supported at both ends by a hinge, and has an axial stiffness EA. The member is subject to extension by the following four distributed axial loads $q(x)$ with top value $\hat{q}$:

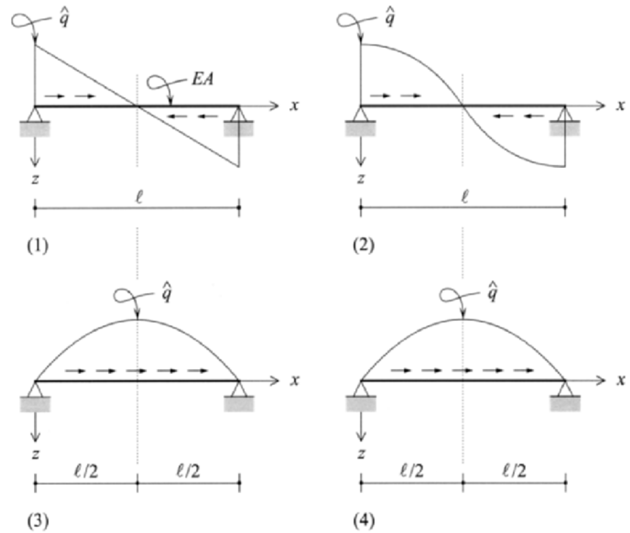
$$(1)\ \ q(x) = \hat{q} \cdot \left(1 - 2\frac{x}{\ell}\right), \qquad (2)\ \ q(x) = \hat{q} \cos\frac{\pi x}{\ell},$$

$$(3)\ \ q(x) = 4\,\hat{q} \cdot \left(\frac{x}{\ell} - \frac{x^2}{\ell^2}\right), \quad (4)\ \ q(x) = \hat{q} \sin\frac{\pi x}{\ell}.$$

In the calculation use $\ell = 5$ mm, $\hat{q} = 2.4$ kN/m and $EA = 2$ MN.

Questions:

a. Use the differential equation for extension to determine both the normal force N and displacement u as a functions of x.
b. Sketch the N diagram and u diagram.
c. Determine the support reactions and draw them as they act on the member.



3 Cross-Sectional Properties

In the calculation of stresses due to extension, the cross-sectional area A plays an important role:

$$\sigma = \frac{N}{A}.$$

We also encounter the cross-sectional area A when calculating the deformation due to extension, namely in the axial stiffness EA of the member (the resistance of the member to axial deformation).

When calculating stresses and deformations due to bending and torsion, we come across other cross-sectional properties.

Below you will find a summary of a number of geometrical characteristics of a cross-section involved in extension, bending and torsion[1] (see Figure 3.1):

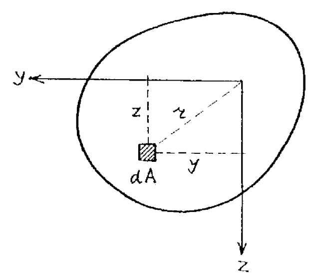


Figure 3.1 Area element dA with its coordinates y and z.

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1 Another (geometric) cross-sectional quantity involved in torsion is the *torsion constant* I_t, or *torsional stiffness factor*. This quantity will be dealt with in Chapter 6.

$$A = \int_A \mathrm{d}A, \qquad S_y = \int_A y \,\mathrm{d}A, \qquad I_{yy} = \int_A y^2 \,\mathrm{d}A,$$

$$S_z = \int_A z \,\mathrm{d}A, \qquad I_{yz} = I_{zy} = \int_A yz \,\mathrm{d}A,$$

$$I_{zz} = \int_A z^2 \,\mathrm{d}A,$$

$$I_p = \int_A r^2 \,\mathrm{d}A.$$

To prepare for the bending problem in Chapter 4, in which cross-sectional properties play an important role, we have devoted this chapter to the geometric characteristics of cross-sections. We cover their definitions and properties, and the ways in which they can be calculated.

The geometric quantities S_y and S_z are called *first moments of area*.[1] They are also referred to as *static moments* (*of area*). They play a role in determining the location of the centroid of the cross-section. We cover this in Section 3.1.

The geometric quantities I_{yy}, $I_{yz} = I_{zy}$ and I_{zz} are called *second moments of area*.[2] These quantities are involved in calculating the stresses and deformations due to bending.[3] I_{yy} and I_{zz} are also referred to as the *moments of inertia* of the cross-section. I_{yz} and I_{zy} are also known as the *product*

1 Other names are *moments of area of the first degree* or *linear moments of area*.

2 Other names are *moments of area of the second degree* or *quadratic moments of area*.

3 See Chapter 4: Members Subject to Extension and Bending.

of inertia of the cross-section. Sometimes, the product of inertia is also referred to as a moment of inertia.

Another quadratic moment of area is the *polar moment of inertia*:

$$I_p = \int_A r^2 \, \mathrm{d}A.$$

We come across the polar moment of inertia in the formulas used for circular cross-sections to determine shear stresses and deformations resulting from torsion.[1]

The moments of inertia are covered in Section 3.2. The difference between the formal approach and engineering practice is explained in Section 3.2.4.

With thin-walled cross-sections, the material can be seen as concentrated in the centre lines, so that the cross-section changes into a line figure. This often simplifies the calculation of cross-sectional properties, and is covered in Section 3.3.

Sections 3.1 to 3.3 end with a number of examples.

In Section 3.4 some remarks are made on the difference that may be noticed between the formal approach used in this book and engineering practice.

The chapter ends with a number of problems in Section 3.5.

[1] See Chapter 6: Members Subject to Torsion.

3.1 First moments of area; centroid and normal centre

In Section 3.1.1 we define the *first moments of area* or *static moments* S_y and S_z, and explain their meaning. The *parallel axis theorem* in Section 3.1.2 gives transformation rules for the static moments due to a *translation* of the coordinate system. The location of the centroid of the cross-sectional area is addressed in Section 3.1.3. The parallel axis theorem plays an important part in this. Finally, Section 3.1.4 includes a number of examples.

3.1.1 Static moments

In a yz coordinate system, the *static moments* or *first moments of area* S_y and S_z for an area A are defined as

$$S_y = \int_A y \, \mathrm{d}A,$$

$$S_z = \int_A z \, \mathrm{d}A.$$

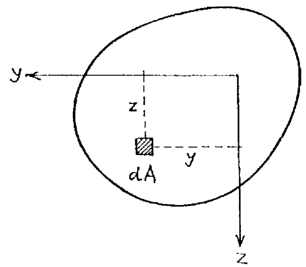

Figure 3.2 The static moments $S_y = \int_A y \, \mathrm{d}A$ and $S_z = \int_A z \, \mathrm{d}A$.

S_y is found by multiplying a small area element dA by its y coordinate (see Figure 3.2) and summing all the contributions over the cross-section. Note that S_y involves the integral of y, and S_z of z. This makes these definitions easy to memorise.

Since small area elements with a positive y coordinate generates positive contributions to S_y and small area elements with a negative y coordinate generates negative contributions, S_y can be interpreted as a measure for the position of the material with respect to the z axis (see Figure 3.3).

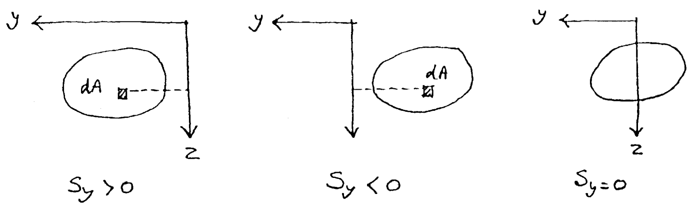

Figure 3.3 The static moment $S_y = \int_A y \, \mathrm{d}A$ is a measure for the position of the material with respect to the z axis.

In the same way S_z is a measure for the position of the material with respect to the y axis.

Note: Pay attention to the fact that

$$S_y = \int_A y \, \mathrm{d}A \text{ is the static moment } \textit{in the } xy \textit{ plane},^1 \text{ and that}$$

$$S_z = \int_A z \, \mathrm{d}A \text{ is the static moment } \textit{in the } xz \textit{ plane}.^2$$

In literature aimed at technical applications, a different notation is sometimes used with S_y and S_z interchanged. We should therefore always be aware of the definitions of S_y and S_z. The notation used here has the benefit that S_y and S_z can be seen as the y and z component of a vector, so that the rules of vector algebra can be applied to these quantities. This is useful, for example, when determining the transformation rules for static moments due to a rotation of the coordinate system.

It is easy to show that the static moment about a line of mirror symmetry is zero. In Figure 3.4, for example, the z axis is a line of symmetry. For each area element[3] $\mathrm{d}A^{(1)}$ there is an equally large and mirror-symmetrical area element $\mathrm{d}A^{(2)}$. Since their y coordinates have opposite signs, their joint contribution to $S_y = \int_A y \, \mathrm{d}A$ is zero. The total contribution of all area elements $\mathrm{d}A^{(1)}$ (to the left of the line of symmetry) therefore cancels the

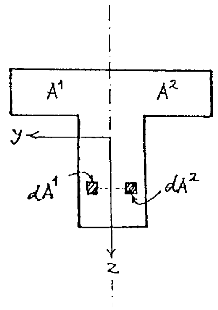

Figure 3.4 $S_y = 0$ when the z axis is a line of symmetry.

[1] Or the moment about the z axis.
[2] Or the moment about the y axis.
[3] Remember that the indices related to an *area* or *region* are applied as *upper index*. Indices related to a *point* or *location* are applied as *sub-index*.

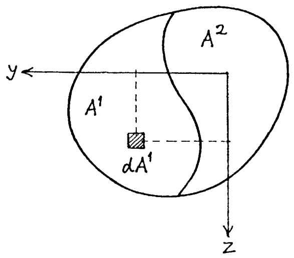

Figure 3.5 A cross-section split into two parts.

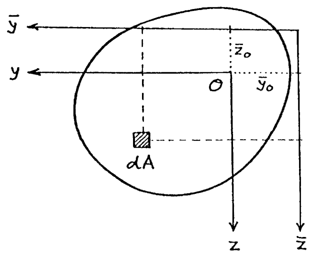

Figure 3.6 Mutually translated coordinate systems.

total contribution of all area elements $\mathrm{d}A^{(2)}$ (to the right of the line of symmetry). Hence

$$S_y = \int_A y\,\mathrm{d}A = \int_{A^{(1)}} y\,\mathrm{d}A^{(1)} + \int_{A^{(2)}} y\,\mathrm{d}A^{(2)} = 0.$$

The static moment of an area divided into two parts can be calculated by summing the static moments of both parts (see Figure 3.5):

$$S_y = \int_A y\,\mathrm{d}A = \int_{A^{(1)}} y\,\mathrm{d}A^{(1)} + \int_{A^{(2)}} y\,\mathrm{d}A^{(2)} = S_y^{(1)} + S_y^{(2)}.$$

In the same way:

$$S_z = S_z^{(1)} + S_z^{(2)}.$$

For complicated cross-sectional shapes, this is a useful tool: split the cross-section in a number of parts that are easy to calculate, such as rectangles, triangles, circles, etc., and sum their separate contributions.

3.1.2 Parallel axis theorem for static moments

For static moments, the so-called *parallel axis theorem* plays an important role in determining the centroid of an area.

For a translated $\overline{yz}$ coordinate system, in which $\overline{y}_0$ and $\overline{z}_0$ are the coordinates of the origin O of the original yz coordinate system (see Figure 3.6), we have

$$\overline{y} = y + \overline{y}_0,$$

$$\overline{z} = z + \overline{z}_0.$$

The static moments in the translated $\overline{yz}$ coordinate system are

$$S_{\overline{y}} = \int_A \overline{y}\,\mathrm{d}A = \int_A y\,\mathrm{d}A + \overline{y}_0 \int_A \mathrm{d}A,$$

$$S_{\overline{z}} = \int_A \overline{z}\,\mathrm{d}A = \int_A z\,\mathrm{d}A + \overline{z}_0 \int_A \mathrm{d}A,$$

or

$$S_{\overline{y}} = S_y + \overline{y}_0 A,$$

$$S_{\overline{z}} = S_z + \overline{z}_0 A.$$

These formulas give the transformation rules due to a translation of the coordinate axes, and are referred to as the *parallel axis theorem for static moments*.

3.1.3 Centroid and normal centre

The *centroid* C is defined as that point of an area A for which the static moments of the area are zero when the origin of the yz coordinate system is chosen there:

$$S_y = \int_A y\,\mathrm{d}A = 0,$$

$$S_z = \int_A z\,\mathrm{d}A = 0.$$

The *normal centre* NC is defined as that point of the cross-sectional area where the resultant of all normal stresses due to extension has its point of application.

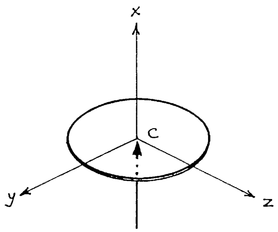

Figure 3.7 A plate in the gravitational field, with a uniformly distributed mass, can be kept in equilibrium by a single force at its centroid C.

In Section 2.4 we showed for a homogeneous cross-section the static moments about the normal centre NC are also zero. In a homogeneous cross-section, the normal centre apparently coincides with the centroid C of the cross-sectional area.[1]

The word *centroid* is derived from the statics of bodies in the gravitational field. A plate, in the shape of the cross-section, with a uniformly distributed mass, can be maintained in equilibrium by a single force at its centroid (see Figure 3.7). If the origin of the coordinate system is chosen at the centroid, as defined above, the moment equilibrium about the z axis requires

$$\int_A \rho g y \, \mathrm{d}A = \rho g \int_A y \, \mathrm{d}A = 0,$$

in which ρ is the mass per area and g the gravitational field strength. Since ρ and g are equal for all area elements $\mathrm{d}A$, they can be left outside the integral. The moment equilibrium about the z axis therefore leads to the condition that the static moment S_y must be zero.

In the same way, the moment equilibrium about the y axis leads to the condition that the static moment S_z must be zero.

1 Since in a *homogeneous cross-section* the centroid C and the normal centre NC coincide, both concepts are often interchanged, even though they are clearly defined differently. But note: for *inhomogeneous cross-sections*, the centroid and the normal centre do *not* coincide and the two concepts may no longer be interchanged! We recommend keeping the two concepts distinct even for homogeneous cross-sections.

In the (non-centroidal) $\overline{yz}$ coordinate system in Figure 3.8, in which $\bar{y}_C$ and $\bar{z}_C$ are the coordinates of the centroid C, we have

$$\bar{y} = y + \bar{y}_C,$$

$$\bar{z} = z + \bar{z}_C.$$

Using the parallel axis theorem from Section 3.1.2 we find

$$S_{\bar{y}} = S_y + \bar{y}_C A,$$

$$S_{\bar{z}} = S_z + \bar{z}_C A.$$

Since the (centroidal) yz coordinate system passes through centroid C,

$$S_y = 0,$$

$$S_z = 0.$$

Hence

$$S_{\bar{y}} = \bar{y}_C A,$$

$$S_{\bar{z}} = \bar{z}_C A.$$

The static moments $S_{\bar{y}}$ and $S_{\bar{z}}$ of an area A are equal to the product of that area and the $\bar{y}$ and $\bar{z}$ coordinate respectively of the centroid C of that area.

Also the converse applies: if $S_{\bar{y}}$ and $S_{\bar{z}}$ are known, the coordinates of the

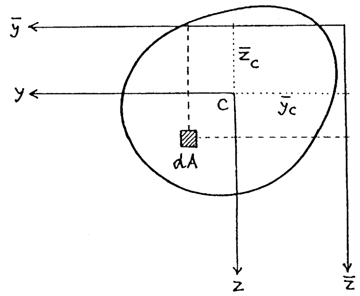

Figure 3.8 Centroidal yz coordinate system and non-centroidal $\overline{yz}$ coordinate system. $\bar{y}_C$ and $\bar{z}_C$ are the coordinates of centroid C in the translated $\overline{yz}$ coordinate system.

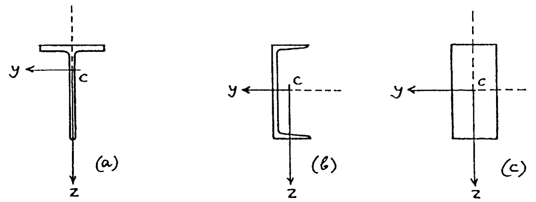

Figure 3.9 In mirror-symmetrical cross-sections, the centroid C is on the line of symmetry.

centroid C[1] are

$$\bar{y}_C = \frac{S_{\bar{y}}}{A}$$

$$\bar{z}_C = \frac{S_{\bar{z}}}{A}.$$

Note: When determining the static moments of an area A, we can consider the area as being concentrated at its centroid C.

Cross-sections having mirror symmetry
A plane figure that by reflection in a line m is reflected on itself is known as having *mirror symmetry*[2] with respect to m. m is known as a *line of (mirror) symmetry*.

For cross-sectional areas having mirror symmetry *the centroid* C *is on the line of symmetry*. In Figure 3.9a, $S_y = 0$. Therefore the centroid C is on the z axis. In Figure 3.9b, $S_z = 0$, and the centroid C is on the y axis. In Figure 3.9c there are two lines of symmetry and therefore the centroid C coincides with the intersection of both lines.

Cross-sections having point symmetry
A plane figure that by reflection in a point C is reflected on itself is said to have *point symmetry*[3] with respect to C. C is known as a *centre of (point) symmetry*.

For cross-sectional areas having point symmetry, *the centroid coincides*

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1 See the derivation in Section 2.4.
2 Mirror symmetry is also referred to as *reflection symmetry* or *line symmetry*.
3 Point symmetry is also referred to as *polar symmetry*.

with the centre of symmetry C. Two examples are given in Figure 3.10.

For each area element $dA^{(1)}$ there exists an equal and point-symmetrical area element $dA^{(2)}$. In a coordinate system with the origin at C, the centre of symmetry, the y and z coordinates of these elements have opposite signs and their joint contributions to respectively $\int_A y\,dA$ and $\int_A z\,dA$ are zero. Therefore, summed over the entire area, the result is

$$S_y = 0 \text{ and } S_z = 0.$$

This affirms that the centroid of the area coincides with the centre of point symmetry.

Cross-sections having rotational symmetry
Plane figures that after rotation through an angle α about a point C coincide with themselves are said to have *rotational symmetry* with respect to C. C is known as the *centre of rotation* and α is known as the *angle of rotation*.

With plane figures, point symmetry is a special case of rotational symmetry (the angle of rotation α is then 180°).

For cross-sectional areas having rotational symmetry, the *centroid coincides with the centre of rotation* C. Two examples are given in Figure 3.11.

For the cross-section in Figure 3.11a, the angle of rotation $\alpha = 72°$. This regular pentagon has five lines of symmetry, of which two are shown. The centroid is on the intersection of the lines of symmetry; that is also in the centre of rotation.

The hollow cross-section with flaps in Figure 3.11b has no line of symmetry. The angle of rotation is $\alpha = 120°$. The shape of the cross-section can be imagined by rotating part PQ twice through 120° about C (see

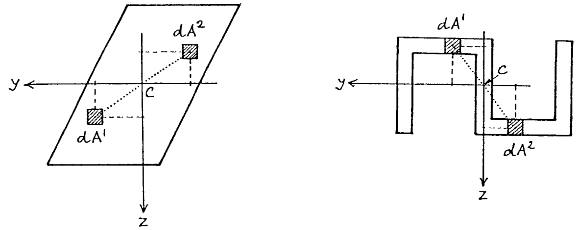

Figure 3.10 In point-symmetrical cross-sections, the centroid C coincides with the centre of point symmetry.

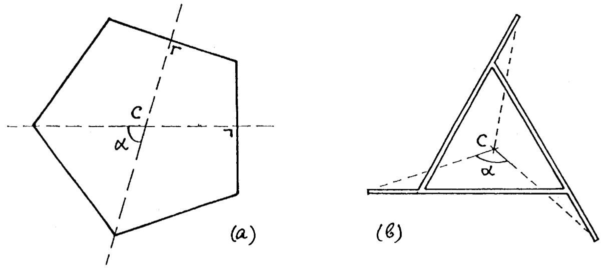

Figure 3.11 In rotation-symmetrical cross-sections, the centroid C coincides with the centre of rotational symmetry.

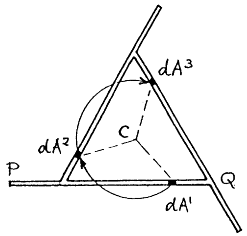

Figure 3.12 The centroid of the three equally large area elements $\mathrm{d}A^{(1)}$, $\mathrm{d}A^{(2)}$ and $\mathrm{d}A^{(3)}$ is at the centre of rotational symmetry.

Figure 3.12). For each area element $\mathrm{d}A^{(1)}$ on PQ there are two equally large rotational symmetric area elements $\mathrm{d}A^{(2)}$ and $\mathrm{d}A^{(3)}$. It is easy to show that the centroid of $\mathrm{d}A^{(1)}$, $\mathrm{d}A^{(2)}$ and $\mathrm{d}A^{(3)}$ is at the centre of rotation C. By repeating this procedure for all other area elements, we find that the centroid of the entire cross-section coincides with the centre of rotation.

To illustrate the derived formulas, we will determine the area of a cross-section and the location of its centroid in six examples.

3.1.4 Examples

Example 1
Given the triangular cross-section in Figure 3.13a.

Questions:
a. Determine the cross-sectional area A.
b. Determine the z coordinate of centroid C.[1]

Solution:
a. The width $b(z)$ of the dark strip in Figure 3.13b is

$$b(z) = \frac{z}{h}\, b.$$

[1] In general, the yz coordinate system is chosen in such a way that the origin of the coordinate system coincides with the centroid (normal centre) of the cross-section. Other yz coordinate systems are generally overlined or accented. Only when there can be no confusion are we allowed to deviate from this rule, as in this example.

For a height $\mathrm{d}z$ the area $\mathrm{d}A$ of the strip is

$$\mathrm{d}A = b(z)\,\mathrm{d}z = \frac{b}{h}z\,\mathrm{d}z.$$

The area A of the triangle is found by summing the areas $\mathrm{d}A$ of all the strips across height h. This is done by integrating:

$$A = \int_0^h b(z)\,\mathrm{d}z = \frac{b}{h}\int_0^h z\,\mathrm{d}z = \frac{b}{h}\cdot \tfrac{1}{2}z^2\Big|_0^h = \tfrac{1}{2}bh.$$

b. For the z coordinate of centroid C,

$$z_\mathrm{C} = \frac{S_z}{A}.$$

Since all area elements $\mathrm{d}A$ on the dark strip have the same z coordinate, S_z is easily calculated:

$$S_z = \int_A z\,\mathrm{d}A = \int_0^h z\cdot\frac{b}{h}z\cdot\mathrm{d}z = \int_0^h \frac{b}{h}z^2\,\mathrm{d}z = \frac{b}{h}\cdot\tfrac{1}{3}z^3\Big|_0^h = \tfrac{1}{3}\,bh^2.$$

Therefore

$$z_\mathrm{C} = \frac{\tfrac{1}{3}\,bh^2}{\tfrac{1}{2}\,bh} = \tfrac{2}{3}\,h.$$

This is in line with the known fact that the medians in a triangle intersect in one point (they are concurrent) and divide each other in the ratio 1:2 (see Figure 3.13c).

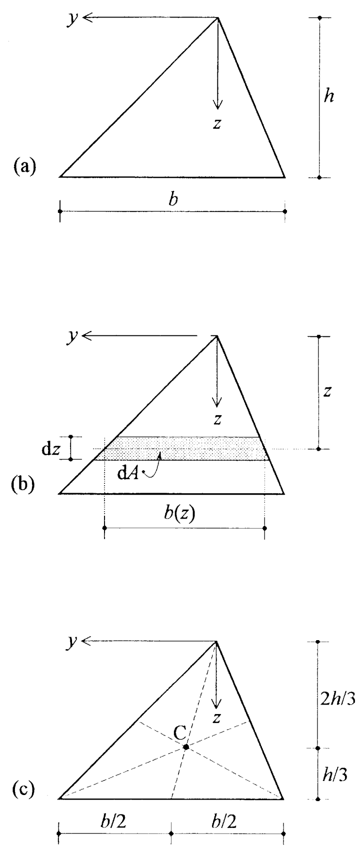

Figure 3.13 The centroid of a triangle.

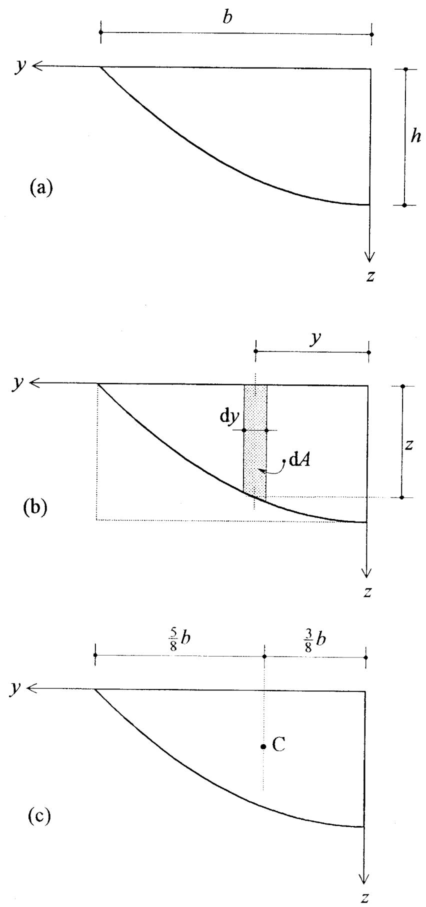

Figure 3.14 The centroid of a quadrant bounded by a parabola.

Example 2

Given the cross-section in Figure 3.14a, bounded by the lines $y = 0$, $z = 0$ and the parabola $z = h(1 - y^2/b^2)$.

Questions:

a. Determine the area A of the cross-section.
b. Determine the y coordinate of centroid C.

Solution:

a. The area of the dark strip in Figure 3.14b is

$$\mathrm{d}A = z\,\mathrm{d}y = h\left(1 - \frac{y^2}{b^2}\right)\mathrm{d}y.$$

The total area A of the cross-section is found by integrating:

$$A = \int_0^b z\,\mathrm{d}y = h\int_0^b \left(1 - \frac{y^2}{b^2}\right)\mathrm{d}y = h\left(y - \frac{1}{3}\frac{y^3}{b^2}\right)\Bigg|_0^b = \frac{2}{3}bh.$$

Note that the area enclosed by the parabola in Figure 3.14b is equal to 2/3 of the rectangular area.

b. The contribution of the dark strip to the static moment S_y is

$$\mathrm{d}S_y = y\,\mathrm{d}A = yz\,\mathrm{d}y = hy\left(1 - \frac{y^2}{b^2}\right)\mathrm{d}y.$$

By summing the contributions of all the strips, or in other words by integrating, we find

$$S_y = \int_A y \, dA = h \int_0^b y \left(1 - \frac{y^2}{b^2}\right) dy = h \left(\tfrac{1}{2} y^2 - \tfrac{1}{4} \frac{y^4}{b^2}\right)\Bigg|_0^b = \tfrac{1}{4} b^2 h.$$

For the y coordinate of the centroid this gives (see Figure 3.14c)

$$y_C = \frac{S_y}{A} = \frac{\tfrac{1}{4} b^2 h}{\tfrac{2}{3} bh} = \tfrac{3}{8} b.$$

Example 3

Given the cross-section in Figure 3.15a in the shape of a thin-walled half ring. The radius of the ring (with respect to its centre line) is R, and the wall thickness is t. Thin-walled means that the wall thickness t is much is smaller than the radius R ($t \ll R$).

Questions:

a. Determine the cross-sectional area A.
b. Determine the location of centroid C.

Solution:

a. The small dark part of the ring in Figure 3.15b, with length $R\, d\varphi$ and thickness t, has an area

$$dA = t \cdot R \, d\varphi.$$

By integrating we find the cross-sectional area:

$$A = \int_0^\pi t R \, d\varphi = t\varphi\big|_0^\pi = \pi R t.$$

Note that the area of the thin-walled half-ring is equal to the product of the

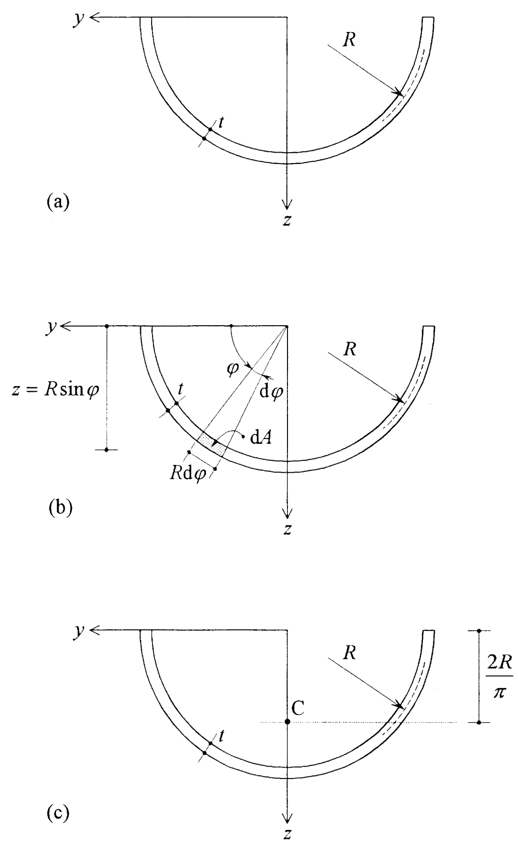

Figure 3.15 The centroid of a thin-walled half-ring.

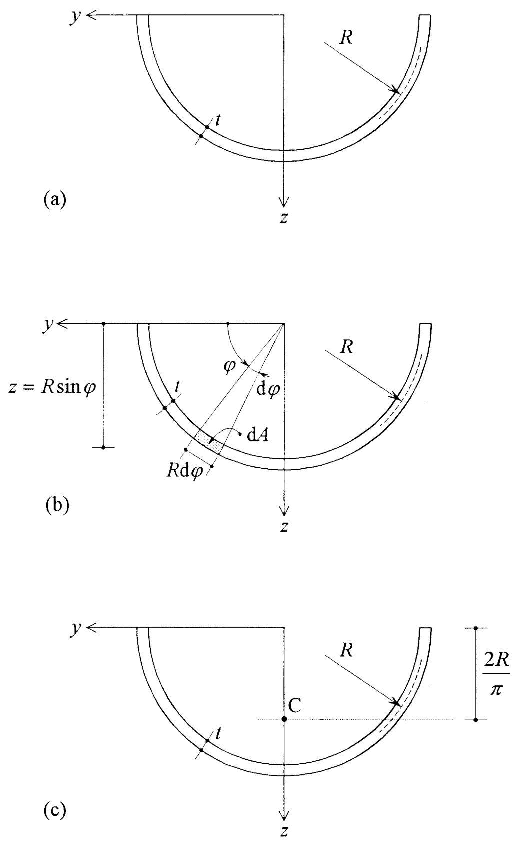

Figure 3.15 The centroid of a thin-walled half-ring.

developed length πR and the wall-thickness t.

b. The centroid of the cross-sectional area is located on the z axis, as this is a line of symmetry. Therefore

$$y_C = 0.$$

The z coordinate of the centroid follows from

$$z_C = \frac{S_z}{A}.$$

The dark area element in Figure 3.15b gives a contribution dS_z to S_z:

$$dS_z = z\,dA = R\sin\varphi \cdot tR\,d\varphi = R^2 t\sin\varphi\,d\varphi.$$

By integrating we find

$$S_z = \int_A z\,dA = R^2 t\int_0^{\pi} \sin\varphi\,d\varphi = -R^2 t\cos\varphi\Big|_0^{\pi} = 2R^2 t$$

and (see Figure 3.15c)

$$z_C = \frac{S_z}{A} = \frac{2R^2 t}{\pi R t} = \frac{2R}{\pi} \approx 0.64R.$$

Example 4
Given the semicircular cross-section with radius R in Figure 3.16a.

Question:
Determine the coordinates y_C and z_C of centroid C.

Solution:
The centroid is on the line of symmetry:

$$y_C = 0.$$

For determining z_C we consider a plate in the gravitational field, in the shape of the cross-section, and subject to a uniformly distributed dead weight. The centroid is the point where the resultant of the uniformly distributed dead weight applies.

The semicircle can be seen as being composed of a very large number of very small triangles. One of these triangles has been darkened (see Figure 3.16b). Of each triangle, the centroid is at "one third of its height". The centroids of all triangles form a half ring with radius r:

$$r = \tfrac{2}{3} R.$$

This means that the total dead weight of the semicircular plate can be seen as being (uniformly) concentrated in a half ring. Using the results from the previous example (the centroid of a thin-walled half ring), the location of the centroid is (see Figure 3.16c)

$$z_C = \frac{2r}{\pi} = \frac{4R}{3\pi} \approx 0.42R.$$

Example 5
Given the L-shaped cross-section in Figure 3.17.

Questions:
a. Determine the cross-sectional area.
b. Determine the location of the centroid.

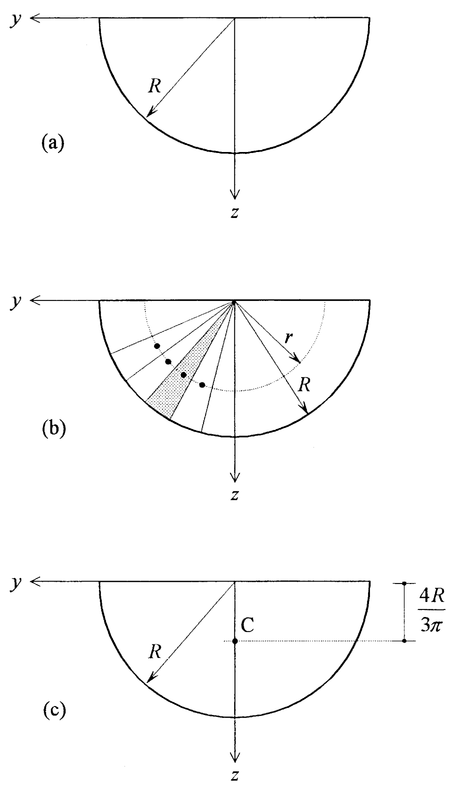

Figure 3.16 The centroid of a semicircular cross-section.

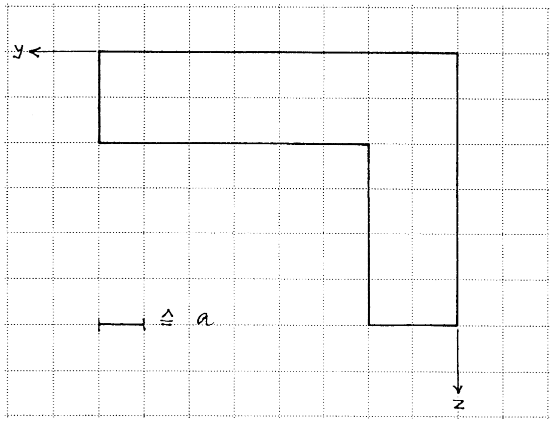

Figure 3.17 An L-shaped cross-section.

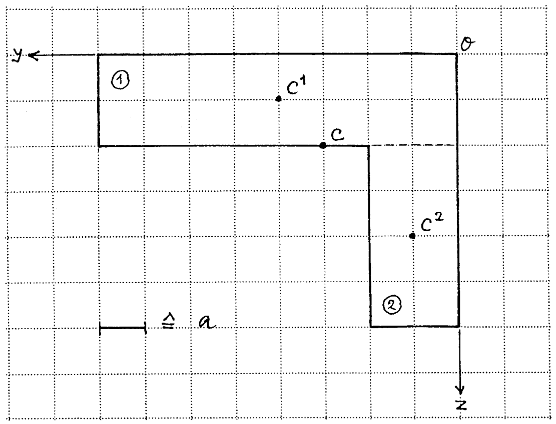

Figure 3.18 The L-shaped cross-section seen as the sum of two rectangles.

Solution:
In the same way as we can determine an area by summing the areas of the constituent parts, we can determine the static moments by summing the static moments of the constituent parts (see Section 3.2.1). This property is useful for complicated cross-sectional shapes. We split the cross-section into n parts that are easy to calculate, and find (with $i = 1, 2, \ldots, n$)

$$A = \sum A^i,$$
$$S_y = \sum S_y^i,$$
$$S_z = \sum S_z^i.$$

The $\sum$ symbol means there has to be summed over all n constituent parts.

Another useful property is that, when calculating a static moment, the area can be considered to be concentrated at its centroid (see Section 3.2.2.). The static moment S_y^i of part i is therefore equal to the product of its area (A^i) and the y coordinate of its centroid (y_C^i):

$$S_y^i = y_\text{C}^i A^i.$$

In the same way

$$S_z^i = z_\text{C}^i A^i.$$

We now have the following formulas for determining the area A and static moments S_y and S_z of a cross-section divided into n parts:

$$A = \sum A^i,$$
$$S_y = \sum S_y^i = \sum y_\text{C}^i A^i,$$

$$S_z = \sum S_z^i = \sum z_C^i A^i.$$

Returning to the example, the L-shaped cross-section can be split into two rectangles (1) and (2), for which we know the area (A^i) and the location of the centroid $(y_C^i; z_C^i)$ (see Figure 3.18). Using the above-mentioned formulas, the area and static moments of the cross-section are calculated in Table 3.1.

The cross-section can also be seen as the difference between two rectangles (see Figure 3.19). The area of rectangle (2) must now be used negatively. Table 3.2 shows that this approach gives the same values for A, S_y and S_z.

The location of the centroid C is

$$y_C = \frac{S_y}{A} = \frac{72a^2}{24a^2} = 3a,$$

$$z_C = \frac{S_z}{A} = \frac{48a^3}{24a^2} = 2a.$$

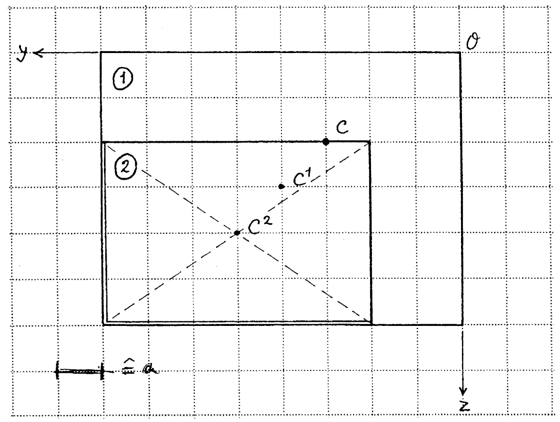

Figure 3.19 The L-shaped cross-section seen as the difference between two rectangles.

Table 3.1

Section i	A^i	y_C^i	z_C^i	$S_y^i = y_C^i A^i$	$S_z^i = z_C^i A^i$
1	$16a^2$	$4a$	a	$64a^3$	$16a^3$
2	$8a^2$	a	$4a$	$8a^3$	$32a^3$
$\sum$	$A = 24a^2$		$\sum$	$S_y = 72a^3$	$S_z = 48a^3$

Table 3.2

Section i	A^i	y_C^i	z_C^i	$S_y^i = y_C^i A^i$	$S_z^i = z_C^i A^i$
1	$48a^2$	$4a$	$3a$	$192a^3$	$144a^3$
2	$-24a^2$	$5a$	$4a$	$-120a^3$	$-96a^3$
$\sum$	$A = 24a^2$		$\sum$	$s_y = 72a^3$	$S_z = 48a^3$

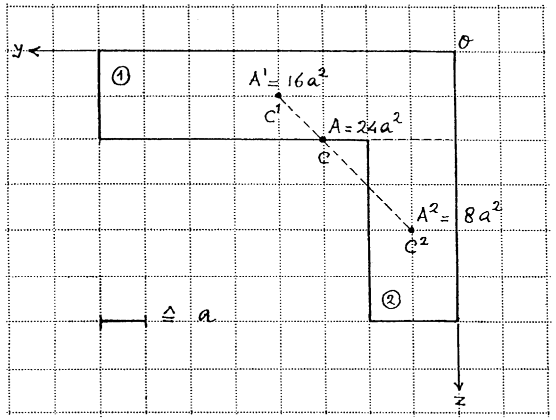

Figure 3.20 The L-shaped cross-section seen as a plate in the gravitational field, with a uniformly distributed mass.

Table 3.3

Circle i	A^i	y^i_C	$S^i_y = y^i_C A^i$
1	$4\pi R^2$	0	0
2	$-\pi R^2$	R	$-\pi R^3$
$\sum$	$A = 3\pi R^2$	$\sum$	$S_y = -\pi R^3$

Alternative solution:
The centroid of the cross-section can be found by considering the cross-section as a plate in the gravitational field with a uniformly distributed dead weight.

Assume that the dead weight of (a part of) the plate is equal to its area (see Figure 3.20). The resultant of the dead weight of rectangle (1) is equal to area $A^{(1)}$ and has its line of action through centroid $C^{(1)}$:

$$A^{(1)} = 16a^2.$$

The resultant of the dead weight of rectangle (2) is equal to area $A^{(2)}$ and has its line of action through centroid $C^{(2)}$:

$$A^{(2)} = 8a^2.$$

The total dead weight is

$$A = A^{(1)} + A^{(2)} = 24a^2.$$

The line of application of A passes through the centroid C we are looking for. The centroid C is on the line segment $C^{(1)}C^{(2)}$. The location of C is determined by the ratio

$$\frac{CC^{(1)}}{CC^{(2)}} = \frac{A^{(2)}}{A^{(1)}} = \frac{8a^2}{16a^2} = \tfrac{1}{2}\,.$$

Using the grid, the location of centroid C can directly be indicated in Figure 3.20.

Example 6
Given the cross-section in Figure 3.21a, where a small circle (2) with radius R has been removed from the large circle (1) with radius $2R$.

Question:
Determine the coordinates y_C and z_C of the centroid C.

Solution:
The centroid is on the y axis because this is a line of symmetry:

$$z_C = 0.$$

The y coordinate follows from

$$y_C = \frac{S_y}{A}.$$

A and S_y have been determined in Table 3.3.

The y coordinate of centroid C is (see Figure 3.21b)

$$y_C = \frac{S_y}{A} = \frac{-\pi R^3}{3\pi R^2} = -\frac{1}{3}R.$$

3.2 Second moments of area

In Section 3.2.1 we define the *second moments of area* I_{yy}, I_{yz}, I_{zy} and I_{zz}, and explain their meaning.

With *Steiner's parallel axis theorem* in Section 3.2.2 we obtain transformation rules for the second moments of area due to a *translation* of the

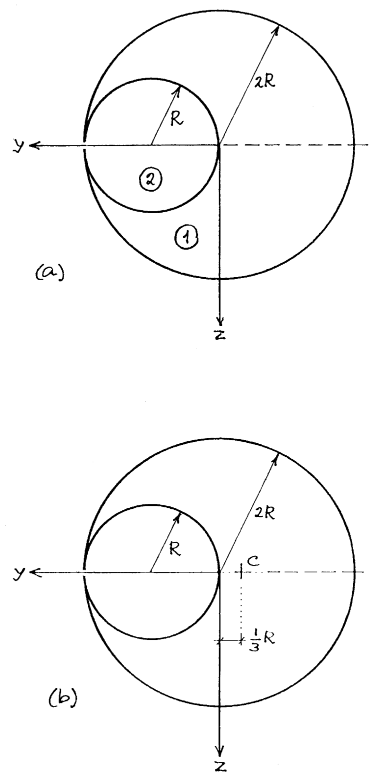

Figure 3.21 The centroid of a circular cross-section with a circular hole.

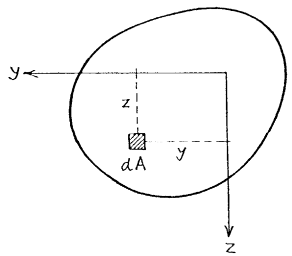

Figure 3.22 An area element dA with its coordinates y and z.

coordinate system. The *polar moment of inertia* I_p, also a second moment of area, is covered in Section 3.2.3.

Unfortunately there is a difference between the formal definitions and those mostly used in engineering practice. These differences are dealt with in Section 3.2.4. To conclude, a number of examples are given in Section 3.2.5.

3.2.1 Moments of inertia

In a yz coordinate system, the *second moments of area* I_{yy}, I_{yz}, I_{zy} and I_{zz} for an area A are defined as (see Figure 3.22)

$$I_{yy} = \int_A y^2 \, \mathrm{d}A,$$

$$I_{yz} = I_{zy} = \int_A yz \, \mathrm{d}A,$$

$$I_{zz} = \int_A z^2 \, \mathrm{d}A.$$

I_{yy} and I_{zz} are generally referred to as the *moments of inertia* of the cross-section and $I_{yz} = I_{zy}$ is known as the *product of inertia*. These geometric quantities are used in determining the stresses and deformations due to bending. Sometimes, the product of inertia is also referred to as a moment of inertia (when generalising).

Note that the double index in I_{yy}, $I_{yz} = I_{zy}$ and I_{zz} returns under the integral symbol. This makes these definitions easy to memorise.

I_{yy} is found by multiplying an area element dA by the square of its y coordinate and by summing all contributions over the cross-section. I_{yy}

is therefore always positive. I_{yy} can be seen as a measure of the amount of area with (in an absolute sense) large y coordinate.

In the same way, I_{zz} is always positive and can be seen as a measure of the amount of area with (in an absolute sense) large z coordinate.

Figure 3.23 shows three cross-sections, each with the same area A. The material in cross-section (1) is far more stretched in the z direction than in the y direction (in an absolute sense the area elements have far larger z coordinates than y coordinates), so that we can conclude that

$$I_{yy}^{(1)} < I_{zz}^{(1)}.$$

For the circular cross-section (2) symmetry implies that

$$I_{yy}^{(2)} = I_{zz}^{(2)}.$$

The circular cross-section (3) has been moved with respect to cross-section (2) in the negative y direction. This displacement does not influence the value of I_{zz}, therefore

$$I_{zz}^{(3)} = I_{zz}^{(2)}.$$

I_{yy} does change, however. Since cross-section (3) has far more material with a large (be it negative) y coordinate than with a large z coordinate,

$$I_{yy}^{(3)} > I_{zz}^{(3)}.$$

The *products of inertia* I_{yz} and I_{zy} are equal by definition. They are found by multiplying all the area elements dA by their y and z coordinate and summing all contributions over the cross-section. The product of inertia is

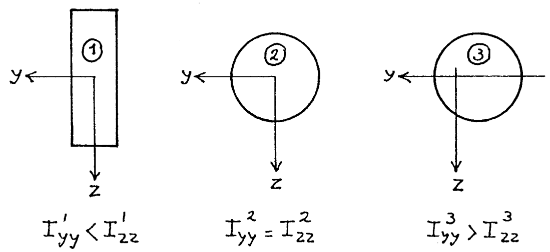

Figure 3.23 The moments of inertia I_{yy} and I_{zz} are a measure of the amount of material with (in an absolute sense) large y coordinate and z coordinate respectively.

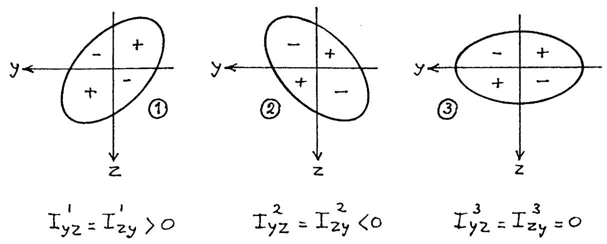

Figure 3.24 The product of inertia $I_{yz} = I_{zy}$ is a measure of the distribution of the material across the quadrants.

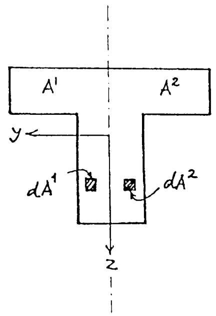

Figure 3.25 In a mirror-symmetrical cross-section, with one of the coordinate axes along the line of symmetry, the product of inertia is $I_{yz} = I_{zy} = 0$.

a measure for the distribution of the material across the quadrants.

In Figure 3.24 the same ellipsoidal cross-section has been placed in three different positions. Cross-section (1) contains more material in the positive quadrants (and moreover with larger y and z coordinates) than in the negative quadrants, so that

$$I_{yz}^{(1)} = I_{zy}^{(1)} > 0.$$

In cross-section (2) the material in the negative quadrants dominates:

$$I_{yz}^{(2)} = I_{zy}^{(2)} < 0.$$

For cross-section (3), symmetry considerations imply

$$I_{yz}^{(3)} = I_{zy}^{(3)} = 0.$$

If the cross-section has a line of symmetry and one of the coordinate axes coincides with the line of symmetry, we can show that the product of inertia is zero.

In Figure 3.25 the z axis is a line of symmetry. For each area element $\mathrm{d}A^{(1)}$ there is an equal mirror-symmetrical area element $\mathrm{d}A^{(2)}$. Both area elements have the same z coordinate, but their y coordinates have opposite signs. As a result their joint contribution to $\int_A yz\,\mathrm{d}A$ is zero.

For all area elements $\mathrm{d}A^{(1)}$ (to the left of the line of symmetry) the contribution to $\int_A yz\,\mathrm{d}A$ cancels those for area elements $\mathrm{d}A^{(2)}$ (to the right of the line of symmetry), so that for the entire cross-section,

$$I_{yz} = \int_A yz\,\mathrm{d}A = \int_{A^{(1)}} yz\,\mathrm{d}A^{(1)} + \int_{A^{(2)}} yz\,\mathrm{d}A^{(2)} = 0.$$

As before, the moments of inertia of an area divided into two parts can be calculated by summing the moments of inertia of the separate parts (see Figure 3.26):

$$I_{yz} = I_{yz}^{(1)} + I_{yz}^{(2)}.$$

In the same way

$$I_{yy} = I_{yy}^{(1)} + I_{yy}^{(2)},$$

$$I_{zz} = I_{zz}^{(1)} + I_{zz}^{(2)}.$$

For more complicated cross-sectional shapes, this is a useful property: divide the cross-section into a number of parts that can be easily calculated, such as rectangles, triangles, circles, etc., and sum their separate contributions.

Note: Pay attention to the fact that

$I_{yy} = \int_A y^2 \, \mathrm{d}A$ is the moment of inertia *in the xy plane*,[1] and that

$I_{zz} = \int_A z^2 \, \mathrm{d}A$ is the moment of inertia *in the xz plane*.[2]

In much of the literature aimed at technical applications, a different notation is used: I_{yy}, I_{zz}, I_{yz} are denoted by I_y, I_z, C_{yz} respectively. We should

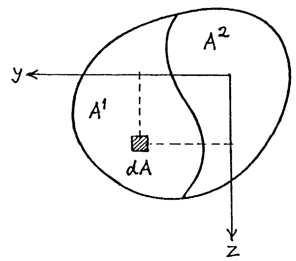

Figure 3.26 A cross-sectional area split into two parts.

1 Or: moment of inertia about the z axis.
2 Or: moment of inertia about the y axis.

therefore always be aware of the definitions of the moments of inertia. See also Section 3.2.4.

The benefit of the notation used here is that it is in line with the notation for the components of a *second-order tensor*. The geometric quantities I_{yy}, $I_{yz} = I_{zy}$ and I_{zz} behave as components of a second-order tensor, so that we can now use the known rules of tensor calculus. This is particularly useful when investigating the transformation of these quantities due to a rotation of the coordinate system.[1]

To conclude this section, we define a number of common concepts.

Centroidal moments of inertia
A centroidal coordinate system is a coordinate system with its origin at the centroid C of the cross-sectional area. The moments of inertia of a cross-sectional area in a centroidal coordinate system are known as the *centroidal moments of inertia*.

Principal axes, principal directions and principal values
If in a certain centroidal yz coordinate system the product of inertia $I_{yz} = I_{zy}$ is zero, then the coordinate axes are referred to as the *principal axes* of the cross-section. The directions of these axes are known as the *principal directions*. The values of the moments of inertia I_{yy} and I_{zz} in the principal coordinate system are known as the *principal values*.

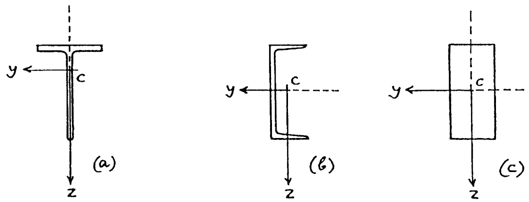

Figure 3.27 Principal axes.

If one of the coordinate axes is a line of symmetry, then $I_{yz} = I_{zy} = 0$, and the coordinate axes are principal axes (see Figure 3.27).

Radius of inertia
Assume that for determining the moments of inertia I_{yy} and I_{zz} the cross-

[1] The benefits mentioned become apparent in a number of subjects covered in Volume 4. See also Sections 9.4 and 9.11.

sectional area A may be considered to be concentrated at the point

$$(y, z) = (r_y, r_z).$$

Then

$$I_{yy} = r_y^2 A,$$
$$I_{zz} = r_z^2 A.$$

Hence

$$r_y = \sqrt{\frac{I_{yy}}{A}},$$
$$r_z = \sqrt{\frac{I_{zz}}{A}}.$$

The quantities r_y and r_z are known as the *radii of inertia* of the cross-section;[1] they have the dimension of a length.

The radii of inertia are used in formulas to check the stability of members subject to compression. They can also play a role in formulas for concrete beams that, in order to avoid tensile stresses, have to be prestressed.

Note: Although the notation of the radii of inertia r_y and r_z suggest that they are components of a vector, this is not the case. Upon rotation of the coordinate system they do not transform like the components of a vector.

1 Also referred to as *radii of gyration*.

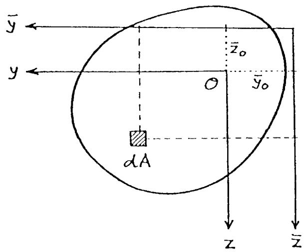

Figure 3.28 Mutually translated coordinate systems.

3.2.2 Steiner's parallel axis theorem for moments of inertia

As with static moments, the transformation formulas due to a translation of the coordinate system are also important for the moments of inertia.

If in a translated $\overline{yz}$ coordinate system $\overline{x}_0$ and $\overline{y}_0$ are the coordinates of the origin O of the initial yz coordinate system (see Figure 3.28), then the relationship between the coordinates in the translated and initial coordinate system is

$$\overline{y} = y + \overline{y}_0,$$

$$\overline{z} = z + \overline{z}_0.$$

For the moments of inertia in the translated $\overline{yz}$ coordinate system we find

$$I_{\overline{yy}} = \int_A \overline{y}^2 \, \mathrm{d}A = \int_A y^2 \, \mathrm{d}A + 2\overline{y}_0 \int_A y \, \mathrm{d}A + \overline{y}_0^2 \int_A \mathrm{d}A,$$

$$I_{\overline{yz}} = I_{\overline{zy}} = \int_A \overline{yz} \, \mathrm{d}A$$

$$= \int_A yz \, \mathrm{d}A + \overline{y}_0 \int_A z \, \mathrm{d}A + \overline{z}_0 \int_A \mathrm{d}A + \overline{y}_0 \overline{z}_0 \int_A \mathrm{d}A,$$

$$I_{\overline{zz}} = \int_A \overline{z}^2 \, \mathrm{d}A = \int_A z^2 \, \mathrm{d}A + 2\overline{z}_0 \int_A z \, \mathrm{d}A + \overline{z}_0^2 \int_A \mathrm{d}A,$$

or

$$I_{\overline{yy}} = I_{yy} + 2\overline{y}_0 S_y + \overline{y}_0^2 A,$$

$$I_{\overline{yz}} = I_{\overline{zy}} = I_{yy} + \overline{y}_0 S_z + \overline{z}_0 S_y + \overline{y}_0 \overline{z}_0 A,$$

$$I_{\overline{zz}} = I_{zz} + 2\overline{z}_0 S_z + \overline{z}_0^2 A.$$

If the origin O of the initial yz coordinate system is chosen at the centroid C (see Figure 3.29), the static moments S_y and S_z are by definition zero and the formulas above can be considerably simplified:

$$I_{\overline{yy}} = I_{yy} + \overline{y}_C^2 A,$$

$$I_{\overline{yz}} = I_{\overline{zy}} = I_{yz} + \overline{y}_C \overline{z}_C A,$$

$$I_{\overline{zz}} = I_{zz} + \overline{z}_C^2 A.$$

In this form they are known as *Steiner's parallel axis theorem*.[1]

Since I_{yy}, I_{yz} and I_{zz} are centroidal moments of inertia,[2] for clarity we recommend writing Steiner's parallel axis theorem in this way:

$$I_{\overline{yy}} = I_{yy(\text{centr})} + \overline{y}_C^2 A,$$

$$I_{\overline{yz}} = I_{\overline{zy}} = I_{yz(\text{centr})} + \overline{y}_C \overline{z}_C A,$$

$$I_{\overline{zz}} = I_{zz(\text{centr})} + \overline{z}_C^2 A.$$

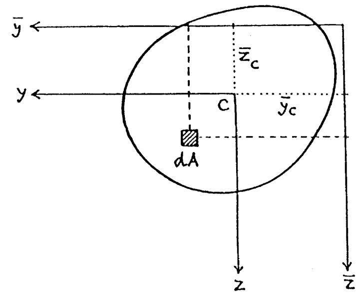

Figure 3.29 Centroidal yz coordinate system and non-centroidal $\overline{yz}$ coordinate system. $\overline{y}_C$ and $\overline{z}_C$ are the coordinates of centroid C in the translated $\overline{yz}$ coordinate system.

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1 Jacob Steiner (1796–1863), Swiss mathematician, one of the great geometricians of the 19th century. He contributed greatly to the development of projective geometry.

2 The moments of inertia in a coordinate system with its origin at centroid C; see the end of Section 3.2.1.

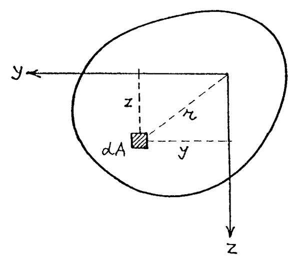


Figure 3.30 Area element dA with its coordinates.

Steiner's theorem implies that $I_{\overline{yy}}$ and $I_{\overline{zz}}$ are minimal when $\overline{y}_{\mathrm{C}} = 0$ and $\overline{z}_{\mathrm{C}} = 0$, or when the $\overline{yz}$ coordinate system has its origin at centroid C. The centroidal moments of inertia $I_{yy(\mathrm{centr})}$ and $I_{zz(\mathrm{centr})}$ are therefore the smallest moments of inertia.

3.2.3 Polar moment of inertia

For an area A the polar moment of inertia I_{p} is defined as (see Figure 3.30)

$$I_{\mathrm{p}} = \int_A r^2 \,\mathrm{d}A.$$

The polar moment of inertia plays a role in the rotation of a body about an axis. The same quantity is found in formulas for determining shear stresses and deformations due to torsion in circular cross-sections.

Since

$$r^2 = y^2 + z^2$$

it also holds that

$$\int_A r^2 \,\mathrm{d}A = \int_A y^2 \,\mathrm{d}A + \int_A z^2 \,\mathrm{d}A,$$

or

$$I_{\mathrm{p}} = I_{yy} + I_{zz}.$$

The polar moment of inertia I_{p} is equal to the sum of the moments of inertia I_{yy} and I_{zz}.

Note that $I_p = I_{yy} + I_{zz}$ does not change when the yz coordinate system rotates. $I_p = I_{yy} + I_{zz}$ is said to be *invariant*.

For an arbitrary cross-sectional shape it is usually more difficult to determine I_p than I_{yy} and I_{zz}. I_p can then be found as the sum of I_{yy} and I_{zz}. An exception is the circular cross-section for which I_p is easier to determine than I_{yy} and I_{zz}. Here I_p is often used to find I_{yy} and I_{zz}.

3.2.4 Examples

Example 1

Given the rectangular cross-section in Figure 3.31a.

Questions:

a. Determine the centroidal moments of inertia.
b. Determine the moments of inertia in the $\overline{yz}$ coordinate system.

Solution:

a. The centroidal moments of inertia are the moments of inertia in a yz coordinate system with its origin at centroid C (see Figure 3.31b). Symmetry implies

$$I_{yz} = I_{zy} = 0.$$

The y and z axes are therefore *principal axes* of the cross-section.

To determine I_{zz} we first look at the hatched strip in Figure 3.31b, with area

$$\mathrm{d}A = b\,\mathrm{d}z.$$

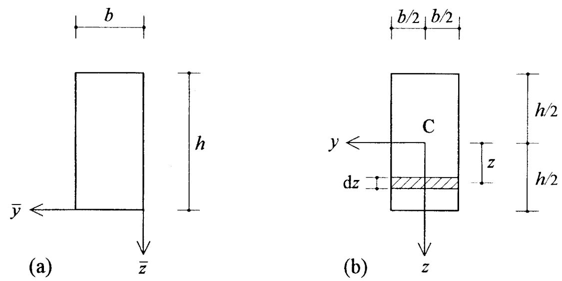


Figure 3.31 Rectangular cross-section with (a) a non-centroidal $\overline{yz}$ coordinate system and (b) a centroidal yz coordinate system.

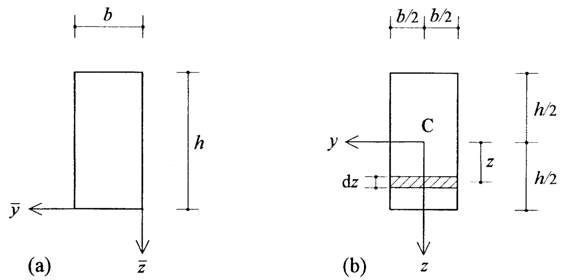


Figure 3.31 Rectangular cross-section with (a) a non-centroidal $\overline{yz}$ coordinate system and (b) a centroidal yz coordinate system.

All area elements on this strip have the same z coordinate. For the contribution of this strip to I_{zz} we have

$$\mathrm{d}I_{zz} = I_{zz}^{\text{strip}} = z^2\,\mathrm{d}A = bz^2\,\mathrm{d}z.$$

The moment of inertia I_{zz} is found by summing the contributions of all the strips over height h, which can be achieved by integrating:

$$I_{zz} = \sum I_{zz}^{\text{strip}} = \int_A z^2\,\mathrm{d}A = b\int_{-h/2}^{+h/2} z^2\,\mathrm{d}z = \tfrac{1}{3}bz^3\Big|_{-h/2}^{+h/2} = \tfrac{1}{12}\,bh^3.$$

This formula for the moment of inertia in the xz plane for a rectangular cross-section,

$$I_{zz} = \tfrac{1}{12}\,bh^3$$

is widely used in building practice.[1]

In the formula for I_{zz} the height h appears to the power of three, and the width b only appears to the first power. I_{zz} can be seen as a measure for the extent of the cross-section in z direction.

When determining I_{yy} we can use the properties derived from the formula for I_{zz}:

$$I_{yy} = \tfrac{1}{12}\,b^3h.$$

[1] I_{zz} is involved in bending in the vertical xz plane.

In summary, the centroidal moments of inertia for a rectangular cross-section are

$$I_{zz} = \tfrac{1}{12}\, bh^3,$$

$$I_{yy} = \tfrac{1}{12}\, b^3h,$$

$$I_{yz} = I_{zy} = 0.$$

b. The moments of inertia in the non-centroidal $\overline{yz}$ coordinate system are found using Steiner's parallel axis theorem (see Figure 3.32):

$$I_{\overline{zz}} = I_{zz(\text{centr})} + \overline{z}_C^2 A = \tfrac{1}{12}\, bh^3 + \left(-\tfrac{1}{2}h\right)^2 bh = \tfrac{1}{3}\, bh^3,$$

$$I_{\overline{yy}} = I_{yy(\text{centr})} + \overline{y}_C^2 A = \tfrac{1}{12}\, b^3h + \left(\tfrac{1}{2}b\right)^2 bh = \tfrac{1}{3}\, b^3h,$$

$$I_{\overline{yz}} = I_{\overline{zy}} = I_{yz(\text{centr})} + \overline{y}_C\overline{z}_C A = 0 + \left(\tfrac{1}{2}b\right)\left(-\tfrac{1}{2}\,h\right)bh = -\tfrac{1}{4}\, b^2h^2.$$

The negative value of $I_{\overline{yz}}$ is in agreement with the fact that the cross-section lies in a negative $\overline{yz}$ quadrant.

Example 2
A compound beam is constructed of two similar beams with rectangular cross-section, as shown in Figure 3.33.

Question:
Determine the centroidal moments of inertia of the compound cross-section.

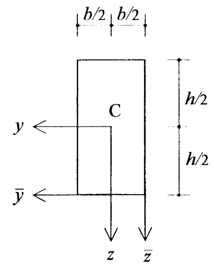


Figure 3.32 When calculating the moments of inertia in the non-centroidal $\overline{yz}$ coordinate system, Steiner's parallel axis theorem is used, with $\overline{y}_C = b/2$ and $\overline{z}_C = -h/2$.

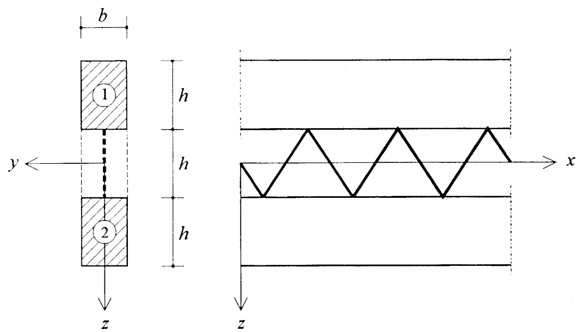


Figure 3.33 A cross-section constructed of two rectangular beams.

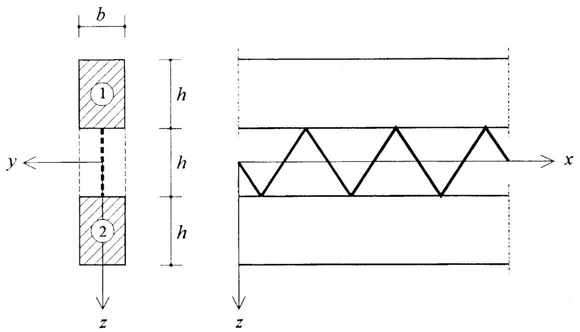


Figure 3.33 A cross-section constructed of two rectangular beams.

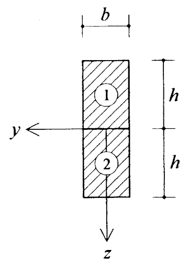


Figure 3.34 The two rectangular beams on top of one another.

Solution:
For each of the two rectangular cross-sections (1) and (2) we have[1]

$$A = bh,$$

$$I_{zz(\mathrm{centr})} = \tfrac{1}{12}\, bh^3,$$

$$I_{yy(\mathrm{centr})} = \tfrac{1}{12}\, b^3h.$$

Using Steiner's parallel axis theorem we find

$$I_{zz} = I^{(1)}_{zz(\mathrm{centr})} + A^{(1)}(-h)^2 + I^{(2)}_{zz(\mathrm{centr})} + A^{(2)}(+h)^2,$$

$$I_{yy} = I^{(1)}_{yy(\mathrm{centr})} + I^{(2)}_{yy(\mathrm{centr})}.$$

This means for the compound cross-section

$$I_{zz} = 2I_{zz(\mathrm{centr})} + 2Ah^2 = 2\cdot\tfrac{1}{12}\, bh^3 + 2\cdot bh\cdot h^2 = \tfrac{13}{6}\, bh^3,$$
$$I_{yy} = 2I_{yy(\mathrm{centr})} = 2\cdot\tfrac{1}{12}\, b^3h = \tfrac{1}{6}\, b^3h.$$

Note the large contribution of the parallel axis theorem to the value of I_{zz}! For the compound cross-section in Figure 3.34, with the two rectangular

---

1 $I_{zz(\mathrm{centr})}$ is I_{zz} in a local yz coordinate system with its origin at the centroid of the rectangular cross-section. With respect to the compound cross-section this yz coordinate system is non-centroidal. Therefore we formally should overline the yz coordinate systems for the rectangles (1) and (2). Since these coordinate systems are not shown and the extra indication "(centr)" is used, there is no possibility of confusion, and the overlining is omitted.

cross-sections on top of one another, we have

$$I_{zz} = \tfrac{1}{12}\, b(2h)^3 = \tfrac{4}{6}\, bh^3.$$

By moving the material in Figure 3.34 apart by the distance h, the value of I_{zz} increases more than three times.

For the compound cross-section symmetry implies

$$I_{yz} = I_{zy} = 0.$$

Example 3
Given the thin-walled strip[1] in Figure 3.35.

Thin-walled means that the wall thickness t is much smaller than the height h:

$$t \ll h.$$

Question:
Determine the centroidal moments of inertia.

Solution:
The formulas for a rectangular cross-section give

$$I_{zz} = \tfrac{1}{12}\, th^3,$$

$$I_{yy} = \tfrac{1}{12}\, t^3 h,$$

$$I_{yz} = I_{zy} = 0.$$

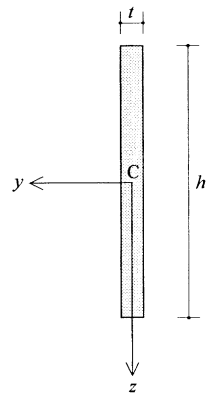


Figure 3.35 A thin-walled strip: $t \ll h$.

1 Other thin-walled cross-sections are covered in Section 3.3.

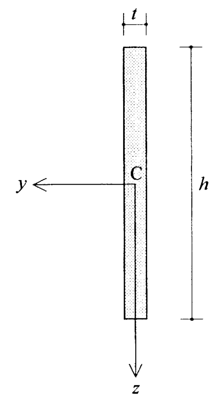


Figure 3.35 A thin-walled strip: $t \ll h$.

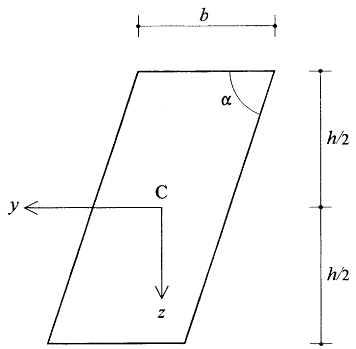


Figure 3.36 A parallelogram-shaped cross-section.

I_{yy} and I_{zz} are related by

$$I_{yy} = \frac{t^2}{h^2} I_{zz}.$$

For a thin-walled strip, with $t \ll h$, I_{yy} is negligibly small compared to I_{zz}, or in practical terms:

$$I_{yy} \approx 0.$$

Summarising for the thin-walled strip in Figure 3.35:

$$I_{zz} = \tfrac{1}{12} t h^3,$$

$$I_{yy} \approx 0,$$

$$I_{yz} = I_{zy} = 0.$$

Example 4

You are given the parallelogram-shaped cross-section in Figure 3.36.

Question:
Determine the centroidal moments of inertia.

Solution:
In Figure 3.37 look at the hatched part of the cross-section and consider it as a thin-walled strip. The centroid C′ of the strip is on the line $y = z \cot\alpha$.[1] The area of the strip is

[1] $\cot\alpha = 1/\tan\alpha$.

$$A^{\text{strip}} = \mathrm{d}A = b\,\mathrm{d}z.$$

We determine the centroidal moments of inertia with the formulas for a thin-walled strip, derived in Example 3. With thickness dz of the strip we find

$$I^{\text{strip}}_{zz(\text{centr})} = 0,$$

$$I^{\text{strip}}_{yy(\text{centr})} = \tfrac{1}{12}\,b^3\,\mathrm{d}z,$$

$$I^{\text{strip}}_{yz(\text{centr})} = 0.$$

The contributions of the hatched strip to the requested centroidal moments of inertia of the cross-section are found by applying Steiner's parallel axis theorem to the strip:

$$\mathrm{d}I_{zz} = I^{\text{strip}}_{zz} = I^{\text{strip}}_{zz(\text{centr})} + z^2 \cdot A^{\text{strip}},$$

$$\mathrm{d}I_{yy} = I^{\text{strip}}_{yy} = I^{\text{strip}}_{yy(\text{centr})} + (z\cot\alpha)^2 \cdot A^{\text{strip}},$$

$$\mathrm{d}I_{yz} = I^{\text{strip}}_{yz} = I^{\text{strip}}_{yz(\text{centr})} + (z\cot\alpha)z \cdot A^{\text{strip}}.$$

This leads to

$$\mathrm{d}I_{zz} = b \cdot z^2\,\mathrm{d}z,$$

$$\mathrm{d}I_{yy} = \tfrac{1}{12}\,b^3 \cdot \mathrm{d}z + b\cot^2\alpha \cdot z^2\,\mathrm{d}z,$$

$$\mathrm{d}I_{yz} = b\cot\alpha \cdot z^2\,\mathrm{d}z.$$

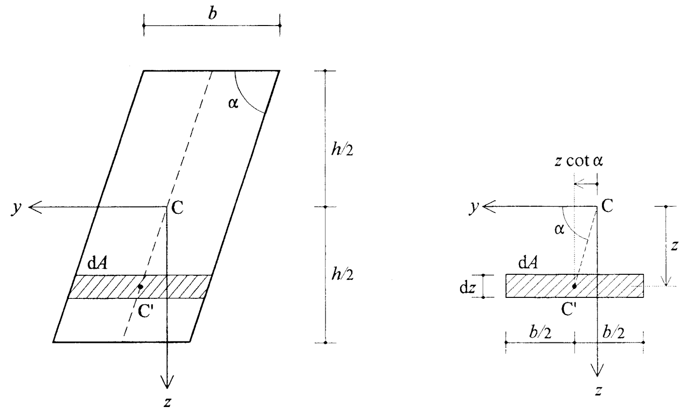


Figure 3.37 The parallelogram-shaped cross-section seen as a stack of thin-walled strips.

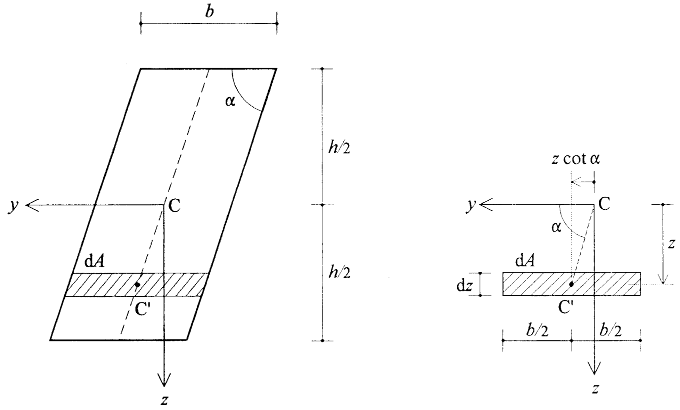


Figure 3.37 The parallelogram-shaped cross-section seen as a stack of thin-walled strips.

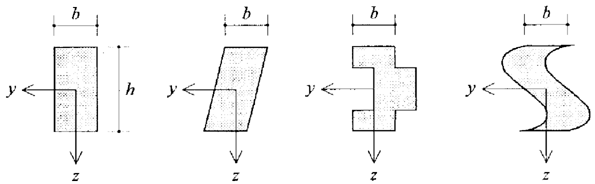


Figure 3.38 All cross-sections have the same distribution of material across the height and therefore also have the same I_{zz}.

The moments of inertia are found by summing the contributions of all strips, that is by integrating over height h:

$$I_{zz} = b \int_{-h/2}^{+h/2} z^2 \, \mathrm{d}z,$$

$$I_{yy} = \tfrac{1}{12} b^3 \int_{-h/2}^{+h/2} \mathrm{d}z + b \cot^2 \alpha \int_{-h/2}^{+h/2} z^2 \, \mathrm{d}z,$$

$$I_{yz} = b \cot \alpha \int_{-h/2}^{+h/2} z^2 \, \mathrm{d}z.$$

This results in

$$I_{zz} = \tfrac{1}{12} bh^3,$$

$$I_{yy} = \tfrac{1}{12} b^3 h + \tfrac{1}{12} bh^3 \cot^2 \alpha,$$

$$I_{yz} = I_{zy} = \tfrac{1}{12} bh^3 \cot \alpha.$$

Check: For $\alpha = 90°$, $\cot \alpha = 0$, and these formulas change into those for a rectangular cross-section (see Example 1):

$$I_{zz} = \tfrac{1}{12} bh^3,$$

$$I_{yy} = \tfrac{1}{12} b^3 h,$$

$$I_{yz} = I_{zy} = 0.$$

Comment: Note that I_{zz} is independent of angle α and equal to the moment of inertia of a rectangular cross-section. Moreover, all cross-sectional shapes shown in Figure 3.38, with a constant width b across height h, have the same I_{zz}:

$$I_{zz} = \tfrac{1}{12} bh^3.$$

All these cross-sectional shapes have the same material distribution in vertical z direction; they consist of the same stack of strips. Moving the strips with respect to one another in the y direction does not influence the magnitude of I_{zz} (but, of course, it affects the magnitude of I_{yy} and I_{yz}).

Example 5

You are given the triangular cross-section in Figure 3.39. The triangle is defined by the height h, base b and direction α of the median from the top.

Questions:

a. Determine the moments of inertia in the $\overline{yz}$ coordinate system through the top.
b. Determine the centroidal moments of inertia.
c. Determine the moment of inertia $I_{\overline{\overline{zz}}}$ in a $\overline{\overline{yz}}$ coordinate system with the $\overline{\overline{yy}}$ axis along the base.

Solution:

a. We follow the same procedure as in Example 4. Take the hatched part of the cross-section in Figure 3.40 and consider it as a thin-walled strip. The centroid C′ of the strip is located on the median $\overline{y} = \overline{z}\cot\alpha$. The difference with respect to Example 4 is that the width of the strip is no longer constant but depends on $\overline{z}$:

$$b(\overline{z}) = \frac{\overline{z}}{h}\, b.$$

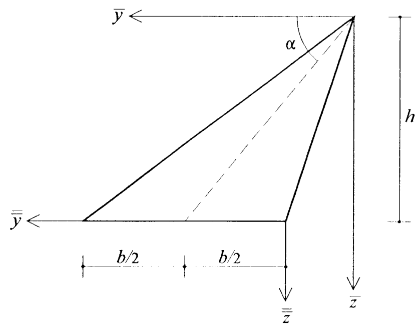


Figure 3.39 A triangular cross-section.

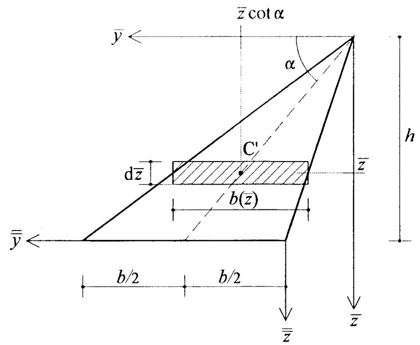


Figure 3.40 The triangular cross-section seen as a stack of thin-walled strips.

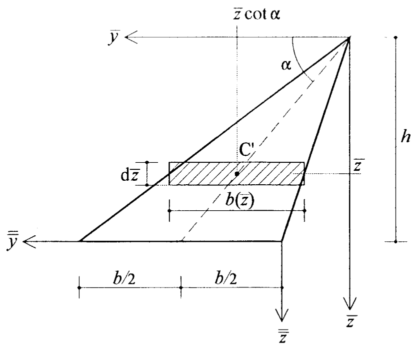


Figure 3.40 The triangular cross-section seen as a stack of thin-walled strips.

The area of the hatched strip is

$$A^{\text{strip}} = b(\bar{z})\,\mathrm{d}\bar{z} = \frac{b}{h}\,\bar{z}\,\mathrm{d}\bar{z}.$$

The centroidal moments of inertia for the thin-walled strip are

$$I^{\text{strip}}_{zz(\text{centr})} = 0,$$

$$I^{\text{strip}}_{yy(\text{centr})} = \tfrac{1}{12}\,\{b(\bar{z})\}^3\,\mathrm{d}z = \tfrac{1}{12}\,\frac{b^3}{h^3}\,\bar{z}^3\,\mathrm{d}\bar{z},$$

$$I^{\text{strip}}_{yz(\text{centr})} = 0.$$

The contributions of the hatched strip to the requested moments of inertia in the $\overline{yz}$ coordinate system through the top of the triangle are found by applying Steiner's parallel axis theorem:

$$\mathrm{d}I_{\overline{zz}} = I^{\text{strip}}_{\overline{zz}} = I^{\text{strip}}_{zz(\text{centr})} + \bar{z}^2 A^{\text{strip}},$$

$$\mathrm{d}I_{\overline{yy}} = I^{\text{strip}}_{\overline{yy}} = I^{\text{strip}}_{yy(\text{centr})} + (\bar{z}\cot\alpha)^2 A^{\text{strip}},$$

$$\mathrm{d}I_{\overline{yz}} = I^{\text{strip}}_{\overline{yz}} = I^{\text{strip}}_{yz(\text{centr})} + (\bar{z}\cot\alpha)\bar{z}A^{\text{strip}}.$$

This leads to

$$\mathrm{d}I_{\overline{zz}} = \frac{b}{h}\,z^3\,\mathrm{d}\bar{z},$$

$$\mathrm{d}I_{\overline{yy}} = \left\{\tfrac{1}{12}\frac{b^3}{h^3} + \frac{b}{h}\,(\cot\alpha)^2\right\}\bar{z}^3\,\mathrm{d}\bar{z},$$

$$\mathrm{d}I_{\overline{yz}} = \frac{b}{h}(\cot\alpha)z^3\,\mathrm{d}\overline{z}.$$

Integration over height h gives:

$$I_{\overline{zz}} = \frac{b}{h}\int_0^h \overline{z}^3\,\mathrm{d}\overline{z} = \tfrac{1}{4}bh^3,$$

$$I_{\overline{yy}} = \left\{\tfrac{1}{12}\frac{b^3}{h^3} + \frac{b}{h}(\cot\alpha)^2\right\}\int_0^h \overline{z}^3\,\mathrm{d}\overline{z} = \tfrac{1}{48}b^3h + \tfrac{1}{4}b^3h(\cot\alpha)^2,$$

$$I_{\overline{yz}} = \frac{b}{h}\cot\alpha\int_0^h \overline{z}^3\,\mathrm{d}\overline{z} = \tfrac{1}{4}bh^3\cot\alpha.$$

Check: For $\alpha = 90°$ the triangle is equilateral with the $\overline{z}$ axis as a line of symmetry (see Figure 3.41). In that case $I_{\overline{yz}} = 0$. This is in conformity with the expression for $I_{\overline{yz}}$, as $\cot\alpha = 0$ for $\alpha = 90°$.

Also note that $I_{\overline{zz}}$ is independent of angle α. All triangles in Figure 3.42, with the same base and height, have the same moment of inertia $I_{\overline{zz}}$ (see also Example 4).

b. The centroidal moments of inertia are found by using Steiner's parallel axis theorem:

$$I_{\overline{zz}} = I_{zz(\text{centr})} + \overline{z}_{\text{C}}^2 A,$$

$$I_{\overline{yy}} = I_{yy(\text{centr})} + \overline{y}_{\text{C}}^2 A,$$

$$I_{\overline{yz}} = I_{zz(\text{centr})} + \overline{y}_{\text{C}}\overline{z}_{\text{C}} A.$$

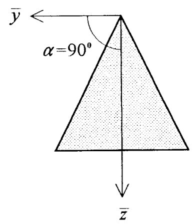


Figure 3.41 A symmetrical cross-section: $I_{\overline{yz}} = I_{\overline{zy}} = 0$.

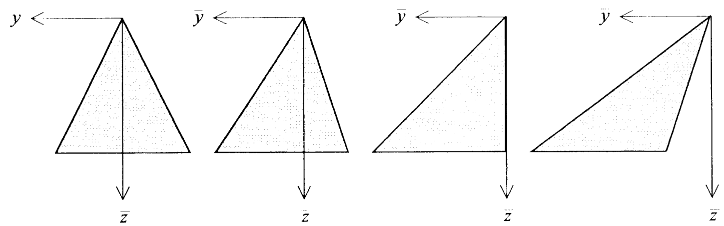


Figure 3.42 All triangular cross-sections have the same distribution of material across the height and therefore also have the same I_{zz}.

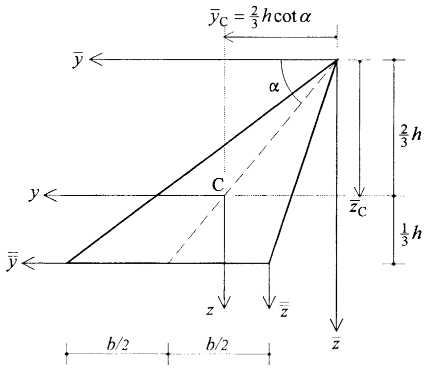


Figure 3.43 A triangle and its centroid C.

$I_{\overline{zz}}$, $I_{\overline{yy}}$ and $I_{\overline{yz}}$ are known quantities. In addition, Figure 3.43 shows that

$$\overline{z}_C = \tfrac{2}{3} h,$$
$$\overline{y}_C = \overline{z}_C \cot\alpha = \tfrac{2}{3} h \cot\alpha,$$
$$A = \tfrac{1}{2} bh.$$

We now find

$$I_{zz(\mathrm{centr})} = I_{\overline{zz}} - \overline{z}_C^2 A = \tfrac{1}{4} bh^3 - \left(\tfrac{2}{3} h\right)^2 \left(\tfrac{1}{2} bh\right),$$

$$I_{yy(\mathrm{centr})} = I_{\overline{zz}} - \overline{y}_C^2 A = \tfrac{1}{48} b^3 h + \tfrac{1}{4} bh^3 (\cot\alpha)^2 - \left(\tfrac{2}{3} h \cot\alpha\right)^2 \left(\tfrac{1}{2} bh\right),$$

$$I_{yz(\mathrm{centr})} = I_{\overline{yz}} - \overline{y}_C \overline{z}_C A = \tfrac{1}{4} bh^3 \cot\alpha - \left(\tfrac{2}{3} h \cot\alpha\right)\left(\tfrac{2}{3} h\right)\left(\tfrac{1}{2} bh\right).$$

These equations simplify to

$$I_{zz(\mathrm{centr})} = \tfrac{1}{36} bh^3,$$
$$I_{yy(\mathrm{centr})} = \tfrac{1}{48} b^3 h + \tfrac{1}{36} bh^3 (\cot\alpha)^2,$$
$$I_{yz(\mathrm{centr})} = \tfrac{1}{36} bh^3 \cot\alpha.$$

c. $I_{\overline{\overline{zz}}}$ can also be found using Steiner's parallel axis theorem (see Figure 3.43):

$$I_{\overline{\overline{zz}}} = I_{zz(\mathrm{centr})} + \overline{\overline{z}}_C^2 A = \tfrac{1}{36} bh^3 + \left(-\tfrac{1}{3} h\right)^2 \left(\tfrac{1}{2} bh\right) = \tfrac{1}{12} bh^3.$$

The moments of inertia I_{zz} (either overlined or not) are independent of the angle α between the median and y axis. The various values are given below (see Figure 3.43):

$$I_{\overline{zz}(\text{top})} = \tfrac{1}{4}\, bh^3,$$

$$I_{zz(\text{centr})} = \tfrac{1}{36}\, bh^3,$$

$$I_{\overline{\overline{zz}}(\text{base})} = \tfrac{1}{4}\, bh^3.$$

Example 6
You are given the cross-section in Figure 3.44a. The dimensions are shown in the figure.

Questions:
a. Determine the centroidal moments of inertia.
b. Determine the centroidal polar moment of inertia.

Solution (units in mm):
a. Symmetry implies

$$I_{yz} = I_{zy} = 0.$$

To determine I_{zz} and I_{yy}, the cross-section is divided into a rectangle and a number of triangles to which the previously derived formulas can be applied.

Calculation for I_{zz} (see Figure 3.44b):

$$I_{zz} = I^{\text{rectangle}}_{zz(\text{centr})} + 2 \times \left(I^{\text{triangle}}_{zz(\text{centr})} + I^{\text{triangle}}_{zz(\text{centr})}\right)$$

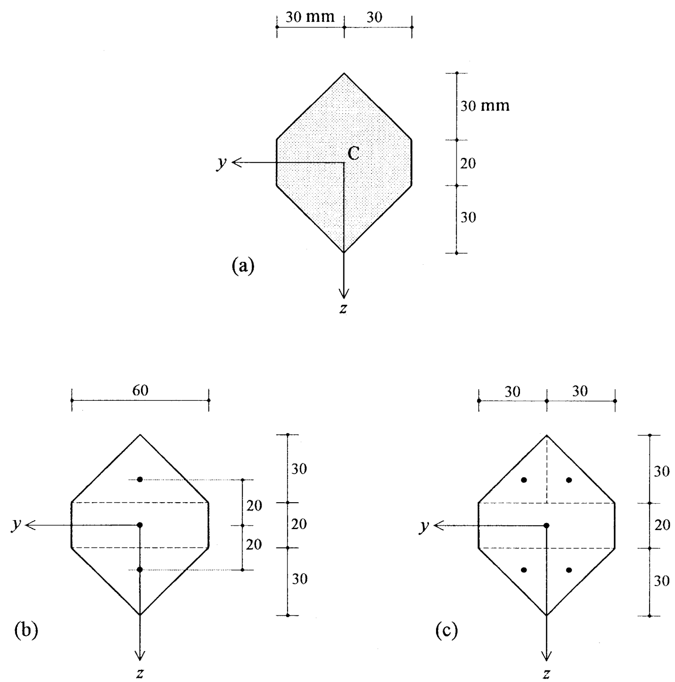


Figure 3.44 A cross-sectional shape that can be seen as being composed of a rectangle and a number of triangles.

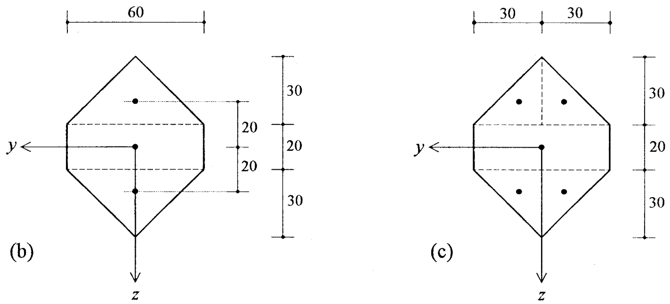


Figure 3.44b and c A cross-sectional shape composed of a rectangle and a number of triangles.

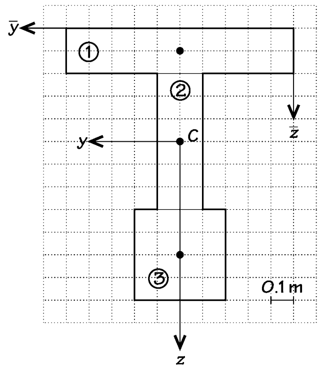


Figure 3.45 A concrete bridge beam modelled as a composition of three rectangles.

$$= \tfrac{1}{12} \times 60 \times 20^3 + 2 \times \left(\tfrac{1}{36}\, 60 \times 30^3 + 20^2 \times \tfrac{1}{2} \times 60 \times 30\right)$$

$$= 850 \times 10^3 \text{ mm}^4.$$

Calculation for I_{yy} (see Figure 3.44c):

$$I_{yy} = I^{\text{rectangle}}_{yy(\text{centr})} + 4 \times I^{\text{triangle}}_{yy(\text{centr})}$$

$$= \tfrac{1}{12} \times 20 \times 60^3 + 4 \times \tfrac{1}{12} \times 30 \times 30^3$$

$$= 630 \times 10^3 \text{ mm}^4.$$

b. The polar moment of inertia is (see Section 3.3.3)

$$I_{\text{p}} = I_{yy} + I_{zz} = 63 \times 10^4 + 85 \times 10^4 = 14.8 \times 10^6 \text{ mm}^4.$$

Example 7

Figure 3.45 shows the simplified cross-section of a concrete bridge girder. The dimensions follow from the figure.

Question:
Determine the centroidal moments of inertia.

Solution:
We first have to determine the centroid C of the cross-section. The centroid is located on the line of symmetry:

$$\overline{y}_{\text{C}} = 0.5 \text{ m}.$$

The $\overline{z}$ coordinate of the centroid follows from

$$\overline{z}_{\mathrm{C}} = \frac{S_{\overline{z}}}{A}.$$

To determine A and $S_{\overline{z}}$ the cross-section has been split up into three rectangles. The calculation is executed in Table 3.4.

From Table 3.4 we find

$$\overline{z}_{\mathrm{C}} = \frac{S_{\overline{z}}}{A} = \frac{0.24 \text{ m}^3}{0.48 \text{ m}^2} = 0.5 \text{ m}$$

the centroid of the cross-section coincides with the centroid of the web (rectangle (2)).

The centroidal moment of inertia I_{zz} is found by summing the contributions of the three rectangles. The contribution of rectangle i to I_{zz} is

$$I^i_{zz} = I^i_{zz(\text{centr})} + I^i_{zz(\text{Steiner})}.$$

The centroidal moment of inertia $I^i_{zz(\text{centr})}$ for rectangle i can be found with the formula derived for a rectangle: "$(1/12)bh^3$".

The contribution $I^i_{zz(\text{Steiner})}$, due to Steiner's parallel axis theorem, is

$$I^i_{zz(\text{Steiner})} = (\overline{z}^i_{\mathrm{C}})^2 A^i.$$

The numerical results with respect to the calculation of I_{zz} are given in Table 3.5.

Note the large contribution of the flanges (1) and (3) to the centroidal moments of inertia, this as a result of Steiner's parallel axis theorem!

Table 3.4

part *i*	A^i (m^2)	$\overline{z}^i_{\mathrm{C}}$ (m)	$S^i_{\overline{z}} = \overline{z}^i_{\mathrm{C}} A^i$ (m^3)
1	0.20	+0.10	+0.02
2	0.12	+0.50	+0.06
3	0.16	+1.00	+0.16
$\sum$	$A = 0.48$ m^2	$\sum$	$S_{\overline{z}} = +0.24$ m^3

Table 3.5

part *i*	A^i(m^2)	z^i_{C}(m)	$I^i_{zz(\text{centr})}$ (m^4)	$I^i_{zz(\text{Steiner})}$ (m^4)	I^i_{zz} (m^4)
1	0.20	–0.40	6.67×10^{-4}	320×10^{-4}	326.67×10^{-4}
2	0.12	0	36×10^{-4}	0	36×10^{-4}
3	0.16	+0.50	21.33×10^{-4}	400×10^{-4}	421.33×10^{-4}

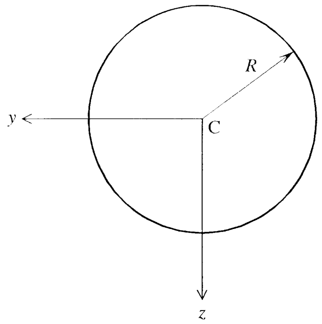


Figure 3.46 A solid circular cross-section.

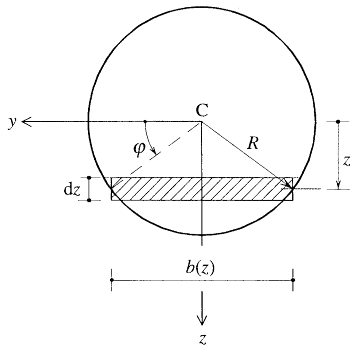


Figure 3.47 The solid circular cross-section seen as a stack of thin-walled strips.

Example 8

In Figure 3.46 you are given the circular cross-section with radius R.

Questions:

a. Determine the centroidal moment of inertia I_{zz}.
b. Determine the centroidal polar moment of inertia I_{p}.

Solution:

a. To determine I_{zz}, the cross-section is again considered to be a stack of thin strips with thickness $\mathrm{d}z$, width $b(z)$ and area $\mathrm{d}A = b(z)\,\mathrm{d}z$ (see Figure 3.47):

$$I_{zz} = \int_A z^2\,\mathrm{d}A = \int_{-R}^{+R} z^2 b(z)\,\mathrm{d}z.$$

Put

$$b(z) = 2R\cos\varphi,$$

$$z = R\sin\varphi,$$

$$\mathrm{d}z = \frac{\mathrm{d}(R\sin\varphi)}{\mathrm{d}\varphi}\,\mathrm{d}\varphi = R\cos\varphi\,\mathrm{d}\varphi,$$

and adjust the integration limits:

$$I_{zz} = \int_{-R}^{+R} z^2 b(z)\,\mathrm{d}z = \int_{-\pi/2}^{+\pi/2} (R\cos\varphi)^2(2R\cos\varphi)R\cos\varphi\,\mathrm{d}\varphi$$

$$= 2R^4 \int_{-\pi/2}^{+\pi/2} \sin^2\varphi\cos^2\varphi\,\mathrm{d}\varphi.$$

Use the trigonometric relationships for the double angle:[1]

$$\sin^2\varphi\cos^2\varphi = \tfrac{1}{4}\sin^2 2\varphi = \tfrac{1}{8}(1-\cos 4\varphi).$$

We now find

$$\begin{aligned} I_{zz} &= \tfrac{1}{4}R^4\int_{-\pi/2}^{+\pi/2}(1-\cos 4\varphi)\,\mathrm{d}\varphi \\ &= \tfrac{1}{4}R^4\big(\varphi - \tfrac{1}{4}\sin 4\varphi\big)\Big|_{-\pi/2}^{+\pi/2} = \tfrac{1}{4}\pi R^4. \end{aligned}$$

b. The centroidal polar moment of inertia I_p is

$$I_\mathrm{p} = \int_A r^2\,\mathrm{d}A = \int_A (y^2+z^2)\,\mathrm{d}A = I_{yy} + I_{zz}$$

in which $I_{yy} = I_{zz}$, so that

$$I_\mathrm{p} = 2I_{zz} = 2\cdot\tfrac{1}{4}\pi R^4 = \tfrac{1}{2}\pi R^4.$$

Alternative solution:
Here we cover questions a and b in reverse order: we first determine the polar moment of inertia I_p and then determine the moments of inertia I_{yy} and I_{zz}.

---

1 The formulas are: $\sin 2\alpha = 2\sin\alpha\cos\alpha$,
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha$.

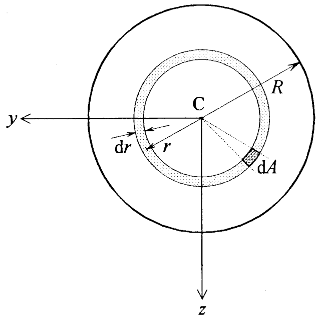


Figure 3.48 The solid circular cross-section seen as a composition of rings that fit together.

b. Assume the cross-section consists of a large number of thin-walled rings that fit together. Figure 3.48 shows one of these rings with radius r and thickness dr. The contribution of the ring to the polar moment of inertia of the cross-section is

$$\mathrm{d}I_\mathrm{p} = I_\mathrm{p}^\mathrm{ring} = \int_{A^\mathrm{ring}} r^2\,\mathrm{d}A.$$

All area elements dA on the ring have the same distance r to C. Hence

$$\mathrm{d}I_\mathrm{p} = I_\mathrm{p}^\mathrm{ring} = \int_{A^\mathrm{ring}} r^2\,\mathrm{d}A = r^2 A^\mathrm{ring}.$$

The area A^ring of the thin-walled ring is equal to the product of circumference $2\pi r$ and thickness dr:

$$A^\mathrm{ring} = 2\pi\,\mathrm{d}r.$$

The contribution of the thin-walled ring to I_p is

$$\mathrm{d}I_\mathrm{p} = I_\mathrm{p}^\mathrm{ring} = 2\pi r^3\,\mathrm{d}r.$$

Through integration we can sum the contributions of all the rings, so that

$$I_\mathrm{p} = \int_0^R 2\pi r^3\,\mathrm{d}r = \tfrac{1}{2}\pi R^4.$$

a. The centroidal polar moment of inertia I_p of a circular cross-section is much easier to determine than the centroidal moments of inertia I_{yy} and I_{zz}. Therefore it is the most plausible way to determine I_{yy} and I_{zz} via I_p.

Symmetry implies

$$I_{yy} = I_{zz}, \quad \text{and} \quad I_p = I_{yy} + I_{zz} = \tfrac{1}{2}\pi R^4.$$

Hence

$$I_{yy} = I_{zz} = \tfrac{1}{2} I_p = \tfrac{1}{4}\pi R^4.$$

Example 9
Figure 3.49 shows a thick-walled ring. The interior radius is R_i, the exterior radius is R_e.

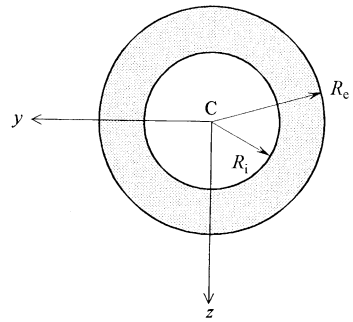


Figure 3.49 A thick-walled ring.

Questions:
a. Determine the centroidal moments of inertia.
b. Determine the centroidal polar moment of inertia.

Solution:
The thick-walled ring can be considered the difference between two circular cross-sections, with radii R_e and R_i, respectively. Using the formulas derived in the previous example we find

$$I_p = \tfrac{1}{2}\pi(R_e^4 - R_i^4)$$

and

$$I_{yy} = I_{zz} = \tfrac{1}{2} I_p = \tfrac{1}{4}\pi(R_e^4 - R_i^4).$$

Example 10
You are given the rectangular cross-section in Figure 3.50, with two circular holes.

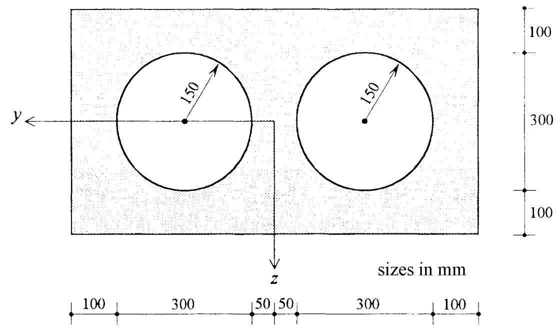


Figure 3.50 A rectangular cross-section with two circular holes.

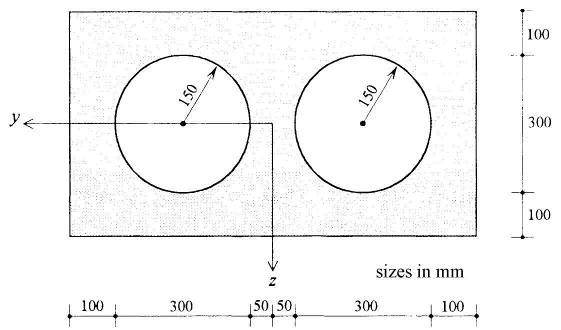


Figure 3.50 A rectangular cross-section with two circular holes.

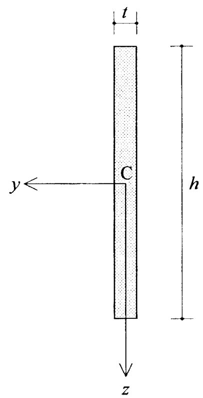


Figure 3.51 A thin-walled strip: $t \ll h$.

Question:
Determine the area A and the centroidal moments of inertia I_{zz} and I_{yy}.

Solution (units in mm):

$$A = 900 \times 500 - 2\pi \times 150^2 = 450 \times 10^3 - 141.4 \times 10^3$$

$$= 308.6 \times 10^3 \text{ mm}^2,$$

$$I_{zz} = \tfrac{1}{12} \times 900 \times 500^3 - 2 \times \tfrac{1}{4}\pi \times 150^4 = 9.375 \times 10^9 - 0.795 \times 10^9$$

$$= 8.58 \times 10^9 \text{ mm}^4,$$

$$I_{yy} = \tfrac{1}{12} \times 900^3 \times 500 - 2 \times \left(\tfrac{1}{4}\pi \times 150^4 + 200^2 \times \pi \times 150^2\right)$$

$$= 30.38 \times 10^9 - 6.45 \times 10^9$$

$$= 23.93 \times 10^9 \text{ mm}^4.$$

Comment: The material around the y axis ($z = 0$) contributes a relatively small amount to I_{zz}. By removing material at this place in the cross-section we can save on material and weight without a major reduction in I_{zz}.[1] In this way the holes allow a saving of over 30% in material while I_{zz} decreases by only 8%.

Since the removed material is eccentric with respect to the z axis, I_{yy} decreases significantly more, namely by 21%.

---

1 Reducing the amount of material reduces the costs for material. Reducing the weight leads to lower foundation costs. One must however take into account the costs for removing the material.

3.3 Thin-walled cross-sections

The strip shown in Figure 3.51 is referred to as being thin-walled, which means that the thickness t is much smaller than the height h:

$$t \ll h.$$

Thin-walled cross-sections are constructed of thin-walled strips (see Section 3.2, Example 3). The strips may be curved or closed. Examples of thin-walled cross-sections are given in Figure 3.52.

With thin-walled cross-sections, the material can be considered concentrated in the centre lines of the strips, so that the cross-section changes into a line figure. This simplifies the calculation of the cross-sectional properties. This is illustrated in Section 3.3.1 for a symmetrical I-section. A number of numerical examples are presented in Section 3.3.2.

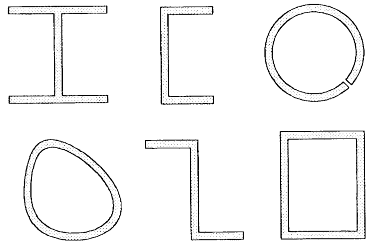


Figure 3.52 Open and closed thin-walled cross-sections.

3.3.1 Symmetrical I-section

Figure 3.53 shows a symmetrical I-section with height h, width b, flange thickness t_f and web thickness t_w. For a numerical example we use the values associated with the standard steel section HE 200A:

$$h = 190 \text{ mm}, \quad t_\mathrm{f} = 10 \text{ mm},$$

$$b = 200 \text{ mm}, \quad t_\mathrm{w} = 6.5 \text{ mm}.$$

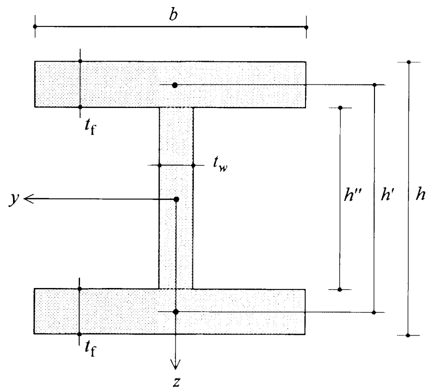


Figure 3.53 A symmetrical I-section.

With these values we find (see Figure 3.53)

$$h' = h - t_\mathrm{f} = 180 \text{ mm},$$

$$h'' = h - 2t_\mathrm{f} = 170 \text{ mm}.$$

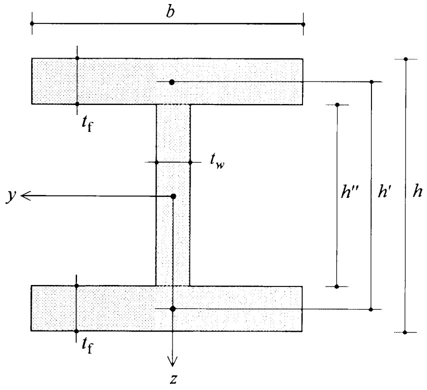


Figure 3.53 A symmetrical I-section.

The area A of the cross-section is

$$A = 2A^{\text{flange}} + A^{\text{web}} = 2bt_{\text{f}} + h''t_{\text{w}}. \tag{3.1a}$$

The moment of inertia I_{yy} is

$$I_{yy} = 2 \times I^{\text{flange}}_{yy(\text{centr})} + I^{\text{web}}_{yy(\text{centr})}$$

in which

$$I^{\text{flange}}_{yy(\text{centr})} = \tfrac{1}{12}\, t_{\text{f}} b^3 = \tfrac{1}{12} \times 10 \times 200^3 = 6.67 \times 10^6 \text{ mm}^4,$$

$$I^{\text{web}}_{yy(\text{centr})} = \tfrac{1}{12}\, h'' t_{\text{w}}^3 = \tfrac{1}{12} \times 170 \times 6.5^3 = 3.89 \times 10^3 \text{ mm}^4.$$

The contribution of the web to I_{yy} is about 0.58‰ of the contribution of the flanges and can therefore be ignored:

$$I_{yy} = 2 \times I^{\text{flange}}_{yy(\text{centr})} = \tfrac{1}{6}\, t_{\text{f}} b^3, \tag{3.2a}$$

$$I_{yy} = 2 \times 6.67 \times 10^6 = 13.34 \times 10^6 \text{ mm}^4.$$

The moment of inertia I_{zz} is

$$I_{zz} = 2 \times \left(I^{\text{flange}}_{zz(\text{centr})} + I^{\text{flange}}_{zz(\text{Steiner})}\right) + I^{\text{web}}_{zz(\text{centr})}$$

in which

$$I^{\text{flange}}_{zz(\text{centr})} = \tfrac{1}{12}\, b t_{\text{f}}^3 = \tfrac{1}{12} \times 200 \times 10^3 = 16.67 \times 10^3 \text{ mm}^4,$$

$$I^{\text{flange}}_{zz(\text{Steiner})} = b t_{\text{f}} \left(\tfrac{1}{2} h''\right) = 200 \times 10 \times \left(\tfrac{1}{2} \times 180\right)^2 = 16.20 \times 10^6 \text{ mm}^4,$$

$$I^{\text{web}}_{zz(\text{centr})} = \tfrac{1}{12}\,t_{\text{w}}(h'')^3 = \tfrac{1}{12} \times 6.5 \times 170^3 = 2.66 \times 10^6\ \text{mm}^4.$$

The contribution of the centroidal moments of inertia of the flanges to I_{zz} is about 1‰ of the contribution due to Steiner's parallel axis theorem and will therefore be ignored:

$$I_{zz} = 2 \times I^{\text{flange}}_{zz(\text{Steiner})} + I^{\text{web}}_{zz(\text{centr})} = \tfrac{1}{2}\,b(h')^2 t_{\text{f}} + \tfrac{1}{12}\,(h'')^3 t_{\text{w}}, \qquad (3.3\text{a})$$

$$I_{zz} = 32.40 \times 10^6 + 2.66 \times 10^6 = 35.06 \times 10^6\ \text{mm}^4.$$

We see that the flanges contribute about 12 times more to I_{zz} than the web.

For the dimensions of a HE 200A section, $t_{\text{f}} \ll b$ and $t_w \ll h$. The cross-section can be considered thin-walled. With the material concentrated in the centre lines the cross-section changes into a line figure (see Figure 3.54).

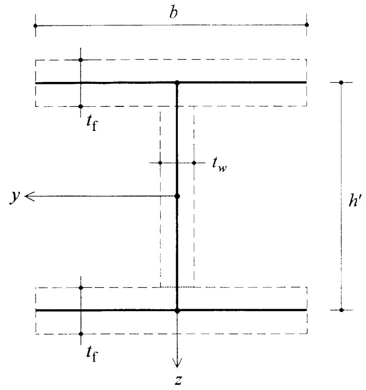


Figure 3.54 The thin-walled I-section modelled as a line figure.

Area A of the cross-section is found (per strip) as the product of wall thickness and length of the strip:

$$A = 2bt_{\text{f}} + h' t_{\text{w}}. \qquad (3.1\text{b})$$

Modelling the cross-section as a line figure means

$$I^{\text{web}}_{yy(\text{centr})} = 0,$$

$$I^{\text{flange}}_{zz(\text{centr})} = 0.$$

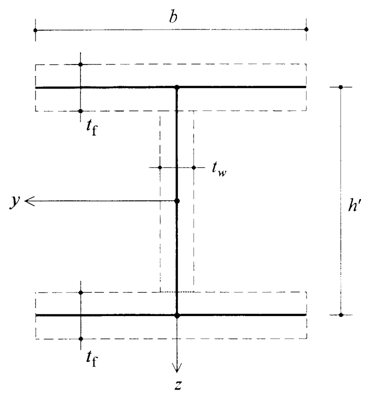


Figure 3.54 The thin-walled I-section modelled as a line figure.

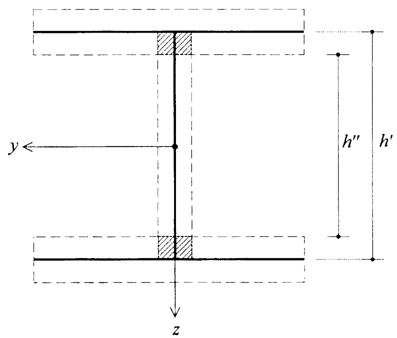


Figure 3.55 In the "thin-walled formulas" for A and I_{zz} the contribution of the hatched areas has been accounted twice.

The moments of inertia of the cross-section as a line figure are now

$$I_{yy} = 2 \times I_{yy(\text{centr})}^{\text{flange}} = \tfrac{1}{6}\, t_{\text{f}} b^3, \tag{3.2b}$$

$$I_{zz} = 2 \times I_{zz(\text{Steiner})}^{\text{flange}} + I_{zz(\text{centr})}^{\text{web}} = \tfrac{1}{2}\, b(h')^2 t_{\text{f}} + \tfrac{1}{12}\,(h')^3 t_{\text{w}}. \tag{3.3b}$$

When determining A and I_{zz} we account the hatched areas in Figure 3.55 twice.[1] Therefore the thin-walled formulas (3.1b) and (3.3b) give slightly higher results than the thick-walled formulas (3.1a) and (3.3a). The difference is about 1.5%, as can be derived from the numerical values in Table 3.6.

Table 3.6

HE 200A	A (mm^2)	I_{yy} (mm^4)	I_{zz} (mm^4)
thick-walled	5105	13.34×10^6	35.06×10^6
thin-walled	5170	13.34×10^6	35.56×10^6
table book	5380	13.34×10^6	36.92×10^6

Table 3.6 also includes the values for a HE 200A section, taken from a table book with the properties of standard steel beams. A and I_{zz} turn out to be some 4% larger than calculated. The reason for this lies in the corners

1 h' is used in the thin-walled formulas (3.1b) and (3.3b) versus h'' in the thick-walled formulas (3.1a) and (3.3a).

between web and flanges that are rounded in rolled steel sections. These dark areas in Figure 3.56 were ignored in the calculation.

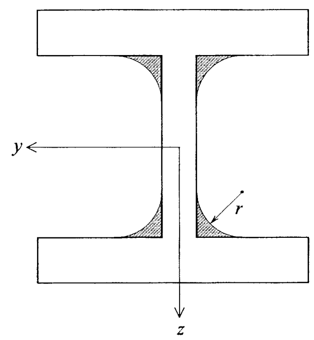


Figure 3.56 A rolled steel section has rounded corners.

3.3.2 Examples

Example 1
You are given the thin-walled Z-section in Figure 3.57a with uniform wall thickness t.

Question:
Determine the centroidal moments of inertia in a yz coordinate system.

Solution:
The cross-section has point symmetry; the centroid C is at half-height in the web. Figure 3.57b shows the thin-walled cross-section as a line figure, with the centroids of the web and the flanges.

For the thin-walled cross-section

$$I^{\text{web}}_{yy(\text{centr})} = 0,$$

$$I^{\text{flange}}_{zz(\text{centr})} = 0.$$

In addition

$$I^{\text{flange}}_{yz(\text{centr})} = I^{\text{web}}_{yz(\text{centr})} = 0.$$

The centroidal moments of inertia are:

$$I_{yy} = 2 \times \left(I^{\text{flange}}_{yy(\text{centr})} + I^{\text{flange}}_{yy(\text{Steiner})}\right)$$

$$= 2 \cdot \left(\tfrac{1}{12}\, ta^3 + at \cdot \left(\tfrac{1}{2}\, a\right)^2\right) = \tfrac{2}{3}\, a^3 t,$$

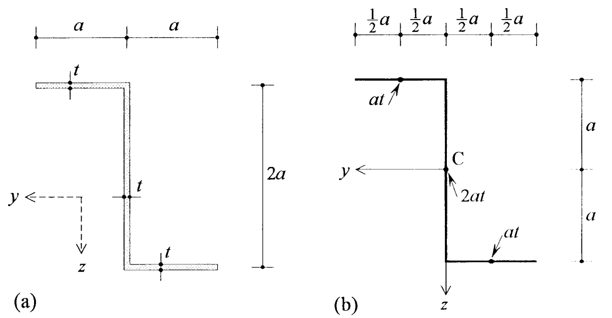


Figure 3.57 (a) A thin-walled Z-section. (b) The cross-section as a line figure, with the centroids of web and flanges.

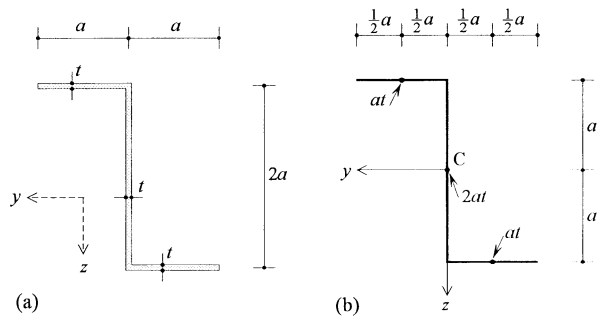


Figure 3.57 (a) A thin-walled Z-section. (b) The cross-section as a line figure, with the centroids of web and flanges.

$$I_{zz} = 2 \times I^{\text{flange}}_{zz(\text{Steiner})} + I^{\text{web}}_{yy(\text{centr})} = 2 \cdot at \cdot a^2 + \tfrac{1}{12}\, t(2a)^3 = \tfrac{8}{3}\, a^3 t,$$

$$I_{yz} = I_{zy} = I^{\text{upper flange}}_{yz(\text{Steiner})} + I^{\text{lower flange}}_{yz(\text{Steiner})}$$

$$= at\big(+\tfrac{1}{2}\, a\big)(-a) + at\big(-\tfrac{1}{2}\, a\big)(+a) = -a^3 t.$$

Check: The fact that $I_{yz} = I_{zy}$ is negative for the cross-section is in agreement with the location of the material in the negative quadrants. We can also see from the shape of the cross-section that I_{zz} is larger than I_{yy}: the material is more extended in the z direction (a measure for I_{zz}) than in the y direction (a measure for I_{yy}).

Example 2

You are given the cross-section in Figure 3.58a, in the shape of an open thin-walled ring with radius R and wall thickness t.

Questions:

a. Determine the area.
b. Determine the centroidal polar moment of inertia.
c. Determine the centroidal moments of inertia.

Solution:

At the gap the ends of the ring are not joined, but are next to one another. In determining the area and moments of inertia of the cross-section it is irrelevant whether the ring is open (with gap) or closed (without gap).

a. The area is equal to the product of the wall thickness and the developed length:

$$A = t \cdot 2\pi R = 2\pi Rt.$$

b. The polar moment of inertia is defined as

$$I_p = \int_A r^2 \, dA.$$

On the ring all the area elements dA are at the same distance $r = R$ to the origin of the centroidal coordinate system (see Figure 3.58b). Hence

$$I_p = \int_A r^2 \, dA = R^2 \int_A dA = R^2 A = 2\pi R^3 t.$$

c. Symmetry about the centroid implies

$$I_{yy} = I_{zz}.$$

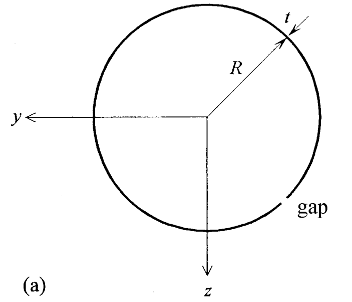
With

$$I_p = \int_A r^2 \, dA = \int_A y^2 \, dA + \int_A z^2 \, dA = I_{yy} + I_{zz}$$

we find

$$I_{yy} = I_{zz} = \tfrac{1}{2} I_p = \pi R^3 t.$$

On the basis of symmetry (the material is uniformly distributed across the



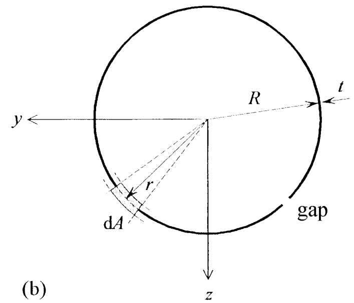


Figure 3.58 An open thin-walled ring.

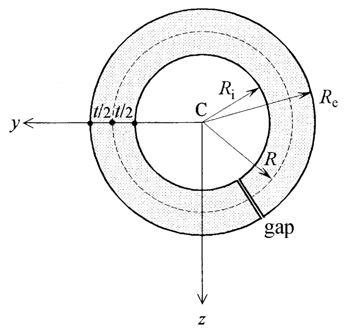


Figure 3.59 An open thick-walled ring.

positive and negative quadrants) the centroidal product of inertia is

$$I_{yz} = I_{zy} = 0.$$

Alternative solution:
The ring-shaped cross-section can also be seen as the difference between two circular cross-sections with radius R_e and R_i (see Figure 3.59). Here

$$R_e = R + \tfrac{1}{2}t,$$

$$R_i = R - \tfrac{1}{2}t.$$

a. The area A is the difference:

$$\begin{aligned} A &= A_e - A_i = \pi(R_e)^2 - \pi(R_i)^2 \\ &= \pi\left(R + \tfrac{1}{2}t\right)^2 - \pi\left(R - \tfrac{1}{2}t\right)^2 \\ &= 2\pi Rt. \end{aligned}$$

b. For the polar moment of inertia we use the formula for a thick-walled ring (see Section 3.2.4, Example 9):

$$\begin{aligned} I_p &= \tfrac{1}{2}\pi(R_e^4 - R_i^4) = \tfrac{1}{2}\pi\left\{\left(R + \tfrac{1}{2}t\right)^4 - \left(R - \tfrac{1}{2}t\right)^4\right\} \\ &= \tfrac{1}{2}\pi\left\{\left(R + \tfrac{1}{2}t\right)^2 + \left(R - \tfrac{1}{2}t\right)^2\right\}\left\{\left(R + \tfrac{1}{2}t\right)^2 - \left(R - \tfrac{1}{2}t\right)^2\right\} \\ &= \tfrac{1}{2}\pi\left(2R^2 + \tfrac{1}{2}t^2\right)(2Rt) = 2\pi Rt\left(R^2 + \tfrac{1}{4}t^2\right). \end{aligned}$$

This expression holds for a thick-walled ring. For thin-walled rings with

$t \ll R$ the term $t^2/4$ may be neglected with respect to R^2. In that case

$$I_p = 2\pi R^3 t.$$

Example 3
You are given the thin-walled triangular cross-section in Figure 3.60a, with uniform wall thickness t.

Questions:
a. Determine the area of the cross-section.
b. Determine the location of the centroid.
c. Determine the centroidal moments of inertia.

Solution:
a. In Figure 3.60b the sides of the triangular cross-section are numbered. In addition, the centroids of the sides are shown. They are at the centre of each side. The area A and static moments $S_{\bar{y}}$ and $S_{\bar{z}}$ have been calculated in Table 3.7.

b. The coordinates of centroid C are

$$\bar{y}_C = \frac{S_{\bar{y}}}{A} = \frac{48a^2t}{24at} = 2a,$$

$$\bar{z}_C = \frac{S_{\bar{z}}}{A} = \frac{120a^2t}{24at} = 5a.$$

The location of the centroid is shown in Figure 3.60c.

Comment: Beware: the centroid of the thin-walled triangle is not at one third of the height!

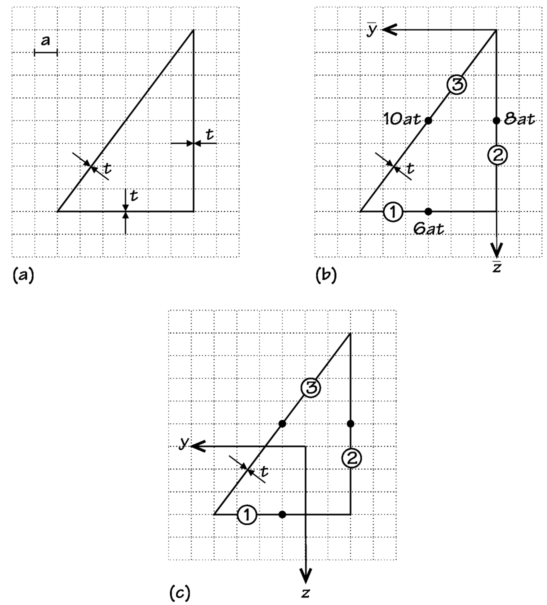


Figure 3.60 A thin-walled triangular cross-section.

Table 3.7

part i	A^i	$\bar{y}^i_C$	$\bar{z}^i_C$	$S^i_{\bar{y}} = \bar{y}^i_C A^i$	$S^i_{\bar{z}} = \bar{z}^i_C A^i$
1	$6at$	$3a$	$8a$	$18a^2t$	$48a^2t$
2	$8at$	0	$4a$	0	$32a^2t$
3	$10at$	$3a$	$4a$	$30a^2t$	$40a^2t$
$\sum$	$A = 24at$		$\sum$	$S_{\bar{y}} = 48a^2t$	$S_{\bar{z}} = 120a^2t$

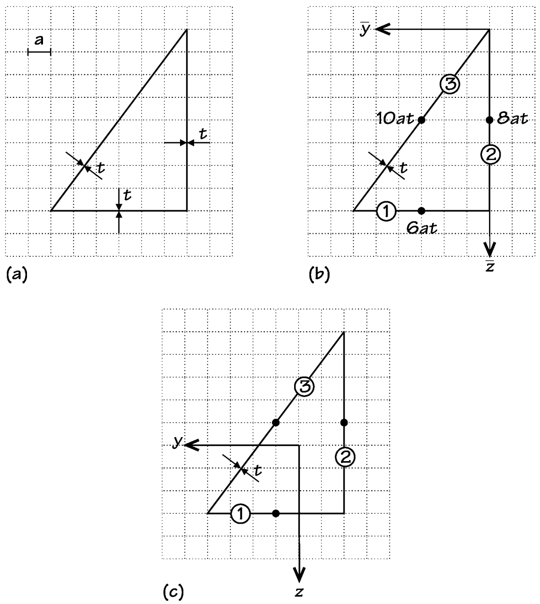


Figure 3.60 A thin-walled triangular cross-section.

c. The centroidal moments of inertia are determined by determining the contributions of the individual sides and summing them.

Calculation for I_{yy}:

$$I_{yy}^{(1)} = I_{yy(\text{centr})}^{(1)} + I_{yy(\text{Steiner})}^{(1)} = \tfrac{1}{12} \cdot t \cdot (6a)^3 + 6at \cdot a^2 = 24a^3t,$$

$$I_{yy}^{(2)} = I_{yy(\text{centr})}^{(2)} + I_{yy(\text{Steiner})}^{(2)} = 0 + 8at \cdot (-2a)^2 = 32a^3t,$$

$$I_{yy}^{(3)} = I_{yy(\text{centr})}^{(3)} + I_{yy(\text{Steiner})}^{(3)} = \tfrac{1}{12} \cdot \tfrac{5}{3}t \cdot (6a)^3 + 10at \cdot a^2 = 40a^3t.$$

Hence

$$I_{yy} = \sum_{i=1}^{3} I_{yy}^{(i)} = 24a^3t + 32a^3t + 40a^3t = 96a^3t.$$

The following formula has been used to determine $I_{\text{yy(centr)}}^{(3)}$ for oblique strip (3):

$$I_{yy} = \text{“}\tfrac{1}{12}bh^3\text{”},$$

in which b is the width of the strip in z direction ($b = 5t/3$) and h is the height of the strip in the y direction ($h = 6a$) (see Figure 3.61). See also Section 3.2.4, Example 4.

Calculation for I_{zz}:

$$I_{zz}^{(1)} = I_{zz(\text{centr})}^{(1)} + I_{zz(\text{Steiner})}^{(1)} = 0 + 6at \cdot (3a)^2 = 54a^3t,$$

$$I_{zz}^{(2)} = I_{zz(\text{centr})}^{(2)} + I_{zz(\text{Steiner})}^{(2)} = \tfrac{1}{12} \cdot t \cdot (8a)^3 + 8at \cdot (-a)^2 = \tfrac{152}{3}a^3t,$$

$$I_{zz}^{(3)} = I_{zz(\text{centr})}^{(3)} + I_{zz(\text{Steiner})}^{(3)} = \tfrac{1}{12} \cdot \tfrac{5}{4}\, t \cdot (8a)^3 + 10at \cdot (-a)^2 = \tfrac{190}{3}\, {}^3t.$$

and

$$I_{zz} = \sum_{i=1}^{3} I_{zz}^{(i)} = 54a^3t + \tfrac{152}{3}\, a^3t + \tfrac{190}{3}\, a^3t = 168a^3t.$$

Again, the following formula has been used to determine $I_{zz(\text{centr})}^{(3)}$ for oblique strip (3):

$$I_{zz} = \text{“}\tfrac{1}{12}\, bh^3\text{”},$$

but now b is the width of the strip in the y direction ($b = 5t/4$) and h is the height of the strip in the z direction ($h = 8a$) (see Figure 3.62).

Calculation for I_{yz}:

$$I_{yz}^{(1)} = I_{yz(\text{centr})}^{(1)} + I_{yz(\text{Steiner})}^{(1)} = 0 + 6at \cdot (+a)(+3a) = 18a^3t,$$

$$I_{yz}^{(2)} = I_{yz(\text{centr})}^{(2)} + I_{yz(\text{Steiner})}^{(2)} = 0 + 8at \cdot (-2a)(-a) = 16a^3t,$$

$$I_{yz}^{(3)} = I_{yz(\text{centr})}^{(3)} + I_{yz(\text{Steiner})}^{(3)}.$$

$I_{yz(\text{centr})}^{(3)}$ for oblique strip (3) is determined with the help of Figure 3.63:

$$I_{yz(\text{centr})}^{(3)} = \int_A yz\, \mathrm{d}A,$$

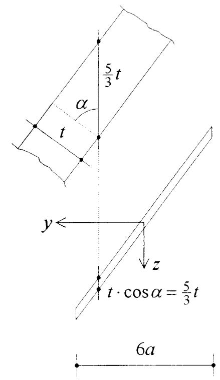


Figure 3.61 $I_{yy(\text{centr})} = \text{“}\frac{1}{12}\, bh^3\text{”} = \frac{1}{12} \cdot \frac{5}{3}\, t \cdot (6a)^3.$

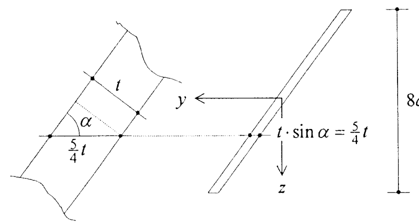


Figure 3.62 $I_{zz(\text{centr})} = \text{“}\frac{1}{12}\, bh^3\text{”} = \frac{1}{12} \cdot \frac{5}{4}\, t \cdot (8a)^3.$

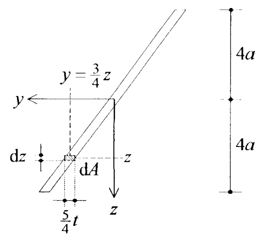


Figure 3.63 $I_{yz} = \int_A yz\,\mathrm{d}A$.

in which

$$y = \tfrac{3}{4}\,z \;\text{ and }\; \mathrm{d}A = \tfrac{5}{4}t\,\mathrm{d}z.$$

This leads to

$$I^{(3)}_{yz(\text{centr})} = \int_{-4a}^{+4a} \tfrac{3}{4}\,z \cdot z \cdot \tfrac{5}{4}\,t\,\mathrm{d}z = \tfrac{5}{16}\,z^3 t\Big|_{-4a}^{+4a} = 40a^3 t,$$

and

$$I^{(3)}_{yz} = I^{(3)}_{yz(\text{centr})} + I^{(3)}_{yz(\text{Steiner})} = 40a^3 t + 10at \cdot (+a)(-a) = 30a^3 t.$$

The centroidal product of inertia of the cross-section is finally

$$I_{yz} = \sum_{i=1}^{3} I^{(i)}_{yz} = 18a^3 t + 16a^3 t + 30a^3 t = 64a^3 t.$$

Comment: $I^{(3)}_{yz(\text{centr})}$ could also have been determined using the formula derived in Section 3.2.4, Example 4, for a parallelogram-shaped cross-section:

$$I^{(3)}_{yz(\text{centr})} = \text{“}\tfrac{1}{12}\,bh^3 \cot\alpha\text{”} = \tfrac{1}{12} \cdot \tfrac{5}{4}\,t \cdot (8a)^3 \cdot \tfrac{3}{4} = 40a^3 t.$$

3.4 Formal approach and engineering practice

In this book a formal definition is used for the *second moments of area* or *moments of inertia* I_{yy}, $I_{yz} = I_{zy}$, and I_{zz}. These geometric cross-sectional properties behave as components of a second-order tensor, and their notation is in line with the notation for the components of a second-order tensor.[1]

The term "second moment of area" is preferred to the term "moment of inertia" frequently used in engineering practice because it has nothing to do with mass. Nevertheless we follow engineering practice in which I_{yy} and I_{zz} are referred to as the *moments of inertia* of the cross-section, and $I_{yz} = I_{zy}$ is known as the *product of inertia*. Sometimes the product of inertia is also said to be a moment of inertia.

In engineering text books and engineering practice usually a different notation is used for the moments of inertia: I_y, $I_{yz} = I_{zy}$, and I_z. Sometimes the *product of inertia* is also denoted as $C_{yz} = C_{zy}$. In a yz coordinate system there definitions are (see Figure 3.64)

Engineering practice:	*Formal definition:*
$I_y = \int_A z^2 \, \mathrm{d}A$ (about the y axis)	$I_{zz} = \int_A z^2 \, \mathrm{d}A$ (in the xz plane)
$I_{yz} = C_{yz} = \int_A yz \, \mathrm{d}A$	$I_{yz} = \int_A yz \, \mathrm{d}A$
$I_z = \int_A y^2 \, \mathrm{d}A$ (about the z axis)	$I_{yy} = \int_A y^2 \, \mathrm{d}A$ (in the xy plane)

In engineering practice, I_y and I_z are generally referred to as the moments of inertia *about the y and z axis* of the cross-section. In the formal definition I_{yy} and I_{zz} are referred to as *the moments of inertia in the xy and xz plane* respectively.

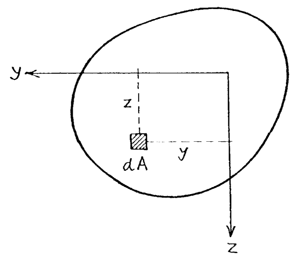


Figure 3.64 An area element dA with its coordinates y and z.

[1] The introduction of tensors and the tensor transformation rules are covered in Section 9.11.

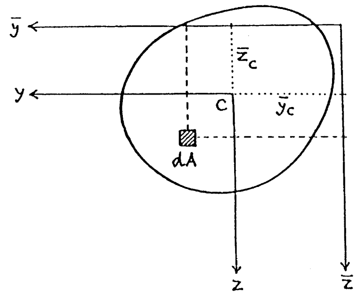


Figure 3.65 Centroidal yz coordinate system and non-centroidal $\overline{yz}$ coordinate system. $\overline{y}_C$ and $\overline{z}_C$ are the coordinates of centroid C in the translated $\overline{yz}$ coordinate system.

Note that

$$I_y = I_{zz} \text{ and } I_z = I_{yy}.$$

In the engineering notation special attention is required for the change in the indices. The moment of inertia about the y axis, I_y, requires integrating z^2, and a similar source for errors arises for the moment of inertia about the z axis.

All manipulations on moments of inertia remain the same. As an example we show the engineering notation for Steiner's parallel axis theorem. If the origin of the initial yz coordinate system is chosen at the centroid C, the moments of inertia in a translated non-centroidal $\overline{yz}$ coordinate system become (see Figure 3.65)

Engineering practice:	*Formal definition:*
$I_{\overline{y}} = I_y + \overline{z}_C^2 A,$	$I_{\overline{zz}} = I_{zz} + \overline{z}_C^2 A$
$I_{\overline{yz}} = C_{\overline{yz}} = I_{yz} + \overline{y}_C \overline{z}_C A,$	$I_{\overline{yz}} = I_{yz} + \overline{y}_C \overline{z}_C A$
$I_{\overline{z}} = I_z + \overline{y}_C^2 A,$	$I_{\overline{yy}} = I_{yy} + \overline{y}_C^2 A.$

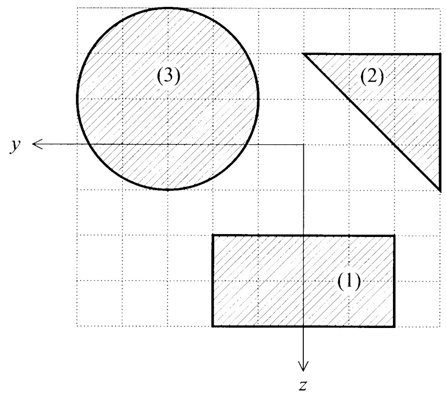
Pay special attention to the changes in the indices.

3.5 Problems

Static moment, centroid and normal centre (Section 3.1)

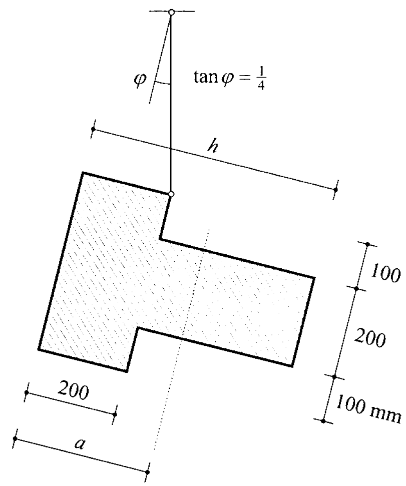
3.1 For the static moments S_y and S_z give both the mathematical definition and the physical meaning.

3.2: 1–3 The figure shows three cross-sections in the shape of (1) a rectangle, (2) a triangle and (3) a circle. The distance between two consecutive grid lines is 10 mm.



Questions:
In the given yz coordinate system determine
a. the static moment S_y.
b. the static moment S_z.

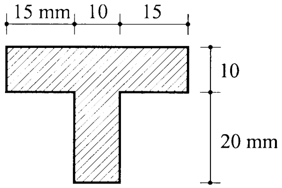
3.3 Due to its dead weight, the homogeneous T-shaped plate with constant thickness hangs in the position as shown.



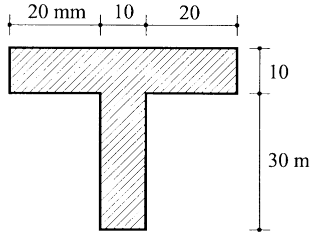
Questions:
a. Determine the distance a from the “top” of the flange to the centroid of the plate.
b. Determine the “height” h of the plate.

3.4 *Questions*:
a. Give the definition of the centroid C of an area.
b. Give the definition of the normal centre NC of a cross-sectional area.
c. When do/don’t the centroid C and normal centre NC in a cross-sectional area coincide?

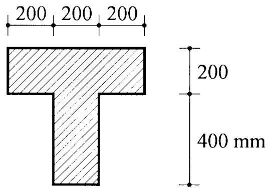
3.5: 1–4 You are given the cross-sectional dimensions of four homogeneous T-beams.



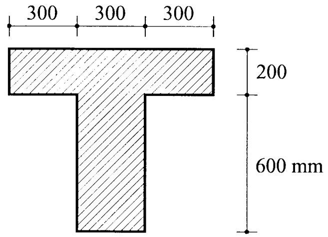
(1)



(2)



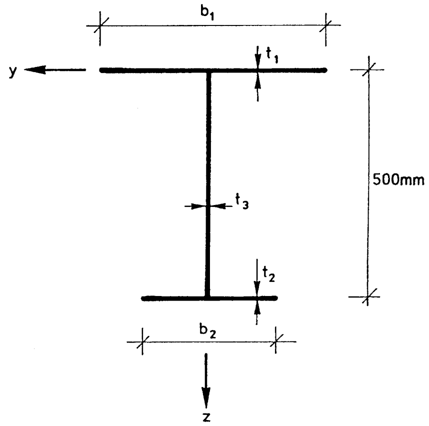
(3)



(4)

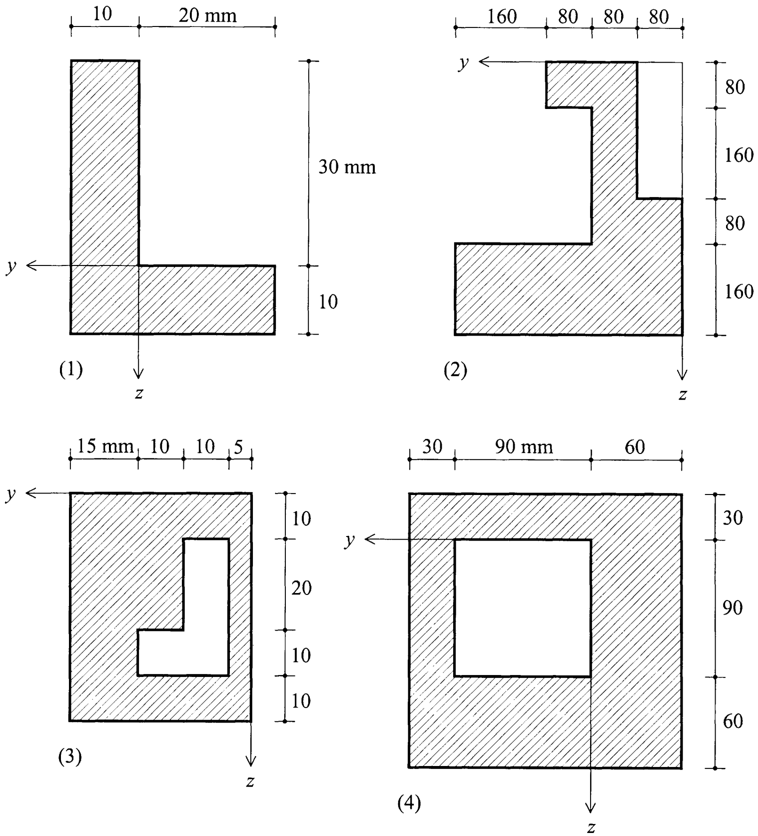
Question:
Determine the location of the normal centre NC.

3.6 You are given a thin-walled I-section with unequal flanges. In the calculation use
$t_1 = t_2 = 25$ mm, $t_3 = 20$ mm, $b_1 = 400$ mm and $b_2 = 200$ mm.



Question:
Determine the z coordinate of the centroid.

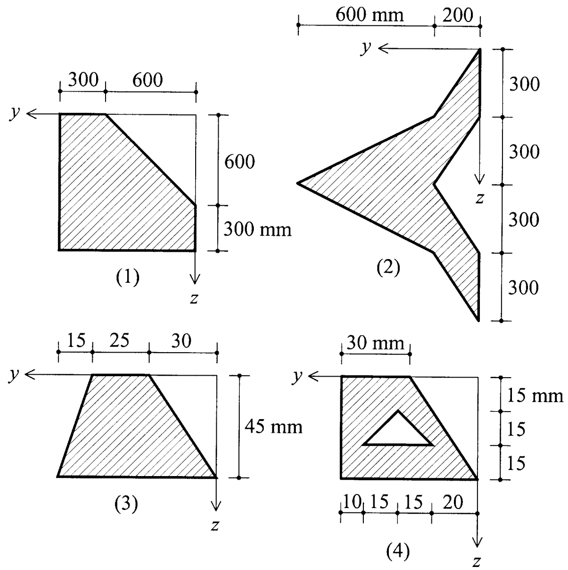
3.7: 1–4 You are given four different plane figures.



Questions:
Determine in the given yz coordinate system

a. the area A;
b. the static moment S_y;
c. the y coordinate of the centroid;
d. the static moment S_z;
e. the z coordinate of the centroid.

3.8: 1–4 You are given four different plane figures.

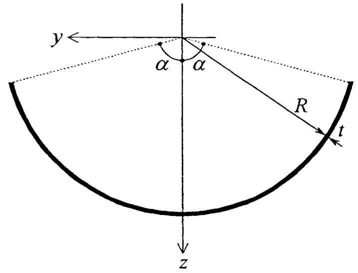


Questions:

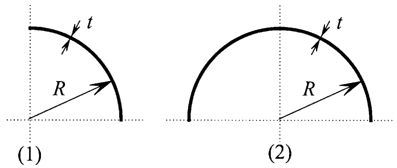
a. Determine the y coordinate of the centroid.
b. Determine the z coordinate of the centroid.

3.9 The thin-walled cross-section in the shape of a circular arch has a radius $R = 200$ mm and wall thickness $t = 25$ mm. The angle of aperture is 2α, in which $\alpha = 73°$.

Question:
Determine the z coordinate of the centroid of the cross-section.

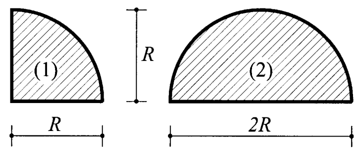


3.10: 1–2 You are given two thin-walled cross-sections in the shape of a quarter ring and half ring respectively, with radius R and wall thickness t.



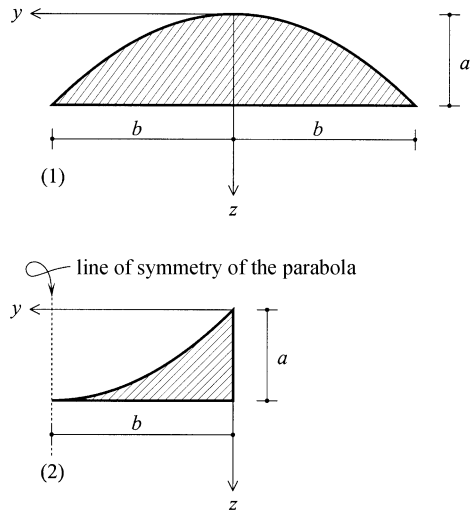
Questions:
a. Determine the area A.
b. Determine the location of the centroid.

3.11: 1–2 You are given two circle sectors with radius R.



Questions:
a. Determine the area A.
b. Determine the location of the centroid.

3.12: 1–2 You are given two parabolic segments.



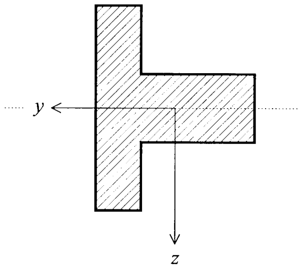
Questions:
a. Determine the area A.
b. Determine the y coordinate of the centroid.
c. Determine the z coordinate of the centroid.

Moments of inertia (Sections 3.2 and 3.3)

3.13 For the following cross-sectional properties, give both the mathematical definition and the physical meaning:
a. The moments of inertia I_{yy} and I_{zz}.
b. The products of inertia I_{yz} and I_{zy}.

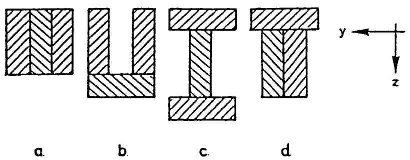
3.14 You are given a mirror symmetrical cross-section with the y axis along the line of symmetry. See for example the cross-section shown.

Question:
Show that $I_{yz} = I_{zy} = 0$.



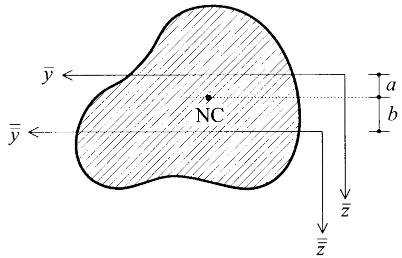
3.15 What do you understand by the *centroidal moments of inertia* of a cross-section?

3.16 Four different profiles are constructed from three battens.



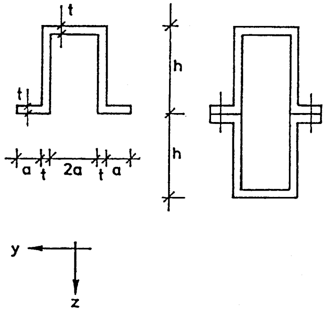
Questions:
a. Which profile has the largest centroidal moment of inertia I_{zz}?
b. Which two profiles have the same centroidal moment of inertia I_{zz}?
c. Which profile has the smallest centroidal moment of inertia I_{yy}?
Answer the questions without extensive calculation.

3.17 NC is the normal centre of an arbitrary cross-section with an area $A = 12 \times 10^3$ mm^2. In the singly-overlined coordinate system, with the $\bar{y}$ axis at a distance $a = 25$ mm above the normal centre, it applies that $I_{\bar{z}\bar{z}} = 47.5 \times 10^6$ mm^4.



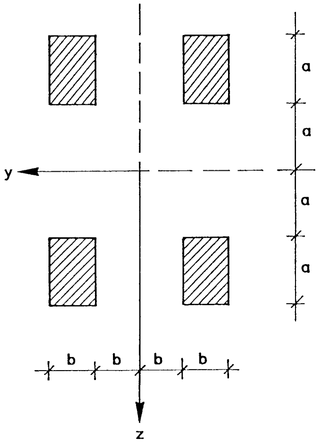
Questions:
a. Determine the moment of inertia $I_{\bar{\bar{z}}\bar{\bar{z}}}$ in the doubly-overlined coordinate system with the $\bar{\bar{y}}$ axis at a distance $b = 50$ mm under the normal centre.
b. Determine the centroidal moment of inertia I_{zz}.

3.18 Two identical profiles, with height h, area A and centroidal moment of inertia I_{zz}, are joined to form a box girder. In the calculation use $h = 100$ mm, $A = 2400$ mm^2 and $I_{zz} = 4 \times 10^6$ mm^4.



Question:
Determine the centroidal moment of inertia I_{zz} of the compound cross-section.

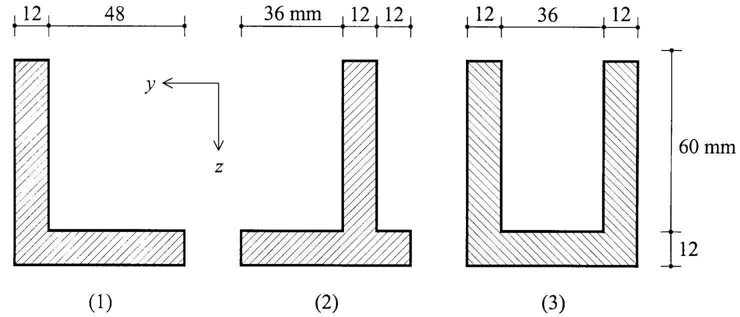
3.19 The compound cross-section shown, with material-free lines of symmetry, is constructed of four rectangular cross-sections. In the calculation use $a = 120$ mm and $b = 60$ mm.



Questions:
a. Determine the moment of inertia I_{zz} of the compound cross-section.
b. Determine the centroidal moment of inertia I_{zz} of the compound cross-section when all the constituent rectangles are located directly next to one another.
c. By what percentage does I_{zz} increase when the touching rectangles from question b are moved apart to the position shown in question a?

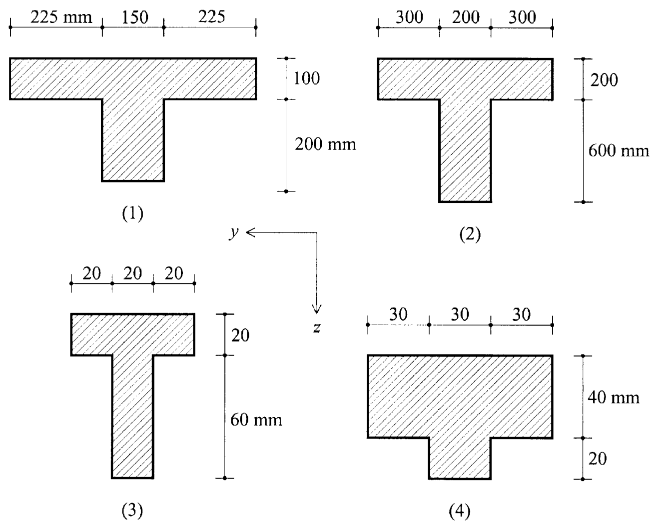
3.20 As Problem 3.19, but now replace in the questions I_{zz} by I_{yy}.

3.21: 1–3 You are given three different cross-sectional shapes.



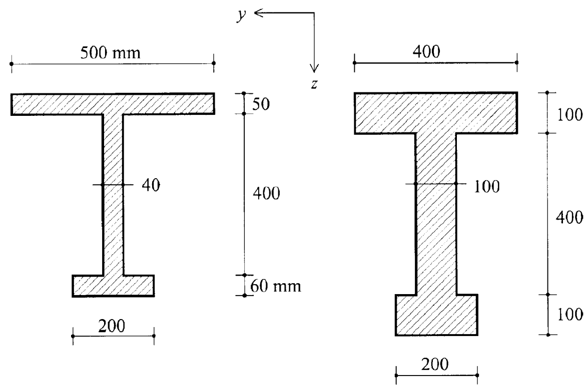
Questions:
a. Determine the centroidal moment of inertia I_{yy}.
b. Determine the centroidal product of inertia I_{yz}.

3.22: 1–4 You are given the cross-sections of four different T-beams.



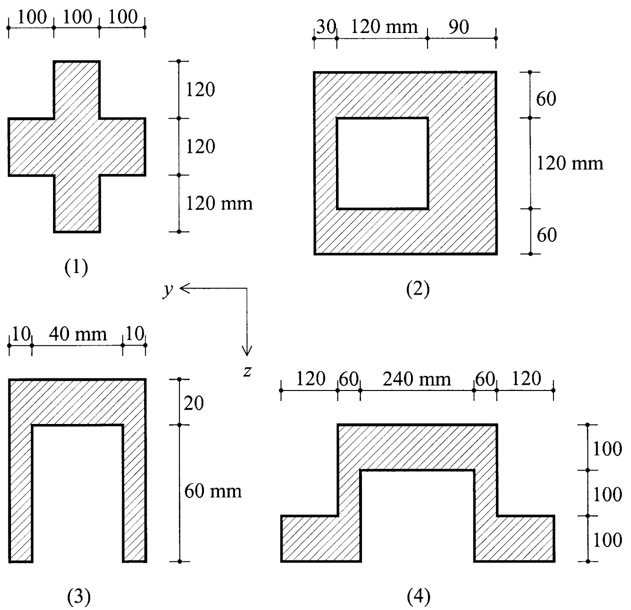
Question:
Determine the centroidal moment of inertia I_{zz}.

3.23: 1–2 You are given the cross-sections of two I-shaped beams with unequal flanges.



Question:
Determine the centroidal moment of inertia I_{zz}.

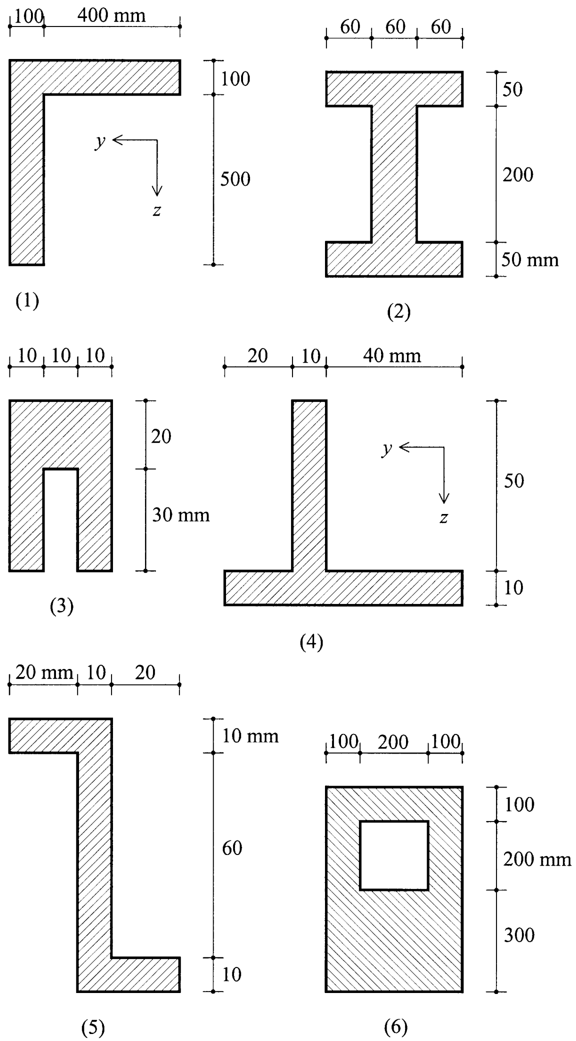
3.24: 1–4 You are given four different cross-sectional shapes.



Question:
Determine the centroidal moment of inertia I_{zz}.

3.25 When does one refer to the moments of inertia I_{yy} and I_{zz} as the *principal values*, and when are the y and z directions the *principal directions* of the cross-section?

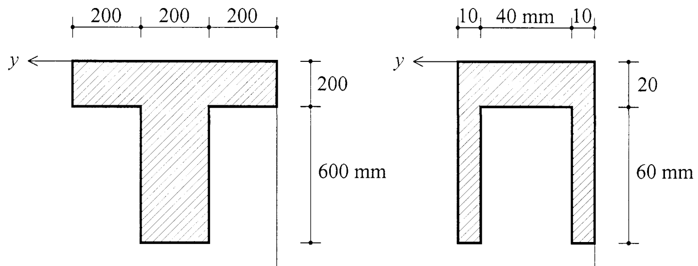
3.26: 1–6 You are given six different cross-sectional shapes.



Questions:

a. Determine the centroidal moment of inertia I_{yy}. Is this a principal value?
b. Determine the centroidal moment of inertia I_{zz}. Is this a principal value?
c. Determine the centroidal product of inertia $I_{yz} = I_{zy}$.

3.27: 1–2 You are given two different symmetric profiles.

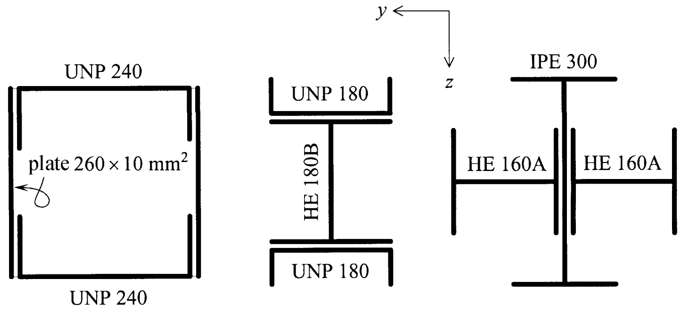


(1) (2)

Questions:
Determine in the given yz coordinate system

a. the moment of inertia I_{yy};
b. the moment of inertia I_{zz};
c. the product of inertia I_{yz}.

3.28: 1–3 You are given three cross-sections constructed of steel sections. Use a book of tables to obtain the cross-sectional properties of the various steel sections.

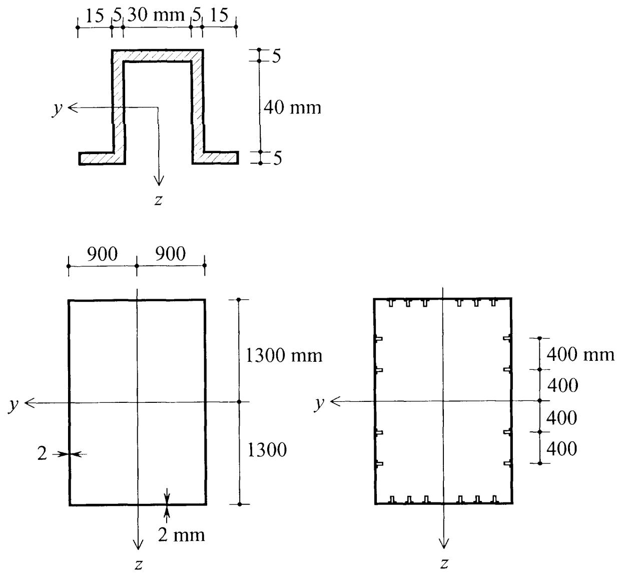


(1) (2) (3)

Questions:
For the compound cross-section determine

a. the area A;
b. the centroidal moment of inertia I_{yy};
c. the centroidal moment of inertia I_{zz}.

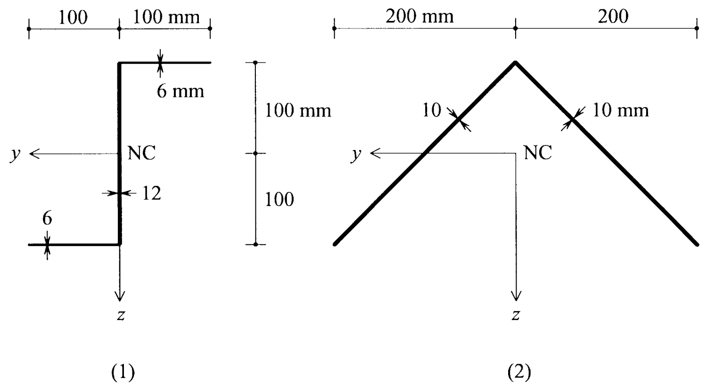
3.29 The figure shows the dimensions of a stiffener. The stiffeners are used on the thin-walled box girder shown with a rectangular cross-section and a uniform wall thickness of 2 mm. The location of the stiffeners is shown in the figure.



Questions:

a. Determine the centroid of a stiffener.
b. Determine the centroidal moments of inertia I_{yy} and I_{zz} of a stiffener.
c. Determine the centroidal moment of inertia I_{zz} of the cross-section of the box girder without stiffeners.
d. Determine the centroidal moment of inertia I_{zz} of the cross-section of the box girder with stiffeners. In percentage terms, what is the contribution of the stiffeners to this moment of inertia?

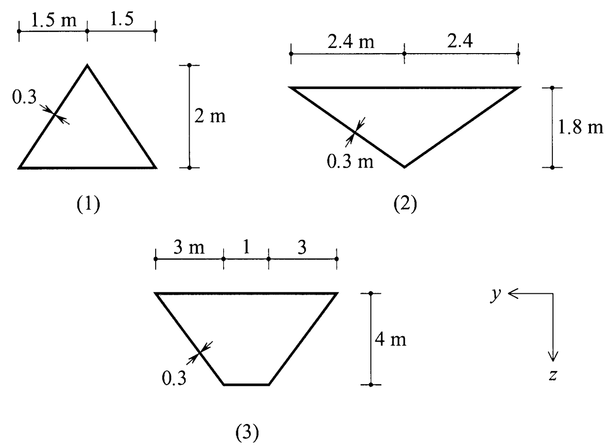
3.30: 1–2 You are given two thin-walled open cross-sections.



Questions:

a. Determine the centroidal moment of inertia I_{yy}.
b. Determine the centroidal moment of inertia I_{zz}.
c. Determine the centroidal product of inertia $I_{yz} = I_{zy}$.

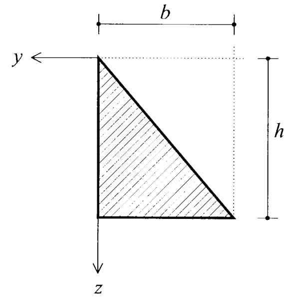
3.31: 1–3 You are given three hollow sections with everywhere the same wall thickness 0.30 m. Assume that the cross-sections are thin-walled.



Questions:

a. Determine the location of the centroid of the cross-section.
b. Determine the centroidal moment of inertia I_{zz}.
c. Determine the centroidal moment of inertia I_{yy}.

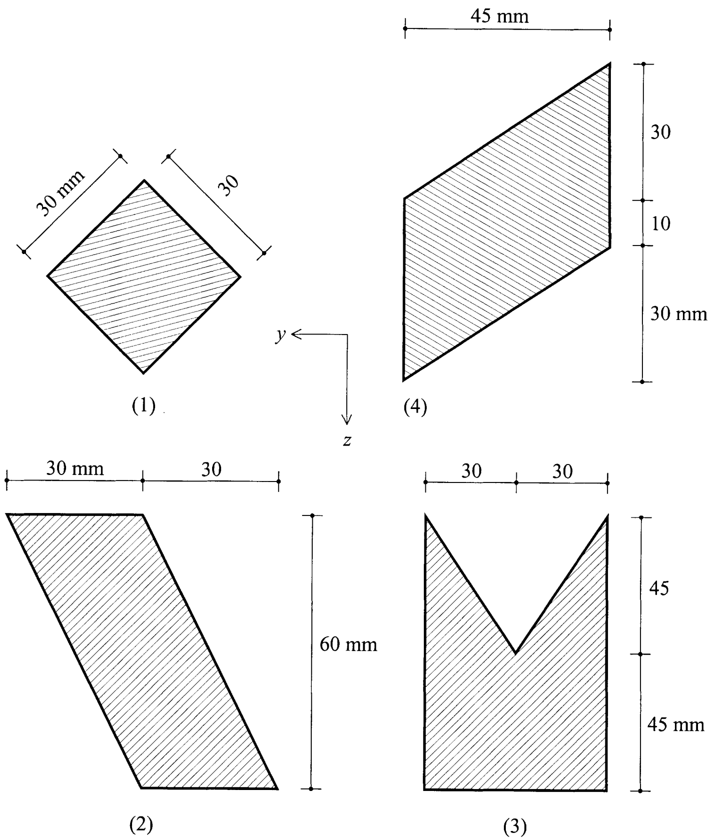
3.32 You are given a cross-section in the shape of a right-angled triangle with width b and height h.



Questions:

a. Through integration, determine the quantities of inertia I_{yy}, I_{zz}, and $I_{yz} = I_{zy}$ in the given yz coordinate system.
b. Use Steiner's parallel axis theorem to determine the centroidal quantities of inertia $I_{yy(\text{centr})}$, $I_{zz(\text{centr})}$ and $I_{yz(\text{centr})} = I_{zy(\text{centr})}$.

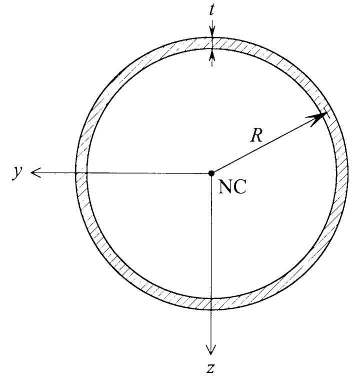
3.33: 1–4 The four homogeneous cross-sections shown can be considered as being built up by a number of rectangles and/or triangles for which it is assumed that the formulas for the centroidal quantities of inertia are known.



Questions:

a. Determine the location of the normal centre NC.
b. Determine the centroidal moment of inertia I_{zz}.
c. Determine the centroidal moment of inertia I_{yy}.
d. Determine the centroidal product of inertia $I_{yz} = I_{zy}$.

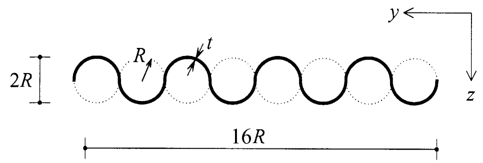
3.34 You are given a thin-walled hollow circular cross-section with radius $R = 400$ mm and wall-thickness $t = 7$ mm.



Questions:

a. Show through integration that for the centroidal moment of inertia $I_{zz} = \pi R^3 t$.
b. Determine the value of I_{zz} for the given numerical dimensions of the cross-section.

3.35 You are given the thin-walled corrugated sheet in the shape of a large number of half circles with radius R and wall thickness t.

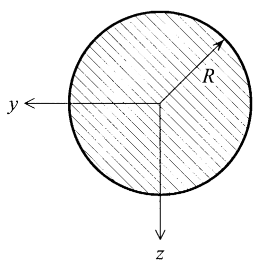


Question:
Determine the centroidal moment of inertia I_{zz}.

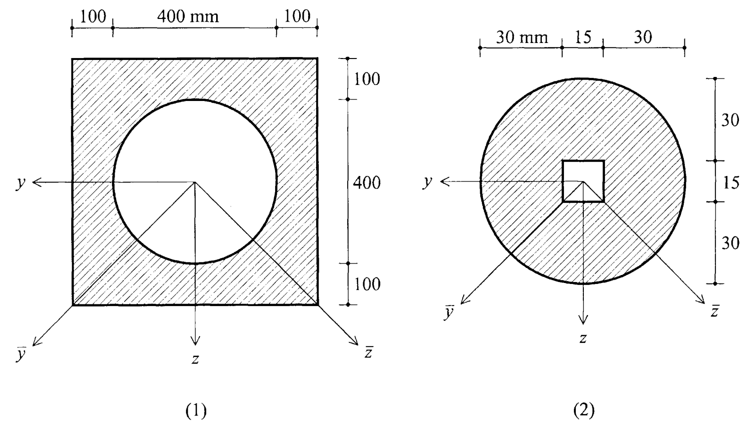
3.36 You are given a solid circular cross-section with radius $R = 75$ mm.

Questions:

a. Show by integration that for the centroidal moment of inertia $I_{zz} = \pi R^4/4$.
b. Show by integration that for the centroidal polar moment of inertia $I_p = \pi R^4/2$.
c. Determine I_{zz} and I_p for the given numerical value of radius R.



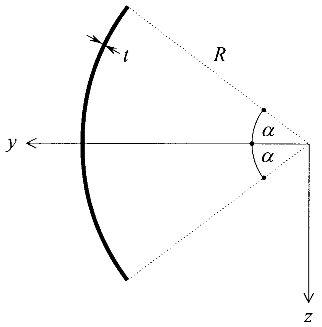
3.37: 1–2 You are given two symmetrical cross-sections with a hole.



Questions:

a. Determine the moment of inertia I_{zz}.
b. Determine the polar moment of inertia I_p.
c. Determine the moment of inertia $I_{\overline{yy}}$.

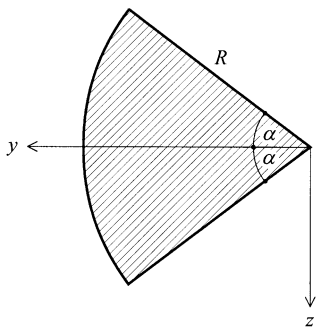
3.38 You are given a thin-walled cross-section in the shape of a circular arch segment with radius R, wall-thickness t, and angle of aperture 2α.



Questions:

a. Determine the area A.
b. Determine the static moments S_y and S_z in the given (non-centroidal) yz coordinate system.
c. Determine the coordinates of the centroid of the cross-section.
   Check these values for $\alpha = \pi/2$ and $\alpha = \pi$.
d. Determine the inertia quantities I_{yy}, I_{zz} and $I_{yz} = I_{zy}$ in the given (non-centroidal) yz coordinate system.
e. Use Steiner's parallel axis theorem, and the answers to questions c and d, to determine the centroidal inertia quantities $I_{yy(\text{centr})}$, $I_{zz(\text{centr})}$ and $I_{yz(\text{centr})} = I_{zy(\text{centr})}$. Check these values for $\alpha = \pi/2$ and $\alpha = \pi$.

3.39 You are given a circle sector with radius R and angle of aperture 2α.

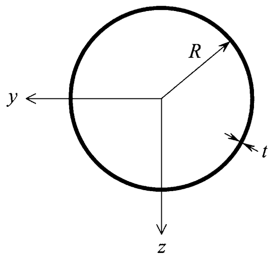


Questions:

a. Determine the area A.
b. Determine the static moments S_y and S_z in the given (non-centroidal) yz coordinate system.
c. Determine the coordinates of the centroid.
   Check these values for $\alpha = \pi/2$ and $\alpha = \pi$.
d. Determine the inertia quantities I_{yy}, I_{zz} and $I_{yz} = I_{zy}$ in the given (non-centroidal) yz coordinate system.
e. Use Steiner's parallel axis theorem, and the answers to questions c and d, to determine the centroidal inertia quantities $I_{yy(\text{centr})}$, $I_{zz(\text{centr})}$ and $I_{yz(\text{centr})} = I_{zy(\text{centr})}$. Check these values for $\alpha = \pi/2$ and $\alpha = \pi$.

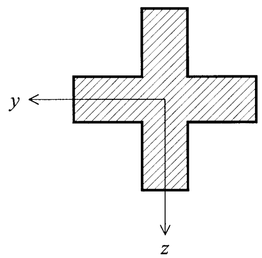
3.40 You are given a thin-walled cross-section in the shape of a ring with radius R and wall-thickness t.

Questions:
a. Show that $I_p = 2\pi R^3 t$.
b. Why is $I_{yy} = I_p/2$?



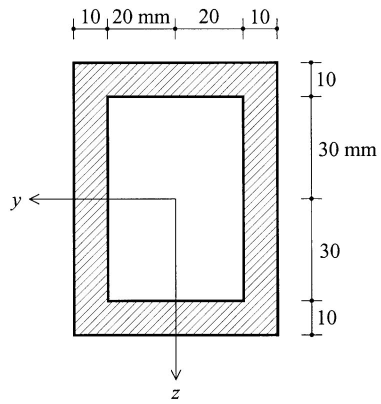
3.41 The given cross-section is rotational-symmetric and has in the given centroidal yz coordinate system a moment of inertia $I_{yy} = 40 \times 10^6$ mm^4.

Question:
Determine the centroidal polar moment of inertia I_p.

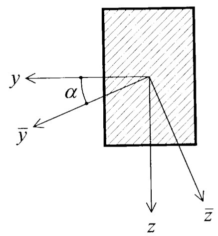


3.42 You are given hollow rectangular cross-section.

Question:
Determine the centroidal polar moment of inertia I_p.



3.43 When rotating the yz coordinate system through an angle α to the overlined $\overline{yz}$ position, the associated moments of inertia I_{yy} and I_{zz} of the given cross-section change into $I_{\overline{yy}}$ and $I_{\overline{zz}}$.



Question:
Show (without extensive calculations) that $I_{yy} + I_{zz} = I_{\overline{yy}} + I_{\overline{zz}}$.

4 Members Subject to Bending and Extension

The behaviour of members subject to bending and extension is analysed with the help of the *fibre model*. In Section 4.1, the various assumptions associated with this model are described.

One of the assumptions is that planar cross-sections remain planar. Section 4.2 shows that the consequence is a linear strain distribution over the cross-section.

When describing the behaviour of members (line elements), we distinguish three types of basic relationships: the *kinematic relationships*, the *constitutive relationships* and the *static* or *equilibrium relationships*. They are covered in Section 4.3.

In Section 4.4, we derive the *stress formula* for bending in combination with extension. The application of the stress formula is illustrated in Section 4.5 in a number of examples.

For bending without extension (without a normal force), the *section modulus* is often used to calculate the extreme bending stresses. This concept is covered in more detail in Section 4.6.

Section 4.7 includes a number of examples of bending without normal forces, in which the concept of the section modulus is applied.

The derived stress formulas so far relate to a situation in which the load acts in one of the principal directions. If this is not the case, the load can be resolved into components according to the principal directions. By superposing the contributions due to extension and to bending in both principal directions, in Section 4.8 we find the *general stress formula with respect to the principal directions*.

In Section 4.9, we investigate in which area the centre of force[1] must lie such that all stresses within the cross-section have the same sign. This area is known as the *core*[2] *of the cross-section*.

Cores play an important role in prestressed concrete beams and spread foundations. Examples are given in Section 4.10.

The mathematical description of the problem of bending with extension is covered in Section 4.11. Here, the three basic relationships from Section 4.3 are combined to form two separate differential equations: one for extension and one for bending.

In Section 4.12 we look at the effect of a change in temperature on the constitutive relationships.

Finally, in Section 4.12, some critical observations are added to the fibre model, and a summary is given of the various formulas in this chapter.

---

1 The centre of force is the point of application of the resultant of all normal stresses in the cross-section, see also Volume 1, Sections 10.1.1. and 14.2.

2 Also reffered to as *kern* or *kernel*.

4.1 The fibre model

To understand the behaviour of a member (line element), we use a *physical model*, for which we can easily describe the member properties. In Section 2.1, we introduced the *fibre model* for members subject to extension (see Figure 4.1). This model is inspired by the fibre structure of wood, and appears to be an effective model for bending with extension also.

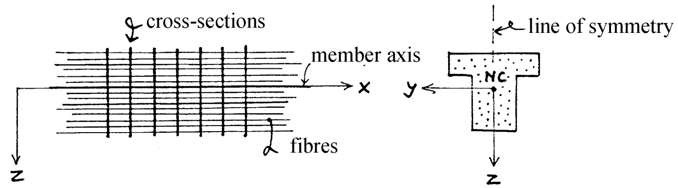


Figure 4.1 The fibre model for a member subject to extension and bending. The model consists of many *fibres* parallel to the axial direction, that are kept together by many rigid planes normal to the fibres. These rigid planes are known as *cross-sections*.

The assumptions for the fibre model are as follows:

- The member consists of many parallel *fibres* in longitudinal direction.
- The fibres are kept together by many rigid planes normal to the direction of the fibres. These rigid planes are called *cross-sections*.
- The cross-sections (rigid planes) are planar and perpendicular to the fibres, before and after the deformation of the member. This assumption is known as *Bernoulli's hypothesis*.[1]

With respect to the *material behaviour* the following assumptions are made:

- The cross-section is *homogeneous*: all fibres consist of the same material and therefore have the same material properties.
- The material behaves *linear-elastically* according to Hooke's Law:

$$\sigma = E\varepsilon.$$

[1] The assumption "planar cross-sections remain planar" is known as Bernoulli's hypothesis, after the Swiss Jacob Bernoulli (1654–1705). However, Bernoulli did not manage to derive the correct stress distribution in the cross-section. The first to achieve that was the French physicist Parent (1666–1716). His work remained unnoticed as a result of his poor presentation. Independently of Parent, the French physicist Charles Augustin de Coulomb (1736–1806) published an article in 1773 in which he suggested the correct normal stress distribution due to bending. Coulomb also addressed the shear stresses resulting from shear forces.

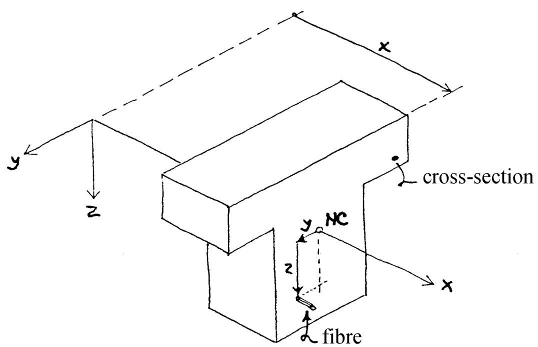


Figure 4.2 The location of a cross-section is defined by its x coordinate, that of a fibre is defined by its y and z coordinates.

The description of the member's behaviour occurs in an xyz coordinate system with the x axis in the direction of the fibres, and the yz plane normal to the fibres, that is parallel to the cross-sections. The location of a cross-section is defined by the x coordinate while that of a fibre is defined by its y and z coordinates (see Figure 4.2).

In principle, the x axis can be chosen along any arbitrary fibre. It is the convention however that the x axis coincides with the member axis. In this choice, the origin of the yz coordinate system in the cross-ection coincides with the normal centre NC. This has great benefits, as will become clear in Section 4.3.2.

At present little can be said about the orientation of the yz coordinate system within the cross-section. We make the following assumption:

- The load and support reactions act in the xz plane and they cause displacements and rotations in the xz plane only.

Such a situation can be expected when the xz plane is a plane of mirror symmetry,[1] as in the cross-sections in Figures 4.1 and 4.2.

This last assumption means that the problem of a member subject to bending and extension is reduced to a two-dimensional problem. We deviate from this in Section 4.8.

---

1 In Section 4.3.2. we shall see that the y and z axis are best taken to coincide with the principal directions of the cross-section.

4.2 Strain diagram and neutral axis

When a member deforms, the fibres change in length, but the planar cross-sections remain planar and normal to the fibres, before and after the deformation of the member (*Bernoulli's hypothesis*).

Since the cross-sections are assumed to be rigid, and the displacements and rotations occur solely in the xz plane, all fibres with the same z coordinate will deform in the same way. Such a *fibre layer* is shown in Figure 4.3a and is indicated hereafter by its z coordinate in the undeformed state.

Let us look at the fibres between the two cross-sections in Figure 4.3b, at a mutual distance Δx ($\Delta x \to 0$). Due to deformation of the member, these cross-sections rotate with respect to one another through a (small) angle $\Delta\varphi$. The deformed state is shown in Figure 4.3c. The change in length $\Delta u(z)$ of the fibres in the z layer is

$$\Delta u(z) = (R + z)\Delta\varphi - \Delta x,$$

in which R is the radius of curvature of the member axis. The change in length $\Delta u(z)$ of the fibres is linear in z.

The strain of the fibres in the z layer is

$$\varepsilon(z) = \lim_{\Delta x \to 0} \frac{\Delta u(z)}{\Delta x} = \frac{\mathrm{d}u(z)}{\mathrm{d}x} = R\frac{\mathrm{d}\varphi}{\mathrm{d}x} + z\frac{\mathrm{d}\varphi}{\mathrm{d}x} - 1.$$

Since all fibres between two consecutive cross-sections have the same initial length Δx, the strain $\varepsilon(z)$ is also linear in z. Assuming

$$R\frac{\mathrm{d}\varphi}{\mathrm{d}x} = 1 + \varepsilon \quad \text{and} \quad z\frac{\mathrm{d}\varphi}{\mathrm{d}x} = z\kappa_z,$$

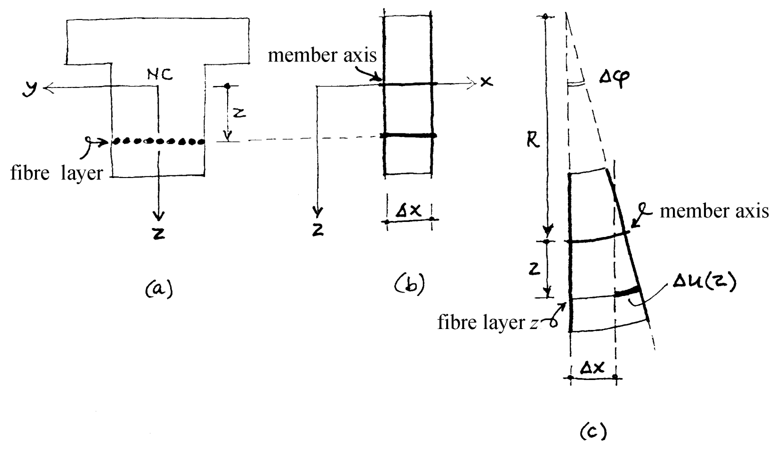


Figure 4.3 (a) Cross-section with a fibre layer in which all fibres have the same z coordinate. When the member deforms the fibres change in length. The rigid cross-sectional planes are planar and perpendicular to the fibres, both before and after deformation of the member (*Bernoulli's hypothesis*). (b) A short member segment from the undeformed member and (c) after the deformation by bending and extension.

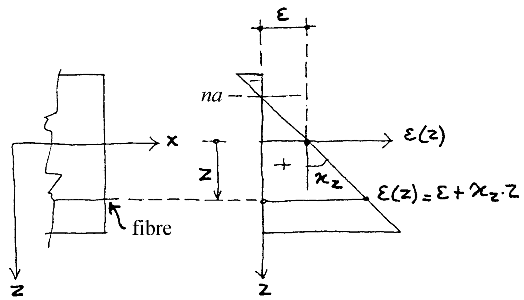


Figure 4.4 Strain diagram and neutral axis (*na*). The strain distribution is linear because planar cross-sections remain planar, and is independent of the material behaviour.

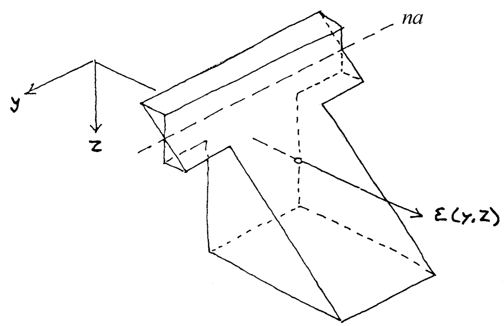


Figure 4.5 Spatial representation of the strain diagram. The neutral axis (*na*) is a straight line that divides the cross-section into two parts, one with only positive strains (the fibres lengthen) and the other with only negative strains (the fibres shorten).

we can write for this *linear strain distribution*

$$\varepsilon(z) = \varepsilon + \kappa_z z. \tag{4.1}$$

Here ε and κ_z are *deformation quantities*. We come back to these quantities in Section 4.3.1.

In Figure 4.4 the strain distribution over the height of the cross-section is sketched in a strain diagram or ε diagram. From the strain diagram we see that ε is the strain in the fibre layer $z = 0$ (through the member axis) and that κ_z is the *slope of the strain diagram*:

$$\kappa_z = \frac{\mathrm{d}\varepsilon(z)}{\mathrm{d}z}.$$

In Section 4.3.1 we show that κ_z is also the curvature of the member axis in the xz plane.

The strain diagram is generally drawn in two-dimensions. The fibre layer in which the strain is zero is known as the *neutral axis*, in Figure 4.4 indicated as *na*.

The neutral axis is a straight line that divides the cross-sectional area into two parts: one part with only positive strains (where all fibres lengthen) and the other with only negative strains (where all fibres shorten). To clarify the concept *neutral axis* there is a spatial representation of the strain diagram in Figure 4.5.

Comment: The fact that the strain distribution is linear follows directly from the assumption that planar cross-sections remain planar and is independent of the material behaviour. The linear strain distribution therefore not only applies for elastic deformations but also for plastic deformations.

4.3 The three basic relationships

As noted in Section 2.2, there are three basic relationships when describing member behaviour:

- *Kinematic relationships*

Kinematic relationships link deformations and displacements. They follow from the lasting cohesion in the member: holes and gaps do not suddenly appear in a member. The kinematic relationships are independent of the material behaviour.

- *Constitutive relationships*

Constitutive relationships link the section forces and the associated deformations. They follow from the (in this case linear-elastic) material behaviour.

- *Static relationships or equilibrium relationships*

Static relationships link the load (due to external forces) and section forces. They follow from the equilibrium.

The three basic relationships that are derived below for bending with extension allow us to link the loads (due to external forces) and the associated displacements. This is shown in the scheme in Figure 4.6 for a member with bending and extension in the xz plane.

The direct link between loads and displacements, mentioned above, is further discussed in Section 4.11.

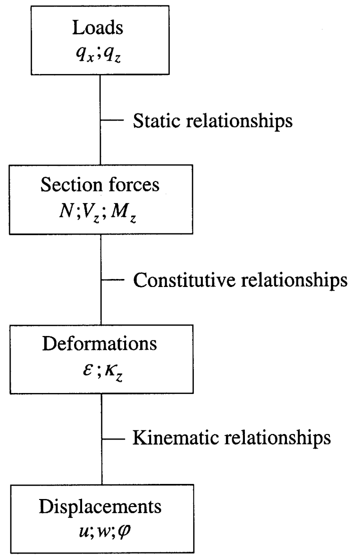


Figure 4.6 Schematic representation of the relationship between load and displacement for bending with extension. In order to obtain this link, we must have three basic relationships: kinematic, constitutive and static (or equilibrium).

4.3.1 Kinematic relationships

We first derive the kinematic relationships. They relate the deformation quantities and displacement quantities.

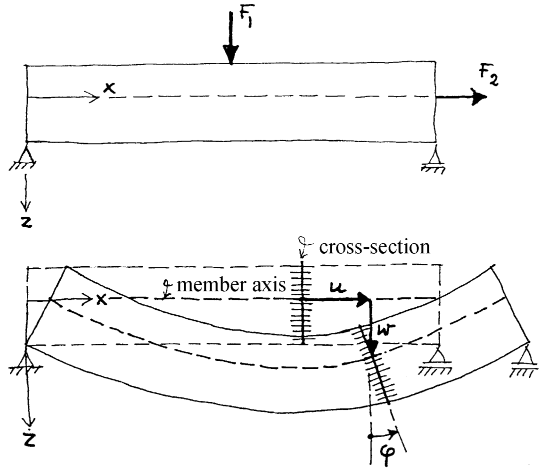


Figure 4.7 The displacement and rotation of a cross-section in a member subject to extension and bending when the deformation is in the xz plane.

Figure 4.7 shows the displacement of a cross-section as the result of a (largely exaggerated) deformation of the member. In Section 4.1 we assumed that displacements and rotations occur only in the xz plane. In that case, the position of the cross-section is defined by three displacement quantities, the two components of translation, and one rotation:

u – the displacement of the cross-section in the x direction;
w – the displacement of the cross-section in the z direction;
φ – the rotation of the cross-section about the y axis.[1]

In Figure 4.8 we see an enlarged representation of the situation for the cross-section from Figure 4.7, together with the fibres that are being kept together by the cross-section.

On the basis of Bernoulli's hypothesis, planar cross-sections remain normal to the fibres, before and after deformation of the member. This means

$$\varphi = -\frac{\mathrm{d}w}{\mathrm{d}x}\,. \tag{4.2}$$

The rotation of the cross-section is equal to the slope of the member axis. The minus sign indicates that the positive direction of φ (φ_y) in the xz coordinate system is opposite to the positive direction of $\mathrm{d}w/\mathrm{d}x$.

As a result of the relationship between φ and w via formula (4.2), there remain only two independent displacement quantities for the cross-section: u and w.

---

1 φ is the shortened notation for φ_y, the rotation about the y axis. Since there is only one rotation, the index y is omitted for simplicity.

Figure 4.8 shows the z fibre layer in the cross-section. If the rotation $\varphi = -\mathrm{d}w/\mathrm{d}x$ of the cross-section is small ($\varphi \ll 1$ and therefore $\sin\varphi \approx \varphi$), then the displacement in the x direction for these fibres in the z layer is

$$u(z) = u + z\sin\varphi \approx u + z\varphi = u - z\frac{\mathrm{d}w}{\mathrm{d}x}\,.$$

The strain in the fibre layer is

$$\varepsilon(z) = \frac{\mathrm{d}u(z)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} + z\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - z\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\,.$$

In Section 4.2 we defined the strain distribution by formula (4.1):

$$\varepsilon(z) = \varepsilon + \kappa_z z.$$

By comparing both expressions for $\varepsilon(z)$ we find the required *kinematic relationships* that link the deformation quantities and displacement quantities:

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}\,, \tag{4.3}$$

$$\kappa_z = \frac{\mathrm{d}\varphi}{\mathrm{d}x} = -\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\,. \tag{4.4}$$

The deformation quantity ε is the strain in the fibre layer $z = 0$ (the member axis); the deformation quantity κ_z is the slope of the strain diagram (see Figure 4.9).

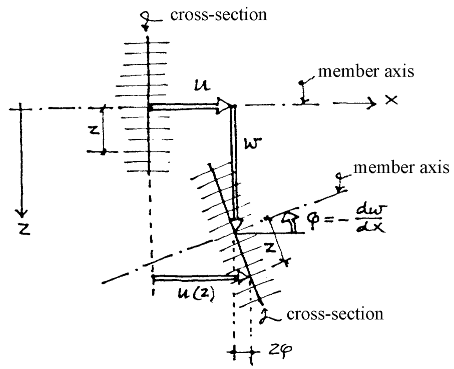


Figure 4.8 The enlarged cross-section from Figure 4.7, with the fibres that are kept together by the cross-section. $u(z)$ is the displacement in the x direction of a point at the fibre layer z.

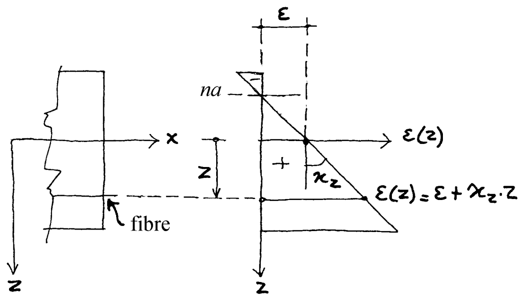


Figure 4.9 Strain diagram and neutral axis (*na*). The deformation quantity ε is the strain of the fibre layer $z = 0$ (the member axis). The deformation quantity κ_z is equal to the slope of the strain diagram. κ_z is also the *curvature of the member axis* in the xz plane.

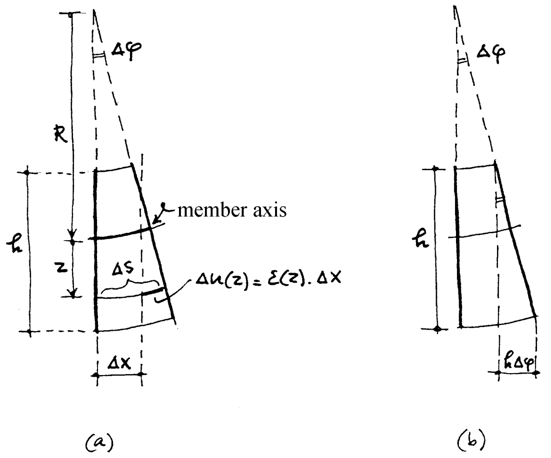


Figure 4.10 (a) The deformation of a small member element between two cross-sections at a mutual distance Δx ($\Delta x \to 0$). After deformation, the cross-sections are at an angle $\Delta\varphi$ to one another and $\varepsilon(z)$ is the strain of the fibre layer at a distance z to the member axis. (b) When deformed, the difference in length between the outer fibre layers is $h\Delta\varphi$.

The deformation quantity κ_z is also the *curvature of the member* in the xz plane. To prove this, we again look at a small member element between two cross-sections at a distance Δx ($\Delta x \to 0$) (see Figure 4.10a). After deformation, the cross-sections are at an angle $\Delta\varphi$ to one another, and ε is the strain of the member axis. In a fibre at a vertical distance z from the member axis the strain is $\varepsilon(z)$. For this fibre, the arc length in the deformed state is

$$\Delta s = \Delta x + \Delta u = \{1 + \varepsilon(z)\}\Delta x.$$

In mathematics, the curvature is defined as the change per arc length of the direction of a tangent to a curve, so

$$\kappa = \lim_{\Delta s \to s} \frac{\Delta\varphi}{\Delta s} = \frac{\mathrm{d}\varphi}{\mathrm{d}s}.$$

For the curvature of the z fibre in the xz plane we find

$$\kappa_z = \lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta s} = \frac{1}{1+\varepsilon(z)} \cdot \lim_{\Delta x \to 0} \frac{\Delta\varphi}{\Delta x} = \frac{1}{1+\varepsilon(z)} \frac{\mathrm{d}\varphi}{\mathrm{d}x}.$$

As a consequence of the differences in strain, the curvature is not the same for all the fibres.

Structural materials, such as concrete, steel, wood and so forth, generally experience only minor strains. For these materials $\varepsilon(z) \ll 1$, and the influence of the strain on the difference in curvature can be neglected:

$$\kappa_z = \lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta s} \approx \lim_{\Delta x \to 0} \frac{\Delta\varphi}{\Delta x} = \frac{\mathrm{d}\varphi}{\mathrm{d}x}.$$

κ_z therefore not only represents the slope of the strain diagram, but also the curvature of the member (axis) in the xz plane.

The reciprocal of the absolute value of the curvature κ is the *radius of curvature* R of the bend member:

$$R = \frac{1}{|\kappa_z|} .$$

Note: The assumption $\varepsilon(z) \ll 1$ implies that the difference in strain between the outermost fibres must also be small (see Figure 4.10b):

$$\frac{h\Delta\varphi}{\Delta x} \approx \frac{h}{R} \ll 1.$$

This means that the height h of the member must be far smaller than the radius of curvature R of the deformed member. The derivation is therefore valid only when the member does not bend too much.

4.3.2 Constitutive relationships

The constitutive relationships link the section forces (the stress resultants in the cross-section) and the deformation quantities. In order to define the constitutive relationships we have to know the material behaviour.

It is assumed that the material behaves *linear-elastically* and follows *Hooke's Law*. According to Hooke's Law (in its simplest form) the normal stresses σ in the fibres are proportional to the strains ε in the fibres:

$$\sigma = E\varepsilon.$$

The *modulus of elasticity* E or *Young's modulus* is a material constant.

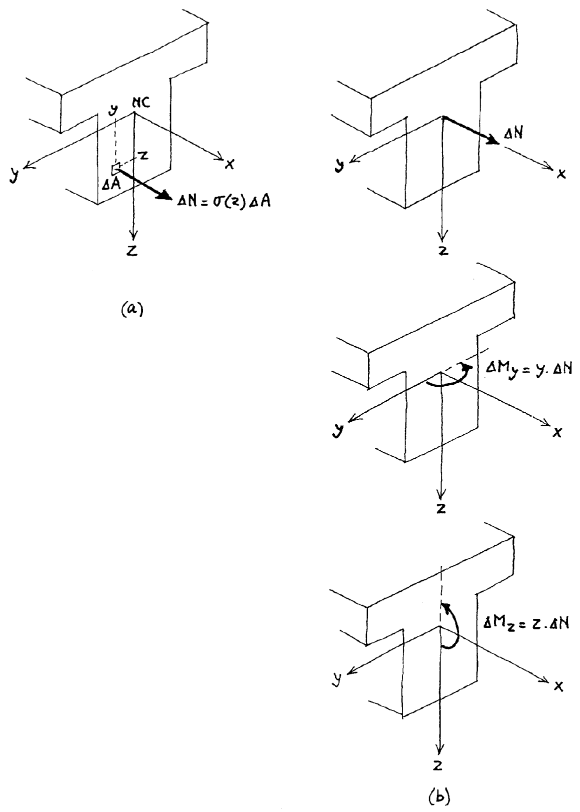


Figure 4.11 (a) In a fibre (y, z) with area ΔA and stress σ_z there is a small force $\Delta N = \sigma(z)\Delta A$. (b) If this force ΔN is moved towards the normal centre NC, the bending moments $\Delta M_y = y\Delta N$ and $\Delta M_z = z\Delta N$ are generated.

It is also assumed that the cross-section is *homogeneous*. This means that all fibres have the same modulus of elasticity E.

From the linear strain distribution in a linear-elastic member with homogeneous cross-section, given by

$$\varepsilon(z) = \varepsilon + \kappa_z z,$$

there is a linear stress distribution:

$$\sigma(z) = E\varepsilon(z) = E(\varepsilon + \kappa_z z). \tag{4.5}$$

From this distribution of the normal stresses we can now determine the section forces (stress resultants). There are three section forces: the normal force N, bending moment M_y in the xy plane, and the bending moment M_z in the xz plane.

In a fibre (y, z) with area ΔA and stress $\sigma(z)$, the resultant is a small force ΔN (see Figure 4.11a):

$$\Delta N = \sigma(z)\Delta A.$$

If this small force is moved to the normal centre NC of the cross-section it produces the following bending moments (see Figure 4.11b):

$$\Delta M_y = y\Delta N = y\sigma(z)\Delta A,$$

$$\Delta M_z = z\Delta N = z\sigma(z)\Delta A.$$

The section forces are found by summing the contributions of all the fibres, achieved by integrating over the area of the cross-section:

$$N = \int_A \sigma(z)\, \mathrm{d}A,$$

$$M_y = \int_A y\sigma(z)\, \mathrm{d}A,$$

$$M_z = \int_A z\sigma(z)\, \mathrm{d}A.$$

Substitute expression (4.5) for $\sigma(z)$ and we find

$$N = E\varepsilon \int_A \mathrm{d}A + E\kappa_z \int_A z\, \mathrm{d}A,$$

$$M_y = E\varepsilon \int_A y\, \mathrm{d}A + E\kappa_z \int_A yz\, \mathrm{d}A,$$

$$M_z = E\varepsilon \int_A z\, \mathrm{d}A + E\kappa_z \int_A z^2\, \mathrm{d}A.$$

With the knowledge gained in Chapter 3 we recognise the following cross-sectional quantities in the surface integrals:

$$\int_A \mathrm{d}A = A, \quad \int_A y\, \mathrm{d}A = S_y, \quad \int_A yz\, \mathrm{d}A = I_{yz},$$

$$\int_A z\, \mathrm{d}A = S_z, \quad \int_A z^2\, \mathrm{d}A = I_{zz}.$$

So the section forces can be written as

$$N = EA\varepsilon + ES_z\kappa_z,$$
$$M_y = ES_y\varepsilon + EI_{yz}\kappa_z,$$
$$M_z = ES_z\varepsilon + EI_{zz}\,\kappa_z.$$

These expressions supply a link between the section forces N; M_y; M_z and deformation quantities ε; κ_y; κ_z, and are therefore the required *constitutive relationships*. They hold for a member subject to bending and extension, with the deformations solely in the xz plane.

Using two assumptions from Section 4.1 that relate to the choice of the coordinate system, we can considerably simplify the constitutive relationships.

- *The first assumption relates to the location of the x axis.*

If the x axis coincides with the member axis, the origin of the yz coordinate system coincides with the normal centre NC of the cross-section. In a coordinate system through the normal centre of a homogeneous cross-section, the static moments S_y and S_z are zero, and we can therefore simplify the constitutive relationships to

$$N = EA\varepsilon,$$
$$M_y = EI_{yz}\kappa_z,$$
$$M_z = EI_{zz}\kappa_z.$$

- *The second assumption relates to the orientation of the y and z axes.*

It was assumed that the load and support reactions act in the xz plane. In that case there are section forces only in the xz plane, and $M_y = EI_{yz}\kappa_z$

(a section force in the xy plane) must be zero. This is possible only if the product of inertia I_{yz} is zero. For that the y and z axes must coincide with the *principal directions* of the cross-section. Only then do the member load and support reactions in the xz plane generate displacements and rotations only in the xz plane.[1]

This leaves the following relationships:

$$N = EA\varepsilon \qquad \text{(strain)}, \tag{4.6}$$

$$M_z = EI_{zz}\kappa_z \quad \text{(bending)}. \tag{4.7}$$

These are the *constitutive relationships* for members subject to bending and extension, in the simplest formulation. The simple form is a result of the choice of the coordinate system.

By selecting the origin of the coordinate system at the normal centre of the (homogeneous) cross-section, the cases of extension and bending can be treated separately:

- A normal force N generates extension (strain ε) only and no bending.
- A bending moment M_z generates bending only in the xz plane (curvature κ_z) and no extension (that is no strain in the member axis).

EA is known as the *axial stiffness* (resistance of the member to a change in

---

1 In Section 4.1 it was originally postulated that the xz plane, the plane in which the load and support reactions are acting, must be a symmetry plane. Now we see that the derivation is valid also for a non-symmetrical cross-section, as long as the y and z axis coincide with the principal directions.

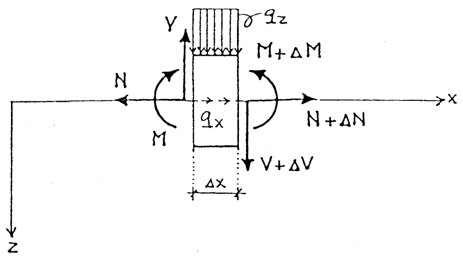


Figure 4.12 The section forces on an isolated member element with small length Δx ($\Delta x \to 0$).

length). EI_{zz} is known as the *bending stiffness* in the xz plane (resistance of the member to bending in the xz plane).

4.3.3 Static relationships

The static or equilibrium relationships supply a link between loads and section forces. They follow from the equilibrium of a small member element and were derived before.[1]

The derivation of the static relationships will be briefly repeated here.

In Figure 4.12, a small element with length Δx has been isolated from the member. The member element is subject to the distributed loads q_x and q_z.

The (unknown) shear forces acting on the left-hand and right-hand cross-sectional planes are shown in their positive directions. Assume that the forces on the left-hand cross-sectional plane are N, V and M.[2] Also assume that these forces increase over a distance Δx by ΔN, respectively ΔV and ΔM. The forces on the right-hand cross-sectional plane are then $(N + \Delta N)$, $(V + \Delta V)$, and $(M + \Delta M)$.

The force equilibrium of the member element in x and z direction respectively implies

$$\Delta N + q_x \Delta x = 0,$$

$$\Delta V + q_z \Delta x = 0.$$

---

[1] See Volume 1, Section 11.1.

[2] For the shear force V and the bending moment M, both acting in the xz plane, we have omitted the z index in this section. It is felt that no misunderstandings are possible.

The moment equilibrium about point A on the right-hand sectional plane implies that[1]

$$\Delta M - V\Delta x = 0.$$

If these three equilibrium equations are divided by Δx, we find in the limit $\Delta x \to 0$

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0 \ \text{(extension)}, \tag{4.8}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0 \ \text{(bending)}, \tag{4.9}$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0 \ \text{(bending)}. \tag{4.10}$$

These are the *static relationships* for members subject to bending and extension. The cases of extension (only normal forces) and bending (bending moments and shear forces, no normal forces) can be treated separately.

For bending, relationships (4.9) and (4.10) can be combined to

$$\frac{\mathrm{d}^2M}{\mathrm{d}x^2} + q_z = 0 \ \text{(bending)}. \tag{4.11}$$

Note: The derivation is not valid when concentrated forces and/or couples act on any member element with length Δx. In that case, the functions of N, V and/or M are no longer continuous and/or continuously differentiable.

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[1] For details, see Volume 1, Section 11.1.

The static relationships are covered again in Section 4.11 where they are applied together with the constitutive and kinematic relationships in order to derive a direct relationship between loading and displacements for a member subject to extension and bending.

4.4 Stress formula and stress diagram

For the stress distribution in a homogeneous cross-section, (4.5) states that

$$\sigma(z) = E\varepsilon(z) = E(\varepsilon + \kappa_z z).$$

Using the constitutive relationships (4.6) and (4.7) we can express the strain ε and curvature κ_z directly in terms of the normal force N and the bending moment M_z respectively:

$$\varepsilon = \frac{N}{EA} \quad \text{and} \quad \kappa_z = \frac{M_z}{EI_{zz}}\,.$$

Substituting these expressions ε and κ_z in that for $\sigma(z)$ leads to the following stress formula:

$$\sigma(z) = E\left(\frac{N}{EA} + \frac{M_z z}{EI_{zz}}\right) = \frac{N}{A} + \frac{M_z z}{I_{zz}}\,. \qquad (4.12)$$

This is an extremely important formula because we now can determine the distribution of the normal stresses in the cross-section directly from the magnitudes of the normal force and the bending moment.

The stress formula applies only when the cross-section is homogeneous, the y and z axes coincide with the principal directions of the cross-section, and the load and support reactions act in the xz plane. Note that the stress distribution in the homogeneous cross-section is independent of the modulus of elasticity E.

Figure 4.13a shows the distribution of the normal stress as a function of z in a (normal) stress diagram or σ diagram. Figure 4.13b shows how the stresses according to the stress diagram act on a (positive) cross-sectional plane, while Figure 4.13c shows the associated section forces (the stress resultants).

When interpreting the stress diagram, we must remember that the stress is constant in a fibre layer over the width of the member. In order to emphasise this, the stress diagram is also shown spatially in Figure 4.14.

In a homogeneous cross-section, stress and strain diagrams have the same shape (stress and strain diagrams are similar); compare Figures 4.5 and 4.14.

In the neutral axis (na), the fibre layer in which the strain is zero, the stress is zero. The neutral axis is therefore a straight line that divides the cross-section into two, a part with only tensile stresses, and one with only compressive stresses (see Figure 4.14).

The location of the neutral axis (z_{na}) can be found with stress formula (4.12) as the line where the stress is zero:

$$\sigma(z_{na}) = \frac{N}{A} + \frac{M_z\, z_{na}}{I_{zz}} = 0 \Rightarrow z_{na} = -\frac{N}{M_z}\frac{I_{zz}}{A}. \tag{4.13}$$

Note that the neutral axis can be located outside the cross-section.

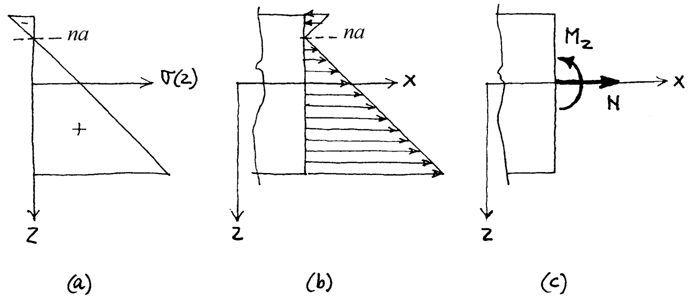

Figure 4.13 (a) Normal stress diagram with neutral axis (na). (b) The normal stresses as they act on the cross-section according to the stress diagram. (c) The associated section forces (stress resultants).

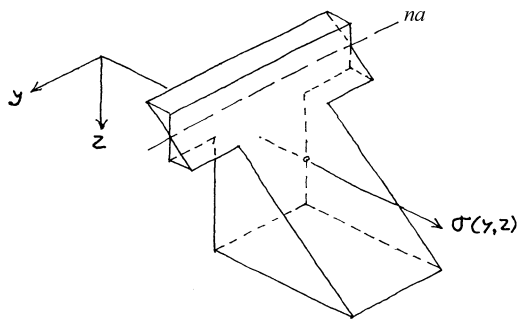

Figure 4.14 Spatial representation of the stress diagram. The neutral axis (na) is a straight line in the cross-section that separates the area with tensile stresses and the area with compressive stresses.

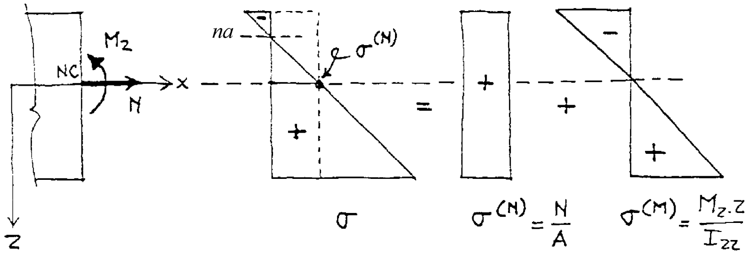

Figure 4.15 The stress diagram σ split into the contributions $\sigma^{(N)}$ and $\sigma^{(M)}$ resulting from the normal force N (extension) and the bending moment M_z (bending) respectively. The normal force N generates a constant normal stress in the cross-section. The bending moment generates a linear stress distribution over the height of the cross-section, with the neutral axis passing through the normal centre NC.

In the stress formula, the stresses resulting from N (extension) and M_z (bending) are determined separately and added. The separate contributions of N and M_z are also recognisable in the stress diagram (see Figure 4.15).

The normal force gives a constant normal stress in the cross-section:

$$\sigma^{(N)} = \frac{N}{A} \text{ (extension).} \tag{4.14}$$

In the total stress diagram, $\sigma^{(N)}$ is the stress at the member axis.

The bending moment gives a linear stress distribution over the height of the cross-section:

$$\sigma^{(M)} = \frac{M_z z}{I_{zz}} \text{ (bending).} \tag{4.15}$$

For bending, the neutral axis passes through the normal centre NC (it coincides with the y axis). The largest bending stresses[1] occur in the outermost fibres of the cross-section and have opposite signs.

Note: In strain, all the fibres are equally loaded and the material in the cross-section is used efficiently. In bending, the outermost fibres of the cross-section are most heavily loaded, while the fibres near the member axis barely participate. For bending, the material in the cross-section is used far less efficiently than for extension.

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[1] The expression “bending stress” is often used for the normal stress due to a bending moment only. But beware: “normal stress” may not be seen as the stress due to a normal force only!

4.5 Examples relating to the stress formula for bending with extension

This section includes five examples. In the first four (Sections 4.5.1 to 4.5.4) we determine the stress distribution due to bending with extension. In the last (Section 4.5.5), we determine the section forces from a given stress distribution.

4.5.1 A column loaded by an eccentric compressive force

The prismatic column with triangular cross-section in Figure 4.16 is loaded by an eccentric compressive force of 600 kN. The cross-sectional dimensions of the column and the eccentricity of the compressive force are shown in the figure. The dead weight of the column is omitted from the calculation.

Questions:

a. Which fibres are most heavily loaded?
b. Draw the normal stress diagram for a cross-section. Also draw the separate stress diagrams due to the (central) normal force and the bending moment respectively.
c. Determine the location of the neutral axis and plot it on the stress diagram.

Solution:

a. In Figure 4.16, the N and M diagrams are shown for the column modelled as a line element. All cross-sections are subject to the same normal force and the same bending moment.

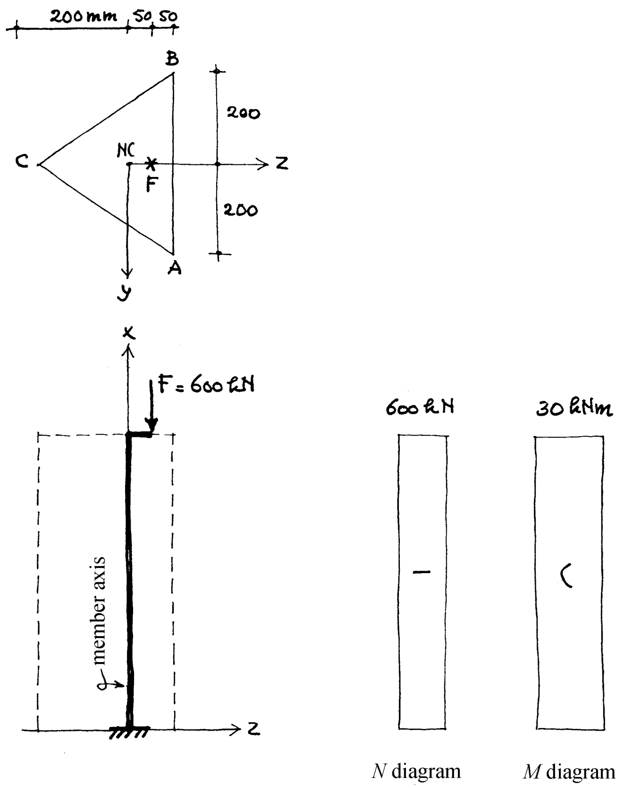


Figure 4.16 A column with triangular cross-section loaded by an eccentric compressive force, with the associated normal force and bending moment diagrams.

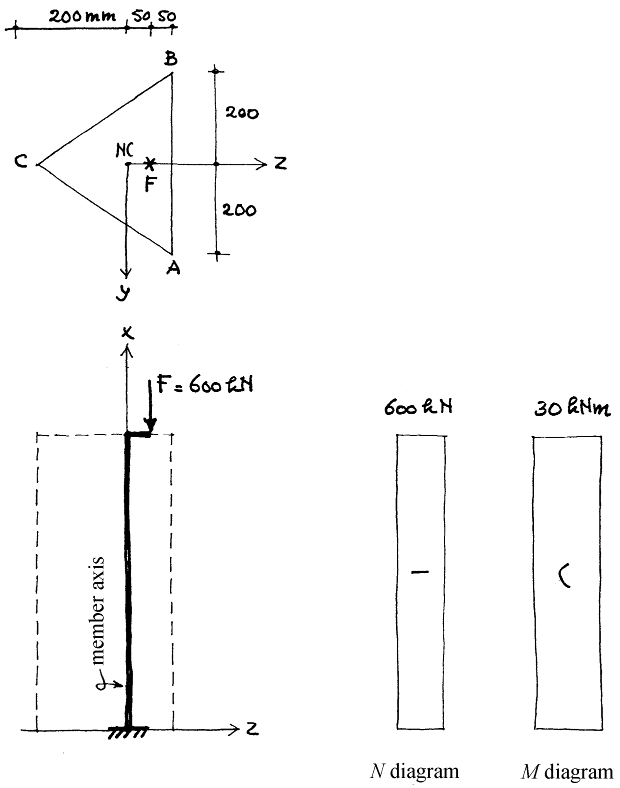


Figure 4.16 A column with triangular cross-section loaded by an eccentric compressive force, with the associated normal force and bending moment diagrams.

For the coordinate system given in Figure 4.16 we have

$$N = -600 \text{ kN} = -600 \times 10^3 \text{ N},$$
$$M_z = (-600 \text{ kN})(+50 \text{ mm}) = -30 \times 10^6 \text{ Nmm}.$$

To determine the distribution of the normal stresses we have to know the area A of the cross-section and the (centroidal) moment of inertia I_{zz}, associated with bending in the xz plane:

$$A = \tfrac{1}{2}(400 \text{ mm})(300 \text{ mm}) = 60 \times 10^3 \text{ mm}^2,$$
$$I_{zz} = \tfrac{1}{36} bh^3 (400 \text{ mm})(300 \text{ mm})^3 = 300 \times 10^6 \text{ mm}^4.$$

The units N and/or mm, in which all quantities will be expressed are not mentioned in the interim calculations below.

For the normal stress distribution equation (4.12) implies

$$\sigma(z) = \frac{N}{A} + \frac{M_z z}{I_{zz}} = \frac{-600 \times 10^3}{60 \times 10^3} + \frac{-30 \times 10^6 \times z}{300 \times 10^6}$$
$$= \left(-10 - \frac{z}{10}\right) \text{N/mm}^2.$$

The extreme stresses occur in the outermost fibres (fibre *layers*) AB and C:

AB: $z = +100 \text{ mm} \Rightarrow \sigma = -20 \text{ N/mm}^2$,

C: $z = -200 \text{ mm} \Rightarrow \sigma = +10 \text{ N/mm}^2$.

b. The normal stress is distributed linearly between AB and C. The normal stress diagram is shown in Figure 4.17, with the separate contributions $\sigma^{(N)}$ and $\sigma^{(M)}$ due to the (central) compressive force of 600 kN and the bending moment of 30 kNm respectively.

Note: Directly applying stress formula (4.12) means that one has to be careful with the signs for N and M. In order to make sure that no errors are made, we recommend that you look at the contributions $\sigma^{(N)}$ due to extension and $\sigma^{(M)}$ due to bending separately, and check that their signs agree with the directions of N and M.

In Figure 4.17, $\sigma^{(N)}$ is indeed a compressive stress and the signs of the bending stresses $\sigma^{(M)}$ agree with the direction of the bending moment M.

c. The location z_{na} of the neutral axis is found from the condition that the stress in that fibre layer is zero. Using the normal stress formula we find

$$\sigma_{(na)} = -10 - \frac{z_{na}}{10} = 0 \Rightarrow z_{na} = -100 \text{ mm}.$$

The neutral axis is therefore 100 mm to the left of the normal centre NC, as is shown in Figure 4.17.

4.5.2 A ceiling hook subject to an eccentric tensile force

The cast-iron ceiling hook in Figure 4.18 is loaded by a vertical force of 5 kN, of which the line of action is shown in the figure. The dimensions of cross-section AB, a symmetrical T-section, can be found from the figure. The T-section should not be considered to be thin-walled.

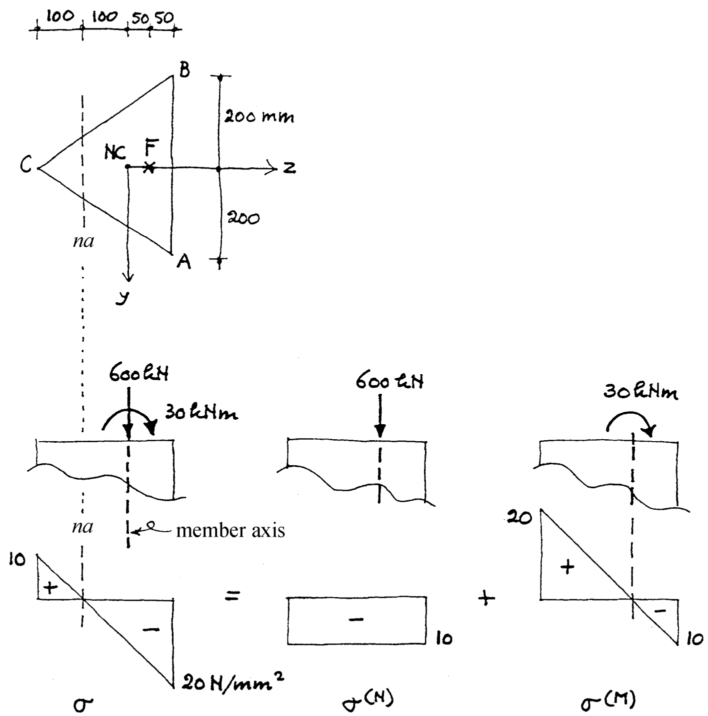


Figure 4.17 The normal stress diagram σ, together with the separate contributions $\sigma^{(N)}$ due to the (central) compressive force of 600 kN, and $\sigma^{(M)}$ due to the bending moment of 30 kNm.

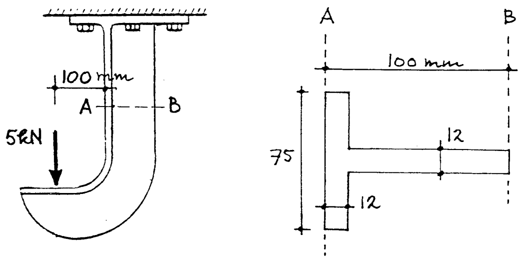


Figure 4.18 A ceiling hook with symmetrical T-section.

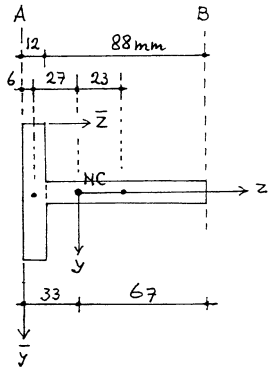


Figure 4.19 The location of the normal centre NC in the cross-section.

Questions:

a. For cross-section AB, determine and plot the normal stress distribution due to the 5 kN force. How large are the maximum stresses?
b. Determine the location of the neutral axis, and plot it on the stress diagram.

Solution (units N and mm):

a. In cross-section AB there is a tensile force of 5000 N. There is also a bending moment, of which the magnitude can be determined only when we know the location of the normal centre of the cross-section. Therefore we calculate first

- the area A of the cross-section,
- the location of the normal centre NC, and
- the (centroidal) moment of inertia I_{zz}, associated with bending in the xz plane for the coordinate system shown in Figure 4.19.

The area A of the cross-section is

$$A = (75 + 88) \times 12 = 1956 \text{ mm}^2.$$

The location of the normal centre NC is (see Figure 4.19)

$$\bar{z}_{\text{NC}} = \frac{S_{\bar{z}}^{\text{flange}} + S_{\bar{z}}^{\text{web}}}{A} = \frac{75 \times 12 \times 6 + 88 \times 12 \times (12 + 44)}{1956} = 33 \text{ mm}.$$

The moment of inertia I_{zz} of the T-section follows from

$$I_{zz} = I_{zz(\text{centr})}^{\text{flange}} + I_{zz(\text{Steiner})}^{\text{flange}} + I_{zz(\text{centr})}^{\text{web}} + I_{zz(\text{Steiner})}^{\text{web}}.$$

This gives

$$I_{zz} = \tfrac{1}{12} \times 75 \times 12^3 + 75 \times 12 \times 27^2 + \tfrac{1}{12} \times 12 \times 88^3 + 88 \times 12 \times 23^2$$

$$= 1.907 \times 10^6 \text{ mm}^4.$$

In addition to the normal force $N = +5000$ N there is a bending moment $M_z = -(5000 \text{ N})(133 \text{ mm}) = -665 \times 10^3$ Nmm in cross-section AB, as can be derived from the equilibrium of the part under section AB (see Figure 4.20). Determining the normal stress distribution is now a question of completing the stress formula(e).

According to (4.14) the contribution due to extension is

$$\sigma^{(N)} = \frac{N}{A} = \frac{+5 \times 10^3}{1956} = +2.56 \text{ N/mm}^2.$$

According to (4.15) the contribution due to bending is

$$\sigma^{(M)} = \frac{M_z z}{I_{zz}} = \frac{-665 \times 10^3 \times z}{1.907 \times 10^6} = \left(-0.35 \times \frac{z}{1 \text{ mm}}\right) \text{ N/mm}^2.$$

The associated stress diagrams are shown in Figure 4.21.

Check: The sign of the stresses $\sigma^{(N)}$ and $\sigma^{(M)}$ agree with the directions of N and M.

The stress diagrams $\sigma^{(N)}$ and $\sigma^{(M)}$ superimposed on one another of course lead to the same result as a direct application of stress formula (4.12):

$$\sigma^{(z)} = \frac{N}{A} + \frac{M_z z}{I_{zz}} = \left(+2.56 - 0.35 \times \frac{z}{1 \text{ mm}}\right) \text{ N/mm}^2.$$

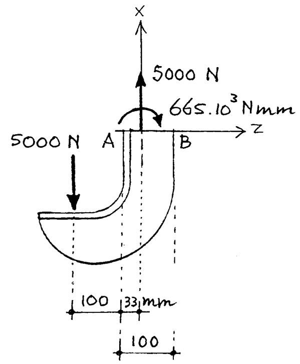


Figure 4.20 The normal force and the bending moment in cross-section AB follow directly from the equilibrium of the isolated part under the section.

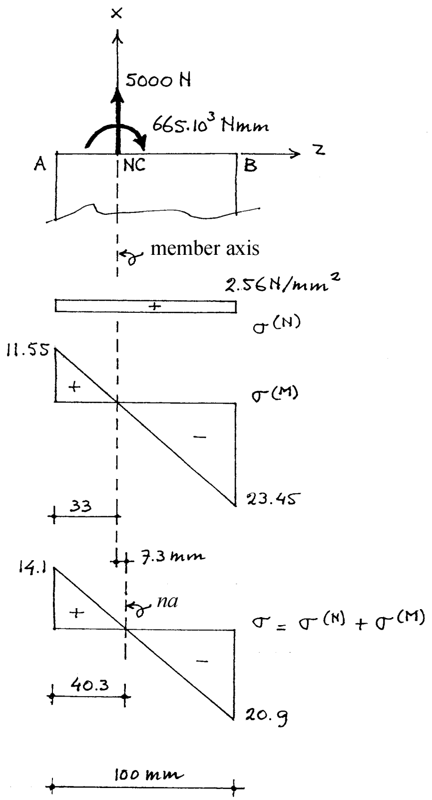


Figure 4.21 The normal stress diagram σ can be found by superposing the contributions $\sigma^{(N)}$ and $\sigma^{(M)}$ due to the normal force and the bending moment respectively.

The stresses are extreme in the outermost fibre layers at A and B:

$$\text{A}: \; z = -33 \text{ mm} \Rightarrow \sigma = +14.1 \text{ N/mm}^2,$$

$$\text{B}: \; z = +67 \text{ mm} \Rightarrow \sigma = -20.9 \text{ N/mm}^2.$$

Figure 4.21 also shows the ultimate normal stress diagram.

Note that the extreme bending stresses $\sigma^{(M)}$, due to the eccentricity of the axial load, are considerably larger than the stress $\sigma^{(N)}$ due to the normal force.

b. The location of the neutral axis can be derived directly from the stress diagram in Figure 4.21. The distance of the neutral axis to the left side A is

$$\frac{14.1}{14.1 + 20.9} \times 100 = 40.3 \text{ mm}.$$

It is also possible to use the derived expression for the stress distribution, using the condition that $\sigma = 0$ at $z = z_{na}$:

$$\sigma(z_{na}) = +2.56 - 0.35 \times z_{na} = 0 \Rightarrow z_{na} = \frac{2.56}{0.35} = 7.3 \text{ mm}.$$

This means that the neutral axis is 7.3 mm to the right of the normal centre NC. This is in line with what was found before.

4.5.3 A crane

The small crane in Figure 4.22 is on a quay wall and carries a load of 188 kN. The lifting cable is transported to the winch at E over two frictionless pulleys C and D. The cable is vertical over DE. At AB, the cross-section is a hollow rectangle with external dimensions of 750×375 mm^2 and a wall thickness of 25 mm.

Questions:

a. For cross-section AB, determine and plot the normal stress distribution and include the values.
b. Determine the location of the neutral axis and plot it on the stress diagram.

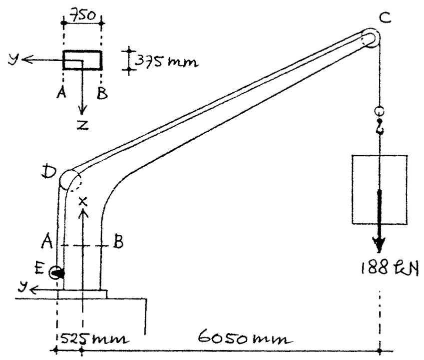


Figure 4.22 The column of the crane has a hollow rectangular cross-section at AB.

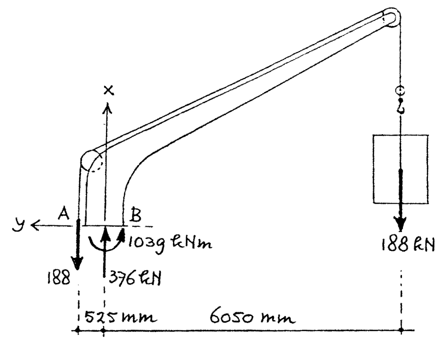


Figure 4.23 The section forces in cross-section AB follow directly from the equilibrium of the isolated part above the section.

Solution (units N and mm):
The equilibrium of the crane above section AB (see Figure 4.23) gives the following normal force N and bending moment M_y in cross-section AB:[1]

$$N = -2 \times 188 \times 10^3 = -376 \times 10^3 \text{ N},$$

$$M_y = +188 \times 10^3 \times 6050 - 188 \times 10^3 \times 525 = 1.039 \times 10^9 \text{ Nmm}.$$

In order to determine the normal stresses, we have to know the area A of the cross-section and, since the bending moment is acting in the xy plane, the moment of inertia I_{yy}. These quantities are determined below for the hollow cross-section as the difference between two rectangles:

$$A = 375 \times 750 - (375 - 50)(750 - 50) = 53.75 \times 10^3 \text{ mm}^2,$$

[1] The bending moment M_y acts in the xy plane.

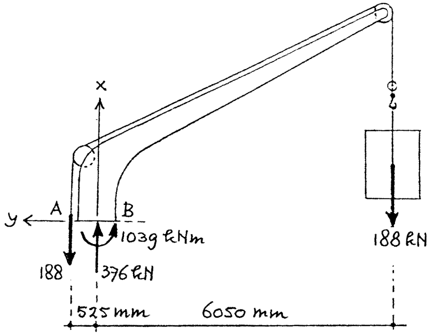


Figure 4.23 The section forces in cross-section AB follow directly from the equilibrium of the isolated part above the section.

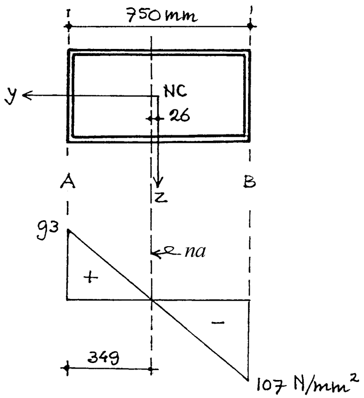


Figure 4.24 The normal force diagram with the neutral axis (*na*).

$$I_{yy} = \tfrac{1}{12} \times 375 \times 750^3 - \tfrac{1}{12} \times (375 - 50)(750 - 50)^3$$
$$= 3.894 \times 10^9 \text{ mm}^4.$$

The stress resulting from the normal force is

$$\sigma^{(N)} = \frac{N}{A} = \frac{-376 \times 10^3}{53.75 \times 10^3} = -7 \text{ N/mm}^2.$$

The maximum bending stresses are

$$\sigma^{(M)}_{\max} = \pm \frac{M_y \cdot \frac{1}{2}h}{I_{yy}} = \pm \frac{1.039 \times 10^9 \times 750/2}{3.894 \times 10^9} = \pm 100 \text{ N/mm}^2$$

with tension in the outermost fibre layer at A and compression in the outermost fibre layer at B.

The resulting normal stresses at A and B are

$$\text{A}: \quad \sigma = -7 + 100 = +93 \text{ N/mm}^2,$$
$$\text{B}: \quad \sigma = -7 - 100 = -107 \text{ N/mm}^2.$$

The stress distribution between A and B is linear. The σ diagram is given in Figure 4.24.

b. From the stress diagram we can derive the location of the neutral axis at a distance

$$\frac{93}{93 + 107} \times 750 \approx 349 \text{ mm}$$

from A, that is 26 mm to the left of the normal centre NC (see Figure 4.24).

4.5.4 A prestressed beam

The prismatic T-beam in Figure 4.25 is simply supported at A and B, and is loaded by two forces of 346 kN. The T-beam is prestressed with a straight tendon at P, 180 mm under the beam axis. The prestressing force F_p is 2400 kN. The dimensions of the beam and the location of the normal centre NC are given in the figure.

The cross-sectional quantities of the beam are:

$$A = 480 \times 10^3 \text{ mm}; \ I_{yy} = 160 \times 10^3 \text{ mm}; \ I_{zz} = 32.4 \times 10^9 \text{ mm}^4$$

Questions:
For materials that are not particularly resistant to tension (such as concrete), one can apply prestressing to suppress possible tensile stresses. Check whether this has been successful here. The following suggestions may help in answering:

a. Model the prestressed beam as a line element and draw all forces acting on it. Plot the normal force and bending moment diagrams with the deformation symbols.
b. Check the normal stresses in the indicative cross-sections. Where in those cross-sections is the neutral axis positioned?

Solution:
a. The tendon exerts an eccentric compressive force of 2400 kN on both beam ends. By moving these eccentric compressive forces to the beam axis, we generate moments on the beam ends:

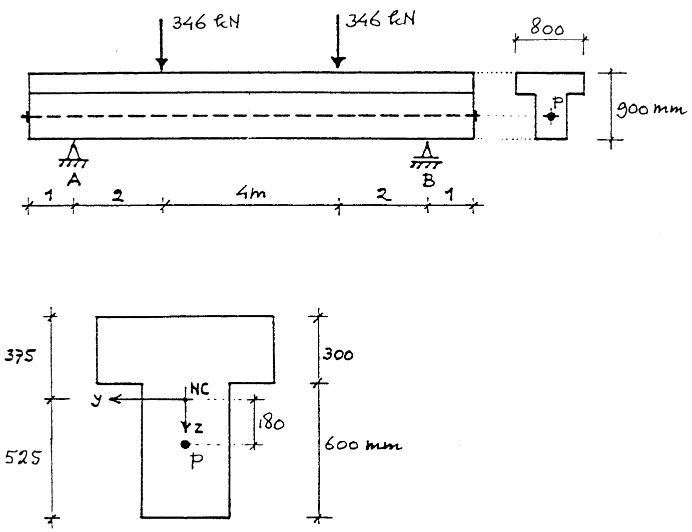


Figure 4.25 A T-beam with its cross-sectional dimensions. The T-beam is prestressed by a straight steel tension rod (tendon) at P, 180 mm under the member axis.

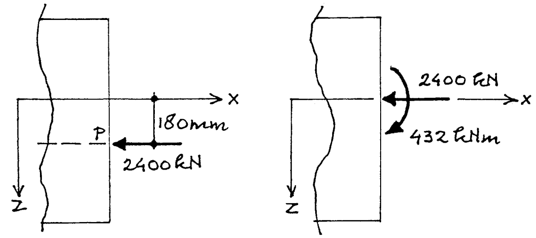


Figure 4.26 The tendon exerts an eccentric compressive force of 2400 kN on both beam ends. By moving these eccentric compressive forces to the member axis we generate moments of 432 kNm on the beam ends.

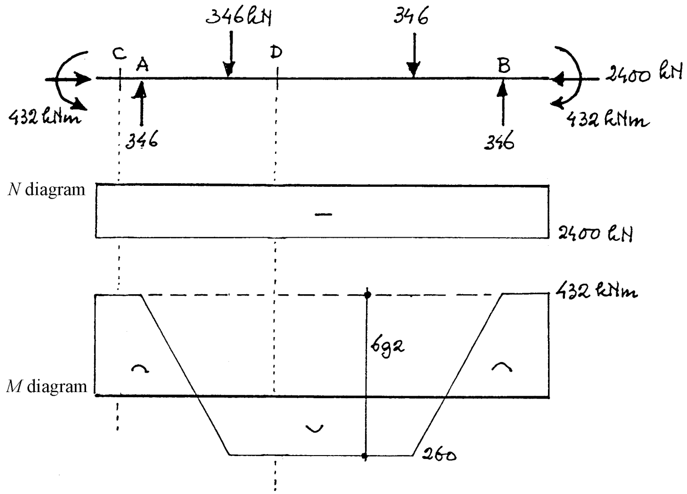


Figure 4.27 The T-beam modelled as a line element with all the forces acting on it and the associated normal force and bending moment diagrams.

$$(180 \text{ mm})(2400 \text{ kN}) = 432 \text{ kNm}.$$

See Figure 4.26 and make sure the directions are correct!

Figure 4.27 shows all the forces acting on the prestressed beam modelled as a line element, as well as the normal force and bending moment diagrams.

b. As the normal force is constant, we need to check for stresses in the cross-sections where the bending moments are largest. Figure 4.27 shows the relevant cross-sections C and D, where the bending moments are 432 kNm and 260 kNm respectively.

- Stress check cross-section C (see Figure 4.28):

$$N = -2400 \text{ mm},$$

$$M_z = -432 \text{ kNm} \text{ (compression at the bottom, and tension at the top).}$$

Due to the normal force, the normal stress is

$$\sigma^{(N)} = \frac{N}{A} = \frac{-244 \text{ kN}}{48 \times 10^3 \text{ mm}^2} = -5.0 \text{ N/mm}^2.$$

The maximum bending stresses occur in the outer fibres at the bottom (b) and at the top (t), with $z_b = +525$ mm and $z_t = -375$ mm respectively:

$$\sigma_\text{b}^{(M)} = \frac{M_z z_\text{b}}{I_{zz}} = \frac{(-432 \text{ kNm})(+525 \text{ mm})}{32.4 \times 10^9 \text{ mm}^4} = -7.0 \text{ N/mm}^2,$$

$$\sigma_\text{t}^{(M)} = \frac{M_z z_\text{t}}{I_{zz}} = \frac{(-432 \text{ kNm})(-3.75 \text{ mm})}{32.4 \times 10^9 \text{ mm}^4} = +5.0 \text{ N/mm}^2.$$

Sign check: In agreement with the direction of the bending moment, there is compression in the bottom fibres and tension in the top fibres.

The resulting normal stresses in the outer fibre layers are:

$$\sigma_b = (-5.0 - 7.0)\ \text{N/mm}^2 = -12.0\ \text{N/mm}^2,$$

$$\sigma_t = (-5.0 + 5.0)\ \text{N/mm}^2 = 0.$$

The stress diagram is shown in Figure 4.28. There is compression over the entire cross-section. The neutral axis coincides with the top fibre layer.

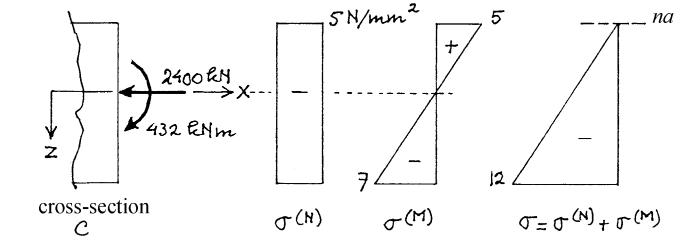


Figure 4.28 The section forces at C and the associated normal stress diagram obtained by superposing the contributions $\sigma^{(N)}$ due to extension and $\sigma^{(M)}$ due to bending. The neutral axis (*na*) coincides with the top fibre layer.

- Stress check cross-section D (see Figure 4.29):

$$N = -2400\ \text{kN},$$

$$M_z = +260\ \text{kN (tension at the bottom and compression at the top).}$$

The normal stress $\sigma^{(N)}$ due to extension (the normal force) is the same as in cross-section C:

$$\sigma^{(N)} = \frac{N}{A} = \frac{-2400\ \text{kN}}{480 \times 10^3\ \text{mm}^2} = -5.0\ \text{N/mm}^2.$$

The maximum bending stresses occur in the outer fibres at the bottom (b) and at the top (t) of the cross-section:

$$\sigma_b^{(M)} = \frac{M_z z_b}{I_{zz}} = \frac{(+260\ \text{kNm})(+525\ \text{mm})}{32.4 \times 10^9\ \text{mm}^4} = +4.2\ \text{N/mm}^2,$$

$$\sigma_t^{(M)} = \frac{M_z z_t}{I_{zz}} = \frac{(+260\ \text{kNm})(-375\ \text{mm})}{32.4 \times 10^9\ \text{mm}^4} = -3.0\ \text{N/mm}^2.$$

Sign check: Here too, the sign of the stress in the outer fibres is in agreement with the direction of the bending moment

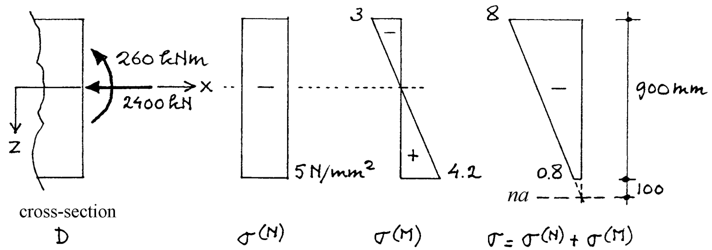


Figure 4.29 The section forces at D and the associated normal stress diagram obtained by superposing the contributions $\sigma^{(N)}$ due to extension and $\sigma^{(M)}$ due to bending. The neutral axis (*na*) is outside the cross-section here.

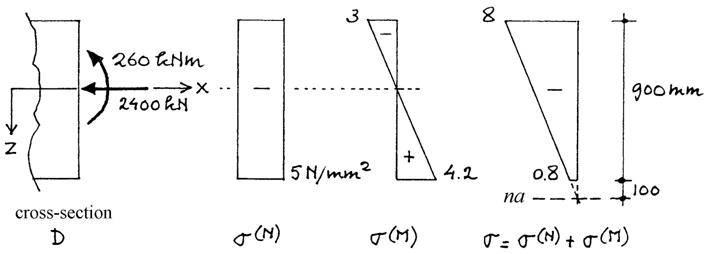


Figure 4.29 The section forces at D and the associated normal stress diagram obtained by superposing the contributions $\sigma^{(N)}$ due to extension and $\sigma^{(M)}$ due to bending. The neutral axis (*na*) is outside the cross-section here.

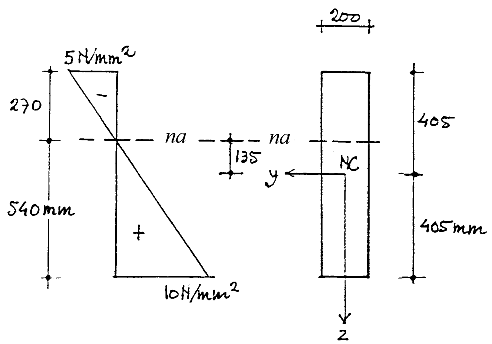


Figure 4.30 The normal stress diagram for a rectangular cross-section of a laminated wood joist.

The resulting stresses in the outer fibre layers are

$$\sigma_b = (-5.0 + 4.2)\ \text{N/mm}^2 = -0.8\ \text{N/mm}^2,$$

$$\sigma_t = (-5.0 - 3.0)\ \text{N/mm}^2 = -8.0\ \text{N/mm}^2.$$

The stress diagram is shown in Figure 4.29. Here too, there is compression in the entire cross-section. The neutral axis is outside the cross-section.

From the stress diagram, we can derive that the neutral axis is

$$\frac{0.8\ \text{N/mm}^2}{(8.0 - 0.8)\ \text{N/mm}^2}(900\ \text{mm}) = 100\ \text{mm}$$

below the bottom of the cross-section.

Comment: When calculating the bending stresses $\sigma^{(M)}$ one has to be careful with the plus and minus signs in the formula. Therefore always check whether the signs of the bending stresses agree with the direction of the bending moment.

4.5.5 Interpreting a normal stress diagram

Figure 4.30 shows the normal stress distribution for a rectangular cross-section from a laminated wood joist.

Questions:

a. Use the given stress diagram to determine the magnitude and point of application of the resultant R_t of all tensile stresses. Do the same for the resultant R_c of all compressive stresses.
b. From the magnitudes and points of application of R_t and R_c, determine the normal force and bending moment in the cross-section.

c. Where in the cross-section is the centre of force?

Solution:
a. Figure 4.31 gives a spatial representation of the stress diagram. The resultant of all the tensile (compressive) stresses is equal to the volume of the stress diagram over the area with tension (compression):

$$R_t = \tfrac{1}{2} \times (10\ \text{N/mm}^2)(540\ \text{mm})(200\ \text{mm}) = 540\ \text{kN},$$

$$R_c = \tfrac{1}{2} \times (5\ \text{N/mm}^2)(270\ \text{mm})(200\ \text{mm}) = 135\ \text{kN}.$$

The points of application of the stress-resultants R_t and R_c are given in Figure 4.32a.

b. The normal force N is by definition positive as a tensile force, and is found from the difference between R_t and R_c:

$$N = R_t - R_c = (540\ \text{kN}) - (135\ \text{kN}) = 405\ \text{kN}.$$

N is a tensile force.

The bending moment is found as the sum of the moments with respect to the normal centre NC of all the small stress resultants $\sigma\,\Delta A$ (see Section 4.3.2). This is equal to the sum of the moments with respect to NC of the stress resultants R_t and R_c. In the given coordinate system

$$\begin{aligned} M_z &= +R_t \times (225\ \text{mm}) + R_d \times (315\ \text{mm}) \\ &= +(540\ \text{kN})(225\ \text{mm}) + (135\ \text{kN})(315\ \text{mm}) = +164\ \text{kNm}. \end{aligned}$$

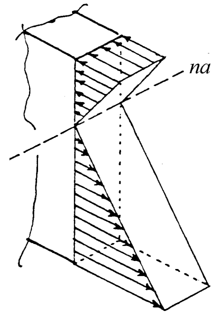


Figure 4.31 Spatial representation of how the stresses from the diagram in Figure 4.30 act on the cross-section.

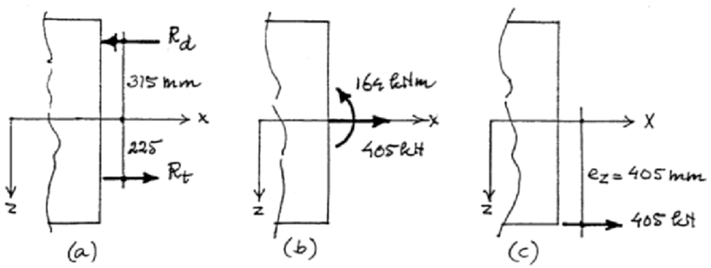


Figure 4.32 (a) The resultants R_t and R_c of the tensile and compressive stresses respectively, (b) the section forces N and M_z, and (c) the centre of force in the cross-section.

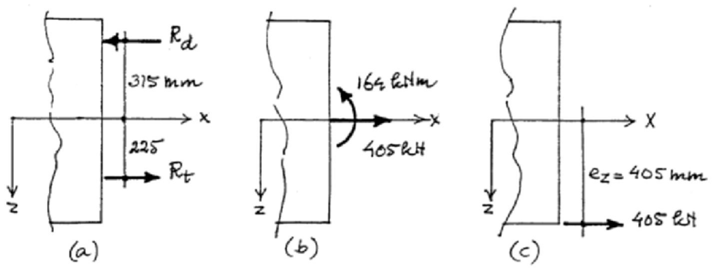


Figure 4.32 (a) The resultants R_t and R_c of the tensile and compressive stresses respectively, (b) the section forces N and M_z, and (c) the centre of force in the cross-section.

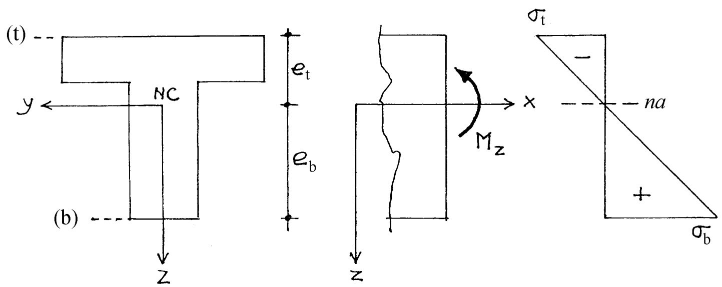


Figure 4.33 Stress diagram for bending without extension; in the outer fibre layers (b) and (t) the bending stresses are maximum and have opposite signs; the neutral axis (*na*) passes through the normal centre NC.

This bending moment gives a tensile stress in the bottom fibre layer and a compressive stress in the top one.

Figure 4.32b shows the section forces N and M_z in magnitude and direction.

c. The combination of a (central) normal force N and a bending moment M_z is statically equivalent to a single eccentric force, in this case a tensile force of 405 kN (see Figure 4.32c). The point of application of this eccentric tensile force is known as the *centre of force*.[1] The eccentricity e_z of the centre of force is

$$e_z = \frac{M_z}{N} = \frac{+164\ \text{kNm}}{+405\ \text{kN}} = +0.405\ \text{m} = +405\ \text{mm}.$$

The centre of force is at the bottom of the cross-section.

4.6 Section modulus

For bending without extension (without a normal force), the neutral axis passes through the normal centre NC, and the maximum normal stresses occur in the outer fibre layers (see Section 4.4, equation (4.15)).

Assume e_b and e_t are the distances from NC to the bottom (b) and top fibre layer (t) respectively. The maximum stresses in the outer fibre layers are then (see Figure 4.33):

---

1 The *centre of force* in the cross-section is the point of application of the resultant of all normal stresses in the cross-section (see Volume 1, Section 14.2). For a tensile force the centre of force is also called the *centre of tension*.

$$\sigma_b = +\frac{M_z e_b}{I_{zz}} \text{ and } \sigma_t = -\frac{M_z e_t}{I_{zz}}.$$

The values I_{zz}/e_b and I_{zz}/e_t are known as the *section moduli* of the cross-section and are indicated by $W_{z;b}$ and $W_{z;t}$:

$$W_{z;b} = \frac{I_{zz}}{e_b} \text{ and } W_{z;t} = \frac{I_{zz}}{e_t}. \tag{4.16}$$

With these notations we can write for the maximum bending stresses

$$\sigma_b = +\frac{M_z}{W_{z;b}} \text{ and } \sigma_t = -\frac{M_z}{W_{z;t}}. \tag{4.17}$$

Often the signs in these formulas are omitted, and one has to deduce them from the direction of the bending moment.

In books of tables with properties for designing beams, you will find the section modulus alongside other cross-sectional quantities, as the location of the centroid and the moments of inertia. With the help of the section modulus one can easily calculate the *maximum bending stresses*.

For cross-sections in which the y axis is a line of symmetry, such as those in Figure 4.34, it holds that $e_b = e_t = h/2$ and $W_{z;b}$ and $W_{z;t}$ have the same magnitude:

$$W_{z;b} = W_{z;t} = W_z = \frac{I_{zz}}{\frac{1}{2}h}. \tag{4.18}$$

The maximum *bending tensile stress and bending compressive stress* now

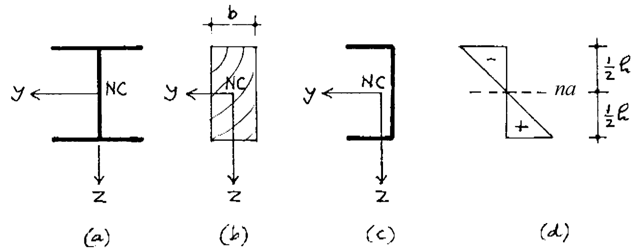


Figure 4.34 (a) to (c) Cross-sections with the y axis as a line of symmetry; (d) with bending in the xz plane, the maximum tensile bending stress and maximum compressive bending stress are equal (but with opposite signs).

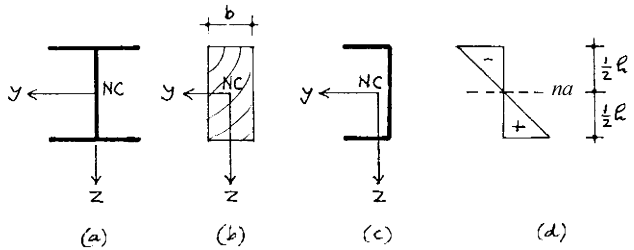


Figure 4.34 (a) to (c) Cross-sections with the y axis as a line of symmetry; (d) with bending in the xz plane, the maximum tensile bending stress and maximum compressive bending stress are equal (but with opposite signs).

have the same magnitude, and we write

$$\sigma_{\max} = \pm \frac{M_z}{W_z}. \tag{4.19}$$

The signs are generally omitted in this formula also.

Note: There is a frequently used formula for the section modulus for a rectangular cross-section (see Figure 4.34b):

$$W_z = \frac{I_{zz}}{\frac{1}{2}h} = \frac{\frac{1}{12}bh^3}{\frac{1}{2}h} = \frac{1}{6}bh^2. \tag{4.20}$$

In the next section, the application of these formulas is illustrated with the help of a number of examples.

4.7 Examples of the stress formula related to bending without extension

This section includes four examples of bending without a normal force, in which we use the section modulus to determine the bending stresses.

In the first example in Section 4.7.1 we address the investigation of the most appropriate cross-sectional shape for bending.

Thereafter we look at two examples in which the beams have symmetrical cross-sections. The beams are subject to bending and have to be dimensioned. Section 4.7.2 concerns a laminated wooden beam; Section 4.7.3 concerns a steel floor joist. In both examples load factors are used.

Finally, in Section 4.7.4, we determine the maximum bending stresses in a cantilevered T-beam.

4.7.1 The most favourable cross-sectional shape for bending

Four prismatic beams (a) to (d) are loaded by the same point load F, at the middle of the span with length $\ell = 1$ m. The beams are made of plain concrete, and have different cross-sections. Figure 4.35, alongside the scheme, shows the cross-sectional dimensions of the beams (a) to (d). The area of the cross-section and therefore the amount of material is (nearly) the same for all four beams.

It is given that cracking of concrete occurs at a bending tensile stress of 3 MPa and that concrete crushes at a bending compressive stress of 40 MPa.[1]

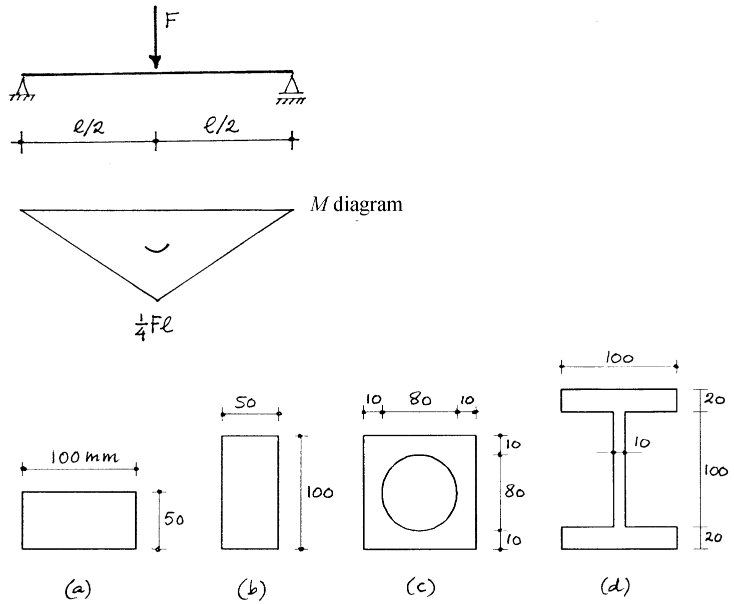


Figure 4.35 A simply supported prismatic beam with a point load at midspan. For the beam, four cross-sectional shapes are investigated that all have (virtually) the same area and therefore the same amount of material.

Question:
Determine the load-bearing capacity for each beam. The dead weight of the beams is ignored.

Solution:
The maximum bending moment occurs at midspan:

$$M_{\max} = \frac{1}{4} F\ell.$$

The maximum bending stresses occur in the outer fibre layers $z = \pm h/2$.

[1] Remember that 3 MPa $= 3 \times 10^6$ N/m^2 $= 3$ N/mm^2.

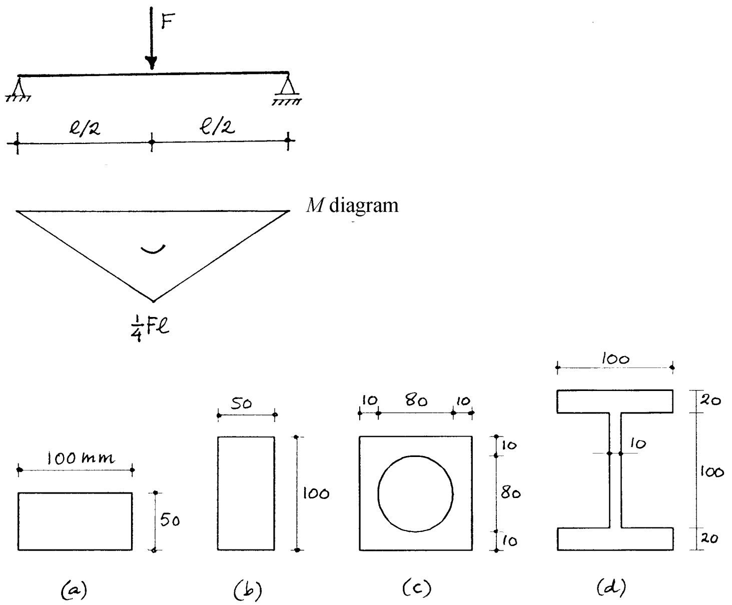


Figure 4.35 A simply supported prismatic beam with a point load at midspan. For the beam, four cross-sectional shapes are investigated that all have (virtually) the same area and therefore the same amount of material.

According to (4.18), it is

$$\sigma_{\max} = \pm\frac{M}{W},$$

in which

$$W = \frac{I}{\frac{1}{2}h}.$$

As there is no possibility of confusion, the index z in M_z, W_z and I_{zz} has been omitted here.

The maximum tensile stress is the critical one, and in this case it occurs at the bottom of the beam. This stress may not exceed the given *limiting value* $\bar{\sigma} = 3$ N/mm^2 for tensile stresses.[1]

The maximum bending moment $\bar{M}$ that the cross-section can transfer follows from (4.18):

$$\bar{M} = \bar{\sigma}W.$$

The maximum admissible force $\bar{F}$, the load-bearing capacity, is therefore

$$\bar{F} = \frac{4\bar{M}}{\ell} = \frac{4\bar{\sigma}W}{\ell}.$$

For all four beams the load-bearing capacity is calculated in Table 4.1.

The table shows that adjusting the cross-sectional shape can considerably

[1] Also referred to as *maximum admissible tensile stress.*

enhance the load-bearing capacity with the same amount of material (the same area A of the cross-section).

For bending without normal force, the outer fibres are loaded maximally. The fibres near the neutral axis are not used optimally as far as strength is concerned. From a structural perspective, it appears to be favourable to remove material here and bring it outward, far from the neutral axis. The result is a larger moment arm, and therefore a larger bending moment that can be transferred with the same maximum stress.

It is for this reason that you will rarely find solid rectangular cross-sections for steel beams, but mainly sections with a maximum of material at the outside, in the outer fibres (see the IPE and H-sections in Figure 4.36). IPE-sections have small flanges. H-sections have broad flanges; such beams are known as *broad-flanged beams*.

If one can achieve a larger load-bearing capacity by adjusting the cross-sectional shape, then one can use lighter profiles. This is very important for structures in which the dead weight is a substantial part of the load, such as for bridge beams with large spans. In addition to the *saving of material*, there is also a *saving on weight*, so that the bending moments due to the dead weight become smaller.

Returning to the example of the plain concrete beams, we note that the stresses in the compression zone are well below the admissible value. Hence, the available material has not been used optimally for the plain concrete cross-sections.

In concrete, the load-bearing capacity can be further increased by *reinforcing* the beam in the tension zone (so-called *reinforced concrete*: the reinforcing steel transfers the tension forces) or by applying prestressing through which the tensile stresses are "suppressed" (so-called *prestressed concrete*; see the example in Section 4.5.4).

Table 4.1

Type	A (mm^2)	I (mm^4)	$\frac{1}{2}h$ (mm)	W (mm^3)	$\bar{M}$ (Nmm)	$\bar{F}$ (N)
a	5000	1.042×10^6	25	41.68×10^3	125.0×10^3	500
b	5000	4.167×10^6	50	83.34×10^3	250.0×10^3	1000
c	4973	6.323×10^6	50	126.5×10^3	379.4×10^3	1518
a	5000	15.367×10^6	70	219.5×10^3	658.6×10^3	2634

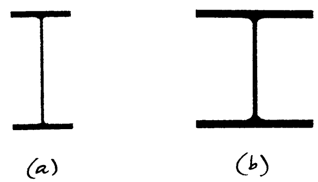


Figure 4.36 Steel beams: (a) an IPE-section and (b) an H-section of a broad-flanged beam.

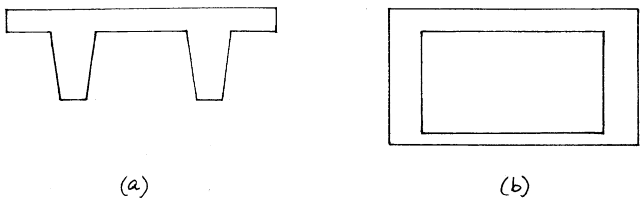
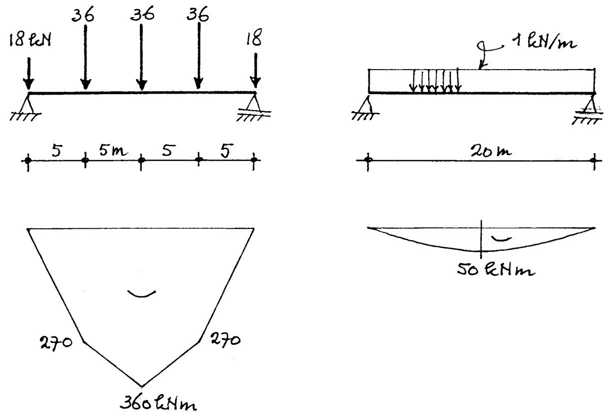


Figure 4.37 Concrete beams with adjusted cross-sectional shapes: (a) a prefab double T-beam, and (b) a box girder.



live load (given)

dead weight (estimated)

Figure 4.38 Loading schemes and bending moment diagrams for the given live load and an estimated dead weight of a wooden main girder.

Also for concrete beams we see that the shape of the cross-section is adjusted to the force flow as far as possible (see for example the T-beam and box girder in Figure 4.37).

4.7.2 Dimensioning a laminated wooden main girder (strength calculation)

Figure 4.38 shows the scheme for a wooden main girder with a span of 20 m. The live load on the girder consists of a number of point loads, shown in the figure. The dead weight of the girder is estimated at 1 kN/m. The figure also shows the bending moment diagrams due to both the live load and dead weight. The main girder is constructed as a laminated beam[1] with a rectangular cross-section, composed of planks that are 196 mm wide and 34 mm thick (see Figure 4.39).

The beam is considered to be sufficiently strong when the bending stress due to the design value of the load nowhere exceeds the design value f of the bending strength. The design value is equal to $\gamma_{\rm dw}$ times the dead weight and $\gamma_{\rm ll}$ times the live load. Here $\gamma_{\rm dw}$ and $\gamma_{\rm ll}$ are the so-called load factors.[2]

Questions:

a. Determine the number of planks in the cross-section so that the beam

---

1 A laminated beam is a beam composed of planks glued together. The cross-section can be considered solid.

2 The regulations state which load combinations have to be taken into account and which load factors γ have to be applied. We do not address this here. Nor do we address the way in which the design value f of the bending strength for wood is determined. See also Volume 1, Section 6.2.5.

just meets the strength demand. Use in the calculation

$$\gamma_{dw} = 1.2, \ \gamma_{ll} = 1.5 \ \text{and} \ f = 20 \, \text{N/mm}^2.$$

b. Determine the maximum bending stress in the serviceability state, for which

$$\gamma_{dw} = \gamma_{ll} = 1.$$

Solution:
The maximum bending moment due to the design value for the load occurs at midspan:

$$M_{max} = 1.5 \times (360 \, \text{kNm}) + 1.2 \times (50 \, \text{kNm}) = 600 \, \text{kNm}.$$

The (extreme) bending stress σ_{max} caused by this moment must remain under the design value $f = 20 \, \text{N/mm}^2$ of the bending strength:

$$\sigma_{max} = \frac{M_{max}}{W} \leq f.$$

This gives the least section modulus $W_{required}$ that is required:

$$W_{required} = \frac{M_{max}}{f} = \frac{600 \, \text{kNm}}{20 \, \text{N/mm}^2} = 3 \times 10^6 \, \text{mm}^3.$$

The cross-section of the laminated beam, consisting of a stack of planks 196 mm wide, must satisfy[1]

$$W_{required} = \tfrac{1}{6} bh^2 = \tfrac{1}{6} \times (196 \, \text{mm}) \times h^2 = 30 \times 10^6 \, \text{mm}^3$$

[1] Remember that, for a rectangular cross-section, $W = \frac{1}{6} bh^2$, see formula (4.20).

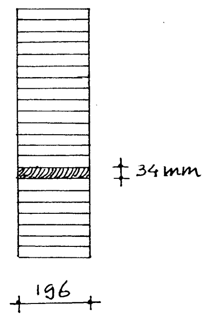


Figure 4.39 The main girder is constructed as a laminated beam, composed of planks 196 mm wide and 34 mm thick.

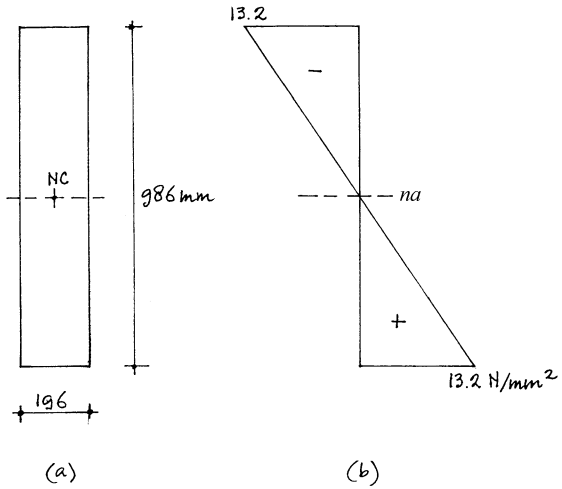


Figure 4.40 (a) The final cross-sectional dimensions of the main girder and (b) the stress diagram resulting from the maximum bending moment in the serviceability state.

in which h is the height of the beam. From this equation we find the minimum height of the beam:

$$h_{\text{min}} = \sqrt{\frac{6 \times W_{\text{required}}}{b}} = \sqrt{\frac{6 \times (30 \times 10^6 \text{ mm}^3)}{196 \text{ mm}}} = 958.3 \text{ mm}.$$

For a thickness of 34 mm per plank, the number of planks we need at least is

$$\frac{958.3 \text{ mm}}{34 \text{ mm}} = 28.2 \approx 29 \,.$$

With 29 planks of 34 mm thickness the height of the laminated beam is $29 \times (34 \text{ mm}) = 986$ mm. The final cross-section is shown in Figure 4.40a. The section modulus W for this cross-section is

$$W = \tfrac{1}{6}\, bh^2 = \tfrac{1}{6}\, (196 \text{ mm})(986 \text{ mm})^2 = 31.76 \times 10^6 \text{ mm}^3.$$

Checking the estimated dead weight:
With a specific weight of 6 kN/m^3 the dead weight of the beam is

$$(196 \text{ mm})(986 \text{ mm})(6 \text{ kN/m}^3) = 1.16 \text{ kN/m}.$$

This is slightly more than the 1 kN/m assumed. The maximum bending moment due to the dead weight is therefore not 50 kNm, but $1.16 \times 50 = 58$ kNm. To be sure (not being experienced designers), we therefore check for the adjusted dead weight.

The design value of the load gives the following maximum bending moment at midspan:

$$M_{\max} = 1.5 \times (360\ \text{kNm}) + 1.2 \times (58\ \text{kNm}) = 609.6\ \text{kNm}.$$

Hence the maximum bending stress is

$$\sigma = \frac{M_{\max}}{W} = \frac{609.6\ \text{kNm}}{31.76 \times 10^6\ \text{mm}^3} = 19.2\ \text{N/mm}^2 \leq f = 20\ \text{N/mm}^2.$$

This cross-section therefore meets the strength demand.

b. The last question relates to the maximum bending stress in the serviceability state.

With $\gamma_{\text{dw}} = \gamma_{\text{ll}} = 1$, the maximum bending moment in the serviceability state is :

$$M_{\max} = (360\ \text{kNm}) + (58\ \text{kNm}) = 418\ \text{kNm},$$

and the maximum bending stress is

$$\sigma = \frac{M_{\max}}{W} = \frac{418\ \text{kNm}}{31.76 \times 10^6\ \text{mm}^3} = 13.2\ \text{N/mm}^2.$$

Figure 4.40b shows the normal stress distribution in the serviceability state.

4.7.3 Dimensioning a steel floor beam (strength calculation)

In Figure 4.41 a steel beam with a span of 8 m is bearing a wooden floor. The total floor load, including the dead weight, is 3.7 kN/m^2.

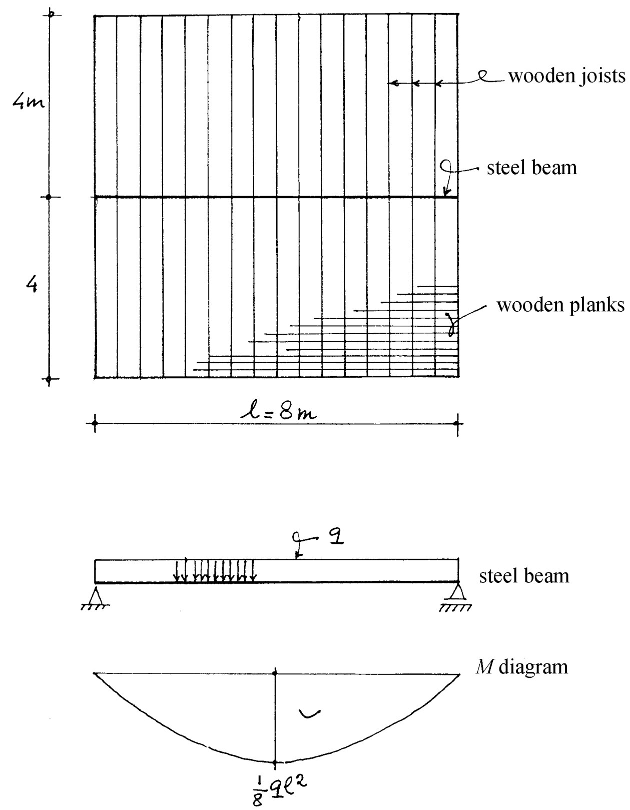


Figure 4.41 A steel floor beam with loading scheme and bending moment diagram.

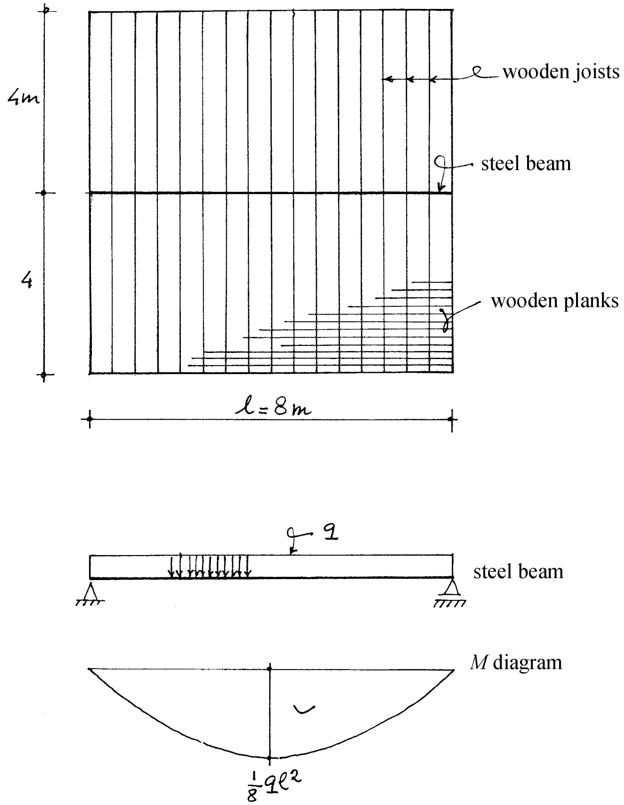


Figure 4.41 A steel floor beam with loading scheme and bending moment diagram.

The steel beam is considered to be sufficiently strong if the bending stress due to the design value of the load nowhere exceeds the yield strength f_y. The design value of the load is equal to the load at serviceability level multiplied by a load factor γ.[1]

Questions:

a. Dimension the steel beam for strength, assuming a load factor $\gamma = 1.5$ and $f_y = 235\ \mathrm{N/mm^2}$.
b. Determine the maximum bending stress in the serviceability state.

Solution:

a. If the steel beam is carrying half the floor load on each side,[2] the load on the beam is (4 m)(3.7 kN/m^2) = 14.8 kN/m. Estimate the dead weight of the steel beam at 1 kN/m, then the total load on the beam is

$$q = 15.8\ \mathrm{kN/m}.$$

With the load factor $\gamma = 1.5$, the design value of the load is

$$\gamma q = 1.5 \times 15.8\ \mathrm{kN/m} = 23.7\ \mathrm{kN/m}.$$

The maximum bending moment in the beam is at midspan. Due to the design load

$$M_{\max} = \tfrac{1}{8}\gamma q \ell^2 = \tfrac{1}{8}(23.7\ \mathrm{kN/m})(8\ \mathrm{m})^2 = 189.6\ \mathrm{kNm}.$$

---

1 Departing from the regulations, we use the same load factor for all loads in this example.

2 This is a reasonable assumption.

The beam meets the strength demand if

$$\sigma_{\max} = \frac{M_{\max}}{W} \leq f_y.$$

This implies the least section modulus W_{required} that is required:

$$W_{\text{required}} = \frac{M_{\max}}{f_y} = \frac{189.6 \text{ kNm}}{235 \text{ N/mm}^2} = 806.8 \times 10^3 \text{ mm}^3.$$

Table 4.2 includes a number of standard sections that meet the strength demand.

We choose IPE 360. The IPE-section has the smallest weight of material as well as the largest height (360 mm). If one wanted to build more slenderly, that would require more material and therefore more money.

b. In the serviceability state the final load on the beam is 14.8 kN/m plus 571 N/m dead weight. Rounded off that makes

$$q = 15.4 \text{ kN/m}.$$

The maximum bending moment in the serviceability state is now

$$M_{\max} = \tfrac{1}{8}\, q\ell^2 = \tfrac{1}{8}\,(15.4 \text{ kN/m})(8 \text{ m})^2 = 123.2 \text{ kNm},$$

and the maximum bending stress is

$$\sigma_{\max} = \frac{M_{\max}}{W} = \frac{123.2 \text{ kNm}}{904 \times 10^3 \text{ mm}^3} = 136 \text{ N/mm}^2.$$

Table 4.2

Section	W (mm^3)	Weight (N/m)
IPE 360	904×10^3	571
HE 200 M	967×10^3	1020
HE 240 B	938×10^3	832
HE 250 A	836×10^3	682

Note: The section chosen meets the strength demand. In addition to the strength demand, there is also a stiffness demand as the beam may not deflect too much. This may mean in certain cases that a heavier section has to be used than was found on the basis of the strength demand. Determining the deflection and checking the stiffness of the beam is covered in Chapter 8.

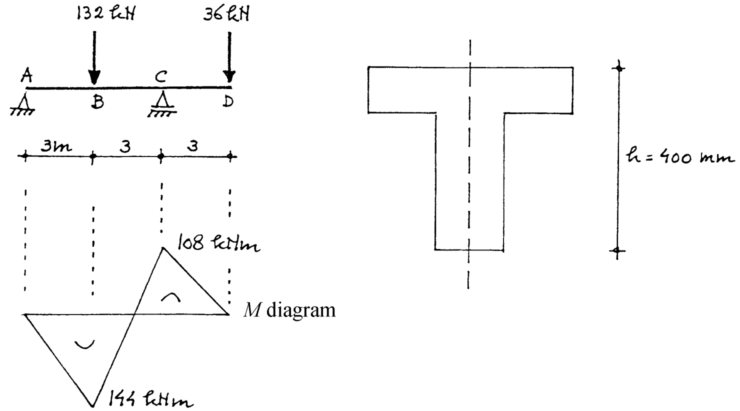


Figure 4.42 Loading scheme for a cantilevered T-beam, and the associated bending moment diagram.

4.7.4 Maximum bending stresses in a cantilevered T-beam

For the cantilevered beam in Figure 4.42 a T-shaped cross-section with a height of 400 mm has been used. There is bending in the vertical symmetry plane of the T-beam. The figure also shows the bending moment diagram for the given load.

The section moduli of the cross-section are

$$W_b = 27 \times 10^6 \text{ mm}^3 \text{ and } W_t = 45 \times 10^6 \text{ mm}^3.$$

Questions:

a. Determine the location of the normal centre NC.
b. Draw the normal stress diagram for the cross-sections in which the bending moment is a maximum.
c. Determine the maximum tensile and compressive bending stresses, and indicate where these stresses occur.

Solution:

a. The section moduli were defined in (4.16):

$$W_b = \frac{I}{e_b} \text{ and } W_t = \frac{I}{e_t},$$

in which e_b is the distance from the normal centre NC to the bottom fibre layer (b) and e_t is the distance to the top fibre layer (t) (see Figure 4.43).

For e_b/e_t we find

$$\frac{e_b}{e_t} = \frac{W_t}{W_b} = \frac{45 \times 10^6 \text{ mm}^3}{27 \times 10^6 \text{ mm}^3} = \frac{5}{3}.$$

For

$$e_b + e_t = h = 400 \text{ mm}$$

this leads to

$$e_b = \frac{5}{5+3} \times 400 \text{ mm} = 250 \text{ mm},$$

$$e_t = \frac{3}{5+3} \times 400 \text{ mm} = 150 \text{ mm},$$

from which the location of the normal centre NC is found.

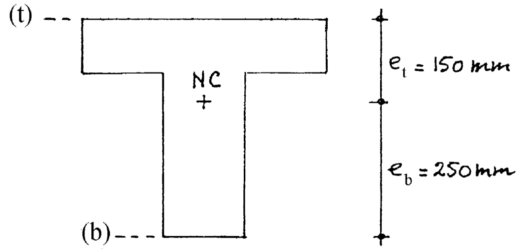


Figure 4.43 The distances e_b and e_t from the normal centre NC to the bottom fibre layer (b) and top fibre layer (t) respectively.

b. The M diagram in Figure 4.42 shows that the bending moment is a maximum at B and C. Since the distances e_b and e_t to the outer fibre layers differ, both fibre layers have to be involved in calculating the maximum tensile and compressive bending stresses.

- Cross-section B: $M = 144$ kN/m (tension at the bottom and compression at the top)

$$\sigma_b = +\frac{M}{W_b} = +\frac{144 \text{ kNm}}{27 \times 10^6 \text{ mm}^3} = +5.3 \text{ N/mm}^2,$$

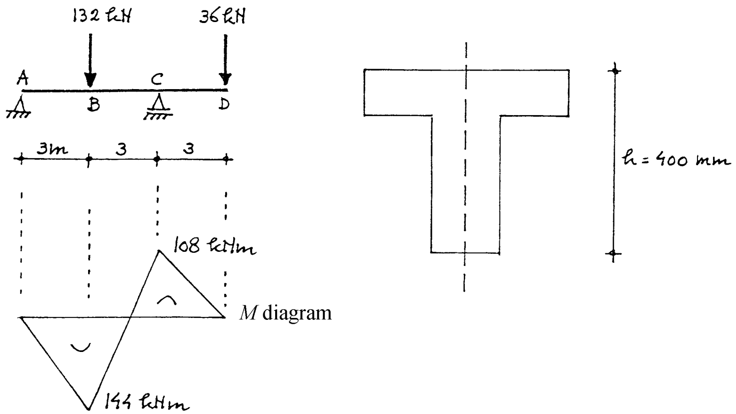


Figure 4.42 Loading scheme for a cantilevered T-beam, and the associated bending moment diagram.

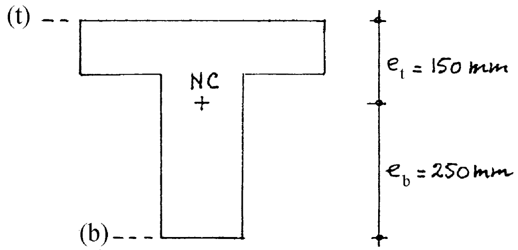


Figure 4.43 The distances e_b and e_t from the normal centre NC to the bottom fibre layer (b) and top fibre layer (t) respectively.

$$\sigma_t = -\frac{M}{W_t} = -\frac{144 \text{ kNm}}{45 \times 10^6 \text{ mm}^3} = -3.2 \text{ N/mm}^2.$$

- Cross-section C: $M = 108$ kNm (tension at the top and compression at the bottom)

$$\sigma_b = -\frac{M}{W_b} = -\frac{108 \text{ kNm}}{27 \times 10^6 \text{ mm}^3} = -4.0 \text{ N/mm}^2,$$

$$\sigma_t = +\frac{M}{W_t} = +\frac{108 \text{ kNm}}{45 \times 10^6 \text{ mm}^3} = +2.4 \text{ N/mm}^2.$$

The stress diagrams are shown in Figure 4.44.

c. The maximum *tensile bending stress* is 5.3 N/mm^2 and occurs in cross-section B, in the bottom fibre layer. The maximum *compressive bending stress* is 4.0 N/mm^2 and occurs at support C, also in the bottom fibre layer.

4.8 General stress formula related to the principal directions

The stress formula

$$\sigma(z) = \frac{N}{A} + \frac{M_z z}{I_{zz}}$$

applies only when the load and support reactions act in the xz plane, and the y and z axes coincide with the principal directions of the cross-section (see Section 4.3.2).

If the load does not act in a principal direction, as with the purlin in the inclined roof plane in Figure 4.45, the load can be resolved into components in the principal directions. In addition to the bending moments M_z in the xz plane there are also bending moments M_y in the xy plane.

The stress formula (with respect to the principal directions) is found by superposing three contributions:

Extension: $\sigma = \dfrac{N}{A}$,

Bending in the xy plane: $\sigma(y) = \dfrac{M_y y}{I_{yy}}$,

Bending in the xz plane: $\sigma(z) = \dfrac{M_z z}{I_{zz}}$.

This results in the following *general stress formula*:

$$\sigma(y,z) = \frac{N}{A} + \frac{M_y y}{I_{yy}} + \frac{M_z z}{I_{zz}}. \tag{4.21}$$

Note how easy this formula is to memorise!

If $N = 0$ the neutral axis always passes through normal centre NC.

If $N \neq 0$ the neutral axis is found from the condition $\sigma(y,z) = 0$:

$$-\frac{M_y}{N}\frac{A}{I_{yy}}y - \frac{M_z}{N}\frac{A}{I_{zz}}z = 1.$$

This equation in y and z is that of a straight line.

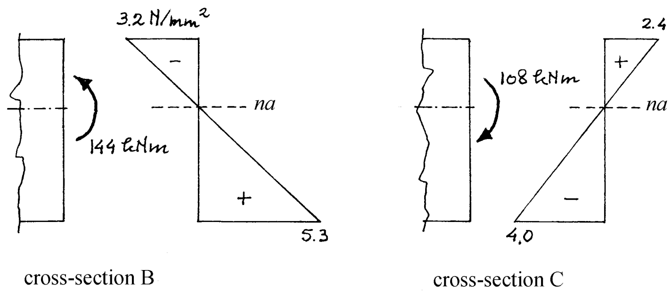


Figure 4.44 The normal stress diagrams for the cross-sections at B and C.

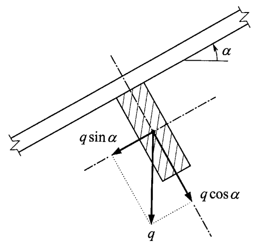


Figure 4.45 A purlin in a inclined roof plane. The load on the purlin can be resolved into components in the principal directions of the cross-section.

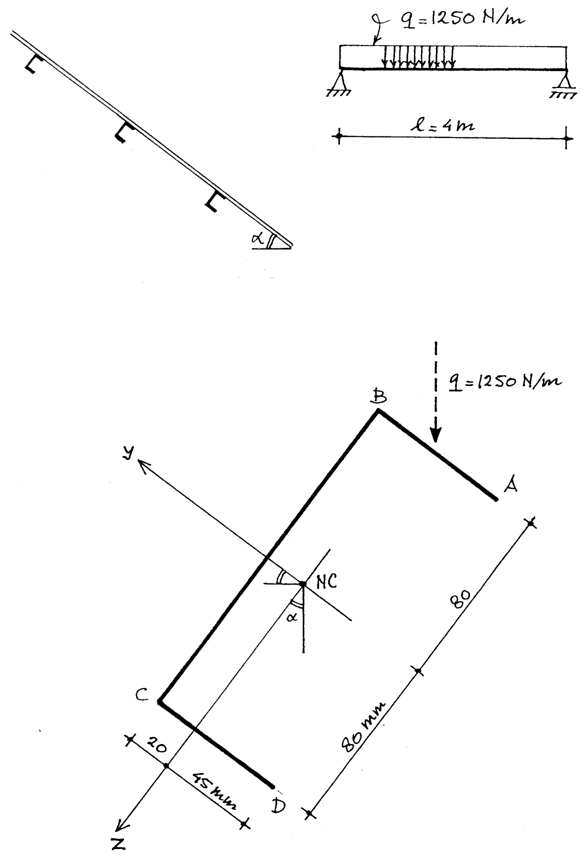


Figure 4.46 Loading scheme and cross-sectional dimensions of a steel roof purlin with a U-section. The purlin can be considered thin-walled.

Formal approach and engineering practice
In the *formal approach* we found that for the general stress formula with respect to the principal directions

$$\sigma_{xx} = \frac{N}{A} + \frac{M_y y}{I_{yy}} + \frac{M_z z}{I_{zz}} . \tag{a}$$

In the *technical notation* often used in *engineering practice* we obtain the following expression:[1]

$$\sigma_{xx} = \frac{N}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y} . \tag{b}$$

Notice not only the difference between the indices in the expressions (a) and (b), but also the difference in sign of one of the terms. The formal approach leads clearly to an expression that is easier to memorise.

In the example below, we follow the formal approach, and use the general stress formula with respect to the principal directions for determining the stress distribution in a steel roof purlin.

---

[1] Remember that the definitions of M_y and M_z are different in the *formal approach* and *engineering practice*, and also that the moments of inertia are defined and denoted in different ways (see Sections 2.8 and 3.4):

$$M_y^{\text{formal}} = -M_z^{\text{technical}} \text{ and } I_{yy}^{\text{formal}} = I_z^{\text{technical}},$$

$$M_z^{\text{formal}} = +M_y^{\text{technical}} \text{ and } I_{zz}^{\text{formal}} = I_y^{\text{tecnical}}.$$

Example

Steel purlins with a U-section have been used in the roof truss in Figure 4.46. The purlins are simply supported and have a span $\ell = 4$ m. They are subject to a uniformly distributed vertical load $q = 1250$ N/m, which includes the dead weight of the purlin.

The dimensions of the U-section and the location of the normal centre NC are given in the figure, and in the principal yz coordinate system

$$I_{yy} = 0.85 \times 10^6 \text{ mm}^4 \text{ and } I_{zz} = 9.25 \times 10^6 \text{ mm}^4.$$

The U-section can be considered thin-walled. The slope α of the roof is given by $\tan\alpha = 3/4$.

Questions:

a. For the cross-section at midspan, determine the normal stresses at the four corners A to D.
b. For this cross-section at midspan, draw the normal stress diagram. Also draw in the diagram the location of the neutral axis. What is the angle between the neutral axis and the y axis?

Solution:

a. The loads q_y and q_z in the principal directions y and z are

$$q_y = -q \sin\alpha = -(1250 \text{ N/m}) \times \tfrac{3}{4} = -750 \text{ N/m},$$

$$q_z = +q \cos\alpha = +(1250 \text{ N/m}) \times \tfrac{4}{5} = +1000 \text{ N/m}.$$

Note: Since the load in the xy plane acts opposite to the positive y direction, q_y is negative (see Figure 4.47).

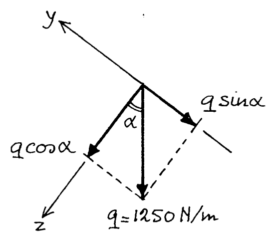


Figure 4.47 The load on the purlin resolved in the principal directions of the cross-section.

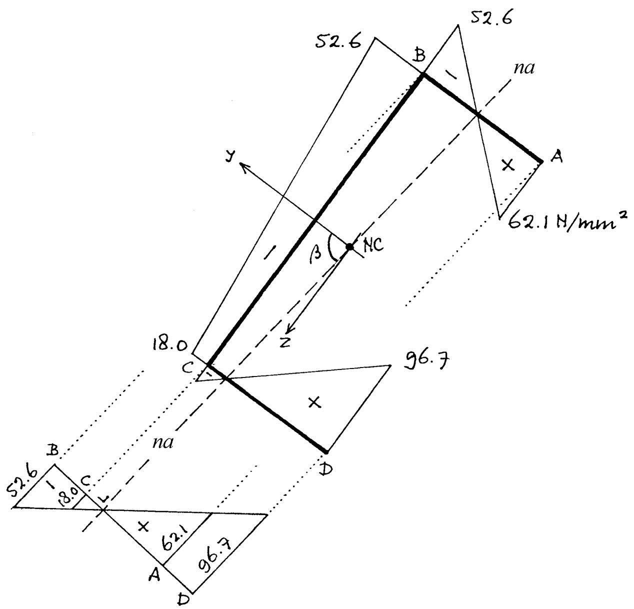


Figure 4.48 The normal stress diagram. The stresses are plotted normal to the centre lines of web and flanges of the U-section. Since there is only bending and no extension, the neutral axis (*na*) must pass through the normal centre NC. Since the normal stress in the cross-section is proportional to the distance to the neutral axis the stress distribution in the cross-section can also be represented by plotting the stress distribution on a line normal to the neutral axis.

At midspan the bending moments are

$$M_y = \tfrac{1}{8} q_y \ell^2 = \tfrac{1}{8}(-750\ \text{N/m})(4\ \text{m})^2 = -1500\ \text{Nm},$$

$$M_z = \tfrac{1}{8} q_z \ell^2 = \tfrac{1}{8}(+1000\ \text{N/m})(4\ \text{m})^2 = +2000\ \text{N/m}.$$

In the stress formula (there is no normal force)

$$\sigma(y,z) = \frac{M_y y}{I_{yy}} + \frac{M_z z}{I_{zz}},$$

these bending moments give the following stress distribution:

$$\sigma(y,z) = \frac{-1500\ \text{Nm}}{0.85 \times 10^6\ \text{mm}^4} \times y + \frac{+2000\ \text{Nm}}{9.25 \times 10^6\ \text{mm}^4} \times z$$

$$= -(1.765\ \text{N/mm}^3) \times y + (0.216\ \text{N/mm}^3) \times z\,.$$

The stresses in the corners A to D are determined in Table 4.3.

In the fourth and fifth column, the stresses due to M_y and M_z are shown separately. The resulting stress is found by summing the contributions of M_y and M_z. This stress is shown in the last column.

b. The stresses in the cross-section are linear. For a thin-walled cross-section, a uniform stress across the wall thickness is assumed. The stress varies only along the wall. The stress distribution along the wall, for example flange AB, can be determined by plotting the stresses at A and B normal to AB and drawing a straight line between these points. In this way, the stresses along flanges and web are shown for the whole thin-walled U-section in Figure 4.48. The maximum tensile bending stress occurs at D and is 96.7 N/mm^2. The maximum compressive bending stress is 52.6 N/mm^2 and occurs at B.

In Figure 4.48 there are two places in the flanges where the stress is zero. These points are located on the neutral axis. Since the neutral axis is a straight line it can be plotted directly on the stress diagram. If the stress diagram is drawn to scale, the neutral axis must pas through the normal centre NC (there is no normal force). This is a check.

The equation for the neutral axis follows from the condition

$$\sigma(y_{na}, z_{na}) = -(1.765\ \text{N/mm}^3) \times y_{na} + (0.216\ \text{N/mm}^3) \times z_{na} = 0.$$

For the angle β between the y axis and the neutral axis we now find

$$\tan\beta = \frac{z_{na}}{y_{na}} = \frac{1.765\ \text{N/mm}^3}{0.216\ \text{N/mm}^3} = 8.17 \Rightarrow \beta = 83°.$$

It can be shown that the normal stress in the cross-section is proportional to the distance to the neutral axis. This means that there is the same normal stress at all points on a line parallel to the neutral axis. The stress diagram can therefore also be represented by plotting the stress distribution on a line normal to the neutral axis. Such a stress diagram is also included in Figure 4.48.

In Figure 4.49, the stress diagram is represented spatially.

4.9 Core of the cross-section

Some materials, especially stony materials such as brickwork and plain concrete, can effectively transfer compressive stresses but offer little or no

Table 4.3

Corner	y coord. (mm)	z coord. (mm)	σ due to M_y (N/mm^2)	σ due to M_z (N/mm^2)	Resulting normal stress $\sigma(y, z)$ (N/mm^2)
A	–45	–80	+79.4	–17.3	+62.1
B	+20	–80	-35.3	–17.3	–52.6
C	+20	+80	–35.3	+17.3	–18.0
D	–45	+80	+79.4	+17.3	+96.7

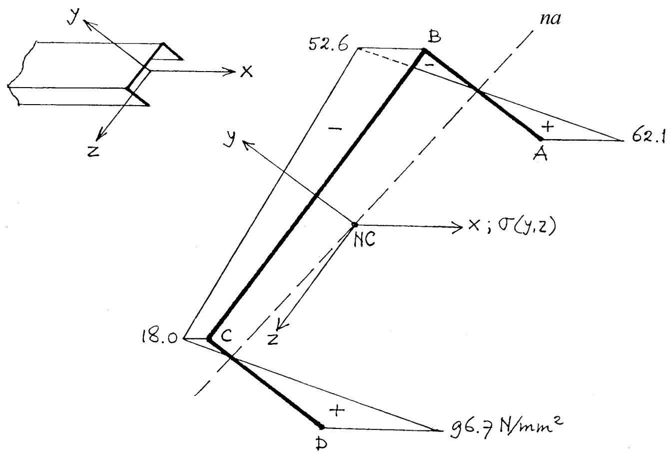


Figure 4.49 Spatial representation of the normal stress distribution in the purlin.

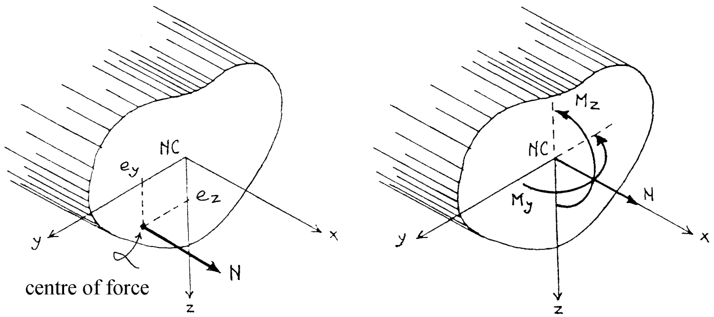


Figure 4.50 If the resultant of all the normal stresses in a cross-section is a force N, the point of application of this force is known as the centre of force (cf). If the eccentric normal force N is moved from the centre of force to the normal centre NC, bending moments M_y and M_z are generated.

resistance to tensile stresses. To prevent cracking, we assume that these materials cannot transfer tensile stresses at all. Consequently there must be compression within the entire cross-section, and the neutral axis must fall outside the cross-section. A borderline case occurs when the neutral axis just touches the boundary of the cross-section.

In this section we determine the area in which the centre of force has to be, in order for all stresses within the cross-section to have the same sign. This area is known as the *core of the cross-section*. The concept of a core is very important in dealing with materials that will resist compression better than tension. Cores play a role in prestressed concrete beams as well in spread foundations.

If the resultant of all normal stresses in a cross-section is a force, the point of application of this force is known as the *centre of force*[1] (see Figure 4.50). If this eccentric normal force N is moved from the centre of force, with coordinates $(e_y; e_z)$, to the normal centre NC, bending moments M_y and M_z are generated:

$$M_y = Ne_y \text{ and } M_z = Ne_z.$$

When N, M_y and M_z are known, the coordinates $(e_y; e_z)$ of the centre of force are

$$e_y = \frac{M_y}{N} \text{ and } e_z = \frac{M_z}{N}.$$

The core of the cross-section is the set of centres of force for which all

---

[1] See Volume 1, Section 14.2. The concept *centre of force* is only relevant when $N \neq 0$.

the stresses within the cross-section have the same sign, and the neutral axis is outside the cross-section or just touches it. In other words, it is that part of the cross-section within which an axial force can be applied without causing a stress of opposite sign at any point.

When determining the core, it is not relevant whether the normal force is a tensile force or a compressive force. It is therefore assumed here that the normal force is a tensile force, even though cores are applied mostly for cross-sections subject to compression.

In the first instance, we will look at cross-sections for which the two principal axes are also lines of symmetry, so that $W_{z;\mathrm{b}} = W_{z;\mathrm{t}} = W_z$. We have chosen a rectangular cross-section.

Assume that the cross-section is subject to a normal force N and bending moment M_z, and that the bending moment M_y is zero. In that case, $e_y = M_y/N = 0$ and the centre of force (cf) is located in the xz plane (see Figure 4.51).

Next we look for the location of the centre of force[1] for which the neutral axis coincides with one of the outer fibre layers. In that case there is either tension or compression in the entire cross-section.

For the stresses in the outer fibre layers applies

$$\sigma_\mathrm{b} = \frac{N}{A} + \frac{M_z}{W_z} = \frac{N}{A} + \frac{Ne_z}{W_z},$$

$$\sigma_\mathrm{t} = \frac{N}{A} - \frac{M_z}{W_z} = \frac{N}{A} - \frac{Ne_z}{W_z}.$$

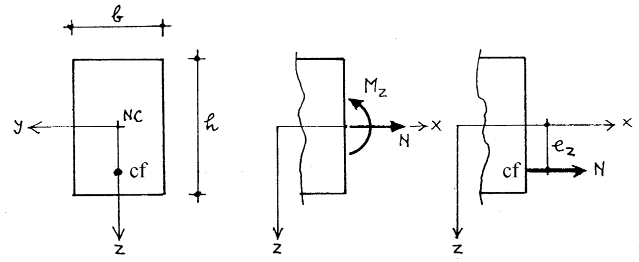


Figure 4.51 If only a normal force N and bending moment M_z act in the cross-section, and the bending moment M_y is zero, the centre of force (cf) is in the xz plane.

[1] Remember: the centre of force is the point of application of the resultant axial force in the cross-section.

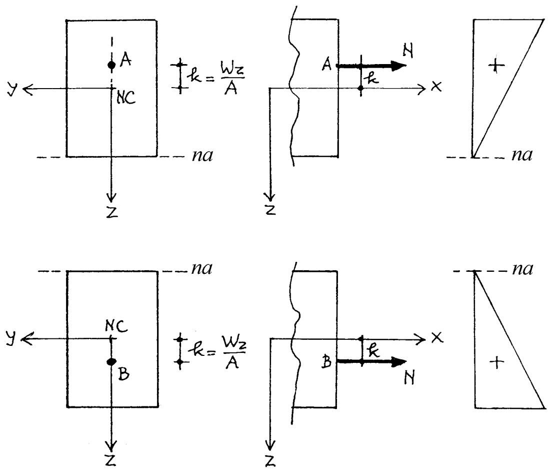


Figure 4.52 A and B are the centres of force for which the neutral axis (*na*) is exactly in one of the outer fibre layers. A is known as the upper core point and B as the lower core point. The distance k is known as the core radius.

If the neutral axis coincides with the bottom fibre layer, the location e_z of the centre of force follows from $\sigma_b = 0$:

$$e_z = -\frac{W_z}{A}.$$

This is point A in Figure 4.52.

If the neutral axis coincides with the top fibre layer, the location e_z of the centre of force follows from $\sigma_t = 0$:

$$e_z = +\frac{W_z}{A}.$$

This is point B in Figure 4.52.

The quotient W_z/A is known as the *core radius* k. For a rectangular cross-section

$$k = \frac{W_z}{A} = \frac{\frac{1}{6}bh^2}{bh} = \tfrac{1}{6}h. \tag{4.22}$$

With the centre of force at A the stress in the bottom fibre layer is zero. With an axial force applied at B, the stress in the top fibre layer is zero. For the rectangular cross-section in Figure 4.53 it can equally be argued that with the centre of force at C the stress in the left-hand edge of the cross-section is zero, and with the centre of force at D the stress in the right-hand edge is zero.

If the centre of force is located on AC the eccentric normal force can be resolved into a force at A and a force at C. The force at A gives a zero stress at the bottom edge and the force at C gives a zero stress at the left-hand edge. Together, this leads to a zero stress at corner E. Hence, if the centre of force is located on AC the normal stress is zero at the corner E.

Conclusion: *If the centre of force is located somewhere on the boundary of rhombus ABCD, the neutral axis touches the boundary of the cross-section. The rhombus ABCD is known as the core of the cross-section.*

If the centre of force is located inside the core, the neutral axis is completely outside the cross-section and all normal stresses in the cross-section have the same sign, namely that of the normal force N.

If the centre of force is located outside the core, the neutral axis cuts the cross-section into areas, one with compressive stresses and the other with tensile stresses.

Another example is the cross-section of the T-beam in Figure 4.54, with only one line of symmetry. The beam is subject to a load in the plane of symmetry. Since the distances e_b and e_t to the outer fibres layers are not equal, we have to distinguish between the section moduli W_b and W_t. For the stress in the bottom fibre layer (b) we have

$$\sigma_\text{b} = \frac{N}{A} + \frac{M_z e_\text{b}}{I_{zz}}.$$

With $M_z = Ne_z$ and $I_{zz}/e_\text{o} = W_{z;\text{o}}$ we can write

$$\sigma_\text{o} = \frac{N}{A} + \frac{Ne_z}{W_{z;\text{o}}} = 0,$$

The neutral axis is located in the bottom fibre layer if the stress there is zero:

$$\sigma_\text{o} = \frac{N}{A} + \frac{Ne_z}{W_{z;\text{b}}} = 0,$$

from which we find

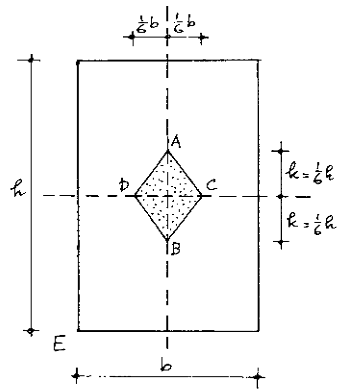


Figure 4.53 The rhombus ABCD is known as the core of the cross-section. If the centre of force is located within the core then the neutral axis is entirely outside the cross-section and all the normal stresses in the cross-section have the same sign.

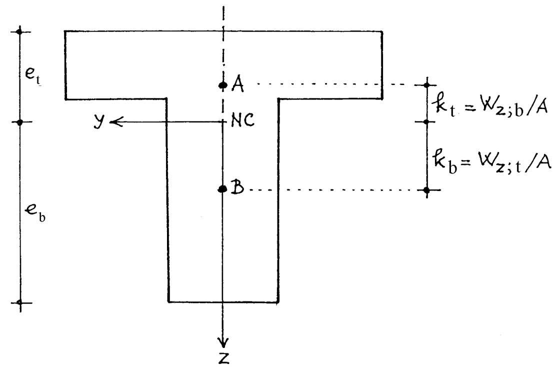


Figure 4.54 For a T-beam the upper core radius k_t and lower core radius k_b have different magnitudes.

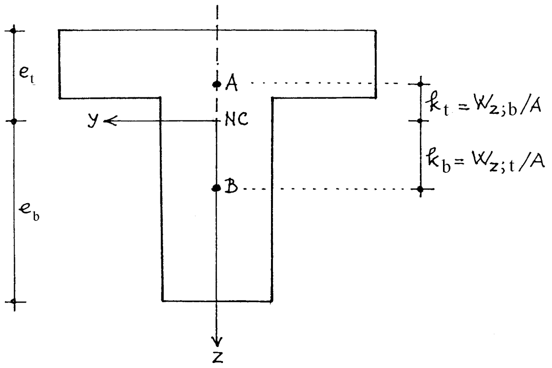


Figure 4.54 For a T-beam the upper core radius k_t and lower core radius k_b have different magnitudes.

$$e_z = -\frac{W_{z;b}}{A}.$$

This point, point A in Figure 4.54, is known as the *upper core point*. $W_{z;b}/A$ is known as the *upper core radius* k_t:

$$k_t = \frac{W_{z;b}}{A}.$$

In the same way, the *lower core point* is found by locating the neutral axis at the top fibre layer:

$$e_z = +\frac{W_{z;t}}{A}.$$

This is point B in Figure 4.54. The *lower core radius*; k_b is given by

$$k_b = \frac{W_{z;t}}{A}$$

As long as, in Figure 4.54, the centre of force due to the load in the plane of symmetry stays between A and B, the neutral axis will fall outside the cross-section, and the stresses will have the same sign everywhere.

4.10 Applications related to the core of the cross-section

In practice, the concept of a core of a cross-section is applied, for example, to determine how to prestress a concrete beam to avoid tensile stresses.

This is covered in the example in Section 4.10.1.

Another application is given in Section 4.10.2, and relates to the calculation of the earth pressure under a rigid foundation plate. It is assumed that the earth pressure on the plate at a certain point is proportional to the vertical displacement of the plate in that point.

4.10.1 Beams with central and eccentric prestressing

The roof in Figure 4.55 is constructed of concrete roof plates supported on prestressed concrete beams. The roof plates are 0.10 m thick, 2 m long and 1 m wide. The beams have a rectangular cross-section, 0.10 m wide and 0.24 m high, and a length of 3 m. Plates and beams are simply supported.

The load on the roof consists of a uniformly distributed (surface) load of 1 kN/m^2. The specific weight of the roof plates is 24 kN/m^3 and that of the beams is 25 kN/m^3.

The compressive stress in the concrete of the beams may not exceed the limiting value of 10 N/mm^2. The concrete cannot resist tensile stresses.

Questions:
If the beams are prestressed by means of a straight tendon parallel to the beam axis, determine the prestressing force required to suppress the tensile stresses due to the dead weight and working load:

a. for central prestressing;
b. for eccentric prestressing.

Check for both cases that the compressive stresses remain under the given limiting value of 10 N/mm^2.

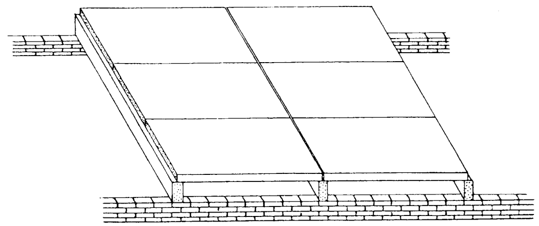


Figure 4.55 A roof constructed of concrete plates supported by prestressed concrete beams.

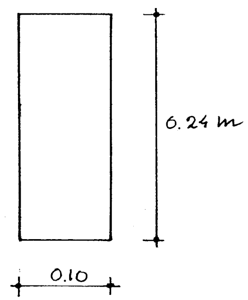


Figure 4.56 The concrete beams have a rectangular cross-section.

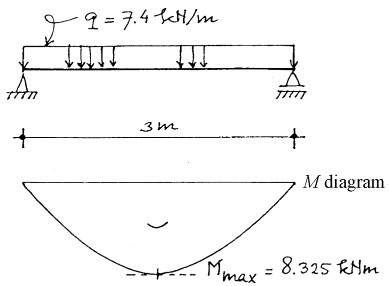


Figure 4.57 The loading scheme and bending moment diagram for a beam.

Solution:
We first determine the area A and section modulus W. For the rectangular cross-section of the beam in Figure 4.56:

$$A = (0.10\text{ m})(0.24\text{ m}) = 24 \times 10^{-3}\text{ m}^2,$$

$$W = \tfrac{1}{6}(0.10\text{ m})(0.24\text{ m})^2 = 960 \times 10^{-6}\text{ m}^3.$$

Next we determine the load q for a single beam resulting from the dead weight and the working load.

The total load per mm^2 plate is

$$(1\text{ kN/m}^2) + (0.10\text{ m})(24\text{ kN/m}^3) = 3.4\text{ kN/m}^2,$$

and gives a line load on the beams of

$$(2\text{ m})(3.4\text{ kN/m}^2) = 6.8\text{ kN/m}.$$

There is also the dead weight of the beams:

$$(0.10\text{ m})(0.24\text{ m})(25\text{ kN/m}^3) = 0.6\text{ kN/m}.$$

The total load on a beam is therefore the uniformly distributed (line) load

$$q = (6.8\text{ kN/m}) + (0.6\text{ kN/m}) = 7.4\text{ kN/m}.$$

A scheme of the loaded beam is shown in Figure 4.57, together with the bending moment diagram The maximum bending moment occurs at midspan and is[1]

$$M_{\max}^{(q)} = \tfrac{1}{8}q\ell^2 = \tfrac{1}{8}(7.7 \text{ kN/m})(3 \text{ m})^2 = 8.325 \text{ kNm}.$$

The associated stress distribution $\sigma^{(q)}$ in the cross-section is shown in Figure 4.59a, with tension at the bottom and compression at the top of the beam. The maximum tensile and compressive stress in the outer fibres are equal in magnitude. This magnitude follows from the formula

$$\sigma = \frac{M}{W} = \frac{8.32 \text{ kNm}}{960 \times 10^{-6} \text{ m}^3} = 8.67 \text{ kN/mm}^2.$$

Hence (see Figure 4.59a)

$$\sigma_b^{(q)} = +8.67 \text{ N/mm}^2 \text{ and } \sigma_b^{(q)} = -8.67 \text{ N/mm}^2.$$

Since the concrete cannot transfer tensile stresses, this stress distribution is unacceptable.

The tensile stresses in the cross-section can be eliminated by prestressing the beam, in which steel bars, cables or wires are installed under tension. In this case the beam is prestressed by a straight cable, the tendon. The tensile force F_p in the tendon is the prestressing force. This tensile force exerts equally large compressive forces F_p on the beam ends via the anchors.

a. *Central prestressing*
With central prestressing, the tendon coincides with the beam axis and the

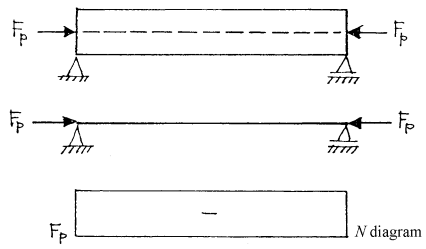


Figure 4.58 The loading case for central prestressing, together with the loading scheme of the beam and the normal force diagram. In central prestressing, the straight tendon coincides with the member axis and the beam ends are subject to central compressive forces. Central prestressing does not generate bending moments.

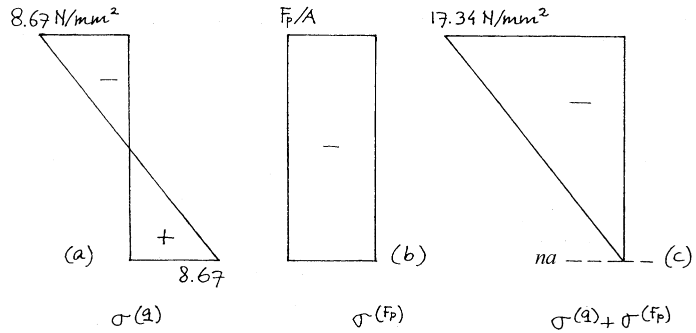


Figure 4.59 Stress diagrams for the critical cross-section at mid-span, due to the load q and central prestressing force F_p.

[1] The upper index indicates the cause of the bending moment, in which we distinguish between the load q (due to dead weight and working load) and the prestressing force F_p.

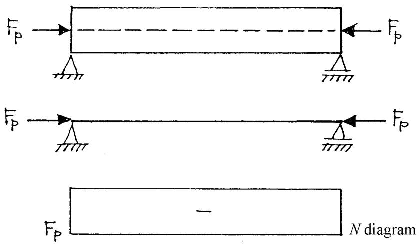


Figure 4.58 The loading case for central prestressing, together with the loading scheme of the beam and the normal force diagram. In central prestressing, the straight tendon coincides with the member axis and the beam ends are subject to central compressive forces. Central prestressing does not generate bending moments.

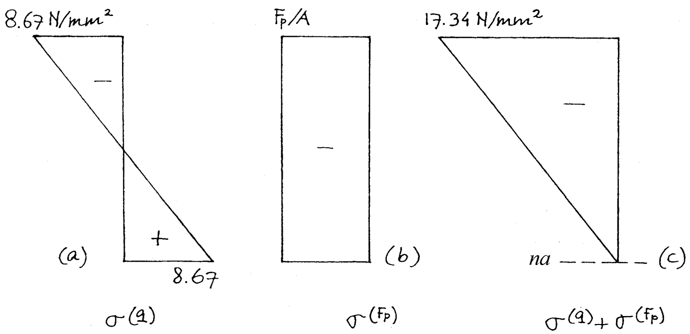


Figure 4.59 Stress diagrams for the critical cross-section at mid-span, due to the load q and central prestressing force F_p.

beam ends are loaded by central compressive forces. Figure 4.58 shows the loading case for central prestressing, together with the loading scheme of the beam and the normal force diagram. Central prestressing does not generate bending moments.

As a result of central prestressing, the stress in the cross-section is constant:

$$\sigma = -\frac{F_p}{A}.$$

The stress diagram $\sigma^{(F_p)}$ is shown in Figure 4.59b.

The critical cross-section is at midspan. If no tensile stresses are allowed, the stress diagrams in Figures 4.59a and 4.59b show that the stress in the bottom fibre layer has to be zero (or negative):

$$\sigma_b = \sigma_b^{(q)} + \sigma_b^{(F_p)} = (+8.67\ \text{N/mm}^2) - \frac{F_p}{A} \leq 0.$$

For the prestressing force F_p we find

$$F_p \geq (8.67\ \text{N/mm}^2) \times A = (8.67\ \text{N/mm}^2)(24 \times 10^{-3}\ \text{m}^2) = 208\ \text{kN}.$$

The minimum required prestressing force is

$$F_p = 208\ \text{kN}.$$

The resulting stress diagram in Figure 4.59c shows that for this prestressing force a compressive stress of 17.34 N/mm^2 occurs at the top of the beam, considerably larger than the limiting value of 10 N/mm^2.

Conclusion: *With central prestressing, it would seem impossible to meet the demands that there are no tensile stresses in the cross-section and that the compressive stresses are below the given limiting value.*

b. *Eccentric prestressing*
A better result is achieved with eccentric prestressing, by shifting the tendon from the member axis towards the tension area. Figure 4.60 shows the prestressed beam, its loading scheme, and the N and M diagrams when the prestressing force F_p has an eccentricity e_p.

For the critical cross-section at midspan, Figure 4.61 shows the stress diagrams due to $M_{max}^{(q)}$ and F_p. For the stress at the bottom of the beam we find

$$\sigma_b = +\frac{M_{max}^{(q)}}{W} - \frac{F_p}{A} - \frac{F_p e_p}{W}.$$

With core radius $k = W/A$ this can also be written as

$$\sigma_b = +\frac{M_{max}^{(q)}}{W} - \frac{F_p}{W}(k + e_p).$$

If no tensile stresses are permitted (in the bottom fibre layer) then $\sigma_b \leq 0$. Hence

$$F_p \geq \frac{M_{max}^{(q)}}{k + e_p}.$$

The minimum required prestressing force F_p becomes smaller with increasing eccentricity e_p.

In addition to the maximum moment, we also have to check for a minimum moment, for example when there is no working load, whether there are

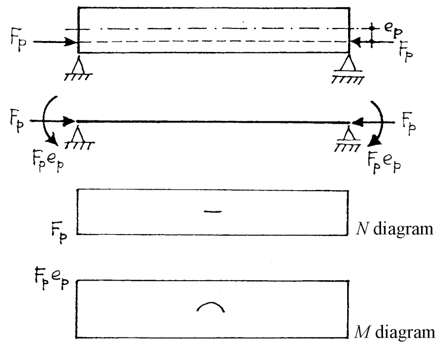


Figure 4.60 The loading case for eccentric prestressing, together with the loading scheme of the beam, and the normal force and bending moment diagrams. The (straight) tendon is now under the member axis. The eccentric compressive forces on the beam ends generate a bending moment in the beam.

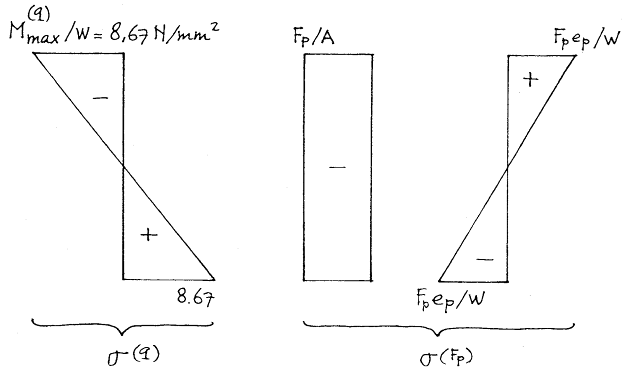


Figure 4.61 Stress diagrams for the cross-section at midspan, resulting from the load q and the eccentric prestressing F_p.

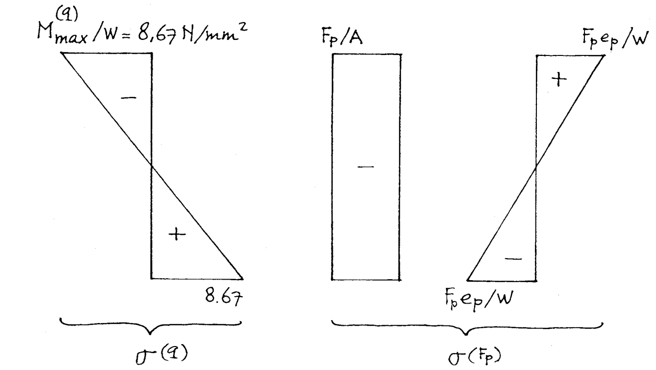


Figure 4.61 Stress diagrams for the cross-section at midspan, resulting from the load q and the eccentric prestressing F_{p}.

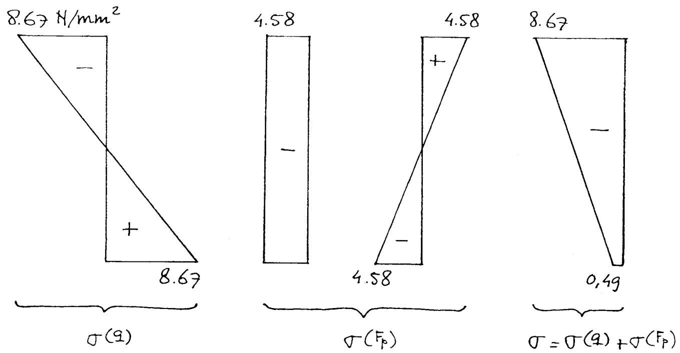


Figure 4.62 Stress diagrams for the cross-section at midspan when the tendon is located at the lower core point, and the prestressing force is 110 kN.

indeed no tensile stresses. For the case in question the ends of the beam are critical.

Here, the bending moment due to the load is zero and the stress distribution in the cross-section is entirely determined by the eccentric compressive force F_{p}. If no tensile stresses may occur, the eccentric compressive force must have its point of application within the core. The maximum eccentricity is therefore achieved by placing the tendon at the lower core point. Then $e_{\mathrm{p}} = k$, and the minimum required prestressing force is

$$F_{\mathrm{p}} = \frac{M_{\max}^{(q)}}{2k}.$$

For a cross-section with differing lower core point k_{b} and upper core point k_{t} we would have found:

$$F_{\mathrm{p}} = \frac{M_{\max}^{(q)}}{k_{\mathrm{b}} + k_{\mathrm{t}}}.$$

The minimum required prestressing force F_{p} appears to be equal to the maximum bending moment due to q divided by the height of the core.

For the beam with rectangular cross-section in the example we have

$$k = \tfrac{1}{6} h = 0.04 \text{ m}.$$

For the minimum required prestressing force F_{p} we find

$$F_{\mathrm{p}} = \frac{M_{\max}^{(q)}}{2k} = \frac{8.325 \text{ kNm}}{2 \times (0.04)} = 104 \text{ kN}.$$

Figures 4.62 and 4.63 show the stress diagrams for the cross-sections at midspan and at the beam ends respectively, when the prestressing cable is located at the lower core point and that the magnitude of the prestressing force is 110 kN (slightly larger than the required value of 104 kN). Checking the correctness of the values in the figures is left to the reader.

Conclusion: *There are no tensile stresses and the compressive stress remains below the given limiting value of 10 N/mm*^{2}.

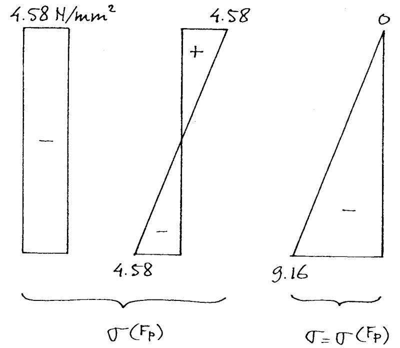


Figure 4.63 Stress diagrams for the end cross-sections when the tendon is located at the lower core point, and the prestressing force is 110 kN.

4.10.2 Earth pressure under a building with spread foundation

For the earth pressure under a foundation plate, it is common practice to assume that the vertical earth pressure on the plate at a certain point is proportional to the vertical displacement of the plate at that point. In this way, the earth pressures can be found rather quickly and are relatively accurate.

For a rigid foundation plate, the vertical displacement is linear, and so is the earth pressure. In that case, we can use the stress formulas for cross-sections subject to bending and extension.

Since soil is incapable of transferring tensile stresses, it is usual to define compressive stresses as positive, a practice that is adopted in this section.

The (symmetrical) foundation plate in Figure 4.64 is loaded in the plane of symmetry by a couple T, and a force F at the normal centre (or centroid) of the plate. On the analogy of cross-sections subject to bending and extension, the extreme earth pressures $\sigma_{e;extr}$ at the edges of the plate are (*note: earth pressures are positive here*)

$$\sigma_{e;extr} = \frac{F}{A} \pm \frac{T}{W}. \tag{4.23}$$

Here A is the area of the foundation plate and W is the section modulus.

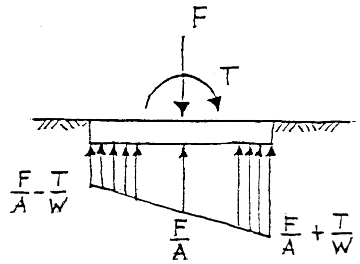


Figure 4.64 It is common practice to assume that the vertical earth pressure under a foundation plate at a certain point is proportional to the vertical displacement of the plate at that point. For a rigid foundation plate, the earth pressure will be linearly distributed and we can use the stress formulas for a cross-section subject to bending and extension.

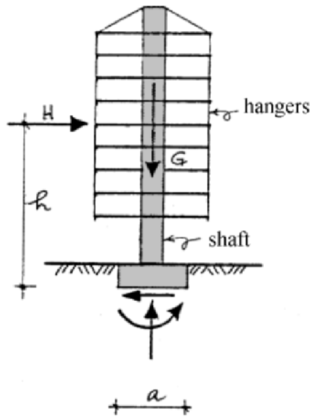


Figure 4.65 The shaft of a suspended building is rigidly connected with a thick concrete plate on a spread foundation. The building, shaft and foundation plate have a square ground plan.

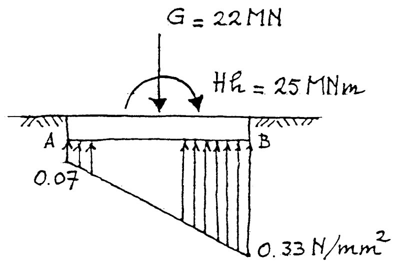


Figure 4.66 The distribution of the earth pressure due to the dead weight G and the moment Hh caused by the wind load.

This formula is used in the example below to determine the earth pressure under a rigid foundation plate.

Example

Figure 4.65 shows the model of a suspended building, constructed of a concrete shaft with cantilevers at the top from which hangers have been suspended that together bear the floors. The shaft is rigidly connected with a thick concrete plate on a spread foundation. The building, shaft and foundation plate have a square ground plan. The side of the square foundation plate has a length $a = 10.5$ m. The total weight G of the building is 22 MN. The resulting wind load H is 1 MN and applies at a distance $h = 25$ m above the underside of the foundation plate. It is assumed that this horizontal force is entirely transferred by the friction under the foundation plate, and that the resistance from the edges of the foundation plate can be neglected.

Questions:

a. Determine the distribution of the earth pressure under the foundation plate for the given load.
b. Determine the magnitude of the horizontal force H for which the earth pressure in one of the edges of the foundation plate is just zero.
c. Determine the distribution of the earth pressure for $H = 2$ MN.

Solution:

For the square foundation plate,

area: $A = a^2 = (10.5 \text{ m})^2 = 110.25 \text{ m}^2,$

section modulus: $W = \frac{1}{6} \cdot a \cdot a^2 = \frac{1}{6} \times (10.5 \text{ m})^3 = 192.94 \text{ m}^3,$

core radius: $k = \frac{1}{6}a = \frac{1}{6} \times (10.5 \text{ m}) = 1.75 \text{ m}.$

The foundation plate is subject to a vertical force G and a couple Hh (see Figure 4.66).

a. The extreme earth pressures at A and B using (4.23) are

$$\sigma_{\mathrm{e;A}} = +\frac{G}{A} - \frac{Hh}{W} = +\frac{22\ \mathrm{MN}}{110.25\ \mathrm{m}^2} - \frac{(1\ \mathrm{MN})(25\ \mathrm{m})}{192.94\ \mathrm{m}^3}$$

$$= (+0.20 - 0.13)\ \mathrm{N/mm}^2 = +0.07\ \mathrm{N/mm}^2\ \ \text{(compression!)}$$

$$\sigma_{\mathrm{e;B}} = +\frac{G}{A} + \frac{Hh}{W} = +\frac{22\ \mathrm{MN}}{110.25\ \mathrm{m}^2} + \frac{(1\ \mathrm{MN})(25\ \mathrm{m})}{192.94\ \mathrm{m}^3}$$

$$= (+0.20 + 0.13)\ \mathrm{N/mm}^2 = +0.33\ \mathrm{N/mm}^2\ \ \text{(compression!)}$$

Figure 4.66 shows the distribution of the earth pressure under the foundation plate.

The location of the centre of force is (see Figure 4.67)

$$e = \frac{Hh}{G} = \frac{(1\ \mathrm{MN})(25\ \mathrm{m})}{22\ \mathrm{MN}} = 1.14\ \mathrm{m}.$$

The eccentricity of the centre of force is smaller than the core radius $k = 1.75$ m; all stresses under the foundation plate therefore have the same sign.

b. When the eccentricity e of the centre of force is equal to the core radius k, $e = k = 1.75$ m, there will be zero stresses at the (left) edge. In that case, the magnitude of the horizontal force H follows from $e = k = Hh/G$:

$$H = \frac{Gk}{h} = \frac{(22\ \mathrm{MN})(1.75\ \mathrm{m})}{25\ \mathrm{m}} = 1.54\ \mathrm{MN}.$$

The associated diagram for the earth pressure is shown in Figure 4.68.

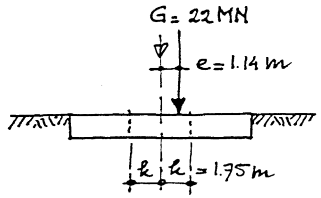


Figure 4.67 The centre of force has an eccentricity $e = Hh/G$ and is outside the core of the square foundation plate. k is the core radius.

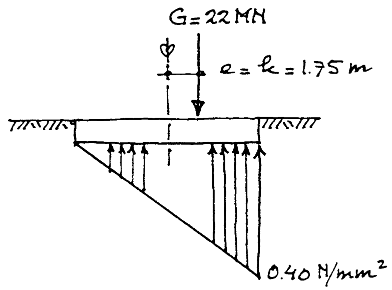


Figure 4.68 When the eccentricity e of the centre of force is equal to the core radius k, there will be zero stresses at the (left) edge.

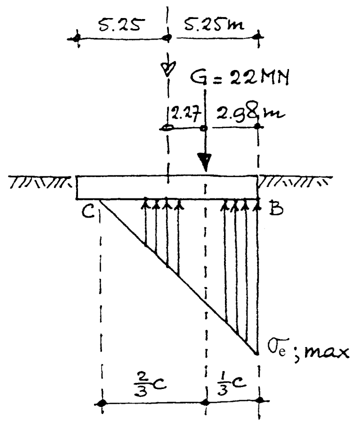


Figure 4.69 If the centre of force falls outside the core of the foundation plate, tensile stresses should occur. Since that is not possible in soil, the stress diagram changes in the sense that the active area of the soil under the foundation plate becomes smaller.

c. If H becomes larger than 1.54 MN, the centre of force falls outside the core of the foundation plate and tensile stresses will occur. Since soil cannot transfer tensile stresses, the stress diagram will change in the sense that the active area of soil under the foundation plate gets smaller.

When $H = 2$ MN the eccentricity of the centre of force is:

$$e = \frac{Hh}{G} = \frac{(2\text{ MN})(25\text{ m})}{22\text{ MN}} = 2.27\text{ m}.$$

The point of application of the eccentric compressive force G is now 2.98 m from the edge B, as indicated in Figure 4.69. Assume that the earth pressure in that case is linear from zero at C to $\sigma_{e;\max}$ at edge B. The distance CB is indicated by c. Since the active part of the foundation plate is rectangular, the line of action of the resultant of the triangularly distributed earth pressure is at $c/3$ from edge B.

From the moment equilibrium of the foundation plate it follows that the line of action of the resulting earth pressure has to coincide with the line of action of the eccentric compressive force G. Hence

$$\tfrac{1}{3}c = 2.98\text{ m} \Rightarrow c = 8.94\text{ m}.$$

From the force equilibrium of the foundation plate,

$$\tfrac{1}{2}ac\,\sigma_{e;\max} = G,$$

we find the maximum earth pressure at edge B (see Figure 4.69):

$$\sigma_{e;\max} = \frac{2G}{ac} = \frac{2 \times (22\text{ MN})}{(10.5\text{ m})(8.94\text{ m})} = 0.47\text{ N/mm}^2.$$

With this, the distribution of the earth pressure under the foundation plate is determined for $G = 22$ MN and $H = 2$ MN.

4.11 Mathematical description of the problem of bending with extension

After Section 4.3 we addressed in detail the various stress formulas and the associated subjects. In this section, we derive the *differential equations* for a member subject to bending with extension.

Using differential equations, we can trace generally applicable properties of the member behaviour. They can be used to determine the distribution of the section forces and displacements, even when the member is statically indeterminately supported. Differential equations also play an essential role in investigating the behaviour of combined systems, such as a railway sleeper in a ballast bed, a suspension bridge, or a block of buildings constructed as a frame linked to a rigid shaft – more advanced subjects from mechanics that fall outside the scope of this book.

The differential equations are derived from the three types of relationships covered in Section 4.3. For bending with extension in the xz plane they are[1]

- The *kinematic relationships* that link the deformations and displacements (see Section 4.3.1):

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{(extension)},$$

[1] Except for the loads q, all the indices are hereafter omitted. They are the index y for φ, the index z for κ, V and M, and the index zz for I.

$$\left.\begin{aligned} \varphi &= -\frac{\mathrm{d}w}{\mathrm{d}x} \\ \kappa &= \frac{\mathrm{d}\varphi}{\mathrm{d}x} \end{aligned}\right\} \Rightarrow \kappa = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \quad \text{(bending)}.$$

- *The constitutive relationships* that link the section forces and associated deformations (see Section 4.3.2):

$$N = EA\varepsilon \quad \text{(extension)},$$

$$M = EI\kappa \quad \text{(bending)}.$$

- *The static* or *equilibrium relationships* that link the load (by external forces) and the section forces (see Section 4.3.3):

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0 \quad \text{(extension)},$$

$$\left.\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}x} + q_z &= 0 \\ \frac{\mathrm{d}M}{\mathrm{d}x} - V &= 0 \end{aligned}\right\} \Rightarrow \frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + q_z = 0 \quad \text{(bending)}.$$

A summary of all the equations is included in Table 4.4.

Table 4.4 The differential equations for extension and bending of a prismatic member.

	kinematic relationships	constitutive relationships	static relationships	differential equations
extension	$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}$	$N = EA\varepsilon$	$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0$	$EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + q_x = 0$
bending	$\varphi = -\frac{\mathrm{d}w}{\mathrm{d}x}$ $\kappa = \frac{\mathrm{d}\varphi}{\mathrm{d}x}$ $\to \kappa = -\frac{\mathrm{d}^2w}{\mathrm{d}x^2}$	$M = EI\kappa$	$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0$ $\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0$ $\to \frac{\mathrm{d}^2M}{\mathrm{d}x^2} + q_z = 0$	$-EI\frac{\mathrm{d}^4w}{\mathrm{d}x^4} + q_z = 0$

The kinematic relationships for extension and bending are independent of one another, due to the assumption that the rotations φ of the cross-sections are small. Since the cross-sections remain perpendicular to the member axis, the rotation φ can be eliminated from the two kinematic equations for bending and one equation in w is left over.

The constitutive relationships for extension and bending are also independent of one another as a result of the choice of the coordinate system, with the x axis through the normal centre of the cross-section and the y and z axes coinciding with the principal directions.

Finally, the static relationships for extension and bending can be treated separately. This is so because the equilibrium equations were applied to the undeformed geometry of the member. By eliminating the shear force V in the equations for bending, one equation in M is left.

Substituting the kinematic relationships in the constitutive relationships, and again substituting the result in the static relationships, we obtain the following two differential equations for a prismatic member subject to extension and bending:

$$\text{extension:}\quad EA\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}+q_x=0,$$

$$\text{bending:}\quad -EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}+q_z=0.$$

For extension, we have a second-order differential equation in the displacement u.

For bending, we have a fourth-order differential equation in the displacement w.

Note: The differential equations can be applied only when all quantities as a function of x are continuous and/or continuously differentiable. If not, then the member has to be split into a number of *fields*, so that the differential equations indeed are valid for each separate field. We neglect the fact that in mathematics there are methods to integrate discontinuous functions.

Solving the differential equations is done through repeated integration. For each integration there is one integration constant. The total number of integration constants in the solution is equal to the order of the differential equation: two for extension and four for bending.

The integration constants follow from the *end conditions* and *joining condi-*

tions. Both the end conditions and joining conditions can be regarded as the boundary conditions for a specific field. These are the conditions that the quantities expressed in the displacements u and w (and/or the relationships between these quantities) have to meet at a field boundary (field end or field joining).

The boundary conditions relate to the following quantities:

$$u,$$
$$N = EA\varepsilon = EA\frac{\mathrm{d}u}{\mathrm{d}x},$$
$$w,$$
$$\varphi = -\frac{\mathrm{d}w}{\mathrm{d}x},$$
$$M = EI\kappa = -EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2},$$
$$V = \frac{\mathrm{d}M}{\mathrm{d}x} = -EI\frac{\mathrm{d}^3w}{\mathrm{d}x^3}.$$

When the boundary conditions for extension relate only to the quantities u and N, and those for bending solely to the quantities w, φ, M and V, the extension and bending effects are independent of one another and can be treated separately.

The differential equation for extension was earlier derived in Section 2.5 and applications were given in Section 2.7.

Applications of the differential equation for bending will be covered in Chapter 8. There we will also cover other methods for determining the displacements due to bending.

Note: We assumed a prismatic member. The axial stiffness EA and bending stiffness EI are then constant, i.e. independent of x. For a non-prismatic member, EA and EI are functions of x and the differential equations for extension and bending respectively are

$$\text{extension:} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q_x = 0,$$

$$\text{bending:} \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(-EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) + q_z = 0.$$

The reader is left to check the correctness of these equations.

4.12 Thermal effects

In this section we investigate the thermal effects at cross-sectional level. The result is a "tool" for determining the influence of a change in temperature of a member on the deformations and the force flow. We will work in an xyz coordinate system with the x axis along the member axis (through the normal centre of the cross-section) and the y and z axis coinciding with the principal directions of the cross-sections.

We also assume that:

- the change in temperature is constant in the y direction and linear in the z direction,[1]
- the cross-section is homogeneous, and
- the member deforms in the xz plane.

[1] Since we restrict ourselves to the cross-sectional thermal effects, the variation of the temperature in the x direction is not relevant here.

We return to the familiar fibre model, and we first look at a single fibre. Under the influence of the stress σ in the fibre there is a strain

$$\varepsilon^{(\sigma)} = \frac{\sigma}{E}.$$

Due to an increase in temperature ΔT there is an additional strain

$$\varepsilon^{(T)} = \alpha \Delta T.$$

Here α is the *coefficient of thermal expansion* of the material. For convenience, we hereafter omit the Δ sign and write

$$\varepsilon^{(T)} = \alpha T,$$

in which T now represents the *increase in temperature* of the fibre.

The total strain is

$$\varepsilon = \varepsilon^{(\sigma)} + \varepsilon^{(T)} = \frac{\sigma}{E} + \alpha T.$$

From this we find that the stress σ in the fibre is

$$\sigma = E(\varepsilon - \alpha T).$$

The modulus of elasticity E and coefficient of thermal expansion α are material constants. In a homogeneous cross-section, all fibres have the same E and the same α. On the contrary, the stress σ, strain ε and temperature increase T can differ per fibre and be functions of the location (y, z) of the fibre. In general, for an arbitrary (y, z) fibre we have

$$\sigma(y, z) = E \cdot [\varepsilon(y, z) - \alpha T(y, z)]. \tag{4.24}$$

If the member deforms in the xz plane and planar cross-sections remain planar, the strain distribution is independent of the y coordinate:

$$\varepsilon(y, z) = \varepsilon(z) = \varepsilon + \kappa_z z. \tag{4.25}$$

ε is the strain of the member axis and κ_z is the curvature of the member (axis) in the xz plane (κ_z is also the slope of the strain diagram) (see Figure 4.70).

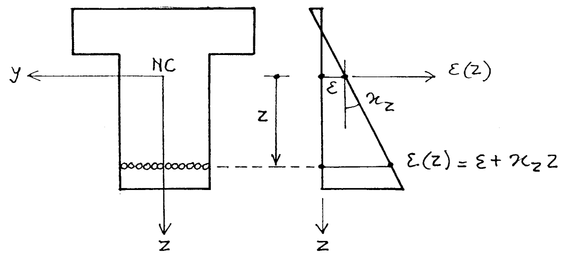


Figure 4.70 The strain diagram for a member deforming in the xz plane. ε is the strain of the member axis and κ_z is the curvature of the member (axis) in the xz plane; κ_z is also the slope of the strain diagram.

If the increase in temperature is linear over the height of the cross-section (the z direction) and constant over the width (the y direction), the temperature increase is also independent of the y coordinate, and can be written as

$$T(y, z) = T(z) = T + z\frac{\mathrm{d}T(z)}{\mathrm{d}z}. \tag{4.26}$$

T is the increase in temperature of the member axis and $\mathrm{d}T(z)/\mathrm{d}z$ is the *temperature gradient*. In a linear distribution $T(z)$, the temperature gradient is independent of z (see Figure 4.71).

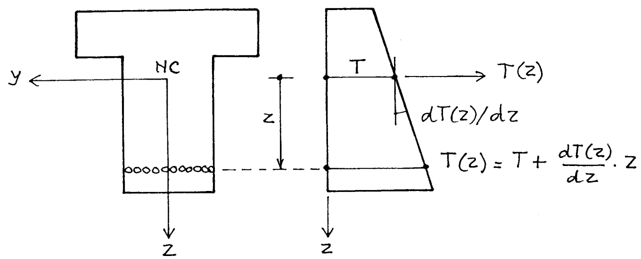


Figure 4.71 It is assumed that the increase in temperature is linear over the height of the cross-section (the z direction) and is constant in the width direction (the y direction). T is the increase in temperature of the member axis and $\mathrm{d}T(z)/\mathrm{d}z$ is the temperature gradient.

The stress in fibre layer z is found by substituting (4.25) and (4.26) in (4.24):

$$\sigma(z) = E\left\{(\varepsilon + \kappa_z z) - \alpha\left(T + z\frac{\mathrm{d}T(z)}{\mathrm{d}z}\right)\right\}$$

or, rewritten,

$$\sigma(z) = E(\varepsilon - \alpha T) + Ez\left(\kappa_z - \alpha\frac{\mathrm{d}T(z)}{\mathrm{d}z}\right). \tag{4.27}$$

The section forces (stress resultants) N, M_y and M_z are

$$N = \int_A \sigma(z)\,\mathrm{d}A, \tag{4.28}$$

$$M_y = \int_A y\sigma(z)\, \mathrm{d}A, \tag{4.29}$$

$$M_z = \int_A z\sigma(z)\, \mathrm{d}A. \tag{4.30}$$

After substituting (4.27) in (4.28) to (4.30) we can work out the integrals. In doing so it is important to remember that in (4.27) the terms in brackets are independent of z. We find

$$N = EA(\varepsilon - \alpha T) + ES_z\left(\kappa_z - \alpha\frac{\mathrm{d}T(z)}{\mathrm{d}z}\right),$$

$$M_y = ES_y(\varepsilon - \alpha T) + EI_{yz}\left(\kappa_z - \alpha\frac{\mathrm{d}T(z)}{\mathrm{d}z}\right),$$

$$M_z = ES_z(\varepsilon - \alpha T) + EI_{zz}\left(\kappa_z - \alpha\frac{\mathrm{d}T(z)}{\mathrm{d}z}\right).$$

If, as usual, we choose the origin of the yz coordinate system at the normal centre of the cross-section, the static moments S_y and S_z are zero, and the expressions above simplify to

$$N = EA(\varepsilon - \alpha T),$$

$$M_y = EI_{yz}\left(\kappa_z - \alpha\frac{\mathrm{d}T}{\mathrm{d}z}\right),$$

$$M_z = EI_{zz}\left(\kappa_z - \alpha\frac{\mathrm{d}T}{\mathrm{d}z}\right).$$

If the y and z direction coincide with the principal directions of the cross-section, the product of inertia I_{yz} is also zero, and so is the bending moment M_y that acts in the xy plane. Now two equations are left:

$$N = EA(\varepsilon - \alpha T), \tag{4.31}$$

$$M_z = EI\left(\kappa_z - \alpha \frac{\mathrm{d}T}{\mathrm{d}z}\right). \tag{4.32}$$

If the member is free to deform and there is no load, there will be no normal forces and bending moments. From the zero value for N and M_z we find the deformation quantities $\varepsilon^{(T)}$ and $\kappa^{(T)}$ associated with a *free deformation* due to a change in temperature:

$$\varepsilon = \varepsilon^{(T)} = \alpha T, \tag{4.33}$$

$$\kappa_z = \kappa_z^{(T)} = \alpha \frac{\mathrm{d}T(z)}{\mathrm{d}z}. \tag{4.34}$$

The constitutive relationships (4.31) and (4.32) therefore can also be written as

$$N = EA(\varepsilon - \varepsilon^{(T)}), \tag{4.35}$$

$$M_z = EI(\kappa_z - \kappa_z^{(T)}). \tag{4.36}$$

Note: The influence of a change in temperature finds expression in the constitutive relationships. This is not surprising if you remember that the coefficient of thermal expansion is a material property. The kinematic and static relationships remain unchanged.

When $\varepsilon^{(T)}$ and $\kappa_z^{(T)}$ are constant over the member length and are not functions of x, the differential equations for bending and extension remain unchanged. When working out the boundary condition (end and/or joining conditions), it is important to remember the constitutive relationships have changed due to the change in temperature. An application of the formulas is given in Section 8.2, Example 4.

4.13 Notes for the fibre model and summary of the formulas

After a number of comments regarding the fibre model, we provide a summary of the different formulas derived in this chapter.

4.13.1 Notes for the fibre model

In the fibre model, only the normal stresses σ due to normal force and bending moment are responsible for the deformations. Experiments and more accurate calculations using elasticity theory show that the fibre model gives a correct description of the real member behaviour for extension and pure bending (bending without shear forces).

For bending with shear forces, i.e. with a varying bending moment, there are also shear stresses (acting in the cross-sectional plane) in addition to normal stresses (acting normal to the cross-sectional plane). The magnitude of the shear stresses can be found from the equilibrium.[1] The fibre model now is less accurate in the description of the member behaviour. There are differences that must be ascribed to the deformation through shear stresses. However, the differences are minor for slender members (the length is very large compared with the cross-sectional dimensions). In that case, the bending stresses are much larger than the shear stresses and the shear deformation is negligible.

If the shear deformation cannot be ignored, planar cross-sections are no longer planar and there is no longer a linear bending stress distribution. Nevertheless, the fibre model (with planar cross-sections remaining planar) can still be used in many cases. The shear deformation due to shear forces

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1 See Chapter 5.

is then expressed in the fibre model as a tilting of the planar cross-sections (d) with respect to the member axis (s), as shown in Figure 4.72. The cross-sections no longer remain perpendicular to the member axis, and the rotation φ of a cross-section is now no longer equal in magnitude to the slope dw/dx of the member axis!

In the above we assumed prismatic members, members with the same cross-section (the same cross-sectional quantities and material properties) everywhere. The fibre model can also be used roughly on members with a cross-section that changes gradually. If the cross-section does not change gradually but in steps, you will have to count on a length with a stress distribution at both sides of the step change that deviates from the linear distribution (a disruption zone). According to *Saint Venant's principle*,[1] the total length with deviant stress distribution is of the same magnitude as the sum of the cross-sectional dimensions at both sides of the joining (see Figure 4.73).

There are also deviant stress distributions at the points of application of concentrated loads and the points where support reactions act. In the fibre model, with its rigid cross-sections, these details are not taken into account.

4.13.2 Summary formulas

Prior warning: All the formulas were derived in a particular context and have limiting conditions.

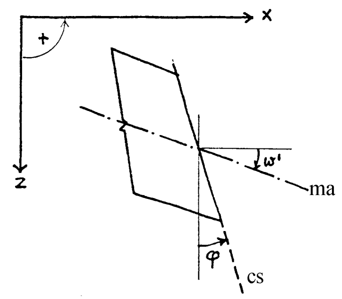


Figure 4.72 If, in the model, the shear deformation under the influence of the shear forces is included, this is expressed in a tilting of the planar cross-sections (cs) with respect to the member axis (ma). The rotation of the cross-section is now no longer equal in magnitude to the slope dw/dx of the member axis.

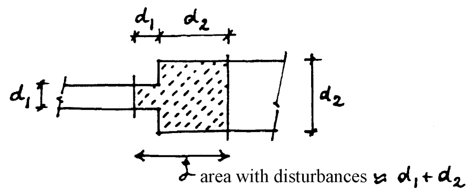


Figure 4.73 If the cross-section of a beam does not vary gradually, but in steps, one has to take into account a certain length where the stress distribution deviates from the linear distribution. According to *Saint Venant's principle*, this length is of the same order as the sum of the cross-sectional dimensions at both sides of the step change.

[1] Named after Adhémar Jean Claude Barré de Saint Venant (1797–1886), French civil engineer. He contributed greatly to the development of the theory of elasticity.

- *Kinematic relationships* (Section 4.3.1):

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{(extension)},$$

$$\left.\begin{array}{l} \varphi = -\dfrac{\mathrm{d}w}{\mathrm{d}x} \\ \kappa = \dfrac{\mathrm{d}\varphi}{\mathrm{d}x} \end{array}\right\} \Rightarrow \kappa = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \quad \text{(bending)}.$$

- *Constitutive relationships* (Section 4.3.2):

$$N = EA\varepsilon \quad \text{(extension)},$$

$$M = EI\kappa \quad \text{(bending)}.$$

- *Static (or equilibrium) relationships* (Section 4.3.3):

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0 \quad \text{(extension)},$$

$$\left.\begin{array}{l} \dfrac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0 \\ \dfrac{\mathrm{d}M}{\mathrm{d}x} - V = 0 \end{array}\right\} \Rightarrow \frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + q_z = 0 \quad \text{(bending)}.$$

- *Differential equations for bending and extension* (Section 4.11) (see Table 4.4):

Table 4.4 The differential equations for extension and bending of a prismatic member.

	kinematic relationships	constitutive relationships	static relationships	differential equations
extension	$\varepsilon = \dfrac{\mathrm{d}u}{\mathrm{d}x}$	$N = EA\varepsilon$	$\dfrac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0$	$EA\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} + q_x = 0$
bending	$\varphi = -\dfrac{\mathrm{d}w}{\mathrm{d}x}$, $\kappa = \dfrac{\mathrm{d}\varphi}{\mathrm{d}x}$ → $\kappa = -\dfrac{\mathrm{d}^2w}{\mathrm{d}x^2}$	$M = EI\kappa$	$\dfrac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0$, $\dfrac{\mathrm{d}M}{\mathrm{d}x} - V = 0$ → $\dfrac{\mathrm{d}^2M}{\mathrm{d}x^2} + q_z = 0$	$-EI\dfrac{\mathrm{d}^4w}{\mathrm{d}x^4} + q_z = 0$

extension: $$EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + q_x = 0,$$

bending: $$-EI\frac{\mathrm{d}^4w}{\mathrm{d}x^4} + q_z = 0.$$

- *Strain formula for bending and extension* (Section 4.2):

$$\varepsilon(z) = \varepsilon + \kappa_z z.$$

- *Stress formula for bending and extension* (Section 4.4):

$$\sigma(z) = \frac{N}{A} + \frac{M_z z}{I_{zz}}.$$

The contributions due to extension and bending are

$$\sigma^{(N)} = \frac{N}{A} \text{ (extension) and } \sigma^{(M)} = \frac{M_z z}{I_{zz}} \text{ (bending).}$$

In extension, the normal stress is constant over the cross-section. In bending, the normal stress is linear over the height of the cross-section and is zero at the normal centre.

- *Section moduli* (Section 4.6):

$$W_{z;\mathrm{b}} = \frac{I_{zz}}{e_\mathrm{b}} \text{ and } W_{z;\mathrm{t}} = \frac{I_{zz}}{e_\mathrm{t}}.$$

e_b and e_t are the distances from the normal centre to the outer fibre layers at the bottom (b) and top (t) of the cross-section.

The extreme bending stresses in the outer fibre layers (b) and (t) are

$$\sigma_{\mathrm{b}} = +\frac{M_z}{W_{z;\mathrm{b}}} \text{ and } \sigma_t = -\frac{M_z}{W_{z;\mathrm{t}}} .$$

If the y axis is a line of symmetry, $W_{z;\mathrm{b}} = W_{z;\mathrm{t}} = W_z$, and the extreme bending stresses are equal in magnitude.

A rectangular cross-section has

$$W_z = \tfrac{1}{6}\, bh^2 \text{ or in brief } W = \tfrac{1}{6}\, bh^2.$$

In a symmetrical cross-section, the extreme stresses due to bending and extension are

$$\sigma_{\mathrm{max}} = \frac{N}{A} \pm \frac{M_z}{W_z} \text{ or in brief } \sigma_{\mathrm{max}} = \frac{N}{A} \pm \frac{M}{W} .$$

- *General stress formula related to the principal directions* (Section 4.8):

$$\sigma(y, z) = \frac{N}{A} + \frac{M_y y}{I_{yy}} + \frac{M_z z}{I_{zz}} .$$

- *Core of the cross-section* (Section 4.9):

$$k_{\mathrm{b}} = \frac{W_{z;\mathrm{t}}}{A} \text{ (lower core radius) and } k_{\mathrm{t}} = \frac{W_{z;\mathrm{b}}}{A} \text{ (upper core radius).}$$

If the y axis is a line of symmetry, then $W_{z;\mathrm{b}} = W_{z;\mathrm{t}} = W_z$, and the lower and upper core radii are equal:

$$k = \frac{W_z}{A} .$$

A rectangular cross-section has

$$k = \tfrac{1}{6}\,h.$$

- *Thermal effects* (Section 4.12):

Free deformation due to a change in temperature:

$$\varepsilon = \varepsilon^{T} = \alpha T \quad \text{and} \quad \kappa_z = \kappa_z^{(T)} = \alpha \frac{\mathrm{d}T}{\mathrm{d}z}\,.$$

The constitutive relationships taking into account a change in temperature:

$$N = EA\big(\varepsilon - \varepsilon^{(T)}\big) \quad \text{and} \quad M_z = EI\big(\kappa_z - \kappa_z^{(T)}\big),$$

or

$$N = EA(\varepsilon - \alpha T) \quad \text{and} \quad M_z = EI\left(\kappa_z - \alpha \frac{\mathrm{d}T}{\mathrm{d}z}\right).$$

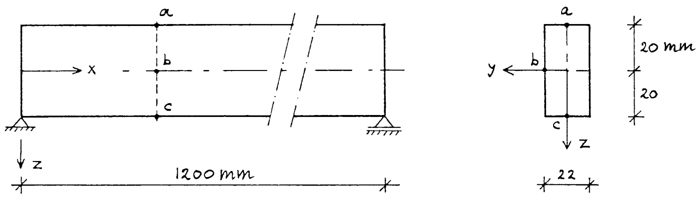
4.14 Problems

General comment: The dead weight of structures is left out of consideration unless expressly indicated otherwise in the questions.

Strain diagram, stress diagram and stress formula (Sections 4.1 to 4.5)

4.1 A simply supported beam with rectangular cross-section is loaded in the xz plane. For four loading cases (I to IV) strain measurements are taken with respect to the fibres a, b, and c in a cross-section. The results are as follows:

Loading case	Fibre *a*	Fibre *b*	Fibre *c*
I	–0.1‰	+0.1‰	+0.3‰
II	–0.2‰	0	+0.2‰
III	–0.3‰	+0.2‰	+0.3‰
IV	+0.3‰	+0.1‰	–0.1‰

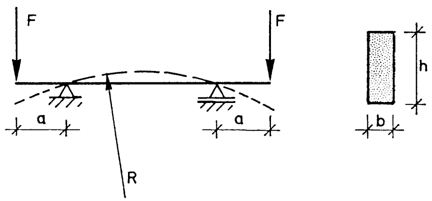


Questions:
a. Assess the reliability of these results using a sketch of the strain diagram. What is your conclusion?
b. For loading case I, determine the value of the strain ε and curvature κ.
c. For loading case IV, determine the value of the strain ε and curvature κ.
d. What are the dimensions of ε and κ?

4.2 *Questions*:
a. Which quantities are linked by the kinematic relationships for members subject to bending and extension.
b. On which assumptions are the kinematic relationships based?

4.3 *Questions*:
a. What is the name for the relationships $N = EA\varepsilon$ and $M = EI\kappa$, and under which conditions do these relationships apply?
b. What is the definition of the quantities N and M?
c. Derive the relationships $N = EA\varepsilon$ and $M = EI\kappa$ clearly showing the meaning of the normal centre.

4.4 A beam with rectangular cross-section $b \times h$ and modulus of elasticity E is subject to a four-point bending test. In doing so, the beam assumes the shape of a circle between the supports with a radius of curvature R. In the calculation use $a = 0.5$ m, $b = 20$ mm, $h = 30$ mm, $E = 210$ GPa and $F = 0.6$ kN.



Questions:
a. Determine the radius of curvature R in mm.
b. Determine the maximum bending stress in the beam.

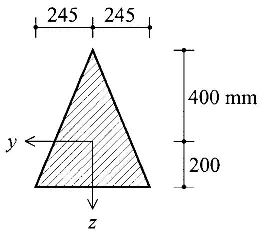
4.5 The stress formula

$$\sigma(z) = \frac{N}{A} + \frac{M_z z}{I_{zz}}$$

may be applied only if the coordinate system used meets certain conditions.

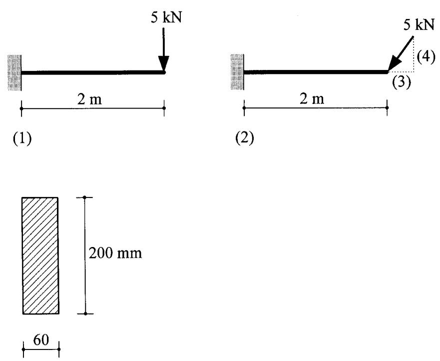
Question:
What are these conditions?

4.6 A normal force $N = -441$ kN and a bending moment $M_z = +88.2$ kNm act in the triangular cross-section shown.



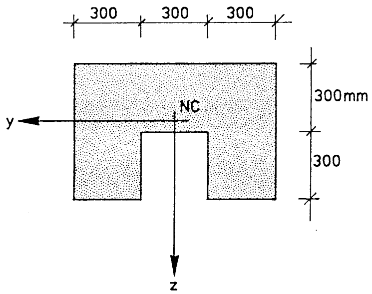
Question:
Plot the normal stress diagram for the cross-section. Clearly indicate the separate contributions by N and M_z.

4.7: 1–2 The same cantilever beam, with the rectangular cross-section shown, is loaded in two ways by a force of 5 kN. The force is applied at the member axis.



Questions:
a. Determine the maximum normal stress in the beam, in an absolute sense.
b. Determine the location where this maximum stress occurs. Is this stress tensile or compressive?

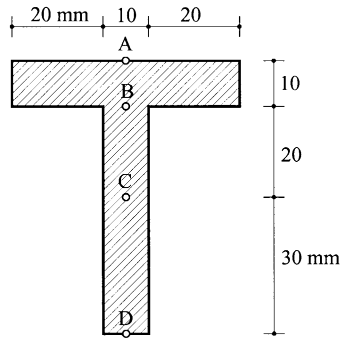
4.8 A normal force $N = -2250$ kN and bending moment $M_z = 436.5$ kNm act in the cross-section shown.



Questions:
a. Determine the normal stress at the point ($y = -200$ mm, $z = +180$ mm).
b. Determine the normal stress at the point ($y = +200$ mm, $z = -120$ mm).
c. Plot the normal stress diagram and clearly indicate the contributions by N and M_z.
d. Determine the z coordinate of the neutral axis.

4.9 *Questions*:
a. What is the definition of the normal centre in a homogeneous cross-section?
b. What is the significance of the normal centre?
c. In question (a), what is meant by the concept "*homogeneous*"?

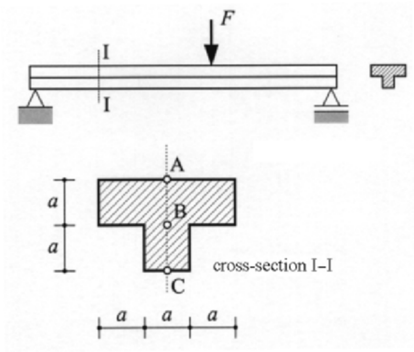
4.10 In the cross-section shown, only a bending moment acts in the vertical plane of symmetry.



Question:
Which of the sign combinations (a) to (e) could be correct for the normal stress at the points A to D?

	A	B	C	D
a.	–	0	+	+
b.	–	–	+	+
c.	–	–	0	+
d.	–	+	+	+
e.	+	+	–	–

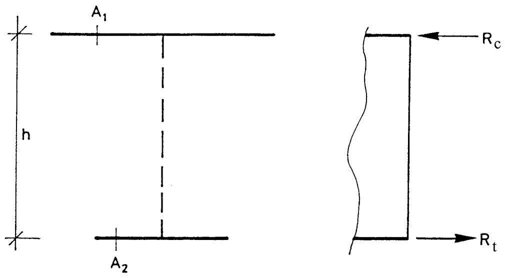
4.11 You are given a simply supported T-beam. For the given load, σ_A, σ_B and σ_C are the normal stresses in cross-section I–I, in the fibres at A, B and C respectively.



Question:
Which of the combinations of normal stresses (a) to (f) could be correct?

	σ_A in N/mm^2	σ_B in N/mm^2	σ_C in N/mm^2
a.	–30	+10	–50
b.	–30	–10	+50
c.	–30	0	+30
d.	+30	–10	–50
e.	–50	+10	+30
f.	–30	+10	+50

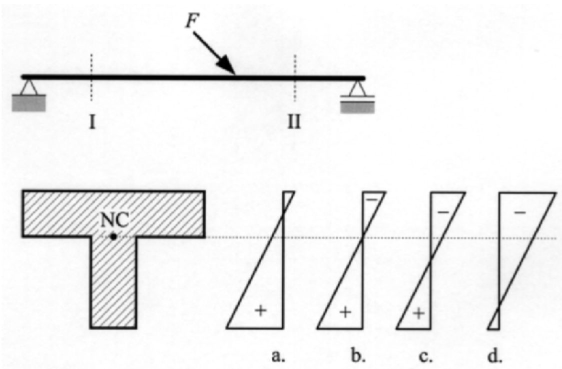
4.12 You are given a thin-walled I-section with height $h = 300$ mm. The area of the upper flange is $A_1 = 3600$ m^2 and of the lower flange $A_2 = 2400$ mm^2. The area of the web is so small that it can be neglected. There is a compressive force $R_c = 180$ kN in the upper flange and a tensile force $R_t = 90$ kN in the lower flange.



Questions:

a. Determine the normal force in the cross-section.
b. Determine the bending moment in the cross-section.

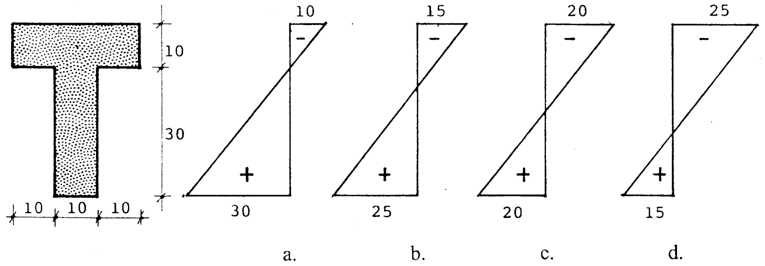
4.13 A T-beam is loaded as shown.



Questions:

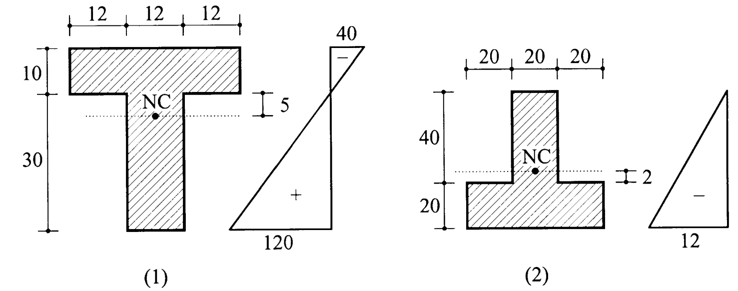
a. Which normal stress diagram could match cross-section I?
b. Which normal stress diagram could match cross-section II?

4.14 In the cross-section shown there is only a bending moment. The cross-sectional dimensions are given in mm and the stresses in N/mm^2.



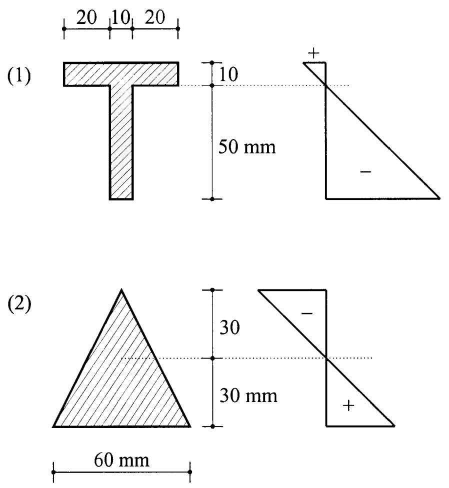
Question:
Which stress diagram is correct?

4.15: 1–2 You are given two cross-sections with associated normal stress diagrams. The cross-sectional dimensions are given in mm and the stresses in N/mm^2.



Question:
Determine the normal force N in the cross-section.

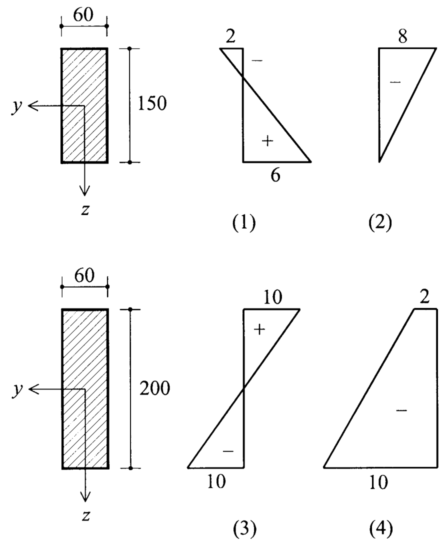
4.16: 1–2 You are given two cross-sections with associated stress diagrams.



Question:
Which of the combinations (a) to (f) is valid for the sign of the normal force and the deformation symbol of the bending moment in the cross-section?

	N	***M***
a.	0	⌣
b.	0	⌢
c.	+	⌣
d.	+	⌢
e.	−	⌣
f.	−	⌢

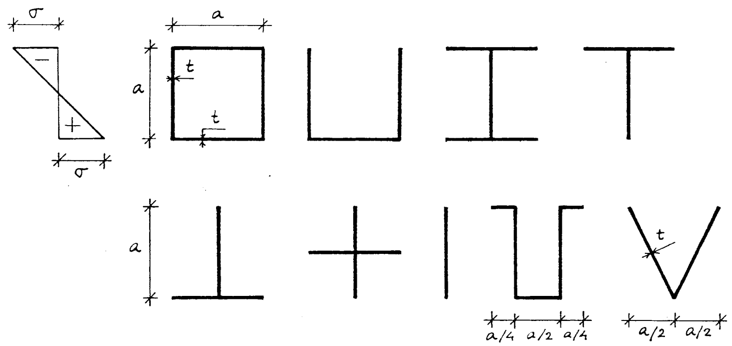
4.17: 1–4 You are given two different rectangular cross-sections with two normal stress diagrams for each cross-section. The cross-sectional dimensions are given in mm, the stresses are in N/mm^2.



Questions:

a. Determine the normal force N in the cross-section, with the right sign.
b. Determine the bending moment M_z in the cross-section, with the right deformation symbol.

4.18 The stress diagram shown applies to all nine thin-walled cross-sections (the wall thickness t is far smaller than the length a).

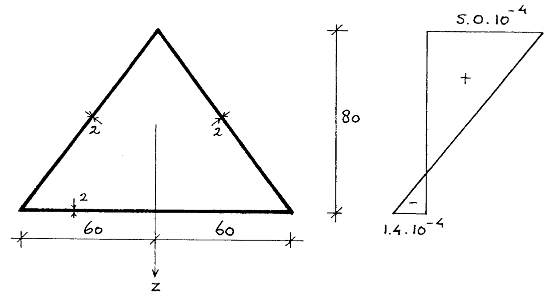


Questions:
a. Which of the cross-sections is subject to bending without a normal force?
b. In which of the cross-sections is the normal force not equal to zero?

4.19: 1–9 See problem 4.18 for the figure and details.

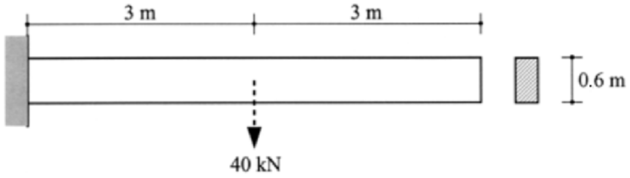
Questions:
a. Determine the normal force N, expressed in a, t and σ, with the right sign.
b. Determine the bending moment M, expressed in a, t and σ, with the right deformation symbol ($\smile$ or $\frown$).

4.20 You are given the strain diagram of a thin-walled triangular box girder with a uniform wall thickness. The cross-sectional dimensions are given in mm.



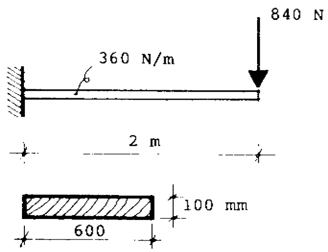
Questions:
a. Determine the location of the normal centre.
b. From the given strain diagram, determine the deformation quantities ε and κ_z.
c. If the modulus of elasticity $E = 210 \times 10^3$ N/mm^2, plot the stress diagram.
d. From the stress diagram, determine the resultant R_t of all tensile stresses.
e. From the stress diagram, determine the resultant R_c of all compressive stresses.
f. From the magnitude and location of R_t and R_c, determine the magnitude of the normal force N and the bending moment M_z in the cross-section, with the correct signs.
g. Where in the cross-section is the centre of force?
h. From the answers to (b) and (f), derive the magnitude of the axial and bending stiffness.

4.21 The cantilever beam has a rectangular cross-section. The dead weight of the beam is 40 kN.



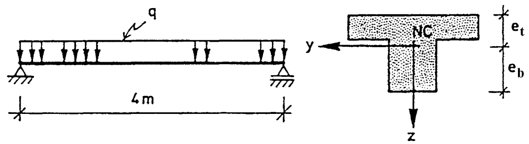
Question:
Determine the resultant of all tensile stresses in the cross-section at the fixed end. Compare its magnitude to the dead weight of the beam.

4.22 The cantilever joist has a cross-section of $600 \times 100\ \text{mm}^2$ and a dead weight of 360 N/m, and is loaded at its free end by a vertical force of 840 kN.



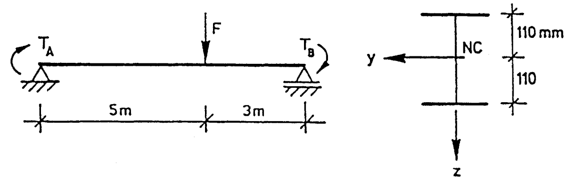
Question:
Determine the maximum bending stress in the cross-section at the fixed end.

4.23 The simply supported beam, with a centroidal moment of inertia $I_{zz} = 800 \times 10^{-6}\ \text{m}^4$, is carrying a uniformly distributed load q over its full length. Assume $e_t = 200$ mm and $e_b = 400$ mm.



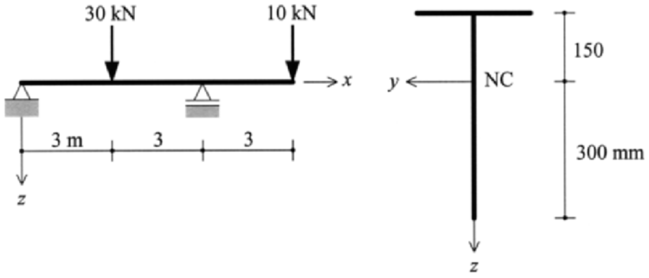
Question:
Determine the load q for which the maximum bending stress in the beam is $6\ \text{N/mm}^2$.

4.24 A simply supported beam is loaded at the ends A and B by couples T_A and T_B and is carrying the point load F. A steel I-section has been used for the beam, 220 mm high, with $I_{zz} = 27.5 \times 10^6\ \text{mm}^4$. Use in the calculation $T_A = 4$ kNm, $T_B = 6$ kNm and $F = 12$ kN.



Question:
Determine the maximum bending stress in the beam.

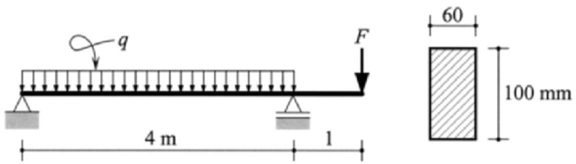
4.25 The simply supported thin-walled cantilevered T-beam has a moment of inertia $I_{zz} = 150 \times 10^6$ mm^4. The load and location of the normal centre NC are shown in the figure.



Questions:

a. Determine the maximum bending tensile stress in the T-beam and the location of the cross-section in which it occurs. Plot the stress diagram for this cross-section.
b. Determine the maximum bending compressive stress in the T-beam and the location of the cross-section in which it occurs. Plot the stress diagram for this cross-section.

4.26: 1–2 The simply supported cantilevered beam has a rectangular cross-section and is loaded by a uniformly distributed load q between the supports and by a force F at the free end.

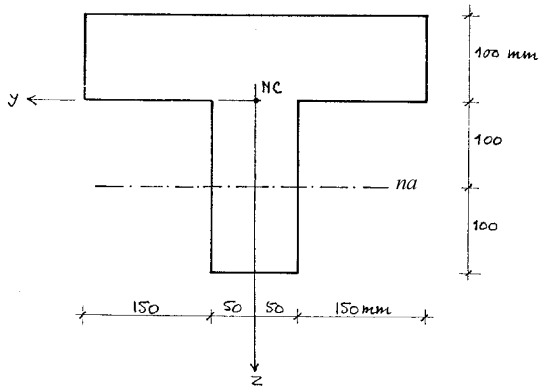


Questions:

Determine the maximum bending stress in the beam when

1. $F = 1$ kN and $q = 0.6$ kN/m.
2. $F = 1$ kN and $q = 0.9$ kN/m.

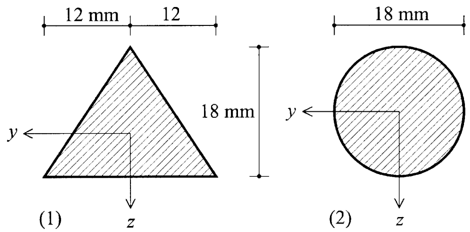
4.27 A bending moment M and normal force N act in the given cross-section of a T-beam. The maximum normal stress in an absolute sense is a compressive stress of 8 N/mm^2. For the cross-section $A = 60 \times 10^3$ mm^2 and $I_{zz} = 400 \times 10^6$ mm^4. The location of the normal centre NC and that of the neutral axis (na) are also given.



Questions:

a. Plot the normal stress diagram for the cross-section.
b. Determine the normal force N, with the right sign.
c. Determine the bending moment M, with the right sign.
d. Where in the cross-section is the centre of force?

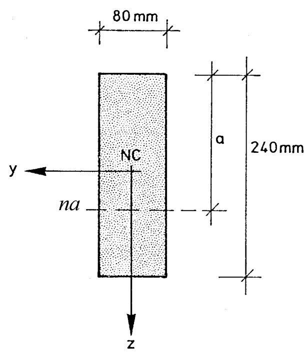
4.28: 1–2 The following applies to the solid triangular and circular cross-sections:
$E = 210 \times 10^3$ N/mm^2, $\varepsilon = 0.25‰$ and $\kappa_z = -83.3 \times 10^{-3}$ m^{-1}.



Questions:

a. Plot the strain and stress diagrams.
b. Determine the radius of curvature of the member at the given cross-section.
c. Determine the axial and bending stiffness of the member at the given cross-section.
d. Determine the normal force with the right sign, and the bending moment with the right deformation symbol.
e. Where in the cross-section is the centre of force?

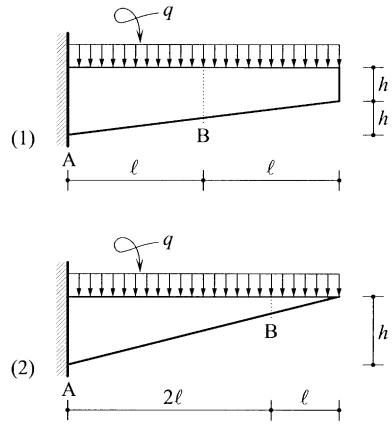
4.29 You are given a rectangular cross-section in which the neutral axis (*na*) is at a distance a below the upper edge.



Questions:
Determine the coordinate e_z of the centre of force when

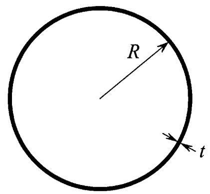
a. $a = 0$.
b. $a = 60$ mm.
c. $a = 240$ mm.
d. $a = 320$ mm.

4.30: 1–2 You are given two tapered consoles with rectangular cross-section and a uniformly distributed load q over the full length. The width of both consoles is constant. The maximum bending stress in cross-section B is 12 N/mm^2. It is allowed to use the stress formula derived for a prismatic member.



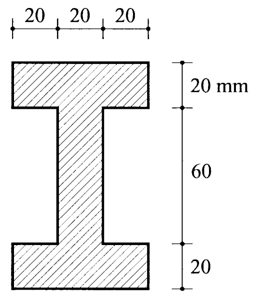
Question:
Determine the maximum bending stress in cross-section A at the fixed end.

4.31 In the thin-walled tube with radius $R = 350$ mm and wall-thickness $t = 13$ mm, the maximum bending stress is 120 N/mm^2.



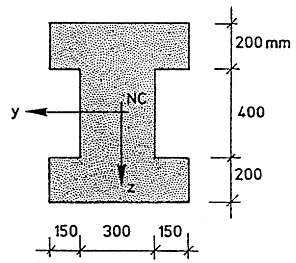
Question:
Determine the bending moment in the tube.

4.32 The section shown is loaded in the vertical plane by a bending moment M.



Question:
Determine the magnitude of the bending moment M when the maximum bending stress in the cross-section is 125 N/mm^2.

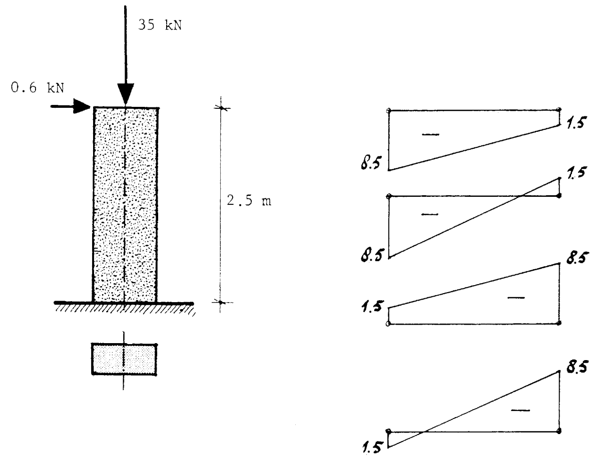
4.33 A central compressive force of 1800 kN acts in the cross-section shown.



Questions:

a. Determine the maximum bending moment that the cross-section can transfer in the xz plane for the given compressive force if the tensile stress may not be larger than 2.5 N/mm^2.
b. Plot the associated stress diagram and determine the z coordinate of the neutral axis.

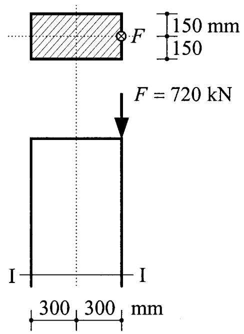
4.34 For the fixed column with rectangular cross-section, loaded by two forces, are four stress diagrams given for the cross-section at the fixed end. The area of the cross-section is $A = 10 \times 10^3$ mm^2; the section modulus is $W = 300 \times 10^3$ mm^3.



Question:
Which stress distribution is correct?

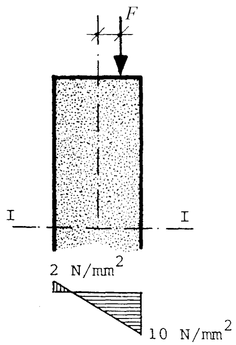
4.35 A column with rectangular cross-section is loaded by an eccentric compressive force $F = 720$ kN. The dead weight of the column is neglected.

Question:
Plot the normal stress diagram for cross-section I–I.

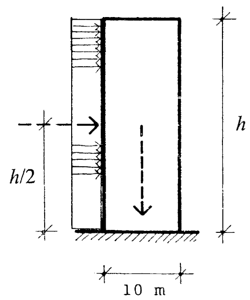


4.36 Due to an eccentric compressive force F, there is the stress distribution in cross-section I–I of the column as shown. The cross-section of the column is square with 400 mm sides.

Questions:
a. Determine the magnitude of force F.
b. Determine the eccentricity of force F.

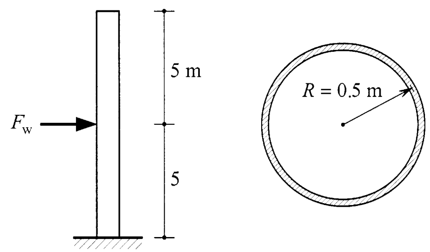


4.37 The tower shown has a square cross-section with 10 m sides and a weight of 3 kN/m^3. The tower is subject to horizontal wind loading of 1 kN/m^2. The tower is on a spread foundations. The foundation plate is square with 10×10 m^2 dimensions. It is assumed that the earth pressure under the foundation plate is linear and that it can be determined using the stress formula for a member subject to bending and extension.



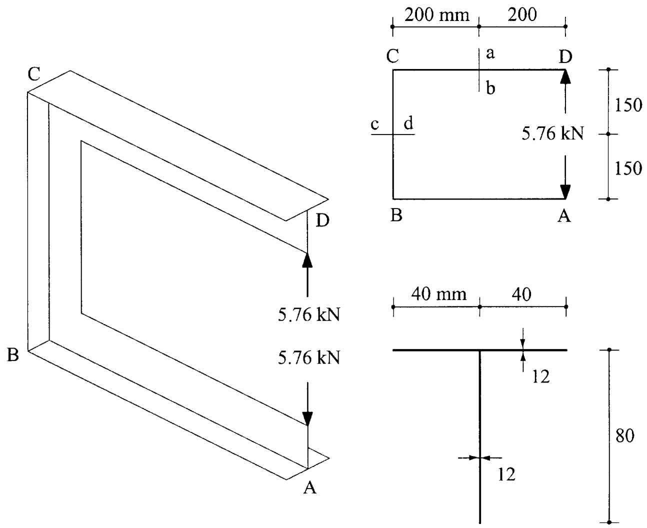
Question:
Determine height h for which there are no tensile stresses under the foundation plate.

4.38 The chimney, that can be understood as a thin-walled circular tube with radius $R = 0.5$ m, is 10 m high and has a dead weight of 60 kN. F_w is the resultant of the horizontal wind loading. Tensile stresses are not permitted in the cross-section at the base.



Question:
Determine the maximum value of F_w.

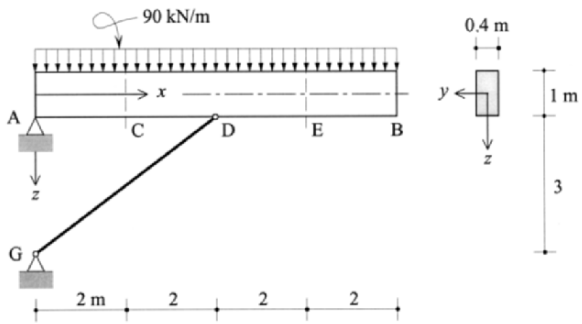
4.39 The figure shows the centre lines of a clamping bracket ABCD, modelled as a line element. The clamping bracket has a T-shaped cross-section. The cross-sectional dimensions are given in the figure. The wall thickness is a uniform 12 mm. The cross-section can be considered thin-walled. The clamping bracket is loaded at A and D by two compressive forces of 5.76 kN.



Questions:

a. Determine the required cross-sectional quantities.
b. Determine and plot the normal stress distribution in cross-section a–b.
c. Determine and plot the normal stress distribution in cross-section c–d.

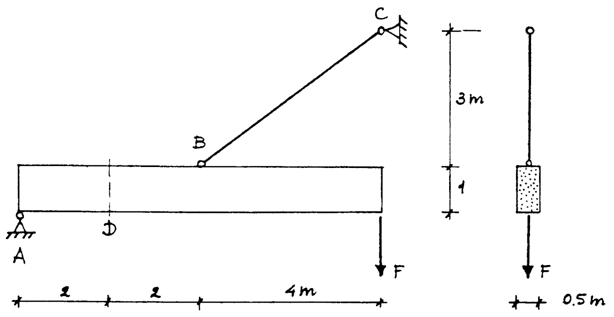
4.40 The beam ADB has a rectangular cross-section and rests with its lower side on a hinged support at A and an oblique bar support DG at D. The dimensions are given in the figure. ADB bears a uniformly distributed load of 90 kN/m.



Questions:

a. Model beam ADB as a line element, and draw all the forces acting on it.
b. For ADB, draw the bending moment, shear force and normal force diagrams.
c. Determine and the plot the normal stress diagram for the cross-sections at C and E.

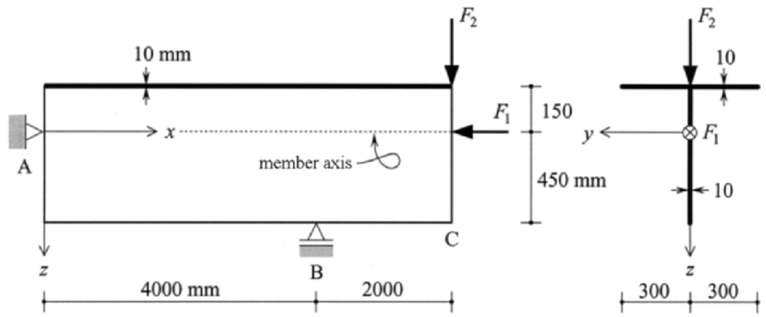
4.41 The beam shown has a rectangular cross-section, and is supported by a hinge at A at the lower side of the beam, and by an oblique bar support at B at the upper side. A vertical force $F = 250$ kN acts at the free end.



Questions:

a. Model the isolated beam as a line element, and draw all the forces acting on it.
b. Plot the M, V and N diagrams.
c. Plot the normal stress diagram for the cross-section at D.
d. Plot the normal stress diagram for the cross-section directly to the left of B.
e. Plot the normal stress diagram for the cross-section directly to the right of B.
f. Which of the diagrams (c, d or e) is closest to reality? Explain your answer.

4.42 The thin-walled T-beam shown has a uniform wall thickness of 10 mm and is loaded as shown by the forces $F_1 = 240$ kN and $F_2 = 30$ kN. For the cross-section $A = 12 \times 10^3$ mm^2 and $I_{zz} = 450 \times 10^6$ mm^4.



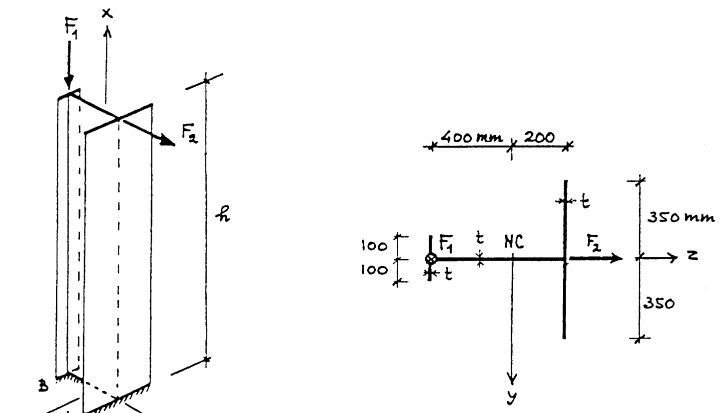
Questions:

a. Model the beam as a line element, and plot the M and N diagrams.
b. Determine and plot the normal stress diagram for the cross-section at support B.
c. Determine and plot the normal stress diagram for the cross-section midway between A and B.

4.43 A thin-walled steel column with height h and a uniform wall thickness t is fixed at its base and is loaded by the forces F_1 and F_2 at its free end, as shown in the figure. In the calculation use $F_1 = 315$ kN, $F_2 = 63$ kN, $h = 3$ m and $t = 10$ mm. The following cross-sectional quantities are additionally given: $A = 15 \times 10^3$ mm^2 and $I_{zz} = 840 \times 10^6$ mm^4. The location of the normal centre NC can be read from the figure.

Questions:

a. Model the column as a line element and draw all the forces acting on it, including the support reactions.
b. Plot the N and M diagrams, with the signs (deformation symbols).

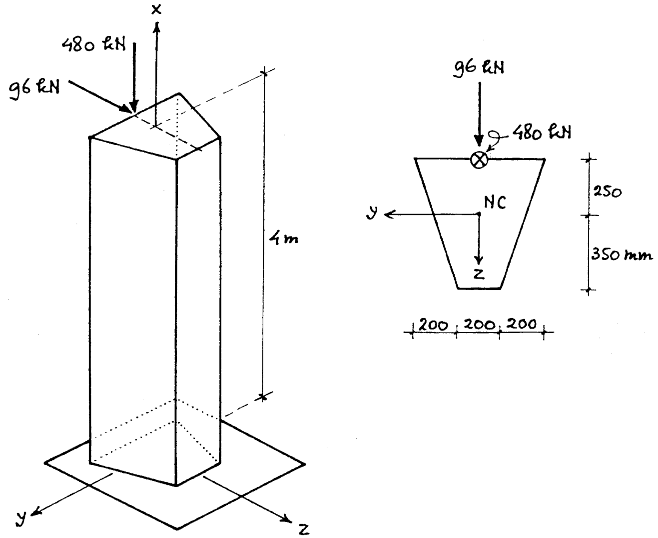


top view

c. Determine and plot the normal stress diagram for the cross-section at the fixed end. At which distance from NC is the neutral axis in this cross-section?
d. Determine and plot the normal stress diagram at half-height. At which distance from NC is the neutral axis in this cross-section?

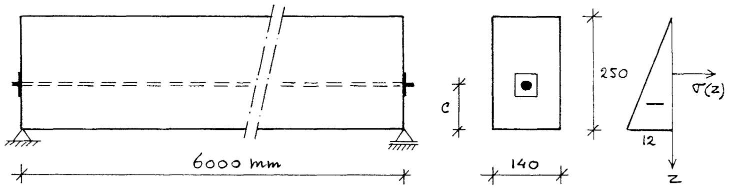
4.44 The concrete column shown, with a height 4 m, is fixed at its base. The column is loaded by an eccentric compressive force of 480 kN and a horizontal force of 96 kN at its free end. The column has a trapezoidal cross-section. The dimensions can be read from the figure. The modulus of elasticity for concrete is 30 GPa.

Questions:
a. Show the correctness of the location of the normal centre NC.



b. Prove that $I_{zz} = 6.6 \times 10^9$ mm^4.
c. Model the column as a line element and plot the M and N diagrams with signs (deformation symbols).
d. Determine the normal stress diagram for the cross-section at the fixed end.
e. Determine the normal stress diagram for the cross-section 0.5 m below the free end.
f. Determine the normal stress diagram for the cross-section at half-height.

4.45 A straight tendon has been placed in a canal in a concrete beam with rectangular cross-section. With a prestressing jack, a tensile stress σ_p is generated in the tendon after which the jack is replaced by an anchorage. The diagram shows the stresses produced by the prestressing in each cross-section. The stress is given in N/mm^2. The beam dimensions are in mm.

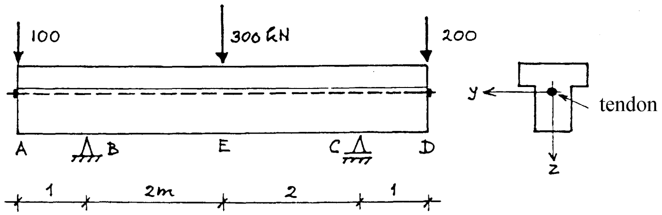


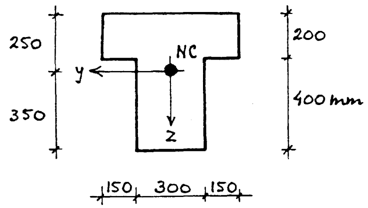
Other data: Cross-section tendon $A_p = 200\ \text{mm}^2$; modulus of elasticity tendon $E_p = 200 \times 10^2\ \text{mm}^2$; modulus of elasticity of concrete $E_c = 35 \times 10^3\ \text{N/mm}^2$. The concrete beam can be considered to have a homogeneous concrete cross-section (the area of the canal can be neglected).

Questions:

a. Determine the magnitude of the prestressing force F_p and the distance c of the tendon to the bottom side of the beam.
b. Determine the lengthening of the tendon due to the prestressing.
c. Determine the shortening of the concrete fibres in the layer at the height of the tendon due to the prestressinging.
d. Determine the "*stroke*" (lengthening) that the prestressing jack has to make.
e. Determine the maximum uniformly distributed load q (including the dead weight) that the beam can carry without tensile stresses in the fibres and with compressive stresses that do not exceed the limiting value of 12 N/mm^2.
f. Determine the change in length that the concrete fibres at the height of the tendon undergo due to this load q.
g. Show that the magnitude of the prestressing force is marginally larger under the influence of the load q calculated under (e).

4.46 You are given a simply supported centrally prestressed cantilevered T-beam. The prestressing force is 1200 kN. The dimensions and load are given in the figure. For the cross-section of the beam use
$A = 240 \times 10^{-3}\ \text{m}^2$, $I_{yy} = 4.5 \times 10^{-3}\ \text{m}^4$ and $I_{zz} = 7.4 \times 10^{-3}\ \text{m}^4$.

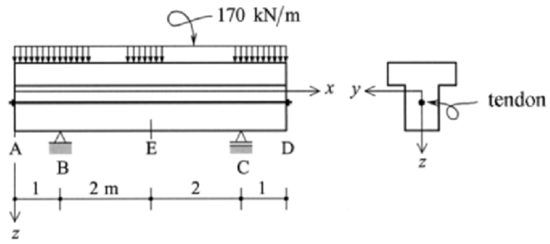


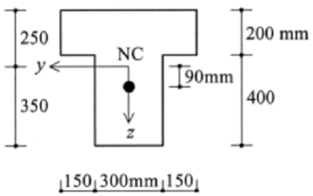


Questions:

a. Plot the M, N and V diagrams as a result of loading and prestressing.
b. In which cross-section is the compressive stress a maximum? Plot the normal stress distribution for that cross-section.
c. In which cross-section is the tensile stress a maximum? Plot the normal stress distribution for that cross-section.
d. Plot the normal stress distribution in the cross-section at support B.

4.47 You are given an eccentrically prestressed T-beam. The straight tendon is 90 mm below the beam axis. The prestressing force is 1200 kN. Dimensions and load are given in the figure. For the beam cross-section use $A = 0.24\ \text{m}^2$, $I_{yy} = 4.5 \times 10^{-3}\ \text{m}^4$ and $I_{zz} = 7.4 \times 10^{-3}\ \text{m}^4$.

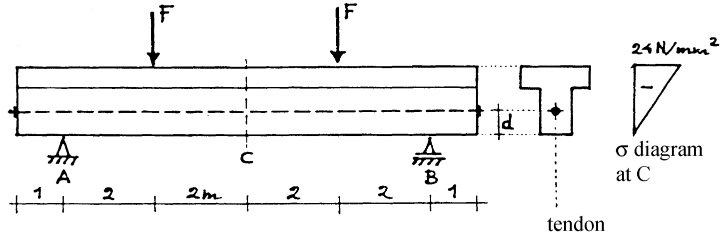


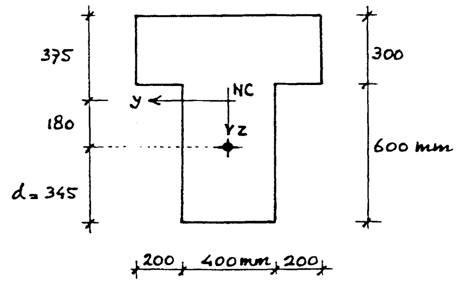


Questions:

a. Plot the M and N diagrams.
b. In which cross-section is the compressive stress a maximum? Plot the normal stress distribution for that cross-section.
c. In which cross-section is the tensile stress a maximum? Plot the normal stress distribution for that cross-section.
d. Where in the cross-section at E is the neutral axis? Determine the distance from the neutral axis to the lower side of the cross-section.

4.48 You are given the normal stress diagram in the cross-section at midspan C of the simply supported cantilevered T-beam, loaded by two equal forces F. The tendon is straight and is placed $d = 345$ mm from the lower side of the beam. The dimensions and location of the normal centre NC in the cross-section are given in the figure. For the cross-section of the beam use $A = 480 \times 10^3\ \text{mm}^2$, $I_{yy} = 16 \times 10^9\ \text{mm}^4$ and $I_{zz} = 33.3 \times 10^9\ \text{mm}^4$.

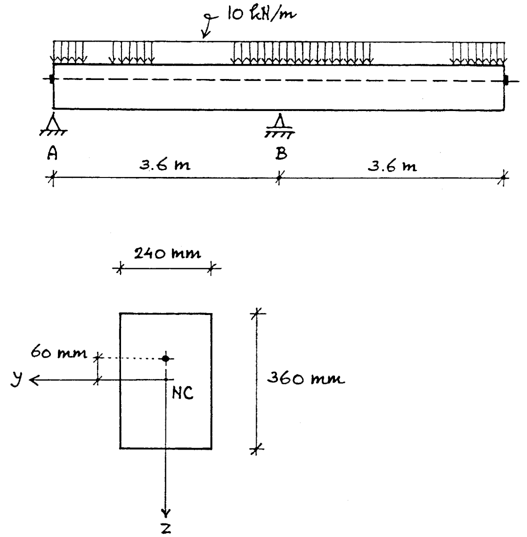




Questions:

a. Determine the normal force and the bending moment in the cross-section at midspan C.
b. Determine the prestressing force F_p.
c. Determine the forces F.
d. Plot the M diagram for the beam, with the deformation symbols.
e. Determine the normal stress diagram for the cross-section at support B.

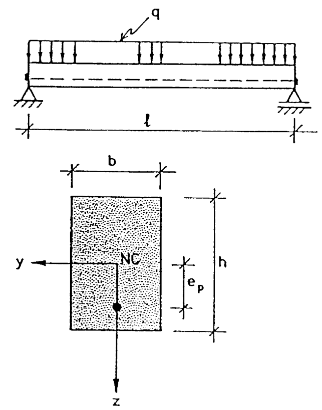
4.49 The cantilevered prestressed beam shown has a rectangular cross-section. The straight tendon is located at ($y = 0$; $z = -60$ mm). The prestressing force F_p is 432 kN. The beam carries a uniformly distributed load $q = 10$ kN/m over its entire length.



Questions:

a. Plot the N and M diagrams due to only the prestressing force F_p.
b. Plot the M diagram due to the distributed load q.
c. Plot the normal stress diagram for a cross-section directly next to support A.
d. Plot the normal stress diagram for the cross-section at support B.
e. Determine the normal stress diagram for the cross-section midway between A and B.

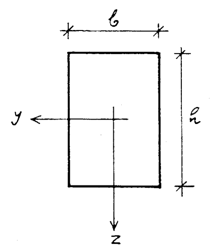
4.50 You are given a simply supported prestressed beam with rectangular cross-section which carries a uniformly distributed load q over its entire length. The straight tendon is located at ($y = 0$, $z = e_p$). The prestressing force is F_p. In the calculation use $\ell = 8$ m, $b = 0.5$ m, $h = 0.9$ m, $q = 100$ kN/m and $e_p = 250$ mm.

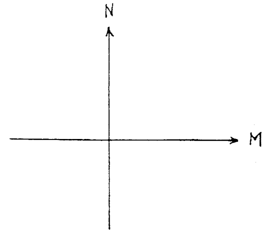


Questions:

a. Determine the minimum prestressing force F_p for which there is no tension in the cross-section at midspan.
b. Plot the normal stress diagram for that cross-section. Clearly indicate the separate contributions due to the distributed load q and the prestressing force F_p.

4.51 A normal force N and bending moment $M_z = M$ are acting in the rectangular cross-section shown, with dimensions $b = 200$ mm and $h = 360$ mm.



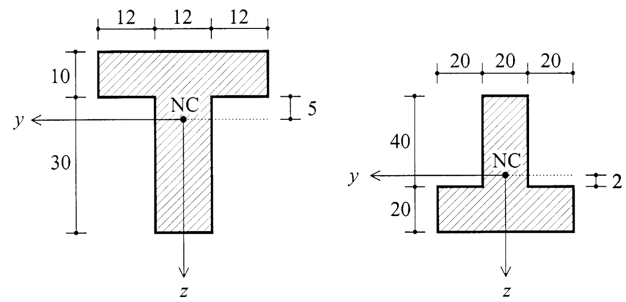


Questions:

a. In the MN diagram, plot all combinations of M and N for which the maximum tensile or compressive stress in the cross-section is 15 N/mm^2.
b. In the diagram, hatch the area with combinations of M and N for which the tensile and compressive stresses in the cross-section do not exceed the value of 15 N/mm^2.
c. In the same diagram, plot all combinations of M and N for which the stress in the upper or lower fibre layer is zero, and the stresses in all the other fibres are compressive and do not exceed the value of 15 N/mm^2. Hatch the area for which there are no tensile stresses and the compressive stress nowhere exceeds the value of 15 N/mm^2.
d. If a force F_x is applied at $(y, z) = (0, -20$ mm$)$, plot the path in the MN diagram as F_x varies. For which value(s) of F_x has the compressive stress reached the limit value of 15 N/mm^2? Check the value(s) by plotting the associated stress diagram.

Section modulus and bending without extension (Sections 4.6 and 4.7)

4.52 You are given two cross-sections.

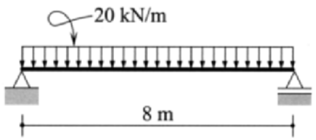


(1) (2)

Questions:

a. Determine the section modulus $W_{z;\mathrm{b}}$.
b. Determine the section modulus $W_{z;\mathrm{t}}$.
c. Determine the maximum bending stress due to $M_z = 408$ Nm.
d. Plot the stress diagram due to $M_z = 408$ Nm.

4.53: 1–5 An IPE-section is used for the five steel beams shown. The strength demand is that the maximum bending stress for the given load may not exceed 160 N/mm^2.

(1)


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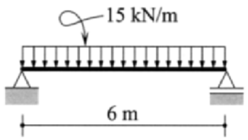

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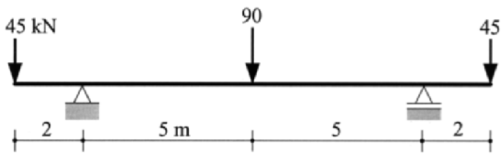

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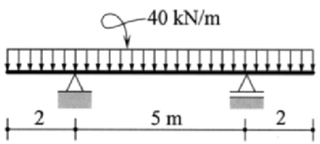

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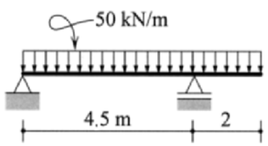
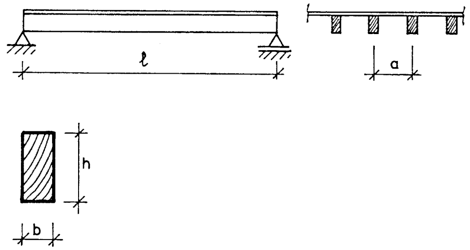


Questions:

a. Determine the lightest section from the table that meets this strength demand.
b. Determine the maximum bending stress upon application of the chosen section.

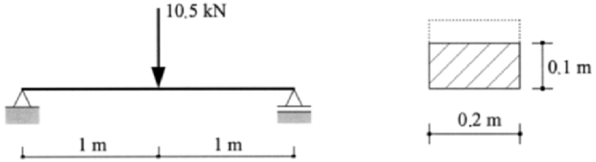
	Section	*W* in mm^3
a.	IPE 270	429×10^3
b.	IPE 300	557×10^3
c.	IPE 330	713×10^3
d.	IPE 360	904×10^3
e.	IPE 400	1160×10^3
f.	IPE 450	1500×10^3
g.	IPE 500	1980×10^3

4.54 You are given a wooden joisting covered by a wooden floor. The beams must be able to carry a floor load of 4 kN/m^2. Use in the calculation $\ell = 6$ m, $a = 0.6$ m, $b = 150$ mm and $h = 300$ mm.



Questions:
Determine the maximum bending stress in a joist for the given floor load.

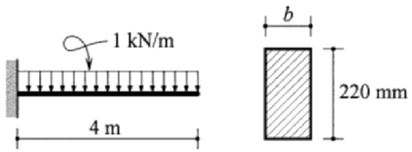
4.55 A simply supported wooden beam, 2 m long, is loaded at midspan by a point load of 10.5 kN. The rectangular cross-section of the beam is 0.2 m wide and 0.1 m high.



Questions:

a. Determine the maximum bending stress in the beam.
b. In order to ensure that the maximum bending stress remains below the limiting value of 7 MPa, a 0.2 m wide wooden strip is glued to the beam. What is the minimum required thickness of this strip?

4.56 In the beam shown, the bending stress due to the given load may not exceed 10 N/mm^2.

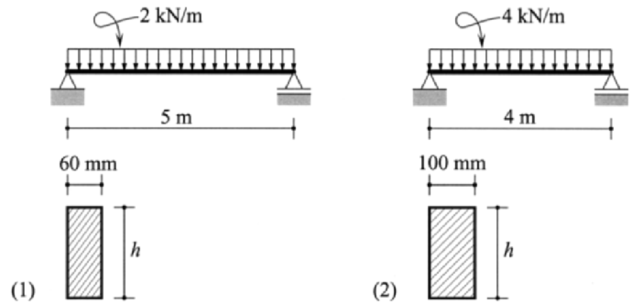


Questions:

a. Select from the table the minimum required width *b*.
b. Determine the maximum bending stress for the chosen width.

	***b* in mm**
a.	90
b.	110
c.	130
d.	150

4.57: 1–2 The two wooden beams shown are designed exclusively for strength. The maximum bending stress due to the given load in the serviceability state may not exceed 10 N/mm^2.

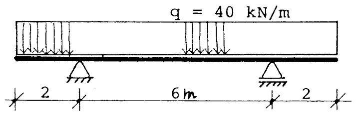


Questions:

a. Determine the minimum required beam height *h* from the table below that meets the given strength demand.
b. Determine the maximum bending stress upon application of the chosen beam height.

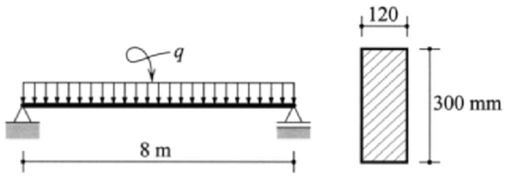
	***h* in mm**
a.	210
b.	230
c.	240
d.	260

4.58 The beam shown has a rectangular cross-section and may not be higher than 500 mm. For the given load, the bending stress must remain below 10 N/mm^2.



Question:
Determine the minimum required width of the beam.

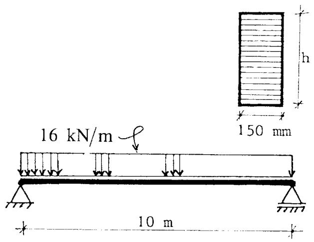
4.59 You are given a simply supported wooden beam with rectangular cross-section and a uniformly distributed load q over the entire length of the beam. The bending stress in the beam may not exceed 10 N/mm^2.



Question:
Determine the maximum admissible load q on the beam.

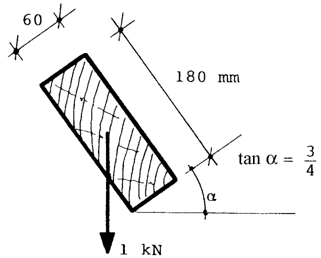
4.60 The wooden beam is constructed of planks glued on top of one another, 150 mm wide and 30 mm thick. For the given load, the maximum bending stress may not exceed 10 N/mm^2.

Questions:
a. Determine the minimum height of the beam.
b. Determine the maximum bending stress for the calculated height of the beam.



General stress formula related to the principal directions (Section 4.8)

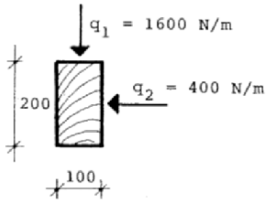
4.61 A simply supported, 4-m long purlin in an inclined roof plane is loaded at midspan by a vertical force of 1 kN.



Questions:
a. Determine the cross-section in which the bending stress is a maximum.
b. Determine the magnitude of this maximum bending stress.
c. For this cross-section, determine the normal stresses at the four corners.
d. Plot the normal stress diagram.
e. Draw the neutral axis in the cross-section.

4.62 A simply supported beam with a length of 6 m and a rectangular cross-section of 100 × 200 mm^2 bears a vertical uniformly distributed load

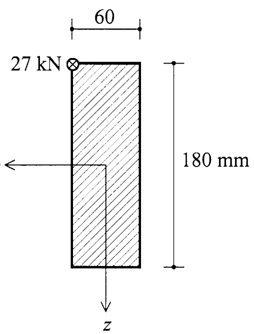
$q_1 = 1600$ N/m and a horizontal uniformly distributed load $q_2 = 400$ N/m, both over the entire length of the beam.



Questions:

a. In which cross-section is the bending stress a maximum?
b. Determine the magnitude of this bending stress.
c. For this cross-section, determine the normal stresses at the four corners.
d. Plot the normal stress diagram.
e. Draw the neutral axis in the cross-section.

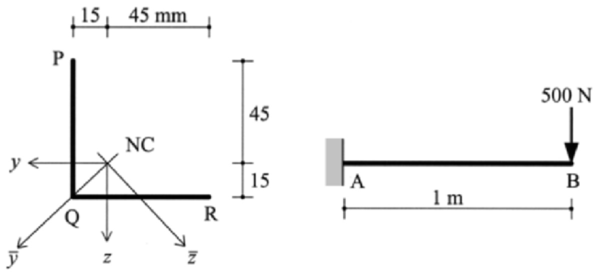
4.63 In the rectangular cross-section shown, the resultant of all normal stresses is a compressive force of 27 kN with its point of application at the upper left-hand corner.



Questions:

a. Determine the normal force N and the bending moments M_y and M_z.
b. Determine the normal stress at each of the corners.
c. Plot the normal stress diagram.
d. Draw the neutral axis in the cross-section.

4.64 An angle steel with equal legs is fixed at A (with one of the legs vertical) and free at B. There is a vertical force of 500 kN at B. The centroidal moments of inertia of the angle steel are $I_{yy} = I_{zz} = 225 \times 10^3$ mm^4 and $I_{yz} = -135 \times 10^3$ mm^4. The principal moments of inertia are $I_{\overline{yy}} = 90 \times 10^3$ mm^4, $I_{\overline{zz}} = 360 \times 10^3$ mm^4 and $I_{\overline{yz}} = 0$. The angle steel is thin-walled.



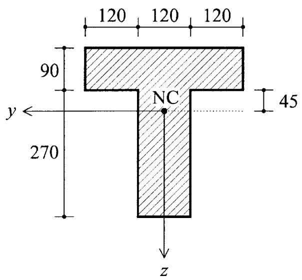
Question:

Determine the normal stresses in P, Q and R of the cross-section at A.

Core of the cross-section (Sections 4.9 and 4.10)

4.65 What do you understand by the *core of the cross-section*?

4.66: 1–3 You are given three different cross-sections.



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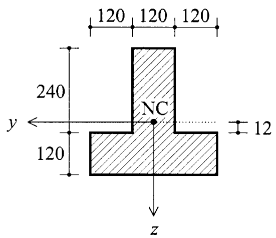

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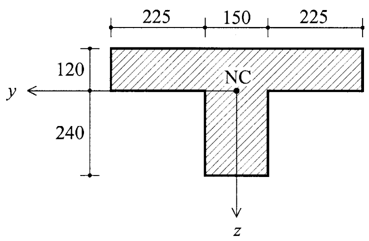


(3)

Questions:

a. Determine the location of the upper core point.
b. Determine the location of the lower core point.

4.67 A column has a solid circular cross-section with a diameter of 400 mm.

Question:
Determine the core radius of the cross-section.

4.68 In the cross-section of a thin-walled tube with an area $A = 2500\ \text{mm}^2$, the resultant of all normal stresses is a tensile force of 200 kN of which the point of application is on the edge of the core.

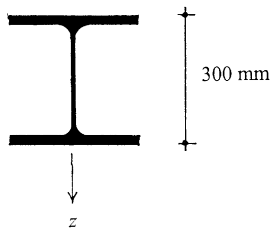
Questions:

a. Determine the maximum normal stress in the cross-section.
b. Plot the normal stress distribution in the cross-section.

4.69 For the steel I-section shown applies $A = 15 \times 10^3\ \text{mm}^2$ and $I_{zz} = 24.75 \times 10^6\ \text{mm}^4$.

Questions:

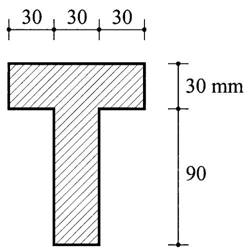
a. Plot the normal stress diagram if the resultant of all normal stresses is a tensile force of 495 kN with its point of application at the lower core point of the cross-section.
b. Determine the location of the lower core point.



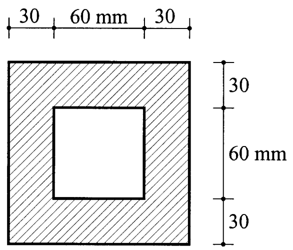
4.70 You are given the cross-section of a T-beam.

Questions:

a. Plot the normal stress diagram if a tensile force of 27 kN is applied at the upper core point.
b. Draw the normal stress diagram if a compressive force of 336 kN is applied at the lower core point.
c. Determine the location of the upper core point.
d. Determine the location of the lower core point.

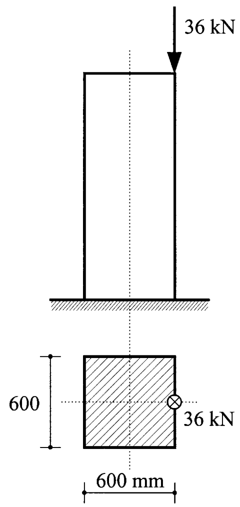


4.71 You are given a square hollow cross-section.



Question: Draw the core of this cross-section.

4.72 A block with square cross-section of 600×600 mm^2 has a mass of 3600 kg and is loaded at the middle of one of its sides by a compressive force of 36 kN. The stress distribution under the block is linear. Use $g = 10$ N/kg for the gravitational field strength.

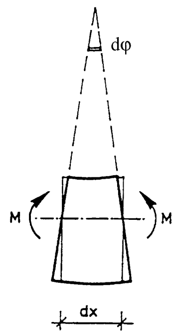


Questions:

a. Determine the maximum compressive stress if tensile stresses are allowed under the block. Plot the stress distribution under the block.
b. Determine the maximum compressive stress if no tensile stresses are allowed under the block. Plot the stress distribution under the block.

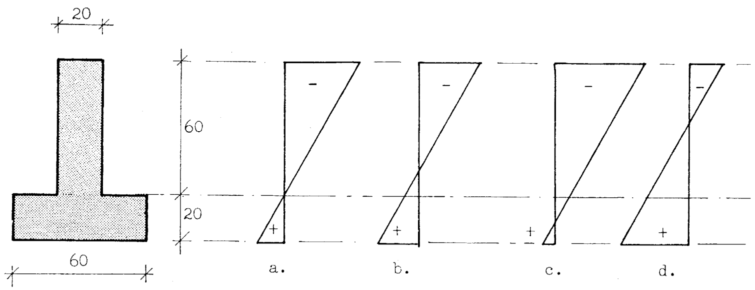
Mixed problems

4.73 You are given a small element with length dx from a member subject to bending. As a result of a bending moment M, the end cross-sections of the element rotate through an angle dφ with respect to one another. In the calculation use $M = 24$ kNm, d$x = 150$ mm and d$\varphi = 3 \times 10^{-3}$ rad.



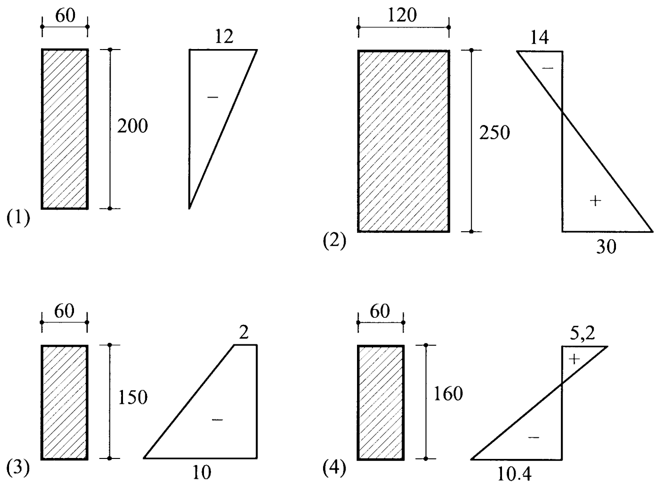
Question:
Determine the bending stiffness EI of the member.

4.74 The cross-section shown is subject to bending. The cross-sectional dimensions are in mm.



Question:
Which stress diagram could be correct?

4.75: 1–4 You are given four different rectangular cross-sections with associated normal stress diagrams. The cross-sectional dimensions are in mm, the stresses are in N/mm^2.

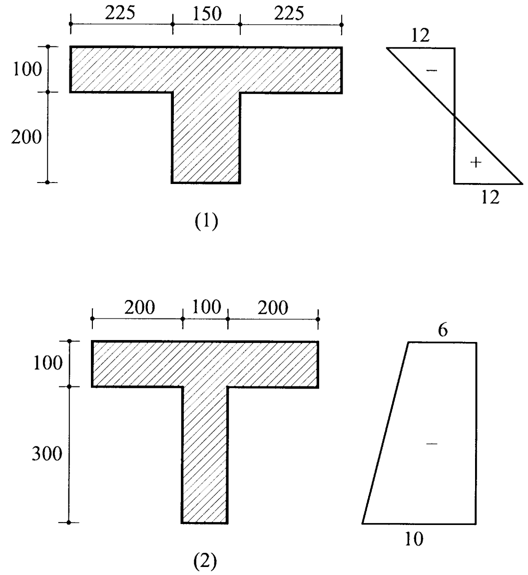


Questions:

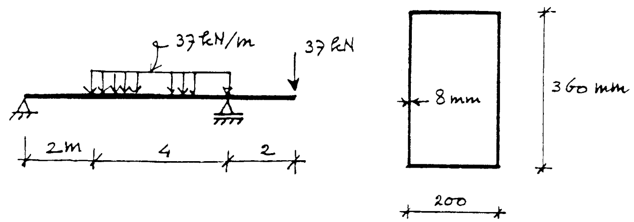
a. Determine the normal force in the cross-section, with the right sign.
b. Determine the bending moment in the cross-section, with the right bending symbol (⌣ or ⌢).

4.76: 1–2 You are given the cross-sections of two T-beams with associated normal stress diagrams. The cross-sectional dimensions are in mm and the stresses are in N/mm^2.

Question:
Determine the normal force N in the cross-section. Is this a tensile or compressive force?



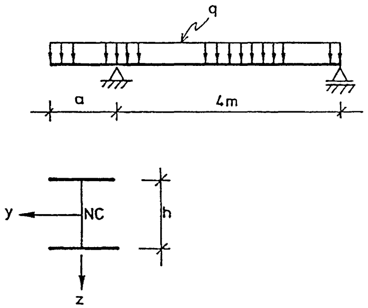
4.77 A thin-walled box with rectangular cross-section and uniform wall thickness has been used for the cantilevered beam shown.



Questions:

a. In which cross-section is the normal stress a maximum for the given load?
b. Plot the normal stress diagram for that cross-section.

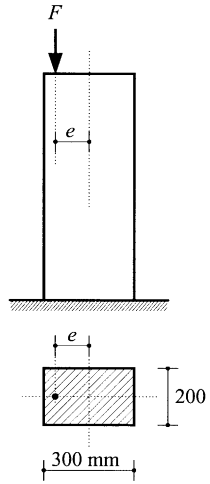
4.78 A uniformly distributed load $q = 26.1$ kN/m acts on the cantilevered beam shown. An I-section is used for the beam, with a height $h = 300$ mm and moment of inertia $I_{zz} = 270 \times 10^6$ mm^4.



Questions:

a. Determine the maximum bending stress in the beam if $a = 1$ m, and the location of the cross-section in which this stress occurs.
b. Determine the maximum bending stress in the beam if $a = 2$ m, and the location of the cross-section in which this stress occurs.

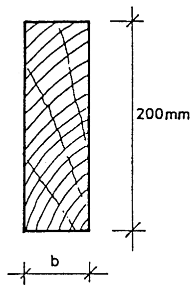
4.79 You are given a column with rectangular cross-section, loaded by an eccentric compressive force.



Questions:

a. How large may the eccentricity e be so that no tension occurs in the column.
b. For that value of e determine the maximum compressive stress in an arbitrary cross-section if $F = 150$ kN.

4.80 The given cross-section of a wooden joist is subject to a bending moment M and normal force N, acting in the vertical plane of symmetry. There are just no tensile stresses in the cross-section.

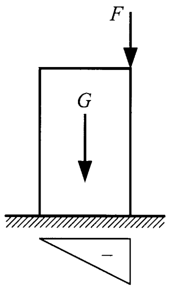


Questions:

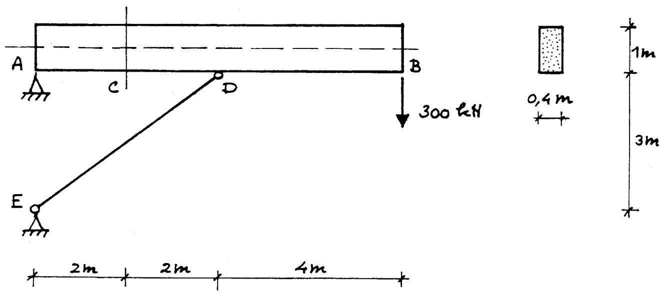
a. Determine the normal force N if $M = 3$ kNm.
b. Determine the maximum normal stress in the cross-section if $b = 75$ mm.

4.81 For the homogeneous block with rectangular cross-section, the given normal stress distribution at the base is the result of the dead weight $G = 48$ kN and the eccentric compressive force F.

Question:
Determine the magnitude of force F.



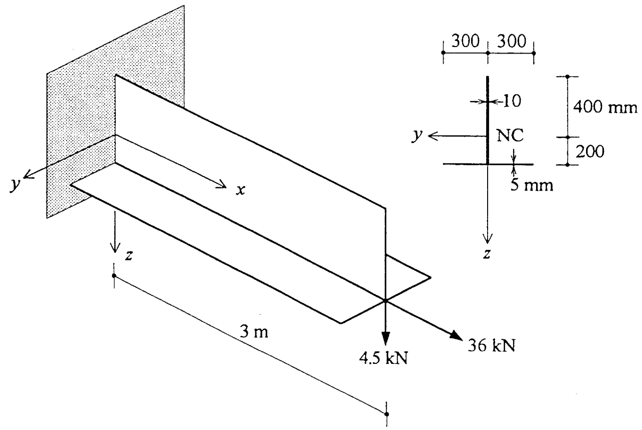
4.82 Beam ACDB has a rectangular cross-section and is supported at its lower side by a hinge at A and an oblique bar at D. The beam is loaded by a vertical force of 300 kN at the free end B.



Questions:

a. Model the isolated beam as a line element and draw all the forces acting on it.
b. Draw the M, V and N diagrams.
c. Draw the normal stress diagram for the cross-section at C.
d. Draw the normal stress diagram for the cross-section directly to the left of D.
e. Draw the normal stress diagram for the cross-section directly to the right of D.
f. Which of the diagrams (c, d or e) is closest to reality? Explain your answer.

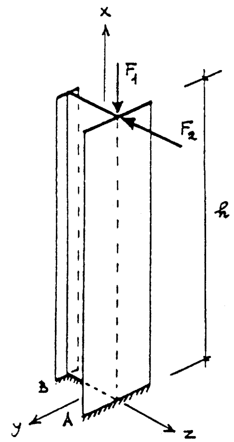
4.83 The cantilever beam with a length of 3 m has a thin-walled T-cross-section. The beam is loaded at its free end by two forces with their point of application at the centre of the flange: a horizontal tensile force of 36 kN and a vertical force $F_z = 4.5$ kN.

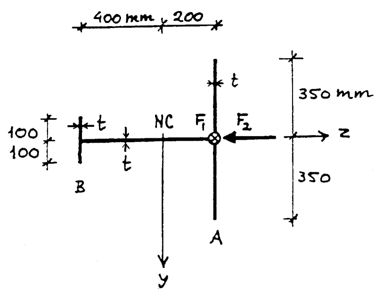


Questions:

a. Demonstrate the correctness of the location of the given normal centre NC.
b. Verify that $I_{yy} = 90 \times 10^6$ mm^4 and $I_{zz} = 360 \times 10^6$ mm^4.
c. Model the beam as a line element, draw all the forces acting on it, and draw the M and N diagrams.
d. Draw the normal stress diagram for the fixed cross-section at $x = 0$ m.
e. Draw the normal stress diagram for the cross-section at $x = 2$ m.

4.84 A thin-walled steel column with height h and uniform wall thickness t is fixed at its base. The column is loaded at its free end by forces F_1 and F_2 (see the figure). In the calculation use $F_1 = 315$ kN, $F_2 = 63$ kN, $h = 3$ m and $t = 10$ mm. For the cross-sectional quantities it is given that $A = 15 \times 10^3$ mm^2 and $I_{zz} = 840 \times 10^6$ mm^4. The location of the normal centre is given in the figure.



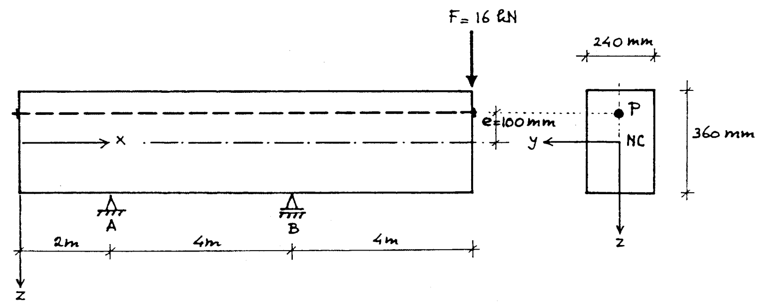


top view

Questions:

a. Model the column as a line element and draw all the forces acting on it, including the support reactions.
b. Draw the N and M diagrams, with the correct signs/symbols.
c. Determine and draw the normal stress diagram for the cross-section at the fixed end. In this cross-section, how far is the neutral axis from NC?
d. Determine and draw the normal stress diagram for the cross-section halfway up. In this cross-section, how far is the neutral axis from NC?

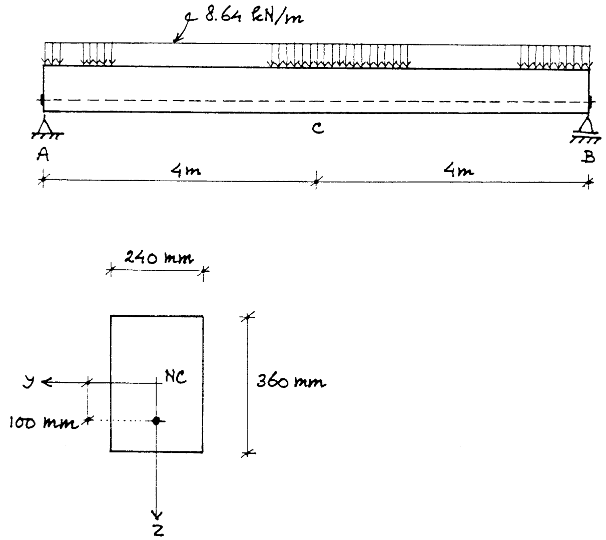
4.85 The cantilevered beam with rectangular cross-section is prestressed by means of a straight tendon at P, with an eccentricity $e = 100$ mm. The prestressing force F_p is unknown. All other data can be found from the figure.



Questions:

a. Draw the M diagram as a result of only force $F = 16$ kN, with the deformation symbols.
b. Draw the N and M diagrams due to the unknown prestressing force F_p (express the values in F_p and e), with the deformation symbols.
c. Determine the minimum prestressing force F_p for which no tension occurs in the cross-section at support B.
d. Draw the N and M diagrams due to $F = 16$ kN and F_p with the value determined in (c).
e. Draw the normal stress diagram for the cross-section at support B.
f. Draw the normal stress diagram for the cross-section at support A.

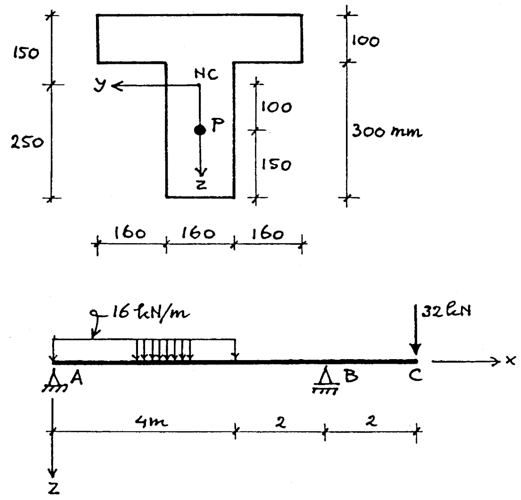
4.86 The simply supported prestressed beam has a rectangular cross-section and carries a uniformly distributed load across its entire length. The prestressing cable is straight and located at ($y = 0$; $y = 100$ mm). The prestressing force F_p is unknown.



Questions:

a. Determine the minimum prestressing force F_p for which no tension occurs in the cross-section at midspan C.
b. Draw the normal stress diagram for the cross-section at C.
c. Draw the normal stress diagram for the cross-section at A.
d. Are both diagrams (in A and C) equally realistic? Explain your answer.

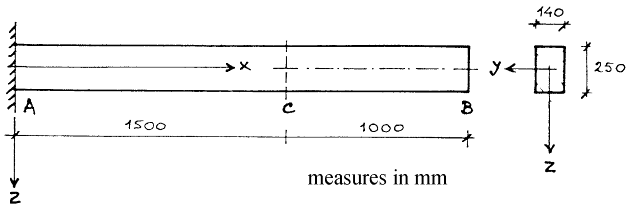
4.87 The T-beam shown is prestressed with a straight prestressing bar at P. The prestressing force is 240 kN. All other data can be found in the figure.



Questions:

a. Determine the N and M diagrams due to the load and prestressing together.
b. In which cross-section is the tensile stress a maximum? Draw the normal stress diagram for that cross-section.
c. In which cross-section is the compressive stress a maximum? Draw the normal stress diagram for this cross-section.

4.88 Cantilever beam AB, with rectangular cross-section, is fixed at A. At its free end B the beam is loaded in the xz plane by a (horizontal) compressive force F. The point of application of F is unknown. The compressive force F generates a compressive stress of 12 N/mm^2 in the top fibre layer of the cross-section at C, and a zero stress in the bottom fibre layer of that cross-section.

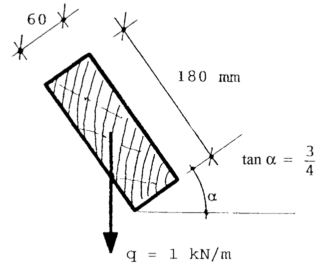


Questions:
a. Draw the normal stress diagram for the cross-section at C.
b. Determine the normal force N and bending moment M for the cross-section at C.
c. How large is the compressive force F and where in the cross-section B is its point of application?
d. Draw the N and M diagrams due to F.

If there is also a uniformly distributed load q_z acting over the entire length of the beam, there are some more questions:
e. Determine the maximum value q_z whereby no tensile stresses occur in the beam.
f. Draw the N and M diagrams due to the values determined for F and q_z.
g. Draw the normal stress distribution for the cross-section at A.
h. Draw the normal stress distribution for the cross-section at C.

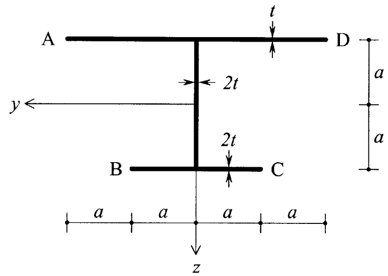
4.89 A simply supported 3.6 m purlin is carrying a uniformly distributed load of 1 kN/m over its entire length.

Questions:
a. In which cross-section is the bending stress a maximum?



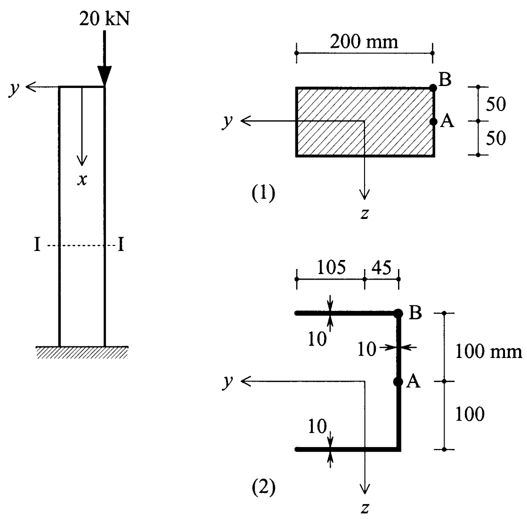
b. Determine the magnitude of this maximum bending stress.
c. For this cross-section, determine the normal stresses in the four corners.
d. Draw the normal stress diagram.
e. Sketch the neutral axis in the cross-section.

4.90 The given thin-walled cross-section is subject to the bending moments $M_y = -80\sigma a^2 t$ and $M_z = +52\sigma a^2 t$.



Questions:
a. Determine the normal stresses at corners A to D, expressed in σ.
b. Draw the normal stress distribution in the cross-section.
c. Sketch the neutral axis in the cross-section.

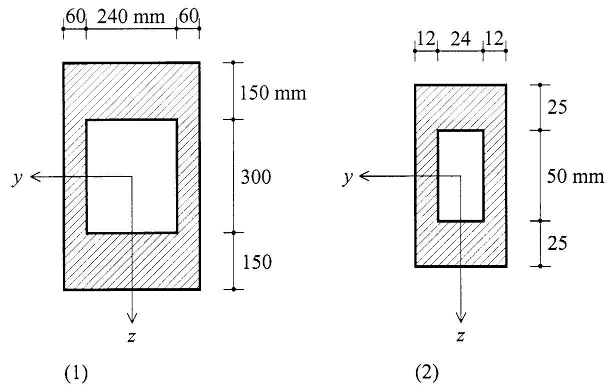
4.91: 1–2 Two columns are loaded by an eccentric compressive force of 20 kN. Column (1) has a rectangular cross-section and column (2) has a thin-walled U-cross-section.



Questions:
a. Determine the normal stress at the corners of cross-section I–I if the compressive force is applied at A. Where in the cross-section is the neutral axis?
b. Determine the normal stress at the corners of cross-section I–I if the compressive force is applied at B. Where in the cross-section is the neutral axis?

4.92: 1–2 You are given two symmetrical rectangular hollow cross-sections. The cross-sections are subject to bending moments M_y and M_z. Use in the calculation the following values for

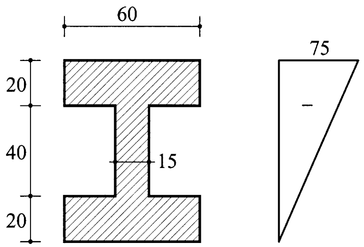
- cross-section (1): $M_y = 712.8$ kNm and $M_z = 648$ kNm.
- cross-section (2): $M_y = 3.6$ kNm and $M_z = 1.8$ kNm.

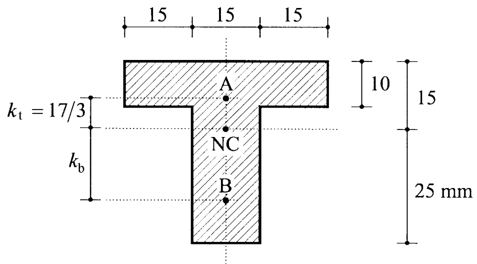


Questions:
a. Determine section modulus W_y.
b. Determine the maximum bending stress due to the bending moment M_y.
c. Determine section modulus W_z.
d. Determine the maximum bending stress due to bending moment M_z.

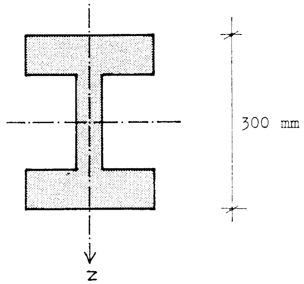
4.93 You are given the cross-section of a beam, with associated normal stress distribution. The cross-sectional dimensions are in mm and the stresses in N/mm^2.

Questions:
a. Determine the section modulus for the cross-section.
b. Determine the normal force N in the cross-section, with the right sign.
c. Determine the bending moment M in the cross-section, with the right deformation symbol ($\smile$ or $\frown$).





4.94 For the given cross-section it applies that $A = 20 \times 10^3$ mm^2 and $I_{zz} = 240 \times 10^6$ mm^4.



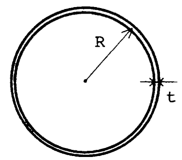
Questions:

a. Draw the normal stress diagram if the resultant of all stresses is a tensile force of 60 kN that has its point of application at the upper core point.
b. Determine the distance from the upper core point to the normal centre of the cross-section.

4.95 The location of the normal centre NC and upper core point A are shown in the given cross-section.

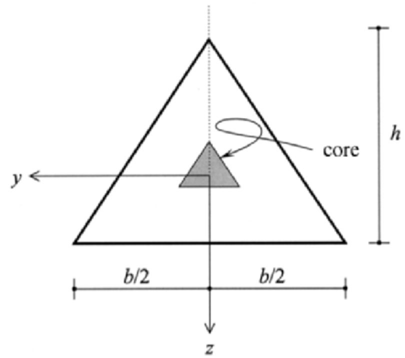
Question:
Determine the location of lower core point B, i.e. the lower core radius k_b.

4.96 You are given the cross-section of a thin-walled circular steel tube, with radius R and wall thickness t.



Question:
Determine the core radius of the cross-section, to be expressed in R and t.

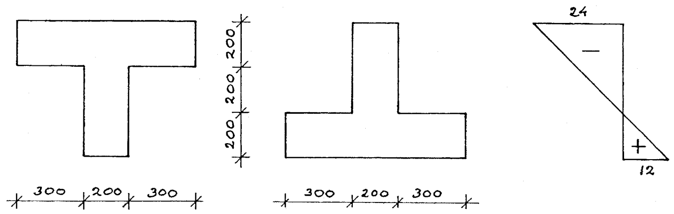
4.97 You are given a solid isosceles triangular cross-section with base b and height h.



Questions:

a. Determine section modulus $W_{z;b}$.
b. Determine section modulus $W_{z;t}$.
c. Determine the location of the upper core point.
d. Determine the location of the bottom edge of the core.

4.98: 1–2 You are given two homogeneous cross-sections with the same normal stress diagram. The cross-sectional dimensions are given in mm and the stresses in N/mm^2.

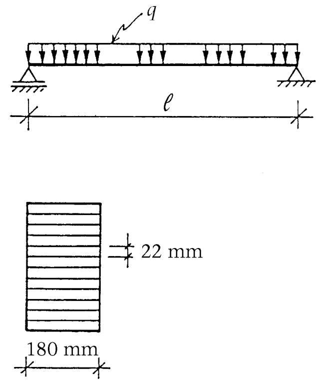


(1) (2)

Questions:

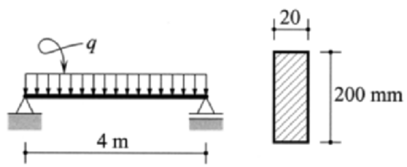
a. Determine the location of the normal centre.
b. From the stress diagram given, determine the magnitude and point of application of resultant R_t of all tensile stresses.
c. From the stress diagram given, determine the magnitude and point of application of resultant R_c of all the compressive stresses.
d. From the magnitudes and the points of application of the stress resultants R_t and R_c determine the normal force N and the bending moment M in the cross-section, with the correct signs or deformation symbols.
e. Where in the cross-section is the centre of force?
f. If $E = 30 \times 10^3$ N/mm^2, draw the deformation diagram and determine from this diagram the deformation quantities ε and κ.
g. Using the answers from (d) and (f), determine the magnitude of the axial and bending stiffness at the cross-section.

4.99 A simply supported wooden beam is constructed of n planks, 22 mm thick, that have been glued together. The beam has a span of $\ell = 3$ m and carries a uniformly distributed load $q = 24$ kN/m.



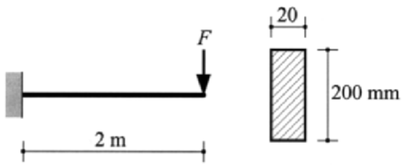
Question:
Determine the minimum number of planks n for which the bending stress is no larger than 10 N/mm^2.

4.100 You are given a simply supported wooden beam with rectangular cross-section and a uniformly distributed load q.



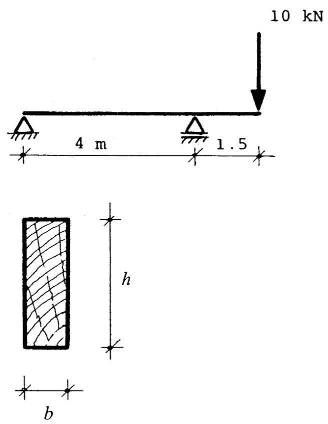
Question:
Determine the uniformly distributed load q for which the maximum bending stress is 6 N/mm^2.

4.101 You are given a wooden cantilever beam with rectangular cross-section, and loaded at its free and by a force F.



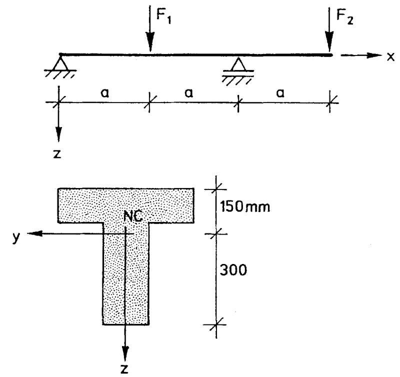
Question:
Determine the force F for which the maximum bending stress in the beam is 10 N/mm^2.

4.102 The cantilevered beam shown has a rectangular cross-section with a height h that is three times the width b. For the given load, the bending stress may not exceed the value of 10 N/mm^2 at any point.



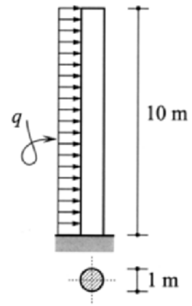
Question:
Determine the minimum height h of the beam.

4.103 A T-beam, with $a = 2$ m, is loaded as shown by forces $F_1 = 195$ kN and $F_2 = 45$ kN. The centroidal moment of inertia I_{zz} of the cross-section is 1.8×10^9 mm^4.



Questions:
a. Determine the maximum bending tensile stress in the beam.
b. Determine the maximum bending compressive stress in the beam.

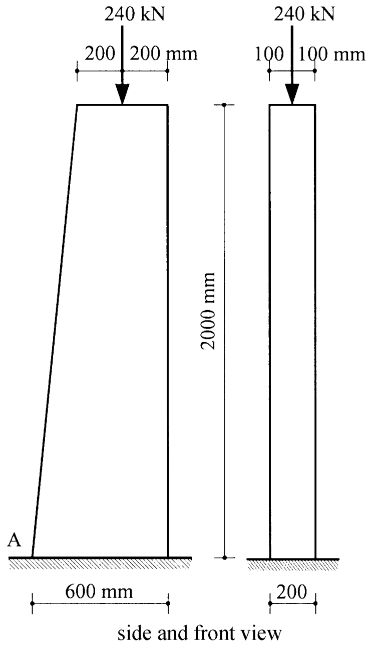
4.104 You are given a 10 m marble column fixed in a foundation and free at the other end. The column has a circular cross-section with diameter $d = 1$ m. The dead weight of marble is 27.5 kN/m^3. No tensile stresses are allowed in marble. The column is subject to a uniformly distributed wind load q.



Questions:

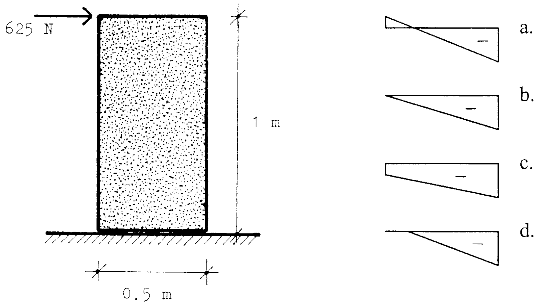
a. Determine the maximum wind load q.
b. Determine the normal stress distribution in the cross-section at the fixed end due to the wind load and the dead weight.
c. Draw the normal stress distribution in the cross-section at half-height due to wind load and the dead weight.

4.105 The tapered column shown is loaded by a compressive force of 240 kN at the centroid of the upper cross-section. The dead weight of the column is ignored. To determine the normal stresses in a cross-section, you can use the stress formula derived for a prismatic member.



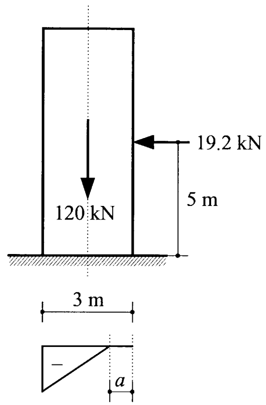
Question:
Determine the normal stress at A.

4.106 A block with $0.5 \times 0.5\ m^2$ base and height of 1 m is standing freely on the floor. The dead weight of the block is 5 kN. The block is pushed by a horizontal force of 625 N. The block cannot slide. The normal stresses between block and floor are assumed linear.



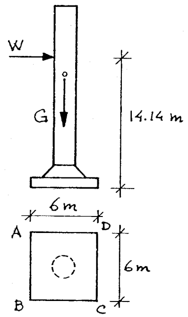
Question:
Which of the diagrams for the stress distribution between block and floor may be correct?

4.107 The soil under the square $3 \times 3\ m^2$ foundation plate cannot transfer tensile stresses. The stress distribution for the given load is shown.



Questions:
a. Determine the distance *a* over which there is no earth pressure.
b. Determine the maximum earth pressure under the foundation plate.

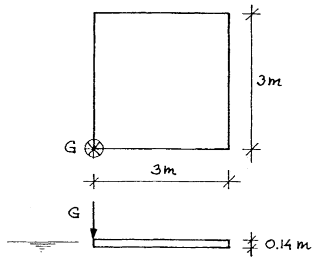
4.108 A tower-like structure is rigidly fixed to a square 6×6 m^2 foundation plate. The foundation plate is assumed to be rigid. The soil should be seen as an elastic layer, generating a counter pressure that is proportional to its compression The total weight of the structure with foundation plate is G. The resultant of the wind load is W and acts 14.14 m above the base of the foundation plate. The earth pressure on the foundation plate due to G and W is 14.65 kN/m^2 along AB and 85.35 kN/m^2 along CD.



Questions:

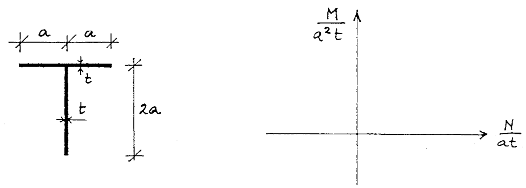
a. Determine G and W from the given earth pressure.
b. Determine the earth pressure under the foundation plate due to G and W when the wind load W acts in the direction AC. Use for G and W the values found in question (a).
c. Which of the wind directions intended under (a) and (b) is least favourable with respect to the strength of the foundation?

4.109 A solid wooden raft with a square surface of 3×3 m^2 and thickness of 0.14 m is floating in still water. Assume that the raft does not deform. A person is standing still (with bare feet) at one of the corners. Mass density water: 1000 kg/m^3. Mass density wood: 500 kg/m^3. Assume the gravitational field strength is 10 N/kg.



Question:
What weight G is allowed so that the person does not get his feet wet?

4.110 You are given the thin-walled T-section ($t \ll a$).



Questions:

a. In a diagram, plot all the combinations of M and N for which the maximum tensile stress and/or compressive stress has reached the limiting value $\bar{\sigma}$.
b. In the diagram, hatch the area with the combinations M and N for which neither the tensile stress nor the compressive stress exceeds the limiting value $\bar{\sigma}$.

5 Shear Forces and Shear Stresses Due to Bending

In the previous chapter, we looked (*inter alia*) at the normal stress distribution in the cross-section of a member subject to extension and bending. The normal stress distribution in a cross-section is directly related to the normal force and the bending moment.

When the bending moment in a beam is not constant the beam must also transfer shear forces. There are not only *shear forces and shear stresses in the cross-sectional planes* of the beam, but also *in longitudinal section planes*.

In Section 5.1 we look at the shear forces and shear stresses in a longitudinal section plane and we derive the associated formulas. Some examples are given in Section 5.2.

In Section 5.3 we derive the formulas for the shear stresses in a cross-sectional plane. The application of these formulas is illustrated by a number of examples in Section 5.4. Each of the examples contains something noteworthy.

In Section 5.5 we address the concept of the *shear force centre* or *shear centre* SC. The shear centre is that point in the cross-section through which the line of action of the shear force must pass so that there will be no torsion.

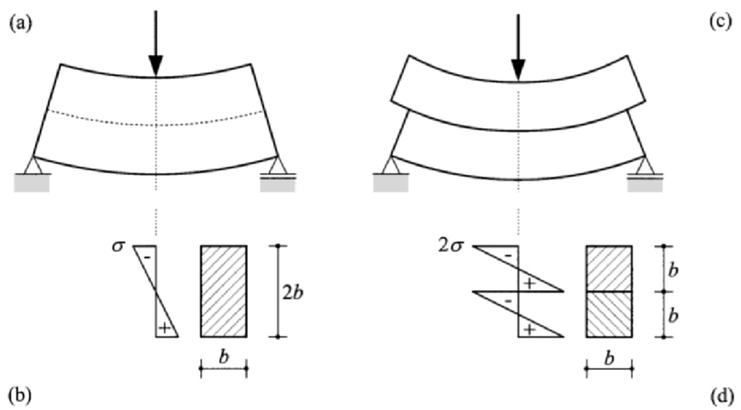


Figure 5.1 (a) A rectangular beam, built up of two square beams joined together rigidly by a glued joint, behaves as a solid beam. (b) The associated bending stress diagram in the cross-section at midspan. (c) The square beams are separate and can move with respect to one another. (d) The associated bending stress diagram for the cross-section at midspan.

In Section 5.6 a summary is given of the various formulas and rules.

Chapter 5 ends with a number of problems in Section 5.7.

When deriving and applying the various formulas, we consistently use signs to indicate the directions of the various quantities. In practice these signs are often omitted and one works with the absolute values. The directions are then (if necessary) determined afterwards on the basis of insight, experience and common sense.

5.1 Shear forces and shear stresses in longitudinal direction

The fact that shear forces can occur in the longitudinal direction of beams subject to bending is illustrated by the simply supported beam in Figure 5.1. The beam with a rectangular cross-section is built up of two square beams, and is loaded by a force at midspan.

In Figure 5.1a both square beams are rigidly joined, for example by means of a glued joint, and they work together fully: the rectangular beam behaves as a solid beam. The normal stress diagram for the cross-section at midspan is shown in Figure 5.1b.

In Figure 5.1c the square beams are separate and can move with respect to one another. The bottom fibres of the upper beam lengthen, while the top fibres of the lower beam shorten. Figure 5.1d shows the normal stress diagram at midspan. If each square beam is carrying half of the total load, the maximum bending stresses are twice as large as those in the stress diagram in Figure 5.1b. The calculation is left to the reader.

If both square beams work together fully, longitudinal shear forces (interaction forces) will have to act in the glued joint in order to eliminate the difference in length between the deformed adjacent fibres of both square beams (see Figure 5.2).

In this section we derive the formulas for the shear force and shear stresses in a longitudinal section plane.

5.1.1 Change of the normal stress in a fibre

In preparation of determining the magnitude and distribution of the shear forces in the longitudinal direction, we first determine how the normal stress in a fibre changes between two consecutive cross-sections a-a and b-b at a small distance Δx ($\Delta x \to 0$) (see Figure 5.3).

The normal stress $\sigma(z)$ in a z fibre layer follows from the stress formula

$$\sigma(z) = \frac{N}{A} + \frac{M_z z}{I_{zz}}. \tag{5.1}$$

Here it must be noted that the formula applies only when

- the x axis passes through the normal centre NC, and
- the z direction is a principal direction.

The normal stress distribution in cross-section a-a will generally differ from that in cross-section b-b. Assume $\sigma(z)$ is the normal stress in a z fibre at cross-section a-a and $\sigma(z) + \Delta\sigma(z)$ is the normal stress in the same fibre at cross-section b-b. In the limit $\Delta x \to 0$, the change per length of the normal stress in the fibre considered is

$$\lim_{\Delta x \to 0} \frac{\Delta\sigma(x)}{\Delta x} = \frac{\mathrm{d}\sigma(z)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{N}{A} + \frac{M_z z}{I_{zz}}\right). \tag{5.2}$$

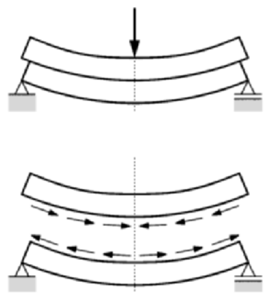


Figure 5.2 If both square beams work together fully, longitudinally shear forces (interaction forces) have to act between both beams to eliminate the difference in length between the adjacent fibres at the glued joint.

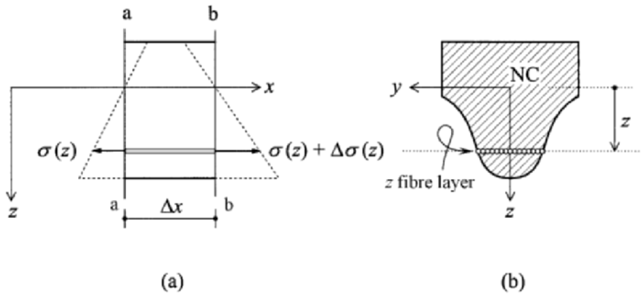


Figure 5.3 (a) A beam segment between the cross-sections a-a and b-b, with small length Δx ($\Delta x \to 0$). The normal stress in a fibre is not constant but changes over the length Δx. (b) The cross-section of the beam segment; all fibres in the z fibre layer have the same normal stress.

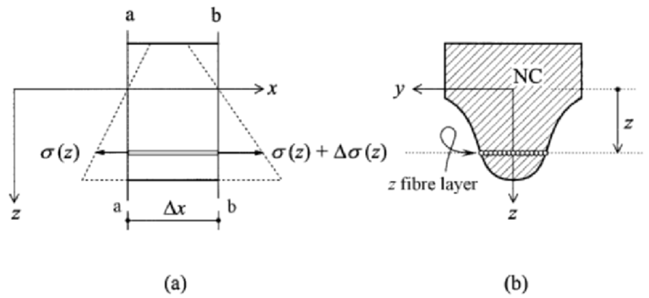


Figure 5.3 (a) A beam segment between the cross-sections a-a and b-b, with small length Δx ($\Delta x \to 0$). The normal stress in a fibre is not constant but changes over the length Δx. (b) The cross-section of the beam segment; all fibres in the z fibre layer have the same normal stress.

Below, we will use the following two assumptions:

- the beam is prismatic, and
- the normal force is constant.

If the beam is prismatic, the cross-sectional properties A and I_{zz} are constant, i.e. independent of the x coordinate. In that case, (5.2) can be written

$$\frac{\mathrm{d}\sigma(z)}{\mathrm{d}x} = \frac{1}{A}\frac{\mathrm{d}N}{\mathrm{d}x} + \frac{z}{I_{zz}}\frac{\mathrm{d}M_z}{\mathrm{d}x}.$$

If N is also constant, i.e. independent of x, then $\mathrm{d}N/\mathrm{d}x = 0$. With $\mathrm{d}M_z/\mathrm{d}x = V_z$ the change per length of the normal stress in a z fibre is now

$$\frac{\mathrm{d}\sigma(z)}{\mathrm{d}x} = \frac{z}{I_{zz}}\frac{\mathrm{d}M_z}{\mathrm{d}x} = \frac{V_z z}{I_{zz}}. \tag{5.3}$$

The change in normal stress in a z fibre is directly related to the change in the bending moment, i.e. to the shear force.

Note: Since the derivation is based on stress formula (5.1), formula (5.3) applies only when

- the x axis passes through normal centre NC, and
- the z direction is a principal direction.

Next we determine the shear force in the longitudinal direction.

5.1.2 Shear force in the longitudinal direction (traditional formula)

From the small beam segment in Figure 5.4a, with length Δx, the lower part has been isolated. This part is referred to as the *shearing element* or *sliding element*. Quantities relating to the sliding element hereafter have an upper index "a". In this way, A^{a} is the cross-sectional area of the sliding element (see Figure 5.4b). Figure 5.4c gives a spatial representation of the sliding element.

Note: The longitudinal section plane may be curved in the transverse direction.

Assume the resultant of all normal stresses on the back of the sliding element is N^{a} and that on the front is $N^{\mathrm{a}} + \Delta N^{\mathrm{a}}$ (see Figures 5.4a and 5.4c). Here

$$N^{\mathrm{a}} = \int_{A^{\mathrm{a}}} \sigma(z)\,\mathrm{d}A. \qquad (5.4)$$

Since the resultants of the normal stresses on the front and back of the sliding element are not equal, a longitudinal shear force must act on the longitudinal section plane.

Assume s_x^{a} is the shear force per length in the longitudinal direction, acting on the sliding element. The total shear force on the sliding element, with small length Δx, then equals $s_x^{\mathrm{a}}\Delta x$.

Note: *Here we assume that the longitudinal shear forces and shear stresses, acting on the sliding element, are positive if they act in the positive x direction.*

The sliding element has to meet the conditions of force equilibrium in x direction:

$$\sum F_x = -N^{\mathrm{a}} + (N^{\mathrm{a}} + \Delta N^{\mathrm{a}}) + s_x^{\mathrm{a}}\Delta x = 0.$$

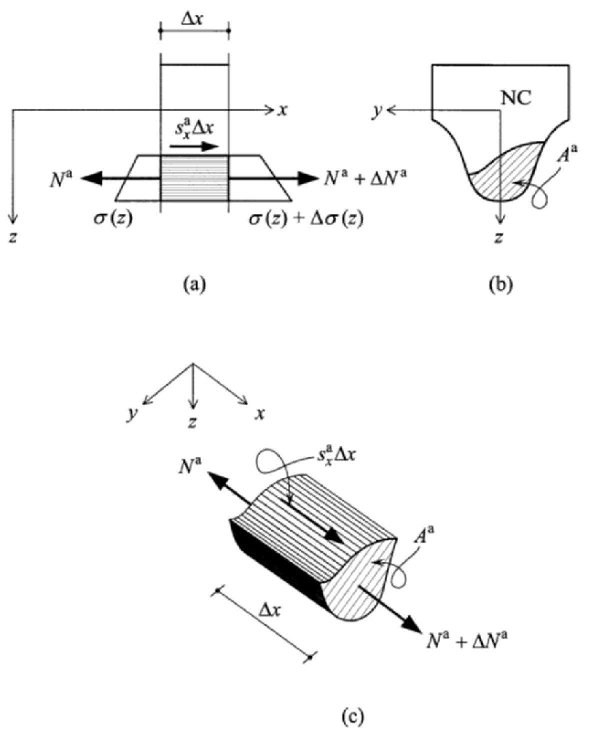


Figure 5.4 (a) The lower part of the beam segment, with small length Δx, has been isolated and is called the sliding element. (b) The cross-section of the sliding element has an area A^{a}. (c) Spatial representation of the sliding element with all forces acting on it. Since the resultant of all the normal stresses on the front and back of the sliding element are not equal, a longitudinal shear force must act on the longitudinal section plane.

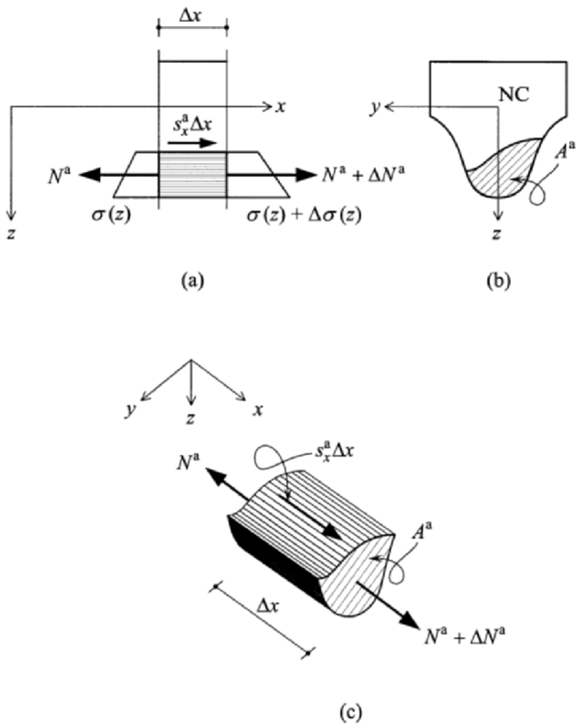


Figure 5.4 (a) The lower part of the beam segment, with small length Δx, has been isolated and is called the sliding element. (b) The cross-section of the sliding element has an area A^{a}. (c) Spatial representation of the sliding element with all forces acting on it. Since the resultant of all the normal stresses on the front and back of the sliding element are not equal, a longitudinal shear force must act on the longitudinal section plane.

In the limit $\Delta x \to 0$ we find

$$s_x^{\mathrm{a}} = \lim_{\Delta x \to 0} \frac{\Delta N^{\mathrm{a}}}{\Delta x} = -\frac{\mathrm{d}N^{\mathrm{a}}}{\mathrm{d}x}. \tag{5.5}$$

$\mathrm{d}N^{\mathrm{a}}/\mathrm{d}x$ is found by differentiating (5.4):

$$\frac{\mathrm{d}N^{\mathrm{a}}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\int_{A^{\mathrm{a}}} \sigma(z)\,\mathrm{d}A = \int_{A^{\mathrm{a}}} \frac{\mathrm{d}\sigma(x)}{\mathrm{d}x}\,\mathrm{d}A.$$

Differentiating with respect to the longitudinal direction and integrating with respect to the cross-sectional area of the sliding element are two independent operations that may be interchanged.

Using (5.3) we now find

$$\frac{\mathrm{d}N^{\mathrm{a}}}{\mathrm{d}x} = \int_{A^{\mathrm{a}}} \frac{\mathrm{d}\sigma(x)}{\mathrm{d}x}\,\mathrm{d}A = \int_{A^{\mathrm{a}}} \frac{V_z z}{I_{zz}}\,\mathrm{d}A = \frac{V_z}{I_{zz}}\int_{A^{\mathrm{a}}} z\,\mathrm{d}A = \frac{V_z S_z^{\mathrm{a}}}{I_{zz}}. \tag{5.6}$$

In (5.6), in the last but one step, V_z and I_{zz} have been placed outside the integral as they apply to the entire cross-section. The remaining integral

$$\int_{A^{\mathrm{a}}} z\,\mathrm{d}A$$

is equal to the *static moment of the cross-sectional area of the sliding element* and is denoted by S_z^{a}.

Substituting (5.6) into (5.5) gives the following formula for s_x^{a}, the shear force per length in the longitudinal direction:

$$s_x^{\mathrm{a}} = -\frac{\mathrm{d}N^{\mathrm{a}}}{\mathrm{d}x} = -\frac{V_z S_z^{\mathrm{a}}}{I_{zz}}. \tag{5.7}$$

The (distributed) longitudinal shear force s_x^{a} is directly related to the magnitude of cross-sectional shear force V_z. The minus sign is the result of the assumption that a longitudinal shear force is positive if it acts on the sliding element in the positive x direction.

The derivation shows that s_x^{a}, the shear force per length in the longitudinal direction, is independent of the (constant) normal force in the beam.

Alternative derivation
The following alternative derivation is based on a beam without normal force. In that case the normal force N^{a} is the resultant of all normal stresses $\sigma^M(z)$ on the sliding element due to bending.

For the normal force N^{a} on the sliding element at cross-section a-a (see Figure 5.5)

$$N^{\mathrm{a}} = \int_{A^{\mathrm{a}}} \sigma^M(z)\,\mathrm{d}A.$$

Substitute the normal stress distribution due to bending in this equation:

$$\sigma^M(z) = \frac{M_z z}{I_{zz}},$$

and we find

$$N^{\mathrm{a}} = \int_{A^{\mathrm{a}}} \frac{M_z z}{I_{zz}}\,\mathrm{d}A = \frac{M_z}{I_{zz}} \int_{A^{\mathrm{a}}} z\,\mathrm{d}A = \frac{M_z S_z^{\mathrm{a}}}{I_{zz}}.$$

Here

$$S_z^{\mathrm{a}} = \int_{A^{\mathrm{a}}} z\,\mathrm{d}A$$

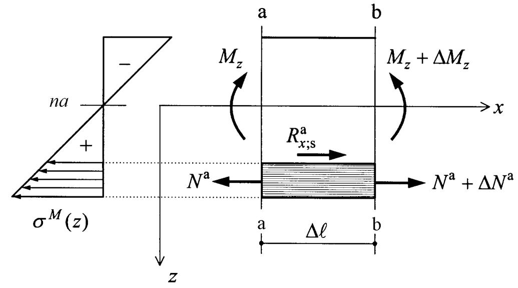


Figure 5.5 The longitudinal shear force $R_{x;s}^{\mathrm{a}}$ is found from the force equilibrium of the sliding element in the x direction. This shear force is proportional to the change in the bending moment between the front and back cross-sections of the beam segment considered.

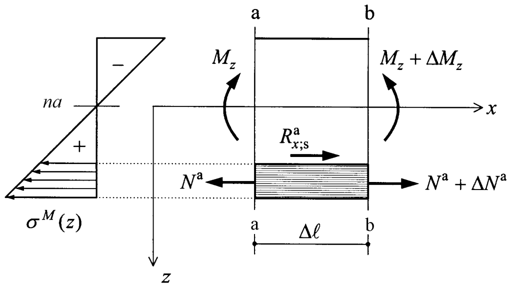


Figure 5.5 The longitudinal shear force $R^{a}_{x;s}$ is found from the force equilibrium of the sliding element in the x direction. This shear force is proportional to the change in the bending moment between the front and back cross-sections of the beam segment considered.

is the static moment of the cross-sectional area of the sliding element.

If at cross-section a-a the normal force N^{a} on the sliding element changes to $N^{a} + \Delta N^{a}$ at cross-section b-b (see Figure 5.5), the change ΔN^{a} is the result of the change in the bending moment M_z by a quantity ΔM_z. In other words:

$$\Delta N^{a} = \frac{\Delta M_z S^{a}_z}{I_{zz}}. \tag{5.8}$$

Figure 5.6 gives a spatial representation of the sliding element of the beam, with a small length $\Delta\ell$. s^{a}_x is the shear force per length in the longitudinal direction, and positive when it acts in the positive x direction on the sliding element (see Figure 5.6a). $R^{a}_{x;s}$ is the resultant shear force in the longitudinal direction, also positive when it acts in the positive x direction on the sliding element (see Figure 5.6b).

The resultant shear force $R^{a}_{x;s}$ follows directly from the force equilibrium in the x direction (see Figures 5.5 and 5.6b):

$$\sum F_x = -N^{a} + R^{a}_{x;s} + (N^{a} + \Delta N^{a}) = 0 \Rightarrow R^{a}_{x;s} = -\Delta N^{a}.$$

Substitute expression (5.8) for ΔN^{a} and we find

$$R^{a}_{x;s} = -\frac{\Delta M_z S^{a}_z}{I_{zz}}. \tag{5.9}$$

Conclusion: *The resultant shear force in longitudinal direction is proportional to the difference between the bending moments on the front and back cross-section of the beam segment considered.*

Note: Formula (5.9) applies only if the x axis passes through the normal centre NC and the z direction is a principal direction.

If the length $\Delta\ell$ of the sliding element is large:

$$R^{\mathrm{a}}_{x;s} = \int_{\Delta\ell} s^{\mathrm{a}}_x \,\mathrm{d}x. \tag{5.10}$$

If $\Delta\ell = \Delta x$ is small:

$$R^{\mathrm{a}}_{x;s} = s^{\mathrm{a}}_x \Delta x = -\frac{\Delta M_z S^{\mathrm{a}}_z}{I_{zz}},$$

or

$$s^{\mathrm{a}}_x = -\frac{\Delta M_z}{\Delta x}\frac{S^{\mathrm{a}}_z}{I_{zz}}.$$

In the limiting case $\Delta x \to 0$ this again leads to (5.7):

$$s^{\mathrm{a}}_x = -\lim_{\Delta x\to 0}\frac{\Delta M_z}{\Delta x}\frac{S^{\mathrm{a}}_z}{I_{zz}} = -\frac{\mathrm{d}M_z}{\mathrm{d}x}\frac{S^{\mathrm{a}}_z}{I_{zz}} = -\frac{V_z S^{\mathrm{a}}_z}{I_{zz}}. \tag{5.7}$$

5.1.3 Shear force in the longitudinal direction (alternative formula)

In Section 5.1.2 we derived

$$s^{\mathrm{a}}_x = -\frac{\mathrm{d}N^{\mathrm{a}}}{\mathrm{d}x}. \tag{5.5}$$

To keep matters simple, it is again assumed that there is no normal force and that N^{a} is the resultant of all normal stresses $\sigma^M(z)$ on the sliding element

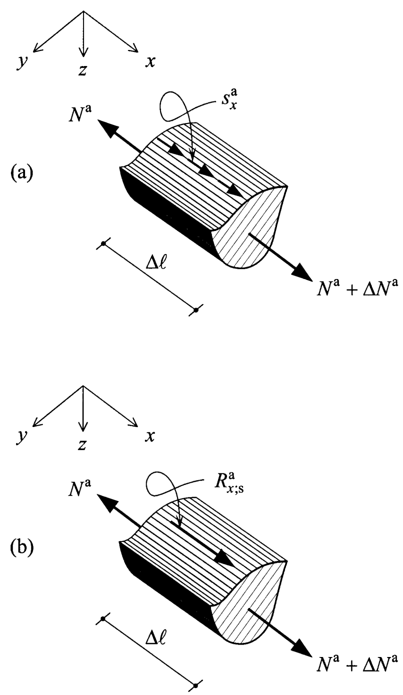


Figure 5.6 Spatial representation of the sliding element with all forces acting on it: (a) s^{a}_x is the shear force per length on the longitudinal section plane; (b) $R^{\mathrm{a}}_{x;s}$ is the resultant shear force on the longitudinal section plane.

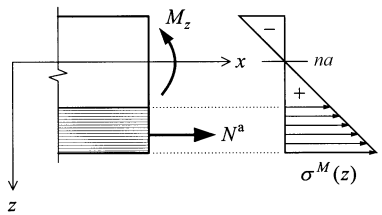


Figure 5.7 If $N^{\rm a}$ is the resultant of all the bending stresses $\sigma^M(z)$ on the sliding element, then this force is proportional to the bending moment M_z. So we can write $N^{\rm a} = k \cdot M_z$. Here k is a proportionality factor determined by the shape of the of the cross-section of the sliding element.

due to bending only (see Figure 5.7):

$$N^{\rm a} = \int_{A^{\rm a}} \sigma^M(z)\,{\rm d}A.$$

This normal force $N^{\rm a}$ on the sliding element is proportional to the magnitude of the bending moment M_z in the cross-section in question. We can therefore say

$$N^{\rm a} = k \cdot M_z. \tag{5.11}$$

Here, k is a proportionality factor of which the magnitude is determined entirely by the shape of the cross-section of the sliding element.

Since $k = N^{\rm a}/M_z$ is constant, this relationship can be determined from (5.11) for any *arbitrary value* of $M_z = M_z^*$. Hence

$$k = \frac{N^{\rm a}\ (\text{due to } M_z^*)}{M_z^*}.$$

By differentiating (5.11) with respect to x we find

$$\frac{{\rm d}N^{\rm a}}{{\rm d}x} = k \cdot \frac{{\rm d}M_z}{{\rm d}x} = k \cdot V_z = \left[\frac{N^{\rm a}\ (\text{due to } M_z^*)}{M_z^*}\right] \cdot V_z.$$

Substituting this in (5.5) leads to the following expression for $s_x^{\rm a}$, the shear force per length in the longitudinal direction:

$$s_x^{\rm a} = -V_z \cdot \left[\frac{N^{\rm a}\ (\text{due to } M_z^*)}{M_z^*}\right]. \tag{5.12}$$

Comment: In deriving this alternative formula for the shear stress per length in the longitudinal direction, we have not used any property bound to the principal directions. Therefore, expression (5.12) also applies if the y,z directions do not coincide with the principal directions of the cross-section. However, there is the limitation that M^*, the *resultant bending moment* in the cross-section, must act in the same plane as the *resultant shear force* V_z, as otherwise the differentiation of (5.11) is no longer valid.[1]

From formula (5.12) we can directly derive the fact that the shear force s_x^a is a maximum when N^a, the resultant of all normal stresses on the sliding element due to the bending moment M_z^*, is a maximum. This is in the planar longitudinal section through the neutral axis associated with this bending moment (see Figure 5.8).

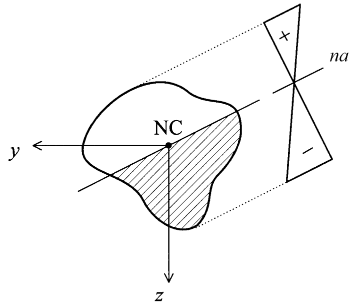


Figure 5.8 For bending without normal force, the shear force per length is a maximum in the planar longitudinal section through the neutral axis. Since there is no normal force, the neutral axis passes through the normal centre NC.

5.1.4 Shear stresses in the longitudinal direction

Figure 5.9 shows the sliding element between two cross-sections at a mutual distance equal to the unit of length.

If the longitudinal shear force s_x^a (force per length) is uniformly smeared over the developed width b^a of the longitudinal section, we find the *average shear stress* τ_{average}^a (force per area) on the longitudinal section plane of the sliding element:

$$\tau_{\text{average}}^a = \frac{s_x^a}{b^a}\,. \tag{5.13}$$

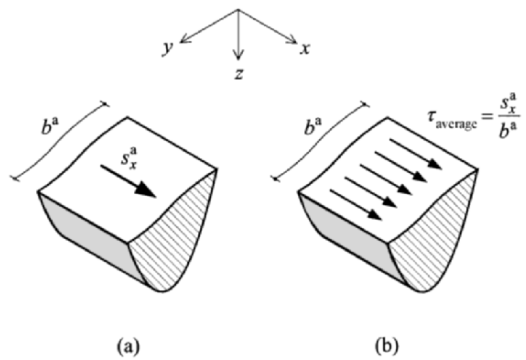


Figure 5.9 If the longitudinal shear force s_x^a (force per length) is smeared uniformly over the developed width b^a of the longitudinal section plane, the average shear stress τ_{average}^a (force per area) on the sliding element is $\tau_{\text{average}}^a = s_x^a/b^a$.

[1] In another way it can be shown that formula (5.12) can be used under all conditions if, without taking account of the actual moment in the cross-section, we take for M_z^* an arbitrary moment acting in the plane of the resultant shear force V_z.

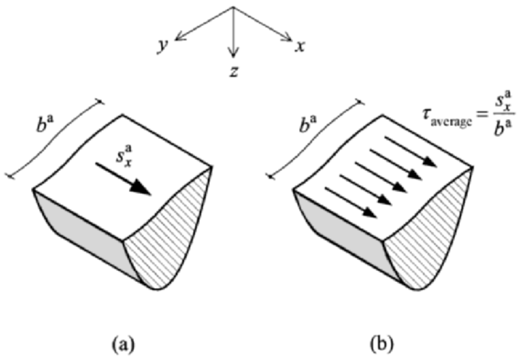


Figure 5.9 If the longitudinal shear force s_x^a (force per length) is smeared uniformly over the developed width b^a of the longitudinal section plane, the average shear stress $\tau^a_{average}$ (force per area) on the sliding element is $\tau^a_{average} = s_x^a/b^a$.

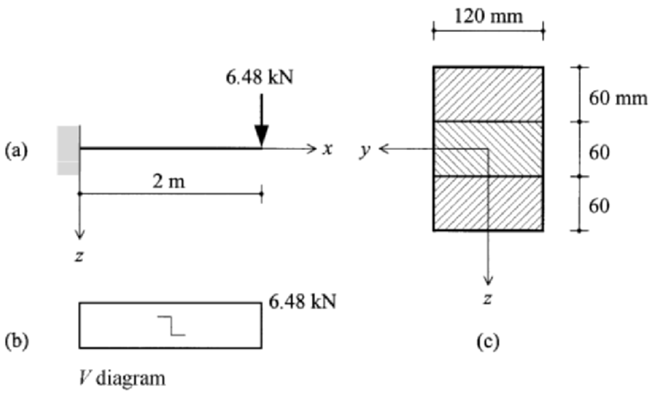


Figure 5.10 (a) Laminated timber beam with (b) shear force diagram. The beam is constructed by gluing together three timber layers, as shown in (c) the cross-section.

The actual shear stress need not be uniformly distributed over the width b^a. This is discussed in Section 5.3.

It can be agreed that the longitudinal shear stress $\tau^a_{average}$ on the sliding element is positive as it, just like the longitudinal shear force, acts in the positive x direction. Usually, however, one works with the absolute values and signs are omitted. In the following, the letter τ is used for the absolute value of a shear stress.

5.2 Examples relating to shear forces and shear stresses in the longitudinal direction

The formulas derived in the previous section for the shear forces and shear stresses in the longitudinal direction are used in this section for determining the forces in the joints for beams of which the cross-section is built up of several parts. We will be looking at a laminated timber beam, a steel beam with a welded I-section, a wooden beam with dowels and a nailed wooden box beam.

5.2.1 Laminated timber beam

The laminated timber beam in Figure 5.10a is loaded at its free end by a force of 6.48 kN. Due to this load the cross-sectional shear force V_z in the beam is constant, see the shear force diagram in Figure 5.10b. The laminated beam is formed by gluing together three timber layers, as indicated in Figure 5.10c.

Questions:

a. Calculate the shear force (force per length) in the glued joints.
b. Calculate the (average) shear stress in the glued joints.

Solution:

a. Below, the shear force per length is determined for the lower glued joint. Figure 5.11a gives a side view of the sliding element isolated from a beam segment with small length Δx. A shear force s_x^a acts on the sliding element in the longitudinal direction. Remember that s_x^a is positive when it acts on the sliding element in the positive x direction. Figure 5.11b shows the cross-section of the sliding element, for which the lowermost of the three timbers has been chosen.

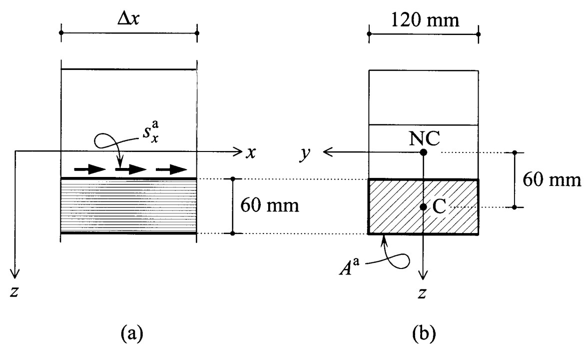


Figure 5.11 (a) Side view of a beam segment with small length Δx. The lower timber has been selected as the sliding element. In the glued joint, a longitudinal shear force s_x^a (force per length) acts on the lower timber. (b) Cross-section of the beam. The sliding area is A^a.

The shear force per length is

$$s_x^a = -\frac{V_z S_z^a}{I_{zz}}.$$

Since the cross-sectional shear force V_z is constant over the length of the beam (see Figure 5.10b), this also holds for s_x^a, the longitudinal shear force per length.

In the given coordinate system

$$V_z = +6480\text{ N}.$$

Note: The deformation symbol has to be translated here into the correct plus/minus sign in the given coordinate system.

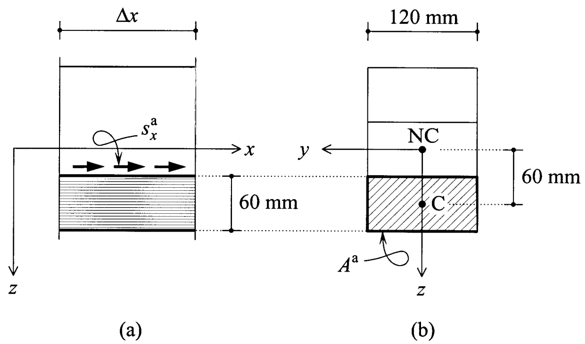


Figure 5.11 (a) Side view of a beam segment with small length Δx. The lower timber has been selected as the sliding element. In the glued joint, a longitudinal shear force s_x^a (force per length) acts on the lower timber. (b) Cross-section of the beam. The sliding area is A^a.

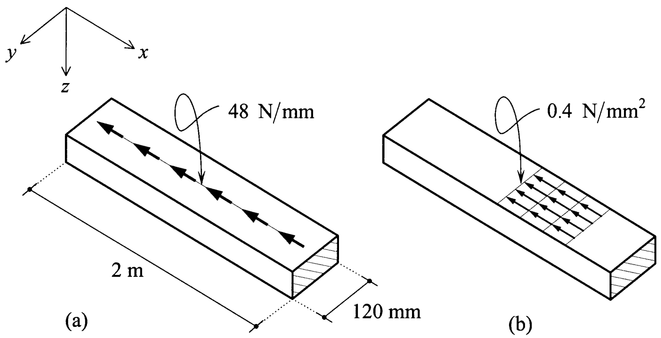


Figure 5.12 (a) A constant longitudinal shear force per length of 48 N/mm acts over the entire length of the lower timber. (b) If the shear force per length of 48 N/mm is uniformly smeared across the 120 mm width of the horizontal section plane, the average shear stress in the glued joint is $\tau_{\text{average}} = 0.4\ \text{N/mm}^2$.

Furthermore,

$$I_{zz} = \tfrac{1}{12} bh^3 = \tfrac{1}{12} (120\ \text{mm})(180\ \text{mm})^3 = 58.32 \times 10^6\ \text{mm}^4.$$

The static moment S_z^a of the sliding area A^a is equal to the product of A^a and z_C^a, in which z_C^a is the z coordinate of the centroid of the sliding area (see Figure 5.11b):

$$S_z^a = A^a z_C^a = (120\ \text{mm})(60\ \text{mm})(+60\ \text{mm}) = +432 \times 10^3\ \text{mm}^3.$$

The shear force per length is now

$$s_x^a = -\frac{V_z S_z^a}{I_{zz}} = -\frac{(+6480\ \text{N})(+432 \times 10^3\ \text{mm}^3)}{58.32 \times 10^6\ \text{mm}^4} = -48\ \text{N/mm}.$$

The minus sign indicates that the longitudinal shear force on the sliding element acts in the negative x direction.

Figure 5.12a shows the sliding element over the entire length of the beam (this is the bottom timber), including the uniformly distributed longitudinal *shear force per length* of 48 N/mm acting on it.

b. If the shear force per length of 48 N/mm is smeared uniformly over the 120 mm width of the horizontal section plane, we find the *average shear stress* τ_{average} in the glued joint:

$$\tau_{\text{average}} = \frac{48\ \text{N/mm}}{120\ \text{mm}} = 0.4\ \text{N/mm}^2.$$

Figure 5.12b shows the (average) shear stresses as they act in the glued joint on the bottom timber of the beam.

Note: If the part of the beam above the longitudinal section is chosen as the sliding element (i.e. the two upper timbers; see Figure 5.13), the only consequence is that the static moment S_z^a changes sign. Since the static moment of the entire cross-section is zero, the static moment of the upper part of the cross-section is equal and opposite to that of the lower part. This can be verified by calculation (see Figure 5.13):

$$S_z^a = A^a z_C^a = (120 \text{ mm})(120 \text{ mm})(-30 \text{ mm}) = -432 \times 10^3 \text{ mm}^3.$$

The shear forces per length on the upper and lower sliding part are therefore equally large, but have opposite signs. This means that the shear force on the upper sliding part has an direction opposite to the shear force on the lower sliding part, entirely in line with the concept of *interaction*.

It is left to the reader to verify that the shear forces and shear stresses in the upper glued joint are equal to those in the lower glued joint.

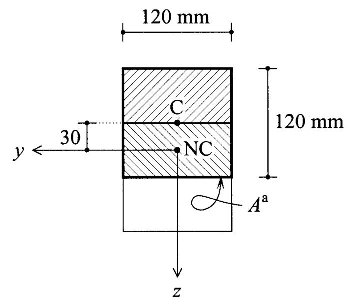


Figure 5.13 Since the static moment of the entire cross-section is zero, the static moment of the hatched upper part of the cross-section is equal to that of the non-hatched lower part, but with an opposite sign. This means that the shear forces on the upper part are equal and opposite to those on the lower part. This is entirely in line with the concept of *interaction*.

5.2.2 Steel beam with a welded I-section

The simply supported steel beam AB in Figure 5.14a has a span ℓ and is carrying a uniformly distributed load q. Figure 5.14b shows the associated V diagram. An I-section has been chosen for the beam. The cross-sectional dimensions are shown in Figure 5.14c. The cross-section is thin-walled. The flanges have a wall thickness t. The web is three times as thick as a flange. It is also given that the height h of the I-section is equal to 1/12 of the span ℓ, i.e. $\ell = 12h$.

This cross-section is not a standardised *rolled steel section* but a *welded section*. Here the flanges and web are made of rectangular steel plates and joined by means of welds.

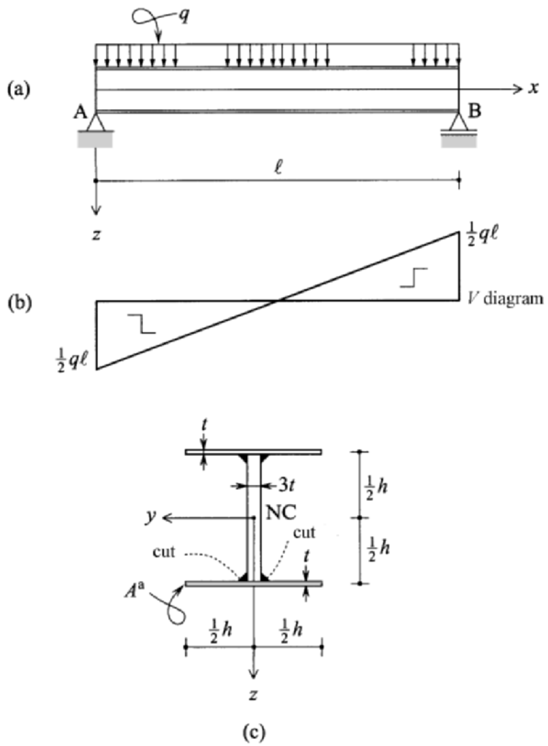


Figure 5.14 (a) A simply supported steel beam with a uniformly distributed load over its entire length, and (b) the associated shear force diagram. (c) The cross-section of the beam is a welded I-section. The connection between flanges and web is realised by double corner welds. The lower flange is chosen as the sliding part of the cross-section for calculating the shear forces in a corner weld.

There are various types of welds. Here the joint between a flange and the web is a so-called *double corner weld*. Figure 5.15a shows the joint between web and bottom flange. In double corner welds, there is always a (small) gap between web and flange. Therefore the web and flange can exert forces on one another only via the corner welds.

For the calculation, we assume that the corner weld is shaped like a isosceles triangle. The thickness of the gap (sketched rather large in Figure 5.15) is ignored.

Questions:

a. Determine the maximum shear force per length that a single corner weld has to transfer.
b. Which cut across the corner weld is *most dangerous*? In other words: how should the cut across the corner weld be chosen such that the (average) shear stress in the weld is as large as possible? And how large is this shear stress?

Solution:

a. The formula for the shear force per length in the longitudinal direction is

$$s_x^{\mathrm{a}} = -\frac{V_z S_z^{\mathrm{a}}}{I_{zz}}\,.$$

We see that the longitudinal shear force s_x^{a} is largest where the cross-sectional shear force V_z is largest, i.e. at the supports A and B.

Below we look at the shear force at B, where

$$V_z = -\tfrac{1}{2}\,q\ell.$$

The (centroidal) moment of inertia I_{zz} is

$$I_{zz} = \tfrac{1}{12} \cdot 3t \cdot h^3 + 2 \cdot ht \cdot \left(\tfrac{1}{2}\,h\right)^2 = \tfrac{3}{4}\,h^3 t.$$

If a cut is introduced across both corner welds at the bottom flange, and the bottom flange is seen as the sliding part of the cross-section, then the static moment S_z^{a} of the sliding part is (see Figure 5.14c)

$$S_z^{\mathrm{a}} = ht \cdot \left(+\tfrac{1}{2}\,h\right) = +\tfrac{1}{2}\,h^2 t.$$

The expressions for V_z, I_{zz} and S_z^{a} substituted into the formula for the shear force per length gives

$$s_x^{\mathrm{a}} = -\frac{\left(-\tfrac{1}{2}\,q\ell\right)\left(+\tfrac{1}{2}\,h^2 t\right)}{\tfrac{3}{4}\,h^3 t} = +\frac{q\ell}{3h}\,.$$

With $\ell = 12h$ this becomes

$$s_x^{\mathrm{a}} = +4q.$$

Note: At the supports, the double corner weld has to transfer a distributed shear force in the longitudinal direction that is four times as large as the distributed load q on the beam!

The shear force is transferred by two (similar) corner welds. Per corner weld, the shear force (per length) is $2q$.

b. The maximum (average) shear stress in the longitudinal direction occurs where the (developed) length b^{a} of the cut over the weld is shortest. This is in the so-called *throat cut* across the weld (see Figure 5.15b). For the double corner weld $b^{\mathrm{a}} = 2a$.

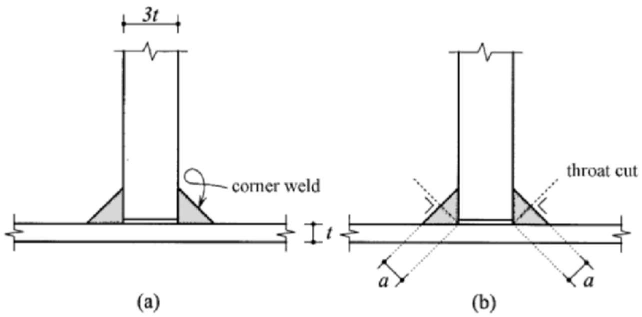


Figure 5.15 (a) With double corner welds, there is always a small gap between the web and the flange. Web and the flanges can therefore exert forces on one another only via the corner welds. (b) The throat cut is the cut in which the average shear stress in longitudinal direction is a maximum.

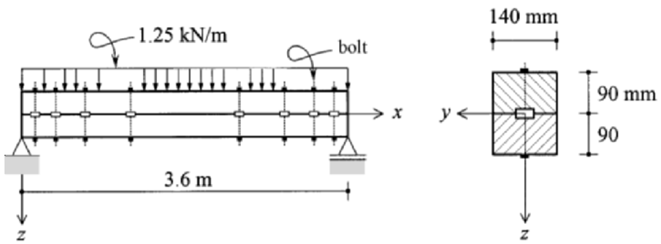


Figure 5.16 A wooden beam is constructed of two timbers that are joined by means of dowels or shear connectors. Dowels can be hardwood blocks (*carpenter dowels*), steel rings (*ring dowels*) that fit tightly in recesses made in the timbers, or *toothed plate connectors*. They prevent the beams from sliding over one another. The timbers are clamped to one another by means of bolts so that the dowels cannot jump away.

At the supports, where the shear forces are largest, the (average) longitudinal shear stress in the double throat cut is largest:

$$\tau_{\text{average}} = \frac{s_x^{\text{a}}}{b^{\text{a}}} = \frac{4q}{2a} = 2\frac{q}{a}\,.$$

5.2.3 Wooden beam with dowels

The simply supported wooden beam in Figure 5.16 has a span of 3.6 m and carries over its entire length a uniformly distributed load of 1.25 kN/m. The cross-section is constructed of two timbers of $90 \times 140\ \text{mm}^2$ that are joined by means of *dowels* or *shear connectors*. The maximum shear force that a dowel can transfer[1] is $\bar{F}_{\text{dowel}} = 5$ kN.

Dowels may be hardwood blocks (*carpenter dowels*), steel rings (*ring dowels*) that fit tightly in recesses made in the timbers, or *toothed plate connectors*. They prevent the timbers from sliding over one another. The timbers are clamped to one another by means of bolts so that the dowels cannot jump away.

Questions:

a. Determine the number of dowels required for the given load.[2]

---

1 This is the *limiting value* of the shear force that the dowel can transfer, also known as the *allowable shear force*. This value is overlined here.

2 Since the timbers are joined locally, at the dowels only, and since there is always some clearance between timbers and dowels, the timbers will not work together fully. In practice this effect has been taken into account by a reduction of the moment of inertia of the cross-section. Below we will ignore this effect and assume that the built-up beam behaves as one piece.

b. Determine the *active area* per dowel if all dowels are equally loaded.

Solution:
a. Figure 5.18 shows the support reactions and the V and M diagrams.

The longitudinal shear force per length between the timbers is

$$s_x^{\mathrm{a}} = -\frac{V_z S_z^{\mathrm{a}}}{I_{zz}},$$

in which

$$I_{zz} = \tfrac{1}{12}(140\text{ mm})(180\text{ mm}) = +68.04 \times 10^6\text{ mm}^4.$$

If the bottom timber is seen as the sliding element, then (see Figure 5.17)

$$S_z^{\mathrm{a}} = (140\text{ mm})(90\text{ mm})(+45\text{ mm}) = +567 \times 10^3\text{ mm}^3.$$

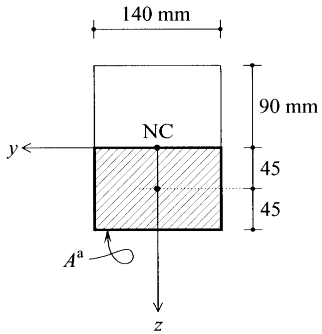


Figure 5.17 The lower timber is chosen as the sliding element.

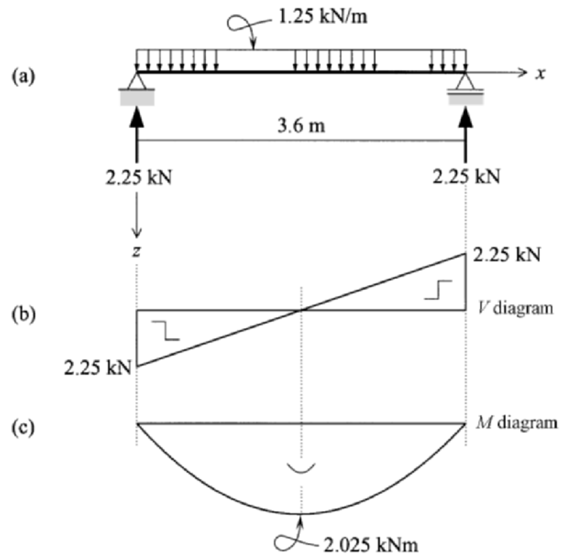


Figure 5.18 (a) The beam modelled as a line element with (b) shear force diagram and (c) bending moment diagram.

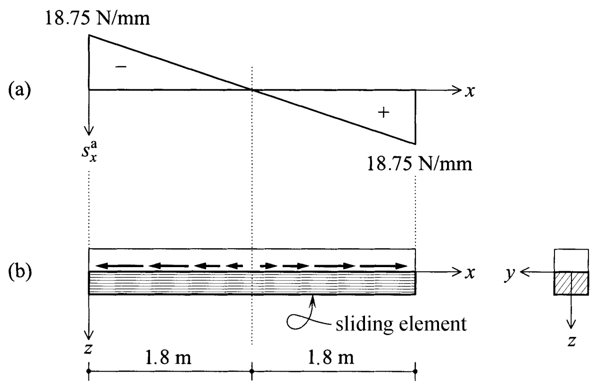


Figure 5.19 (a) Diagram with the distribution of the longitudinal shear force s_x^a (force per length) acting on the lower timber. This longitudinal shear force is proportional to the cross-sectional shear force V_z (see Figure 5.18b). (b) At midspan the cross-sectional shear force changes sign. There the longitudinal shear force also changes sign and therefore direction. The longitudinal shear forces are largest at the supports and decrease towards midspan.

We now find

$$s_x^a = -\frac{V_z S_z^a}{I_{zz}} = -V_z \cdot \frac{567 \times 10^3 \text{ mm}^3}{68.04 \times 10^6 \text{ mm}^4} = -\frac{V_z}{120 \text{ mm}} .$$

Figure 5.19a shows the distribution of the longitudinal shear force s_x^a (force per length) on the bottom timber in a diagram. The maximum value occurs at the supports, where the cross-sectional shear force V_z is a maximum:

$$|s_{x;\max}^a| = \frac{|V_{z;\max}|}{120 \text{ mm}} = \frac{2.25 \times 10^3 \text{ N}}{120 \text{ mm}} = 18.75 \text{ N/mm}.$$

The longitudinal shear force s_x^a (force per length) in Figure 5.19 has the same distribution as the cross-sectional shear force V_z in Figure 5.18b. To find the sign (direction) of the longitudinal shear force, the deformation symbols in the V diagram have to be translated into the correct plus- and minus signs in the xz coordinate system in which we work.

The cross-sectional shear force V is positive for the left-hand side of the beam, and negative for the right-hand side. As a consequence the longitudinal shear force s_x^a is negative on the left-hand side (acting in the negative x direction) and positive on the right-hand side (acting in the positive x direction) (see Figure 5.19b).

Since it is of no consequence for the dowels in which direction the longitudinal shear force acts, it is common practice to neglect the signs in the calculation and to work with the absolute values. For the rest of this problem we will also use only absolute values.

Figure 5.20 shows a spatial representation of the distribution of the longitudinal shear force s_x^a (force per length) on the lower timber. The resultant longitudinal shear force R_s^a over the half beam length is equal to the area of the triangular diagram:

$$R_s^a = \frac{1}{2} \times (18.75 \text{ N/mm}) \times (1.8 \times 10^3 \text{ mm}) = 16.875 \text{ kN}.$$

To find R_s^a we can also use formula (5.8) from Section 5.1.2:

$$R_{x;s}^a = -\frac{\Delta M_z S_z^a}{I_{zz}},$$

in which ΔM is the increase of the bending moment over the half beam length. From the M diagram in Figure 5.18c it follows that

$$|\Delta M| = 2.025 \text{ kNm}.$$

For the (absolute value of the) resultant shear force over the half beam length we now find

$$R_s^a = \left| -\frac{\Delta M_z S_z^a}{I_{zz}} \right| = \frac{(2.025 \text{ kNm})(567 \times 10^3 \text{ mm}^3)}{68.04 \times 10^4 \text{ mm}^4} = 16.875 \text{ kN}.$$

This is the value we found earlier.

Assume the number of dowels for a half beam is n. If each dowel can transfer a maximum shear force $\bar{F}_{\text{dowel}} = 5$ kN, the value of n follows from

$$n \geq \frac{R_s^a}{\bar{F}_{\text{dowel}}} = \frac{16.875 \text{ kN}}{5 \text{ kN}} \approx 3.4 \Rightarrow n = 4.$$

This means that for each half of the beam four dowels are required. So the

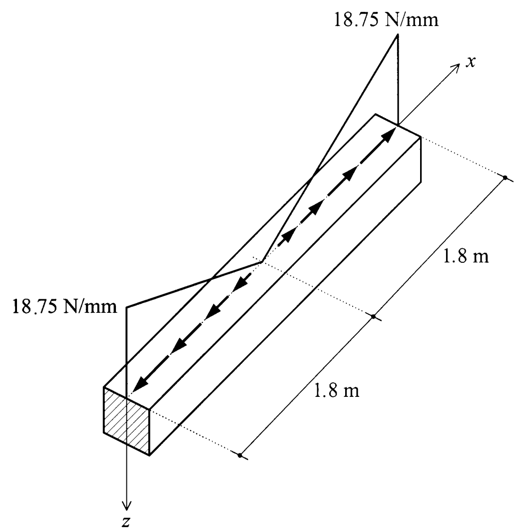


Figure 5.20 Spatial representation of the distribution of the longitudinal shear forces on the lower timber.

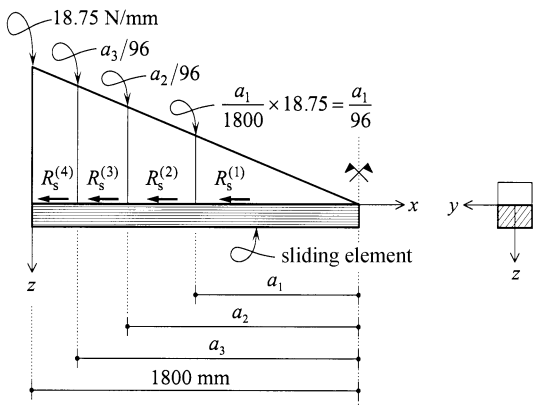


Figure 5.21 s_x^a diagram for the longitudinal shear forces on the lower timber at the left-hand side of the beam. The shear force interaction is concentrated in a dowel. If four dowels per half beam length are sufficient, and one wants to apply them such that all four are equally loaded, the active area per dowel is found by dividing the s_x^a diagram into four equal areas.

entire beam requires $n_{tot} = 8$ dowels.

Note: Per area for which the shear force has the same sign, one must always apply a whole number of dowels. One could also have said that the total shear force in the *entire beam* is equal to $2R_s^a$, and that the total number of dowels required is

$$n_{tot} \geq \frac{2R_s^a}{\bar{F}_{dowel}} = \frac{2 \times (16.875 \text{ kN})}{5 \text{ kN}} \approx 6.8 \Rightarrow n_{tot} = 7.$$

It now appears that seven instead of eight dowels will suffice. This is incorrect, however. With a symmetrical placement of seven dowels, one of the dowels is at midspan. Since the shear force at midspan is zero, this dowel is useless; only six dowels are actually effective, and not seven dowels.

b. The longitudinal shear forces are largest at the supports and decrease towards midspan. Therefore the dowels are more tightly spaced near the supports of the beam than near midspan (see Figure 5.16).

In a dowel, the shear force interaction is concentrated. The force on a dowel is equal to the resultant of the longitudinal shear forces in the "*active area*" of the dowel and can be found from the corresponding area of the s_x^a diagram or longitudinal shear force diagram.

Figure 5.21 shows the s_x^a diagram for the lower timber at the left-hand side of the beam.

If the four dowels per half beam length are applied in such a way that all four are equally loaded, the active area per dowel is found by dividing the s_x^a diagram into four equal areas. In that case

$$R_s^{(1)} = R_s^{(2)} = R_s^{(3)} = R_s^{(4)} = F_{dowel}.$$

F_{dowel} (now not overlined) is the actual shear force per dowel and is equal to the resultant shear force divided by the required number of dowels:

$$F_{dowel} = \frac{R_s^a}{n} = \frac{16.875 \times 10^3 \text{ N}}{4} = 4219 \text{ N}.$$

The lengths a_1 to a_3 are found from the s_x^a diagram in Figure 5.21:

$$\tfrac{1}{2} \times a_1 \times \frac{a_1}{1800 \text{ mm}} \times (18.75 \text{ N/mm}) = R_s^{(1)}$$
$$= F_{dowel} = 4219 \text{ N}.$$

In the same way

$$\tfrac{1}{2} \times a_2 \times \frac{a_2}{1800 \text{ mm}} \times (18.75 \text{ N/mm}) = R_s^{(1)} + R_s^{(2)}$$
$$= 2F_{dowel} = 8438 \text{ N},$$

and

$$\tfrac{1}{2} \times a_3 \times \frac{a_3}{1800 \text{ mm}} \times (18.75 \text{ N/mm}) = R_s^{(1)} + R_s^{(2)} + R_s^{(3)}$$
$$= 3F_{dowel} = 12657 \text{ N}.$$

This leads to

$$a_1 = 900 \text{ mm},$$
$$a_2 = 1273 \text{ mm},$$
$$a_3 = 1559 \text{ mm}.$$

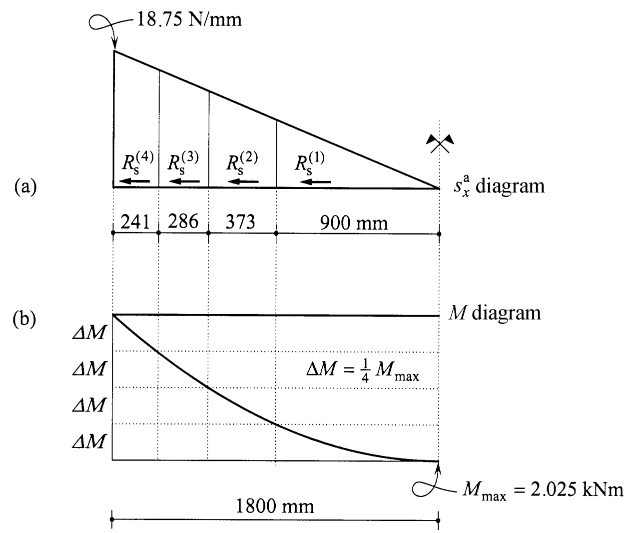


Figure 5.22 (a) The active areas of the four dowels for the left-hand side of the beam. (b) Since the resultant shear force over a certain length is proportional to the change in the bending moment ΔM over that length, the active areas of the four dowels can also be determined via the intersections of the M diagram with a number of parallel lines at mutually equal distances $\Delta M = \frac{1}{4} M_{max}$.

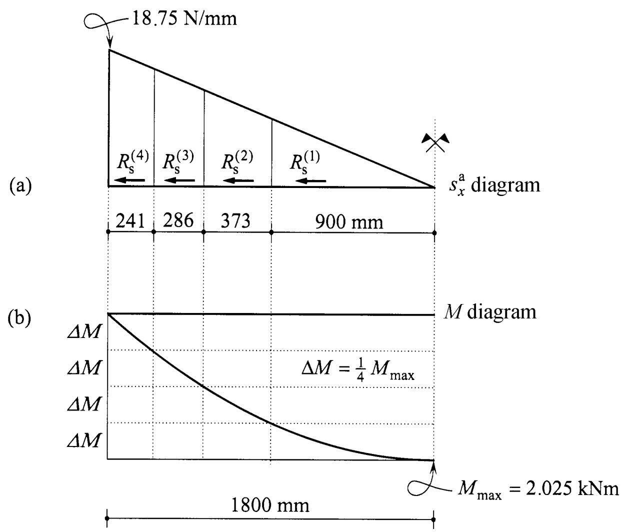


Figure 5.22 (a) The active areas of the four dowels for the left-hand side of the beam. (b) Since the resultant shear force over a certain length is proportional to the change in the bending moment ΔM over that length, the active areas of the four dowels can also be determined via the intersections of the M diagram with a number of parallel lines at mutually equal distances $\Delta M = \frac{1}{4} M_{\text{max}}$.

The *active areas* of the four dowels for the left-hand side of the beam are shown in the s_x^a diagram in Figure 5.22a.

Figure 5.22b also shows the bending moment diagram for the left-hand side of the beam. According to formula (5.9) in Section 5.1.2 the resultant of the longitudinal shear forces over a certain length is proportional to ΔM, the change in the bending moment over that length.

If all dowels are equally loaded, the change ΔM over all *active areas* of the dowels must be equal. For four dowels per half beam length we have $\Delta M = \frac{1}{4} M_{\text{max}}$.

The *active areas* of the dowels could also be determined via the intersections of the M diagram with a number of parallel lines at equal distances ΔM (see Figure 5.22b). However, this approach is more laborious than the one using the shear force diagram, and is omitted here.

5.2.4 Nailed wooden box beam

The simply supported wooden box beam in Figure 5.23, with a span of 5.6 m, carries a uniformly distributed load of 2.1 kN/m over its full length. The beam is constructed of two battens that serve as flanges, and two sheets of plywood that serve as webs. The cross-sectional dimensions are shown in Figure 5.23. The plywood is fastened to the battens by wire nails. Each nail can transfer a maximum shear force of $\bar{F}_{\text{nail}} = 300$ N.[1]

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[1] Also said to be the *allowable load in shear*.

Questions:

a. Determine the required number of nails in the whole beam.
b. How should these nails be distributed over the length of the beam?

Solution:

a. Figure 5.24 shows the support reactions and the V and M diagrams. The calculation is left to the reader.

The hollow cross-section can be seen as the difference between two rectangles, hence

$$I_{zz} = \tfrac{1}{12}\,(200\text{ mm})(240\text{ mm})^3 - \tfrac{1}{12}\,(160\text{ mm})(120\text{ mm})^3$$

$$= 207.36 \times 10^6\text{ mm}^4.$$

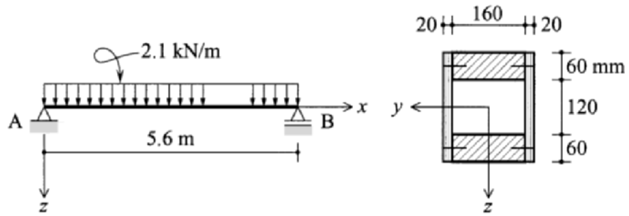


Figure 5.23 A simply supported wooden box beam. The beam is constructed of two battens that serve as flanges, and two sheets of plywood that serve as webs. The plywood is fastened to the battens by wire nails.

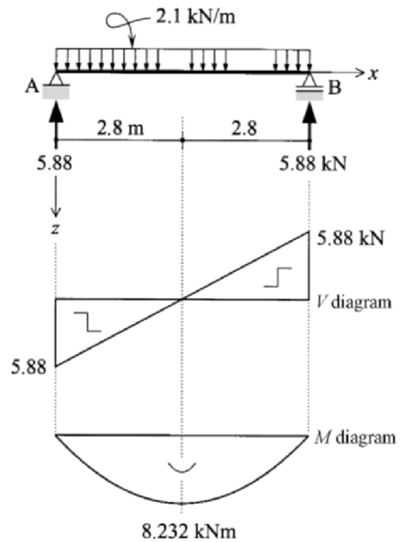


Figure 5.24 The beam modelled as a line element with shear force diagram and bending moment diagram.

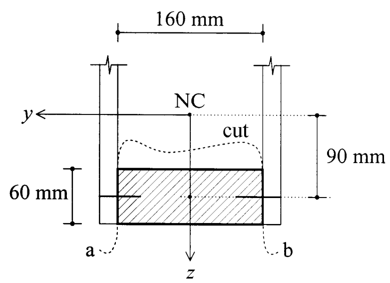


Figure 5.25 To determine the required number of wire nails, a symmetrical double cut a-b is applied over the lower batten. The lower batten is considered the sliding element.

Assume the bottom batten is the sliding element (see Figure 5.25), then the static moment of the sliding element is

$$S_z^{\mathrm{a}} = A^{\mathrm{a}} z_{\mathrm{C}}^{\mathrm{a}} = (160 \text{ mm})(60 \text{ mm})(+90 \text{ mm}) = 864 \times 10^3 \text{ mm}^3.$$

According to formula (5.7) from Section 5.1.2 the total distributed longitudinal shear force (force per length) in the *double cut* a-b is

$$s_x^{\mathrm{a}} = -\frac{V_z S_z^{\mathrm{a}}}{I_{zz}}. \tag{5.7}$$

This longitudinal shear force s_x^{a} is a maximum at the supports, where the cross-sectional shear force V_z (in absolute sense) is largest: $|V_z| = 5.88$ kN. So

$$|s_{x;\max}^{\mathrm{a}}| = \frac{(5.88 \times 10^3 \text{ N})(864 \times 10^3 \text{ mm}^3)}{207.36 \times 10^6 \text{ mm}^4} = 24.5 \text{ N/mm}.$$

The distribution of the longitudinal shear force s_x^{a} (force per length) is plotted in a diagram in Figure 5.26a. This s_x^{a} diagram is similar to the V diagram in Figure 5.24. At A the cross-sectional shear force is positive, therefore s_x^{a} is negative and acts in the negative x direction. Check this! The opposite holds at B.

Note: For determining the required number of nails, the signs are less important than the shape of the longitudinal shear force diagram.

The shear force distribution in Figure 5.26a applies to the double cut a-b (see Figures 5.25 and 5.26b). On the basis of mirror symmetry, we can expect half the shear force to act in each of the cuts a and b.

In Figure 5.26b the bottom batten (the sliding element) is shown spatially, with the distribution of the longitudinal shear forces (forces per length) as they act in the cuts a and b on the batten. At A the longitudinal shear force s_x^a is negative and acts on the bottom batten in the negative x direction. At B the longitudinal shear force is acting in the positive x direction.

The resultant longitudinal shear force over the half beam length AC in the joint cuts a and b is equal to the area of the half shear force diagram in Figure 5.26a:

$$R_s^a = \frac{1}{2}(2.8 \times 10^3 \text{ mm})(24.5 \text{ N/mm}) = 34300 \text{ N}.$$

R_s^a can also be found with formula (5.9) from Section 5.1.2:

$$R_s^a = \left|\frac{\Delta M \cdot S_z^a}{I_{zz}}\right| = \frac{M_{\max} \cdot S_z^a}{I_{zz}}$$

$$= \frac{(8.232 \times 10^6 \text{ Nmm})(864 \times 10^3 \text{ mm}^3)}{207.36 \times 10^6 \text{ mm}^4} = 34300 \text{ N}.$$

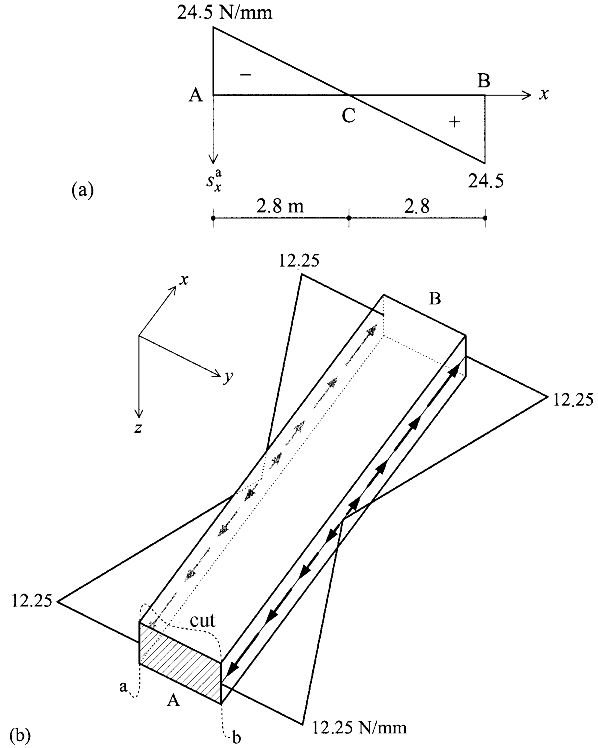


Figure 5.26 (a) The distribution of the longitudinal shear force per length in double cut a-b. On the basis of mirror symmetry, half this shear force acts in each of the sections a and b. (b) The sliding batten in a spatial representation, with the longitudinal shear forces (forces per length) in the cuts a and b as they actually act.

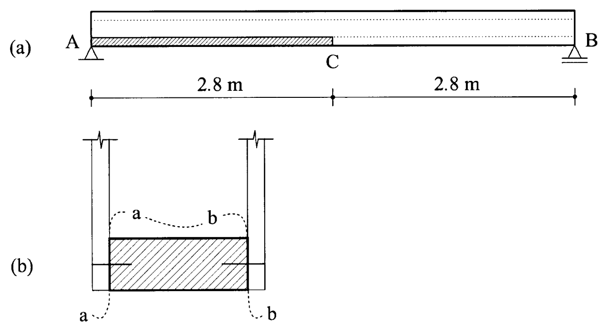


Figure 5.27 (a) The resultant longitudinal shear force over half the beam length AC in double cut a-b is equal to the area of half the $s_x^{\rm a}$ diagram in Figure 5.26a. (b) Since the same number of wire nails are required for cut a-a as for cut b-b the total number of nails in the hatched area AC in (a) must be even.

With $\bar{F}_{\rm nail} = 300$ N, the total number of nails n that is required for the hatched area of the beam in Figure 5.27a is

$$n \geq \frac{R_{\rm s}^{\rm a}}{\bar{F}_{\rm nail}} = \frac{34300 \text{ N}}{300 \text{ N}} = 114.3.$$

Since the same number of nails needed for both cut a and cut b (see Figure 5.27b), n must be even. Therefore

$$n = 116.$$

In other words, 58 wire nails are needed per cut. For the whole beam, the total number $n_{\rm tot}$ of nails required is

$$n_{\rm tot} = 4n = 4 \times 116 = 464.$$

b. Assume all wire nails are equally loaded. The shear force per nail is then

$$F_{\rm nail} = \frac{R_{\rm s}^{\rm a}}{n} = \frac{34300 \text{ N}}{116} = 295.7 \text{ N}.$$

Figure 5.28a shows the $s_{\rm x}^{\rm a}$ diagram for AC relating to double cut a-b. To determine the distribution of the wire nails over the length of the beam, we look in this diagram for the length a_1 for which the hatched area is exactly equal to the shear force that can be transferred by two nails (one in cut a and one in cut b):

$$\tfrac{1}{2} \times a_1 \times \frac{a_1}{2800 \text{ mm}} \times (24.5 \text{ N/mm}) = 2F_{\rm nail} = 2 \times (295.7 \text{ N}).$$

We find

$$a_1 = 368 \text{ mm}.$$

In Figure 5.28b, the s_x^{a} diagram for AC is divided into 7 fields of 368 mm and an end field of 224 mm. In the fields i ($i = 1, \ldots, 7$) we can now directly read off the number of wire nails n^i from the areas divided into rectangles and triangles. A triangle stands for two nails. A rectangles has double the area and represents four nails. Fields 1 to 7 amount to 98 nails, as shown in Table 5.1. The end field therefore needs $n - 98 = 116 - 98 = 18$ wire nails.

Table 5.1

field i	n^i
1	2
2	6
3	10
4	14
5	18
6	22
7	26
$\sum =$	98

Check of the number of nails in the end field:

$$n^{(8)} = \frac{R_{\mathrm{s}}^{(8)}}{F_{\mathrm{nail}}} = \frac{\frac{1}{2}(24.5 + 22.54)(\mathrm{N/mm})(224 \text{ mm})}{295.7 \text{ N}} = 17.8 \approx 18.$$

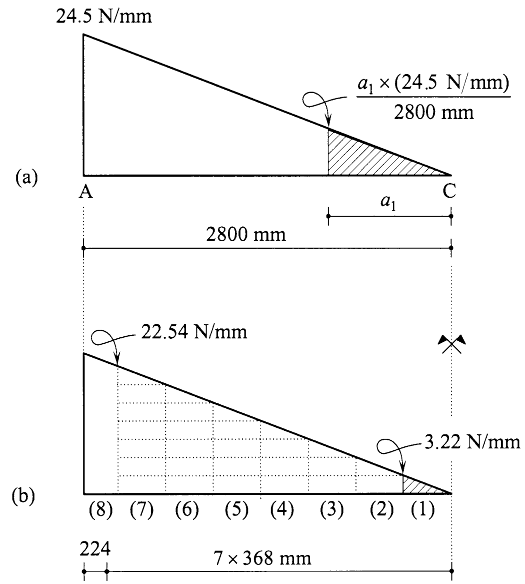


Figure 5.28 (a) The longitudinal shear force distribution (s_x^{a} diagram) for the left-hand side AC of the beam, relating to double cut a-b. To determine the distribution of the wire nails over the length of the beam, length a_1 follows from the condition that the area of the hatched triangle is equal to the shear force that can be transferred by two nails (one in cut a-a and one in cut b-b; see Figure 5.27). (b) With $a_1 = 368$ mm, the s_x^{a} diagram for AC can be divided into seven fields of 368 mm and an end field of 224 mm. In fields 1 to 7 we find the number of nails directly from the areas divided into rectangles and triangles. A triangle represents two nails. A rectangle represents four nails. Fields 1 to 7 amount to 98 wire nails. The remaining wire nails are used in the end field.

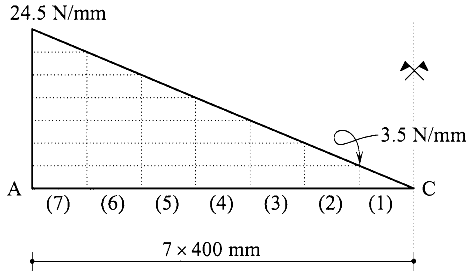


Figure 5.29 The longitudinal shear force distribution ($s_x^{\rm a}$ diagram) for the left-hand side AC of the beam, relating to double cut a-b. To find the number of required wire nails and their distribution over the length of the beam in one go, length AC can be divided directly into a number of fields of equal length, after which the (even) number of nails per field can be determined.

Table 5.2

field i	$R_{\rm s}^i$ (N)	$R_{\rm s}^i/\bar{F}_{\rm nail}$	n^i
1	700	2.3	4
2	2100	7.0	8
3	3500	11.7	12
4	4900	16.3	18
5	6300	21.0	22
6	7700	25.7	26
7	9100	30.3	32
$\sum =$	34300	114.3	122

Alternative solution:
Divide the length of the beam into a number of fields of equal length, e.g. seven fields of 400 mm (see Figure 5.29). Per field i determine the resultant shear force $R_{\rm s}^i$. For field (1) the shear force diagram is a triangle:

$$R_{\rm s}^{(1)} = \tfrac{1}{2} \times (400 \text{ mm}) \times \left(\tfrac{1}{7} \times 24.5 \text{ N/mm}\right) = 700 \text{ N}.$$

Using the distribution in triangles and rectangles, the resultant shear forces can now be easily determined per field; this is shown in Table 5.2. The number of nails n^i required per field i follows from

$$n^i \geq \frac{R_{\rm s}^i}{\bar{F}_{\rm nail}}.$$

The results of the calculation are included in the last two columns of Table 5.2. Since the $s_x^{\rm a}$ diagram relates to the double cut a-b, and both cuts need the same number of nails, n^i must be even.

Table 5.2 shows that using this alternative approach 122 wire nails are required in the hatched part of the box beam in Figure 5.27, instead of 116. The total number of wire nails in the beam is 488, that is 5% more than the 464 we found previously.

5.3 Cross-sectional shear stresses

In this section we derive the formulas for the shear stresses on a cross-sectional plane.

Using the moment equilibrium for a small parallelepiped, we first show that the shear stresses on two mutually perpendicular planes are equal. Next we determine the shear stresses on a cross-sectional plane (cross-sectional shear stresses) from the shear stresses on a longitudinal plane (longitudinal shear stresses). We assume that the shear stresses, normal to the intersection line of both planes, are uniformly distributed over the width of that line.

For cross-sections (or parts thereof) for which the width of the sliding element is constant, we show a relationship between the shape of the shear stress diagram and that of the bending stress diagram. This leads to a set of rules with which we can quickly sketch the shear stress distribution.

5.3.1 Shear stresses on two mutually perpendicular planes

Figure 5.30a shows a small rectangular block with dimensions Δx; Δy; Δz. The stresses are shown only for the visible planes without naming them. The dimensions of the block are so small that all stresses on the planes are uniformly distributed and their resultants therefore apply at the centres of the planes. The arrows in Figure 5.30a can therefore be considered stress resultants.

Below we look at the moment equilibrium of the block parallel to the xz plane. In the equation $\sum T_y|A = 0$ (the equation for the moment equilibrium about the a-a axis through A, parallel to the y axis) only the shear stress resultants in Figure 5.30b are relevant. All other stress resultants make a zero contribution as they either pass through the a-a axis or act parallel to it.

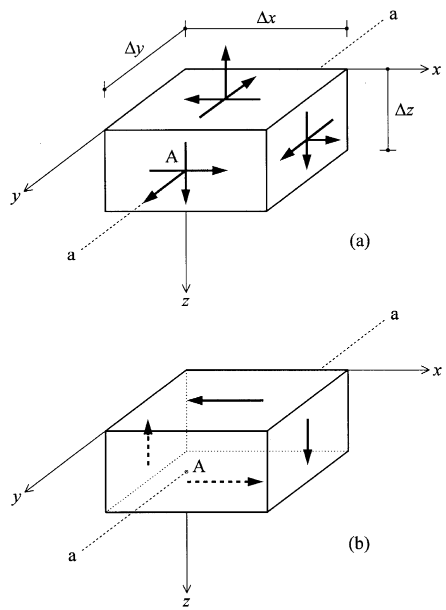


Figure 5.30 (a) The stresses on a small rectangular block with dimensions Δx; Δy; Δz. The stresses are shown only for the visible sides, without naming them. The dimensions of the block are so small that all the stresses on the sides are uniformly distributed and their resultants apply at the centres of the sides. The arrows shown can be interpreted as the stress resultants. (b) The equation for the moment equilibrium about axis a-a through A, parallel to the y axis, only includes the shear stress resultants shown. All other stress resultants make a zero contribution as they either pass through axis a-a or are parallel to it.

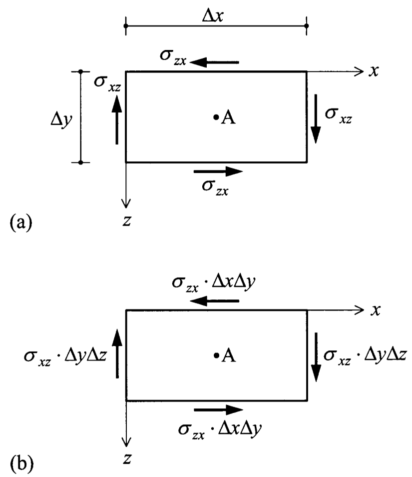


Figure 5.31 (a) Side view of the small block, with the shear stresses that play a role in the equations for the moment equilibrium about A. For the shear stresses, the kernel-index notation has been used. In the limiting case in which the dimensions of the block approach zero, the stresses on opposite sides are equal. (b) The shear stress resultants on the block. Moment equilibrium implies $\sigma_{zx} = \sigma_{xz}$: the shear stresses on the two perpendicular planes are equal.

Figure 5.31a gives a side view of the block with the shear stresses that play a role in the equations for the moment equilibrium about A. The kernel-index notation has been used for the shear stresses.[1] In the limiting case that the dimensions of the block approach zero, the stresses on the opposite planes are equal.

Figure 5.31b shows the resultants of the shear stresses. The resultants on the upper and lower plane, $\sigma_{zx}\Delta x\Delta y$, form a couple $(\sigma_{zx}\Delta x\Delta y)\Delta z$ that acts anti-clockwise about A. The resultants $\sigma_{xz}\Delta y\Delta z$ on the side planes form a couple $(\sigma_{xz}\Delta y\Delta z)\Delta x$ that acts clockwise.

From the moment equilibrium we find:

$$\sum T_y|A = +(\sigma_{zx}\Delta x\Delta y)\Delta z - (\sigma_{xz}\Delta y\Delta z)\Delta x = 0$$

so that

$$\sigma_{zx} = \sigma_{xz}.$$

The shear stresses in the two mutually perpendicular x and z planes are therefore equal.

In the same way, the moment equilibrium $\sum T_z = 0$ gives

$$\sigma_{xy} = \sigma_{yx},$$

[1] See Volume 1, Section 10.1.2.

and $\sum T_x = 0$ gives

$$\sigma_{yz} = \sigma_{zy}.$$

Conclusion: *From the moment equilibrium of a small rectangular element it follows that the shear stresses in two perpendicular directions are equal.* As a formula this means:

$$\sigma_{ij} = \sigma_{ji} \;\text{ with }\; i, j = x, z, y \;\text{ and }\; i \neq j.$$

Figure 5.32 shows the above for the situation in which there is no coordinate system. The shear stress is now denoted by the Greek letter τ.

The shear stresses in two perpendicular planes are equal, and their directions are such that either the arrowheads or the tails are pointed to one another.

Since the situation in Figure 5.32c does not meet these conditions it is not correct: there is no moment equilibrium!

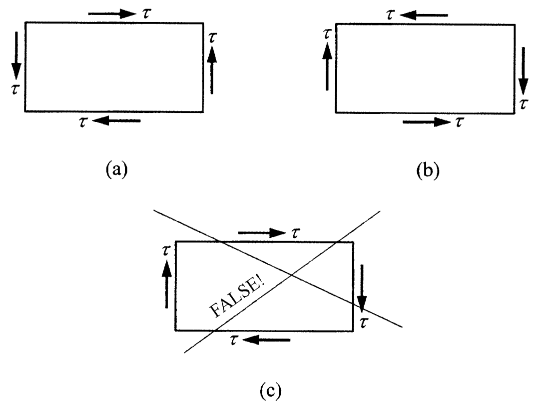


Figure 5.32 (a) and (b) The shear stresses on two perpendicular planes are equal and their directions are such that either the arrowheads or the arrow tails point to one another. Since there is no coordinate system, the shear stresses are denoted by τ. (c) With these shear stresses there is no moment equilibrium, so this picture is false.

5.3.2 Shear stress formulas

Figure 5.33a shows a rectangular cross-section that has to transfer a shear force V_z. To keep the figure simple, the shear force has been drawn outside the cross-section.[1]

The sliding element of the cross-section is hatched in Figure 5.33a. The plane cut PQ has a width b^{a}, and is normal to the sides of the cross-section. To name the shear stresses, an m axis is introduced, normal to the cut PQ,

[1] Attention: V_z is not a "compressive force".

with the arrowhead for the positive direction pointing out of the (hatched) material of the sliding element.

In Figure 5.33b the sliding element has been isolated and s_x^a is shown, the shear force per length in longitudinal direction. This can be determined using the traditional formula (5.7) from Section 5.1.2:

$$s_x^a = -\frac{V_z S_z^a}{I_{zz}} \tag{5.7}$$

in which z must be a principal direction of the cross-section.

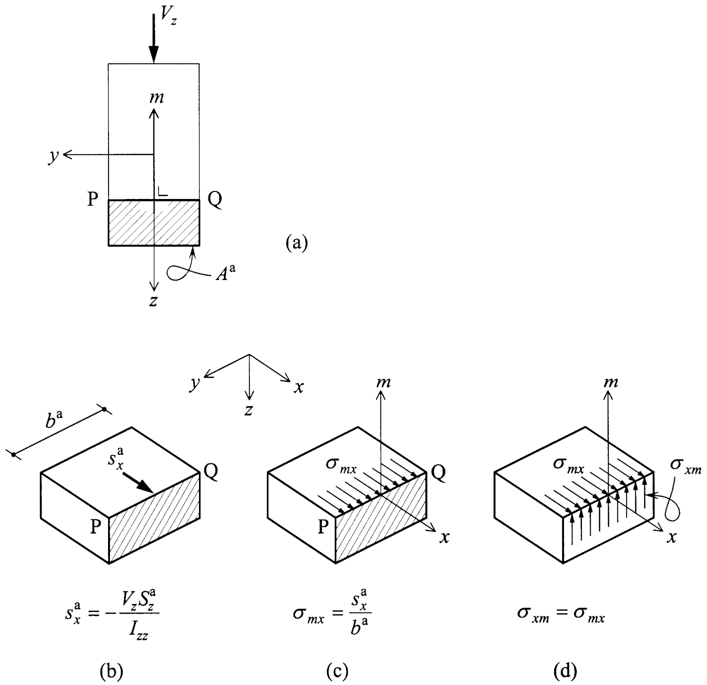


Figure 5.33

Figure 5.33 (a) A rectangular cross-section subject to a shear force V_z. To ensure clarity in the figure, the shear force is placed outside the cross-section. The sliding element of the cross-section is hatched. Plane cut PQ has a width b^a and is normal to the edges of the cross-section. In order to label the shear stresses, the m axis has been introduced, perpendicular to the cut PQ and in such a way that the positive direction, indicated by the arrowhead, points out of the (hatched) material of the sliding part of the cross-section. (b) The sliding element with the longitudinal shear force s_x^a (force per length) on the longitudinal section plane. (c) Smearing the shear force s_x^a (force per length) uniformly over width b^a leads to the longitudinal shear stress σ_{mx} (force per area). (d) In the cut PQ, the longitudinal section plane and the cross-sectional plane are perpendicular to one another. Since the shear stresses on two perpendicular planes are equal, $\sigma_{xm} = \sigma_{mx}$.

s_x^{a} can also be determined using the alternative formula (5.12) from Section 5.1.3:

$$s_x^{\mathrm{a}} = -V_z \cdot \left[\frac{N^{\mathrm{a}} \text{ (due to } M_z^*)}{M_z^*} \right]. \tag{5.12}$$

Here z need not be a principal direction.

The shear force s_x^{a} (force per length) uniformly smeared over the width b^{a} leads to the shear stress σ_{mx} (force per area)[1] (see Figure 5.33c):

$$\sigma_{mx} = \frac{s_x^{\mathrm{a}}}{b^{\mathrm{a}}}.$$

In PQ, the longitudinal section plane and cross-sectional plane are perpendicular to one another, and since the shear stresses on two perpendicular planes are equal to one another, we also know the shear stress on the cross-sectional plane (see Figure 5.33d):

$$\sigma_{xm} = \sigma_{mx} = \frac{s_x^{\mathrm{a}}}{b^{\mathrm{a}}}.$$

With (5.7) this leads to the traditional shear stress formula:

$$\sigma_{xm} = -\frac{V_z S_z^{\mathrm{a}}}{b^{\mathrm{a}} I_{zz}}, \tag{5.14}$$

[1] See also formula (5.13) in Section 5.1.4 in which no xm coordinate system was available.

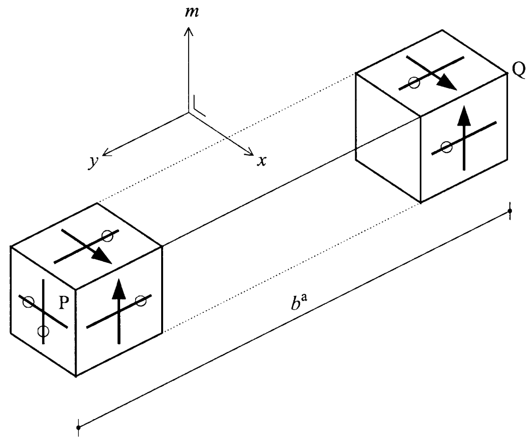


Figure 5.34 The two small boundary elements at P and Q on the rib PQ. Since the outside of the beam is unloaded, there are no shear stresses here. This means that the shear stresses at P and Q parallel to PQ are zero. Only the shear stresses $\sigma_{xm} = \sigma_{mx}$, normal to PQ, can act here.

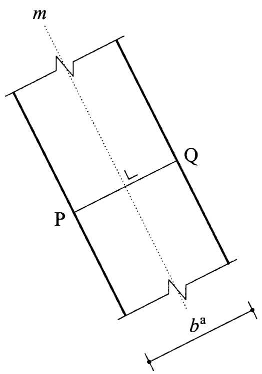


Figure 5.35 When the plane cut PQ is chosen to be normal to the centre line m, and the width b^a is not too large, we can assume that the shear stresses (1) are normal to PQ (and have no component parallel to PQ) over the entire width b^a, and (2) are uniformly distributed over the entire width b^a.

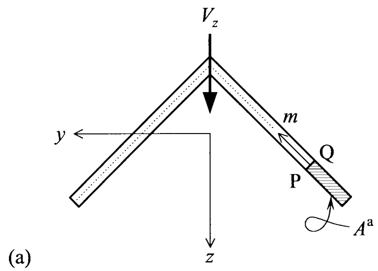


Figure 5.36 (a) A thin-walled angle steel with equal legs has to transfer a shear force V_z in the plane of mirror symmetry. The sliding element is hatched. The cut PQ is normal to the centre line of the leg. The m axis has been chosen along the centre line in such a way that the arrowhead is directed out of the hatched sliding area. The arrowhead indicates the positive m direction.

that only applies when z is a principal direction.

With (5.12) we find an alternative shear stress formula, independent of the principal directions:

$$\sigma_{xm} = -\frac{V_z}{b^a}\left[\frac{N^a \text{ (due to } M_z^*)}{M_z^*}\right]. \tag{5.15}$$

In Figure 5.34 rib PQ has been enlarged and the two small boundary elements at P and Q are shown. Since the *outside of the beam* is unloaded, there are no shear stresses here. This means that *within the beam* the shear stresses at P and Q, parallel to PQ, are zero. Only the shear stresses perpendicular to PQ, $\sigma_{xm} = \sigma_{mx}$, are present.

If the cut PQ is plane, and the width b^a is not too large, we may assume that

- the shear stresses over the entire width b^a are perpendicular to PQ, and have no component parallel to PQ;
- the shear stresses are uniformly distributed over the width b^a.

The fact that the shear stress formulas (5.14) and (5.15) lead to shear stresses that are normal to the cut PQ, and are constant over the width b^a of this cut is purely the result of the assumptions. In many cases these assumptions give a good representation of reality.

The stress formulas (5.14) and (5.15) apply to all profiles (or parts thereof) of which the edges of the cross-section are parallel to one another, such as in Figure 5.35, where only a part of the cross-section is shown.

A condition is that the cut PQ is chosen normal to the centre line. Only in that case are the small boundary elements at P and Q right-angled and, in the same way as above, can we make it plausible that the shear stresses in the cross-section are normal to PQ and parallel to the centre line.

An example is the angle steel with equal legs in Figure 5.36a, with a shear force V_z. The sliding element is hatched. The cut PQ is normal to the centre line. The m axis is parallel to the centre line such that the arrow for the positive direction points out of the material of the hatched sliding element.

Figure 5.36b gives a spatial representation of the sliding element, with the longitudinal shear force s_x^{a} (force per length). Figure 5.36c shows the shear stresses:

$$\sigma_{xm} = \sigma_{mx} = \frac{s_x^{\mathrm{a}}}{b^{\mathrm{a}}} \, .$$

Note that, although the cross-sectional shear force V_z acts vertically, the cross-sectional shear stresses do not act vertically, but are parallel to the centre line(s) of the profile.

Check option: The resultant force due to the shear stresses in the cross-section equals to the cross-sectional shear force V_z by definition.[1]

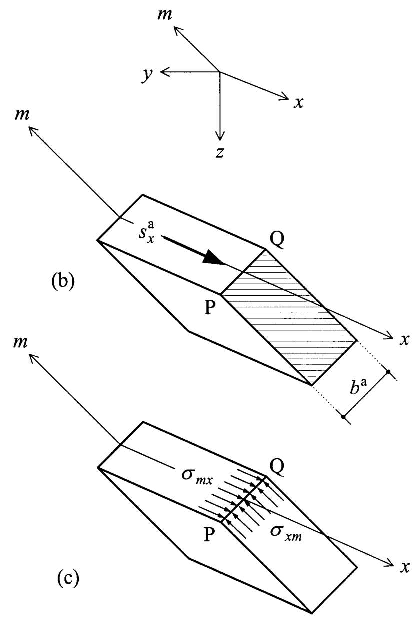


Figure 5.36 (b) A spatial representation of the sliding element, together with the longitudinal shear force s_x^{a} (force per length). (c) The shear stresses $\sigma_{xm} = \sigma_{mx} = s_x^{\mathrm{a}}/b^{\mathrm{a}}$ act normal to PQ. Although the shear force V_z acts vertically, the shear stresses in the cross-section are not vertical but are parallel to the centre lines of the legs.

1 Figure 5.36 shows all quantities in their positive direction. If the shear stress due to the shear force V_z is determined, we find a negative value for σ_{xm}. This means that the actual shear stresses are opposite to the shear stresses in Figure 5.36c. In the cross-sectional plane therefore the actual shear stress acts downwards, which is entirely in line with the direction of the shear force. See also the examples in Section 5.4.

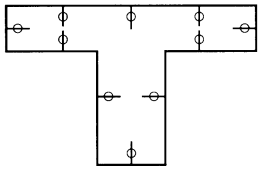


Figure 5.37 The shear stresses perpendicular to the boundary of a cross-section are zero.

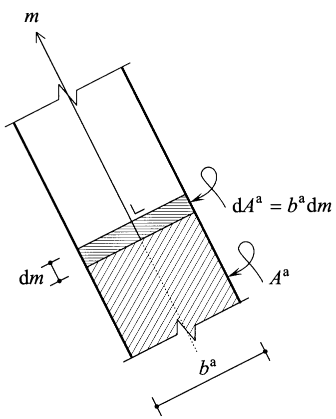


Figure 5.38 Part of a cross-section in which the width b^a of the sliding element is constant (i.e. independent of m).

5.3.3 Rules relating to the shear stress distribution in a cross-section

The fact that there act no shear stresses on the outside of the beam and that the shear stresses in two mutually perpendicular planes are always equal to one another leads to the first of a set of generally applicable rules:

Rule 1. All shear stresses perpendicular to the boundary of a cross-section are zero.

See for example the T-beam in Figure 5.37.

Using the alternative shear stress formula (5.15), derived in Section 5.3.2, it is possible to derive a further three rules for cross-sections (or parts thereof) in which the width b^a is constant. Figure 5.38 shows a part of a cross-section in which b^a is constant, or in other words, is independent of m.

In a slightly changed notation, the shear stress formula (5.15) is

$$\tau = \frac{V_z}{b^a}\left[\frac{N^a \text{ (due to } M_z^*)}{M_z^*}\right]. \tag{5.15}$$

Here τ is the shear stress due to the shear force V_z. The sign is neglected.

Remember that the asterisk indicates that the magnitude of M_z is not relevant here.

By differentiating (5.15) for τ with respect to m we find (b^a is independent of m)

$$\frac{d\tau}{dm} = \frac{V_z}{b^a}\frac{1}{M_z^*}\left[\frac{dN^a}{dm}\text{(due to } M_z^*)\right]. \tag{5.16}$$

If σ is the normal stress at centre line m due to a bending moment M_z^*, again without taking note of the sign, then (see Figure 5.38)

$$\mathrm{d}N^a = \sigma \cdot \mathrm{d}A^{\mathrm{a}} = \sigma \cdot b^{\mathrm{a}}\,\mathrm{d}m,$$

so that

$$\frac{\mathrm{d}N^{\mathrm{a}}}{\mathrm{d}m} = b^{\mathrm{a}}\sigma.$$

Substituting in (5.16) we find

$$\frac{\mathrm{d}\tau}{\mathrm{d}m} = \frac{V_z}{M_z^*}\,\sigma. \tag{5.17}$$

From differential equation (5.17) we can now derive the rules 2 to 4 for a cross-section (or parts thereof) with a constant width b^{a}.

Rule 2. If the bending stress σ is constant, the shear stress τ must be linear (in m).

Rule 3. If the bending stress σ is linear, the shear stress τ must be parabolic (quadratic in m).

Rule 4. The shear stress τ has an extreme in the cut through the normal centre NC; at this point $\sigma = 0$ and so $\mathrm{d}\tau/\mathrm{d}m = 0$.

Rules 1 to 4 allow us to predict the general shape of the shear stress diagram, without extensive calculations. Determining the shear stresses in a limited number of cuts is then generally sufficient for a good sketch of the shear stress diagram.

When determining the shear stresses, it is common practice to neglect the signs and to use absolute values. In that case, it is usual to indicate the shear stress by means of τ.[1] If necessary, the directions of the shear stresses can de derived afterwards from the direction of the shear force. The cross-sectional shear force is after all always the resultant of all shear stresses in the cross-section.

The use of the shear stress formulas, whether or not in combination with the rules derived here, is illustrated in Section 5.4 with the help of a number of examples.

5.4 Examples relating to the shear stress distribution in a cross-section

Below, the shear stress formulas derived in the previous section are applied to a number of simple cross-sectional shapes. In addition to a beam with rectangular cross-section and a T-beam we will look at open thin-walled cross-sections and hollow thin-walled cross-sections. In all these cases, the cross-section is mirror symmetrical and the shear force acts in the plane of symmetry.

Next, we determine the shear stress distribution in a solid triangular cross-section and a solid circular cross-section. In both cross-sections the width of the sliding element is not constant but varies. In such situations the shear stress formulas have to be used with care.

If the shear force is not acting in the plane of mirror symmetry, the line of action of the resultant of all shear stresses in the cross-section (the

[1] If one is not consistent with the sign convention $\sigma_{xm} = \sigma_{mx}$ in the kernel-index notation, this soon leads to errors. In such a case, it is preferable to indicate the normal stress with the letter σ and a shear stress with the letter τ.

shear force) usually does not pass through the normal centre NC. This leads to a new special point in the cross-section: the *shear force centre* or *shear centre* SC. The shear centre is the point in the cross-section through which the line of action of the shear force must pass so that there will be no torsion.

This section is closed with a number of examples in which we determine the location of the shear centre.

5.4.1 A beam with rectangular cross-section and a T-beam

Example 1: Beam with rectangular cross-section
The beam with rectangular cross-section in Figure 5.39a has to transfer the shear force V shown.

Questions:
a. Determine the shear stress distribution in the cross-section.
b. Determine the resultant of all shear stresses in the cross-section.

Solution:
a. The hatched sliding element in Figure 5.39a has been shown separately in Figure 5.39b. The shear stress σ_{xm} at a distance z under the normal centre NC can be found using formula (5.14) from Section 5.3.2:

$$\sigma_{xm} = -\frac{V_z S_z^{\mathrm{a}}}{b^{\mathrm{a}} I_{zz}}. \tag{5.14}$$

Here

$$V_z = V, \; b^{\mathrm{a}} = b, \; \text{ and } \; I_{zz} = \tfrac{1}{12} bh^3.$$

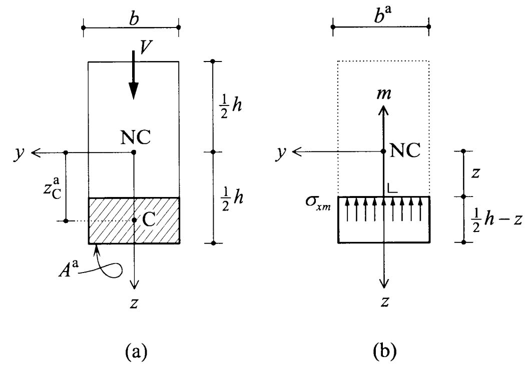


Figure 5.39 (a) A beam with a rectangular cross-section has to transfer shear force V shown. The sliding element of the cross-section is hatched. (b) The sliding element of the cross-section with the shear stress σ_{xm} at the cut.

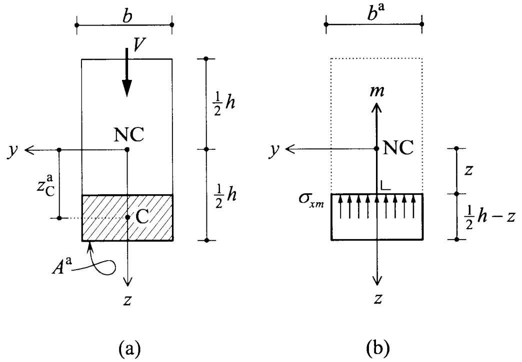


Figure 5.39 (a) A beam with a rectangular cross-section has to transfer shear force V shown. The sliding element of the cross-section is hatched. (b) The sliding element of the cross-section with the shear stress σ_{xm} at the cut.

Furthermore,

$$S_z^a = A^a \cdot z_C^a$$

in which

$$A^a = b \cdot \left(\tfrac{1}{2}h - z\right),$$

and

$$z_C^a = z + \tfrac{1}{2} \cdot \left(\tfrac{1}{2}h - z\right) = \tfrac{1}{2}\left(\tfrac{1}{2}h + z\right).$$

This gives

$$S_z^a = A^a \cdot z_C^a = b\left(\tfrac{1}{2}h - z\right) \cdot \tfrac{1}{2}\left(\tfrac{1}{2}h + z\right) = \tfrac{1}{2}b\left(\tfrac{1}{4}h^2 - z^2\right).$$

Substitution in shear stress formula (5.14) leads to

$$\sigma_{xm} = -\frac{V \cdot \frac{1}{2}b\left(\frac{1}{4}h^2 - z^2\right)}{b \cdot \frac{1}{12}bh^3} = -\frac{3}{2}\frac{V}{bh}\left(1 - 4\frac{z^2}{h^2}\right).$$

The shear stress σ_{xm} is quadratic in z: the distribution of the shear stresses over the height of the cross-section is parabolic.

Figure 5.40a shows the parabolic shear stress distribution in a *shear stress diagram*. Significant values include those at the boundaries $z = \pm\frac{1}{2}h$ and at the centre $z = 0$:

$z = \pm\frac{1}{2}h \Rightarrow \sigma_{xm} = 0$ (see also Section 5.3.3, rule 1),

$z = 0 \quad \Rightarrow \sigma_{xm} = -\frac{3}{2}\frac{V}{bh}$ (vertex of the parabola; see Section 5.3.3, rule 4).

The shear stress σ_{xm} is negative across the entire height of the beam ($-\frac{1}{2}h < z < +\frac{1}{2}h$). This means that the shear stresses act in the negative m direction everywhere, in line with the direction of the shear force in Figure 5.40b.

Plus and minus signs are not used in the shear stress diagram in Figure 5.40a. Instead thereof a *visual notation* has been used with arrows indicating the direction of the shear stresses on the cross-sectional plane. In a such case, one usually omits the m direction in the diagram and refers to the shear stress as τ.

The average vertical shear stress τ_{average} is found by uniformly smearing the shear force V over the area A of the cross-section:

$$\tau_{\text{average}} = \frac{V}{A} = \frac{V}{bh}.$$

However, the vertical shear stresses cannot be uniformly distributed as they have to be zero at the top and bottom of the rectangular cross-section (see Section 5.3.3, rule 1). Therefore the shear stresses near the centre of the cross-section have to be larger than the average shear stress.

The maximum shear stress $\tau_{\max}$ occurs in the cut through the normal centre NC (see Section 5.3.3, rule 4):

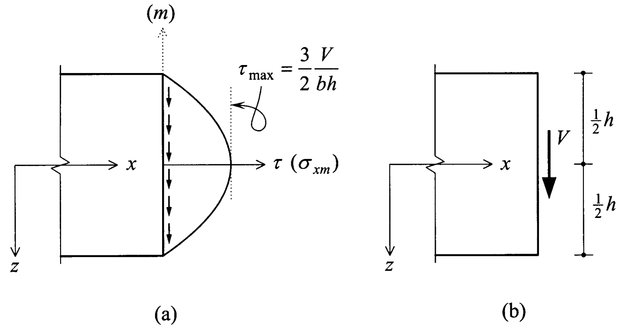


Figure 5.40 (a) The distribution of the shear stresses over the height of the cross-section can be plotted in a shear stress diagram. Here we have used a visual notation: arrowheads indicate the direction in which the shear stresses actually act. In such a case, the m direction is generally omitted and the shear stresses are denoted by τ. (b) The direction of the shear stresses is in line with the direction of the shear force V.

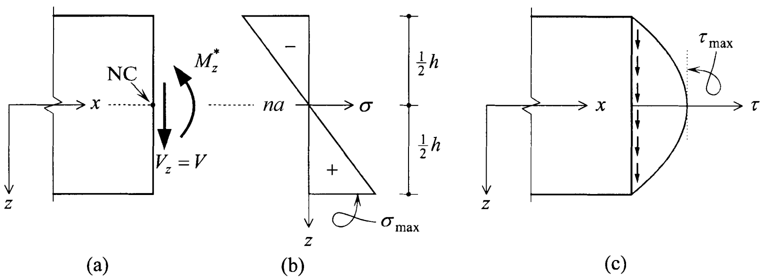


Figure 5.41 (a) Side view of a small beam segment to the left of a cross-section in which a shear force $V_z = V$ acts and a bending moment M_z^* of arbitrary value. (b) The normal stress distribution due to the bending moment M_z^*. (c) Since the normal stress is linear over the height of the cross-section, the shear stress is parabolic (rule 3). The shear stress is largest at the level of the normal centre NC (rule 4). This is the vertex of the parabola. The direction of the shear stresses follows from the direction of the shear force. The shear stresses are zero at the upper and lower edges (rule 1). These rules are sufficient to quickly sketch the shear stress diagram. To complete the diagram only τ_{max} has to be calculated.

$$\tau_{max} = \frac{3}{2}\frac{V}{bh}.$$

The maximum shear stress turns out to be 50% larger than the average shear stress.

Alternative solution to question a:
a. If you do not have to know the shear stress distribution as a function of m, a quick alternative solution is possible using the rules in Section 5.3.3.

Figure 5.41a shows a small beam segment to the left of the cross-section in which the shear force $V_z = V$ and an arbitrary bending moment M_z^* act. The normal stress distribution due to M_z^* is linear and is shown in Figure 5.41b. The shear stress distribution due to $V_z = V$ is therefore parabolic (rule 3) (see Figure 5.41c). The shear stresses are vertical and act in the direction of the shear force in Figure 5.41a. In addition, the shear stresses are zero at the top and bottom of the cross-section (rule 1). The maximum shear stress (the vertex of the parabola) is located at the level of the normal centre (rule 4).

The maximum shear stress can be determined using Figure 5.42:

$$\tau_{max} = \left|\frac{V_z S_z^a}{b^a I_{zz}}\right| = \frac{V \cdot \frac{1}{2}bh \cdot \frac{1}{4}h}{b \cdot \frac{1}{12}bh^3} = \frac{3}{2}\frac{V}{bh}.$$

τ_{max} can also be determined from shear stress formula (5.15), using the normal stress diagram in Figure 5.41b:

$$\tau = \frac{V_z}{b^a}\left[\frac{N^a \text{ (due to } M_z^*)}{M_z^*}\right]. \tag{5.15}$$

If we are concerned with the maximum shear stress, at the level of the normal centre NC, then N^a (due to M_z^*) is the resultant of all normal stresses at one side of the neutral axis (na) (see Figure 5.41b):

$$N^a \text{ (due to } M_z^*) = \tfrac{1}{2} \cdot \sigma_{max} \cdot \tfrac{1}{2} h \cdot b = \tfrac{1}{4} bh\sigma_{max}.$$

With

$$\sigma_{max} = \frac{M_z^*}{W} = \frac{M_z^*}{\frac{1}{6} bh^2}$$

we find

$$N^a \text{ (due to } M_z^*) = \tfrac{1}{4} bh \cdot \frac{M_z^*}{\frac{1}{6} bh^2} = \frac{3M_z^*}{2h}.$$

Formula (5.15) now gives

$$\tau_{max} = \frac{V}{b} \cdot \frac{\frac{3M_z^*}{2h}}{M_z^*} = \frac{3}{2} \frac{V}{bh}.$$

b. The resultant R of all shear stresses in the cross-section is equal to the volume of the spatial stress diagram in Figure 5.43. Since the area under a parabola is equal to two-thirds of the area of the rectangular base, we find:

$$R = \tfrac{2}{3} \cdot h\tau_{max} \cdot b = \tfrac{2}{3} \cdot h \cdot \frac{3}{2} \frac{V}{bh} \cdot b = V.$$

In a cross-section the resultant of all shear stresses indeed is equal to the shear force.

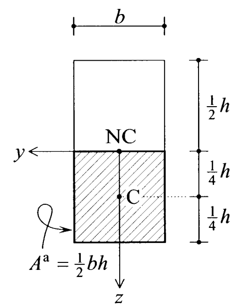


Figure 5.42 The maximum shear stress occurs in the plane cut through the normal centre NC, coinciding with the neutral axis in the case of bending without extension.

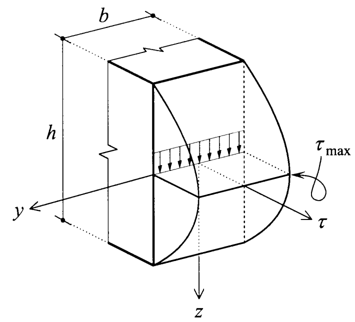


Figure 5.43 The resultant of all shear stresses in the cross-section (this is the cross-sectional shear force) is equal to the volume of the spatially shown shear stress diagram.

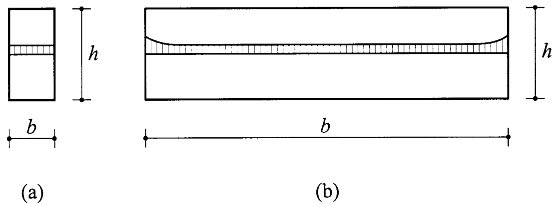


Figure 5.44 The shear stress formulas are based on the assumption that the shear stresses are uniformly distributed over the width $b^a = b$. This is a good assumption for narrow cross-sections with $b \ll h$. For wider cross-sections the shear stress along the edges is larger than that at the centre: (a) with $b/h = 0.5$ the difference is only 4% and (b) with $b/h = 4$ it is 100%.

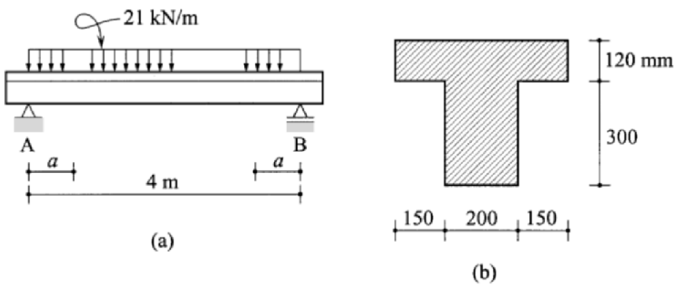


Figure 5.45 (a) A simply supported concrete T-beam with a uniformly distributed load. (b) The cross-sectional dimensions.

Comment: When deriving the shear stress formulas in Section 5.3.2 we *assumed* that the shear stresses are uniformly distributed over the width $b^a = b$ of the cut (see Figure 5.44a). This turns out to be correct only for narrow cross-sections with $b \ll h$. For wider cross-sections the shear stress along the edges is larger than at the centre (see Figure 5.44b), and the maximum shear stresses are larger than those calculated with the formulas derived here. For a rectangular cross-section with $b/h = 0.5$ the difference is only 4%, with $b/h = 1$ this is 13% and with $b/h = 4$ the difference is 100%.

Example 2: T-beam

The simply supported concrete T-beam in Figure 5.45 has a span of 4 m and carries a uniformly distributed load of 21 kN/m over the full span. The cross-sectional dimensions are given in Figure 5.45b. The vertical shear stresses in the beam may not exceed the value $\bar{\tau} = 0.5$ kN/m, also called the allowable shear stress.

Questions:

a. For the cross-section directly to the left of support B, draw the distribution of the vertical shear stresses due to the shear force.
b. Over which length a are the shear stresses too large, and will additional components be required, such as extra stirrups or bent up reinforcing bars.

Solution:

a. First, the location of the normal centre NC and the magnitude of the centroidal moment of inertia I_{zz} are determined for the cross-section. For that purpose the cross-section is split into a web with area A_1 and a flange with area A_2 (see Figure 5.46):

$$A_1 = (200 \text{ mm})(300 \text{ mm}) = 60 \times 10^3 \text{ mm}^2,$$

$$A_2 = (500 \text{ mm})(120 \text{ mm}) = 60 \times 10^3 \text{ mm}^2.$$

Since the areas A_1 and A_2 are equally large the normal centre NC is located precisely between the centroids of rib and flange.

This location can also be determined formally in the $\overline{yz}$ coordinate system given in Figure 5.46:

$$\bar{z}_{\text{NC}} = \frac{S_{\bar{z}}}{A} = \frac{(270 \text{ mm}) \times A_1 + (60 \text{ mm}) \times A_2}{A_1 + A_2} = 165 \text{ mm}.$$

For the (centroidal) moment of inertia I_{zz} we now find:

$$I_{zz} = \tfrac{1}{12}(200 \text{ mm})(300 \text{ mm})^3 + (200 \text{ mm})(300 \text{ mm})(105 \text{ mm})^2$$

$$+ \tfrac{1}{12}(500 \text{ mm})(120 \text{ mm})^3 + (500 \text{ mm})(120 \text{ mm})(105 \text{ mm})^2$$

$$= 1845 \times 10^6 \text{ mm}^4.$$

Next we determine the shear stress distribution in the cross-section.

The V diagram for the beam modelled as a line element is given in Figure 5.47.

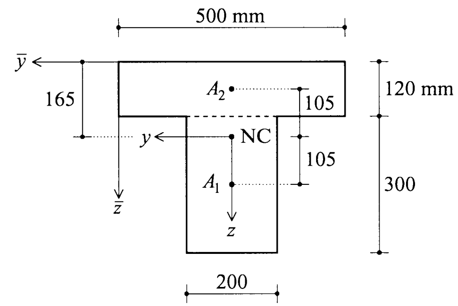


Figure 5.46 The location of the normal centre NC in the cross-section.

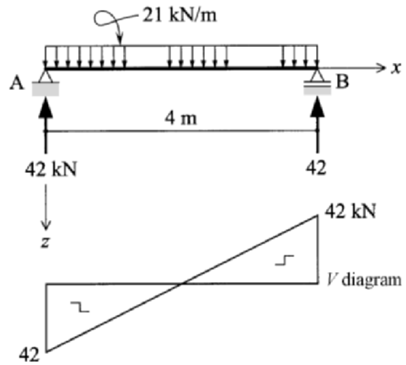


Figure 5.47 The T-beam modelled as a line element, with its V diagram.

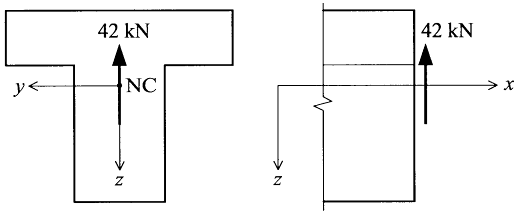


Figure 5.48 The (cross-sectional) shear force directly to the left of support B.

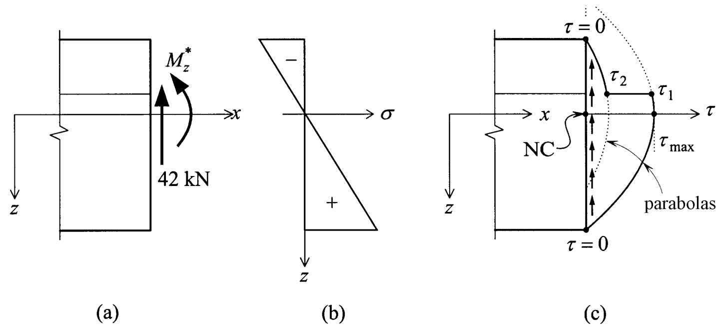


Figure 5.49 (a) The (cross-sectional) shear force directly to the left of support B, together with a bending moment M_z^* of arbitrary value. (b) The normal stress distribution due to M_z^*. (c) A sketch of the shear stress distribution. The vertical shear stresses are zero at the upper and lower edges of the T-beam. Since the bending stress is linear over the height, the shear stress distribution will be parabolic. Due to the difference in width b^a, the parabolic distributions for web and flange are not the same. Both parabolas have their vertex at the level of the normal centre NC.

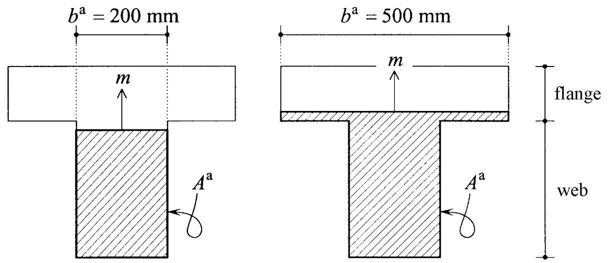


Figure 5.50 Due to the step change of the width b^a at the joint between flange and web, the shear stress distribution "*jumps*" from one parabola to the other.

Figure 5.48a shows the cross-section at support B, on which a shear force of 42 kN acts. Figure 5.48b gives the side view of a part of the beam directly to the left of support B, together with the shear force of 42 kN.

Figure 5.49a shows this part again, but this time there is also a bending moment M_z^*. Note: this is not the actual bending moment; this bending moment with an arbitrary value is introduced only for the sake of rules 2 to 4 in Section 5.3.3. These rules make it possible to make an adequate sketch of the shear stress distribution with relative little calculation. Since rules 2 to 4 apply only when the width b^a is constant, the web and flange of the T-beam have to be treated separately.

Applying rule 1:
The vertical shear stresses are zero at the bottom edge of the web, and also at the top and bottom edges of the flange.

Applying rule 3:
Figure 5.49b shows the normal stress diagram due to the bending moment

M_z^*. The normal stresses vary linearly across the height of the beam, i.e. linearly across the web and linearly across the flange. This means that the shear stresses due to the shear force V_z vary parabolically. As a result of the difference in width b^a the parabolic shear stress distributions for web and flange are not the same.

Applying rule 4:
The shear stress is a maximum at the level of the normal centre NC; this is the vertex of the parabolic shear stress diagram.

Using this information, we can now make a good sketch of the shear stress diagram (see Figure 5.49c). The direction of the shear stresses follows from the direction of the shear force in Figure 5.49a. As a result of the step change in the width b^a where the flange meets the web (see Figure 5.50), the shear stress distribution "jumps" from one parabola to the other (see Figure 5.49c).

To complete the sketch we have to determine the values of τ_{max}, τ_1 and τ_2. For that we use shear stress formula (5.14) with absolute values:

$$\tau = \left| \frac{V_z S_z^a}{b^a I_{zz}} \right| .$$

Calculation of τ_{max}, the maximum shear stress at the level of the normal centre NC (see Figure 5.51a):

$$\tau_{max} = \frac{(42 \times 10^3 \text{ N})(200 \text{ mm})(255 \text{ mm})(252/2 \text{ mm})}{(200 \text{ mm})(1845 \times 10^6 \text{ mm}^4)} = 0.74 \text{ N/mm}^2.$$

Calculation of τ_1, the shear stress at the upper boundary of the web (see Figure 5.51b):

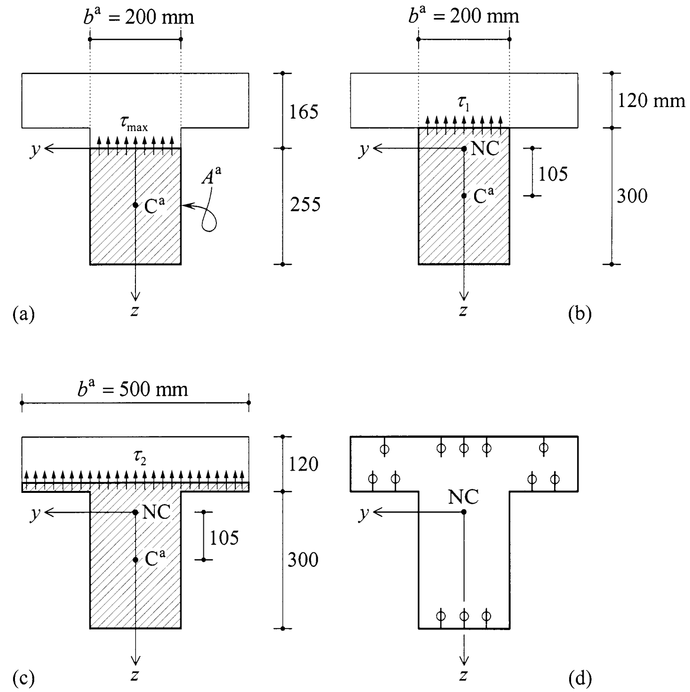


Figure 5.51 (a) Calculation of τ_{max}, the maximum shear stress at the level of normal centre NC: $b^a = 200$ mm. (b) Calculation of τ_1, the shear stress at the upper boundary of the web: $b^a = 200$ mm. (c) Calculation of τ_2, the shear stress at the lower boundary of the flange: $b^a = 500$ mm. (d) The actual shear stresses at the lower boundary of the flange have to be zero over a large proportion of the width b^a.

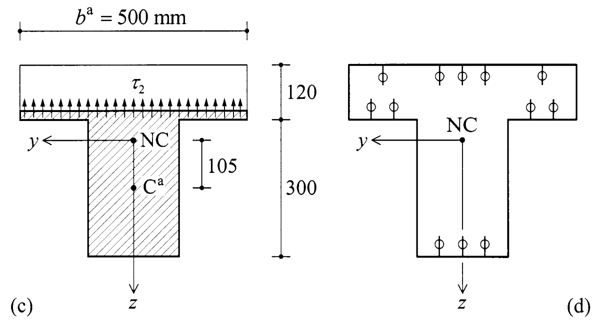


Figure 5.51c and d (c) Calculation of τ_2, the shear stress at the lower boundary of the flange: $b^a = 500$ mm. (d) The actual shear stresses at the lower boundary of the flange have to be zero over a large proportion of the width b^a.

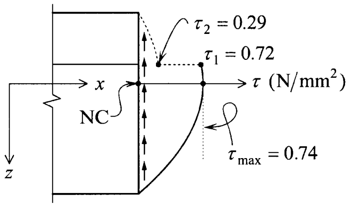


Figure 5.52 The shear stress diagram. The shear stress τ_2 at the lower boundary of the flange is based on the assumption that the shear stress is constant over the total width of the flange. This is not realistic as it is zero for a large proportion of the width. Here the shear stress formula fails and indicates only an average value. If one is interested in the actual values, the shear stress diagram for the flange is of little practical significance. This part of the diagram has therefore been shown by a dotted line.

$$\tau_1 = \frac{(42 \times 10^3 \text{ N})(200 \text{ mm})(300 \text{ mm})(105 \text{ mm})}{(200 \text{ mm})(1845 \times 10^6 \text{ mm}^4)} = 0.72 \text{ N/mm}^2.$$

Calculation of τ_2, the shear stress at the lower boundary of the flange (see Figure 5.51c):

$$\tau_2 = \frac{(42 \times 10^3 \text{ N})(200 \text{ mm})(300 \text{ mm})(105 \text{ mm})}{(500 \text{ mm})(1845 \times 10^6 \text{ mm}^4)} = 0.29 \text{ N/mm}^2.$$

Figure 5.52 shows the shear stress diagram with the values determined above.

When determining τ_1 and τ_2, at the upper boundary of the web and the lower boundary of the flange, we use the same values of V_z, S_z^a and I_{zz} in the shear stress formula. Only the values for b^a are different: τ_1 is related to the width of the web and τ_2 to the width of the flange (see Figures 5.51b and 5.51c).

The shear stress formula is based on the *assumption* that the shear stress is constant over width b^a. In Figure 5.51c a uniform distribution of the shear stresses τ_2 over the width of the flange is not realistic as, except at the web, no vertical shear stresses can act at the bottom edge of the flange (rule 1) (see Figure 5.51d).

It is not known how the vertical shear stresses vary across the width of the flange. Here the shear stress formula fails and gives only the average shear stress. For the actual values, this part of the shear stress diagram is of little practical significance. Therefore it is shown by a dotted line in Figure 5.52.

Comment: The formulas (5.7) and (5.12), for the *shear force per length* in a cut, still remain valid.

b. The maximum vertical shear stress τ_{max} in the cross-section occurs at the level of the normal centre NC and is proportional to the magnitude of the cross-sectional shear force V. With the solution of question a we found that, due to a shear force $V = 42$ kN, the maximum vertical shear stress is $\tau_{max} = 0.74$ N/mm^2. Hence it follows that

$$\tau_{max} = \frac{V}{42 \text{ kN}} \times (0.74 \text{ N/mm}^2).$$

Figure 5.53a shows the V diagram for the T-beam. If we plot τ_{max} as a function of the location x of the cross-section, we find a diagram that is similar to the V diagram (see Figure 5.53b).

In the hatched areas in Figure 5.53b, the distance to the zero line is larger than the allowable shear stress $\bar{\tau}$:

$$\tau_{max} > \bar{\tau} = 0.5 \text{ N/mm}^2.$$

Here the shear stresses are too large and additional measures will have to be taken.

The distance a follows from the similarity of the hatched triangle and triangle PQR:

$$\frac{a}{2 \text{ m}} = \frac{(0.74 - 0.5) \text{ N/mm}^2}{0.74 \text{ N/mm}^2} \Rightarrow a = 0.65 \text{ m}.$$

Note: If a concentrated force acts on a beam, such as in this example at the supports, there is a disruption in the calculated stress distribution. The disruption occurs across a length of the order of the beam height and is a result of the fact that the concentrated force "needs some length to be introduced"

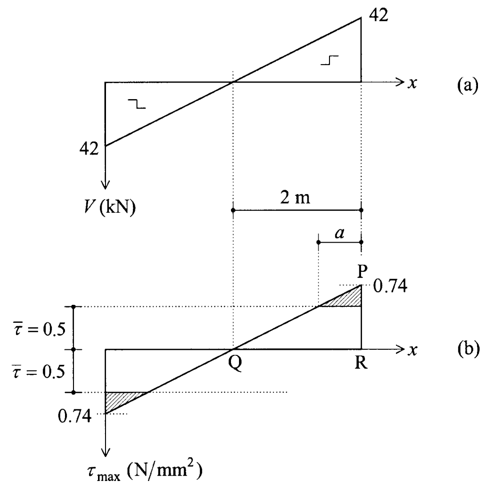


Figure 5.53 (a) Shear force diagram (V diagram) for the T-beam. (b) Plotting the maximum shear stress τ_{max} as a function of the location x of the cross-section, gives a diagram that is similar to the shear force diagram. In the hatched areas, the actual shear stress is too large and additional measures will have to be taken, such as extra stirrups or bent up reinforcing bars.

into the structure. This phenomenon is not covered in elementary beam theory, nor in this example.

Comment: Using computer programs based on the so-called *finite element method* we can learn more about the stress distribution in areas with disruptions or, as in question a, about the actual shear stress distribution in the flange of a T-beam.

5.4.2 Thin-walled open cross-sections

In this section we determine the shear stress distribution due to shear for a number of thin-walled cross-sections. In all cases, the cross-section is mirror symmetrical and the shear force acts in the plane of mirror symmetry.

Example 1: An I-section

A thin-walled I-section, with the cross-sectional dimensions given in Figure 5.54, has to transfer the shear force $V_z = V$ as shown. The shear force has been drawn outside the cross-section for sake of clarity.[1] In the calculation use $b = h$ and $t_f = t_w = t$.[2]

Questions:

a. Sketch the shear stress distribution in the web, and determine the relevant values.
b. Sketch the shear stress distribution in the flanges, and determine the relevant values.

---

1 In Figure 5.54, the force V is not a compressive force.

2 The index “f” for the wall thickness t relates to the flange; the index “w” relates to the web.

c. Determine the resultant of all shear stresses in the cross-section.

Solution:
a. In Figure 5.55, a cut has been introduced in the web, normal to the centre line. The location of the cut is determined by an auxiliary coordinate m_1. The sliding element of the cross-section has been hatched.

The shear stress is determined using formula (5.15):

$$\sigma_{xm} = -\frac{V_z S_z^{\mathrm{a}}}{b^{\mathrm{a}} I_{zz}}, \qquad (5.15)$$

in which

$$I_{zz} = \tfrac{1}{12} t_{\mathrm{w}} h^3 + 2 \cdot t_{\mathrm{f}} b \cdot \left(\tfrac{1}{2} h\right)^2 = \tfrac{7}{12} t h^3,$$

and

$$b^{\mathrm{a}} = t_{\mathrm{w}} = t.$$

In addition

$$S_z^{\mathrm{a}} = t_{\mathrm{f}} b \cdot \tfrac{1}{2} h + t_{\mathrm{w}} m_1 \cdot \left(\tfrac{1}{2} h - \tfrac{1}{2} m_1\right) = \tfrac{1}{2} t h^2 + \tfrac{1}{2} t h m_1 - \tfrac{1}{2} t m_1^2.$$

Substitute these expressions into (5.15) and we find

$$\sigma_{xm} = -\frac{V \cdot \left(\tfrac{1}{2} t h^2 + \tfrac{1}{2} t h m_1 - \tfrac{1}{2} t m_1^2\right)}{t \cdot \tfrac{7}{12} t h^3}$$
$$= -\frac{6}{7} \frac{V}{th} \left(1 + \frac{m_1}{h} - \frac{m_1^2}{h^2}\right).$$

The shear stresses in the web are quadratic in m_1 (parabolic).

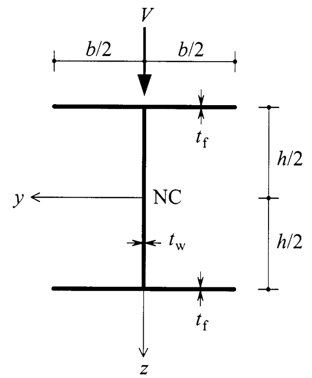


Figure 5.54 A thin-walled I-section has to transfer shear force $V_z = V$ in the vertical plane of mirror symmetry.

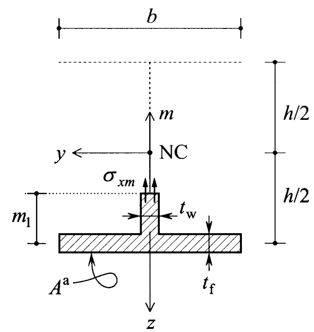
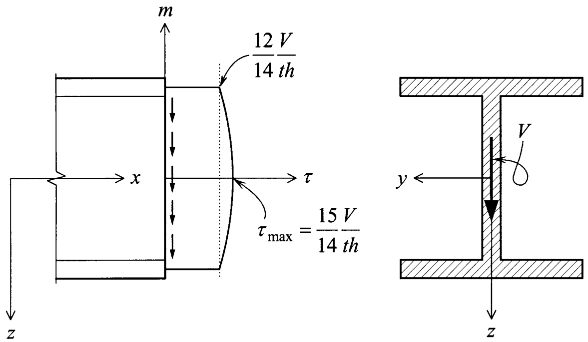


Figure 5.55 To determine the shear stresses in the web, a cut normal to the centre line m is introduced. The location of the cut is determined by the auxiliary coordinate m_1. The sliding part of the cross-section is hatched.



(a) (b)

Figure 5.56 (a) The shear stress diagram for the web (determined with respect to the centre lines of the I-section): the shear stresses are parabolic and have their top values at the level of the normal centre NC. (b) The direction of the shear stresses in the web is in line with the direction of the cross-sectional shear force.

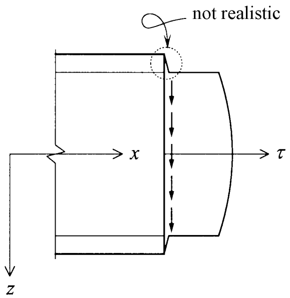


Figure 5.57 The diagram for the vertical shear stresses in the I-section as often seen in the literature. The jump in the shear stress distribution is caused by the difference in width of web and flange. For the actual magnitude of the shear stresses, the values in the flange are not realistic, and are best omitted.

The location of the maximum (the vertex of the parabola) follows from

$$\frac{\mathrm{d}\sigma_{xm}}{\mathrm{d}m_1} = -\frac{6}{7}\frac{V}{th}\left(0 + \frac{1}{h} - \frac{2m_1}{h^2}\right) = 0 \Rightarrow m_1 = \tfrac{1}{2}h.$$

This, as can be expected from rule 4, is at the level of the normal centre NC.

Relevant values for creating a good sketch of the shear stress distribution are found in

$$m_1 = 0 \quad \Rightarrow \sigma_{xm} = -\frac{6}{7}\frac{V}{th}$$

$$m_1 = \tfrac{1}{2}h \Rightarrow \sigma_{xm} = -\frac{15}{14}\frac{V}{th} \quad \text{(vertex of the parabola).}$$

Figure 5.56a shows the shear stress diagram for the web. Here we have used the *visual notation*: the shear stresses are represented by arrows with their heads in the directions in which they actually act on the cross-sectional plane.

The three calculated shear stresses are negative and act opposite to the m direction, that is in the positive z direction. This direction holds for the whole web ($0 < m_1 < h$), and is in line with the direction of the shear force V in Figure 5.56b.

Comment: In the literature, the diagram in Figure 5.57 is often used for the vertical shear stress distribution in an I-section. The step change in the shear stress distribution is caused by the difference in width between web and flange. From the previous section we know that these values in the flange are *not realistic* and therefore are best omitted!

b. In Section 5.3.2 it was stated in general that in (those parts of) a cross-section with a constant width b^a measured normal to the centre line, the following apply:

- the shear stresses are parallel to the centre line;
- the shear stresses are uniformly distributed across the width b^a.

A condition is that the width b^a is relatively small compared to the length of the centre line of (the relevant part of) the cross-section. *Thin-walled cross-sections* meet this condition by definition.

For a thin-walled I-section, this means that the shear stresses in the flanges are horizontal, regardless of the (vertical) direction of the shear force. The background is illustrated briefly in Figure 5.58.

In Figure 5.58a a cut has been introduced at P normal to the centre line of the upper flange. The hatched part to the right of the cut is considered the sliding element. This is spatially shown in Figure 5.58b for a beam segment with small length Δx.

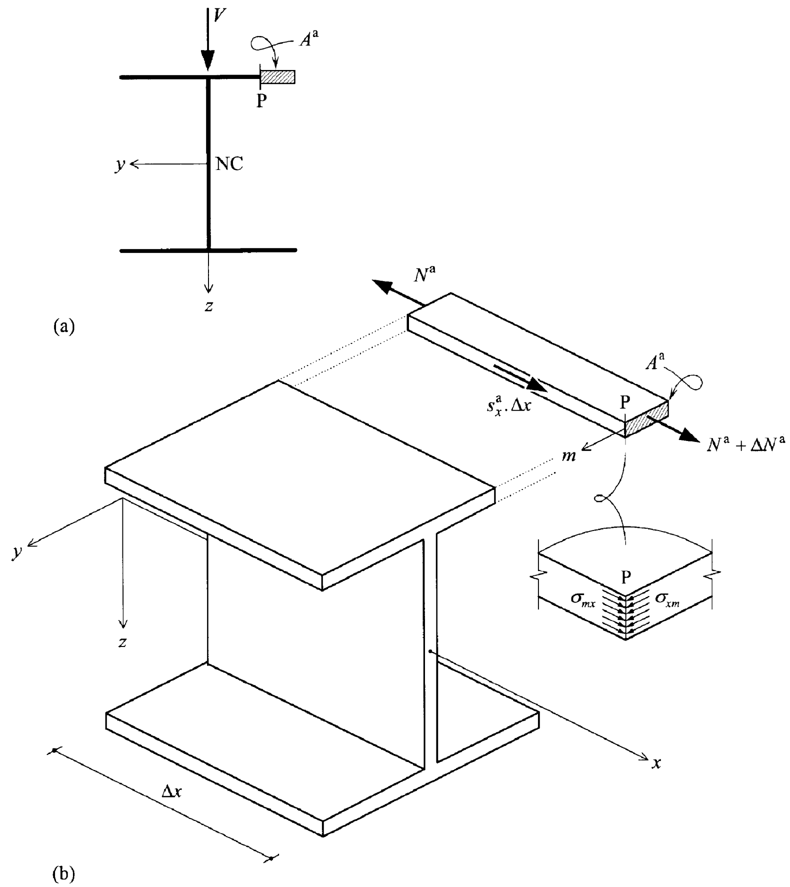


Figure 5.58 (a) To calculate the shear stresses in a flange, a cut normal to the centre line is introduced at P. The part to the right of the cut is chosen as the sliding part of the cross-section, and is hatched. (b) The sliding element of the flange has been isolated from the rest of the small beam segment with length Δx. The longitudinal shear force s_x^a (force per length) follows from the force equilibrium in x direction of the sliding element. At P, the shear force s_x^a leads to the longitudinal shear stress σ_{mx}. Since the shear stresses on two mutually perpendicular planes are equal, there is a horizontal shear stress $\sigma_{xm} = \sigma_{mx}$ in the flange. In the cross-section, the shear stresses in the flanges are horizontal, even though the shear force acts in the vertical direction.

Figure 5.58

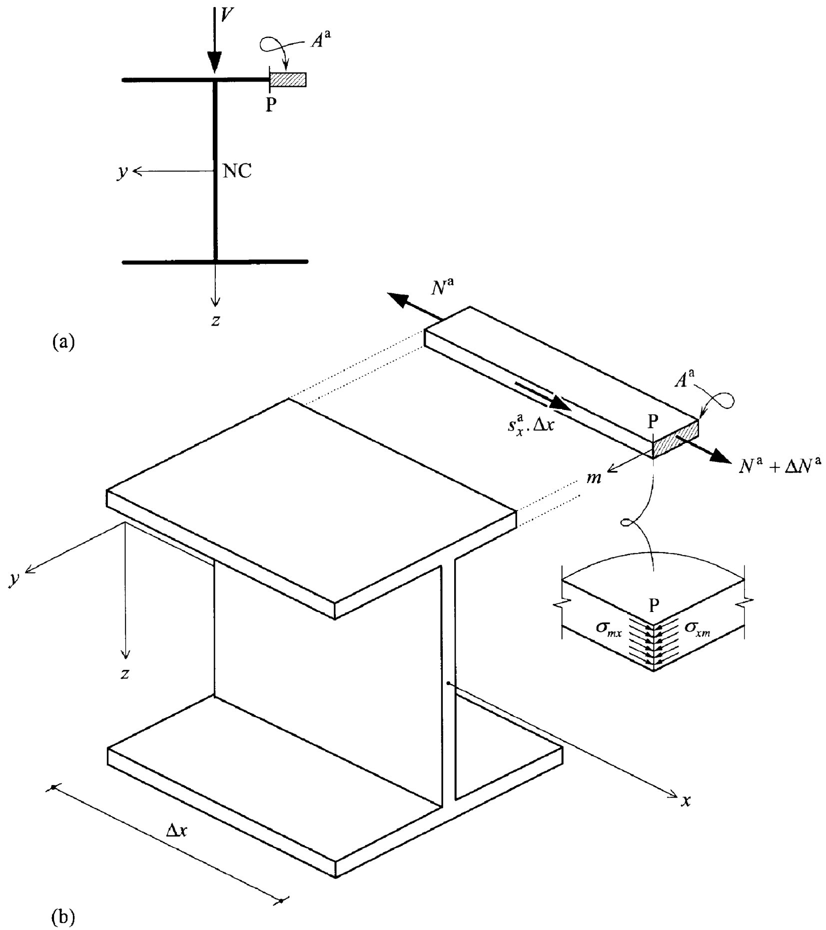


Figure 5.58

If there is a shear force, the bending moments on the front and back cross-sectional plane of the small beam segment are not of the same magnitude. This follows from

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V \quad \text{or} \quad \Delta M = \int V\,\mathrm{d}x.$$

Hence the bending stresses[1] on the front and back cross-sectional planes are different.

Assume that the resultant of all normal stresses on the back plane of the sliding element is N^{a}, and that on the front plane is $N^{\mathrm{a}} + \Delta N^{\mathrm{a}}$. The sliding element can be in equilibrium only if a longitudinal shear force acts in the cut. With s_x^{a} as shear force per length, the resultant shear force in the cut with small length Δx is $s_x^{\mathrm{a}}\Delta x$.

For a small wall thickness t, it can be assumed that s_x^{a}, the shear force per length, is the resultant of a shear stress σ_{mx} which is uniformly distributed across the wall thickness:

$$\sigma_{mx} = \frac{s_x^{\mathrm{a}}}{t}\,.$$

Since the shear stresses on two mutually perpendicular planes are equal[2] the horizontal cross-sectional shear stress σ_{xm} in the flange is equal to the shear stress σ_{mx} in the longitudinal cut.

We will now determine the shear stress distribution in the top flange.

1 Bending stresses are normal stresses due to bending.
2 This follows from the moment equilibrium of a small parallelepiped, see Section 5.3.1.

In order to speed up matters, a symmetrical *double cut* has been introduced across both flange halves in Figure 5.59. The location of the double cut is defined by the distance m_2 to the flange edges.

Since the z axis is a line of mirror symmetry with respect to the cross-section and the load, the shear force in the double cut will distribute itself evenly across the single cuts a-a and b-b. Hence, the shear stresses in a-a and b-b will be equal and opposite.

Figure 5.60 is a magnified representation of the sliding elements, with the m directions and the shear stress σ_{xm}. Remember that for the calculation we have to use $b = h$ and $t_f = t$.

The following hold:

$$V_z = V,$$

$$S_z^a = 2 \cdot m_2 t_f \cdot \left(-\tfrac{1}{2}h\right) = -thm_2,$$

$$b^a = 2t_f = 2t \quad \text{(Note: this is a double cut!)},$$

$$I_{zz} = \tfrac{7}{12}th^3.$$

Substitute these values in shear stress formula (5.15) and we find

$$\sigma_{xm} = -\frac{V \cdot (-thm_2)}{2t \cdot \frac{7}{12}h^3} = +\frac{6}{7}\frac{V}{th}\frac{m_2}{h}.$$

In both flange halves the shear stress distribution is linear in m_2. Significant values for a good sketch are found in the edges of the flange ($m_2 = 0$) and at the web ($m_2 = \frac{1}{2}b$):

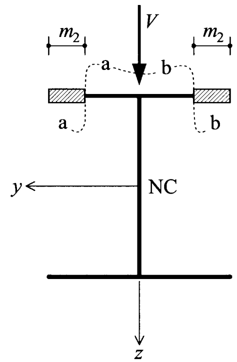


Figure 5.59 To calculate the shear stress distribution in a flange, one can use a symmetrical double cut across both flange halves. The location of the double cut is defined by the distance m_2 to the flange edges.

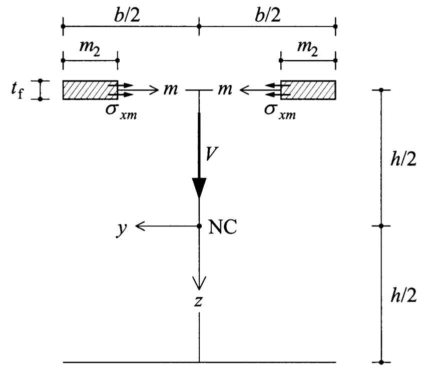


Figure 5.60 The sliding parts of the cross-section, with the m directions and shear stresses σ_{xm}.

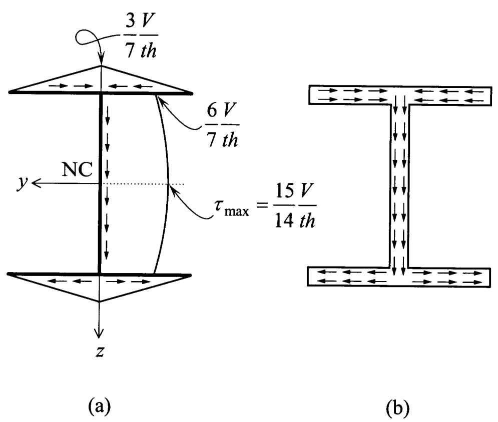


Figure 5.61 (a) The shear stress diagram, determined for the I-section and plotted on the centre lines. For thin-walled cross-sections plotting on the centre lines is a useful way of presenting the shear stress distribution in both the web and flanges in a single figure. The values are plotted in such a way that the figure remains orderly. (b) The shear stresses in the cross-section are uniformly distributed across the wall thickness. It is as if the shear stresses flow from the edges of the upper flange towards the web, and flow out below to the edges of the lower flange. This *continuity in shear flow* is a characteristic for the shear stress distribution in a thin-walled cross-sections due to a cross-sectional shear force.

$$m_2 = 0 \qquad\qquad \Rightarrow \sigma_{xm} = 0,$$

$$m_2 = \tfrac{1}{2}b\left(=\tfrac{1}{2}h\right) \Rightarrow \sigma_{xm} = +\frac{3}{7}\frac{V}{th}.$$

The shear stresses in the upper flange are all positive and therefore act in the positive m direction(s).

The determination of the shear stress distribution in the lower flange occurs in the same way and is left to the reader.

For thin-walled cross-sections, it is useful to plot the shear stresses in the yz plane, along the centre lines of the cross-section (see Figure 5.61a). In this way, we can present the shear stress distribution in both web and flanges in a single figure. Arrows indicate the actual directions of the shear stresses. It is irrelevant on which side of the centre lines the values are plotted. They are plotted in such a way that the shear stress diagram is easy to read.

Figure 5.61b shows again how the shear stresses act in the thin-walled I-section, without including the values. The shear stresses are *constant across the wall thickness*. It is as if the shear stresses "*flow towards*" the web from the edges of the upper flange, and "*flow out*" at the lower flange. This *continuity in flow direction* of the shear stresses is characteristic for the shear stress distribution in thin-walled cross-sections due to a cross-sectional shear force.

The product of shear stress τ and wall thickness t is called the *shear flow* (see Figure 5.62):

$$s = \tau t.$$

If we look somewhat more closely at the join between web and flange, hereafter referred to as the "joint", the total shear flow towards the joint (the *flow-in*, s_{in}) is equal to the total shear flow from the joint (the *flow-out*, s_{out}), or

$$s_{\text{in}} = s_{\text{out}}.$$

So the following hold for the joint between web and upper flange (see Figure 5.63):

$$\text{flow-in:} \quad s_{\text{in}} = 2 \times \frac{3}{7}\frac{V}{th} \cdot t = \frac{6}{7}\frac{V}{h},$$

$$\text{flow-out:} \quad s_{\text{out}} = \frac{6}{7}\frac{V}{th} \cdot t = \frac{6}{7}\frac{V}{h}.$$

The flow-in equals the flow-out.

This is not a coincidence but a generally valid property that results from the force equilibrium of the joint in the longitudinal direction.

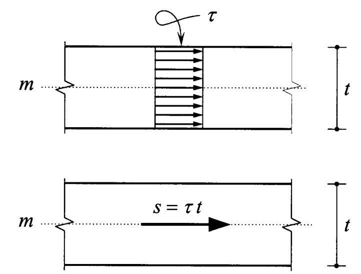


Figure 5.62 The product of shear stress τ and wall thickness t is known as the shear flow s: $s = \tau t$.

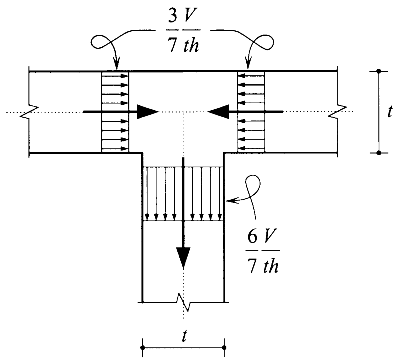


Figure 5.63 Where web and upper flange meet, the total shear flow towards the joint (the *flow-in*) is equal to the total shear flow from the joint (the *flow-out*).

To demonstrate the general applicability, we look at a beam element with small length Δx, and isolate the joint between web and upper flange (see Figure 5.64a).

If $s_x^{(1)}$, $s_x^{(2)}$ and $s_x^{(3)}$ are the shear forces per length on the sliding elements (1) to (3), the resultant shear forces are $s_x^{(1)}\Delta x$, $s_x^{(2)}\Delta x$ and $s_x^{(3)}\Delta x$. There are equal and opposite forces acting on the joint. The force equilibrium of the joint in x direction implies

$$-s_x^{(1)}\Delta x - s_x^{(2)}\Delta x - s_x^{(3)}\Delta x = 0,$$

or

$$s_x^{(1)} + s_x^{(2)} + s_x^{(3)} = 0. \tag{5.18}$$

We have assumed that the dimensions of the joint in the transverse direction are so small that the resultants of the normal stresses on the front and back (cross-sectional) planes of the joint can be neglected.

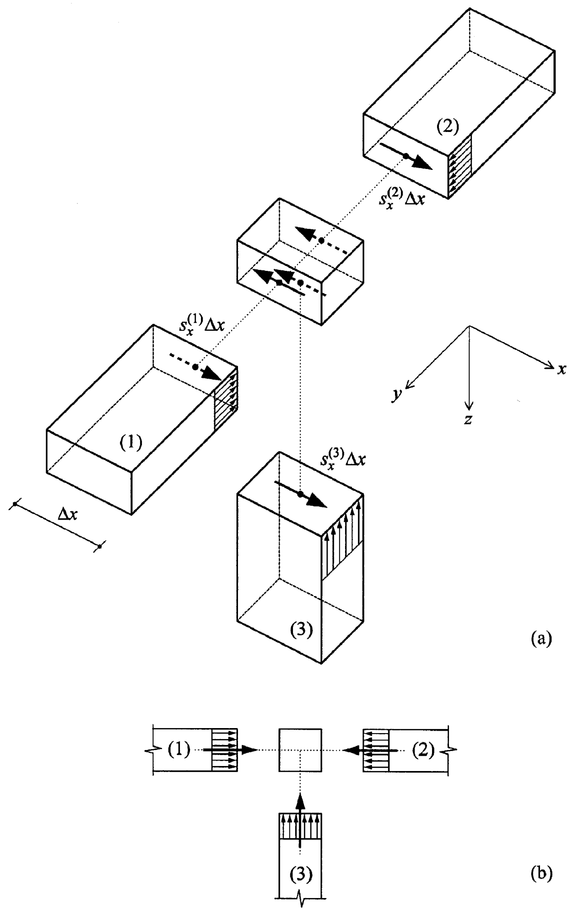


Figure 5.64

Figure 5.64 (a) The joint between web and upper flange has been isolated for a beam segment with small length Δx. The force equilibrium of the joint in x direction implies that, in the joint, the sum of the shear forces per length must equal zero. (b) In the plane of the cross-section, this means that the total shear flow towards the joint must be zero: $\sum \tau t = 0$. In other words: the total *flow-in* and *flow-out* must be equal at a joint.

Expression (5.18) means that, at a joint, the sum of the shear forces per length must be zero. The shear force per length is equal to the product of the longitudinal shear stress τ (on the longitudinal section plane) and the wall thickness t (the width of the cut). Expression (5.18) can therefore also be formulated as

$$\sum \tau t = 0. \tag{5.19}$$

In expression (5.19), since the shear stresses in two mutually normal planes are equal, τ can also be seen as the shear stress in the plane of the cross-section, so that τt is the shear flow s.

Expression (5.19) can be interpreted as follows:

The total shear flow towards a joint must be zero (see Figure 5.64b),

or in other words:

At a joint, the total flow-in must be equal to the total flow-out.

For a thin-walled cross-section we can now add three new rules to the four rules mentioned in Section 5.3.3 and relating to the shear stress distribution in a cross-section due to shear.

Rule 5. There is continuity in the "*flow direction*" of the shear stresses.

Rule 6. The *shear flow* s equals the product of the shear stress τ and thickness t:

$$s = \tau t.$$

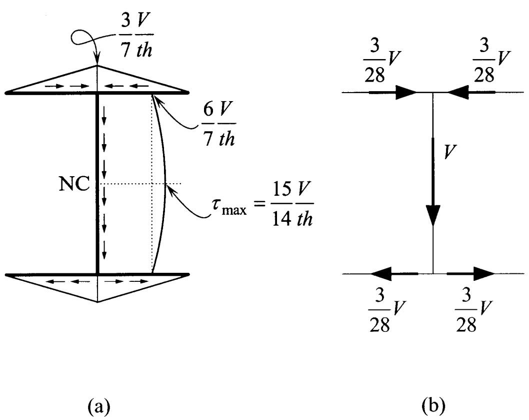


Figure 5.65 (a) The shear stress distribution in the cross-section with (b) the shear stress resultants in web and flanges. The resultant of the vertical shear stresses in the web is equal to the vertical shear force V in the cross-section. The horizontal shear stress resultants in the flanges form an equilibrium system together.

Rule 7. At a joint, the total flow-in is equal to the total flow-out:

$$s_{\text{in}} = s_{\text{out}}.$$

At the end of this example, we will use the rules 1 to 7 to present a quick alternative solution to the questions a and b. First, however, we will answer question c and determine the resultant of all shear stresses in the cross-section.

c. Figure 5.65a shows the shear stress diagram for the thin-walled I-section. Figure 5.66 gives a spatial representation of the shear stress distribution in the web. The resultant of all shear stresses in the web is a vertical force that is equal to the volume of the stress diagram. This in turn is equal to the area of the stress diagram in Figure 5.65a, multiplied by the web thickness $t_{\text{w}} = t$. The area of the stress diagram of the web is most easily determined by splitting the diagram into a rectangle and a parabola. In this way we find the following shear stress resultant in the web:

$$R^{\text{web}} = \left\{ \underbrace{\frac{6}{7}\frac{V}{th} \cdot h}_{\text{rectangle}} + \underbrace{\tfrac{2}{3} \cdot \left(\frac{15}{14}\frac{V}{th} - \frac{6}{7}\frac{V}{th} \right) \cdot h}_{\text{parabola}} \right\} \times t$$

$$= \tfrac{6}{7} V + \tfrac{2}{3} \cdot \tfrac{3}{14} V = V.$$

The resultant of the shear stresses in a flange half is a horizontal force equal to the area of the triangular shear stress diagram in Figure 5.65a, multiplied by the web thickness $t_{\text{f}} = t$:

$$R^{\text{half flange}} = \left\{ \tfrac{1}{2} \cdot \frac{3}{7}\frac{V}{th} \cdot \frac{h}{2} \right\} \times t = \tfrac{3}{28} V.$$

Figure 5.65b shows the shear stress resultants for the I-section. The forces in the flange halves form an equilibrium system. The resultant of the vertical shear stresses in the web, as can be expected, is equal to the vertical shear force V to be transferred by the cross-section.

Note with respect to the magnitude of the shear stresses in the web that the shear force V is transferred fully by the web of the I-section. The average shear stress in the web is

$$\tau_{\text{average}} = \frac{V}{A_{\text{web}}} \text{ with } A_{\text{web}} = th.$$

In the I-section in question, the maximum shear stress is

$$\tau_{\max} = \frac{15}{14}\frac{V}{th} = \frac{15}{14}\frac{V}{A_{\text{web}}} = 1.07\tau_{\text{average}}.$$

Here the maximum shear stress in the web of the I-section is just 7% larger than the average shear stress in the web.

The deviation depends on the exact cross-sectional dimensions, but varies little for the thin-walled I-sections used in practice. Therefore the following global rule can be used for I-sections in the design phase:

$$\tau_{\max} \approx \tau_{\text{average}},$$

or, more realistically:

$$\tau_{\max} \approx 1.1 \times \tau_{\text{average}}.$$

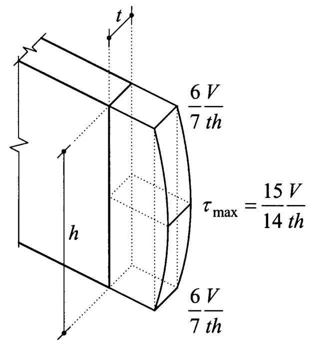


Figure 5.66 Spatial representation of the shear stress distribution in the web. The shear force is equal to the resultant of all shear stresses in the web, and is found from the volume of the stress diagram.

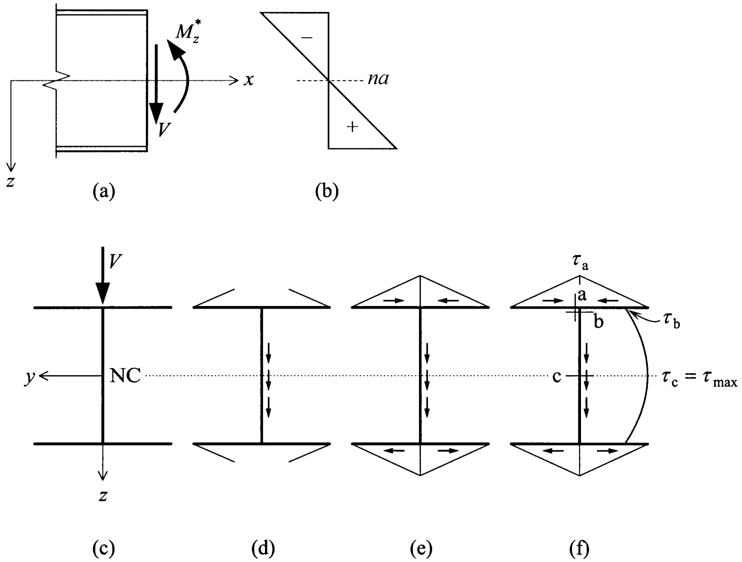


Figure 5.67 (a) Side view of the beam segment to the left of a cross-section, with the shear force V in the xz plane. An arbitrary bending moment M_z^* has been added, acting in the same plane as the shear force. (b) The bending stress distribution due to M_z^*: the bending stress is constant in the flanges and linear in the web. This means that the shear stress is linear in the flanges and parabolic in the web. (c) The shear stresses in the web have the same direction as the shear force V. (d) The horizontal shear stresses in the flanges are linear, and zero at the edges. (e) From the continuity of the shear flow it follows that the shear stresses in the upper flange must flow towards the web and that those in the lower flange must flow away from the web. (f) The shear stresses in the web are parabolic. The vertex of the parabola is located at the level of the normal centre NC. For the joins between the web and flanges, *flow-in* = *flow-out*.

Alternative solution to the questions a and b:
Using rules 1 to 7 we can make a good sketch of the shear stress distribution with little calculation.

Figure 5.67a gives the side view of a small part of the beam, with a shear force V in the xz plane. A bending moment M_z^* has been added in the plane in which the shear force acts. Note that this moment is not actually present, but is introduced for the sake of rules 2 to 4 from Section 5.3.3.

Figure 5.67b shows the bending stress diagram due to M_z^*. Figure 5.67c shows the cross-section, with the shear force V that it has to transfer. Due to this shear force, the following apply for the shear stresses in the thin-walled I-section:

- The shear stresses are constant across the wall thicknesses. They act vertically in the web and horizontally in the flanges.
- From the direction of the shear force it follows that the shear stresses in the web are directed downwards (see Figure 5.67d).
- The shear stresses are zero at the edges of the flanges (rule 1).
- From Figure 5.67b we see that there is a constant bending stress in the flanges, so the shear stress distribution here is linear (rule 2) (see Figure 5.67d).
- Based on mirror symmetry, the shear stress distributions in the left-hand and right-hand flange halves are equal and opposite.
- From the continuity of the flow direction it follows that the shear stresses in the upper flange must *flow towards* the web and that the shear stresses in the lower flange must *flow away* from the web (rule 5) (see Figure 5.67e).
- From Figure 5.67b we see that the bending stresses in the web are linearly distributed, so the shear stress distribution here is parabolic (rule 3) (see Figure 5.67f).

- The vertex of the parabola (the location of the maximum shear stress in the web) is at the level of the normal centre NC (rule 4).
- At the joins between web and flanges, "flow-in" = "flow-out" (rule 7) (see Figure 5.67f):

$$2 \times \tau_a \cdot t_f = \tau_b \cdot t_w.$$

With $t_f = t_w = t$ we find

$$\tau_b = 2\tau_a.$$

We therefore have to know only two values to draw a complete sketch of the shear stress diagram: τ_a (or τ_b) and τ_c (see Figure 5.67f).

Example 2: An angle steel with equal legs

A thin-walled angle steel, with equal legs of 200 mm and a wall thickness of 15 mm, has to transfer a shear force of $40\sqrt{2}$ kN in the plane of mirror symmetry (see Figure 5.68).

Questions:

a. Determine the shear stress distribution in the cross-section, including the maximum shear stress.
b. Determine the resultant of all the shear stresses in the cross-section.

Solution (units in N and mm):

a. The location of the normal centre NC is given in Figure 5.68. It is left to the reader to check its correctness.

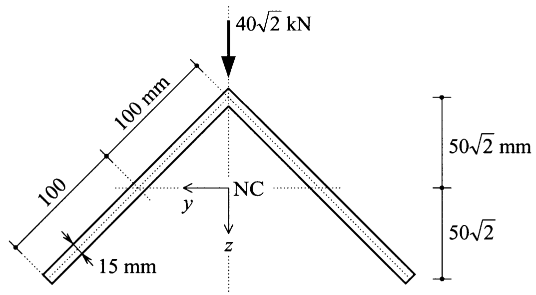


Figure 5.68 A thin-walled angle steel with equal legs must transfer a shear force of $40\sqrt{2}$ kN in the plane of symmetry.

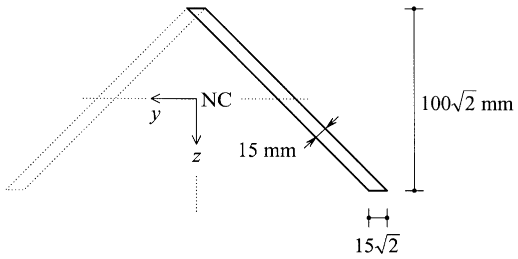


Figure 5.69 To determine the moment of inertia I_{zz}, the legs of the thin-walled angle steel are considered as strips. For a thin-walled strip, it is not important whether it is a rectangle or a parallelogram.

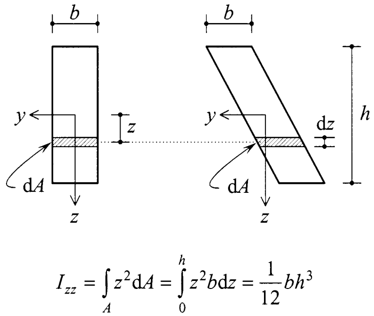


Figure 5.70 The moment of inertia I_{zz} of a parallelogram is equal to that of a rectangle with the same (horizontally measured) width b and height h.

To determine the shear stress distribution, we will use the following formula:

$$\sigma_{xm} = -\frac{V_z S_z^{\mathrm{a}}}{b^{\mathrm{a}} I_{zz}}.$$

For determining the (centroidal) moment of inertia I_{zz}, the angle steel is considered to consist of two strips, of which the right-hand strip is shown in Figure 5.69. For a thin-walled strip it makes little difference whether it is seen as a rectangle or as a parallelogram. A parallelogram has the benefit that we can use its property that the moment of inertia is equal to that of a rectangle with the same (horizontally measured) width and height (see Figure 5.70). In this way, we find for the moment of inertia of the angle steel the following value:

$$I_{zz} = 2 \times \tfrac{1}{12} bh^3 = 2 \times \tfrac{1}{12} \times 15\sqrt{2} \times (100\sqrt{2})^3 = 10 \times 10^6 \text{ mm}^4.$$

Figure 5.71 shows the sliding part of the cross-section. The location of the cut where the shear stress σ_{xm} acts is defined by the distance m (measured along the centre line) to the edge at the lower side of the angle steel ($0 < m \leq 200$ mm).

The static moment S_z^{a} of the sliding element is

$$S_z^{\mathrm{a}} = 15 \times m \times \left(50\sqrt{2} - \tfrac{1}{4} m\sqrt{2}\right) = (750m - 3.75m^2) \times \sqrt{2} \text{ mm}^3.$$

Furthermore

$$V_z = 40\sqrt{2} \times 10^3 \text{ N}$$

and

$$b^{\mathrm{a}} = 15 \text{ mm}.$$

If these values are substituted in the shear stress formula, we find the following for the right-hand leg of the angle steel:

$$\sigma_{xm} = -\frac{V_z S_z^a}{b^{\mathrm{a}} I_{zz}} = -\frac{40\sqrt{2} \times 10^3 \times (750m - 3.75m^2)\sqrt{2}}{15 \times 10 \times 10^6}$$
$$= (2m^2 - 400m) \times 10^{-3} \text{ N/mm}^2.$$

The shear stress distribution is quadratic in m and therefore a parabola. The location of the maximum shear stress (the vertex of the parabola) we find from

$$\frac{\mathrm{d}\sigma_{xm}}{\mathrm{d}m} = (4m - 400) = 0 \Rightarrow m = 100 \text{ mm}.$$

The shear stress is a maximum at the level of the normal centre NC.

Three values are sufficient for a good sketch of the shear stress distribution in the right-hand leg of the angle steel:

$$m = 0 \Rightarrow \sigma_{xm} = 0,$$
$$m = 100 \text{ mm} \Rightarrow \sigma_{xm} = -20 \text{ N/mm}^2 \text{ (the maximum shear stress)},$$
$$m = 200 \text{ mm} \Rightarrow \sigma_{xm} = 0.$$

Since the shear stresses are negative across the entire height, they act opposite to the direction of σ_{xm} in Figure 5.71.

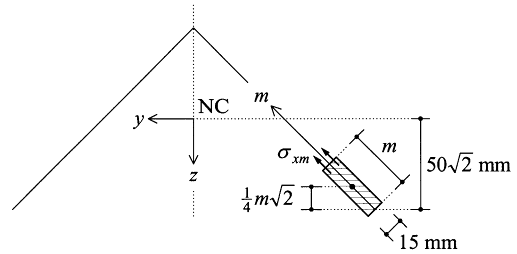


Figure 5.71 The sliding part of the cross-section is hatched. The location of the cut where the shear stress σ_{xm} acts is defined by the distance m measured along the centre line to the lower side of the profile.

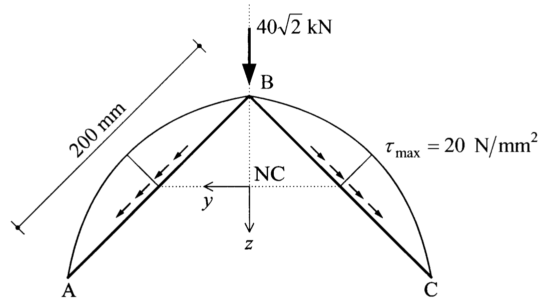


Figure 5.72 The shear stress diagram. The shear stress distribution is parabolic. The maximum shear stress occurs at the level of the normal centre NC; this is the vertex of the parabola.

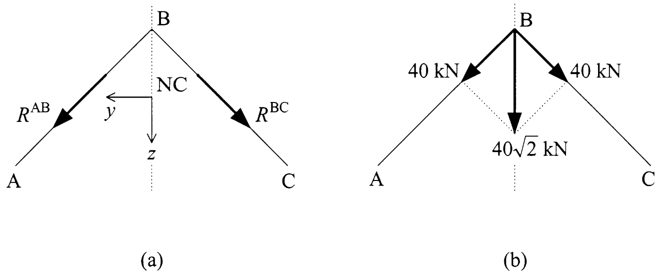


Figure 5.73 (a) The shear stress resultants R^{AB} and R^{BC} for the legs AB and BC respectively. (b) The resultant of R^{AB} and R^{BC} is equal to the vertical shear force of $40\sqrt{2}$ kN.

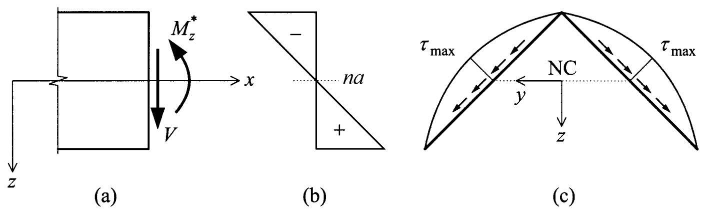


Figure 5.74 (a) Side view of the beam segment to the left of a cross-section, with the shear force V in the xz plane. An arbitrary bending moment M_z^* has been added, acting in the same plane as the shear force. (b) The bending stress diagram due to M_z^*. The bending stresses are linear in both legs of the angle steel so that the shear stress distributions here are parabolic. (c) The shear stress diagram. The parabolic shear stresses are a maximum at the level of the normal centre NC. The shear stresses are zero at the lower edges of the legs. With this the complete shear stress diagram can be sketched. The direction of the shear stresses follow from the direction of the shear force.

The mirror symmetry of the cross-section and load implies that the shear stress distributions in the left-hand and right-hand legs of the angle steel are equal. Figure 5.72 shows the complete shear stress diagram.

b. The resultant force R^{AB} of the shear stresses in leg AB of the angle steel (see Figure 5.72) is equal to the area of the parabolic shear stress diagram, multiplied by the wall thickness of 15 mm:

$$R^{AB} = \tfrac{2}{3} \times (200 \text{ mm})(20 \text{ N/mm}^2) \times (15 \text{ mm}) = 40 \times 10^3 \text{ N} = 40 \text{ kN}.$$

The shear stress resultant in BC is equal to that in AB, hence

$$R^{AB} = R^{BC} = 40 \text{ kN}.$$

The shear stress resultants for AB and BC are shown in Figure 5.73a. The resultant of all shear stresses in the cross-section is indeed equal to the vertical shear force of $40\sqrt{2}$ kN (see Figure 5.73b).

Alternative solution:
Figure 5.74a shows the side view of a small part of the beam, with the shear force $V_z = V$. A bending moment M_z^* of arbitrary value has been added, acting in the same plane as the shear force.

Figure 5.74b shows the bending stress diagram due to M_z^*. The bending stresses are linear in both legs of the angle steel, so the shear stress distribution is parabolic here (rule 3), with a maximum at the level of the normal centre NC (rule 4). In addition, the shear stresses are zero at the lower end of the legs (rule 1). With this we can plot the whole shear stress diagram (see Figure 5.74c). The direction of the shear stresses follow from the direction of the shear force (components in the legs). In the end we have to determine only τ_{max}.

Example 3: A steel U-section

A steel U-section (or channel profile) has been used as a column in a temporary support. The cross-section can be considered thin-walled. Figure 5.75a shows the cross-sectional dimensions. Figure 5.75b shows the side view of a part of the column, with the shear force of 9.5 kN that the column has to transfer in the xy plane.

Questions:

a. Check the correctness of the location of the normal centre NC in Figure 5.75a.
b. Determine the shear stress distribution.
c. Determine the maximum shear stress in the cross-section and the location where it occurs.

Solution (units in N and mm):

a. In a yz coordinate system through the normal centre NC, the following applies per definition:

$$S_y = \int_A y\,\mathrm{d}A = 0 \quad \text{and} \quad S_z = \int_A z\,\mathrm{d}A = 0.$$

Since the y axis is a line of symmetry, the condition $S_z = 0$ is met. There remains to show that

$$S_y = S_y^{\text{web}} + 2 \times S_y^{\text{flange}} = 0.$$

Figure 5.76 indicates what is understood by web and flanges in this situation.[1]

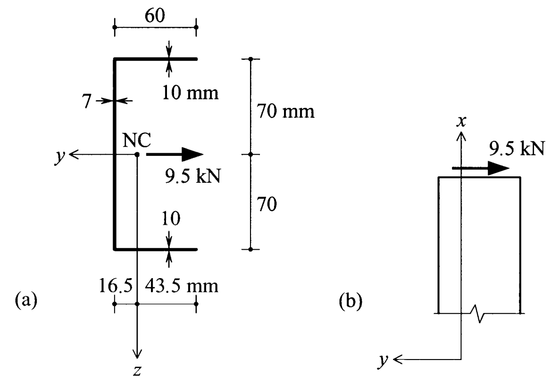


Figure 5.75 (a) The cross-sectional dimensions of a thin-walled steel U-section, used as a column. (b) The column has to transfer a shear force of 47.5 kN in the plane of symmetry as shown.

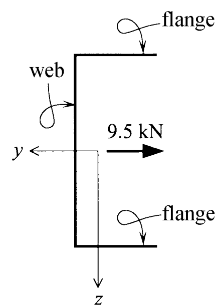
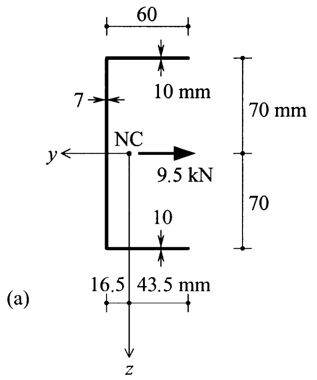


Figure 5.76 Web and flanges of the U-section.

[1] In this situation, with a shear force parallel to the "flanges", it is disputable what is flange is and what is web.



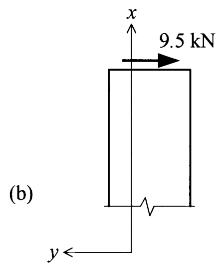


Figure 5.75 (a) The cross-sectional dimensions of a thin-walled steel U-section, used as a column. (b) The column has to transfer a shear force of 47.5 kN in the plane of symmetry as shown.

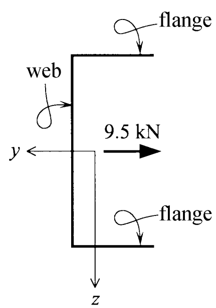


Figure 5.76 Web and flanges of the U-section.

$$S_y = 7 \times 140 \times (+16.5) + 2 \times \{10 \times 60 \times (-30 + 16.5)\}$$
$$= 16170 - 16200 = -30 \text{ mm}^3.$$

Note that the flanges have a negative contribution to S_y.

S_y is not exactly zero. The actual location of the normal centre NC is a little more to the right and follows from

$$y_{\text{NC}} = \frac{S_y}{A} = \frac{-30}{7 \times 140 + 2 \times 10 \times 60} = -0.014 \text{ mm}.$$

So the values 16.5 and 43.5 in Figure 5.75a are rounded off from the more accurate values 16.514 and 43.486.

b. Since the shear force acts in the xy plane, the shear stress formula is

$$\sigma_{xm} = -\frac{V_y S_y^{\text{a}}}{b^{\text{a}} I_{yy}}.$$

For the thin-walled U-section, the following applies:

$$I_{yy} = I_{yy(\text{Steiner})}^{\text{web}} + 2 \times \left\{I_{yy(\text{centr})}^{\text{flange}} + I_{yy(\text{Steiner})}^{\text{flange}}\right\}$$
$$= 7 \times 140 \times (+16.5)^2 +$$
$$+ 2 \times \left\{\tfrac{1}{12} \times 10 \times 60^3 + 10 \times 60 \times (-30 + 16.5)^2\right\}$$
$$= 845.5 \times 10^3 \text{ mm}^4.$$

First we determine the shear stresses in the upper flange. Figure 5.77a shows the sliding element. The location of the cut with shear stress σ_{xm} is defined by distance m_1 to the edge of the flange.

The static moment S_y^{a} of the sliding part of the flange is

$$S_y^{\mathrm{a}} = 10 \times m_1 \times \left(-43.5 + \tfrac{1}{2}m_1\right) = 5m_1^2 - 435m_1 \quad (0 \le m_1 \le 60).$$

With $V_y = -9.5 \times 10^3$ N and $b^{\mathrm{a}} = 10$ mm we find

$$\sigma_{xm} = -\frac{V_y S_y^{\mathrm{a}}}{b^{\mathrm{a}} I_{yy}} = -\frac{(-9.5 \times 10^3)(5m_1^2 - 435m_1)}{10 \times (845.5 \times 10^3)}$$
$$= (m_1^2 - 87m_1) \times 5.618 \times 10^{-3}\ \mathrm{N/mm^2}.$$

This is a parabolic shear stress distribution. The shear stress is a maximum when $m_1 = 43.5$ mm, that is at the level of the normal centre NC.

A number of values are

$$m_1 = 0 \quad \Rightarrow \sigma_{xm} = 0,$$
$$m_1 = 43.5\ \mathrm{mm} \Rightarrow \sigma_{xm} = -10.64\ \mathrm{N/mm^2}\ \text{(the top value)},$$
$$m_1 = 60\ \mathrm{mm} \quad \Rightarrow \sigma_{xm} = -9.1\ \mathrm{N/mm^2}.$$

σ_{xm} is negative everywhere and is therefore acting opposite to the direction of σ_{xm} shown in Figure 5.77a. This is in line with the direction of the shear force.

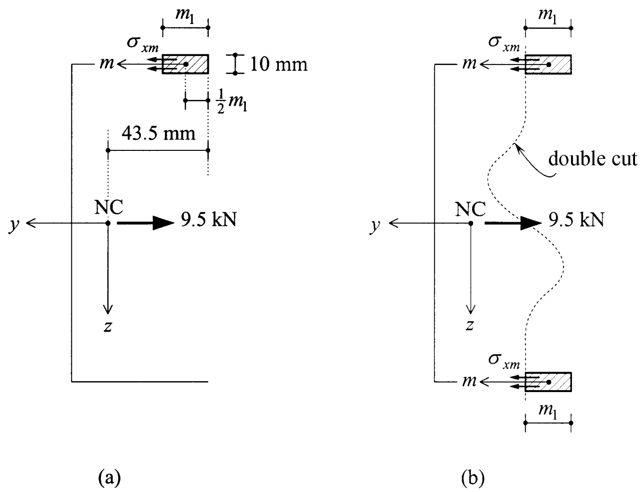


Figure 5.77 (a) The sliding part of the cross-section for determining the shear stresses in the upper flange. (b) Since the shear stress distribution is symmetrical, we can also use a symmetrical double cut.

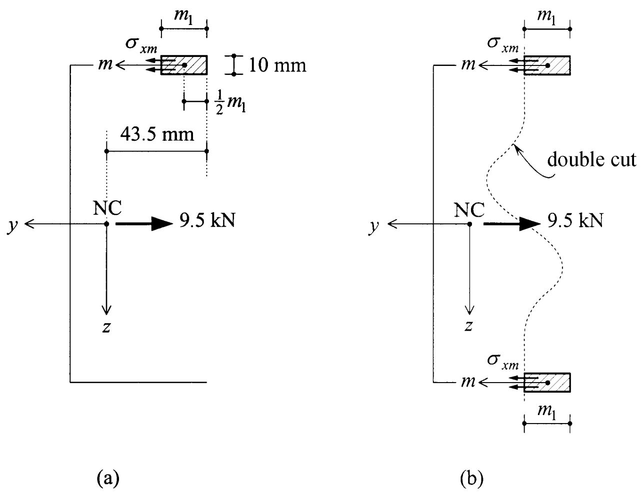


Figure 5.77 (a) The sliding part of the cross-section for determining the shear stresses in the upper flange. (b) Since the shear stress distribution is symmetrical, we can also use a symmetrical double cut.

In the same way, we can find the shear stress distribution in the lower flange. However, it is possible to determine the shear stress distributions in both flanges at one go. Therefore we use the symmetrical double cut in Figure 5.77. Since the cross-section is mirror symmetrical and the shear force is acting in the plane of mirror symmetry, the shear stress distribution is also symmetrical.

Figure 5.78a shows the shear stress distribution for both flanges. It could have been predicted that the shear stresses in the flanges would be parabolic, founded on the bending stress diagram in Figure 5.79a due to the arbitrary bending moment M_y^* in the xy plane, the same plane in which the shear force acts (see Figure 5.79c). The bending stresses are linear in the flanges, so the shear stress distribution is parabolic here (rule 3).

Figure 5.79 shows that the bending stresses in the web are constant and that the shear stresses here are therefore linear (rule 2). The shear stresses at the top and bottom of the web can be determined using the property *flow-in* = *flow-out* (rule 7) (see Figure 5.78a):

$$\tau_a \times 7 = \tau_b \times 10.$$

This implies

$$\tau_a = \tfrac{10}{7}\,\tau_b = \tfrac{10}{7} \times 9.5 = 13.0\ \text{N/mm}^2.$$

Due to mirror symmetry, the shear stresses are zero on the line of mirror symmetry (the y axis).

Figure 5.78b shows the complete shear stress diagram.

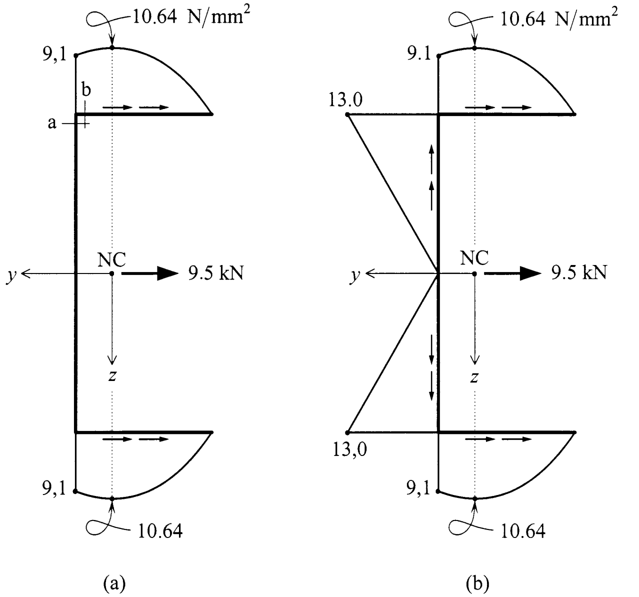


Figure 5.78 (a) The parabolic shear stress distribution in both flanges. The maximum shear stress occurs at the level of the normal centre NC. (b) The complete shear stress diagram.

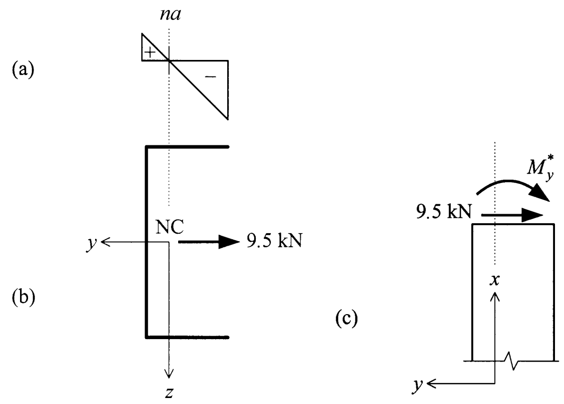


Figure 5.79 (a) The bending stress diagram for (b) the U-section, due to (c) an arbitrary bending moment M_y^* in the xy plane, the same plane in which the shear force acts.

c. The maximum shear stress in the flanges is 10.64 N/mm^2 and occurs at the level of the normal centre NC. However, the largest shear stress in the cross-section is not in the flanges, but in the web at the join with the flanges, and is 13.0 N/mm^2.

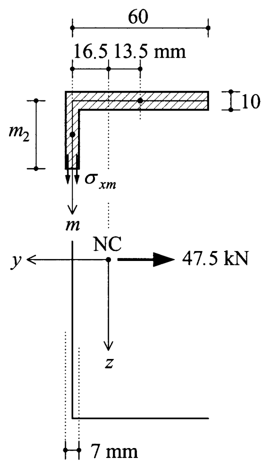


Figure 5.80 The sliding part of the cross-section for determining the shear stresses in the web.

Comment in relation to the shear stress distribution in the web: If we want to determine the shear stress distribution in the web in the same way as that for the flanges, we can use the sliding element in Figure 5.80. The location of the cut with shear stress σ_{xm} is defined by the distance m_2 to (the centre line of) the upper flange.

For the static moment S_y^{a} of the sliding element,

$$S_y^{\mathrm{a}} = 10 \times 60 \times (-13.5) + 7 \times m_2 \times (+16.5) = 115.5 m_2 - 8100.$$

With $V_y = -9.5 \times 10^3$ N and $b^{\mathrm{a}} = 7$ mm we find

$$\sigma_{xm} = -\frac{V_y S_y^{\mathrm{a}}}{b^{\mathrm{a}} I_{yy}} = -\frac{(-9.5 \times 10^3)(115.5 \times m_2 - 8100)}{7 \times (845.5 \times 10^3)}$$
$$= (115.5 \times m_2 - 8100) \times 1.605 \times 10^{-3} \text{ N/mm}^2.$$

This is indeed a linear distribution.

A number of values:

$$m_2 = 0 \quad \Rightarrow \sigma_{xm} = -13.0 \text{ N/mm}^2,$$
$$m_2 = 70 \text{ mm} \quad \Rightarrow \sigma_{xm} = -0.02 \text{ N/mm}^2,$$
$$m_2 = 140 \text{ mm} \Rightarrow \sigma_{xm} = +12.95 \text{ N/mm}^2.$$

The fact that the shear stress at the centre of the web is not exactly zero is a consequence of the fact that the distance from the normal centre NC to the web in Figure 5.75a has been rounded off from 16.514 mm to 16.5 mm. This is also the reason for the difference in shear stress values at the top of the web (13.0 N/mm^2) and bottom of the web (12.95 N/mm^2).

5.4.3 Thin-walled hollow cross-sections

Figure 5.81a shows the cross-section of a box girder bridge with the shear force V. Since the cross-section is mirror symmetrical and the shear force acts in the plane of symmetry, the shear stress distribution in the cross-section will be mirror symmetrical.

In the symmetrical double cut, as indicated in Figure 5.81b, the shear stresses are assumed uniformly distributed across the total thickness of both walls. Now the shear stress distribution can be determined using the formulas (5.14) or (5.15):

$$\sigma_{xm} = -\frac{V_z S_z^{\mathrm{a}}}{b^{\mathrm{a}} I_{zz}}, \tag{5.14}$$

$$\sigma_{xm} = -\frac{V_z}{b^{\mathrm{a}}} \left[\frac{N^{\mathrm{a}}\ (\text{due to})\ M_z^*)}{M_z^*} \right]. \tag{5.15}$$

In these formulas the total wall thickness of the double cut has to be used for width b^{a} of the sliding element. For Figure 5.81b

$$b^{\mathrm{a}} = 2 \times t.$$

More generally, we can say that the shear stress formulas (5.14) and (5.15) can be applied successfully to thin-walled mirror symmetrical *unicellular* hollow cross-sections with the shear force acting in the plane of mirror symmetry.

In many other cases, where we do not know whether the shear stresses are uniformly distributed over the total width of the multiple cut, the shear stress formulas (5.14) and (5.15) cannot not be used. Two examples are given in Figures 5.82 and 5.83.

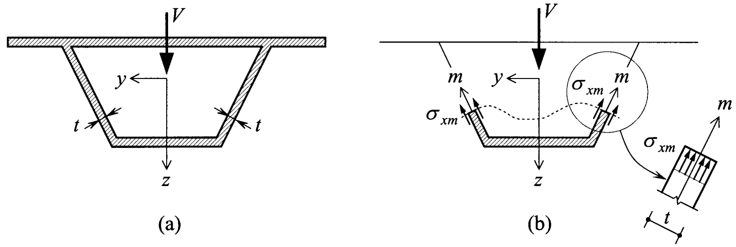


Figure 5.81 (a) The cross-section of a thin-walled box girder bridge. Since the cross-section is mirror symmetrical and the shear force acts in the plane of symmetry the shear stress distribution is also mirror symmetrical. (b) The shear stresses can be determined by introducing a symmetrical double cut (normal to the centre lines). In the shear stress formula we have to use the total wall thickness across the double cut for the width of the sliding element: $b^{\mathrm{a}} = 2 \times t$.

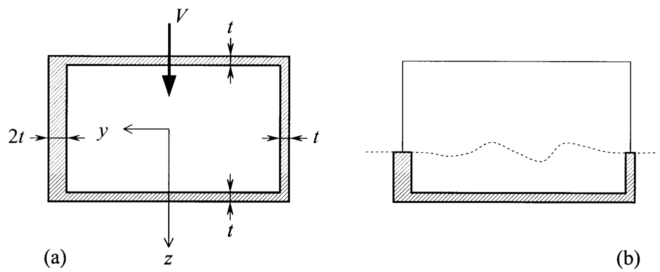


Figure 5.82 (a) A thin-walled hollow cross-section, subject to a shear force which does not act in a plane of symmetry. (b) For the given double cut it is possible to determine the total shear force per length (shear flow), but it is unknown how this shear force is distributed between the webs.

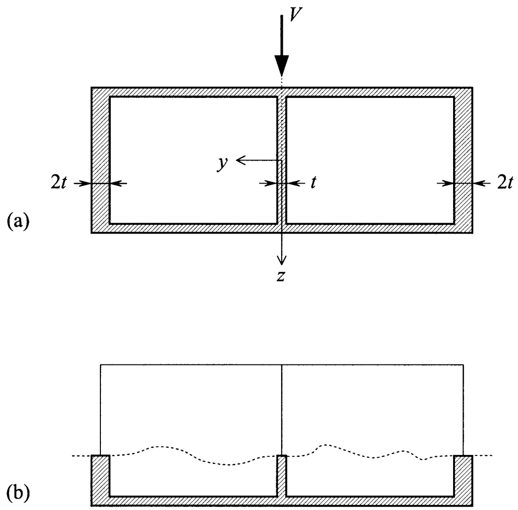


Figure 5.83 (a) A mirror symmetrical hollow cross-section with two cells, subject to a shear force in the plane of symmetry. The shear stress distribution will be symmetrical. (b) For the given cut over three webs we can to determine the total shear force per length (shear flow) on the sliding element, but we do not know how this shear flow is distributed over the webs. The shear stresses in the outer webs may be different from those in the centre web.

For the hollow cross-section in Figure 5.82a, the vertical shear force V does not act in a plane of symmetry. With the double cut in Figure 5.82b, we can determine the total shear force per length (shear flow) in the double cut, but we do not know how it is distributed between the cuts. In other words, it is unknown which part of the shear force is transferred by each of the webs. The shear stresses therefore need not be the same in the two cuts. This means that the shear stress formulas (5.14) and (5.15) cannot be used here.

Another example is the symmetrical hollow cross-section in Figure 5.83a, with *two cells*. The shear stress distribution due to shear force V in the plane of symmetry will be symmetrical. It is possible to determine the total shear force per length (shear flow) for the sliding element in Figure 5.83b, but it is unknown how this shear force is distributed between the three webs. The shear stresses in the outer webs may be different from those in the centre web. Here too, the shear stress formulas (5.14) and (5.15) cannot be used.

Next you will find two examples in which the formulas (5.14) and (5.15) suffice to determine the shear stress distribution.

Example 1: A rectangular hollow cross-section

A thin-walled box girder, with the cross-sectional dimensions given in Figure 5.84, has to transfer a shear force of 60 kN.

Question:

Determine the shear stress distribution in the cross-section.

Solution (units in N and mm):

The shear stress distribution can be determined by using formula (5.14):

$$\sigma_{xm} = -\frac{V_z S_z^{\mathrm{a}}}{b^{\mathrm{a}} I_{zz}}, \tag{5.14}$$

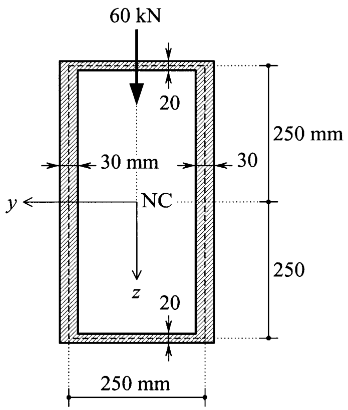


Figure 5.84 A thin-walled rectangular hollow cross-section has to transfer the shear force of 60 kN as shown.

in which

$$V_z = 60 \times 10^3 \text{ N},$$

and

$$\begin{aligned} I_{zz} &= 2 \times I^{\text{web}}_{zz(\text{centr})} + 2 \times I^{\text{flange}}_{zz(\text{Steiner})} \\ &= 2 \times \tfrac{1}{12} \times 30 \times 500^3 + 2 \times 20 \times 250 \times (\pm 250)^2 \\ &= 1250 \times 10^6 \text{ mm}^4. \end{aligned}$$

We first determine the shear stress distribution in the top flange. For the sliding element in Figure 5.85a it holds that

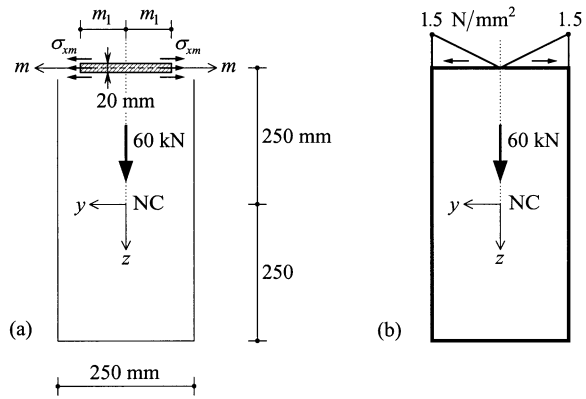


Figure 5.85 (a) To determine the shear stresses in the upper flange, we work with a symmetrical double cut. (b) The shear stress distribution in the upper flange.

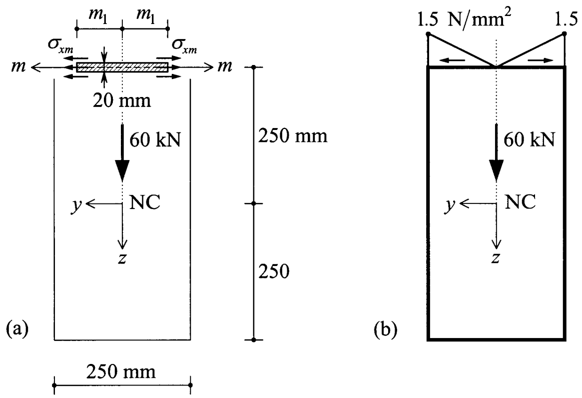


Figure 5.85 (a) To determine the shear stresses in the upper flange, we work with a symmetrical double cut. (b) The shear stress distribution in the upper flange.

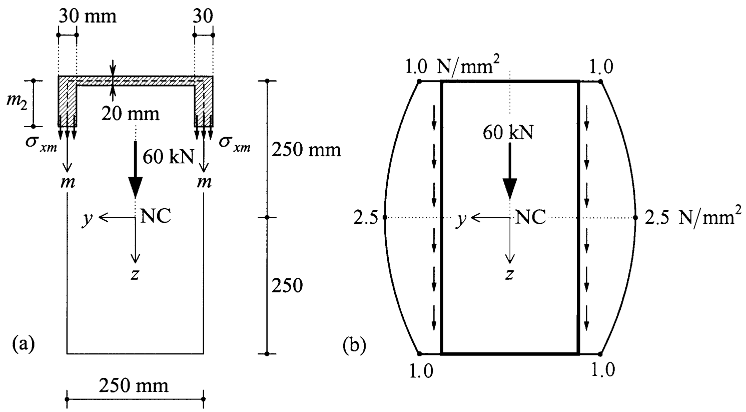


Figure 5.86 (a) The symmetrical double cut for determining the shear stresses in the webs. (b) The shear stress distribution in the webs.

$$b^{\mathrm{a}} = 2 \times 20 = 40 \text{ mm},$$

and

$$S_z^{\mathrm{a}} = 2 \times 2 \times m_1 \times (-250) = -10 m_1 \times 10^3 \text{ mm}^3.$$

So we find for the top flange

$$\sigma_{xm} = -\frac{(+60 \times 10^3)(-10 m_1 \times 10^3)}{40 \times (1250 \times 10^6)} = +12 m_1 \times 10^{-3} \text{ N/mm}^2.$$

This is a linear distribution. All shear stresses act in the positive m direction.

A number of values:

$$m_1 = 0 \qquad \Rightarrow \sigma_{xm} = 0,$$
$$m_1 = 125 \text{ mm} \Rightarrow \sigma_{xm} = +1.5 \text{ N/mm}^2.$$

The shear stress distribution is shown in Figure 5.85b.

The shear stress distribution in the webs is determined using the sliding element in Figure 5.86a. Here

$$b^{\mathrm{a}} = 2 \times 30 = 60 \text{ mm}$$

and

$$S_z^{\mathrm{a}} = 20 \times 250 \times (-250) + 2 \times 30 \times m_2 \times \left(-250 + \tfrac{1}{2} m_2\right)$$
$$= (30 m_2^2 - 15 m_2 \times 10^3 - 1.25 \times 10^6) \text{ mm}^3.$$

For the shear stress distribution in the webs we now find

$$\sigma_{xm} = -\frac{(+60 \times 10^3)(30m_2^2 - 15m_2 \times 10^3 - 1.25 \times 10^6)}{40 \times (1250 \times 10^6)}$$
$$= (-24m_2^2 \times 10^{-6} + 12m_2 \times 10^{-3} + 1)\ \text{N/mm}^2.$$

This is a parabolic distribution. The location of the top follows from $d\sigma_{xm}/dm_2 = 0$ and is located at $m_2 = 250$ mm, at the level of the normal centre NC.

A number of values:

$$m_2 = 0 \quad \Rightarrow \sigma_{xm} = +1\ \text{N/mm}^2,$$
$$m_2 = 250\ \text{mm} \Rightarrow \sigma_{xm} = +2.5\ \text{N/mm}^2\ \text{(the top value)},$$
$$m_2 = 500\ \text{mm} \Rightarrow \sigma_{xm} = +1\ \text{N/mm}^2.$$

The shear stress distributions in the webs are shown in Figure 5.86b. The shear stresses are positive and therefore act in the positive m direction, or downwards. This is in line with the direction of the shear force in the cross-section.

The determination of the shear stresses in the bottom flange is left to the reader.

Figure 5.87 shows the complete shear stress diagram. A number of checks, which are left to the reader, include the following:

- In the corners *flow-in* = *flow-out*.
- The resultant of all shear stresses in the webs is equal to the shear force.

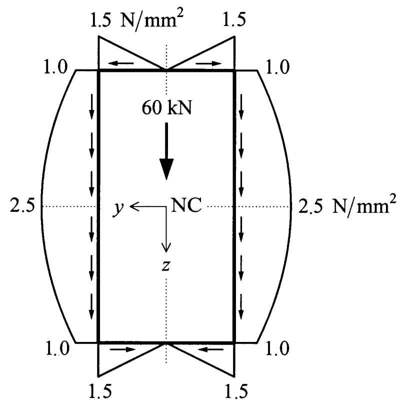


Figure 5.87 The shear stress diagram for the whole cross-section.

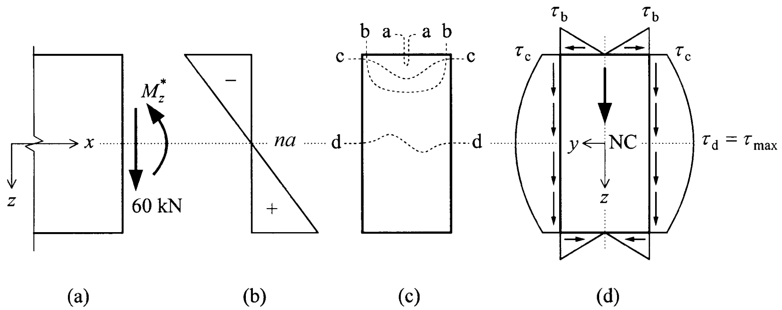


Figure 5.88 (a) Side view of the box girder to the left of a cross-section subject to a shear force of 60 kN and an arbitrary bending moment M_z^*, both acting in the xz plane. (b) The bending stress diagram due to M_z^*. Since the bending stress is constant in the flanges, the shear stresses here are linear. The bending stress is linear in the webs, so the shear stresses here are parabolic. (c) For a good sketch of the shear stress diagram, determining the values in the four symmetrical double sections a-a to d-d suffices. (d) The complete shear stress diagram.

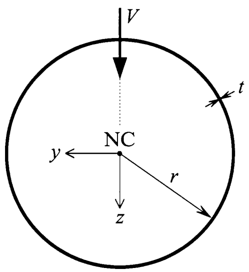


Figure 5.89 The cross-section of a thin-walled tube subject to a shear force $V_z = V$.

Alternative solution:
Again an alternative and quicker solution is possible by first predicting the shear stress distribution on the basis of the bending stress diagram, after which the values at a number of relevant places are determined to get a good sketch of the shear stress distribution.

Figure 5.88b shows the bending stress diagram due to the arbitrary bending moment M_z^*, acting in the same plane the shear force of 60 kN is acting. The bending stresses are constant in the flanges, here the shear stresses will be linear (rule 2). In the webs the bending stresses are linear and the shear stress distribution will be parabolic (rule 3). The vertex of the parabola is located at the level of the normal centre NC (rule 4).

To make a good sketch of the shear stress diagram, it is not necessary to determine the shear stress as a function of the location of the cut. It suffices to determine the values in the four symmetrical double cuts a-a to d-d shown in Figure 5.88c.

Since in the double cut a-a the area of the sliding element is zero, the static moment of the sliding element[1] and therefore also the shear stress is zero. Furthermore, we can use the property *flow-in* = *flow-out* for the shear stresses in the double cuts b and c. This greatly reduces the amount of calculation required.

Example 2: A thin-walled hollow circular tube
Figure 5.89 shows the cross-section of a thin-walled hollow circular tube with radius r and wall thickness t. The cross-section has to transfer a vertical shear force $V_z = V$.

---

[1] If one chooses the sliding element of the cross-section on the other side of section a-a, the area is equal to that of the entire cross-section and the static moment is also zero.

Questions:

a. Determine the shear stress distribution.
b. Determine the maximum shear stress.
c. Show that the resultant of all shear stresses in the cross-section is equal to the shear force.

Solution:

a. The shear stress is determined using formula (5.14):

$$\sigma_{xm} = -\frac{V_z S_z^{\mathrm{a}}}{b^{\mathrm{a}} I_{zz}} \,. \qquad (5.14)$$

Here (see also Section 3.3.2, example 2)

$$I_{zz} = I_{yy} = \tfrac{1}{2}\, I_{\mathrm{p}} = \pi r^3 t \,.$$

Figure 5.90 shows the sliding part of the cross-section. The location of the symmetrical double cut is defined by the angle φ.

The width b^{a} of the double cut is

$$b^{\mathrm{a}} = 2t \,.$$

We now have to determine only the static moment S_z^{a} of the sliding part of the cross-section. To do so we look at a small element from the tube wall with length $r\mathrm{d}\theta$ (see Figure 5.91). With wall thickness t, the area $\mathrm{d}A$ of this element is

$$\mathrm{d}A = t \cdot r \,\mathrm{d}\theta \,.$$

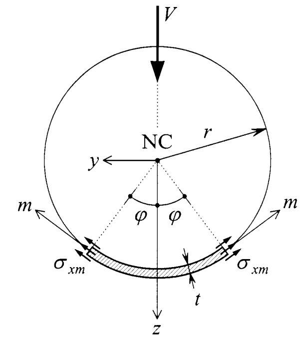


Figure 5.90 The sliding part of the cross-section. The location of the symmetrical double cut is indicated by the angle φ.

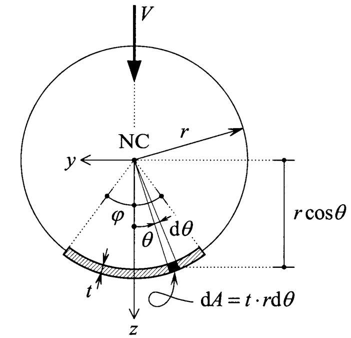


Figure 5.91 To find the static moment S_z^{a} of the sliding part of the cross-section, the contribution of a small area element with length $r\,\mathrm{d}\theta$ and wall thickness t is determined first.

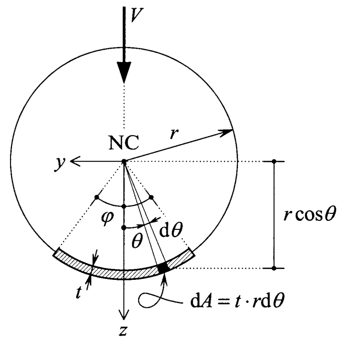


Figure 5.91 To find the static moment S_z^a of the sliding part of the cross-section, the contribution of a small area element with length $r\,d\theta$ and wall thickness t is determined first.

The z coordinate of this small area is

$$z = r\cos\theta.$$

The contribution of this small area to the static moment S_z^a is

$$dS_z^a = z \cdot dA = r\cos\theta \cdot rt\,d\theta = r^2 t\cos\theta\,d\theta.$$

The static moment S_z^a of the sliding element is found by adding all contributions dS_z^a between $\theta = -\varphi$ and $\theta = +\varphi$, that means by integrating between these values:

$$S_z^a = \int_{-\varphi}^{+\varphi} r^2 t\cos\theta\,d\theta = r^2 t\sin\varphi\Big|_{-\varphi}^{+\varphi} = 2r^2 t\sin\varphi.$$

We have now determined all quantities that occur in the shear stress formula (5.14). The shear stress distribution is therefore

$$\sigma_{xm} = -\frac{V \cdot 2r^2 t\sin\varphi}{2t \cdot \pi r^3 t} = -\frac{V}{\pi r t}\sin\varphi.$$

In Figure 5.92a the value of the shear stress has been plotted radially as a function of φ. The direction is shown by means of arrows.

A simpler picture can be obtained by plotting the shear stress as a function of $y = \pm r\sin\varphi$. The shear stress diagram then consists of two straight lines (see Figure 5.92b).

Note that the shear stress is zero on the vertical line of symmetry.

b. The maximum shear stress τ_{max} occurs in $\varphi = \pi/2$ (or $y = \pm r$):

$$\tau_{max} = \frac{V}{\pi r t}\,.$$

With $A = 2\pi r t$ we can also write

$$\tau_{max} = 2\frac{V}{A}\,.$$

The average (vertical) shear stress in the cross-section is

$$\tau_{average} = \frac{V}{A}\,.$$

For a thin-walled tube, the maximum shear stress is twice as large as the average shear stress:

$$\tau_{max} = 2 \times \tau_{average}\,.$$

c. To calculate the resultant of all shear stresses in the cross-section we first look at the two mirror symmetrical located area elements dA in Figure 5.93a. With a length $r\mathrm{d}\varphi$ and wall thickness t the area of each element is

$$\mathrm{d}A = t \cdot r\,\mathrm{d}\varphi.$$

Both elements are subject to the same shear stress τ:

$$\tau = |\sigma_{xm}| = \frac{V}{\pi r t}\sin\varphi.$$

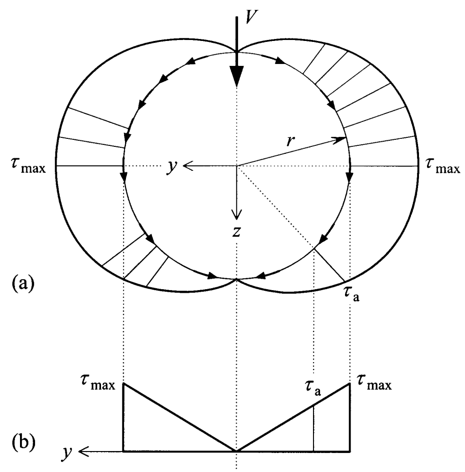


Figure 5.92 (a) The shear stresses due to the vertical shear force V plotted radially as a function of φ. The direction of the shear stresses is indicated by arrows. (b) A simpler representation is possible by plotting the shear stresses as a function of $y = \pm r \sin\varphi$. In that case the diagram is built up of two straight lines.

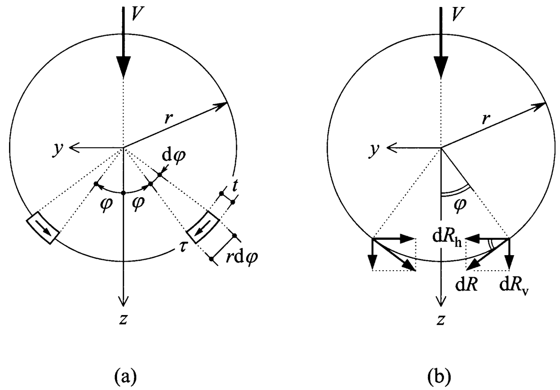


Figure 5.93 (a) To find the resultant of all shear stresses in the cross-section we first investigate the contribution of the shear stresses in two small, symmetrically-located area elements dA, with length $r\mathrm{d}\varphi$ and wall thickness t. (b) The horizontal components dR_h in the two symmetrically-located area elements together form an equilibrium system and have a zero resultant. The vertical components dR_v remain. By summing up these components with respect to both halves of the cross-section (that means by integrating) we find that the resultant of all shear stresses in the cross-section is indeed equal to the shear force V.

The resultant of the shear stresses on a each of the area elements is (see Figure 5.93b):

$$\mathrm{d}R = \tau \cdot \mathrm{d}A = \frac{V}{\pi r t} \sin\varphi \cdot rt\,\mathrm{d}\varphi = \frac{V}{\pi} \sin\varphi \cdot \mathrm{d}\varphi.$$

The horizontal components dR_h of both symmetrical area elements form an equilibrium system and have a zero resultant. Only the vertical components dR_v remain:

$$\mathrm{d}R_\mathrm{v} = \mathrm{d}R \cdot \sin\varphi = \frac{V}{\pi} \sin^2\varphi \cdot \mathrm{d}\varphi.$$

By adding the contributions dR_v for all small area elements across both halves of the cross-section we find the vertical resultant R_v:

$$R_\mathrm{v} = 2\int_0^\pi \mathrm{d}R_\mathrm{v} = 2\int_0^\pi \frac{V}{\pi} \sin^2\varphi \cdot \mathrm{d}\varphi = \frac{2V}{\pi}\int_0^\pi \sin^2\varphi \cdot \mathrm{d}\varphi.$$

With

$$\int_0^\pi \sin^2\varphi \cdot \mathrm{d}\varphi = \frac{\pi}{2}$$

we find

$$R_\mathrm{v} = V.$$

The resultant of all shear stresses in the cross-section indeed equals the shear force V.

Comment: The integral

$$\int_0^{\pi} \sin^2\varphi \cdot \mathrm{d}\varphi$$

can easily be determined if one bears in mind that the areas enclosed by the functions $\sin^2\varphi$ and $\cos^2\varphi$ between $\varphi = 0$ and $\varphi = \pi$ are equal in magnitude:

$$\int_0^{\pi} \sin^2\varphi \cdot \mathrm{d}\varphi = \int_0^{\pi} \cos^2\varphi \cdot \mathrm{d}\varphi, \tag{5.20}$$

and that

$$\begin{aligned}\int_0^{\pi} \sin^2\varphi \cdot \mathrm{d}\varphi + \int_0^{\pi} \cos^2\varphi \cdot \mathrm{d}\varphi &= \int_0^{2} (\sin^2\varphi + \cos^2\varphi) \cdot \mathrm{d}\varphi \\ &= \int_0^{\pi} \mathrm{d}\varphi = \pi.\end{aligned} \tag{5.21}$$

From (5.20) and (5.21) we directly find

$$\int_0^{\pi} \sin^2\varphi \cdot \mathrm{d}\varphi = \int_0^{\pi} \cos^2\varphi \cdot \mathrm{d}\varphi = \frac{\pi}{2}\,.$$

5.4.4 Cross-sections with varying width of the sliding element

So far, we have only determined the shear stresses for cross-sections with a constant width of the sliding element. In this section, we look at two examples in which the width of the sliding element varies, namely a solid triangular cross-section and a solid circular cross-section. Here too, we as-

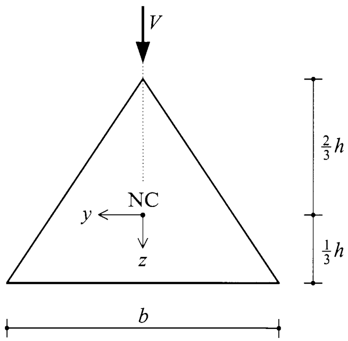


Figure 5.94 A solid equilateral triangular cross-section subject to a shear force V in the vertical plane of mirror symmetry.

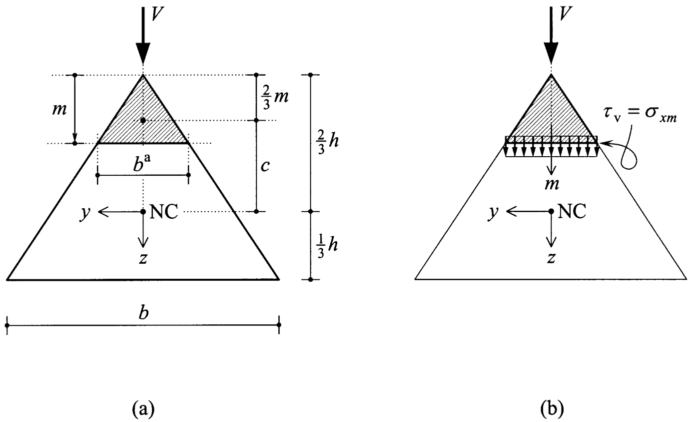


Figure 5.95 (a) The hatched triangle is selected as the sliding part of the cross-section. The width b^a of the sliding part depends on the location m of the cut. (b) The shear stress σ_{xm} at the cut.

sume that the shear stresses normal to the cut are constant and that they can be determined using the known shear stress formulas. However, the actual shear stresses also have components parallel to the cut.

In a third example we illustrate that the approach used for the first two examples must be used with care.

Example 1: Solid triangular cross-section
The equilateral triangular cross-section in Figure 5.94 is subject to a shear force V in the vertical plane of mirror symmetry.

Questions:
a. Determine the distribution of the vertical shear stresses.
b. Determine the distribution of the shear stresses along the oblique edges.

Solution:
a. The hatched triangle in Figure 5.95a is selected as the sliding element. In Figure 5.95b, this part has been shown with the shear stresses σ_{xm} at the cut.

The shear stresses are determined using formula (5.14):

$$\sigma_{xm} = -\frac{V_z S_z^a}{b^a I_{zz}}, \tag{5.14}$$

in which

$$V_z = +V,$$

$$b^a = b\frac{m}{h}, \quad \text{and}$$

$$I_{zz} = \tfrac{1}{36}\,bh^3 \quad \text{(see Section 3.2.4, Example 5).}$$

The static moment S_z^{a} is

$$S_z^{\mathrm{a}} = A^{\mathrm{a}} \cdot z_{\mathrm{C}}^{\mathrm{a}},$$

in which A^{a} is the area of the sliding element,

$$A^{\mathrm{a}} = \tfrac{1}{2} \cdot b^{\mathrm{a}} \cdot m = \tfrac{1}{2}\, b \frac{m^2}{h}\,,$$

and $z_{\mathrm{C}}^{\mathrm{a}}$ is the z coordinate of the centroid of the hatched triangle (see Figure 5.95a):

$$z_{\mathrm{C}}^{\mathrm{a}} = -c = -\tfrac{2}{3}\,(h - m) \quad \text{(note the minus signs!)}.$$

This leads to

$$S_z^{\mathrm{a}} = A^{\mathrm{a}} \cdot z_{\mathrm{C}}^{\mathrm{a}} = \tfrac{1}{2}\, b \frac{m^2}{h} \cdot \left\{-\tfrac{2}{3}\,(h - m)\right\} = -\tfrac{1}{3}\, bh^2 \left(\frac{m^2}{h^2} - \frac{m^3}{h^3}\right).$$

We have now determined all the necessary quantities, and using shear stress formula (5.14) we find

$$\sigma_{xm} = -\frac{(+V)\left\{-\tfrac{1}{3}\, bh^2 \left(\dfrac{m^2}{h^2} - \dfrac{m^3}{h^3}\right)\right\}}{b\dfrac{m}{h} \cdot \tfrac{1}{36}\, bh^3} = 12 \frac{V}{bh}\left(\frac{m}{h} - \frac{m^2}{h^2}\right).$$

Figure 5.96b shows the shear stress distribution in a diagram.

The vertical shear stresses σ_{xm} are quadratic in m and the shear stress diagram therefore is parabolic. The location of the vertex of the parabola is

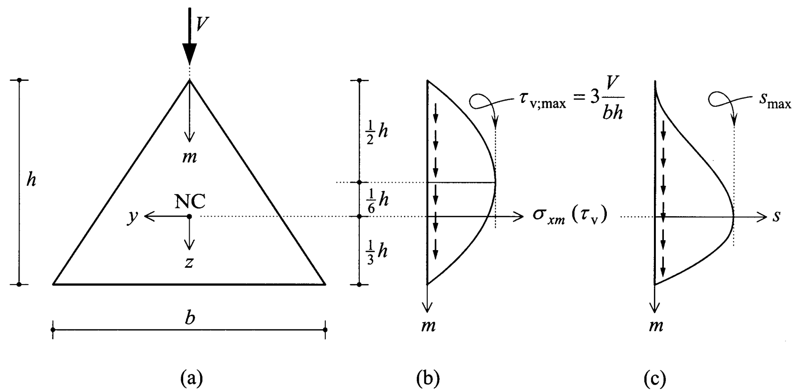


Figure 5.96 (a) The normal centre NC is at one third of the height. (b) The maximum vertical shear stress (the vertex of the parabola) is not at the level of the normal centre, but at half-height. (c) On the other hand, the vertical shear flow $s = \tau_{\mathrm{v}} t = |\sigma_{xm}| b^{\mathrm{a}}$ is largest at the cut through the normal centre.

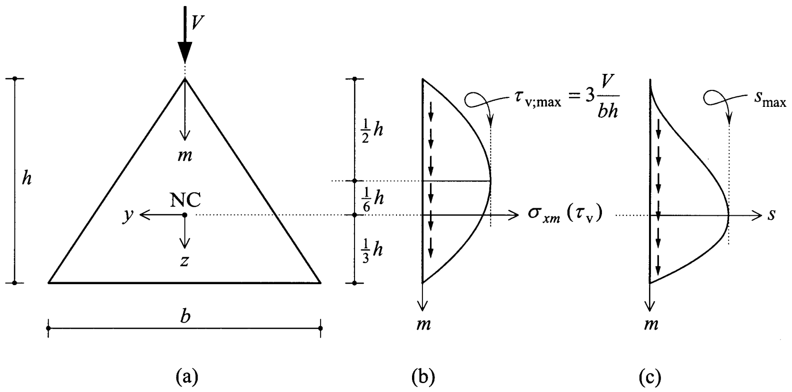


Figure 5.96 (a) The normal centre NC is at one third of the height. (b) The maximum vertical shear stress (the vertex of the parabola) is not at the level of the normal centre, but at half-height. (c) On the other hand, the vertical shear flow $s = \tau_v t = |\sigma_{xm}|b^a$ is largest at the cut through the normal centre.

found from

$$\frac{\mathrm{d}\sigma_{xm}}{\mathrm{d}m} = 12\frac{V}{bh}\left(\frac{1}{h} - \frac{2m}{h^2}\right) = 0 \Rightarrow m = \tfrac{1}{2}h.$$

The maximum shear stress (the vertex of the parabola) is not at the level of the normal centre but at half-height of the triangular cross-section!

The shear stresses are positive across the entire height and, as expected, act in the direction of the shear force V.

The maximum vertical shear stress $\tau_{v;max}$ is

$$\tau_{v;max} = 3\frac{V}{bh} = 1.5\frac{V}{A}.$$

Here, $A = \frac{1}{2}bh$ is the area of the cross-section.

The average vertical shear stress is

$$\tau_{v;average} = \frac{V}{A}.$$

As in a rectangular cross-section, the maximum vertical shear stress appears to be 50% larger than the average vertical shear stress:

$$\tau_{v;max} = \tfrac{3}{2}\tau_{v;average}.$$

Comment: The vertical shear stress is a maximum at half-height and not at the level of the normal centre NC. The shear flow $s = \tau t = |\sigma_{xm}|b^a$ on the other hand is largest in the cut along the y axis, through the normal centre NC. See Figure 5.96c, which shows the distribution of the shear flow s over the height of the cross-section. It is up to the reader to check the shape of this shear flow diagram.

For each arbitrary cross-sectional shape, the shear flow is largest in a cut through the normal centre NC. Prove it.[1]

b. The shear stresses σ_{xm} are determined on the assumption that they are uniformly distributed across width b^a and that they act vertically. See Figure 5.97a, in which the sliding element has been magnified and in which $\tau_v = \sigma_{xm}$.

On the oblique edges, the shear stresses cannot be vertical; they have to run parallel to the edges (rule 1). Apparently τ_v is the vertical component of the shear stress τ_{edge} directed along the edge (see Figure 5.97b):

$$\tau_{edge} = \frac{\tau_v}{\cos\alpha}\,.$$

The horizontal component of the shear stress along the edge is

$$\tau_{h;edge} = \tau_v \tan\alpha.$$

It is assumed that the shear stresses between the edges change direction gradually as shown in Figure 5.97c. The shear stress τ_φ at an angle φ is

$$\tau_\varphi = \frac{\tau_v}{\cos\varphi}\,,$$

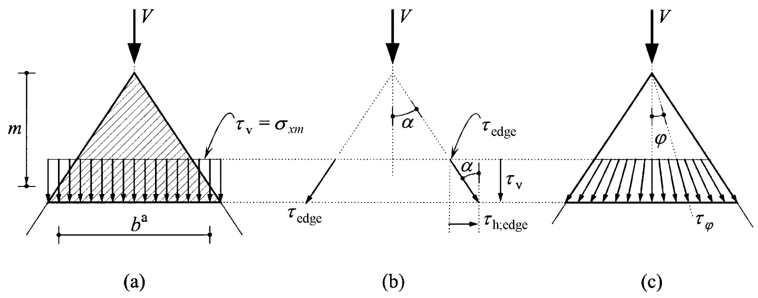


Figure 5.97 (a) The shear stresses $\tau_v = \sigma_{xm}$ were determined on the assumption that they are uniformly distributed over the width b^a of the plane cut and act vertically. (b) At the edges, the shear stresses cannot be vertical; they have to be parallel to the edge. We assume that τ_v is the vertical component of the shear stress τ_{edge} directed along the edge. (c) We also assume that the shear stress between the edges gradually changes direction as shown.

1 Use Section 5.1.3 and Figure 5.8.

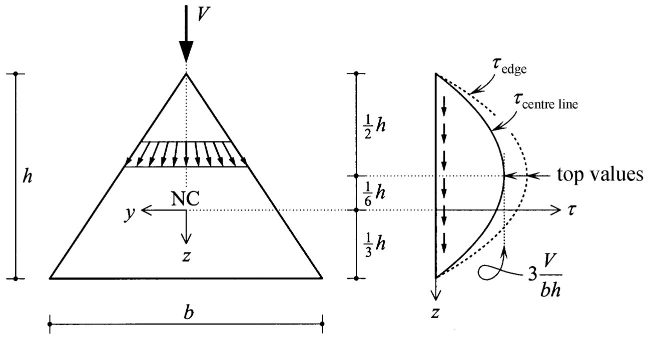


Figure 5.98 Distribution across the height of the cross-section of the shear stresses $\tau_{\text{centre line}}$ in the line of symmetry (solid line) and of the shear stresses τ_{edge} along the edges (dotted line), both in one diagram.

with horizontal component $\tau_{\text{h};\varphi}$:

$$\tau_{\text{h};\varphi} = \tau_{\text{v}} \tan\varphi.$$

In Figure 5.98, the shear stress distribution across the height has been plotted in a diagram for $\tau_{\text{centre line}} = \tau_{\varphi=0}$ along the line of symmetry, and $\tau_{\text{edge}} = \tau_{\varphi=\alpha}$ along the edges. Table 5.3 presents the ratio between τ_{edge} and $\tau_{\text{centre line}}$ for three values of b/h:

$$\frac{\tau_{\text{edge}}}{\tau_{\text{centre line}}} = \frac{\tau_{\varphi=\alpha}}{\tau_{\varphi=0}} = \frac{1}{\cos\alpha} = \sqrt{\tfrac{1}{4}(b/h)^2 + 1}\,.$$

With a width b that is twice the height h, the shear stresses along the edge are 41% larger than those along the line of symmetry. However, the shear stress distribution we determined is an approximation. If the width b is relatively large with respect to the height h we can start doubting the accuracy of the values found. Are the vertical shear stress components really uniformly distributed across the width?

Table 5.3

b/h	$\tau_{\text{edge}}/\tau_{\text{centre line}}$
0.5	1.03
1.0	1.12
2.0	1.41

More complicated calculations using the theory of elasticity or a computer-based approach can answer this question. This is outside the scope of this book.

Example 2: Solid circular cross-section
The solid circular cross-section in Figure 5.99, with radius r, transfers the vertical shear force V as shown.

Questions:
a. Determine the shear stress distribution.
b. Determine the maximum shear stress.

Solution:
In Figure 5.100a the sliding element has been hatched. The location of the cut is defined by the angle φ. Figure 5.100b shows the sliding element separately with the shear stresses σ_{xm} at the cut.

These stresses are determined using shear stress formula (5.14):

$$\sigma_{xm} = -\frac{V_z S_z^{\mathrm{a}}}{b^{\mathrm{a}} I_{zz}}\,. \tag{5.14}$$

Here

$$V_z = +V,$$

$$b^{\mathrm{a}} = 2r\sin\varphi, \quad \text{and}$$

$$I_{zz} = \tfrac{1}{4}\pi r^4 \quad \text{(see Section 3.2.4, Example 8).}$$

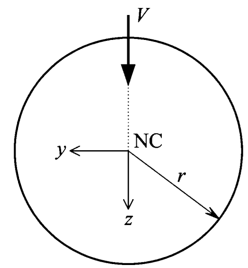


Figure 5.99 A solid circular cross-section, with radius r, subject to a vertical shear force V.

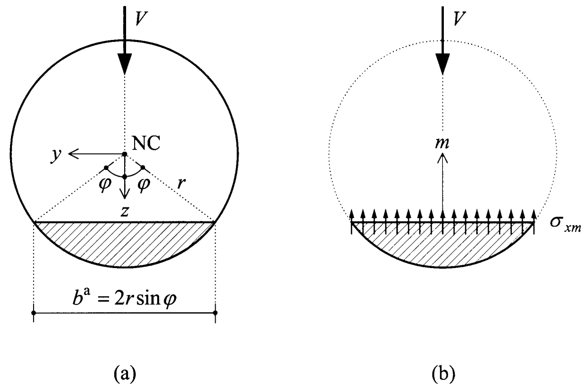
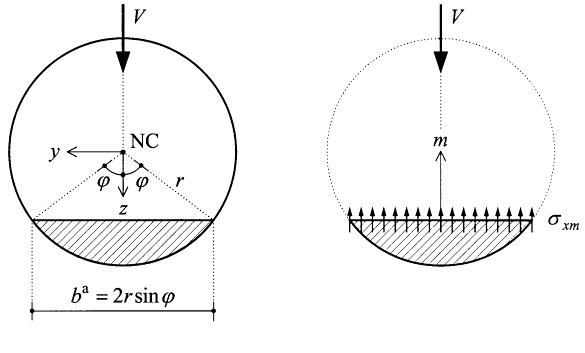


Figure 5.100 (a) The sliding part of the cross-section. The location of the cut is indicated by the angle φ. (b) The shear stresses σ_{xm} at the cut.



(a) (b)

Figure 5.100 (a) The sliding part of the cross-section. The location of the cut is indicated by the angle φ. (b) The shear stresses σ_{xm} at the cut.

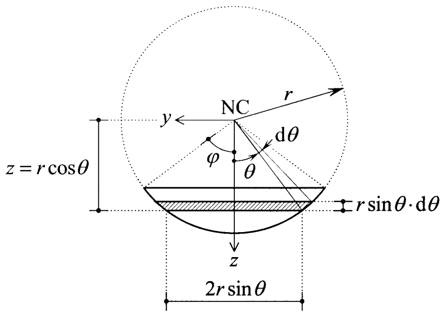


Figure 5.101 To find the static moment S_z^a of the sliding part of the cross-section we sum up (by integrating) the contributions of the thin strips between the cuts set by the angles θ and $\theta + d\theta$.

A difficult part of the calculation is the static moment S_z^a of the sliding element. To find this, a very narrow strip between two cuts has been created, defined by the angles θ and $\theta + d\theta$ in Figure 5.101, in which $d\theta$ is very small.

The width of this very narrow strip is $2r\sin\theta$ and the thickness[1] is $r\sin\theta \cdot d\theta$. The area of the strip is

$$dA = 2r\sin\theta \cdot r\sin\theta \cdot d\theta = 2r^2\sin\theta \cdot d\theta,$$

and has z coordinate

$$z = r\cos\theta.$$

The contribution of this strip to S_z^a is

$$dS_z^a = z \cdot dA = r\cos\theta \cdot 2r^2\sin^2\theta \cdot d\theta = 2r^3\sin^2\theta\cos\theta \cdot d\theta.$$

By summing the contributions of all the strips, or integrating between $\theta = 0$ and $\theta = \varphi$, we find

$$S_z^a = \int_0^{\varphi} 2r^3\sin^2\theta\cos\theta \cdot d\theta = 2r^3\int_0^{\varphi}\sin^2\theta \cdot d(\sin\theta)$$

[1] In Volume 1, Section 15.3.2, we derived that the vertical displacement due to a small rotations equals the rotation multiplied by the horizontal distance to the centre of rotation. In the same way we can say for small values of $d\theta$: the vertical distance equals the angle $d\theta$ multiplied by the horizontal distance $r\sin\theta$ to the centre of rotation.

$$= 2r^3\left(\tfrac{1}{3}\sin^3\theta\right)\Big|_0^{\varphi} = \tfrac{2}{3}r^3\sin^3\varphi.$$

Using shear stress formula (5.14) we find

$$\sigma_{xm} = -\frac{V \cdot \frac{2}{3}r^3\sin^3\varphi}{2r\sin\varphi \cdot \frac{1}{4}\pi r^4} = -\frac{4}{3}\frac{V}{\pi r^2}\sin^2\varphi.$$

Since

$$\sin^2\varphi = 1 - \cos^2\varphi = 1 - \frac{z^2}{r^2}$$

we can also write σ_{xm} as a function of z:

$$\sigma_{xm} = -\frac{4}{3}\frac{V}{\pi r^2}\left(1 - \frac{z^2}{r^2}\right).$$

The vertical shear stress σ_{xm} is quadratic in z and the distribution is therefore parabolic across the height of the cross-section. Figure 5.102 shows the distribution of the vertical shear stresses $\tau_v = |\sigma_{xm}|$ in a shear stress diagram.

σ_{xm} is negative across the entire height of the cross-section. This means that the vertical shear stresses act opposite to the m direction in Figure 5.100, i.e. downwards. This agrees with the direction of the shear force.

The vertex of the parabola is at $z = 0$, at the level of the normal centre NC. Here the maximum vertical shear stress $\tau_{v;max}$ occurs:

$$\tau_{v;max} = \frac{4}{3}\frac{V}{\pi r^2} = \frac{4}{3}\frac{V}{A},$$

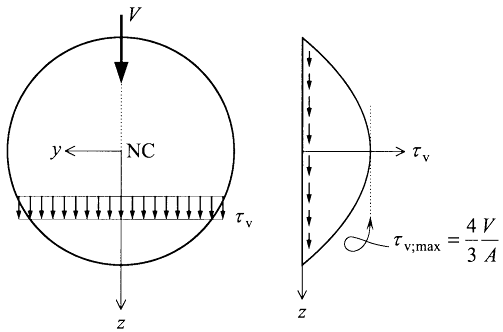


Figure 5.102 The vertical shear stresses τ_v, uniformly distributed across the width of the cross-section, are parabolically distributed across the height of the cross-section.

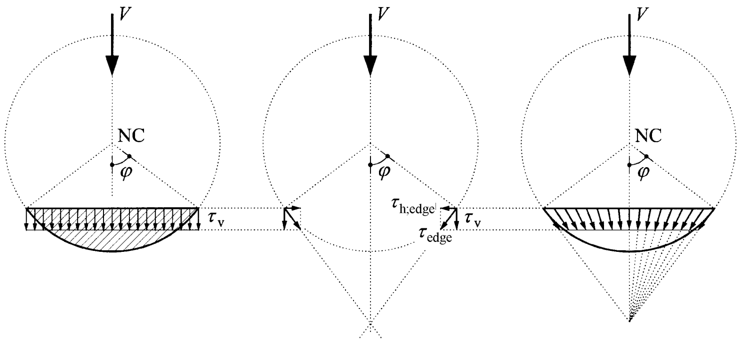


Figure 5.103 (a) The vertical shear stresses determined with the shear stress formula are uniformly distributed across the width of the cut. (b) Except at half-height, the shear stresses at the edges of the cross-section cannot be vertical, but must be directed along the edge. At the edge, τ_v is the vertical component of the shear stress τ directed along the edge. (c) Between the edges, we assume that the shear stress changes direction as shown: all shear stresses in a cut pass through the intersection of the tangents at the end points of that cut.

in which $A = \pi r^2$ is the area of the cross-section.

The average vertical shear stress is

$$\tau_{v;\text{average}} = \frac{V}{A}.$$

The maximum vertical shear stress is therefore 33% larger than the average vertical shear stress:

$$\tau_{\max} = \tfrac{4}{3}\,\tau_{v;\text{average}}.$$

b. The vertical shear stresses determined before are uniformly distributed across width b^a; see Figure 5.103a, in which $\tau_v = |\sigma_{xm}|$. Except at half-height, the shear stress in an edge of the cross-section cannot be vertical but must be parallel to the edge. At an edge, τ_v is the vertical component of the shear stress τ directed along the edge (see Figure 5.103b). For the shear stress τ_{edge} in an edge we have

$$\tau_{\text{edge}} = \frac{\tau_v}{\sin\varphi}.$$

The horizontal component is

$$\tau_{h;\text{edge}} = \frac{\tau_v}{\tan\varphi}.$$

Between the left and right-hand edges we assume that the direction of the shear stress changes in the manner shown in Figure 5.103c: all shear stresses at the cut pass through the intersection of the tangents at the end points of the cut. The largest shear stress at a cut therefore occurs at the edges.

In the shear stress diagram in Figure 5.104, the shear stress distributions of $\tau_{\text{centre line}}$ along the line of symmetry and τ_{edge} along the edges are plotted across the height of the cross-section. The maximum shear stress is still the vertical shear stress at half-height of the cross-section, and acting across the entire width.

Comment: The shear stress distribution shown is an approximation. Using the theory of elasticity we can show that the vertical component of the shear stress is not uniformly distributed, but rather slightly larger at the centre and slightly less at the edges. The maximum shear stress occurs at the centre of the cross-section and is about 4% larger than the value determined before.

A third example will illustrate that the strategy used for Examples 1 and 2 cannot always be used successfully.

Example 3: Solid square cross-section
The solid square cross-section in Figure 5.105 has to transfer a shear force V in diagonal direction.

Question:
How can the shear stress distribution in the cross-section be determined?

Solution:
The approach in the previous examples presupposes that the vertical component of the shear stresses at a cut is constant across the width b^a. Figure 5.106a shows the vertical shear stress components τ_v for cut AC with this assumption.

The shear stresses in this figure cannot be correct, however. Since the shear stress at the corners A and B has a zero component in two directions (those

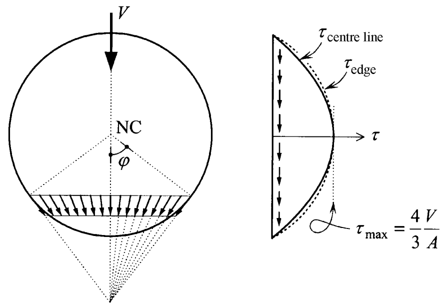


Figure 5.104 The distributions across the height of the shear stresses $\tau_{\text{centre line}}$ in the line of symmetry (solid line) and τ_{edge} along the edges (dotted line). The maximum shear stress is the vertical shear stress at half height, acting across the entire width of the cross-section.

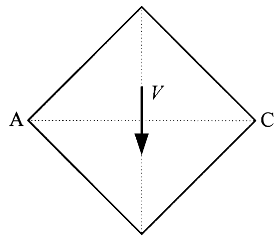


Figure 5.105 A square cross-section subject to a shear force V in diagonal direction.

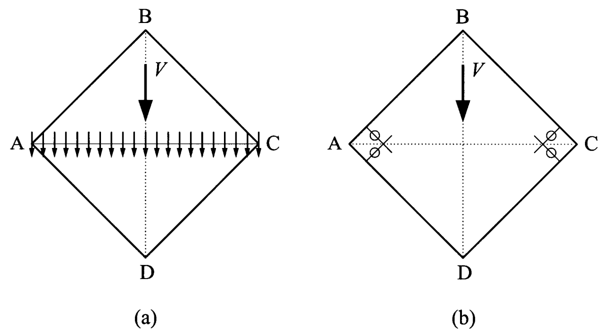


Figure 5.106 (a) The vertical component of the shear stresses cannot, as shown here, be constant across the width of cut AC. (b) Since at corners A and B the shear stresses in the two directions normal to the edges have zero components, the shear stress must be zero here.

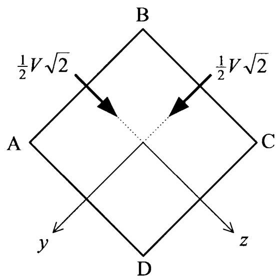


Figure 5.107 The diagonal shear force can be resolved into components in the y and z direction, parallel to the edges of the cross-section. So we can use the shear stress formula for a rectangular cross-section to find the shear stresses σ_{xy} and σ_{xz} as a function of y and z respectively.

normal to the edges, see Figure 5.106b), the shear stress there must be zero here. Therefore also τ_{v} must be zero at A and B. So the vertical component of the shear stresses at cut AC is clearly not constant across the width of the cut. Here the shear stress formula gives inaccurate results.

Another and better way to find the shear stress distribution is to resolve the shear force in the y and z directions parallel to the sides of the cross-section (see Figure 5.107). Using the shear stress formula for a rectangular cross-section, we can determine the shear stresses σ_{xy} and σ_{xz} as a function of y and z respectively. These shear stresses are parabolic.

The total shear stress is found from

$$\tau = \sqrt{\sigma_{xy}^2 + \sigma_{xz}^2}\,.$$

We will not perform the calculation here but only provide the result:

- The shear stress is zero at the corners A, B, C and D.
- The shear stresses at the horizontal cut AC act vertically and are parabolically distributed (see Figure 5.108a).
- Along diagonal BD the shear stresses act vertically and are parabolically distributed (see Figure 5.108b).
- The shear stresses outside diagonals AC and BD have a horizontal component.
- The maximum shear stress occurs at the centre of the cross-section and is

  $$\tau_{\max} = \frac{3}{2}\frac{V}{A}$$

  in which A is the area of the cross-section.

Conclusion: *The shear stress formula will usually lead to incorrect results for a cut through a corner of the cross-section. Since the shear stress must be zero at the corner, the shear stresses cannot be uniformly distributed along the width of the cut. This is also true when the corners are rounded with a small radius.*

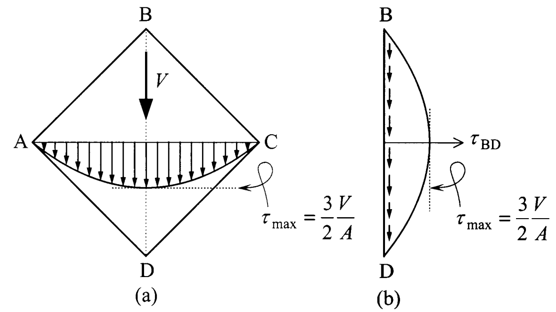


Figure 5.108 The actual shear stress is zero at the corners A, B, C and D. In the diagonals AC and BD the shear stresses are vertically directed and parabolically distributed.

5.5 Shear centre

So far, we have looked exclusively at mirror symmetrical cross-sections, with the shear force as the resultant of all shear stresses in the cross-section acting in the plane of mirror symmetry. In the following examples, the cross-sections are thin-walled and mirror symmetrical, but the shear force is now acting in a principal direction that does not coincide with the line of symmetry. See the examples in Figure 5.109.

In these cases we can determine the shear stress distribution due to the shear force. When we next determine the resultant of all shear stresses in the cross-section, this resultant indeed is equal in magnitude and direction to the shear force, but its line of action does not pass through the normal centre NC, as one possibly might have expected. This leads to a new point in the cross-section: the so-called *shear force centre* or *shear centre* SC.

The shear force centre SC *is that point in the cross-section through which the line of action of the shear force has to pass so that there will to be no torsion.*

In the following examples we look for the shear centre.

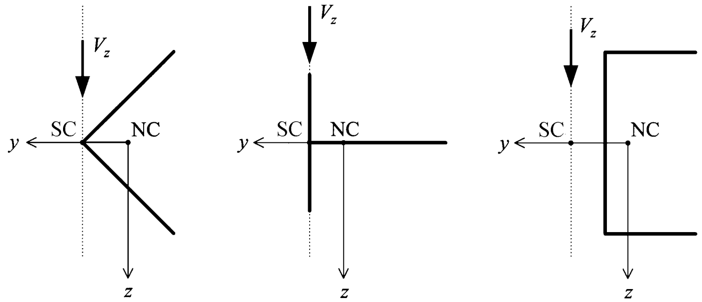


Figure 5.109 Examples of mirror symmetrical open thin-walled cross-sections, subject to shear forces in a principal direction that does not coincide with the line of symmetry. The line of action of the resultant of all shear stresses due to the shear force does not to pass through the normal centre NC. This leads to a new special point in the cross-section: the so-called *shear centre* SC.

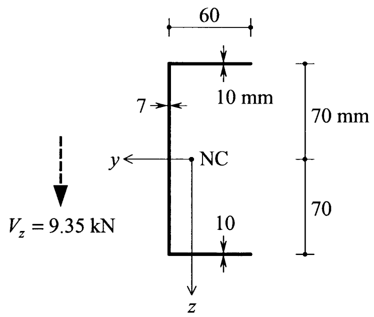


Figure 5.110 A U-section on its side, subject to a vertical shear force $V_z = 9.35$ kN. The position of the line of action is unknown.

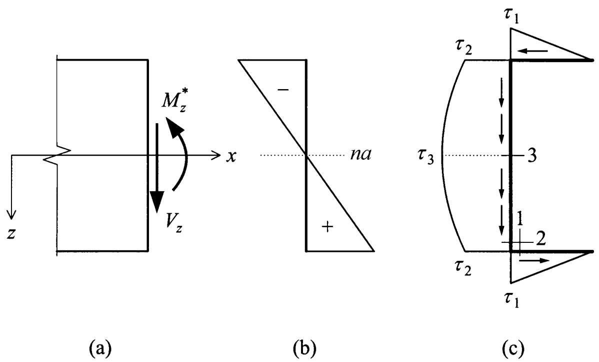


Figure 5.111 (a) Side view of the beam to the left of the cross-section with the shear force V_z and an arbitrary bending moment M_z^*, both acting in the xz plane. (b) The bending stress diagram due to M_z^*. (c) A sketch of the shear stress diagram due to V_z.

Example 1: A steel U-section

The U-section in Figure 5.110 has to transfer a shear force $V_z = 9.35$ kN. This U-section was used previously in Section 5.4.2, Example 3.

Questions:

a. Determine the shear stress distribution.
b. Determine the resultant of all shear stresses in a flange.
c. Determine the resultant of all shear stresses in the web.
d. Determine the magnitude, direction and line of action of the resultant of all shear stresses in the cross-section.
e. Determine the location of the shear centre SC.

Solution (units in N and mm):

a. The shear stresses are determined using (5.14):

$$\sigma_{xm} = -\frac{V_z S_z^{\mathrm{a}}}{b^{\mathrm{a}} I_{zz}}, \qquad (5.14)$$

in which

$$I_{zz} = I_{zz(\mathrm{centr})}^{\mathrm{web}} + 2 \times I_{zz(\mathrm{Steiner})}^{\mathrm{flange}}$$
$$= \tfrac{1}{12} \times 7 \times 140^3 + 2 \times (60 \times 10) \times 70^2 = 7.48 \times 10^6 \text{ mm}^4.$$

The shear stress distribution is determined using the quick method. For that we have drawn the side view of the beam with the shear force V_z and a bending moment M_z^* in Figure 5.111a. Figure 5.111b shows the bending stress diagram due to M_z^* and Figure 5.111c gives a sketch of the shear stress diagram due to V_z.

In the flanges, the bending stress is constant, so the shear stress distribution is linear. On the (free) flange edges, the shear stress is zero. At the joints between the flanges and the web, *flow-in = flow-out*. The shear stresses at the top and bottom of the web are therefore not zero. In the web, the bending stress is linear, so the shear stress is parabolic. The top value is at the level of the normal centre NC. The direction of the shear stress in the web is equal to that of the shear force. The direction of the shear stresses in the flanges now follows from *flow-in = flow-out* at the joints.

For the complete shear stress diagram, we have to determine only the shear stresses τ_1 to τ_3 at the cuts 1 to 3 (see Figure 5.111c).

Calculation τ_1:
For the lower flange as sliding element

$$S_z^a = 60 \times 10 \times (+70) = +42 \times 10^3 \text{ mm}^3.$$

Hence

$$\tau_1 = \left| -\frac{V_z S_z^a}{b^a I_{zz}} \right| = \frac{(9.35 \times 10^3)(42 \times 10^3)}{10 \times (7.48 \times 10^6)} = 5.25 \text{ N/mm}^2.$$

Calculation τ_2:
flow-in = flow-out:

$$\tau_1 \cdot t_f = \tau_2 \cdot \tau_w$$

in which $t_f = 10$ mm is the flange thickness and $t_w = 7$ mm is the web thickness.

$$\tau_2 = \frac{t_f}{t_w}\,\tau_1 = \frac{10}{7} \times 5.25 = 7.5 \text{ N/mm}^2.$$

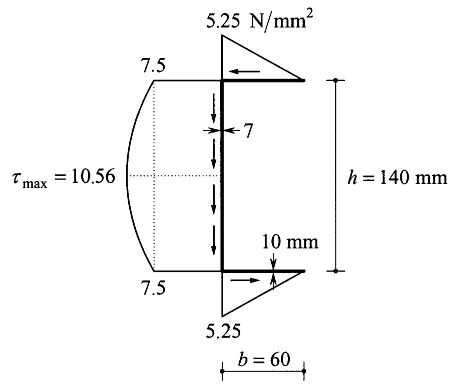


Figure 5.112 The complete shear stress diagram.

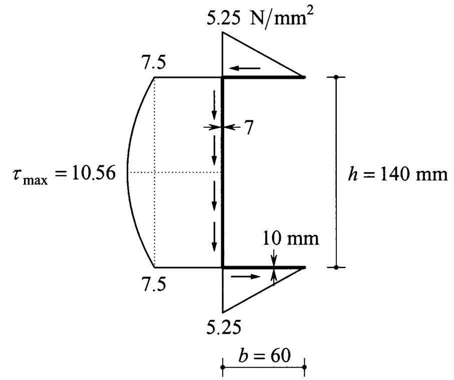


Figure 5.112 The complete shear stress diagram.

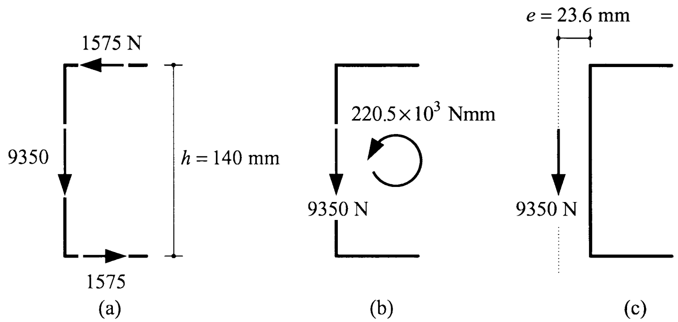


Figure 5.113 (a) The shear stress resultants in web and flanges. (b) The forces in the flanges form a couple. (c) By combining the force in the web and the couple produced by the forces in the flanges, we find that the force in the web shifts by a distance e. The resultant of all shear stresses therefore has its line of action at a distance e from the centre line of the web.

Calculation $\tau_3 = \tau_{max}$:
For the lower half of the cross-section, chosen as sliding element,

$$S_z^a = 60 \times 10 \times (+70) + 7 \times 70 \times (+35) = 59.15 \times 10^3 \text{ mm}^3.$$

So

$$\tau_3 = \tau_{max} = \left| -\frac{V_z S_z^a}{b^a I_{zz}} \right| = \frac{(9.35 \times 10^3)(59.15 \times 10^3)}{7 \times (7.48 \times 10^6)} = 10.56 \text{ N/mm}^2.$$

Figure 5.112 shows the complete shear stress diagram, with the values and directions.

b. The resultant of all shear stresses in a flange is

$$R^{\text{flange}} = \tfrac{1}{2} b\tau_1 \cdot t_f = \tfrac{1}{2} \times 60 \times 5.25 \times 10 = 1575 \text{ N}.$$

c. The resultant of all shear stresses in the web is most easily determined by splitting the shear stress diagram into a rectangle and a parabola:

$$R^{\text{web}} = \left\{ h\tau_2 + \tfrac{2}{3} h(\tau_{max} - \tau_2) \right\} \cdot t_w$$
$$= \left\{ 140 \times 7.5 + \tfrac{2}{3} \times 140 \times (10.56 - 7.5) \right\} \times 7 \approx 9350 \text{ N}.$$

d. Figure 5.113a shows the shear stress resultants in web and flanges. The resultant in the web is equal in direction and magnitude to the shear force. The resultants in the flanges form a couple with moment (see Figure 5.113b)

$$R^{\text{flange}} \times h = 1575 \times 140 = 220.5 \times 10^3 \text{ Nmm}.$$

By combining the force of 9350 N in the web with the couple[1] of 220.5×10^3 Nmm from the flanges, we find that the force shifts over a distance e (see Figure 5.113c):

$$e = \frac{220.5 \times 10^3 \text{ Nmm}}{9350 \text{ N}} = 23.6 \text{ mm}.$$

The resultant of all shear stresses due to a shear force V_z is a force equal in magnitude and direction to the shear force, but with its line of action outside the cross-section at a distance e from the centre line of the web.

Alternative solution question d:
The line of action of the shear force can also be found from the condition that the moment of the shear force about an arbitrary point is equal to the sum of moments about the same point of all shear stress resultants in the cross-section. If that point is chosen on the line of action of the shear force, the sum of moments of all shear stress resultants must be zero. This gives the equation of the line of action of the shear force, as shown below.

Assume point A with coordinates (y,z) is a point on the line of action of the shear force (see Figure 5.114). The y coordinate of the normal centre NC is given here.[2] Then

$$\sum T_x|A = +1575 \times (70 - z) + 1575 \times (70 + z) - 9350 \times (y - 16.5)$$
$$= 374.8 \times 10^3 - 9350 \times y = 0,$$

so

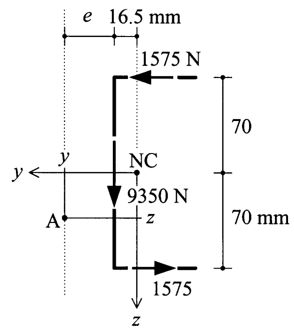


Figure 5.114 When there is no torsional moment, the sum of the moments of all shear stress resultants must be zero with respect to any arbitrary point $A(y, z)$ on the line of action of the shear force.

[1] See Volume 1, Section 3.1 and Section 3.5, Problem 3.2-2.
[2] It was determined previously in Section 5.4.2, Example 3.

$$y = +40.1 \text{ mm and } z \text{ is undetermined.}$$

The line of action of the shear force is vertical at a distance e from the centre line of the web:

$$e = y - 16.5 = 40.1 - 16.5 = 23.6 \text{ mm.}$$

e. If the vertical shear force does not have the line of action shown in Figure 5.113c, there will be torsion. We can check this experimentally by holding a U-shaped curtain rail on its side. Under the influence of its dead weight, that applies within the cross-section, the rail will *twist*. As a result, additional shear stresses are generated in the cross-section. Shear stresses due to torsion are covered in Chapter 6.

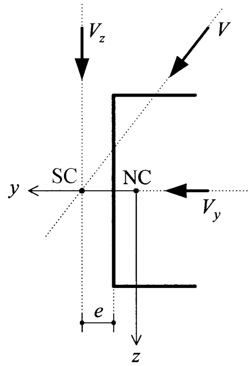


Figure 5.115 The shear centre SC is that point in the cross-section through which the line of action of the shear force V must pass so that there will be no torsion.

If torsion is unwanted, and there should be no shear stresses due to torsion, the shear force V_z must have the line of action given in Figure 5.115. For a shear force V_y, the line of action must coincide with the line of mirror symmetry. The intersection of both lines of action is called the shear centre SC.

The shear centre SC is that point in the cross-section through which the line of action of the shear force has to pass so that there will be no torsion.

Note the analogy with the definition of the normal centre NC:

The normal centre NC is that point in the cross-section at which the normal force must act so that there will be no bending.

If the cross-section has a line of symmetry, the shear centre SC will be located on that line. If the cross-section does not have a line of symmetry, first determine the shear stress distribution and the line of action for the shear force V_y in one principal direction, and repeat the procedure for the shear force V_z in the other principal direction. The intersection of the lines of action of V_y and V_z is then the shear centre SC (see Figure 5.115).

Formulas for the thin-walled U-section
In the thin-walled U-section in Figure 5.116a, A_w is the area of the web:

$$A_w = ht_w,$$

and A_f is the area of a flange:

$$A_f = bt_f.$$

For the (centroidal) moment of inertia I_{zz} we find

$$I_{zz} = \tfrac{1}{2} A_f h^2 (1 + \alpha),$$

in which

$$\alpha = \frac{1}{6} \frac{A_w}{A_f}.$$

For the shear stress distribution in Figure 5.116b, due to the shear force $V_z = V$, applies

$$\tau_1 = \frac{V}{A_w} \frac{t_w}{t_f} \frac{1}{1+\alpha},$$

$$\tau_2 = \frac{V}{A_w} \frac{1}{1+\alpha},$$

$$\tau_3 = \tau_{max} = \frac{V}{A_w} \frac{1+1.5\alpha}{1+\alpha}.$$

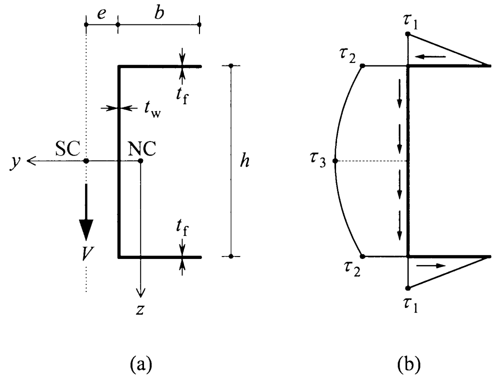


Figure 5.116 (a) The thin-walled U-section, subject to a shear force V_z with its line of action through the shear (force) centre SC. (b) A sketch of the associated shear stress diagram.

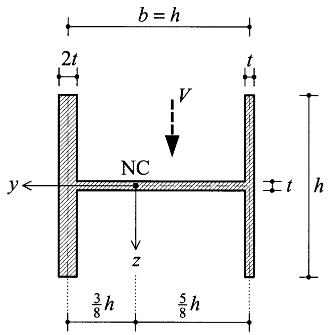


Figure 5.117 A thin-walled I-section on its side, subject to a vertical shear force $V_z = V$. The line of action of V is unknown.

In these equations, V/A_w is the average vertical shear stress in the web.

For the distance e from the shear centre SC to the centre line of the web we can derive

$$e = \frac{b}{2(1-\alpha)} \,.$$

This shows that the location of the shear centre SC is determined entirely by the shape of the cross-section and is independent of the magnitude and direction of the shear force.

The reader is asked

- to derive the given expressions;
- to check that these expressions, applied to the example, indeed lead to the same values.

Example 2: An I-section on its side

In Figure 5.117 an I-section on its side is loaded by a shear force $V_z = V$. The location of the normal centre NC can be found from the figure. Note that the two vertical flanges have different wall thicknesses.

Questions:

a. Determine the shear stress distribution.
b. Determine the shear centre.

Solution:

a. The shear stresses are determined using (5.14):

$$\sigma_{xm} = -\frac{V_z S_z^\mathrm{a}}{b^\mathrm{a} I_{zz}} \,, \tag{5.14}$$

in which

$$I_{zz} = \tfrac{1}{12}(2t + t)h^3 = \tfrac{1}{4}th^3.$$

Due to an arbitrary bending moment M_z^* in the plane of the shear force V_z the bending stress distribution in the vertical flanges will be linear, so the shear stress distribution here must be parabolic.

Calculation τ_1 in the cuts 1 (see Figure 5.118a):

$$|S_z^{\mathrm{a}}| = \tfrac{1}{2}h \cdot 2t \cdot \tfrac{1}{4}h = \tfrac{1}{4}th^2,$$

and therefore

$$\tau_1 = \frac{V \cdot \frac{1}{4}th^2}{2t \cdot \frac{1}{4}th^3} = \frac{1}{2}\frac{V}{th}.$$

Calculation τ_2 in the cuts 2 (see Figure 5.118a)

$$|S_z^{\mathrm{a}}| = \tfrac{1}{2}h \cdot t \cdot \tfrac{1}{4}h = \tfrac{1}{8}th^2.$$

This gives

$$\tau_2 = \frac{V \cdot \frac{1}{8}th^2}{t \cdot \frac{1}{4}th^3} = \frac{1}{2}\frac{V}{th}.$$

The shear stresses are equal in both vertical flanges. Figure 5.118b shows their shear stress distributions.

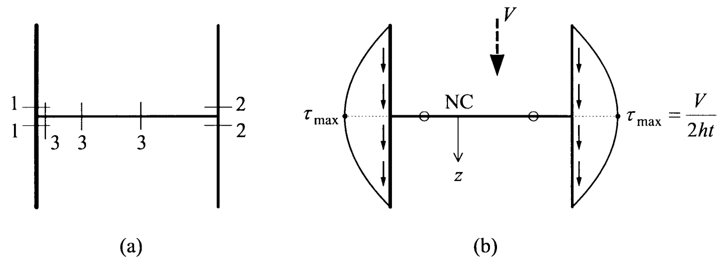


Figure 5.118 (a) The cuts where the shear stresses are determined. (b) The shear stress diagram.

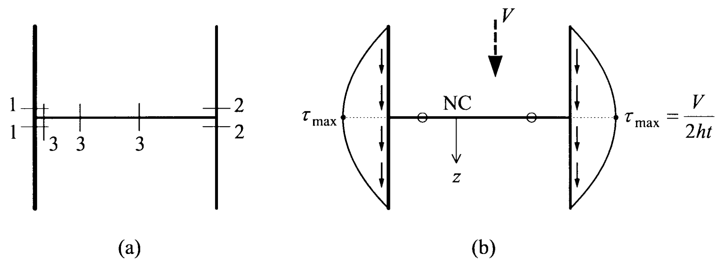


Figure 5.118 (a) The cuts where the shear stress is determined. (b) The shear stress diagram.

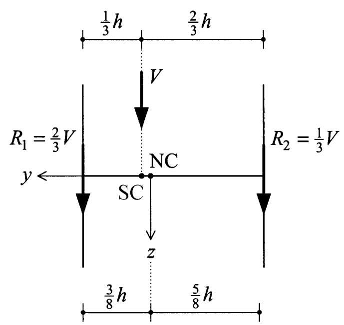


Figure 5.119 The shear centre SC is located on the intersection of the line of symmetry and the line of action of the shear force. The shear centre SC and normal centre NC are clearly two different points in the cross-section.

Calculation τ_3 in the cuts 3 (see Figure 5.118a)

$$|S_z^{\mathrm{a}}| = 0,$$

and so

$$\tau_3 = 0.$$

The shear stress in the horizontal web is zero (see Figure 5.118b). At the joint with the horizontal flanges this is in line with the demand *flow-in* = *flow-out*.

b. Figure 5.119 shows the resultant shear forces R_1 and R_2 in the vertical flanges:

$$R_1 = \tfrac{2}{3} \cdot h \cdot \tfrac{1}{2} \frac{V}{th} \cdot 2t = \tfrac{2}{3} V,$$

$$R_2 = \tfrac{2}{3} \cdot h \cdot \tfrac{1}{2} \frac{V}{th} \cdot t = \tfrac{1}{3} V.$$

The resultant of R_1 and R_2 is the shear force:

$$R = R_1 + R_2 = \tfrac{2}{3} V + \tfrac{1}{3} V = V.$$

The shear centre SC is located on the line of action of the shear stress resultant $R = V$, and on the line of symmetry. It therefore coincides with the intersection of both lines (see Figure 5.119).

The shear centre SC and normal centre NC are clearly two different points in the cross-section.

Example 3: Thin-walled T- and L-sections
You are given the thin-walled sections in Figure 5.120.

Question:
Determine the shear centre SC.

Solution:
For these sections the shear centre SC can be found without calculation.

The shear force V is the resultant of the shear stress resultants R_1 and R_2 in Figure 5.121. The line of action of the shear force V always passes through the intersection of the lines of action of R_1 and R_2, regardless of the magnitude and direction of the shear force. This point is therefore the shear centre SC.

Conclusion: *In thin-walled T- and L-sections the shear centre SC is on the intersection between "flange" and "web"*.

5.6 Other cases of shear

So far, we have looked at situations in which shear is combined with bending. We derived the formulas for the shear forces and shear stresses from the change in the bending stresses between two consecutive cross-sections. This means that the formulas apply only when the material behaves linear elastically and when all the other assumptions on which the bending stress formulas are based are met.

In this section, we look at a number of examples for which the theory we have covered so far does not suffice, and reality is clearly more complex.

An example is given in Figure 5.122. It shows the loading scheme for a steel plate that is being cut with scissors. The distance a between the forces F exerted by the cutting tool is considerably smaller than the thickness t

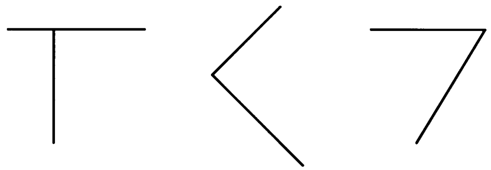


Figure 5.120 Thin-walled T- and L-sections.

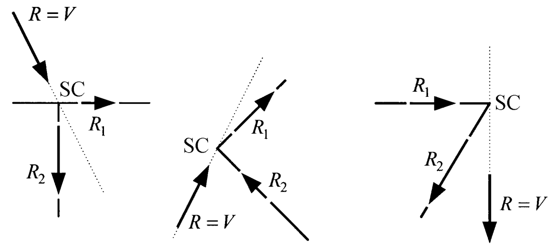


Figure 5.121 In thin-walled T- and L-sections the shear forces centre SC is located on the intersection of "flange" and "web".

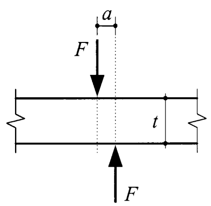


Figure 5.122 The loading scheme for a steel plate that is being cut with scissors. The distance a between the forces F exerted by the cutting tool is considerably smaller than the thickness t of the plate. In this area the shear stress formulas derived cannot be used.

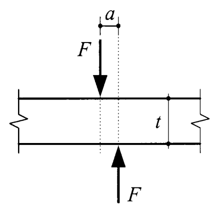


Figure 5.122 The loading scheme for a steel plate that is being cut with scissors. The distance a between the forces F exerted by the cutting tool is considerably smaller than the thickness t of the plate. In this area the shear stress formulas derived cannot be used.

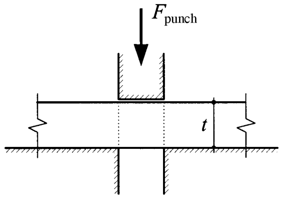


Figure 5.123 Model of a punching system, where a cutting stamp made of hard steel is pressed through the plate material with large force.

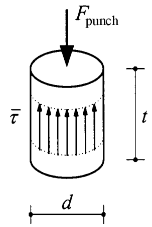


Figure 5.124 The stresses and forces acting on a cylindrical punching slug. The slug is the material that is stamped out of the plate.

of the plate. In this area, the elementary beam theory no longer applies, and the shear stress formulas cannot be used. Moreover, if the forces F are increased to such an extent that the plate is actually cut through, the plate material is clearly no longer linear-elastic.

Another example is punching or die-cutting, where a cutting stamp of hard steel is pressed through the plate material with a large force. The plate is on a rigid foundation that includes a cut out that is only slightly larger than the stamp. A model of the punch setup is shown in Figure 5.123.

When performing calculations for such complicated situations, it is common practice to use an average shear stress τ_{average} (for which one simply writes τ) and then compare this to a limiting value $\bar{\tau}$, which could be referred to as the (*allowable*) *shear strength*. This shear strength is generally determined experimentally and is attuned to the practical situation in question. Often, the shear stresses in such situations should not be considered real stresses, but rather design quantities. An example is given below.

Example 1: Punching a plate
Round holes have to be punched into an aluminium plate with thickness $t = 7.5$ mm. The maximum punching force that the press can exert is 40 kN. The shear strength of aluminium is $\bar{\tau} = 60$ MPa.

Question:
Determine the maximum diameter d of the holes that can be punched.

Solution:
The material that is punched out of the plate is known as the punching slug. Figure 5.124 shows all the stresses and forces acting on the cylindrical slug.

Shear strength $\bar{\tau}$ acts on the on the lateral surface of the cylinder with area $\pi d \cdot t$. The top of the slug is subject to punching force F_{punch}. From the vertical equilibrium of the slug it follows that

$$F_{\text{punch}} = \pi d \cdot t \cdot \bar{\tau},$$

from which we find

$$d_{\max} = \frac{F_{\text{punch;max}}}{\pi \cdot t \cdot \bar{\tau}} = \frac{40 \times 10^3\ \text{N}}{\pi \times (7.5\ \text{mm})(60\ \text{N/mm}^2)} = 28.3\ \text{mm}.$$

Another example relates to the bolted connections in Figure 5.125. These so-called *lap joints* are used in steel and wooden structures. A distinction is made between the (asymmetrical) *single shear joint* in Figure 5.125a and the (generally symmetrical) *double shear joint* in Figure 5.125b. Figure 5.126 shows the joints in a spatial representation. Of course, more than one bolt can be used.

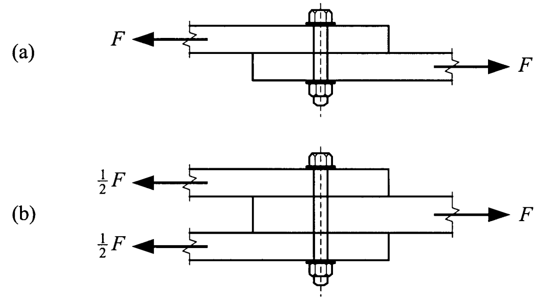


Figure 5.125 Lap joints. A distinction can be made between (a) the asymmetrical single shear joint and (b) the generally symmetrical double shear joint. Lap joints are used in steel and wooden structures.

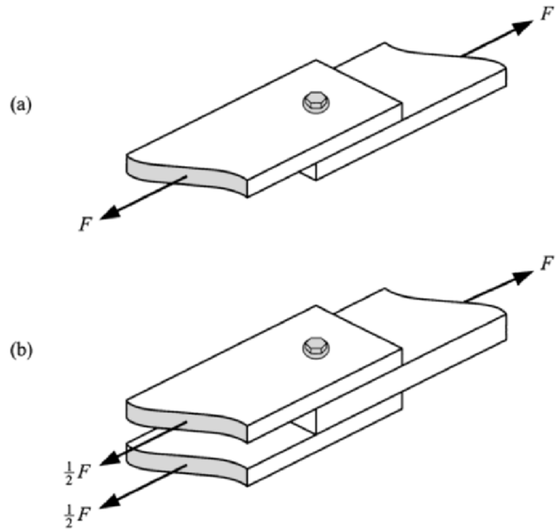


Figure 5.126 Spatial representation of (a) the single shear joint and (b) the double shear joint.

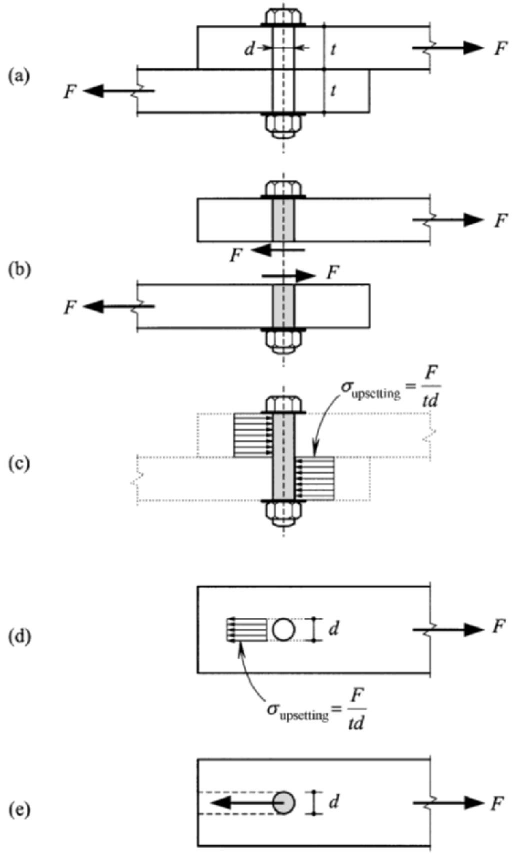


Figure 5.127

If the friction between the overlapping strips can be ignored, the transfer of force F from one strip to the other strip(s) occurs entirely via the bolt.[1] In the single shear joint in Figure 5.127a, the bolt shank has to transfer a shear force F (see Figure 5.127b). With a diameter d of the bolt shank, the average shear stress in the shank is

$$\tau_{\text{bolt}} = \frac{F}{A_{\text{bolt}}} = \frac{F}{\frac{1}{4}\pi d^2}\,.$$

The contact stress between the bolt and the strip is actually extremely complicated. In practice it is simplified by introducing the *upsetting stress* $\sigma_{\text{upsetting}}$. This is the average normal stress on area τd that is found from the projection of the bolt shank on the strip:

Figure 5.127 (a) A single shear lap joint. (b) The bolt (shank) has to transfer a shear force F. (c) The contact stress between the bolt and the strip is very complicated. In practice the situation is simplified by introducing the upsetting stress $\sigma_{\text{upsetting}}$. This is the average normal stress on the area τd, found from the projection of the bolt shank on the strip. (d) The upsetting stress acting on the upper strip. (e) The distance from the bolt hole to the end of the strip must be large enough to prevent the strip material from sliding. The dotted line gives a simple model of the slide shape.

---

[1] In so-called *high tensile bolts*, applied on steel joints, this is not the case. Here the joining parts are screwed so tightly together that the force F is transferred entirely by friction.

$$\sigma_{\text{upsetting}} = \frac{F}{\tau d}.$$

Figure 5.127c shows the upsetting stress on the bolt, while Figure 5.127d shows that on the upper strip.

The distance from the bolt hole to the end of the strip has to be large enough to prevent the strip material from sliding. Figure 5.127e shows a simple model of the slide shape. In regulations, the demands with respect to sliding are often combined with those related to the upsetting stress.

To end this section, we provide second example.

Example 2: A bolted double shear lap joint

The double shear lap joint with two bolts in Figure 5.128 has to transfer a tensile force of 78 kN. All the information required can be found from the figure.

Questions:

a. Determine the (average) shear stress in the cross-section of the bolt shank.
b. Determine the upsetting stress on each of the strips.

Solution:

a. Figure 5.129 shows that the two bolts together have to transfer a shear force of 39 kN. The (average) sheer stress in the bolt shank is then

$$\tau_{\text{bolt}} = \frac{39 \times 10^3\ \text{N}}{2 \times \frac{1}{4}\pi \times (16\ \text{mm})^2} = 97\ \text{N/mm}^2.$$

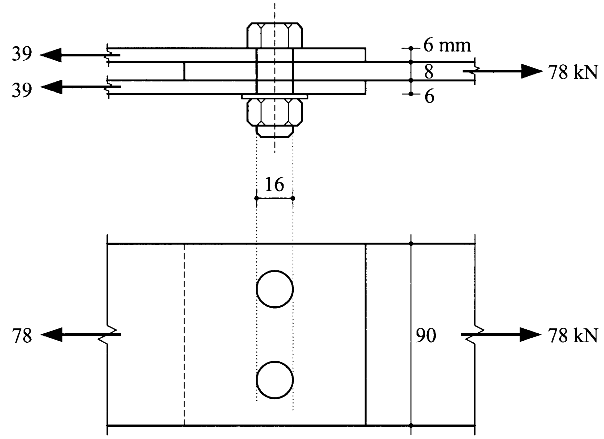


Figure 5.128 A double shear lap joint in which two bolts have to transfer a tensile force of 78 kN.

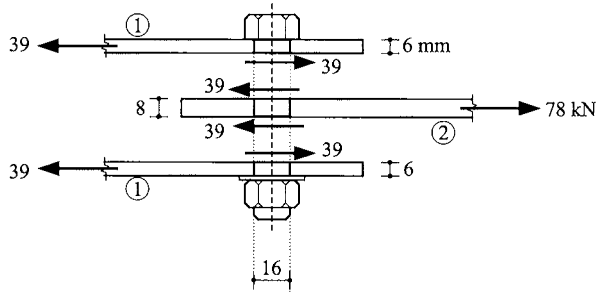


Figure 5.129 The two bolts together have to transfer shear forces of 39 kN.

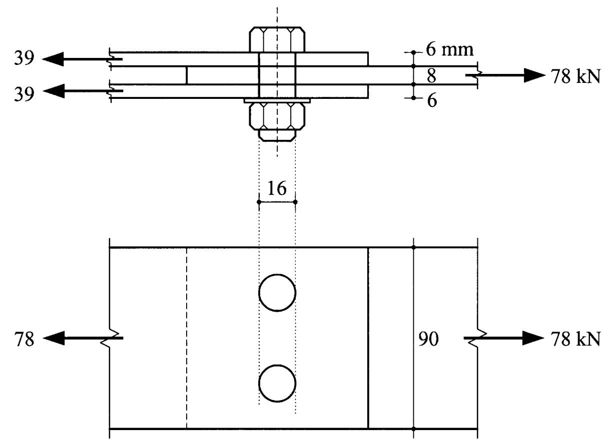


Figure 5.128 A double shear lap joint in which two bolts have to transfer a tensile force of 78 kN.

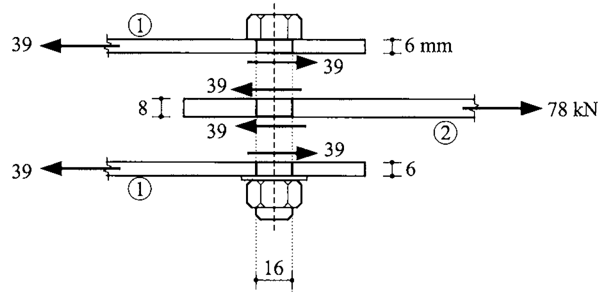


Figure 5.129 The two bolts together have to transfer shear forces of 39 kN.

b. The upsetting stress on the outer strips (5.1) is

$$\sigma^{(1)}_{\text{upsetting}} = \frac{39 \times 10^3 \text{ N}}{2 \times (16 \text{ mm})(6 \text{ mm})} = 203 \text{ N/mm}^2.$$

The upsetting stress on the centre strip (2) is

$$\sigma^{(2)}_{\text{upsetting}} = \frac{78 \times 10^3 \text{ N}}{2 \times (16 \text{ mm})(8 \text{ mm})} = 305 \text{ N/mm}^2.$$

5.7 Summary of the formulas and rules

Section 5.7.1 is a summary of the most important formulas for determining the shear forces and shear stresses in both a longitudinal direction and in the plane of the cross-section.

In Section 5.7.2 there are a number of rules for sketching the shear stress distribution in the plane of the cross-section.

5.7.1 Formulas

- Shear force per length in the longitudinal direction (also known as the shear flow in the longitudinal direction) (Sections 5.1.2 and 5.1.3):

$$s^{\text{a}}_x = -\frac{V_z S^{\text{a}}_z}{I_{zz}} \quad \text{(the } z \text{ direction is a principal direction),} \tag{5.7}$$

$$S^{\text{a}}_x = -V_z \cdot \left[\frac{N^{\text{a}}(\text{due to } M^*_z)}{M^*_z} \right]. \tag{5.12}$$

In formula (5.12) M^*_z is a bending moment of arbitrary value, acting in the same plane as in which the shear force is acting. Here the z direction need not be a principal direction.

- Resultant shear force in the longitudinal direction (Section 5.1.2):

$$R^{a}_{x;s} = -\frac{\Delta M_z S^{a}_{z}}{I_{zz}} \text{ (the } z \text{ direction is a principal direction).} \quad (5.9)$$

- Average shear stress in the longitudinal direction (Section 5.1.4):

$$\tau^{a}_{\text{average}} = \frac{s^{a}_{x}}{b^{a}} . \quad (5.13)$$

- The shear stresses in two mutually perpendicular planes are equal (Sections 5.3.1 and 5.3.2):

$$\sigma_{ij} = \sigma_{ji} \text{ with } i, j = x, y, z \text{ and } i \neq j.$$

For the relationship between the shear stress σ_{xm} in the plane of the cross-section and the shear stress σ_{mx} in a longitudinal section this means

$$\sigma_{xm} = \sigma_{mx} = \frac{s^{a}_{x}}{b^{a}} .$$

- Shear stress formulas (Section 5.3.2):
The shear formulas apply on the *assumption* that the shear stresses are acting normal to the cut and are constant over the width b^{a} of that cut:

$$\sigma_{xm} = -\frac{V_z S^{a}_{z}}{b^{a} I_{zz}} \text{ (the } z \text{ direction is a principal direction),} \quad (5.14)$$

$$\sigma_{xm} = -\frac{V_z}{b^{a}} \left[\frac{N^{a}(\text{due to } M^{*}_{z})}{M^{*}_{z}} \right] . \quad (5.15)$$

Formula (5.15) applies even when the z direction is not a principal direction.

- Definition shear centre:
  The *shear centre* SC is that point in the cross-section through which the line of action of a shear force must pass so that there will be no torsion.

5.7.2 Rules for the shear stress distribution in the cross-sectional plane

1. The shear stress normal to the edge of a cross-section is zero (Section 5.3.3).

Rules 2 to 4 apply only to cross-sections (or parts thereof) in which the width b^a of the sliding element is constant (Section 5.3.3):

2. If the bending stress σ is constant, the shear stress τ must be linear (in m).
3. If the bending stress σ is linear, the shear stress τ must be parabolic (quadratic in m).
4. The shear stress τ is extreme at the cut through the normal centre NC.

Other rules (Section 5.4.2, Example 1 and Section 5.4.4, Example 1):

5. The *flow direction* of the shear stresses is continuous.
6. The *shear flow* s is equal to the product of the shear stress τ and thickness t: $s = \tau t$.
7. Where flanges and webs join, the shear flow must meet the demand that the total flow-in is equal to the total flow-out: *flow-in* = *flow-out*.
8. The shear flow is always largest in a cut through the normal centre NC.

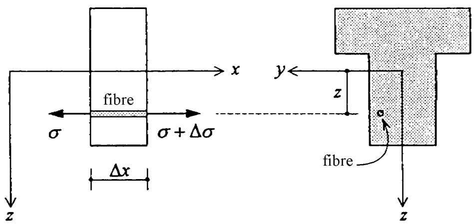
5.8 Problems

General comment:

- All problems are without torsion.
- In a number of questions, you are asked to provide both the shear stress and the normal stress.
- The dead weight of the structure is ignored unless explicitly mentioned otherwise.

Shear forces and shear stresses in longitudinal direction (Sections 5.1 and 5.2)

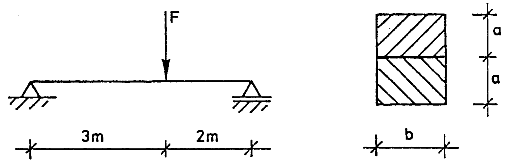
5.1 A prismatic beam is subject to bending and shear in the xz plane. There is a constant normal force in the beam. The normal stress in a fibre is a function of x.



Question:
Show that for the change per length of the normal stress σ in a fibre at a distance z from the xy plane the following applies:

$$\lim_{\Delta x \to 0} = \frac{\Delta\sigma}{\Delta x} = \frac{\mathrm{d}\sigma}{\mathrm{d}x} = \frac{V_z z}{I_{zz}}.$$

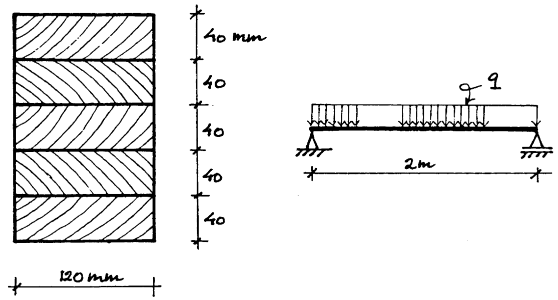
5.2 A beam is constructed of two timbers glued together, and is loaded by the point load F as shown. In the calculation use $a = 90$ mm, $b = 120$ mm and $F = 6$ kN.



Questions:

a. Determine the maximum shear flow (force per length) in the glued joint. Where does this maximum shear flow occur?
b. Determine the maximum shear stress in the glued joint. Where does this maximum occur?

5.3 A beam is built up of 5 planks of 40×120 mm^2 glued together. The beam is simply supported with a span of 2 m and carries a uniformly distributed load q over the entire length. The normal stress may not exceed the limiting value $\bar{\sigma} = 7$ N/mm^2 (the allowable normal stress). The shear stress in the glued joints may not exceed the limiting value $\bar{\tau} = 0.6$ N/mm^2 (the allowable shear stress).



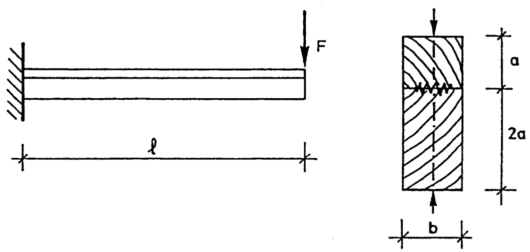
Questions:

a. For which load q does the normal stress reach its limiting value?
b. For which load q does the shear stress in one of the glued joints reach

its limiting value?

c. Which limiting value (for the normal stress or shear stress) is indicative of the bearing capacity of the beam?

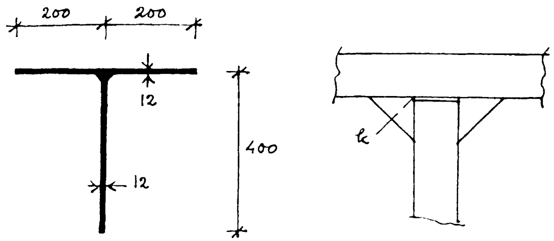
5.4 A cantilever beam is built up of two timbers connected by toothed plate shear connectors. The timbers are clamped to one another by bolts. Each connector can transfer a maximum shear force of 7 kN. In the calculation use $a = 100$ mm, $b = 120$ mm, $\ell = 3$ m and $F = 4.5$ kN.



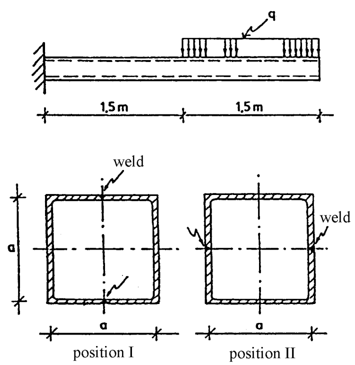
Question:
Determine the minimum number of shear connectors required in the beam.

5.5 The web and flange of a thin-walled T-section are connected by a double corner weld. The cross-sectional dimensions in the figure are given in mm. A shear force of 10 kN acts in the plane of mirror symmetry of the cross-section.

Question:
Determine the shear force per length (in N/mm) in the throat cut k of one of the corner welds.



5.6 The cantilever box girder has a square cross-section and is constructed of two U-sections welded together. The cross-section is thin-walled with a uniform wall thickness t. The cross-section can be placed in the positions I or II. In the calculation use $q = 8$ kN/m, $a = 180$ mm and $t = 10$ mm.



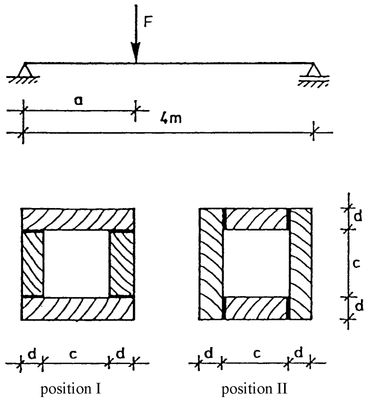
Questions:

a. Determine the maximum shear force per length in a single weld for the cross-section in position I.
b. Determine the maximum shear force per length in a single weld for the cross-section in position II.

5.7 A simply supported beam with a square hollow cross-section, is constructed of four planks glued together. It is assumed that the shear stress in the glued joints is uniformly distributed over the width of the joint. The beam can be placed in position I or position II. The beam is subject to the vertical load as shown.

Question:
Which of the statements below is correct? The shear stress in the glued joints is:

a. larger in position I than in position II.
b. equal in the positions I and II.
c. smaller in position I than in position II.

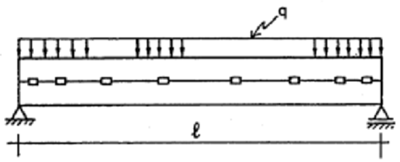


5.8 See problem 5.7. In the calculation use $a = 1.5$ m, $F = 23.2$ kN, $c = 120$ mm and $d = 40$ mm.

Questions:

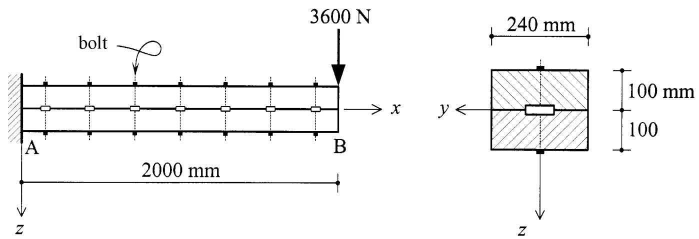
a. Determine the maximum shear stress in the glued joints for the beam in position I.
b. Determine the maximum shear stress in the glued joints for the beam in position II.

5.9 A simply supported beam with length $\ell = 4$ m is built up of two square beams of 120×120 mm^2 that are connected by means of dowels. Each dowel can transfer a maximum shear force of 5 kN. The beam carries a uniformly distributed load $q = 1.8$ kN/m over its full length.



Question:
Determine the minimum number of dowels required.

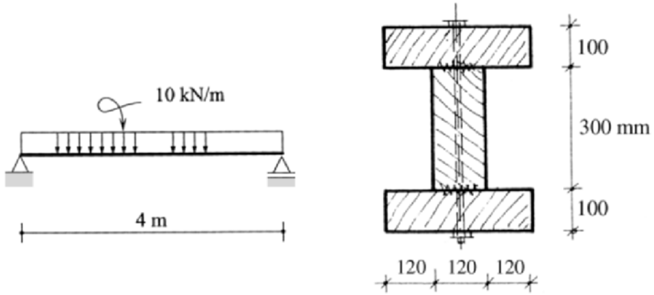
5.10 A cantilever beam AB is loaded at its free end B by a vertical force of 3600 N. The beam is constructed of two timbers of 100×240 mm^2 that are joined by means of dowels, as shown in the figure. The maximum shear force that a dowel can transfer is 5.8 kN.



Questions:

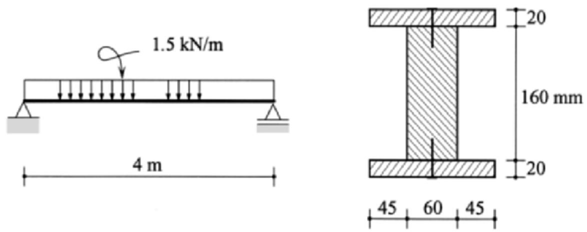
a. Determine the minimum number of dowels required.
b. Determine the maximum force per dowel.

5.11 A simply supported beam consists of three battens that are joined by means of toothed plate shear connectors, and clamped by bolts. Each connector can transfer a maximum shear force of 6 kN. For the moment of inertia in the vertical plane use $I = 3.21 \times 10^9$ mm^4.



Question:
How many connectors are needed in the given situation?

5.12 The wooden beam shown is constructed of a batten and two planks that are connected by means of wire nails. Each nail can transfer a maximum shear force of 300 N. The beam is simply supported with a span of 4 m and carries a uniformly distributed load of 1.5 kN/m over the full length. Use for the moment of inertia in the calculation $I_{zz} = 69.28 \times 10^6$ mm^4.

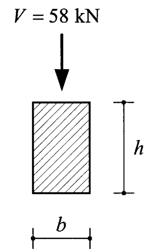


Question:
For the given load, the total number of wire nails required is closest to

a. 40.
b. 80.
c. 160.
d. 320.

Shear stresses in the cross-sectional plane (Sections 5.3 and 5.4)

5.13 A rectangular cross-section of width b and height h transfers a shear force $V = 58$ kN in the vertical plane. The area of the cross-section is $A = 14.5 \times 10^3$ mm^2.

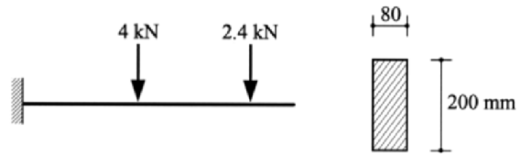


Questions:

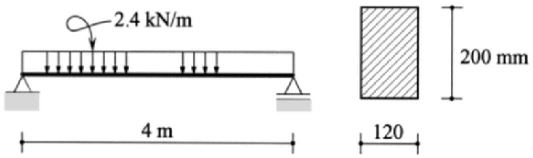
a. Determine the average vertical shear stress.
b. Determine the maximum shear stress.
c. Determine the shear stress at quarter-height.
d. How does the maximum shear stress change if the shear force does not act vertically but horizontally?

5.14: 1–2 You are given two different beams with rectangular cross-section.

(1)

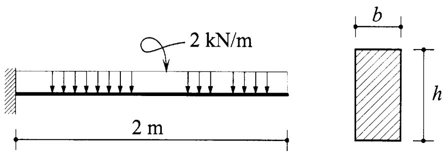


(2)



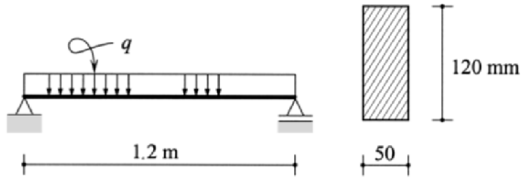
Question:
Determine the maximum shear stress in the beam.

5.15 The cantilever beam has a rectangular cross-section and caries a uniformly distributed load over the entire length.



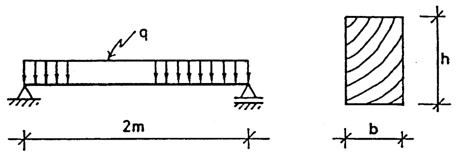
Question:
Determine area A of the cross-section in order that the shear stress nowhere exceeds the limiting value $\bar{\tau} = 0.5$ N/mm^2.

5.16 A simply supported beam with a span of 1.2 m carries a uniformly distributed load q over the full length. The beam has a rectangular cross-section with dimensions 50×120 mm^2.



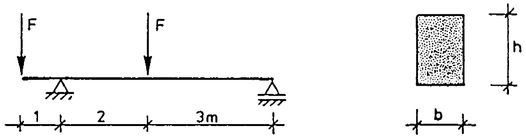
Question:
Determine the uniformly distributed load q for which the maximum shear stress in the beam is 0.6 N/mm^2.

5.17 When dimensioning the wooden beam shown, the shear stress appears to be indicative. The limiting value for the shear stress (the allowable shear stress) is $\bar{\tau} = 0.6$ N/mm^2. The beam has a rectangular cross-section with width $b = 80$ mm and carries a uniformly distributed load $q = 0.8$ kN/m over the entire length of the beam.



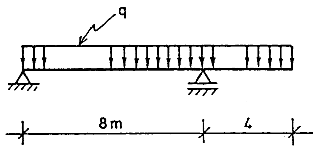
Question:
Determine the minimum height h of the beam.

5.18 A beam with rectangular cross-section is loaded as shown. In the calculation use $F = 42$ kN, $b = 120$ mm and $h = 300$ mm.



Question:
Determine the maximum vertical shear stress in the beam.

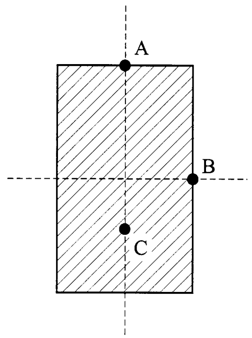
5.19 A prismatic beam with rectangular cross-section is supported as shown and is loaded by a uniformly distributed load $q = 40$ kN/m.



Question:
Determine the minimum area A of the cross-section required so that the shear stress is nowhere larger than $\bar{\tau} = 1.2$ N/mm^2.

5.20 The rectangular cross-section shown has to transfer a vertical shear force.

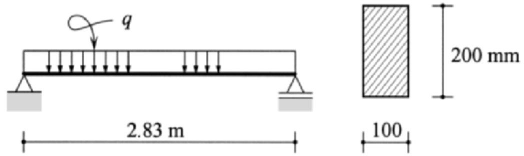
Questions:
a. At which of the points A, B and/or C is the shear stress largest?
b. At which of these points is the shear stress smallest?



5.21 A horizontal shear force is transferred by the rectangular cross-section from problem 5.20.

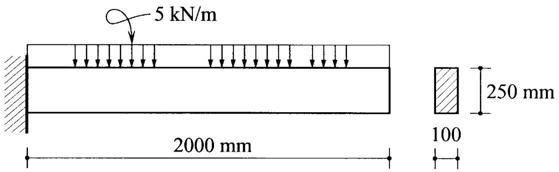
Questions:
a. In which of the points A, B and/or C is the shear stress smallest?
b. At which of these points is the shear stress largest?

5.22 A simply supported beam with a span of 2.83 m carries a uniformly distributed load q over the entire length. The beam has a rectangular cross-section with dimensions 100×200 mm^2. The limiting value for the bending stress (the allowable bending stress) is $\bar{\sigma} = 7.5$ N/mm^2; the limiting value for the shear stress (the allowable shear stress) is $\bar{\tau} = 0.6$ N/mm^2.



Question:
Determine the maximum load q that the beam can carry.

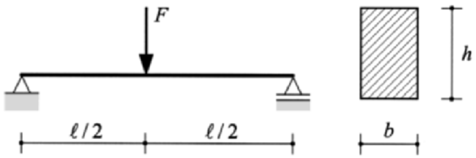
5.23 The wooden cantilever beam has a rectangular cross-section and carries a uniformly distributed load $q = 5$ kN/m. The limiting value for the bending stress is $\bar{\sigma} = 7$ N/mm^2; the limiting value for the shear stress is $\bar{\tau} = 1$ N/mm^2.



Questions:

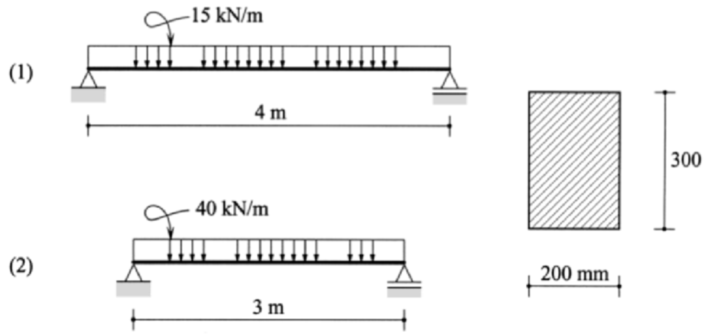
a. Investigate whether the maximum bending stress remains below the limiting value.
b. Investigate whether the maximum shear stress remains below the limiting value.
c. Determine the maximum load q that the beam can carry without exceeding the limiting values for the bending and shear stresses.

5.24 A simply supported beam with rectangular cross-section carries a point load F at midspan.



Question:
For the given situation, determine the ratio between the maximum bending stress and maximum shear stress if $\ell = 15h$.

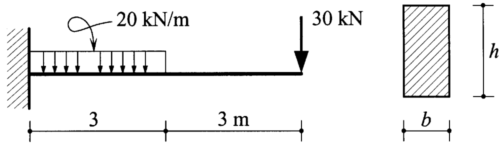
5.25: 1–2 You are given two different beams with the same rectangular cross-section.



Questions:

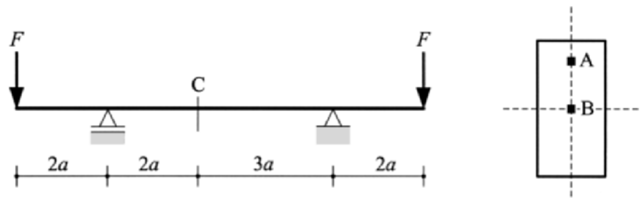
a. Determine the maximum bending stress in the beam.
b. Determine the maximum shear stress in the beam.

5.26 The cantilever beam shown has a rectangular cross-section $b \times h$ with $b = h/2$. The limiting value for the shear stress is $\bar{\tau} = 0.33$ N/mm^2; the limiting value for the bending stress is $\bar{\sigma} = 15$ N/mm^2. The load follows from the figure.



Question:
Determine the minimum beam height h.

5.27: 1–2 A beam with rectangular cross-section is subject to a so-called four-point bending test.



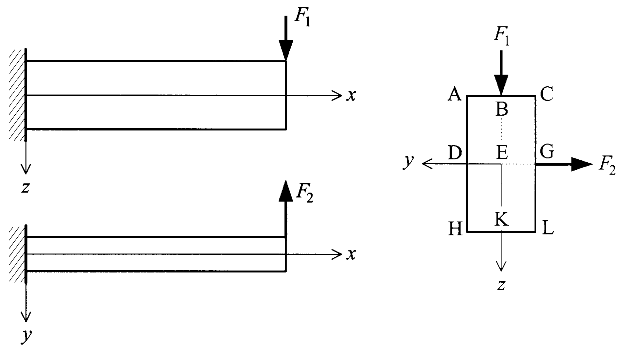
Questions:

1. Which claim is correct for the normal stress σ_A and shear stress τ_A at point A of cross-section C?
   a. $\sigma_A \neq 0$ and $\tau_A \neq 0$.
   b. $\sigma_A = 0$ and $\tau_A \neq 0$.
   c. $\sigma_A \neq 0$ and $\tau_A = 0$.
   d. $\sigma_A = 0$ and $\tau_A = 0$.

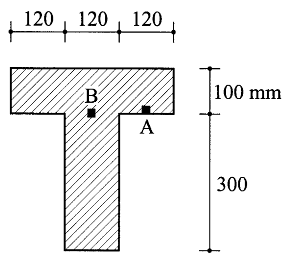
2. Which claim is correct for the normal stress σ_B and shear stress τ_B at point B of cross-section C?
   a. $\sigma_B \neq 0$ and $\tau_B \neq 0$.
   b. $\sigma_B = 0$ and $\tau_B \neq 0$.
   c. $\sigma_B \neq 0$ and $\tau_B = 0$.
   d. $\sigma_B = 0$ and $\tau_B = 0$.

5.28 A cantilever beam with rectangular cross-section is loaded at its free end by the two forces F_1 and F_2 as shown.

Questions:
a. In which of the given points is the tensile bending stress largest?
b. In which of the given points is the compressive bending stress largest?
c. In which of the given points is the shear stress largest?

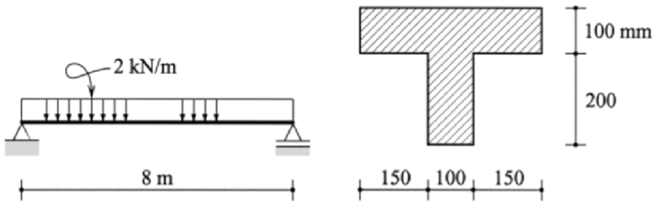


5.29 A vertical shear force of 42.8 kN acts in the cross-section. The moment of inertia in the vertical plane is $I = 102 \times 10^3\ \text{mm}^4$.



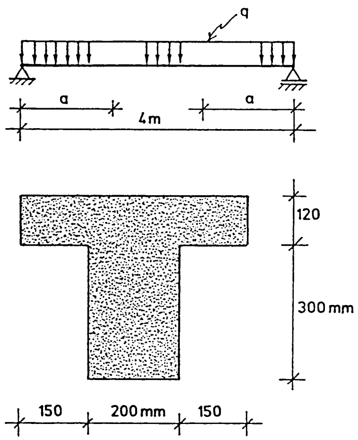
Questions:
a. Determine the vertical shear stress at A.
b. Determine the vertical shear stress at B.
c. Determine the maximum vertical shear stress in the cross-section and the location where it occurs.

5.30 A glued wooden T-beam is simply supported and carries a uniformly distributed load over the full length.



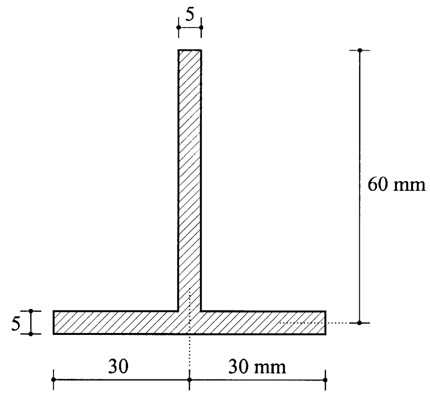
Question:
Determine the maximum shear stress in the beam.

5.31 A simply supported concrete T-beam is carrying a uniformly distributed load of 40 kN/m. The shear stress in the beam may not exceed the limiting value $\bar{\tau} = 0.6$ N/mm^2.



Question:
Over which length *a* must provisions be made (such as extra stirrups or bent up reinforcing bars) in order to transmit excessively large shear stresses?

5.32 The T-section shown has to transfer a downward shear force of 1 kN in the vertical plane of symmetry. The cross-section should be considered thin-walled in the calculation.



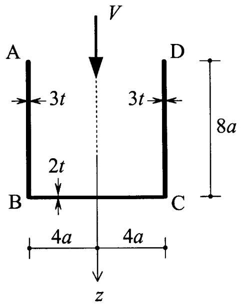
Questions:

a. Determine the distribution of the shear stresses as a function of the location in the cross-section.
b. Draw the shear stress diagram, including relevant values and the direction of the shear stresses.
c. Determine the shear force per length in the longitudinal direction at the join of the web to the flange.

5.33 A thin-walled symmetrical U-profile has to transfer the shear force $V_z = V$ in the plane of symmetry. The cross-sectional dimensions are shown in the figure.

Questions:

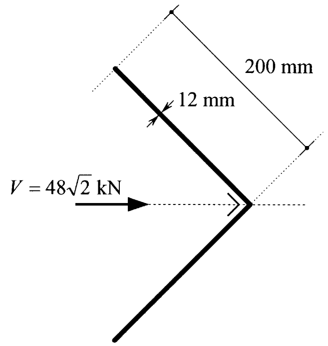
a. Determine the location of the normal centre NC and the magnitude of the (centroidal) moment of inertia I_{zz}.
b. Sketch the bending stress diagram due to an arbitrary bending moment M_z^*.
c. On the basis of the bending stress distribution in the cross-section, what can you say about the shear stress distribution? Sketch the shear stress diagram (without calculation). Also indicate in which direction the shear stresses are acting.
d. Where in the cross-section is the shear stress a maximum? Determine its value.
e. Also determine the shear stresses at C, in both the web and the flange.



5.34 A thin-walled angle steel with equal legs and a uniform wall thickness of 12 mm has to transfer a shear force $V = 48\sqrt{2}$ kN in the horizontal plane of symmetry.

Questions:

a. Plot the shear stress diagram. Indicate the direction of the shear stresses.
b. Determine the maximum shear stress. Where in the cross-section does it occur?

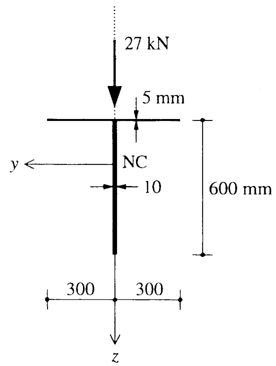


c. Verify that the resultant of all shear stresses in the cross-section is equal to the shear force.

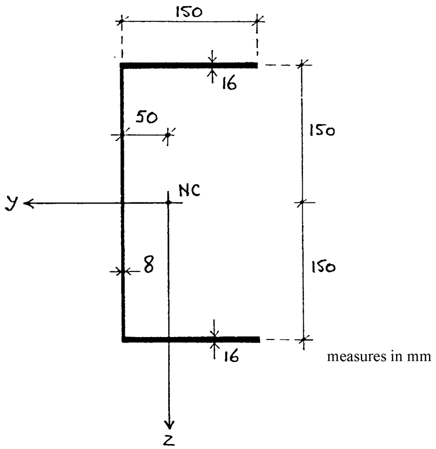
5.35 A thin-walled T-section with a web thickness of 10 mm and a flange thickness of 5 mm has to transfer a shear force of 18 kN in the plane of symmetry. The cross-sectional dimensions are given in the figure.

Questions:

a. Determine the location of the normal centre NC.
b. Determine the shear stresses in the web as a function of z.
c. Determine the shear stresses in the flange as a function of y.
d. For the entire cross-section plot the shear stress diagram. Indicate the direction of the shear stresses and include a number of relevant values.



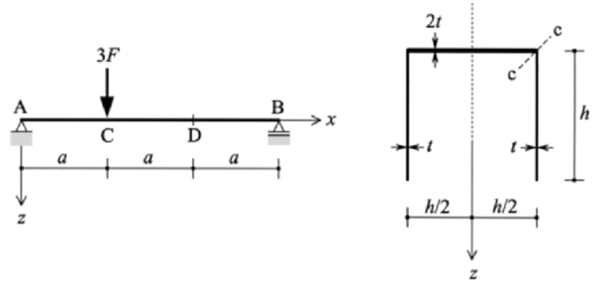
5.36 The cross-sectional dimensions of a thin-walled U-section are given in the figure. The cross-section has to transfer a shear force $V_y = 7200$ N.



Questions:

a. Verify that the moment of inertia in the xy plane is $I_{yy} = 18 \times 10^6$ mm^4.
b. Sketch the shear stress diagram (without calculation). Indicate the directions.
c. Determine the values for a number of relevant places. What is the maximum shear stress and where does it occur?

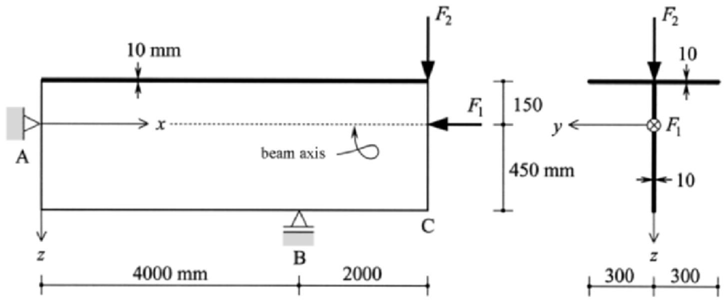
5.37 The simply supported beam AB is subject to a vertical force $3F$ at C, at one-third of the span. The beam has a thin-walled cross-section (a U-section); the web thickness is t and the flange thickness $2t$. Both height and width of the profile are h.



Questions:

a. Determine the location of the normal centre NC and the magnitude of the (centroidal) moment of inertia I_{zz}.
b. Determine and plot the distribution of the shear stresses acting in cross-section D on the left-hand part of the beam. Indicate the direction of the shear stresses and a number of relevant values.
c. Determine the longitudinal shear force per length at cut c-c at the join between web and flange.

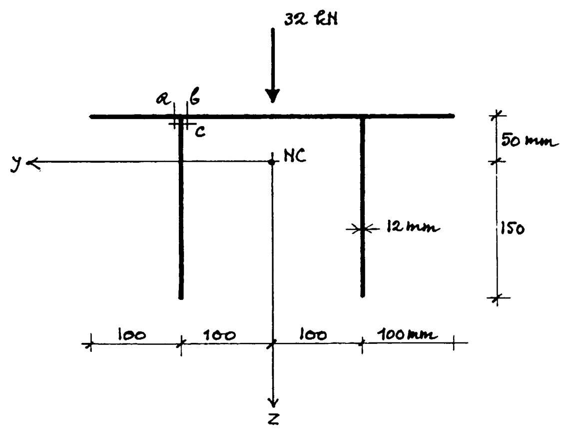
5.38 A thin-walled T-beam with uniform wall thickness of 10 mm is loaded by the forces $F_1 = 240$ kN and $F_2 = 40$ kN as shown in the figure. For the cross-section of the T-beam applies $A = 12 \times 10^3$ mm and $I_{zz} = 450 \times 10^6$ mm^4.



Questions:

a. Model the beam as a line element and plot the M, V and N diagrams.
b. In which cross-section is the normal stress a maximum? Determine and plot the normal stress diagram for that cross-section.
c. Determine and plot the shear stress diagram for a cross-section to the left of support B. How large is the maximum shear stress in that cross-section and where does it occur?
d. Determine and plot the shear stress diagram for a cross-section to the right of support B. How large is the maximum shear stress in that cross-section and where does it occur?

5.39 A thin-walled double T-section has a uniform wall thickness of 12 mm. The location of the normal centre NC is given in the figure. You are also given $I_{zz} = 40 \times 10^6$ mm^4. The cross-section has to transfer a shear force of 32 kN in the vertical plane of symmetry.



Questions:

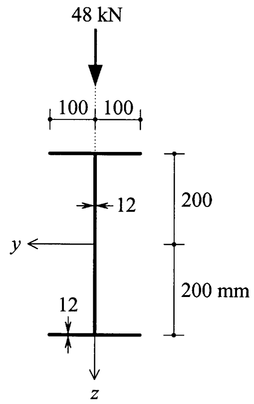
a. Determine the shear stress at cut a.
b. Determine the shear stress at cut b.
c. Determine the shear stress at cut c.
d. Determine the maximum shear stress in a web. Where does it occur?
e. For the entire cross-section plot the shear stress diagram, indicate the direction of the shear stresses and include a number of relevant values.

5.40 A shear force of 48 kN acts in the vertical plane of symmetry of the thin-walled I-section shown. The I-section has a uniform wall thickness of 12 mm. The moment of inertia in the vertical plane is $I_{zz} = 256 \times 10^6$ mm^4.

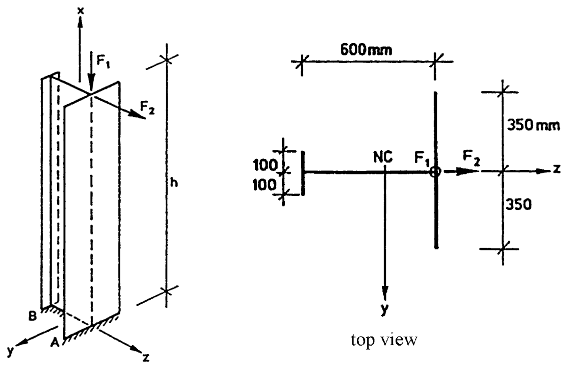
Questions:

a. Verify that $I_{zz} = 256 \times 10^6$ mm^4.
b. Determine the maximum shear stress in one of the flanges.

c. Determine the maximum shear stress in the web.
d. Plot the shear stress diagram for the entire cross-section. Include a number of relevant values and indicate the direction of the shear stresses by means of arrows.

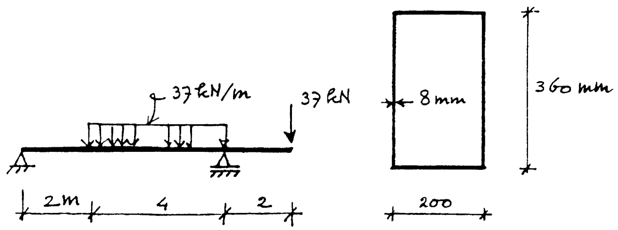


5.41 A thin-walled steel column with height h and a uniform wall thickness t is fixed at the base and loaded at its free end by the forces F_1 and F_2 as shown in the figure. In the calculation use $F_1 = F_2 = 66.15$ kN, $h = 3$ m and $t = 10.5$ mm. For the cross-section you are also given $A = 15.75 \times 10^3$ mm^2 and $I_{zz} = 882 \times 10^6$ mm^4.



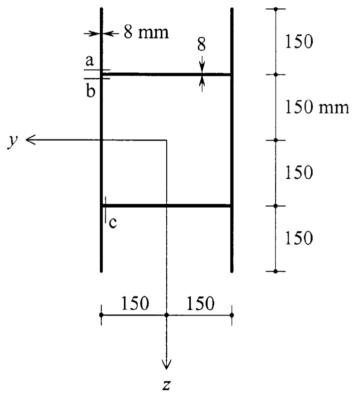
Questions:
a. Determine the location of the normal centre NC.
b. Model the column as a line element and draw all the forces acting on it, including the support reactions.
c. Draw the N, V and M diagrams, with the deformation symbols.
d. For the cross-section 1 m above the fixed end, determine and plot the normal stress diagram.
e. For the cross-section 1 m above the fixed end, determine and plot the shear stress diagram.

5.42 A thin-walled box girder has a rectangular cross-section and uniform wall thickness. Load and dimensions are given in the figure.



Questions:
a. In which cross-section does the maximum normal stress occur? For this cross-section plot the normal stress diagram, with the correct signs for tension and compression.
b. In which cross-section does the maximum shear stress occur? Draw the shear stress diagram for that cross-section. Indicate in the cross-section the direction of the shear force and the direction of the shear stresses.
c. Verify that the resultant of all shear stresses in the cross-section is equal to the shear force.

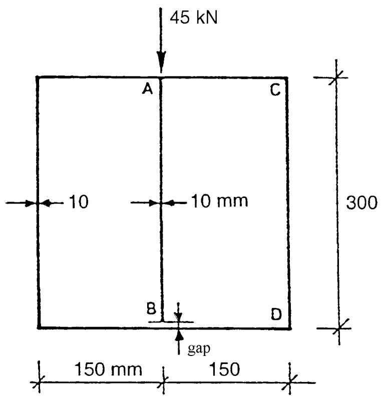
5.43 A shear force $V_z = 44$ kN is acting in the thin-walled cross-section shown with a uniform wall thickness of 8 mm. The moment of inertia is $I_{zz} = 396 \times 10^6$ mm^4.



Questions:

a. Determine the shear stress at cut a.
b. Determine the shear stress at cut b.
c. Determine the shear stress at cut c.
d. Determine the maximum shear stress in the cross-section.
e. Sketch the complete shear stress diagram. Write down the relevant values and use arrows to indicate the direction of the shear stresses.

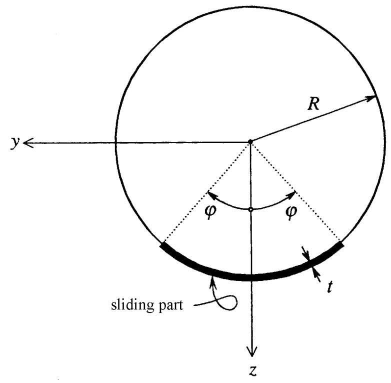
5.44 The thin-walled cross-section shown has a uniform wall thickness of 10 mm and has to transfer a shear force of 45 kN in the plane of symmetry.



Questions:

a. Determine and plot the shear stress diagram for AB. How large is the maximum shear stress in AB?
b. Determine the resultant of all shear stresses in AB.
c. How large is the resultant of all shear stresses in CD?
d. Determine and plot the shear stress diagram for CD. How large is the maximum shear stress in CD?
e. Determine that the resultant of all shear stresses in CD from the shear stress diagram, and verify that the answer agrees with the answer to question c.

5.45 You are given the cross-section of a thin-walled tube with radius R and wall thickness t. The cross-section has to transfer a shear force V_z.

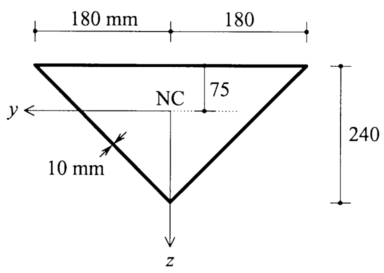


Questions:

a. Determine I_{zz} for this cross-section.
b. For the static moment of the sliding part of the cross-section, shown in the figure, verify that $S_z^a = 2R^2 t \sin\varphi$.

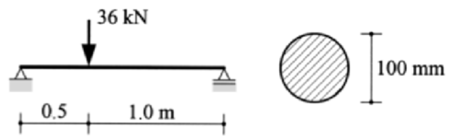
c. Determine and plot the shear stress distribution due to the shear force V_z as a function of φ. In the calculation use $R = 150$ mm, $t = 8.6$ mm and $V_z = 40.5$ kN.
d. How large is the maximum shear stress (in N/mm^2) and where does it occur?

5.46 The thin-walled cross-section shaped as an isosceles triangle has a uniform wall thickness of 10 mm. A shear force of 25 kN is acting in the vertical plane of symmetry. The moment of inertia in the vertical plane is $I_{zz} = 61.2 \times 10^6$ mm^4.



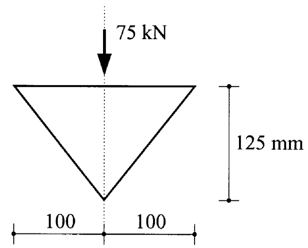
Questions:
a. Verify the location of the normal centre NC.
b. Verify that $I_{zz} = 61.2 \times 10^6$ mm^4.
c. Sketch the shear stress diagram for the entire cross-section (without calculation!). Use arrows to indicate the direction of the shear stresses.
d. Determine the values in the shear stress diagram for a number of relevant places.
e. Determine the maximum shear stress in the cross-section and the place where it occurs.
f. Verify that the resultant of all shear stresses in the cross-section is equal to the shear force of 25 kN.

5.47 A simply supported member with a solid circular cross-section is loaded as shown by a force of 36 kN. The diameter of the circular cross-section is 100 mm.



Questions:
a. Determine the maximum bending stress in the member.
b. Determine the maximum shear stress in the member.

5.48 The solid cross-section in the shape of an isosceles triangle transfers a shear force of 75 kN in the vertical plane of symmetry.

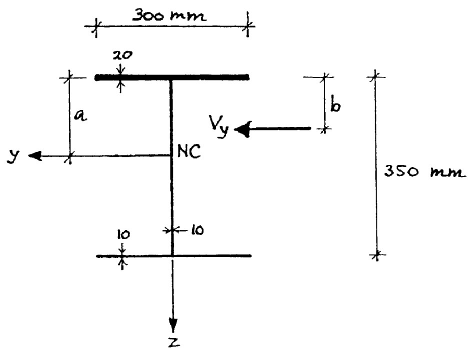


Questions:
a. Determine the shear stress at the level of the normal centre of the cross-section.
b. Determine the maximum shear stress in the cross-section.
c. Sketch the shear stress distribution across the height of the cross-section.

Shear centre (Section 5.5)

5.49 *Question*:
What are the properties of the shear centre SC in a cross-section?

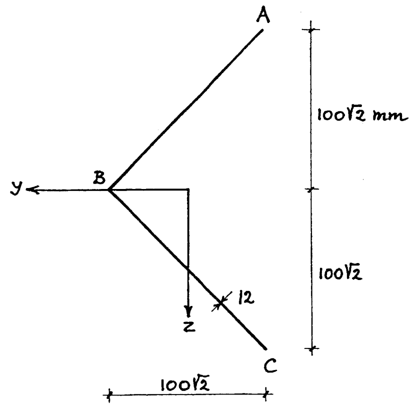
5.50 A horizontal shear force $V_y = 24$ kN acts in the thin-walled cross-section shown. The cross-sectional dimensions are given in the figure.



Questions:

a. Determine the distance a between the normal centre NC and the (centre line of the) upper flange.
b. For the entire cross-section determine and plot the shear stress distribution due to $V_y = 24$ kN. Indicate the directions and include a number of relevant values.
c. Determine the shear stress resultants for the web and flanges individually.
d. Verify that the resultant of all shear stresses in the cross-section is equal to the shear force $V_y = 24$ kN. Determine distance b from the line of action of V_y to the upper flange.
e. Where in the cross-section is the shear centre SC?

5.51 A shear force $V_z = 40\sqrt{2}$ kN in a angle steel with equal legs acts normal to the plane of symmetry of the cross-section. The profile is thin-walled with a uniform wall thickness of 12 mm. The dimensions are given in the figure.



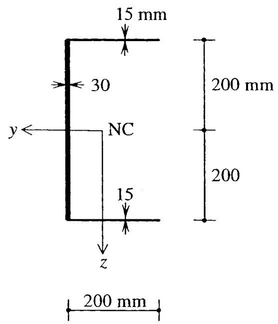
Questions:

a. Determine the shear stress distribution as a function of z.
b. Plot the shear stress diagram. Indicate the directions and include a number of relevant values.
c. Determine the resultant of all shear stresses in AB and BC respectively.
d. Verify that the resultant of all shear stresses in the cross-section is equal to the shear force $V_z = 40\sqrt{2}$ kN. Where is the line of action of V_z?
e. Where is the shear centre SC?

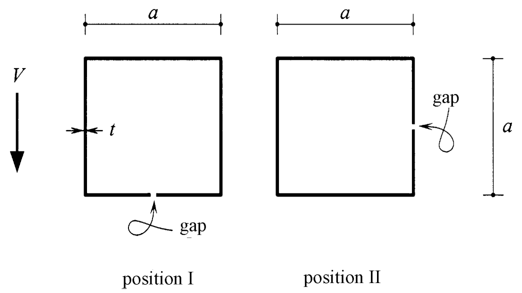
5.52 A vertical shear force $V_z = 20$ kN acts in the thin-walled cross-section shown. The cross-sectional dimensions are given in the figure.

Questions:

a. Determine the location of the normal centre NC.
b. Determine cross-sectional properties needed to determine the shear stress distribution.
c. Determine and plot the shear stress distribution. Indicate the directions and include a number of relevant values.
d. Determine and plot the resultant of all shear stresses in the upper flange, the lower flange and the web respectively.
e. Where is the line of action of the resultant of all shear stresses in the cross-section (that is the line of action of the shear force)?
f. Where is the shear centre SC of the cross-section?



5.53 A vertical shear force V acts in a thin-walled square cross-section with a small gap. The cross-section has a uniform wall thickness t. In position I the gap is at the centre of the bottom flange. In position II the cross-section has turned through 90° and the gap is at the centre of the right-hand web.



Questions (without calculation):

a. Sketch the shear stress distribution for the cross-section in position I. Use arrows to indicate the direction of the shear stresses.
b. Sketch the shear stress distribution for the cross-section in position II. Use arrows to indicate the direction of the shear stresses.

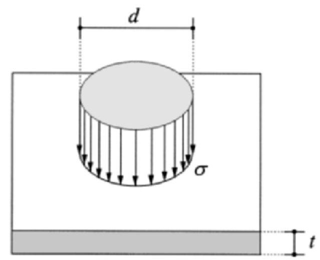
5.54 See the cross-sections from problem 5.53, but now use in the calculation $a = 240$ mm, $t = 12.5$ mm and $V = 32$ kN.

Questions:

a. Determine and plot the shear stress diagram for the cross-section in position I.
b. Verify that the resultant of all shear stresses in the cross-section is equal to the shear force. Where is the line of action of the shear force?
c. Determine and plot the shear stress diagram for the cross-section in position II.
d. Verify that the resultant of all shear stresses in the cross-section is equal to the shear force. Where is the line of action of the shear force?
e. Where in the cross-section is the shear centre SC?

Other cases of shear (Section 5.6)

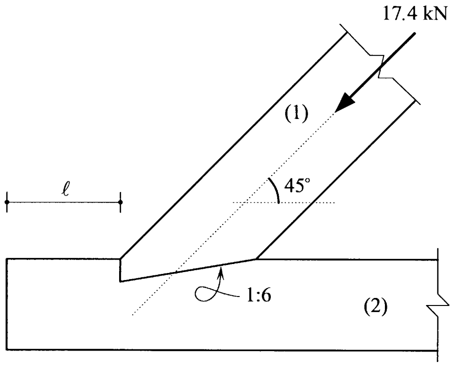
5.55 Round holes with diameter d are punched into an aluminium plate with thickness $t = 8$ mm. The shear strength is $\bar{\tau} = 60$ MPa.



Questions:

a. Determine the punching pressure σ required for a diameter $d = 30$ mm.
b. Determine the punching pressure σ required for a diameter $d = 40$ mm.

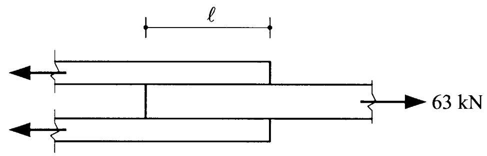
5.56 Rafter (1) is connected to joist (2) by means of a toothed joint. Both rafter and joist are 80 mm wide. There is a compressive force of 17.4 kN in rafter (1). All other data can be found in the figure. Assume that the toothed joint is smooth (frictionless).



Questions:

a. Determine the upsetting force exerted by the rafter on the edge of the joist.
b. Determine the required length ℓ of the edge so that the shear stress remains below limiting value $\bar{\tau} = 0.9$ MPa.

5.57 The glued lap joint shown connects three planks of width 100 mm and has to transfer a tensile force of 63 kN. The average shear stress in the glued joint may not exceed the limiting value $\bar{\tau} = 1.75$ MPa.

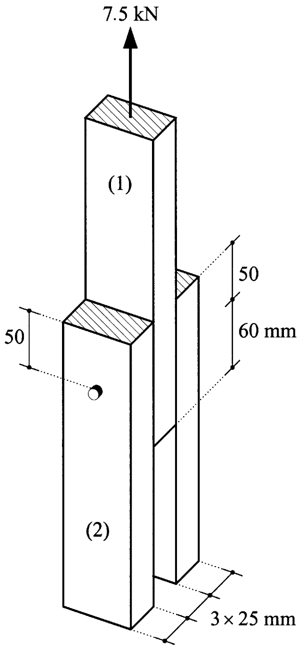


Question:
Determine the required length ℓ of the lap joint.

5.58 The bolted lap joint between the three planks of width 25 mm has to transfer a tensile force of 7.5 kN. The bolt has a diameter of 10 mm. The length of the edge is 60 mm for the inner plank (1) and 50 mm for the outer planks (2).

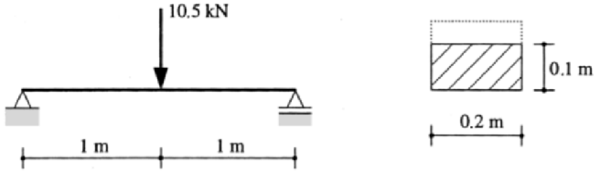
Questions:

a. Determine the shear stress in the bolt.
b. Determine the shear stress in the sliding part of plank (1).
c. Determine the upsetting stress on plank (1).
d. Determine the shear stress in the sliding parts of the planks (2).
e. Determine the upsetting stress on the planks (2).



Mixed questions

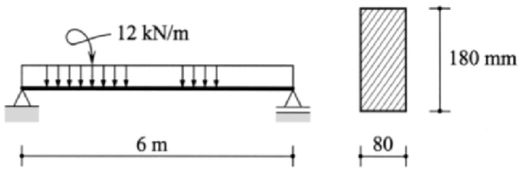
5.59 A simply supported wooden beam, 2 m long, is loaded by a point load of 10.5 kN at midspan. The rectangular cross-section of the beam is 0.2 m wide and 0.1 m high, as shown in the figure.



Questions:

a. Determine the maximum bending stress in the beam.
b. For the given load, the maximum bending stress has to remain below a limiting value of 7 MPa. To ensure that, a 0.2 m wide wooden strip is glued on top of the beam. What minimum thickness is required for this strip?
c. For case b determine the maximum shear force per length that the glued joint has to transfer. Where does this maximum occur?

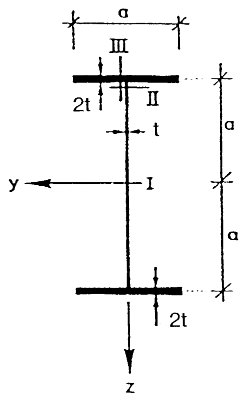
5.60 A simply supported beam with a rectangular cross-section of $80 \times 180\ \text{mm}^2$ has a span of 6 m and carries a uniformly distributed load of 12 kN/m over the full length.



Questions:

a. Determine the maximum bending stress in the beam.
b. Determine the maximum shear stress in the beam.

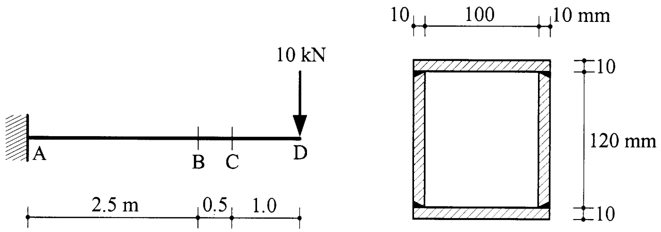
5.61 A thin-walled I-section has to transfer a shear force of 42 kN in the vertical plane of symmetry. The cross-sectional dimensions are given in the figure. In the calculation use $a = 300$ mm and $t = 15$ mm.



Questions:

a. Determine the shear stress in the web at cut I.
b. Determine the shear stress in the web at cut II, directly beneath the upper flange.
c. Determine the shear stress in the upper flange at cut II, directly adjacent to the web.
d. Plot the complete shear stress diagram, include a number of relevant values and indicate the direction of the shear stresses.

5.62 Given the welded steel box beam AD, fixed at A and free at D. The beam is loaded by a vertical force of 10 kN at D. The moment of inertia in the vertical plane (the plane in which the load acts) is $I = 13 \times 10^6$ mm^4.

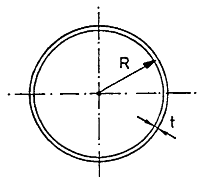


Question:
Determine the longitudinal shear force that a single weld has to transfer along length BC.

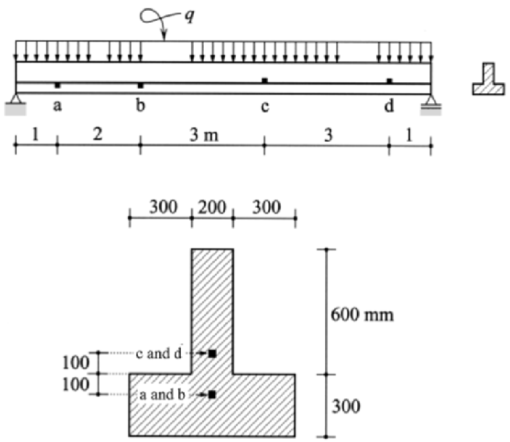
5.63 A thin-walled circular tube with radius R and wall thickness t has to transfer a vertical shear force $V = 7$ kN. The area of the cross-section is $A = 1000$ mm^2.

Questions:

a. Determine the average vertical shear stress.
b. Determine the maximum shear stress.
c. Determine the shear stress at a quarter-height.



5.64 A simply supported upside down T-beam carries a uniformly distributed full load q.



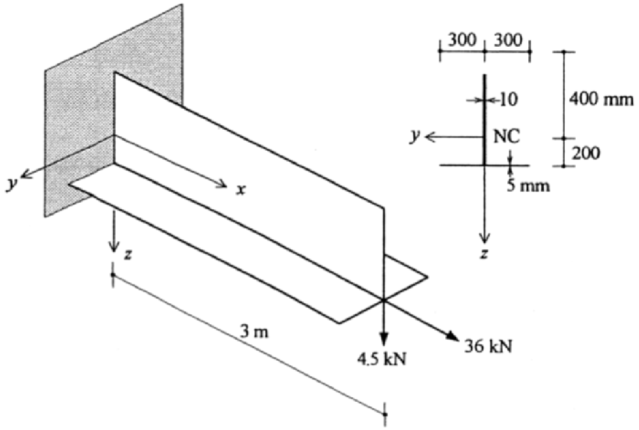
Questions:

a. In which of the four given points is the vertical shear stress largest?
b. In which of those points is the vertical shear stress smallest?

5.65 The cantilever beam shown, with a length of 3 m, has a thin-walled cross-section. The cross-sectional dimensions and the location of the normal centre NC are shown in the figure. The moments of inertia are $I_{yy} = 90 \times 10^6$ mm^4 and $I_{zz} = 360 \times 10^6$ mm. At the free end acts a vertical force of 4.5 kN. There is also a horizontal tensile force of 36 kN with its point of application at the centre of the flange.

Questions:

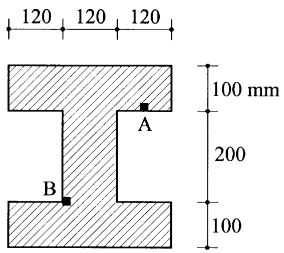
a. Model the beam as a line element and draw all the forces acting on it. Draw the M, V and N diagrams, with the deformation symbols. Include a number of relevant values.
b. Plot the normal stress diagram for the cross-section at $x = 1$ m.



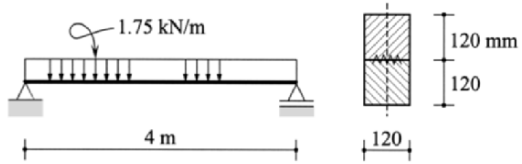
c. For cross-section at $x = 1$ m, determine the shear stress distribution in the web as a function of z.
d. For the entire cross-section plot the shear stress diagram. Include a number of relevant values. Indicate the direction of the shear stresses acting on the positive cross-sectional plane.

5.66 A vertical shear force of 42.8 kN acts in the cross-section shown.

Questions:
a. Determine the vertical shear stress at A.
b. Determine the vertical shear stress at B.
c. Determine the maximum vertical shear stress in the cross-section and the place where it occurs.



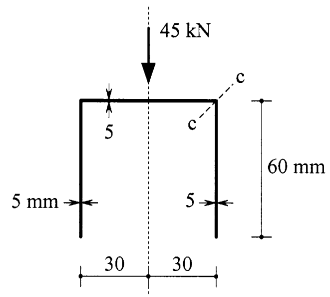
5.67 The simply supported wooden beam with a uniformly distributed load is constructed of two timbers, coupled by means of toothed plate shear connectors and clamped to one another by means of bolts. Each shear connector can transfer a maximum shear force of 5 kN.



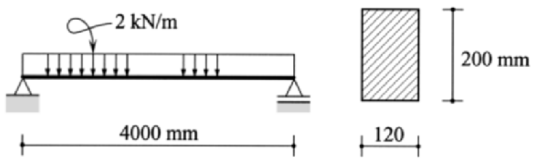
Question:
How many shear connectors are needed for the entire beam?

5.68 A thin-walled U-section with uniform wall thickness $t = 5$ mm has to transfer a shear force of 45 kN in the vertical plane of symmetry.

Questions:
a. Determine the shear stress distribution as a function of the place in the cross-section.
b. Plot the shear stress diagram. Include a number of relevant values and indicate the direction in which the shear stresses are actually acting.
c. Determine the longitudinal shear force per length at cut c-c, where web and flange are connected.

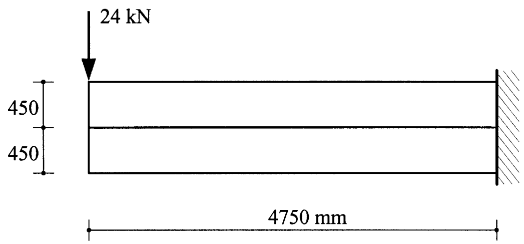


5.69 A simply supported beam with rectangular cross-section carries a uniformly distributed load over its entire length.



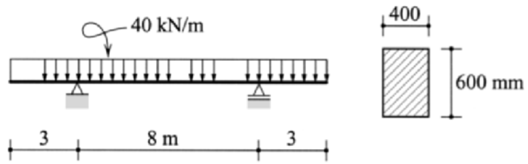
Question:
For the given situation, determine the ratio between the maximum shear stress and maximum bending stress.

5.70 The cantilever beam shown is loaded by a vertical force of 24 kN at the free end. The beam is constructed of two identical rectangular beams that are connected by means of ring dowels. Each dowel can transfer a maximum shear force of 20 kN.



Question:
Determine the minimum number of ring dowels required.

5.71 The simply supported beam with overhangs has a rectangular cross-section and carries a uniformly distributed load over its entire length.

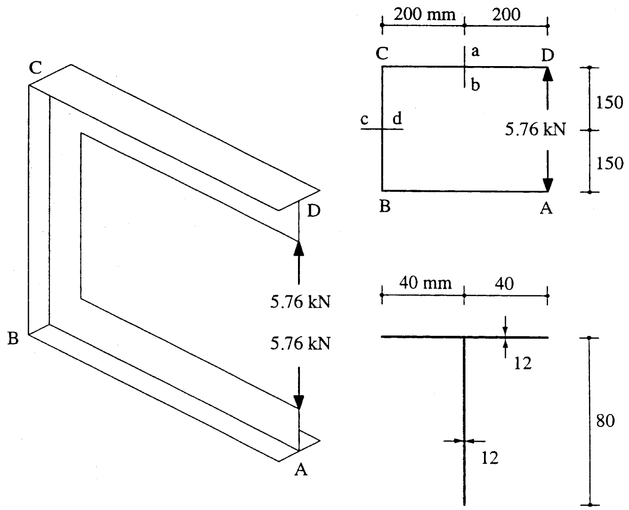


Question:
Determine the maximum shear stress in the beam and the place where it occurs.

5.72 In the figure the dimensions are given for the clamping frame ABCD modelled as a line element. The clamping has a T-shaped cross-section, the dimensions of which are shown in the figure. The wall thickness is a uniform 12 mm. The cross-section must be considered thin-walled. The clamp is loaded at A and D by two compressive forces of 5.76 kN.

Questions:

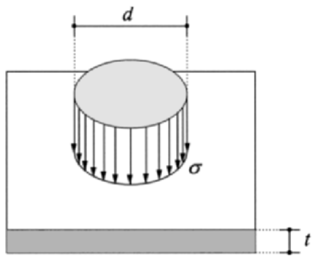
a. Determine and plot the shear stress distribution in (web and flange of) cross-section a-b. Indicate the direction of the shear stresses and include a number of relevant values.
b. In the same way determine and plot the shear stress distribution in (web and flange of) cross-section c-d.



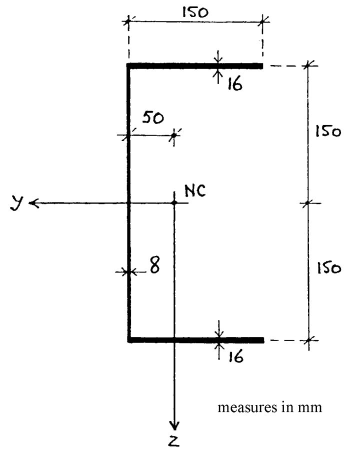
5.73 Round holes are punched into a plate.

Question:
Which of the statements below is correct?

1. The punch pressure required is larger for large holes than for small holes.
2. The punch pressure required is larger for small holes than for large holes
3. The punch pressure required is independent of the diameter of the holes.



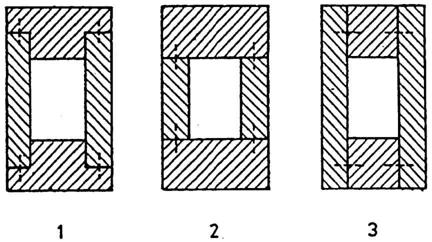
5.74 The cross-sectional dimensions for a thin-walled U-section are given in the figure. The cross-section has to transfer a shear force $V_z = 11.2$ kN.



Questions:

a. Verify that the moment of inertia is $I_{zz} = 126 \times 10^6$ mm^4.
b. Sketch the shear stress diagram (without calculation). Indicate the directions.
c. Determine the values for a number of relevant points. How large is the maximum shear stress and where does it occur?
d. Verify that the resultant of all shear stresses is equal to the shear force.
e. Where is the line of action of the shear force V_z?
f. Where is the shear centre SC of the cross-section?

5.75 A simply supported hollow wooden beam caries a uniformly distributed load. The beam can be built up in three different ways: see the cross-sections 1, 2 and 3 as shown in the figure. The various parts of the beam are connected by means of hidden pins.



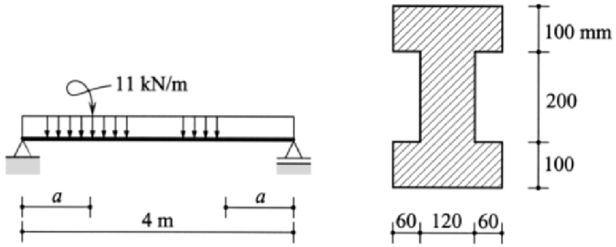
Question:
Order the cross-sections according to the number of pins required, starting with the cross-section needing the most pins. The answer is a three-figure number.

5.76 A simply supported wooden beam with a span of 5 m carries a uniformly distributed load q over its entire length. The beam has a rectangular cross-section with area $A = 20 \times 10^3$ mm^2.

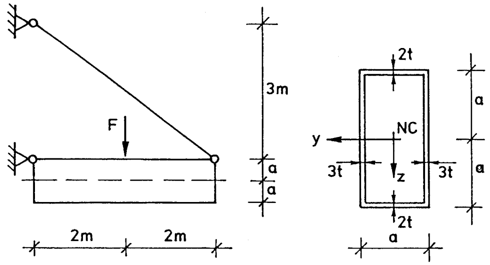
Question:
For which load q is the maximum shear stress in the beam 1.2 N/mm^2.

5.77 The simply supported concrete I-beam has a span of 4 m and carries a uniformly distributed load of 11 kN/m. The limiting value for the shear stress (the allowable shear stress) is $\bar{\tau} = 0.7$ N/mm^2.

Question:
Over which length a are the shear stresses too large and are extra provisions required (such as stirrups or bent up reinforcement)?



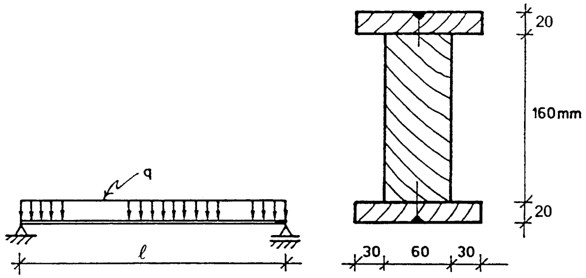
5.78 A thin-walled box beam with rectangular cross-section is supported and loaded as shown. The wall thickness of the flanges is $2t$ and that of the webs is $3t$. In the calculation use $a = 250$ mm, $t = 10$ mm and $F = 60$ kN.



Questions:

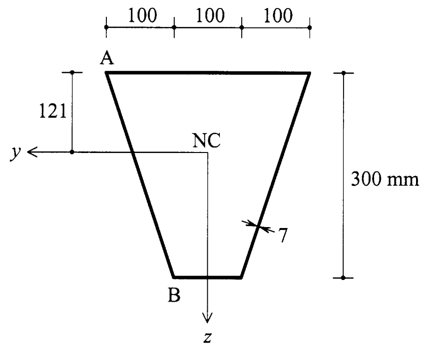
a. Determine the normal stress diagram for the cross-section with the maximum normal stress.
b. Determine the shear stress diagram for the cross-section with the maximum shear stress.

5.79 A simply supported wooden beam with length $\ell = 4$ m carries a uniformly distributed load $q = 2$ kN/m. As shown in the figure, the beam is built up from three parts connected by wire nails. The maximum shear force that a nail can transfer is 200 N. For the moment of inertia of the cross-section, in the plane of loading, use $I = 59.52 \times 10^6$ mm^4.



Question:
What is the minimum number of wire nails required?

5.80 The thin-walled cross-section in the shape of an isosceles trapezium has a uniform wall thickness of 7 mm. The cross-section has to transfer a shear force of 99.3 kN in the vertical plane of symmetry. The moment of inertia in the xz plane is $I_{zz} = 90.1 \times 10^6$ mm^4.



a. Verify the location of the normal centre NC.
b. Verify that $I_{zz} = 90.1 \times 10^6$ mm^4.
c. Determine the shear stress in the upper flange at corner A.
d. Determine the shear stress in the lower flange at corner B.
e. Determine the maximum shear stress in one of the webs.
f. Plot the shear stress diagram for the entire cross-section. Include a number of relevant values and use arrows to indicate the direction of the shear stresses.

6 Bar Subject to Torsion

The previous chapter addressed the shear stress distribution due to a shear force. In this chapter we look at the shear stresses caused by torsional moments. We will also look at the deformation due to torsion.

In most cases, determining the stresses and deformations due to torsion is rather complicated. A fully comprehensive and general approach is not within the scope of this book. Here we will cover a number of simple cases. The theory introduced here can therefore not be used for more complicated situations.

In Section 6.1 we again look at the material behaviour, but now with a particular focus on the deformation of the material under the influence of shear stresses.

Subsequently we look at the loading case of torsion for a number of members with simple cross-sectional shapes: circular cross-sections in Section 6.2 and thin-walled cross-sections in Section 6.3. Here we not only discuss the shear stress distribution in the cross-section, but we also look at the deformation of the member subject to torsion.

In Section 6.4 we present some numerical examples. After a summary of the various formulas in Section 6.5, the chapter ends with a number of problems in Section 6.6.

6.1 Material behaviour in shear

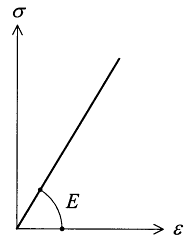


Figure 6.1 In the linear-elastic range, according to Hooke's Law there is a linear relationship between the normal stress σ and extension ε: $\sigma = E\varepsilon$. The modulus of elasticity E characterises the resistance (stiffness) of the material against deformation in extension.

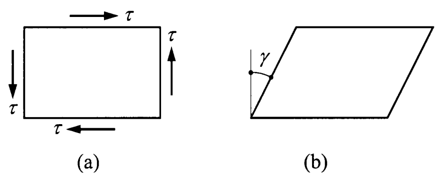


Figure 6.2 (a) Side view of a small rectangular volume element subject to pure shear in the plane of the drawing. (b) The shear stresses τ will deform the rectangle: the rectangle changes into a parallelogram. The change γ of the right angle is called the *shear strain* (or sometimes the *angle of shear*).

To obtain information about the material behaviour, in Chapter 1 we looked only at the tension test. The tension test provides a stress-strain diagram or σ-ε diagram, representing the relationship between the normal stress σ and the strain ε. In the linear-elastic range, *Hooke's law in extension* applies (see Figure 6.1):

$$\sigma = E\varepsilon.$$

The modulus of elasticity E characterises the resistance (stiffness) against deformation of the material in *extension*.

Shear forces and torsional moments are transferred by shear stresses. Shear stresses also cause deformations. In members subject to a shear force, these deformations are generally so small that they can be ignored. Furthermore, they are not required to find the shear stress distribution. For, in the elementary beam theory the shear stress distribution due to a shear force is derived from the difference between the bending stresses on two adjacent cross-sections (see Sections 5.1 to 5.4).

To find the shear stresses due to torsion, we must know, however, in which way the material deforms under the influence of shear. Figure 6.2a shows the side view of a small rectangular volume element that is subject to pure shear in the plane of the drawing. The shear stress τ on one of the sides cannot exist alone but always occurs in combination with equally large shear stresses on the other three sides (see Section 5.3.1).

The shear stresses τ will cause the rectangular volume element to distort: the rectangle in Figure 6.2a changes into the parallelogram in Figure 6.2b. The change of the right angle is often denoted by γ. The small angle γ is a measure of the deformation of the element due to shear and is called the *shear strain* (or sometimes the *angle of shear*).

If we plot the magnitude of the shear stress τ against that of the shear strain γ we get a *shear stress-strain diagram* or *τ-γ diagram*.

In the linear-elastic range there is a linear relationship between the shear stress τ and shear strain γ (see Figure 6.3):

$$\tau = G\gamma.$$

This relationship is known as *Hooke's law in shear*.[1]

G is a constant of proportionality and is called the *shear modulus of elasticity*. The shear modulus is a material quantity that characterises the resistance (stiffness) against deformation of the material in shear. In the τ-γ diagram the shear modulus is the slope $G = \tau/\gamma$ in the linear-elastic range.

The shear strain γ is expressed in radians and is therefore dimensionless. The shear modulus has the same dimension as a stress, thus force/area.

To gain an idea of the order of magnitude of the shear strain γ, the value γ_y is determined for structural steel at the moment that the *yield shear stress* τ_y is reached. For structural steel, the shear stress at which yield occurs is

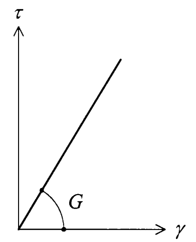


Figure 6.3 In the linear-elastic range, Hooke's Law states that there is a linear relationship between the shear stress τ and shear strain γ: $\tau = G\gamma$. The shear modulus (of elasticity) G characterises the resistance (stiffness) of the material against the deformation in shear.

[1] In *Engineering Mechanics*, Volume 4, we cover the complete Hooke's law.

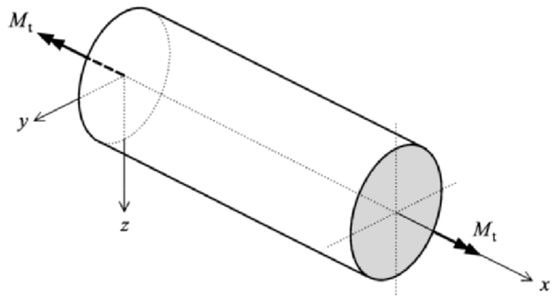


Figure 6.4 The τ-γ diagram and the shear modulus G can be derived from the *torsion test*.

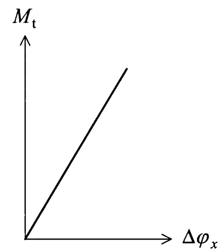


Figure 6.5 If, for various values of the torsional moment M_t, we measure the associated rotation $\Delta\varphi_x$ of one end cross-section with respect to the other and plot the results against one another, we generate the M_t-$\Delta\varphi_x$ diagram.

approximately half the yield strength f_y in the case of tension,[1] so

$$\tau_y = 120\ \text{N/mm}^2.$$

In addition, for structural steel

$$G = 80\ \text{GPa}.$$

This gives

$$\gamma_y = \frac{\tau_y}{G} = \frac{120\ \text{N/mm}^2}{80 \times 10^3\ \text{N/mm}^2} = 1.5 \times 10^3\ \text{rad} \approx 0.09°.$$

This change of the right angle is so small that it cannot be seen with the naked eye.

The τ-γ diagram and the shear modulus G can be derived using the *torsion test*. In this test, a straight prismatic member is loaded by two equal and opposite twisting moments M_t at both ends (see Figure 6.4).[2] A member loaded in this manner is said to be in *pure torsion*. If for various values of M_t the associated rotation $\Delta\varphi_x$ of one end cross-section with respect to the other is measured, the results can be plotted in a M_t-$\Delta\varphi_x$ diagram. See Figure 6.5, where only the linear-elastic path is shown.

---

1 See Section 1.2.

2 Twisting or torsional moments act in the plane of the cross-section. They are shown by means of bent arrows in the plane of the cross-section, but also, as in Figure 6.4, by means of straight arrows with a double arrow head, normal to the plane of the cross-section. See *Engineering Mechanics*, Volume 1, Sections 3.3.1 and 10.1.3.

The translation of this M_t-$\Delta\varphi_x$ diagram into a τ-γ diagram requires some effort. This will be clear in the next section where the inverse route is followed: starting from the τ-γ diagram we derive the relationship between M_t and $\Delta\varphi_x$.

6.2 Torsion of bars with circular cross-section

The simplest torsion problem is probably that of a *thin-walled circular tube*. This is what we start with in Section 6.2.1. The *kinematic and constitutive equations for torsion*, derived in this section, however are generally valid.

In Section 6.2.2 we look at torsion of a *bar with solid circular cross-section*. Here, we consider the member to consist of a large number of thin-walled circular tubes that fit together perfectly.

We use the same approach in Section 6.2.3 for a *thick-walled circular tube*.

In these cases we look not only at the shear stress distribution, but also at the deformation due to torsion.

6.2.1 Thin-walled circular tube

The thin-walled tube in Figure 6.6 has a length ℓ and a circular cross-section with radius R and wall thickness t. At its ends, the tube is subject to two equal and opposite torsional moments M_t.

- *Shear stress formula*

The tube is split in the longitudinal direction into a large number of slices with small length dx (see Figure 6.6). All these slices are subject to torsion in the same way, so there will be the same shear stress distribution everywhere.

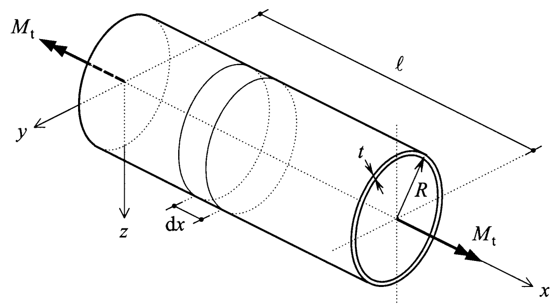


Figure 6.6 A thin-walled circular tube is loaded at its ends by two equal and opposite torsional moments M_t. If the tube in the longitudinal direction is divided into a large number of segments with small length dx, all these segments are subject to torsion in the same way. So there will be the same shear stress distribution everywhere.

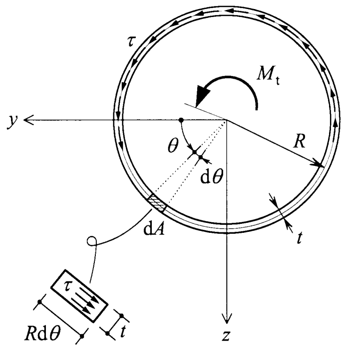


Figure 6.7 For thin-walled cross-sections, it can be assumed that the shear stresses are parallel to the centre line and constant across the wall thickness. Since the tube and load are axially symmetric, it can also be expected that the shear stresses τ are constant in the circumferential direction.

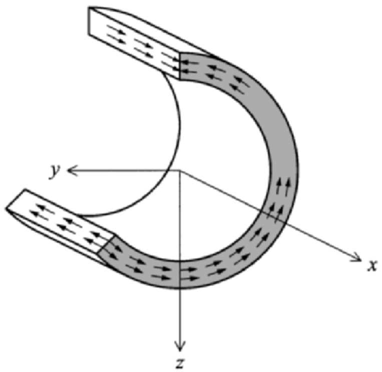


Figure 6.8 In addition to shear stresses in the plane of the cross-section, there are also shear stresses in the longitudinal direction.

For thin-walled cross-sections, it can be assumed that the shear stresses are parallel to the centre line and are constant across the wall thickness. In addition, both tube and the load are axially symmetric. Therefore it can be expected that the shear stresses τ are constant in the circumferential direction (see Figure 6.7).

From Figure 6.7 it can be derived that the shear stress τ on a small cross-sectional element with area $\mathrm{d}A = t \cdot R\,\mathrm{d}\theta$ makes the following contribution to the torsional moment M_t:

$$\mathrm{d}M_t = R \cdot \tau \mathrm{d}A = \tau \cdot R^2 t\,\mathrm{d}\theta.$$

By integration we find

$$M_t = \int_0^{2\pi} \tau \cdot R^2 t\,\mathrm{d}\theta = 2\pi R^2 t \cdot \tau.$$

Since all shear stresses τ act in circumferential direction and are constant, and the small shear forces (shear stress $\times$ area) on all small areas $\mathrm{d}A$ have the same arm R, we can write M_t down directly:

$$M_t = \tau \cdot A \cdot R = \tau \cdot 2\pi Rt \cdot R = 2\pi R^2 t \cdot \tau.$$

This leads to the following shear stress formula for a thin-walled tube subject to torsion:

$$\tau = \frac{M_t}{2\pi R^2 t}, \tag{6.1a}$$

or

$$\tau = \frac{M_t R}{I_p}. \tag{6.1b}$$

Here I_p is the polar moment of inertia of the thin-walled circular cross-section (see also Section 3.3.2, Example 2):

$$I_p = 2\pi R^3 t.$$

Comment: There are not only shear stresses in the cross-sectional planes, but also shear stresses in longitudinal planes (see Figure 6.8).

- *Deformation due to torsion*

To get more knowledge about the deformation of the tube in Figure 6.6, subject to torsion by the torsional moments M_t, a small slice with length dx has been isolated from the tube (see Figure 6.9a). Next a small rectangular element ABCD on the slice has been isolated (see Figure 6.9b).

Element ABCD is subject to pure shear. Due to the shear stress τ, the rectangle ABCD changes into a parallelogram (see Figure 6.10).

The right angle between AB and AD changes by γ

$$\gamma = \frac{\tau}{G}.$$

In consequence, point B moves in the circumferential direction to B′ over a distance $\gamma\, dx$. This means that the two cross-sections at a distance dx of one another rotate about an angle $d\varphi_x$ with respect to one another:

$$d\varphi_x = \frac{\gamma\, dx}{R} = \frac{\tau\, dx}{GR}. \qquad (6.2)$$

All the rectangular elements on the slice in Figure 6.9 behave like the rectangular element ABCD. Using the fact that all these elements are deformed in the same way into parallelograms that fit together perfectly, we can state the following:

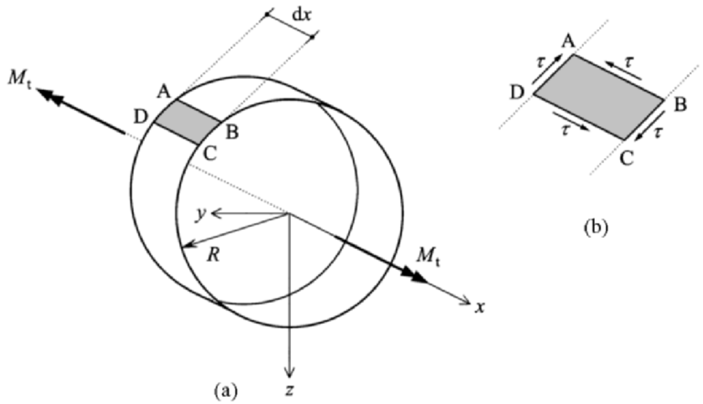


Figure 6.9 (a) To find out more about the deformation of the tube due to the torsional moment M_t, a small segment of length dx has been isolated. (b) A small rectangular element ABCD has been isolated from the segment.

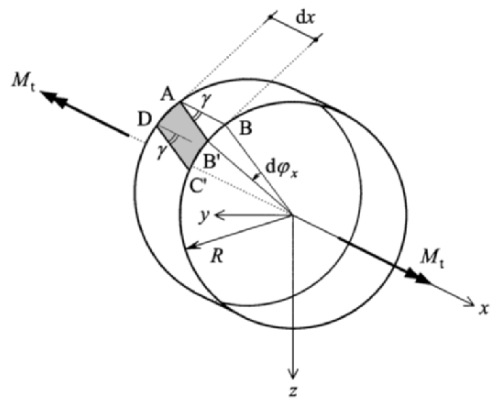


Figure 6.10 Element ABCD is subject to pure shear. Due to the shear stresses τ the rectangle ABCD changes into a parallelogram.

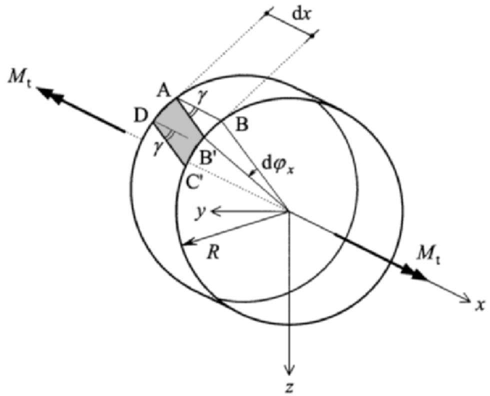


Figure 6.10 Element ABCD is subject to pure shear. Due to the shear stresses τ the rectangle ABCD changes into a parallelogram.

The cross-sections of a thin-walled circular tube subject to torsion remain plane after deformation and maintain their circular shape.

The change of the rotation $d\varphi_x$ per length dx is called the *torsional strain* χ:

$$\chi = \frac{d\varphi_x}{dx}. \tag{6.3}$$

The torsional strain χ has the dimension radian/length (length^{-1}).

The torsional strain χ is the *deformation quantity* in torsion, similar to the strain ε in extension and curvature κ in bending.

Expression (6.3) represents the *kinematic relationship for torsion*. It links the torsional strain χ of the member (a deformation quantity) and the rotation φ_x of the cross-section (a displacement quantity).

By combining the shear stress formula (6.1) with the formulas (6.2) and (6.3) we find for the thin-walled circular tube

$$\chi = \frac{M_t}{GI_p} \tag{6.4a}$$

or

$$M_t = GI_p\chi. \tag{6.4b}$$

GI_p is referred to as the *torsional stiffness* of the tube. This quantity characterises the resistance of the tube against the deformation due to *torsion*. The torsional stiffness GI_p has the dimension force × length^2.

Expression (6.4) links the torsional moment M_t (a section force) and the torsional strain χ (a deformation quantity) and is the *constitutive equation for torsion*.

Comments:

- It is customary to indicate the *torsional stiffness* by means of GI_t, in which I_t is known in general as the *torsion constant* of the cross-section. For expression (6.4) one should therefore properly write

  $$\chi = \frac{M_t}{GI_t} \quad \text{or} \quad M_t = GI_t\chi .$$

  For circular cross-sections, the torsion constant I_t is equal to the polar moment of inertia I_p.
- In the shear stress formula (6.1) the shear modulus G is missing. In the linear-elastic range the shear stress is independent of the material. If two homogeneous tubes of different material have the same dimensions and are loaded by equal torsional moments, the same shear stresses occur. However, the torsional strain is not the same!
- The derived *kinematic* and *constitutive equation for torsion* are generally valid[1] and fit in the array of equations for extension and bending, as shown in Table 6.1.

Table 6.1

	Kinematic equation	**Constitutive equation**
Extension	$\varepsilon = \frac{du}{dx}$	$N = EA\varepsilon$
Bending	$\kappa_z = \frac{d\varphi_y}{dx} = -\frac{d^2w}{dx^2}$	$M_z = EI_{zz}\kappa_z$
Torsion	$\chi = \frac{d\varphi_x}{dx}$	$M_t = GI_t\chi$

[1] They apply not only to thin-walled tubes but also to members with other cross-sections.

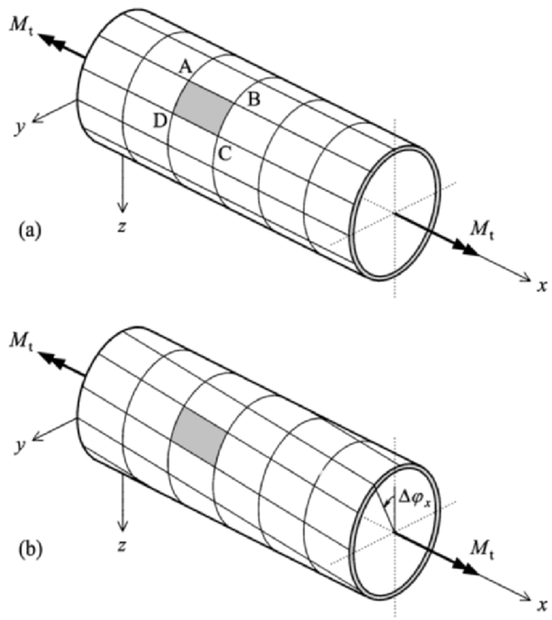


Figure 6.11 The thin-walled tube (a) before and (b) after the deformation due to torsion. All rectangular elements on the tube are deformed into parallelograms. It can clearly be seen that planar cross-sections remain planar and that straight lines parallel to the x axis remain straight.

Figure 6.11a shows the tube divided into a large number of rectangular elements. Figure 6.11b shows the *distorted* tube. Under the influence of the torsional moment M_t all rectangles on the tube change into parallelograms.

We can clearly see that the planar cross-sections remain planar, and that initially straight lines parallel to the x axis remain straight after deformation through torsion.

These observations are confirmed by experiments.

To find the rotation $\Delta\varphi_x$ of the right-hand cross-section with respect to the left-hand cross-section we use formulas (6.3) and (6.4):

$$\chi = \frac{\mathrm{d}\varphi_x}{\mathrm{d}x} = \frac{M_t}{GI_p}\,.$$

Hence

$$\mathrm{d}\varphi_x = \frac{M_t}{GI_p}\,\mathrm{d}x\,.$$

Since M_t is constant over the entire length ℓ of the tube this gives

$$\Delta\varphi_x = \int_0^\ell \mathrm{d}\varphi_x = \frac{M_t}{GI_p}\int_0^\ell \mathrm{d}x = \frac{M_t\ell}{GI_p}\,.$$

Comment: Notice the resemblance to the expression for the extension of a truss member:

$$\Delta\ell = \frac{N\ell}{EA}\,.$$

6.2.2 Bar with solid circular cross-section

The bar in Figure 6.12, with length ℓ and a solid circular cross-section with radius R, is subject to torsion by the moments M_t at the ends. Below we first look at the deformation of the bar and then at the shear stress distribution.

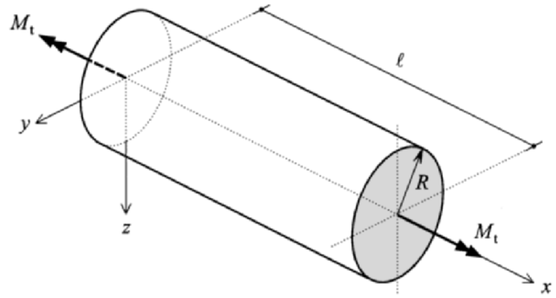


Figure 6.12 Bar with a solid circular cross-section, subject to torsion.

- *Deformation due to torsion*

To handle this problem, we consider the solid bar to consist of a large number of perfectly fitting thin-walled circular tubes. Assume the shear stress in a thin-walled tube with radius r and thickness dr is $\tau(r)$ (see Figure 6.13). Also assume dM_t is the contribution of the shear stresses in this thin-walled tube to the total torsional moment M_t in the solid cross-section.

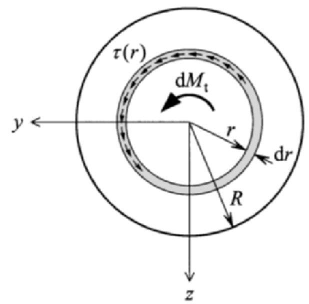


Figure 6.13 It is assumed that the solid bar is built up of a large number of perfectly fitting thin-walled circular tubes, all with the same torsional strain.

Applying formula (6.4b) on a thin-walled tube with a torsional strain $\chi = \mathrm{d}\varphi/\mathrm{d}x$ we find

$$\mathrm{d}M_t = G \cdot 2\pi r^3 \,\mathrm{d}r \cdot \chi .$$

Here $2\pi r^3 \,\mathrm{d}r$ is the polar moment of inertia of the thin-walled tube with radius r and thickness dr.

Since all thin-walled tubes are part of one and the same solid cross-section they have to fit together perfectly after deformation. This means that the torsional strain $\chi = \mathrm{d}\varphi/\mathrm{d}x$ has to be the same for all constituent thin-walled tubes.

The resultant torsional moment M_t in the solid cross-section is found by summing the contributions dM_t of all the thin-walled tubes, that is by integration with respect to r:

$$M_t = G\chi \int_0^R 2\pi r^3 \,\mathrm{d}r = G\chi \cdot \tfrac{1}{2}\pi R^4. \tag{6.5a}$$

In the term $\frac{1}{2}\pi R^4$ we recognise the polar moment I_p of the solid circular cross-section:

$$I_\mathrm{p} = \tfrac{1}{2}\pi R^4.$$

So (6.5a) we can also write as

$$M_\mathrm{t} = G I_\mathrm{p} \chi, \qquad (6.5b)$$

or, in reverse,

$$\chi = \frac{M_\mathrm{t}}{G I_\mathrm{p}}. \qquad (6.5c)$$

$G I_\mathrm{p}$ is the torsional stiffness of the solid circular cross-section.

Using the same method as in Section 6.2.1, we find the following for the rotation $\Delta\varphi_x$ of the end cross-sections with respect to one another:

$$\Delta\varphi_x = \frac{M_\mathrm{t}\ell}{G I_\mathrm{p}}.$$

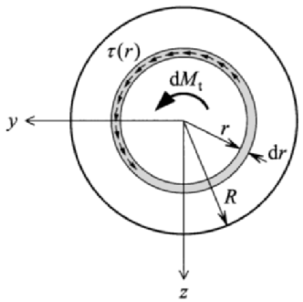


Figure 6.13 It is assumed that the solid bar is built up of a large number of perfectly fitting thin-walled circular tubes, all with the same torsional strain.

- *Shear stress formula for torsion*

Assume the thin-walled tube in Figure 6.13, with radius r, wall thickness dr and shear stress $\tau(r)$, makes a contribution dM_t to the torsional moment M_t in the solid cross-section. According to (6.1)

$$\mathrm{d}M_\mathrm{t} = 2\pi r^2\,\mathrm{d}r \cdot \tau(r).$$

Next assume a torsional strain χ. For the thin-walled tube the constitutive equation (6.4) applies:

$$\mathrm{d}M_\mathrm{t} = G \cdot 2\pi r^3 \,\mathrm{d}r \cdot \chi .$$

By eliminating $\mathrm{d}M_\mathrm{t}$ from the two above-mentioned equations we find

$$\tau(r) = rG\chi . \tag{6.6a}$$

The shear stress due to torsion is proportional to the distance r to the axis of the bar, (see Figure 6.14). Substitute the previously derived expression (6.5c) for the torsional strain χ in (6.6a) and we find the shear stress formula for torsion:

$$\tau(r) = \frac{M_\mathrm{t} r}{I_\mathrm{p}} . \tag{6.6b}$$

Comment: Notice that this shear stress formula for torsion is very similar to the bending stress formula:

$$\sigma(z) = \frac{Mz}{I} .$$

The bar with solid circular cross-section was considered to be built up of a large number of perfectly fitting thin-walled tubes, all with the same torsional strain. In Section 6.2.1 we found that in thin-walled circular tubes, subject to torsion, planar cross-sections remain planar and maintain their circular shape. If we assume further that the change in length of a thin-walled tube subject to torsion is negligible, we can conclude for the bar with solid cross-section – as a built-up of thin-walled circular tubes – that planar cross-sections remain planar and maintain their circular shape after the deformation due to torsion.

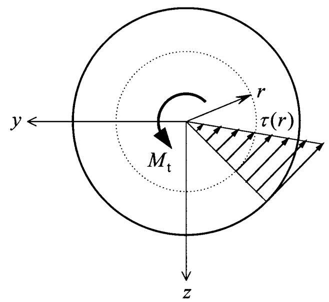


Figure 6.14 The shear stress due to torsion is proportional to distance r to the axis of the bar.

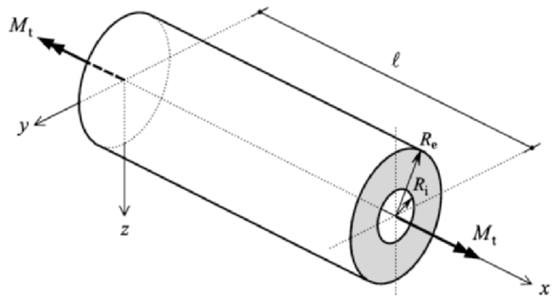


Figure 6.15 Thick-walled circular tube, subject to torsion.

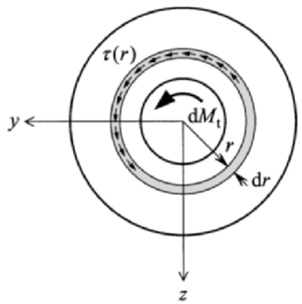


Figure 6.16 The thick-walled circular tube is assumed to consist of a large number of perfectly fitting thin-walled circular tubes, all with the same torsional strain.

Since all the thin-walled tubes have the same torsional strain, i.e. the same rotation per length measured along the axis of the bar, they will not rotate with respect to one another within the same cross-section. This means that straight radial lines in the solid cross-section remain straight and radial after the deformation due to torsion.

Conclusions for a bar with solid circular cross-section, subject to torsion:

a. *Planar cross-sections remain planar and maintain their circular shape.*
b. *Straight radial lines in the cross-section remain straight and radial.*

These conclusions are confirmed by experiments.

6.2.3 Thick-walled circular tube

In Figure 6.15 a thick-walled tube with length ℓ is loaded by torsional moments M_t at the ends. The tube has a circular cross-section with inner radius R_i and outer radius R_e.

The approach used in Section 6.2.2 can also be used successfully for this thick-walled tube. Again the cross-section is considered to consist of a large number of thin-walled tubes that fit together perfectly.

- *Deformation due to torsion*

Assume the shear stresses in a thin-walled tube with radius r and wall thickness dr make a contribution dM_t to the torsional moment M_t in the thick-walled cross-section (see Figure 6.16).

Assuming a torsional strain $\chi = d\varphi/dx$, we find with formula (6.4b) for a thin-walled tube

$$dM_t = G \cdot 2\pi r^3 \, dr \cdot \chi .$$

Here $2\pi r^3 \, dr$ is the polar moment of inertia of a thin-walled tube with radius r and thickness $\mathrm{d}r$.

Since all thin-walled tubes are part of one and the same cross-section they have to fit together perfectly after deformation. This means that the torsional strain $\chi = \mathrm{d}\varphi/\mathrm{d}x$ has to be the same for all thin-walled tubes.

The torsional moment M_t in the thick-walled cross-section is found by summing the contribution $\mathrm{d}M_\mathrm{t}$ of all thin-walled cross-sections, that is by integrating with respect to r:

$$M_\mathrm{t} = G\chi \int_{R_\mathrm{i}}^{R_\mathrm{e}} 2\pi r^3 \,\mathrm{d}r = G\chi \cdot \tfrac{1}{2}\pi(R_\mathrm{e}^4 - R_\mathrm{i}^4). \tag{6.7a}$$

Notice that in (6.7a) only the lower limit of integration differs from (6.5a).

For a thick-walled circular cross-section the polar moment of inertia I_p is (see Section 3.2.4, Example 9)

$$I_\mathrm{p} = \tfrac{1}{2}\pi(R_\mathrm{e}^4 - R_\mathrm{i}^4).$$

So we can write (6.7a) as

$$M_\mathrm{t} = GI_\mathrm{p}\chi, \tag{6.7b}$$

or, in reverse,

$$\chi = \frac{M_\mathrm{t}}{GI_\mathrm{p}}. \tag{6.7c}$$

Here GI_p is again the torsional stiffness of the cross-section.

- *Shear stress formula*

In thick-walled circular cross-sections also, the shear stresses due to torsion

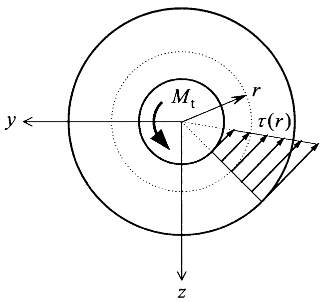


Figure 6.17 The shear stress due to torsion is proportional to distance r to the tube axis.

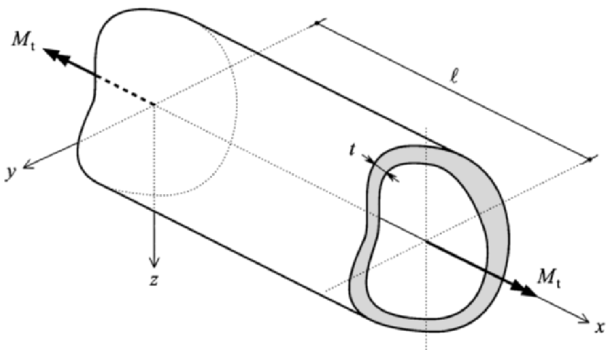


Figure 6.18 A thin-walled tube subject to torsion. The hollow cross-section is of an arbitrary shape. The wall thickness t need not be constant but may vary (gradually).

are proportional to the distance r to the tube axis (see Figure 6.17). We find the same shear stress formula as for solid circular cross-sections:

$$\tau(r) = \frac{M_t r}{I_p} \quad \text{with} \quad R_i \leq r \leq R_e. \tag{6.8}$$

The derivation is identical to that in Section 6.2.2 and is not repeated here.

6.3 Torsion of thin-walled cross-sections

In Section 6.3.1, we derive the shear stress formula and the torsion constant for *thin-walled closed cross-sections of arbitrary shape*. These formulas are known as Bredt's formulas.

In Section 6.3.2 a *thin-walled strip* is considered to consist of a large number of thin-walled rectangular tubes that fit together perfectly and to which Bredt's formulas apply.

The formulas derived for a thin-walled strip are used in Section 6.3 to find the torsion constant and shear stress formula for *thin-walled open cross-sections*.

Barré de Saint Venant[1] found that for members with non-circular cross-section, subject to torsion, the planar cross-sections no longer remain planar but warp. In Section 6.3.1 this will be proved for a non-circular thin-walled tube. The formulas derived here apply only if the cross-sections are free to warp. If the cross-sections are not free to warp, but have to remain plane,

[1] Adhémar Jean Claude Barré de Saint Venant (1797–1886) published many important papers on the theory of elasticity and on the strength of materials. In 1853 he developed the fundamental differential equation for elastic torsion.

there will be also axial normal stresses. Torsion with restrained warping is outside the scope of this book.

6.3.1 Thin-walled tubes

In Figure 6.18 a thin-walled tube is subject to torsion. The torsional moment is M_t. The hollow cross-section is of an arbitrary shape. The wall thickness t need not be constant but may change gradually.

First we derive the shear stress formula and next we look at the deformation due to torsion.

- *Shear stress formula*

In Figure 6.19 the m axis has been chosen in the peripheral direction, along the *centre line* or *median line* of the thin-walled cross-section. The origin of the m axis can be an arbitrary point on the centre line.

A cross-section is considered thin-walled when the wall thickness is small with respect to the other cross-sectional dimensions. In that case, we can expect that the shear stresses in the wall are parallel to the centre line m. In addition we can expect that these stresses σ_{xm} are constant across the wall thickness t, as shown in Figure 6.19.

In Figure 6.20 an arbitrary part has been isolated from a tube segment with small length dx. The cuts (1) and (2) are normal to the centre line m. Since in two mutually normal planes the shear stresses are equal, the shear stresses in the cross-sectional plane are equal to the shear stresses in the longitudinal direction. These shear stresses are shown in the cuts (1) and (2).

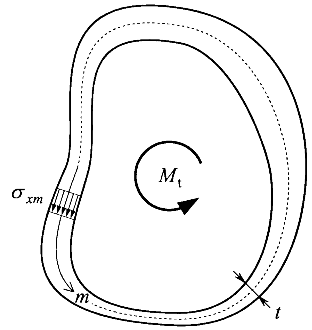


Figure 6.19 If the wall thickness is small, it can be assumed that the shear stresses σ_{xm} are parallel to the centre line m. It may also be expected that they are constant across the wall thickness t.

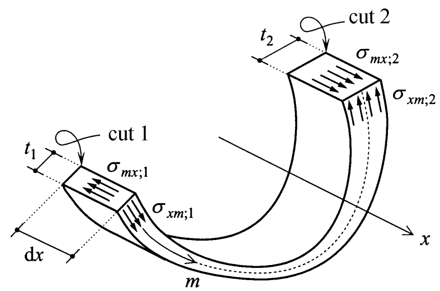


Figure 6.20 An arbitrary part isolated from a tube segment with small length dx. The cuts (1) and (2) are normal to the centre line m.

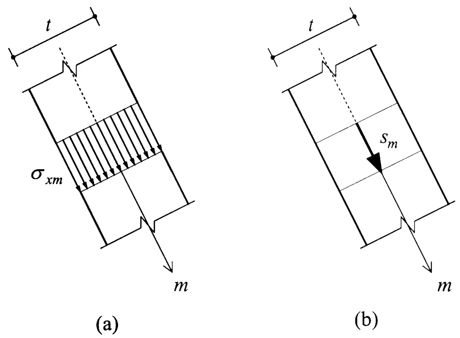


Figure 6.21 (a) The product of shear stress σ_{xm} and wall thickness t is called (b) the shear flow s_m. In a thin-walled tube subject to torsion the shear flow is constant: $s_m = \sigma_{xm} \cdot t =$ constant.

If there are no normal stresses in the longitudinal direction,[1] the force equilibrium in the x direction of the isolated part in Figure 6.20 implies that

$$\sigma_{mx;1} \cdot t_1 \cdot \mathrm{d}x = \sigma_{mx;2} \cdot t_2 \cdot \mathrm{d}x.$$

Since $\sigma_{mx} = \sigma_{xm}$ we can switch over to the shear stresses in the cross-sectional plane. After dividing by the length dx we find

$$\sigma_{xm;1} \cdot t_1 = \sigma_{xm;2} \cdot t_2.$$

Since cuts (1) and (2) have been arbitrarily chosen we can conclude that the product of shear stress σ_{xm} and wall thickness t is constant. This product is called the *shear flow* s_m (see Figure 6.21):

$$s_m = \sigma_{xm} \cdot t = \text{constant}. \qquad (6.9a)$$

We omit the suffix m, and write

$$s = \tau \cdot t = \text{constant}. \qquad (6.9b)$$

Formula (6.9) implies that the shear stress τ in a thin-walled tube is a maximum where the wall thickness t is a minimum, and *vice versa*.

Comment: The name for the concept *shear flow*, which was already used in earlier sections for the product of shear stress and wall thickness, can now be explained properly. The inner and outer boundaries of the thin-

1 Warping is not restrained.

walled cross-section can be thought of as being the boundaries of a belt canal (without side branches) with constant depth d and variable width t. Next assume a constant quantity of water is steadily circulating through the canal. The rate of flow of the water is τ. The amount of water that passes through a plane across the canal per unit of time is the *volume speed* or *volume flow rate*, and is equal to the product of rate of flow τ, depth d and width t. In the belt canal with steadily circulating water the volume speed is constant, or

$$\tau \cdot t \cdot d = \text{constant.}$$

If the depth d of the channel is constant, then

$$\tau \cdot t = \text{constant.}$$

Because of the *flow analogy* this quantity has been called the *shear flow* s:

$$s = \tau \cdot t = \text{constant.}$$

The shear flow s is constant. The shear stress (rate of flow) τ is inversely proportional to the wall-thickness t (width of the belt canal). The shear stress (rate of flow) τ is largest where the wall-thickness (width of the canal) t is smallest, and *vice versa*.

In the thin-walled cross-section in Figure 6.22a we look more closely at a small area element with length dm in the peripheral direction. A small shear force acts on this small area element; its magnitude is

$$\text{shear stress} \times \text{wall thickness} \times \text{length} = \text{shear flow} \times \text{length} = s_m \, \mathrm{d}m.$$

For clarity, this small force has been exaggerated in Figure 6.22.

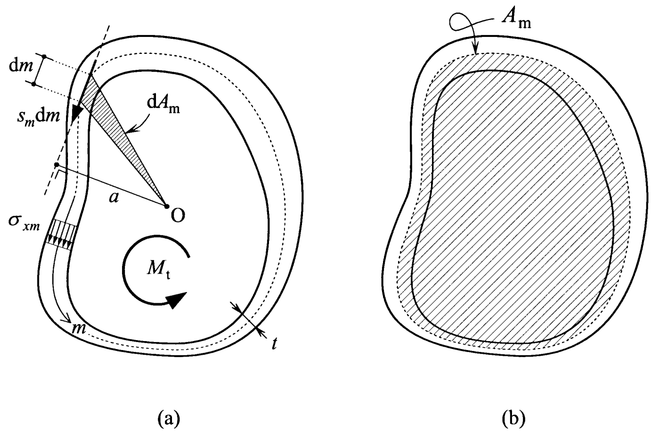


Figure 6.22 (a) The contribution to the torsional moment M_t of the shear stresses on a small area element with length dm is $\mathrm{d}M_t = a \cdot s_m \, \mathrm{d}m = 2 s_m \, \mathrm{d}A_m$. Here $\mathrm{d}A_m = \frac{1}{2} a \, \mathrm{d}m$ is the area of the hatched triangle. (b) A_m is the area enclosed by the centre line m of the thin-walled cross-section.

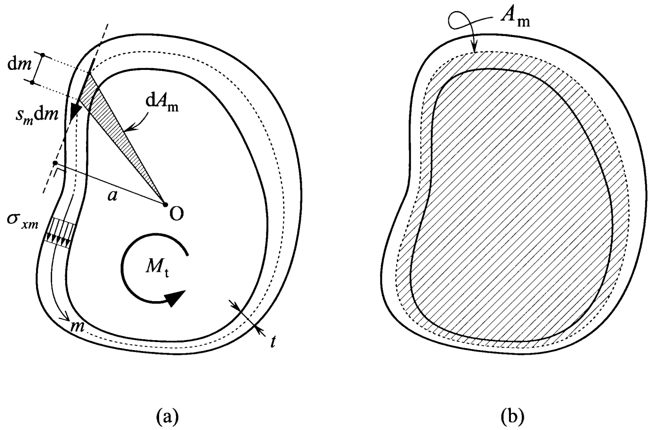


Figure 6.22 (a) The contribution to the torsional moment M_t of the shear stresses on a small area element with length dm is $\mathrm{d}M_t = a \cdot s_m \,\mathrm{d}m = 2s_m \,\mathrm{d}A_m$. Here $\mathrm{d}A_m = \frac{1}{2} a \,\mathrm{d}m$ is the area of the hatched triangle. (b) A_m is the area enclosed by the centre line m of the thin-walled cross-section.

Assume dM_t is the contribution of this shear force to the torsional moment M_t in the cross-section:

$$\mathrm{d}M_t = a s_m \,\mathrm{d}m.$$

Here a is the distance from the line of action of the small force s_m dm to the point O with respect to which the torsional moment is determined.

By summing or integrating all contributions dM_t over the total peripheral length (contour) c of the cross-section we find the torsional moment in the cross-section:

$$M_t = \int_0^c a s_m \,\mathrm{d}m.$$

Integrating along the contour c, we find

$$M_t = \oint a s_m \,\mathrm{d}m = s_m \oint a \,\mathrm{d}m. \tag{6.10}$$

The integral sign with a circle means that the integration process is carried out around the hollow cross-section along the centre line m. Since the shear flow s_m is constant (independent of m) this term has been taken outside the integral. The integral

$$\oint a \,\mathrm{d}m.$$

is completely determined by the geometry of the cross-section and has a very simple geometrical interpretation. The product adm is equal to twice the area dA_m of the hatched triangle in Figure 6.22a, with base dm and height a.

If we sum all the contributions $a\mathrm{d}m$ in the peripheral direction we find twice the area A_m that is enclosed by the centre line of the thin-walled cross-section (see Figure 6.22b):

$$\oint a\,\mathrm{d}m = 2A_\mathrm{m}. \tag{6.11}$$

Substituting (6.11) in (6.10) leads to the following relationship between the torsional moment M_t and the shear flow s_m:

$$M_\mathrm{t} = 2A_\mathrm{m}s_m. \tag{6.12a}$$

With

$$s_m = \sigma_{xm}t \tag{6.9a}$$

we find the following shear stress formula:

$$\sigma_{xm} = \frac{s_m}{t} = \frac{M_\mathrm{t}}{2A_\mathrm{m}t}\,. \tag{6.13a}$$

This shear stress formula is known as *Bredt's first formula*,[1] and its derivation is entirely based on equilibrium considerations.

When working without m direction, we may omit the suffix m in s_m, and write

$$M_\mathrm{t} = 2A_\mathrm{m}s, \tag{6.12b}$$

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[1] Rudolph Bredt (1842–1900), German engineer, developed the theory for a thin-walled tube subject to torsion and published it in 1896.

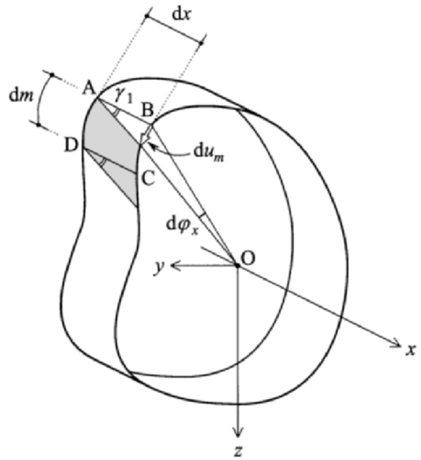


Figure 6.23 A narrow strip of small length dx has been isolated from the thin-walled tube. We look at the small rectangular element ABCD, located on this strip. Following the rotation $d\varphi_x$, the rectangular element will change into a parallelogram. The change of the right angle at A is caused by the displacement of B in the *m* direction (the peripheral direction).

$$s = \tau t, \tag{6.9b}$$

$$\tau = \frac{s}{t} = \frac{M_t}{2A_m t}. \tag{6.13b}$$

Notice that A_m is the area enclosed by the centre line of the cross-section (see Figure 6.22b). Do not confuse this with the actual area A of the cross-section!

Comment: In the derivation of Bredt's first formula (the shear stress formula), the location of the point O with respect to which the moment is determined, is not important.

• *Deformation due to torsion*

In Figure 6.23 a narrow strip with small length dx has been isolated from the thin-walled cylinder. Below we will look at the deformation of the small rectangular element ABCD, located on this strip.

Assume the back and front cross-sections of the strip, at mutual distance dx, rotate with respect to one another through an angle $d\varphi_x$. For this rotation, we have (see Section 6.2.1)

$$d\varphi_x = \chi \, dx, \tag{6.3}$$

in which χ is the *torsional strain*. The rectangular element ABCD will change into a parallelogram (see Figure 6.23).

The change of the right angle at A is caused by the displacement of B (with respect to A) in the *m* direction (that is in the peripheral direction).

The displacement of B due to rotation $d\varphi_x$ is normal to the joining line OB if O is the point about which the cross-section rotates. This displacement has a component $\mathrm{d}u_m$ in the m direction:[1]

$$\mathrm{d}u_m = a\,\mathrm{d}\varphi_x .$$

Here a is the distance from O to the tangent at B to the centre line m (see Figure 6.24).

In Figure 6.25a element ABCD has been shown separately. For the change γ_1 of the right angle at A we have

$$\gamma_1 = \frac{\mathrm{d}u_m}{\mathrm{d}x} = \frac{a\,\mathrm{d}\varphi_x}{\mathrm{d}x} = a\chi .$$

For the shear stresses τ on the small rectangular element ABCD we now find

$$\tau = G\gamma_1 = Ga\chi . \tag{6.14}$$

We should have

$$\tau = \frac{s}{t} . \tag{6.15}$$

According to (6.15) the shear stress is inversely proportional to the wall thickness t and does not depend on the distance a, as in (6.14). This means the shear strain γ_1 cannot be the correct.

Reality is more complicated: planar cross-sections no longer remain planar but start to *warp*. Points on the cross-section also undergo displacements u_x in the x direction. The warping u_x is the same for all cross-sections, and

[1] See also *Engineering Mechanics*, Volume 1, Section 15.3.2.

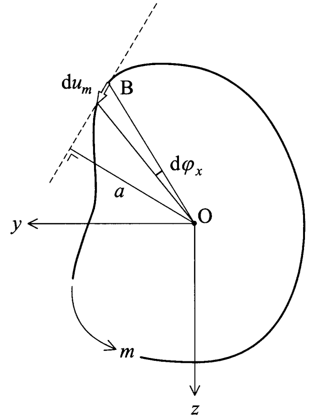


Figure 6.24 The displacement of B due to rotation $\mathrm{d}\varphi_x$ is normal to the joining line OB if O is the point about which the cross-section rotates. This displacement has a component $\mathrm{d}u_m$ in m direction for which $\mathrm{d}u_m = a\,\mathrm{d}\varphi_x$ applies.

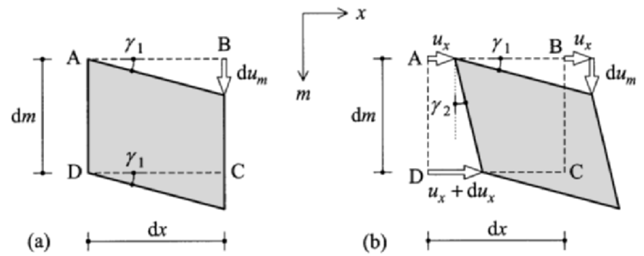


Figure 6.25 (a) The deformation of the rectangular element. Closer investigation shows that γ_1 cannot be the correct change of the right angle at A. (b) Reality is more complicated: since there are also displacements u_x in the x direction, the change of the right angle at A is $\gamma_1 + \gamma_2$.

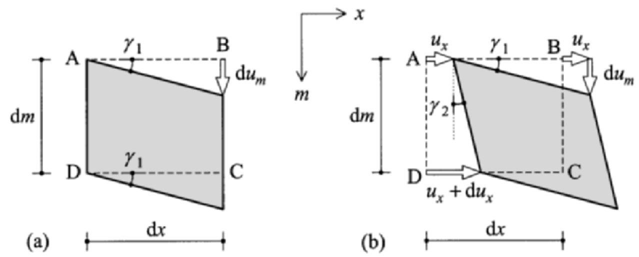


Figure 6.25 (a) The deformation of the rectangular element. Closer investigation shows that γ_1 cannot be the correct change of the right angle at A. (b) Reality is more complicated: since there are also displacements u_x in the x direction, the change of the right angle at A is $\gamma_1 + \gamma_2$.

depends on m only (and not on x).

Figure 6.26 shows how the cross-section can warp, and Figure 6.25b shows what happens to element ABCD:

$$\gamma_1 = \frac{\mathrm{d}u_m}{\mathrm{d}x} = \frac{a\,\mathrm{d}\varphi_x}{\mathrm{d}x} = a\chi,$$

$$\gamma_2 = \frac{\mathrm{d}u_x}{\mathrm{d}m}.$$

The total change of the right angle at A is

$$\gamma = \gamma_1 + \gamma_2 = a\chi + \frac{\mathrm{d}u_x}{\mathrm{d}m}.$$

The constant shear flow equation is

$$s = \tau t = G\gamma t = Gt\left(a\chi + \frac{\mathrm{d}u_x}{\mathrm{d}m}\right).$$

From this we find

$$\mathrm{d}u_x = \frac{s}{Gt}\,\mathrm{d}m - a\chi\,\mathrm{d}m,$$

and so

$$\Delta u_x = \int \mathrm{d}u_x = \frac{s}{G}\int \frac{1}{t}\,\mathrm{d}m - \chi \int a\,\mathrm{d}m. \tag{6.16}$$

Since the shear flow s, shear modulus G and torsional strain χ are constant (independent of m) they have been placed outside the integral.

Expression (6.16) can be used to determine u_x as a function of m, representing the *warping of the cross-section*.

If all contributions $\mathrm{d}u_x$ all around the periphery are summed, for example from B round to B, the value for u_x must not change:

$$\oint \mathrm{d}u_x = 0.$$

In that case it follows from (6.16) that

$$\frac{s}{G}\oint \frac{1}{t}\,\mathrm{d}m - \chi \oint a\,\mathrm{d}m = 0.$$

With

$$s = \frac{M_\mathrm{t}}{2A_\mathrm{m}} \qquad (6.12)$$

and

$$\oint a\,\mathrm{d}m = 2A_\mathrm{m} \qquad (6.11)$$

we can write

$$\frac{M_\mathrm{t}}{2GA_\mathrm{m}}\oint \frac{1}{t}\,\mathrm{d}m - \chi \cdot 2A_\mathrm{m} = 0.$$

This gives the following relationship between the torsional moment M_t and the torsional strain χ:

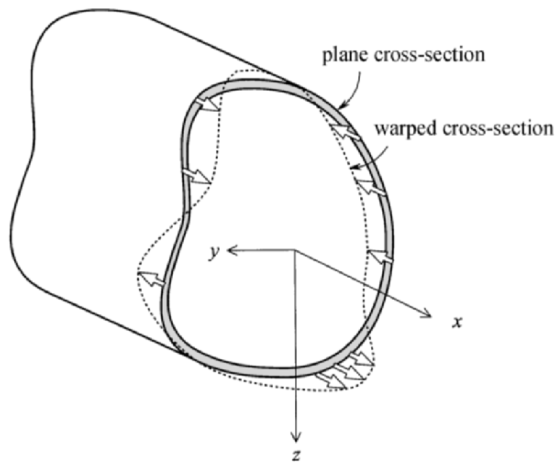


Figure 6.26 Planar cross-sections do not remain planar but will warp. The warping u_x is the same for all cross-sections and depends only on the m coordinate in the peripheral direction (and not on the x coordinate).

$$M_\mathrm{t} = G\frac{4A_\mathrm{m}^2}{\displaystyle\oint \frac{1}{t}\,\mathrm{d}m}\cdot\chi\,.$$

For this constitutive equation we write

$$M_\mathrm{t} = GI_\mathrm{t}\chi \tag{6.4}$$

in which GI_t is the *torsional stiffness*.

The quantity I_t is known as the *torsion constant*.[1] For a thin-walled tube

$$I_\mathrm{t} = \frac{4A_\mathrm{m}^2}{\displaystyle\oint \frac{1}{t}\,\mathrm{d}m}\,. \tag{6.17}$$

This expression is known as *Bredt's second formula*.

Comment: In the derivation of Bredt's second formula, the result is independent of the location of point O, the point about which the cross-section rotates.

Comment: Equations (6.4) and (6.12) can be used to determine the warping of the cross-section:

$$\Delta u_x = \chi\left(\frac{I_\mathrm{t}}{2A_\mathrm{m}}\int_0^m \frac{1}{t}\,\mathrm{d}m - \int_0^m a\,\mathrm{d}m\right).$$

---

[1] Sometimes I_t is called the *torsional moment of inertia*. This unfortunate nomenclature is due to the analogy of the constitutive relationships $M_\mathrm{t} = GI_\mathrm{t}\chi$ and $M = EI\kappa$ for torsion and bending respectively. For bending, the quantity I is known as the moment of inertia.

It is preferable to use the warping function $\Delta u_x/\chi$ as this function depends only on the m coordinate in the peripheral direction and therefore only on the shape of the cross-section. The warping function is independent of the magnitude of the torsional moment.

Check: We can check Bredt's formulas using the thin-walled circular cross-section in Figure 6.27, with radius R and constant wall thickness t. For this cross-section the area within the centre line is

$$A_{\mathrm{m}} = \pi R^2.$$

In addition

$$\oint \frac{1}{t}\,\mathrm{d}m = \frac{1}{t}\oint \mathrm{d}m = \frac{2\pi R}{t}.$$

With the shear stress formula (6.13) we find

$$\tau = \frac{M_{\mathrm{t}}}{2A_{\mathrm{m}}t} = \frac{M_{\mathrm{t}}}{2\pi R^2 t} = \frac{M_{\mathrm{t}}R}{2\pi R^3 t} = \frac{M_{\mathrm{t}}R}{I_{\mathrm{p}}}.$$

And with formula (6.17) we find for the torsion constant

$$I_{\mathrm{t}} = \frac{4A_{\mathrm{m}}^2}{\oint \frac{1}{t}\,\mathrm{d}m} = \frac{4\times(\pi R^2)^2}{2\pi R/t} = 2\pi R^3 t = I_{\mathrm{p}}.$$

These results agree with those found earlier in Section 6.2.1.

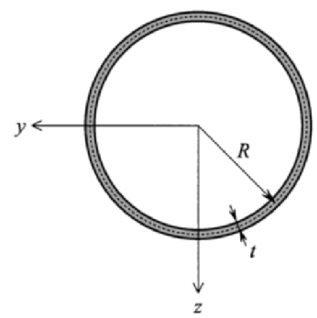


Figure 6.27 Bredt's formulas derived for thin-walled tubes can be checked using a thin-walled circular tube.

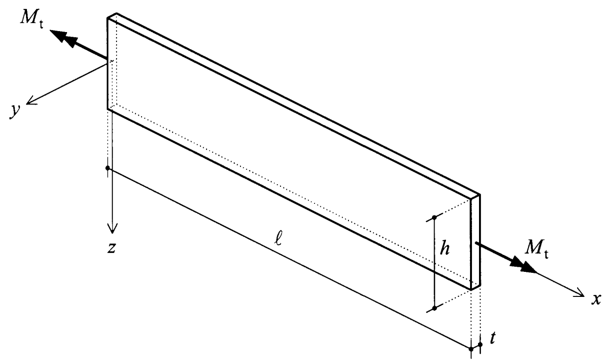


Figure 6.28 A thin-walled strip subject to torsion. A thin-walled strip is a member with rectangular cross-section of which the width or wall thickness t is far smaller than the height h: $t \ll h$.

6.3.2 Thin-walled strip

A thin-walled strip is a member with rectangular cross-section of which the width or wall thickness t is far smaller than the height h: $t \ll h$ (see Figure 6.28).

An exact calculation of the shear stress distribution and the torsion constant is outside the scope of this book. Below, the calculation proceeds on the basis of a number of assumptions that are justified through experiments and more accurate calculations based on the theory of elasticity.

- *Shear stress formula*

A torsional moment M_t causes shear stresses that, as it were, flow around within the cross-section (see Figure 6.29). Over a large part of the height, the shear stresses are parallel to the centre line of the strip. This picture changes only near the ends.

It is assumed that the magnitude of the shear stresses σ_{xz}, parallel to the centre line, are proportional to the distance y to the centre line:

$$\sigma_{xz} = ky.$$

Here k is a yet unknown proportionality constant.

To determine the constant k the thin-walled strip is thought to be built up of a large number of perfectly fitting rectangular thin-walled tubes with constant wall thickness. Figure 6.30 shows one of these tubes. The width of the tube is $2y$. The area A_m enclosed by the centre lines of this tube is approximately

$$A_m \approx 2hy.$$

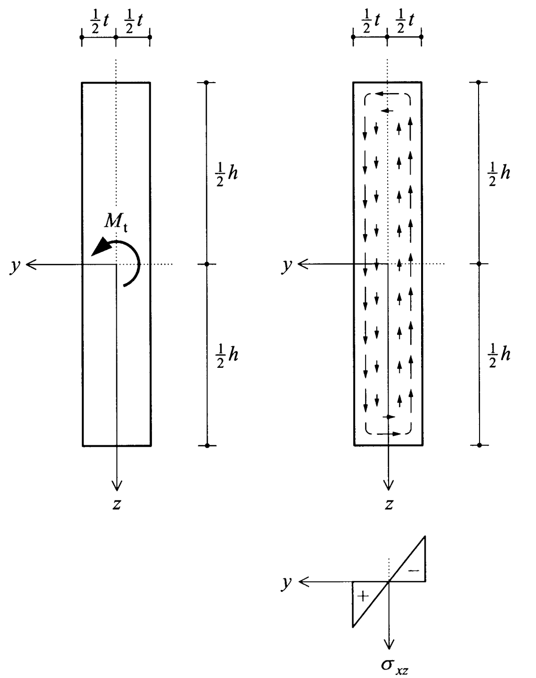


Figure 6.29 A torsional moment M_t causes shear stresses in the strip that, as it were, flow around within the cross-section. Over a large part of the height, the shear stresses are parallel to the centre line of the strip. This only changes near the ends. It is assumed that the shear stress σ_{xz}, is parallel to the centre line and that its magnitude is proportional to the distance y to the centre line.

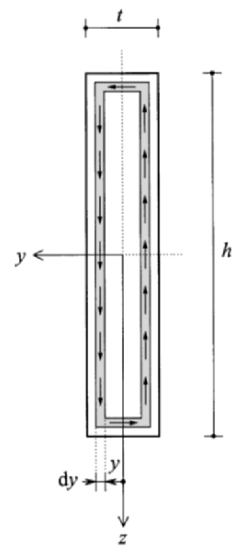


Figure 6.30 The thin-walled strip is assumed to consist of a large number of perfectly fitting rectangular tubes with a constant wall thickness. To these tubes we can apply Bredt's formulas.

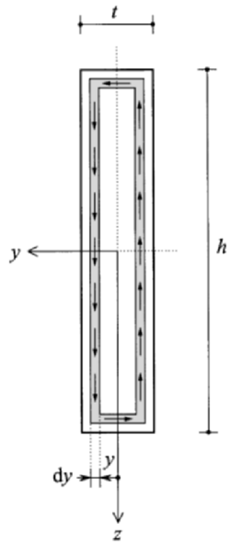


Figure 6.30 The thin-walled strip is assumed to consist of a large number of perfectly fitting rectangular tubes with a constant wall thickness. To these tubes we can apply Bredt's formulas.

Comment: The actual height of the tube will be between h and $(h-t)$. Since $t \ll h$ the height for all thin-walled tubes is approximated by the same value h. It should also be noted that the wall thickness $\mathrm{d}y$ of the cylinder is small with respect to the width $2y$.

The shear stresses in the tube are

$$\sigma_{xz} = ky.$$

Assume the contribution of these shear stresses to the torsional moment in the tube is $\mathrm{d}M_\mathrm{t}$. Bredt's first formula (6.13) states the following:

$$M_\mathrm{t} = 2A_\mathrm{m}t\tau.$$

Applied to the tube in Figure 6.30 this becomes

$$\mathrm{d}M_\mathrm{t} = 2 \cdot 2hy \cdot \mathrm{d}y \cdot ky = 4khy^2\,\mathrm{d}y.$$

The total torsional moment M_t in the strip follows from summing the contributions of all individual thin-walled tubes. This is done by integration with respect to y:

$$M_\mathrm{t} = \int_0^{t/2} 4khy^2\,\mathrm{d}y = \tfrac{1}{6}\,kht^3.$$

With this we have found the constant k:

$$k = \frac{M_\mathrm{t}}{\frac{1}{6}\,ht^3}\,.$$

The shear stress formula for the strip is now

$$\sigma_{xz} = \frac{M_t y}{\frac{1}{6}\,ht^3}\,. \tag{6.18a}$$

If we do not use a coordinate system, we write

$$\tau = \frac{M_t e_m}{\frac{1}{6}\,ht^3}\,. \tag{6.18b}$$

Here e_m is the distance to centre line m. The direction of τ can be deduced directly from the direction of the torsional moment.

The maximum shear stress τ_{max} occurs at the boundaries of the cross-section, at $y = \pm t/2$ or, without coordinate system, at $e_m = t/2$:

$$\tau_{max} = \frac{M_t}{\frac{1}{3}\,ht^2}\,. \tag{6.19}$$

• *Deformation due to torsion*
For the thin-walled strip, the constitutive equation (6.4) gives

$$M_t = GI_t\chi\,.$$

To find the torsion constant I_t the strip is again thought to consist of a large number of perfectly fitting rectangular thin-walled tubes with constant wall thickness.

For the thin-walled tube in Figure 6.30 its area is approximately

$$A_m = 2ht\,,$$

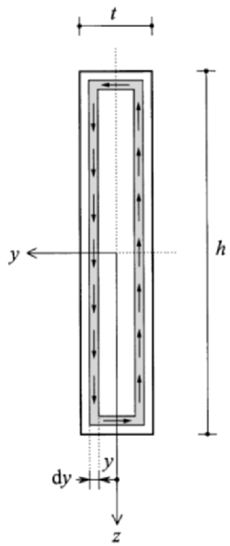


Figure 6.30 The thin-walled strip is assumed to consist of a large number of perfectly fitting rectangular tubes with a constant wall thickness. To these tubes we can apply Bredt's formulas.

and

$$\oint \frac{1}{t}\,\mathrm{d}m = \frac{1}{\mathrm{d}y}\,(4y+2h) \approx \frac{2h}{\mathrm{d}y}\,.$$

Furthermore assume that $\mathrm{d}I_\mathrm{t}$ is the contribution of this thin-walled tube to the torsion constant I_t of the cross-section of the strip. According to Bredt's second formula (6.17)

$$I_\mathrm{t} = \frac{4A_\mathrm{m}^2}{\displaystyle\oint \frac{1}{t}\,\mathrm{d}m}\,.$$

Applied to the thin-walled tube in Figure 6.30 we find

$$\mathrm{d}I_\mathrm{t} = \frac{4\cdot(2hy)^2}{2h/\mathrm{d}y} = 8hy^2\,\mathrm{d}y.$$

The torsion constant I_t of the thin-walled strip is found by summing the contributions of all thin-walled tubes. Through integration we find

$$I_\mathrm{t} = \int_0^{t/2} 8hy^2\,\mathrm{d}y = \tfrac{8}{3}\,hy^3\Big|_0^{t/2} = \tfrac{1}{3}\,ht^3. \tag{6.20}$$

Comment: Knowing the torsion constant for the strip, we can write the shear stress formula (6.18) as follows:

$$\sigma_{xz} = \frac{M_t y}{\frac{1}{6}ht^3} = \frac{M_t y}{\frac{1}{2}I_t}, \tag{6.18c}$$

or without a coordinate system

$$\tau = \frac{M_t e_m}{\frac{1}{2}I_t}. \tag{6.18d}$$

Comment: For the thin-walled strip the factor 1/2 in the denominator destroys the analogy with the bending stress formula:

$$\sigma = \frac{M_z z}{I_{zz}}.$$

Comment: Figure 6.31 shows the shear stress distribution σ_{xz}, determined before, due to the torsional moment M_t. If, in reverse order, we calculate the torsional moment from these shear stresses, we find

$$\sum T_x = h\int_{-1/2}^{+1/2} y\sigma_{xz}\,\mathrm{d}y = h\int_{-1/2}^{+1/2} y\frac{M_t y}{\frac{1}{6}ht^3}\,\mathrm{d}y$$

$$= \frac{M_t}{\frac{1}{6}t^3}\left.\frac{y^3}{3}\right|_{-t/2}^{+t/2} = \frac{1}{2}M_t.$$

The shear stresses σ_{xz} seem only to provide half the torsional moment M_t! The other half of the torsional moment is provided by the shear stresses σ_{xy}. These shear stresses normal to the centre line act only within the small

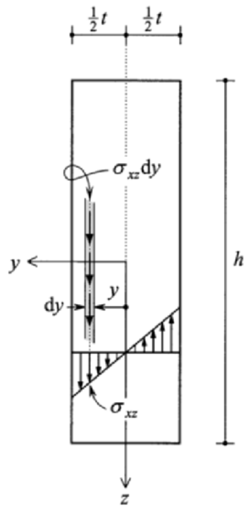


Figure 6.31 The shear stresses σ_{xz} determined appear to provide only half the torsional moment M_t.

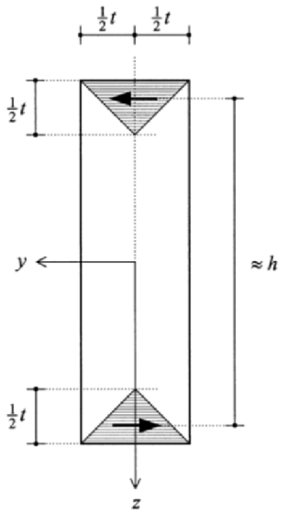


Figure 6.32 The other half of the torsional moment M_t is provided by the shear stresses σ_{xy}, acting in the small hatched triangular areas, and normal to the centre line of the strip. Although their resultants are small, they still provide half the torsional moment because of the large arm h.

Figure 6.33 Open thin-walled cross-sections can be considered to be built up of thin-walled strips. So the formulas derived for a thin-walled strip can also be applied to open thin-walled cross-sections. Circular or other curved parts can be considered as strips, on the condition that the wall thickness is far smaller than the radius of curvature of the strip .

triangular areas, hatched in Figure 6.32. Although their resultants are small, they provide half the torsional moment, as a result of the large arm h.

6.3.3 Thin-walled open cross-sections

Thin-walled open cross-sections such as those in Figure 6.33 can be thought to be built up from a number of thin-walled strips. Circular or other curved parts can also be viewed as strips, on the condition that the wall thickness is far smaller than the radius of the curved strip.

If the wall thickness t for all constituent strips of a thin-walled open cross-section is uniform, then the torsion constant I_t of this cross-section is according to (6.20):

$$I_t = \tfrac{1}{3}\, ht^3. \tag{6.21}$$

Here h is the developed length of the thin-walled cross-section.

If the wall thickness is not equal for all constituent strips, then

$$I_t = \sum_i \tfrac{1}{3}\, h_i t_i^3. \tag{6.22}$$

The summation has to be performed over the number of constituent strips; each strip i has a thickness t_i and length h_i. The shear stress τ due to torsion again is parallel to the centre line m in each of the constituent strips and proportional to the distance e_m to the centre line:

$$\tau = \frac{M_t e_m}{\tfrac{1}{2}\, I_t}. \tag{6.23}$$

The direction of τ can be deduced directly from the direction of the torsional moment M_t.

The formulas (6.21) to (6.23), based on those of a thin-walled strip, are justified through experiments and also through more accurate calculations based on the theory of elasticity.

Comment: Near re-entrant angles of a thin-walled open cross-section there can be considerable stress concentrations, depending on the radius of curvature r (see Figure 6.34). The shear stress formulas derived do not take into account these stress concentrations.

Comment: From (6.23) it follows that in a *thin-walled open cross-section* the shear stress due to torsion is a maximum where e_m is a maximum; this is at the boundary of the strip, where the *wall thickness is largest*. But beware: in a *thin-walled closed cross-section* the shear stress is constant across the wall thickness and is a maximum where the *wall thickness is smallest*.

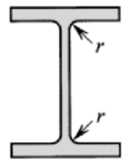


Figure 6.34 Considerable stress concentrations may occur in the re-entrant corners of thin-walled cross-sections, depending on the radius of curvature r. The derived shear stress formulas do not apply to these stresses.

6.4 Numerical examples

In this section we present eight numerical examples with respect to torsion.

In Example 1 we determine the contribution of the shear stresses in the outer shell of a solid circular cross-section to the torsional moment. Example 2 addresses dimensioning a shaft with solid circular cross-section. In addition the rotation due to torsion is determined. In Example 3 the behaviour under torsion of a rectangular hollow cross-section and a circular hollow cross-section are compared. Both cross-sections are thin-walled and have the same use of material. Deformation due to torsion is discussed again in Example 4: here one of the four supports of a box girder bridge undergoes a settlement. In Example 5 the shear stress distribution is determined for a

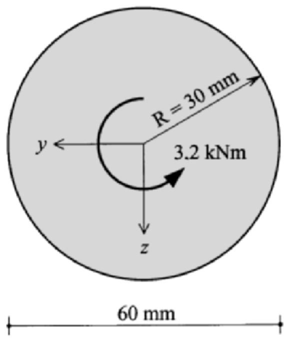


Figure 6.35 A shaft with solid circular cross-section has to transfer a torsional moment of 3.2 kNm.

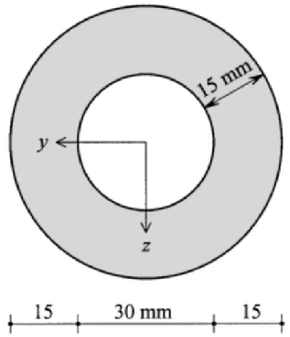


Figure 6.36 It is investigated which part of the torsional moment is transferred by the outer circular shell with a thickness of 15 mm.

rectangular thin-walled hollow cross-section subject to an eccentric shear force. This example combines the shear stresses due to a shear force and a torsional moment. Next, in Example 6, the behaviour of a square thin-walled hollow cross-section is compared to that of a thin-walled strip. Both cross-sections have the same area. In Example 7 the torsion stresses in a closed thin-walled cross-section are compared to those in an open thin-walled cross-section. Finally, in Example 8 we determine the shear stress distribution for a U-profile due to a shear force of which the line of action does not pass through the shear centre. Also this example combines the shear stresses due to a shear force and a torsional moment.

Example 1: Shear stresses in a shaft with solid circular cross-section subject to torsion

A shaft with solid circular cross-section has to transfer a torsional moment of 3.2 kNm. The diameter of the shaft is 60 mm (see Figure 6.35).

Questions:

a. Determine the maximum shear stress in the shaft.
b. Which part of the torsional moment is transferred by the outer circular shell with a thickness of 15 mm (see Figure 6.36).
c. Determine the ratio between the torsional stiffness of the thick-walled tube in Figure 6.36 and that of the solid cross-section in Figure 6.35.

Solution:

a. The solid cross-section has a radius of $R = 30$ mm. The polar moment of inertia is

$$I_{\text{p}} = \tfrac{1}{2}\pi R^4 = \tfrac{1}{2} \times \pi \times (30\text{ mm})^4 = 1.272 \times 10^6\text{ mm}^4.$$

The shear stresses due to torsion are proportional to the distance to the centre of the cross-section. The maximum shear stress occurs at the boundary (see Figure 6.37) and is

$$\tau_{max} = \frac{M_t R}{I_p} = \frac{(3.2 \times 10^6 \text{ Nmm})(30 \text{ mm})}{1.272 \times 10^6 \text{ mm}^4} = 75.7 \text{ N/mm}^2.$$

b. Figure 6.38 shows the shear stress distribution in the outer shell of 15 mm. In fact this is the shear stress distribution in a thick-walled tube. The polar moment of inertia of the tube is

$$I_p = \tfrac{1}{2}\pi(R^4 - R_i^4) = \tfrac{1}{2} \times \pi \times \{(30 \text{ mm})^4 - (15 \text{ mm})^4\}$$
$$= 1.193 \times 10^6 \text{ mm}^4.$$

The relationship between the torsional moment M_t^{tube} and the maximum shear stress τ_{max} in the tube is

$$M_t^{tube} = \frac{\tau_{max} I_p}{R_e} = \frac{(75.5 \text{ N/mm}^2)(1.193 \times 10^6 \text{ mm}^4)}{30 \text{ mm}} = 3.0 \text{ kNm}.$$

The outer shell of the tube therefore transfers

$$\tfrac{3.0}{3.2} \times 100\% = 93.7\%$$

of the torsional moment in the solid cross-section.

This is not surprising if one knows that the larger shear stresses occur in the outer shell, and moreover have larger “moment levers”.

c. For circular cross-sections the torsion constant I_t is equal to the polar moment of inertia I_p. The requested ratio between the torsional stiffness of

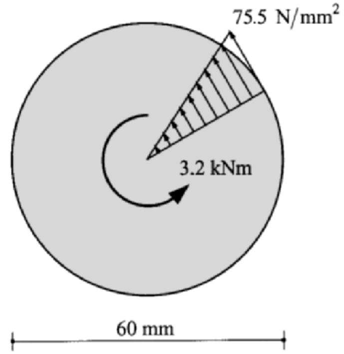


Figure 6.37 The shear stress distribution in the solid shaft, due to torsion .

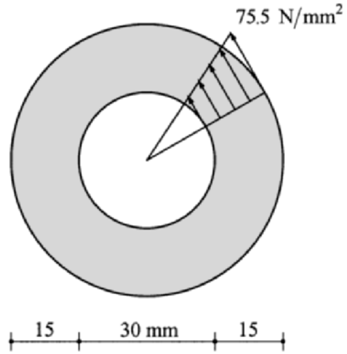


Figure 6.38 The outer 15 mm-thick circular shell transfers some 94% of the total torsional moment in the solid cross-section. This contribution is so large because the larger shear stresses occur in the outer shell and have moreover the larger “moment levers”.

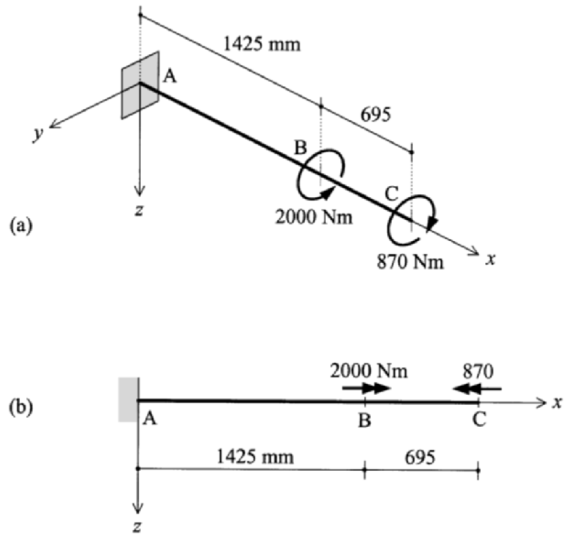


Figure 6.39 (a) Spatial representation of a member fixed at A, loaded by torsional moments at B and C. (b) The member in the xz plane with the torsional moments represented by moment vectors (double headed arrows).

the tube and the solid cross-section is

$$\frac{GI^{\text{tube}}}{GI_{\text{t}}^{\text{solid}}} = \frac{I_{\text{p}}^{\text{tube}}}{I_{\text{p}}^{\text{solid}}} = \frac{1.193 \times 10^6 \text{ mm}^4}{1.272 \times 10^6 \text{ mm}^4} = 0.937.$$

Comment: The hollow cross-section in Figure 6.36, with a wall-thickness of 15 mm (equal to half the radius of the solid cross-section), uses 25% less material then the solid cross-section (check it!), but transfers about 94% of the torsional moment in the solid cross-section. Also it contributes 94% to the torsional stiffness of the solid cross-section. From this we can conclude that, as far as the use of material is concerned, hollow cross-sections are more efficient than solid cross-sections.

Example 2: Dimensioning and deformation of a member with solid circular cross-section

The prismatic member ABC in Figure 6.39 is fixed at A, and has a solid circular cross-section. The member is loaded by torsional moments of 2000 Nm and 870 Nm at B and C respectively. Figure 6.39a gives a spatial representation of the member with loading. Figure 6.39b shows the member in the xz plane with the torsional moments represented by moment vectors.[1] The lengths AB and BC can be read from the figure. The shear stress may not exceed the limiting value $\bar{\tau} = 90$ N/mm^2 (the allowable shear stress). The shear modulus G is 80 MPa.

Questions:

a. What does it mean that "*member ABC is prismatic*"?

---

1 The moment vector has a double arrow head and is normal to the plane in which the moment acts. The direction of the moment vector and the direction of rotation of the moment are related via the corkscrew rule or right-hand rule, see *Engineering Mechanics*, Volume 1, Section 3.3.1.

b. Sketch the M_t diagram.
c. Determine for the cross-section the minimum diameter required, rounded off to 1 mm.
d. Determine the rotations of the cross-sections at B and C.

Solution:
a. A prismatic member is a member in which the cross-sectional quantities are the same for all cross-sections, so that they are independent of the x coordinate chosen along the member axis. These cross-sectional quantities include the area A of the cross-section, the moment of inertia I, the torsion constant I_t, the axial stiffness EA, the bending stiffness EI and the torsional stiffness GI_t.

b. The M_t diagram
In Figure 6.40a member ABC has been isolated. The as yet unknown fixed-end moment A_t (a torsional moment), for which the direction has been assumed in Figure 6.40a, follows from the moment equilibrium of the member about the x axis:

$$\sum T_x = A_t + (2000\,\text{Nm}) - (870\,\text{Nm}) = 0 \Rightarrow A_t = -1130\,\text{Nm}.$$

Figure 6.40b shows the fixed-end moment as it actually acts at A on the member.

The torsional moment in AB is found from the moment equilibrium about the x axis of the part to the right or left of an arbitrary section between A and B. Figure 6.40c shows the part to the left of the section. The unknown section force M_t is pictured in accordance with its positive direction.

Remember that the torsional moment in a cross-section is positive when it acts on the positive section plane in the positive direction of rotation about

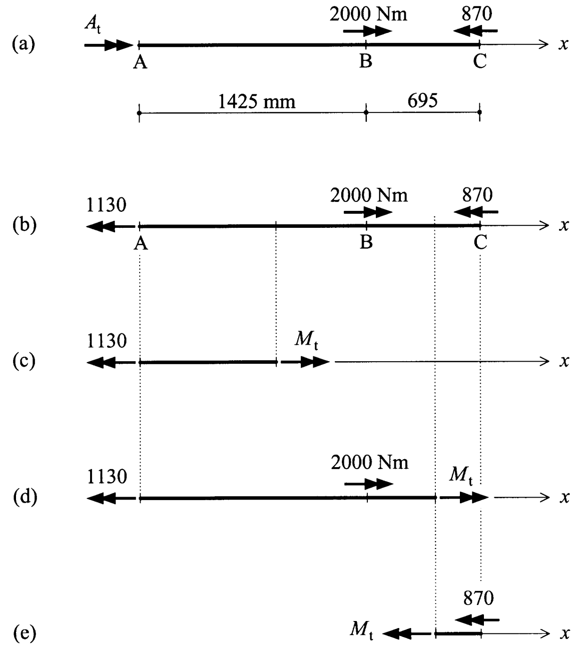


Figure 6.40 (a) The isolated member ABC. The fixed-end moment (torsional moment) A_t follows from the moment equilibrium of the member about the x axis. (b) The isolated member with all the torsional moments acting on it. (c) The torsional moment in AB can be found from the moment equilibrium about the x axis of the part to the left of any arbitrary section between A and B. (d) The torsional moment in BC can be found from the moment equilibrium about the x axis of the part to the left of any arbitrary section between B and C. (e) Check: the part to the right of the section must be in equilibrium.

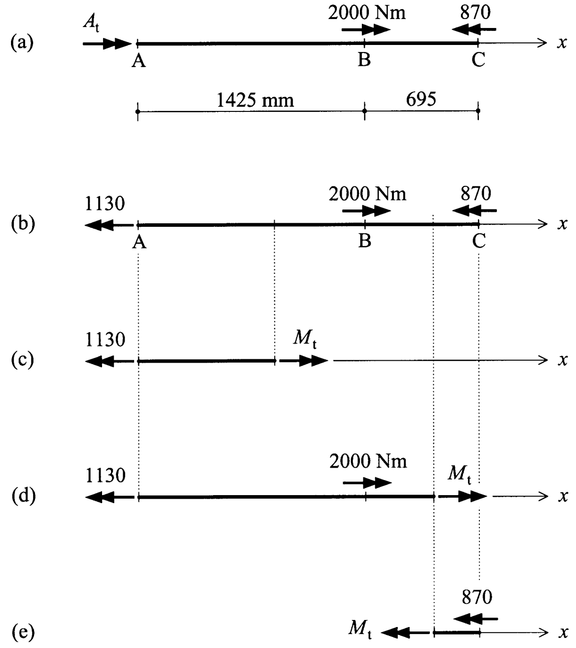


Figure 6.40 (a) The isolated member ABC. The fixed-end moment (torsional moment) A_t follows from the moment equilibrium of the member about the x axis. (b) The isolated member with all the torsional moments acting on it. (c) The torsional moment in AB can be found from the moment equilibrium about the x axis of the part to the left of any arbitrary section between A and B. (d) The torsional moment in BC can be found from the moment equilibrium about the x axis of the part to the left of any arbitrary section between B and C. (e) Check: the part to the right of the section must be in equilibrium.

the x axis, and when it acts on the negative section plane in the negative direction of rotation about the x axis.[1]

We can also say that a torsional moment is positive when the arrowheads of the moment vector on the positive section plane points in the positive x direction and on the negative section plane points in the negative x direction.

The torsional moment in AB is found from the moment equilibrium about the x axis of the part to the left or to the right of an arbitrary section between A and B. From the equilibrium of the part to the left of the section, shown in Figure 6.40c, we find

$$\sum T_x = -(1130 \text{ Nm}) + M_t = 0 \Rightarrow M_t = +1130 \text{ Nm}.$$

The torsional moment in AB is positive.

The torsional moment in BC is found from the moment equilibrium of the part to the left of an arbitrary section between B and C (see Figure 6.40d):

$$\sum T_x = -(1130 \text{ Nm}) + (2000 \text{ Nm}) + M_t = 0 \Rightarrow M_t = -870 \text{ Nm}.$$

The same result is found from the equilibrium of the part to the right of the section (see Figure 6.40e):

$$\sum T_x = -M_t - (870 \text{ Nm}) = 0 \Rightarrow M_t = -870 \text{ Nm}.$$

The torsional moment in BC is negative.

The distribution of the torsional moments is shown in the M_t diagram in

1 See *Engineering Mechanics*, Volume 1, Section 10.1.3.

Figure 6.41. In Figure 6.41a with *plus and minus signs* and in Figure 6.41b with *deformation symbols*.

Comment: In a *vector addition*, here the addition of moment vectors in the x direction, it is irrelevant whether the vectors represent moments or forces. It is therefore not surprising that the calculation of the torsional moments represented by vectors is very similar to the determination of the normal forces in a member subject to extension. Even the positive direction of the torsional moment as section force, represented by its vector, is equal to the positive direction of the normal force (a tensile force) (see Figure 6.42).

c. The minimum member diameter required
The maximum shear stress in the cross-section has to remain below the limiting value $\bar{\tau} = 90 \text{ N/mm}^2$. This means that

$$\tau_{\max} = \frac{M_t R}{I_p} = \frac{M_t R}{\frac{1}{2}\pi R^4} = \frac{2M_t}{\pi R^3} \leq \bar{\tau}.$$

From this we find

$$R^3 \geq \frac{2M_t}{\pi\bar{\tau}} = \frac{2 \times (1130 \times 10^3 \text{ Nmm})}{\pi \times (90 \text{ N/mm}^2)} = 7.781 \times 10^3 \text{ mm}^3.$$

The minimum required member diameter d is therefore

$$d = 2R = 2 \times \sqrt[3]{7.781 \times 10^3 \text{ mm}^3} = 39.6 \text{ mm} \approx 40 \text{ mm}.$$

d. The rotation of the cross-sections at B and C
The torsional stiffness of the solid circular cross-section with a diameter $d = 40$ mm is

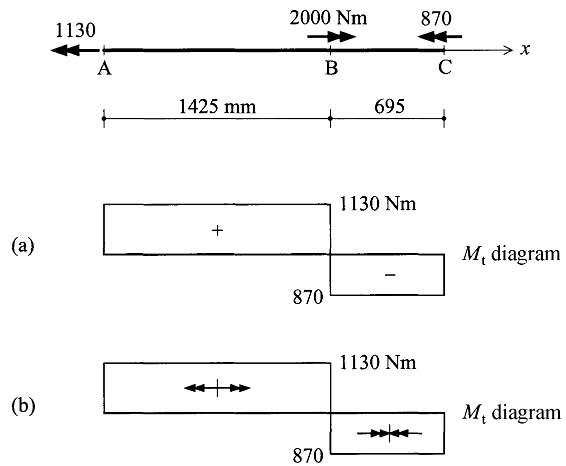


Figure 6.41 The distribution of the torsional moments represented in an M_t diagram: (a) with plus and minus signs and (b) with deformation symbols.

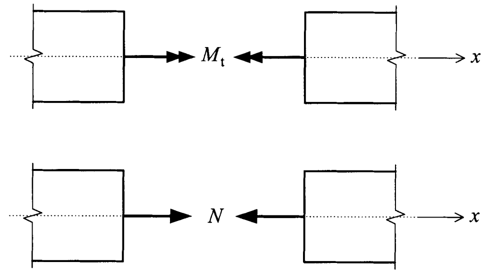


Figure 6.42 The positive direction of the torsional moment in a cross-section, in vector presentation (vectors with double headed arrows), is the same as the positive direction of a normal force (a tensile force).

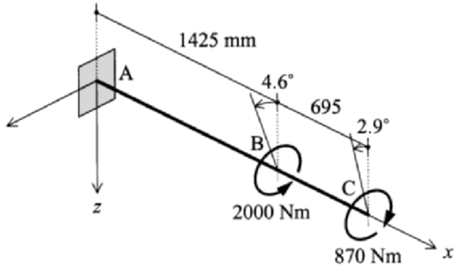


Figure 6.43 The member subject to torsion, with the rotations at B and C.

$$GI_t = GI_p = G \times \tfrac{1}{2}\pi R^4 = (80 \times 10^3\ \text{N/mm}^2) \times \tfrac{1}{2}\pi \times (40/2\ \text{mm})^2$$
$$= 20.11 \times 10^9\ \text{Nmm}^2.$$

The rotation of the cross-section at B with respect to the cross-section at A is

$$\Delta\varphi_x^{AB} = \varphi_{x;B} - \varphi_{x;A} = \frac{M_t^{AB}\ell^{AB}}{GI_t}.$$

Since the member is fixed at A, $\varphi_{x;A} = 0$. The rotation of the cross-section at B is therefore

$$\varphi_{x;B} = \frac{M_t^{AB}\ell^{AB}}{GI_t} = \frac{(+1130 \times 10^3\ \text{Nmm})(1425\ \text{mm})}{20.11 \times 10^9\ \text{Nmm}^2}$$
$$= +80 \times 10^{-3}\ \text{rad} = (+80 \times 10^{-3}\ \text{rad}) \times \frac{360°}{2\pi\ \text{rad}} = +4.6°.$$

The rotation of the cross-section at C with respect to the cross-section at B is

$$\Delta\varphi_x^{BC} = \varphi_{x;C} - \varphi_{x;B} = \frac{M_t^{BC}\ell^{BC}}{GI_t},$$

from which we find

$$\varphi_{x;C} = \varphi_{x;B} + \frac{M_t^{BC}\ell^{BC}}{GI_t}$$
$$= (+80 \times 10^{-3}\ \text{rad}) + \frac{(-870 \times 10^3\ \text{Nmm})(695\ \text{mm})}{20.11 \times 10^9\ \text{Nmm}^2}$$
$$= +50 \times 10^{-3}\ \text{rad} = (+50 \times 10^{-3}\ \text{rad}) \times \frac{360°}{2\pi\ \text{rad}} = 2.9°.$$

The rotations at B and C are shown in Figure 6.43.

Example 3: Rectangular thin-walled tube versus circular thin-walled tube

The rectangular thin-walled tube 1, for which the dimensions are given in Figure 6.44a, is replaced by the circular thin-walled tube 2 as pictured in Figure 6.44b. Both tubes are made of the same material and have the same shear modulus G. In addition, it is given that

- the material use for tube 2 is half the material use for tube 1, and
- a torsional moment M_t gives the same maximum shear stress in both tubes.

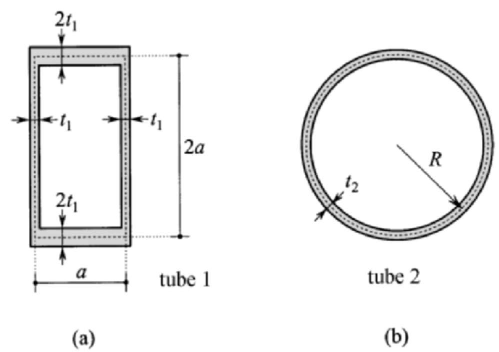


Figure 6.44 (a) Rectangular thin-walled tube and (b) circular thin-walled tube.

Questions:

a. Determine radius R (expressed in a) and wall thickness t_2 (expressed in t_1) of tube 2.
b. Determine the ratio between the torsional stiffnesses of both tubes. Use for tube 2 the values determined under a.

Solution:

a. From the information on the use of material it follows that area A_2 of tube 2 must be half that of area A_1 of tube 1:

$$A_2 = \tfrac{1}{2} A_1.$$

With

$$A_1 = 2 \times 2a \times t_1 + 2 \times a \times 2t_1 = 8at_1,$$

$$A_2 = 2\pi R t_2,$$

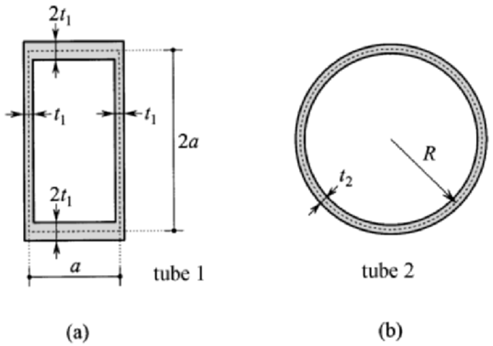


Figure 6.44 (a) Rectangular thin-walled tube and (b) circular thin-walled tube.

we find

$$2\pi R t_2 = 4 a t_1 . \tag{a}$$

The maximum shear stress $\tau_{\max;1}$ in rectangular tube 1 occurs where the wall thickness is smallest, i.e. in the webs with thickness t_1:

$$\tau_{\max;1} = \frac{M_t}{2A_m t_1} .$$

Here A_m is the area within the centre lines of the rectangular tube:

$$A_m = 2a^2,$$

so that

$$\tau_{\max;1} = \frac{M_t}{4a^2 t_1} .$$

The maximum shear stress $\tau_{\max;2}$ in circular tube 2 is

$$\tau_{\max;2} = \frac{M_t R}{I_p} = \frac{M_t R}{2\pi R^3 t_2} = \frac{M_t}{2\pi R^2 t^2} .$$

It is given that the torsional moment M_t causes the same maximum shear stress in both tubes:

$$\tau_{\max;1} = \frac{M_t}{4a^2 t_1} = \tau_{\max;2} = \frac{M_t}{2\pi R^2 t_2} .$$

Hence

$$2\pi R^2 t_2 = 4a^2 t_1. \tag{b}$$

From (a) and (b) we find the required dimensions of the circular tube cross-section:

$$R = a,$$

$$t_2 = \frac{2}{\pi} t_1 \approx 0.64 t_1.$$

b. The torsional stiffness of rectangular tube 1 is

$$GI_{t;1} = G\frac{4A_m^2}{\oint \frac{1}{t}\,dm} = G\frac{4 \times (2a^2)^2}{2\frac{2a}{t_1} + 2 \cdot \frac{a}{2t_1}} = \tfrac{16}{5}\,Ga^3 t_1.$$

With $R = a$ and $t_2 = 2t_1/\pi$ the torsional stiffness of circular tube 2 is

$$GI_{t;2} = GI_p = G \cdot 2\pi R^3 t_2 = G \cdot 2\pi a^3 \cdot \frac{2}{\pi} t_1 = 4Ga^3 t_1.$$

The ratio between the torsional stiffnesses is

$$\frac{GI_{t;2}}{GI_{t;1}} = \frac{4Ga^3 t_1}{\frac{16}{5}\,Ga^3 t_1} = 1.25.$$

The circular cross-section offers with respect to the rectangular cross-section a 50% reduction in the use of material, and besides the torsional stiffness is even 25% larger!

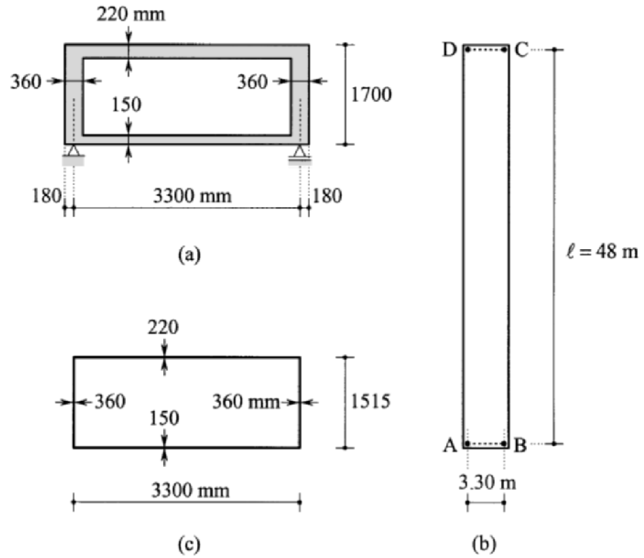


Figure 6.45 (a) Rectangular cross-section of (b) a concrete box girder bridge, simply supported at A, B, C and D. (c) The centre lines of the cross-section that is considered thin-walled.

Example 4: Settlement of a box girder bridge

A concrete box girder bridge has the rectangular cross-section shown in Figure 6.45a The box girder is prismatic and can be considered thin-walled. Figure 6.45b shows the top view of the bridge with a span of $\ell = 48$ m. The bridge is simply supported at the points A, B, C and D. The supports are assumed to be able to transfer tensile forces. Diaphragms (cross-beams) have been constructed at AB and CD to transfer the support reactions into the hollow cross- section.

Support A undergoes a settlement as a result of which a torsional moment is induced in the beam. The maximum shear stress due to this torsional moment is 1 N/mm^2.

In the calculation use $G = 13.5$ GPa for the shear modulus.

Questions:

a. Determine the torsional moment in the hollow cross-section.
b. Determine the settlement of support A.
c. Determine the support reactions due to the settlement of support A.

Solution:

First we determine the cross-sectional quantities necessary for answering the questions. Figure 6.45c shows the centre lines of the cross-section. The distance between the centre lines between the upper and lower flange is

$$(1700 \text{ mm}) - \frac{220 \text{ mm}}{2} - \frac{150 \text{ mm}}{2} = 1515 \text{ mm}.$$

The area A_m within the centre lines of the cross-section is

$$A_m = (3300 \text{ mm})(1515 \text{ mm}) = 5 \times 10^6 \text{ mm}^2.$$

The torsion constant I_t is

$$I_t = \frac{4A_m^2}{\oint \frac{dm}{t}} = \frac{4 \times (5 \times 10^6 \text{ mm}^2)^2}{\frac{3300 \text{ mm}}{150 \text{ mm}} + \frac{3300 \text{ mm}}{220 \text{ mm}} + 2 \times \frac{1515 \text{ mm}}{360 \text{ mm}}}$$
$$= 2.2 \times 10^{12} \text{ mm}^4 = 2.2 \text{ m}^4.$$

a. The shear stress due to torsion is

$$\tau = \frac{M_t}{2A_m t}.$$

The maximum shear stress occurs in the lower flange as the wall thickness is smallest there:

$$\tau_{max} = \frac{M_t}{2A_m t_{min}}.$$

With $\tau_{max} = 1 \text{ N/mm}^2$ we find for the torsional moment

$$M_t = 2A_m t_{min} \tau_{max} = 2 \times (5 \times 10^6 \text{ mm}^2)(150 \text{ mm})(1 \text{ N/mm}^2)$$
$$= 1.5 \times 10^9 \text{ Nmm} = 1500 \text{ kNm}.$$

b. Due to the settlement $u_{z;A}$ of support A cross-section AB rotates with respect to cross-section CD through an angle φ_x (see Figure 6.46):

$$\varphi_x = \frac{u_{z;A}}{b},$$

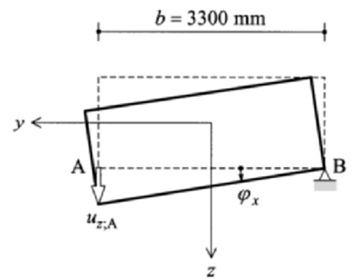


Figure 6.46 If support A undergoes a settlement of $u_{z;A}$ cross-section AB will rotate through an angle $\varphi_x = u_{z;A}/b$.

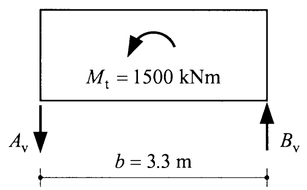


Figure 6.47 The torsional moment $M_t = 1500$ kNm acting on the end cross-section AB is statically equivalent to (the moment caused by) the support reactions A_V and B_V, acting on the box girder at A and B respectively.

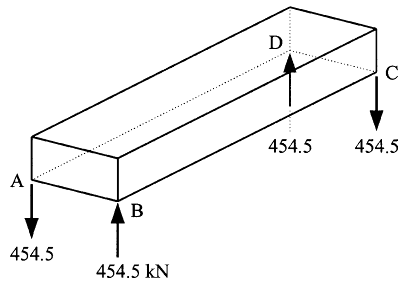


Figure 6.48 The support reactions due to a settlement of support A, assuming that the supports can transfer tensile forces.

in which

$$\varphi_x = \frac{M_t \ell}{G I_t}.$$

For settlement in A we now find

$$u_{z;A} = \varphi_x b = \frac{M_t \ell b}{G I_t} = \frac{(1500 \times 10^3 \text{ Nm})(48 \text{ m})(3.3 \text{ m})}{(13.5 \times 10^9 \text{ N/mm}^2)(2.2 \text{ m}^4)}$$

$$= 8 \times 10^{-3} \text{ m} = 8 \text{ mm}.$$

c. The torsional moment $M_t = 1500$ kNm to which the box girder is subject at cross-section AB is statically equivalent to the support reactions at A and B. The vertical support reactions A_v and B_v together therefore form the couple M_t (see Figure 6.47), from which it follows that

$$A_v = B_v = \frac{M_t}{b} = \frac{1500 \text{ kNm}}{3.3 \text{ m}} = 454.5 \text{ kN}.$$

Figure 6.48 shows the vertical support reactions at A en B, and also at C and D.

Example 5: Rectangular hollow cross-section with an eccentric shear force

The thin-walled tube in Figure 6.49 has a rectangular cross-section and transfers an eccentric shear force $V_z = V = 60$ kN. The cross-sectional dimensions are given in the figure.

Question:
Determine the shear stress distribution in the cross-section.

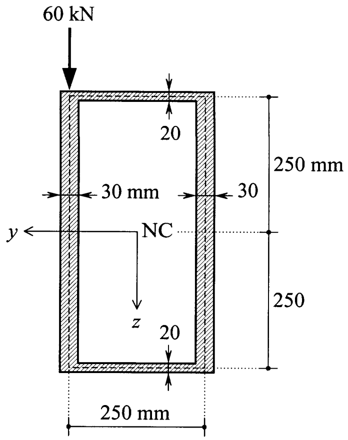


Figure 6.49 The thin-walled rectangular tube has to transfer an eccentric shear force.

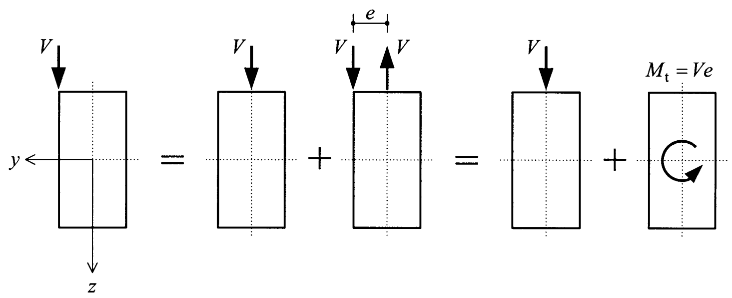


Figure 6.50 The shear force V with eccentricity e is statically equivalent to a central shear force combined with a torsional moment $M_t = Ve$.

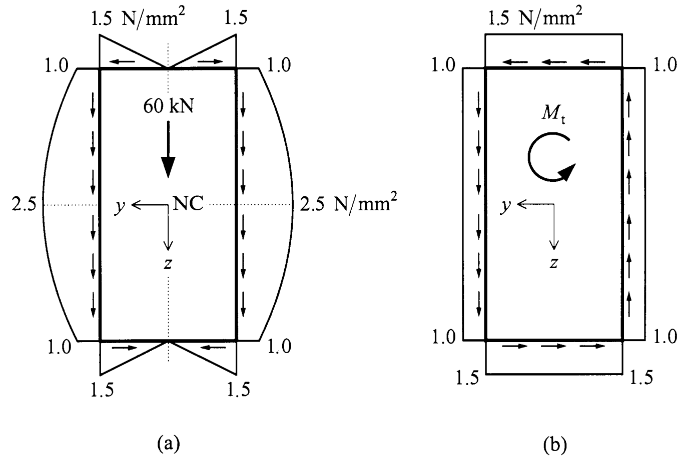


Figure 6.51 The shear stress distributions due to (a) the central shear force and (b) the torsional moment.

Solution:

The eccentric shear force V can be replaced by a central shear force and a torsional moment M_t (see Figure 6.50):

$$M_t = Ve = (60 \times 10^3 \text{ N})(125 \text{ mm}) = 7.5 \times 10^6 \text{ Nmm} = 7.5 \text{ kNm}.$$

The shear stress distribution due to the shear force in the vertical plane of symmetry was determined previously in Section 5.4.3, Example 1, and is shown in Figure 6.51a. The shear stresses are uniformly distributed across the wall thickness.

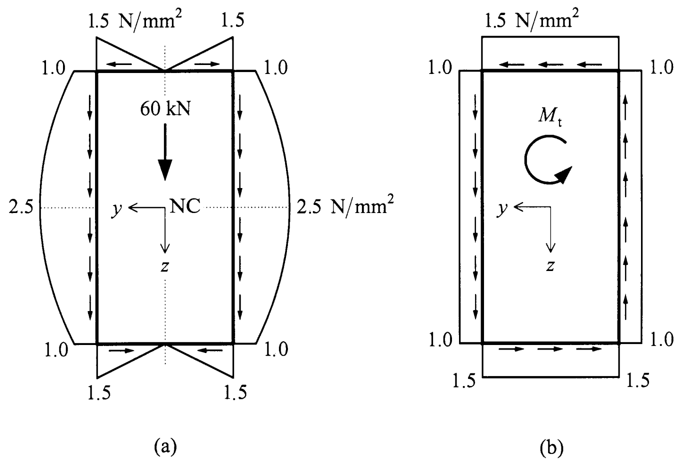


Figure 6.51 The shear stress distributions due to (a) the central shear force and (b) the torsional moment.

The shear stresses due to the torsional moment are also uniformly distributed across the wall thickness. The magnitude of these shear stresses follows from

$$\tau = \frac{M_t}{2A_m t},$$

in which

$$A_m = (250 \text{ mm})(500 \text{ mm}) = 125 \times 10^3 \text{ mm}^2.$$

In the flanges, with $t = 20$ mm, we find

$$\tau_{\text{flange}} = \frac{7.5 \times 10^6 \text{ Nmm}}{2 \times (125 \times 10^3 \text{ mm}^2)(20 \text{ mm})} = 1.5 \text{ N/mm}^2,$$

and in the webs, with $t = 30$ mm,

$$\tau_{\text{web}} = \frac{7.5 \times 10^6 \text{ Nmm}}{2 \times (125 \times 10^3 \text{ mm}^2)(30 \text{ mm})} = 1.0 \text{ N/mm}^2.$$

Since the shear flow $s = \tau t$ is constant, the shear stresses are inversely proportional to the wall thickness, so the shear stress in the web can also be derived from that in the flange:

$$\tau_{\text{web}} = \frac{t_{\text{flange}}}{t_{\text{web}}} \tau_{\text{flange}} = \frac{20 \text{ mm}}{30 \text{ mm}} (1.5 \text{ N/mm}^2) = 1.0 \text{ N/mm}^2.$$

The shear stress distribution due to torsion is pictured in Figure 6.51b.

The resultant shear stress distribution is shown in Figure 6.52 and is found by superposing both diagrams in Figure 6.51.

Example 6: Square thin-walled tube versus thin-walled strip

The square thin-walled tube in Figure 6.53a has a side $a = 200$ mm, measured along the centre lines, and a uniform wall thickness $t = 2.5$ mm. The thin-walled strip in Figure 6.53b has the same height $a = 200$ mm, but a wall thickness that is four times as large: $4t = 10$ mm. Tube and strip are made of the same material, with the same use of material. The shear modulus is G. Both cross-sections are subject to the same torsional moment M_t.

Questions:

a. Determine the ratio between the torsional stiffness of tube and strip.
b. Determine the ratio between the maximum shear stress in tube and strip.

Solution:

a. According to (6.17) the torsional stiffness of the tube is

$$GI_{t;\text{tube}} = G \cdot \frac{4 \cdot (a^2)^2}{\frac{4a}{t}} = Ga^3 t.$$

According to (6.20) the torsional stiffness of the strip is

$$GI_{t;\text{strip}} = G \cdot \tfrac{1}{3}\, a \cdot (4t)^3 = \frac{64}{3}\, Gat^3.$$

The ratio requested between the torsional stiffness of tube and strip is

$$\frac{GI_{t;\text{tube}}}{GI_{t;\text{strip}}} = \frac{Ga^3 t}{\frac{64}{3}\, Gat^3} = \frac{3}{64}\,\frac{a^2}{t^2}\,.$$

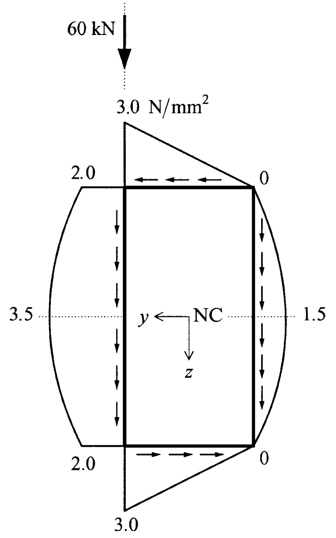


Figure 6.52 The shear stress distribution due to the eccentric shear force.

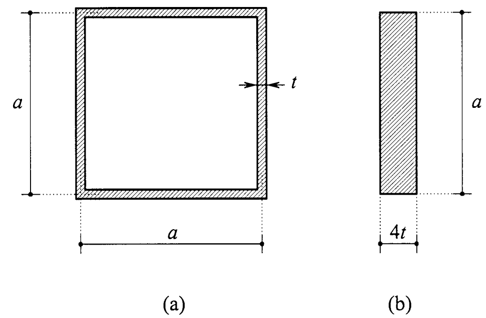


Figure 6.53

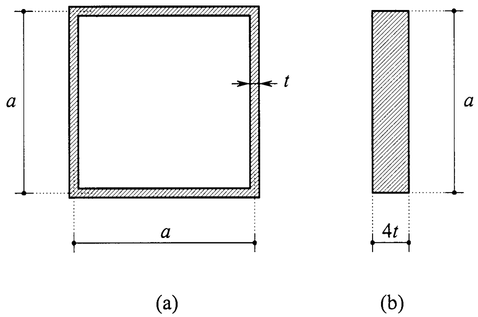


Figure 6.53 (a) A square thin-walled tube and (b) a thin-walled strip. Tube and strip are made of the same material and have the same use of material. For $a/t = 80$ and compared with the strip, the tube has a torsional stiffness that is 300 times as large and a torsion stress that is a factor of 30 smaller.

With $a = 200$ mm, $t = 2.5$ mm and $a/t = 80$ we find

$$\frac{GI_{\mathrm{t;tube}}}{GI_{\mathrm{t;strip}}} = \tfrac{3}{64} \times 80^2 = 300.$$

For the given dimensions and with respect to torsion, the tube is 300 times as stiff as the strip.

b. The shear stress in the thin-walled tube is uniform (see Section 6.3.1). According to (6.13)

$$\tau_{\mathrm{tube}} = \frac{M_{\mathrm{t}}}{2A_{\mathrm{m}}t} = \frac{M_{\mathrm{t}}}{2a^2t}\,.$$

The shear stress in the strip is linearly distributed across thickness $4t$ (see Section 6.3.2). According to (6.19) the maximum shear stress in the strip is

$$\tau_{\mathrm{strip;max}} = \frac{M_{\mathrm{t}}}{\tfrac{1}{3}\cdot a\cdot(4t)^2} = \frac{3}{16}\,\frac{M_{\mathrm{t}}}{16at^2}\,.$$

The ratio between the maximum shear stress in tube and strip is

$$\frac{\tau_{\mathrm{tube;max}}}{\tau_{\mathrm{strip;max}}} = \frac{M_{\mathrm{t}}}{2a^2t}\cdot\frac{16}{3}\,\frac{at^2}{M_{\mathrm{t}}} = \frac{8}{3}\,\frac{t}{a}\,.$$

With $a = 200$ mm, $t = 2.5$ mm and $a/t = 80$ we find

$$\frac{\tau_{\mathrm{tube;max}}}{\tau_{\mathrm{strip;max}}} = \tfrac{8}{3} \times \tfrac{1}{80} = \tfrac{1}{30}\,.$$

The shear stress in the tube with given dimensions is 30 times smaller than the maximum shear stress in the strip.

Comment: With respect to torsion the maximum shear stress in a closed thin-walled cross-section is much smaller than in a thin-walled strip, and the torsional stiffness is much larger.

Example 7: Hollow thin-walled cross-section versus open thin-walled cross-section

Figure 6.54 shows two square thin-walled cross-sections with dimensions relating to the centre lines of 160 × 160 mm^2. Cross-section I is closed and cross-section II is open (see the small gap at the bottom flange). The wall thickness of the flanges is 10 mm and that of the webs is 5 mm. Both cross-sections are subject to the same torsional moment $M_t = 768$ Nm.

Questions:

a. Determine the maximum shear stress in cross-section I.
b. Determine the maximum shear stress in cross-section II.

Solution:

a. In the closed cross-section the shear stress is constant across the wall thickness:

$$\tau = \frac{M_t}{2A_m t}.$$

Here A_m is the area enclosed by the centre lines of the cross-section:

$$A_m = (160\text{ mm})(160\text{ mm}) = 25.6 \times 10^3\text{ mm}^2.$$

The shear stress is a maximum where the wall thickness is smallest, that is in the webs with $t = 5$ mm:

$$\tau_{max}^{(I)} = \tau_{web} = \frac{768 \times 10^3\text{ Nmm}}{2 \times (25.6 \times 10^3\text{ mm}^2)(5\text{ mm})} = 3.0\text{ N/mm}^2.$$

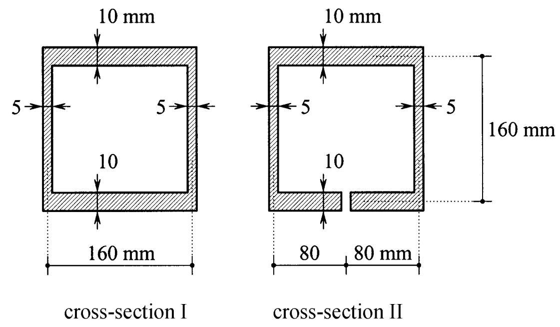


Figure 6.54 Two thin-walled cross-sections with the same dimensions. Section I is closed and section II is open.

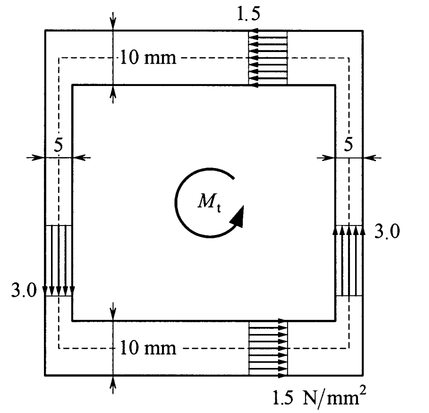


Figure 6.55 Torsional stresses in the closed cross-section. The shear stresses are constant over the wall thickness. The shear stress is inversely proportional to the wall thickness. This follows from the property that the shear flow (shear stress × wall thickness) is constant. The maximum shear stress occurs where the wall thickness is *smallest*.

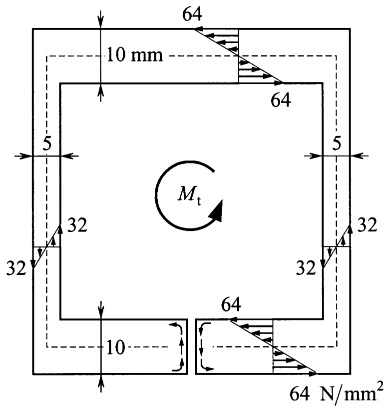


Figure 6.56 Shear stresses due to torsion of the open cross-section. The shear stresses are linear across the wall thickness. The maximum shear stress is proportional to the wall thickness. The maximum shear stress occurs where the wall thickness is *largest*.

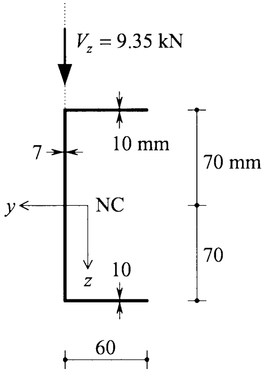


Figure 6.57 The thin-walled U-section transfers a shear of which the line of action coincides with the web.

In the flanges the shear stress is half the magnitude. Check this!

Figure 6.55 shows the shear stress distribution.

b. In the open cross-section the shear stresses are linear across the wall thickness and the shear stress formula is

$$\tau = \frac{M_t e_m}{\frac{1}{2} I_t}.$$

Here e_m is the distance to centre line m and I_t is the torsion constant:

$$I_t = \sum \tfrac{1}{3} ht^3 = \tfrac{1}{3}\{2 \times (80 \text{ mm})(10 \text{ mm})^3 + (160 \text{ mm})(10 \text{ mm})^3 + 2 \times (160 \text{ mm})(5 \text{ mm})^3\} = 120 \times 10^3 \text{ mm}^4.$$

Since the shear stress is proportional to the distance to the centre lines the largest shear stress occurs where the wall thickness is a maximum, namely in the flanges with $t = 10$ mm and $e_{m;\max} = t/2 = 5$ mm:

$$\tau_{\max}^{(II)} = \tau_{\text{flange}} = \frac{(768 \times 10^3 \text{ Nmm})(5 \text{ mm})}{\frac{1}{2} \times (120 \times 10^3 \text{ mm}^4)} = 64 \text{ N/mm}^2.$$

In the webs the maximum shear stress is half the magnitude. Check this!

Figure 6.56 shows the shear stresses in webs and flanges.

Note that the shear stresses in the thin-walled open cross-section are considerably larger than those in the thin-walled closed cross-section! For the given dimensions the maximum shear stress in the open cross-section is more than 42 times as large as that in the closed cross-section.

Comment: For a thin-walled closed cross-section the maximum shear stress is inversely proportional to the wall thickness. For a thin-walled open cross-section the maximum shear stress is proportional to the wall thickness.

This means that in a closed cross-section the shear stress is a maximum where the wall thickness is smallest, while in an open cross-section the shear stress is a maximum where the wall thickness is largest.

Example 8: U-section subject to an eccentric shear force
The thin-walled U-section in Figure 6.57 transfers a shear force of 9.35 kN. The line of action of the shear force coincides with the web.

Questions:
a. Determine the shear stress distribution due to shear (without torsion).
b. Determine the shear stress distribution due to torsion (without shear).
c. Determine the maximum shear stress in the cross-section.

Solution:
a. In Section 5.5, example 1, the shear force distribution was determined for the given cross-section due to a vertical shear force $V = 9.35$ kN with an unknown line of action. That shear stress distribution is given in Figure 6.58a.

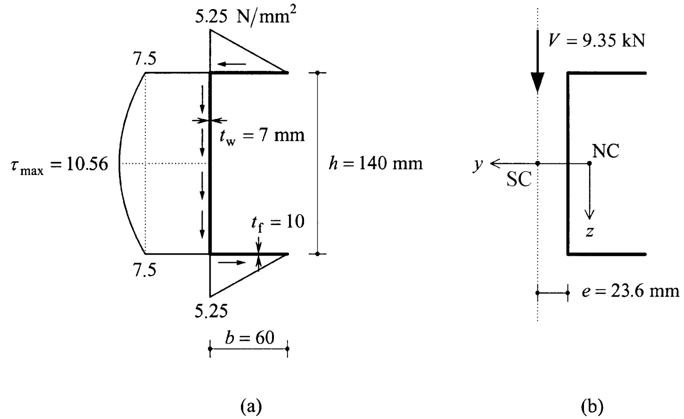


Figure 6.58 (a) Shear stress diagram due to the shear force; there is no torsion. (b) The shear force V, as a resultant of all these shear stresses, has its line of action at a distance $e = 23.6$ mm from the centre line of the web. The shear centre SC is located on this line. The shear centre is defined as the point of the cross-section through which the line of action of the shear force must pass so that there will be no torsion.

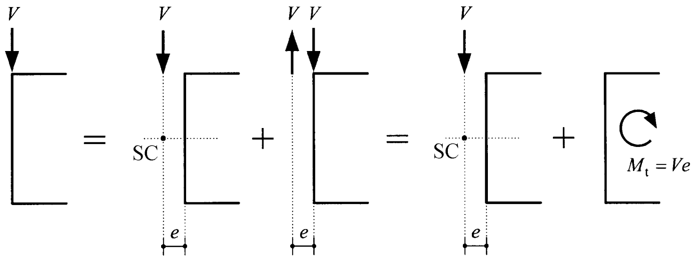


Figure 6.59 If the shear force V is shifted through a distance e from the web to shear centre SC, a torsional moment $M_t = Ve$ is generated.

In the same example it was shown that the shear force V as a resultant of all these shear stresses has its line of action at a distance $e = 23.6$ mm from the centre line of the web. Shear centre SC is located on this line (see Figure 6.59b).

The shear centre SC is that point of the cross-section through which the line of action of the shear force must pass so that there will be no torsion.

b. In this question the line of action of the shear force coincides with the web, and does not pass through the shear centre. Therefore there is a torsional moment $M_t = Ve$ (see Figure 6.59):

$$M_t = Ve = (9.35 \times 10^3 \text{ N})(23.6 \text{ mm}) = 220.66 \times 10^3 \text{ Nmm}.$$

In the thin-walled open cross-section the torsional moment gives shear stresses that are linear across the wall thickness (see Figure 6.60):

$$\tau = \frac{M_t e_m}{\frac{1}{2} I_t}.$$

The shear stresses are largest in the edges of flanges and web, where $e_m = t/2$.

The torsion constant I_t is

$$\begin{aligned} I_t &= \sum \tfrac{1}{3} h t^3 \\ &= \tfrac{1}{3}\{2 \times (60 \text{ mm})(10 \text{ mm})^3 + (140 \text{ mm})(7 \text{ mm})^3\} \\ &= 56.0 \times 10^3 \text{ mm}^4. \end{aligned}$$

The maximum shear stress in the flanges is

$$\tau_{f;max} = \frac{M_t \cdot \frac{1}{2} t_f}{\frac{1}{2} I_t} = \frac{(220.66 \times 10^3 \text{ Nmm})(\frac{1}{2} \times 10 \text{ mm})}{\frac{1}{2} \times 56.0 \times 10^3 \text{ mm}^4}$$
$$= 39.40 \text{ N/mm}^2.$$

The maximum shear stress in the web is

$$\tau_{w;max} = \frac{M_t \cdot \frac{1}{2} t_w}{\frac{1}{2} I_t} = \frac{(220.66 \times 10^3 \text{ Nmm})(\frac{1}{2} \times 7 \text{ mm})}{\frac{1}{2} \times 56.0 \times 10^3 \text{ mm}^4}$$
$$= 27.58 \text{ N/mm}^2.$$

The magnitude and direction of the maximum shear stresses due to the torsional moment M_t are shown in Figure 6.60.

c. The maximum shear stress in the cross-section is found by superposing the shear stresses due to the shear force (Figure 6.58a) on that due to the torsional moment (Figure 6.60). Here one must remember that, in a thin-walled open cross-section, the shear stresses due to a shear force are constant across the wall thickness while those due to a torsional moment are linearly distributed!

The maximum shear stress in the flanges occurs at the inside of the cross-section, at the join to the web (see Figure 6.61), and amounts to

$$\tau_{f;max} = (5.25 \text{ N/mm}^2) + (39.40 \text{ N/mm}^2) = 44.65 \text{ N/mm}^2.$$

The maximum shear stress in the web occurs at half height, also at the

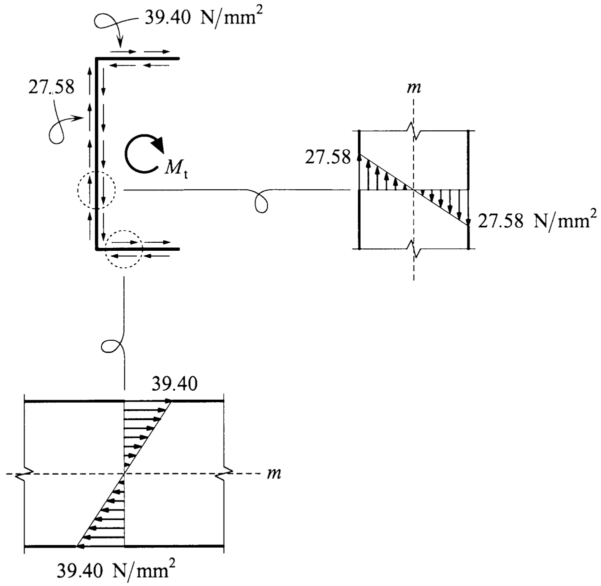


Figure 6.60 The shear stresses due to the torsional moment are linear across the wall thickness. The maximum shear stress is proportional to the wall thickness.

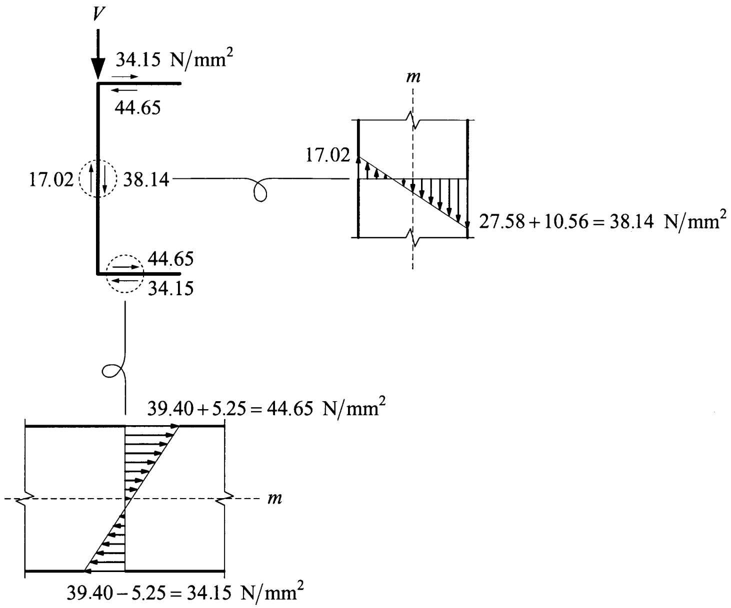


Figure 6.61 The maximum shear stresses due to the shear force with its line of action along the web occur at the inside of the section: in the flanges at the join to the web, and in the web at half height.

inside of the cross-section (see Figure 6.61):

$$\tau_{w;\max} = (10.5\ \text{N/mm}^2) + (27.58\ \text{N/mm}^2) = 38.14\ \text{N/mm}^2.$$

Figure 6.61 shows how at these locations the shear stress is distributed across the wall thickness.

If the line of action of the shear force does not pass through the shear centre SC a torsional moment is induced. Due to this torsional moment the shear stresses can increase considerably, especially in open cross-sections.

Comment: We have not taken into account possible stress concentrations in the corners. These cannot be determined with the formulas we have derived.

6.5 Summary of the formulas

Here we provide a brief summary of the most important formulas for determining stresses and deformations due to torsion.

- Constitutive and kinematic equations for torsion (Section 6.2.1)

Constitutive equation

$$M_t = GI_t \cdot \chi.$$

Kinematic equation

$$\chi = \frac{d\varphi_x}{dx}, \quad \varphi_x = \int \chi \cdot dx.$$

Here G is the shear modulus, I_t the torsion constant, GI_t the torsional stiff-

stiffness, χ the torsional strain and φ_x is the rotation of the cross-section.

- Circular cross-sections

In circular cross-sections the torsion constant I_t is equal to the polar moment of inertia I_p.

Thin-walled circular cross-section (Section 6.2.1)

$$\tau = \frac{M_\mathrm{t} R}{I_\mathrm{t}}\,, \quad I_\mathrm{t} = I_\mathrm{p} = 2\pi R^3 t\,.$$

Thick-walled circular cross-section (Section 6.2.3)

$$\tau = \frac{M_\mathrm{t} r}{I_\mathrm{t}}\,, \quad I_\mathrm{t} = I_\mathrm{p} = \tfrac{1}{2}\pi(R_\mathrm{e}^4 - R_\mathrm{i}^4)\,.$$

Solid circular cross-section (Section 6.2.2)

$$\tau = \frac{M_\mathrm{t} r}{I_\mathrm{t}}\,, \quad I_\mathrm{t} = I_\mathrm{p} = \tfrac{1}{2}\pi R^4\,.$$

- Closed thin-walled cross-sections (Section 6.3.1)

$$s = \tau t = \text{constant} \ \ \text{(the shear flow is constant)},$$

$$\tau = \frac{M_\mathrm{t}}{2A_\mathrm{m} t}\,, \quad I_\mathrm{t} = \frac{4A_\mathrm{m}^2}{\displaystyle\oint \frac{1}{t}\,\mathrm{d}m}\,.$$

Here A_m is the area that is enclosed by the centre line m of the closed thin-walled cross-section.

- Thin-walled strip (Section 6.3.2)

$$\tau = \frac{M_{\mathrm{t}} e_m}{\frac{1}{2} I_{\mathrm{t}}}, \quad I_{\mathrm{t}} = \frac{1}{3} h^3 t.$$

- Open thin-walled cross-sections (Section 6.3.3)

$$\tau = \frac{M_{\mathrm{t}} e_m}{\frac{1}{2} I_{\mathrm{t}}}, \quad I_{\mathrm{t}} = \sum_i \frac{1}{3} h_i t_i^3.$$

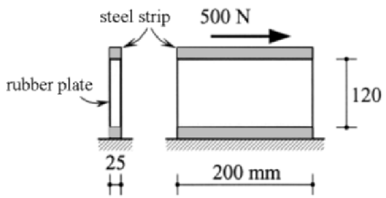
6.6 Problems

General comments:

- In a number of problems, not only the shear stress due to a torsion will be required, but also the shear stress due to a shear force and sometimes also the normal stress.
- The dead weight of the structure is ignored unless indicated otherwise.

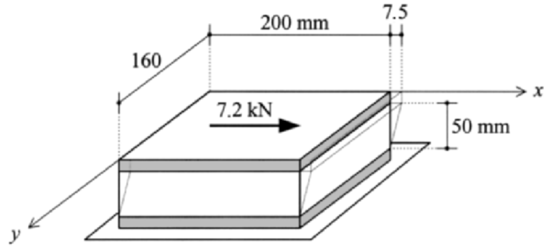
Material behaviour in the case of shear (Section 6.1)

6.1 A 25 mm thick rubber plate, 200 mm long and 120 mm high, is firmly glued to two steel strips at top and bottom. A force of 500 N acts on the upper strip.



Question:
Determine the displacement of the upper strip with respect to the lower strip if the shear modulus for rubber is 3 MPa.

6.2 A rubber block of $200 \times 160 \times 50\ \text{mm}^3$ is firmly glued at the top and the bottom to two rigid steel plates. Due to the force of 7.2 kN as shown, the upper plate moves 7.5 mm in the x direction.

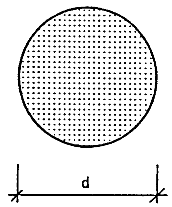


Questions:

a. Determine the shear modulus for the type of rubber used.
b. Determine the displacement of the upper plate if the force of 7.2 kN acts in the y direction and not the x direction.

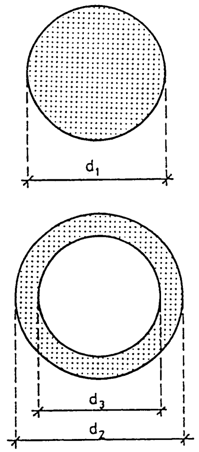
Torsion of circular cross-sections (Section 6.2)

6.3 A solid circular cross-section has to transfer a torsional moment of 1 kNm. The shear stress in the cross-section may not exceed 10 MPa.



Question:
Determine the minimum required diameter d of the cross-section.

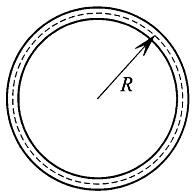
6.4 A solid circular cross-section with diameter $d_1 = 150$ mm and a hollow circular cross-section with external diameter $d_2 = 180$ mm and an as yet unknown inner diameter d_3 are subject to the same torsional moment M_t. Due to M_t the same maximum shear stress $\tau_{max} = 80$ MPa occurs in both cross-sections.



Questions:

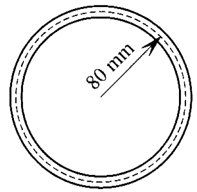
a. Determine the magnitude of the torsional moment M_t.
b. Determine the inner diameter d_3 of the hollow cross-section.
c. Determine the shear stress at the inner boundary of the hollow cross-section.

6.5 A thin-walled circular tube of which the cross-section has a radius $R = 150$ mm and an area $A = 8000$ mm^2 is subject to a torsional moment $M_t = 30$ kNm.



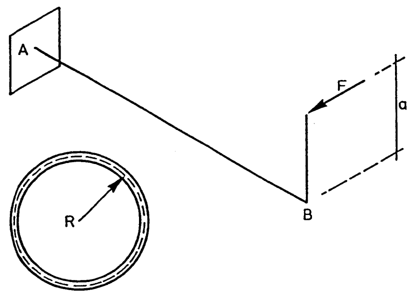
Question:
Determine the shear stress in the cross-section.

6.6 The cross-section of a thin-walled steel tube has an area $A = 1000$ mm^2. The shear stress may not exceed the limiting value $\bar{\tau} = 90$ MPa.



Question:
Determine the maximum torsional moment that the cross-section can transfer.

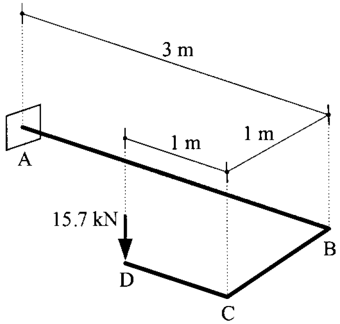
6.7 The thin-walled circular tube AB is loaded by a force F with eccentricity a. The radius of the circular cross-section is R and the wall thickness is t. In the calculation use $F = 30$ kN, $a = 1.65$ m, $R = 150$ mm and $t = 7$ mm.

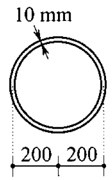


Questions:

a. Determine the shear stress distribution in the cross-section of the tube due to the torsional moment.
b. Determine the shear stress distribution due to the shear force.
c. Determine the shear stress distribution due to the torsional moment and the shear force.
d. Determine the maximum shear stress and the place where it occurs.

6.8 The double bent member in the horizontal plane is constructed from a thin-walled tube of radius $R = 200$ mm and wall thickness $t = 10$ mm. The structure is loaded by a vertical force of 15.7 kN at D. The force acts in the axis of symmetry of the cross-section.

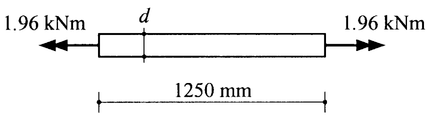




Questions with respect to the cross-section at the fixed support A:

a. Determine the shear stress distribution due to the torsional moment.
b. Determine the shear stress distribution due to the shear force.
c. Determine the location and magnitude of the maximum shear stress.
d. Determine the normal stress distribution due to the bending moment.

6.9 A solid circular shaft has to transfer a torsional moment of 1.96 kNm. The limiting value of the shear stress is $\bar{\tau} = 80$ N/mm^2. The shear modulus is $G = 80$ GPa.



Questions:

a. Determine the required diameter d of the shaft.
b. Determine the rotation of the end cross-sections with respect to one another (use for d the value found in (a)).

6.10 The solid shaft in problem 6.9 is replaced by a hollow shaft with an external diameter of 75 mm. All other data remain the same.

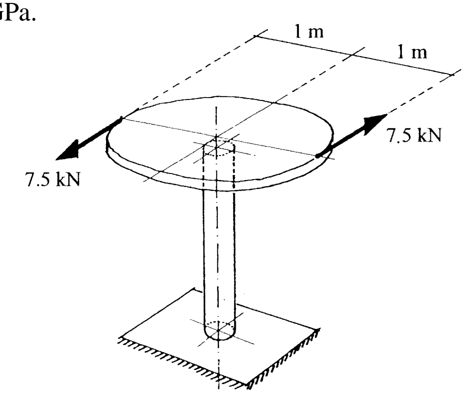
Questions:

a. Determine the required wall thickness of the shaft.
b. Determine the rotation of the end cross-sections with respect to one another (use the wall-thickness found in (a)).

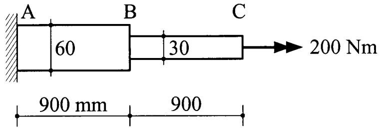
6.11 The horizontal forces in the plane of the roof cause torsion in the column. The column is rigidly connected to the roof and the foundation. A steel tube has been used for the column with an external diameter of 180 mm. The polar moment of inertia is $I_p = 60 \times 10^6$ mm^4. For the shear modulus of steel use $G = 80$ GPa.

Questions:

a. Determine the maximum shear stress in the column.
b. Determine the minimum shear stress in the column.
c. Determine the difference in rotation $\Delta\varphi$ between both end cross-sections of the column (in degrees) if the column length is 2.40 m.



6.12 Solid member ABC consists of two equally long parts AB and BC with circular cross-section of different diameter. The member, is fixed at A and is loaded at the free end C by a torsional moment of 200 Nm. The dimensions are shown in the figure. The shear modulus is $G = 80$ GPa.



Questions:

a. Determine the maximum shear stress.
b. Determine the rotation at B (in radians).
c. Determine the rotation at C (in degrees).

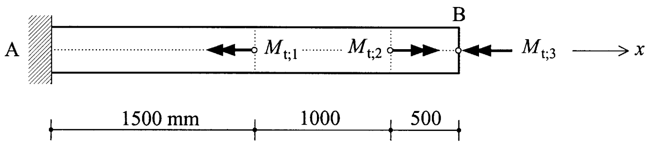
6.13: 1–3 Prismatic member AB has a solid circular cross-section and is subject to torsion by the three moments $M_{t;1}$, $M_{t;2}$ and $M_{t;3}$. The polar moment of inertia is $I_p = 2.578 \times 10^6$ mm^4. The shear modulus is $G =$ 80 GPa.

There are three different loading cases:

(1) $M_{t;1} = 5$ kNm, $M_{t;2} = 3$ kNm and $M_{t;3} = 6$ kNm.

(2) $M_{t;1} = 5$ kNm, $M_{t;2} = 9$ kNm and $M_{t;3} = 5$ kNm.

(3) $M_{t;1} = 3.2$ kNm, $M_{t;2} = 12$ kNm and $M_{t;3} = 2.4$ kNm.

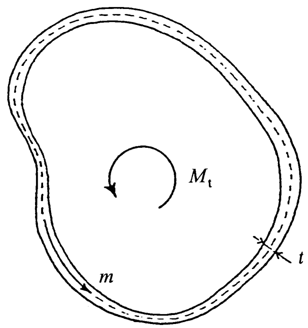


Questions:

a. Draw the M_t diagram.
b. Determine the distribution of the torsional strain χ over the length of the member and draw the χ diagram.
c. Determine the rotation at B (in degrees).

Torsion of thin-walled cross-sections (Section 6.3)

6.14 A thin-walled tube with a gradually changing wall thickness t is loaded by a torsional moment M_t.



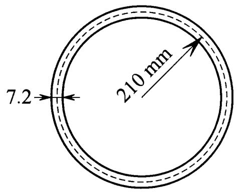
Questions:

a. What do you understand by the *shear flow* in a cross-section?
b. Prove that the shear flow is constant.
c. For the shear stress due to torsion, derive the formula below:

$$\sigma_{xm} = \frac{M_t}{2A_m t} .$$

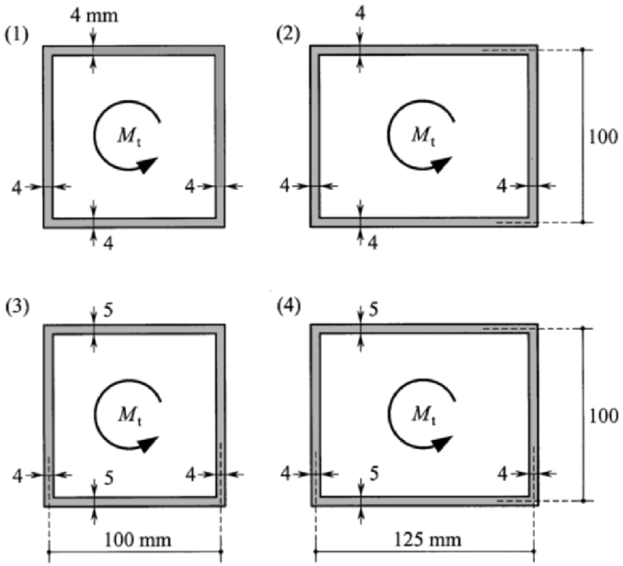
d. What does the quantity A_m mean in this formula?

6.15 A shear stress of 100 N/mm^2 is caused by a torsional moment in a thin-walled circular tube. The cross-sectional dimensions are given in the figure.



Question:
Determine the magnitude of the torsional moment.

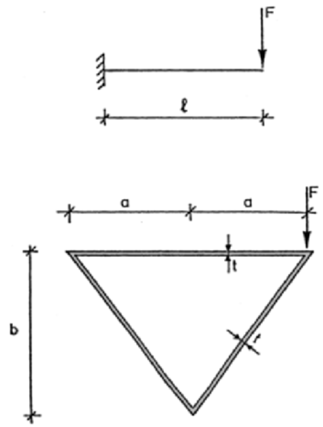
6.16: 1–4 Four thin-walled closed cross-sections transfer the same torsional moment $M_t = 1000$ Nm.



Questions:

a. Determine the shear stress distribution in the cross-section.
b. Determine the contribution of the shear stresses in the flanges to the torsional moment.
c. Determine the contribution of the shear stresses in the webs to the torsional moment.

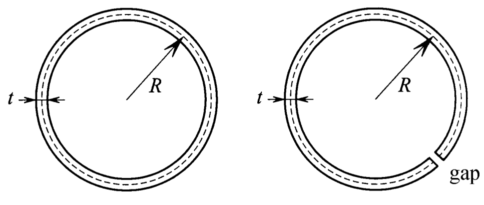
6.17 A cantilever beam with a thin-walled triangular cross-section is loaded at its free end by an eccentric force $F = 45$ kN. The cross-section has a uniform wall thickness $t = 24$ mm. For the rest use $a = 260$ mm and $b = 375$ mm.



Questions:

a. Determine the magnitude of the torsional moment in the beam.
b. Determine the maximum shear stress due to only the torsional moment.

6.18 You are given two circular thin-walled cross-sections with radius R and wall thickness t. Cross-section I is closed and cross-section II has a gap at S. The same torsional moment M_t acts in both cross-sections.



cross-section I cross-section II

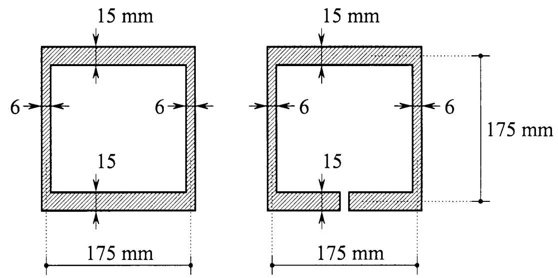
Questions:

a. Determine the expression for the maximum shear stress $\tau_{max;I}$ in cross-section I.
b. Determine the expression for the maximum shear stress $\tau_{max;II}$ in cross-section II.
c. Determine the ratio $\tau_{max;II}/\tau_{max;I}$. What does this mean numerically if $R = 60$ mm and $t = 3$ mm?

6.19 You are given two square thin-walled cross-sections: cross-section I is closed and cross-section II is open (it has a small gap at the centre of the lower flange). The wall thickness of the flanges is 15 mm and that of the webs is 6 mm. The same torsional moment $M_t = 735$ Nm is acting in both cross-sections.

Questions:

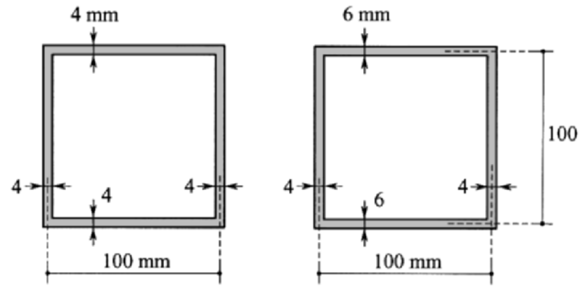
a. Draw the shear stress distribution in cross-section I.
b. Determine the maximum shear stress in cross-section I.



cross-section I cross-section II

c. Draw the shear stress distribution in cross-section II.
d. Determine the maximum shear stress in cross-section II.

6.20: 1–2 You are given two square thin-walled cross-sections.

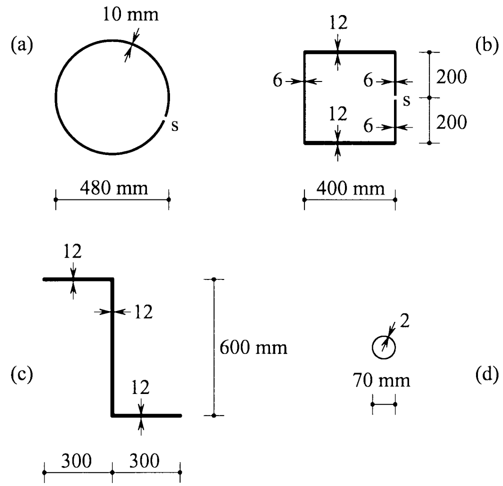


(1) (2)

Questions:

a. Determine the torsion constant of the cross-section.
b. Determine the torsion constant if the cross-section is no longer closed but has a small gap at the centre of the right-hand web.

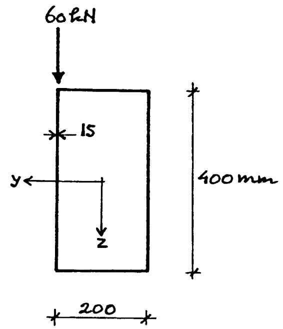
6.21 You are given three thin-walled open cross-sections (a), (b) and (c) and one thin-walled closed cross-section (d). The cross-sectional dimensions are given in the figure.



Questions:

a. Arrange (in ascending order) the open cross-sections according to the magnitude of the torsion constant.
b. Compare the torsion constant of the closed cross-section (d) with the torsion constant of the open cross-sections.

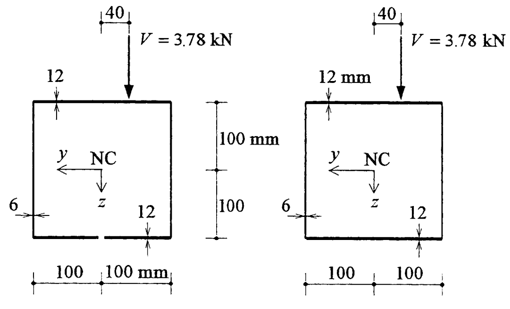
6.22 The rectangular hollow cross-section is thin-walled with uniform wall thickness of 15 mm. The cross-section has to transfer an eccentric vertical force of 60 kN. The line of application of the shear force coincides with the left web. The cross-sectional dimensions are given in the figure.



Questions:

a. Determine the shear force and the torsional moment in the cross-section.
b. Draw the shear stress distribution in the cross-section due to only the shear force. Indicate the direction of the shear stresses and include the values at a number of relevant places.
c. Determine the shear stress distribution due to only the torsional moment.
d. Determine the shear stress distribution due to the combination of shear force and torsional moment.
e. Determine the maximum shear stress and the point where this occurs.

6.23 You are given two thin-walled square cross-sections with a flange thickness of 12 mm and a web thickness of 6 mm. One cross-section is open, with a gap at the centre of the lower flange, while the other cross-section is closed. Both cross-sections have to transfer the same eccentric shear force. All necessary information can be found in the figure.

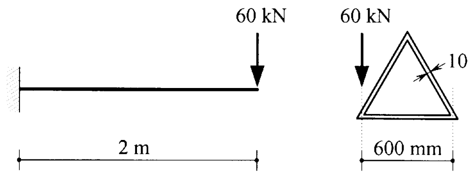


open cross-section closed cross-section

Questions:

a. For both cross-sections, sketch the shear stress distribution due to the shear force only. Indicate the direction of the shear stresses and include a number of relevant values. For each of the cross-sections indicate the location where the shear stress due to shear is a maximum and determine these values.
b. In the same way, for both cross-sections sketch the shear stress distribution due to the torsional moment only. For each of the cross-sections, indicate the location where the shear stress due to torsion is largest and determine these values.
c. For the open cross-section, indicate the location and magnitude of the maximum shear stress due to the combination of shear and torsion.
d. For the closed cross-section, indicate the location and magnitude of the maximum shear stress due to the combination of shear and torsion.

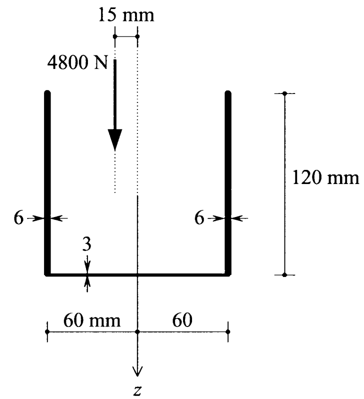
6.24 The cross-section of the cantilever beam shown is shaped like a thin-walled isosceles triangle with a uniform wall thickness. The cross-sectional dimensions are given in the figure. The beam is loaded by an eccentric force of 60 kN at its free end.



Questions with respect to the cross-section at the fixed support:

a. Sketch the shear stress diagram due to the torsional moment.
b. Sketch the shear stress diagram due to the shear force.
c. Determine the location and magnitude of the maximum shear stress.
d. Determine the normal stress diagram due to the bending moment.

6.25 You are given a thin-walled U-section. The force of 4800 N is the resultant of all shear stresses in the cross-section. The dimensions are given in the figure.

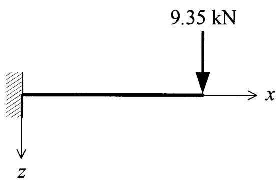


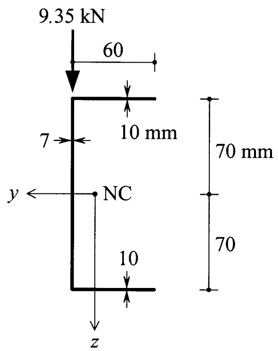
Questions:

a. Determine the magnitude and direction of the shear force and the torsional moment in the cross-section.
b. Determine the moment of inertia I_{zz}.

c. Determine the shear stress distribution due to the shear force. Sketch this distribution, indicate the direction of the shear stresses and include a number of relevant values.
d. Determine the torsion constant I_t.
e. Determine the shear stress distribution due to the torsional moment. Draw this distribution, indicate the direction of the shear stresses and include a number of relevant values.
f. Determine the location and magnitude of the maximum shear stress due to the combination of shear force and torsional moment.

6.26 A thin-walled U-section has been used for a cantilever beam. The beam is loaded at its free end by a vertical force of 9.35 kN of which the line of action coincides with the web. The dimensions are given in the figure.



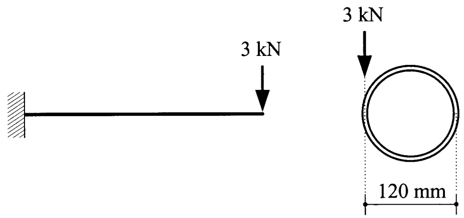


Questions with respect to the middle cross-section of the beam:

a. Determine the shear stress distribution due to the shear force. Sketch this distribution, indicate the direction of the shear stresses and include a number of relevant values.
b. Determine the magnitude and direction of the torsional moment.
c. Determine the shear stress distribution due to the torsional moment. Sketch this distribution, indicate the direction of the shear stresses and include a number of relevant values.
d. Determine the location and magnitude of the maximum shear stress in the cross-section.

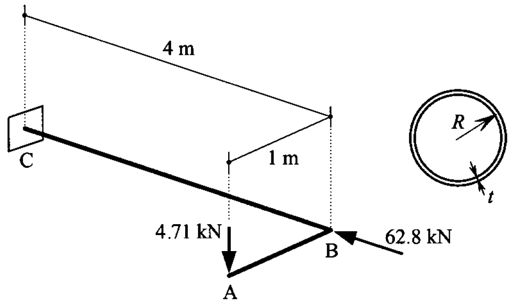
Mixed problems

6.27 A thin-walled tube is loaded as shown by an eccentric shear force of 3 kN. The tube has a circular cross-section with a diameter of 120 mm and an area of 1250 mm^2.



Question:
Determine the maximum shear stress in the cross-section.

6.28 The bent bar type structure in the horizontal plane is constructed of a thin-walled circular tube with radius $R = 100$ mm and wall thickness $t = 5$ mm. The structure is loaded by a vertical force at A and a horizontal force at B. Both forces act on the centre lines of the structure.



Questions with respect to the cross-section at the fixed support C:

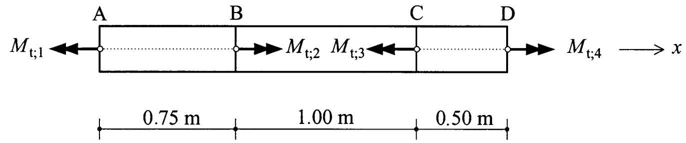
a. Determine the shear stress distribution due to the torsional moment.
b. Determine the shear stress distribution due to the shear force.
c. Determine the location and magnitude of the maximum shear stress.
d. Determine the normal stress distribution due to the normal force.
e. Determine the normal stress distribution due to the bending moment.
f. Determine the locations and magnitudes of the maximum tensile and maximum compressive stress, and the location of the neutral axis.

6.29 A prismatic circular tube with an internal diameter of 63 mm and a wall thickness of 3 mm is loaded by a torsional moment M_t. The torsional strain of the tube may not exceed $\bar{\chi} = 0.25°/\text{m}$ and the shear stress may not exceed $\bar{\tau} = 20$ MPa. The shear modulus is $G = 38$ GPa.

Questions:

a. Determine the value of M_t for which the limiting value $\bar{\chi}$ is reached.
b. Determine the value of M_t for which the limiting value $\bar{\tau}$ is reached.
c. Determine the maximum torsional moment that the tube can transfer.

6.30: 1–2 Member ABCD consists of three parts with different torsional stiffnesses:
$GI_t^{(AB)} = 4\ \text{kNm}^2$, $GI_t^{(BC)} = 1.6\ \text{kNm}^2$ and $GI_t^{(CD)} = 2.5\ \text{kNm}^2$.

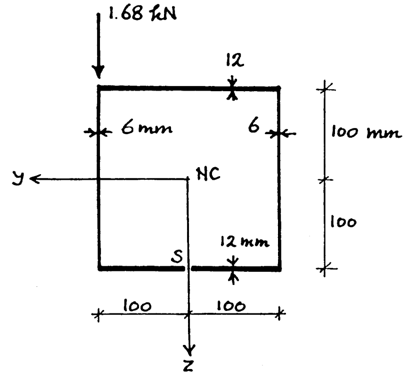


There are two loading cases:
(1) $M_{t;1} = 120$ Nm, $M_{t;2} = 40$ Nm, $M_{t;3} = 20$ Nm and $M_{t;4} = 100$ Nm.
(2) $M_{t;1} = 80$ Nm, $M_{t;2} = 120$ Nm, $M_{t;3} = 90$ Nm and $M_{t;4} = 50$ Nm.

Questions:

a. Plot the M_t diagram.
b. Determine the torsional strain distribution across along the length of the member (the χ diagram).
c. Determine the change in rotation $\Delta\varphi_x$ across AB, BC and CD.
d. Determine the rotation of cross-section B with respect to the cross-section at A.
e. Determine the rotation of cross-section C with respect to the cross-section at A.
f. Determine the rotation of cross-section D with respect to the cross-section at A.

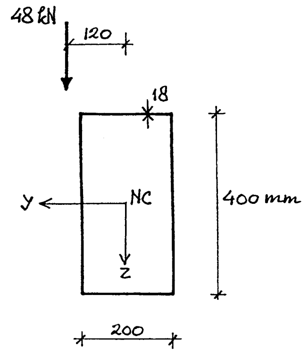
6.31 The dimensions of the square open cross-section, with a gap at S, are given in the figure. An eccentric shear force of 1.68 kN is transferred by the cross-section. The line of application of the shear force coincides with the left web.



Questions:

a. Determine the shear force and the torsional moment in the cross-section.
b. Draw the shear stress distribution due to only the shear force. Indicate the direction of the shear stresses and include a number of relevant values.
c. In the same way draw the shear stress distribution due to only the torsional moment.
d. Determine the maximum shear stress due to the eccentric shear force and the location(s) where it occurs.

6.32 The thin-walled tube with rectangular cross-section and uniform wall thickness of 18 mm, has to transfer an eccentric vertical force of 48 kN as shown in the figure. Also the cross-sectional dimensions follow from the figure.



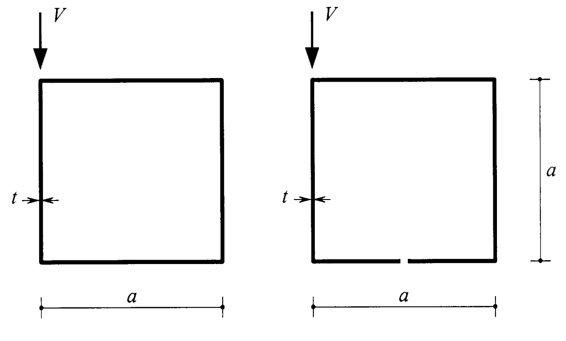
Questions:

a. Determine the shear force and the torsional moment in the cross-section.
b. Prove that the moments of inertia of the cross-section are:

$$I_{yy} = 168 \times 10^6 \text{ mm}^4 \text{ and } I_{zz} = 480 \times 10^6 \text{ mm}^4.$$

c. Draw the shear stress distribution in the cross-section due to the shear force only. Indicate the direction of the shear stresses and include the values in a relevant number of places.
d. In the same way draw the shear stress distribution due to only the torsional moment.
e. Sketch the shear stress distribution due to the combination of shear force and bending moment.
f. Where in the cross-section is the shear stress a maximum and how large is this maximum value?

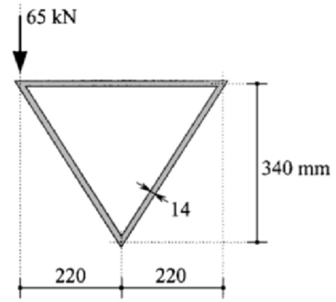
6.33 You are given two square thin-walled cross-sections with uniform wall thickness t. Cross-section I is closed and cross-section II is open with a small gap at the centre of the lower flange. The (shear) force V shown is the resultant of all shear stresses in the cross-section. In the calculation use $V = 31$ kN, $a = 360$ mm and $t = 20$ mm.



Questions:

a. Determine the location and magnitude of the maximum shear stress in cross-section I.
b. Determine the location and magnitude of the maximum shear stress in cross-section II.

6.34 The force of 65 kN shown is the resultant of all shear stresses in the triangular cross-section. The cross-section is thin-walled with a uniform wall thickness of 14 mm.



Questions:

a. Determine the magnitude and direction of the shear force and the torsional moment in the cross-section.
b. Determine the shear stress distribution due to the shear force only.
c. Determine the shear stress distribution due to the torsional moment only.
d. Determine the maximum shear stress in the cross-section and the place where it occurs.

7 Deformation of Trusses

In trusses all members are subject to extension. If the truss is statically determinate, all member forces follow directly from the equilibrium, and the change in length of all members can be determined using Section 2.6.

This chapter describes how to determine the joint displacements in a truss from the change in length of the members. Here we assume that the deformation of the truss is exclusively the result of the change in length of the members and not of any deformation in the joints. Hence the joints are considered non-deformable. The analysis applies only if the deformations are small, i.e. the change in length of the members must be small with respect to the original length of the member. In Section 7.1 we show that this condition is nearly always met in practice. The section continues by assessing the influence of a small member rotation on the joint displacements.

Section 7.2 takes a graphical approach to determine the joint displacements using a so-called *Williot diagram*. This method is successfully only on the condition that it is always possible to find a joint that is directly linked to two other joints for which the displacements are known.

If this is not the case, the calculation becomes more complicated. Such situations are covered in Sections 7.3 and 7.4; in Section 7.3 by means of a

Williot diagram corrected with a *rigid-body rotation*, and in Section 7.4 by means of a *Williot diagram with a Mohr correction diagram*, also known as a *Williot–Mohr diagram*. The different methods are illustrated by some examples.

Chapter 7 ends with a number of problems in Section 7.5.

7.1 The behaviour of a single truss member

In this section we show that the change in length of a truss member is always very small in practice. Next we will point out again that the small displacement along the arc of an circle, due to a small rotation, can be replaced by an equal displacement along the tangent to the arc.[1]

7.1.1 Change in length of a truss member

When determining the changes in length $\Delta\ell$ we assume that

- the members are prismatic;
- the material exhibits linear elastic behaviour and therefore follows Hooke's law.

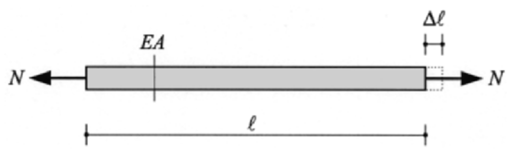


Figure 7.1 The change in length $\Delta\ell$ of a prismatic member loaded by extension is $\Delta\ell = N\ell/EA$.

For the prismatic and linear elastic member in Figure 7.1, subject to extension, we have (see Section 2.6.1)

$$\Delta\ell = \frac{N\ell}{EA}.$$

[1] See also *Engineering Mechanics*, Volume 1, Section 15.3.2.

If the change in length with respect to the original member length has to be small, this means that

$$\frac{\Delta\ell}{\ell} = \frac{N}{EA} = \varepsilon \ll 1.$$

Hence the strain in the member has to be small.

Below we will look at steel, aluminium and wood and to which degree they will be strained at serviceability level.

Steel
Steel Fe360 has the modulus of elasticity

$$E = 210\ \text{GPa} = 210 \times 10^3\ \text{N/mm}^2,$$

and a yield point

$$f_y = 235\ \text{MPa} = 235\ \text{N/mm}^2.$$

The yield strain is (see Figure 7.2)

$$\varepsilon_y = \frac{f_y}{E} = \frac{235\ \text{N/mm}^2}{210 \times 10^3\ \text{N/mm}^2} \approx 1.1 \times 10^{-3} = 1.1‰.$$

A steel member Fe360, one metre long, will therefore yield at an elongation of approximately 1.1 mm. In practice we will remain clearly below the yield point so that the strains are less than 1‰.

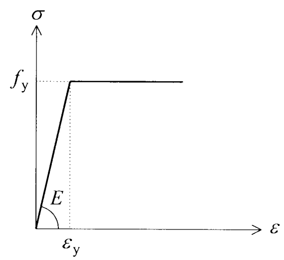


Figure 7.2 The σ-ε diagram for an elastic-plastic material with yield point f_y and yield strain ε_y.

Aluminium

The modulus of elasticity of aluminium is about one third of that of steel:

$$E = 70 \text{ GPa}$$

but the ultimate stresses are lower. At the 0.2% offset yield strength[1]

$$f_{0.2} = 150 \text{ MPa}$$

the strain is

$$\varepsilon = \frac{f_{0.2}}{E} = \frac{150 \text{ N/mm}^2}{70 \times 10^3 \text{ N/mm}^2} \approx 2.1 \times 10^3 = 2.1‰.$$

The deformation of an aluminium truss will therefore roughly be twice as large as that of a steel truss, but is still small.

Wood

For wood, with a modulus of elasticity E of about 10 MPa and a tensile strength $f_\mathrm{t} = 50$ MPa, the fracture strain is

$$\frac{f_\mathrm{t}}{E} = \frac{50 \text{ N/mm}^2}{10 \times 10^3 \text{ N/mm}^2} = 5 \times 10^{-3} = 5‰.$$

The deformation of a wooden truss is about five times that of a steel truss, but is still small.

It can be assumed that at serviceability level the condition that the deformations are small is practically always met.

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1 See Section 1.2.

7.1.2 Rotation of a truss member

If the joints in a truss move, the members will generally also rotate. For small displacements, the rotations are also small. This implies a linear relationship between the member rotation and the associated joint displacements, as shown below.

In Figure 7.3a member AB with length $\Delta\ell$ is subject to a rotation θ about A. As a result, B moves along a circle with its centre at A and ends up at B′. In Figure 7.3 the displacements are expressed in ℓ and θ.

If the rotation θ remains small, e.g. $\theta < 0.05$ rad $\approx 3°$, then

$$\sin\theta \approx \tan\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1,$$

and the arc cannot be distinguished from a straight line normal to AB, the member axis in its original position. For small rotations, the displacement along the arc of the circle can therefore be replaced by an equal displacement along the tangent of the circle[1] (see Figure 7.3b).

We will use this property frequently below.

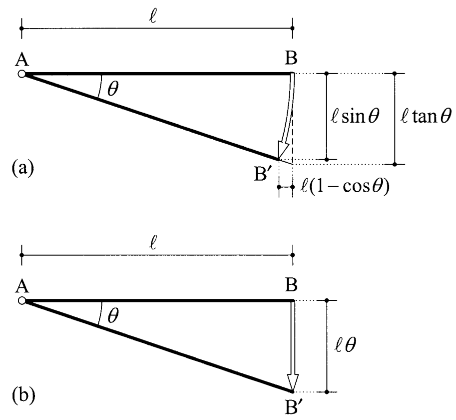

Figure 7.3 (a) When member AB rotates about A, B undergoes a circular movement about A and ends up at B′. (b) For small rotations, the displacement along the arc can be replaced by a displacement along the tangent to the circle.

7.2 Williot diagram

In this section, we use a number of examples to present a graphical method for determining the joint displacements in a truss due to a change in length of the truss members.

1 See also *Engineering Mechanics*, Volume 1, Section 15.3.2.

Table 7.1

Member i	$N^{(i)}$ (kN)	$\ell^{(i)}$ (mm)	$EA^{(i)}$ (kN)	$\Delta\ell^{(i)}$ (mm)
1	$+20$	2000	40×10^3	$+1$
2	$-10\sqrt{2}$	$2000\sqrt{2}$	$20\sqrt{2} \times 10^3$	$-\sqrt{2}$

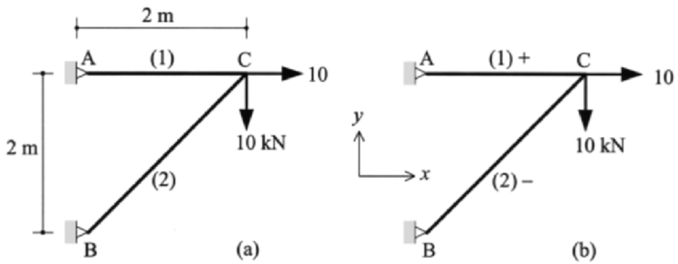

Figure 7.4 (a) A simple truss consisting of two members. (b) The tension and compression members for the given load.

Example 1
Figure 7.4a shows a simple truss, consisting of two members. The dimensions and loading are given in the figure. By convention, the joints are labelled by capital letters and the members by numbers. The two members have different axial stiffnesses:

$$EA^{(1)} = 40 \times 10^3 \text{ kN} \text{ and } EA^{(2)} = 20\sqrt{2} \times 10^3 \text{ kN}.$$

Question:
Determine the displacement of joint C for the given load.

Solution:
The calculation consists of two phases. In the first phase, we determine the member forces and changes in length, and in the second the displacement of joint C.

We first have to calculate the member forces. This calculation is left to the reader. The result is shown in the second column of Table 7.1.

In Figure 7.4b the truss has been shown again indicating for each member whether it is in tension or compression.

With member length $\ell^{(i)}$ in the third column and the axial stiffness $EA^{(i)}$ in the fourth column, we can determine the change in length $\Delta\ell^{(i)}$ for member (i):

$$\Delta\ell^{(1)} = \frac{N^{(1)}\ell^{(1)}}{EA^{(1)}} = \frac{(+20 \text{ kN})(2000 \text{ mm})}{40 \times 10^3 \text{ kN}} = +1 \text{ mm},$$

$$\Delta\ell^{(2)} = \frac{N^{(2)}\ell^{(2)}}{EA^{(2)}} = \frac{(-10\sqrt{2} \text{ kN})(2000 \text{ mm})}{20\sqrt{2} \times 10^3 \text{ kN}} = -\sqrt{2} \text{ mm}.$$

The result is included in the last column of Table 7.1.

Comments:

- Tensile forces ($N > 0$) cause a lengthening of the member ($\Delta\ell > 0$) and compressive forces ($N < 0$) a shortening ($\Delta\ell < 0$). An error in the sign therefore immediately leads to incorrect joint displacements. *So be aware of the signs!*
- Also pay close attention to the *units* in which you are working. It is recommended that you determine the units in which you wish to perform the calculation beforehand. Here we have chosen kN units and mm units.

The second phase consists of calculating the displacement of joint C. Joint C is fixed to joints A and B via the members (1) and (2), for which we know the displacements: zero in this case.

To find the displacement of joint C, we temporarily cut the members at C, and let them keep their original direction. Next we plot at C the changes of length of the members (1) and (2) along the member axes.

Member (1) elongates by 1 mm, as a result of which C on member (1) moves 1 mm to the right. Member (2) shortens by $\sqrt{2}$ mm, as a result of which C on member (2) moves $\sqrt{2}$ mm to the left and down. These displacements have been exaggerated in Figure 7.5a, and drawn to the following scale: 1 square $\equiv$ 0.5 mm.

After deformation, the members are no longer connected: there is a gap. The location of C on member (1) no longer coincides with the location of C on member (2). This is the result of keeping the members (1) and (2) in their original directions. In reality, these directions are not fixed but free: the members may still rotate about A and B. The next step therefore is to release the directions of the members (1) and (2) and to rotate them about A and B respectively. By rotating member (1) about A, end C moves along

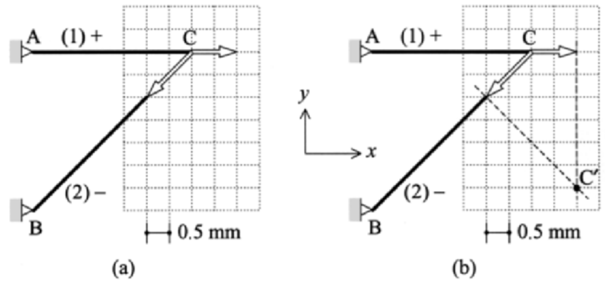

Figure 7.5 (a) The members are isolated from one another in C and their directions are temporarily fixed. Member AC lengthens so that C displaces to the right. Member BC shortens so that C moves downwards to the left. The members are no longer contiguous. (b) Next, members AC and BC are rotated about A and B respectively until they join up again in C′. $\overline{CC'}$ is now the displacement vector of joint C. The displacements in the figure are shown some 570 times larger than the structural dimensions.

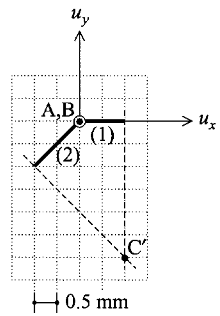

Figure 7.6 If the joint displacements are drawn in a separate figure, we can create a Williot diagram or displacement diagram.

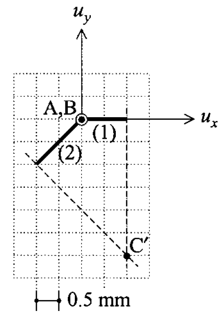

Figure 7.6 If the joint displacements are drawn in a separate figure, we can create a Williot diagram or displacement diagram.

Table 7.2

Member i	$N^{(i)}$ **(kN)**	$\ell^{(i)}$ **(mm)**	$EA^{(i)}$ **(kN)**	$\Delta\ell^{(i)}$ **(mm)**
1	$+20$	2000	40×10^3	$+1$
2	$-10\sqrt{2}$	$2000\sqrt{2}$	$20\sqrt{2} \times 10^3$	$-\sqrt{2}$
3	-10	2000	40×10^3	-0.5
4	0	$2000\sqrt{2}$	40×10^3	0
5	-10	2000	40×10^3	-0.5
6	$+10\sqrt{2}$	$2000\sqrt{2}$	$20\sqrt{2} \times 10^3$	$+\sqrt{2}$

a line perpendicular to AC, indicated in Figure 7.5b by a dashed line. In the same way, the rotation of member (2) about B gives a displacement of C along the perpendicular to BC. Both members are rotated until they meet in C′, the intersection of both dashed perpendiculars.

In Figure 7.5b, $\overline{\mathrm{CC'}}$ is therefore the displacement vector[1] of joint C due to the change in length of members (1) and (2).

Since the displacements are very small with respect to the structural dimensions, it is not possible to draw the structure and the displacements to the same scale. In Figure 7.5 this has been solved by choosing different scales for structure and displacements: the structural dimensions have been reduced and the displacements have been magnified. However, for trusses with more than two members this approach is not practicable. For this reason, the displacements are drawn in a separate figure (see Figure 7.6); this (*relative*) *displacement diagram* is also referred to as a *Williot diagram*.[2]

The joints that do not move, in this case A and B, are at the origin of the diagram, indicated by an encircled dot. The changes in length of the members are represented by heavy line segments without an arrow head.[3] Each line segment is labelled by the member number to which the change in length refers. The displacements due to a rotation of the members are represented by dashed lines. The displaced joints are given an accent in the figure.

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1 The displacement vector has not been drawn separately in the figure.

2 Named after the French engineer Williot (1843–1907) who presented this method in 1877.

3 The arrow directions are omitted in the Williot diagram as they can be confusing in certain situations.

The magnitude of the displacement can be found in the Williot diagram by *measurement* or *calculation*. From the Williot diagram in Figure 7.6, using the squares, we can read off that

$$u_{x;\mathrm{C}} = +1 \text{ mm} \text{ and } u_{y;\mathrm{C}} = -3 \text{ mm}.$$

Example 2
For the truss in Figure 7.7a is the axial stiffness of the diagonals $20\sqrt{2}$ MN and of the other members 40 MN.

Question:
Determine the joint displacements.

Solution (using kN and mm units):
Calculating the forces in and changes in length of the members is left to the reader. The results are shown in Table 7.2.

In Figure 7.7b, the truss has been drawn again. The figure shows whether each member is in tension or compression, or is a zero-force member.

When drawing the Williot diagram, we always look for a joint that is connected to two other joints for which the displacements are known.

At first we can determine the displacement of joint C; joint C is connected, via the members (1) and (2), to the joints A and B for which the displacements are zero. Having found the displacement of joint C, we can determine the displacement of joint D; joint D is connected to the joints B and C via the members (3) and (4). And finally we can determine the displacement of E.

The order in which we plot the joint displacements in the Williot diagram is therefore A, B → C → D → E.

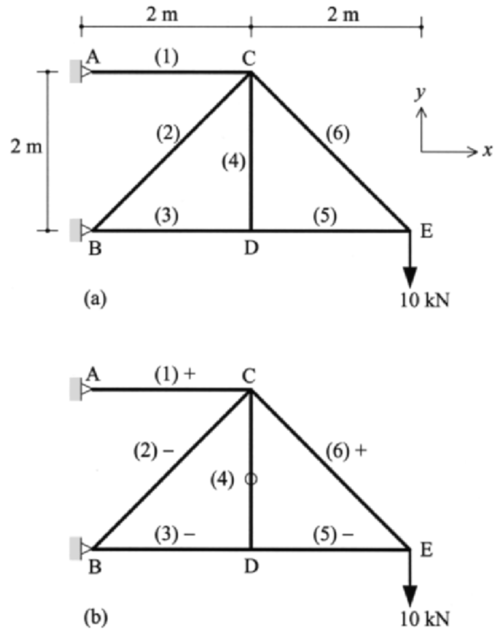

Figure 7.7 (a) A truss, loaded by a vertical force at E. (b) The tension, compression and zero-force members for the given load.

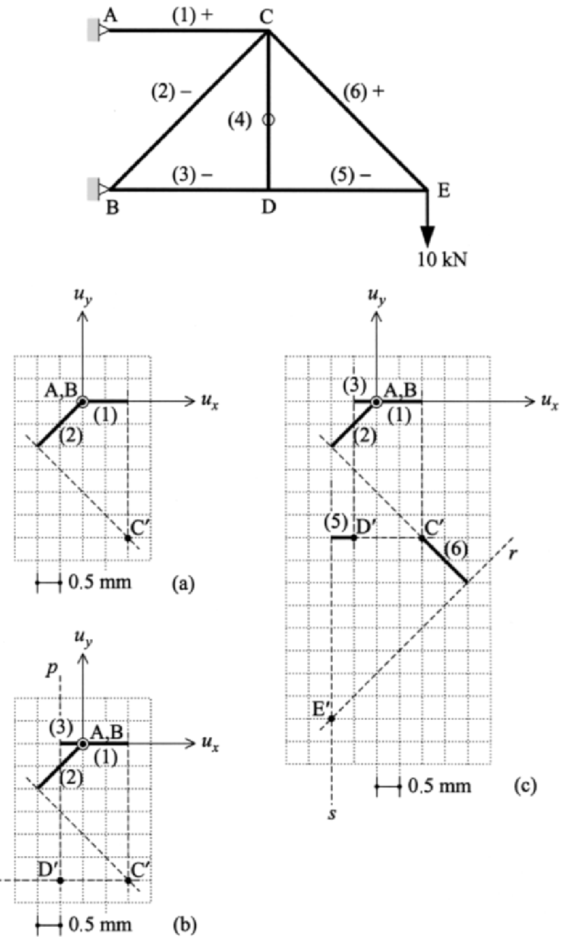

Figure 7.8 The various stages in drawing the Williot diagram. From fixed points A and B the order is (a) C, (b) D and (c) E.

Joint C

The configuration of part ABC of the truss and the changes in length of the members (1) and (2) are equal to that of the truss in Example 1. The Williot diagram for joint C in Figure 7.8a is therefore equal to that in Example 1 and is found in exactly the same way.

Joint D

Joint D is connected to B via member (3). Member (3) shortens by 0.5 mm. In the Williot diagram, D therefore moves 0.5 mm (1 square) to the left with respect to B (see Figure 7.8b). By rotating member (3) about B, D ends up somewhere on the dashed vertical line p.

Joint D is also connected to C via member (4), or properly speaking, to the displaced joint C′. Since member (4) is a zero-force member, D does not move vertically with respect to C′. Member (4) can however rotate about C′. This means that D in the Williot diagram must be somewhere on the dashed horizontal line q through C′. The new position of joint D is found as the intersection D′ of dashed lines p and q.

Note that when drawing the Williot diagram we need Figure 7.7b. From this figure we can read the direction of the current member and tell whether the member in question lengthens or shortens.

Joint E

Members (5) and (6) connect joint E to joints C and D, for which we have already determined the displacements.

Member (5) lengthens. In the Williot diagram, E moves with respect to C′, by a distance of $\sqrt{2}$ mm downwards to the right (see Figure 7.8c). After member (5) rotates about C′, E ends up somewhere on the dashed oblique line r. Member (6) shortens. In the Williot diagram, E moves with respect to D′ by 0.5 mm to the left. After member (6) rotates about D′, E ends up somewhere on the dashed vertical line s. The new position of E is found as

Table 7.3

Joint	u_x (mm)	u_y (mm)
C	+1	−3
D	−0.5	−3
E	−1	−7

the intersection E′ of the dashed lines r and s.

Using the square grid, we can read off the joint displacements directly from the Williot diagram in Figure 7.8c. The values are collected in Table 7.3.

To gain an impression of the deformation of the truss, we have drawn the displacements and structural dimensions on different scales in Figure 7.9; here the displacements are drawn 200 times as large as the structural dimensions.

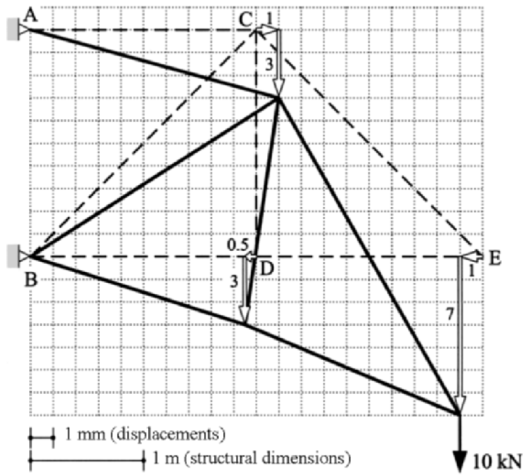

Figure 7.9 The deformed truss; the displacements have been scaled up 200 times as large as the structural dimensions.

Example 3

The truss in Figure 7.10a is dimensioned so that for the given load, all members have the same strain (in an absolute sense): $|\varepsilon| = 1/1500$.

Question:
Determine the joint displacements.

Solution:
If the strain is given, we have to know only the sign of the normal force[1] to be able to determine the change in length, as $\Delta\ell = \varepsilon\ell$. Figure 7.10b shows

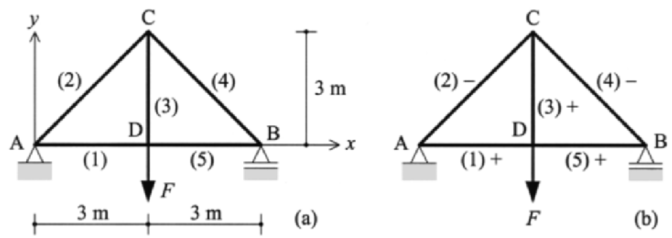

Figure 7.10 (a) A truss with (b) the tension and compression members due to the given load.

1 Its magnitude is not relevant now.

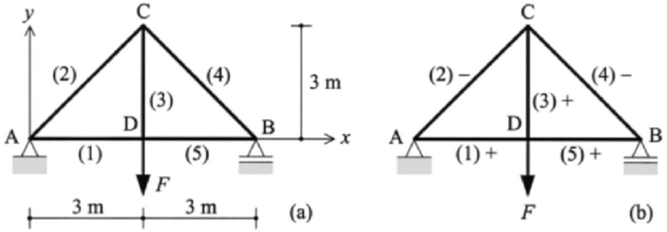

Figure 7.10 (a) A truss with (b) the tension and compression members due to the given load.

Table 7.4

Member i	$N^{(i)}$ (sign)	$\ell^{(i)}$ (mm)	$\Delta\ell^{(i)}$ (mm)
1	+	3000	+2
2	–	$3000\sqrt{2}$	$-2\sqrt{2}$
3	+	3000	+2
4	–	$3000\sqrt{2}$	$-2\sqrt{2}$
5	+	3000	+2

the tension and compression members in the truss. The changes in length are shown in Table 7.4.

When drawing the Williot diagram, we always look in a truss for a joint that is connected to two other joints for which the displacements are known. Here this is not possible since B is a roller and can move horizontally.

The solution is found in the members (1) and (5) that are in the same line, and connect joint A (hinged support) with joint B (roller support). Due to the elongation of members (1) and (5), the roller B will move to the right along the horizontal roller track by a distance $\Delta\ell^{(1)} + \Delta\ell^{(2)}$. The fact that D can still move vertically is irrelevant, as explained below in the Williot diagram in Figure 7.11a.

Joint D
Member (1) lengthens by 2 mm, so D moves 2 mm (two squares in the Williot diagram) to the right with respect to fixed point A. After member (1) rotates about A, the new location of D ends up somewhere on the dashed vertical line p. At this stage it is not possible to say anything else about the final location of D.

Joint B
Member (5) lengthens by 2 mm. As a result, B moves 2 mm (two squares) to the right with respect to D′, i.e. 2 mm to the right with respect to dashed line p. After of member (5) rotates about D′, the new location of B must be on the dashed vertical line q. Note that the location of dashed line q can be found without knowing the vertical displacement of D.

B is located on the horizontal roller track and can move only horizontally with respect to A. In the Williot diagram B must therefore be located on the dashed horizontal line r through A. The final location of B is now found as the intersection B′ of the lines q and r.

Conclusion: *If joints A and B are connected by two (or more) members in the same line, and we look for the displacement of A with respect to B (or vice versa), then the members between A and B can be considered a continuous member with a change in length that is equal to the sum of the changes in length of the individual members.*

This means that, in the Williot diagram, the changes in length of members which are in the same line can be plotted directly behind one another.

In the Williot diagram in Figure 7.11b, the displacements have been plotted in the order A → B → C → D.

Joint C
Joint C is connected to joints A and B via the members (2) and (4). Member (2) shortens by $2\sqrt{2}$ mm, so C moves downwards to the left with respect to A (diagonally across two squares). After member (2) rotates about A, the location of C ends up somewhere on dashed line s.

Member (4) shortens by $2\sqrt{2}$ mm. With respect to B′, C moves downwards to the right (diagonally across two squares). After member (5) rotates, C ends up somewhere on dashed line t. The new position of C is found as the intersection C′ of lines s and t.

Joint D
Joint D is connected to joints A and C by members (1) and (3). The displacements of joints A and C are known Due to the lengthening of member (1), D moves with respect to A by 2 mm (two squares) to the right. After rotating member (1) about A, the new location of D ends up somewhere on the dashed vertical line p. This part of the Williot diagram was drawn previously. New is now the effect of member (3).

Due to the lengthening of member (3), D moves with respect to C′ by 2 mm (two squares) downwards. After rotating member (3) about C′, the new

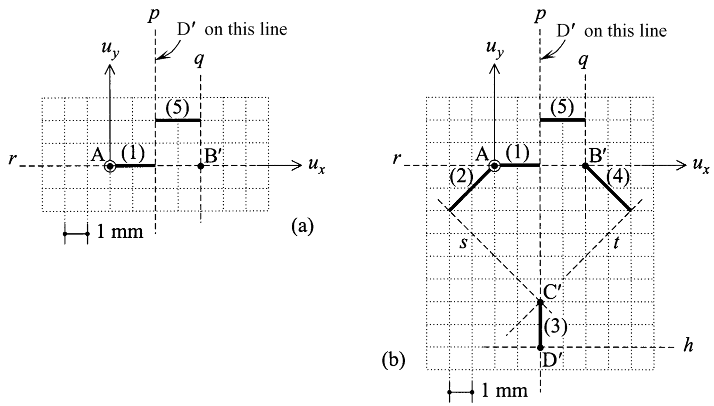

Figure 7.11 (a) Since B can move only along the horizontal roller track, B′, the new position of B, must lie on the horizontal line r through fixed point A in the Williot diagram. The changes in lengths of the members (1) and (5), which are in the same line, can be drawn directly behind one another in the diagram. (b) The full Williot diagram for the joint displacements, to be constructed from A and B in the order C and D.

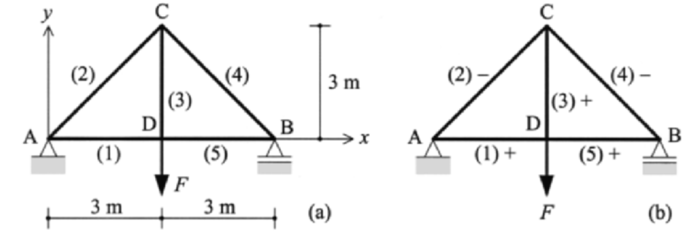

Figure 7.10 (a) A truss with (b) the tension and compression members due to the given load.

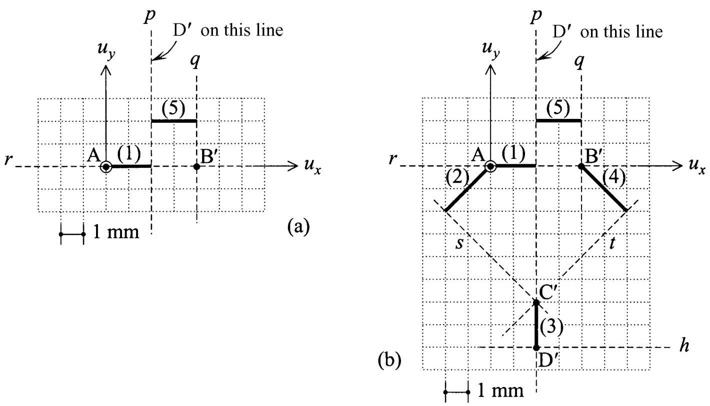

Figure 7.11 (a) Since B can move only along the horizontal roller track, B′, the new position of B, must lie on the horizontal line r through fixed point A in the Williot diagram. The changes in lengths of the members (1) and (5), which are in the same line, can be drawn directly behind one another in the diagram. (b) The full Williot diagram for the joint displacements, to be constructed from A and B in the order C and D.

location of D will be somewhere on the dashed horizontal line h. The final location of D is the intersection D′ of the lines p and h.

Note: Point D′ in the Williot diagram can of course also be found via the members (5) and (3) that connect joint D with joints B and C. This is left to the reader.

Using the square grid, the joint displacements can be read directly off the Williot diagram. The values are shown in Table 7.5.

Table 7.5

Joint	u_x (mm)	u_y (mm)
B	+4	0
C	+2	−6
D	+2	−8

Example 4

Member AB in Figure 7.12 is supported on a hinge at A and a roller with vertical roller track at B. The member is loaded by a vertical force $F = 50$ kN at B. The axial stiffness of the member is $EA = 43$ MN.

Question:
Determine the settlement of B.

Solution:
With

$$\cos\beta = \frac{2}{\sqrt{5}},$$

the normal force N and the elongation $\Delta\ell$ of member AB are

$$N = -\frac{F}{\cos\beta} = -\tfrac{1}{2}F\sqrt{5} = -25\sqrt{5}\text{ kN},$$

$$\Delta\ell = \frac{N\ell}{EA} = \frac{(-25\sqrt{5}\text{ kN})(2\sqrt{5}\text{ m})}{43\times 10^3\text{ kN}} = -5.81\times 10^{-3}\text{ m}.$$

Subject to the compressive force, member AB shortens by $|\Delta\ell| = 5.81$ mm (see Figure 7.13a). As a result, the shortened member AB ends up with its end B at P, no longer on the vertical roller track. As the shortened member rotates about A, B moves along dashed line p. Finally the end B of the shortened member AB is back on the roller track at B′. For the vertical displacement of B we now find

$$w_\text{B} = \frac{|\Delta\ell|}{\cos\beta} = \tfrac{1}{2}|\Delta\ell|\sqrt{5} = \tfrac{1}{2}\sqrt{5}\times(5.81\text{ mm}) = 6.5\text{ mm}.$$

The answer was found by drawing the displacements in Figure 7.13a. The displacements can also be drawn in a Williot diagram, as shown in Figure 7.13b. A does not move and is found at the origin (the encircled dot) of the Williot diagram. Due to the shortening $|\Delta\ell|$ of AB, B moves with respect to A to the right and downwards. After the member rotates about A, B moves along the dashed line p. The vertical roller B allows only a vertical displacement at B. In the Williot diagram, this is a displacement

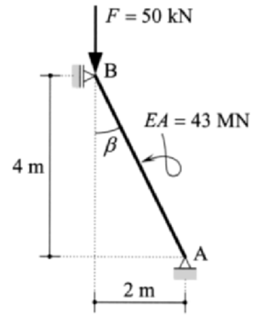

Figure 7.12 Member AB is supported on a hinge in A and a roller with vertical roller track in B and is loaded by a vertical force in B.

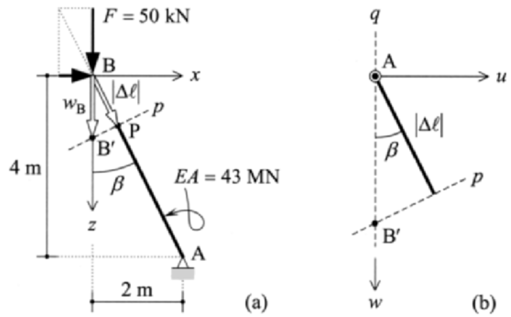

Figure 7.13 (a) Subject to the compressive force in the member, AB shortens by $|\Delta\ell|$. After rotating about A, B returns to B′ on the vertical roller track. (b) The Williot diagram for the displacement of joint B.

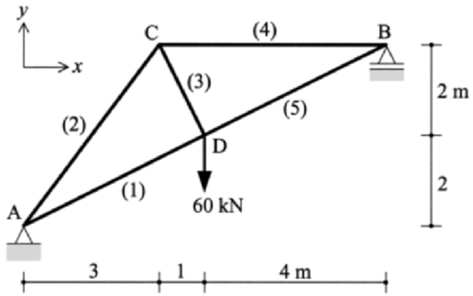

Figure 7.14 A truss with hinged support A and roller support B with horizontal roller track. The supports are at uneven heights. The members (1) and (5) are in the same line.

Table 7.6

Member i	$N^{(i)}$ (kN)	$\ell^{(i)}$ (mm)	$EA^{(i)}$ (kN)	$\Delta\ell^{(i)}$ (mm)
1	$+18\sqrt{5}$	$2000\sqrt{5}$	$24\sqrt{5}\times 10^3$	$+1.5\sqrt{5}$
2	-60	5000	60×10^3	$+5$
3	$+24\sqrt{5}$	$1000\sqrt{5}$	$24\sqrt{5}\times 10^3$	$+\sqrt{5}$
4	-60	5000	60×10^3	$+5$
5	$+30\sqrt{5}$	$2000\sqrt{5}$	$24\sqrt{5}\times 10^3$	$+2.5\sqrt{5}$

along the dashed vertical line q through the origin of the diagram. The final location of B is the intersection B′ of lines p and q.

From the Williot diagram we find, as derived earlier,

$$w_{\mathrm{B}} = \frac{|\Delta\ell|}{\cos\beta}\,.$$

Example 5

For the truss in Figure 7.14, the axial stiffness of the even members is 60 MN and that of the odd members is $24\sqrt{5}$ MN.

Question:
Determine the joint displacements for the given load.

Solution (using kN and mm units):
The calculation of the member forces and the changes in length of the members is again left to the reader. The result is given in Table 7.6.

In Figure 7.15, the tension and compression members in the truss are indicated by means of plus and minus signs so that the Williot diagram can be drawn quickly.

To start drawing the Williot diagram, we have the problem that there is no joint that is directly connected to two other joints for which the displacements are known.

The solution is found in the members (1) and (5) that are in the same line, connecting joint A (hinged support) with joint B (roller support). For the displacement of B with respect to A, the changes in length of members (1) and (5) can be plotted in the Williot diagram directly behind one another (see Example 3).

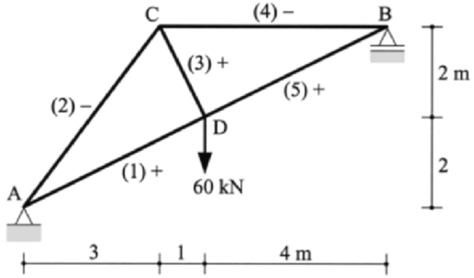

Figure 7.15 The tension and compression members for the given load.

Joint B
As a result of the elongation of members (1) and (5), B moves with respect to A to the right and upwards by a distance $\Delta\ell^{(1)} + \Delta\ell^{(5)}$ (see Figure 7.16):

$$\Delta\ell^{(1)} + \Delta\ell^{(5)} = (1.5\sqrt{5} + 2.5\sqrt{5})\ \text{mm} = 4\sqrt{5}\ \text{mm}.$$

After member (1) rotates, the new location for D is somewhere on dashed line p and that for B is somewhere on dashed line q. B can move only along the horizontal roller track. This displacement is represented in the Williot diagram by the dashed horizontal line r through A. The final location of B is therefore the intersection B′ of the lines q and r.

Now that the displacement of B is known, we can determine the displacement of C, and subsequently from A and C (or B and C) determine the displacement of D (see the Williot diagram in Figure 7.16).

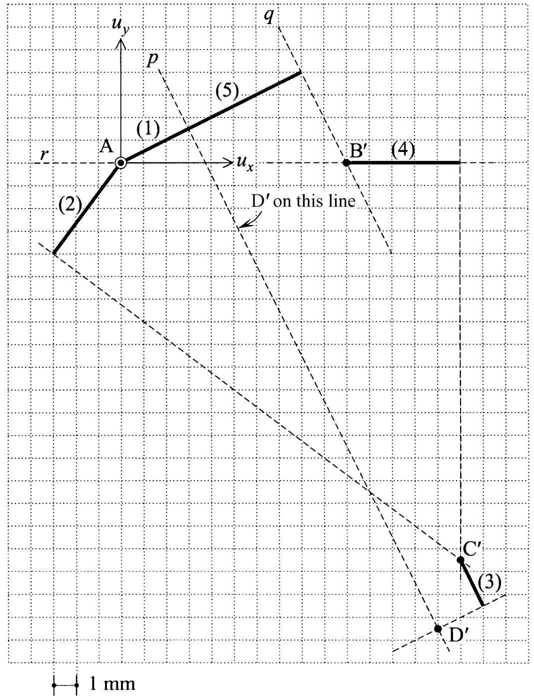

Figure 7.16 The Williot diagram for the joint displacements. In order to find the displacement of B we can plot the change in length of the members (1) and (5), which are in the same line, directly behind one another in the diagram. The Williot diagram can then be completed from A and B in the order C and D.

Table 7.7

Joint	u_x (mm)	u_y (mm)
B	+10	0
C	+15	−17.5
D	+14	−20.5

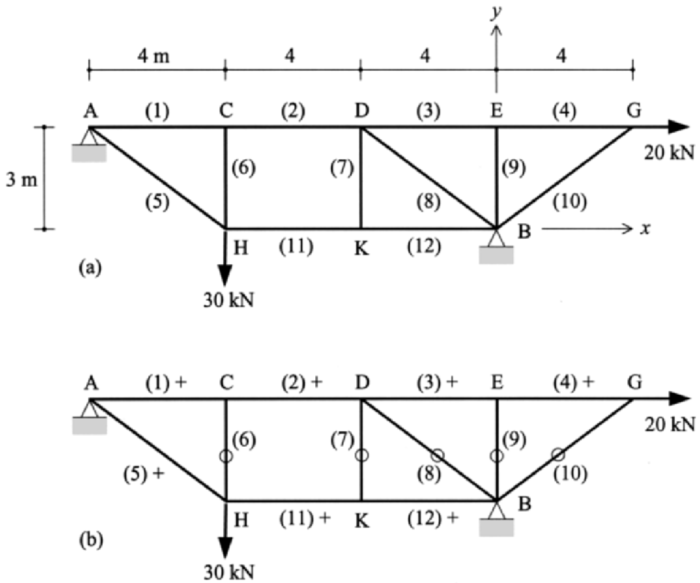

Figure 7.17 (a) A truss with (b) the tension, compression and zero-force members for the given load.

A summary of the joint displacements is given in Table 7.7.

Example 6

The truss in Figure 7.17a is dimensioned in such a way that for the given load in all loaded members there is a strain $|\varepsilon| = 0.75‰$.

Question:

Determine the joint displacements.

Solution:

The member forces are given in Table 7.8. The calculation is left to the reader.

The change in length of member (i) is

$$\Delta\ell^{(i)} = |\varepsilon| \times \ell \times \text{sign}(N^{(i)}),$$

in which $\text{sign}(N^{(i)})$ stands for the sign of the normal force $N^{(i)}$ in member i. Hence

$$\begin{aligned} \text{sign}(N^{(i)}) &= +1 \quad \text{for } N^{(i)} > 0, \\ \text{sign}(N^{(i)}) &= -1 \quad \text{for } N^{(i)} < 0, \\ \text{sign}(N^{(i)}) &= 0 \quad \text{for } N^{(i)} = 0. \end{aligned}$$

The values found for $\Delta\ell$ are given in the last column of Table 7.8.

Members (6) to (10) are zero-force members and maintain their original length.

Joints A and B do not move: in the Williot diagram they coincide with the origin. When calculating the joint displacements we use the property that the changes in length of members that are in the same line can be plotted behind one another in the Williot diagram.

The displacements of joints D, E, G and H are found in this way directly from A and B. Next we can find the displacement of C from A and H and that of K from B and D.

The Williot diagram is shown in Figure 7.18. In this diagram the dashed line due to the rotation of zero-force member i is indicated with $\perp(i)$, and represents a displacement normal to the referred member.

Table 7.8

Member i	**$N^{(i)}$ (kN)**	**$\ell^{(i)}$ (m)**	**$\Delta\ell^{(i)}$ (mm)**	**Member i**	**$N^{(i)}$ (kN)**	**$\ell^{(i)}$ (m)**	**$\Delta\ell^{(i)}$ (mm)**
1	+20	4	+3	7	0	3	0
2	+20	4	+3	8	0	5	0
3	+20	4	+3	9	0	3	0
4	+20	4	+3	10	0	5	0
5	+50	5	+3.75	11	+40	4	+3
6	0	3	0	12	+40	4	+3

Table 7.9

Joint	**u_x (mm)**	**u_y (mm)**	**Joint**	**u_x (mm)**	**u_y (mm)**
C	+3	−14.25	G	+12	−16
D	+6	+8	H	−6	−14.25
E	+9	0	K	−3	+8

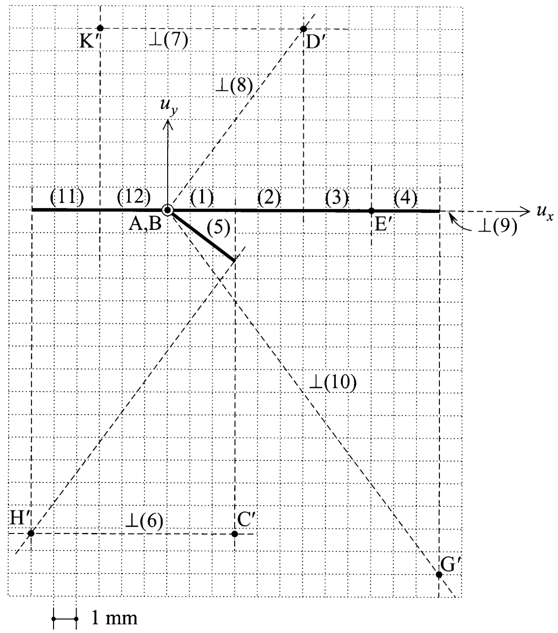

Figure 7.18 The Williot diagram for the joint displacements. Joints A and B do not move, and coincide with the origin of the diagram. When determining the joint displacements we use the property that the changes in length of members that are in the same line can be plotted behind one another in the Williot diagram. The displacements of the joints D, E, G and H are found directly from A and B. Next, the displacement of C is found from A and H while that of K is found from B and D.

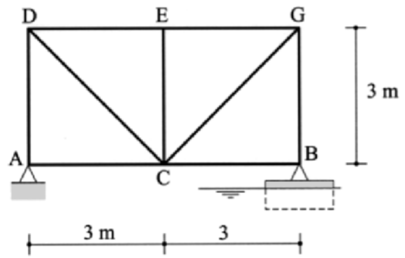

Figure 7.19 A truss is supported in a hinge at A on a land abutment while B is on a pontoon. Both supports are at equal height. By changing the water level, support B submerges by 0.2 m.

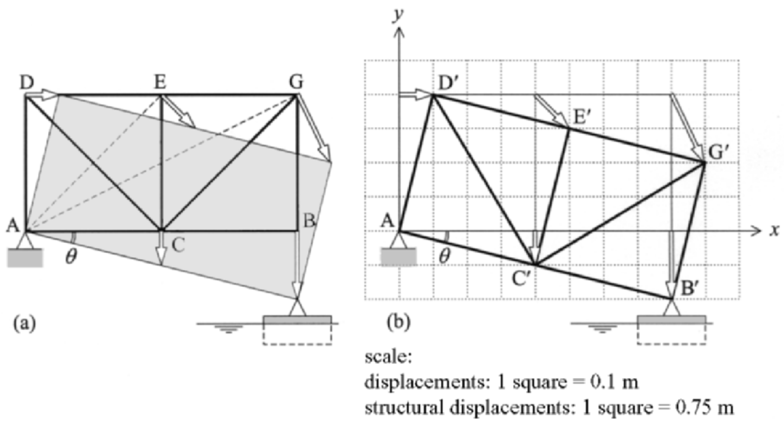

Figure 7.20 (a) If B submerges, the truss, as a rigid body, rotates about A. As a result of the rotation the points of the truss are displaced normal to the joining line with A, and the magnitude of the displacement is proportional to the distance to A. (b) The truss after the rotation about A. The displacements are drawn 7.5 times as large as the structural dimensions.

The joint displacements from the Williot diagram are given in Table 7.9.

Example 7

The truss in Figure 7.19 is supported by a hinge on an abutment at A, and on a pontoon at B. Both supports are at the same height. As a result of a change in the water level, support B submerges by a distance of 0.2 m.

Question:

a. Determine the joint displacements due to the submerging of B.
b. Draw these displacements in a Williot diagram.

Solution:

If support B submerges, the truss does not deform but behaves as a rigid body, and undergoes a rotation about A (see Figure 7.20a).

Intermezzo:[1]

If a rigid body rotates about a point P through a small angle θ, the displacement of a point Q at a distance ℓ from P is equal to $u = \ell\theta$ and this displacement will be perpendicular to the joining line PQ (see Figure 7.21).

For u_h, the horizontal component of the displacement of Q, and u_v, the vertical component, we have

$$u_\text{h} = \ell_\text{v} \cdot \theta,$$

$$u_\text{v} = \ell_\text{h} \cdot \theta.$$

Here ℓ_h is the horizontal component of the distance ℓ between P and Q, and ℓ_v is the vertical component.

[1] See also *Engineering Mechanics*, Volume 1, Section 15.3.2.

To summarise, for a small rotation (ignoring the sign):

- the horizontal displacement is equal to "rotation × vertical distance to the centre of rotation";
- the vertical displacement is equal to "rotation × horizontal distance to the centre of rotation".

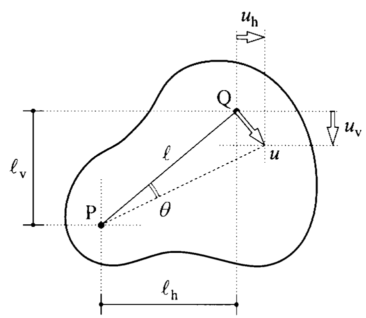

Figure 7.21 For a small rotation we have (ignoring the sign):

- the horizontal displacement u_h is equal to "the rotation $\theta \times$ the vertical distance ℓ_v to the centre of rotation";
- the vertical displacement u_v is equal to "the rotation $\theta \times$ the horizontal distance ℓ_h to the centre of rotation".

Using the rules in the intermezzo we can calculate the rotation θ of the truss from the vertical displacement $u_{v;B}$ of B (see Figure 7.20):

$$\theta = \frac{u_{v;B}}{\ell_h^{AB}} = \frac{0.2 \text{ m}}{6 \text{ m}} = \frac{1}{30} \text{ rad} = 1.9°.$$

In column 2 to 5 of Table 7.10, we use this value of θ to determine the joint displacements u_h and u_v (without signs).

In Figure 7.20 the displacements of the truss are drawn to scale, but 7.5 as large as the structural dimensions.

The joint displacements of the truss due to the rotation as a rigid body are perpendicular to the joining lines between the referred joints and the centre of rotation A, and are proportional to their distances to A.

The displacement in G is therefore perpendicular to joining line AG, that in E is perpendicular to AE, that in C is perpendicular to AC, etc. The direction of the displacements can be determined easily using Figure 7.20. In the xy coordinate system in Figure 7.20b, all horizontal displacements u_x are positive and all vertical displacements u_y are negative; see the last two columns of Table 7.10.

Note that all points on a horizontal line have the same vertical distance ℓ_v and therefore undergo the same horizontal displacement u_h (u_x); see for example the joints D, E and G. In the same way, all points on a vertical line

Table 7.10

Joint	ℓ_v (mm)	$u_h = \ell_v \cdot \theta$ (m)	ℓ_h (m)	$u_v = \ell_h \cdot \theta$ (m)	u_x (m)	u_y (m)
B	0	0	6	0.2	0	–0.2
C	0	0	3	0.1	0	–0.1
D	3	0.1	0	0	+0.1	0
E	3	0.1	3	0.1	+0.1	–0.1
G	3	0.1	6	0.2	+0.1	–0.2

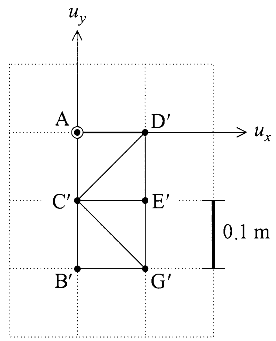

Figure 7.22 The displacements due to a rotation θ about A, set down in a Williot diagram, form a figure that is similar to the shape of the truss, but rotated about an angle of 90° about A (in the direction of θ).

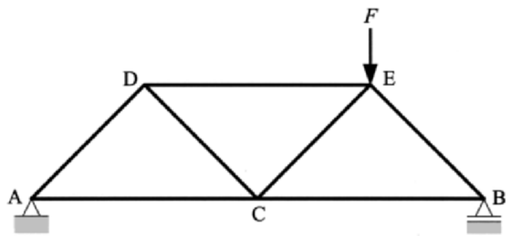

Figure 7.23 A truss for which we cannot draw the Williot diagram: from the fixed point A we do not get any further than the displacement of B along the roller track.

undergo the same vertical displacement; see for example C and E or A and D.

In Figure 7.22 the displacements from Table 7.10 are plotted in a Williot diagram. In the diagram, the displaced joints form a figure that is similar to the shape of the truss, yet rotated through an angle of 90° about A (in the direction of θ). The scale depends on the magnitude of the rotation. We use this property in the next section.

Table 7.10

Joint	ℓ_v (mm)	$u_h = \ell_v \cdot \theta$ (m)	ℓ_h (m)	$u_v = \ell_h \cdot \theta$ (m)	u_x (m)	u_y (m)
B	0	0	6	0.2	0	–0.2
C	0	0	3	0.1	0	–0.1
D	3	0.1	0	0	+0.1	0
E	3	0.1	3	0.1	+0.1	–0.1
G	3	0.1	6	0.2	+0.1	–0.2

7.3 Williot diagram with rigid-body rotation

The success of the graphical method in which the joint displacements of a truss are constructed in a Williot diagram depends on the availability of a joint that is directly attached to two other joints for which the displacements are known. This is not always the case. If, for example, we want to draw the Williot diagram for the truss in Figure 7.23, we start from the fixed joint A, but cannot get any further than the displacement of joint B along the roller track. Then it stops: there is no joint that is directly attached to joints A and B.

A way out of this problem is found by temporarily assuming that one of the members attached to the fixed point A does not rotate but keeps its original direction. Assume member AD is fixed in direction. Based on this situation we can construct a Williot diagram for the joint displacements. In general we will find that the displaced joint B is no longer located on the roller track. We therefore need to make a correction, a rigid-body rotation of the truss about the fixed point A until B is on the roller track again.[1]

The displacements due to the rigid-body rotation of the truss can be determined analytically, or we can draw a separate Williot diagram (see Section 7.1, Example 7) from which we can read off the values.

The resulting joint displacements of the truss are found by superimposing

- the displacements from the Williot diagram, assuming AD is fixed in its original direction at A, and
- the displacements due to the rigid-body rotation of the truss about A.

This method is illustrated below using two examples.

Example 1
Figure 7.23 shows the dimensions of a truss loaded by a vertical force at E. Figure 7.24 shows which members are subject to tension and which to compression. The members are dimensioned in such a way that for the given load $|\varepsilon| = 1/1500$.

Question:
Determine the joint displacements.

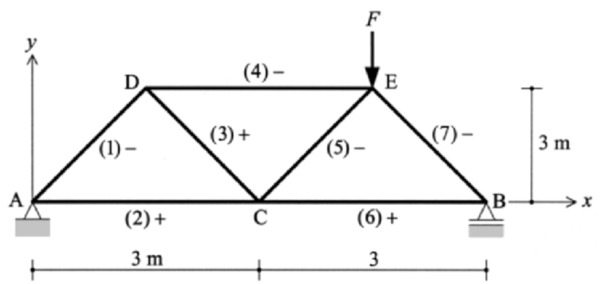

Figure 7.24 The tension and compression members for the given load.

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1 The temporary assumption that AD is fixed in direction has been abandoned.

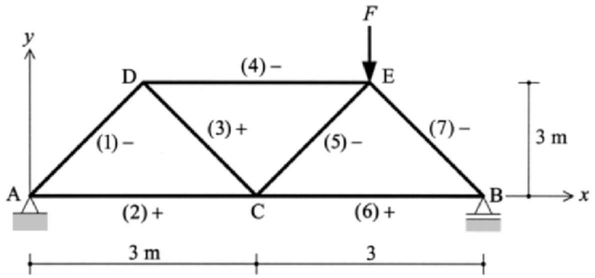

Figure 7.24 The tension and compression members for the given load.

Table 7.11

Member i	$N^{(i)}$ **sign**	$\ell^{(i)}$ **(m)**	$\Delta\ell^{(i)}$ **(mm)**
1	+	$1.5\sqrt{2}$	$-\sqrt{2}$
2	–	3	+2
3	+	$1.5\sqrt{2}$	$+\sqrt{2}$
4	–	3	–2
5	–	$1.5\sqrt{2}$	$-\sqrt{2}$
6	+	3	+2
7	–	$1.5\sqrt{2}$	$-\sqrt{2}$

Solution:
For the change in length of a member i we have (see also Section 7.1, Example 6)

$$\Delta\ell^{(i)} = |\varepsilon| \times \ell \times \text{sign}(N^{(i)}).$$

The values of $\Delta\ell$ are calculated in Table 7.11.

There are four steps in determining the joint displacements:

1. draw the Williot diagram with one of the members fixed in direction;
2. calculate the angle through which the truss has to be rotated;
3. calculate the displacements due to the rigid-body rotation of the truss;
4. superimpose the displacements found with steps (1) and (3).

First step: Draw the Williot diagram for a fixed member direction
First the truss is (temporarily) loosened from the roller support at B, so that B can move freely. Subsequently we assume that member AD is fixed in its original direction and (in the order C, E, B) we can construct all joint displacements in a Williot diagram (see Figure 7.25). These values are given in Table 7.12.

These displacements actually represent the deformation (change in shape) of the truss. Figure 7.26 shows the deformed truss with member AD fixed in its original direction. It should be noted that the displacements have been scaled up 100 times as large as the structural dimensions.

Comment: Instead of fixing the direction of member AD, one could fix the direction of member AC. In general this leads to a different Williot diagram.

From the deformed truss in Figure 7.26 it appears that, if AD is fixed in direction, member AC does not rotate. In this specific example, we therefore find (coincidentally) the same Williot diagram when member AC is fixed in direction as when member AD is fixed in direction. It is left to the reader to check this.

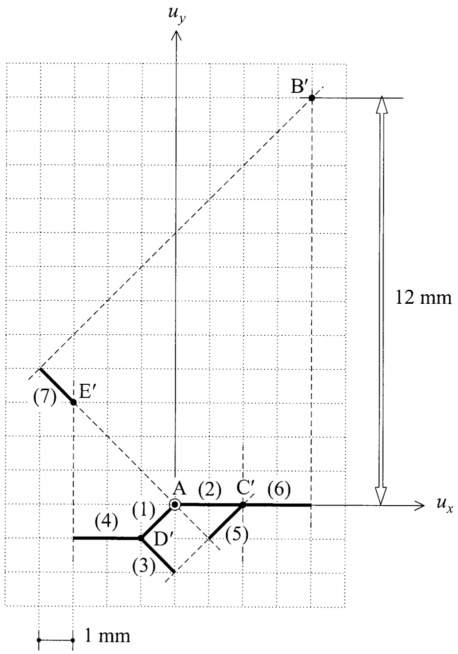

Figure 7.25 With a fixed direction of member AD, the Williot diagram can be drawn in the order C, E, B. The diagram shows that B is displaced by 12 mm upwards and is therefore no longer located on the roller track.

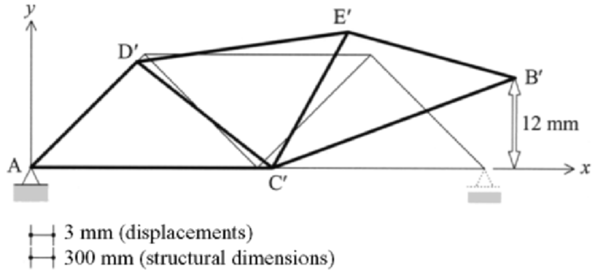

Figure 7.26 The deformed truss if we (temporarily) fix the direction of AD. The displacements have been scaled up 100 times as large as the structural dimensions. In order to get joint B back on the roller track, the deformed truss as a rigid body is rotated about point A.

Table 7.12

Joint	u_x (mm)	u_y (mm)
B	+4	+12
C	+2	0
D	−1	−1
E	−3	+3

Second step: Calculate the angle through which the truss has to be rotated
The Williot diagram in Figure 7.25 shows that B has moved 12 mm upwards. See also the deformed truss in Figure 7.26. This cannot be correct as B is resting on a roller with horizontal roller track. B can move only

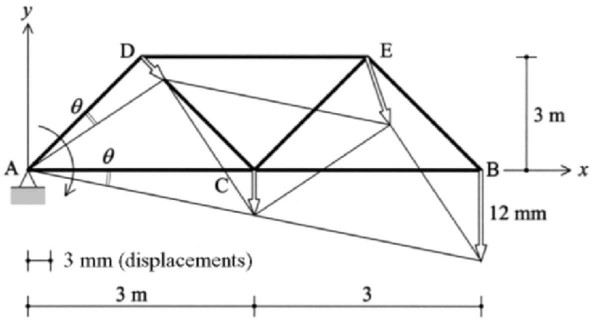

Figure 7.27 The joint displacements due to the rigid-body rotation. Since the joint displacements in the deformed truss are small with respect to the structural dimensions the calculation of the joint displacements due to the rigid-body rotation can be related to the undeformed truss.

Table 7.13

Joint	ℓ_v (m)	ℓ_h (m)	u_x (mm)	u_y (mm)
B	0	6	0	−12
C	0	3	0	−6
D	15	1.5	+3	−3
E	1.5	4.5	−3	−9

horizontally and must therefore be on the horizontal line through A in the Williot diagram.

In order to correct the displacement of joint B, the deformed truss as a rigid body is rotated about fixed point A, so that B moves 12 mm downwards. Since the joint displacements in the deformed truss are small with respect to the structural dimensions, the calculation of the joint displacements due to the rigid-body rotation can be related to the undeformed truss. Due to the rigid-body rotation, B therefore moves perpendicular to line AB in the undeformed truss (see Figure 7.27), and not normal to line AB′ in the deformed truss in Figure 7.26, which gives an inaccurate picture of reality!

The angle θ through which the truss has to rotate is (see Figure 7.27)

$$\theta = \frac{12 \text{ mm}}{6 \text{ m}} = 2 \times 10^{-3} \text{ rad}.$$

Third step: Calculate the displacements due to the rigid-body rotation of the truss

Figure 7.27 shows the joint displacements due to the rigid-body rotation of the truss as a whole. All displacements are perpendicular to the joining line with fixed point A and are proportional to the distance to A.

With $u_h = |u_x| = \ell_v \cdot \theta$ and $u_v = |u_y| = \ell_h \cdot \theta$ we can now calculate the horizontal and vertical displacement as a result of the rotation θ. For the displacements u_x and u_y, related to the xy coordinate system, the signs can be derived from the directions in Figure 7.27. The results are shown in Table 7.13.

Of course we can also draw a separate Williot diagram for the displacements due to the rigid-body rotation of the truss. This diagram is similar to the shape of the truss, but rotated through 90° (see Section 7.1, Example 7).

In the diagram we know the points A and B: A is the fixed point (zero displacements) and B moves downwards by 12 mm. Since the Williot diagram has the same shape as the truss, it can easily be drawn between the known points A and B (see Figure 7.28). The displacements from the Williot diagram are the same as those shown in Table 7.13.

Fourth step: Superimposing the displacements found with steps 1 and 3
Table 7.14 shows the resultant joint displacements. They are found by superimposing the displacements due to the *deformation of the truss*, assuming member AD is fixed in direction, and the displacements due to the *rigid-body rotation of the truss*, to correct for the consequences of the fixed member direction.

Table 7.14

	Displacement due to deformation		**Displacement due to rigid-body rotation**		**Resulting displacement**	
Joint	u_x **(mm)**	u_y **(mm)**	u_x **(mm)**	u_y **(mm)**	u_x **(mm)**	u_y **(mm)**
B	+4	+12	0	–12	+4	0
C	+2	0	0	–6	+2	–6
D	–1	–1	+3	–3	+2	–4
E	–3	+3	+3	–9	0	–6

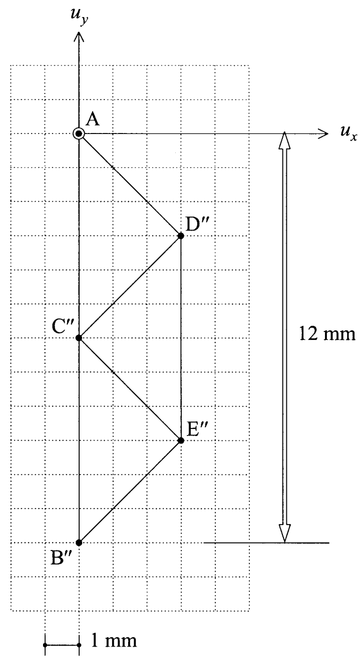

Figure 7.28 The Williot diagram for the displacements due to the rigid-body rotation of the truss. In the diagram, A is the fixed point while the displacement of B is known, namely 12 mm downwards. Since the Williot diagram has the same shape as the truss, with the exception that it is rotated by 90° (in the direction of the rigid-body rotation), it can easily be drawn between the known points A and B.

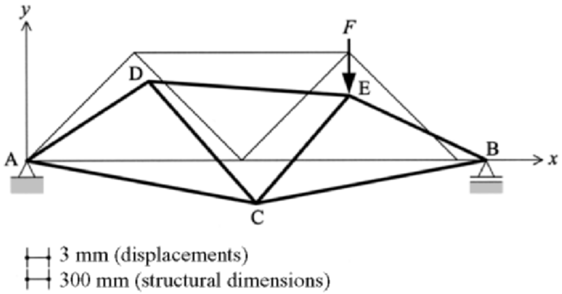

Figure 7.29 The deformed truss after the rigid-body rotation. The displacements are scaled up 100 times as large as the structural dimensions.

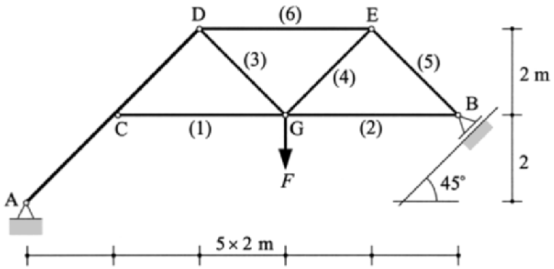

Figure 7.30 A structure consisting of the truss BCDE joined to the rigid member ACD.

Figure 7.29 shows the truss in a deformed condition, with the displacements scaled up 100 times as large as the structural dimensions.

Example 2

The structure in Figure 7.30 consists of a rigid member ACD and truss BCDE. The roller track at B is at an incline of 45°. A vertical force F acts at G. The truss is dimensioned in such a way that all loaded members have a strain $|\varepsilon| = 1‰$.

Table 7.15

Member i	$N^{(i)}$	$\ell^{(i)}$ **(m)**	$\Delta\ell^{(i)}$ **(mm)**	**Member** i	$N^{(i)}$	$\ell^{(i)}$ **(m)**	$\Delta\ell^{(i)}$ **(mm)**
1	0	4	0	4	$+\frac{1}{2}F\sqrt{2}$	$2\sqrt{2}$	$+2\sqrt{2}$
2	0	4	0	5	$-\frac{1}{2}F\sqrt{2}$	$2\sqrt{2}$	$-2\sqrt{2}$
3	$+\frac{1}{2}F\sqrt{2}$	$2\sqrt{2}$	$+2\sqrt{2}$	6	$-F$	4	–4

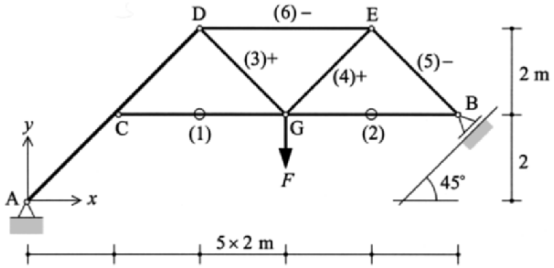

Figure 7.31 The tension, compression and zero-force members in the truss section of the structure for the given load.

Question:
Determine the joint displacements.

Solution:
The member forces are shown in the second column of Table 7.15. The calculation is left to the reader. Figure 7.31 shows the tension, compression and zero-force members.

The changes in length of the members are

$$\Delta\ell^{(i)} = |\varepsilon| \times \ell \times \text{sign}(N^{(i)}),$$

in which $\varepsilon = 0.001$. The values calculated are shown in the last column of Table 7.15.

Figure 7.32 shows the Williot diagram for the case in which the direction of ACD is fixed. Since the member is rigid, C and D do not move, and in the Williot diagram they coincide with fixed point A. The Williot diagram is plotted in the order G, E, B.

The Williot diagram shows that B undergoes an upward vertical displacement of 12 mm (vector $\overline{\text{AB}'}$ in the Williot diagram[1]).

In reality, B can move only along the roller track at a 45° angle. In the Williot diagram this is a displacement along line r through A. In order to get B back to roller track r, the (deformed) structure is rotated through an angle θ about fixed point A. In this rigid-body rotation, B moves in a direction perpendicular to AB. In the Williot diagram B moves from B′ to B″. This is a displacement $u_{\text{h;B}} = 2$ mm to the right and $u_{\text{v;B}} = 10$ mm downwards (see Figure 7.32).

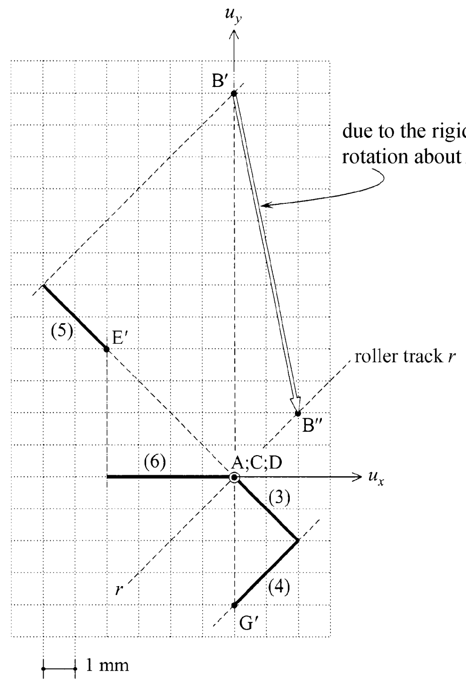

Figure 7.32 The Williot diagram for the joint displacements can be drawn for a fixed direction of member ACD in the order G, E, B. The diagram shows that B is subject to an upward vertical displacement of 12 mm ($\overline{\text{AB}'}$ in the Williot). In reality, B can move only along the roller track at an angle of 45°, a displacement in the diagram along line r through A. In order to get B on the roller track r, the (deformed) truss is rotated through an angle θ about the fixed point A. As a result of this rigid-body rotation, B moves in a direction normal to the joining line AB in the truss. In the diagram, B moves from B′ to B″.

[1] Displacement vector $\overline{\text{AB}'}$ is a vector pointing from A to B′. This vector is not shown in Figure 7.32.

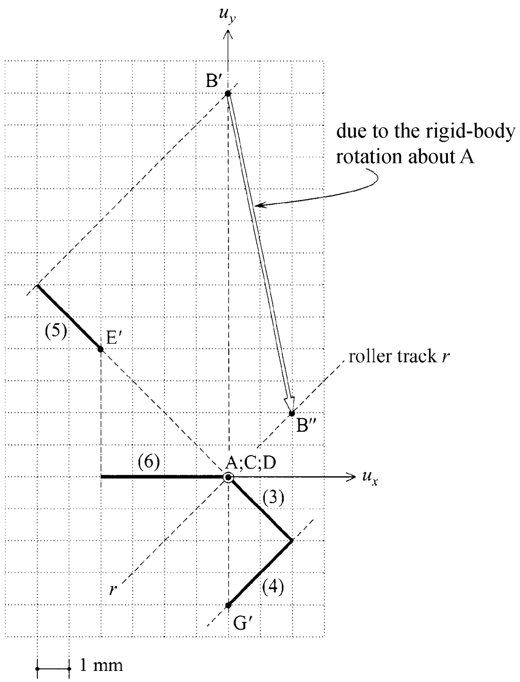

Figure 7.32 The Williot diagram for the joint displacements can be drawn for a fixed direction of member ACD in the order G, E, B. The diagram shows that B is subject to an upward vertical displacement of 12 mm ($\overline{AB'}$ in the Williot). In reality, B can move only along the roller track at an angle of 45°, a displacement in the diagram along line r through A. In order to get B on the roller track r, the (deformed) truss is rotated through an angle θ about the fixed point A. As a result of this rigid-body rotation, B moves in a direction normal to the joining line AB in the truss. In the diagram, B moves from B′ to B″.

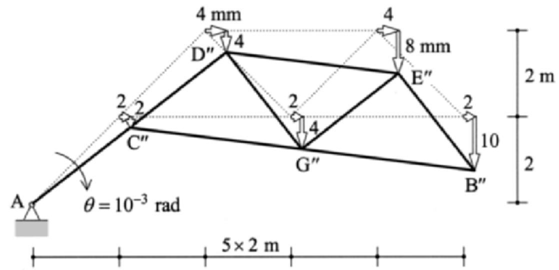

Figure 7.33 The displacements due to the rigid-body rotation are proportional to the distance to the centre of rotation A, and can be determined from $u_h = |u_x| = \ell_v \cdot \theta$ and $u_v = |u_y| = \ell_h \cdot \theta$. The displacements have been scaled up here 125 times as large as the structural dimensions.

Table 7.16

	Displacement due to deformation		**Displacement due to rigid-body rotation**		**Resulting displacement**	
Joint	u_x **(mm)**	u_y **(mm)**	u_x **(mm)**	u_y **(mm)**	u_x **(mm)**	u_y **(mm)**
B	0	+12	+2	–10	+2	+2
C	0	0	+2	–2	+2	–2
D	0	0	+4	–4	+4	–4
E	–4	+4	+4	–8	0	–4
G	0	–4	+2	–6	+2	–10

The magnitude of the rotation θ is (see Figure 7.33)

$$\theta = \frac{u_{v;B}}{\ell_h^{AB}} = \frac{10 \text{ mm}}{10 \text{ m}} = 10^{-3} \text{ rad.}$$

From the horizontal displacement at B we find the same direction and magnitude of θ:

$$\theta = \frac{u_{\mathrm{h;B}}}{\ell_{\mathrm{v}}^{\mathrm{AB}}} = \frac{2\ \mathrm{mm}}{2\ \mathrm{m}} = 10^{-3}\ \mathrm{rad}.$$

Figure 7.33 shows all joint displacements due to the rigid-body rotation. The displacements are proportional to the distance to the centre of rotation A, and can be determined by using $u_{\mathrm{h}} = |u_x| = \ell_{\mathrm{v}} \cdot \theta$ and $u_{\mathrm{v}} = |u_{\mathrm{v}}| = \ell_{\mathrm{h}} \cdot \theta$. In Figure 7.33 the displacements have been scaled up 125 times as large as the structural dimensions.

We could also draw a Williot diagram for the displacements caused by the rigid-body rotation (see Figure 7.34). In the Williot diagram, fixed point A is known, as is point B″, 2 mm to the right of A and 10 mm below A. Between A and B″ we can draw a figure that has the same shape as the structure, but rotated through an angle of 90°. This figure is the Williot diagram we are looking for.

The displacements due to the *deformation of the structure*, assuming member ACD is fixed in direction, can be found from the Williot diagram in Figure 7.32. They have been placed in the second and third column of Table 7.16.

The displacements due to the *rigid-body rotation of the structure* can be found from Figure 7.33, or from the Williot diagram in Figure 7.34. They are given in columns four and five of Table 7.16. The latter two columns of the Table show the resulting joint displacements, found by superimposing the displacements due to the *deformation* and the *rigid-body rotation*.

It is left to the reader to make a sketch of the deformed structure. Use squared paper and a scale of 1 square ≡ 0.5 m for the structural dimensions and 1 square ≡ 4 mm for the displacements.

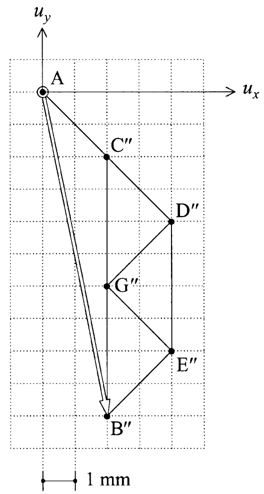

Figure 7.34 The Williot diagram for the displacements due to the rigid-body rotation. In the diagram, we know the fixed point A and the point B″ due to a displacement of B by 2 mm to the right and 10 mm downwards (see Figure 7.33). Between A and B″ we can draw a figure that has the same shape as the structure but which has been rotated through 90°. This is the Williot diagram we are looking for.

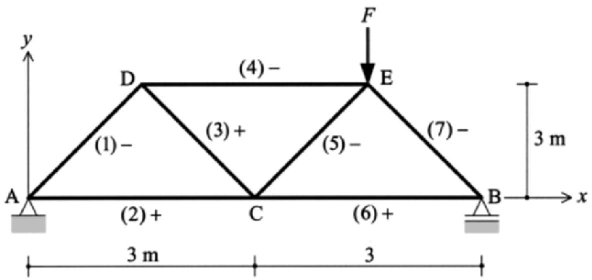

Figure 7.35 A truss with the tension and compression members for the given load.

Table 7.17

Member i	$\Delta\ell^{(i)}$ **(mm)**	**Member** i	$\Delta\ell^{(i)}$ **(mm)**
1	$-\sqrt{2}$	5	$-\sqrt{2}$
2	$+2$	6	$+2$
3	$+\sqrt{2}$	7	$-\sqrt{2}$
4	-2		

7.4 Williot–Mohr diagram

The correction by the rigid-body rotation of the structure can also be directly plotted using a so-called *correction diagram* within the Williot diagram that is based on a fixed member direction. The method is then fully graphical. The method with the correction diagram was devised by Otto Mohr.[1] A Williot diagram with correction diagram is therefore also referred to as a *Williot–Mohr diagram*.

The method is illustrated by two examples.

Example 1
We start with the truss in Figure 7.35, for which the changes in length of the members are given in Table 7.17. We determined the joint displacements of this truss previously in Section 7.2, Example 1.

Question:
Determine the joint displacements using a Williot–Mohr diagram.

Solution:
In a Williot–Mohr diagram we actually draw two Williot diagrams in one figure.

The first Williot diagram relates to the deformation of the structure with one of the members fixed in direction. See Figure 7.36a, in which the direction of member AD is fixed. The points in the first Williot diagram are indicated by means of B′, C′, etc.

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1 Otto Christian Mohr (1835–1918), German engineer, contributed greatly to the development of structural mechanics. He was particularly well known for his graphical methods.

From the Williot diagram we see that B′ is 12 mm too high and is no longer on the horizontal roller track. In order to correct this, we have to rotate the structure about A so that B′ moves downwards by 12 mm.

So far, the approach is identical to that in Section 7.2, Example 1. The difference occurs when we draw the second Williot diagram related to the rigid-body rotation.

The second Williot diagram is related to the joint displacements as a result of the rigid-body rotation of the structure, *but beware*!: *in this Williot diagram the displacements are plotted in the opposite direction*. The displacement of B is therefore not downwards by 12 mm, but upwards by 12 mm! The second Williot diagram is shown in Figure 7.36b. The displaced joints are shown by B_0, C_0, etc.

The second Williot diagram is a figure with the same shape as the structure, but rotated through 90°, and can therefore be easily drawn from points A and B_0.

If both Williot diagrams are drawn on top of one another we find the *Williot–Mohr diagram* (see Figure 7.37). The figure $AB_0E_0D_0$ is known as the Mohr (correction) diagram.

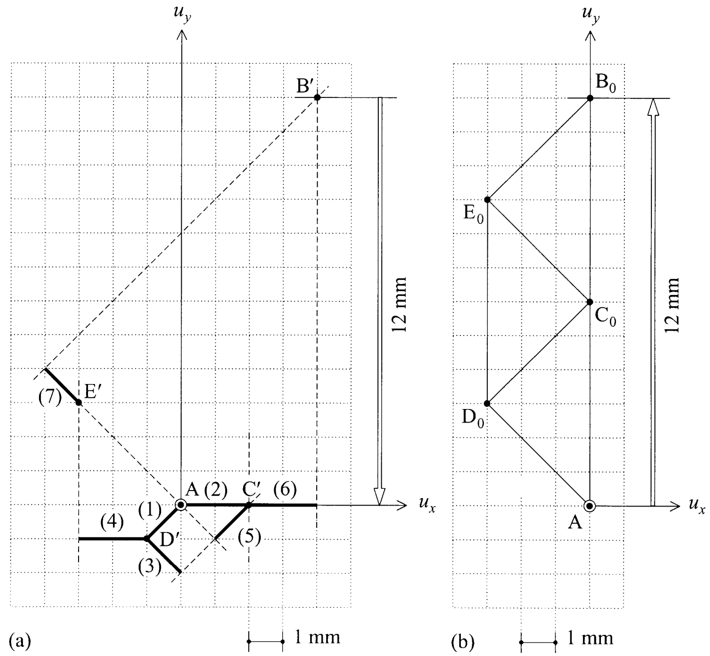


Figure 7.36 (a) The Williot diagram for the deformation of the truss if the direction of member AD is fixed. (b) The Williot diagram for the rigid-body rotation of the truss, but *plotted in the opposite direction*!

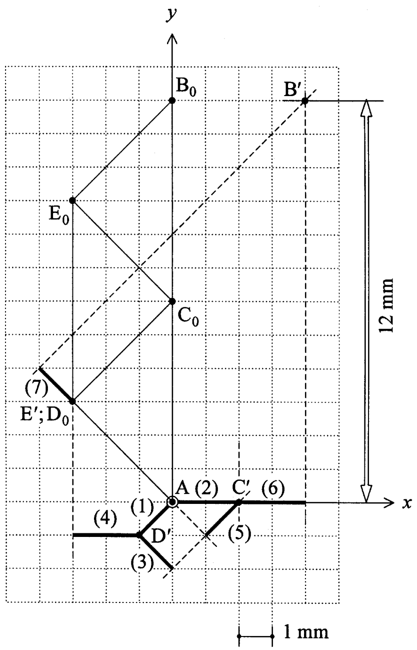


Figure 7.37 If both Williot diagrams from Figure 7.36 are superimposed on one another we find the Williot–Mohr diagram. Figure $AB_0E_0D_0$ is known as the correction diagram or the Mohr diagram. In the Williot–Mohr diagram the joint displacements are no longer measured from the fixed point A, but each joint has its own origin on the correction diagram. The displacement vector of B is $\overline{B_0B'}$, for C it is $\overline{C_0C'}$, etc.

In a Williot–Mohr diagram, the joint displacements are no longer measured from the origin at fixed point A, but each joint has its own origin on the Mohr correction diagram.[1]

Working with a Mohr correction diagram means that we do not rotate the deformed truss, but rotate the undeformed truss in the opposite sense. The displacement vector[2] of B is (see Figure 7.37)

$$-\overline{AB_0} + \overline{AB'} = \overline{B_0A} + \overline{AB'} = \overline{B_0B'}.$$

Here $-\overline{AB_0}$ is the displacement due to the rigid-body rotation and $\overline{AB'}$ is the displacement due to the deformation of the truss with a fixed direction of (in this case) member AD. The displacement vector of C is

$$-\overline{AC_0} + \overline{AC'} = \overline{C_0A} + \overline{AC'} = \overline{C_0C'}.$$

Through measurements in the figure, taking the scale into account, we find that joint C (with displacement vector $\overline{C_0C'}$) moves 2 mm to the right ($u_x = +2$ mm) and 6 mm downwards ($u_y = -6$ mm).

In the same way we find the displacements of D and E.

The reader is asked to check whether the joint displacements found in the Williot–Mohr diagram agree with the values in Table 7.18, found previously in Section 7.2, Example 1.

1 The $u_x; u_y$ coordinate system, that had its origin at fixed point A, has disappeared. Instead, the x and y directions are given.

2 Displacement vector $\overline{AB_0}$ is a vector that points from A to B_0. $\overline{B_0A}$ is a vector that points from B_0 to A. In other words: $\overline{AB_0} = -\overline{B_0A}$.

Table 7.18

	Displacement	
Joint	u_x **(mm)**	u_y **(mm)**
B	+4	0
C	+2	−6
D	+2	−4
E	0	−6

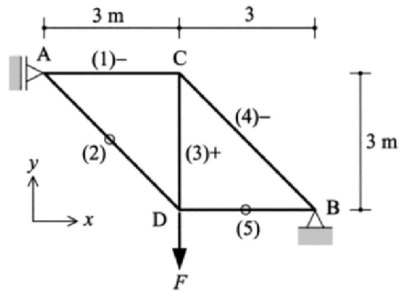


Figure 7.38 A truss with the tension, compression and zero-force members for the given load.

Example 2
The truss in Figure 7.38 is loaded by a vertical force F at D. The figure shows which members are subject to compression and tension and which are zero-force members. All loaded members have a strain $|\varepsilon| = 1/1500$.

Question:
Determine the joint displacements using a Williot–Mohr diagram.

Solution:
The changes in length of the members are $\Delta\ell^{(i)} = |\varepsilon| \times \ell \times \text{sign}(N^{(i)})$, and are shown in Table 7.19.

Table 7.19

Member i	$N^{(i)}$ **(sign)**	$\ell^{(i)}$ **(m)**	$\Delta\ell^{(i)}$ **(mm)**
1	–	3	2
2	0	$3\sqrt{2}$	0
3	+	3	+2
4	–	$3\sqrt{2}$	$-2\sqrt{2}$
5	0	3	0

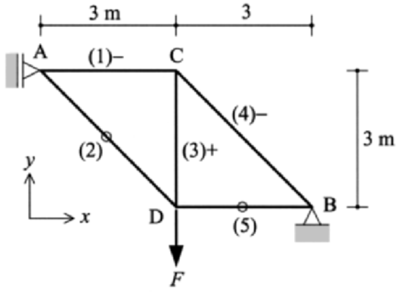


Figure 7.38 A truss with the tension, compression and zero-force members for the given load.

Table 7.19

Member i	$N^{(i)}$ (sign)	$\ell^{(i)}$ (m)	$\Delta\ell^{(i)}$ (mm)
1	–	3	2
2	0	$3\sqrt{2}$	0
3	+	3	+2
4	–	$3\sqrt{2}$	$-2\sqrt{2}$
5	0	3	0

First, from fixed point B we draw the Williot diagram, with a fixed direction of one of the members BC or BD. Figure 7.39a shows the Williot diagram, assuming zero-force member BD fixed in direction.

The Williot diagram shows that A moves 8 mm to the right (the horizontal component of displacement vector $\overline{BA'}$) and is therefore no longer on the vertical roller track. Since A can move only vertically, a correction has to be made.

If we use the Mohr correction diagram, the displacement vector $\overline{A_0A'}$ in the Williot–Mohr diagram will have to be aimed vertically. This means that A_0 must be located on the vertical line a through A′, parallel to the roller track (see Figure 7.39b).

The correction diagram is found by rotating the truss about fixed point B. In doing so, A moves in a direction perpendicular to joining line AB in the truss (see Figure 7.38). This means that A_0 must be on line b in the Williot–Mohr diagram, through the fixed point B and perpendicular to the joining line AB (see Figure 7.39b).

Conclusion: A_0 *is at the intersection of the lines a and b.*

In the Mohr correction diagram we now know A_0 and the fixed point B. Since the correction diagram arises from a rotation about B, the diagram has the same shape as the truss, but rotated through 90°. It is now relatively simple to find the points C_0 and D_0 between A_0 and B. Figure 7.40 shows the complete Williot–Mohr diagram.

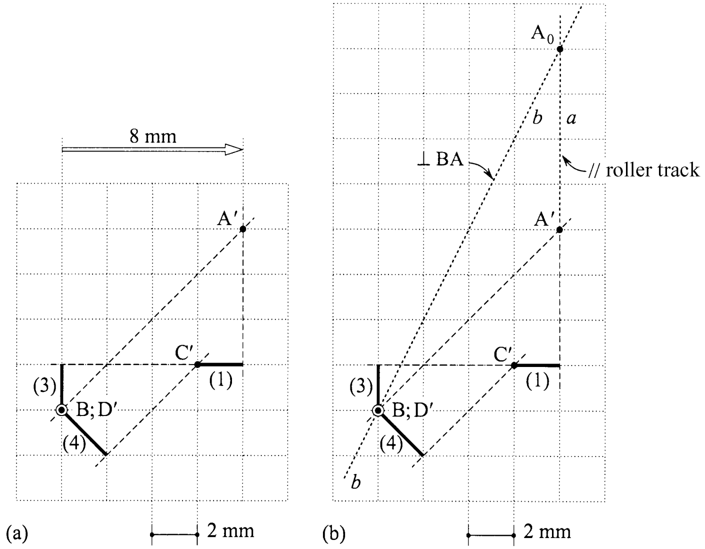


Figure 7.39 (a) The Williot diagram for the deformation of the truss if the direction of zero-force member BD is fixed. The diagram can be drawn in the order B, D, C, A. (b) Since A can move only vertically, the displacement vector $\overline{A_0A'}$ in the Williot–Mohr diagram must be aimed vertically. This means that the point A_0 of the correction diagram must be on the vertical line a through A′, parallel to the roller track. The correction diagram is formed by rotating the truss about the fixed point B. In doing so, A moves in a direction normal to the joining line AB in the truss. This means that A_0 in the Williot–Mohr diagram must be located on the line b, through the fixed point B and normal to the joining line AB. A_0 is at the intersection of the lines a and b.

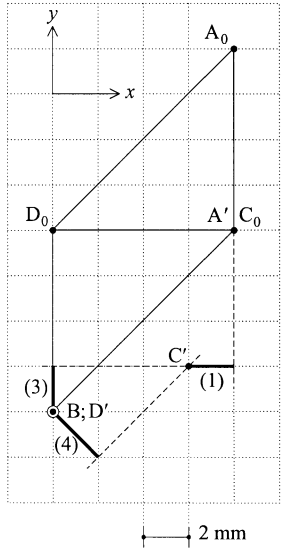


Figure 7.40 The Williot diagram with correction diagram or the Williot–Mohr diagram. The Williot diagram is shown for a fixed direction of member BD. If we fix the direction of member BC (instead of BD), we arrive at a different Williot–Mohr diagram. The joint displacements remain the same of course.

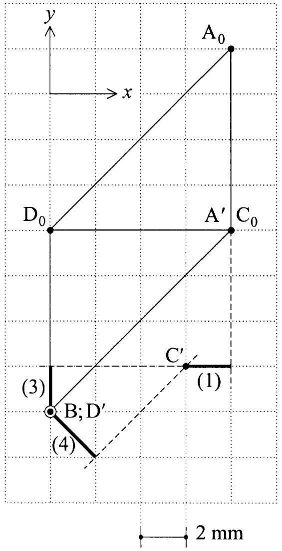


Figure 7.40 The Williot diagram with correction diagram or the Williot–Mohr diagram. The Williot diagram is shown for a fixed direction of member BD. If we fix the direction of member BC (instead of BD), we arrive at a different Williot–Mohr diagram. The joint displacements remain the same of course.

Table 7.20

Joint	Displacement	
	u_x (mm)	u_y (mm)
A	0	−8
C	−2	−6
D	0	−8

The diagram shows that, for example, joint C (with displacement vector $\overline{C_0C'}$) moves 2 mm to the left ($u_x = -2$ mm) and 6 mm downwards ($u_y = -6$ mm). All joint displacements are shown in Table 7.20.

Figure 7.41 shows the deformed truss: the displacements have been drawn 125 times as large as the structural dimensions.

Comment: If we fix the direction of member BC instead of BD, we will get a different Williot–Mohr diagram. The joint displacements found from this diagram remain the same however. It is left to the reader to check this.

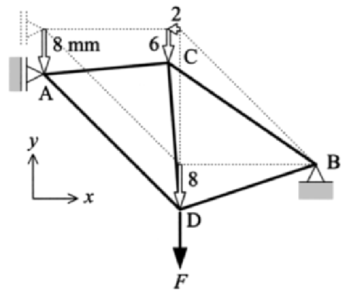


Figure 7.41 The deformed truss. The displacements have been scaled up 125 times as large as the structural dimensions.

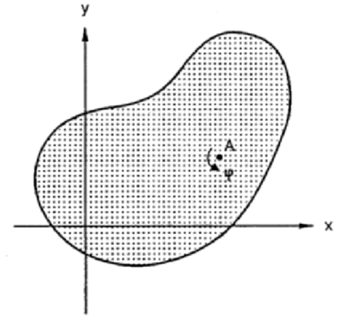
7.5 Problems

General comments:

- The material behaves linear elastically, and the stress always remains beneath the yield point.
- The dead weight of the structure is ignored unless indicated otherwise.

Williot diagram (Section 7.1)

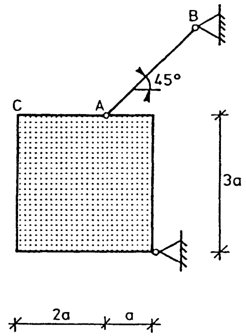
7.1 A rigid body undergoes a (small) rotation in the xy plane of $\varphi = 3.5°$ about point A with coordinates $(x_A; y_A) = (+4.585 \text{ m}; +2.290 \text{ m})$. A point B on the body, with coordinates $(x_B; y_B) = (+0.525 \text{ m}; -0.755 \text{ m})$, is displaced by a distance u_B as a result of this rotation.



Questions:

a. Sketch point B and the direction of the displacement u_B in the figure.
b. Determine the horizontal component $u_{x;B}$ of the displacement of B.
c. Determine the vertical component $u_{y;B}$ of the displacement of B.

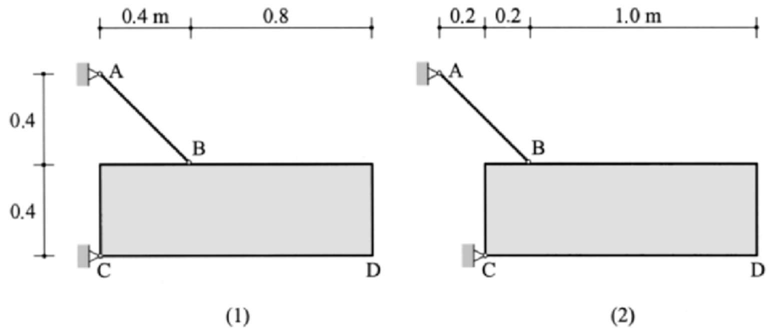
7.2 As a result of the dead weight of the rigid block, member AB is extended by 8.5 mm.



Questions:

a. Determine the horizontal displacement of C.
b. Determine the vertical displacement of C.

7.3: 1–2 A rigid block is supported in two different ways. Subject to an increase in temperature, the wire AB extends by $1.5\sqrt{2}$ mm.



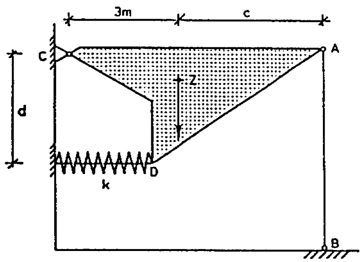
Questions:

a. Determine the horizontal displacement of corner D.
b. Determine the vertical displacement of corner D.

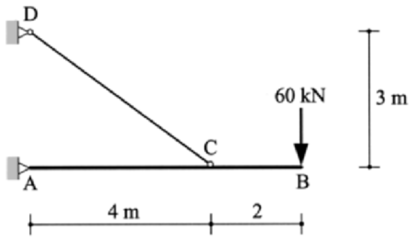
7.4 A rigid block with weight 60 kN is supported as shown. In the situation shown, the spring is stress-free. The stiffness of the spring is $k = 5000$ kN/m. In the calculation use $c = 5$ m and $d = 4$ m.

Questions:

a. Determine the compression of the spring once member AB is removed.
b. Determine the vertical displacement of A.
c. Determine the rotation of the body about C in degrees.



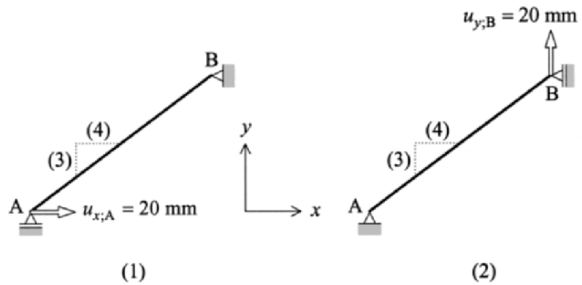
7.5 A rigid member AB is supported and loaded as shown. Member CD, which is deformable, has a cross-section $A = 1500$ mm^2 and modulus of elasticity $E = 200 \times 10^3$ N/mm^2.



Questions:

a. Determine the change in length of member CD.
b. Determine the settlement of B.

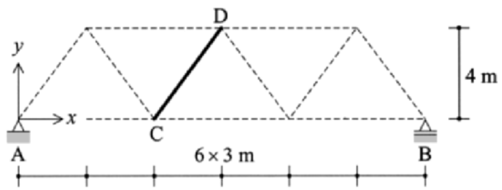
7.6: 1–2 Member AB is supported in two different ways. In both cases the roller support undergoes a prescribed displacement of 20 mm. The member is so long that the rotation due to the prescribed displacement remains small.



Question:
Determine the change in length of member AB.

7.7 As a result of a certain load (not shown in the figure) joints C and D of the truss undergo the following displacements:

- Joint C: $u_{x;C} = +25$ mm; $u_{y;C} = -30$ mm.
- Joint D: $u_{x;D} = 0$; $u_{y;D} = -15$ mm.

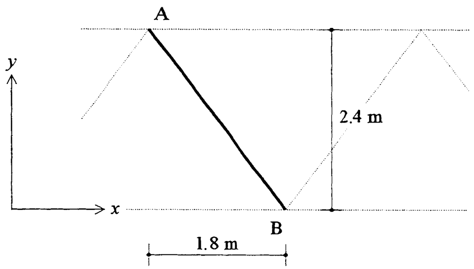


Questions:

a. Determine the change in length of member CD due to only the joint displacement $u_{x;C} = +25$ mm.
b. Determine the change in length of member CD due to only the joint displacement $u_{x;D} = -15$ mm.
c. Determine the change in length of member CD in the truss for the given joint displacements of C and D.

7.8 In the figure shown, AB is a member isolated from a truss. The length of the member is 3 m and the axial stiffness 750 MN. The displacements of the member ends A and B are

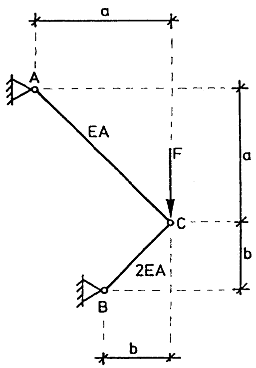
- $u_{x;\text{A}} = +10$ mm;
- $u_{x;\text{A}} = -15$ mm;
- $u_{x;\text{B}} = -10$ mm;
- $u_{x;\text{B}} = -25$ mm.



Questions:

a. Determine the change in length of member AB.
b. Determine the normal force in member AB, with the correct sign.

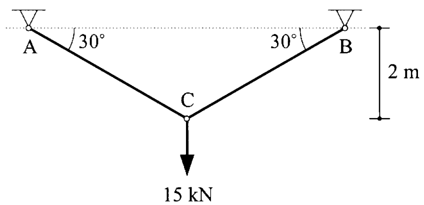
7.9 In the structure shown, member BC has twice the axial stiffness as member AC. The structure is loaded at C by a vertical force F. In the calculation use $a = 2.5$ m, $b = 1.25$ m, $EA = 11.3$ MN and $F = 25.6$ kN.



Questions:

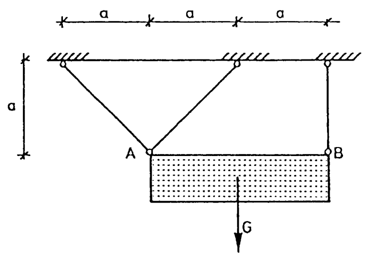
a. Determine the vertical displacement of C.
b. Determine the horizontal displacement of C.

7.10 A 15 kN load is suspended from two steel wires. The wire cross-section has area $A = 100$ mm^2. The modulus of elasticity of steel is set at $E = 200$ GPa.



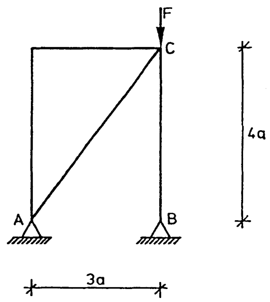
Question:
Determine the vertical displacement of C.

7.11 A block with weight G is suspended from three wires. All wires have the same axial stiffness EA. In the calculation use $a = 2$ m, $G = 50$ kN and $EA = 12.5$ MN.



Question:
Which of the vertical vertical displacements at A and B is larger? Determine this displacement.

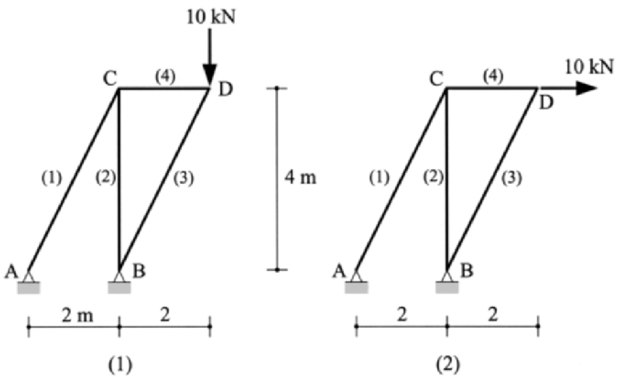
7.12 In the truss shown all members have the same axial stiffness $EA = 32$ MN. The truss is loaded at C by the vertical force F. In the calculation use $a = 1$ m and $F = 20$ kN.



Questions:

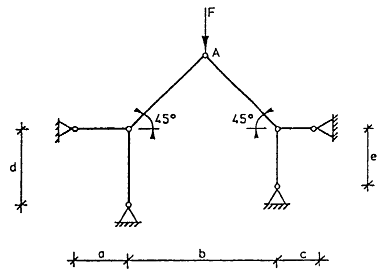
a. Determine the vertical displacement of joint C.
b. Determine the horizontal displacement of joint C.

7.13 You are given the same truss loaded in two different ways by a force of 10 kN at D. The member cross-sections are $A^{(1)} = A^{(3)} = 50\sqrt{5}$ mm^2 and $A^{(2)} = A^{(4)} = 50$ mm^2. The modulus of elasticity is $E = 200$ GPa.



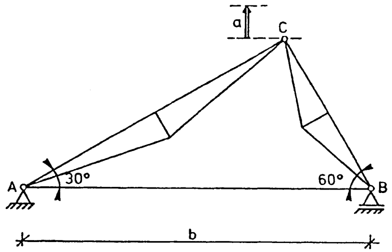
Question:
Using a Williot diagram, determine the displacement of joint D. Draw the displacements in the diagram 5 times as large as they are in reality (use a scale of 1 cm ≡ 2 mm).

7.14 The structure shown is dimensioned in such a way that, due to the vertical force F at A, all loaded members are subject to the same absolute strain $|\varepsilon| = 0.001$. In the calculation use $a = 1.5$ m, $b = 4.0$ m, $c = 1.0$ m, $d = 2.0$ m and $e = 1.5$ m.



Question:
Use a Williot diagram to determine the vertical displacement of point A. Draw the displacements in the diagram 10 times as large as they are in reality using a scale 1 cm ≡ 1 mm.

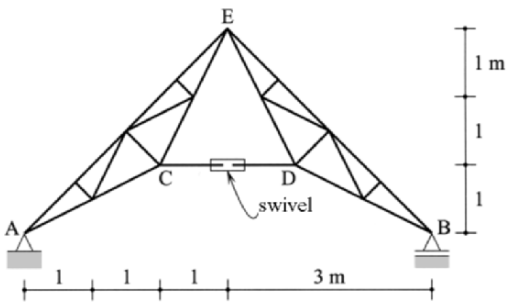
7.15 For the truss shown, the top hinge has to be raised by $a = 13$ mm by shortening tie rod AC.



Questions:

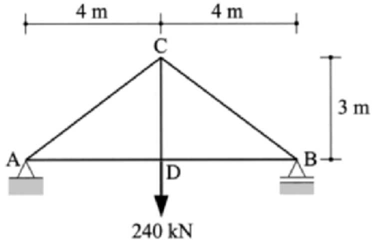
a. Determine the length by which the tie rod has to be shortened if $b = 6$ m.
b. Determine the length by which the tie rod has to be shortened if $b = 2$ m.

7.16 The truss frame shown has a swivel in member CD. By shortening member CD the height of the frame can be adapted.



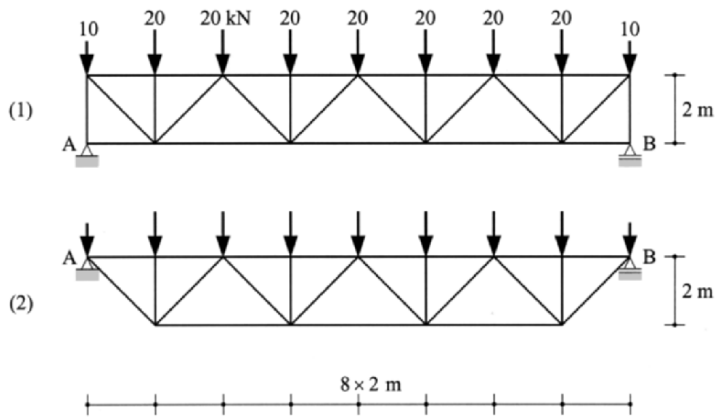
Question:
By how much must member CD be shortened if the joint E is to be raised by 30 mm?

7.17 In the truss shown, all members have the same axial stiffness $EA = 216$ MN. The load consists of a vertical force of 240 kN at joint D.



Question:
Determine the vertical displacement of joint D.

7.18: 1–2 The trusses shown are dimensioned in such a way that the given loading (which is equal for both trusses) generates a stress of 100 N/mm^2 in all tension members and a stress of 50 N/mm^2 in all compression members. Use $E = 200$ GPa for the modulus of elasticity.

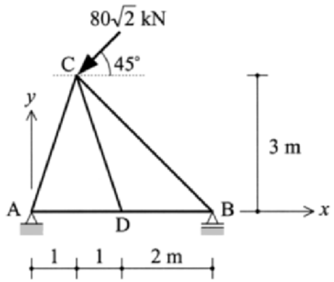


Question:
Determine the displacement of joint B.

7.19 In the structure shown, all joints are hinged. For the given load, all loaded members have the same absolute strain $|\varepsilon| = 0.001$.

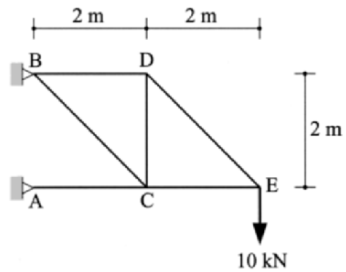
Questions:

a. Determine the member forces.
b. Determine the change in length of the members.
c. Determine the Williot diagram for the joint displacements. Draw the displacements in the diagram 10 times as large as they are in reality (so use a scale of 1 cm ≡ 1 mm).
d. For all joints, write down the displacements u_x and u_y in a table.

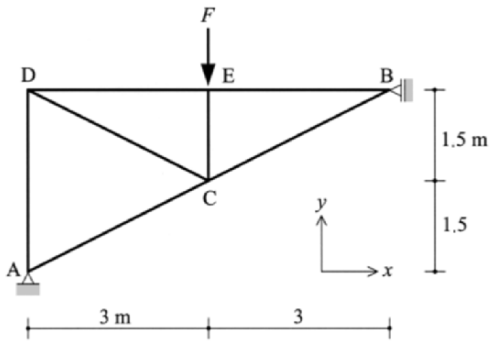


7.20 In the truss shown, all members have the same axial stiffness $EA = 40$ MN.

Question:
Determine the horizontal displacement of joint E due to the given load.



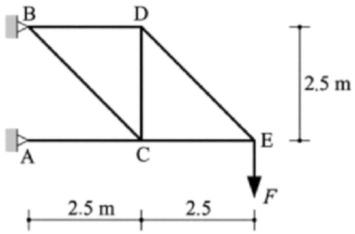
7.21 For the load shown, all loaded members in the truss have the same absolute strain $|\varepsilon| = 1/1500$.



Questions:

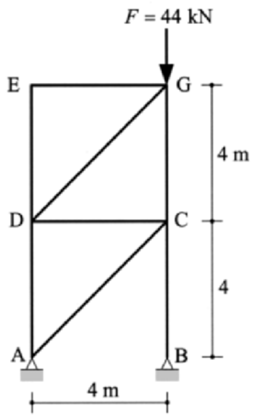
a. In the truss indicate all tension, compression and zero-force members by means of +, − and 0, without directly determining the magnitude of the member forces.
b. Use a Williot diagram to determine the displacement of joints B to E. Draw the diagram on squared paper and use a scale of 1 square ≡ 1 mm.

7.22 In the truss shown, all members have the same absolute strain for the given load: $|\varepsilon| = 0.08\%$.



Question:
Use a Williot diagram to determine the vertical displacement of joint E. Draw the diagram on squared paper using a scale of 1 square ≡ 1 mm.

7.23 The truss, in which all members have the same axial stiffness $EA = 35.2$ MN, is loaded at G by a vertical force $F = 44$ kN.



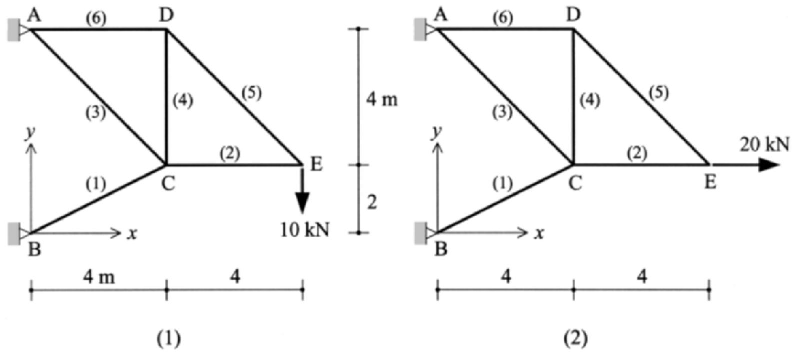
Questions:

a. Draw the Williot diagram for the joint displacements. Use squared paper and draw the displacements at full size.
b. Draw the deformed structure on squared paper. For the structural dimensions use a scale of 1 cm ≡ 1 m; draw the displacements at full size.

7.24: 1–2 The same truss is loaded on two different ways by a force at D. For the member cross-sections:

$$A^{(1)} = 100\sqrt{5}\ \text{mm}^2;$$
$$A^{(2)} = A^{(4)} = A^{(6)} = 100\ \text{mm}^2;$$
$$A^{(3)} = A^{(5)} = 100\sqrt{2}\ \text{mm}^2.$$

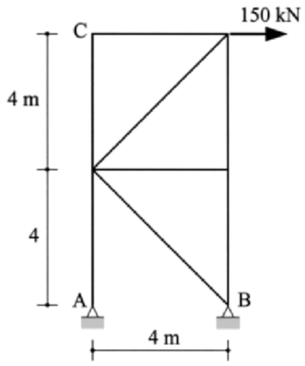
The modulus of elasticity is $E = 200$ GPa.



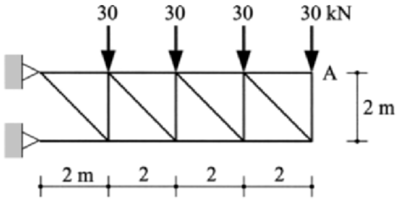
Question:
Use a Williot diagram to determine the joint displacements. Draw the diagram on squared paper (with 5 mm squares) and use a scale of 1 square ≡ 2/3 mm. Put the values u_x and u_y in a table.

7.25 In the truss shown, all members have the same axial stiffness $EA = 400$ MN.

Question:
Determine the vertical displacement of joint C due to the given load.

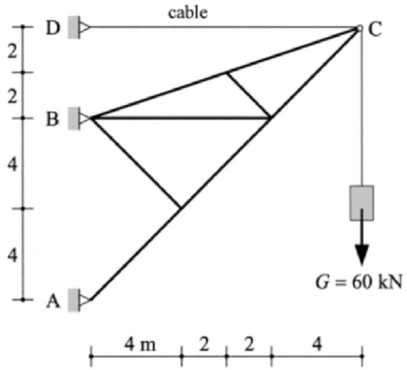


7.26 In the truss shown, all members have the same cross-sectional area $A = 2000$ mm^2 and modulus of elasticity $E = 200$ GPa.



Question:
Determine the horizontal displacement of joint A for the given load.

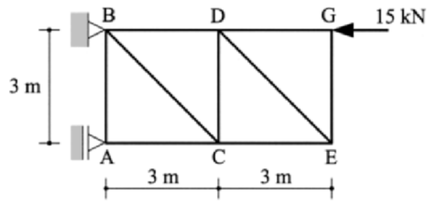
7.27 Truss ABC has a pulley at C of which the diameter is negligible compared to the other dimensions of the structure. A load of weight $G = 60$ kN is suspended from a cable that at C runs over the pulley without friction and is fixed at D. All members have the same axial stiffness $EA = 84.85$ MN. Neglect the change in length of the cable.



Questions:

a. Determine the member forces and the changes in length of the members.
b. Determine the displacement of joint C.
c. Determine the settlement of load G due to the displacement of joint C.

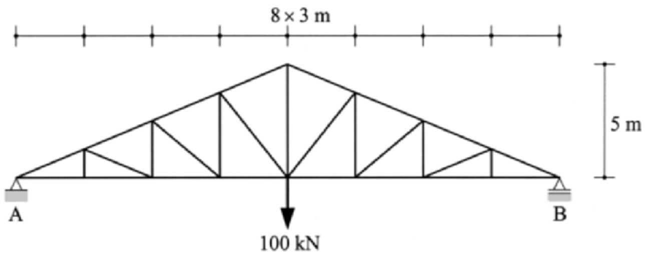
7.28 The truss shown is loaded by a horizontal force of 15 kN at G. All members have the same axial stiffness $EA = 22.5$ MN.



Questions:

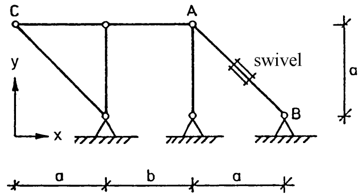
a. Determine the horizontal displacement of joint G.
b. Determine the vertical displacement of joint G.
c. Sketch the deformed truss.

7.29 In the truss shown, all members have the same cross-sectional area $A = 2000\ \text{mm}^2$ and modulus of elasticity $E = 200$ GPa.



Question:
Determine the displacement of joint B for the given load.

7.30 In the structure shown, member AB is shortened by $15\sqrt{2}$ mm by means of a swivel.



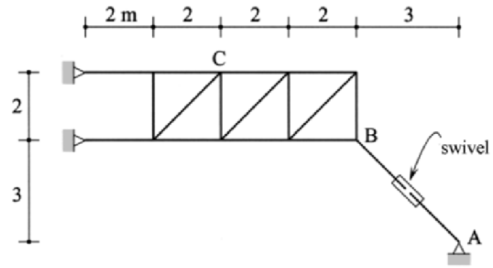
Questions:

a. Determine the vertical displacement $u_{y;C}$ of joint C.
b. Sketch the deformed structure.

7.31 The truss shown has a swivel in member AB.

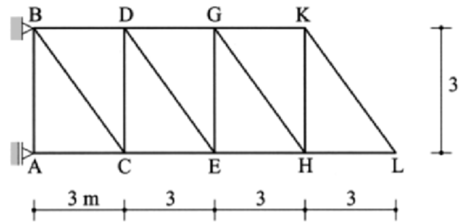
Questions:

a. Determine the vertical displacement of joint C if member AB is lengthened by 20 mm by using the swivel.



b. Sketch the deformed structure.

7.32 In the cantilever structure, joint L must be raised by 90 mm as compared to the position shown in the figure. This is achieved by changing the length of one of the members.



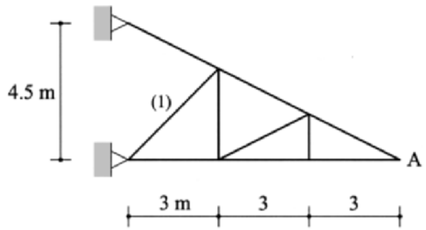
Questions:

a. By which amount must the length of member CE change such that joint L is raised by 90 mm. Is this a lengtening or shortening? Sketch the deformed structure.
b. By which amount must the length of member DE change such that joint L is raised by 90 mm? Is this a lengtening or shortening? Sketch the deformed structure.

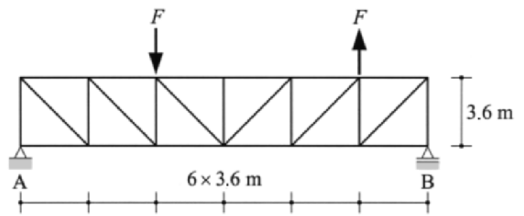
7.33 In the truss shown, member (1), which is damaged, is replaced by a new member that is 30 mm too long.

Questions:

a. By how much does the horizontal position of A change?
b. By how much does the vertical position of A change?
c. Draw the deformed truss.



7.34 The truss shown is dimensioned in such a way that for the given load there is a tensile stress of 140 N/mm^2 in all tension members and a compressive stress of 70 N/mm^2 in all compression members. For all members the modulus of elasticity is $E = 210$ GPa.

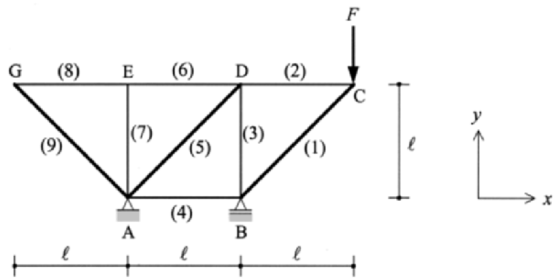


Question:
Determine the displacement of the roller at B.

7.35 In the truss shown, all diagonal members have an axial stiffness $EA\sqrt{2}$; the other members have axial stiffness EA. In the calculation use $\ell = 5$ m, $EA = 125$ MN and $F = 25$ kN.

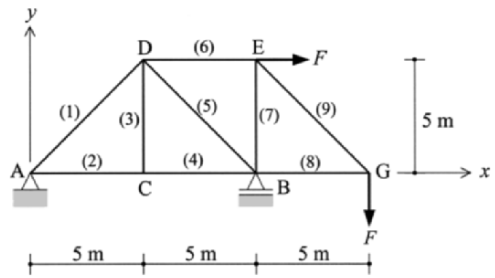
Questions:

a. Determine all member forces.



b. Determine the changes in length for all members.
c. Use a Williot diagram to determine the displacements of all joints. In the diagram draw the displacements 10 times as large as they actually are (use a scale of 1 cm ≡ 1 mm).

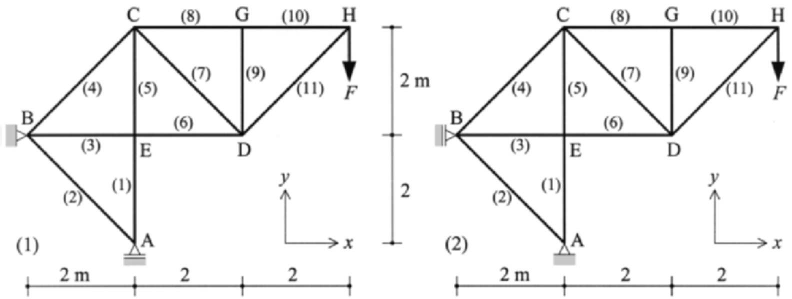
7.36 The truss shown is dimensioned in such a way that for the given load all loaded members have the same absolute strain $|\varepsilon| = 0.6$‰.



Questions:

a. For all members, find the changes in length $\Delta\ell$ and put them in a table.
b. Use a Williot diagram to determine the horizontal and vertical displacements for all joints and collect the values in a table. Draw the diagram on squared paper using a scale of 1 square ≡ 1.5 mm.

7.37: 1–2 The same truss is supported in two different ways. The truss is dimensioned in such a way that for the given load $|\varepsilon| = 0.001$ for all loaded members.



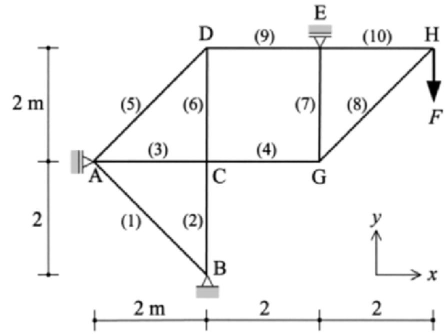
Questions:

a. For each of the members, indicate whether they are tension, compression or zero-force members (without determining the member forces).
b. For all members, collect the changes in length in a table.
c. In order to find the joint displacements, draw the Williot diagram on squared paper. Use a scale of 1 square ≡ 2 mm.
d. Collect the displacements u_x and u_y for all joints in a table.
e. Draw the deformed truss. For the structural dimensions use a scale of 1 cm ≡ 0.5 m and for the displacements use 1 cm ≡ 20 mm.

7.38 The truss shown is dimensioned so that for the given load all loaded members have the same absolute strain $|\varepsilon| = 0.5 \times 10^{-3}$.

Questions:

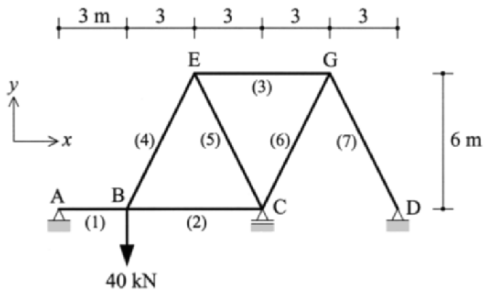
a. Determine all member forces expressed in F.
b. For all members determine the change in length $\Delta\ell$.



c. Use a Williot diagram to determine the displacements of the joints. In the diagram draw the displacements five times as large as they actually are (using a scale of 5 mm ≡ 1 mm).

7.39 In the truss shown, joint B is loaded by a vertical force of 40 kN. All members have the same modulus of elasticity $E = 2 \times 10^5$ N/mm^2. The cross-sectional areas of the members are

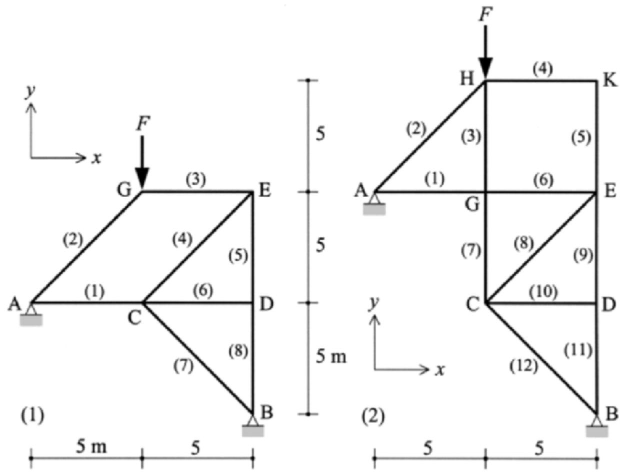
- member 1: $A = 150$ mm^2;
- members 2 and 3: $A = 600$ mm^2;
- members 4 to 7: $A = 300\sqrt{5}$ mm^2.



Questions:

a. Collect all the member forces in a table alongside the changes in lengths of the members.
b. Use a Williot diagram to determine the displacement in the x and y directions for all joints. Collect these values in a table. Draw the Williot diagram on squared paper (with 5 mm squares) and for the scale of the displacements use 1 square $\equiv$ 1 mm.

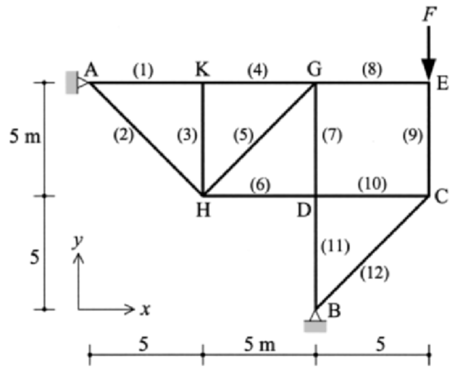
7.40: 1–2 In the two trusses shown, the members are dimensioned in such a way that for the given load all members have the same absolute strain $|\varepsilon| = 0.001$.



Questions:

a. Determine all the member forces, expressed in F, and place them in a table.
b. For all members determine the change in length and collect the results in a table.
c. Draw the Williot diagram for the joint displacements. Draw the displacements in the diagram full size.
d. For all joints, read off the horizontal and vertical displacements u_x and u_y and collect the values in a table.
e. If, with unchanged load and axial stiffness, all member lengths are doubled, what impact does this have on the stresses in the truss members (strength) and on the joint displacements (stiffness)? Substantiate your answer.

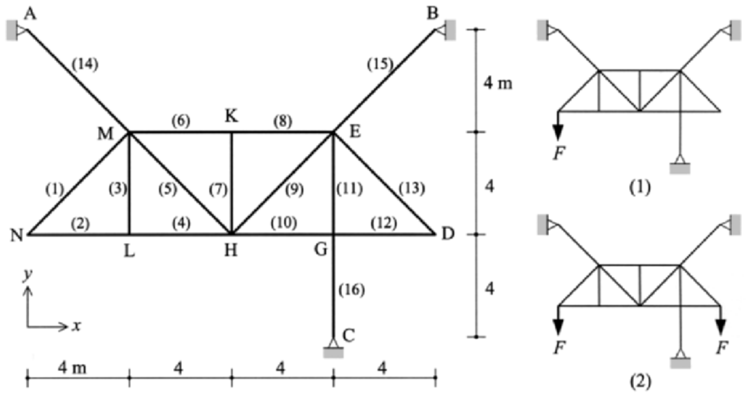
7.41 In the truss shown, all loaded members have the same absolute strain $|\varepsilon| = 0.001$.



Questions:

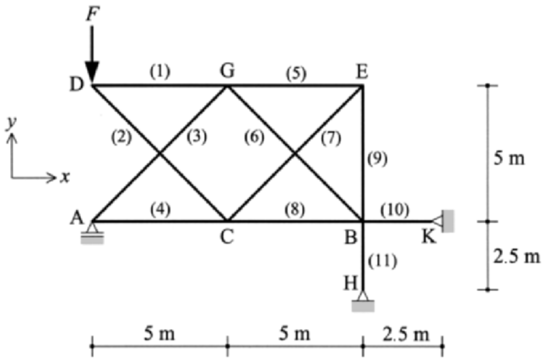
a. For all members, determine the change in length in mm and collect all the values in a table.
b. Use a Williot diagram to determine the displacement of joint E. In the diagram, draw the displacements full size. Indicate in which order you determine the joint displacements in order to reach the requested result as quickly as possible.

7.42: 1–2 The same truss is loaded in two different ways. The truss is dimensioned in such a way that in both loading cases $|\varepsilon| = 0.5‰$ for all loaded members.



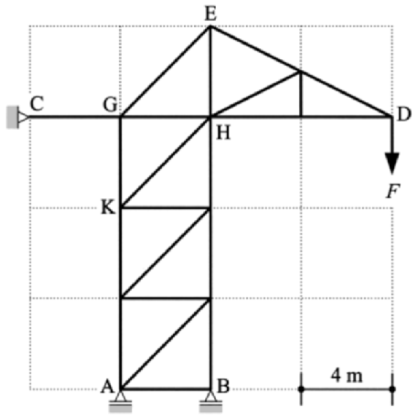
Questions:
a. In the truss, indicate all tension, compression and zero-force members by +, − and 0, without directly determining the magnitude of the member forces.
b. Use a Williot diagram to determine the horizontal and vertical displacements of joint D in the most efficient way.
c. Use a Williot diagram to determine the horizontal and vertical displacements of joint N in the most efficient way.

7.43 In the truss shown, members (2) and (3) cross one another as do members (6) and (7). For the given load all loaded members have the same absolute strain $|\varepsilon| = 0.8‰$.



Questions:
a. Collect all the member forces and changes in length of the members in a table.
b. On squared paper (using 5 mm squares), draw a Williot diagram for the joint displacements. Use a scale of 1 square ≡ 4 mm.
c. From the Williot diagram, read off the joint displacements u_x and u_y, and collect the values in a table.

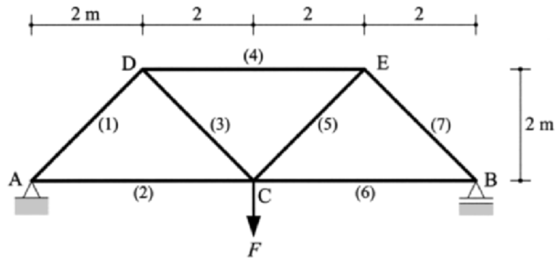
7.44 For the truss, drawn on a squared grid, and the given load, all loaded members have the same absolute strain $|\varepsilon| = 0.5‰$.



Questions:
a. In the truss, indicate the tension, compression and zero-force members.
b. Use a Williot diagram to determine the displacement of joint D. Draw the Williot on squared paper with 5 mm squares and use a scale of 1 square ≡ 2 mm.
Hint: from C, first determine the displacement of joints G and H.

Williot diagram with rigid-body rotation (Section 7.2)

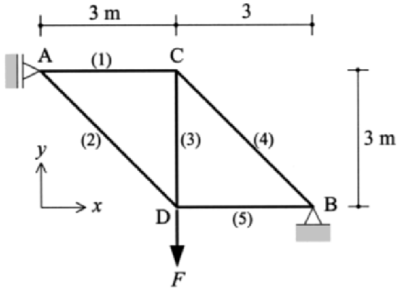
7.45 For the given load it holds for all loaded members in the truss that
- diagonal members (odd numbers): $|\varepsilon| = 0.50 \times 10^{-3}$;
- other members (even numbers): $|\varepsilon| = 0.25 \times 10^{-3}$.



Questions:
a. Indicate which members are tension, compression and zero-force members. Determine the changes in length of the members.
b. If the truss is isolated from the roller support at B and the direction of member AC is fixed, use a Williot diagram to determine the displacements of joints B up to E. For the displacements in the diagram, use a scale of 1 cm ≡ 1 mm.
c. Through which angle should the truss as a rigid body be rotated? Determine the displacements of B up to E due to the rigid-body rotation.
d. Collect the final displacements of B up to E in a table.

7.46 As problem 7.45, but now fix the direction of member AD.

7.47 For the truss shown, the vertical force F at D causes in all loaded members the same absolute strain of $|\varepsilon| = 1/1500$.



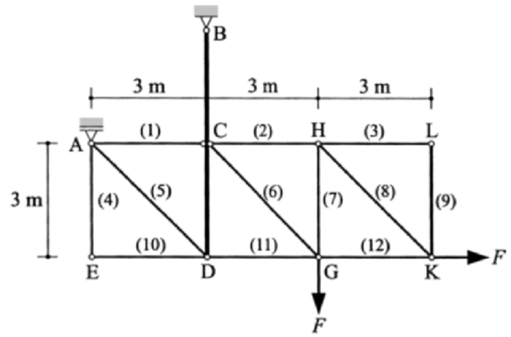
Questions:

a. On squared paper, draw the Williot diagram when the direction of member BC is fixed. For the displacements use a scale of 1 square $\equiv$ 2 mm.
b. Determine the joint displacements due to the rigid-body rotation.
c. Determine the final joint displacements. Check the answers using Section 7.4, Example 2, Table 7.20.

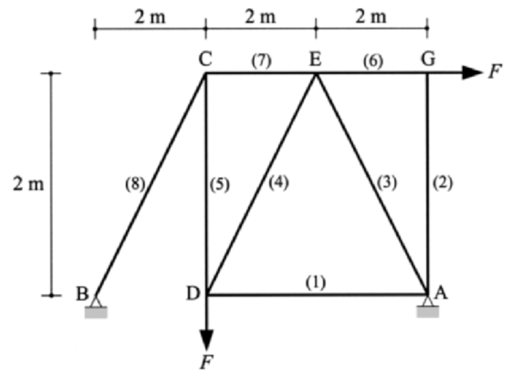
7.48 In the structure shown, member BCD is rigid. If the structure is loaded at G and K by forces F, the loaded members have an absolute strain $|\varepsilon| = 1‰$.

Questions:

a. Determine the member forces, expressed in terms of F, and the changes in length of the members.
b. Draw the Williot diagram for the joint displacements if the direction of member BCD is fixed. Use a scale of 1 cm $\equiv$ 3 mm for the displacements in the diagram.
c. Determine the displacements due to the rigid-body rotation.
d. Determine the final joint displacements.



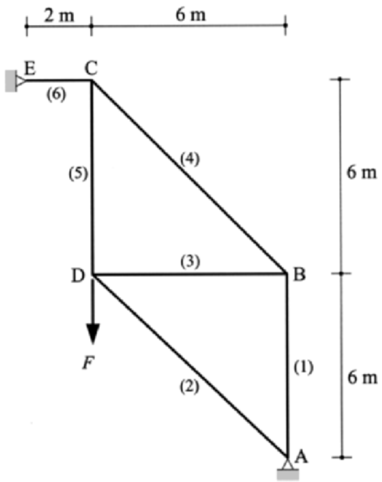
7.49 For the given load by the forces F at D and G, all the loaded members in the truss have the same absolute strain $|\varepsilon| = 1‰$.



Questions:

a. Draw the Williot diagram when the direction of member AG is fixed. For the displacements in the diagram, use a scale of 1 cm $\equiv$ 2 mm.
b. Determine the angle through which the truss as a rigid body has to be rotated, and find the joint displacements due to the rigid-body rotation.
c. Determine the resultant joint displacements.

7.50 All loaded members in the truss have $|\varepsilon| = 1‰$, except for member EC which is rigid and therefore does not deform.

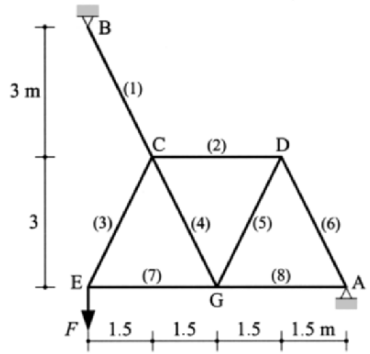


Questions:

a. Draw the Williot diagram when the direction of member AB is fixed. Draw the displacements in the diagram five times as large as they actually are, using a scale of 1 cm ≡ 2 mm.
b. Determine the joint displacements due to the rigid-body rotation.
c. Determine the resultant joint displacements.

7.51 As problem 7.50, but now member EC is no longer rigid, but has an absolute strain $|\varepsilon| = 1‰$.

7.52 The truss shown is supported by a hinge at A and suspended at C from member BC. If the vertical force F acts at E, all loaded members have $|\varepsilon| = 1/1500$, except for member BC which is rigid and therefore does not deform.

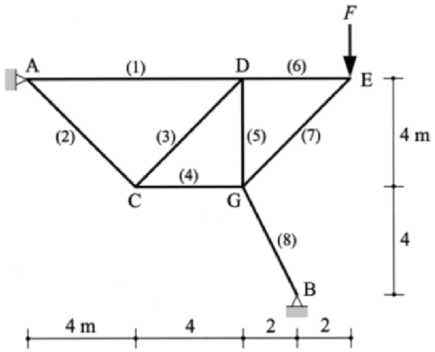


Questions:

a. Determine the member forces, expressed in terms of F, and the changes in length in the members.
b. Since joint G is directly linked to the fixed points A and B, all joint displacements can be found in one go using a Williot diagram. If you do not see this, isolate the structure at B and draw the Williot diagram for a fixed direction of member AG. Do so on squared paper with 5 mm squares and use a scale of 1 square ≡ 1 mm.
c. Determine the angle through which the truss as a rigid body has to be rotated, and find the displacements due to the rigid-body rotation.
d. Determine the resultant joint displacements.

7.53 As problem 7.52, but now member BC has also $|\varepsilon| = 1/1500$ and is no longer rigid.

7.54 For given load F at E, all loaded members in the truss have the same absolute strain $|\varepsilon| = 0.1\%$.



Questions:

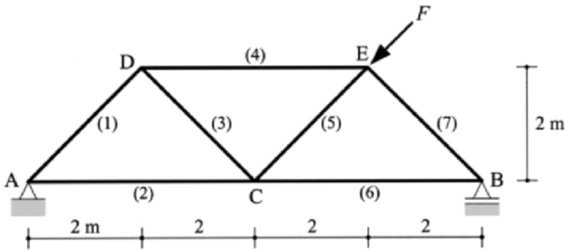
a. Determine all the member forces, expressed in terms of F, and the changes in length of the members.
b. Draw the Williot diagram for the joint displacements, fixing the direction of member AC. Draw the diagram on squared paper (with 5 mm squares) and use a scale of 1 square ≡ 1 mm.
c. Determine the angle through which the truss as a rigid body has to be rotated and find the joint displacements due to the rigid-body rotation.
d. Determine the final joint displacements.

7.55 As problem 7.54, but now fix the direction of member AD.

Comment: Problems 7.56 to 7.66 are also suitable for the method based on a Williot diagram with rigid-body rotation.

Williot–Mohr diagram (Section 7.3)

7.56 The truss is loaded by a force F at D that acts at an angle of 45°. All loaded members have the same absolute strain $|\varepsilon| = 1‰$.

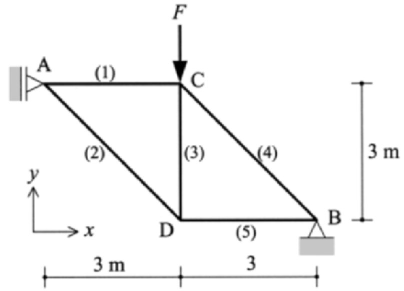


Questions:

a. Indicate which members are tension, compression and zero-force members. Determine the changes in length of the members.
b. If the truss is isolated from the roller support at B and you fix the direction of member AC, determine the displacements of joints B to E using a Williot diagram. In the diagram, use a scale of 1 cm ≡ 4 mm for the displacements.
c. Draw the Mohr diagram and, from the Williot–Mohr diagram, read off the displacements of the joints B to E.

7.57 As problem 7.56, but now fix the direction of member AD.

7.58 For the given load, all loaded members in the given truss have the same absolute strain $|\varepsilon| = 1/1500$.



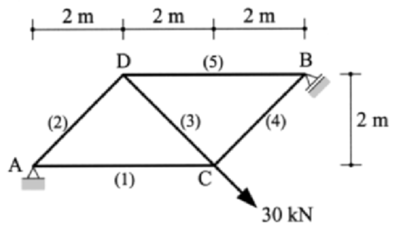
Questions:

a. Draw the Williot diagram when the direction of member BC is fixed. Draw the displacements in the diagram five times the actual size (choose a scale of 1 cm ≡ 2 mm).
b. Correct the Williot diagram with the Mohr diagram and read off the final displacements of the joints from the figure.

7.59 As problem 7.58, but now fix the direction of member BD.

7.60 The truss shown is supported on a hinge at A and on a roller at B. The roller track is at 45°. The truss is loaded by a force of 30 kN that acts at C at 45°. All members have the same modulus of elasticity $E = 200$ GPa. The cross-sectional areas of the members are

- members 1 and 5: $A = 100\sqrt{2}$ mm^2;
- members 2 to 4: $A = 100$ mm^2.

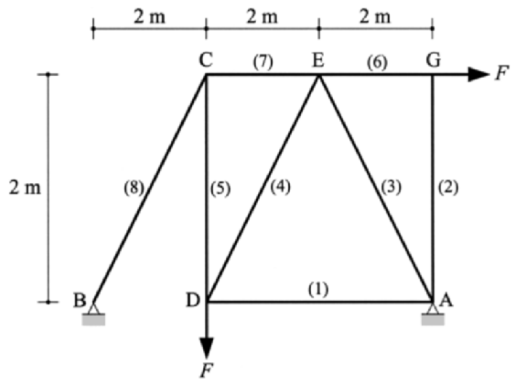


Questions:

a. Draw the Williot diagram when the direction of member AC is fixed. For the displacements in the diagram, use a scale of 1 cm ≡ 1.5 mm.
b. Correct the Williot diagram with the Mohr diagram, and read off all joint displacements from the Williot–Mohr diagram.

7.61 As problem 7.60, but now fix the direction of member AD.

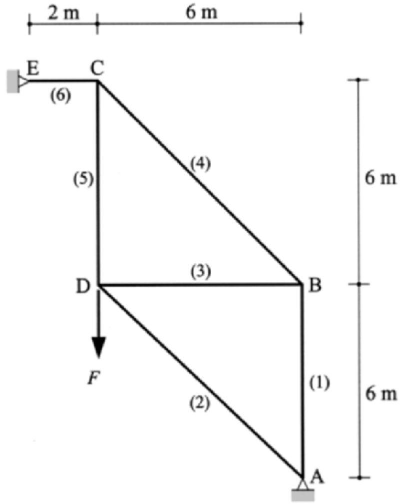
7.62 For the given loading by the forces F at D and G, all the loaded members in the truss are subject to the same absolute strain $|\varepsilon| = 1‰$.



Questions:

a. Draw the Williot diagram when the direction of member AD is fixed. For the displacements in the diagram, use a scale of 1 cm ≡ 2 mm.
b. Correct the Williot diagram with the Mohr diagram, and read off all joint displacements from the Williot–Mohr diagram.

7.63 For all loaded members in the truss shown, it holds under the given load that $|\varepsilon| = 1‰$. An exception is member EC, which is rigid and therefore does not deform.

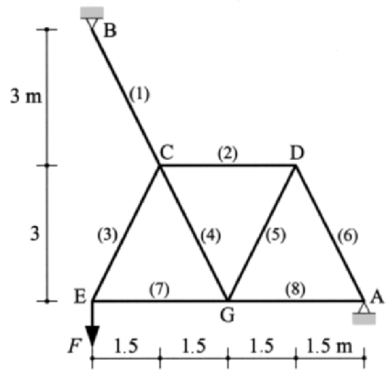


Questions:

a. Draw the Williot diagram when the direction of member AD is fixed. Draw the displacements in the Williot diagram five times as large as they actually are (with a scale of 1 cm ≡ 2 mm).
b. Correct the Williot diagram with the Mohr diagram, and read off all joint displacements from the Williot–Mohr diagram.

7.64 As problem 7.63, but now member EC is no longer rigid and is subject to an absolute strain $|\varepsilon| = 1‰$.

7.65 The truss shown is supported in a hinge at A. At C it is suspended from member BC. If the vertical force F acts at E, all loaded members have $|\varepsilon| = 1/1500$, except member BC, which is rigid and therefore does not deform.



Questions:

a. Determine the member forces, expressed in terms of F, and find the changes in length of the members.
b. Since joint G is directly joined to the fixed points A and B, a Williot diagram can be used to find all the joint displacements in one go. If you do not see this, isolate the structure at B and draw the Williot diagram for a fixed direction of member AG. Do so on squared paper with 5 mm squares and use a scale of 1 square ≡ 1 mm.
c. Correct the Williot diagram with the Mohr diagram.
d. Determine the final joint displacements.

7.66 As problem 7.65, but now member BC has $|\varepsilon| = 1/1500$ and is no longer rigid.

Comment: Problems 7.45 to 7.55 are suitable also for the method based on a Williot diagram that is corrected by a Mohr diagram.

8 Deformation Due to Bending

In this chapter we discuss how to determine the deflection due to bending. Using a number of examples, we distinguish four methods:

1. The method starting directly from the *moment distribution* (Section 8.1);
2. The method based on the *differential equation for bending* (Section 8.2);
3. The method using *forget-me-nots* (Section 8.3);
4. The *moment-area method* (Section 8.4).

The chapter ends with two properties related to the moment-area theorems for a simply supported beam, namely the rotation of the beam at the supports, and a formula to approximate the maximum deflection (Section 8.5).

A brief description is given of the four methods to determine the deflection due to bending.

The first two methods are based on *differential equations*, and have an analytical character.

The first method starts directly from the moment distribution, and is discussed in Section 8.1. Since we have to know the moment distribution beforehand, this method is applicable only for statically determinate beams.

This is not the case for the second method covered in Section 8.2, based on the differential equation for bending. This method can be used for both statically determinate and statically indeterminate beams. Moreover it has the benefit that, once the deflections have been determined, they can be used to determine the distribution of the section forces, and the magnitude of the support reactions. A disadvantage of the second method is that it is more laborious than the first.

The first two methods, based on differential equations, can actually be used only for straight prismatic beams with a relatively simple loading. With these methods the requested quantities are found as functions of the location x. However, in practice it is usually sufficient to know a limited number of relevant values at certain locations. This aspect is met by the latter two methods: the method using *forget-me-nots* and the *moment-area method*.

Forget-me-nots are formulas for the deflections and rotations associated with a limited number of simple standard cases. By combining the forget-me-nots (superposition) it is often easy to determine the deflections and rotations for many cases that are not standard. A number of examples are given in Section 8.3.

A disadvantage of the method using forget-me-nots is that one has to have the formulas at hand, or know them off by heart.

In principle, forget-me-nots can be applied to bent bars and portals also, although it is not always particularly easy to do so. The method covered in Section 8.4 based on the *moment-area theorems* is more appropriate for that case. Since the moment-area theorems can be applied only when the moment distribution is known, the method is usually limited to statically determinate structures.

A *visual interpretation*[1] is used with the moment-area formulae.

The methods based on forget-me-nots and moment-area theorems have a strong *visual* focus: when applying them, we need to have a good picture of the deformation of the members in the structure.

All four methods use the basic equations derived in Section 4.3 for a member subject to bending in the xz plane. These equations and the formulae to be derived from them in this chapter apply only if the x axis coincides with the member axis (and therefore passes through the normal centre of the cross-section) and the z axis coincides with one of the principal directions of the cross-section.[2]

Comment: The deflection w in the z direction, as a function of the location x, describes the shape of the member axis after deformation through bending. The member axis deformed through bending is also known as the *bending curve* or *elastic curve*.[3]

8.1 Direct determination from the moment distribution

The deflection due to bending can be determined directly from a known moment distribution by using the constitutive equation found in Section 4.3.2:

$$M_z = EI_{zz}\kappa_z,$$

---

1 The moment-area formulae can also be treated analytically, an approach not followed here.

2 See Section 4.3.2.

3 The name “elastic curve” applies only to elastic beams.

and the kinematic equation found in Section 4.3.1:

$$\kappa_z = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,.$$

By substituting the expression for the curvature κ_z in the constitutive equation we find the following relationship between the bending moment M_z and the second derivative of the deflection w:

$$M_z = -EI_{zz}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,.$$

Written as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{M_z}{EI_{zz}}\,,$$

this formula can be used to determine the deflection w directly from the moment distribution by twice integrating the quantity M_z/EI_{zz} (curvature κ_z). For of a prismatic member the bending stiffness EI_{zz} is independent of x and we can find $EI_{zz}w$ by twice integrating the bending moment M_z:

$$EI_{zz}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -M_z.$$

The importance of this formula is demonstrated using four examples. To simplify the notation, the z indices are omitted.

Example 1: Cantilever beam with uniformly distributed load

The cantilever beam AB in Figure 8.1a, with length ℓ and bending stiffness EI, is fixed at A, and carries a uniformly distributed load q along its entire length.

Questions:

a. Determine the deflection w and rotation φ as functions of x.
b. Determine the deflection w_B and rotation φ_B at the free end B.

Solution:

a. The differential equation

$$EI\frac{d^2w}{dx^2} = -M$$

is relevant only in a coordinate system. If the bending moment diagram is given with deformation symbols, as in Figure 8.1b, these symbols will have to be translated into signs related to the coordinate system. In Figure 8.1b, the sign of the bending moment is shown between brackets.

For the xz coordinate system in Figure 8.1a the bending moment is

$$M = -\tfrac{1}{2}q(\ell - x)^2 = -\tfrac{1}{2}qx^2 + q\ell x - \tfrac{1}{2}q\ell^2.$$

This expression is found by writing down the moment equilibrium for the part of the beam to the right of the cross-section at a distance x from fixed end A (see Figure 8.1c). The bending moment to be determined in the cross-section is assumed to act in a positive sense. It is therefore important to take account of the applicable sign conventions.

Substitution of the moment distribution in the differential equation leads to

$$EI\frac{d^2w}{dx^2} = -M = +\tfrac{1}{2}qx^2 - q\ell x + \tfrac{1}{2}q\ell^2.$$

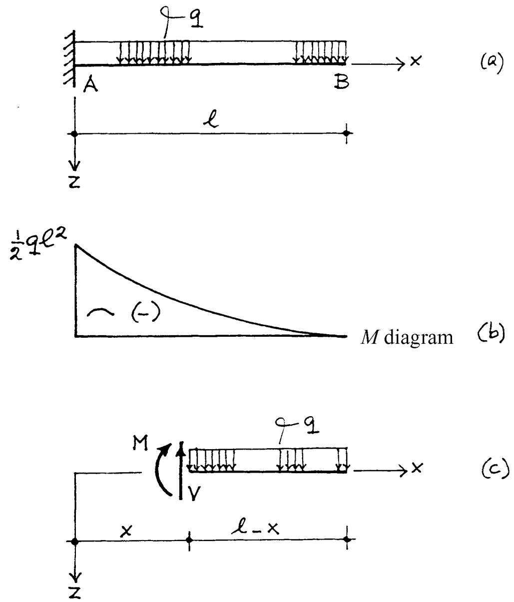


Figure 8.1 (a) A cantilever beam with uniformly distributed load and (b) associated bending moment diagram. (b) The bending moment M as a function of x is found directly from the moment equilibrium of the part of the beam to the right of the section at a distance x from fixed end A.

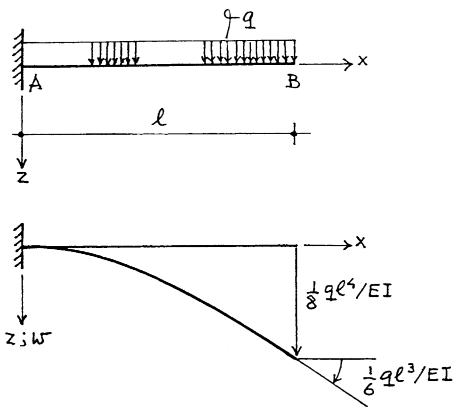


Figure 8.2 Deformation of the cantilever beam due to a uniformly distributed load. To make the image clearer, the deformations have been magnified with respect to the length of the beam. The deformed beam axis is also referred to as the bending curve or elastic curve.

After a single integration we find

$$EI\frac{\mathrm{d}w}{\mathrm{d}x} = +\tfrac{1}{6}qx^3 - \tfrac{1}{2}q\ell x^2 + \tfrac{1}{2}q\ell^2 + C_1,$$

and after a second integration

$$EIw = +\tfrac{1}{24}qx^4 - \tfrac{1}{6}q\ell x^3 + \tfrac{1}{4}q\ell^2x^2 + C_1x + C_2.$$

C_1 and C_2 are integration constants. These constants follow from the boundary conditions with respect to w and/or $\varphi = -\mathrm{d}w/\mathrm{d}x$. In this example we know that at the fixed end A ($x = 0$) both the vertical deflection w and the rotation φ of the cross-section (or the slope $\mathrm{d}w/\mathrm{d}x$ of the member axis) are zero:

$$x = 0,\ \ w = 0;$$

$$x = 0,\ \ \varphi = -\frac{\mathrm{d}w}{\mathrm{d}x} = 0.$$

The boundary conditions lead to

$$C_1 = C_2 = 0.$$

The distribution of the deflection w and rotation φ are

$$w = \frac{q\ell^4}{24EI}\left(+\frac{x^4}{\ell^4} - 4\frac{x^3}{\ell^3} + 6\frac{x^2}{\ell^2}\right),$$

$$\varphi = -\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{q\ell^3}{6EI}\left(-\frac{x^3}{\ell^3} + 3\frac{x^2}{\ell^2} - 3\frac{x}{\ell}\right).$$

The expressions for w and φ as functions of x can be written down in various ways. We have selected a form in which the term between brackets is dimensionless.

b. The requested deflection and rotation at the free end B ($x = \ell$) are found by substituting $x/\ell = 1$ in the expressions for w and φ:

$$w_\mathrm{B} = \frac{q\ell^4}{8EI},$$

$$\varphi_\mathrm{B} = -\frac{q\ell^3}{6EI}.$$

Figure 8.2 shows the deformation of the beam modelled as a line element. The deformed beam axis is known as the *bending curve* or *elastic curve*. To ensure the image is legible, the deflections have been magnified with respect to the length of the beam.

According to the calculation, φ_B is negative. This means that the rotation at B is opposite to the positive direction in the given xz coordinate system. Figure 8.2 shows the actual rotation at B with its magnitude (i.e. the absolute value).

Example 2: Simply supported beam with uniformly distributed load
The simply supported beam AB in Figure 8.3a has a length ℓ and bending stiffness EI. The beam carries a uniformly distributed load q over the entire length ℓ.

Questions:

a. Determine the maximum deflection.
b. Determine the rotations φ_A and φ_B at the supports.

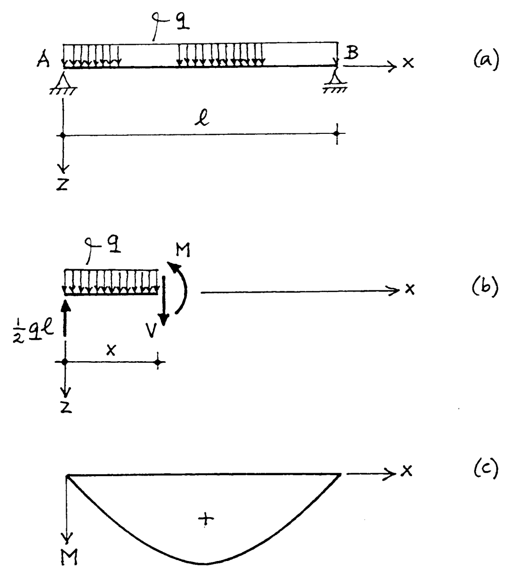


Figure 8.3 (a) A simply supported beam with uniformly distributed load. (b) The bending moment M as a function of x is found from the moment equilibrium of the part of the beam to the left of the cross-section at a distance x from support A. (c) Bending moment diagram.

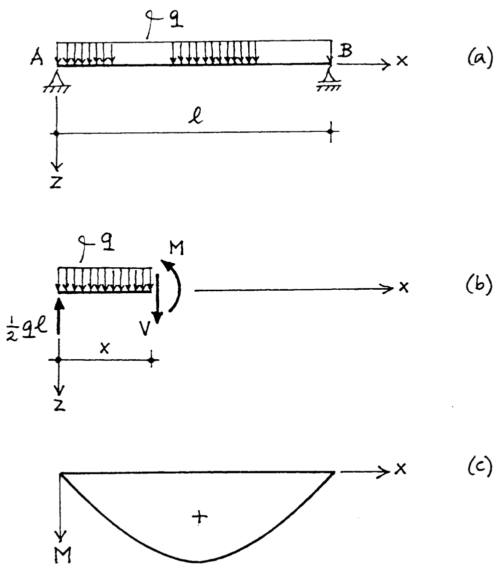


Figure 8.3 (a) A simply supported beam with uniformly distributed load. (b) The bending moment M as a function of x is found from the moment equilibrium of the part of the beam to the left of the cross-section at a distance x from support A. (c) Bending moment diagram.

Solution:

a. In the given xz coordinate system the moment distribution is

$$M = -\tfrac{1}{2}qx^2 + \tfrac{1}{2}q\ell x.$$

This can be derived from the equilibrium of the part of the beam to the left or right of the cross-section at a distance x from A (see Figure 8.3b). The M diagram is shown in Figure 8.3c.

Using the differential equation

$$EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -M = +\tfrac{1}{2}qx^2 - \tfrac{1}{2}q\ell x,$$

we find by repeated integration

$$EI\frac{\mathrm{d}w}{\mathrm{d}x} = +\tfrac{1}{6}qx^3 - \tfrac{1}{4}qx^2\ell + C_1,$$

$$EIw = +\tfrac{1}{24}qx^4 - \tfrac{1}{12}qx^3\ell + C_1x + C_2.$$

The two boundary conditions required are found at the supports A ($x = 0$) and B ($x = \ell$), where the deflection is zero:

$$x = 0,\ w = 0;$$

$$x = \ell,\ w = 0.$$

From the first boundary condition it follows that

$$C_2 = 0.$$

The second boundary condition gives

$$+\tfrac{1}{24}q\ell^4 - \tfrac{1}{12}q\ell^3 \cdot \ell + C_1\ell + C_2 = 0,$$

from which we find

$$C_1 = +\tfrac{1}{24}q\ell^3.$$

With C_1 and C_2 we have found the following expressions for w and φ:

$$w = \frac{q\ell^4}{24EI}\left(+\frac{x^4}{\ell^4} - 2\frac{x^3}{\ell^3} + \frac{x}{\ell}\right),$$

$$\varphi = -\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{q\ell^3}{24EI}\left(-4\frac{x^3}{\ell^3} + 6\frac{x^2}{\ell^2} - 1\right).$$

Figure 8.4 shows the deformed beam. The deflections have again been magnified with respect to the length of the beam.

The deflection w is largest where $\varphi = -\mathrm{d}w/\mathrm{d}x$ is zero. Formally, we have to look for the value of x for which $\varphi = 0$. On the basis of the mirror symmetry we can expect this to be the case at midspan C, so for $x/\ell = 1/2$:

$$w_\mathrm{C} = \frac{q\ell^4}{24EI}\left\{+\left(\tfrac{1}{2}\right)^4 - 2\times\left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)\right\} = \frac{5}{384}\frac{q\ell^4}{EI}.$$

Check:

$$\varphi_\mathrm{C} = \left(-\frac{\mathrm{d}w}{\mathrm{d}x}\right)_\mathrm{C} = \frac{q\ell^3}{24EI}\left\{-4\times\left(\tfrac{1}{2}\right)^3 + 6\times\left(\tfrac{1}{2}\right)^2 - 1\right\} = 0.$$

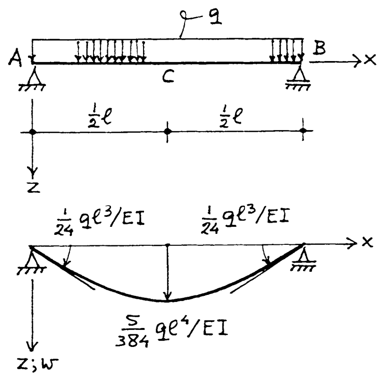


Figure 8.4 Deformation of the simply supported beam subject to a uniformly distributed load. The displacements have been magnified with respect to the length of the beam.

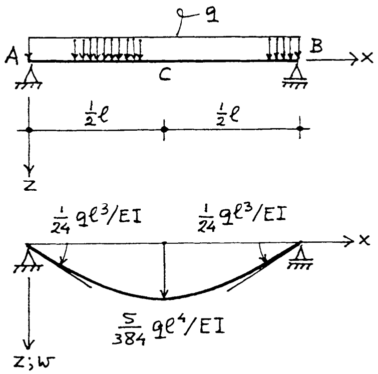


Figure 8.4 Deformation of the simply supported beam subject to a uniformly distributed load. The displacements have been magnified with respect to the length of the beam.

The slope of the member axis is indeed zero at midspan.

b. At the supports A ($x/\ell = 0$) and B ($x/\ell = 1$) the rotations are

$$\varphi_A = -\frac{q\ell^3}{24EI},$$

$$\varphi_B = +\frac{q\ell^3}{24EI}.$$

In line with the mirror symmetry of the loading case, the rotations at A and B are of equal magnitude, but have opposite directions.

Example 3: Simply supported beam loaded by a uniformly distributed load and a couple at one of the supports

The simply supported beam AB in Figure 8.5a, with a span of 4 m, carries a uniformly distributed load of 24 kN/m along its entire length and is additionally loaded by a couple of 96 kNm at B. The beam has a bending stiffness $EI = 2000$ kNm^2.

Questions:

a. Determine the equation of the elastic curve.
b. Determine the maximum deflection.
c. Determine the rotations φ_A and φ_B at the supports.

Solution (units in kN and m):

a. In Figure 8.5b the part of the beam to the left of the cut at a distance x from A has been isolated. A shear force V and a bending moment M are acting in the cut (cross-section). Both section forces are shown in the figure in their positive directions according to the coordinate system given. The support reaction at A is 24 kN. The calculation is left to the reader.

From the moment equilibrium of the isolated part of the beam

$$\sum T_y|\text{cut} = -(24\text{ kN}) \cdot x + (24\text{ kN/m}) \cdot x \cdot \tfrac{1}{2}x + M = 0$$

it follows that

$$M = -(12\text{ kN/m}) \cdot x^2 + (24\text{ kN}) \cdot x.$$

The M diagram is shown in Figure 8.5c.

From the differential equation

$$EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -M = (12\text{ kN/m}) \cdot x^2 - (24\text{ kN}) \cdot x$$

we find after repeated integration

$$EI\frac{\mathrm{d}w}{\mathrm{d}x} = (4\text{ kN/m}) \cdot x^3 - (12\text{ kN}) \cdot x^2 + C_1,$$

$$EIw = (1\text{ kN/m}) \cdot x^4 - (4\text{ kN}) \cdot x^3 + C_1 x + C_2.$$

Figure 8.5 (a) A simply supported beam loaded by a uniformly distributed load and a couple at the right-hand support B. (b) The isolated part of the beam for determining the bending moment M as a function of x. (c) Bending moment diagram. (d) Deformed beam axis or elastic curve. The displacements have been magnified at least 100 times as compared to the structural dimensions. The elastic curve has a point of inflection where the bending moment is zero and the deformation sign changes sign.

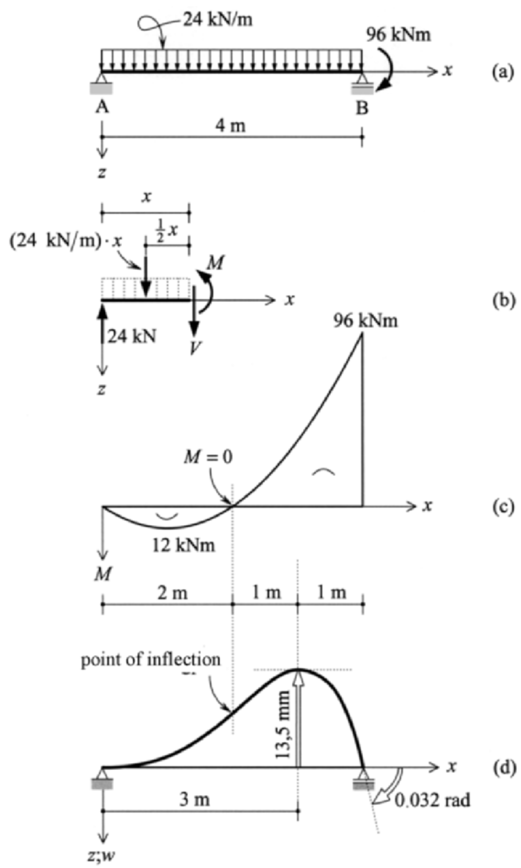


Figure 8.5

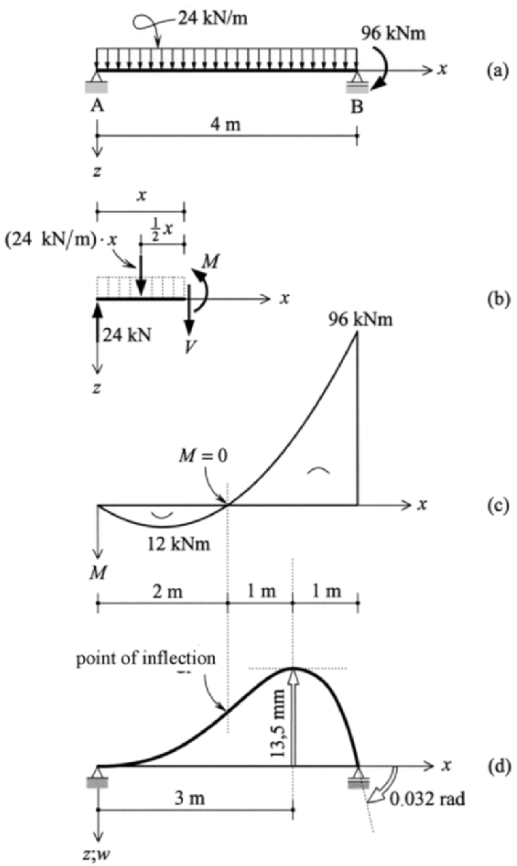


Figure 8.5

The integration constants C_1 and C_2 follow from the boundary conditions at the supports A ($x = 0$) and B ($x = 4$ m), where deflection w is zero:

$$x = 0, \quad w = 0;$$
$$x = 4\text{ m}, \quad w = 0.$$

Elaborating the boundary conditions gives

$$C_1 = 0 \text{ and } C_2 = 0.$$

With $EI = 2000\,\text{kNm}^2$ we find for the deflection w and rotation φ the following functions of x:

$$w = \frac{x^4}{2000\text{ m}^3} - \frac{4x^3}{2000\text{ m}^2},$$

$$\varphi = -\frac{\text{d}w}{\text{d}x} = -\frac{x^3}{500\text{ m}^3} + \frac{3x^2}{500\text{ m}^2}.$$

The beam axis deformed by bending (the elastic curve) is shown in Figure 8.5d. The deflections are magnified by more than 100 times the structural dimensions. The effect of the couple at B is apparently so large that the beam does not bend downwards, but rather bends upwards everywhere.

Comment: The bending moment is zero at $x = 2$ m (see the M diagram in Figure 8.5c). At this point the bending moment changes sign. The curvature, proportional to the bending moment, also changes sign here. This is found in the reversal of the deformation symbols in the M diagram, as well in the shape of the elastic curve (see Figure 8.5d). Where the bending moment changes sign, the elastic curve has *a point of inflection*.

b. The deflection w is extreme where the slope $\mathrm{d}w/\mathrm{d}x$ of the elastic curve is zero (or where the rotation $\varphi = -\mathrm{d}w/\mathrm{d}x$ of the cross-section is zero):

$$\varphi = -\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{x^3}{500\ \mathrm{m}^3} + \frac{3x^2}{500\ \mathrm{m}^2} = 0 \Rightarrow x = 3\ \mathrm{m}.$$

This value of x substituted in the expression for w leads to

$$w_{(x=3\ \mathrm{m})} = \frac{(3\ \mathrm{m})^4}{2000\ \mathrm{m}^3} - \frac{4 \times (3\ \mathrm{m})^3}{2000\ \mathrm{m}^2} = -13.5 \times 10^{-3}\ \mathrm{m}.$$

The maximum deflection is therefore a deflection upwards of 13.5 mm at $x = 3$ m (see Figure 8.5d).

c. At the supports A ($x = 0$) and B ($x = 4$ m) the rotations are

$$\varphi_\mathrm{A} = 0,$$

$$\varphi_\mathrm{B} = -\frac{(4\ \mathrm{m})^3}{500\ \mathrm{m}^3} + \frac{3 \times (4\ \mathrm{m})^2}{500\ \mathrm{m}^2} = -0.032\ \mathrm{rad}\ (\approx 1.83^\circ).$$

Since $\varphi_\mathrm{A} = 0$ the elastic curve at A has a horizontal tangent (see Figure 8.5d).

Comment: For numerical calculations, it must be noted that the quantities φ and $\mathrm{d}w/\mathrm{d}x$ are expressed in radians.

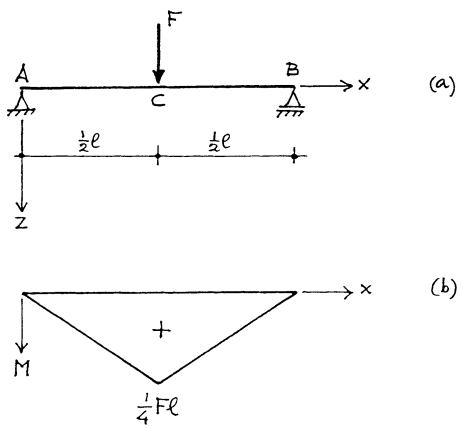


Figure 8.6 (a) A simply supported beam loaded by a force F at midspan with (b) the associated bending moment diagram.

Example 4: Simply supported beam loaded by a point load at midspan

The simply supported beam AB in Figure 8.6a, with length ℓ and bending stiffness EI, is loaded by a force F at midspan C.

Questions:

a. Determine the deflection w_{C} at midspan.
b. Determine the rotations φ_{A} and φ_{B} at the supports.

Solution:

a. The bending moment diagram is shown in Figure 8.6b. The maximum bending moment occurs at the point load and is $\frac{1}{4}F\ell$. From here the bending moment varies linearly to zero at the supports.

Since the bending moment diagram cannot be described by means of a single function for the entire beam, the fields to the left and right of the point load have to be examined separately. In this case it is possible to use *symmetry considerations* so that it is sufficient to consider only one half of the beam. Below we consider the left half of the beam. In the given coordinate system, the bending moment is

$$M = +\tfrac{1}{2}Fx \quad \left(0 \le x \le \tfrac{1}{2}\ell\right).$$

With

$$EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -M = -\tfrac{1}{2}Fx \quad \left(0 \le x \le \tfrac{1}{2}\ell\right)$$

we find after integrating

$$EI\frac{\mathrm{d}w}{\mathrm{d}x} = -\tfrac{1}{4}Fx^2 + C_1,$$

$$EIw = -\tfrac{1}{12}Fx^3 + C_1 x + C_2.$$

The integration constants follow from the end condition at A ($x = 0$) and the joining condition at C ($x = \frac{1}{2}\,\ell$):

$$x = 0,\ w = 0 \qquad \Rightarrow C_2 = 0;$$

$$x = \tfrac{1}{2}\,\ell,\ \frac{\mathrm{d}w}{\mathrm{d}x} = 0 \Rightarrow C_1 = +\tfrac{1}{16}\,F\ell^2.$$

The joining condition at C ($x = \frac{1}{2}\,\ell$) follows from the *mirror symmetry* of the loaded beam: the beam will not rotate at midspan so that the tangent to the member axis remains horizontal there.

In the left-hand halve of the beam ($0 \leq x \leq \frac{1}{2}\,\ell$) the deflection w and rotation φ are

$$w = \frac{F\ell^3}{48EI}\left(-4\frac{x^3}{\ell^3} + 3\frac{x}{\ell}\right),$$

$$\varphi = -\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{F\ell^2}{16EI}\left(+4\frac{x^2}{\ell^2} - 1\right).$$

The deformation of the beam is shown in Figure 8.7.

The deflection at C ($x/\ell = 1/2$), at the point load, is also the maximum deflection of the beam:

$$w_\mathrm{C} = w_\mathrm{max} = \frac{F\ell^3}{48EI}\,.$$

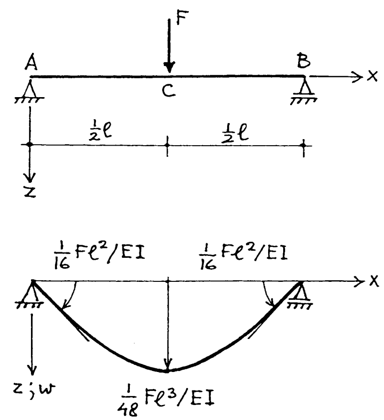


Figure 8.7 Bending curve of the simply supported beam carrying a point load at midspan.

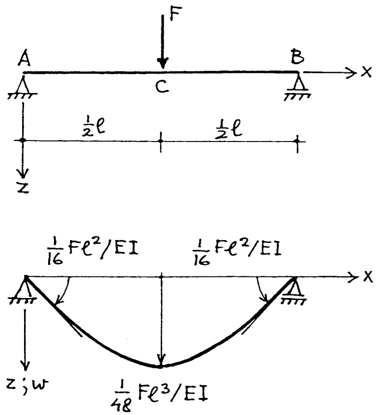


Figure 8.7 Bending curve of the simply supported beam carrying a point load at midspan.

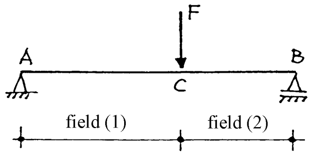


Figure 8.8 If in this case we want to determine the bending curve using the differential equation, we have to distinguish between two fields, each with their own differential equation.

b. For the rotation at support A ($x/\ell = 0$) we find

$$\varphi_A = -\frac{F\ell^2}{16EI}.$$

The rotation at support B is equal and opposite to that at A. In the given coordinate system that means

$$\varphi_B = +\frac{F\ell^2}{16EI}.$$

Figure 8.7 shows the deflections and rotations as they actually occur, and their magnitudes (i.e. their absolute value).

Comment: If the point of application C of the force F is not at midspan, the beam will have to be divided into two *fields*: field (1) to the left of C and field (2) to the right of C, as shown in Figure 8.8.

Now, for each field we have to determine the equation of the M diagram. By integrating twice we can then find the distribution of the deflections $w^{(1)}$ for field (1) and $w^{(2)}$ for field (2).

Two integration constants occur per field, so that a total of four integration constants have to be found from the two end conditions and the two joining conditions.

The end conditions state that the deflection at the supports is zero:

$$w_A^{(1)} = 0,$$

$$w_B^{(2)} = 0.$$

The two joining conditions state that the fields (1) and (2) are rigidly connected to one another at C, and therefore must have the same deflection and rotation:

$$w_C^{(1)} = w_C^{(2)},$$

$$\varphi_C^{(1)} = \varphi_C^{(2)}.$$

Determining the deflections for this relatively simple loading case is starting to become quite laborious. Other methods, such as those with *forget-me-nots* (Section 8.3) or *moment-area theorems* (Section 8.4), are preferable here.

8.2 Differential equation for bending

In Section 4.13, we derived the fourth-order differential equation for bending in the xz plane for a prismatic member (see also Table 8.1):

$$-EI\frac{d^4w}{dx^4} + q = 0,$$

or written otherwise:

$$EI\frac{d^4w}{dx^4} = q.$$

If the distributed load q is known, we can integrate four times to find the deflection w. After each integration a single integration constant appears. The total number of integration constants in the general solution is therefore four.

Table 8.1 The differential equations for bending in the xz plane for a prismatic beam with bending stiffness EI.

	kinematic relationships	constitutive relationship	static relationships	differential equation
bending	$\varphi = -\frac{dw}{dx}$ $\kappa = \frac{d\varphi}{dx}$ $\rightarrow \kappa = -\frac{d^2w}{dx^2}$	$M = EI\kappa$	$\frac{dV}{dx} + q_z = 0$ $\frac{dM}{dx} - V = 0$ $\rightarrow \frac{d^2M}{dx^2} + q_z = 0$	$-EI\frac{d^4w}{dx^4} + q_z = 0$

The integration constants follow from the *boundary conditions* (*end conditions* at an end and *joining conditions* at a joint). These are the conditions that certain quantities, expressed in terms of w (or relationships between these quantities) have to meet at the member ends and joints between the fields.

A member end always provides two end conditions and a joint between fields always provides four joining conditions. They can relate to

Table 8.1 The differential equations for bending in the xz plane for a prismatic beam with bending stiffness EI.

	kinematic relationships	constitutive relationship	static relationships	differential equation
bending	$\varphi=-\frac{dw}{dx}$ $\kappa=\frac{d\varphi}{dx}$ → $\kappa=-\frac{d^2w}{dx^2}$	$M=EI\kappa$	$\frac{dV}{dx}+q_z=0$ $\frac{dM}{dx}-V=0$ → $\frac{d^2M}{dx^2}+q_z=0$	$-EI\frac{d^4w}{dx^4}+q_z=0$

$$w,$$

$$\varphi=-\frac{\mathrm{d}w}{\mathrm{d}x}\,,$$

$$M=EI\kappa=-EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\,,$$

$$V=\frac{\mathrm{d}M}{\mathrm{d}x}=-EI\frac{\mathrm{d}^3w}{\mathrm{d}x^3}\,.$$

The expressions for the bending moment M and the shear force V can be simply derived from the basic relationships in Table 8.1.

Determining the deflection w using the fourth-order differential equation for bending is more labour-intensive than the previous method based on a second-order differential equation, but is more effective for a non-uniformly distributed load and can also be used for statically indeterminate beams. We illustrate this using four examples.

To simplify the notation, the derivative of x is indicated by means of a prime ($'$):

$$\frac{\mathrm{d}(\ldots)}{\mathrm{d}x}=(\ldots)'.$$

In this new notation, the differential equation for bending is

$$EIw'''' = q,$$

and the oter relations are

$$\varphi = -w',$$

$$M = -EIw'',$$

$$V = M' = -EIw'''.$$

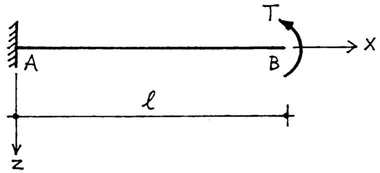


Figure 8.9 A cantilever beam, loaded at its free end by a couple.

Example 1: Cantilever beam loaded by a couple at its free end
Beam AB in Figure 8.9, with length ℓ and bending stiffness EI, is fixed at A and loaded by a couple T at the free end B.

Questions:
a. Determine the deflection w as a function of x.
b. Determine the deflection w_B and rotation φ_B at the free end B.

Solution:
a. There is no distributed load, so $q = 0$. In this case the differential equation for bending is

$$EIw'''' = 0.$$

Through repeated integration we find

$$EIw''' = C_1,$$

$$EIw'' = C_1 x + C_2,$$

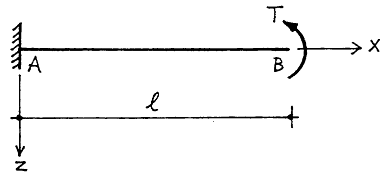


Figure 8.9 A cantilever beam, loaded at its free end by a couple.

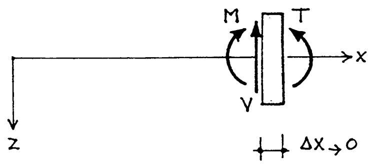


Figure 8.10 The boundary conditions $V = 0$ and $M = +T$ at the free end follow from the equilibrium of a small beam element with length Δx ($\Delta x \to 0$).

$$EIw' = \tfrac{1}{2}C_1x^2 + C_2x + C_3,$$

$$EIw = \tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4.$$

There are four boundary conditions: two at A and two at B. At A ($x = 0$) we know that the deflection w and rotation $\varphi = -w'$ are zero:

$$x = 0,\ w = 0 \qquad \Rightarrow C_4 = 0,$$

$$x = 0,\ \varphi = -w' = 0 \Rightarrow C_3 = 0.$$

At B ($x = \ell$) we know that the shear force V is zero:

$$x = \ell,\ V = -EIw''' = 0 \Rightarrow C_1 = 0.$$

We also know the bending moment at B: $M = +T$.

$$x = \ell,\ M = -EIw'' = +T \Rightarrow C_2 = -T.$$

The fact that $V = 0$ and $M = +T$ at B can also be checked by looking at the equilibrium of a small member element with length Δx ($\Delta x \to 0$) (see Figure 8.10).

Now that we have determined the integration constants we can write down w, φ, M and V as functions of x:

$$w = -\frac{T\ell^2}{2EI}\frac{x^2}{\ell^2},$$

$$\varphi = -w' = +\frac{T\ell}{EI}\frac{x}{\ell},$$

$$M = -EIw'' = T,$$

$$V = -EIw''' = 0.$$

Figure 8.11 shows the deformed beam.

b. The deflection and rotation at B ($x/\ell = 1$) is

$$w_{\mathrm{B}} = -\frac{T\ell^2}{2EI},$$

$$\varphi_{\mathrm{B}} = \left(-\frac{\mathrm{d}w}{\mathrm{d}x}\right)_{\mathrm{B}} = +\frac{T\ell}{EI}.$$

Comment: Checking whether the expressions for M and V agree with the M and V diagrams is left to the reader.

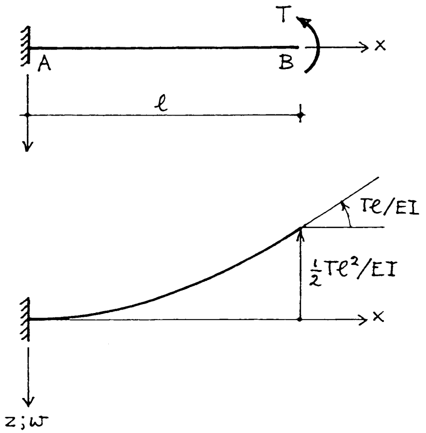


Figure 8.11 Elastic curve of a cantilever beam loaded by a couple at its free end.

Example 2: Water-retaining sheet-pile wall fixed in a concrete floor
The water-retaining sheet-pile wall, fixed in a concrete floor, is modelled in Figure 8.12 as the cantilever beam AB. The beam has a length ℓ and bending stiffness EI. The water pressure is linearly distributed from q at the fixed end A to zero at the free end B. In the given xz coordinate system

$$q(x) = -q\frac{x}{\ell} + q.$$

Questions:
a. Determine the expressions for w, φ, M and V as functions of x.
b. Determine the deflection w_{B} and rotation φ_{B} at the free end B.

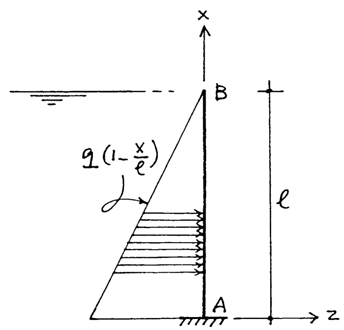


Figure 8.12 The water pressure on a water-retaining sheet-pile wall, fixed in a concrete floor and modelled as a cantilever beam.

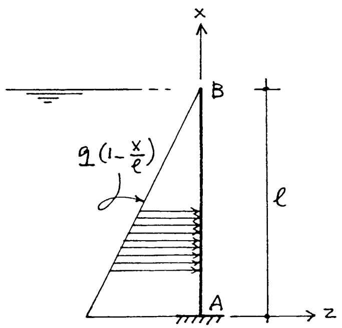


Figure 8.12 The water pressure on a water-retaining sheet-pile wall, fixed in a concrete floor and modelled as a cantilever beam.

Solution:
a. The differential equation for bending is

$$EIw'''' = q(x) = -q\frac{x}{\ell} + q.$$

Through integration we find

$$EIw''' = -\tfrac{1}{2}\frac{q}{\ell}x^2 + qx + C_1,$$

$$EIw'' = -\tfrac{1}{6}\frac{q}{\ell}x^3 + \tfrac{1}{2}qx^2 + C_1x + C_2,$$

$$EIw' = -\tfrac{1}{24}\frac{q}{\ell}x^4 + \tfrac{1}{6}qx^3 + \tfrac{1}{2}C_1x^2 + C_2x + C_3,$$

$$EIw = -\tfrac{1}{120}\frac{q}{\ell}x^5 + \tfrac{1}{24}qx^4 + \tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4.$$

Each boundary gives two boundary conditions. The boundary conditions at A ($x = 0$) lead to

$$x = 0,\ w = 0 \qquad \Rightarrow C_4 = 0,$$

$$x = 0,\ \varphi = -w' = 0 \Rightarrow C_3 = 0.$$

From the boundary conditions at B ($x = \ell$) we find the following using $C_3 = C_4 = 0$:

$$x = \ell,\ M = -EIw'' = 0 \Rightarrow -\tfrac{1}{6}q\ell^2 + \tfrac{1}{2}q\ell^2 + C_1\ell + C_2 = 0,$$

$$x = \ell,\ V = -EIw''' = 0 \Rightarrow -\tfrac{1}{2}q\ell + q\ell + C_1 = 0.$$

These are two equations with C_1 and C_2 as the unknowns. The solution is

$$C_1 = -\tfrac{1}{2}q\ell,$$
$$C_2 = +\tfrac{1}{6}q\ell^2.$$

Together with $C_3 = C_4 = 0$ we now find

$$w = \frac{q\ell^4}{120EI}\left(-\frac{x^5}{\ell^5} + 5\frac{x^4}{\ell^4} - 10\frac{x^3}{\ell^3} + 10\frac{x^2}{\ell^2}\right),$$

$$\varphi = -w' = \frac{q\ell^3}{120EI}\left(+5\frac{x^4}{\ell^4} - 20\frac{x^3}{\ell^3} + 30\frac{x^2}{\ell^2} - 20\frac{x}{\ell}\right),$$

$$M = -EIw'' = \frac{q\ell^2}{120}\left(+20\frac{x^3}{\ell^3} - 60\frac{x^2}{\ell^2} + 60\frac{x}{\ell} - 20\right),$$

$$V = -EIw''' = \frac{q\ell}{120}\left(+60\frac{x^2}{\ell^2} - 120\frac{x}{\ell} + 60\right).$$

Check (after differentiating again):

$$q(x) = EIw'''' = \frac{q}{120}\left(-120\frac{x}{\ell} + 120\right) = -q\frac{x}{\ell} + q.$$

This is indeed the expression for the distributed load on the sheet-pile wall.

b. The deflection w and rotation φ at B ($x/\ell = 1$) are

$$w_B = \frac{q\ell^4}{30EI},$$

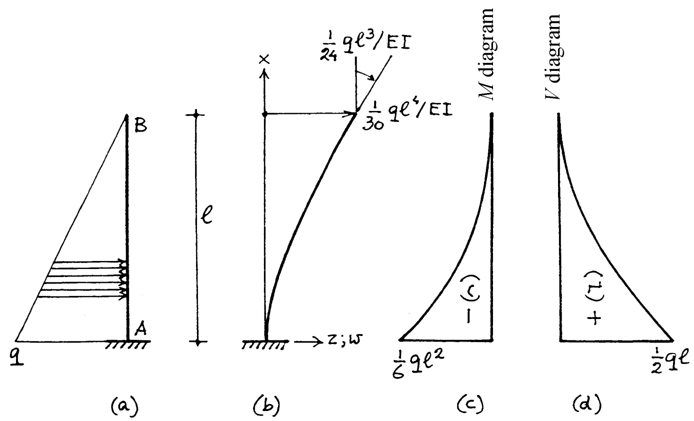


Figure 8.13 A water-retaining sheet piling, fixed in a concrete floor and modelled as a cantilever beam: (a) loading, (b) elastic curve, (c) bending moment diagram and (d) shear force diagram.

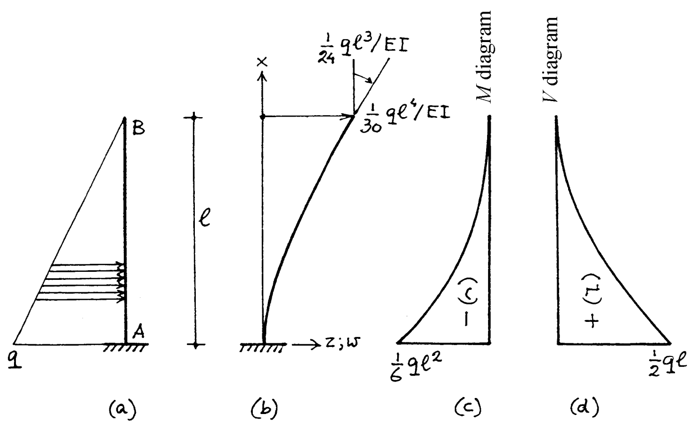


Figure 8.13 A water-retaining sheet piling, fixed in a concrete floor and modelled as a cantilever beam: (a) loading, (b) elastic curve, (c) bending moment diagram and (d) shear force diagram.

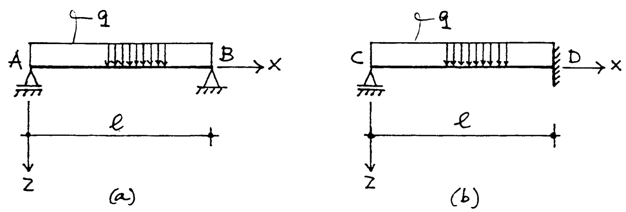


Figure 8.14 A beam with uniformly distributed load, supported in two different ways: (a) statically determinate and (b) statically indeterminate.

$$\varphi_\mathrm{B} = -\frac{q\ell^3}{24EI}.$$

A sketch of the deformed sheet-pile wall is given in Figure 8.13b.

The M and V diagrams are shown in Figures 8.13c and 8.13d. The deformation symbols are given between brackets.

The bending moment and shear force are largest at the fixed end A ($x/\ell = 0$):

$$M_\mathrm{A} = -\tfrac{1}{6}q\ell^2,$$

$$V_\mathrm{A} = +\tfrac{1}{2}q\ell.$$

It is left to the reader to check the correctness of these values by considering the equilibrium of the beam as a whole. In doing so, pay attention to both the magnitude and the sign.

Example 3: Beam with a uniformly distributed load, supported in two different ways

The third example relates to the two beams AB and CD in Figure 8.14. Both beams have the same span ℓ and bending stiffness EI, and carry a uniformly distributed load q over the entire length ℓ. Both beams are supported on a roller at the left-hand side, but the supports at the right-hand side are different: beam AB is supported by a hinge at the right end while beam CD is fixed at the right end.

Questions:

a. For both beams determine the deflection w as a function of x.
b. For beam CD draw the bending curve and the M and V diagrams.

Intermezzo: For beam CD the number of available equilibrium equations is insufficient to determine all the support reactions. Unlike beam AB, which is *statically determinate*, beam CD is *statically indeterminate*.

The force distribution in a statically indeterminate beam is such that the deformed beam "keeps fitting" between the supports. In order to find this force distribution, also the constitutive and kinematic relationships have to be involved in the calculation in addition to the equilibrium equations. In Figure 8.15, the relationship between the deflection w and the load q_z is shown for bending in the xz plane. Since all three relationships mentioned are used to derive the fourth-order differential equation for bending, this differential equation for bending can be used for statically indeterminate beams also.

Solution:
a. For both beams in Figure 8.14, the calculation is the same up to and including the boundary conditions at the roller support $x = 0$. There are no differences until applying the boundary conditions at the right-hand support $x = \ell$.

The same differential equation applies to both beams:

$$EIw'''' = q,$$

and so does the same general solution, found by integrating four times:

$$EIw''' = qx + C_1,$$
$$EIw'' = +\tfrac{1}{2}qx^2 + C_1x + C_2,$$
$$EIw' = +\tfrac{1}{6}qx^3 + \tfrac{1}{2}C_1x^2 + C_2x + C_3,$$
$$EIw = +\tfrac{1}{24}qx^4 + \tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4.$$

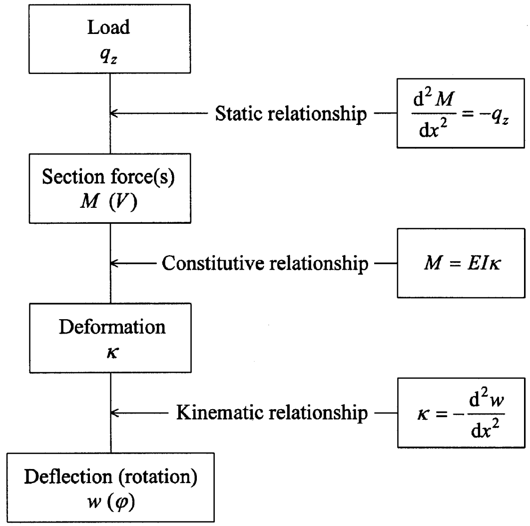


Figure 8.15 Schematic representation of the relationship between the load and displacement w for bending in the xz plane. For statically determinate structures, the moment distribution can be determined directly from the static relationship. This is not possible for statically indeterminate structures, and one needs all three types of basic relationships.

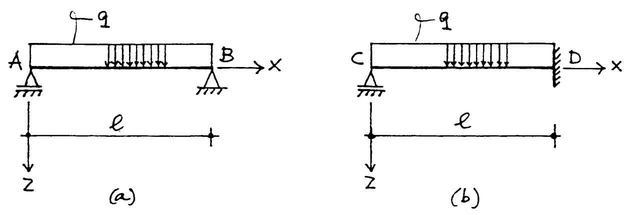


Figure 8.14 A beam with uniformly distributed load, supported in two different ways: (a) statically determinate and (b) statically indeterminate.

The boundary conditions at the roller supports A and D ($x = 0$) are the same:

$$x = 0,\ w = 0 \qquad\qquad \Rightarrow C_4 = 0,$$
$$x = 0,\ M = -EIw'' = 0 \Rightarrow C_2 = 0.$$

The boundary conditions at hinged support B and fixed support D ($x = \ell$) differ. Here the solutions for beam AB and CD start to deviate.

We first look for the solution for the statically determinate beam AB in Figure 8.14a.

Using $C_2 = C_4 = 0$, we find at B ($x = \ell$)

$$x = \ell,\ w = 0 \qquad\qquad \Rightarrow \tfrac{1}{24}\,q\ell^4 + \tfrac{1}{6}\,C_1\ell^3 + C_3\ell = 0,$$
$$x = \ell,\ M = -EIw'' = 0 \Rightarrow \tfrac{1}{2}\,q\ell^2 + C_1\ell = 0.$$

The solution of the two equations in C_1 and C_3 is

$$C_1 = -\tfrac{1}{2}\,q\ell,$$
$$C_3 = +\tfrac{1}{24}\,q\ell^3.$$

For beam AB we therefore find

$$w = \frac{q\ell^4}{24EI}\left(+\frac{x^4}{\ell^4} - 2\frac{x^3}{\ell^3} + \frac{x}{\ell}\right).$$

This is in line with what we found before in Section 8.1, Example 2, directly from the moment distribution using a second-order differential equation.

Finally we look at the statically indeterminate beam CD in Figure 8.14b.

At the fixed end D ($x = \ell$) we find, using $C_2 = C_4 = 0$.

$$x = \ell,\ w = 0 \qquad \Rightarrow \tfrac{1}{24}\,q\ell^4 + \tfrac{1}{6}\,C_1\ell^3 + C_3\ell = 0,$$

$$x = \ell,\ \varphi = -w = 0 \Rightarrow \tfrac{1}{6}\,q\ell^3 + \tfrac{1}{2}\,C_1\ell^2 + C_3 = 0.$$

The solution of these two equations in C_1 and C_3 is

$$C_1 = -\tfrac{3}{8}\,q\ell,$$

$$C_3 = +\tfrac{1}{48}\,q\ell^3.$$

The deflection of beam CD is therefore

$$w = \frac{q\ell^4}{48EI}\left(+2\frac{x^4}{\ell^4} - 3\frac{x^3}{\ell^3} + \frac{x}{\ell}\right),$$

and for φ, M and V we find

$$\varphi = -w' = \frac{q\ell^3}{48EI}\left(-8\frac{x^3}{\ell^3} + 9\frac{x^2}{\ell^2} - 1\right),$$

$$M = -EIw'' = \frac{q\ell^2}{48}\left(-24\frac{x^2}{\ell^2} + 18\frac{x}{\ell}\right),$$

$$V = -EIw''' = \frac{q\ell}{48}\left(-48\frac{x}{\ell} + 18\right).$$

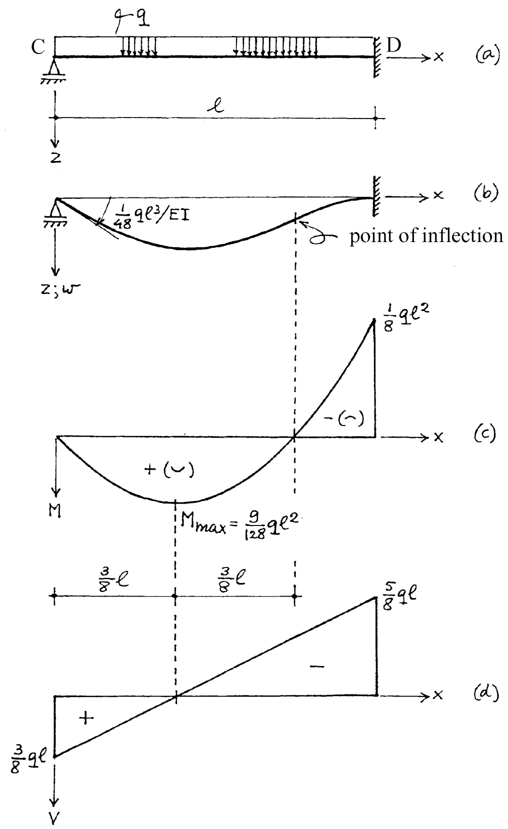


Figure 8.16 (a) Statically indeterminate beam loaded by a uniformly distributed load with (b) elastic curve, (c) bending moment diagram and (d) shear force diagram. The elastic curve has a point of inflection where the bending moment is zero. The field moment is largest where the shear force is zero.

b. The deformed beam CD is shown in Figure 8.16b.

The rotation at C ($x = 0$) is

$$\varphi_C = -\frac{q\ell^3}{48EI}.$$

In Figures 8.16c and 8.16d, the moment and shear force distributions are shown. The fixed-end moment at D ($x = \ell$) is

$$M_D = -\tfrac{1}{8}q\ell^2.$$

The shear forces at C ($x = 0$) and D ($x = \ell$) are

$$V_C = +\tfrac{3}{8}q\ell,$$

$$V_D = -\tfrac{5}{8}q\ell.$$

The field moment is a maximum where the shear force V is zero, namely at $x = \frac{3}{8}\ell$:

$$M_{max} = \tfrac{9}{128}q\ell^2.$$

At $x = \frac{3}{4}\ell$ the bending moment changes sign and therefore also the curvature. This is found in the reversal of the deformation symbols for bending, in the M diagram indicated in brackets. There is a point of inflection in the elastic curve.

Comment: The deformation symbols in the bending moment diagram can be an important help in correctly describing the deformation of the beam.

Comment: The reader is asked to draw the support reactions in the direction in which they act and to check the equilibrium of the beam as a whole.

Example 4: Prestressed plate bridge subject to solar radiation
A prestressed plate bridge across three supports is modelled in Figure 8.17 as line element ABC. The spans AB and BC have the same length $\ell = 30$ m. The plate thickness is constant at $h = 1$ m.

Subject to solar radiation, the temperature in the plate increases with a linear distribution across the plate thickness h by $T = 15°$ K at the upper side to zero at the underside. The distribution is shown in Figure 8.18.

The coefficient of thermal expansion is $\alpha = 10^{-5}$ K^{-1}. The modulus of elasticity of prestressed concrete is $E = 30 \times 10^3$ N/mm^2. The dead weight is 25 kN/m^3.

Questions:

a. Derive the differential equation for bending, using the constitutive relationship in which the influence of a change in temperature has been taken into account.
b. Using this differential equation, determine the deflection due to the change in temperature. Use symmetry considerations and work in the given xz coordinate system.
c. For the plate bridge modelled as a line element ABC sketch the bending curve. How large is the maximum deflection and where does it occur?
d. For a strip from the plate of 1-metre width, draw the bending moment and shear force diagrams. Find the support reactions.
e. Compare the previously determined values of the bending moments, shear forces and support reactions to those due to the dead weight.

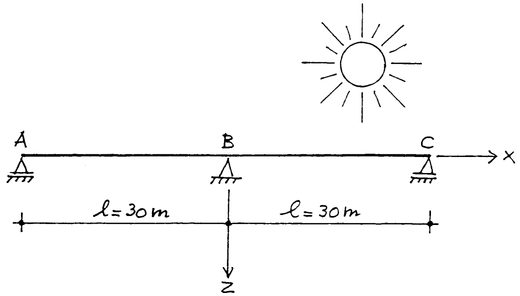


Figure 8.17 A prestressed plate bridge modelled as a line element across three supports, subject to solar radiation.

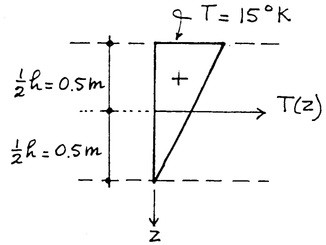


Figure 8.18 Due to the solar radiation, the increase in temperature in the plate is assumed to be linear across the thickness h, from $T = 15°$ K at the upper side to zero at the underside.

Table 8.2 The differential equations for bending in the xz plane for a prismatic beam with bending stiffness EI, taking into account thermal effects.

	kinematic relationships	constitutive relationship	static relationships	differential equation
bending	$\varphi=-w'$ $\kappa=\varphi'$ → $\kappa=-w''$	$M=EI(\kappa-\kappa^T)$	$V'+q_z=0$ $M'-V=0$ → $M''+q_z=0$	$-\{EI(w''+\kappa^T)\}''+q_z=0$

Solution:

a. Table 8.2 includes the various relationships for bending, taking into account thermal effects.[1]

The kinematic relationships remain unchanged:

$$\varphi = -w',$$

$$\kappa = \varphi' = -w''.$$

The effect of a temperature change finds expression only in the constitutive relationship. The following was derived in Section 4.12:

$$M = EI(\kappa - \kappa^T) = -EI(w'' + \kappa^T),$$

in which

$$\kappa^T = \alpha \frac{\mathrm{d}T(z)}{\mathrm{d}z} = -\alpha \frac{T}{h}\,.$$

The equilibrium equations also remain unchanged, so that

$$V = M' = \{-EI(w'' + \kappa^T)\}'$$

and

$$q = -V' = \{EI(w'' + \kappa^T)\}''$$

1 See Section 4.12.

or

$$\{EI(w'' + \kappa^T)\}'' = q.$$

This is the fourth-order differential equation for bending in its most general form.

If the member is prismatic (the bending stiffness EI is independent of x) then EI can be left outside the brackets. If κ^T is also independent of x, as it is here, this term disappears from the differential equation. Under these conditions it holds that

$$EIw'''' = q.$$

Comment: With this fourth-order differential equation the effect of the temperature change would appear to have disappeared from the problem. But appearances are deceptive: the temperature effect reappears in the relationships applicable for the boundary conditions.

b. Due to the mirror symmetry of the bridge about B, the computation can be restricted to half the bridge, with support B acting as a fixed end. The solution to the differential equation follows for beam BC (see Figure 8.19). We will initially work in symbols. At a later stage, we will enter the numerical values.

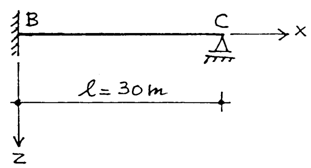


Figure 8.19 Due to the mirror symmetry of the bridge about B, the computation can be restricted to half the bridge, with support B acting as a fixed end.

There is no distributed load q in the example, so

$$EIw'''' = 0.$$

The general solution is found by integrating four times:

$$EIw''' = C_1,$$

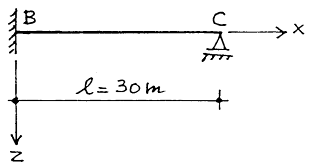


Figure 8.19 Due to the mirror symmetry of the bridge about B, the computation can be restricted to half the bridge, with support B acting as a fixed end.

$$EIw'' = C_1 x + C_2,$$

$$EIw' = \tfrac{1}{2} C_1 x^2 + C_2 x + C_3,$$

$$EIw = \tfrac{1}{6} C_1 x^3 + \tfrac{1}{2} C_2 x^2 + C_3 x + C_4.$$

The boundary conditions at the "fixed end" B are

$$x = 0,\ w = 0 \qquad \Rightarrow C_4 = 0,$$

$$x = 0,\ \varphi = -w' = 0 \Rightarrow C_3 = 0.$$

Using $C_3 = C_4 = 0$ we find the boundary conditions at support C:

$$x = \ell,\ w = 0 \qquad \Rightarrow \tfrac{1}{6} C_1 \ell^3 + \tfrac{1}{2} C_2 \ell^2 = 0,$$

$$x = \ell,\ M = -EIw'' - EI\kappa^T = 0 \Rightarrow -C_1 \ell - C_2 - EI\kappa^T = 0.$$

The solution of the two equations in C_1 and C_2 is

$$C_1 = -\tfrac{3}{2} EI \frac{\kappa^T}{\ell},$$

$$C_2 = \tfrac{1}{2} EI\kappa^T.$$

The deflection w is

$$w = \tfrac{1}{4} \kappa^T \ell^2 \left(-\frac{x^3}{\ell^3} + \frac{x^2}{\ell^2} \right).$$

The derivatives of w are

$$w' = \tfrac{1}{4}\,\kappa^T \ell \left(-3\frac{x^2}{\ell^2} + 2\frac{x}{\ell}\right),$$

$$w'' = \tfrac{1}{4}\,\kappa^T \ell \left(-6\frac{x}{\ell} + 2\right),$$

$$w''' = \tfrac{1}{4}\,\frac{\kappa^T}{\ell}(-6) = -\tfrac{3}{2}\,\frac{\kappa^T}{\ell},$$

and finally (as a check):

$$w'''' = 0.$$

c. In these expressions

$$\kappa^T = -\alpha\frac{T}{h} = -(10^{-5}\ \mathrm{K}^{-1})\frac{15\ \mathrm{K}}{1\ \mathrm{m}} = -150 \times 10^{-6}\ \mathrm{m}^{-1}$$

and

$$\ell = 30\ \mathrm{m}.$$

The bending deformation due to the solar radiation is shown in Figure 8.20. The deflection w is an extreme where w' is zero, this occurs at

$$x = \tfrac{2}{3}\,\ell = 20\ \mathrm{m},$$

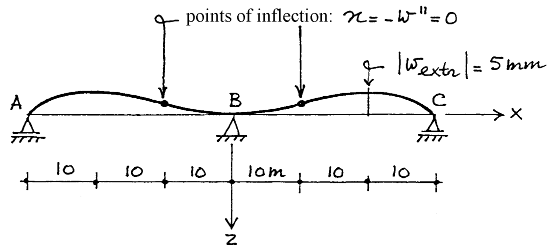


Figure 8.20 Elastic curve due to the effect of solar radiation. The beam bends upwards across its entire length. The curvature at the points of inflection is zero.

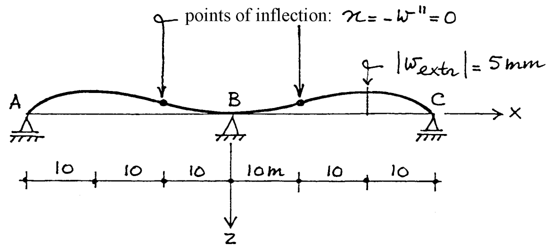


Figure 8.20 Elastic curve due to the effect of solar radiation. The beam bends upwards across its entire length. The curvature at the points of inflection is zero.

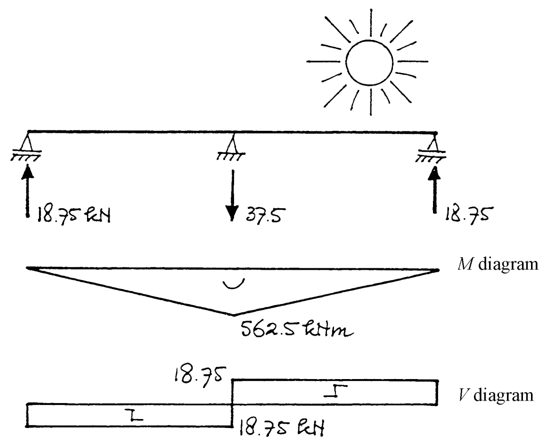


Figure 8.21 Support reactions, bending moment diagram and shear force diagram for a 1 metre wide plate strip, due to the solar radiation only. Note that the deformation sign in the M diagram near the end supports no longer agrees with the actual curvature in Figure 8.20.

and is

$$w_{\text{extr}} = \tfrac{1}{4}\,\kappa^T \ell^2 \left\{-\left(\tfrac{2}{3}\right)^3 + \left(\tfrac{2}{3}\right)^2\right\} = \tfrac{1}{27}\,\kappa^T \ell^2$$

$$= \tfrac{1}{27} \times (-1.5 \times 10^{-4}\ \text{m}^{-1})(30\ \text{m})^2 = -5 \times 10^{-3}\ \text{m}.$$

This is a displacement upwards of 5 mm.

The deformed structure has a *point of inflection* at $x = \frac{1}{3}\,\ell = 10$ m, where the curvature is zero. The reader is asked to check this.

d. To determine the bending moments and shear forces in a strip of the plate with width $b = 1$ m, we first determine the bending stiffness EI for this strip:

$$EI = E \cdot \tfrac{1}{12}\,bh^3 = (30 \times 10^3\ \text{N/mm}^2) \times \tfrac{1}{12} \times (1\ \text{m})(1\ \text{m})^3$$

$$= 2.5 \times 10^6\ \text{kNm}^2.$$

Be aware of the units!

The bending moment is

$$M = -EIw'' - EI\kappa^T = -\tfrac{1}{4}\,EI\kappa^T \left(-6\frac{x}{\ell} + 2\right) - EI\kappa^T$$

$$= -\tfrac{3}{2}\,EI\kappa^T \left(-\frac{x}{\ell} + 1\right)$$

$$= -\tfrac{3}{2} \times (2.5 \times 10^6\ \text{kNm}^2)(-150 \times 10^{-6}\ \text{m}^{-1}) \left(-\frac{x}{\ell} + 1\right)$$

$$= +562.5 \times \left(-\frac{x}{\ell} + 1\right)\ \text{kNm}.$$

The shear force is

$$V = M' = +\tfrac{3}{2}\, EI \frac{\kappa^T}{\ell}$$

$$= +\tfrac{3}{2} \times (2.5 \times 10^6 \text{ kNm}^2) \times \frac{-150 \times 10^{-6} \text{ m}^{-1}}{30 \text{ m}} = -18.75 \text{ kN}.$$

Figure 8.21 shows the support reactions, bending moment diagram, and shear force diagram due to the solar radiation, in which the plus and minus signs have been translated into deformation symbols.

Comment: The deformation symbol in the M diagram in Figure 8.21 is not in agreement with the actual curvature of the beam near the end supports, as shown in Figure 8.20. This is due to the fact that the actual curvature consists of two components, namely a curvature M/EI due to the bending moments that occur, and a curvature κ^T associated with a free deformation due to the change in temperature.

e. In Figure 8.22, as a comparison, the support reactions, bending moments and shear forces due to the dead weight of $q = 25$ kN/m are shown for the 1 metre wide strip of the plate. To do so, we used the results from Example 3, as presented in Figure 8.16.

The support moment at B decreases by 20% under the influence of the solar radiation. The maximum field moment on the other hand increases by some 14% from 1582 kNm in $x = 0.625\ell = 18.75$ m to 1800 kNm in $x = 0.6\ell = 18$ m.

Comment: The reader is asked to draw the bending moment and shear force diagrams for the 1 metre wide strip due to the combination of the dead weight and solar radiation, and to check the above-mentioned values.

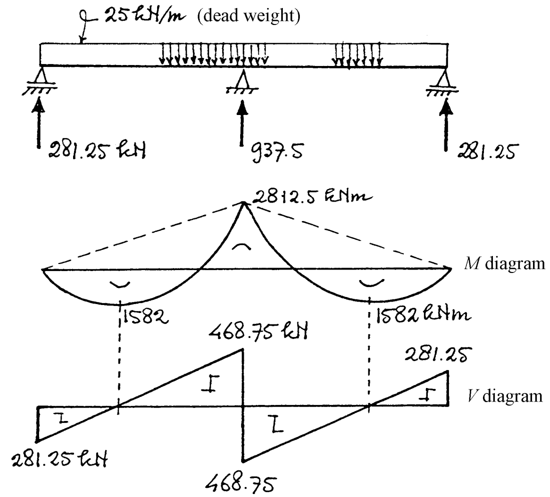
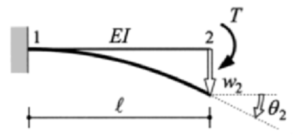
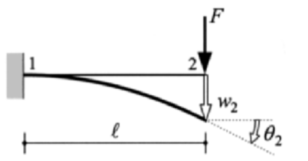
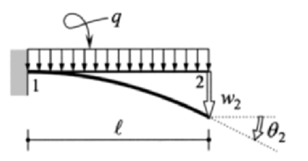


Figure 8.22 Support reactions, bending moment diagram and shear force diagram for a 1-metre wide plate strip, due to its dead weight.

Table 8.3 Forget-me-nots for a prismatic cantilever beam with length ℓ and bending stiffness EI.

(1)		$\theta_2 = \dfrac{T\ell}{EI}$	$w_2 = \dfrac{T\ell^2}{2EI}$
(2)		$\theta_2 = \dfrac{F\ell^2}{2EI}$	$w_2 = \dfrac{T\ell^3}{3EI}$
(3)		$\theta_2 = \dfrac{q\ell^3}{6EI}$	$w_2 = \dfrac{T\ell^4}{8EI}$

8.3 Forget-me-nots

A number of loading cases are so common that, like the Greek scientist Myosotis Palustris, we can include them in a table. In this way, Table 8.3 includes for three loading cases the expressions for the rotation and deflection at the free end of a prismatic cantilever beam with length ℓ and bending stiffness EI.[1]

The translation of the Greek word "myosotis" is "forget-me-not". The formulas in Table 8.3 are therefore known as the *forget-me-nots*.

They are easy to memorise if you remember the numerical pairs $(1, 2)$, $(2, 3)$, $(6, 8)$ for the coefficients in the denominator. If you forget the power of the length ℓ, this can be found from a dimension analysis.

By making clever use of these simple formulas, it is possible to determine more complicated loading cases also. A disadvantage of the method with forget-me-nots is that you have to know the formulas off by heart or have the table within reach.

The working method with forget-me-nots is strongly visual; we usually work without a coordinate system or do not use one in the first instance.[2] The deflection w is still a displacement normal to the member axis, but the positive direction of w is no longer derived from a coordinate system but rather from the picture that includes the sketch of the deformed beam. The positive direction of the rotation θ is also derived from that picture. See for

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[1] These expressions can be determined in the way shown in Sections 8.1 and/or 8.2; see Section 8.2, Example 1 for forget-me-not (1) and Section 8.1, Example 1, for forget-me-not (3). It is left to the reader to determine forget-me-not (2).

[2] For this reason there is no coordinate system in Table 8.3.

example the deformation due to bending of a fixed member with slope α in Figure 8.23a.

Sometimes the displacements[1] have to be named in a coordinate system. In order to avoid confusion with the letter w, in a xz coordinate system the displacements in the x and z directions respectively are no longer denoted by means of u and w, but rather with u_x and u_z, and the rotation with φ (actually φ_y).[2]

Figure 8.23b shows the deflection w at the free end of the fixed member with a slope, translated into the (*global*[3]) xz coordinate system:

$$u_x = +w \sin \alpha,$$

$$u_z = +w \cos \alpha.$$

Since the direction of the rotation θ at the free end is opposite to the positive sense of rotation in the xz coordinate system it also holds that

$$\varphi = -\theta.$$

The forget-me-nots are nearly always used in combination with a sketch of the deformed beam (the *elastic curve*). Before sketching the deformation of the beam it is advisable to first sketch the bending moment diagram, including the deformation symbols; calculations are generally not necessary (see Figure 8.24). With the deformation symbols, you can immediately see how

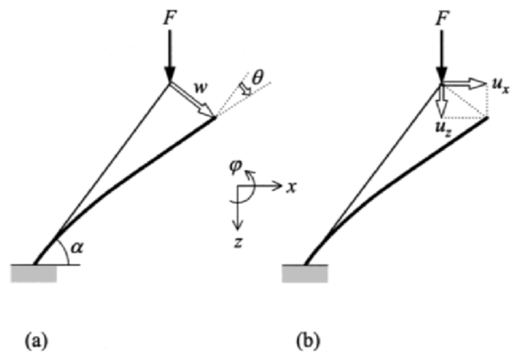


Figure 8.23 (a) The method with forget-me-nots has a strong visual orientation. One generally works without a coordinate system. The positive directions of the displacement w, normal to the member axis, and the rotation θ are derived from a picture in which the deformed beam is sketched. (b) In a (global) xz coordinate system, to avoid confusion with the letter w, the displacements are no longer named u and w, but u_x and u_z.

---

1 Rotations are also referred to as "displacements" when generalising.

2 Since there is no possibility of confusion, the index y is omitted for simplicity.

3 The position of a global coordinate system is related to the direction of gravity in the vast majority of cases.

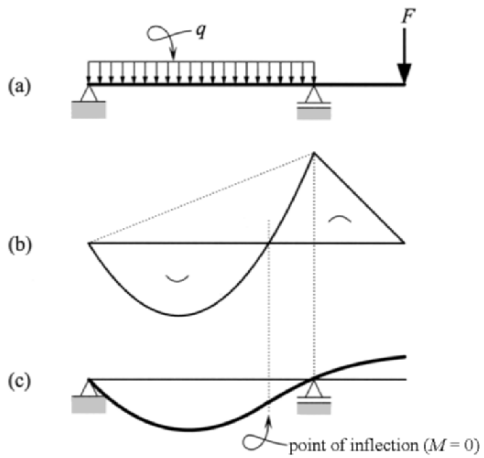


Figure 8.24 (a) Cantilever beam. Forget-me-nots are nearly always used with (c) a sketch of the deformed beam or elastic curve. For a good sketch, it is advisable to draw first (b) the bending moment diagram, including the deformation signs; calculations are often not required. The deformation signs in the bending moment diagram give a direct clue about the way in which the beam bends. The elastic curve has a point of inflection where the bending moment changes sign.

the beam curves. At the point where the bending moment changes signs, the elastic curve has a point of inflection. When sketching the elastic curve, we should take account of the limited freedom of movement in the supports. The beam in Figure 8.24, for example, cannot move vertically at its supports.

Comment: The elastic curve in Figure 8.24c is only a rough sketch. Calculations will have to show whether the beam does indeed move upwards at the free end. If not, the sketch will have to be changed.

Below we include 10 examples with forget-me-nots. For the first six, we use the forget-me-nots in Table 8.3. After Example 6, we present Table 8.4 with eight new forget-me-nots, to be used in the last four examples.

The application is limited to straight beams. All the beams are prismatic and have bending stiffness EI, unless indicated otherwise.

In principle, the forget-me-nots can also be used for bent and non-prismatic beams, although this is not generally very practicable. The method presented in Section 8.4, based on the moment-area theorems, is far more suitable for such cases.

Example 1: Tail-wagging effect
Beam ABC in Figure 8.25a is fixed at A and is loaded by the force F at B.

Question:
Determine the rotation and deflection at the free end C.

Solution:
Figure 8.25b shows a sketch of the bending moment diagram and Figure 8.25c shows a sketch of the elastic curve. Since the bending moment along BC is zero, this part of the beam remains straight (the curvature is

zero). The rotation θ_C at C is therefore equal to the rotation θ_B at B:

$$\theta_C = \theta_B.$$

The deflection w_C at C is equal to the deflection w_B at B, to which the deflection $b\theta_B$ due to the rotation θ_B at B has to be added:

$$w_C = w_B + b\theta_B.$$

The deflection $b\theta_B$ due to the rotation at B is known as the *tail-wagging effect*.

When determining the deflection due to the tail-wagging effect we again use the fact that for a small rotation θ_B the deflection $b\theta_B$ along the arc of the circle (with centre B and radius BC) can be replaced by a deflection of the same magnitude along the tangent to this circle, i.e. perpendicular to BC.[1]

Using forget-me-not (2), we find

$$\theta_B = \frac{Fa^2}{2EI} \text{ and } w_B = \frac{Fa^3}{3EI}.$$

The rotation and deflection at C are

$$\theta_C = \theta_B = \frac{Fa^2}{2EI},$$

$$w_C = w_B + b\theta_B = \frac{Fa^3}{3EI} + b\frac{Fa^2}{2EI} = \frac{Fa^2(2a+3b)}{6EI}.$$

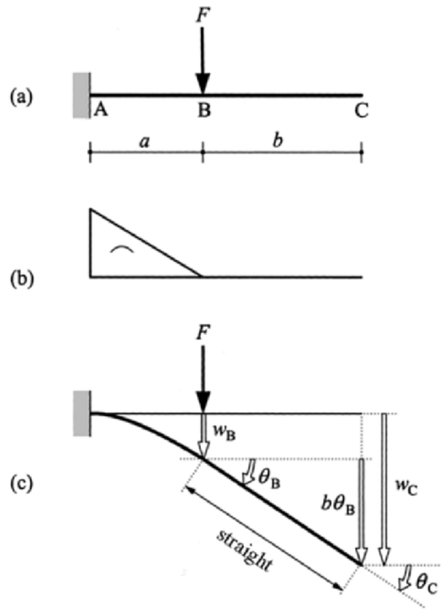


Figure 8.25 (a) Cantilever beam ABC loaded by a force F at B, with (b) the bending moment diagram and (c) the elastic curve. Since the bending moment along BC is zero, this part of the beam remains straight. The displacement at C due to the rotation θ_B at B is known as the *tail-wagging effect*.

[1] See also Section 7.1.2 and *Engineering Mechanics*, Volume 1, Section 15.3.2.

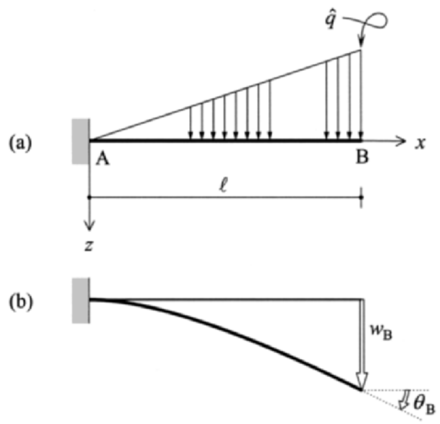


Figure 8.26 (a) A cantilever beam with triangular load and (b) a sketch of the elastic curve.

Example 2: Cantilever beam with triangular load

Cantilever beam AB in Figure 8.26a is fixed at A and carries a linearly distributed load $q(x) = \hat{q}x/\ell$ over its entire length ℓ.

Question:

Determine the deflection and rotation of the beam at the free end B.

Solution:

Figure 8.26b shows a sketch of the expected deformation of the beam with the rotation θ_B and deflection w_B at B. The sketch is based on common sense; it is not necessary to draw the M diagram first.

In order to determine the deflection at B, the distributed load is split into a large number of small forces $q(x)\Delta x$ (see Figure 8.27a):

$$q(x)\Delta x = \hat{q}\frac{x}{\ell}\,\Delta x.$$

For each of these forces, the contribution $\Delta\theta_B$ to the rotation at B and the contribution Δw_B to the deflection at B are calculated separately (see Figure 8.27b). In fact we now have the same situation as in Example 1. Using forget-me-not (2) we find

$$\Delta\theta_B = \frac{\hat{q}\frac{x}{\ell}\,\Delta x \cdot \Delta x^2}{2EI},$$

$$\Delta w_B = \frac{\hat{q}\frac{x}{\ell}\,\Delta x \cdot x^3}{3EI} + \frac{\hat{q}\frac{x}{\ell}\,\Delta x \cdot x^2}{2EI} \cdot (\ell - x).$$

The second term in the expression for Δw_{B} is the contribution from the tail-wagging effect.

The rotation and deflection at B are found by summing the contributions from all the small forces, i.e. integrating over the length ℓ:

$$\theta_{\mathrm{B}} = \sum \frac{\hat{q}\frac{x}{\ell}\,\Delta x \cdot x^2}{2EI} = \frac{\hat{q}}{2\ell EI}\int_0^{\ell} x^3\,\mathrm{d}x = \frac{1}{8}\,\frac{\hat{q}\ell^3}{EI}$$

$$w_{\mathrm{B}} = \sum \left(\frac{\hat{q}\frac{x}{\ell}\,\Delta x \cdot x^3}{3EI} + \frac{\hat{q}\frac{x}{\ell}\,\Delta x \cdot x^2}{2EI} \cdot (\ell - x) \right)$$

$$= \frac{\hat{q}}{6\ell EI}\int_0^{\ell} [2x^4 + 3x^3(\ell - x)]\,\mathrm{d}x = \frac{11}{120}\,\frac{\hat{q}\ell^4}{EI}\,.$$

These displacements are in line with the expectation in Figure 8.26b.

Translated to the xz coordinate system the displacements are

$$\varphi_{y;\mathrm{B}} = -\theta_{\mathrm{B}} = -\frac{1}{8}\,\frac{\hat{q}\ell^3}{EI}\,,$$

$$w_{z;\mathrm{B}} = w_B = +\frac{11}{120}\,\frac{\hat{q}\ell^4}{EI}\,.$$

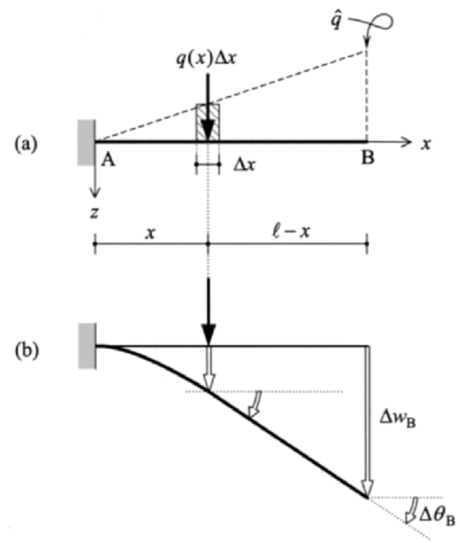


Figure 8.27 (a) To determine the displacement at B, the distributed load is split into a large number of small forces $q(x)\Delta x$. (b) The influence of a single force on the displacement and rotation at B.

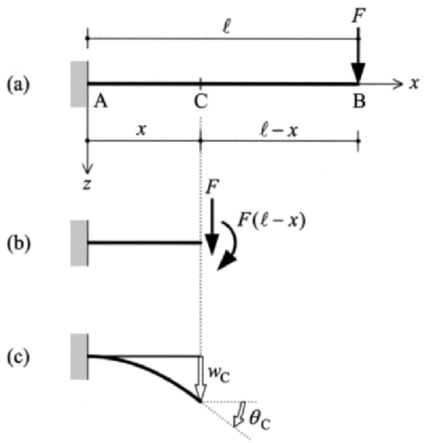


Figure 8.28 (a) A cantilever beam with a point load at the free end. (b) The load on part AC of the beam. (c) The deflection and rotation at C is determined by the deformation of AC only.

Example 3: Rotation and deflection as functions of *x*

Cantilever beam AB, with length ℓ, is loaded at the free end by a force F (see Figure 8.28a).

Question:
Determine the deflection u_z and rotation φ_y as functions of x.

Solution:
The deflection at C, at a distance x from A, is determined by the deformation of AC. In Figure 8.28b, part AC has been released from CB. The load on AC consists only of the section forces at C: a shear force F and a bending moment $F(\ell - x)$. The deformation due to this load is sketched in Figure 8.28c.

By applying the forget-me-nots (1) and (2) and superposing their contributions, the rotation and deflection at C are found to be

$$\theta_C = \frac{F(\ell - x)\cdot x}{EI} + \frac{F\cdot x^2}{2EI} = \frac{F\ell^2}{2EI}\left(-\frac{x^2}{\ell^2} + 2\frac{x}{\ell}\right),$$

$$w_C = \frac{F(\ell - x)\cdot x^2}{2EI} + \frac{F\cdot x^3}{3EI} = \frac{F\ell^3}{6EI}\left(-\frac{x^3}{\ell^3} + 3\frac{x^2}{\ell^2}\right).$$

These expressions can be written in various ways; here we have selected a form for which the term between brackets is dimensionless.

The requested displacements, formulated in the given xz coordinate system, are

$$\varphi_y(x) = -\theta_C = \frac{F\ell^2}{2EI}\left(+\frac{x^2}{\ell^2} - 2\frac{x}{\ell}\right),$$

$$u_z(x) = +w_C = \frac{F\ell^3}{6EI}\left(-\frac{x^3}{\ell^3} + 3\frac{x^2}{\ell^2}\right).$$

Example 4: Superposing deformations
Cantilever beam ABC is fixed at A, and carries a uniformly distributed load q over length BC (see Figure 8.29a).

Questions:
a. Determine the deflection at the free end C.
b. Determine the rotation at the free end.

Solution:
a. In Figure 8.29b the elastic curve has been sketched. The deflection at C is found by determining the deformation of AB and BC separately and superposing them.

Deformation AB
AB, released from BC, is a cantilever beam loaded at B by a shear force $q\ell$ and a bending moment $\frac{1}{2}q\ell^2$ (see Figure 8.29c). The forget-me-nots (1) and (2) give the rotation and deflection at B as

$$\theta_B = \frac{\frac{1}{2}q\ell^2 \cdot \ell}{EI} + \frac{q\ell \cdot \ell^2}{2EI} = \frac{q\ell^3}{EI},$$

$$w_B = \frac{\frac{1}{2}q\ell^2 \cdot \ell^2}{2EI} + \frac{q\ell \cdot \ell^3}{3EI} = \frac{7}{12}\frac{q\ell^4}{EI}.$$

The deformation of only AB (BC remains straight) results in the contributions $w_{C;1}$ and $w_{C;2}$ in Figure 8.29c. Contribution $w_{C;1}$ is equal to the

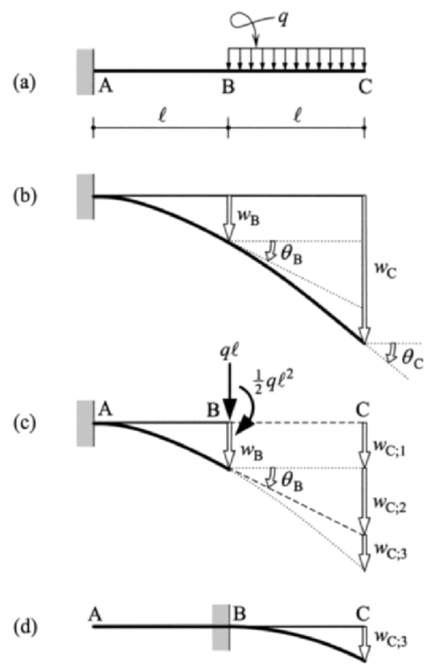


Figure 8.29 (a) Cantilever beam ABC with a uniformly distributed load on BC. (b) Sketch of the elastic curve. The displacement at C is found by determining the deformations of AB and BC separately and superposing them. (c) The deformation of only AB (BC remains straight) results in the contributions $w_{C;1}$ and $w_{C;2}$. (d) The deformation of only BC (AB remains straight) results in the contribution $w_{C;3}$.

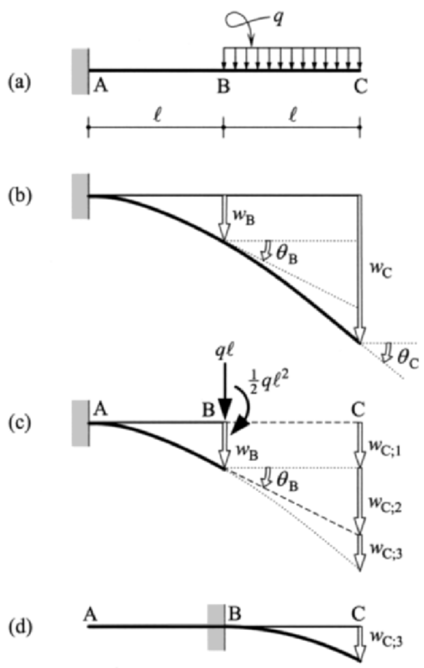


Figure 8.29 (a) Cantilever beam ABC with a uniformly distributed load on BC. (b) Sketch of the elastic curve. The displacement at C is found by determining the deformations of AB and BC separately and superposing them. (c) The deformation of only AB (BC remains straight) results in the contributions $w_{C;1}$ and $w_{C;2}$. (d) The deformation of only BC (AB remains straight) results in the contribution $w_{C;3}$.

deflection at B:

$$w_{C;1} = w_B = \frac{7}{12}\frac{q\ell^4}{EI}$$

and contribution $w_{C;2}$ is the tail-wagging effect due to the rotation at B:

$$w_{C;2} = \ell \cdot \theta_B = \ell \cdot \frac{q\ell^3}{EI} = \frac{q\ell^4}{EI}\,.$$

Deformation BC
The deflection due to the deformation of BC is shown in Figure 8.29c as contribution $w_{C;3}$. Since AB is straight, this deflection can be determined using forget-me-not (3) for the situation shown in Figure 8.29d:

$$w_{C;3} = \frac{q\ell^4}{8EI}\,.$$

Final deflection
The final deflection of C is equal to the sum of the three contributions $w_{C;1}$ to $w_{C;3}$:

$$w_C = w_{C;1} + w_{C;2} + w_{C;3} = \frac{7}{12}\frac{q\ell^4}{EI} + \frac{q\ell^4}{EI} + \frac{q\ell^4}{8EI} = \frac{41}{24}\frac{q\ell^4}{EI}\,.$$

Comment: The deflection at C is only 15% less than that for a load along the entire length of the beam (check this!). This is not surprising if one imagines that a load near the free end causes relatively the largest moments and, moreover, that the influence on the deflection is larger due to the larger lever in the tail-wagging effect.

b. The rotation at C can also be found by superposing the individual contributions due to the deformation of AB (see Figure 8.29c) and BC (see Figure 8.29d):

$$\theta_C = \underbrace{\frac{\frac{1}{2}q\ell^2\cdot\ell}{EI} + \frac{q\ell\cdot\ell^2}{2EI}}_{\substack{(=\theta_B)\text{ due to the}\\ \text{deformation of AB}}} + \underbrace{\frac{q\ell^3}{6EI}}_{\substack{\text{due to the}\\ \text{deformation of BC}}} = \frac{7}{6}\frac{q\ell^3}{EI}\,.$$

Example 5: Superposing loading cases

The loading case in Figure 8.30, covered earlier in Example 4, is elaborated here using an alternative method.

Question:
Determine the rotation and deflection at the free end C.

Solution:
In Figure 8.31a the load is split into two loading cases that can easily be determined using forget-me-nots: (1) a downward load q over the entire length of the beam and (2) an upward load q over half the beam length AB.

In Figure 8.31b, the elastic curve is sketched for each of the loading cases.[1] For determining the rotation and deflection in C, only forget-me-not (3) suffices.

Loading case (1)

$$\theta_{C;1} = \frac{q(2\ell)^3}{6EI} = \frac{4}{3}\frac{q\ell^3}{EI}\,,$$

[1] The sketch of the elastic curves in Figure 8.31b provides information only on the directions of the rotations and deflections that occur, and not on their magnitude. The deflections are not shown to scale.

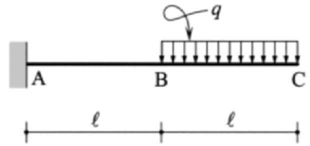


Figure 8.30 Beam ABC, fixed at A, with a uniformly distributed load on BC.

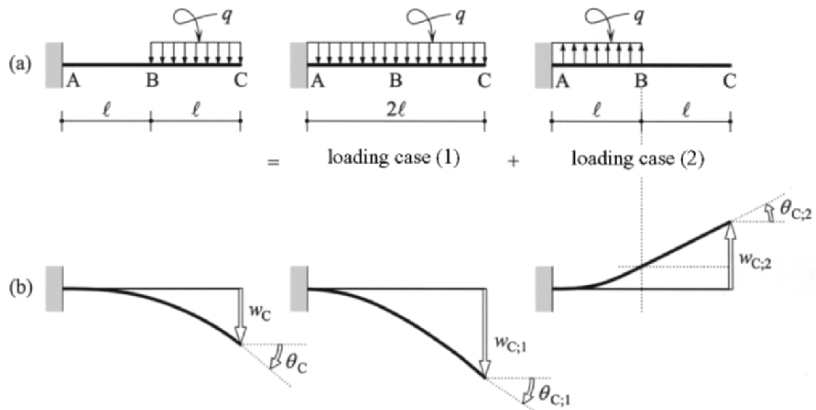


Figure 8.31 (a) The load can be split into two loading cases that are easy to determine using forget-me-nots: (1) a downward load q along the entire length of the beam and (2) an upward load q along half the beam length AB. (b) A sketch of the elastic curve for each of the loading cases. The sketches provide information only on the directions of the rotations and deflections that occur, and not on their magnitude.

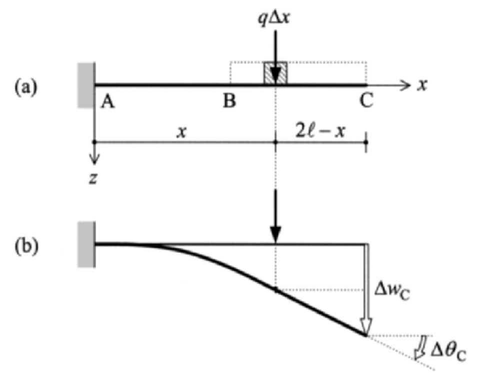


Figure 8.32 (a) In order to determine the deflection at C, the distributed load can be split into many small forces $q(x)\Delta x$. (b) The influence of one of those forces on the deflection and rotation at C.

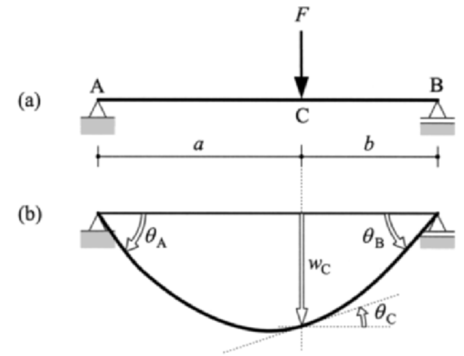


Figure 8.33 (a) A simply supported beam with an eccentric force F. (b) Sketch of the elastic curve.

$$w_{\mathrm{C};1} = \frac{q(2\ell)^4}{8EI} = 2\frac{q\ell^4}{EI}.$$

Loading case (2)

$$\theta_{\mathrm{C};2} = \frac{q\ell^3}{6EI},$$

$$w_{\mathrm{C};2} = \frac{q\ell^4}{8EI} + \frac{q\ell^3}{6EI} \cdot \ell = \frac{7}{24}\frac{q\ell^4}{EI}.$$

The second term in the expression for $w_{\mathrm{C};2}$ is the one due to the tail-wagging effect.

Final rotation and deflection
The final rotation and deflection at C is found by superposing the loading cases (1) and (2). Taking into account the directions in Figure 8.31b we find:

$$\theta_{\mathrm{C}} = \theta_{\mathrm{C};1} - \theta_{\mathrm{C};2} = \frac{4}{3}\frac{q\ell^3}{EI} - \frac{q\ell^3}{6EI} = \frac{7}{6}\frac{q\ell^3}{EI},$$

$$w_{\mathrm{C}} = w_{\mathrm{C};1} - w_{\mathrm{C};2} = 2\frac{q\ell^4}{EI} - \frac{7}{24}\frac{q\ell^4}{EI} = \frac{41}{24}\frac{q\ell^4}{EI}.$$

These values are in line with what we found previously in Example 4.

Comment: The deflections can also be determined as in Example 2, where we first determined the influence of a small force $q\Delta x$ on the rotation and deflection (see Figure 8.32) and then sum the influences of all these small forces through integrating along the length BC. The elaboration is left to the reader. With this approach we also have a superposition of loading cases.

Example 6: Simply supported beam with an eccentric point load
The simply supported beam AB in Figure 8.33a is loaded by the force F at C.

Question:
a. Determine the deflection and rotation at C, where the point load is applied.
b. Determine the rotations at the supports A and B.
c. Determine the maximum deflection and the place where it occurs.

Solution:
a. Figure 8.33b shows a sketch of the elastic curve. The rotation and deflection at C are θ_C and w_C respectively. In order to determine these values, the undeformed beam is released from the supports at A and B. Next, the beam is picked up at C, translated over a distance w_C and rotated through an angle θ_C, and fixed in this position (see Figure 8.34a). The resultant deflections at A and B are $(w_C + \theta_C \cdot a)$ $(\downarrow)$ and $(w_C - \theta_C \cdot b)$ $(\downarrow)$ respectively.

The beam fixed at an angle does not remain straight, but will deform due to the support reactions $Fb/(a+b)$ at A and $Fa/(a+b)$ at B. The deflections at A and B, shown in Figure 8.34b by w_A $(\uparrow)$ and w_B $(\uparrow)$, can be determined with forget-me-not (2). Since the beam is supported at A and B, the resultant deflections must be zero there. At A therefore

$$w_C + \theta_C \cdot a - w_A = w_C + \underbrace{\theta_C \cdot a}_{\text{tail-wagging effect due to the rotation at C}} - \underbrace{\frac{F\frac{b}{a+b} \cdot a^3}{3EI}}_{\text{due to the deformation of AC}} = 0.$$

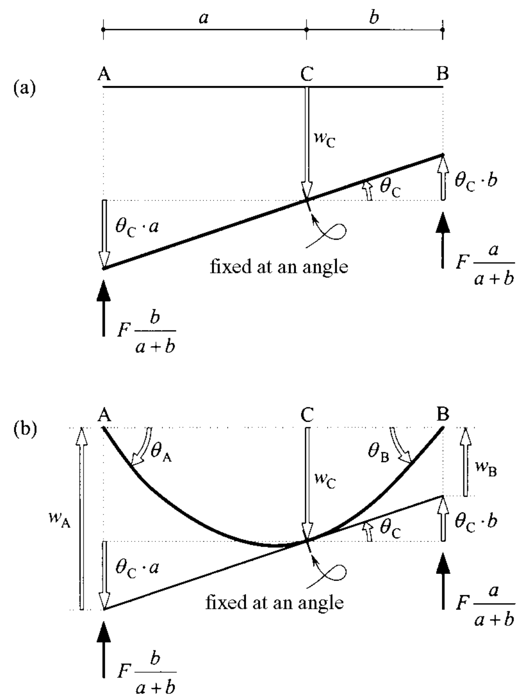


Figure 8.34 In order to determine θ_C and w_C, the undeformed beam is released from the supports at A and B. Subsequently, the beam is picked up at C, translated over a distance w_C, rotated through an angle θ_C, and fixed in this position. Beam ACB does not remain straight but will deform due to the support reactions $Fb/(a+b)$ at A and $Fa/(a+b)$ at B; the deflections are w_A and w_B. Since the beam is supported at A and B, the resultant deflections at A and B (due to the tail-wagging effect at C and the deformation of ACB) will be zero.

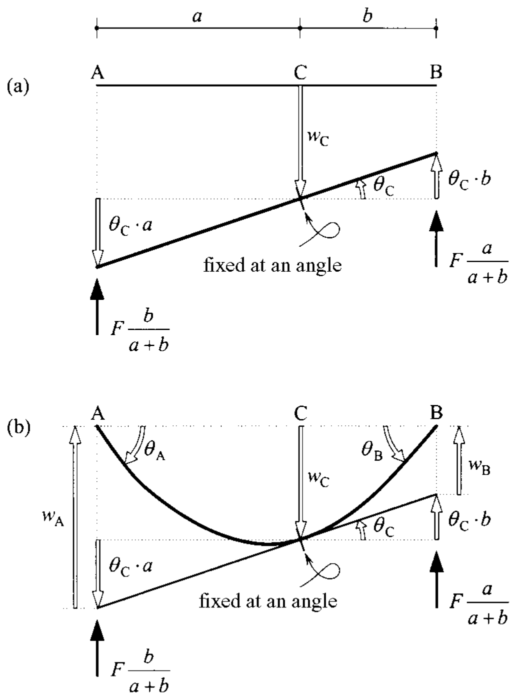


Figure 8.34 In order to determine θ_C and w_C, the undeformed beam is released from the supports at A and B. Subsequently, the beam is picked up at C, translated over a distance w_C, rotated through an angle θ_C, and fixed in this position. Beam ACB does not remain straight but will deform due to the support reactions $Fb/(a+b)$ at A and $Fa/(a+b)$ at B; the deflections are w_A and w_B. Since the beam is supported at A and B, the resultant deflections at A and B (due to the tail-wagging effect at C and the deformation of ACB) will be zero.

In the same way, at B

$$w_C - \theta_C \cdot b - w_B = w_C - \underbrace{\theta_C \cdot b}_{\substack{\text{tail-wagging effect due}\\ \text{to the rotation at C}}} - \underbrace{\frac{F\frac{a}{a+b} \cdot b^3}{3EI}}_{\substack{\text{due to the}\\ \text{deformation of AC}}} = 0.$$

This gives two equations with θ_C and w_C as the unknowns. The solution is

$$w_C = \frac{F}{3EI}\frac{a^2b^2}{a+b},$$

$$\theta_C = \frac{Fab}{3EI}\frac{a-b}{a+b}.$$

Comment: If $a < b$ then θ_C is negative. This means that the rotation at C is opposite to the direction of θ_C given in Figure 8.34.

Comment: If $\theta_C \neq 0$, the deflection at C, where the point load is applied, is not the maximum deflection. The maximum deflection only occurs at C if $\theta_C = 0$. In that case $a = b$, and the point load is applied at midspan. With $a = b = \frac{1}{2}\,\ell$ the maximum deflection is

$$w_{\text{max}} = \frac{F\ell^3}{48EI}.$$

This value is in line with that found earlier in Section 8.1, Example 4.

b. The rotations θ_A and θ_B at A and B respectively are found by superposing the influences of the now known rotation at C, and the deformation of the beam (see Figure 8.34):

$$\theta_A = -\theta_C + \frac{F\frac{b}{a+b}\cdot a^2}{2EI} = \frac{Fab}{6EI}\,\frac{a+2b}{a+b}\,, \tag{8.1}$$

$$\theta_B = +\theta_C + \frac{F\frac{a}{a+b}\cdot b^2}{2EI} = \frac{Fab}{6EI}\,\frac{2a+b}{a+b}\,.$$

In both expressions the second term gives the contribution due to the deformation of the beam fixed at C at an angle. Beware of the signs here.

c. Assume the maximum deflection occurs at D at a distance d from A; we assume that $d \leq a$ (see Figure 8.35a). In Figure 8.35b the elastic curve is shown for part AD. AD can be seen as a beam horizontally fixed at D, loaded at the free end A by the support reaction $Fb/(a+b)$. With a length d of the beam we have

$$\theta_A = \frac{F\frac{b}{a+b}\cdot d^2}{2EI}\,, \tag{8.2}$$

$$w_{max} = \frac{F\frac{b}{a+b}\cdot d^3}{3EI}\,. \tag{8.3}$$

We previously derived

$$\theta_A = \frac{Fab}{6EI}\,\frac{a+2b}{a+b}\,. \tag{8.1}$$

By equating the expressions (8.1) and (8.2) we find the location where the

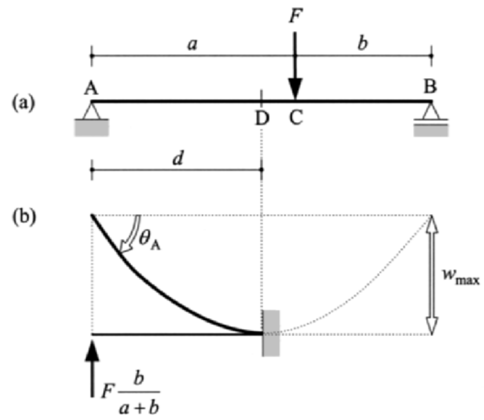


Figure 8.35 (a) A beam with (b) a sketch of the elastic curve, assuming the maximum deflection occurs at D.

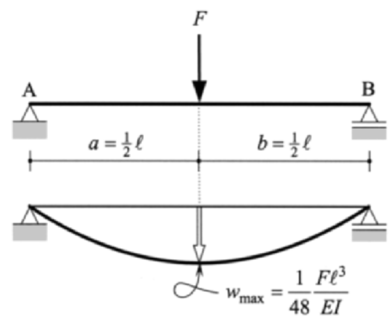
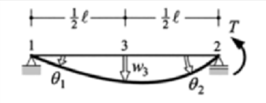
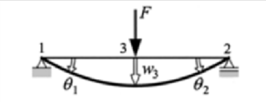
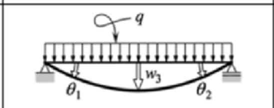
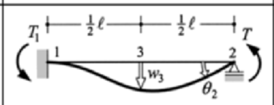
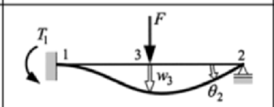
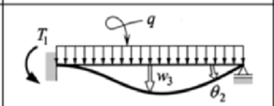
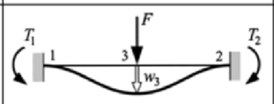
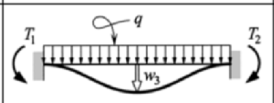


Figure 8.36 The maximum deflection of a simply supported prismatic beam with the point load F at the centre of the span ℓ is $F\ell^3/48EI$.

Table 8.4 Additional forget-me-nots for a prismatic beam supported at both ends with length ℓ and bending stiffness EI.

(4)		$\theta_1 = \frac{1}{6}\frac{T\ell}{EI};\ \theta_2 = \frac{1}{3}\frac{T\ell}{EI};\ w_3 = \frac{1}{16}\frac{T\ell^2}{EI}$
(5)		$\theta_1 = \theta_2 = \frac{1}{16}\frac{F\ell^2}{EI};\ w_3 = \frac{1}{48}\frac{F\ell^3}{EI}$
(6)		$\theta_1 = \theta_2 = \frac{1}{24}\frac{q\ell^3}{EI};\ w_3 = \frac{5}{384}\frac{q\ell^4}{EI}$
(7)		$\theta_2 = \frac{1}{4}\frac{T\ell}{EI};\ w_3 = \frac{1}{32}\frac{T\ell^2}{EI};\ T_1 = \frac{1}{2}T$
(8)		$\theta_2 = \frac{1}{32}\frac{F\ell^2}{EI};\ w_3 = \frac{7}{768}\frac{F\ell^3}{EI};\ T_1 = \frac{3}{16}F\ell$
(9)		$\theta_2 = \frac{1}{48}\frac{q\ell^3}{EI};\ w_3 = \frac{1}{192}\frac{q\ell^4}{EI};\ T_1 = \frac{1}{8}q\ell^2$
(10)		$w_3 = \frac{1}{192}\frac{F\ell^3}{EI};\ T_1 = T_2 = \frac{1}{8}F\ell$
(11)		$w_3 = \frac{1}{384}\frac{q\ell^4}{EI};\ T_1 = T_2 = \frac{1}{12}q\ell^2$

deflection is a maximum:

$$d = \sqrt{\tfrac{1}{3}\, a(a + 2b)}\,. \tag{8.4}$$

By substituting this value of d in (8.3), we find the expression for the maximum deflection:

$$w_{\max} = \frac{Fb(a^2 + 2ab)^{3/2}}{9\sqrt{3}\, EI(a + b)} \quad \text{as long as } a \geq b. \tag{8.5}$$

Comment: Expression (8.4) was derived assuming that $d \leq a$. From (8.4) in that case it follows $a \geq b$. Expression (8.5) therefore applies only under this condition.

Check formula (e): It is left to the reader to check whether for Figure 8.36 this formula with $a = b = \frac{1}{2}\,\ell$ indeed leads to

$$w_{\max} = \frac{F\ell^3}{48EI}\,.$$

The loading case in Figure 8.36 was discussed earlier in Section 8.1, Example 4. It could also be considered a forget-me-not as in Table 8.4.

Table 8.4 supplements the three forget-me-nots from Table 8.3 with eight loading cases. Of course the table can be expanded as required. The forget-me-nots (7) to (11) relate to statically indeterminate beams.[1] In these cases the fixed-end moments are also given. The vertical support reactions can be determined from the moment equilibrium of the beam as a whole.

---

1 These loading cases can be calculated using the method given in Section 8.2 by solving the fourth-order differential equation for bending.

Below there are four more examples in which the forget-me-nots from Table 8.4 are used.

Example 7: Hinged beam

The hinged beam ABCD in Figure 8.37a has hinged joints at B and C, and is fixed at A and D. AB has a bending stiffness EI, CD has a bending stiffness $2EI$ and BC has an infinite bending stiffness. The beam is loaded by a vertical force $6F$ at E, the centre of BC.

Questions:

a. Determine the vertical deflection at E, and sketch the elastic curve of ABCD.
b. How do the deflection and elastic curve asked for in (a) change if BC is not rigid, but has a finite bending stiffness EI?

Solution:

a. The forces at the hinges B and C on parts AB and CD of the beam are shown in Figure 8.37b as the forces $3F$. For AB and CD the figure also includes a sketch of the associated elastic curve. The deflections at B and C can be determined with forget-me-not (2):

$$w_B = \frac{3F \cdot a^3}{3EI} = \frac{Fa^3}{EI},$$

$$w_C = \frac{3F \cdot (2a)^3}{3 \times 2EI} = 4\frac{Fa^3}{EI}.$$

Figure 8.37c shows the deflections to scale. BC remains straight as the bending stiffness is infinite. The deflection at E in the given xz coordinate

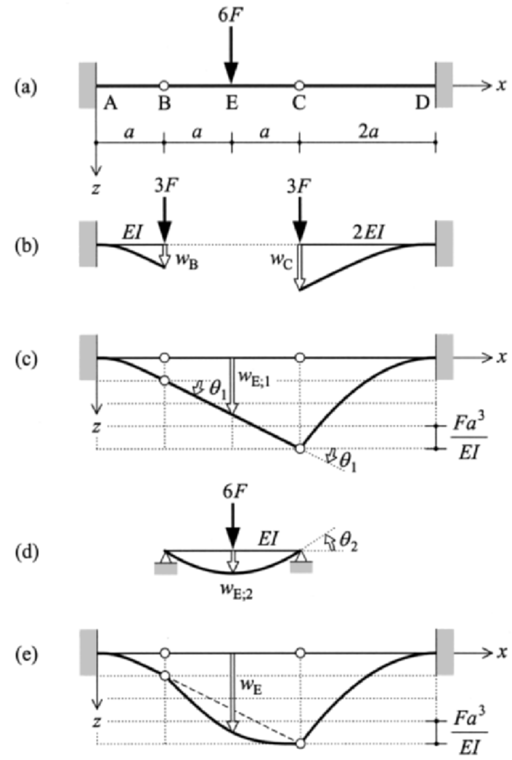


Figure 8.37 (a) Hinged beam with a point load F on centre field BC. (b) Load on and deformation of the end fields AB and CD. (c) The elastic curve if centre field BC has an infinite bending stiffness. (d) Load on and deformation of centre field BC for a finite bending stiffness. (e) The elastic curve if centre field BC has a finite bending stiffness.

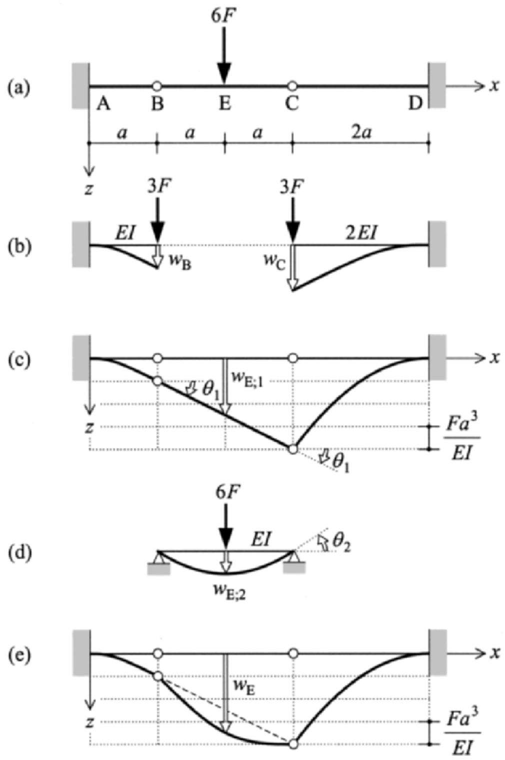


Figure 8.37 (a) Hinged beam with a point load F on centre field BC. (b) Load on and deformation of the end fields AB and CD. (c) The elastic curve if centre field BC has an infinite bending stiffness. (d) Load on and deformation of centre field BC for a finite bending stiffness. (e) The elastic curve if centre field BC has a finite bending stiffness.

system is

$$u_{z;\mathrm{E}} = w_{\mathrm{E};1} = \frac{w_\mathrm{B} + w_\mathrm{C}}{2} = \frac{5}{2}\frac{Fa^3}{EI}.$$

b. If BC is not infinitely stiff, the effect of the deformation of BC has to be superposed on the deflection at E found above (see Figure 8.37d). The additional deflection is, using forget-me-not (5),

$$w_{\mathrm{E};2} = \frac{6F \cdot (2a)^3}{48EI} = \frac{Fa^3}{EI}.$$

The final deflection at E is now

$$u_{z;\mathrm{E}} = w_\mathrm{E} = w_{\mathrm{E};1} + w_{\mathrm{E};2} = \frac{5}{2}\frac{Fa^3}{EI} + \frac{Fa^3}{EI} = \frac{7}{2}\frac{Fa^3}{EI}.$$

Figure 8.37e shows the new sketch of the elastic curve.

Comment: If we investigate the rotation $\varphi_\mathrm{C}^\mathrm{BC}$ at end C of BC[1] more closely, it turns out to be zero (see Figures 8.37c to 8.37e):

$$\varphi_\mathrm{C}^\mathrm{BC} = -\theta_1 + \theta_2 = -\frac{w_\mathrm{C} - w_\mathrm{B}}{2a} + \frac{6F \cdot (2a)^2}{16EI}$$
$$= -\frac{3}{2}\frac{Fa^3}{EI} + \frac{3}{2}\frac{Fa^3}{EI} = 0.$$

The tangent to the elastic curve of BC is therefore horizontal at C.

[1] The rotation at C is different for beam BC and CD. For this reason an upper index shows to which part of the beam the rotation at C is related.

Example 8: Statically indeterminate beam

The statically indeterminate beam in Figure 8.38a has a span ℓ and carries a uniformly distributed load q along its entire length. For this loading case, forget-me-not (9) in Table 8.4 applies.

Question:
Check the value given in Table 8.4 for the deflection at C.

Solution:
Table 8.4 shows that the fixed-end moment at A is $\frac{1}{8}q\ell^2$ (see Figure 8.38b).[1] The figure includes a sketch of the elastic curve. The bending moment diagram and therefore the deformation of the beam do not change if at A the fixed end is replaced by a hinged support and at the same time a couple is applied with the same magnitude and direction as the fixed-end moment (see Figure 8.38c).

The deflection w_C at C is found by determining separately the contributions $w_{C;1}$ due to the distributed load on AB and $w_{C;2}$ due to the couple at A. In doing so we assume that a downward deflection at C is positive, as shown in Figure 8.38b.

The deflection due to the load q is positive (points downwards) and is determined using forget-me-not (6):

$$w_{C;1} = \frac{5}{384}\frac{q\ell^4}{EI}\,.$$

The deflection due to the couple $\frac{1}{8}q\ell^2$ at A is negative (points upwards)

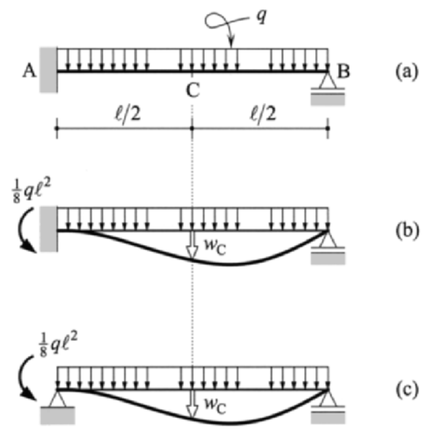


Figure 8.38 (a) A statically indeterminate beam with a uniformly distributed load for which we want to check the deflection at C. (b) Elastic curve of the beam. The fixed-end moment at A is $\frac{1}{8}q\ell^2$ and can be found from Table 8.4. (c) The bending moment diagram and elastic curve do not change if at A the fixed end is replaced by a hinged support and at the same time a couple is applied with the same magnitude and direction as the fixed-end moment. The displacement at C can now be found by determining the contributions due to the distributed load on AB and the couple at A separately with forget-me-nots, and superposing them.

[1] This was derived in Section 8.3 using the fourth-order differential equation for bending.

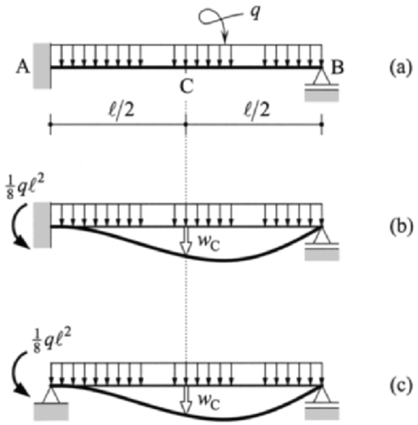


Figure 8.38 (a) A statically indeterminate beam with a uniformly distributed load for which we want to check the deflection at C. (b) Elastic curve of the beam. The fixed-end moment at A is $\frac{1}{8}q\ell^2$ and can be found from Table 8.4. (c) The bending moment diagram and elastic curve do not change if at A the fixed end is replaced by a hinged support and at the same time a couple is applied with the same magnitude and direction as the fixed-end moment. The displacement at C can now be found by determining the contributions due to the distributed load on AB and the couple at A separately with forget-me-nots, and superposing them.

and is determined using forget-me-not (4):

$$w_{C;2} = -\frac{\frac{1}{8}q\ell^2 \cdot \ell^2}{16EI} = -\frac{1}{128}\frac{q\ell^4}{EI}.$$

The final deflection at C is:

$$w_C = w_{C;1} + w_{C;2} = \frac{5}{384}\frac{q\ell^4}{EI} - \frac{1}{128}\frac{q\ell^4}{EI} = \frac{1}{192}\frac{q\ell^4}{EI}.$$

Example 9: Simply supported beam with overhang and uniformly distributed load

The simply supported beam ABC with overhang in Figure 8.39a carries a uniformly distributed load $q = 16$ kN/m along its entire length. The bending stiffness of the beam is $EI = 20$ MNm^2. The dimensions are given in the figure.

Questions:

a. Determine the deflection at D.
b. Determine the deflection at A.
c. Draw a sketch of the elastic curve.

Solution:

a. The deflection at D is found by looking at the influences of the loads on BC and AB separately and by superposing their contributions.

The distributed load on BC causes a deflection $w_{D;1}$ at D (see Figure 8.39b). With forget-me-not (6) we find

$$w_{D;1} = \frac{5}{384}\frac{(16\text{ kN/m})(6\text{ m})^4}{20\text{ MNm}^2} = 13.5\text{ mm }(\downarrow)$$

The distributed load on AB exerts a moment M_B at B on BC (see Figure 8.39c):

$$M_B = \frac{1}{2} \times (16\ \text{kN/m})(2\ \text{m})^2 = 32\ \text{kNm}.$$

As a result D undergoes an upward deflection $w_{D;2}$, to be determined using forget-me-not (5):

$$w_{D;2} = \frac{1}{16} \frac{(32\ \text{kNm})(6\ \text{m})^2}{20\ \text{MNm}^2} = 3.6\ \text{mm}\ (\uparrow)$$

Since $w_{D;1} > w_{D;2}$ the final deflection w_D at D points downwards:

$$w_D = w_{D;1} - w_{D;2} = (13.5\ \text{mm}) - (3.6\ \text{mm}) = 9.9\ \text{mm}\ (\downarrow)$$

or, in the given xz coordinate system,

$$u_{z;D} = +9.9\ \text{mm}.$$

b. The deflection at A is found by adding the influence of the deformation of AB to the tail-wagging effect due to the rotation at B. When determining the tail-wagging effect, we have to consider the effects of the loads on both AB and BC; they are calculated separately.

The distributed load on BC gives an upward deflection $w_{A;1}$ at A (see Figure 8.39b). Using forget-me-not (5) we find the tail-wagging effect at A:

$$w_{A;1} = (2\ \text{m}) \times \theta_{B;1} = (2\ \text{m}) \times \frac{1}{24} \frac{(16\ \text{kN/m})(6\ \text{m})^3}{20\ \text{MNm}^2} = 14.4\ \text{mm}\ (\uparrow)$$

The moment $M_B = 32$ kNm at B due to the distributed load on AB, and

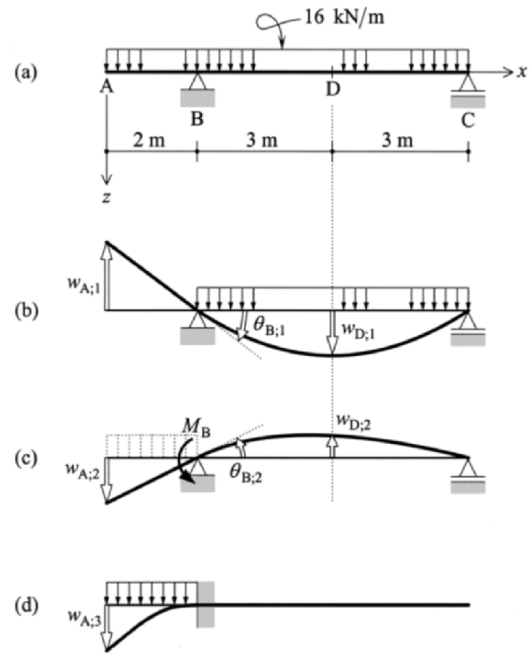


Figure 8.39 (a) A simply supported beam with overhang and uniformly distributed load. (b) Elastic curve due to the deformation of BC under the influence of the load on BC. (c) Elastic curve due to the deformation of BC under the influence of the load on AB. (d) Deflection due to the deformation of AB only.

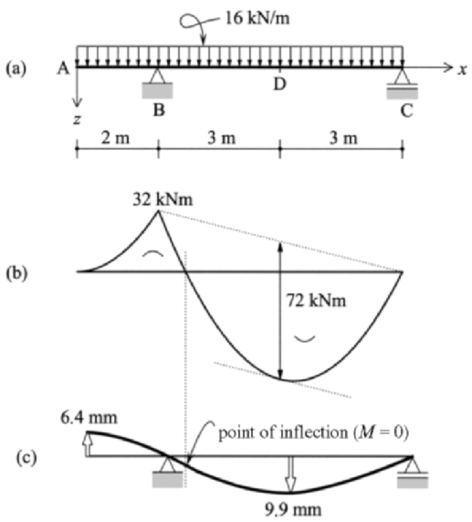


Figure 8.40 (a) A simply supported beam with overhang and uniformly distributed load with (b) bending moment diagram and (c) elastic curve. The elastic curve has a point of inflection where the bending moment is zero.

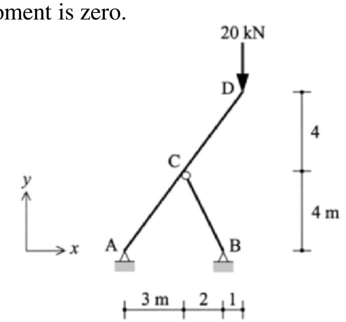


Figure 8.41 An inclined shored bar with a vertical force of 20 kN at the top.

acting on BC, gives a downward deflection $w_{A;2}$ at A (see Figure 8.39c). Using forget-me-not (4) we find the tail-wagging effect at A:

$$w_{A;2} = (2\text{ m}) \times \theta_{B;2} = (2\text{ m}) \times \frac{1}{3}\,\frac{(32\text{ kNm})(6\text{ m})}{20\text{ MNm}^2} = 6.4\text{ mm }(\downarrow)$$

The deflection $w_{A;3}$ due to the deformation of AB only follows from Figure 8.39d, and is determined using forget-me-not (3):

$$w_{A;3} = \frac{1}{8}\,\frac{(16\text{ kN/m})(2\text{ m})^4}{20\text{ MNm}^2} = 1.6\text{ mm }(\downarrow)$$

The final deflection w_A points upwards:

$$w_A = w_{A;1} - w_{A;2} - w_{A;3}$$
$$= (14.4\text{ mm}) - (6.4\text{ mm}) - (1.6\text{ mm}) = 6.4\text{ mm }(\uparrow)$$

or, in the given xz coordinate system:

$$u_{z;A} = -6.4\text{ mm}.$$

c. In Figure 8.40, the bending moment diagram and elastic curve are shown for the beam. The elastic curve has a point of inflection where the bending moment changes sign.

Example 10: An inclined shored bar

The inclined shored bar ACD in Figure 8.41 is loaded by a vertical force of 20 kN at D. The bar has a bending stiffness of 40 MNm^2. Deformation due to the normal force is ignored.

Question:

Determine the vertical displacement of D in the given xy coordinate system.

Solution:
The bending moment diagram for ACD is shown in Figure 8.42a. The force of 20 kN at D has a component of 12 kN normal to the member axis. This force causes the bending deformation of ACD.

Since there is no axial force deformation, C will not move and we can consider C a "fixed" point. Due to the bending of ACD, D will move normal to the member axis. A sketch of the elastic curve is given in Figure 8.42b.

When determining the deflection w at D we consider the contributions due to the deformation of AC and CD separately. They are shown in Figure 8.42b as w_1 and w_2 respectively.

At C a moment of 60 kNm acts on AC (see Figure 8.42c). As a result AC is deformed, and a rotation θ_C occurs at C, to be determined with forget-me-not (4):

$$\theta_C = \frac{1}{3}\frac{(60\text{ kNm})(5\text{ m})}{40\text{ MNm}^2} = 2.5 \times 10^{-3}\text{ rad.}$$

The rotation at C causes the tail-wagging effect w_1 at D:

$$w_1 = \theta_C \times (5\text{ m}) = (2.5 \times 10^{-3}\text{ rad})(5\text{ m}) = 12.5\text{ mm.}$$

The deflection w_2 due to the deformation of CD (see Figure 8.42d) has to be added to the deflection w_1. w_2 is determined using forget-me-not (2):

$$w_2 = \frac{1}{3}\frac{(12\text{ kN})(5\text{ m})^3}{40\text{ MNm}^2} = 12.5\text{ mm.}$$

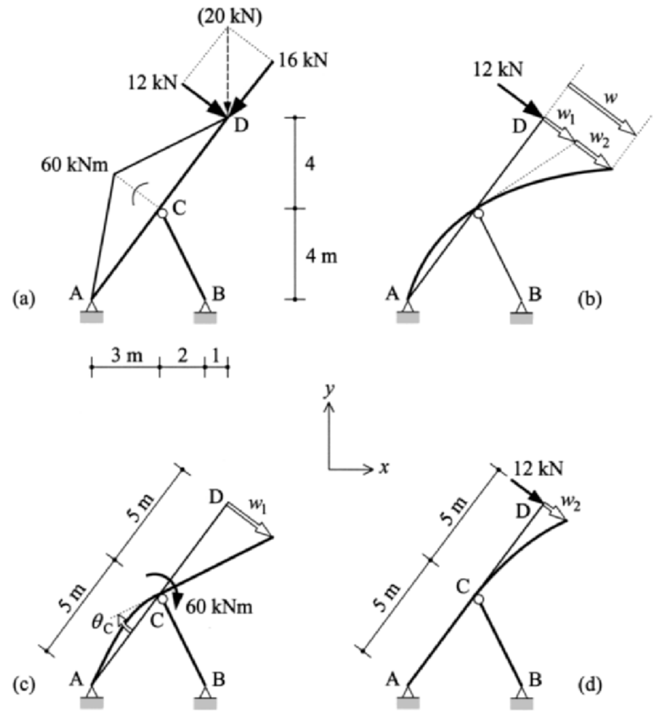


Figure 8.42 (a) Bending moment diagram for the inclined shored bar. (b) Sketch of the elastic curve. Due to the bending of ACD, D is displaced normal to the member axis. (c) w_1 is the deflection at D due to the deformation of AC. (d) w_2 is the deflection at D due to the deformation of CD.

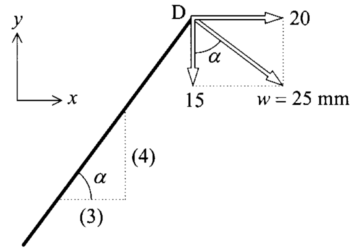


Figure 8.43 The deflection at D, normal to the member axis, resolved into a horizontal and vertical component.

The final deflection at D is (see Figure 8.42b)

$$w = w_1 + w_2 = (12.5 \text{ mm}) + (12.5 \text{ mm}) = 25 \text{ mm}.$$

Figure 8.43 shows a magnified representation around D. The deflection w at D of 25 mm, normal to ACD, has a horizontal component of 20 mm (→) and a vertical component of 15 mm (↓). In the given yz coordinate system the horizontal displacement is

$$u_{x;\text{D}} = +20 \text{ mm}$$

and the vertical displacement

$$u_{y;\text{D}} = -15 \text{ mm}.$$

8.4 Moment-area theorems

The moment-area theorems are a powerful method for determining the rotations and displacements due to bending, for structures in which the moment distribution is known.

The moment-area theorems are based on the kinematic and constitutive relationships derived in Section 4.3 for a member subject to bending in the xz plane.

In Section 8.4.1 we derive the moment-area theorems and present a visual interpretation of the result, after which Section 8.4.2 demonstrates the power of the visual interpretation using a number of examples.

8.4.1 Derivation

There are two moment-area theorems: one relating to the rotation φ and one to the deflection w. When deriving them, we use the kinematic relationships (see Section 4.3.1)

$$\varphi = -\frac{dw}{dx}, \tag{8.6}$$

$$\kappa = \frac{d\varphi}{dx}, \tag{8.7}$$

and the constitutive relationship (see Section 4.3.2)

$$M = EI\kappa \text{ or } \kappa = \frac{M}{EI}. \tag{8.8}$$

We derive the moment-area theorems for beam segment AB in Figure 8.44a, for which the bending moment diagram is shown in Figure 8.44b.

When deriving the propositions, we use the xz coordinate system in which the kinematic and constitutive formulas were also derived. In the bending moment diagram the deformation symbol is therefore placed in brackets. After their derivation, we present a visual interpretation of the moment-area theorems. In doing so, the deformation symbol provides information on the way in which a beam bends.

Derivation of the first moment-area theorem
The first moment-area theorem relates to the rotation φ. By eliminating the curvature κ from (8.7) and (8.8) we find:

$$\frac{d\varphi}{dx} = \frac{M}{EI}. \tag{8.9}$$

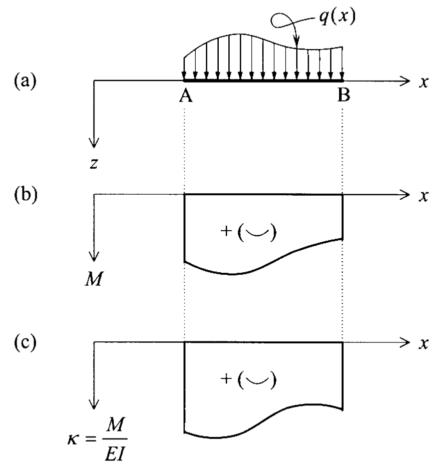


Figure 8.44 (a) Segment AB of a beam with a distributed load. (b) The associated bending moment diagram, and (c) the M/EI diagram, curvature diagram or reduced moment diagram.

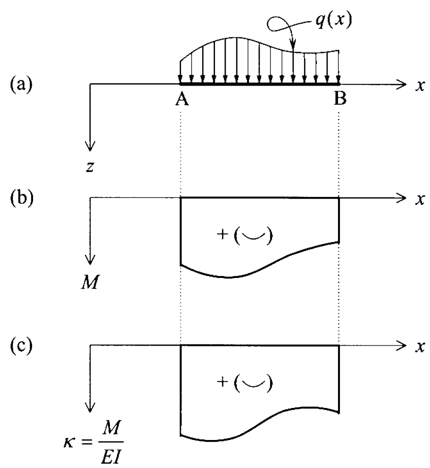


Figure 8.44 (a) Segment AB of a beam with a distributed load. (b) The associated bending moment diagram, and (c) the M/EI diagram, curvature diagram or reduced moment diagram.

The change in rotation φ is determined by the value of the quantity M/EI. This quantity is also referred to as the *reduced moment*. In fact it is simply the *curvature* κ, according to constitutive relationship (8.8): $\kappa = M/EI$.

The M/EI diagram in Figure 8.44c is also referred to as the *reduced moment diagram* or *curvature diagram*.

From (8.9) it follows that

$$\mathrm{d}\varphi = \frac{M}{EI}\,\mathrm{d}x.$$

If we integrate this equation along the length AB we find

$$\Delta\varphi = \varphi_{\mathrm{B}} - \varphi_{\mathrm{A}} = \underbrace{\int_{\mathrm{A}}^{\mathrm{B}} \frac{M}{EI}\,\mathrm{d}x}_{\substack{\text{area } M/EI \\ \text{diagram}}}\,. \tag{8.10a}$$

According to (8.10a), the increase $\Delta\varphi$ of the rotation between A and B is equal to the area of the M/EI or curvature diagram between A and B. This is known as the *first moment-area theorem*.

If the rotation at A is known, the rotation at B is found with

$$\varphi_{\mathrm{B}} = \varphi_{\mathrm{A}} + \underbrace{\int_{\mathrm{A}}^{\mathrm{B}} \frac{M}{EI}\,\mathrm{d}x}_{\substack{\text{area } M/EI \\ \text{diagram}}}\,. \tag{8.10b}$$

Comment 1: Depending on the sign of M, the integral, which represents the area of the M/EI diagram, can be either positive or negative.

Comment 2: The first moment-area theorem (8.10) applies only if the rotation φ is continuous and continually differentiable along the length AB. This means that there may be no hinges between A and B.

Derivation of the second moment-area theorem
The second moment-area theorem relates to the deflection w. From the kinematic relationship (8.6)

$$\varphi = -\frac{\mathrm{d}w}{\mathrm{d}x} \tag{8.6}$$

it follows that

$$\mathrm{d}w = -\varphi\,\mathrm{d}x.$$

Integrating between the limits A and B we obtain

$$\Delta w = w_\mathrm{B} - w_\mathrm{A} = -\int_\mathrm{A}^\mathrm{B} \varphi\,\mathrm{d}x.$$

Using partial integration we find

$$\begin{aligned}\Delta w = w_\mathrm{B} - w_\mathrm{A} &= -\varphi x\big|_\mathrm{A}^\mathrm{B} + \int_\mathrm{A}^\mathrm{B} \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,x\,\mathrm{d}x \\ &= -\varphi_\mathrm{B} x_\mathrm{B} + \varphi_\mathrm{A} x_\mathrm{A} + \int_\mathrm{A}^\mathrm{B} \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,x\,\mathrm{d}x.\end{aligned}$$

According to (8.9), the differential under the integral sign is

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{M}{EI} \tag{8.9}$$

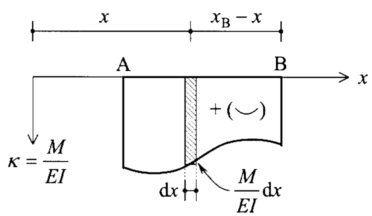


Figure 8.45 In the integral $\int_A^B \frac{M}{EI}(x_B - x)\,dx$, the term under the integral is equal to the static moment of the hatched area with respect to B.

so that Δw can also be defined as

$$\Delta w = w_B - w_A = -\varphi_B x_B + \varphi_A x_A + \int_A^B \frac{M}{EI}\, x\, dx. \tag{8.11}$$

Using the first moment-area theorem (8.10b) we find the rotation φ_B:

$$\varphi_B = \varphi_A + \int_A^B \frac{M}{EI}\, dx. \tag{8.10b}$$

Making use of this formula we can now rewrite (8.11):

$$\Delta w = w_B - w_A = -\varphi_A(x_B - x_A) - \underbrace{\int_A^B \frac{M}{EI}(x_B - x)\, dx}_{\substack{\text{static moment } M/EI \\ \text{diagram wrt B}}}\ . \tag{8.12a}$$

This is known as the *second moment-area theorem.*

The significance of the integral in the right-hand term is illustrated by means of the M/EI or curvature diagram in Figure 8.45. Here the area of the hatched strip, with width dx, is equal to $(M/EI)\,dx$. This value multiplied by the distance $(x_B - x)$ is equal to the static moment (linear area moment) of the hatched area with respect to B. The integral in (8.12a) therefore represents the static moment of the M/EI diagram (curvature diagram) between A and B, with respect to B.

If the deflection w_A and rotation φ_A at A are known, the deflection at B is found from

$$w_B = w_A - \underbrace{\varphi_A(x_B - x_A)}_{\substack{\text{tail-wagging} \\ \text{effect pq due to} \\ \text{the rotation at A}}} - \underbrace{\int_A^B \frac{M}{EI}(x_B - x)\,dx}_{\substack{\text{qr due to the} \\ \text{deformation of AB}}} . \qquad (8.12b)$$

See Figure 8.46 for the visual interpretation.

Comment: The second moment-area theorem (8.12) is applicable only if the rotation φ and deflection w are continuous and continuously differentiable over AB.

Visual interpretation of the moment-area theorems
Figure 8.46a shows the M/EI or curvature diagram for beam AB, where it is assumed that the curvature between A and B is the result of a positive bending moment, so that the integral in the right-hand term of (8.12) is positive.

Figure 8.46b shows a sketch of the associated deformation of beam segment AB (the *elastic curve*).

The distance |pr| in the figure is the difference in deflection between A and B. We distinguish the contributions |pq| and |qr|. The distance |pq| is the result of the tail-wagging effect due to the rotation φ_A at A:

$$|\text{pq}| = \varphi_A(x_B - x_A).$$

The distance |qr| is the result of the deformation of AB and is equal to the static moment of the M/EI diagram with respect to B:

$$|\text{qr}| = \int_A^B \frac{M}{EI}(x_B - x)\,dx.$$

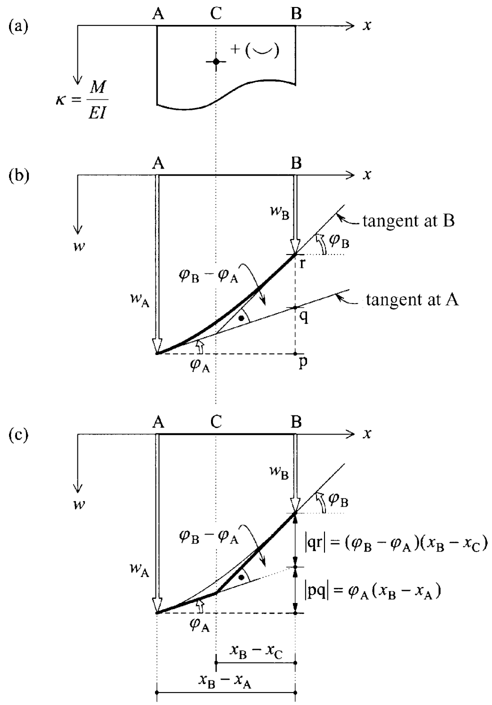


Figure 8.46 (a) The M/EI diagram for beam segment AB. (b) The tangents at A and B to the deformed beam segment AB (elastic curve) intersect at C, at the height of the centroid of the M/EI diagram, and there form an angle that is equal to the area of the M/EI diagram. (c) The displacement at B due to the deformation of AB can be determined from A by superposing the tail-wagging effect |pq| due to the rotation at A and the tail-wagging effect |qr| due to the bend at C. The latter is equal to the area of the M/EI diagram.

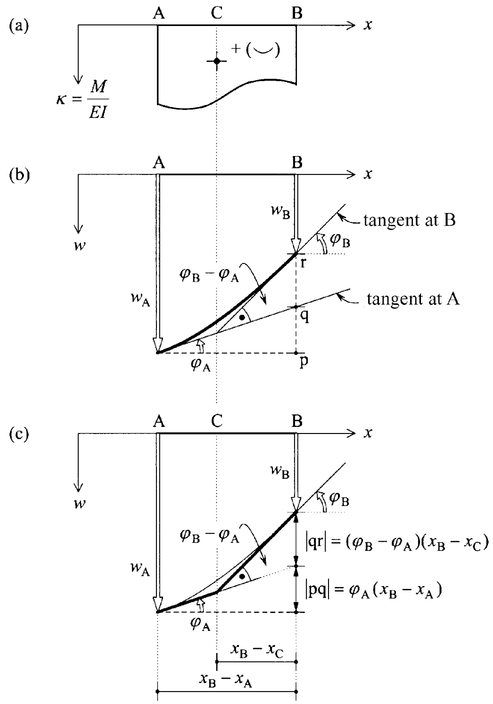


Figure 8.46 (a) The M/EI diagram for beam segment AB. (b) The tangents at A and B to the deformed beam segment AB (elastic curve) intersect at C, at the height of the centroid of the M/EI diagram, and there form an angle that is equal to the area of the M/EI diagram. (c) The displacement at B due to the deformation of AB can be determined from A by superposing the tail-wagging effect |pq| due to the rotation at A and the tail-wagging effect |qr| due to the bend at C. The latter is equal to the area of the M/EI diagram.

The static moment with respect to B can also be written as the product of the area of the M/EI diagram and the distance $(x_B - x_C)$ from B to the centroid C of the M/EI diagram (see Figure 8.46a):

$$|\mathrm{qr}| = \underbrace{\int_A^B \frac{M}{EI}(x_B - x)\,\mathrm{d}x}_{\text{static moment } M/EI \text{ diagram wrt B}} = (x_B - x_C)\underbrace{\int_A^B \frac{M}{EI}\,\mathrm{d}x}_{\text{area } M/EI \text{ diagram}}.$$

According to the first moment-area theorem, the area of the M/EI diagram is equal to the increase in rotation between A and B:

$$\int_A^B \frac{M}{EI}\,\mathrm{d}x = \varphi_B - \varphi_A.$$

Making use of this, we find that the distance |qr| (equal to the static moment of the M/EI diagram with respect to B) is

$$|\mathrm{qr}| = (x_B - x_C)\int_A^B \frac{M}{EI}\,\mathrm{d}x = (x_B - x_C)(\varphi_B - \varphi_A).$$

The second moment-area theorem (8.12b) can therefore be written as

$$w_B = w_A - \underbrace{\varphi_A(x_B - x_A)}_{\substack{\text{tail-wagging effect pq}\\ \text{due to the rotation at A}}} - \underbrace{(x_B - x_C)(\varphi_B - \varphi_A)}_{\substack{\text{qr due to the}\\ \text{deformation of AB}}}. \qquad (8.12c)$$

See Figure 8.46c for the significance of each of the terms.

Conclusion: *The tangents at A and B to the deformed beam segment (elastic curve) AB intersect at C, at the height of the centroid of the M/EI diagram.*

The intersection at C is at an angle that is equal to the area of the M/EI diagram (see Figure 8.46b).

If we wish to determine from A the rotation and deflection at B we could model the elastic curve as two straight lines: the tangents at A and B. These lines intersect at the location of the centroid of the M/EI diagram, and are at an angle that is equal to the area of the M/EI diagram (see Figure 8.46c).

The deflection at B due to the deformation of AB can be determined as the tail-wagging effect due to both the rotation at A and the angle between the tangents at A and B.

Comments: One can imagine that the bend at C is caused by concentrating the curvature of AB at a single point. The bend must therefore agree with the deformation symbol in the M diagram (or the M/EI diagram).

The derived moment-area theorems (8.10a/b) and (8.12a/b) can be elaborated analytically. In doing so, one has to take the coordinate system into account for all the quantities. In that case, the integrals for the area and the static moment of the M/EI diagram can be either positive or negative.

An objection to the analytical approach is that it is really practicable only for straight beams.[1] Another disadvantage is that we cannot "*see what is happening*".

When applying the moment-area theorems, we shall use the visual interpretation as explained with the help of formula (8.12c). Not only can we then "see what is happening", but we can also use them for bent beams and frames.

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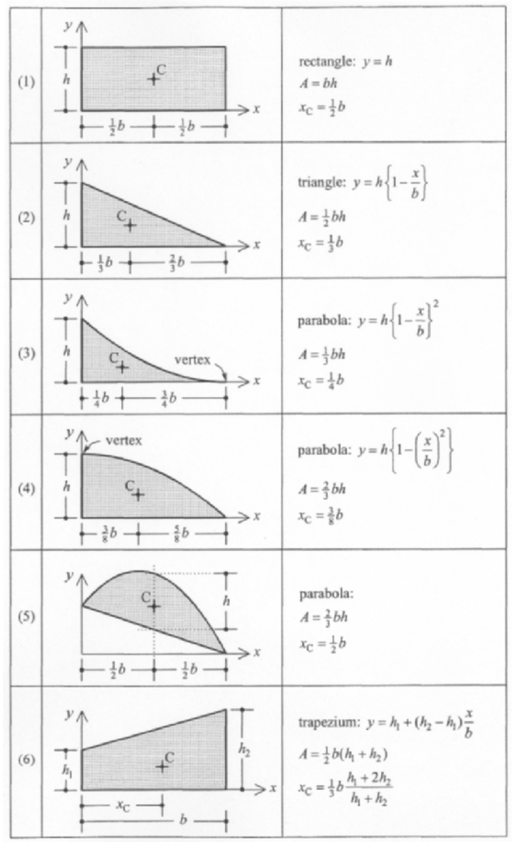
1 In bent beams, problems arise due to the fact that at each bend we have to shift from one (local) coordinate system to another.

Table 8.5 Plane figures with their area A and x coordinate of the centroid C.



In the visual approach, the area and centroid of the M/EI diagram play an important role. These properties are included in Table 8.5 for a number of common types of plane figures.

8.4.2 Examples

In this section we illustrate the efficiency of the moment-area theorems by means of ten examples.

Examples 1 to 4 relate to beams with one fixed and one free end. The beam in Example 3 is non-prismatic. Example 5 addresses a fixed bent beam. In Examples 6 to 9 the structure is simply supported. The structure is a straight beam, with or without overhang (Examples 6 and 7) or consists of several rigidly joined straight bars (Examples 8 and 9). Finally, in Example 10, we determine the displacements of a three-hinged frame.

Comment: In applying the moment-area theorems to the first five examples, we start from a fixed end where the rotation φ and deflection w are known.[1] This changes at Example 6. In Examples 6 to 9, the rotation in the starting point is unknown and has to be determined from a deflection known elsewhere. Example 10, where we determine the displacements of a three-hinged frame, goes even further as the necessary rotations at the supports have to be determined from the joining conditions at the hinge.

The method with moment-area theorems is strongly visual. Therefore, for straight beams we often work without a coordinate system, or initially ignore it. A deflection due to bending, normal to the beam axis, is denoted as w and a rotation as θ. The positive direction of w, as with the forget-me-nots,[2] is derived from the picture that includes a sketch of the deformed

---

1 At a fixed end $\varphi = 0$ and $w = 0$.
2 See Section 8.3.

beam (elastic curve). The positive direction of the rotation θ is also derived from the picture.

With bent members (Example 5) and for other, more complicated member structures (Examples 8 to 10), it is preferable to use a *global coordinate system*[1] and to name the deflections in this system. In order to avoid confusion with the letter w in a global xz coordinate system, the displacements in the x and z direction respectively are no longer denoted by u and w, but by u_x and u_z, and the rotation is denoted by φ (actually φ_y).[2]

Example 1: A forget-me-not
Forget-me-not (2) from Table 8.3 applies to the prismatic cantilever beam in Figure 8.47a, with length ℓ and bending stiffness EI.

Question:
Verify the expressions for the rotation and deflection at B using the moment-area theorems.

Solution:
Figure 8.47b shows the M/EI diagram, including the deformation symbol. To find the deflection at B, the deformed beam can be modelled as two straight lines (the tangents at A and B to the elastic curve) that at the location of the centroid of the M/EI diagram intersect at an angle that is equal to the area of the M/EI diagram.

The centroid of the M/EI diagram is at C, at $\ell/3$ from the fixed end A. Figure 8.47c shows the bend θ. We can imagine that the bend is the result

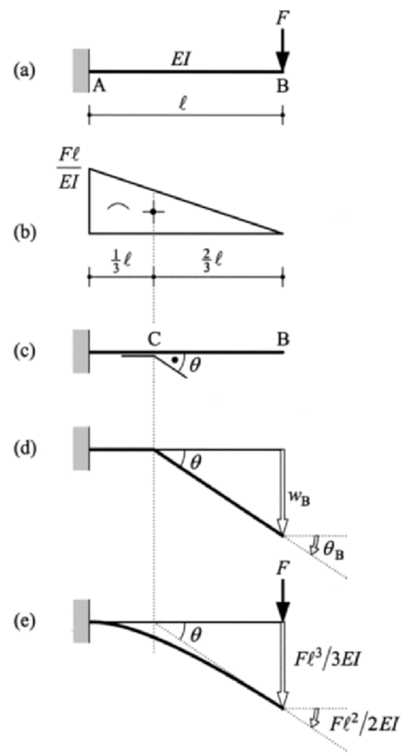


Figure 8.47 Deflection of a cantilever beam loaded by a force.

[1] A global coordinate system is a system that applies to the structure as a whole. The orientation of a global coordinate system is usually related to the direction of gravity.

[2] Since there is no possibility of confusion, the index y is omitted for simplicity.

of concentrating the curvature of the beam at one single point. The bend therefore has to match the deformation symbol.

The bend θ is set down with one of the legs along the beam axis, in such a way that the angle of the bend is open to the direction in which we want to know the rotation and/or deflection. With a certain amount of imagination and good will, you can see an "eye" in the bend, looking in the direction that we are working towards, in this case the direction of B.

Comment: The bend is generally drawn in the M/EI diagram, i.e. in Figure 8.47b. If we draw the M/EI diagram correctly, i.e. at the tension side of the beam axis, the bend will always be outside the M/EI diagram, and will never penetrate it. The bend θ is equal to the area of the M/EI diagram:

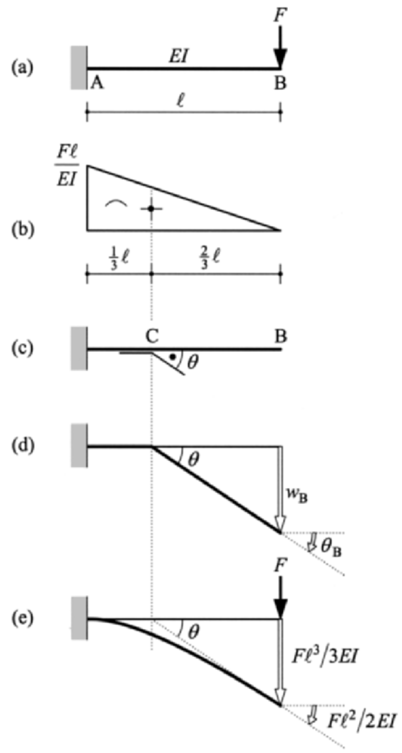


Figure 8.47

Figure 8.47 (a) A cantilever beam loaded by a force F at the free end, with (b) the M/EI diagram. (c) The deformation of the beam can be considered concentrated in a bend at C, at the location of the centroid of the M/EI diagram. The magnitude of the bend θ is equal to the area of the M/EI diagram. If the M/EI diagram is drawn correctly, i.e. at the tension side of the beam axis, the bend will always be outside the M/EI diagram, and never penetrate it. The open side of the bend θ points in the direction in which we want to know the rotation and/or displacement, in this case in the direction of B. With a certain amount of imagination, you can see an "eye" in the bend looking in the direction towards which we are working. (d) The displacement at B is found as the tail-wagging effect due to the bend θ at C. (e) The straight lines at (d) are the tangents to the deformed beam axis (elastic curve) at A and B. Using these lines we can make a quick and accurate sketch of the deformed beam (elastic curve).

$$\theta = \tfrac{1}{2} \cdot \ell \cdot \frac{F\ell}{EI} = \frac{F\ell^2}{2EI}\,.$$

The rotation and deflection at B can be determined from Figure 8.47d. The rotation at B is

$$\theta_B = \theta = \frac{F\ell^2}{2EI}\,.$$

The deflection at B is found as the tail-wagging effect due to the bend θ at C:

$$w_B = \theta \cdot \tfrac{2}{3}\,\ell = \frac{F\ell^2}{2EI} \cdot \tfrac{2}{3}\,\ell = \frac{F\ell^3}{3EI}\,.$$

The values found are in agreement with forget-me-not (2) from Table 8.3.

The straight lines in Figure 8.47d, the tangents to the deformed beam at A and B, can be used for a quick and accurate sketch of the elastic curve (see Figure 8.47e).

Example 2: Deflection at the top of a beam fixed at an angle

Member AB in Figure 8.48a is fixed at an angle at A and is loaded by a vertical force of 30 kN at the free end. The bending stiffness of the member is $EI = 5.2$ MNm^2. Deformation due to normal forces is ignored.

Question:
Determine the horizontal displacement at B due to bending.

Solution:
The bending moment at A is (30 kN)(1 m) = 30 kNm. Figure 8.48b shows the M/EI diagram, with the deformation symbol. The centroid of the M/EI diagram is at C, at one-third of the member length from A. Here the bend θ is shown.

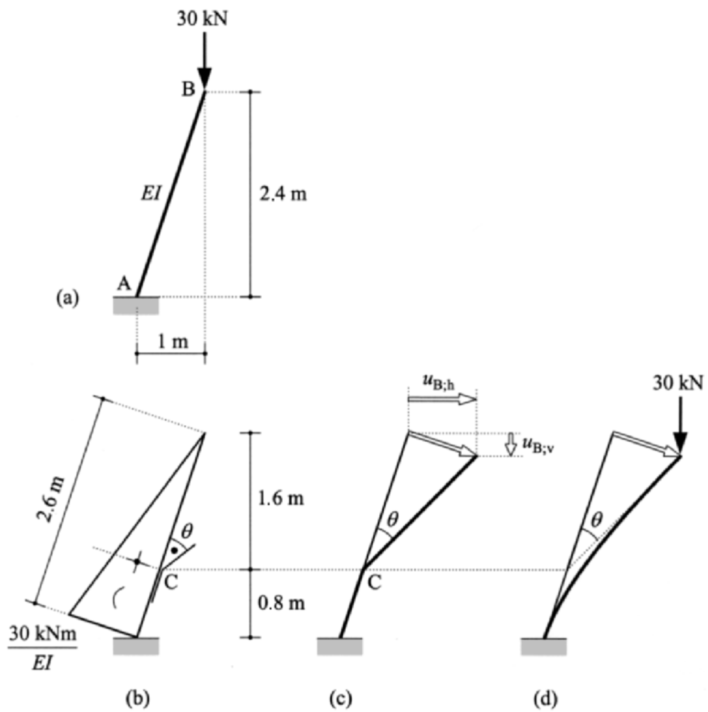


Figure 8.48 (a) Beam fixed at an angle, loaded by a vertical force at the top, with (b) the M/EI diagram. The deformation due to bending of AB can be considered concentrated in the bend θ at C, at the centroid of the M/EI diagram. (c) The displacement at B is found by rotating the upper part CB through an angle θ about C. (d) Sketch of the elastic curve. The displacement due to bending is always normal to the member axis.

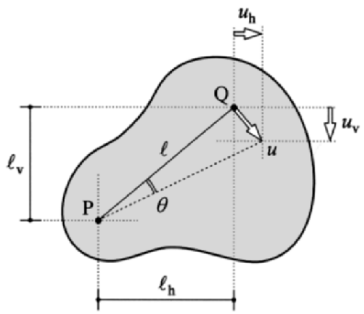


Figure 8.49 For the displacement at Q due to a small rotation θ about P we have

- the horizontal component $u_{\mathrm{h}} = \theta \cdot \ell_{\mathrm{v}}$,
- the vertical component $u_{\mathrm{v}} = \theta \cdot \ell_{\mathrm{h}}$.

Intermezzo:[1]

For the displacement due to a small rotation θ (ignoring signs) Figure 8.49 shows that

- The horizontal component of the displacement at Q is equal to "rotation × vertical distance to the centre of rotation P":

  $$u_{\mathrm{h}} = \theta \cdot \ell_{\mathrm{v}}.$$

- The vertical component of the displacement at Q is equal to "rotation × horizontal distance to the centre of rotation P":

  $$u_{\mathrm{v}} = \theta \cdot \ell_{\mathrm{h}}.$$

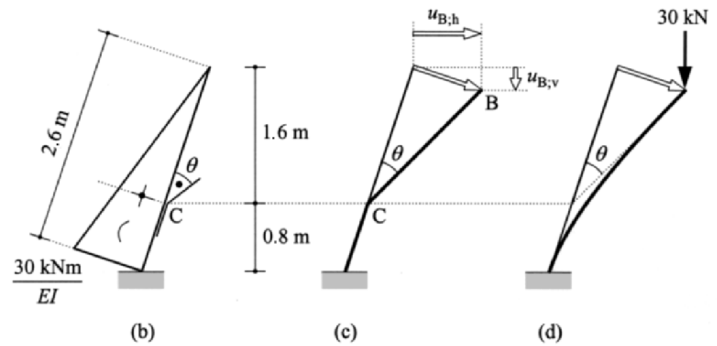


Figure 8.48b–d

The magnitude of the bend θ is equal to the area of the triangular M/EI diagram. For a member length of $\sqrt{(1\ \mathrm{m})^2 + (2.4\ \mathrm{m})^2} = 2.6$ m and a bending stiffness EI of 5.2 MNm^2 this area is

$$\theta = \tfrac{1}{2} \times (2.6\ \mathrm{m}) \times \frac{30\ \mathrm{kNm}}{5.2 \times 10^3\ \mathrm{kNm}^2} = 7.5 \times 10^{-3}\ \mathrm{rad}.$$

At C, the bend θ indicates the angle over which the upper part CB has to be rotated to find the deflection at B (see Figure 8.48c).

With $\theta = 7.5 \times 10^{-3}$ rad and $\ell_{\mathrm{v}} = \frac{2}{3} \times (2.4\ \mathrm{m}) = 1.6$ m, the horizontal component of the deflection at B is (see Figure 8.48c)

$$u_{\mathrm{B;h}} = \theta \cdot \ell_{\mathrm{v}} = (7.5 \times 10^{-3}\ \mathrm{rad})(1.6\ \mathrm{m}) = 12 \times 10^{-3}\ \mathrm{m} = 12\ \mathrm{mm}.$$

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[1] See also Section 7.2, Example 7, and *Engineering Mechanics*, Volume 1, Section 15.3.2.

Figure 8.48d shows the elastic curve. Please note that the deflection due to bending (for small deflections) is always normal to the member axis.

Comment: The method using the moment-area theorems is far more efficient here than that with the forget-me-nots. To verify this, the reader is asked to perform the same calculation with the forget-me-nots.

Example 3: Non-prismatic cantilever beam

The non-prismatic cantilever beam ABC in Figure 8.50a is loaded by a force F at the free end C. The bending stiffness is $2EI$ for AB and EI for BC.

Questions:

a. Determine the rotation and deflection at B and C.
b. Draw a sketch of the elastic curve.
c. By how many percent does the deflection increase at C if the beam is prismatic with bending stiffness EI?

Solution:

a. Figure 8.50b shows the M diagram and Figure 8.50c the M/EI diagram, both with the deformation symbols.

When determining the deflections, we can consider the deformation of AB and BC concentrated at the bends θ_1 and θ_2 at the centroids of respectively the trapezoid M/EI diagram for AB, and the triangular M/EI diagram for BC. For the trapezoid M/EI diagram for AB, the magnitude and location of the bend can be found using the formulas in Table 8.5.

The magnitude of the bends θ are

$$\theta_1 = \tfrac{1}{2} \cdot \ell \cdot \left(\frac{F\ell}{EI} + \frac{F\ell}{2EI} \right) = \frac{3}{4} \frac{F\ell^2}{EI} \quad \text{(area of a trapezium)},$$

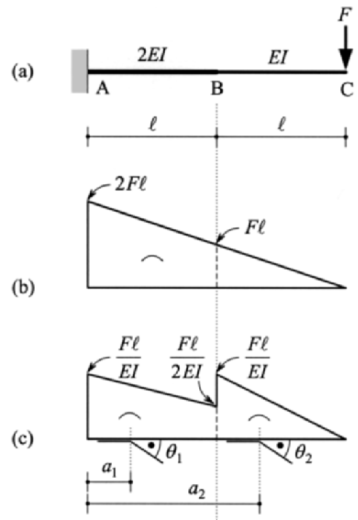

Figure 8.50 (a) A non-prismatic cantilever beam, loaded by the force F at the free end, with (b) the bending moment diagram and (c) the M/EI or κ diagram. The deformation of AB and BC can be considered concentrated at the bends θ_1 and θ_2 at the centroids of respectively the trapezoid M/EI diagram for AB, and the triangular M/EI diagram for BC.

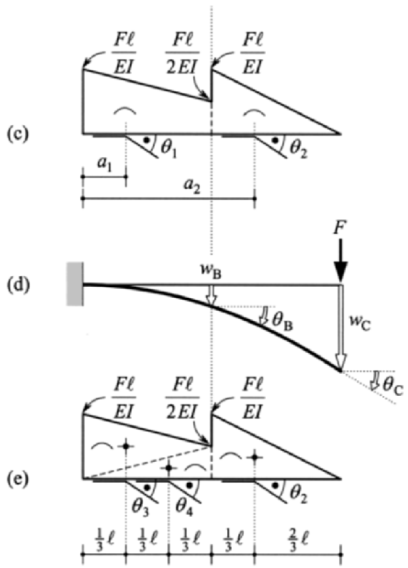

Figure 8.50 (c) The M/EI or curvature diagram. The deformation of AB and BC can be considered concentrated at the bends θ_1 and θ_2 at the centroids of respectively the trapezoid M/EI diagram for AB and the triangular M/EI diagram for BC. (d) Sketch of the elastic curve. The deflections at B and C are found as the tail-wagging effect due to θ_1, and θ_1 and θ_2 respectively. (e) To simplify the calculation, the trapezoid part of the M/EI diagram can also be split into two triangles.

$$\theta_2 = \tfrac{1}{2} \cdot \ell \cdot \frac{F\ell}{EI} = \frac{1}{2}\frac{F\ell^2}{EI} \quad \text{(area of a triangle).}$$

The locations of these bends are defined by (see Figure 8.50c)

$$a_1 = \tfrac{1}{3} \cdot \ell \cdot \frac{\frac{F\ell}{EI} + 2 \times \frac{F\ell}{2EI}}{\frac{F\ell}{EI} + \frac{F\ell}{2EI}} = \tfrac{4}{9}\,\ell,$$

$$a_2 = \ell + \tfrac{1}{3}\,\ell = \tfrac{4}{3}\,\ell.$$

At the bends θ_1 and θ_2 in Figure 8.50c we can again see the "eyes" looking in the direction in which we want to know the deflection.

Figure 8.50d shows the expected shape of the elastic curve.

The deflection at B is found as the tail-wagging effect due to θ_1:

$$w_B = (\ell - a_1) \cdot \theta_1 = \tfrac{5}{9}\,\ell \cdot \frac{3}{4}\frac{F\ell^2}{EI} = \frac{5}{12}\frac{F\ell^3}{EI}.$$

The deflection at C is found as the tail-wagging effect due to both θ_1 and θ_2:

$$w_C = (2\ell - a_1) \cdot \theta_1 + (2\ell - a_2) \cdot \theta_2$$
$$= \tfrac{14}{9}\,\ell \cdot \frac{3}{4}\frac{F\ell^2}{EI} + \tfrac{2}{3}\,\ell \cdot \frac{1}{2}\frac{F\ell^2}{EI} = \frac{3}{2}\frac{F\ell^3}{EI}.$$

The rotations at B and C are

$$\theta_B = \theta_1 = \frac{3}{4}\frac{F\ell^2}{EI},$$

$$\theta_C = \theta_1 + \theta_2 = \frac{3}{4}\frac{F\ell^2}{EI} + \frac{1}{2}\frac{F\ell^2}{EI} = \frac{5}{4}\frac{F\ell^2}{EI}.$$

Comment: If we do not have the formulas for the area and the centroid of a trapezium, we can split the trapezoid M/EI diagram for AB into two simpler shapes, such as the two triangles in Figure 8.50e. In that case the following holds:

$$\theta_2 = \theta_3 = \tfrac{1}{2}\cdot\ell\cdot\frac{F\ell}{EI} = \frac{1}{2}\frac{F\ell^2}{EI},$$

$$\theta_4 = \tfrac{1}{2}\cdot\ell\cdot\frac{F\ell}{2EI} = \frac{1}{4}\frac{F\ell^2}{EI}.$$

The deflection and rotation at B are now found as the tail-wagging effect due to bends θ_3 and θ_4; those at C are found as the tail-wagging effect due to θ_2 to θ_4. For example:

$$w_C = \tfrac{5}{3}\,\ell\cdot\theta_3 + \tfrac{4}{3}\,\ell\cdot\theta_4 + \tfrac{2}{3}\,\ell\cdot\theta_2$$

$$= \tfrac{5}{3}\,\ell\cdot\frac{1}{2}\frac{F\ell^2}{EI} + \tfrac{4}{3}\,\ell\cdot\frac{1}{4}\frac{F\ell^2}{EI} + \tfrac{2}{3}\,\ell\cdot\frac{1}{2}\frac{F\ell^2}{EI} = \frac{3}{2}\frac{F\ell^2}{EI}.$$

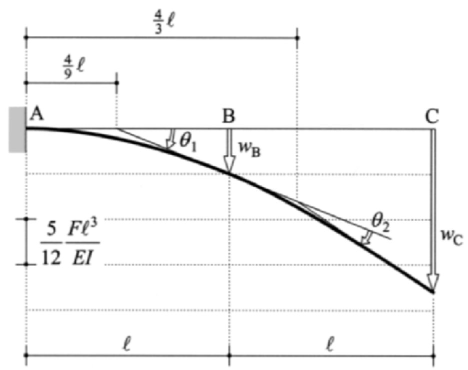

Figure 8.51 The elastic curve with the tangents at A, B and C and the bends θ_1 and θ_2. The deflections are scaled.

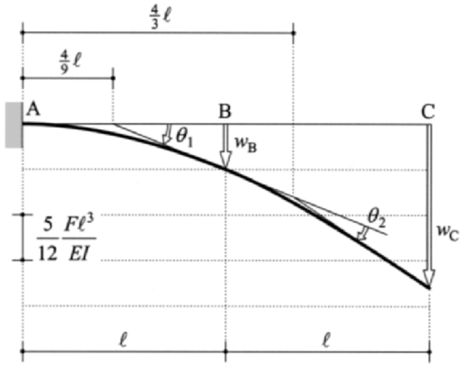

Figure 8.51 The elastic curve with the tangents at A, B and C and the bends θ_1 and θ_2. The deflections are scaled.

b. In Figure 8.51 the deflections are scaled. They have been magnified with respect to the structural dimensions. Using the tangents at A, B and C, we can easily draw an accurate sketch of the elastic curve.

c. If the beam is prismatic, with bending stiffness EI, the deflection at C according to forget-me-not (2) is

$$w_C = \frac{F(2\ell)^3}{3EI} = \frac{8}{3}\frac{F\ell^3}{EI}.$$

With respect to the value $w_C = \frac{3}{2}\frac{F\ell^3}{EI}$ found earlier in part (a), the deflection at C increases by

$$\frac{\frac{8}{3}-\frac{3}{2}}{\frac{3}{2}} \times 100\% = 78\%.$$

Comment: In the area AB the bending moments are largest, as is the influence on the deflection at the free end C (the tail-wagging effect due to θ_1, see Figure 8.50c). A magnification of the bending stiffness in this area therefore causes a relatively strong reduction in the deflection at C.

Example 4: Cantilever beam with distributed load and point load

In Figure 8.52a cantilever beam ABC carries a uniformly distributed load $q = 50$ kN/m along AB and a point load $F = 15$ kN at C. The bending stiffness is $EI = 7.36$ MNm^2.

Question:

Determine the deflection at C.

Solution:
Figure 8.52b shows the M/EI diagram. In order to recognise the shape of the M diagram in the M/EI diagram, the numerical value of the bending stiffness EI has not been included in the M/EI diagram. Figure 8.52c shows a sketch of the expected elastic curve.

When determining the deflection w_C at C we split the M/EI diagram into the parts (1) due to the distributed load q and (2) due to point load F (see Figures 8.52d and 8.52e). With $EI = 7.36$ MNm^2 the bends θ_1 and θ_2 are

$$\theta_1 = \tfrac{1}{3} \times (1.2\text{ m}) \times \frac{36\text{ kNm}}{EI} = 1.957 \times 10^{-3} \quad \text{(area parabola)},$$

$$\theta_2 = \tfrac{1}{2} \times (2.4\text{ m}) \times \frac{36\text{ kNm}}{EI} = 5.870 \times 10^{-3} \quad \text{(area triangle)}.$$

The deflections $w_{C;1}$ due to the distributed load and $w_{C;2}$ due to the point load are determined as the tail-wagging effects due to the bends θ_1 and θ_2 respectively

$$w_{C;1} = \theta_1 \times (2.1\text{ m}) = (1.957 \times 10^{-3}) \times (2.1\text{ m}) = 4.109 \times 10^{-3}\text{ m},$$

$$w_{C;2} = \theta_2 \times (1.6\text{ m}) = (5.870 \times 10^{-3}) \times (1.6\text{ m}) = 9.391 \times 10^{-3}\text{ m}.$$

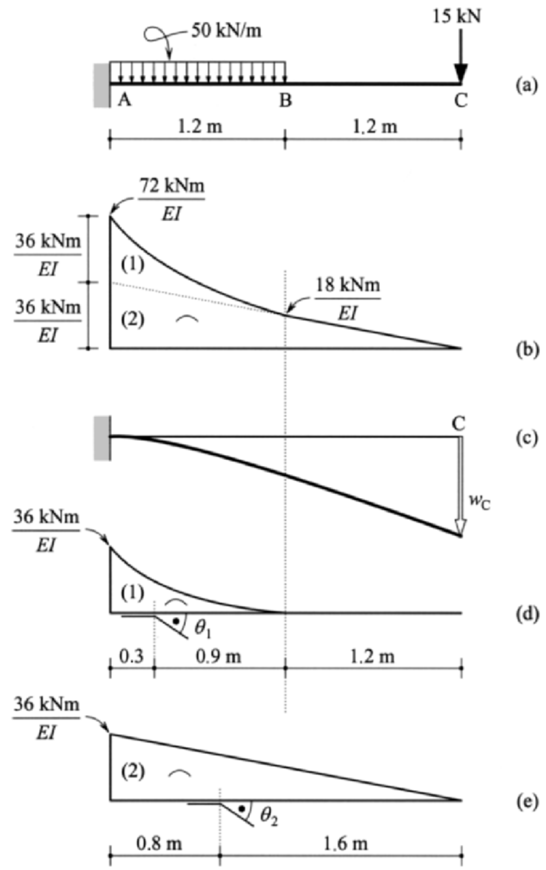

Figure 8.52

Figure 8.52 (a) A cantilever beam with a uniformly distributed load and a point load, (b) the M/EI diagram and (c) a sketch of the elastic curve. (d) The deformation due to the parabolic part (1) of the M/EI diagram can be considered concentrated at the bend θ_1. (e) The deformation due to the triangular part (2) of the M/EI diagram can be considered concentrated at the bend θ_2.

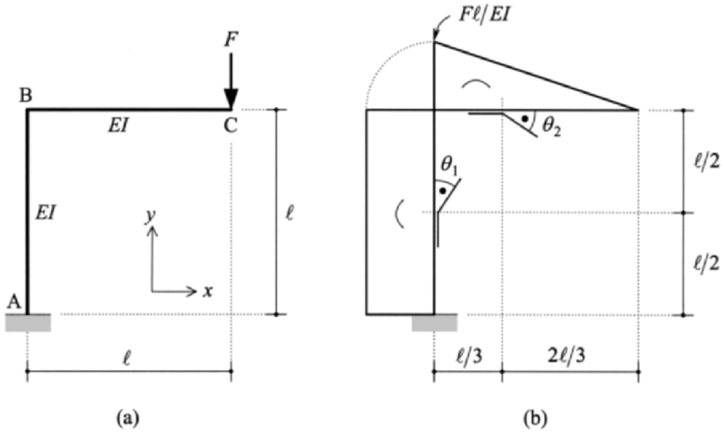

Figure 8.53 (a) A bent beam loaded by the force F at the free end C. (b) The M/EI diagram. The bends θ_1 and θ_2 represent the deformations of AB and BC respectively and are drawn in such a way that their open sides look in the direction that we are working towards, i.e. from A (where the rotation and displacement are known) to B and C (where we wish to know the rotation and displacement).

By superposing the contributions of the distributed load and point load we find for the final deflection at C

$$w_C = w_{C;1} + w_{C;2} = 13.5 \times 10^{-3} \text{ m} = 13.5 \text{ mm}.$$

Comment: Seventy percent of the deflection at C is caused by the point load of 15 kN at C, and 30% by the distributed load of 50 kN/m over AB. This could have been expected as the point load causes larger moments over the length of the beam than the distributed load.

Example 5: Fixed bent beam

The bent prismatic beam ABC, with a right angle at B, is fixed at A and loaded by a force F at the free end C. The bending stiffness of the bent beam is EI. The deformation due to normal forces is ignored.

Questions:

a. Determine the displacements at B and C in the given xy coordinate system.
b. Sketch the elastic curve with scaled displacements.

Solution:

a. Figure 8.53b shows the M/EI diagram. The deformation of AB can be considered concentrated at bend θ_1, that of BC at bend θ_2. The location of these bends (at the centroids of the respective parts of the M/EI diagram) are indicated in the figure. The magnitudes are

$$\theta_1 = \ell \cdot \frac{F\ell}{EI} = \frac{F\ell^2}{EI} \quad \text{(area of a rectangle)},$$

$$\theta_2 = \tfrac{1}{2} \cdot \ell \cdot \frac{F\ell}{EI} = \frac{1}{2}\frac{F\ell^2}{EI} \quad \text{(area of a triangle)}.$$

The bends are drawn in such a way again that the open side looks in the direction towards which we are working, i.e. from A (where the deflection and rotation are known) toward B and C (where we want to determine the deflection and rotation).

When determining the deflections at B and C we look at the following separately: (1) the influence of the deformation of AB and (2) the influence of the deformation of BC.

The deformation of AB causes the tail-wagging effect due to the bend θ_1 (see Figure 8.54a). In the given coordinate system

$$u^{(1)}_{x;\mathrm{B}} = u^{(1)}_{x;\mathrm{C}} = +\theta_1 \cdot \tfrac{1}{2}\,\ell = +\frac{F\ell^2}{EI} \cdot \tfrac{1}{2}\,\ell = +\frac{1}{2}\,\frac{F\ell^3}{EI}\,.$$

B and C have the same vertical distance to the bend θ_1 and therefore experience the same horizontal displacement.[1]

Since the horizontal distance from B to the bend θ_1 is zero, the vertical displacement of B is also zero:

$$u^{(1)}_{y;\mathrm{B}} = 0.$$

In addition

$$u^{(1)}_{y;\mathrm{C}} = -\theta_1 \cdot \ell = -\frac{F\ell^2}{EI} \cdot \ell = -\frac{F\ell^3}{EI}$$

The vertical displacement of C due to θ_1 is aimed downwards, against the positive y direction: hence the minus sign.

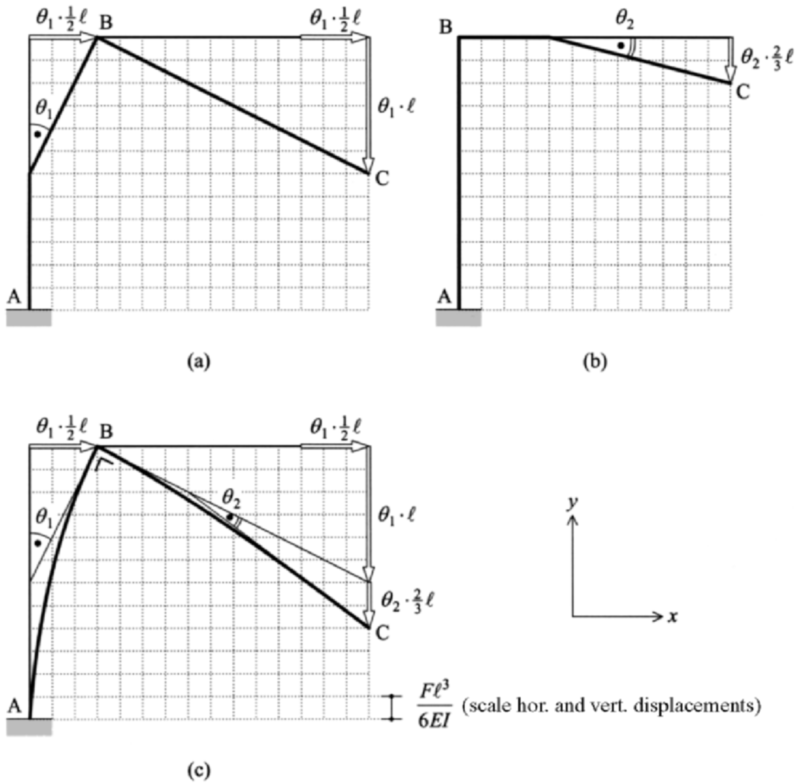

Figure 8.54 (a) Displacements and rotations at B and C due to the deformation of AB. (b) Displacement and rotation at C due to the deformation of BC. (c) The elastic curve with scaled displacements. Since the right angled bent at B is rigid, the deformation of the structure is due to the deformation of the members only, and the right angle at B remains a right angle in the deformed structure.

[1] See also the *Intermezzo* in Example 2.

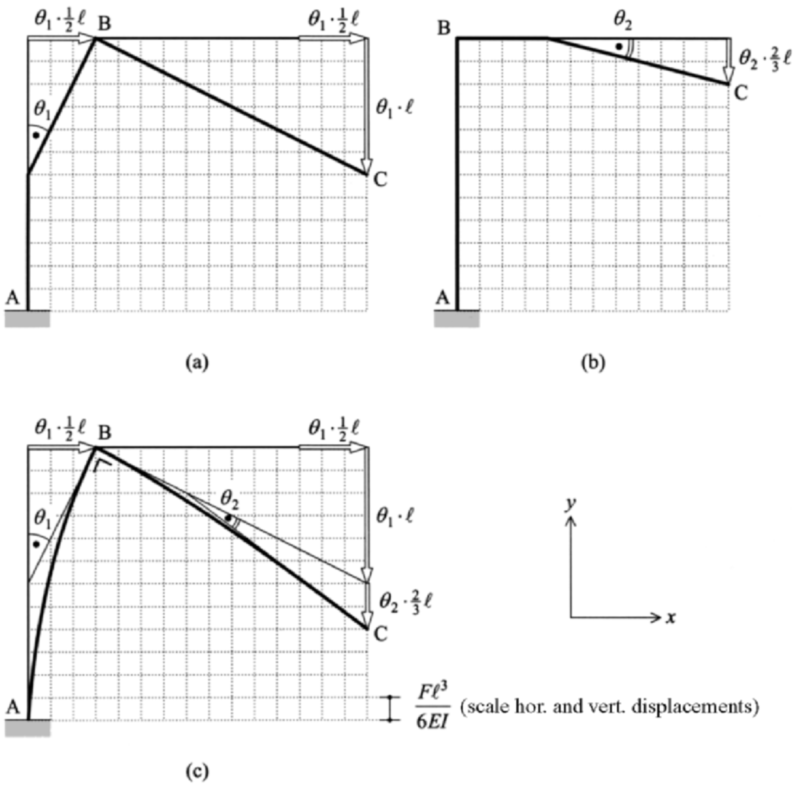

Figure 8.54 (a) Displacements and rotations at B and C due to the deformation of AB. (b) Displacement and rotation at C due to the deformation of BC. (c) The elastic curve with scaled displacements. Since the right angled bent at B is rigid, the deformation of the structure is due to the deformation of the members only, and the right angle at B remains a right angle in the deformed structure.

The deformation of BC causes the tail-wagging effect due to the bend θ_2 (see Figure 8.54b). The deformation of BC influences only the deflection of C. In the given coordinate system

$$u_{x;\mathrm{C}}^{(2)} = 0,$$

$$u_{y;\mathrm{C}}^{(2)} = -\theta_2 \cdot \tfrac{2}{3}\,\ell = -\frac{1}{2}\frac{F\ell^2}{EI} \cdot \tfrac{2}{3}\,\ell = -\frac{1}{3}\frac{F\ell^3}{EI}\,.$$

The results are

$$u_{x;\mathrm{B}} = u_{x;\mathrm{B}}^{(1)} = +\frac{1}{2}\frac{F\ell^3}{EI}\,,$$

$$u_{y;\mathrm{B}} = u_{y;\mathrm{B}}^{(1)} = 0,$$

$$u_{x;\mathrm{C}} = u_{x;\mathrm{C}}^{(1)} + u_{x;\mathrm{C}}^{(2)} = +\frac{1}{2}\frac{F\ell^3}{EI} + 0 = +\frac{1}{2}\frac{F\ell^3}{EI}\,,$$

$$u_{y;\mathrm{C}} = u_{y;\mathrm{C}}^{(1)} + u_{y;\mathrm{C}}^{(2)} = -\frac{F\ell^3}{EI} - \frac{1}{3}\frac{F\ell^3}{EI} = -\frac{4}{3}\frac{F\ell^3}{EI}\,.$$

b. In Figure 8.54c the displacements at B and C are scaled. They have been magnified again with respect to the structural dimensions. In addition, the tangents to the elastic curve at A, B and C are shown. These tangents can be used to draw a quick and accurate sketch of the elastic curve.

Comment: The deformation of the structure is the result of the deformation of the members AB and BC. Rigid corner joints[1] do not change shape. In

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[1] If the corner joint at B is not rigid but, for example, a spring joint, the angle between the members AB and BC will change. This has an effect on the displace-

the deformed structure, the right angle at B is therefore still a right angle (see Figure 8.54c).

Comment: In the first five examples in applying the moment-area theorems, we started at the a fixed end A for which the rotation φ_A and w_A were known (zero). This changes at Example 6. In Examples 6 to 9 the rotation at the start is not known but has to be determined from a deflection known elsewhere. Example 10, where we calculate the displacements for a three-hinged frame, goes a step further in that the necessary rotations at the supports have to be determined from the joining conditions at the hinge.

Example 6: Simply supported beam with overhang

Beam ABC, simply supported at A and B, carries a force F at the free end C of the overhang (see Figure 8.55a). The beam is prismatic with bending stiffness EI.

Questions:

a. Determine the deflection at C.
b. Determine the location and magnitude of the maximum deflection in field AB.

Solution:

a. Figure 8.55b shows the M diagram. Figure 8.55c shows a sketch of the elastic curve. An as yet unknown rotation θ_A occurs at A. The unknown rotation θ_A can be found from the fact that the deflection at B is zero. Starting at A, we can find the deflection at B by considering the deformation of AB. Figure 8.55d shows the M/EI diagram for this part of the beam. For the bend θ_1 we have

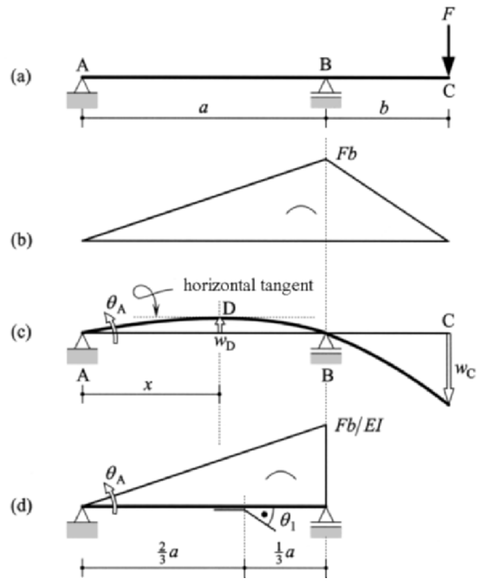


Figure 8.55 (a) A simply supported beam with overhang, loaded by a force F at the free end of the overhang, with (b) the bending moment diagram and (c) a sketch of the elastic curve. The maximum deflection in field AB occurs at D. Here the elastic curve has a horizontal tangent. (d) The M/EI diagram for AB. The unknown rotation θ_A at A is determined from the moment area theorem and the fact that the displacement at B is zero. To determine the displacement at B, we start at A and have to deal only with the deformation of AB.

ment at C. The calculation of the structures with non-rigid joints falls outside the scope of this book.

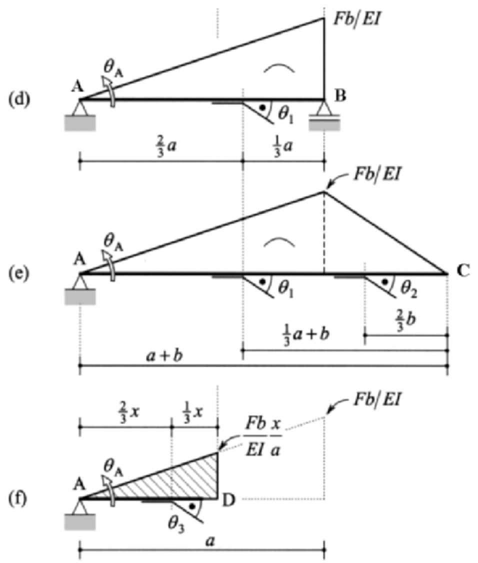


Figure 8.55 (d) The M/EI diagram for AB. The unknown rotation θ_A at A is determined from the moment area theorem and the fact that the displacement at B is zero. To determine the displacement at B, we start at A and have to deal only with the deformation of AB. (e) For the displacement at C, the M/EI diagram for the entire beam has to be taken into account. The deformations of AB and BC are concentrated in the bends θ_1 and θ_2 respectively. (f) If the elastic curve at D has a horizontal tangent, then $\theta_3 = \theta_A$. With this information we can determine the distance x from D to support A.

$$\theta_1 = \tfrac{1}{2} \cdot a \cdot \frac{Fb}{EI} = \frac{1}{2}\frac{Fab}{EI}\,.$$

From

$$w_B\ (\downarrow) = -\theta_A \cdot a + \theta_1 \cdot \tfrac{1}{3}\,a = 0$$

it follows that

$$\theta_A = \tfrac{1}{3}\,\theta_1 = \frac{1}{6}\frac{Fab}{EI}\,.$$

For the deflection at C we have to consider the M/EI diagram for the entire beam (see Figure 8.55e).

With

$$\theta_2 = \tfrac{1}{2} \cdot b \cdot \frac{Fb}{EI} = \frac{1}{2}\frac{Fb^2}{EI}$$

we find

$$\begin{aligned} w_C\ (\downarrow) &= -\theta_A \cdot (a+b) + \theta_1 \cdot \left(\tfrac{1}{3}\,a + b\right) + \theta_2 \cdot \tfrac{2}{3}\,b \\ &= -\frac{1}{6}\frac{Fab}{EI} \cdot (a+b) + \frac{1}{2}\frac{Fab}{EI} \cdot \left(\tfrac{1}{3}\,a + b\right) + \frac{1}{2}\frac{Fb^2}{EI} \cdot \tfrac{2}{3}\,a \\ &= \frac{1}{3}\frac{F(a+b)b^2}{EI}\,. \end{aligned}$$

b. Assume the maximum deflection in field AB occurs at D, at a distance x from A (see Figure 8.55c). The elastic curve has a horizontal tangent at

D, so that $\theta_D = 0$. Using this fact we can determine the distance x. For the length x of the hatched area of the M/EI diagram, Figure 8.55f shows that

$$\theta_3 = \tfrac{1}{2} \cdot x \cdot \frac{Fb}{EI} \cdot \frac{x}{a} = \frac{1}{2} \frac{Fab}{EI} \frac{x^2}{a^2}.$$

From

$$\theta_D\,(\circlearrowright) = \theta_A - \theta_3 = \frac{1}{6} \frac{Fab}{EI} - \frac{1}{2} \frac{Fab}{EI} \frac{x^2}{a^2} = 0$$

it follows that

$$\theta_3 = \theta_A \text{ and } x = \tfrac{1}{3} a\sqrt{3} = 0.577a.$$

This can be used to find the maximum deflection at D:

$$w_D\,(\uparrow) = \theta_A \cdot x - \theta_3 \cdot \tfrac{1}{3} x = \theta_A \cdot \tfrac{2}{3} x$$
$$= \frac{1}{6} \frac{Fab}{EI} \cdot \tfrac{2}{9} a\sqrt{3} = \frac{1}{27} \frac{Fa^2b}{EI} \sqrt{3}.$$

Example 7: Simply supported beam with uniformly distributed load
The simply supported beam in Figure 8.56a has a span of 12 m and carries a uniformly distributed load of 15 kN/m. The beam is prismatic with a bending stiffness $EI = 320$ MNm^2.

Question:
Determine the deflections at C and D, the points at one-third of the span.

Solution:
Figure 8.56b shows a sketch of the elastic curve, and Figure 8.56c shows

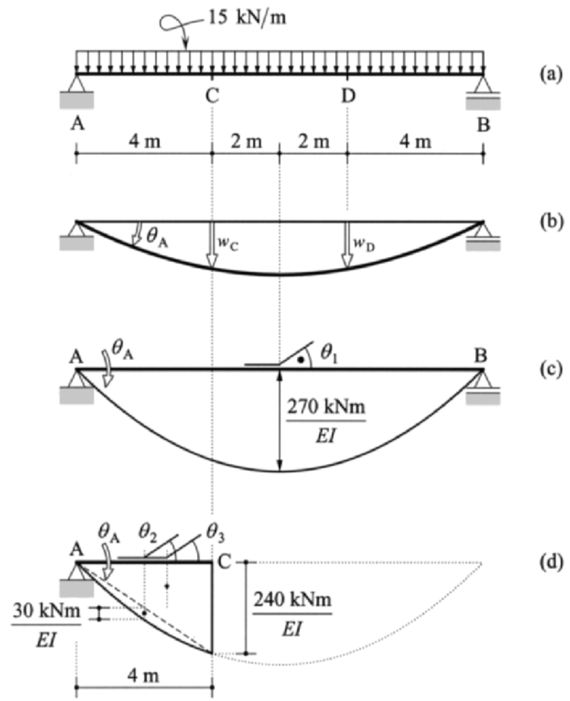


Figure 8.56 (a) A simply supported beam with uniformly distributed load. (b) Sketch of the elastic curve. (c) The M/EI diagram. (d) Part of the M/EI diagram required for determining the deflection at C. In the calculation, the M/EI diagram for AC is split into a parabolic section (bend θ_2) and a triangular section (bend θ_3).

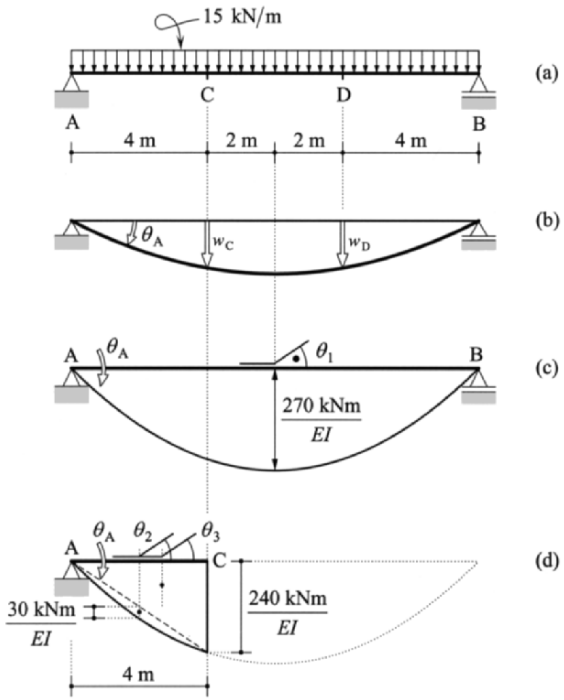


Figure 8.56 (a) A simply supported beam with uniformly distributed load. (b) Sketch of the elastic curve. (c) The M/EI diagram. (d) Part of the M/EI diagram required for determining the deflection at C. In the calculation, the M/EI diagram for AC is split into a parabolic section (bend θ_2) and a triangular section (bend θ_3).

the M/EI diagram.[1]

To determine the deflection w_C at C we first have to know the rotation at one of the supports.

The rotation θ_A at A follows from the fact that the deflection at B is zero. Working from A to B, the bend θ_1 is set down in the M/EI diagram with the open side looking in the direction towards B (see Figure 8.56b). The magnitude of θ_1 is

$$\theta_1 = \tfrac{2}{3} \cdot (12\text{ m}) \cdot \frac{(270\text{ kNm})}{EI} = \frac{2160\text{ kNm}^2}{EI} \quad \text{(area parabola)}.$$

From

$$w_B\,(\downarrow) = \theta_A \cdot (12\text{ m}) - \theta_1 \cdot (6\text{ m}) = 0$$

it follows that

$$\theta_A = \tfrac{1}{2}\,\theta_1 = \tfrac{1}{2} \cdot \frac{2160\text{ kNm}^2}{EI} = \frac{1080\text{ kNm}^2}{EI}.$$

Starting at A, we can now determine the deflection at C. Here the deformation of only AC is relevant. The M/EI diagram for AC is shown in Figure 8.56d. The calculation is left to the reader. The M/EI diagram is split into a parabolic area and a triangular area, of which the contributions

1 In order to recognise the shape of the M diagram in the M/EI diagram the numerical value of the bending stiffness EI has not been included in the diagram.

are calculated separately below. With

$$\theta_2 = \tfrac{2}{3} \cdot (4\ \text{m}) \cdot \frac{(30\ \text{kNm})}{EI} = \frac{80\ \text{kNm}^2}{EI}\ \ \text{(area of a parabola)},$$

$$\theta_3 = \tfrac{1}{2} \cdot (4\ \text{m}) \cdot \frac{(240\ \text{kNm})}{EI} = \frac{480\ \text{kNm}^2}{EI}\ \ \text{(area of a triangle)},$$

we find

$$\begin{aligned} w_\text{C}\ (\downarrow) &= \theta_\text{A} \cdot (4\ \text{m}) - \theta_2 \cdot (2\ \text{m}) - \theta_3 \cdot \left(\tfrac{1}{3} \times 4\ \text{m}\right) \\ &= \frac{1080\ \text{kNm}^2}{EI} \cdot (4\ \text{m}) - \frac{80\ \text{kNm}^2}{EI} \cdot (2\ \text{m}) \\ &\quad - \frac{480\ \text{kNm}^2}{EI} \cdot \left(\tfrac{1}{3} \times 4\ \text{m}\right) \\ &= \frac{3520\ \text{kNm}^3}{EI}. \end{aligned}$$

With $EI = 220\ \text{MNm}^2$, the numerical value of the deflection at C is

$$w_\text{C} = \frac{3520\ \text{kNm}^3}{320\ \text{MNm}^2} = 11 \times 10^{-3}\ \text{m} = 11\ \text{mm}\ (\downarrow)$$

On the basis of symmetry, the deflection at D is equal to that at C:

$$w_\text{D} = w_\text{C} = 11\ \text{mm}\ (\downarrow)$$

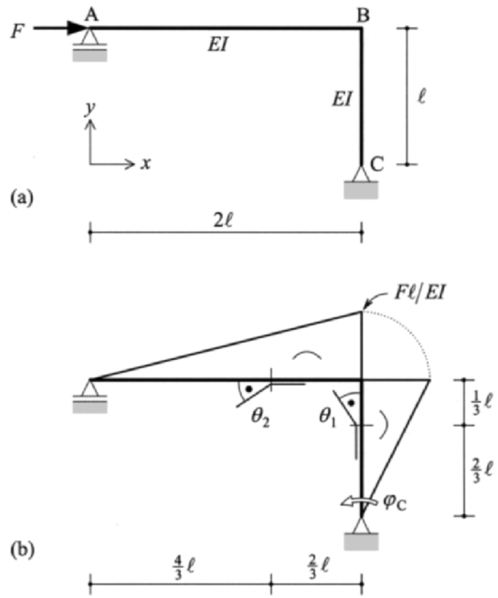


Figure 8.57 (a) The simply supported right-angled beam is loaded by a horizontal force F at roller support A. (b) The M/EI diagram. The bends θ_1 and θ_2 represent the deformations of CB and BA respectively. The open sides of the bends look in the direction in which we are working: from C over B to A. The unknown rotation at C is found from the joining condition that the vertical displacement at A must be zero.

Example 8: Simply supported right-angled beam

The right-angled beam in Figure 8.57a is supported by a hinge at C and supported on a roller with horizontal roller track at A. The corner joint at B is rigid. The structure is loaded by a horizontal force F at A. AB and BC have the same bending stiffness EI. Normal force deformation is ignored.

Questions:

a. Determine the horizontal displacement of the roller support at A.
b. Sketch the elastic curve for ABC.

Solution:

a. Figure 8.57b shows the M/EI diagram. The calculation is left to the reader.

C is a fixed point where the rotation is still unknown. This rotation can be determined by using the fact that the vertical deflection at A is zero.

In order to avoid errors, the calculations below are all performed in the xy coordinate system shown in Figure 8.57a.

Assume the rotation at C is φ_C (see Figure 8.57b). Working from C to A, we draw the bends θ_1 and θ_2, which represent the deformations of BC and AB respectively, with the the open side in the viewing direction, from C over B to A. For these bends we have

$$\theta_1 = \tfrac{1}{2} \cdot \ell \cdot \frac{F\ell}{EI} = \frac{1}{2}\frac{F\ell^2}{EI},$$

$$\theta_2 = \tfrac{1}{2} \cdot 2\ell \cdot \frac{F\ell}{EI} = \frac{F\ell^2}{EI}.$$

The vertical displacement at A is determined as the tail-wagging effect due to φ_C, θ_1 and θ_2:

$$u_{y;A} = -\varphi_C \cdot 2\ell - \theta_1 \cdot 2\ell - \theta_2 \cdot \tfrac{4}{3}\,\ell.$$

From $u_{y;A} = 0$ it follows that

$$\varphi_C = -\theta_1 - \tfrac{2}{3}\,\theta_2 = -\frac{1}{2}\,\frac{F\ell^2}{EI} - \tfrac{2}{3} \times \frac{F\ell^2}{EI} = -\frac{7}{6}\,\frac{F\ell^2}{EI}\,.$$

φ_C is negative; so the rotation at C is opposite to the direction assumed in Figure 8.57b.

The horizontal displacement at A is determined as the tail-wagging effect due to φ_C and θ_1. It should be noted that the tail-wagging effect due to θ_2 does not influence the horizontal displacement of A (again see Figure 8.57b)

$$u_{x;A} = -\varphi_C \cdot \ell - \theta_1 \cdot \tfrac{1}{3}\,\ell$$
$$= -\left(-\frac{7}{6}\,\frac{F\ell^2}{EI}\right) \cdot \ell - \left(\frac{1}{2}\,\frac{F\ell^2}{EI}\right) \cdot \tfrac{1}{3}\,\ell = +\frac{F\ell^3}{EI}\,.$$

b. Figure 8.58 shows the displacements of A and B to scale. With a certain amount of effort, we can also draw the tangents at A, B and C. This provides sufficient information for a good sketch of the elastic curve.

Comment:
Since the corner joint at B is rigid, the right angle at B remains a right angle in the deformed structure.

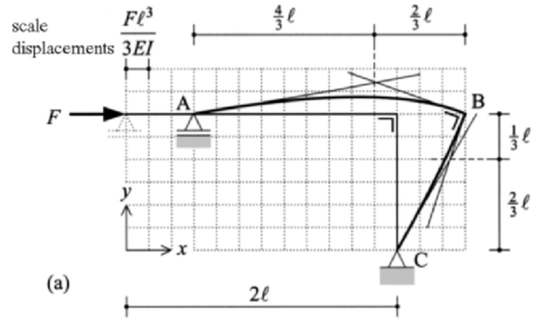


Figure 8.58 The elastic curve drawn to scale, with the tangents at A, B and C. The right angle at B remains a right angle in the deformed structure.

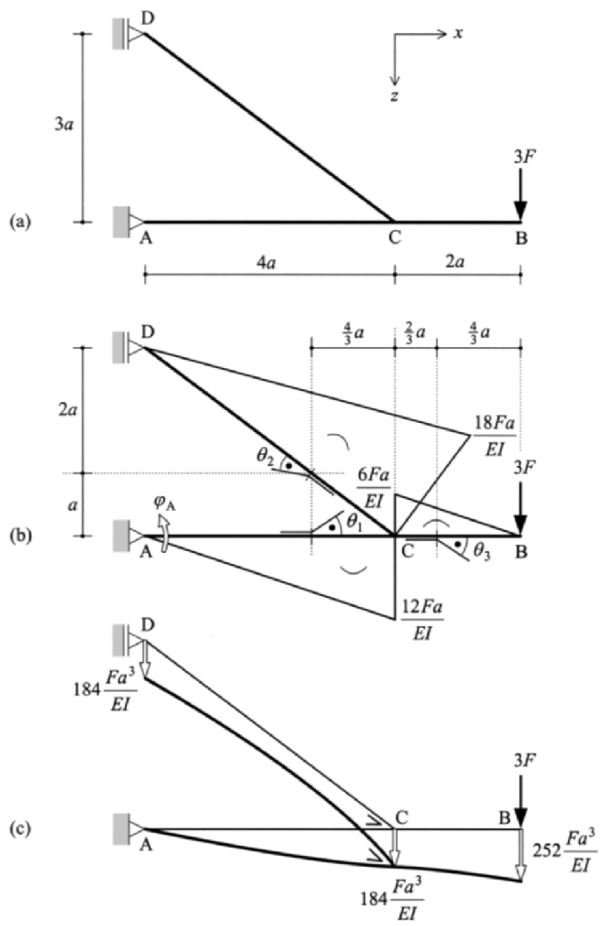


Figure 8.59

Example 9: Simply supported structure composed of two rigidly joined beams

The structure shown in Figure 8.59a is supported by a hinge at A and on a roller with vertical roller track at D. The corner joint at C is rigid. All members have the same bending stiffness EI.

For the given load $3F$ at B we only take account of the deformation due to bending; normal force deformation is ignored.

Use the xz coordinate system given in Figure 8.59a.

Questions:

a. Determine the rotation at A.
b. Determine the displacement at C.
c. Determine the displacement at D.
d. Determine the displacement at B.

Solution:

a. Figure 8.59b shows the M/EI diagram. Since all members have the same bending stiffness EI the M/EI diagram is the same shape as the M diagram. The calculation is left to the reader.

Figure 8.59 (a) Simply supported structure composed of two beams rigidly joined at C, and loaded by a vertical force $3F$ at the free end B. (b) The M/EI diagram. The unknown rotation φ_A can be determined from the joining condition that the horizontal displacement in D is zero. Here only the deformation of ACD is relevant, concentrated in the bends θ_1 and θ_2, that look with their open ends in the direction in which we are working: from A over C to D. (c) Sketch of the elastic curve. The rigid angle at C does not change.

A is a fixed point where the rotation φ_A is still unknown (see Figure 8.59b). This rotation can be found from the condition that the roller cannot move horizontally at D:

$$u_{x;D} = 0.$$

Looking from A only the rotation φ_A and the deformations of AC and CD are of influence on the deflection at D. The deformations of AC and CD are represented by the bends θ_1 and θ_2 respectively. The open side of these bends are again pointed in the viewing direction, from A via C to D. The bends θ_1 and θ_2 are

$$\theta_1 = \tfrac{1}{2} \cdot 4a \cdot \frac{12Fa}{EI} = 24\frac{Fa^2}{EI},$$

$$\theta_2 = \tfrac{1}{2} \cdot 5a \cdot \frac{18Fa}{EI} = 45\frac{Fa^2}{EI}.$$

From

$$u_{x;D} = -\varphi_A \cdot 3a - \theta_1 \cdot 3a - \theta_2 \cdot 2a = 0$$

it follows that

$$\varphi_A = -\theta_1 - \tfrac{2}{3}\theta_2 = -54\frac{Fa^2}{EI}.$$

The minus sign points to the fact that the rotation at A is opposite to the direction assumed in Figure 8.59b (see the sketch of the elastic curve in Figure 8.59c).

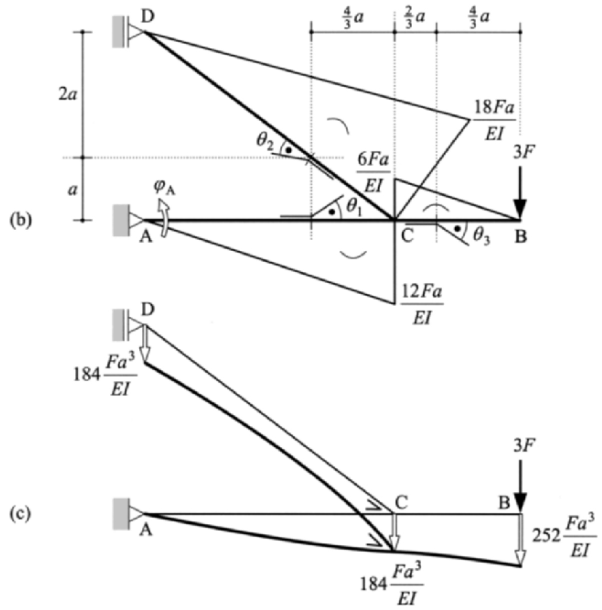


Figure 8.59 (b) The M/EI diagram. The unknown rotation φ_A can be determined from the joining condition that the horizontal displacement in D is zero. Here only the deformation of ACD is relevant, concentrated in the bends θ_1 and θ_2, that look with their open ends in the direction in which we are working: from A over C to D. (c) Sketch of the elastic curve. The rigid angle at C does not change.

b. The vertical deflection at C is

$$u_{z;\mathrm{C}} = -\varphi_\mathrm{A} \cdot 4a - \theta_1 \cdot \tfrac{4}{3}\,a = +184\frac{Fa^3}{EI}\,.$$

c. The vertical deflection at D is

$$u_{z;\mathrm{D}} = +\theta_1 \cdot \tfrac{8}{3}\,a + \theta_2 \cdot \tfrac{8}{3}\,a = +184\frac{Fa^3}{EI}\,.$$

Comment: Since D is directly above A and the horizontal component of the distance between A and D is zero, a rotation at A has no effect on the vertical displacement at D.

d. To determine the vertical deflection at B we also have to know the bend θ_3:

$$\theta_3 = \tfrac{1}{2} \cdot 2a \cdot \frac{6Fa}{EI} = 6\frac{Fa^2}{EI}\,.$$

Hence

$$u_{z;\mathrm{C}} = -\varphi_\mathrm{A} \cdot 6a - \theta_1 \cdot \left(\tfrac{4}{3}\,a + 2a\right) + \theta_3 \cdot \tfrac{4}{3}\,a = +252\frac{Fa^3}{EI}\,.$$

Figure 8.59c shows a sketch of the elastic curve. Since the corner joint at C is rigid, the deformation of the structure has no effect on the angle between the members ACB and CD.

Example 10: Three-hinged frame

In the three-hinged frame in Figure 8.60a, ACSD has a bending stiffness EI and BD has a bending stiffness $EI\sqrt{5}$. The frame is loaded by a vertical force $8F$ at D. Normal force deformation is ignored.

Questions:

a. Determine the displacements at S, C and D.
b. Sketch the elastic curve.

Solution:

a. Figure 8.60b shows the M/EI diagram. The calculation is left to the reader.

Note that the displacements and rotations are consistently denoted in the xz coordinate system given in Figure 8.59a. With complicated structures, working in a coordinate system reduces the probability of mistakes, even when the visual approach is used as with the use of moment-area theorems.

A and B are fixed points with unknown rotations. Assume φ_A is the rotation at A and φ_B the rotation at B (see Figure 8.60c).

Since the rotation directly to the left of S need not be the same as that directly to the right of S, the moment-area theorem cannot be used to work from A to B in one go.[1] Sections AS and BS are therefore released at S for the time being, and addressed separately.

From A and B, we can determine the horizontal and vertical deflection at S for both AS and BS. The unknown rotations φ_A and φ_B in these expressions follow from the *joining condition* that the displacements directly to the left and right of S must be equal. So

$$u_{x;S}^{(AS)} = u_{x;S}^{(BS)},$$

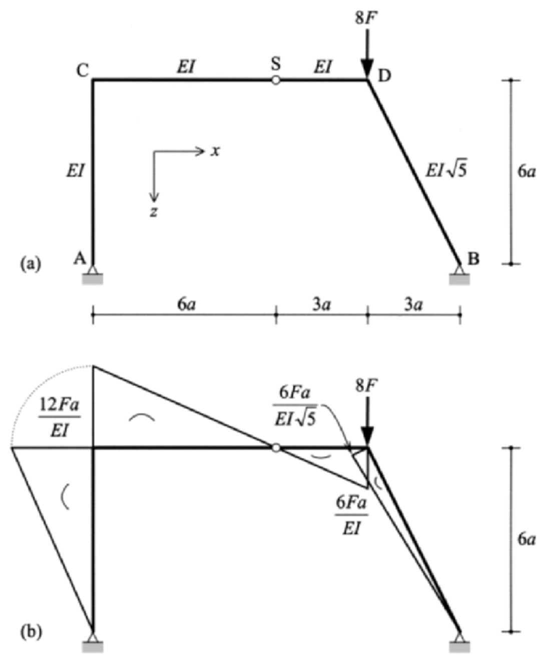


Figure 8.60 (a) Three-hinged frame, loaded by a vertical force $8F$ at D. (b) The M/EI diagram.

[1] If the rotations directly to the left and right of hinge S are not equal, then the rotation φ is discontinuous at S. However, when deriving the moment-area theorems in Section 8.4.1 we stated that the moment-area theorems are valid only if the rotation is continuous.

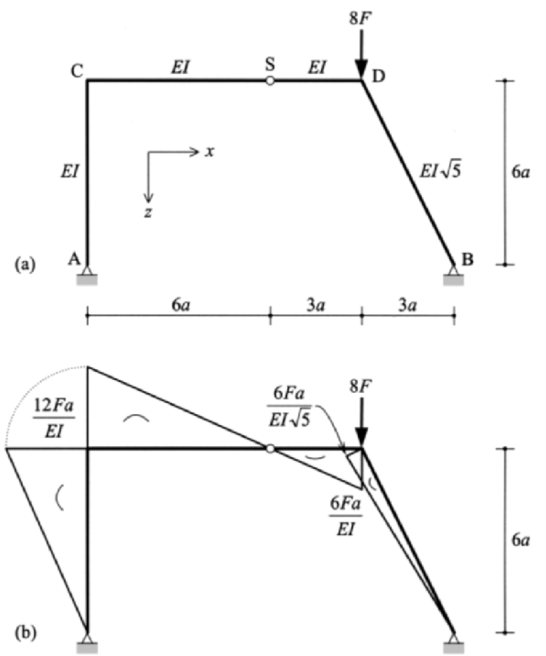


Figure 8.60 (a) Three-hinged frame, loaded by a vertical force $8F$ at D. (b) The M/EI diagram.

$$u_{z;\mathrm{S}}^{(\mathrm{AS})} = u_{z;\mathrm{S}}^{(\mathrm{BS})}.$$

Figure 8.60c shows the bends θ_1 to θ_4 that represent the deformations of the various parts of the three-hinged frame. All the bends θ_1 to θ_4 look in the direction in which we work, that is towards S.

To simplify the notation, we introduce the quantity θ:

$$\theta = \frac{Fa^2}{EI}.$$

The bends θ_1 to θ_4 can be expressed in terms of θ:

$$\theta_1 = \tfrac{1}{2} \cdot 6a \cdot \frac{12Fa}{EI} = 36\frac{Fa^2}{EI} = 36\theta,$$

$$\theta_2 = \theta_1 = 36\theta,$$

$$\theta_3 = \tfrac{1}{2} \cdot 3a\sqrt{5} \cdot \frac{6Fa}{EI\sqrt{5}} = 9\frac{Fa^2}{EI} = 9\theta,$$

$$\theta_4 = \tfrac{1}{2} \cdot 3a \cdot \frac{6Fa}{EI} = 9\frac{Fa^2}{EI} = 9\theta.$$

The displacement directly to the left of S (on AS) is

$$u_{x;\mathrm{S}}^{(\mathrm{AS})} = -\varphi_\mathrm{A} \cdot 6a + \theta_1 \cdot 2a = -6\varphi_\mathrm{A}a + 72\theta a,$$

$$u_{z;\mathrm{S}}^{(\mathrm{AS})} = -\varphi_\mathrm{A} \cdot 6a + \theta_1 \cdot 6a + \theta_2 \cdot 4a = -6\varphi_\mathrm{A}a + 360\theta a.$$

The deflection directly to the right of S (on BS) is

$$u_{x;\mathrm{S}}^{(\mathrm{BS})} = -\varphi_\mathrm{B} \cdot 6a + \theta_3 \cdot 2a = -6\varphi_\mathrm{B} a + 18\theta a,$$

$$u_{z;\mathrm{S}}^{(\mathrm{BS})} = +\varphi_\mathrm{B} \cdot 6a - \theta_3 \cdot 4a - \theta_4 \cdot 2a = +6\varphi_\mathrm{B} a - 54\theta a.$$

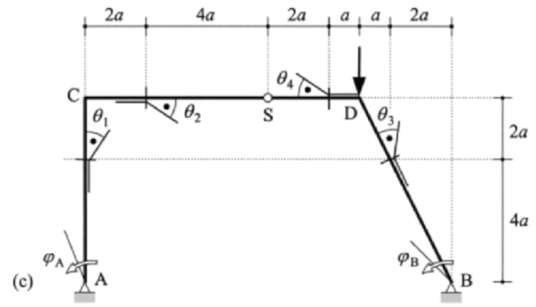


Figure 8.60 (c) The unknown rotations φ_A and φ_B at the supports are determined from the joining condition that the displacements directly to the left and right of hinge S have to be equal. For AS, working from A to S, and for BS, working from B to S, all bends look towards S.

From the equation $u_{x;\mathrm{S}}^{(\mathrm{AS})} = u_{x;\mathrm{S}}^{(\mathrm{BS})}$ it follows that

$$-6\varphi_\mathrm{A} a + 72\theta a = -6\varphi_\mathrm{B} a + 18\theta a \Rightarrow \varphi_\mathrm{A} - \varphi_\mathrm{B} = +9\theta. \quad (8.13)$$

In the same way, it follows from $u_{z;\mathrm{S}}^{(\mathrm{AS})} = u_{z;\mathrm{S}}^{(\mathrm{BS})}$ that

$$-6\varphi_\mathrm{A} a + 360\theta a = +6\varphi_\mathrm{B} a - 54\theta a \Rightarrow \varphi_\mathrm{A} + \varphi_\mathrm{B} = +69\theta. \quad (8.14)$$

(8.13) and (8.14) are two equations with φ_A and φ_B as the two unknowns. The solution is

$$\varphi_\mathrm{A} = 39\theta = +39\frac{Fa^2}{EI},$$

$$\varphi_\mathrm{B} = 30\theta = +30\frac{Fa^2}{EI}.$$

The rotations at A and B are both positive; their directions therefore agree with the directions assumed in Figure 8.60c.

From A we can now determine the following displacements and rotations to the left of S:[1]

1 The intermediate calculations are left to the reader.

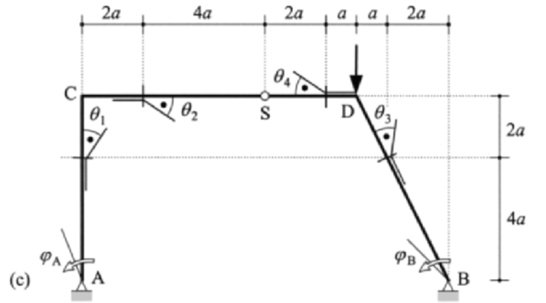


Figure 8.60 (c) The unknown rotations φ_A and φ_B at the supports are determined from the joining condition that the displacements directly to the left and right of hinge S have to be equal. For AS, working from A to S, and for BS, working from B to S, all bends look towards S.

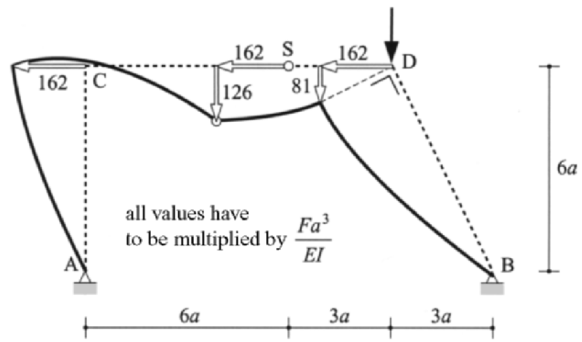


Figure 8.61 Elastic curve drawn to scale. Deformation occurs only in the members. The rigid corner joints at C and D do not change. The displacement at C is normal to AC and the displacement of D normal to BD. The elastic curve for CSD shows a bend at hinge S.

$$u_{x;C} = -\varphi_A \cdot 6a + \theta_1 \cdot 2a = -162\frac{Fa^3}{EI} \text{ and } u_{z;C} = 0,$$

$$u_{x;S} = -\varphi_A \cdot 6a + \theta_1 \cdot 2a = -162\frac{Fa^3}{EI} \ (u_{x;S} = u_{x;C}),$$

$$u_{z;S} = -\varphi_A \cdot 6a + \theta_1 \cdot 6a + \theta_2 \cdot 4a = +126\frac{Fa^3}{EI},$$

$$\varphi_C = \varphi_A - \theta_1 = +3\frac{Fa^2}{EI} \text{ and } \varphi_S^{(AS)} = \varphi_C - \theta_2 = -33\frac{Fa^2}{EI}.$$

From B we can determine the displacements and rotations to the right of S:

$$u_{x;D} = -\varphi_B \cdot 6a + \theta_3 \cdot 2a = -162\frac{Fa^3}{EI} \ (u_{x;D} = u_{x;S} = u_{x;C}),$$

$$u_{x;D} = +\varphi_B \cdot 3a - \theta_3 \cdot a = +81\frac{Fa^3}{EI},$$

$$\varphi_D = \varphi_B - \theta_3 = +21\frac{Fa^2}{EI} \text{ and } \varphi_S^{(BS)} = \varphi_D - \theta_4 = +12\frac{Fa^2}{EI}.$$

Comment: The displacement at S is no longer determined from B, as it is equal to that determined from A. The reader is requested to check this by means of calculation.

b. Figure 8.61 shows the elastic curve. Please note the following:

- The corner joints at C and D are rigid and therefore remain unchanged in the deformed structure.
- The elastic curve for CSD has a bend at hinge S. Here the rotation

φ (the derivative of the vertical displacement) is therefore indeed discontinuous.

- Since the normal force deformation is ignored, the horizontal distance between C, S and D does not change. All points on CSD therefore have the same horizontal displacement. For CSD, the deformation due to bending causes only vertical displacements.
- The displacement at D is normal to BD, since – we repeat – the deformation due to bending gives deflections only normal to the member axis.

8.5 Simply supported beams and the *M/EI* diagram

In this last section in Chapter 8 we look at two properties related to the M/EI diagram for a simply supported beam. In Section 8.5.1 we show that the rotations at the supports can be found as the support reactions due to a distributed load that is equal in shape and magnitude to the M/EI diagram. In Section 8.5.2 we derive an approximate formula for determining the maximum deflection of a simply supported beam. This formula is related to the area of the M/EI diagram.

8.5.1 Rotation at the supports of a simply supported beam; the conjugate-beam method

In this section we show that the rotations at the supports of a simply supported beam can be found as the support reactions due to a distributed load equal in magnitude and shape to the M/EI diagram.

Figure 8.62b shows the M/EI diagram of the simply supported beam AB in Figure 8.62a. The beam is subject to a distributed load, for which the details are not further given. Figure 8.62c shows the deflections and rotations due

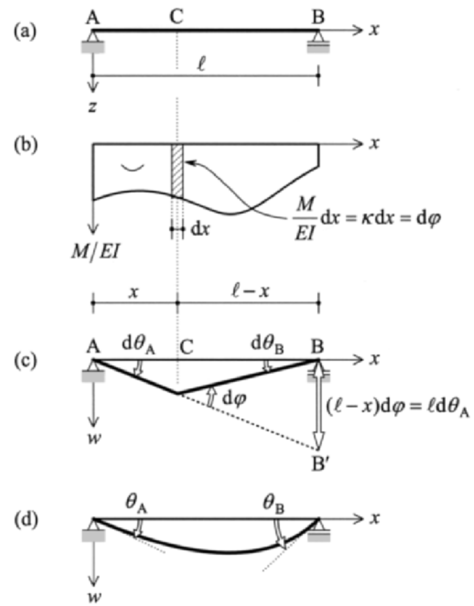


Figure 8.62 (a) A simply supported beam with (b) the M/EI or curvature diagram due to a distributed load for which the details are not given. (c) Elastic curve due to the deformation of only a small beam element at C. AC and BC remain straight and meet at an angle which is equal to the area of the hatched strip in the M/EI diagram. (d) The final elastic curve associated with the complete M/EI diagram in (b).

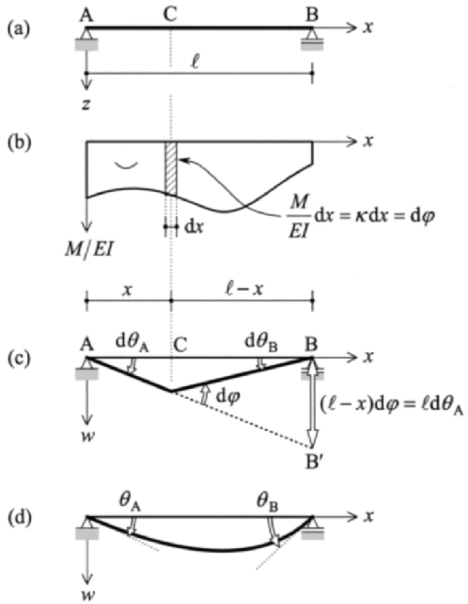


Figure 8.62 (a) A simply supported beam with (b) the M/EI or curvature diagram due to a distributed load for which the details are not given. (c) Elastic curve due to the deformation of only a small beam element at C. AC and BC remain straight and meet at an angle which is equal to the area of the hatched strip in the M/EI diagram. (d) The final elastic curve associated with the complete M/EI diagram in (b).

to the deformation of only a small beam element dx at C, at a distance x from A and $(\ell - x)$ from B. Since the deformation of AC and BC is ignored, these parts remain straight. At C they are at an angle dφ to one another, of which the magnitude is equal to the area of the hatched strip of the M/EI diagram in Figure 8.62b:

$$\mathrm{d}\varphi = \frac{M}{EI}\,\mathrm{d}x.$$

The rotation dθ_A at A due to only the deformation dφ of the beam at C is found from the geometry (kinematic relationship) in Figure 8.62c. The distance BB′ is

$$\mathrm{BB}' = \ell\,\mathrm{d}\theta_\mathrm{A} = (\ell - x)\,\mathrm{d}\varphi,$$

so

$$\mathrm{d}\theta_\mathrm{A} = \left(1 - \frac{x}{\ell}\right)\mathrm{d}\varphi = \left(1 - \frac{x}{\ell}\right)\frac{M}{EI}\,\mathrm{d}x. \qquad (8.13)$$

Since

$$\mathrm{d}\varphi = \mathrm{d}\theta_\mathrm{A} + \mathrm{d}\theta_\mathrm{B},$$

the rotation at B is

$$\mathrm{d}\theta_\mathrm{B} = \mathrm{d}\varphi - \mathrm{d}\theta_\mathrm{A} = \frac{x}{\ell}\frac{M}{EI}\,\mathrm{d}x. \qquad (8.14)$$

The final rotations θ_A in A and θ_B at B (see Figure 8.62d) is found by summing the deformation contributions of all small beam elements dx between $x = 0$ and $x = \ell$, i.e. by integrating all contributions over the length ℓ:

$$\theta_A = \int_0^\ell \left(1 - \frac{x}{\ell}\right) \frac{M}{EI}\, dx, \tag{8.15}$$

$$\theta_B = \int_0^\ell \frac{x}{\ell} \frac{M}{EI}\, dx. \tag{8.16}$$

Figure 8.63a shows the same simply supported beam with an arbitrarily distributed load q. The support reactions at A and B are A_v and B_v respectively. We determine these support reactions by using Figure 8.63b, in which the support reactions dA_v and dB_v are shown due to only the resultant $q dx$ of the distributed load q over the small length dx at C. From the equilibrium of beam AB it follows that

$$dA_v = \left(1 - \frac{x}{\ell}\right) q\, dx, \tag{8.17}$$

$$dB_v = \frac{x}{\ell} q\, dx. \tag{8.18}$$

The support reactions A_v at A and B_v in B are found by integrating the contributions of all beam elements dx over the length ℓ:

$$A_v = \int_0^\ell \left(1 - \frac{x}{\ell}\right) q\, dx, \tag{8.19}$$

$$B_v = \int_0^\ell \frac{x}{\ell} q\, dx. \tag{8.20}$$

If we compare the equilibrium equations (8.17) to (8.20) for the support reactions at A and B to kinematic relationships (8.13) to (8.16) for the rotations at A and B, there is a clear similarity in structure. If, in the equilibrium

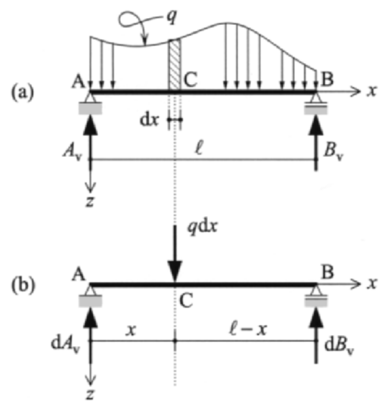


Figure 8.63 (a) A simply supported beam with a distributed load $q = q(x)$. (b) The support reactions dA_v and dB_v due to the resultant $q\, dx$ of the distributed load q over a small beam element dx at C.

equations (8.17) to (8.20) we replace the distributed load q by M/EI, the associated support reactions are exactly equal to the rotations.

Conclusion: *For a simply supported beam, the rotations at the supports can be found as the support reactions due to a distributed load that is equal in shape and magnitude to the M/EI diagram.*

The direction of the rotations in Figure 8.62d can be related to the direction of the support reactions in Figure 8.63a, however they can often be predicted beforehand using common sense.

Comment: If the analogy is applied correctly, the real M/EI diagram and the M/EI diagram as load diagram are plotted on *opposite sides of the member axis*: compare Figures 8.62b and 8.63a. In Figure 8.62b, M/EI is positive in the given xz coordinate system. In the same coordinate system, a positive M/EI diagram as a load acts downwards, as shown in Figure 8.63a.

Comment: Equations (8.17) to (8.20) are based on the *equilibrium equations*, and (8.13) to (8.16) on the *kinematic relationships*. Both types of relationship are shown below as they were derived previously in an xz coordinate system:[1]

Kinematic relationships	Equilibrium equations
$\dfrac{\mathrm{d}(-\varphi)}{\mathrm{d}x} = -\kappa = -\dfrac{M}{EI}$	$\dfrac{\mathrm{d}V}{\mathrm{d}x} = -q$
$\dfrac{\mathrm{d}w}{\mathrm{d}x} = (-\varphi)$	$\dfrac{\mathrm{d}M}{\mathrm{d}x} = V$
$\dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\kappa = -\dfrac{M}{EI}$	$\dfrac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -q$

1 See Section 4.3.

The kinematic relationships have the same structure as the equilibrium equations. If the curvature $\kappa = M/EI$ is seen as a distributed load q, the associated shear force V is equal to the rotation φ apart from the sign, and the bending moment is equal to the deflection w. The shear force diagram (apart from the sign) therefore gives the distribution of the rotation φ and the bending moment diagram gives the bending curve. This analogy is known as the conjugate-beam method.

When further elaborating this analogy, we have to take account, however, of the fact that the boundary conditions may be changed. For the beam AB, fixed at A in Figure 8.64a, and loaded by a point load at the free end B, both the deflection and rotation at A are zero. In Figure 8.64d the M/EI diagram has been introduced as the loading diagram. At A the shear force (that in an absolute sense represents the rotation) must be zero, and the bending moment (that represents the deflection) must be zero. Apparently, A has become a free end in the analogy, and B has become a fixed end. In this situation the bending moment diagram, as a result of the triangular load (see Figure 8.64e) is the same as the elastic curve of the beam due to the point load (see Figure 8.64c).[1]

Since all in all this is quite confusing, this approach is not elaborated for a general case, but its application remains limited to the simply supported beam in which case the boundary conditions in the analogy do not change. This is illustrated using three examples, the elaboration of which is to a great extent left to the reader.

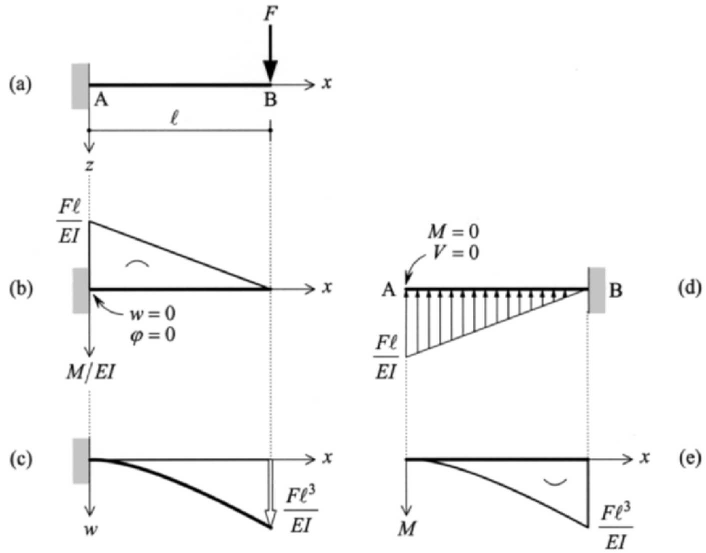


Figure 8.64 (a) Fixed beam loaded by the force F at the free end, with (b) the M/EI diagram and (c) the elastic curve. (d) If the M/EI diagram in (b) is seen as the load diagram – plotted on the opposite side of the beam axis, and acting upwards – then the associated bending moment diagram (e) is the same shape as the elastic curve (c). It is confusing, however, that the boundary conditions at A and B have to be adapted.

[1] In Figure 8.64b M/EI in the given xz coordinate system is negative and, as a distributed load, it acts upwards. Therefore the real M/EI diagram and the M/EI diagram as load diagram are plotted on opposite sides of the member axis.

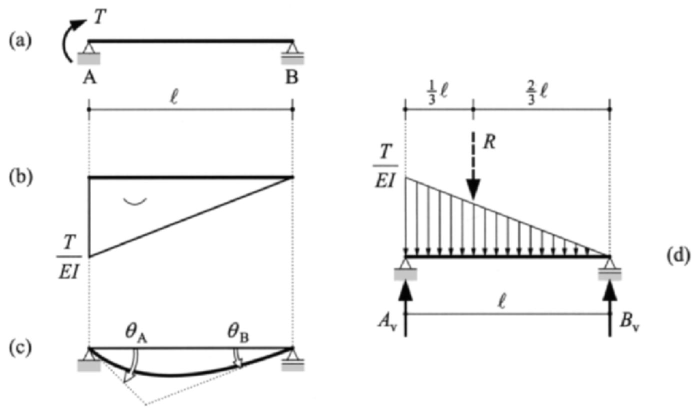


Figure 8.65 (a) A simply supported beam loaded by a couple at the left-hand end, with (b) the M/EI diagram and (c) the elastic curve. The rotations θ_A and θ_B at the supports are equal to the support reactions A_v and B_v due to (d) a distributed load of which the load diagram is equal to the M/EI diagram in (b).

Example 1: Simply supported beam loaded by a couple at one end
The simply supported beam in Figure 8.65a is loaded by the couple T at A. The beam is prismatic with bending stiffness EI.

Question:
Determine the rotations at A and B.

Solution:
Figure 8.65b shows the M/EI diagram and Figure 8.65c shows a sketch of the elastic curve. The rotations θ_A and θ_B are equal to the support reactions A_v and B_v of the simply supported beam in Figure 8.65d, when the M/EI diagram has been applied as the load diagram.

The resultant R is

$$R = \tfrac{1}{2} \cdot \frac{T}{EI} \cdot \ell = \frac{T\ell}{2EI}.$$

From the moment equilibrium about B and A respectively we find

$$\theta_A = A_v = \tfrac{2}{3} R = \tfrac{2}{3} \cdot \frac{T\ell}{2EI} = \frac{T\ell}{3EI},$$

$$\theta_B = B_v = \tfrac{1}{3} R = \tfrac{1}{3} \cdot \frac{T\ell}{2EI} = \frac{T\ell}{6EI}.$$

This result is in agreement with forget-me-not (4) in Table 8.4.

Example 2: Simply supported, non-prismatic beam with uniformly distributed load

In Figure 8.66a, the simply supported beam carries a uniformly distributed load q along the entire length. The beam is non-prismatic with a bending stiffness EI for AC and $2EI$ for BC.

Question:
Determine the rotations at A and B.

Solution:
Figure 8.66b shows the M/EI diagram and Figure 8.66c a sketch of the elastic curve. In Figure 8.66d, the M/EI diagram is considered to be the loading diagram. The support reactions A_v and B_v are equal to the rotations θ_A and θ_B respectively, and can determined from the equilibrium of the beam in Figure 8.66d. The resultants $R^{(AC)}$ and $R^{(BC)}$ are

$$R^{(AC)} = \tfrac{2}{3} \cdot \tfrac{1}{2}\,\ell \cdot \frac{\tfrac{1}{8} q\ell^2}{EI} = \frac{q\ell^3}{24EI}\,,$$

$$R^{(BC)} = \tfrac{2}{3} \cdot \tfrac{1}{2}\,\ell \cdot \frac{\tfrac{1}{8} q\ell^2}{2EI} = \frac{q\ell^3}{48EI}\,.$$

Using the moment equilibrium about B and A respectively, we find

$$\theta_A = A_v = \tfrac{11}{16} \cdot R^{(AC)} + \tfrac{5}{16} \cdot R^{(BC)} = \frac{9}{256}\,\frac{q\ell^3}{EI}\,,$$

$$\theta_B = B_v = \tfrac{5}{16} \cdot R^{(AC)} + \tfrac{11}{16} \cdot R^{(BC)} = \frac{7}{256}\,\frac{q\ell^3}{EI}\,.$$

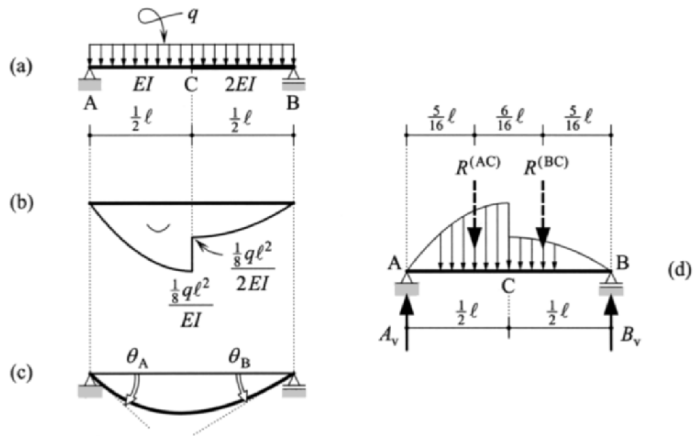


Figure 8.66 (a) A simply supported, non-prismatic beam with uniformly distributed load. (b) The M/EI diagram and (c) the elastic curve. The rotations θ_A and θ_B at the supports are equal to the support reactions A_v and B_v due to (d) a distributed load, of which the load diagram is equal to the M/EI diagram in (b).

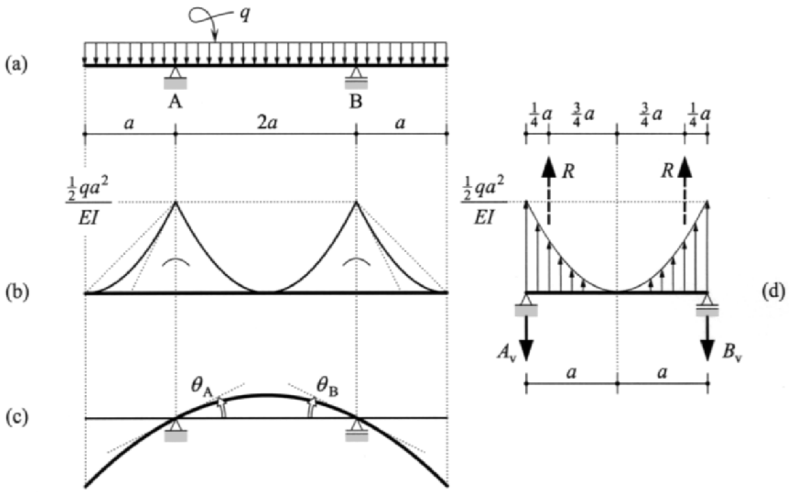


Figure 8.67 (a) A simply supported beam with overhangs and a uniformly distributed load. (b) The M/EI diagram and (c) the elastic curve. The rotations θ_A and θ_B at the supports are equal to the support reactions A_v and B_v due to (d) a distributed load on AB, of which the load diagram is equal to the M/EI diagram in (b). Note that the M/EI diagrams of the overhangs are ignored!

Example 3: Simply supported beam with overhangs and a uniformly distributed load

The beam in Figure 8.67a, simply supported at A and B, has two overhangs of equal length, and carries a uniformly distributed load q along the entire length. The beam is prismatic and has a bending stiffness EI.

Question:
Determine the rotations at the supports.

Solution:
Figure 8.67b shows the M/EI diagram. The calculation is left to the reader. Figure 8.67c shows a sketch of the expected elastic curve.

The rotations at the supports A and B depend on the deformation of AB only. The effect of the overhangs is expressed in the bending moment diagram and consequently in the M/EI diagram for AB. When determining the rotations at A and B, we therefore ignore the M/EI diagram of the overhangs!

In Figure 8.67d, the M/EI diagram for AB is considered to be the load diagram.[1] The resultant R of the distributed load on half the beam AB is

$$R = \tfrac{1}{3} \cdot \frac{\tfrac{1}{2} qa^2}{EI} \cdot a = \frac{qa^3}{6EI} .$$

On the basis of symmetry, the vertical support reactions at A and B are equal:

$$A_v = B_v = R = \frac{qa^3}{6EI} .$$

[1] Here the upward distributed load is not plotted on the underside but on the upperside of the member axis.

The rotations θ_A and θ_B in Figure 8.67c are equal to the vertical support reactions at A and B respectively:

$$\theta_A = \theta_B = \frac{qa^3}{6EI}\,.$$

8.5.2 Maximum deflection of a simply supported beam

In this section we derive an approximate formula for the maximum deflection of a simply supported beam. This formula is related to the area of the M/EI diagram.

The same simply supported prismatic beam with span ℓ and bending stiffness EI carries in Figure 8.68a a uniformly distributed load q, and in Figure 8.68b a point load F at midspan. Figure 8.68 shows the M/EI diagram for both cases.

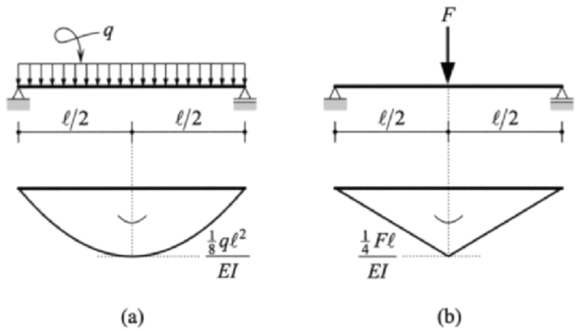


Figure 8.68 The M/EI diagram for a simply supported beam with (a) a uniformly distributed load and (b) a point load at midspan.

The maximum bending of the beam with uniformly distributed load q is:[1]

$$w_{\max}^{(q)} = \frac{5}{384}\frac{q\ell^4}{EI}\,.$$

The area of the M/EI diagram is

$$A_{M/EI}^{(q)} = \tfrac{2}{3}\cdot\frac{\frac{1}{8}q\ell^2}{EI}\cdot\ell = \frac{1}{12}\frac{q\ell^3}{EI}\,.$$

[1] See Section 8.1, Example 2 and Section 8.2, Example 3.

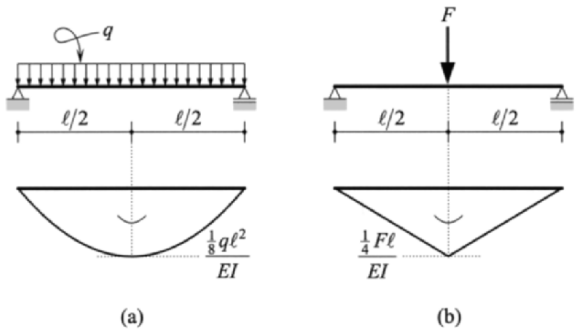


Figure 8.68 The M/EI diagram for a simply supported beam with (a) a uniformly distributed load and (b) a point load at midspan.

The maximum deflection $w_{\max}^{(q)}$ can also be written as

$$w_{\max}^{(q)} = \frac{5}{384}\frac{q\ell^4}{EI} = \frac{5}{32}\cdot \mathrm{A}_{M/EI}^{(q)}\cdot \ell. \tag{8.21}$$

Comment: A dimension check shows us that the area of the M/EI diagram is dimensionless!

In the same way, the maximum bending of the beam with a point load F at the centre of the span can be expressed in the area of the M/EI diagram.

The maximum deflection[1] is

$$w_{\max}^{(F)} = \frac{1}{48}\frac{F\ell^3}{EI}.$$

The area of the M/EI diagram is

$$\mathrm{A}_{M/EI}^{(F)} = \tfrac{1}{2}\cdot \frac{\frac{1}{4}F\ell}{EI}\cdot \ell = \frac{1}{8}\frac{F\ell^2}{EI},$$

so $w_{\max}^{(F)}$ can be written as

$$w_{\max}^{(F)} = \frac{1}{48}\frac{F\ell^3}{EI} = \frac{1}{6}\cdot \mathrm{A}_{M/EI}^{(F)}\cdot \ell. \tag{8.22}$$

The coefficients $5/32 = 15/96$ and $1/6 = 16/96$ in equations (8.21) and (8.22) differ by some 6.5%. Expression (8.22) could therefore be used as

[1] See Section 8.1, Example 4.

an approximation for the maximum deflection:

$$w_{\max} = \frac{1}{6} \cdot \mathrm{A}_{M/EI} \cdot \ell. \tag{8.23}$$

This formula also works quite well when the beam is not prismatic.

Comment: The approximate formula (8.23) is less effective if the bending moment at the supports is not zero.

Below we give four examples.

Example 1: Simply supported beam with two point loads
The simply supported prismatic beam in Figure 8.69a carries two point loads F.

Question:
Find an approximation to the maximum deflection.

Solution:
Figure 8.69b shows the M/EI diagram with area $2Fa/EI$. According to the approximate formula (8.23), the maximum deflection is

$$w_{\max} = \frac{1}{6} \times 2\frac{Fa^2}{EI} \times 3a = \frac{Fa^3}{EI}\,.$$

This is approximately 4.3% larger than the exact value of $\frac{23}{24}\frac{Fa^3}{EI}$.

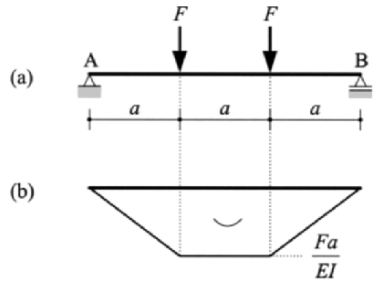


Figure 8.69 (a) A simply supported beam with two point loads and (b) the associated M/EI diagram.

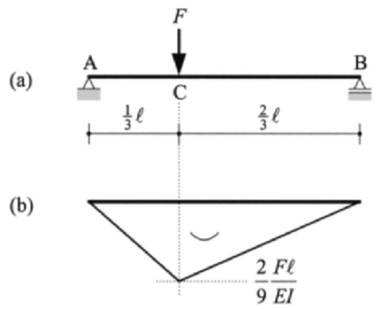


Figure 8.70 (a) A simply supported beam with a point load on one-third of the span and (b) the associated M/EI diagram.

Example 2: Simply supported prismatic beam with an eccentric point load

The simply supported beam AB in Figure 8.70a has a span ℓ and bending stiffness EI. The beam carries a point load F at C at a distance $\ell/3$ from the support A.

Question:
Find an approximation to the maximum deflection.

Solution:
Figure 8.70b shows the M/EI diagram with area

$$\mathrm{A}_{M/EI} = \tfrac{1}{2} \times \tfrac{2}{9}\,\frac{F\ell}{EI} \times \ell = \frac{1}{9}\,\frac{F\ell^2}{EI}\,.$$

The approximate formula (8.23) gives the maximum deflection:

$$w_{\max} = \tfrac{1}{6} \times \frac{1}{9}\,\frac{F\ell^2}{EI} \times \ell = \frac{1}{54}\,\frac{F\ell^3}{EI} = 18.5 \times 10^{-3} \times \frac{F\ell^3}{EI}\,.$$

Note: the maximum deflection does not occur at C, where the point load is applied, nor at midspan.

Comment: The exact value of the deflection can be determined using formula (8.5), derived in Section 8.3, Example 6:

$$w_{\max} = \frac{Fb(a^2 + 2ab)^{3/2}}{9\sqrt{3}\,EI(a+b)} \quad \text{in which } a + b = \ell \text{ and } a > b.$$

In the situation in Figure 8.64 we have to use $a = \frac{2}{3}\,\ell$ and $b = \frac{1}{3}\,\ell$, so

$$w_{\max} = \frac{F \cdot \frac{1}{3}\,\ell\left\{\left(\frac{2}{3}\,\ell\right)^2 + 2\left(\frac{2}{3}\,\ell\right)\left(\frac{1}{3}\,\ell\right)\right\}^{3/2}}{9\sqrt{3} \cdot EI\ell} = 17.92 \times 10^{-3} \times \frac{F\ell^3}{EI}\,.$$

The approximate formula gives a maximum deflection some 3% higher than the exact value.

Example 3: Simply supported prismatic beam with a uniformly distributed load over half the span

The simply supported beam ABC in Figure 8.71a has a length 2ℓ and bending stiffness EI. The beam carries a uniformly distributed load q over AB.

Question:
Find an approximation to the maximum deflection.

Solution:
Figure 8.71b shows the M/EI diagram, the calculation of which is left to the reader. The area of the M/EI diagram can be determined as the sum of the hatched area of the parabola over AB and the area of the triangle over ABC:

$$\mathrm{A}_{M/EI} = \tfrac{2}{3} \times \frac{\frac{1}{8}q\ell^2}{EI} \times \ell + \tfrac{1}{2} \times \frac{\frac{1}{4}q\ell^2}{EI} \times 2\ell = \frac{1}{3}\frac{q\ell^3}{EI}.$$

With approximate formula (8.23) the maximum deflection is

$$w_{\max} = \frac{1}{6} \times \frac{1}{3}\frac{q\ell^3}{EI} \times 2\ell = \frac{1}{9}\frac{q\ell^4}{EI} = 0.111\frac{q\ell^4}{EI}.$$

A computer calculation of the maximum deflection gives $0.015\frac{q\ell^4}{EI}$.

The approximate formula therefore overestimates the maximum deflection by some 5.5%.

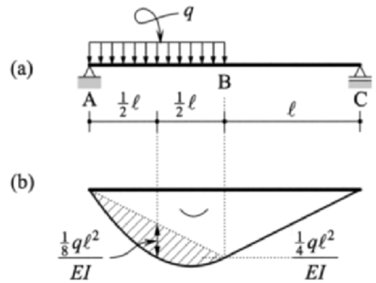


Figure 8.71 (a) A simply supported beam with a uniformly distributed load on half span AB and (b) the associated M/EI diagram. The area of the M/EI diagram is easily determined as the sum of the hatched area of the parabola over AB and the area of the triangle over ABC.

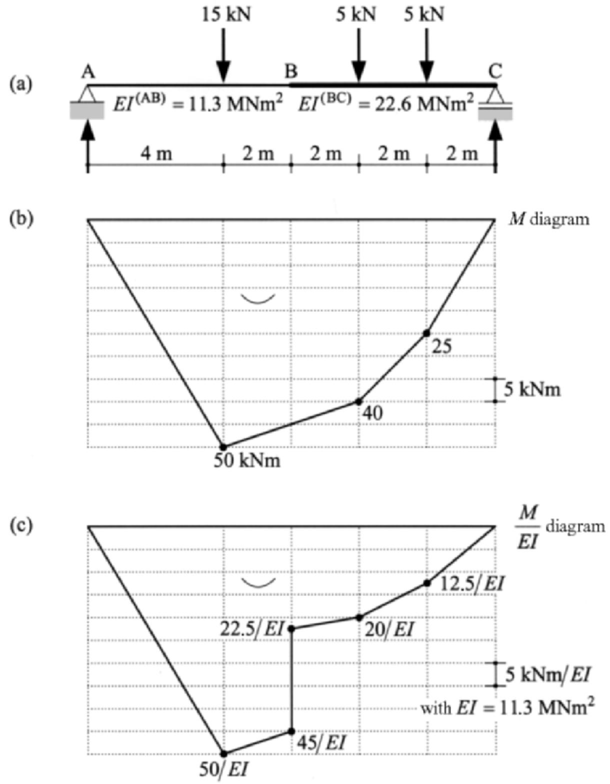


Figure 8.72 (a) A simply supported non-prismatic beam, loaded by a series of concentrated forces, with (b) the bending moment diagram and (c) the M/EI diagram. In the M/EI diagram, $EI = EI^{(AB)} = 11.3$ MNm^2.

Example 4: Simply supported non-prismatic beam with three point loads

The simply supported beam ABC in Figure 8.72a has a bending stiffness $EI = 11.3$ MNm^2 over AB, and a bending stiffness $2EI = 22.6$ MNm^2 over BC. The location and magnitude of the point loads are given in the figure.

Question:
Find an approximation to the maximum deflection.

Solution:
Figure 8.72b shows the M diagram and Figure 8.72c shows the M/EI diagram. The calculation is left to the reader.

The area of the M/EI diagram can be determined by splitting it into triangles and rectangles (or trapeziums). This gives

$$\begin{aligned} \mathrm{A}_{M/EI} &= \frac{282.5 \text{ kNm}^2}{EI} = \frac{282.5 \text{ kNm}^2}{11.3 \text{ MNm}^2} \\ &= 25 \times 10^{-3}. \end{aligned}$$

Comment: From this numerical example we can see that the area of the M/EI diagram is dimensionless.

An approximation for the maximum bending with formula (8.23) leads to

$$\begin{aligned} w_{\max} &= \frac{1}{6} \cdot \mathrm{A}_{M/EI} \cdot \ell \\ &= \tfrac{1}{6} \times (25 \times 10^{-3}) \times (12 \text{ m}) = 50 \text{ mm}. \end{aligned}$$

This value is approximately 4% larger than the maximum deflection of 48 mm that is found with an exact calculation. Figure 8.73 shows a sketch of the elastic curve determined with a computer program. The maximum deflection of 48 mm is to the left of B.

Comment: The authors would like to emphasise that the outlined method of Section 8.5 is strictly limited to simply supported beams. For any other structure, the other general outlined methods should be deployed.

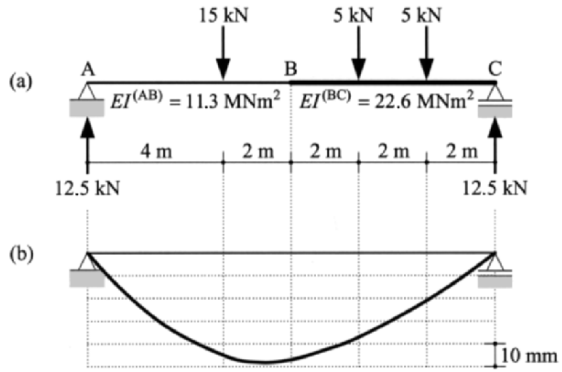


Figure 8.73 (a) The simply supported non-prismatic beam, loaded by a series of concentrated forces, with (b) the elastic curve according to an exact calculation. The maximum deflection occurs to the left of B and is 48 mm.

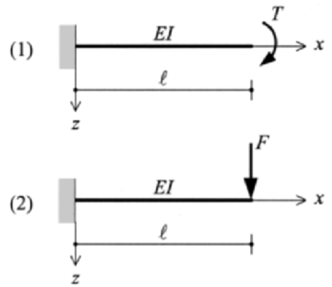
8.6 Problems

General comments:

- In all problems, the material behaviour is linearly elastic.
- Unless stated otherwise, the members are prismatic with bending stiffness EI.
- Deformation due to normal forces is ignored unless stated otherwise.
- The structure's dead weight is ignored unless clearly stated otherwise.
- So-called "second-order effects" are ignored.[1]

Determining displacements directly from the moment distribution (Section 8.1)

8.1: 1–2 The cantilever beam shown is loaded at its free end by a couple T and force F respectively.

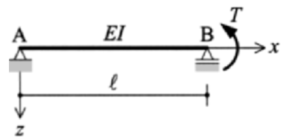


1 "Second-order effects" include the change in force distribution due to a change in the geometry of the structure. These can be important, particularly for members in compression. This subject is covered in *Engineering Mechanics*, Volume 4.

Questions:

a. Use the method described in Section 8.1 to determine the equation for the elastic curve from the moment distribution and plot it.
b. Determine the displacement and rotation at the free end $x = \ell$. Compare these values regarding magnitude and direction with the relevant loading case in Table 8.3.

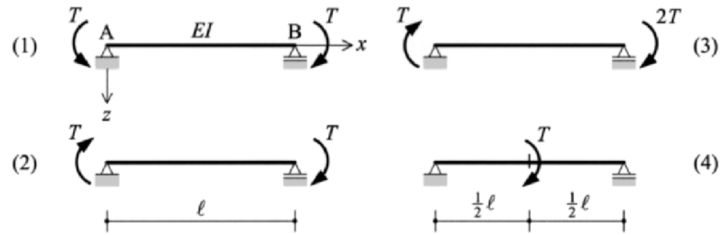
8.2 The simply supported beam AB is loaded by a couple T at support B.



Questions:

a. Determine the bending moment diagram and sketch the elastic curve.
b. Determine the equation for the elastic curve from the moment distribution.
c. Determine the rotations at the supports, and the displacement at mid-span. Compare these values regarding magnitude and direction with the relevant loading case in Table 8.4.

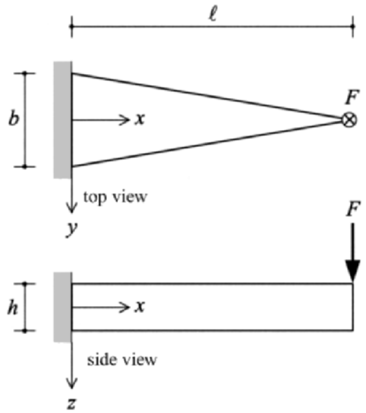
8.3: 1–4 The simply supported beam AB is loaded in four different ways by couples.



Questions:

a. Determine the bending moment diagram and sketch the elastic curve.
b. Determine the equation for the elastic curve from the moment distribution.
c. Determine the rotations and the displacement at midspan.

8.4 A cantilever beam with length ℓ has a constant height h and a linearly varying width $b(x) = b(1 - x/\ell)$. The beam is loaded by a force F at the free end. The material behaviour is linearly elastic with modulus of elasticity E.



Questions:

a. Determine the equation for the elastic curve from the moment distribution.
b. Determine the deflection and rotation at the free end. Compare these values with those of a prismatic beam with height h and (constant) width b.

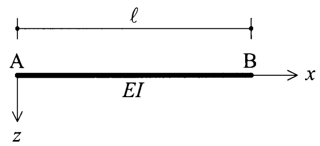
Differential equation for bending (Section 8.2)

8.5 *Questions*:

a. Which three types of equation are at the basis of the fourth-order differential equation for bending? Describe their significance in words.
b. For a prismatic member, derive the fourth-order differential equation for bending.
c. To what order does this differential equation change if the member is not prismatic?

8.6 For an initially straight prismatic beam AB with length ℓ and bending stiffness EI, the elastic curve is given by

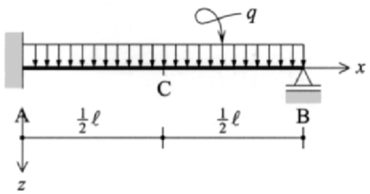
$$w = \frac{q}{48EI}x^3(2x - 3\ell).$$



Questions:

a. Draw the elastic curve.
b. Show that the beam is subject to a uniformly distributed load q.
c. Draw the bending moment diagram and shear force diagram.
d. Draw the forces and moments acting on the beam ends and show that the beam as a whole is in equilibrium.
e. How could the beam be supported and what, in that case, does the load consist of?

8.7 The statically indeterminate beam AB with length ℓ and bending stiffness EI carries a uniformly distributed load q.



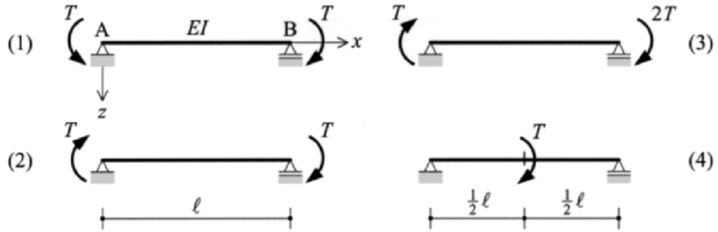
Questions:

a. Determine the displacement w as a function of x.
b. Determine the rotation at B and the deflection at midspan C. Check the correctness of the results using Table 8.4.
c. Determine the bending moment M and shear force V as functions of x.
d. Draw the bending moment and shear force diagrams with the deformation signs if $q = 10$ kN/m and $\ell = 4$ m. At A and B also draw the tangents to the bending moment diagram.
e. Determine the location and magnitude of the maximum bending moment.
f. Determine the support reactions at A and B as they are actually acting; write down their values. Check the correctness of the results using Table 8.4.

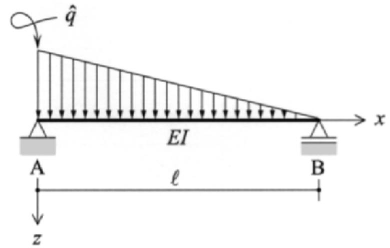
8.8: 1–4 The simply supported beam AB is loaded by couples in four different ways.

Questions:

a. Use the differential equation for bending to determine the equation for the elastic curve and plot it.
b. Use the elastic curve to determine the bending moment and shear force as functions of x, and draw these functions.
c. Determine the expressions for the rotations at the supports and the displacement at midspan.



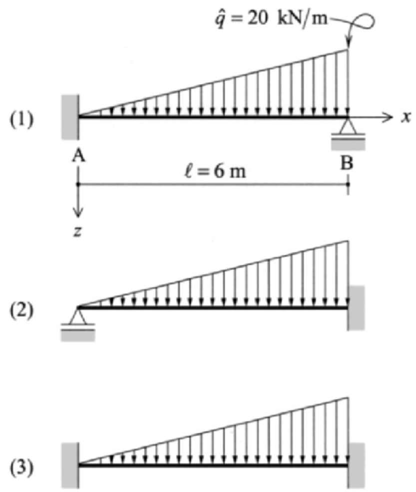
8.9 The simply supported beam AB carries a linearly distributed load $q_z = \hat{q}(1 - \frac{x}{\ell})$. The bending stiffness of the beam is EI.



Questions:

a. Determine the equation of the elastic curve.
b. Determine the rotations at A and B.
c. Determine the maximum deflection.
d. Determine the bending moment and shear force diagrams, with the deformation signs and the tangents at A and B. In the calculation use $\hat{q} = 80$ kN/m and $\ell = 4.5$ m.
e. Determine the magnitude and location of the maximum bending moment.

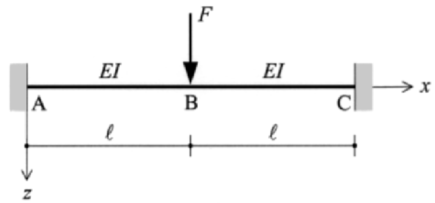
8.10: 1–3 A statically indeterminate beam AB is loaded in three different ways and carries the same linearly distributed load $q_z = \hat{q}\frac{x}{\ell}$. In the calculation use $\hat{q} = 20$ kN/m and $\ell = 6$ m.



Questions:

a. Use the differential equation for bending to determine the bending moment and the shear force as functions of x.
b. Draw the bending moment and shear force diagrams, with the deformation signs.
c. Draw the support reactions at A and B as they are actually acting and write down their values.
d. Determine the maximum field moment.

8.11 The beam ABC is fixed at both ends, has a length 2ℓ and a bending stiffness EI. It carries a point load F at midspan B.

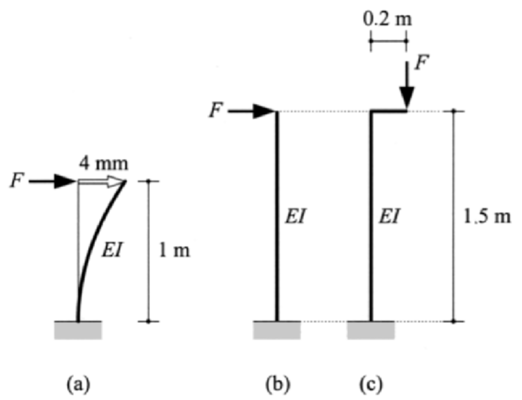


Questions:

a. Solve the differential equation for bending for the half beam AB. Which boundary and joining conditions are used?
b. Draw the elastic curve for the entire beam.
c. Determine the deflection at B. Check the correctness of the answer using Table 8.4.
d. For AB determine the bending moment and the shear force as functions of x.
e. For the entire beam, draw the bending moment and shear force diagrams.
f. Draw the support reactions at A and B as they are actually acting and write down their values. Check the correctness of the results using Table 8.4.

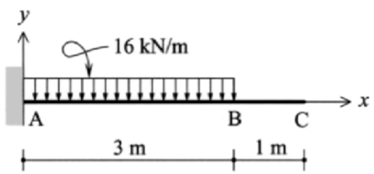
The three forget-me-nots (Section 8.3, Table 8.3)

8.12 The three columns shown have the same bending stiffness EI. In case (a) the displacement at the top is 4 mm.



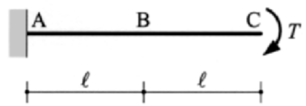
Questions:
a. Determine the displacement at the top in case (b).
b. Determine the displacement at the top in case (c).

8.13 The cantilever beam ABC carries a uniformly distributed load $q = 16$ kN/m over AB. The bending stiffness is $EI = 23.4$ MNm^2.



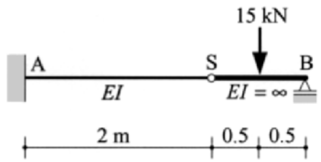
Questions (with the right sign in the given xy coordinate system):
a. Determine the rotation at C in radians.
b. Determine the rotation at C in degrees.
c. Determine the deflection at C.

8.14 You are given a prismatic cantilever beam, loaded at the free end by a couple T. The deflection at B is 12 mm.



Question:
Determine the deflection at the free end C.

8.15 In the hinged beam ASB, AS has a bending stiffness $EI = 5$ MNm^2, while SB has an infinite bending stiffness. The dimensions and load are given in the figure.

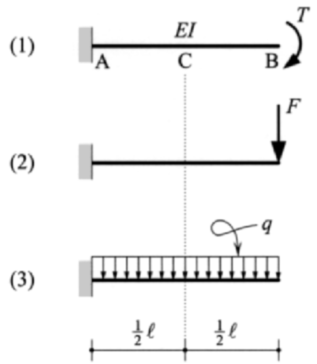


Question:
Determine the displacement at S.

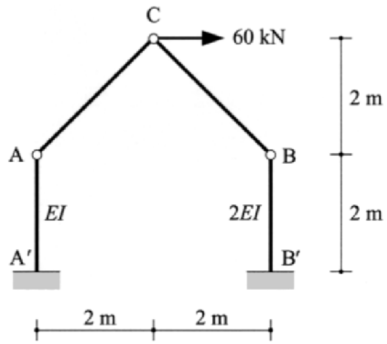
8.16: 1–3 The cantilever beam ACB is loaded in three different ways.

Questions:

a. Sketch the bending curve.
b. Use forget-me-nots to determine the deflection and rotation at C.



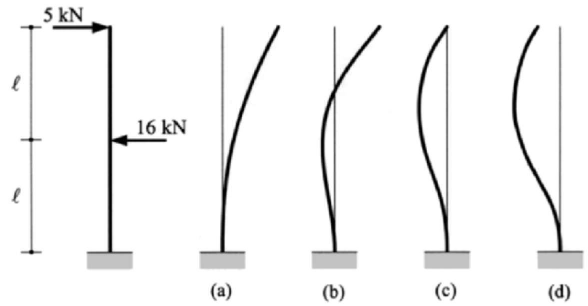
8.17 Member AA′ has a bending stiffness $EI = 15\,\text{MNm}^2$, and member BB′ has a bending stiffness $2EI = 30\,\text{MNm}^2$, as shown in the figure. All other members are rigid. Dimensions and loading are given in the figure.



Questions:

a. Determine the displacement at A.
b. Determine the displacement at B.

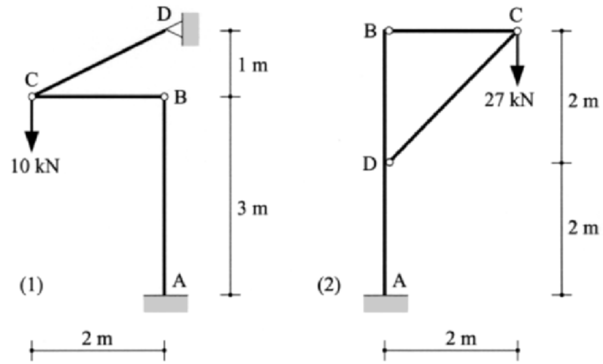
8.18 You are given a prismatic column with bending stiffness EI.



Question:

Determine how the column is deformed for the given load.

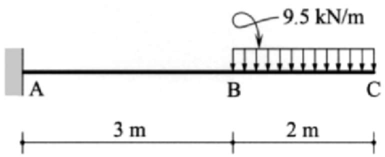
8.19: 1–2 You are given two structures with the dimensions and loading shown in the figure. Both members AB have the same bending stiffness $EI = 18\,\text{MNm}^2$. All other members are rigid.



Question:

Determine the displacement at B.

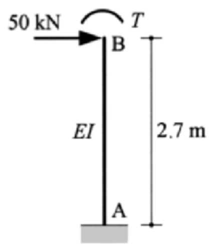
8.20 The beam ABC is fixed at A and carries a uniformly distributed load $q = 9.5$ kN/m over AC. The bending stiffness of the beam is $EI = 19$ MNm^2.



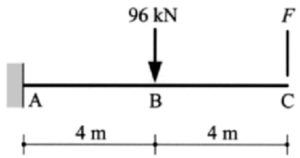
Questions:
a. Determine the deflection at C due to the deformation of AB.
b. Determine the deflection at C due to the deformation of BC.
c. Determine the final deflection at C.

8.21 The column AB fixed at A is loaded at its free end by a horizontal force of 50 kN and a couple T. The direction of the couple is unknown. The bending stiffness is $EI = 4.1$ MNm^2.

Questions:
a. Determine the direction and magnitude of T such that B is not displaced horizontally.
b. Determine the bending moment and shear force diagrams; include the values.
c. Sketch the elastic curve.
d. Determine the rotation at B.
e. Determine the displacement at the centre of AB.
f. Can you find this loading case in Table 8.4? Check the correctness of your values using this table.



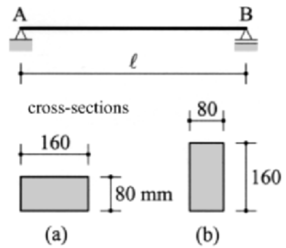
8.22 The cantilever beam ABC fixed at A, and is loaded by a force of 96 kN at B and another unknown force F at the free end C. The direction of force F is not given.



Questions:
a. Determine the direction and magnitude of force F so that the displacement at the free end B is zero.
b. Determine the bending moment diagram for AB and sketch the elastic curve.
c. Determine the deflection at C if the bending stiffness of the beam is $EI = 16$ MNm^2.
d. Can you find this loading case in Table 8.4? Are the values you found in line with the values in the table?

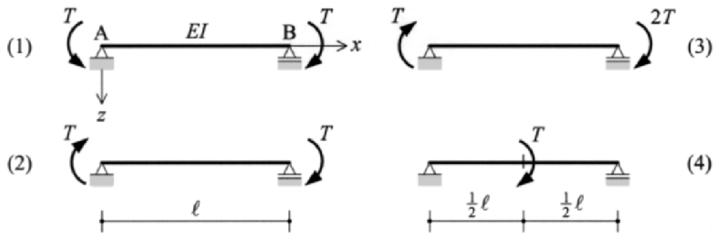
Other forget-me-nots (Section 8.3, Tables 8.3 and 8.4)

8.23 You are given a simply supported beam AB. In position (a) the beam deflects 12 mm due to its dead weight.



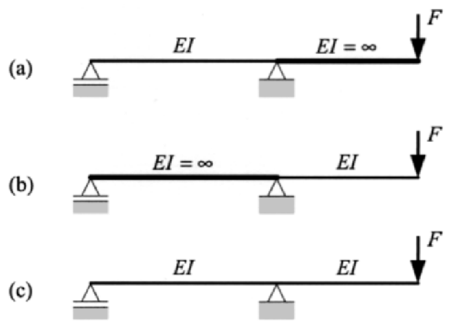
Question:
Determine the deflection due to its dead weight in position (b).

8.24: 1–4 The simply supported beam AB is loaded in four different ways by couples.



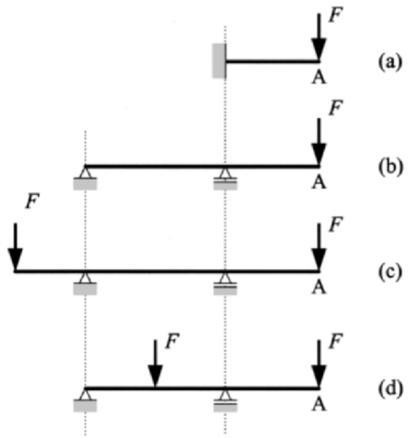
Questions:
a. Determine the bending moment diagram and sketch the elastic curve.
b. Determine the rotations at the supports.
c. Determine the deflection at midspan.

8.25 You are given three beams. At the free end the beam deflects 15 mm in case (a) and 10 mm in case (b).



Question:
Determine the deflection at the free end of the beam in case (c).

8.26 You are given four different beams with the same bending stiffness EI.



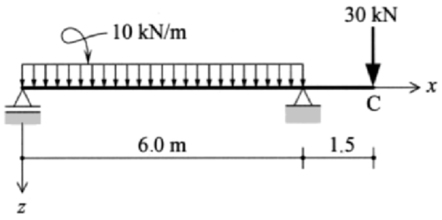
Question:
For which beam is the deflection at the free end A largest?

8.27 See problem 8.26 for details.

Questions:
a. For each of the beams sketch the elastic curve.
b. Order the beams according to the magnitude of the deflection at A, starting with the beam with the smallest deflection.

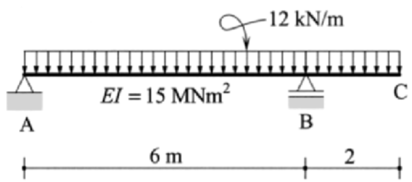
Note: you need not present extensive calculations.

8.28 The simply supported beam has a bending stiffness $EI = 4.5$ MNm^2. Dimensions and loading are given in the figure.



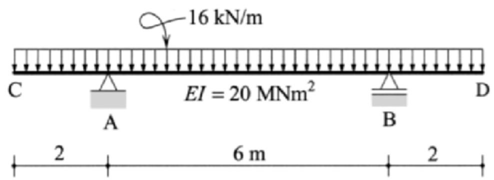
Questions:
a. Determine the bending moment diagram with deformation signs.
b. Sketch the elastic curve with the points of inflection.
c. Determine the vertical displacement $u_{z;\mathrm{C}}$ at C using forget-me-nots.

8.29 The beam ABC is simply supported at A and B with an overhang at B; it carries a uniformly distributed load $q = 12$ kN/m. The bending stiffness of the beam is $EI = 15$ MNm^2.



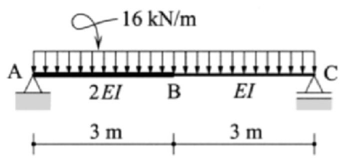
Questions:
a. Determine the deflection at the centre of span AB.
b. Determine the deflection at the free end C.
c. Sketch the elastic curve. In the sketch indicate the location of the points of inflection.

8.30 The beam, simply supported at A and B has two overhangs, and carries a uniformly distributed load $q = 16$ kN/m. The bending stiffness of the beam is $EI = 20$ MNm^2.



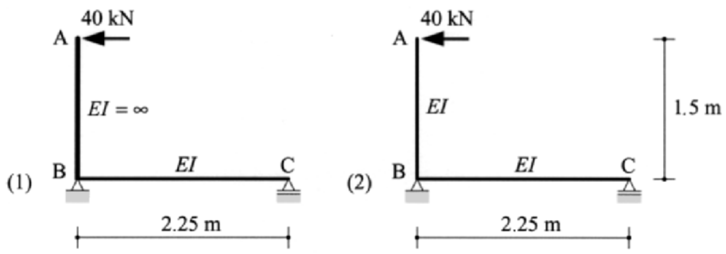
Questions:
a. Determine the deflection at the centre of span AB.
b. Determine the deflection at the free end C.
c. Sketch the elastic curve with the points of inflection.

8.31 The simply supported beam ABC has a bending stiffness $2EI$ over AB, twice as large as the bending stiffness EI of BC. The beam carries a uniformly distributed load $q = 16$ kN/m. For the calculation use $EI = 15$ MNm^2.



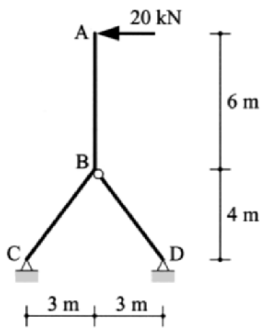
Questions:
a. Determine the deflection at B.
b. Determine the rotation at B.
c. Determine the rotation at A.
d. Determine the rotation at C.

8.32: 1–2 In structure (1) AB has an infinite bending stiffness, while BC has a finite bending stiffness $EI = 40$ MNm^2. This is also the bending stiffness of both AB and BC in structure (2). Dimensions and loading are given in the figure.



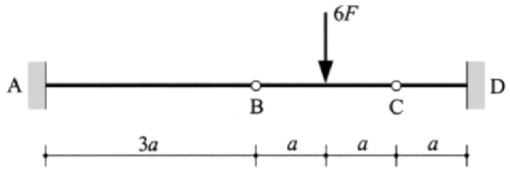
Question:
Determine the displacement at A.

8.33 In the structure shown, the bent member ABC is supported on a hinge at C, and a two-force member at B. Member ABC has a bending stiffness $EI = 176$ MNm^2.



Question:
Determine the displacement at A for the given load.

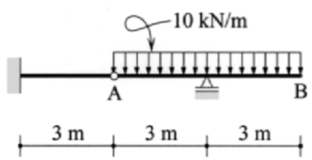
8.34 The hinged beam ABCD has hinges at B and C and is fixed at A and D. All beam segments have the same bending stiffness EI. The beam is loaded by a vertical force $6F$ at the centre of BC.



Questions:

a. Determine the bending moment diagram.
b. Sketch the elastic curve.
c. Determine the deflection at B.
d. Determine the deflection at C.
e. Determine the deflection at the centre of BC.

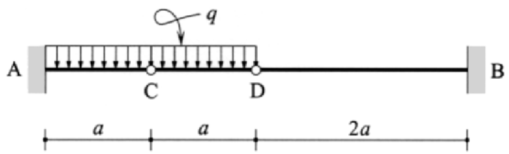
8.35 You are given the hinged beam shown with bending stiffness $EI = 13.5$ MNm^2. The dimensions and load are given in the figure.



Questions:

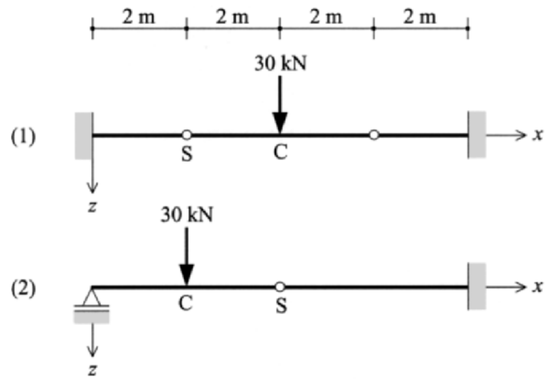
a. Sketch the elastic curve.
b. Determine the deflection at A.
c. Determine the deflection at B.

8.36 The hinged beam ACDB has hinges at C and D and is fixed at A and B. All beam segments have the same bending stiffness EI. A uniformly distributed load q acts over the length ACD.



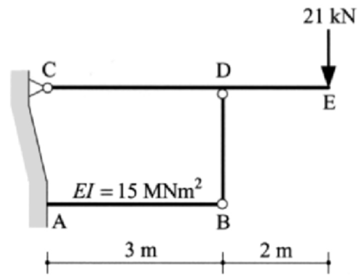
Questions:
a. Determine the bending moment diagram.
b. Sketch the elastic curve.
c. Determine the deflection at C.
d. Determine the deflection at D.
e. Determine the deflection at the centre of CD.
f. Draw the elastic curve to scale.

8.37: 1–2 You are given two hinged beams. In both cases the bending stiffness is $EI = 10\ \text{MNm}^2$. Dimensions and loading are given in the figures.



Questions:
a. Sketch the elastic curve.
b. Determine the deflection at C.
c. Determine the gap $\Delta\varphi$ at hinge S.

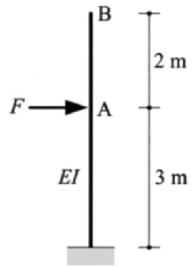
8.38 In the structure shown, AB has a bending stiffness $EI = 15\ \text{MNm}^2$. All other members are infinitely stiff. A is a fixed end, C is a hinged support and B and D are hinged joints. Dimensions and loading are given in the figure.



Questions:
a. Determine the vertical displacement of E.
b. Determine the horizontal displacement of E.

Moment-area theorems (Section 8.4)

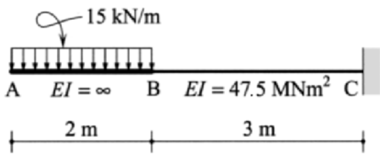
8.39 As a result of the force F at A, B undergoes a displacement of 15 mm.



Question:
Determine the displacement at A.

8.40 Beam ABC, fixed at C, has an infinite bending stiffness over AB and a finite bending stiffness $EI = 47.5$ MNm^2 over BC. AB is subject to a uniformly distributed load $q = 15$ kN/m.

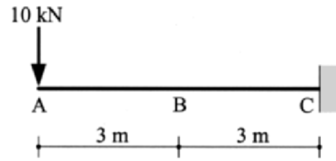
Question:
Determine the deflection at A.



8.41 The cantilever beam ABC is loaded by a force of 10 kN at the free end A. The beam has a bending stiffness of 45 MNm^2.

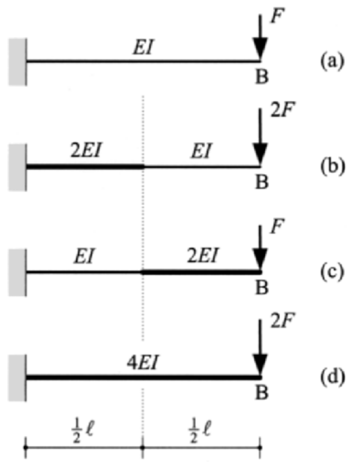
Questions:

a. Determine the deflection at A due to the deformation of BC.
b. Determine the deflection at A due to the deformation of AB.



c. Determine the resultant deflection at A. Explain in words the large difference between the contributions found in (a) and (b).

8.42 You are given four different cantilever beams, loaded at the free end B by point loads F and $2F$.

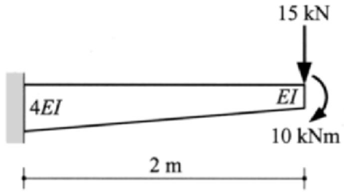


Question:
Order the beams according to levels of deflection at the free end B, starting with the beam with the largest deflection.

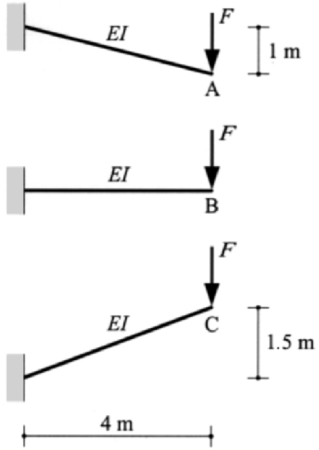
Comment: You need not provide extensive calculations.

8.43 The bending stiffness of the beam shown varies linearly from $4EI$ at the fixed end to EI at the free end. Dimensions and loading are given in the figure. In the calculation use $EI = 2.5$ MNm^2.

Question:
Determine the deflection at the free end.



8.44 You are given three members with the same bending stiffness EI and the same load from a vertical force F at the free end.

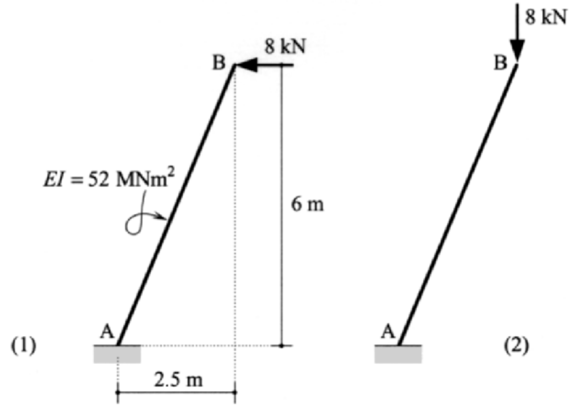


Questions:
If in these three cases one compares the vertical displacement at the free end, which statement is correct?

a. The vertical displacement is largest at A.
b. The vertical displacement is largest at B.
c. The vertical displacement is largest at C.
d. The vertical displacements at A, B and C are equal.

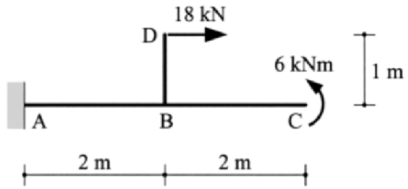
Comment: You need not provide extensive calculations.

8.45: 1–2 Member AB is fixed at an angle at A, and is loaded in two ways at the free end B by a force of 8 kN. The bending stiffness is $EI = 52$ MNm^2.



Questions:
a. Sketch the elastic curve.
b. Use the moment-area theorems to determine the horizontal component of the displacement at B.
c. In the same way, determine the vertical component of the displacement at B.

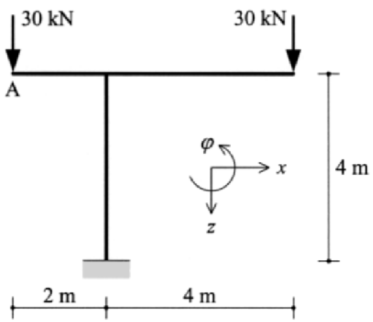
8.46 In the structure shown, all members have the same bending stiffness $EI = 3$ MNm^2. Dimensions and loading are given in the figure.



Questions:

a. Determine the vertical displacement at C.
b. Determine the vertical displacement at C in the case that the bending stiffness of AB is not EI, but $4EI = 12$ MNm^2.

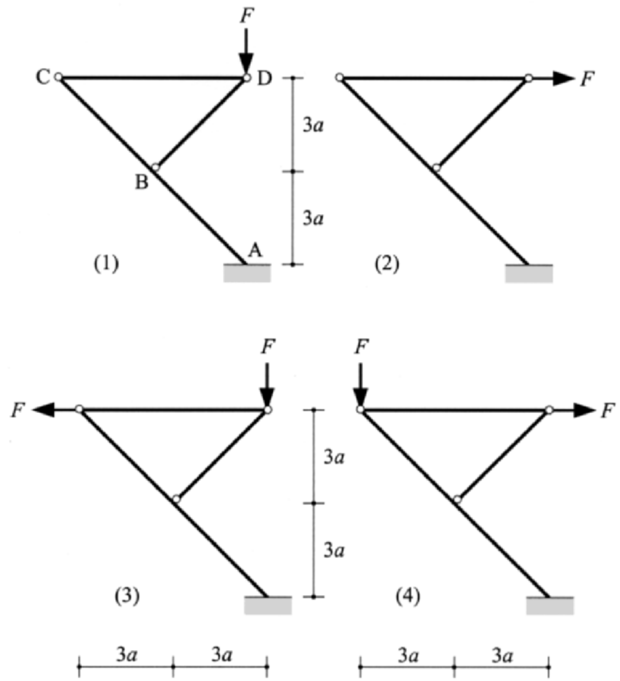
8.47 In the structure shown, all members have the same bending stiffness $EI = 40$ MNm^2. Dimensions and loading are given in the figure.



Questions (in the given xz coordinate system):

a. Determine the vertical displacement at A.
b. Determine the rotation at A.

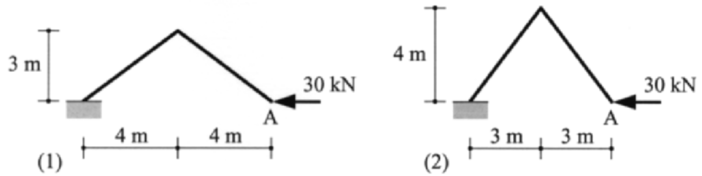
8.48: 1–4 The structure shown is loaded in four different ways by forces F. Member ABC has a bending stiffness EI. Deformation by normal forces is ignored.



Questions:

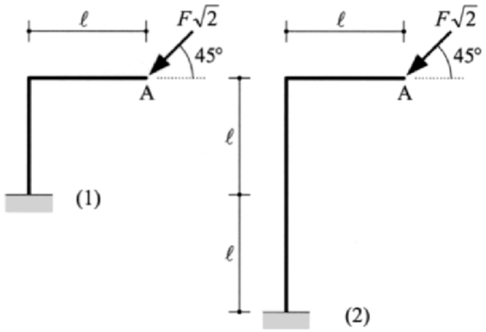
a. Determine the magnitude and direction of the displacement at B, expressed in terms of a, F and EI.
b. Determine the magnitude and direction of the displacement at C.
c. Sketch the elastic curve of ABC.

8.49: 1–2 You are given two fixed bent members. The dimensions and loading are given in the figure. The bending stiffness is $EI = 50$ MNm^2.



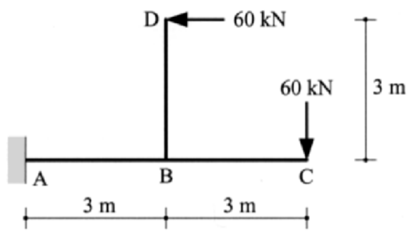
Questions:
a. Determine the vertical displacement of A.
b. Determine the horizontal displacement of A.

8.50: 1–2 You are given two fixed bent members with the same bending stiffness EI. The dimensions and loading are given in the figure.



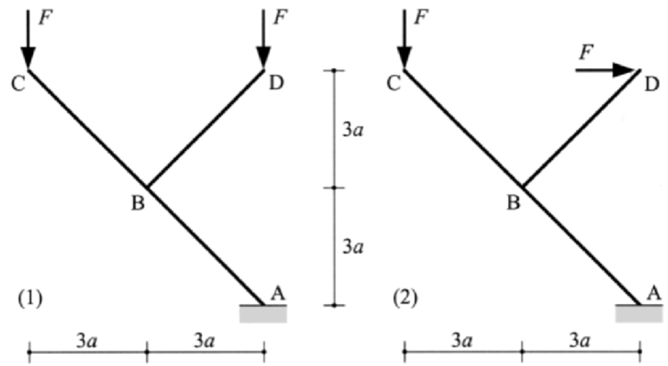
Questions:
a. Determine the horizontal displacement at A.
b. Determine the vertical displacement at A.
c. Determine the rotation at A.

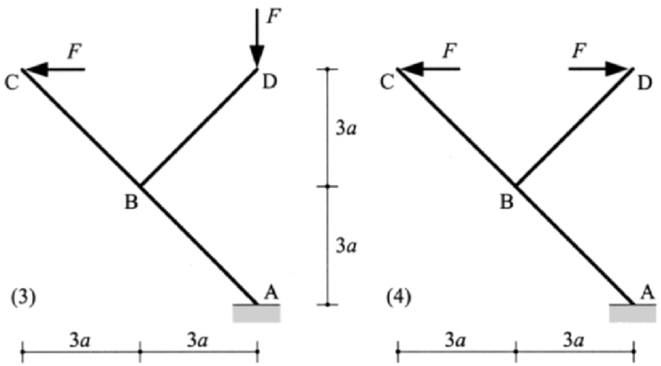
8.51 In the structure shown, all members have the same bending stiffness $EI = 54 \times 10^3$ kNm^2. The dimensions and loading are given in the figure.



Questions:
a. Determine the magnitude and direction of the displacement at C.
b. Determine the magnitude and direction of the displacement at D.

8.52: 1–4 In the structure shown all members have the same bending stiffness $EI\sqrt{2}$. The structure is loaded in four different ways by forces F.

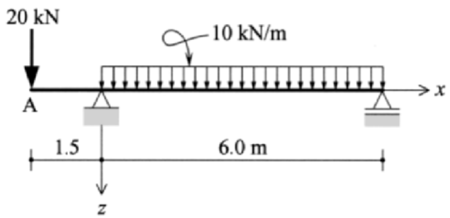




Questions:
Determine the horizontal and vertical component of the displacement

a. at B.
b. at C.
c. at D.

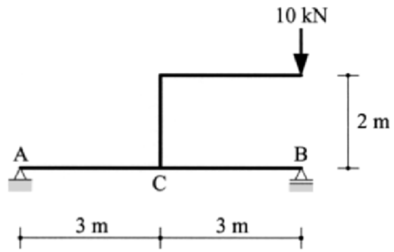
8.53 The free supported overhanging beam has a bending stiffness $EI = 4.5$ MNm^2. The dimensions and loading are shown in the figure.



Questions:

a. Determine the bending moment diagram with deformation signs.
b. Sketch the elastic curve with the points of inflection.
c. Determine the vertical displacement $u_{z;\mathrm{A}}$ at A using the moment-area theorems.

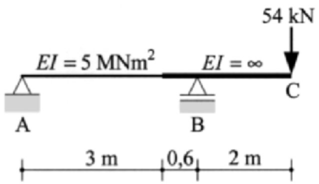
8.54 In the structure shown all members have the same bending stiffness $EI = 7.5$ MNm^2. The dimensions and loading are given in the figure.



Questions:

a. Determine the displacement of C.
b. Sketch the elastic curve for ACB.

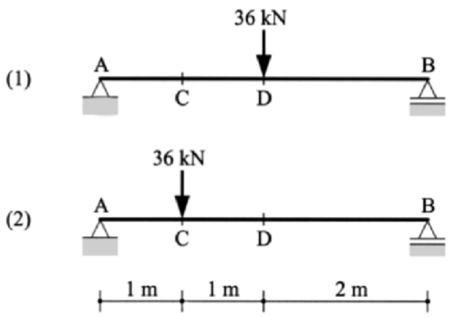
8.55 You are given a free supported overhanging beam. A part of the beam has infinite bending stiffness. For the part with finite bending stiffness $EI = 5$ MNm^2.



Questions:

a. Sketch the elastic curve.
b. Determine the rotations at A and B.
c. Determine the displacement at C.
d. Determine the location and magnitude of the maximum displacement for AB.

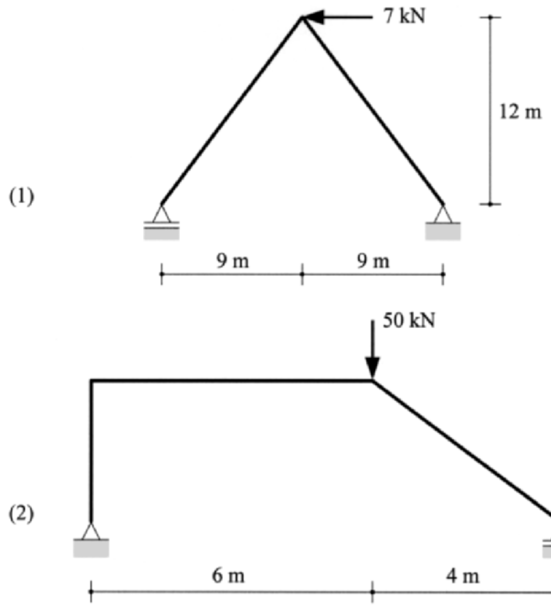
8.56: 1–2 You are given a beam with bending stiffness $EI = 3\ \text{MNm}^2$, loaded in two different ways. The dimensions and loading are given in the figure.



Questions:
a. Determine the rotation at A.
b. Determine the deflection at C.
c. Determine the deflection at D.
d. Determine the location and magnitude of the maximum deflection.

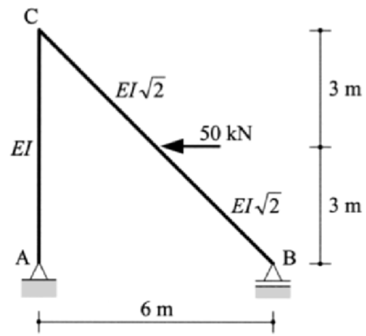
8.57: 1–2 You are given two structures for which the dimensions and loading are given in the figure. All members have the same bending stiffness $EI = 140\ \text{MNm}^2$.

Questions:
a. Determine the displacement of the roller.
b. Determine the displacement of the joint where the force is applied.

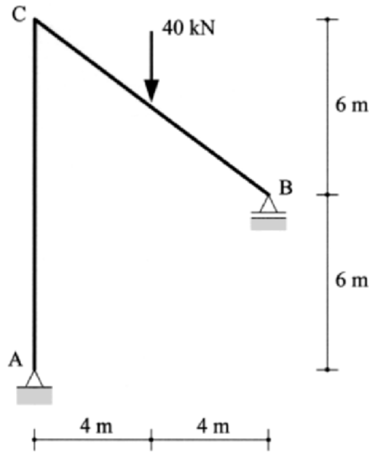


8.58 In the structure shown, AC has a bending stiffness EI, and BC has a bending stiffness $EI\sqrt{2}$. In the calculation use $EI = 261\ \text{MNm}^2$. The dimensions and loading are given in the figure.

Questions:
a. Determine the displacement of B.
b. Determine the displacement of C.
c. Sketch the elastic curve of ACB.

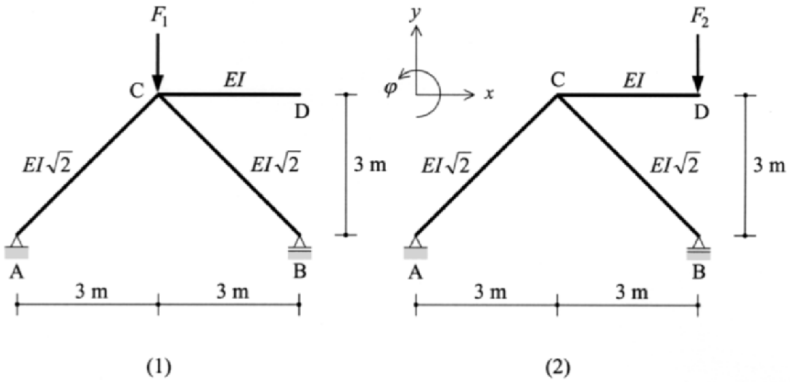


8.59 In the structure shown, all members have the same bending stiffness $EI = 96$ MNm^2. Dimensions and loading can be found in the figure.



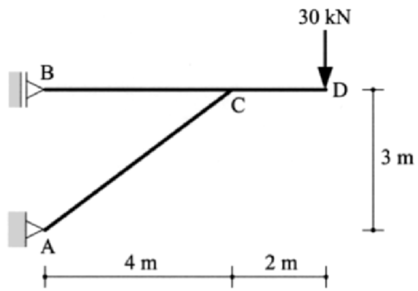
Questions:
a. Determine the displacement of the roller at B.
b. Determine the displacement of C.
c. Sketch the elastic curve.

8.60: 1–2 You are given the same structure, loaded in two different ways. In the calculation use $F_1 = 30$ kN, $F_2 = 15$ kN and $EI = 9$ MNm^2. Use the given xy coordinate system.



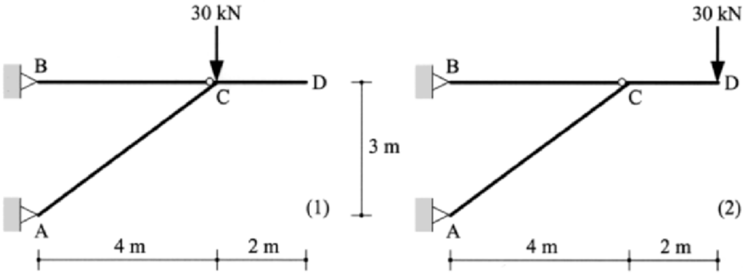
Questions:
a. Determine the rotation at A.
b. Determine the displacement of roller support B.
c. Determine the displacement of D.
d. Determine the displacement of C.
e. Sketch the elastic curve.

8.61 In the given structure, all members have the same bending stiffness $EI = 10$ MNm^2. Dimensions and loading can be found in the figure.



Questions:
a. Sketch the (expected) elastic curve.
b. Determine the displacement at B.
c. Determine the displacement at C.
d. Determine the displacement at D.

8.62: 1–2 In the structure shown, ACD and BC are joined by a hinge at C. All members have the same bending stiffness $EI = 10$ MNm^2. The structure is loaded in two different ways by a force of 30 kN.



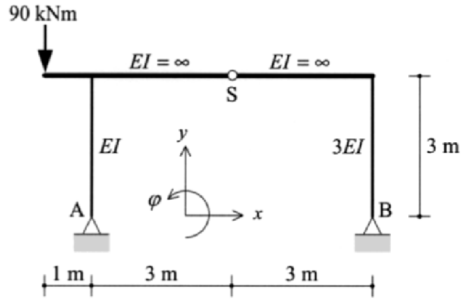
Questions:

a. Sketch the (expected) elastic curve.
b. Determine the displacement at C.
c. Determine the displacement at D.

8.63 In the three-hinged frame, the girder has an infinite bending stiffness. The bending stiffness of the left-hand column is $EI = 5$ MNm^2, that of the right-hand column is $3EI = 15$ MNm^2. Dimensions and loading are given in the figure. Use the given xy coordinate system.

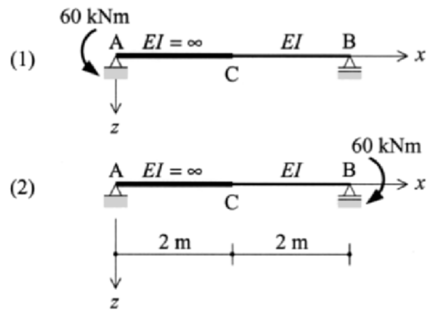
Questions:

a. Determine the rotations at A and B.
b. Determine the horizontal and vertical displacement at hinge S.
c. Determine the gap $\Delta\varphi$ at hinge S.
d. Sketch the elastic curve.



Simply supported beams and the M/EI diagram (Section 8.5)

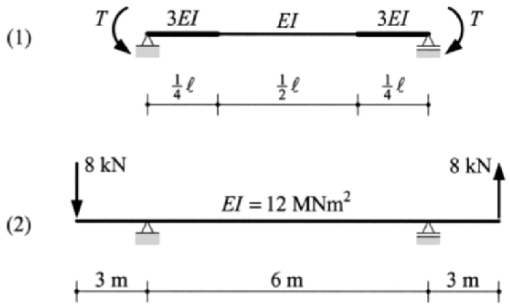
8.64: 1–2 The beam ACB is rigid over AC, and has a finite bending stiffness $EI = 2$ MNm^2 over CB. The load consists of a couple of 60 kNm at one of the supports.



Questions (in the given xz coordinate system):

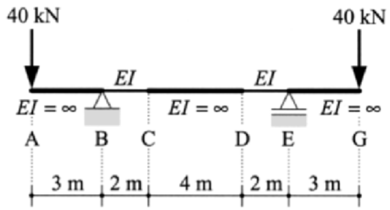
a. Sketch the elastic curve.
b. Determine the rotations at A and B, in both radians and degrees.
c. Determine the deflection at C.

8.65: 1–2 You are given two different beams. All the necessary information can be found in the figures.



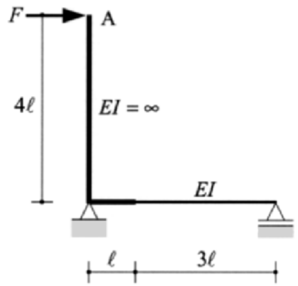
Questions:
a. Make a rough sketch of the elastic curve.
b. Determine the rotations at A and B, in both degrees and radians.

8.66 You are given a non-prismatic beam in which the segments BC and DE have a finite bending stiffness $EI = 36$ MNm^2 and all other segments are rigid.



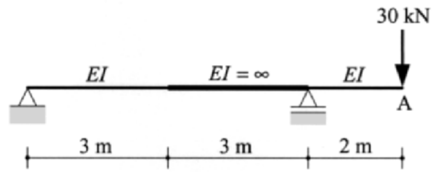
Questions:
a. Sketch the elastic curve.
b. Determine the rotation at B.
c. Determine the rotation at C.
d. Determine the displacement at A.

8.67 All the given information can be derived from the figure.



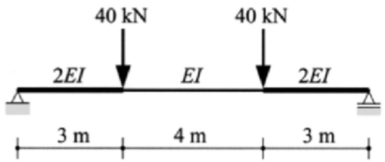
Question:
Determine the displacement at A.

8.68 You are given a beam with overhang. All the necessary information is given in the figure. In the calculation use $EI = 5$ MNm^2.



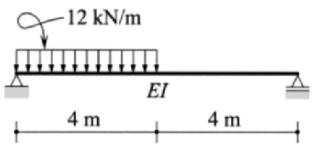
Question:
Determine the deflection at A.

8.69 You are given the free supported non-prismatic beam shown. In the calculation, use $EI = 50$ MNm^2.



Questions:
a. Give an approximation to the maximum deflection.
b. Provide an accurate determination of the maximum deflection.

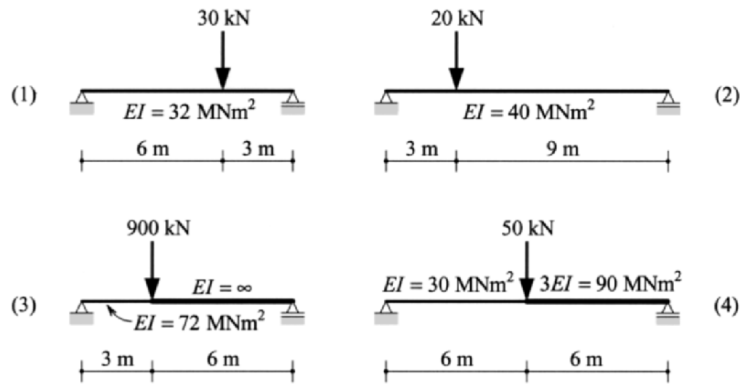
8.70 A simply supported prismatic beam carries a uniformly distributed load of 12 kN/m on its left-hand side. The bending stiffness of the beam is 31 MNm^2.



Questions:
a. Give an approximation to the maximum deflection.
b. Provide an accurate determination of the maximum deflection.

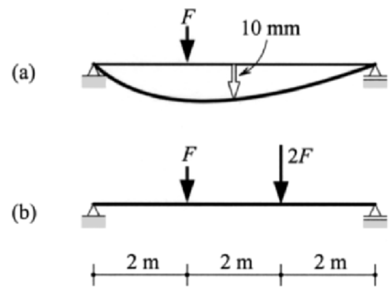
8.71: 1–4 You are given four different beams loaded by a point load. All the necessary information is given in the figure.

Questions:
a. Give an approximation to the maximum deflection.
b. Provide an accurate determination of the maximum deflection.



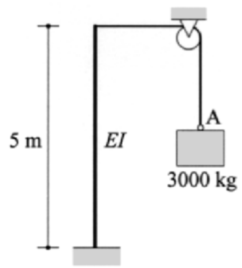
Mixed problems

8.72 You are given two loading cases (a) and (b) for the same beam. In case (a) the deflection at the centre of the span is 10 mm.



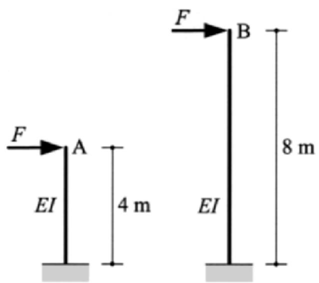
Question:
Determine the deflection at the centre of the span for case (b).

8.73 A block with mass 3000 kg is suspended on a cable that is joined to a fixed column by means of a trolley. The bending stiffness of the column is $EI = 12.5$ MNm^2. Under the influence of the block's weight, the cable stretches by 10 mm. For the gravitational field strength, use $g = 10$ N/kg.



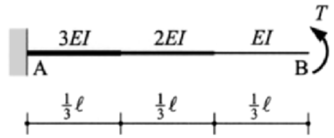
Question:
Determine the vertical displacement of the block.

8.74 Two fixed columns of different lengths but the same bending stiffness EI are loaded at the top by an equal horizontal force F. The horizontal displacement at A is 10 mm.



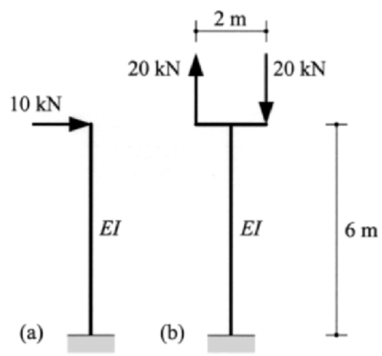
Question:
Determine the horizontal displacement at B.

8.75 The non-prismatic beam AB is fixed at A and loaded by a couple T at its free end B.



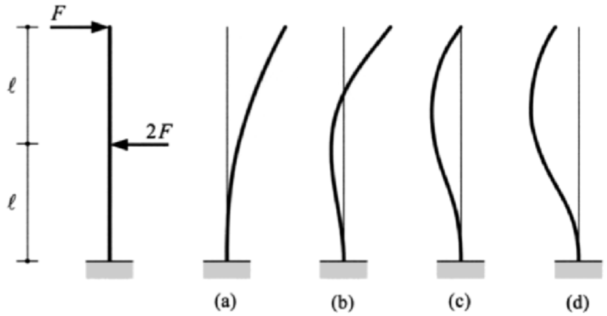
Questions:
a. Determine the rotation at B in radians.
b. Determine the rotation at B in degrees.
c. Determine the displacement at B.

8.76 The top of column (a) bends by 20 mm.



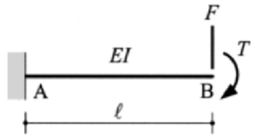
Question:
How much does the top of column (b) bend?

8.77 You are given a prismatic column.



Question:
In which way will the column deform for the given load? Substantiate your answer without extensive calculations.

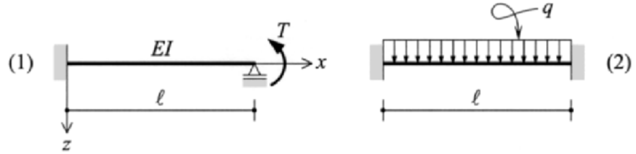
8.78 The beam AB fixed at A is loaded at its free end by a couple T and a force F. The direction of the force is unknown.



Questions:

a. Determine the direction and magnitude of the force F, expressed in terms of T and ℓ, such that the displacement at the free end B is zero.
b. Determine the bending moment and shear force diagrams for AB.
c. Sketch the elastic curve.
d. Determine the rotation at B.
e. Determine the deflection at the centre of AB.
f. Can you find this loading case in Table 8.4? Are the values you found in line with those in the table?

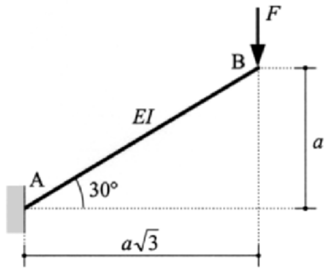
8.79: 1–2 You are given two statically indeterminate beams with the same length ℓ and bending stiffness EI, but with different loads.



Questions:

a. Without calculations sketch the bending moment diagram and elastic curve.
b. Use the differential equation for bending to find the equation for the elastic curve.
c. Now determine the bending moment and shear force as functions of x and draw the bending moment and shear force diagrams.
d. Table 8.4 provides a number of values for this loading case. Compare these to the values you have found.

8.80 The member AB is fixed at an angle of 30° at A, and is loaded by a vertical force F at the free end B.

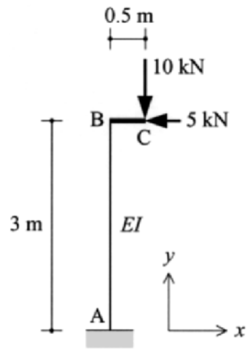


Questions:

a. Sketch the elastic curve.
b. Use forget-me-nots to determine the vertical component of the displacement at B.
c. In the same way determine the horizontal component of the displacement at B.

8.81 As problem 8.80, but now perform the calculation using the moment-area theorems.

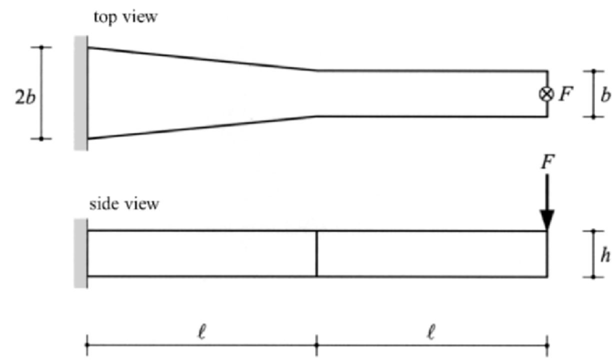
8.82 In the structure shown, AB has a bending stiffness $EI = 3.75$ MNm^2 and BC has infinite bending stiffness. Deformation due to normal forces is ignored.



Questions (using the correct signs in the given xy coordinate system):

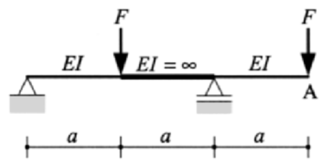
a. Determine the vertical displacement of C.
b. Determine the horizontal displacement of C.

8.83 You are given a non-prismatic cantilever beam loaded by a force F at the free end. The dimensions are given in the figure. In the calculation use $F = 4$ kN, $b = h = 200$ mm, $\ell = 1$ m and $E = 11$ GPa.



Question:
Determine the deflection at the free end.

8.84 You are given an overhanging beam of which a part has an infinite bending stiffness. The loading and dimensions are given in the figure.

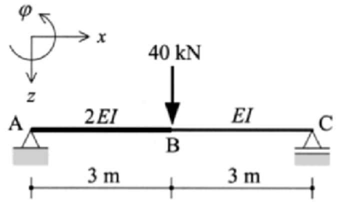


Questions:

a. Sketch the elastic curve.
b. Determine the deflection at A, expressed in terms of F, a and EI.

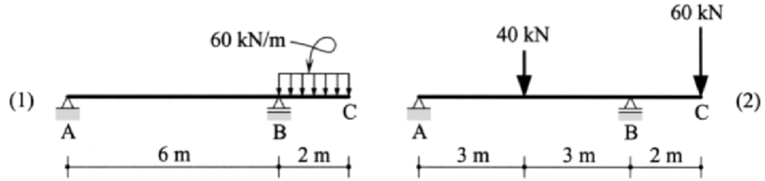
8.85 The simply supported beam ABC shown, with a span of 6 m, is loaded by a force of 40 kN at midspan B. The bending stiffness of AB is $2EI$, twice as large as the bending stiffness EI of BC. In the calculation use $EI = 15$ MNm^2. Perform the calculation using the moment-area theorems.

Questions:
a. Determine the rotation at A.
b. Determine the rotation at B.
c. Determine the rotation at C.
d. Determine the deflection at B.



8.86 As problem 8.85, but now perform the calculation using forget-me-nots.

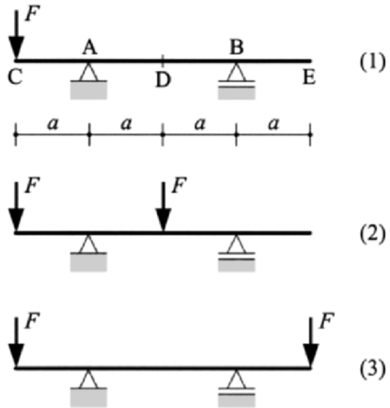
8.87: 1–2 The same beam with overhang is loaded in two different ways. The bending stiffness is $EI = 40$ MNm^2.



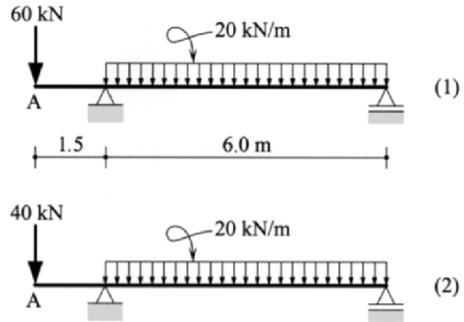
Questions:
a. Determine the rotation at A.
b. Determine the deflection at the centre of AB.
c. Determine the deflection at C.
d. Sketch the elastic curve.

8.88: 1–3 The beam with two overhangs is loaded in three different ways by forces F.

Questions:
a. Sketch the elastic curve.
b. Determine the deflection at C.
c. Determine the deflection at D.
d. Determine the deflection at E.



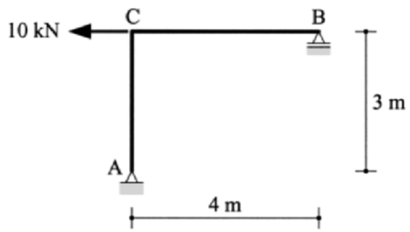
8.89: 1–2 A beam with overhang has a bending stiffness $EI = 5$ MNm^2. The beam is loaded in two different ways. Dimensions and loading are given in the figures.



Questions:
a. Determine the bending moment diagram with the deformation signs.

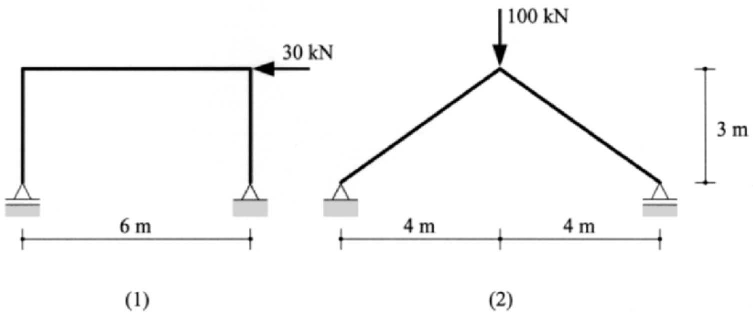
b. Sketch the elastic curve with the points of inflection.
c. Determine the deflection of the overhang at A.

8.90 In the structure shown, all members have the same bending stiffness $EI = 15\ \text{MNm}^2$. Dimensions and loading are shown in the figure.



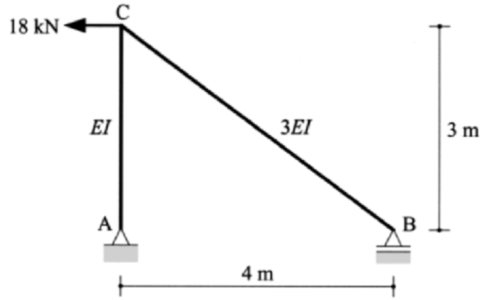
Question:
Determine the displacement of C.

8.91: 1–2 You are given two structures with the dimensions and loading given in the figure. All members have the same bending stiffness $EI = 40\ \text{MNm}^2$.



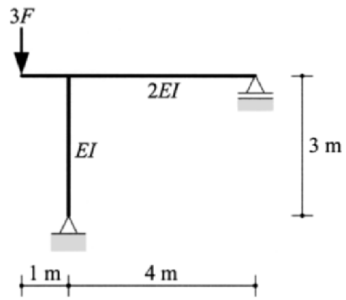
Questions:
a. Determine the displacement of the roller.
b. Determine the displacement of the joint where the force is applied.

8.92 In the structure shown, AC has a bending stiffness EI and BC has a bending stiffness $3EI$. In the calculation use $EI = 28\ \text{MNm}^2$. Dimensions and loading can be found in the figure.



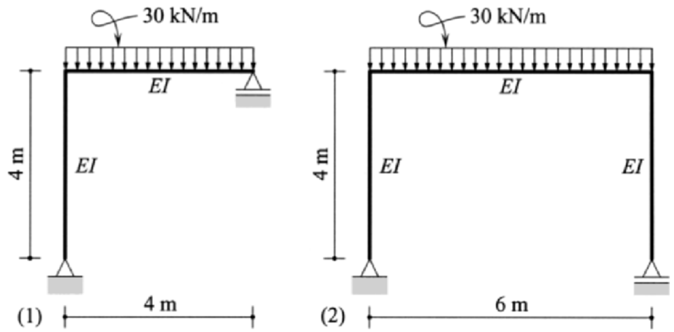
Questions:
a. Determine the displacement of B.
b. Determine the displacement of C.
c. Sketch the elastic curve.

8.93 In the structure shown the beam has bending stiffness $2EI$ and the column has a bending stiffness EI. Loading and dimensions are shown in the figure.



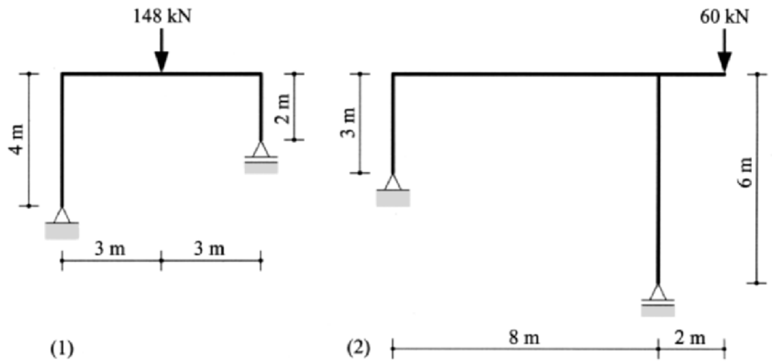
Question:
Determine the displacement of the roller.

8.94: 1–2 You are given two different structures in which all members have the same bending stiffness EI. Both structures carry a uniformly distributed load of 30 kN/m. The dimensions are given in the figures.



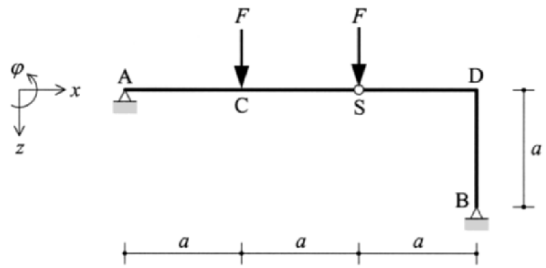
Questions:
a. Determine the displacement of the roller.
b. Sketch the elastic curve.

8.95: 1–2 You are given two structures in which the members all have the same bending stiffness $EI = 80\,\text{MNm}^2$. Dimensions and loading are given in the figures.



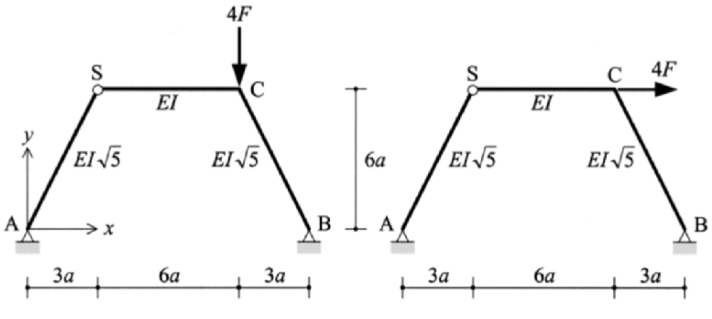
Questions:
a. Determine the displacement of the roller.
b. Determine the horizontal displacement of the girder.
c. Sketch the elastic curve.

8.96 The structure shown is constructed of the parts AS and SB joined by a hinge at S. All parts have the same bending stiffness EI. Loading and dimensions are given in the figure. Use the given xz coordinate system.



Questions:
a. Sketch the (expected) elastic curve.
b. Determine the displacement at S.
c. Determine the gap $\Delta\varphi$ at S.
d. Determine the displacement at C.

8.97: 1–2 In the given three-hinged frame, the girder CS has a bending stiffness EI and the slanted posts have a bending stiffness $EI\sqrt{5}$. The structure is loaded in two ways by a force $4F$ at C. Use the given xy coordinate system.



(1) (2)

Questions:

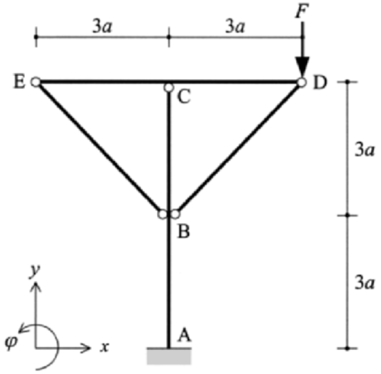
a. Determine the rotations at A and B.
b. Determine the displacement of S.
c. Determine the displacement of C.
d. Sketch the elastic curve.

Mixed exercises including elements from Chapter 7

8.98 In the given structure, all members have the same bending stiffness EI. Dimensions and loading can be found in the figure. Name the displacements in the given xy coordinate system. Deformation due to normal forces is ignored.

Questions:

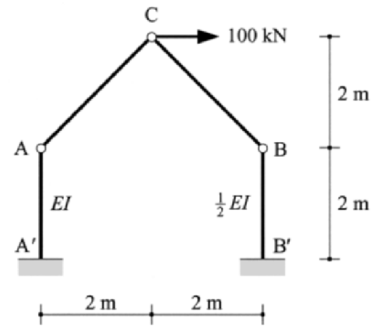
a. Determine the bending moment diagram for ABC.
b. Determine the displacement of B.
c. Determine the displacement of C.
d. Determine the displacement of D.



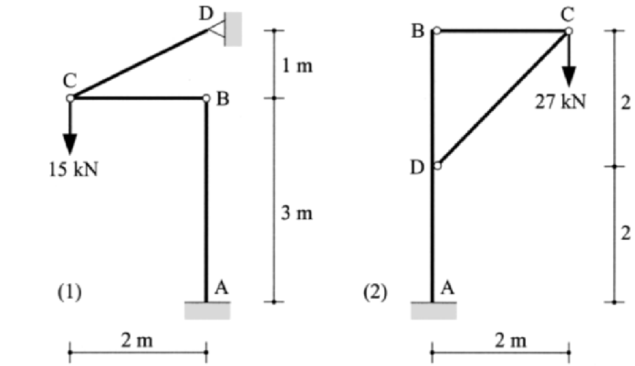
8.99 In the given structure, member AA′ has a bending stiffness $EI = 40\ \text{MNm}^2$, and member BB′ has a bending stiffness $\frac{1}{2}EI = 20\ \text{MNm}^2$. All other members are rigid. Dimensions and loading can be found in the figure.

Question:

Determine the displacement of C.

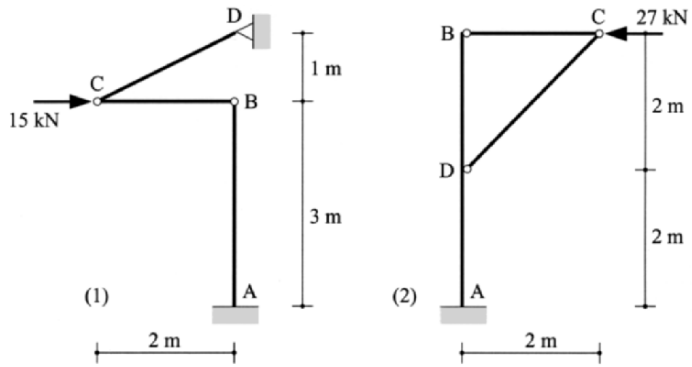


8.100: 1–2 You are given two structures, of which the dimensions and loading are given in the figure. Both members AB have the same bending stiffness $EI = 18$ MNm^2. All other members are rigid.



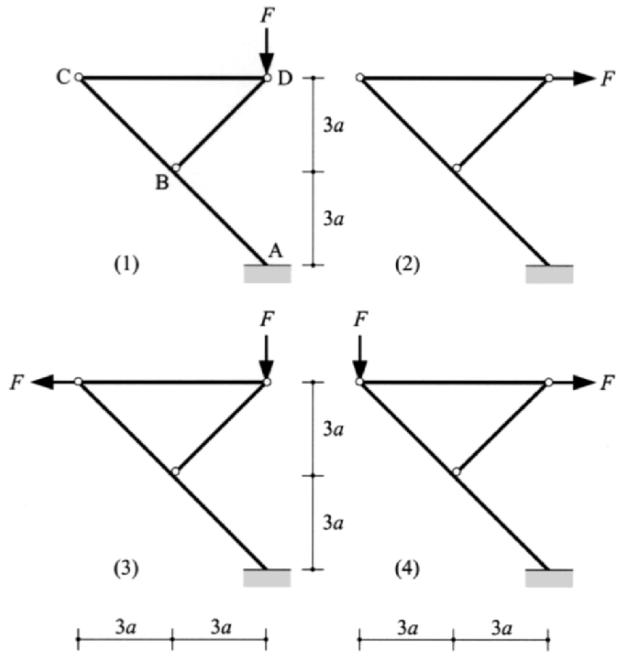
Question:
Determine the displacement of C.

8.101: 1–2 You are given the same structure as in problem 8.100, here loaded by horizontal forces.



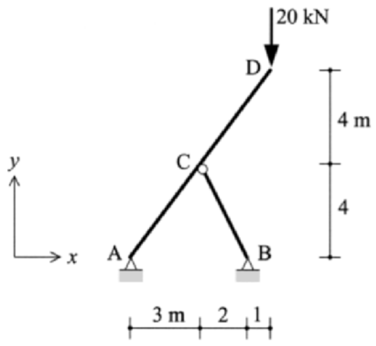
Question:
Determine the displacement of C.

8.102: 1–4 The structure given is loaded in four different ways by forces F. Member ABC has a bending stiffness EI. Deformation due to normal forces is ignored.



Question:
Determine the displacement at D, expressed in terms of a, F and EI.

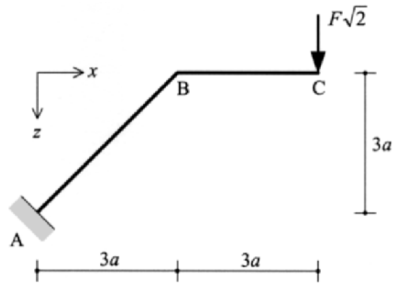
8.103 The inclined shored member ACD is loaded by a vertical force of 20 kN at D. ACD has bending stiffness 40 MNm^2.



Questions:

a. Determine the vertical displacement of D if the deformation due to normal forces is ignored.
b. In the same way determine the horizontal displacement of D.
c. How do the vertical and horizontal displacement of D change if the deformation due to the normal force in (only) member BC is taken into account. The axial stiffness of BC is $12\sqrt{5}$ MN.

8.104 The fixed bent member ABC is loaded by a vertical force $F\sqrt{2}$ at C. The bending stiffness is EI and the axial stiffness is EA.



Questions:

a. Determine the bending moment diagram with the deformation signs and include the values.
b. Determine the horizontal displacement $u_{x;\mathrm{b}}$ of C due to bending only.
c. Determine the horizontal displacement $u_{x;\mathrm{e}}$ of C due to extension only.
d. Determine the final horizontal displacement of C due to both bending and extension.
e. If the member has a rectangular cross-section with width b and (in the plane of bending) depth $h = 0.6a$, how large is the ratio $u_{x;\mathrm{e}}/u_{x;\mathrm{b}}$ at C?

Unsymmetrical and Inhomogeneous Cross-Sections

9

The fibre model for beams introduced in Chapter 4 was limited to symmetrical and homogeneous cross-sections. In this chapter the model is extended in a straightforward manner to be used for unsymmetrical and/or inhomogeneous cross-sections subject to extension and bending.

9.1 Sketch of the problem and required assumptions

The cross-sections used so far always contained at least one line of symmetry and the cross-section itself was always homogeneous, i.e. made of one single material. With the fibre model as introduced in Chapter 4 the beam is modelled as a collection of a large number of parallel and initially straight *fibres*. The fibres are kept together by a large number of rigid planes, called *cross-sections*, which are by definition perpendicular to the fibres and beam axis. The *beam axis* coincides with the fibre through the *normal* (*force*) *centre* NC of the cross-section. In a homogeneous cross-section the normal centre coincides with the centroid of the cross-section.

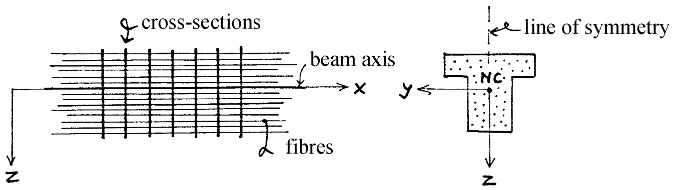


Figure 9.1 The fibre model for a member subject to extension and bending. The model consists of many *fibres* parallel to the axial direction, that are kept together by many rigid planes normal to the fibres. These rigid planes are known as *cross-sections*.

In Figure 9.1 this model is shown together with the coordinate system used. The origin of the *yz* coordinate system in the cross-section per definition

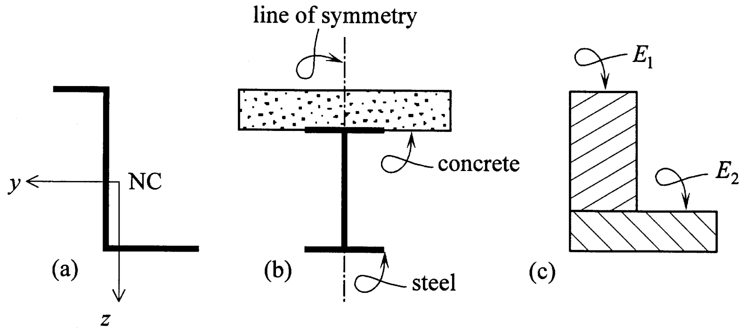


Figure 9.2 (a) An unsymmetrical and homogeneous cross-section, (b) an inhomogeneous cross-section with one line of symmetry, and (c) an unsymmetrical and inhomogeneous cross-section.

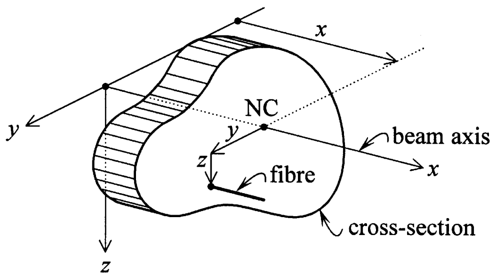


Figure 9.3 The location of a cross-section is defined by its x coordinate, that of a fibre by its y and z coordinates.

coincides with the normal centre NC, located on the beam axis.

Based on this model, formulas for calculating stresses and strains for combined bending and extension have been derived in Chapter 4. For unsymmetrical and/or inhomogeneous cross-sections, as shown in Figure 9.2, these formulas cannot be used. Figure 9.2a shows a homogeneous unsymmetrical cross-section. In Figure 9.2b the cross-section is inhomogeneous with one line of symmetry. In Figure 9.2c the unsymmetrical cross-section is inhomogeneous.

A number of the assumptions introduced in Chapters 2 and 4 also hold for unsymmetrical and inhomogeneous cross-sections subject to bending and extension:

The assumptions with respect to the *fibre model* are as follows:
- The member consists of many parallel *fibres* in longitudinal direction.
- The fibres are kept together by many rigid planes normal to the direction of the fibres. These rigid planes are called *cross-sections*.
- The cross-sections (rigid planes) are planar and perpendicular to the fibres, before and after the deformation of the member. This assumption is known as *Bernoulli's hypothesis*.
- Cross-sectional rotations remain small: $\varphi \ll 1$.

With respect to the *material behaviour* the following assumption is made:
- All fibres behave *linear-elastically* according to Hooke's Law. This implies a linear relationship between the stresses σ and strains ε:

$$\sigma = E\varepsilon.$$

New is the following assumption with respect to inhomogeneous cross-sections:

- In a beam with inhomogeneous cross-section the fibres are of different materials, with each material having its own modulus of elasticity E.

In the figures shown we use a xyz coordinate system (see Figure 9.3). The x axis is always chosen in longitudinal direction, parallel to the fibres. The position of a cross-section is given with its x coordinate. In the cross-section itself the position of a fibre is given with the y and z coordinate. By definition the x axis is chosen along the beam axis, defined as the fibre through the normal centre NC. So the origin of the coordinate system of the cross-section coincides always with the normal centre NC. This special choice of the origin of the coordinate system of the cross-section is a priori unknown. In Chapters 2 and 3 we found that for homogeneous cross-sections the location of the normal centre NC coincides with the centroid.[1] With some simple calculus the location of NC can also be obtained for inhomogeneous cross-sections, as will be shown in Section 9.7.

The positive normal force and bending moments[2] are shown in Figure 9.4; the positive displacements and rotations[3] are shown in Figure 9.5. Their directions are related to the given coordinate system.

All quantities that vary over the cross-section will be presented as functions of y and z. As an example we mention the strain ε and stress σ:

$$\varepsilon(y,z),$$

$$\sigma(y,z).$$

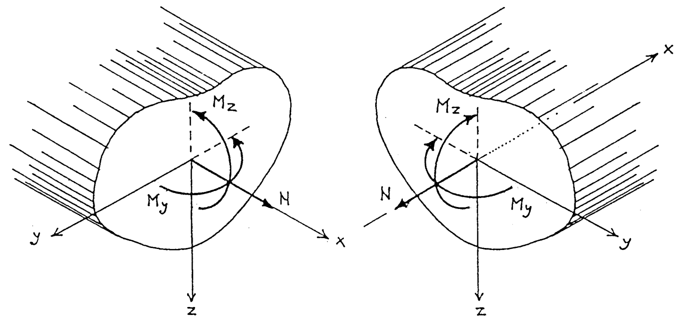


Figure 9.4 The positive directions of the section forces that are transmitted via normal stresses. N is the normal force, M_y the bending moment in the xy plane, and M_z the bending moment in the xz plane.

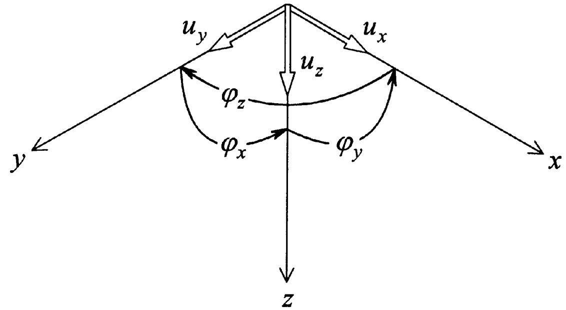


Figure 9.5 The positive displacements and rotations in the xyz coordinate system.

1 See Sections 2.4 and 3.1.3 in this volume.

2 Their definitions were given in *Engineering Mechanics*, Volume 1, Section 10.1.3.

3 For the positive rotations, see *Engineering Mechanics*, Volume 1, Section 1.3.2.

In case of an inhomogeneous cross-section, the modulus of elasticity E can vary over the cross-section and a function of y and z will be used:

$$E(y, z).$$

Most quantities may also vary with respect to the x coordinate. However, in equations which hold for a specific cross-section the x coordinate is omitted. The *fibre model* describes only the strains and stresses due to bending and extension. The influence of torsional moments are therefore excluded. Shear forces will not cause any strains in the fibre model. However with a simple model based on the equilibrium, as introduced in Chapter 5 for symmetrical homogeneous cross-sections, we can also obtain the shear flow and stresses in unsymmetrical and/or inhomogeneous cross-sections. At the end we will discuss the *shear* (*force*) *centre* SC for unsymmetrical thin-walled cross-sections.

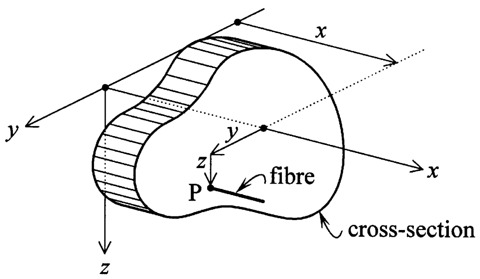


Figure 9.6 A point P(x, y, z) of cross-section x on fibre (y, z).

9.2 Kinematic relationships

The kinematic relationships link the displacements of a cross-section to the fibre strains at that cross-section. In space, the displacement of a cross-section as a rigid plane can be described by three translations u_x, u_y, u_z and three rotations φ_x, φ_y, φ_z. Consider a point $P(x, y, z)$ of cross-section x on fibre (y, z) (see Figure 9.6). With respect to the displacement $u(x, y, z)$ of P, in the direction of the fibre, only three displacement quantities are directly relevant: the translation u_x of the cross-section in the x direction and the rotations φ_y about the y axis and φ_z about the z axis. On the assumption of small rotations[1] we can write

$$u(x, y, z) = u_x - y\varphi_z + z\varphi_y. \tag{9.1}$$

Since the rotations are small their influences can be superimposed. The displacement quantities u_x, φ_y and φ_z are cross-sectional related and therefore only functions of x.

In Figure 9.7 expression (9.1) is clarified with some sketches: the displaced cross-section is translated by u_x and rotated about the y and z axis. Both the top and side view show the influence of the rotations upon the displacement u of point P.

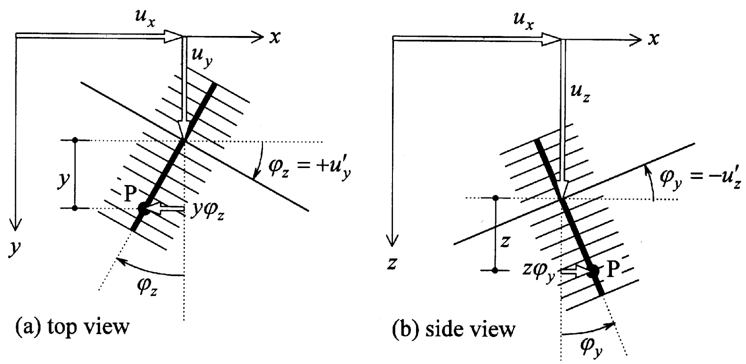


Figure 9.7 A rigid cross-section is translated by u_x in the x direction, and rotated through φ_y and φ_z about the y and z axis respectively. Both the (a) top view and (b) side view show the effect of the rotations upon the displacement u of point P.

With the displacement of point P also the strain of the fibre at P can be obtained:

$$\varepsilon(y, z) = \lim_{\Delta x \to 0} \frac{\Delta u(x, y, z)}{\Delta x} = \frac{\partial u(x, y, z)}{\partial x}$$

$$= \frac{\mathrm{d}u_x}{\mathrm{d}x} - y\frac{\mathrm{d}\varphi_z}{\mathrm{d}x} + z\frac{\mathrm{d}\varphi_y}{\mathrm{d}x},$$

or written in an simplified notation[2]

$$\varepsilon(y, z) = u'_x - \varphi'_z + z\varphi'_y. \tag{9.2}$$

[1] See *Engineering Mechanics*, Volume 1, Section 15.3.2.
[2] We simplify the notation by assuming $\mathrm{d}(\ldots)/\mathrm{d}x = (\ldots)'$.

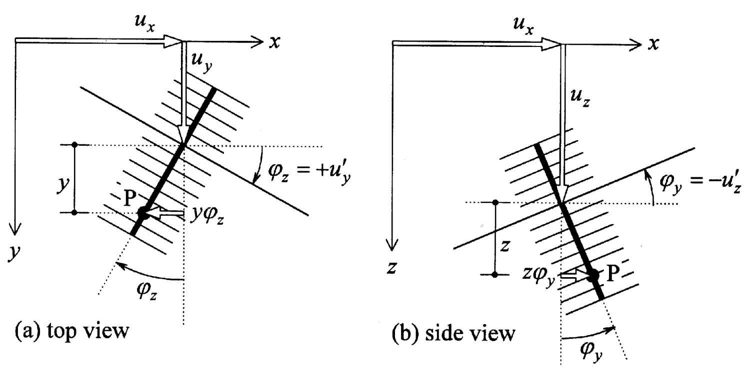


Figure 9.7 A rigid cross-section is translated by u_x in the x direction, and rotated through φ_y and φ_z about the y and z axis respectively. Both the (a) top view and (b) side view show the effect of the rotations upon the displacement u of point P.

The rotations φ_y and φ_z can be expressed in the displacements u_y and u_z (see Figure 9.7):

$$\varphi_y = -\frac{\mathrm{d}u_z}{\mathrm{d}x} = -u'_z,$$

$$\varphi_z = +\frac{\mathrm{d}u_y}{\mathrm{d}x} = u'_y.$$

Comment: The difference in sign between the expressions for φ_y and φ_z is the consequence of the definition of a positive rotation (it is left to the reader to check this).

The strain according to (9.2) in the fibre through P can now be written as

$$\varepsilon(y, z) = u'_x - yu''_y - zu''_z. \tag{9.3}$$

We can rewrite expression (9.3) by introducing the following three *cross-sectional deformation quantities*:

$$\varepsilon = u'_x,$$

$$\kappa_y = -u''_y = -\varphi'_z,$$

$$\kappa_z = -u''_z = +\varphi'_y. \tag{9.4}$$

These equations are known as the *kinematic equations* and link the *cross-sectional deformation quantities* to the *cross-sectional displacement quantities*.

With the kinematic equations (9.4) the strain in a fibre according to (9.3) can be written as

$$\varepsilon(y, z) = \varepsilon + y\kappa_y + z\kappa_z. \tag{9.5}$$

The strain $\varepsilon(y, z)$ at P is equal to the strain ε of the fibre coinciding with the x axis, added to the strain due to bending (curvature) of the beam in the xy plane and xz plane respectively.

In Figure 9.8 the strain distribution over the cross-section is visualised in a spatial strain diagram. From the assumption that plane cross-sections remain plane follows a linear strain distribution, represented by a plane. ε is the strain in the fibre coinciding with the x axis ($y = z = 0$). The slopes of this plane in the y and z directions are κ_y and κ_z:

$$\kappa_y = \frac{\partial \varepsilon(y, z)}{\partial y} \quad \text{slope of the strain diagram in the } y \text{ direction,}$$

$$\kappa_z = \frac{\partial \varepsilon(y, z)}{\partial z} \quad \text{slope of the strain diagram in the } z \text{ direction.}$$

The deformation quantities κ_y and κ_z are the components of the curvature of the beam in the xy and xz planes respectively.

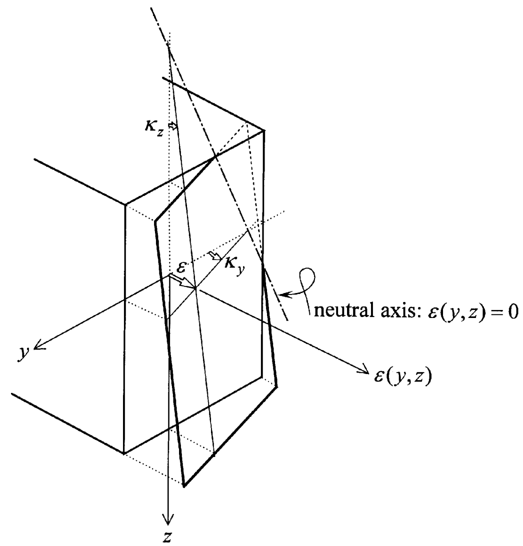


Figure 9.8 The strain distribution over the cross-section in a spatial strain diagram. Since plane cross-sections remain plane, the strain distribution is linear and can be represented by a plane. The slopes of this plane in y and z direction are the curvatures κ_y and κ_z. ε is the strain in the fibre coinciding with the x axis ($y = z = 0$). The fibres with zero strain form a straight line in the cross-section. This straight line is called the *neutral axis*.

9.3 Curvature and neutral axis

In Section 4.3.1 we derived for the curvature of a beam in the xz plane[1]

$$\kappa_z = -u_z''.$$

In the same way we can find the curvature of the beam in the xy plane

$$\kappa_y = -u_y''.$$

Both expressions are valid under the assumption of small strains.

In

$$\varepsilon(y, z) = \varepsilon + y\kappa_y + z\kappa_z, \tag{9.5}$$

the curvatures κ_y and κ_z appear to be the components of a vector κ. From formula (9.5) follows that a positive curvature causes a positive strain for positive values of y and z.

The prove that the curvature κ behaves like a *vector* or *first-order tensor* is given below, where we investigate the effect of a rotation of the cross-sectional yz coordinate system on the components of κ.

Between the coordinates of point P in the yz coordinate system and in a $\overline{yz}$ coordinate system, rotated through an angle α, we have the following

---

1 In Section 4.3.2 the displacement in the z direction is denoted by the letter w. Here we use the kern-index notation and denote the displacement in the z direction by u_z.

relationship (see Figure 9.9):

$$\bar{y} = +y\cos\alpha + z\sin\alpha,$$
$$\bar{z} = -y\sin\alpha + z\cos\alpha. \tag{9.6a}$$

The inverse is

$$y = +\bar{y}\cos\alpha - \bar{z}\sin\alpha,$$
$$z = +\bar{y}\sin\alpha + \bar{z}\cos\alpha. \tag{9.6b}$$

These are the *transformation rules* of the components of a *vector* or *first-order tensor*, in this case the position vector of P.

Using (9.6) we can rewrite the strain distribution in a cross-section according to (9.5):

$$\begin{aligned}\varepsilon(y,z) &= \varepsilon + y\kappa_y + z\kappa_z,\\ &= \varepsilon + (\bar{y}\cos\alpha - \bar{z}\sin\alpha)\kappa_y + (\bar{y}\sin\alpha + \bar{z}\cos\alpha)\kappa_z\\ &= \varepsilon + \bar{y}(\kappa_y\cos\alpha + \kappa_z\sin\alpha) + \bar{z}(-\kappa_y\sin\alpha + \kappa_z\cos\alpha),\end{aligned}$$

or

$$\varepsilon(y,z) = \varepsilon + \bar{y}\kappa_{\bar{y}} + \bar{z}\kappa_{\bar{z}},$$

in which

$$\kappa_{\bar{y}} = +\kappa_y\cos\alpha + \kappa_z\sin\alpha,$$
$$\kappa_{\bar{z}} = -\kappa_y\sin\alpha + \kappa_z\cos\alpha. \tag{9.7a}$$

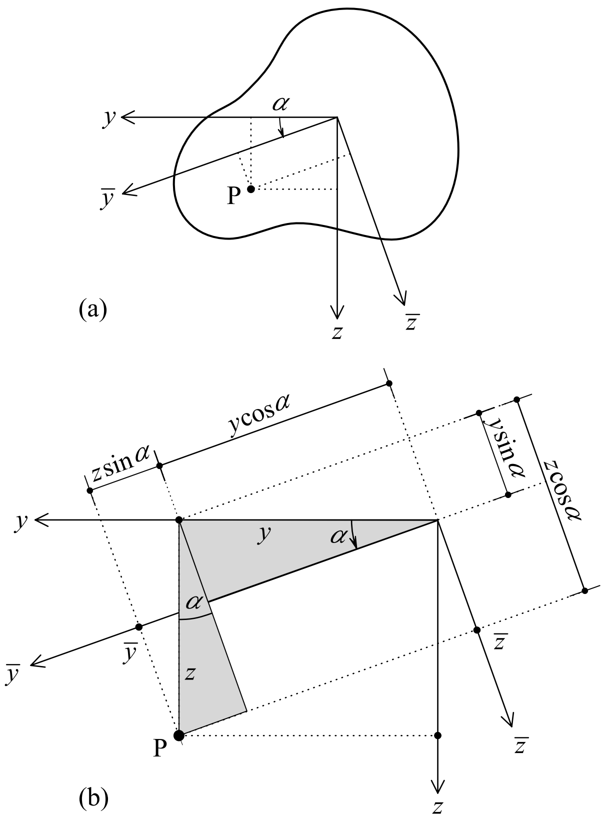


Figure 9.9 (a) The coordinates of a point P in the yz coordinate system and in a $\overline{yz}$ coordinate system, rotated through an angle α, are related by the transformation rules of a *vector* or *first-order tensor*. (b) These transformation rules can be derived from the two shaded triangles.

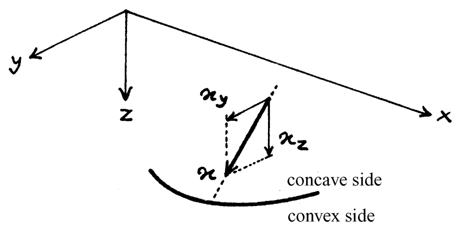


Figure 9.10 The curvature κ behaves as a vector, and can be visualised by an arrow. As a consequence of the used sign conventions, the arrow for κ always points from the concave side to the convex side of the curved beam.

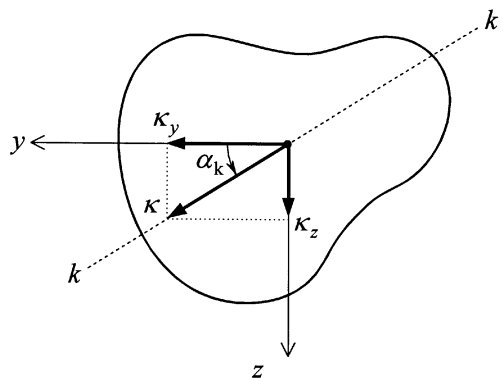


Figure 9.11 The curvature κ and its components, represented by arrows in the plane of the cross-section. The straight line k is the intersection of the *plane of curvature* with the cross-section.

The inverse is

$$\kappa_y = +\kappa_{\bar{y}} \cos\alpha - \kappa_{\bar{z}} \sin\alpha,$$
$$\kappa_z = -\kappa_{\bar{y}} \sin\alpha + \kappa_{\bar{z}} \cos\alpha. \tag{9.7b}$$

Comparing the transformation rules (9.7) with (9.6) for the position vector of P, we see that κ_y and κ_z indeed behave as the components of a vector or first-order tensor. This vector is the curvature κ.

Being a vector means that the curvature has a magnitude and a direction. So the curvature κ can be visualised by an arrow which, as a consequence of the used sign conventions, points from the concave side to the convex side of the curved beam (see Figure 9.10). In Figure 9.11 the arrow representing the curvature κ is shown in the plane of the cross-section. The straight line k is the intersection of the plane of curvature with the cross-section. For convenience sake, although not quite correct, the line k will be called plane of curvature.

The magnitude of the curvature κ is

$$\kappa = \sqrt{\kappa_y^2 + \kappa_z^2}. \tag{9.8}$$

The angle between the curvature κ and the y axis is defined by

$$\tan\alpha_k = \frac{\kappa_z}{\kappa_y}. \tag{9.9}$$

The fibres with zero strain form a straight line in the cross-section. This straight line is called the *neutral axis*, abbreviated as *na* (see Figures 9.8 and 9.12). The expression for the neutral axis *na* can be found from (9.5):

$$\varepsilon(y,z) = \varepsilon + y\kappa_y + z\kappa_z = 0. \tag{9.10}$$

In order to draw the neutral axis in the cross-section, we only need two points, for which we can choose the points of intersection with the coordinate axes. Substituting $z = 0$ in (9.8) we find the intersection y_1 of the neutral axis with the y axis:

$$y_1 = -\frac{\varepsilon}{\kappa_y}.$$

Substituting $y = 0$ in (9.8) the intersection z_1 with the z axis is found:

$$z_1 = -\frac{\varepsilon}{\kappa_z}.$$

In Figure 9.12 the neutral axis is drawn in the cross-sectional coordinate system, together with the plane of curvature k of the beam. The arrow representing the curvature κ points from the concave side (with negative strain: $\varepsilon < 0$) to the convex side (with positive strain: $\varepsilon > 0$). The figure shows that the plane of curvature k of the beam is perpendicular to the neutral axis na. It is left to the reader to proof this.

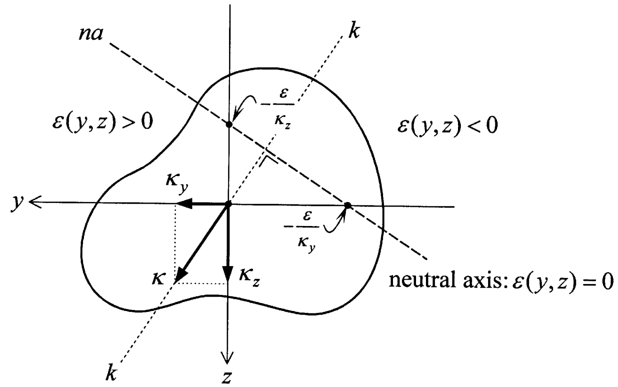


Figure 9.12 The fibres with zero strain form a straight line in the cross-section. This straight line is defined as the *neutral axis*, abbreviated as *na*. The neutral axis divides the cross-section in areas with positive and negative strains, and is perpendicular to the plane of curvature k.

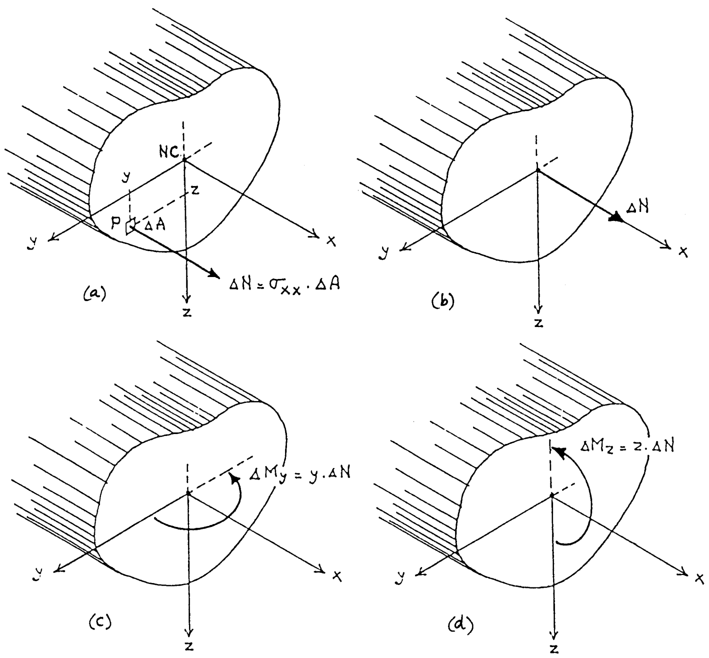


Figure 9.13 (a) The resultant of the normal stresses on small area ΔA around a point P is a small force ΔN. This force at P is statically equivalent to (b) a small force ΔN at the normal centre NC (the intersection of the member axis with the plane of the cross-section), together with (c) a small moment ΔM_y in the xy plane and (d) a small moment ΔM_z in the xz plane.

9.4 Normal force and bending moments – centre of force

Assume the normal stress distribution in a cross-section is given by a yet unknown function $\sigma(y, z)$. At an arbitrary point $\mathrm{P}(y, z)$ of the cross-section the normal stress is $\sigma(y, z)$. The resultant of all normal stresses on a small area ΔA at P is a small (normal) force ΔN (see Figure 9.13a):

$$\Delta N = \sigma(y, z)\Delta A. \tag{9.11}$$

This small force ΔN with its point of application at P is statically equivalent with a small force ΔN at the origin of the cross-sectional yz coordinate system, together with two small (bending) moments ΔM_y and ΔM_z (see Figures 9.13b–d):[1]

$$\Delta M_y = y\Delta N = y\sigma(y, z) \cdot \Delta A,$$

$$\Delta M_z = z\Delta N = z\sigma(y, z) \cdot \Delta A. \tag{9.12}$$

Summing the contributions of all small forces ΔN by integrating over the whole cross-sectional area A leads to the following section forces:

$$N = \int_A \sigma(z)\,\mathrm{d}A, \tag{9.13}$$

$$M_y = \int_A y\sigma(z)\,\mathrm{d}A, \tag{9.14a}$$

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[1] See also *Engineering Mechanics*, Volume 1, Section 10.1.3, and in this volume Section 4.3.2.

$$M_z = \int_A z\sigma(z)\,\mathrm{d}A. \tag{9.14b}$$

- N is the normal force and has its point of application at the origin of the cross-sectional yz coordinate system.
- M_y is the bending moment in the xy plane.
- M_z is the bending moment in the xz plane.

N, M_y and M_z are section forces (interaction forces). Their positive directions are related to the normal stresses in the cross-section.

From (9.11) we find a positive normal force when $\sigma(y, z)$ is a tensile stress; so a positive normal force is a tensile force.

A tensile stress ($\sigma > 0$) at a small area ΔA results in positive contributions to M_y for $y > 0$ and M_z for $z > 0$, as can be seen from (9.12). So a bending moment M_y (M_z) is positive when it causes tensile stresses at the positive y side (z side) of the x axis and compressive stresses at the negative y side (z side).

Figure 9.14 shows the positive directions of N, M_y and M_z for both a positive and a negative cross-sectional plane.[1]

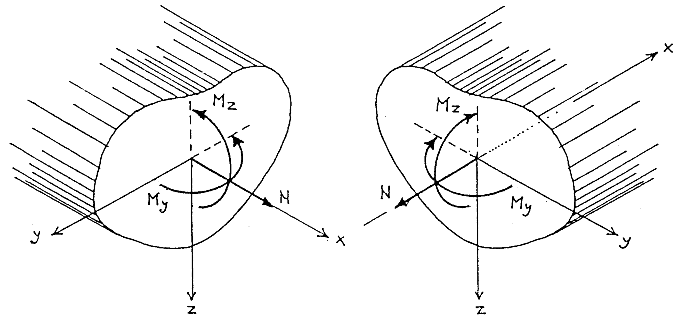

Figure 9.14 The positive directions of N, M_y and M_z for both a positive and a negative cross-sectional plane.

The section forces N, M_y and M_z are defined in the cross-sectional yz coordinate system. Below we will investigate how section forces will change when the yz coordinate system is transformed into a new $\overline{yz}$ coordinate system by rotating it about an angle α (see Figure 9.15). Therefore we investigate the contribution of the small force ΔN with its point of application at $\mathrm{P}(y, z) = \mathrm{P}(\bar{y}, \bar{z})$:

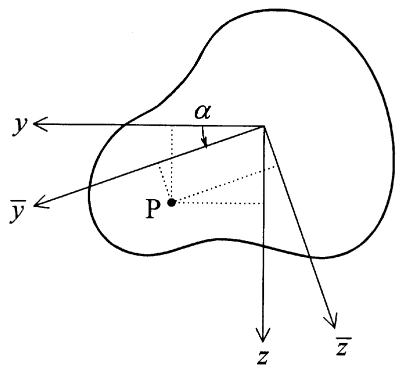

Figure 9.15 In order to investigate how the section forces will change when the yz coordinate system is transformed into a new $\overline{yz}$ coordinate system by rotating it about an angle α, we consider the behaviour of a small force ΔN with its point of application at $\mathrm{P}(y, z) = \mathrm{P}(\bar{y}, \bar{z})$.

$$\Delta N = \sigma(y, z)\Delta A = \sigma(\bar{y}, \bar{z})\Delta A.$$

[1] For the bending moments the formal definitions are used; see Section 2.8.

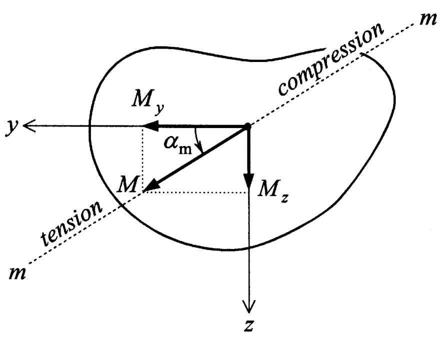

Figure 9.16 The bending moment M behaves as a *vector* or *first-order tensor*, so both M and its components M_y and M_z can be represented by *single pointed arrows* in the cross-section. The arrow for M points from the area with compressive stresses to the area with tensile stresses. The line m is the intersection of the cross-sectional plane with the plane in which the bending moment M acts. This line is called the *plane of loading*, since in this plane the load is transferred.

The magnitude of this small force is not influenced by a rotation of the coordinate system, nor is the magnitude of the normal force N. For the bending moments M_y and M_z this is quite different, as will be shown below.

In the rotated $\overline{xy}$ coordinate system the bending moments are defined by

$$M_{\bar{y}} = \int \bar{y}\sigma(\bar{y}, \bar{z})\,\mathrm{d}A,$$

$$M_{\bar{z}} = \int \bar{z}\sigma(\bar{y}, \bar{z}))\,\mathrm{d}A.$$

With the transformation formulae (9.6b) for the position vector of $\mathrm{P}(y, z)$

$$y = +\bar{y}\cos\alpha - \bar{z}\sin\alpha,$$

$$z = +\bar{y}\sin\alpha + \bar{z}\cos\alpha.$$

we find

$$\begin{aligned} M_y &= \int y\sigma(y, z)\,\mathrm{d}A \\ &= \int (+\bar{y}\cos\alpha - \bar{z}\sin\alpha)\sigma(\bar{y}, \bar{z})\,\mathrm{d}A \\ &= \left(\int \bar{y}\sigma(\bar{y}, \bar{z})\,\mathrm{d}A\right)\cos\alpha - \left(\int \bar{z}\sigma(\bar{y}, \bar{z})\,\mathrm{d}A\right)\sin\alpha, \end{aligned}$$

or

$$M_y = -M_{\bar{y}}\cos\alpha - M_{\bar{z}}\sin\alpha.$$

In the same way we find

$$M_z = -M_{\bar{y}} \sin\alpha + M_{\bar{z}} \cos\alpha.$$

These transformation formulae due to a rotation of the coordinate system are similar to (9.6b), the transformation formulae for the position vector of point $P(y, z) = P(\bar{y}, \bar{z})$: M_y and M_z behave as the components of a *vector* or *first-order tensor*. This vector is the cross-sectional bending moment M.

Both M and its components M_y and M_z can be represented by *single pointed arrows* in the cross-section (see Figure 9.16). The line m is the intersection of the cross-sectional plane with the plane in which the bending moment M acts. In this plane the load is transferred. For convenience, although not quite correct, we call line m the *plane of loading*.

The magnitude of the bending moment M is

$$M = \sqrt{M_y^2 + M_z^2}. \tag{9.15}$$

The bending moment M acts in a plane m through the x axis, making an angle α_m with the y axis:

$$\tan\alpha_m = \frac{M_z}{M_y}\,. \tag{9.16}$$

Figure 9.17 shows two presentations for the bending moment M in the cross-section: with a *bent arrow* and with a *straight single pointed ar-*

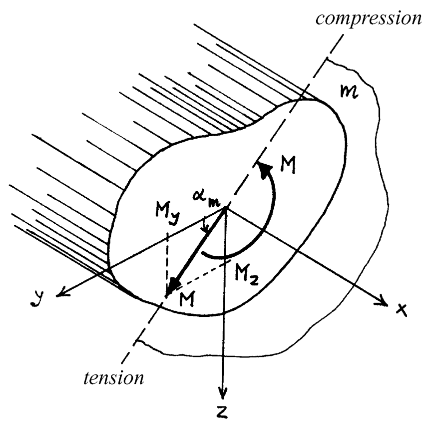

Figure 9.17 Two presentations for the bending moment M: with a *bent arrow* and with a *straight single pointed arrow*. The straight arrow always points from the area with compressive stresses to the area with tensile stresses. m is the plane of loading. Note that the vector presentation with a single arrow is different from the often used *angular vector* presentation with a double arrow (perpendicular to the plane m)

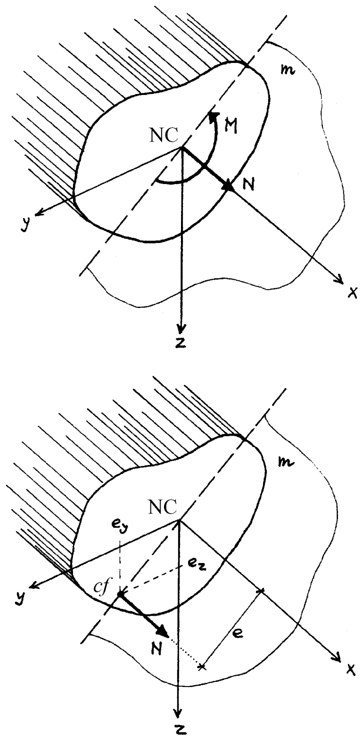

Figure 9.18 (a) Both the normal force N and the bending moment M are section forces that are transferred by the normal stresses in the cross-section. (b) If N is not zero, the resultant of M and N is a single force N, with its point of application at (e_y, e_z) on m. This point of application is called the *centre of force* (*cf*) of the cross-section and is the point of application of the resultant of all normal stresses in the cross-section.

row. The straight arrow[1] always points from the cross-sectional area with compressive stresses to the cross-sectional area with tensile stresses.

Both the normal force N and the bending moment M are section forces that are transferred by the normal stresses in the cross-section. If N is not zero, the resultant of M and N is a single force N, with its point of application on m and with an eccentricity e (see Figure 9.18):

$$e = \frac{M}{N}. \tag{9.17}$$

The eccentricity e has components e_y and e_z:

$$e_y = \frac{M_y}{N} \text{ and } e_z = \frac{M_z}{N}. \tag{9.18}$$

Point (e_y, e_z) is called the *centre of force* of the cross-section.[2] It is the point of application of the resultant of all normal stresses in the cross-section. The centres of force in all consecutive cross-sections form a line known as the *line of force* of the member.

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1 The vector presentation with a single arrow is different from the often used *angular vector* presentation with a double arrow (perpendicular to the plane m). See *Engineering Mechanics*, Volume 1, Section 3.3.

2 See *Engineering Mechanics*, Volume 1, Section 14.2.

9.5 Constitutive relationships for unsymmetrical and/or inhomogeneous cross-sections

In this section we will derive the constitutive relationships that link the section forces N, M_y and M_z to the cross-sectional deformation quantities ε, κ_y and κ_z.

The fibre model used so far, assumes a linear elastic stress-strain relation. We will restrict ourselves to this simple model, using Hooke's Law:

$$\sigma = E\varepsilon.$$

If the cross-section is not made of one single material, the cross-section is called *inhomogeneous*. In an inhomogeneous cross-section the modulus of elasticity E may vary between fibres based on the material used for each fibre or part of the cross-section. The inhomogeneous character can be implemented by using a function for the modulus of elasticity E. In the cross-sectional yz coordinate system this function is denoted as $E(y, z)$.

At a certain point (y, z) of the cross-section, we assume a strain $\varepsilon = \varepsilon(y, z)$ and a stress $\sigma = \sigma(y, z)$. For a homogeneous cross-section all fibres have the same modulus of elasticity E and the stress and strain at the point (y, z) of the cross-section are related by

$$\sigma(y, z) = E \cdot \varepsilon(y, z).$$

For inhomogeneous cross-sections the constitutive relationship which links the stresses to the strains has to be modified slightly:

$$\sigma(y, z) = E(y, z) \cdot \varepsilon(y, z).$$

The kinematic relationships are independent of the material properties. So we can use expression (9.5) to describe the linear strain distribution in both homogeneous and inhomogeneous cross-sections:

$$\varepsilon(y, z) = \varepsilon + y\kappa_y + z\kappa_y.$$

Note that the cross-sectional quantities ε, κ_y and κ_z are independent of y and z. According to Hooke's Law, the stress in a specific fibre (y, z) becomes

$$\sigma(y, z) = E(y, z) \cdot (\varepsilon + y\kappa_y + z\kappa_z). \tag{9.19}$$

For inhomogeneous cross-sections the modulus of elasticity is a function of y and z, and the stress distribution will not be linear anymore.

The normal force N can be found with (9.13):

$$\begin{aligned} N &= \int_A \sigma(y, z)\,\mathrm{d}A = \int_A E(y, z) \cdot (\varepsilon + y\kappa_y + z\kappa_z)\mathrm{d}A \\ &= \varepsilon \int_A E(y, z)\,\mathrm{d}A + \kappa_y \int_A E(y, z) y\,\mathrm{d}A + \kappa_z \int_A E(y, z) z\,\mathrm{d}A. \end{aligned}$$

The bending moments M_y and M_z can be found with (9.14):

$$\begin{aligned} M_y &= \int_A y\sigma(y, z)\,\mathrm{d}A = \int_A E(y, z) \cdot (\varepsilon + y\kappa_y + z\kappa_z) y\,\mathrm{d}A \\ &= \varepsilon \int_A E(y, z) y\,\mathrm{d}A + \kappa_y \int_A E(y, z) y^2\,\mathrm{d}A + \kappa_z \int_A E(y, z) yz\,\mathrm{d}A, \\ M_z &= \int_A z\sigma(y, z)\,\mathrm{d}A = \int_A E(y, z) \cdot (\varepsilon + y\kappa_y + z\kappa_z) z\,\mathrm{d}A \end{aligned}$$

$$= \varepsilon \int_A E(y,z)z\,\mathrm{d}A + \kappa_y \int_A E(y,z)yz\,\mathrm{d}A + \kappa_z \int_A E(y,z)z^2\,\mathrm{d}A.$$

Since the modulus of elasticity $E(y,z)$ is a function of y and z, this quantity remains under the integral.

In order to obtain expressions which can be handled more easily we will introduce a number of new cross-sectional quantities which will be denoted with so-called *double letter symbols*:

$$\int_A E(y,z)\,\mathrm{d}A = EA, \int_A E(y,z)y\,\mathrm{d}A = ES_y, \int_A E(y,z)y^2\,\mathrm{d}A = EI_{yy},$$
$$\int_A E(y,z)z\,\mathrm{d}A = ES_z, \int_A E(y,z)yz\,\mathrm{d}A = EI_{yz} = EI_{zy},$$
$$\int_A E(y,z)z^2\,\mathrm{d}A = EI_{zz}. \quad (9.20)$$

Using these double letter symbols the expressions for N, M_y and M_z can now be rewritten:

$$N = EA\varepsilon + ES_y\kappa_y + ES_z\kappa_z,$$
$$M_y = ES_y\varepsilon + EI_{yy}\kappa_y + EI_{yz}\kappa_z,$$
$$M_z = ES_z\varepsilon + EI_{zy}\kappa_y + EI_{zz}\kappa_z,$$

or in matrix notation:

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & ES_y & ES_z \\ ES_y & EI_{yy} & EI_{yz} \\ ES_z & EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix}. \quad (9.21)$$

These are the cross-sectional constitutive equations we were looking for.

The matrix shown is called the *cross-sectional stiffness matrix* linking the section forces N, M_y and M_z to the sectional deformation quantities ε, κ_y and κ_z.

Comment: Formula (9.21) applies for both homogeneous and inhomogeneous cross-sections. For inhomogeneous cross-sections the double letter symbols represent the integrals defined in (9.20). Only in case of homogeneous cross-sections we can read the double letter symbols as products:

$$EA = E \cdot A, \quad ES_y = E \cdot S_y, \quad EI_{yy} = E \cdot I_{yy}, \quad \ldots$$

For homogeneous cross-sections the constitutive relationships (9.21) can be written as

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = E \cdot \begin{bmatrix} A & S_y & S_z \\ S_y & I_{yy} & I_{yz} \\ S_z & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix}. \tag{9.22}$$

From the constitutive equations (9.21) we can conclude that if we know the strain ε at the fibre coinciding with the x axis and both curvatures κ_y and κ_z, the section forces N, M_y and M_z can be computed.

Moreover we can conclude that all sectional properties of a beam can be assigned to a single fibre which coincides with the x axis. This is why we are allowed to represent beams according to the beam theory[1] as single line elements in frame models.

So far we worked with the x axis chosen along an arbitrary fibre, and there-

[1] We restrict ourselves to the so-called Euler–Bernoulli beam theory.

fore with an arbitrary cross-sectional yz coordinate system. However the constitutive equations (9.21) can be significantly simplified by choosing the x axis along a special fibre, called the *member axis*. The origin of the cross-sectional yz coordinate system then coincides with a special point, called the *normal* (*force*) *centre* NC of the cross-section. The position of the normal centre NC is defined by

$$ES_y = \int_A E(y,z)y \, \mathrm{d}A = 0, \tag{9.23a}$$

$$ES_z = \int_A E(y,z)z \, \mathrm{d}A = 0. \tag{9.23b}$$

Choosing the origin of the cross-sectional yz coordinate system at the normal centre NC the coupling terms ES_y and ES_z between extension and bending in (9.21) will vanish since these become zero due tot the definition of the normal centre NC:

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{zy} & EI_{zz} \end{bmatrix}. \tag{9.24}$$

From (9.24) we conclude that a normal force N, applied at the *normal centre* NC, causes only extension (and a *strain* ε at the member axis) and no bending (no *curvatures* κ_y and κ_z), and that the bending moments M_y and M_z causes only bending (and *curvatures* κ_y and κ_z) and no extension (no *strain* ε at the member axis).[1] This means that there is no interaction between extension and bending. In the constitutive equations the extension

[1] See also Section 2.4.

part of the equations is fully uncoupled from the bending part. The system of constitutive equations can therefore also be written as

$$N = EA\varepsilon \quad \text{(extension)}, \tag{9.26}$$

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \quad \text{(bending)}. \tag{9.27}$$

In (9.26) EA is the axial stiffness of the member at the current cross-section (the resistance of the member to a change in length). In (9.27) the quantities EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz} together represent the stiffness against bending of the member at the current cross-section.

Both $M_y; M_z$ and $\kappa_y; \kappa_z$ are *vectors* or *first-order tensors* and can be expressed in one another by the linear relationship (9.27). This means that the stiffness quantities EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz} are the components of a *second-order tensor*. The matrix

$$\begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix}$$

is called the *bending stiffness tensor*. Since $EI_{yz} = EI_{zy}$ the stiffness tensor is a symmetrical tensor. The properties of the bending stiffness tensor will be investigated in Section 9.11.

From (9.27) we see a coupling between bending in the xy and xz plane by the terms $EI_{yz} = EI_{zy}$. If $EI_{yz} = EI_{zy} = 0$,

$$\begin{bmatrix} EI_{yy} & 0 \\ 0 & EI_{zz} \end{bmatrix}.$$

Bending in the xy and xz plane is uncoupled. Here the y and z directions are called the *principal directions* for the bending stiffness of the cross-section. The associated stiffness quantities EI_{yy} and EI_{zz} are called the *principal values*. The principal values and principal directions are discussed in more detail in Section 9.11.3.

9.6 Plane of loading and plane of curvature – neutral axis

In this section we investigate the relationship between the directions of the vectors representing the bending moment M and curvature κ (see Figure 9.19). This relationship is given by (9.27), the constitutive equations for bending:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}. \tag{9.27}$$

These equations apply only in a cross-sectional yz coordinate system with its origin at the normal centre NC.

m is the intersection of the cross-sectional plane and the plane in which the bending moment acts. This plane can also be called *plane of loading*. The orientation of m is defined by the angle α_m (see Figure 9.19b). The vector representing the bending moment M is pointing from the area with compressive stresses to the area with tensile stresses.

k is the intersection of the cross-sectional plane and the *plane of curvature*. Its orientation is defined by the angle α_k (see Figure 9.19a). The vector representing the curvature κ is pointing from the area with negative strains to the area with positive strains.

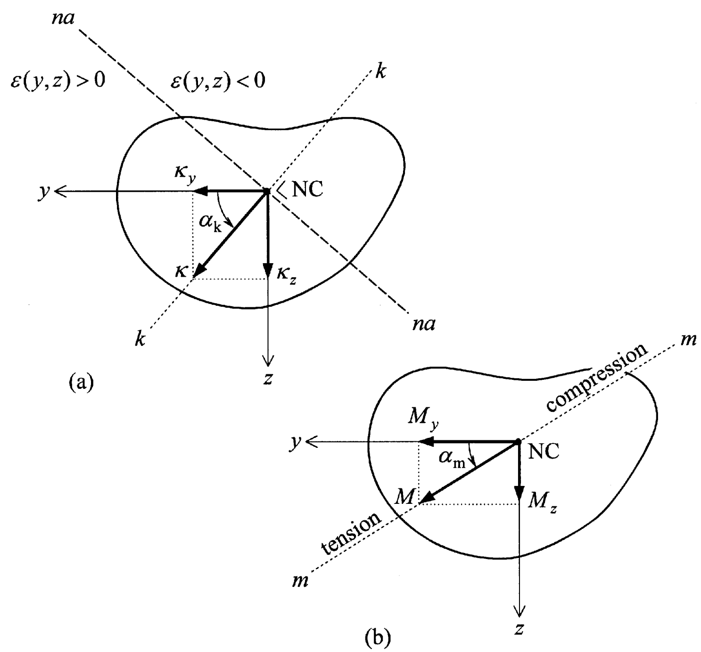
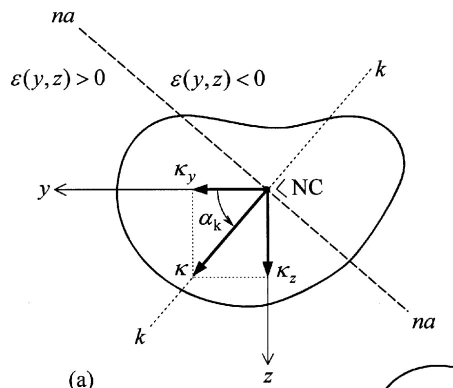


Figure 9.19 (a) The line k is the intersection of the cross-sectional plane and the *plane of curvature*. Its orientation is defined by the angle α_k. The vector representing the curvature κ points from the area with negative strains to the area with positive strains. The plane of curvature k is perpendicular to the neutral axis *na*. For bending without extension the neutral axis passes through the normal centre NC. (b) The line m is the intersection of the cross-sectional plane and the *plane of loading*. The orientation of m is defined by the angle α_m. The vector representing the bending moment M is pointing from the area with compressive stresses to the area with tensile stresses.



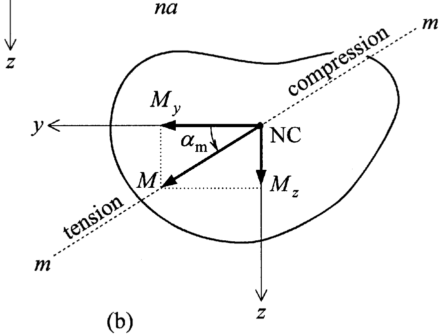


Figure 9.19 (a) The line k is the intersection of the cross-sectional plane and the *plane of curvature*. Its orientation is defined by the angle α_k. The vector representing the curvature κ points from the area with negative strains to the area with positive strains. The plane of curvature k is perpendicular to the neutral axis *na*. For bending without extension the neutral axis passes through the normal centre NC. (b) The line m is the intersection of the cross-sectional plane and the *plane of loading*. The orientation of m is defined by the angle α_m. The vector representing the bending moment M is pointing from the area with compressive stresses to the area with tensile stresses.

The intersection of the cross-sectional plane and the neutral plane (plane with zero strain) is the *neutral axis na*. The neutral axis divides the cross-section in two parts with positive and negative strains respectively. The strain distribution is defined by (9.5):

$$\varepsilon(y, z) = \varepsilon + y\kappa_y + z\kappa_z,$$

and the neutral axis by

$$\varepsilon + y\kappa_y + z\kappa_z = 0.$$

Below we restrict ourselves to *bending without extension* ($\varepsilon = 0$), so the equation of the neutral axis becomes[1]

$$y\kappa_y + z\kappa_z = 0, \tag{9.28}$$

and the neutral axis passes through the normal centre NC. The neutral axis *na* is perpendicular to the vector representing the curvature κ (see Figure 9.19b).

The direction of the neutral axis depends on the load condition of the cross section. Given the plane of loading m, the question is to find the plane of curvature k and the direction of the neutral axis *na* perpendicular to k, and *vice versa*.

Example 1

The triangular shaped homogeneous cross-section in Figure 9.20 is loaded

---

[1] In case of *bending with extension* ($\varepsilon \neq 0$) the neutral axis *na* does not pass through the normal centre NC anymore, but still remains perpendicular to the curvature κ (see Section 9.3, Figure 9.12).

in the vertical plane by a bending moment $M_z = M$. So the plane of loading m makes an angle $\alpha_m = 90°$ with the y axis.

Question:
Determine the orientation of the neutral axis.

Solution:
First we have to calculate the components of the stiffness tensor. The moments of inertia of this triangular cross-section can be found with the formulae derived in Section 3.2.4, Example 5:

$$I_{yy} = I_{zz} = \tfrac{1}{36}\,bh^3 = \tfrac{1}{36}\,(3a)^4 = \tfrac{9}{4}\,a^4 \;\Rightarrow EI_{yy} = EI_{zz} = \tfrac{9}{4}\,Ea^4,$$

$$I_{zz} = I_{zy} = \tfrac{1}{72}\,b^2h^2 = \tfrac{1}{72}\,(3a)^4 = \tfrac{9}{8}\,a^4 \Rightarrow EI_{yz} = EI_{zy} = \tfrac{9}{8}\,Ea^4.$$

These values, together with $M_y = 0$ and $M_z = M$, substituted in (9.27) leads to

$$\begin{bmatrix} 0 \\ M \end{bmatrix} = \tfrac{9}{8}\,Ea^4 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}.$$

The solution of these two equations, with κ_y and κ_z as the unknown quantities, is

$$\kappa_y = -\frac{8}{27}\frac{M}{Ea^4}\,,$$

$$\kappa_z = +\frac{16}{27}\frac{M}{Ea^4}\,.$$

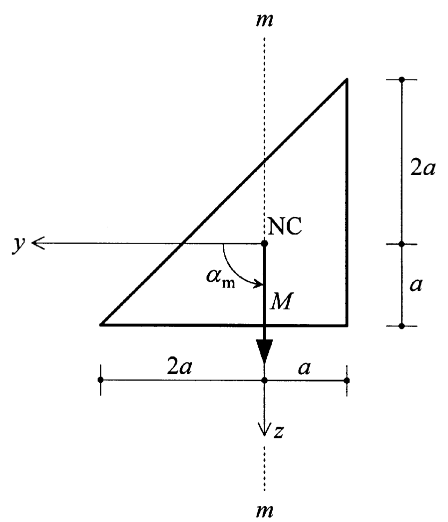


Figure 9.20 A triangular shaped homogeneous cross-section is loaded in the vertical plane by a bending moment $M_z = M$. The plane of loading m thus makes an angle $\alpha_m = 90°$ with the y axis.

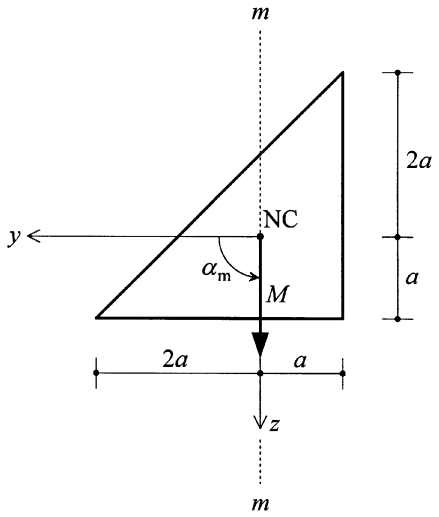


Figure 9.20 A triangular shaped homogeneous cross-section is loaded in the vertical plane by a bending moment $M_z = M$. The plane of loading m thus makes an angle $\alpha_m = 90°$ with the y axis.

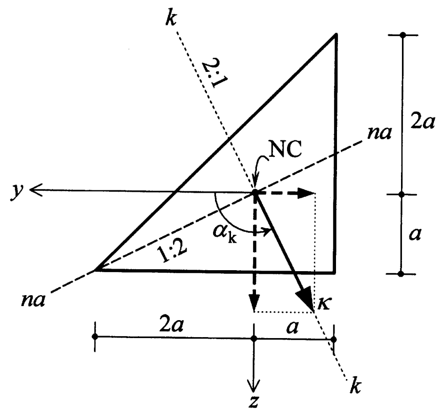


Figure 9.21 The plane of curvature k. The neutral axis na is perpendicular to the line k and, since there is no extension, passes through the normal centre NC. There is a distinct difference between the plane of loading m and the plane of curvature k.

The solution can also be found by inverting the constitutive matrix:

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{1}{\frac{9}{8}Ea^4 \times 3} \times \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ M \end{bmatrix} = \frac{8}{27}\frac{M}{Ea^4} \times \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \kappa_y = -\dfrac{8}{27}\dfrac{M}{Ea^4}, \\ \kappa_z = +\dfrac{16}{27}\dfrac{M}{Ea^4}. \end{cases}$$

The direction of the plane of curvature k is defined by

$$\tan\alpha_k = \frac{\kappa_z}{\kappa_y} = \frac{+16/27}{-8/27} = -2 \Rightarrow \alpha_k = 116.57°.$$

Figure 9.20 shows the plane of loading m. Figure 9.21 shows the plane of curvature k. The neutral axis na is perpendicular to line k and passes through the normal centre NC since there is no extension.

Comment: From this example we observe a distinct difference between the plane of loading m and the plane of curvature k.

Example 2

A beam with the triangular homogeneous cross-section in Figure 9.22 is compelled to curve in the vertical plane with $\kappa_z = \kappa$, so the neutral axis na is horizontal.

Question:

Determine the plane of loading, or in other words, the plane in which the resultant bending moment acts.

Solution:

The direction of the plane of curvature k is defined by $\alpha_k = +90°$ (see Figure 9.22). To find the direction of the plane of loading m we substitute $\kappa_y = 0$ and $\kappa_z = \kappa$ in (9.27), together with the values

$$EI_{yy} = EI_{zz} = \tfrac{9}{4}\,Ea^4 \text{ and } EI_{yz} = EI_{zy} = \tfrac{9}{8}\,Ea^4$$

determined in the previous example:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \tfrac{9}{8}\,Ea^4 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ \kappa \end{bmatrix} = \tfrac{9}{8}\,Ea^4\kappa \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} M_y = \tfrac{9}{8}\,Ea^4\kappa, \\ M_z = \tfrac{9}{4}\,Ea^4\kappa. \end{cases}$$

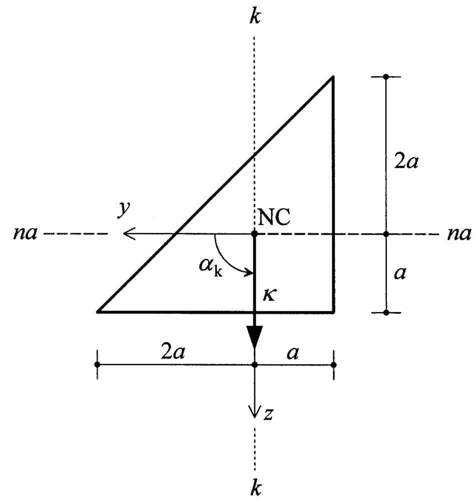


Figure 9.22 A beam with homogeneous triangular cross-section is compelled to curve in the vertical plane with $\kappa_z = \kappa$, so the neutral axis na is horizontal.

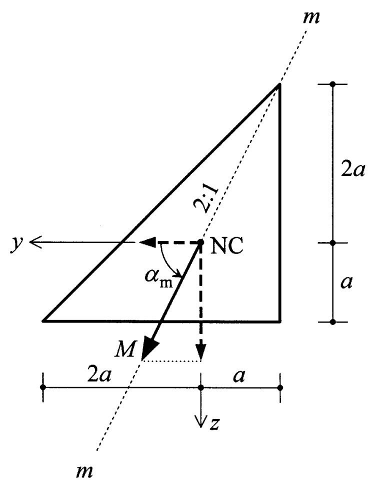


Figure 9.23 The plane of loading m associated with a vertical plane of curvature and a horizontal neutral axis.

The direction of the plane of loading m is defined by (see Figure 9.23)

$$\tan \alpha_m = \frac{M_z}{M_y} = 2.$$

This plane of loading is associated with a vertical plane of curvature and a horizontal neutral axis.

Comments:

- The examples given are related to homogeneous cross-sections. For inhomogeneous cross-sections the strategy remains the same on the understanding that EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz}, the components of the bending stiffness tensor, have to be read as double letter symbols.
- Note that in both examples the plane of loading m does not coincide with the plane of curvature k. An interesting question is: when does the plane of loading coincide with the plane of curvature? This question will be answered in Section 9.11.3.

9.7 The normal centre NC for inhomogeneous cross-sections

The special location of the origin of the cross-sectional yz coordinate system for which the coupling terms ES_y and ES_z between extension and bending are zero is by definition called the *normal* (*force*) *centre* NC. The fibre through NC is called the *beam axis*.

Selecting the origin of the cross-sectional yz coordinate system at the normal centre NC has the advantage that extension and bending can be treated separately:

- A normal force generates solely extension (a strain ε at the beam axis) and no bending (no curvatures κ_y and κ_z).

- A bending moment M with components M_y and M_z generates solely bending (with curvatures κ_y and κ_z) and no extension (no strain ε at the beam axis).

The coupling terms ES_y and ES_z are referred to as the *weighted static moments* (*of area*) or *weighted first-order moments of area*.

To find the position of the normal centre NC of the cross-section, or the origin of the yz coordinate system for which the coupling terms ES_y and ES_z become zero, we assume that $\bar{y}_{\mathrm{NC}}$ and $\bar{z}_{\mathrm{NC}}$ are the coordinates of NC in an arbitrarily chosen $\overline{yz}$ coordinate system (see Figure 9.24). Next we choose the origin of a yz coordinate system at NC. The coordinates of an arbitrary point $\mathrm{P}(\bar{y}, \bar{z}) = \mathrm{P}(y, z)$ in the two coordinate systems are related by

$$\bar{y} = y + \bar{y}_{\mathrm{NC}}\,,$$

$$\bar{z} = z + \bar{z}_{\mathrm{NC}}\,.$$

For the weighted static moments with respect to the $\overline{yz}$ coordinate system we now find

$$\begin{aligned} ES_{\bar{y}} &= \int_A E(y,z)\cdot \bar{y}\,\mathrm{d}A = \int_A E(y,z)\cdot y\,\mathrm{d}A + \bar{y}_{\mathrm{NC}}\int_A E(y,z)\,\mathrm{d}A \\ &= ES_y + EA\cdot \bar{y}_{\mathrm{NC}}, \\ ES_{\bar{z}} &= \int_A E(y,z)\cdot \bar{z}\,\mathrm{d}A = \int_A E(y,z)\cdot z\,\mathrm{d}A + \bar{z}_{\mathrm{NC}}\int_A E(y,z)\,\mathrm{d}A \\ &= ES_z + EA\cdot \bar{z}_{\mathrm{NC}}. \end{aligned} \tag{9.29}$$

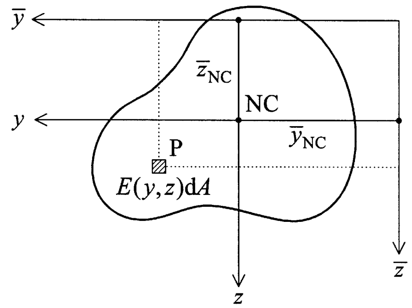


Figure 9.24 The weighted small area $E(y,z)\,\mathrm{dA}$ about point $\mathrm{P}(y,z) = \mathrm{P}(\bar{y}, \bar{z})$ is the product of the area dA and the modulus of elasticity $E(y,z) = E(\bar{y}, \bar{z})$ at P.

These formulae give the transformation rules due to a translation of the coordinate system, and are referred to as the *parallel axis theorem for the weighted static moments*.[1]

By definition the weighted static moments with respect to the yz coordinate system with its origin at NC are zero:

$$ES_y = 0 \text{ and } ES_z = 0,$$

so (9.29) simplifies into

$$\begin{aligned} ES_{\bar{y}} &= EA \cdot \bar{y}_{\mathrm{NC}}, \\ ES_{\bar{z}} &= EA \cdot \bar{z}_{\mathrm{NC}} \end{aligned} \tag{9.30}$$

and the coordinates of NC in an arbitrarily chosen $\overline{yz}$ coordinate system can be found from

$$\bar{y}_{\mathrm{NC}} = \frac{ES_{\bar{y}}}{EA} \text{ and } \bar{z}_{\mathrm{NC}} = \frac{ES_{\bar{z}}}{EA} . \tag{9.31}$$

Comment: For homogeneous cross-sections the normal centre NC coincides with the centroid C of the cross-section, and the double letter symbols EA, ES_y and ES_z may be read as the products $E \cdot A$, $E \cdot S_y$ and $E \cdot S_z$. So (9.31) changes into

$$\bar{y}_{\mathrm{NC}} = \bar{y}_{\mathrm{C}} = \frac{S_{\bar{y}}}{A} \text{ and } \bar{z}_{\mathrm{NC}} = \bar{z}_{\mathrm{C}} = \frac{S_{\bar{z}}}{A} . \tag{9.32}$$

---

[1] For homogeneous cross-sections formulae (29) simplify into those derived in Section 3.1.3.

This is in agreement with the results found before in Section 3.1.3.

Below the newly derived formulae are illustrated by two examples.

Example 1: Normal centre versus mass centre (centre of gravity)
A composite beam with rectangular cross-section is constructed by the two glued materials 1 and 2 (see Figure 9.25a). Both materials have a different modulus of elasticity E and a different mass density ρ. The modulus of elasticity for material 2 is four times larger than for material 1: $E_2 = 4E_1$. The mass density is two times larger for material 2 than for material 1: $\rho_2 = 2\rho_1$.

Questions:

a. Determine the location of the normal centre NC of the cross-section.
b. Determine the location of the mass centre MC (or centre of gravity) of the cross-section.

Solution:
a. In Figure 9.25b for each homogeneous part i the location of the centroid C_i is given. In this figure also the $\overline{yz}$ coordinate system in which we will work is given.

The coordinates of the *normal centre* NC with respect to this coordinate system can be found with the method outlined before, using *double letter symbols* for the inhomogeneous cross-section. We do not need integral calculus here since for each material simple geometrical shapes can be recognised.

The weighted areas (axial stiffnesses) for the homogeneous parts 1 and 2 of the cross-section are

$$(EA)_1 = E_1 a^2,$$

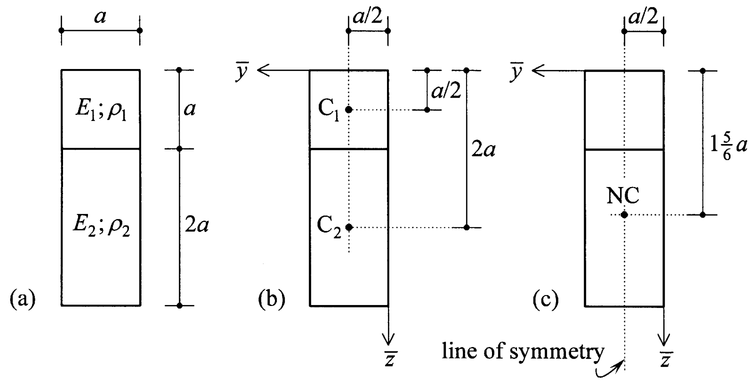


Figure 9.25 (a) A composite beam with rectangular cross-section is constructed of two materials 1 and 2. Both materials have a different modulus of elasticity E and a different mass density ρ. (b) The centroids C_1 and C_2 of the homogeneous parts of the cross-section. (c) The location of the normal centre NC of the cross-section.

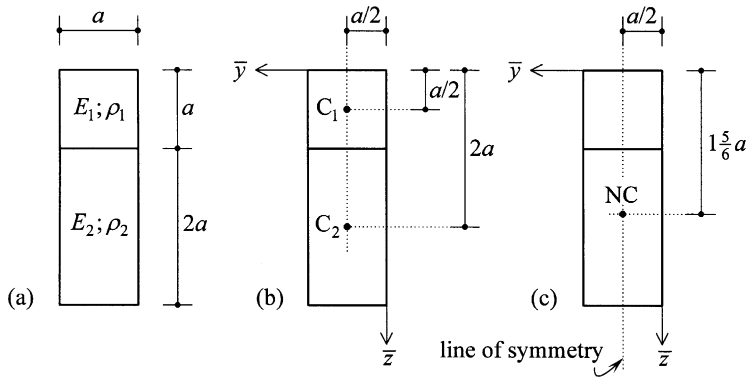


Figure 9.25 (a) A composite beam with rectangular cross-section is constructed of two materials 1 and 2. Both materials have a different modulus of elasticity E and a different mass density ρ. (b) The centroids C_1 and C_2 of the homogeneous parts of the cross-section. (c) The location of the normal centre NC of the cross-section.

$$(EA)_2 = E_2 \cdot 2a^2 = 2a^2 E_2,$$

and the weighted static moments:

$$(ES_{\bar{y}})_1 = (EA)_1 \cdot \tfrac{1}{2}a = \tfrac{1}{2}a^3 E_1, \quad (ES_{\bar{z}})_1 = (EA)_1 \cdot \tfrac{1}{2}a = \tfrac{1}{2}a^3 E_1,$$

$$(ES_{\bar{y}})_2 = (EA)_2 \cdot \tfrac{1}{2}a = a^3 E_1, \quad (ES_{\bar{z}})_2 = (EA)_2 \cdot 2a = 4a^3 E_2.$$

Using these expressions we find with (9.30)

$$\bar{y}_{\text{NC}} = \frac{ES_{\bar{y}}}{EA} = \frac{(ES_{\bar{y}})_1 + (ES_{\bar{y}})_2}{(EA)_1 + (EA)_2} = \frac{\frac{1}{2}a^3 E_1 + a^3 E_2}{a^2 E_1 + 2a^2 E_2} = a\frac{E_1 + 2E_2}{2E_1 + 4E_2},$$

$$\bar{z}_{\text{NC}} = \frac{ES_{\bar{z}}}{EA} = \frac{(ES_{\bar{z}})_1 + (ES_{\bar{z}})_2}{(EA)_1 + (EA)_2} = \frac{\frac{1}{2}a^3 E_1 + 4a^3 E_2}{a^2 E_1 + 2a^2 E_2} = a\frac{E_1 + 8E_2}{2E_1 + 4E_2}.$$

Substituting $E_2 = 4E_1$ results in the following coordinates of the normal centre NC:

$$\bar{y}_{\text{NC}} = \tfrac{1}{2}a,$$

$$\bar{z}_{\text{NC}} = 1\tfrac{5}{6}a.$$

The location of NC is shown in Figure 9.25c.

Since both the geometrical shape and the elastic properties (i.e. the modulus of elasticity E) of the cross-section are mirror symmetric with respect to the line $\bar{y} = a/2$, the normal centre NC will be on this line of symmetry.

b. To find the location of the mass centre MC of the cross-section we consider a beam slice of unit thickness and place the slice horizontal in the vertical gravity field. The point of application of the total weight of the slice is the mass centre MC (or centre of gravity) of the cross-section.

Figure 9.26a shows the weights per homogeneous part with their points of application at the associated centroids. The moment equilibrium about the $\bar{z}$ and $\bar{y}$ axis lead to respectively

$$\underbrace{\rho_1 g \cdot a^2 \cdot \tfrac{1}{2}\,a}_{\text{part 1}} + \underbrace{\rho_2 g \cdot 2a^2 \cdot \tfrac{1}{2}\,a}_{\text{part 2}} = (\rho_1 g \cdot a^2 + \rho_2 g \cdot 2a^2) \cdot \bar{y}_{\mathrm{MC}},$$

$$\underbrace{\rho_1 g \cdot a^2 \cdot \tfrac{1}{2}\,a}_{\text{part 1}} + \underbrace{\rho_2 g \cdot 2a^2 \cdot 2a}_{\text{part 2}} = (\rho_1 g \cdot a^2 + \rho_2 g \cdot 2a^2) \cdot \bar{z}_{\mathrm{MC}},$$

in which g is the gravitational field strength.

With $\rho_2 = 2\rho_1$ we find

$$\bar{y}_{\mathrm{MC}} = \tfrac{1}{2}\,a,$$

$$\bar{z}_{\mathrm{MC}} = 1\tfrac{7}{10}\,a.$$

The location of MC is shown in Figure 9.26b.

For inhomogeneous cross-sections the normal centre NC in general does not coincide with the mass centre MC. This is a vital aspect when encountering inhomogeneous cross-sections.

Example 2: An inhomogeneous and unsymmetrical cross-section
The cross-section of a composite member consists of four squares of different materials (see Figure 9.27a). In the calculation use $E_1 = 30$ GPa, $E_2 = 60$ GPa, $E_3 = 100$ GPa and $E_4 = 200$ GPa.

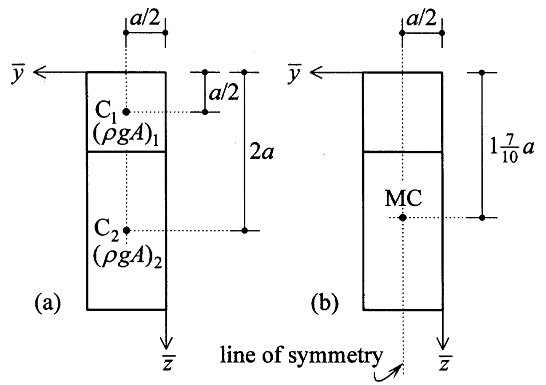


Figure 9.26 To find the location of the mass centre MC of the cross-section, a beam slice of unit thickness is considered, and placed in horizontal position in the vertical gravity field. The point of application of the total weight of the slice is the mass centre MC of the cross-section (or centre of gravity). (a) The weights per homogeneous part, with their points of application at the associated centroids. (b). The location of the mass centre MC.

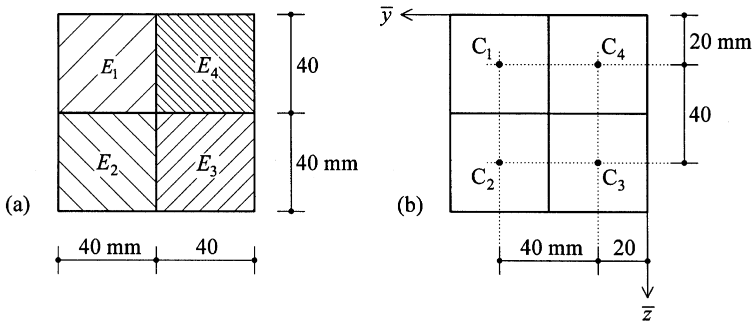


Figure 9.27 (a) The cross-section of a composite member consisting of four squares of different materials glued firmly together. (b) The centroids of the four homogeneous parts.

Question:
Determine the point of application of the resultant of all normal stresses in the cross-section if there is extension only, and no bending.

Solution:
If there is no bending the resultant of all normal stresses is a force, the normal force N, with its point of application at the normal centre NC. In order to calculate the location of NC we will use the $\overline{yz}$ coordinate system with its origin at the top right corner of the cross-section (see Figure 9.27b). Per homogeneous part a number of relevant values are calculated in Table 9.1.

From Table 9.1 we find

$$EA = \int E(\bar{y}, \bar{z})\,\mathrm{d}A = \sum_{i=1}^{4} (EA)_i = 624 \times 10^6 \text{ N} = 624 \text{ MN}.$$

EA is the *axial stiffness of the beam* at the cross-section given.

Table 9.1 Calculation of the weighted area and weighted static moment per homogeneous area.

i	E_i (N/mm^2)	A_i (mm^2)	$(EA)_i$ (N)	$(\bar{y}_C)_i$ (mm)	$(\bar{z}_C)_i$ (mm)	$(ES_{\bar{y}})_i$ (Nmm)	$(ES_{\bar{z}})_i$ (Nmm)
1	30×10^3	1.6×10^3	48×10^6	60	20	2.88×10^9	0.96×10^9
2	60×10^3	1.6×10^3	96×10^6	60	60	5.76×10^9	5.76×10^9
3	100×10^3	1.6×10^3	160×10^6	20	60	3.20×10^9	9.60×10^9
4	200×10^3	1.6×10^3	320×10^6	20	20	6.40×10^9	6.40×10^9
		$\sum$	624×10^6		$\sum$	18.24×10^9	22.72×10^9

In Table 9.1, $(\bar{y}_C)_i$ and $(\bar{z}_C)_i$ are the coordinates of the centroid C_i of the homogeneous square i of the cross-section (see Figure 9.27b).

For the weighted static moments, calculated in the two last columns of Table 9.1, applies[1]

$$(ES_{\bar{y}})_i = (EA \cdot \bar{y}_C)_i \quad \text{and} \quad (ES_{\bar{z}})_i = (EA \cdot \bar{z}_C)_i .$$

For the inhomogeneous cross-section we find

$$ES_{\bar{y}} = \int E(\bar{y},\bar{z})\bar{y}\,\mathrm{d}A = \sum_{i=1}^{4}(ES_{\bar{y}})_i = 18.24 \times 10^9 \text{ Nmm},$$

$$ES_{\bar{z}} = \int E(\bar{y},\bar{z})\bar{z}\,\mathrm{d}A = \sum_{i=1}^{4}(ES_{\bar{y}})_i = 22.72 \times 10^9 \text{ Nmm}.$$

The coordinates of the normal centre NC are found with (9.31):

$$\bar{y}_{NC} = \frac{ES_{\bar{y}}}{EA} = \frac{18.24 \times 10^9 \text{ Nmm}}{624 \times 10^6 \text{ N}} = 29.23 \text{ mm},$$

$$\bar{z}_{NC} = \frac{ES_{\bar{z}}}{EA} = \frac{22.72 \times 10^9 \text{ Nmm}}{624 \times 10^6 \text{ N}} = 36.41 \text{ mm}.$$

Figure 9.28 shows the location of the normal centre NC in the cross-section.

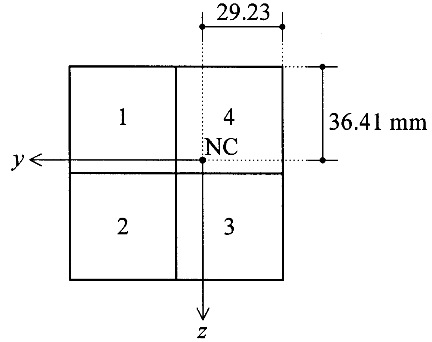


Figure 9.28 The location of the normal centre NC.

[1] For the static moment of each homogeneous part of the cross-section we can apply the formulae derived in Section 3.1.3.

9.8 Stresses due to extension and bending – a straightforward method

In this section a straightforward strategy is presented to find the stresses due to extension and bending. In Section 9.9 this strategy is illustrated by a number of examples.

In Section 9.4 we derived the constitutive equations (9.26) and (9.27):

$$N = EA\varepsilon \qquad \text{(extension)}, \tag{9.26}$$

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \qquad \text{(bending)}. \tag{9.27}$$

Note that these equations are valid only in a yz coordinate system with its origin at the normal centre NC.

When the section forces N, M_y and M_z are known, we can calculate from the constitutive equations the cross-sectional deformation quantities ε, κ_y and κ_z.

In Section 9.2 we derived formula (9.5) for the strain distribution in the cross-section:

$$\varepsilon(y,z) = \varepsilon + y\kappa_y + z\kappa_z, \tag{9.5}$$

in which we can substitute the solutions of (9.26) and (9.27).

From the strain we find the stress according to Hooke's Law:

$$\sigma(y,z) = E(y,z) \cdot \varepsilon(y,z) = E(y,z)\{\varepsilon + y\kappa_y + z\kappa_z\}. \tag{9.33}$$

The strategy is summarised in the scheme of Figure 9.29.

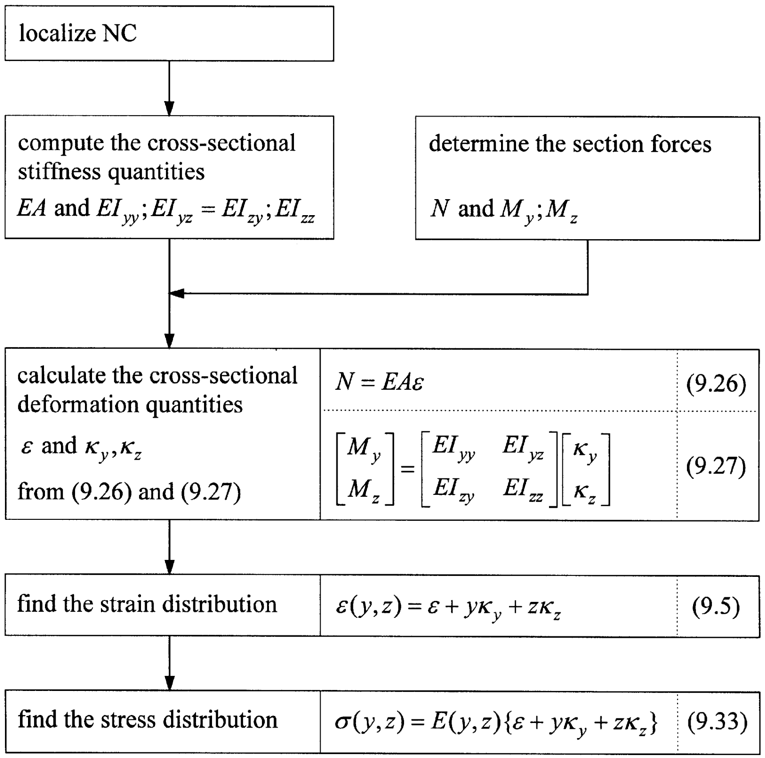


Figure 9.29 The procedure to find the stress distribution in an inhomogeneous cross-section.

In inhomogeneous cross-sections the modulus of elasticity will vary across the cross-section, and the stress distribution will not be similar to the strain distribution. One single expression for the stress distribution can therefore not be found for inhomogeneous cross-sections.

Comment: Since extension and bending are uncoupled in the constitutive equations (9.26) and (9.27), the stresses due to extension and bending are also uncoupled.

9.9 Applications of the straightforward method

The calculation of stresses with the straightforward method is illustrated by some examples: stresses due to extension in Section 9.9.1, due to bending in Section 9.9.2, and due to bending with extension in Section 9.9.3.

9.9.1 Stresses due to extension

The cross-section of a composite member is built up out of four squares of different materials (see Figure 9.30). In the calculation use $E_1 = 30$ GPa, $E_2 = 60$ GPa, $E_3 = 100$ GPa and $E_4 = 200$ GPa.

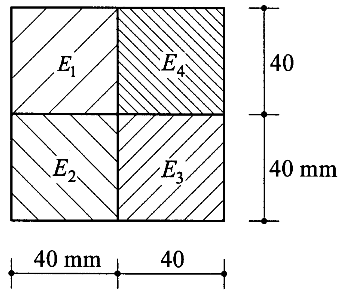


Figure 9.30 The cross-section of a composite member consisting of four squares of different materials glued firmly together.

Question:
Draw the strain and stress diagram due to a normal force $N = 312$ kN.

Solution:
The location of the normal centre NC was calculated in Section 9.7, Example 2, and is given in Figure 9.31a. In the same example the axial stiffness EA of the cross-section was determined:

$$EA = 624 \text{ MN}.$$

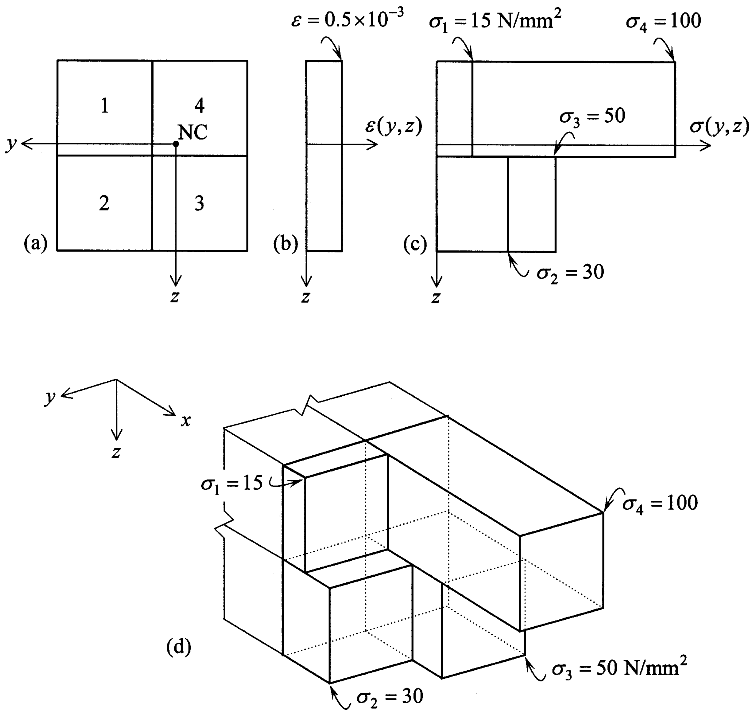


Figure 9.31 (a) The location of the normal centre NC of the composite cross-section. (b) The strain diagram due to a tensile force of 312 kN. The associated stress diagram in (c) projected on the xz plane, and (d) the stress diagram in 3-D.

In case of extension without bending $\kappa_y = \kappa_z = 0$ and, according to (9.5), all fibres have the same strain ε. The value of ε is found from (9.26):

$$\varepsilon(y, z) = \varepsilon = \frac{N}{EA} = \frac{312 \times 10^3 \text{ N}}{624 \times 10^6 \text{ N}} = 0.5 \times 10^{-3} = 0.5‰.$$

Figure 9.31b shows the strain diagram.

The stresses are found with Hooke's Law. Within each homogeneous part of the cross-section the stresses are constant:

part 1: $\sigma_1 = E_1\varepsilon = (30 \times 10^3 \text{ N/mm}^2)(0.5 \times 10^{-3}) = 15 \text{ N/mm}^2$,

part 2: $\sigma_2 = E_2\varepsilon = (60 \times 10^3 \text{ N/mm}^2)(0.5 \times 10^{-3}) = 30 \text{ N/mm}^2$,

part 3: $\sigma_3 = E_3\varepsilon = (100 \times 10^3 \text{ N/mm}^2)(0.5 \times 10^{-3}) = 50 \text{ N/mm}^2$,

part 4: $\sigma_4 = E_4\varepsilon = (200 \times 10^3 \text{ N/mm}^2)(0.5 \times 10^{-3}) = 100 \text{ N/mm}^2$.

In Figure 9.31c the projection of the stress diagram on the xz plane is given. Figure 9.31d shows the stress diagram in 3-D.

Note that the stresses are a maximum/minimum in the material with the largest/smallest stiffness (i.e. modulus of elasticity E).

9.9.2 Stresses due to bending

Here again we use the straightforward method in which all work will be done in the yz coordinate system only. According to the authors this method is to be preferred. However in engineering practice often a different method is being used, in which the coordinate system is chosen in the so called principal directions of the cross-section. This method will be discussed in Section 9.10.

Example 1: Bending stresses in thin-walled homogeneous cross-section
You are given the thin-walled cantilever beam in Figure 9.32. The beam is loaded by a force $F_z = F$ at B. The wall-thickness of the cross-section is $3t$ everywhere. The modulus of elasticity is E.

Questions:
a. Calculate the components of the bending stiffness tensor.
b. Determine the bending moment at the fixed end A.
c. Determine the components of the curvature at the fixed end A.
d. Draw the strain and stress diagram for the cross-section at A.
e. Calculate the extreme bending stresses at A.

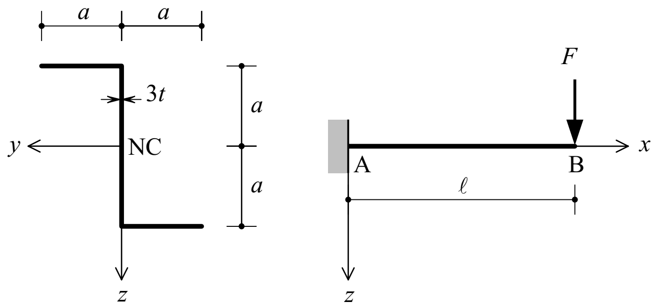


Figure 9.32 A prismatic homogeneous cantilever beam with a thin-walled Z-section, loaded by a vertical force F at the free end B.

Solution:
a. Since the homogeneous cross-section has point symmetry, the normal centre NC coincides with the centre of point symmetry. Here is the origin of the yz coordinate system in which we calculate the components of the bending stiffness tensor (see Figure 9.32):

$$EI_{yy} = E \cdot \tfrac{1}{12} \cdot 3t \cdot (2a)^3 = 2Ea^3t,$$

$$EI_{yz} = EI_{zy} = E \cdot \left\{3at \cdot a \cdot \left(-\tfrac{1}{2}\,a\right) + 3at \cdot (-a) \cdot \tfrac{1}{2}\,a\right\} = -3Ea^3t,$$

$$EI_{zz} = E \cdot \left\{\tfrac{1}{12} \cdot 3t \cdot (2a)^3 + 2 \cdot 3at \cdot a^2\right\} = 8Ea^3t.$$

b. The load acts in the vertical xz plane. So the components of the bending moment at the fixed end A are

$$M_y = 0 \;\text{ and }\; M_z = -F\ell.$$

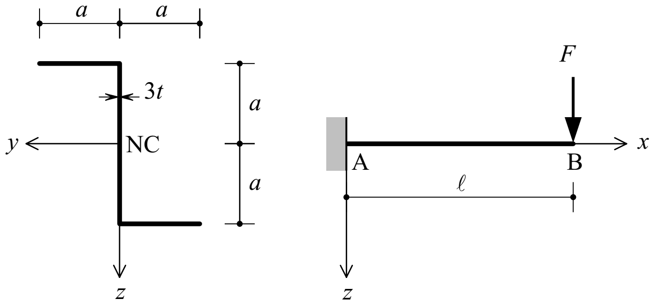


Figure 9.32 A prismatic homogeneous cantilever beam with a thin-walled Z-section, loaded by a vertical force F at the free end B.

c. The constitutive equations for bending are given by (9.27):

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}. \qquad (9.27)$$

Substituting the values calculated, we find

$$\begin{bmatrix} 0 \\ -F\ell \end{bmatrix} = Ea^3t \times \begin{bmatrix} 2 & -3 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}.$$

These are two equations with the two unknowns κ_y and κ_z to solve. The solution can be found by inverting the matrix:

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{1}{7Ea^3t} \times \begin{bmatrix} 8 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -F\ell \end{bmatrix} \Rightarrow \begin{cases} \kappa_y = -\dfrac{3}{7}\dfrac{F\ell}{Ea^3t}, \\ \kappa_z = -\dfrac{2}{7}\dfrac{F\ell}{Ea^3t}. \end{cases} \qquad (9.34)$$

The curvature of the beam at A is

$$\kappa = \sqrt{\kappa_y^2 + \kappa_z^2} = \frac{3F\ell\sqrt{3}}{Ea^3t}.$$

The curvature occurs in a plane k that makes an angle α_k with the y axis:[1]

$$\tan\alpha_k = \frac{\kappa_z}{\kappa_y} = +\frac{2}{3} \Rightarrow \alpha_k = 33.69^\circ + 180^\circ.$$

[1] α_k positive in the direction of a rotation from the y axis to the z axis.

By definition the vector κ points to the area with positive strains (i.e. tensile stresses). The neutral axis *na* is perpendicular to the curvature vector κ and the plane of curvature k. Since there is no extension the neutral axis passes through the normal centre NC. Figure 9.33 shows the plane of loading m, the plane of curvature k, and the neutral axis *na*.

d. The strain distribution in the cross-section follows from

$$\varepsilon(y, z) = \varepsilon + \kappa_y y + \kappa_z z.$$

Substituting $\varepsilon = 0$ (there is no extension), and the values of κ_y and κ_z found before, we find

$$\varepsilon(y, z) = -\frac{F\ell}{7Ea^3t}(3y + 2z). \tag{9.35}$$

Applying Hooke's Law, we find the stress distribution from the strain distribution:

$$\sigma(y, z) = E\varepsilon(y, z) = -\frac{F\ell}{7a^3t}(3y + 2z). \tag{9.36}$$

The stresses and strains are proportional with the distance to the neutral axis. Therefore in a diagram the stresses and strains are plotted on a line perpendicular to the neutral axis (see Figure 9.33). In a homogeneous cross-section, as here, there is a similarity in shape between the stress and strain diagram.

e. The strains are extreme at the points with the largest distance to the neutral axis, that is in Q (maximum positive strain) and R (maximum negative strain). Since the cross-section is homogeneous, the stresses are extreme at the same points Q (maximum tensile stress) and R (maximum compressive

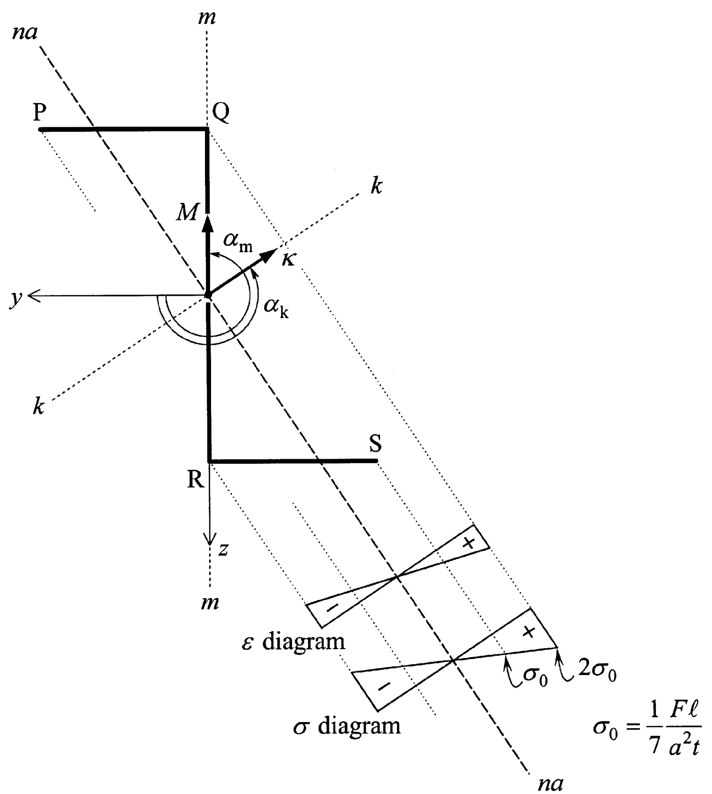


Figure 9.33 For the cross-section at A: the plane of loading m, the plane of curvature k, the neutral axis *na*, the strain (ε) diagram, and the stress (σ) diagram. Since the stresses and strains are proportional with the distance to the neutral axis *na*, they are plotted on a line perpendicular to the neutral axis *na*.

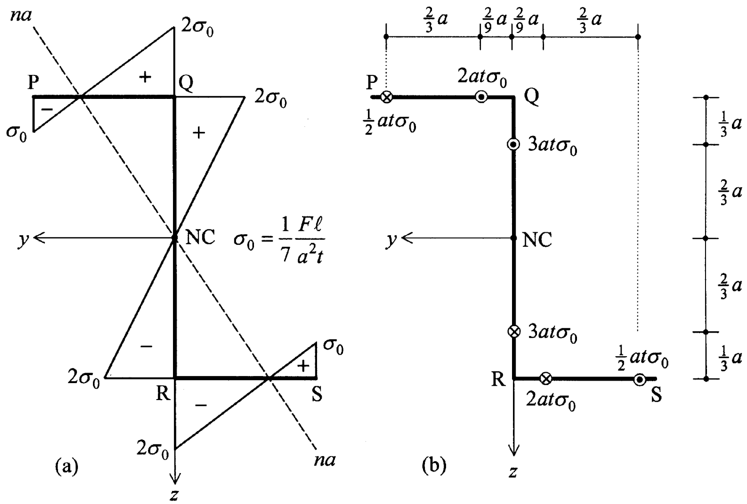


Figure 9.34 (a) The normal stresses plotted along the centre line of the thin-walled Z-section. (b) The magnitudes and points of application of the tensile and compressive stress resultants (forces) in the cross-section.

Table 9.2

	y	z	$\sigma(y,z)$	
P	$+a$	$-a$	$-\sigma_0$	
Q	0	$-a$	$+2\sigma_0$	← maximum tensile stress
R	0	$+a$	$-2\sigma_0$	← maximum compressive stress
S	$-a$	$+a$	$+\sigma_0$	

stress). The stresses at P, Q, R and S are calculated with formula (9.36); their values are given in Table 9.2 and expressed in terms of a reference value σ_0:

$$\sigma_0 = \frac{1}{7}\frac{F\ell}{a^2 t}. \tag{9.37}$$

The stress distribution in the cross-section is linear. For a thin-walled cross-section, it may be assumed that the stress across the wall thickness is constant. So the stress varies only along the wall. The stress distribution along the wall, for example upper flange AB, can be determined by plotting the stresses at A and B normal to AB, and drawing a straight line between these points. In the same way, the stress distribution along web and lower flange can be plotted. The result is given in Figure 9.34a.

Note that the bending moment, acting in the vertical xz plane, causes both tensile and compressive stresses in the flanges.

From the stress diagram in Figure 9.34a we see three points with a zero stress. These points are located on the neutral axis *na*. The neutral axis is a straight line. If the stress diagram is drawn to scale, these three points have to be on a straight line (i.e. the neutral axis) which must pass through the normal centre NC (since there is no normal force). The latter can be regarded as a check.

Another check is to verify the bending moments due to the stress diagram in Figure 9.34a. Figure 9.34b shows the magnitudes and points of application of the tensile and compressive stress resultants in the cross-section. Using expression (9.37) for the reference value σ_0, we find

$$M_y = \left\{-\tfrac{1}{2}at\sigma_0 \cdot \left(\tfrac{2}{3}a + \tfrac{2}{9}a\right) + 2at\sigma_0 \cdot \tfrac{2}{9}a\right\} \times 2 = 0,$$

$$M_z = \left\{+\tfrac{1}{2}at\sigma_0 \cdot a - 2at\sigma_0 \cdot a - 3at\sigma_0 \cdot \tfrac{2}{3}a\right\} \times 2 = -7a^2 t\sigma_0 = -F\ell.$$

These are indeed the components of the bending moment for which we calculated the stress distribution.

Example 2: Bending stresses in an inhomogeneous cross-sections
The cantilever beam in Figure 9.35, with an inhomogeneous cross-section, is constructed of three parts, numbered by 1 to 3. The parts are firmly glued together. Different materials are used for flanges and web. The beam is loaded by a force of 250 N at B. The moduli of elasticity are $E_1 = E_3 = 12000\ \text{N/mm}^2$ and $E_2 = 6000\ \text{N/mm}^2$.

Questions:
a. Calculate the components of the bending stiffness tensor.
b. Determine the bending moment at the fixed end A.
c. Determine the components of the curvature at the fixed end A.
d. Draw the strain and stress diagram for the cross-section at A.
e. Calculate the extreme bending stresses at A.

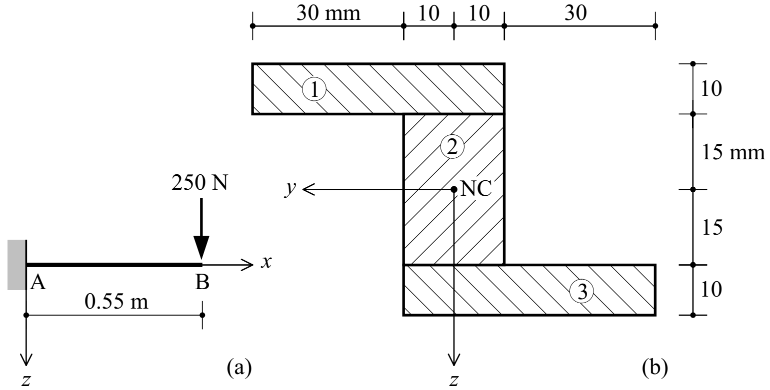


Figure 9.35 (a) Cantilever beam with (b) inhomogeneous cross-section.

Solution (units in N and mm):
a. We will use the solution technique given in the scheme of Figure 9.29. First we have to find the normal centre NC. Since $E_1 = E_3$, the given cross-section has *point symmetry* with respect to both its geometry and elastic material properties (i.e. moduli of elasticity). The centre of point symmetry coincides with the centroid of part 1, so here is the normal centre NC of the cross-section.

The components of the bending stiffness tensor, EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz} are defined in the yz coordinate system with its origin at NC. These sectional bending stiffness quantities represented by *double letter symbols* can be found using the fact that the cross-section is composed of three homogeneous parts of simple geometry for which the local centroids are known (see Figure 9.36).

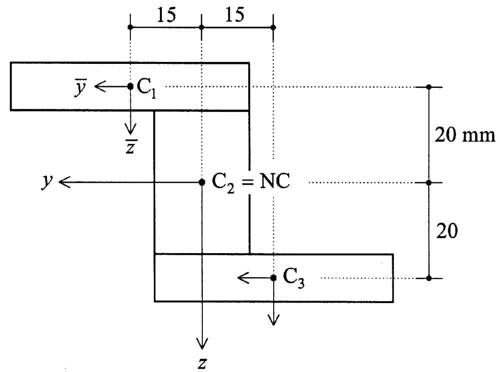


Figure 9.36 Centroids of the homogeneous parts of the inhomogeneous cross-section.

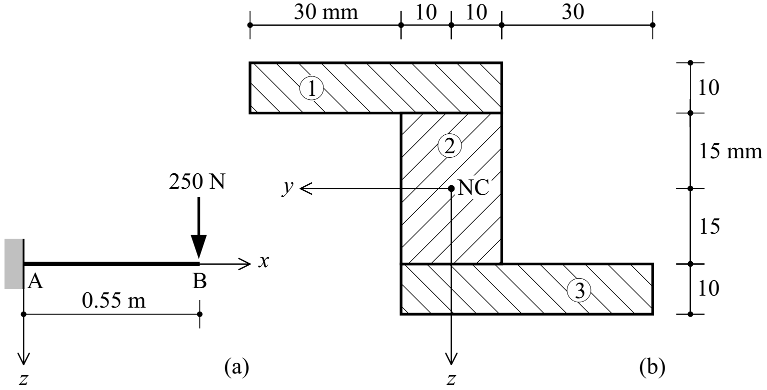


Figure 9.35 (a) Cantilever beam with (b) inhomogeneous cross-section.

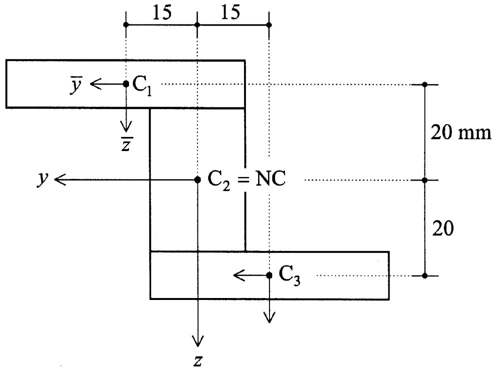


Figure 9.36 Centroids of the homogeneous parts of the inhomogeneous cross-section.

For component EI_{yy} applies

$$EI_{yy} = \sum_{i=1}^{3} (EI_{yy})_i = \sum_{i=1}^{3} \{(EI_{\overline{yy}})_i + (y_C^2)_i (EA)_i)\}.$$

$(EI_{yy})_i$ is the contribution of homogeneous part i to the bending stiffness component EI_{yy} in the central yz coordinate system.

$(EI_{\overline{yy}})_i$ is the product of modulus of elasticity E and centroidal moment of inertia I_{yy} of homogeneous part i of the cross-section in a local centroidal $\overline{yz}$ coordinate system.

$(y_C)_i$ is the y coordinate of the centroid C of homogeneous part i.

$(EA)_i$ is the product of modulus of elasticity E and area A of homogeneous part i.

In the same way

$$EI_{yz} = EI_{zy} = \sum_{i=1}^{3} (EI_{yz})_i = \sum_{i} \{\underbrace{(EI_{\overline{yz}})_i}_{0} + (y_C z_C)_i (EA)_i)\},$$

$$EI_{zz} = \sum_{i=1}^{3} (EI_{zz})_i = \sum_{i=1}^{3} \{(EI_{\overline{zz}})_i + (z_C^2)_i (EA)_i\}.$$

Note that in determining the components of the bending stiffness tensor we use the parallel axis theorem for each of the homogeneous parts.

Comment: For each rectangular homogeneous part in its local centroidal $\overline{yz}$ coordinate system $EI_{\overline{yz}} = EI_{\overline{zy}} = 0$.

It is assumed that the reader is familiar with the calculation of the geometrical properties of a homogeneous cross-section, according to Chapter 3. For instance (measures in mm)

$$EI_{yy} = +E_1\Big(\underbrace{\tfrac{1}{12}\times 10\times 50^3}_{(I_{\overline{yy}})_1} + \underbrace{10\times 50\times 15^2}_{\text{par axis theorem}}\Big) + E_2\underbrace{\tfrac{1}{12}\times 30\times 20^3}_{(I_{\overline{yy}})_2}$$

$$+\,E_3\Big(\underbrace{\tfrac{1}{12}\times 10\times 50^3}_{(I_{\overline{yy}})_3} + \underbrace{10\times 50\times 15^2}_{\text{par axis theorem}}\Big).$$

The ultimate calculation of EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz} is presented in two tables. In Table 9.3 the weighted area and the weighted local centroidal moments of inertia for the homogeneous parts of the cross-section are determined. Since $(EI_{\overline{yz}})_i = (EI_{\overline{zy}})_i = 0$, these values are left out of the table. In Table 9.4 the effect of the parallel axis theorem is calculated. Using a spread sheet may speed up the calculus.

From Tables 9.3 and 9.4 we find

$$EI_{yy} = (2.62\times 10^9\ \text{Nmm}^2) + (2.7\times 10^9\ \text{Nmm}^2) = 5.32\times 10^9\ \text{Nmm}^2,$$

$$EI_{yz} = EI_{zy} = -3.6\times 10^9\ \text{Nmm}^2,$$

$$EI_{zz} = (0.37\times 10^9\ \text{Nmm}^2) + (4.8\times 10^9\ \text{Nmm}^2) = 5.17\times 10^9\ \text{Nmm}^2.$$

b. The load acts in the xz plane. The components of the bending moment M at A are (see Figure 9.37a)

$$M_y = 0 \ \text{ and } \ M_z = -(250\ \text{N})(0.55\ \text{m}) = -137.5\times 10^3\ \text{Nmm}.$$

The bending moment vector M makes an angle $\alpha_m = 270°$ with the y axis.[1]

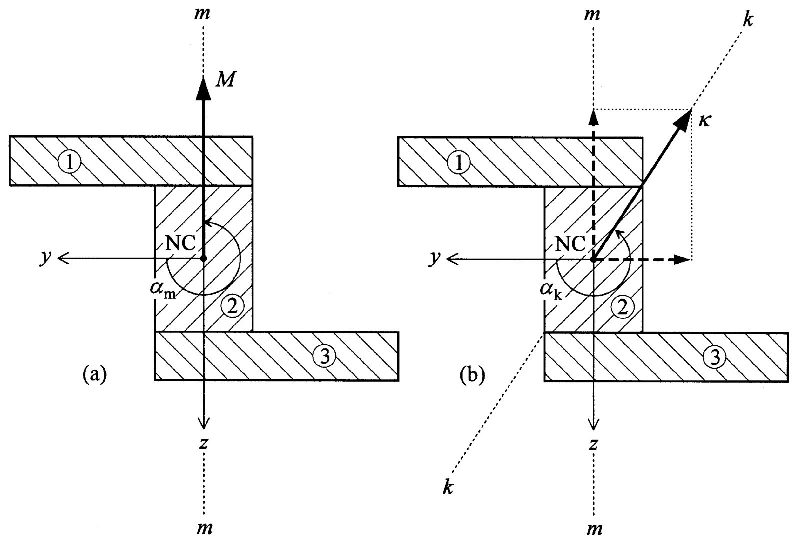


Figure 9.37 (a) Bending moment M and plane of loading m; (b) curvature κ and plane of curvature k.

[1] α_m is positive in the direction from the y axis to the z axis.

Table 9.3 Calculation of the weighted areas and weighted local centroidal moments of inertia for the three homogeneous parts of the cross-section.

part i	E_i (N/mm^2)	A_i (mm^2)	$(I_{\overline{yy}})_i$ (mm^4)	$(I_{\overline{zz}})_i$ (mm^4)	$(EA)_i$ (N)	$(EI_{\overline{yy}})_i$ (Nmm^2)	$(EI_{\overline{zz}})_i$ (Nmm^2)
1	12×10^3	500	104.167×10^3	4.167×10^3	6×10^6	1.25×10^9	0.05×10^9
2	6×10^3	600	20×10^3	45×10^3	3.6×10^6	0.12×10^9	0.27×10^9
3	12×10^3	500	104.167×10^3	4.167×10^3	6×10^6	1.25×10^9	0.05×10^9
				$\sum$	1.5×10^6	2.62×10^9	0.37×10^9

Table 9.4 Calculation of the contributions due to the parallel axis theorem.

part i	$(EA)_i$ (N)	$(y_C)_i$ (mm)	$(z_C)_i$ (mm)	$(y_C^2)_i(EA)_i$ (Nmm^2)	$(y_C z_C)_i(EA)_i$ (Nmm^2)	$(z_C^2)_i(EA)_i$ (Nmm^2)
1	6×10^6	+15	−20	1.35×10^9	-1.8×10^9	2.4×10^9
2	3.6×10^6	0	0	0	0	0
3	6×10^6	−15	−20	1.35×10^9	-1.8×10^9	2.4×10^9
			$\sum$	2.7×10^9	-3.6×10^9	4.8×10^9

c. The curvatures can be obtained from the constitutive equations (9.27):

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 \\ -0.137.5 \times 10^3 \end{bmatrix} = 10^9 \times \begin{bmatrix} 5.32 & -3.6 \\ -3.6 & 5.17 \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}$$

$$\Rightarrow \begin{cases} \kappa_y = -34.03 \times 10^{-6}\ \text{mm}^{-1}, \\ \kappa_z = -50.29 \times 10^{-6}\ \text{mm}^{-1}. \end{cases}$$

The curvature of the beam at A is

$$\kappa = \sqrt{\kappa_y^2 + \kappa_z^2} = 60.72 \times 10^{-6}\ \text{mm}^{-1}.$$

The curvature occurs in a plane k that makes an angle α_k with the y axis:[1]

Figure 9.37b shows the vector κ and the plane of curvature k. Also the plane of loading m is shown. We have to remember that vector κ by definition points to the area with positive strains.

d. The strain distribution over the cross-section is given by (9.5):

$$\varepsilon(y, z) = \varepsilon + y\kappa_y + z\kappa_z,$$

in which

$$\varepsilon = 0,\quad \kappa_y = -34.03 \times 10^{-6}\ \text{mm}^{-1} \ \text{and}\ \kappa_z = -50.29 \times 10^{-6}\ \text{mm}^{-1}.$$

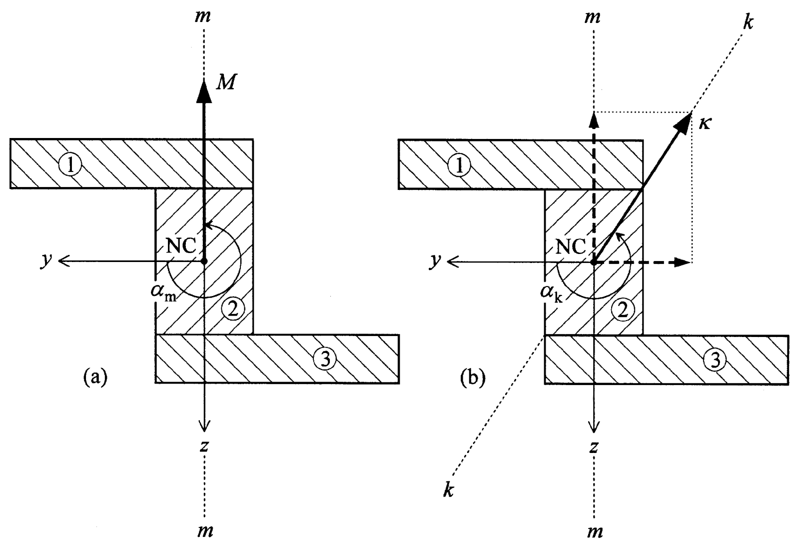


Figure 9.37 (a) Bending moment M and plane of loading m; (b) curvature κ and plane of curvature k.

[1] The vector κ points to the area with positive strains. α_k is positive in the direction of a rotation from the y axis to the z axis.

Table 9.5 Calculation of the strain ε and stress σ at key points in the cross-section.

part i	**point**	y **(mm)**	z **(mm)**	E_i **(N/mm^2)**	**Strain** ε **($\times 10^{-3}$)**	**Stress** σ **(N/mm^2)**	
1	O	+40	−25	12×10^3	−0.10	−1.25	
	P	−10	−25	12×10^3	+1.60	+19.17	
	Q	+40	−15	12×10^3	−0.61	−7.28	
	S	−10	−15	12×10^3	+1.09	**+13.14**	←
2	R	+10	−15	6×10^3	+0.41	+2.48	
	S	−10	−15	6×10^3	+1.09	**+6.57**	←
	T	+10	+15	6×10^3	−1.09	−6.57	
	U	−10	+15	6×10^3	−0.41	−2.48	
3	T	+10	+15	12×10^3	−1.09	−13.14	
	V	−40	+15	12×10^3	+0.61	+7.28	
	W	+10	+25	12×10^3	−1.60	−19.17	
	X	−40	+25	12×10^3	+0.10	1.25	

e. The stress at any point can be found by multiplying the strain of the considered fibre with its corresponding modulus of elasticity:

$$\sigma(y, z) = E(y, z) \times \varepsilon(y, z).$$

The calculation of the strain and stress at key points in the cross-section is given in Table 9.5 (see Figure 9.38).

At the four points R, S, T and U two different stresses are possible, depending on the material considered. The consequence of the difference in modulus of elasticity will be a step in the stress distribution at these points. For point S this is shown in bold in the last column of Table 9.5.

The strain and stress distribution can be presented graphically in diagrams. To draw these diagrams it is important to know the position of the *neutral axis*. The expression for the neutral axis is found from

$$\varepsilon(y, z) = \varepsilon + y\kappa_y + z\kappa_z = 0.$$

Substituting

$$\varepsilon = 0, \quad \kappa_y = -34.03 \times 10^{-6} \text{ mm}^{-1} \quad \text{and} \quad \kappa_z = -50.29 \times 10^{-6} \text{ mm}^{-1}$$

results in

$$-34.03 \times 10^{-6} y - 50.29 \times 10^{6} z = 0$$

or

$$-34.03y - 50.29z = 0$$

in which y and z have to be expressed in mm.

In Figure 9.38 the neutral axis *na* is drawn. The neutral axis is a straight line perpendicular to the plane of curvature k. Since there is no extension, the strain at the normal centre NC is zero ($\varepsilon = 0$), and the neutral axis passes through NC.

Figure 9.38 shows also the strain and stress diagram.

Since the strains are proportional to the distance to the neutral axis, it is usual to plot their values along a line perpendicular to the neutral axis. The strain diagram is always represented by a straight line.

Per homogeneous part of the cross-section also the stresses are proportional with the distance to the neutral axis. Therefore the stresses are also plotted along a line perpendicular to the neutral axis. To avoid any ambiguity the stress distribution is plotted outside the cross-section in separate drawings for the two materials.

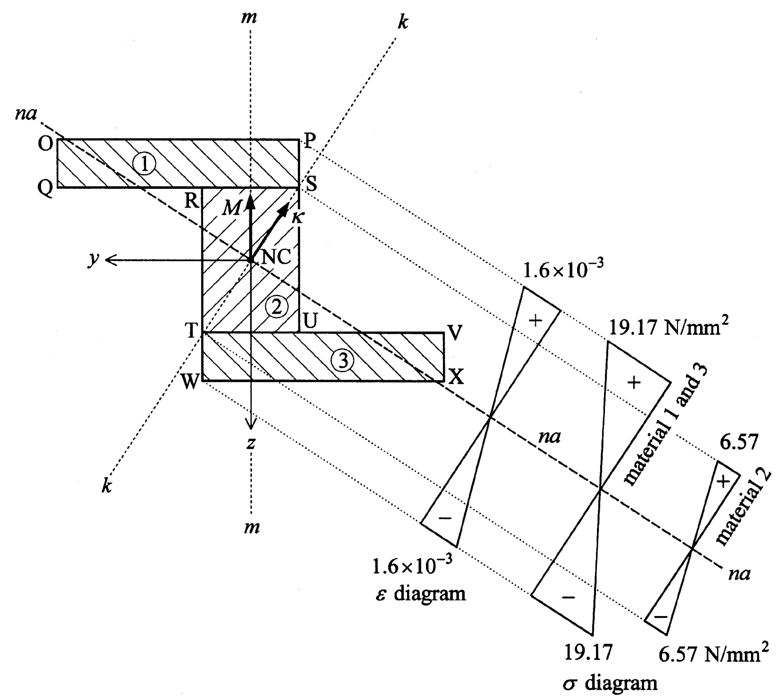


Figure 9.38 The strain and stress diagrams, plotted on a line perpendicular to the neutral axis *na*.

9.9.3 Stresses due to bending and extension

Figure 9.39a shows the cross-section of a composite member, built up out of four squares of different materials. The cross-section transfers an eccentric tensile force $N = 139$ kN for which the centre of force *cf* coincides with the point where the four materials meet.

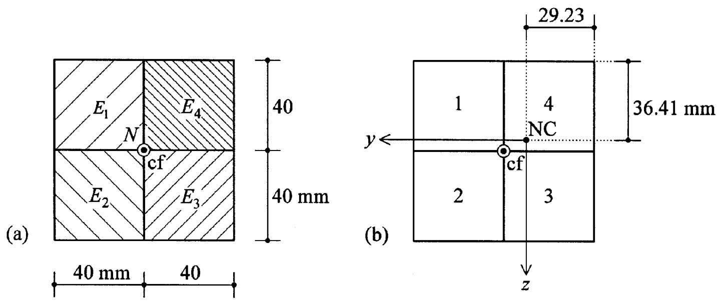


Figure 9.39 (a) The cross-section of a composite member, built up out of four squares of different materials. The cross-section transfers an eccentric tensile force for which the centre of force *cf* coincides with the point where the four materials meet. (b) Location of the normal centre NC of the cross-section.

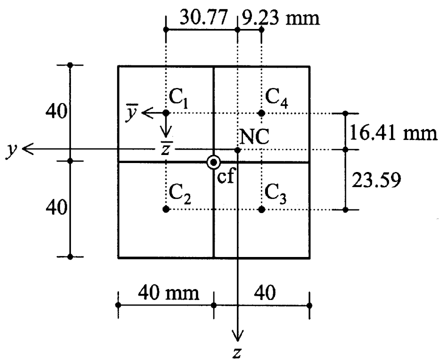


Figure 9.40 The weighted area and the weighted local centroidal moments of inertia of the homogeneous parts of the cross-section are determined with the help of this figure.

In the calculation use $E_1 = 30$ GPa, $E_2 = 60$ GPa, $E_3 = 100$ GPa and $E_4 = 200$ GPa. The axial stiffness of the cross-section is $EA = 624$ MN. The location of the normal centre NC is given in Figure 9.39b.[1]

Question:
Plot the associated strain and stress diagram.

Solution:
We start to calculate the components of the bending stiffness tensor:

$$EI_{yy} = \sum_{i=1}^{4} (EI_{yy})_i = \sum_{i=1}^{4} \{(EI_{\overline{yy}})_i + (y_C^2)_i (EA)_i\},$$

$$EI_{yz} = EI_{zy} = \sum_{i=1}^{4} (EI_{yz})_i = \sum_{i=1}^{4} \{\underbrace{(EI_{\overline{yz}})_i}_{0} + (y_C z_C)_i (EA)_i\},$$

$$EI_{zz} = \sum_{i=1}^{4} (EI_{zz})_i = \sum_{i=1}^{4} \{(EI_{\overline{zz}})_i + (z_C^2)_i (EA)_i\}.$$

Note that in determining the bending stiffness components we use the parallel axis theorem for each of the homogeneous parts.

Comment: For each square homogeneous part in the local centroidal $\overline{yz}$ coordinate system $EI_{\overline{yz}} = EI_{\overline{zy}} = 0$.

The ultimate calculation of EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz} is presented in two tables. In Table 9.6 the weighted areas and the weighted local centroidal moments of inertia for the homogeneous parts of the cross-section are determined (see also Figure 9.40). Since $(EI_{\overline{yz}})_i = (EI_{\overline{zy}})_i = 0$, these values are left out of the table. In Table 9.7 the contributions due to the parallel axis theorem are calculated.

---

[1] The location of the normal centre NC and the value of the axial stiffness *EA* were determined earlier in Section 9.7, Example 2.

Table 9.6 Calculation of the weighted areas and weighted local centroidal moments of inertia for the four homogeneous parts of the cross-section.

part i	E_i (N/mm^2)	A_i (mm^2)	$(I_{\overline{yy}})_i = (I_{\overline{zz}})_i$ (mm^4)	$(EA)_i$ (N)	$(EI_{\overline{yy}})_i = (EI_{\overline{zz}})_i$ (Nmm^2)
1	30×10^3	1.6×10^3	213.33×10^3	48×10^6	6.40×10^9
2	60×10^3	1.6×10^3	213.33×10^3	96×10^6	12.80×10^9
3	100×10^3	1.6×10^3	213.33×10^3	160×10^6	21.33×10^9
4	200×10^3	1.6×10^3	213.33×10^3	320×10^6	42.67×10^9
			$\sum$	624×10^6	83.20×10^9

Table 9.7 Calculation of the contributions due to the parallel axis theorem.

part i	$(EA)_i$ (N)	$(y_C)_i$ (mm)	$(z_C)_i$ (mm)	$(y_C^2)_i(EA)_i$ (Nmm^2)	$(y_C z_C)_i(EA)_i$ (Nmm^2)	$(z_C^2)_i(EA)_i$ (Nmm^2)
1	48×10^6	$+30.77$	-16.41	45.45×10^9	-24.24×10^9	12.93×10^9
2	96×10^6	$+30.77$	$+23.59$	90.89×10^9	$+69.68 \times 10^9$	53.42×10^9
3	160×10^6	-9.23	$+23.59$	13.63×10^9	-34.84×10^9	89.04×10^9
4	320×10^6	-9.23	-16.41	27.26×10^9	$+48.47 \times 10^9$	86.17×10^9
			$\sum$	177.23×10^9	$+59.07 \times 10^9$	241.56×10^9

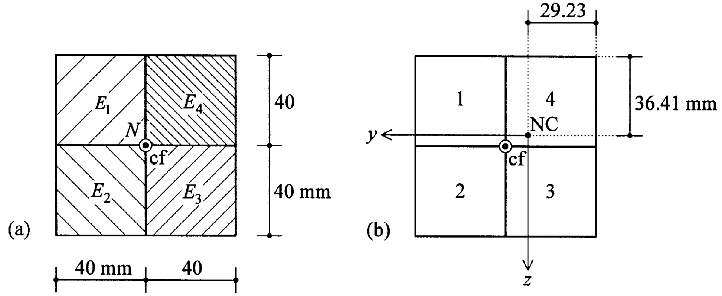


Figure 9.39 (a) The cross-section of a composite member, built up out of four squares of different materials. The cross-section transfers an eccentric tensile force for which the centre of force *cf* coincides with the point where the four materials meet. (b) Location of the normal centre NC of the cross-section.

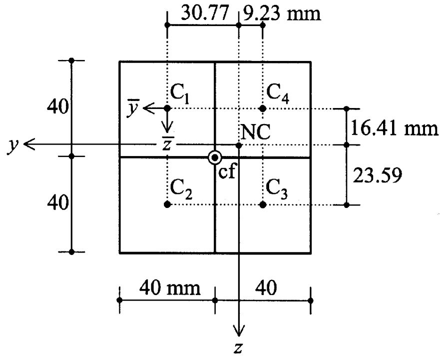


Figure 9.40 The weighted area and the weighted local centroidal moments of inertia of the homogeneous parts of the cross-section are determined with the help of this figure.

From Tables 9.6 and 9.7 we find

$$EI_{yy} = (83.20 \times 10^9 \text{ Nmm}^2) + (177.23 \times 10^9 \text{ Nmm}^2)$$

$$= 260.43 \times 10^9 \text{ Nmm}^2,$$

$$EI_{yz} = EI_{zz} = 59.07 \times 10^9 \text{ Nmm}^2,$$

$$EI_{zz} = (83.20 \times 10^9 \text{ Nmm}^2) + (241.56 \times 10^9 \text{ Nmm}^2)$$

$$= 324.76 \times 10^9 \text{ Nmm}^2.$$

The coordinates of the centre of force *cf* are (see Figure 9.40)

$$e_y = +10.77 \text{ mm},$$

$$e_z = +3.59 \text{ mm}.$$

The centre of force is the point of application of the tensile force $N = 139$ kN the cross-section has to transfer. Due to the eccentricity of the tensile force the bending moments are

$$M_y = Ne_y = (139 \times 10^3 \text{ N})(10.77 \text{ mm}) = 1.497 \times 10^6 \text{ Nmm},$$

$$M_z = Ne_z = (139 \times 10^3 \text{ N})(3.59 \text{ mm}) = 0.499 \times 10^6 \text{ Nmm}.$$

The plane of loading m, the plane in which the bending moment acts, is defined by the angle α_m (see Figure 9.41):

$$\tan \alpha_m = \frac{M_z}{M_y} = \frac{e_z}{e_y} = \frac{3.59}{10.77} = \frac{1}{3} \Rightarrow \alpha_m = 18.43°.$$

The strain distribution is given by (9.5)

$$\varepsilon(y, z) = \varepsilon + y\kappa_y + z\kappa_z,$$

in which the strain ε and curvatures κ_y and κ_z can be found from the constitutive equations (9.26) and (9.27):

$$N = EA\varepsilon \qquad \text{(extension)}, \tag{9.26}$$

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \qquad \text{(bending)}. \tag{9.27}$$

Substitute all known numerical values in (9.26):

$$(139 \times 10^3 \text{ N}) = (624 \times 10^6 \text{ N}) \times \varepsilon, \tag{9.38}$$

and in (9.27):

$$\begin{bmatrix} 1.497 \\ 0.499 \end{bmatrix} \times (10^6 \text{ Nmm}) = (10^9 \text{ Nmm}^2) \times \begin{bmatrix} 260.43 & 59.07 \\ 59.07 & 324.76 \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}. \tag{9.39}$$

From (9.38) we find the strain ε:

$$\varepsilon = \frac{139 \times 10^3 \text{ N}}{624 \times 10^6 \text{ N}} = 222.756 \times 10^{-6}.$$

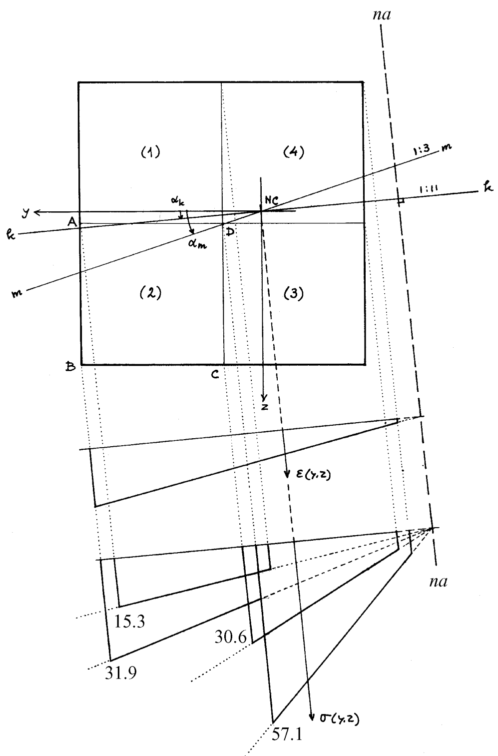


Figure 9.41 The strain diagram and stress diagram. The strain diagram is linear. The stress diagram is more complicated, but linear per homogeneous part of the cross-section. To find the maximum stress, not only the distance to the neutral axis (the magnitude of the strain) is relevant but also the modulus of elasticity of the related fibre.

κ_y and κ_z can be found by inverting (9.39):

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{10^{-3}\ \text{mm}^{-1}}{81088} \times \begin{bmatrix} 324.76 & -59.07 \\ -59.07 & 260.43 \end{bmatrix} \begin{bmatrix} 1.497 \\ 0.499 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \kappa_y = 5.632 \times 10^{-6}\ \text{mm}^{-1}, \\ \kappa_z = 0.512 \times 10^{-6}\ \text{mm}^{-1}. \end{cases}$$

The direction of the curvature κ and the plane of curvature k are defined by the angle α_k:

$$\tan\alpha_k = \frac{\kappa_z}{\kappa_y} = \frac{0.512 \times 10^{-6}\ \text{mm}^{-1}}{5.632 \times 10^{-6}\ \text{mm}^{-1}} = 0.0909 = \frac{1}{11} \Rightarrow \alpha_k = 5.19°.$$

Now the stress distribution can be written as

$$\varepsilon(y, z) = (222.756 + 5.632 \times y + 0.512 \times z) \times 10^{-6},$$

in which y and z have to be expressed in mm.

The equation of the neutral axis *na* is

$$222.761 + 5.632 \times y + 0.512 \times z = 0.$$

The neutral axis intersects the y and z axis in y_1 and z_1 respectively:

$$y_1 = \frac{-222.756\ \text{mm}}{5.632} = -39.55\ \text{mm}$$

and

$$z_1 = \frac{-222.756 \text{ mm}}{0.512} = -435.02 \text{ mm}.$$

The neutral axis is outside the cross-section.

Figure 9.41 shows sketches of the strain diagram and stress diagram. The strain diagram is linear. The stress diagram is more complicated, but linear per homogeneous part of the cross-section. To find the maximum stress, not only the distance to the neutral axis (the magnitude of the strain) is relevant but also the modulus of elasticity of the related fibre.

For each material (homogeneous part) the maximum stress is computed in Table 9.8. The points where these maximum stresses occur are given in column two.

Comment: Since the neutral axis is outside the cross-section the entire cross-section is subject to the same sign of strain, e.g. a positive strain. Therefore the entire cross-section is in tension.

Table 9.8 Calculation of the maximum stress per homogeneous part of the cross-section.

part i	point j	y_j (mm)	z_j (mm)	$\varepsilon(y_j, z_j)$ ($\times 10^{-6}$)	E_i (N/mm^2)	$\sigma(y_j, z_j)$ (N/mm^2)
1	A	50.77	3.59	510.5	30×10^3	15.3
2	B	50.77	43.59	531.0	$60\times^3$	31.9
3	C	10.77	43.59	305.7	100×10^3	30.6
4	D	10.77	3.59	285.3	200×10^3	57.1

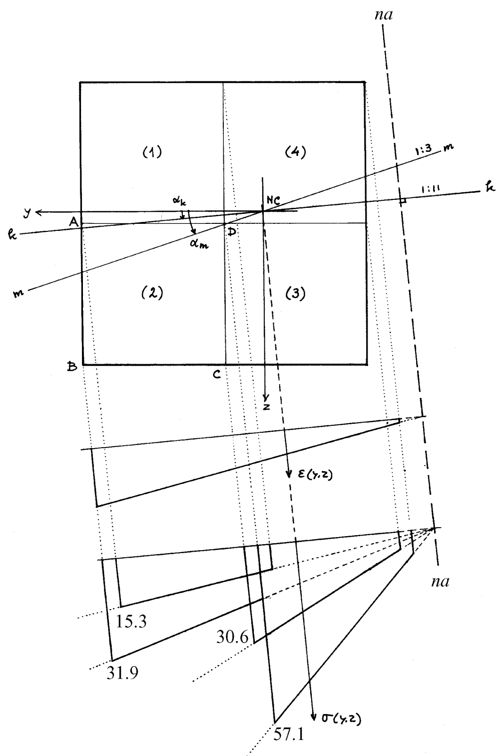


Figure 9.41 The strain diagram and stress diagram. The strain diagram is linear. The stress diagram is more complicated, but linear per homogeneous part of the cross-section. To find the maximum stress, not only the distance to the neutral axis (the magnitude of the strain) is relevant but also the modulus of elasticity of the related fibre.

9.10 Stresses in the principal coordinate system – alternative method

In the previous section the bending stresses in the cross-section were calculated for an arbitrary yz coordinate system. The disadvantage of this method is the coupling between the bending components. The advantage is the straightforward method. According to the authors this method is to be preferred. However in engineering practice the calculation is often based on the principal coordinate system.

If the yz coordinate system is chosen according to the principal directions of the cross-section, then the non-diagonal terms $EI_{yz} = EI_{zy}$ are zero by definition. So the constitutive relations for bending in the principal directions y and z are uncoupled:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & 0 \\ 0 & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}. \tag{9.40}$$

For the components of the curvature we find

$$\kappa_y = \frac{M_y}{EI_{yy}}, \tag{9.41a}$$

$$\kappa_z = \frac{M_z}{EI_{zz}}. \tag{9.41b}$$

The strain distribution in a cross-section subject to combined bending and extension becomes

$$\begin{aligned} \varepsilon(y, z) &= \varepsilon + \kappa_y y + \kappa_z z \\ &= \frac{N}{EA} + \frac{M_y y}{EI_{yy}} + \frac{M_z z}{EI_{zz}}. \end{aligned} \tag{9.42}$$

The stress distribution is found by

$$\begin{aligned} \varepsilon(y,z) &= E(y,z)\cdot\varepsilon(y,z) \\ &= E(y,z)\cdot\left(\frac{N}{EA}+\frac{M_y y}{EI_{yy}}+\frac{M_z z}{EI_{zz}}\right). \end{aligned} \tag{9.43}$$

For homogeneous cross-sections the modulus of elasticity E is independent of the coordinates y and z, so (9.43) simplifies into

$$\sigma(y,z) = E\cdot\left(\frac{N}{EA}+\frac{M_y y}{EI_{yy}}+\frac{M_z z}{EI_{zz}}\right) = \frac{N}{A}+\frac{M_y y}{I_{yy}}+\frac{M_z z}{I_{zz}}. \tag{9.44}$$

Note that in a homogeneous cross-section, the stress distribution due to combined bending and extension is independent of the modulus of elasticity.

The advantage of the very simple and easy to memorise formulae (9.43) and (9.44) is however small. Additional work has to be done since all quantities used have to be referred to the principal coordinate system, as shown by the scheme in Figure 9.42. The additional work consists of computing the principal directions and the principal bending stiffness values and calculating the components of the bending moment in the principal directions. The strain and stress distribution are found in the principal coordinate system and, if needed, have to be transformed to the initially given coordinate system.

In Section 9.11 the transformation rules will be discussed for the components of a vector or first-order tensor (the bending moment M and curvature κ) and of a second-order tensor (the bending stiffness tensor EI). Examples of calculating the stresses in a principal coordinate system are given in Section 9.12.

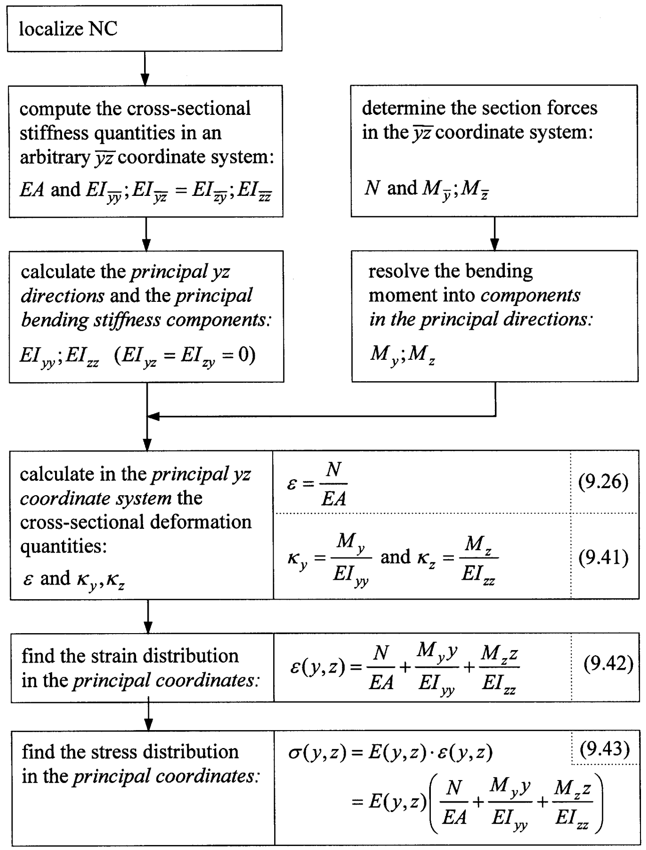


Figure 9.42 Procedure to find the stress distribution in a principal coordinate system.

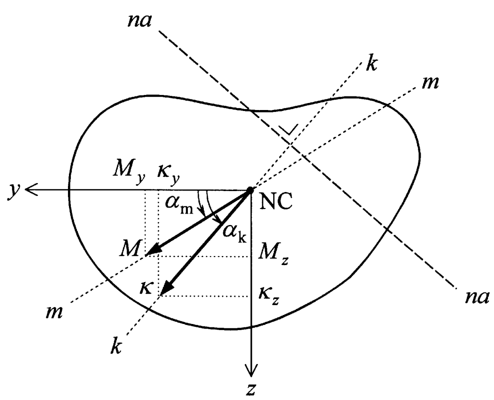


Figure 9.43 Both the bending moment M and the curvature κ are *vectors* or *first-order tensors*. They can be represented by single pointed arrows. m is the plane in which the bending moment M acts, also called the *plane of loading*, and k is the *plane of curvature*. The vectors M and κ are linked by a *second-order tensor*, the bending stiffness tensor EI.

9.11 Transformation formulae for the bending stiffness tensor

The constitutive relationship for bending is given by (9.27):

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}, \qquad (9.27)$$

or in matrix notation

$$\{M\} = [EI]\{\kappa\}.$$

Both the bending moment M and the curvature κ are *vectors* or *first-order tensors*. They can be represented by the arrows in Figure 9.43. Here m is the plane in which the bending moment acts, also called the *plane of loading*, and k is the *plane of curvature*. The vectors M and κ are linked by the bending stiffness matrix which is a *second-order tensor*.

The characteristics of a first-order tensor and second-order tensor are explained in Sections 9.11.1 and 9.11.2.

In Section 9.11.3 we look for the so-called *principal values* and *principal directions* with respect to the bending stiffness of the cross-section, and investigate when the plane of loading m coincides with the plane of curvature k.

The principal directions and principal values of the bending stiffness tensor can also be found with a simple *graphical method*. This method is based on what is called *Mohr's circle* and is presented in Section 9.11.4. With Mohr's circle we can easily find the components of the bending stiffness tensor in any arbitrary coordinate system.

9.11.1 First-order tensor

A *vector* or *first-order tensor* is characterised by[1]

- its magnitude,
- its direction, and
- the transformation rules with respect to its components when the co-ordinate system is rotated.

When the yz coordinate system is transformed by rotation into the $\overline{yz}$ coordinate system (see Figure 9.44), the components of both the bending moment M and curvature κ change according to the same transformation rules:[2]

$$\begin{bmatrix} M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} M_y \\ M_z \end{bmatrix}$$

and

$$\begin{bmatrix} \kappa_{\bar{y}} \\ \kappa_{\bar{z}} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix},$$

or in matrix notation

$$\{\overline{M}\} = [R]\{M\} \text{ and } \{\bar{\kappa}\} = [R]\{\kappa\},$$

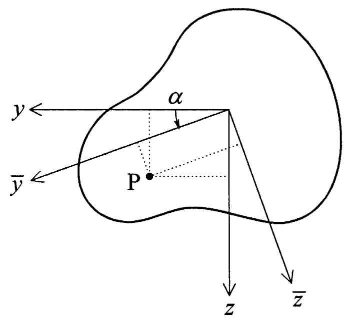


Figure 9.44 The yz coordinate system is transformed into a new $\overline{yz}$ coordinate system by rotating it about an angle α.

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1 These vectors are physical quantities with certain properties, and are more than a column matrix in linear algebra.

2 See Sections 9.3 and 9.4.

in which $[R]$ is called the *transformation matrix*:

$$[R] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

The backward transformation (from $\overline{yz}$ to yz) will be

$$\{M\} = [R]^{-1}\{\overline{M}\} \text{ and } \{\kappa\} = [R]^{-1}\{\bar{\kappa}\}.$$

For the transformation matrix holds that the inverse matrix is equal to the transposed matrix:[1]

$$[R]^{-1} = [R]^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Matrices with this property are called *orthogonal*.

A vector or first-order tensor can be identified by the transformation rules when rotating the coordinate system.

An important fact is the independency of the magnitude of the vector to the coordinate system used. Both in a yz and $\overline{yz}$ coordinate system,

$$M = \sqrt{M_x^2 + M_y^2} = \sqrt{\overline{M}_x^2 + \overline{M}_y^2}$$

and

$$\kappa = \sqrt{\kappa_x^2 + \kappa_y^2} = \sqrt{\bar{\kappa}_x^2 + \bar{\kappa}_y^2}.$$

[1] This can be concluded from the fact that the determinant of the matrix is 1.

We say that the magnitudes M and κ are *invariant*.

9.11.2 Second-order tensor

The components of the first-order tensors M and κ are linearly linked by the bending stiffness matrix

$$\begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix}.$$

If a linear relationship between two first-order tensors is represented by a matrix, this matrix is by definition a second-order tensor. So the bending stiffness matrix is a second-order tensor, referred to as the *bending stiffness tensor*. Since $EI_{zy} = EI_{yz}$, the *bending stiffness tensor* is a *symmetrical tensor*.

In scientific publications the matrix notation is not used very often. The standard notation used is the tensor notation:

$$M_i = EI_{ij}\kappa_j \text{ with } i, j = x, z,$$

or simplified

$$M = EI\kappa. \tag{9.45}$$

A second-order tensor can be identified by the transformation rules for its components when rotating the coordinate system (see Figure 9.44).

In the rotated $\overline{yz}$ coordinate system we find by repeated application of the transformation rules

$$\overline{M} = RM = R\,EI\,\kappa = R\,EI\,R^{-1}\bar{\kappa},$$

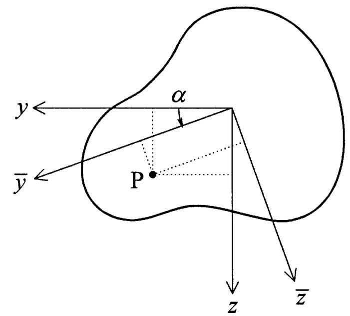

Figure 9.44 The yz coordinate system is transformed into a new $\overline{yz}$ coordinate system by rotating it about an angle α.

or

$$\overline{M} = \overline{EI}\bar{\kappa},$$

in which $\overline{EI}$ is the transformed bending stiffness tensor:

$$\overline{EI} = R\,EI\,R^{-1}.$$

The components of the bending stiffness tensor in the rotated coordinate system follow from

$$\overline{EI} = \begin{bmatrix} EI_{\overline{yy}} & EI_{\overline{yz}} \\ EI_{\overline{zy}} & EI_{\overline{zz}} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix}}_{EI} \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{R^{-1}},$$

which gives

$$\begin{aligned} EI_{\overline{yy}} &= +EI_{yy}\cos^2\alpha + EI_{yz}\sin\alpha\cos\alpha + EI_{zy}\sin\alpha\cos\alpha + EI_{zz}\sin^2\alpha, \\ EI_{\overline{yz}} &= -EI_{yy}\sin\alpha\cos\alpha + EI_{yz}\cos^2\alpha - EI_{zy}\sin^2\alpha + EI_{zz}\sin\alpha\cos\alpha, \\ EI_{\overline{zy}} &= -EI_{yy}\sin\alpha\cos\alpha - EI_{yz}\sin^2\alpha + EI_{zy}\cos^2\alpha + EI_{zz}\sin\alpha\cos\alpha, \\ EI_{\overline{zz}} &= +EI_{yy}\sin^2\alpha - EI_{yz}\sin\alpha\cos\alpha - EI_{zy}\sin\alpha\cos\alpha + EI_{zz}\cos^2\alpha. \end{aligned} \tag{9.46}$$

These are the transformation rules for the components of a second-order tensor. Since $EI_{yz} = EI_{zy}$ (the bending stiffness tensor is a symmetric tensor), the transformation rules simplify into

$$\begin{aligned} EI_{\overline{yy}} &= +EI_{yy}\cos^2\alpha + 2EI_{yz}\sin\alpha\cos\alpha + EI_{zz}\sin^2\alpha, \\ EI_{\overline{yz}} = EI_{\overline{zy}} &= -(EI_{yy} - EI_{zz})\sin\alpha\cos\alpha + EI_{yz}(\cos^2\alpha - \sin^2\alpha), \\ EI_{\overline{zz}} &= +EI_{yy}\sin^2\alpha - 2EI_{yz}\sin\alpha\cos\alpha + EI_{zz}\cos^2\alpha. \end{aligned} \tag{9.47}$$

9.11.3 Principal values and principal directions

A *second-order tensor* describes the linear relation between the components of two vectors or first-order tensors. Most likely these two vectors will not have the same direction (see Figure 9.45). So M and κ will usually not have the same direction. This results in a *plane of curvature* which does not coincide with the *plane of loading*. The bending moment M and curvature κ act only in the same plane if the following relation holds:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \lambda \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}.$$

If we substitute this into the constitutive relationship we find:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \lambda \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}$$

which gives

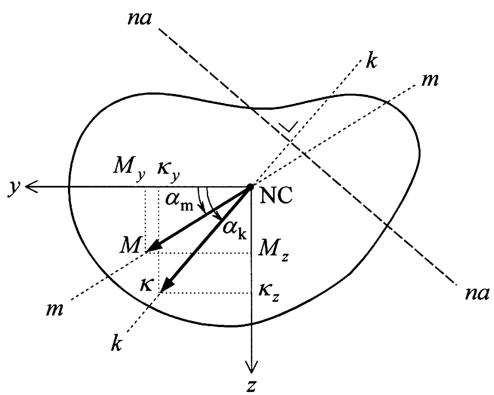

Figure 9.45 Most likely the vectors M and κ will not have the same direction, and the plane of curvature k will not coincide with the plane of loading m.

$$\underbrace{\begin{bmatrix} EI_{yy}-\lambda & EI_{yz} \\ EI_{zy} & EI_{zz}-\lambda \end{bmatrix}}_{EI-\lambda I}\underbrace{\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}}_{\kappa}=0 \tag{9.48}$$

or

$$(EI-\lambda I)\kappa=0 \tag{9.49}$$

in which I is the *unity matrix*.

Equation (9.49) represents an *eigenvalue problem*. Only for certain values of λ this system will have a non-zero solution. The values of λ are the *eigenvalues* and the associated solutions of κ the *eigenvectors*. The homogeneous system of equations (9.48) will have a non-zero solution if the determinant of the matrix is zero:

$$\text{Det}[EI-\lambda I]=(EI_{yy}-\lambda)(EI_{zz}-\lambda)-EI_{yz}EI_{zy}=0.$$

Since the non-diagonal terms are equal ($EI_{yz}=EI_{zy}$), we find the following *characteristic polynomial* which has to be zero:

$$(EI_{yy}-\lambda)(EI_{zz}-\lambda)-EI_{yz}^2=0,$$

or

$$\lambda^2-(EI_{yy}+EI_{zz})\lambda+(EI_{yy}EI_{zz}-EI_{yz}^2)=0. \tag{9.50}$$

The solution of this equation is

$$\lambda_1,\lambda_2=\tfrac{1}{2}(EI_{yy}+EI_{zz})\pm\tfrac{1}{2}\sqrt{(EI_{yy}-EI_{zz})^2+4EI_{yz}^2}\,.$$

The two *eigenvalues* λ_1, λ_2 are also known as the *principal values* EI_1, EI_2.

$$EI_1, EI_2 = \tfrac{1}{2}\,(EI_{yy} + EI_{zz}) \pm \tfrac{1}{2}\sqrt{(EI_{yy} - EI_{zz})^2 + 4EI_{yz}^2}\,. \quad (9.51)$$

The words *eigenvalue* and *principal value* describe the same concept.

The eigenvalues will always be the same, independent of the choice of the coordinate system. This means that the characteristic polynomial will always be the same regardless the choice of the coordinate system. In order to obtain in every coordinate system the same characteristic polynomial, the constants of polynomial (9.50) have to be invariant. So we can write

$$\lambda^2 - I_1\lambda + I_2 = 0$$

in which I_1 and I_2 are two invariants:[1]

$$\begin{aligned} I_1 &= EI_{yy} + EI_{zz}, \\ I_2 &= EI_{yy}EI_{zz} - EI_{yz}^2. \end{aligned} \quad (9.52)$$

For each solution of the eigenvalue λ_i an eigenvector κ^i can be found. The eigenvectors κ^i are independent of one another, which means that they are mutual perpendicular to one another. Their directions are called the *principal directions*.

---

[1] Note: I_1 and I_2 are internationally used notations for the invariants and have nothing to do with the moment of inertia for which the same symbol is used.

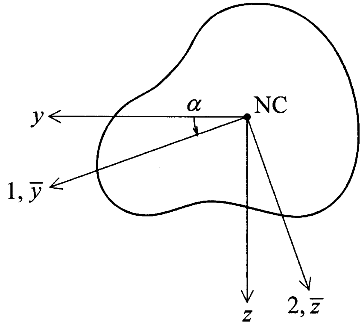

Figure 9.46 If the rotated $\overline{yz}$ coordinate system is a principal coordinate system, then $EI_{\overline{yz}} = EI_{\overline{zy}} = 0$

If we rotate the yz coordinate system to the principal $\overline{yz}$ coordinate system as shown in Figure 9.46, the constitutive relationship in this coordinate system becomes

$$\begin{bmatrix} M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} EI_{\overline{yy}} & 0 \\ 0 & EI_{\overline{zz}} \end{bmatrix} \begin{bmatrix} \kappa_{\bar{y}} \\ \kappa_{\bar{z}} \end{bmatrix}, \tag{9.53}$$

in which $EI_{\overline{yy}}$ and $EI_{\overline{zz}}$ are the principal bending bending stiffness values, also denoted as EI_1 and EI_2.[1]

In (9.53) both non-diagonal terms are zero which is by definition the case for a principal tensor. From this relation we can also see that bending in the principal directions is fully uncoupled. This means that both vectors M and κ only coincide if the plane of loading m or the plane of curvature k coincides with one of the principal directions.

To find the principal directions we can substitute the eigenvalues (9.51) in (9.48) and determine the directions of the eigenvectors. However to avoid extensive calculation we follow another route and return to one of the transformation formulae (9.47) and look for the direction for which both non-diagonal terms $EI_{\overline{yz}} = EI_{\overline{zy}}$ in the rotated $\overline{yz}$ coordinate system are zero (see Figure 9.46):

$$\begin{aligned} EI_{\overline{yz}} &= EI_{\overline{zy}} \\ &= (-EI_{yy} + EI_{zz}) \sin\alpha \cos\alpha + EI_{yz}(\cos^2\alpha - \sin^2\alpha) = 0. \end{aligned}$$

[1] It is usual to denote the larger principal bending stiffness value as EI_1 and the smaller as EI_2.

Introducing the double angle 2α we can write

$$EI_{\overline{yz}} = -\tfrac{1}{2}\,(EI_{yy} - EI_{zz})\sin 2\alpha + EI_{yz}\cos 2\alpha = 0,$$

from which we find

$$\tan 2\alpha = \frac{EI_{yz}}{\tfrac{1}{2}\,(EI_{yy} - EI_{zz})}\,. \tag{9.54}$$

The solution is

$$\alpha_1 = \alpha_0,$$

but since the tangent is a periodic function with period π (180°) there is also a second solution:

$$\alpha_2 = \alpha_0 + 90^\circ.$$

As mentioned before the two principal directions α_1 and α_2 are perpendicular to one another.

Next we look for situations in which the plane of loading m coincides with the plane of curvature k, assuming the yz directions of the coordinate system coincide with the principal directions:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & 0 \\ 0 & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}, \tag{9.55}$$

Here EI_{yy} and EI_{zz} are the principal bending bending stiffness values.

For the principal coordinate system the following relationship between the direction α_m of the bending moment vector (plane of loading) and the di-

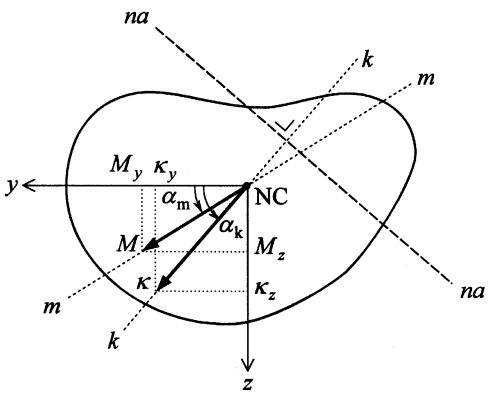

Figure 9.47 For a principal yz coordinate system exists the following relationship between the direction α_m of the bending moment vector (plane of loading m) and the direction α_k of the curvature vector (plane of curvature k): $\tan\alpha_m = (EI_{zz}/EI_{yy})\tan\alpha_k$.

rection α_k of the curvature vector (plane of curvature) can be derived from (9.55) (see Figure 9.47):

$$\tan\alpha_m = \frac{M_z}{M_y} = \frac{EI_{zz}}{EI_{yy}}\frac{\kappa_z}{\kappa_y} = \frac{EI_{zz}}{EI_{yy}}\tan\alpha_k. \tag{9.56}$$

The directions of the vectors M and κ coincide only if

- $\alpha_m = \alpha_k = 0$, both M and κ act along the y axis (a principal axis);
- $\alpha_m = \alpha_k = \pi/2$, M and κ act along the z axis (the other principal axis);
- $EI_{yy} = EI_{zz} \Rightarrow \alpha_m = \alpha_k$.

In the last case, with two equal principal bending stiffness values, all directions are principal directions.

9.11.4 Mohr's circle

In this section we discuss a simple *graphical method* to find the principal directions and principal values of the bending stiffness tensor, and also the stiffness values in any arbitrary direction. This method is based on what is called *Mohr's circle*.[1]

---

[1] Mohr's circle gives a graphical representation of the transformation formulae for the components of a second-order tensor. Here we discuss the bending stiffness tensor. But Mohr's circle can also be used for other second-order tensors. Examples are the stress tensor, strain tensor (see *Engineering Mechanics*, Volume 4). The first of this idea was made by Culmann in 1866. About 20 years later, Mohr made a more complete study. Christian Otto Mohr (1835–1918) was a German civil engineer active in railway and bridge design, and later became professor at the Stuttgart Polytechnikum (1868–1873) and the Dresden Polytechnikum (1873–1900).

In Figure 9.48 the yz coordinate system is rotated through an angle α with respect to the principal directions, simply denoted by 1 and 2. In order to express the components of the bending stiffness tensor for the rotated yz coordinate system into the principal stiffness values, the transformation formulae (9.47) are used:

$$
\begin{aligned}
EI_{yy} &= EI_1 \cos^2\alpha + EI_2 \sin^2\alpha,\\
EI_{yz} &= EI_{zy} = -(EI_1 - EI_2)\sin\alpha\cos\alpha,\\
EI_{zz} &= EI_1 \sin^2\alpha + EI_2 \cos\alpha.
\end{aligned} \tag{9.57}
$$

With

$$
\begin{aligned}
\sin^2\alpha &= \tfrac{1}{2}(1-\cos 2\alpha),\\
\cos^2\alpha &= \tfrac{1}{2}(1+\cos 2\alpha),\\
\sin\alpha\cos\alpha &= \frac{1}{2}\sin 2\alpha,
\end{aligned}
$$

the stiffness values (9.57) can be written as functions of the double angle 2α:

$$
\begin{aligned}
EI_{yy} &= \tfrac{1}{2}(EI_1 + EI_2) + \tfrac{1}{2}(EI_1 - EI_2)\cos 2\alpha,\\
EI_{yz} &= EI_{zy} = -\tfrac{1}{2}(EI_1 - EI_2)\sin 2\alpha,\\
EI_{zz} &= \tfrac{1}{2}(EI_1 + EI_2)\cos^2\alpha - \tfrac{1}{2}(EI_1 - EI_2)\cos 2\alpha.
\end{aligned} \tag{9.58}
$$

Assume

$$
\begin{aligned}
c &= \tfrac{1}{2}(EI_1 + EI_2),\\
r &= \tfrac{1}{2}(EI_1 - EI_2).
\end{aligned} \tag{9.59}
$$

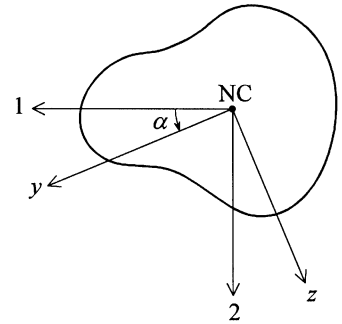

Figure 9.48 The yz coordinate system is rotated through an angle α with respect to the principal directions, simply denoted by 1 and 2.

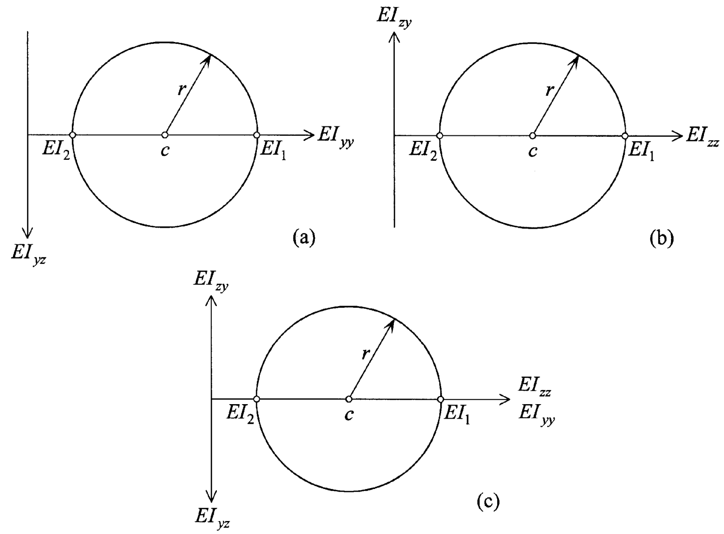

Figure 9.49 (a) *Mohr's circle* in a $(EI_{yy}; EI_{yz})$ coordinate system, with centre $(EI_{yy}; EI_{yz}) = (c; 0)$ and radius r. (b) *Mohr's circle* in a $(EI_{zz}; EI_{zy})$ coordinate system, with centre $(EI_{zz}; EI_{zy}) = (c; 0)$ and radius r. (c) Since both circles represent the transformation formulae for the same bending stiffness tensor, they have to coincide.

It is usual to denote the larger principal bending stiffness value as EI_1 and the smaller as EI_2, so $r > 0$.

Note that c and r are invariant, which means independent of the rotation of the coordinate system.

Substitution of (9.59) in (9.58) gives

$$EI_{yy} = c + r\cos 2\alpha, \tag{9.60a}$$

$$EI_{yz} = EI_{zy} = -r\sin 2\alpha, \tag{9.60b}$$

$$EI_{zz} = c - r\cos 2\alpha. \tag{9.60c}$$

Squaring and adding equations (9.60a) and (9.60b) shows that

$$(EI_{yy} - c)^2 + (EI_{yz})^2 = r^2. \tag{9.61a}$$

This equation can be interpreted as the equation of a circle (*Mohr's circle*) in a $(EI_{yy}; EI_{yz})$ coordinate system, with centre $(EI_{yy}; EI_{yz}) = (c; 0)$ and radius r (see Figure 9.49a).

Squaring and adding equations (9.60b) and (9.60c) gives

$$(EI_{zz} - c)^2 + (EI_{zy})^2 = r^2. \tag{9.61b}$$

This is the equation of Mohr's circle in a $(EI_{zz}; EI_{zy})$ coordinate system, with centre $(EI_{zz}; EI_{zy}) = (c; 0)$ and radius r (see Figure 9.49b). Since the circles in Figures 9.49a and 9.49b are related to the transformation formulae for the same bending stiffness tensor, they have to coincide (see Figure 9.49c).

The diagonal terms of the bending stiffness tensor, EI_{yy} and EI_{zz}, are plotted on the horizontal axis, the non-diagonal terms $EI_{yz} = EI_{zy}$ are plotted on the vertical axis. The positive directions of the vertical EI_{yz} axis and EI_{zy} axis are opposite, and are chosen in such a way that it is very simple to read from Mohr's circle the stiffness values in other (rotated) coordinate systems.

To memorise the positive EI_{yz} and EI_{zy} directions along the vertical axis the following hints are given:

- Rotate the original yz coordinate system so that you can position it between the horizontal and vertical axes as shown in Figure 9.50.
- In the lower corner the y axis points downward: this is the positive EI_{yz} direction. The first index y of EI_{yz} corresponds with the y axis of the rotated coordinate system.
- In the upper corner the z axis points upwards: this is the positive EI_{zy} direction. The first index z of EI_{zy} corresponds with the z axis of the rotated coordinate system.

The use of Mohr's circle will be explained by two examples.

Example 1

In the first example the principal values EI_1 and EI_2 are given, together with the angle α, which defines the position of the yz coordinate system. Below we are looking for the components of the bending stiffness tensor in the yz coordinate system.

For two mutual perpendicular axes y and z the pairs of bending stiffness values (EI_{yy}; EI_{yz}) and (EI_{zz}; EI_{zy}) are represented by two diametrical points of Mohr's circle, A and B respectively. This is shown in Figure 9.51. Point A is found by plotting de angle 2α at the centre c of the circle. The location of A defines the values of EI_{yy} (to be read from the horizontal axis) and EI_{yz} (to be read from the vertical axis). The location of point B

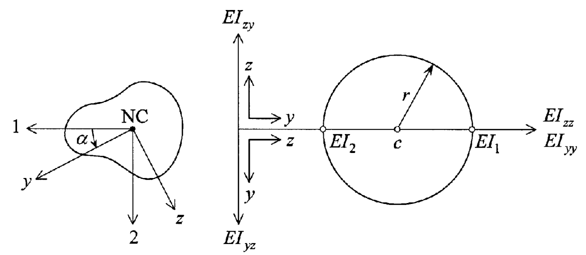

Figure 9.50 In Mohr's circle, the diagonal terms of the bending stiffness tensor, EI_{yy} and EI_{zz}, are plotted on the horizontal axis, the non-diagonal terms $EI_{yz} = EI_{zy}$ are plotted on the vertical axis. The positive directions of the vertical EI_{yz} axis and EI_{zy} axis are opposite, and are chosen in such a way that it is very simple to read from Mohr's circle the stiffness values in other (rotated) coordinate systems.

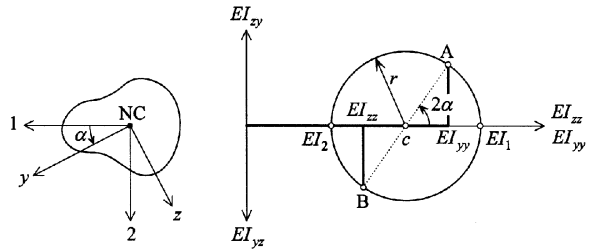

Figure 9.51 For two mutual perpendicular axes y and z, the pairs of bending stiffness values (EI_{yy}; EI_{yz}) and (EI_{zz}; EI_{zy}) are represented by two diametrical points of Mohr's circle, A and B respectively. Point A is found by plotting the angle 2α at the centre c of the circle. The location of A defines the values of EI_{yy} (to be read from the horizontal axis) and EI_{yz} (to be read from the vertical axis). The location of point B is diametrical with respect to A, and defines the values of EI_{zz} and EI_{zy}.

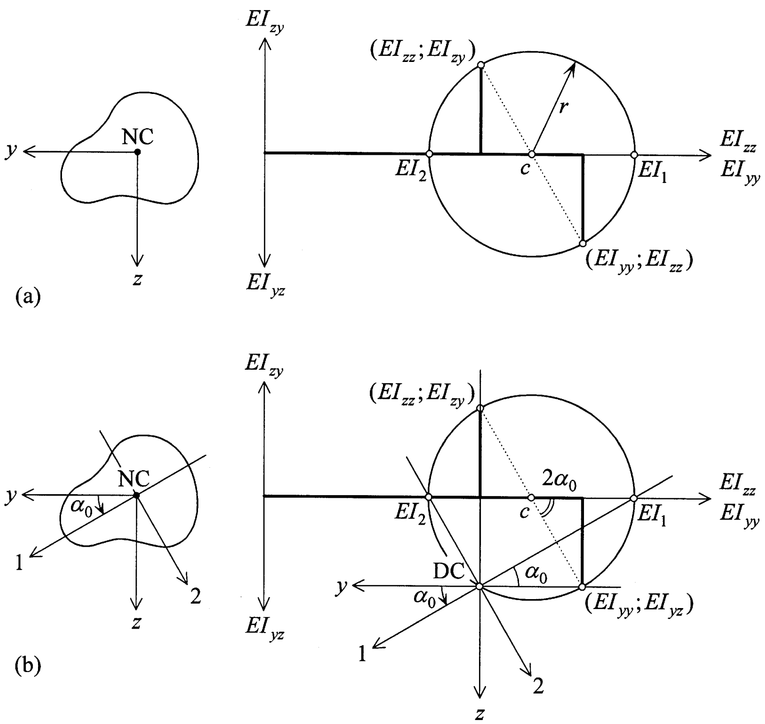

Figure 9.52 (a) From Mohr's circle we immediately can read the principal values: $EI_1 = c + r$ and $EI_2 = c - r$. (b) The *pole* or *direction centre* DC has the special property that

- a line drawn through DC and parallel to the i axis of the ij coordinate system intersects Mohr's circle at the point $(EI_{ii}; EI_{ij})$;
- a line drawn through DC and parallel to the j axis of the ij coordinate system intersects Mohr's circle at the point $(EI_{jj}; EI_{ji})$.

is diametrical with respect to A, and defines the values of EI_{zz} and EI_{zy}. EI_{yz} and EI_{zy} appear to be negative for the given values of EI_1, EI_2 and α ($\alpha \approx 28°$); their values have to be read from the negative axes.

Comment: The expressions (9.60) can be derived easily from Mohr's circle in Figure 9.51:

$$EI_{yy} = c + r\cos 2\alpha, \tag{9.60a}$$

$$EI_{yz} = EI_{zy} = -r\sin 2\alpha, \tag{9.60b}$$

$$EI_{zz} = c - r\cos 2\alpha. \tag{9.60c}$$

Example 2

In the second example the components of the bending stiffness tensor in a yz coordinate system are given, and the principal values and principal directions are asked.

For two mutual perpendicular axes y and z the pairs of bending stiffness values $(EI_{yy}; EI_{yz})$ and $(EI_{zz}; EI_{zy})$ are represented by two diametrical points on Mohr's circle (see Figure 9.52a). In this example, all stiffness values are assumed to be positive.

The centre c and radius r of Mohr's circle can be derived from the graph:

$$c = \tfrac{1}{2}(EI_{yy} + EI_{zz}),$$

$$r = \sqrt{\left\{\tfrac{1}{2}(EI_{yy} + EI_{zz})\right\}^2 + EI_{yz}^2}\,.$$

Both c and r can be related to the two invariants I_1 and I_2 as defined in (9.47):

$$c = \tfrac{1}{2} I_1 \qquad \text{in which } I_1 = EI_{yy} = EI_{zz},$$

$$r = \sqrt{\left(\tfrac{1}{2} I_1\right)^2 - I_2} \quad \text{in which } I_2 = EI_{yy}EI_{zz} - EI_{yz}^2.$$

From Mohr's circle we immediately read the principal values:

$$EI_1 = c + r \text{ and } EI_2 = c - r.$$

To find the principal directions we make use of a special point on Mohr's circle, which is called the *pole* or *direction centre*, and is denoted as DC (see Figure 9.52b). The direction centre DC has the special property that

- a line drawn through DC and parallel to the i axis of the ij coordinate system intersects Mohr's circle at the point $(EI_{ii}; EI_{jj})$;
- a line drawn through DC and parallel to the j axis of the ij coordinate system intersects Mohr's circle at the point $(EI_{jj}; EI_{ji})$.

So the location of the direction centre DC can be found as follows:

- Draw a line parallel to the y axis through point $(EI_{yy}; EI_{yz})$.
- Draw a line parallel to the z axis through point $(EI_{zz}; EI_{zy})$.
- The intersection of the two lines with Mohr's circle is the direction centre DC.

Using the direction centre DC, we can find the principle directions as follows:

- Draw a line through DC and EI_1. This line is parallel to the principal axis denoted as 1, and is the first principal direction.
- Draw a line through DC and EI_2. This line is parallel to the principal axis denoted as 2, and is the second principal direction.

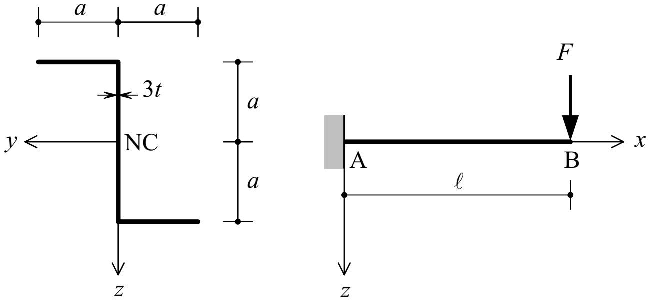

Figure 9.53 Cantilever beam with homogeneous Z-section.

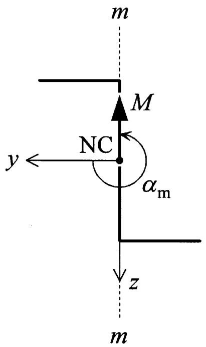

Figure 9.54 The cross-section at A is subject to a bending moment M which acts in the vertical plane m. The vector M points upwards, therefore $\alpha_m = 270°$.

From Mohr's circle in Figure 9.52 we directly read

$$\tan\alpha_0 = \frac{EI_{yz}}{EI_1 - EI_{zz}},$$

and for the central angle $2\alpha_0$ at centre c

$$\tan 2\alpha_0 = \frac{EI_{yz}}{EI_{yy} - c} = \frac{EI_{yz}}{EI_{yy} - \frac{1}{2}(EI_{yy} + EI_{zz})} = \frac{EI_{yz}}{\frac{1}{2}(EI_{yy} - EI_{yy})}.$$

This is in accordance with equation (9.54), derived before.

9.12 Application of the alternative method based on the principal directions

The calculation of stresses in the principal coordinate system, as outlined in Section 9.10, will be illustrated by two examples.

Example 1

Here we use the homogeneous Z-section from Example 1 in Section 9.9.2 (see Figure 9.53), and consider the cross-section at fixed end A. This cross-section is subject to a bending moment M with components $M_y = 0$ and $M_z = -F\ell$. The bending moment acts in the vertical plane m; the vector M points upwards: $\alpha_m = 270°$ (see Figure 9.54).

Questions:

a. Calculate the bending stress distribution in the principal coordinate system.
b. Find, in the principal coordinate system, the directions of the vectors M and κ for the bending moment and curvature respectively.

Solution:
a. In the given yz coordinate system we found

$$EI_{yy} = 2Ea^3t,$$

$$EI_{yz} = EI_{zy} = -3Ea^3t,$$

$$EI_{zz} = 8Ea^3t.$$

With these values we have two points of Mohr's circle. From Figure 9.55a we find

$$c = \frac{2+8}{2}\,Ea^3t,$$

$$r = Ea^3t\sqrt{(8-c)^2+3^2} = 3Ea^3t\sqrt{2} = 4.2426Ea^3t.$$

In Figure 9.55b Mohr's circle is plotted. From the circle we find the principal bending stiffness quantities as

$$EI_1 = c + r = (5 + 3\sqrt{2})Ea^3t = 9.2426Ea^3t,$$

$$EI_2 = c - r = (5 - 3\sqrt{2})Ea^3t = 0.7574Ea^3t.$$

A particular point on the circle is the direction centre DC, which is found by drawing a line parallel to the y axis through point (EI_{yy}; EI_{yz}) and a line parallel to the z axis through point (EI_{zz}; EI_{zy}). Using the direction centre DC we find the principal directions. In a principal $\overline{yz}$ coordinate system with the positive $\bar{y}$ axis in the first quadrant the principal values are (see Figure 9.55b)

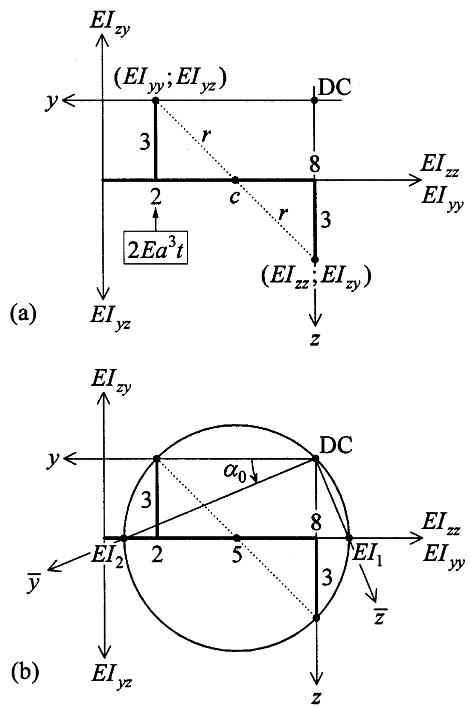

Figure 9.55 (a) The pairs of bending stiffness values (EI_{yy};EI_{yz}) and (EI_{zz}; EI_{zy}) represent two points of Mohr's circle. With the help of these points the radius r and centre c of Mohr's circle can be found. (b) The complete circle. A particular point on the circle is the *direction centre* DC, which is found by drawing a line parallel to the y axis through point (EI_{yy}; EI_{yz}) and a line parallel to the z axis through point (EI_{zz}; EI_{zy}). The intersection of these lines is the direction centre DC. $\bar{y}$ and $\bar{z}$ are the principal coordinate axes. EI_1 and EI_2 are the principal bending stiffnesses.

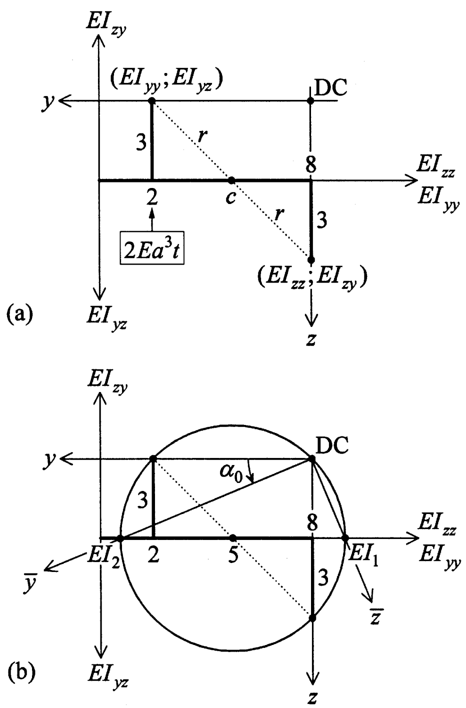

Figure 9.55 (a) The pairs of bending stiffness values $(EI_{yy}; EI_{yz})$ and $(EI_{zz}; EI_{zy})$ represent two points of Mohr's circle. With the help of these points the radius r and centre c of Mohr's circle can be found. (b) The complete circle. A particular point on the circle is the *direction centre* DC, which is found by drawing a line parallel to the y axis through point $(EI_{yy}; EI_{yz})$ and a line parallel to the z axis through point $(EI_{zz}; EI_{zy})$. The intersection of these lines is the direction centre DC. $\bar{y}$ and $\bar{z}$ are the principal coordinate axes. EI_1 and EI_2 are the principal bending stiffnesses.

$$EI_{yy} = EI_2 = 0.7574Ea^3t, \tag{9.62a}$$

$$EI_{zz} = EI_1 = 9.2426Ea^3t. \tag{9.62b}$$

The orientation of the principal $\overline{yz}$ coordinate system is defined by the angle α_0, which can directly be found from Mohr's circle (see Figure 9.55b):

$$\tan\alpha_0 = \frac{|EI_{yz}|}{EI_{zz} - EI_2} = \frac{3}{(8 - (5 - 3\sqrt{2})} = 0.4142 \Rightarrow \alpha_0 = 22.5°.$$

According to (9.44), the stress formula in the principal $\overline{yz}$ coordinate system is very simple:

$$\sigma(\bar{y}, \bar{z}) = E\left(\frac{M_{\bar{y}}\bar{y}}{EI_{\overline{yy}}} + \frac{M_{\bar{z}}\bar{z}}{EI_{\overline{zz}}}\right) = \frac{M_{\bar{y}}\bar{y}}{I_{\overline{yy}}} + \frac{M_{\bar{z}}\bar{z}}{I_{\overline{zz}}}. \tag{9.63}$$

However, all quantities have to be expressed in the $\overline{yz}$ coordinate system.

So the components of M in the principal directions have to be determined. We use the transformation formulae for a first-order tensor:[1]

$$\begin{bmatrix} M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} \cos\alpha_0 & \sin\alpha_0 \\ -\sin\alpha_0 & \cos\alpha_0 \end{bmatrix} \begin{bmatrix} M_y \\ M_z \end{bmatrix}.$$

The result is (see Figure 9.56)

$$\begin{bmatrix} M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} +0.9239 & +0.3827 \\ -0.3827 & +0.9239 \end{bmatrix} \begin{bmatrix} 0 \\ -F\ell \end{bmatrix} = \begin{bmatrix} -0.3827F\ell \\ -0.9239F\ell \end{bmatrix}. \tag{9.64}$$

[1] See Section 9.11.1.

By substituting (9.64) and (9.62) in (9.63) we find the following stress distribution:

$$\sigma(\bar{y},\bar{z}) = -\frac{F\ell}{a^3 t}(0.5053\bar{y} + 0.1000\bar{z}). \tag{9.65}$$

Note that the location of the points in the cross-section also has to be expressed in the principal coordinates $\bar{y}$ and $\bar{z}$. Here again we can use the transformation formulae for a first-order tensor:[1]

$$\begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \cos\alpha_0 & \sin\alpha_0 \\ -\sin\alpha_0 & \cos\alpha_0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} +0.9239 & +0.3827 \\ -0.3827 & +0.9239 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}. \tag{9.66}$$

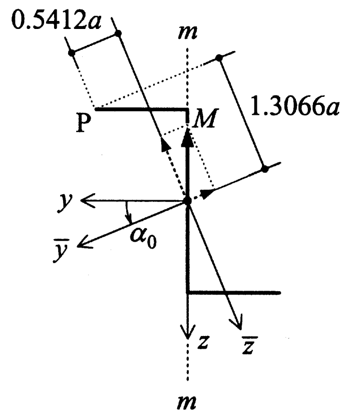

Figure 9.56 The coordinates $(y,z) = (a,-a)$ of P are in the principal $\overline{yz}$ coordinate system $(\bar{y},\bar{z}) = (0.5412a, -1.3066a)$.

To find the bending stress at point P, the coordinates $(y,z) = (a,-a)$ have to be changed into $(\bar{y},\bar{z}) = (0.5412a, -1.3066a)$ (see Figure 9.56). Substituting the $(\bar{y},\bar{z})$ coordinates in (9.64) we find

$$\sigma(\mathrm{P}) = -\frac{F\ell}{a^3 t}(0.5053 \times 0.5412a - 0.1000 \times 1.3066a)$$
$$= -0.1428\frac{F\ell}{a^2 t} = -\frac{1}{7}\frac{F\ell}{a^2 t}.$$

This is in agreement with the result found in Section 9.9.2, Example 1, Table 9.2.

[1] See (9.6a) in Section 9.3.

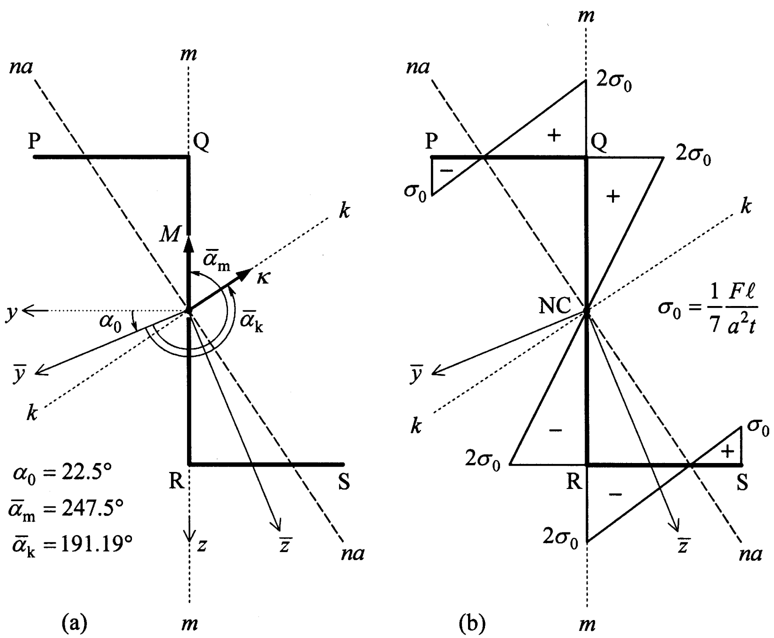

Figure 9.57 (a) The plane of loading m, the plane of curvature k, and the neutral axis *na* perpendicular to the plane of curvature k. (b) The bending stress diagram; the normal stresses are plotted along thin-walled flanges and web. The three points with zero stress are on a straight line: the neutral axis *na*.

Comment: To express the stresses in the initially given yz coordinate system, we can substitute (9.66) in (9.65). After some elaboration we find

$$\sigma(y,z) = -\frac{F\ell}{a^3 t}(0.4286y + 0.2858z) = -\frac{1}{7}\frac{F\ell}{a^3 t}(3y + 2z).$$

This stress distribution is the same as we found in Section 9.9.2, Example 1, formula (9.36).

b. In the principal $\overline{yz}$ coordinate system the direction of vector M is defined by

$$\tan\bar{\alpha}_m = \frac{M_{\bar{z}}}{M_{\bar{y}}} = \frac{-0.9239F\ell}{-0.3827F\ell} = 2.414 \Rightarrow \bar{\alpha}_m = 67.5^\circ + 180^\circ = 247.5^\circ.$$

To find the direction of the curvature vector κ in the principal coordinate system, we use equation (9.56):

$$\tan\bar{\alpha}_k = \frac{EI_{\overline{yy}}}{EI_{\overline{zz}}}\tan\bar{\alpha}_m = \frac{0.7574Ea^3}{9.2426Ea^3 t} \times 2.414 = 0.1978$$

$$\Rightarrow \bar{\alpha}_k = 11.19^\circ + 180^\circ = 191.19^\circ.$$

These directions are shown in Figure 9.57. It can easily be verified that they are in accordance with the directions found in Section 9.11.1, Example 1:

$$\bar{\alpha}_m = \alpha_m - \alpha_0 = 270° - 22.5° = 247.5°,$$

$$\bar{\alpha}_k = \alpha_k - \alpha_0 = 213.69° - 22.5° = 191.19°.$$

Example 2
The next example concerns the unsymmetrical and inhomogeneous cross-section in Figure 9.58. The cross-sectional dimensions and the location of the normal centre NC can be read from the figure. The moduli of elasticity of the web and flange are E and $2E$ respectively, in which $E = 40 \times 10^3$ N/mm^2.

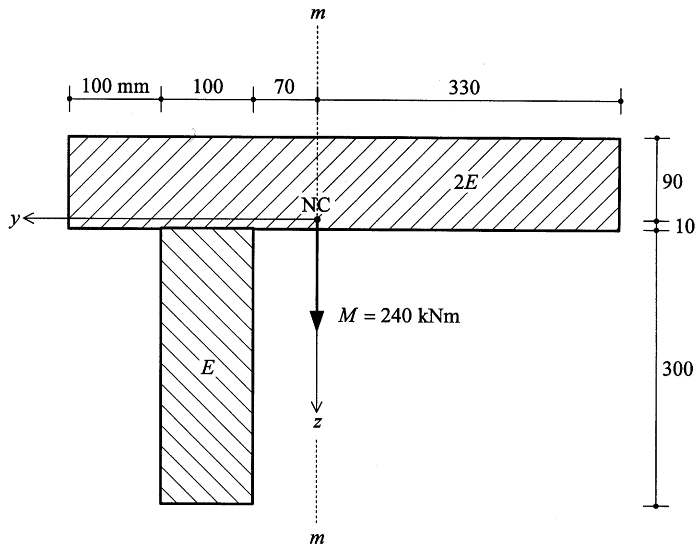

Figure 9.58 An unsymmetrical and inhomogeneous cross-section, subject to a bending moment M in the vertical plane m.

The bending stiffness quantities in the given yz coordinate system, with its origin at NC, are given:

$$EI_{yy} = 166.6 \times 10^{12} \text{ Nmm}^2,$$

$$EI_{yz} = EI_{zy} = 28.8 \times 10^{12} \text{ Nmm}^2,$$

$$EI_{zz} = 51.4 \times 10^{12} \text{ Nmm}^2.$$

The cross-section is subject to a bending moment M in the vertical plane m. The components of M are (see Figure 9.58)

$$M_y = 0 \text{ and } M_z = +240 \text{ kNm}.$$

Questions:

a. Calculate the bending stress distribution in the principal coordinate system.
b. Find the plane of curvature k.
c. Draw the σ diagram by plotting the values perpendicular to k.

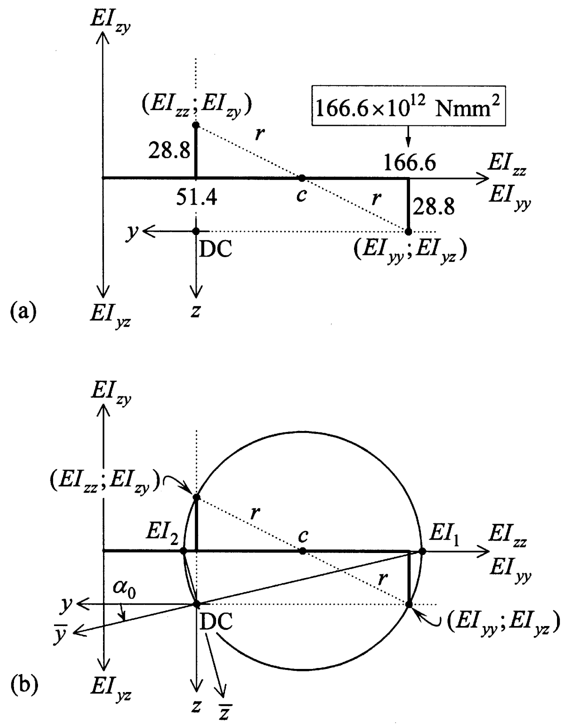

Figure 9.59 (a) The pairs of bending stiffness values $(EI_{yy}; EI_{yz})$ and $(EI_{zz}; EI_{zy})$ represent two points of Mohr's circle. With the help of these points the radius r and centre c of Mohr's circle can be found, and also the direction centre DC. (b) The complete circle. EI_1 and EI_2 are the principal bending stiffnesses. $\bar{y}$ and $\bar{z}$ are the principal coordinate axes.

Solution:

a. To calculate the bending stresses with the formulae based on the principal directions, we first have to find these directions.

The bending stiffness quantities in the given yz coordinate system provide two points of Mohr's circle, as shown in Figure 9.59a. From this figure we find the centre c and radius r of Mohr's circle:

$$c = 109 \times 10^{-12}\ \text{Nmm}^2,$$

$$r = 64.4 \times 10^{-12}\ \text{Nmm}^2.$$

The principal values of the bending stiffness tensor are

$$EI_1 = c + r = 173.4 \times 10^{12}\ \text{Nmm}^2,$$

$$EI_2 = c - r = 44.6 \times 10^{12}\ \text{Nmm}^2.$$

The direction centre DC is found by drawing a line parallel to the y axis through point $(EI_{yy}; EI_{yz})$ and a line parallel to the z axis through point $(EI_{zz}; EI_{zy})$.

Using point DC we can easily find the principal directions. In a principal $\overline{yz}$ coordinate system with the positive $\bar{y}$ axis in the first quadrant the principal values are (see Figure 9.59b)

$$EI_{\overline{yy}} = EI_1 = 173.4 \times 10^{12}\ \text{Nmm}^2, \qquad (9.67a)$$

$$EI_{\overline{zz}} = EI_2 = 44.6 \times 10^{12}\ \text{Nmm}^2. \qquad (9.67b)$$

The orientation of the principal $\overline{yz}$ coordinate system is defined by the angle α_0, which can be found directly from Mohr's circle:

$$\tan\alpha_0 = \frac{|EI_{yz}|}{EI_1 - EI_{zz}} = \frac{28.8}{173.4 - 51.4} = 0.236 \Rightarrow \alpha_0 = 13.28°.$$

For an inhomogeneous cross-section subject to bending the stress distribution in the principal $\overline{yz}$ coordinate system is given by (9.43):

$$\sigma(\bar{y},\bar{z}) = E(\bar{y},\bar{z})\left(\frac{M_{\bar{y}}\bar{y}}{EI_{\overline{yy}}} + \frac{M_{\bar{z}}\bar{z}}{EI_{\overline{zz}}}\right). \tag{9.68}$$

All quantities have to be related to the $\overline{yz}$ coordinate system.

The components of M in the principal $\overline{yz}$ coordinate system are found with the transformation formulae for a first-order tensor:[1]

$$\begin{bmatrix} M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} \cos\alpha_0 & \sin\alpha_0 \\ -\sin\alpha_0 & \cos\alpha_0 \end{bmatrix} \begin{bmatrix} M_y \\ M_z \end{bmatrix}.$$

With $\alpha_0 = 13.28°$, the result is (see Figure 9.60a)

$$\begin{bmatrix} M_{\bar{y}} \\ M_{\bar{z}} \end{bmatrix} = \begin{bmatrix} +0.9733 & +0.2297 \\ -0.2297 & +0.9733 \end{bmatrix} \begin{bmatrix} 0 \\ 240 \times 10^6 \text{ Nmm} \end{bmatrix}$$

$$= \begin{bmatrix} 55.13 \times 10^6 \text{ Nmm} \\ 233.59 \times 10^6 \text{ Nmm} \end{bmatrix}. \tag{9.69}$$

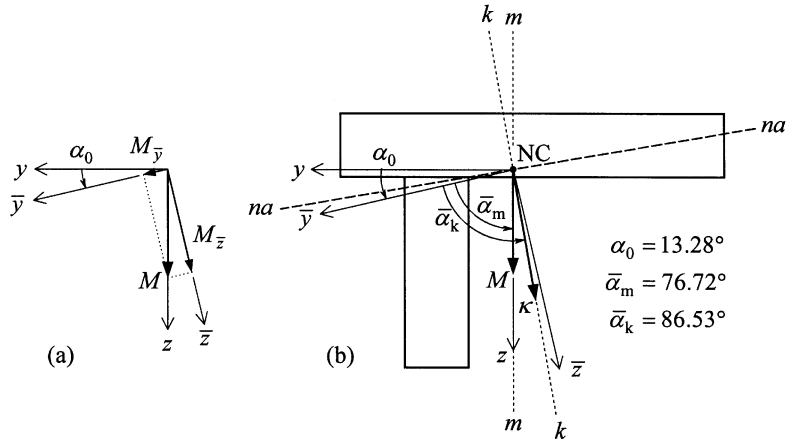

Figure 9.60 (a) The principal $\overline{yz}$ coordinate system. (b) The vectors M and κ and the planes of loading m and curvature k. The neutral axis na passes through the normal centre NC and is perpendicular to k.

[1] See Section 9.11.1.

Table 9.9

E (N/mm^2)	point	y (mm)	z (mm)	σ (N/mm^2)
flange				
80×10^3	P	+270	−90	−56.5
80×10^3	Q	−330	−90	−13.6
80×10^3	V	+270	+10	−15.2
80×10^3	W	−330	+10	+27.7
web				
40×10^3	R	+170	+10	−4.0
40×10^3	S	+70	+10	−0.4
40×10^3	T	+170	+310	+58.0
40×10^3	U	+70	+310	+61.6

By substituting (9.67) and (9.69) in (9.68) we find the following stress distribution:

$$\sigma(\bar{y},\bar{z}) = E(\bar{y},\bar{z})\left(\frac{M_{\bar{y}}\bar{y}}{EI_{\overline{yy}}} + \frac{M_{\bar{z}}\bar{z}}{EI_{\overline{zz}}}\right)$$

$$= \frac{E(\bar{y},\bar{z})}{10^9\text{ mm}}(317.9\bar{y} + 5237.5\bar{z}). \quad (9.70)$$

Substituting

$$\begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \cos\alpha_0 & \sin\alpha_0 \\ -\sin\alpha_0 & \cos\alpha_0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} +0.9733 & +0.2297 \\ -0.2297 & +0.9733 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

in (9.70) we find, after some elaboration, the stress distribution as function of the coordinates in the initially given yz coordinate system:

$$\sigma(\bar{y},\bar{z}) = \frac{E(\bar{y},\bar{z})}{10^9\text{ mm}}(-893.6y + 5170.7z).$$

In Table 9.9 the bending stress is calculated for all corners of the cross-section.

b. In the principal $\overline{yz}$ coordinate system the direction of vector M is defined by

$$\tan\bar{\alpha}_m = \frac{M_{\bar{z}}}{M_{\bar{y}}} = \frac{233.59 \times 10^6\text{ Nmm}}{55.13 \times 10^6\text{ Nmm}} = 4.237 \Rightarrow \bar{\alpha}_m = 76.72^\circ.$$

Check: $\alpha_m = 90^\circ - \alpha_0 = 90^\circ - 13.28^\circ = 76.72^\circ.$

To find the direction of curvature vector κ in the principal coordinate system, we use equation (9.56):

$$\tan\bar{\alpha}_k = \frac{EI_{\overline{yy}}}{EI_{\overline{zz}}}\tan\bar{\alpha}_m$$

$$= \frac{173.4\times10^{12}\ \text{Nmm}^2}{44.6\times10^{12}\ \text{Nmm}^2}\times 4.237 = 16.473 \Rightarrow \bar{\alpha}_k = 86.53°.$$

Figure 9.60b shows the vectors M and κ and the planes of loading m and curvature k. The neutral axis na passes through the normal centre NC and is perpendicular to k.

c. The bending stress diagram is plotted in Figure 9.61. The maximum bending stresses occur at P and U: the maximum compressive bending stress is at P (56.5 N/mm^2) and the maximum tensile bending stress is at U (61.6 N/mm^2).

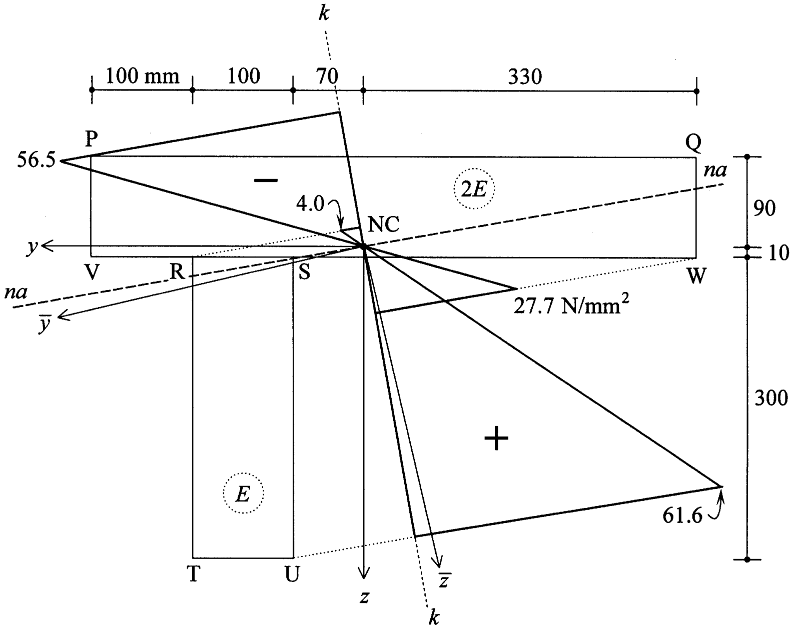

Figure 9.61 The σ diagram. The maximum bending stresses occur at P and U: the maximum compressive bending stress is at P (56.5 N/mm^2) and the maximum tensile bending stress is at U (61.6 N/mm^2.

9.13 Displacements due to bending

In Chapter 8 we discussed various methods to find the deflection of members with a homogeneous cross-section subject to bending in one of the principal planes.

The *forget-me-nots* were discussed in Section 8.3. These formulae are valid only when the member is prismatic and bending occurs in one of the principal planes. So the stiffness EI in the *forget-me-nots* has to be a principal value. The *forget-me-nots* can be used both for homogeneous and inhomogeneous cross-sections.

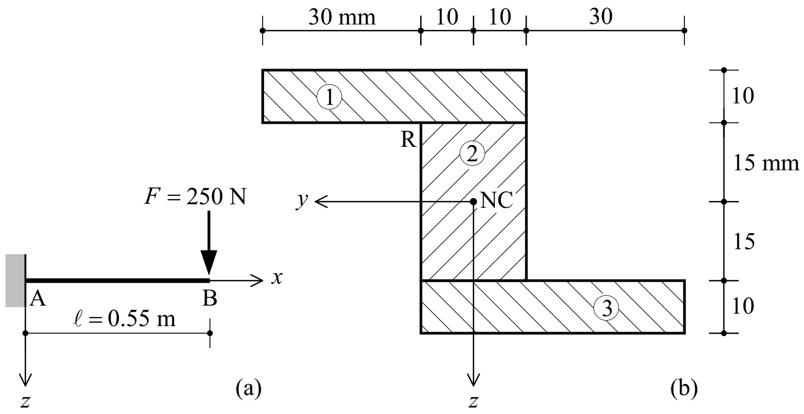

Figure 9.62 Cantilever beam with an inhomogeneous cross-section.

Section 8.4 discussed a method based on the *moment-area formulae*. The "moment-area" refers to the area of the M/EI diagram or, since $M/EI = \kappa$, the curvature diagram. The moment-area formulae are valid for both homogeneous and inhomogeneous cross-sections. An important advantage is that the bending stiffness EI does not need to be a principal value. In a yz coordinate system, not coinciding with the principal directions, we can find the displacement w_y in y direction with the moment-area formulae, in which the "moment-area" refers to the area of the κ_y diagram (curvature diagram in the xy plane). The displacement w_z in z direction can be found with the moment-area formulae, in which the "moment-area" refers to the area of the κ_z diagram (curvature diagram in the xz plane).

Both the method with forget-me-nots and the method based on the moment-area formulae will be illustrated by two examples.

Example 1

Consider the cantilever beam in Figure 9.62, with an inhomogeneous cross-section and constructed of three firmly glued parts, numbered by 1 to 3. For the flanges and web, different materials are used. The beam is loaded by a point load of 250 N at C. The load is applied in such a way that no torsion occurs. The moduli of elasticity are $E_1 = E_3 = 6000$ N/mm^2 and $E_2 = 12000$ N/mm^2. See also Section 9.9.2, Example 2.

Question:

Determine the deflection at C.

Solution:

Figure 9.63 shows the M diagram, and the plane of loading m. The components of the bending moment at the fixed end A are

$$M_{y;A} = 0,$$

$$M_{z;A} = -(250\text{ N})(0.55\text{ mm}) = -137.5 \times 10^3\text{ Nmm}.$$

At the free end B the bending moment is zero. Between A and B the moment varies linearly.

For $M_{y;A} = 0$ and $M_{z;A} = -137.5 \times 10^3$ Nmm we found the y and z components of the curvature at A in Section 9.9.2, Example 2:

$$\kappa_{y;A} = -34.03 \times 10^{-6}\text{ mm}^{-1},$$

$$\kappa_{z;A} = -50.29 \times 10^{-6}\text{ mm}^{-1}.$$

Note that the vertical load causes curvatures in both the horizontal and vertical plane.

The components of the curvature are proportional to the bending moment and therefore varies linearly from the value at A to zero at B. Figure 9.64 shows the curvature diagram with the deformation symbols.

The orientation of the plane of curvature k, defined by the angle β in Figure 9.64, follows from[1]

$$\tan\beta = \frac{\kappa_{z;A}}{\kappa_{y;B}} = \frac{-50.29 \times 10^{-6}\text{ mm}^{-1}}{-34.03 \times 10^{-6}\text{ mm}^{-1}} = 1.478 \Rightarrow \beta = 55.9°.$$

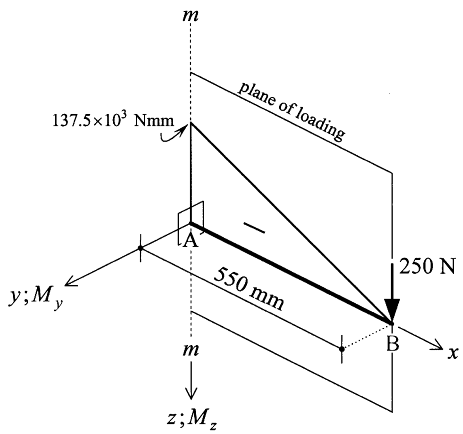

Figure 9.63 The plane of loading.

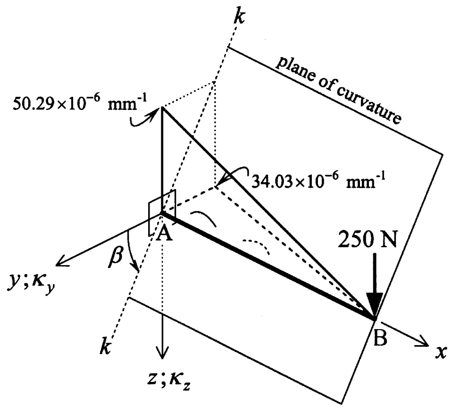

Figure 9.64 The plane of curvature.

[1] More precisely: $\alpha_k = 235.9°$ and $\beta = \alpha_k - 180° = 55.9°$.

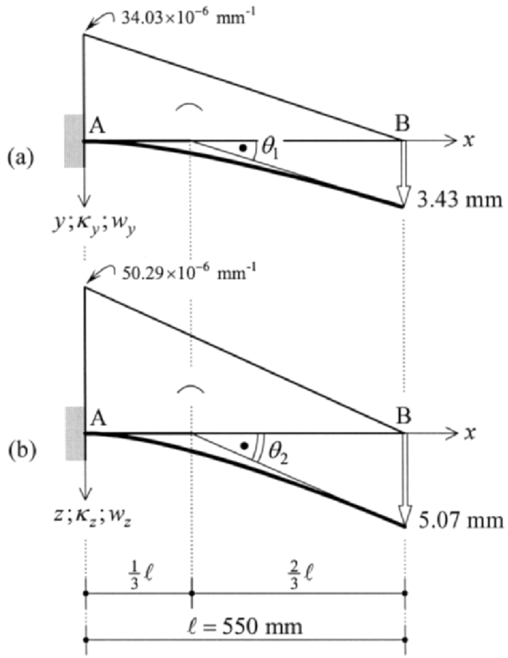

Figure 9.65 (a) Deflection in the xy plane and (b) in the xz plane. The displacements are calculated with the moment-area theorems.

The displacement of B in the y direction can be derived from the curvature diagram with help of the moment-area formulae (see Figure 9.65a):

$$\theta_1 = \tfrac{1}{2}\,\ell|\kappa_{y;\mathrm{A}}|,$$

$$w_{y;\mathrm{B}} = \theta_1 \cdot \tfrac{2}{3}\,\ell = \tfrac{1}{3}\,\ell^2|\kappa_{y;\mathrm{A}}|$$

$$= \tfrac{1}{3}\,(550 \text{ mm})^2(34.03 \times 10^{-6} \text{ mm})^{-1} = 3.43 \text{ mm}.$$

In the same way we find the displacement of B in the z direction (see Figure 9.65b)

$$\theta_2 = \tfrac{1}{2}\,\ell|\kappa_{z;\mathrm{A}}|,$$

$$w_{z;\mathrm{B}} = \theta_2 \cdot \tfrac{2}{3}\,\ell = \tfrac{1}{3}\,\ell^2|\kappa_{z;\mathrm{A}}|$$

$$= \tfrac{1}{3}\,(550 \text{ mm})^2(50.29 \times 10^{-6} \text{ mm})^{-1} = 5.07 \text{ mm}.$$

The total displacement at B is

$$w = \sqrt{(w_{y;\mathrm{B}})^2 + (w_{z;\mathrm{B}})^2} = \sqrt{(3.43 \text{ mm})^2 + (5.07 \text{ mm})^2} = 16.2 \text{ mm}.$$

Figure 9.66 shows a global sketch of the deflection of beam AB.

Comment: The deflection of the beam is in the plane of curvature k:

$$\tan\beta = \frac{w_{z;\mathrm{B}}}{w_{y;\mathrm{B}}} = \frac{5.07 \text{ mm}}{3.43 \text{ mm}} = 1.478 \Rightarrow \beta = 55.9^\circ.$$

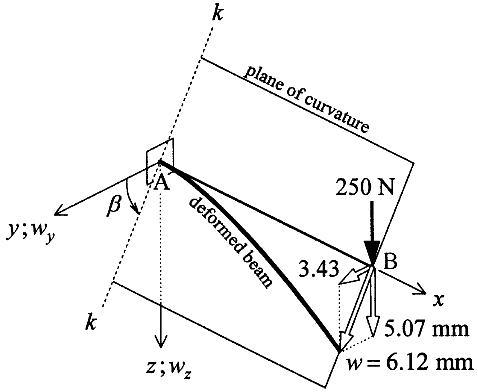

Figure 9.66 The deflection of the beam is in the plane of curvature. The deflection at B is 6.12 mm.

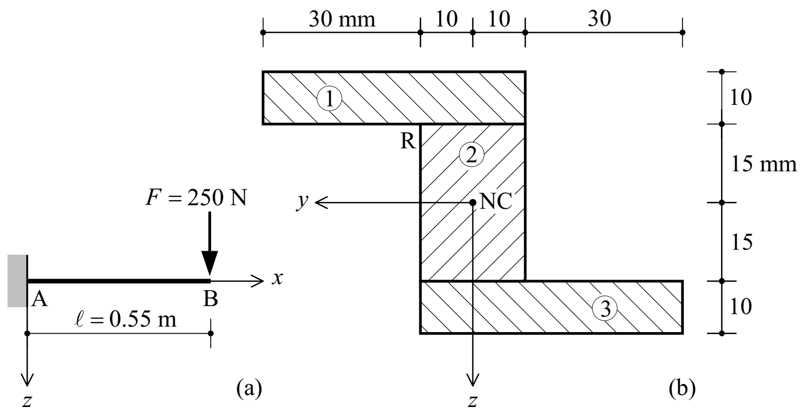

Figure 9.67 Cantilever beam with an inhomogeneous cross-section.

Alternative solution:
An alternative solution is possible by using *forget-me-nots* after resolving the force $F = 250$ N in components according to the *principal directions* of the cross-section. In Section 9.9.2, Example 2, we derived the components of the bending stiffness tensor in the yz coordinate given (see Figure 9.67):

$$EI_{yy} = 5.32 \times 10^9 \text{ Nmm}^2,$$

$$EI_{yz} = EI_{zy} = -3.6 \times 10^9 \text{ Nmm}^2,$$

$$EI_{zz} = 5.17 \times 10^9 \text{ Nmm}^2.$$

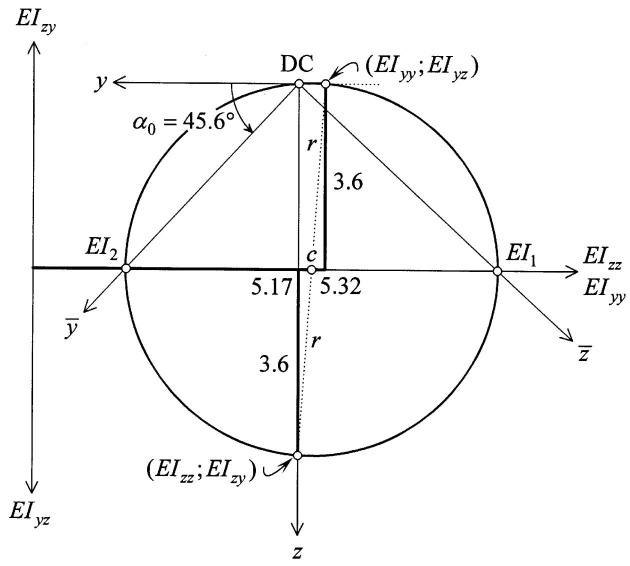

Figure 9.68 Calculation of the principal bending stiffnesses EI_1 and EI_2, using Mohr's circle. All stiffness values in Mohr's circle have to be multiplied by 10^9 Nmm^2.

These values are represented by two points on Mohr's circle: $(EI_{yy};\ EI_{yz})$ and $(EI_{zz};\ EI_{zy})$, as shown in Figure 9.68. In the same figure the direction centre DC is constructed. The centre c of Mohr's circle follows from

$$c = \frac{EI_{yy} + EI_{zz}}{2} = 5.245 \times 10^9 \text{ Nmm}^2,$$

and its radius r from

$$r = \sqrt{(EI_{yy} - c)^2 + (EI_{yz})^2} = 3.601 \times 10^9 \text{ Nmm}^2.$$

From the complete circle of Mohr we find the principal values:

$$EI_{\overline{yy}} = EI_2 = c - r = 1.644 \times 10^9 \text{ Nmm}^2,$$

$$EI_{\overline{zz}} = EI_1 = c + r = 8.846 \times 10^9 \text{ Nmm}^2.$$

The $\bar{y}$ and $\bar{z}$ coordinate axes are the principal axes of the cross-section. The principal directions are defined by the angle α_0:[1]

$$\tan\alpha_0 = \frac{|EI_{zz}|}{EI_{zz} - EI_2} = 1.021 \Rightarrow \alpha_0 = 45.6°.$$

Figure 9.69a shows the principal $\overline{yz}$ coordinate system in the cross-section. The bending stiffness of the cross-section is a minimum in the $\bar{y}$ direction and a maximum in the $\bar{z}$ direction.

[1] It is usual to choose one of the positive principal coordinate axes in the first quadrant.

In Figure 9.69b the load F at the free end B is resolved in components in the principal directions:

$$F_{\bar{y}} = F \sin\alpha_0 = (250\text{ N}) \times 0.7145 = 178.6\text{ N},$$

$$F_{\bar{z}} = F \cos\alpha_0 = (250\text{ N}) \times 0.6997 = 174.9\text{ N}.$$

Next we apply the forget-me-nots:

$$w_{\bar{y}} = \frac{F_{\bar{y}}\ell^3}{3EI_{\bar{y}\bar{y}}} = \frac{(178.6\text{ N})(550\text{ mm})^3}{3 \times (1.644 \times 10^9\text{ Nmm})^2} = 6.02\text{ mm},$$

$$w_{\bar{z}} = \frac{F_{\bar{z}}\ell^3}{3EI_{\bar{z}\bar{z}}} = \frac{(174.9\text{ N})(550\text{ mm})^3}{3 \times (8.846 \times 10^9\text{ Nmm})^2} = 1.10\text{ mm}.$$

Note that $F_{\bar{y}}$ and $F_{\bar{z}}$ are nearly equal in magnitude. However, since the bending stiffness is smaller in the $\bar{y}$ direction than in the $\bar{z}$ direction, the displacement $w_{\bar{y}}$ is larger than $w_{\bar{y}}$.

The resultant displacement w at B is

$$w = \sqrt{(w_{\bar{y};\text{B}})^2 + (w_{\bar{z};\text{B}})^2} = \sqrt{(6.02\text{ mm})^2 + (1.10\text{ mm})^2} = 6.12\text{ mm}.$$

The same value was found before.

Figure 9.70a shows the components of the displacements at the free end B in the yz coordinate system and in the principal $\overline{yz}$ coordinate system. The displacement occurs in the plane of bending k.

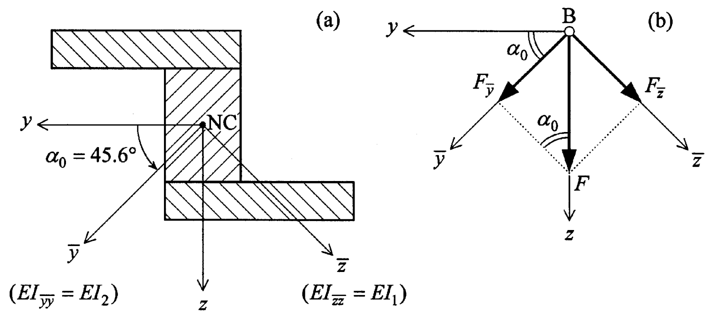

Figure 9.69 (a) The principal $\overline{yz}$ coordinate system. (b) The load F resolved into $\bar{y}$ and $\bar{z}$ components.

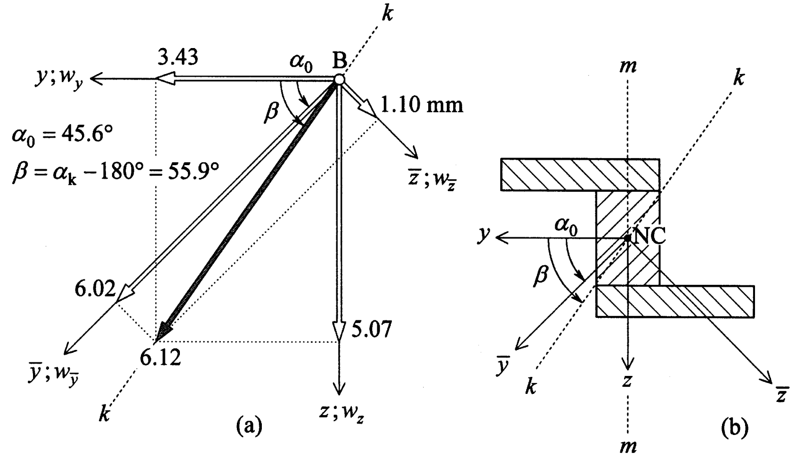

Figure 9.70 (a) The components of the displacement at B. The resultant displacement is in the plane of curvature k. (b) The cross-section with the principal $\overline{yz}$ coordinate system, the plane of loading m and the plane of curvature k.

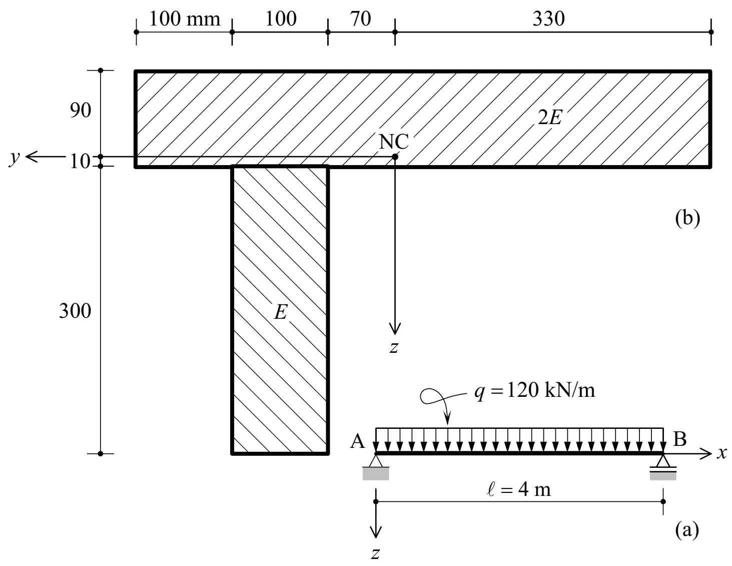

Figure 9.71 (a) Simply supported beam with a uniformly distributed load. (b) The inhomogeneous cross-section.

Example 2

The simply supported prismatic beam AB in Figure 9.71a has a span $\ell = 4$ m and carries a uniformly distributed load $q = 120$ kN/m acting in the vertical xz plane. The load is applied in such a way that no torsion occurs.

The beam has an unsymmetrical and inhomogeneous cross-section. The cross-sectional dimensions and the location of the normal centre NC are given in Figure 9.71b. The moduli of elasticity of web and flange are E and $2E$ respectively, in which $E = 40 \times 10^3$ N/mm^2. The bending stiffness quantities in the given yz coordinate system, with its origin at NC, are given:

$$EI_{yy} = 166.6 \times 10^{12} \text{ Nmm}^2,$$

$$EI_{yz} = EI_{zy} = 28.8 \times 10^{12} \text{ Nmm}^2,$$

$$EI_{zz} = 51.4 \times 10^{12} \text{ Nmm}^2.$$

Question:
Find the displacement of beam AB at midspan.

Solution:
Figure 9.72 shows the plane of loading and the bending moment diagram. The bending moment is parabolic, and has its maximum at midspan C:

$$M_{\max} = \tfrac{1}{8}q\ell^2 = \tfrac{1}{8}(120 \text{ kN/m})(4 \text{ m})^2 = 240 \text{ kNm}.$$

More precisely:

$$M_{y;\text{C}} = 0,$$

$$M_{z;\mathrm{C}} = +240 \text{ kNm} = +240 \times 10^6 \text{ Nmm}.$$

To find the curvature at C we use the constitutive relationship $M = EI\kappa$:

$$\begin{bmatrix} M_{y;\mathrm{C}} \\ M_{z;\mathrm{C}} \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_{y;\mathrm{C}} \\ \kappa_{y;\mathrm{C}} \end{bmatrix}.$$

Substituting all known values in this equation, we have

$$\begin{bmatrix} 0 \\ 240 \end{bmatrix} \times (10^6 \text{ Nmm}) = (10^{12} \text{ Nmm}^2) \times \begin{bmatrix} 166.6 & 28.8 \\ 28.8 & 51.4 \end{bmatrix} \begin{bmatrix} \kappa_{y;\mathrm{C}} \\ \kappa_{z;\mathrm{C}} \end{bmatrix}.$$

By inverting the matrix we find

$$\begin{bmatrix} \kappa_{y;\mathrm{C}} \\ \kappa_{z;\mathrm{C}} \end{bmatrix} = \frac{10^{-6} \text{ mm}^{-1}}{7733.8} \times \begin{bmatrix} 51.4 & -28.8 \\ -28.8 & 166.6 \end{bmatrix} \begin{bmatrix} 0 \\ 240 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \kappa_{y;\mathrm{C}} = -0.894 \times 10^{-6} \text{ mm}^{-1}, \\ \kappa_{z;\mathrm{C}} = +5.170 \times 10^{-6} \text{ mm}^{-1}. \end{cases}$$

Note that the vertical load causes curvatures in both the horizontal and vertical plane.

Since the curvature is proportional to the bending moment, κ_y and κ_z are also parabolic along the beam, with their top values at C. Figure 9.73 shows a sketch of the curvature diagrams with the deformation symbols. The deformation of the beam will occur in the plane of curvature k. The orientation

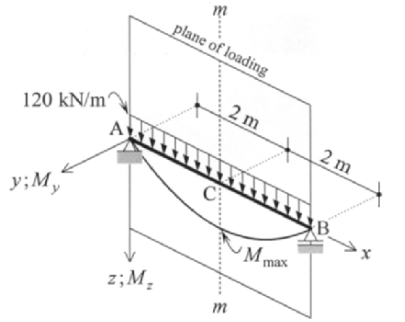

Figure 9.72 Bending moment diagram and the plane of loading m.

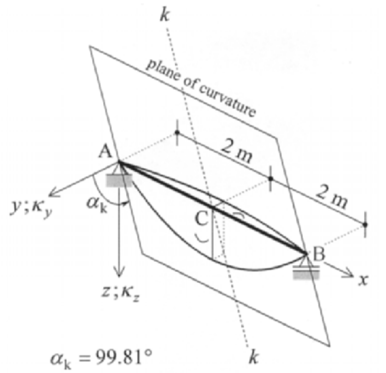

Figure 9.73 κ_y diagram (curvature in the xy plane), κ_z diagram (curvature in the xz plane), and the plane of curvature k.

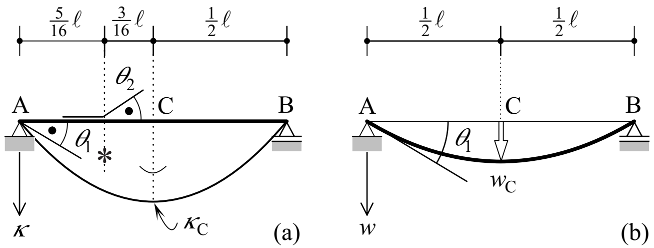

Figure 9.74 (a) Parabolic curvature diagram for the simply supported beam AB. The maximum curvature is κ_C at midspan C. To find the displacement w_C at C we will use the moment-area formulae, in which the "moment-area" in fact refers to the area of the curvature diagram. (b) Sketch of the deformed beam.

orientation of the plane of curvature is defined by the angle α_k:

$$\tan\alpha_k = \frac{\kappa_{z;\text{C}}}{\kappa_{y;\text{C}}} = \frac{+5.170 \times 10^{-6}\ \text{mm}^{-1}}{-0.894 \times 10^{-6}\ \text{mm}^{-1}} = -5.783 \Rightarrow \alpha_k = 99.81°.$$

Comment: For the given load, the plane of curvature k is constant for the entire beam.

Figure 9.74a shows a parabolic curvature diagram for the simply supported beam AB. The maximum curvature is at midspan C. Figure 9.74b shows a sketch of the deformed beam. To find the displacement w_C at midspan C we will use the moment-area formulae, in which the "moment-area" in fact refers to the area of the curvature diagram.[1]

When considering the left-hand part AC of the curvature diagram, the displacement at C can be found by superimposing the tail-wagging effects due to the rotations θ_1 and θ_2.

θ_1 is the rotation of the beam at support A. It can be found as the "support reaction" due to a "distributed load" that is equal to the curvature diagram:[2]

$$\theta_1 = \tfrac{1}{2} \times \underbrace{\tfrac{2}{3}\kappa_\text{C}\ell}_{\substack{\text{area parabolic}\\ \text{curvature diagram}}} = \tfrac{1}{3}\kappa_\text{C}\ell.$$

θ_2 represents the influence of the deformation of AC, which is concentrated

---

1 The "moment-area theorems" are not based on the bending moment M, but on the curvature M/EI. Therefore a more correct name should be "curvature-area theorems".

2 See Section 8.5.1.

at the centroid of part AC of the curvature diagram. The location of the centroid is given in Figure 9.74a.[1]

$$\theta_1 = \underbrace{\tfrac{1}{2} \times \tfrac{2}{3}\kappa_C \ell}_{\text{area curvature diagram AC}} = \tfrac{1}{3}\kappa_C \ell.$$

Using the expressions derived for θ_1 and θ_2 we find the displacement w_C at C:

$$w_C = +\theta_1 \times \tfrac{1}{2}\ell - \theta_2 \times \tfrac{3}{16}\ell = +\tfrac{5}{48}\kappa_C \ell^2.$$

This formula can be applied to both the curvature diagram in the xy plane and the curvature diagram in the xz plane:

$$w_{y;C} = \tfrac{5}{48}\kappa_{y;C}\ell^2 = \tfrac{5}{48}(-0.894 \times 10^{-6}\ \text{mm}^{-1})(4000\ \text{mm})^2$$
$$= -1.49\ \text{mm},$$
$$w_{z;C} = \tfrac{5}{48}\kappa_{z;C}\ell^2 = \tfrac{5}{48}(+5.170 \times 10^{-6}\ \text{mm}^{-1})(4000\ \text{mm})^2$$
$$= +8.62\ \text{mm}.$$

The resultant displacement w_C occurs in the plane of curvature k (see Figure 9.75):

$$w_C = \sqrt{(-1.49\ \text{mm})^2 + (8.62\ \text{mm})^2} = 8.75\ \text{mm}.$$

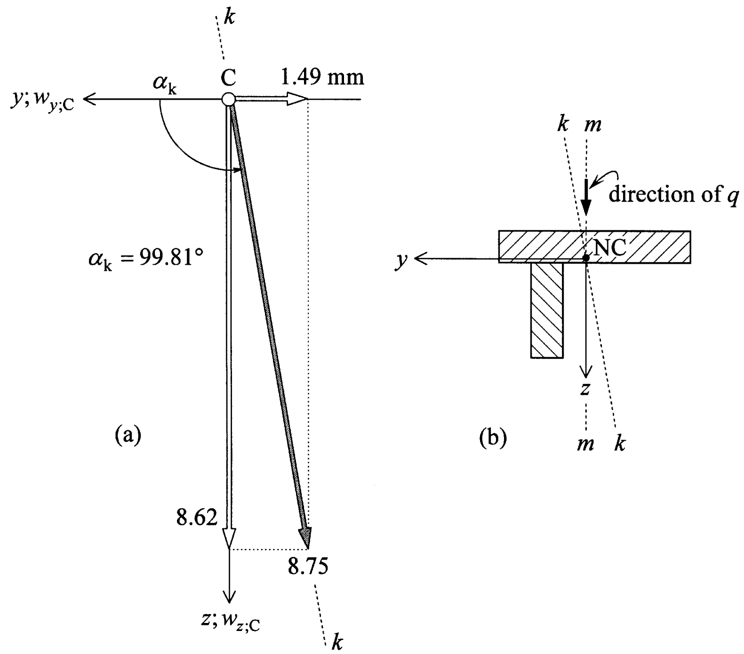

Figure 9.75 The displacement at C, and the components in the y and z directions. The displacement of 8.75 mm is in the plane of curvature k. (b) The cross-section with the plane of loading m and the plane of curvature k.

[1] See Section 8.4.1 and Tabel 8.5.

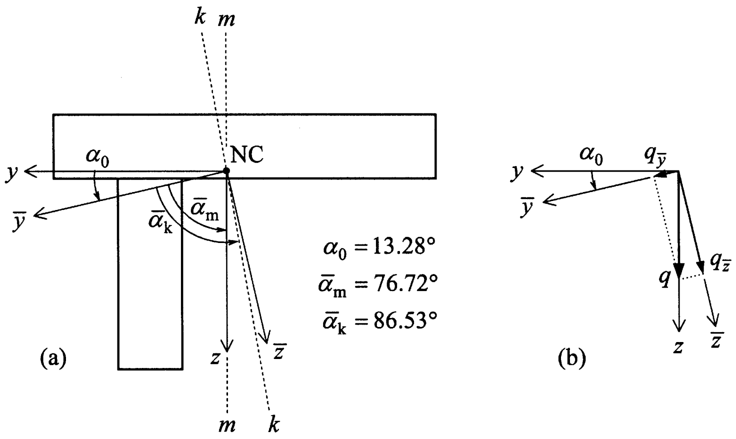

Figure 9.76 (a) The principal $\overline{yz}$ coordinate system. (b) The load q resolved into $\bar{y}$ and $\bar{z}$ components.

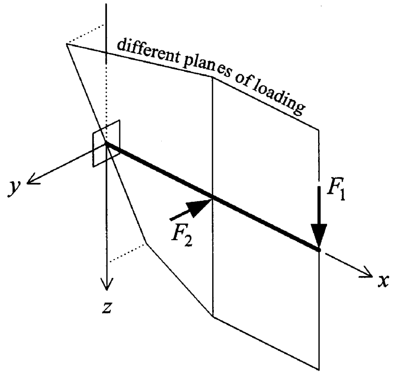

Figure 9.77 A situation with different planes of loading for the beam. The direction of the plane of loading is defined by the direction of the resultant bending moment in the cross-section.

Alternative solution:
For a simply supported beam with a uniformly distributed load we use the following *forget-me-not* to find the deflection at midspan:

$$\omega = \frac{5}{384}\frac{q\ell^4}{EI}.$$

This formula holds only for a uniformly distributed load in one of the principal directions. To apply this forget-me-not, we have to know the principal directions of the cross-section, the principal bending stiffness values and the components of the uniformly distributed load in these directions.

We are fortunate that the principal directions and principal bending stiffness values were derived in Section 9.12, Example 2. With respect to the yz coordinate system, the principal $\overline{yz}$ coordinate system is rotated through an angle $\alpha_0 = 13.28°$ (see Figure 9.76a). The principal bending stiffness values are

$$EI_{\overline{yy}} = EI_1 = 173.4 \times 10^{12}\ \text{Nmm}^2,$$
$$EI_{\overline{zz}} = EI_2 = 44.6 \times 10^{12}\ \text{Nmm}^2.$$

In Figure 9.76b the uniformly distributed load q is resolved in components in the principal directions:

$$q_{\bar{y}} = q\sin\alpha_0 = (120\ \text{kN/m}) \times 0.2297$$
$$= 27.56\ \text{kN/m} = 27.67\ \text{N/mm},$$
$$q_{\bar{z}} = q\cos\alpha_0 = (120\ \text{kN/m}) \times 0.9733$$
$$= 116.79\ \text{kN/m} = 116.79\ \text{N/mm}.$$

We now find

$$w_{\bar{y};\mathrm{C}} = \frac{5}{384}\frac{q_{\bar{y}}\ell^4}{EI_{\overline{yy}}} = \frac{5}{384}\frac{(27.56\ \mathrm{N/mm})(4000\ \mathrm{m})^4}{173.4\times 10^{12}\ \mathrm{Nmm}^2} = 0.53\ \mathrm{mm},$$

$$w_{\bar{z};\mathrm{C}} = \frac{5}{384}\frac{q_{\bar{z}}\ell^4}{EI_{\overline{zz}}} = \frac{5}{384}\frac{(116.79\ \mathrm{N/mm})(4000\ \mathrm{m})^4}{44.6\times 10^{12}\ \mathrm{Nmm}^2} = 8.73\ \mathrm{mm}.$$

The resultant displacement w_C is (see Figure 9.78):

$$w_\mathrm{C} = \sqrt{(0.53\ \mathrm{mm})^2 + (8.73\ \mathrm{mm})^2} = 8.75\ \mathrm{mm}.$$

The value and direction are in agreement with the results found before.

Comment: Again the planes of loading and curvature are constant over the entire length of the beam. Figure 9.77 shows a situation in which there are different planes of loading for the beam. The direction m of the plane of loading is defined by the direction of the resultant bending moment in the cross-section, and not by the direction of the resultant shear force!

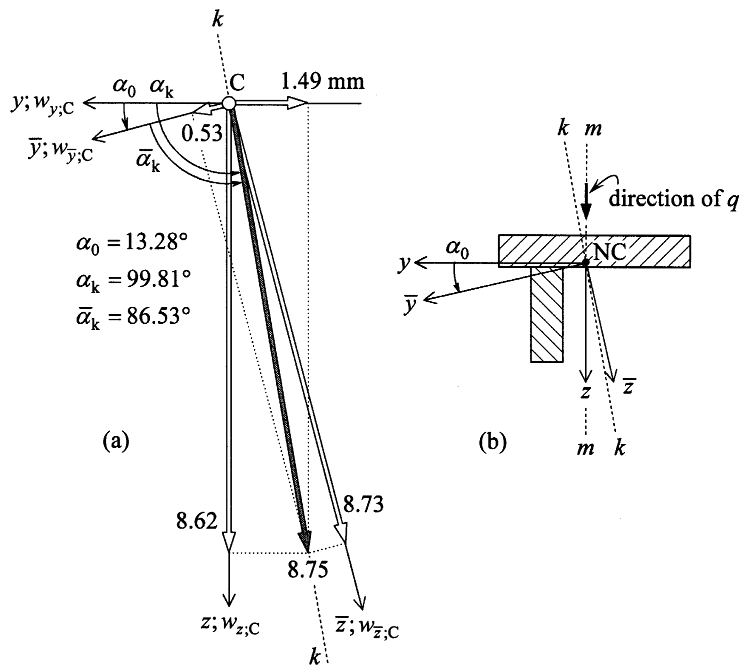

Figure 9.78 (a) The calculation of the displacement at C in the principal $\overline{yz}$ coordinate system gives the same result as the calculation in the non-principal yz coordinate system. (b) The cross-section with the principal $\overline{yz}$ coordinate system, the plane of loading m and the plane of curvature k.

9.14 Maxwell's reciprocal theorem

The strain formula, expressed in the cross-sectional deformation quantities, was derived in Section 9.2 (see equation (9.5)):

$$\varepsilon(y,z) = \varepsilon + \kappa_y y + \kappa_z z. \qquad (9.5)$$

It is common to choose a yz coordinate system with its origin coinciding with the normal centre NC of the cross-section. In that case, ε is the strain at the normal centre NC (at the beam axis), and is caused by extension only; κ_y and κ_z are the curvatures of the beam in the xy plane and xz plane respectively, and are caused by bending only.

In a *principal yz coordinate system*, the constitutive relationships are

$$\varepsilon = \frac{N}{EA} \quad \text{(extension)}, \tag{9.26}$$

$$\kappa_y = \frac{M_y}{EI_{yy}} \quad \text{and} \quad \kappa_z = \frac{M_z}{EI_{zz}} \quad \text{(bending)}. \tag{9.41}$$

Here, EI_{yy} and EI_{zz} are the principal bending stiffness values of the cross-section.

Strain formula (9.5) can also be expressed in terms of the section forces N, M_y and M_z. We then need to substitute the constitutive relationships for the cross-section into the formula for the strain distribution. In a principal yz coordinate system,[1]

$$\varepsilon(y, z) = \frac{N}{EA} + \frac{M_y y}{EI_{yy}} + \frac{M_z z}{EI_{zz}}. \tag{9.42}$$

For a non-zero normal force N, the three section forces N, M_y and M_z can be replaced with one single eccentric normal force N, which acts in a point (e_y, e_z). This point is referred to as the *centre of force* of the cross-section; it is the point of application of the resultant of all normal stresses in the cross-section.

[1] See also Section 9.10.

The coordinates e_y and e_z of the centre of force (the eccentricity of the normal force N) follow from[1]

$$e_y = \frac{M_y}{N} \text{ and } e_z = \frac{M_z}{N}\,.$$

For the bending moments in the xy and xz plane we now can write

$$M_y = N \cdot e_y \text{ and } M_z = N \cdot e_z.$$

Substitute these expressions in (9.42), and we find

$$\begin{aligned}\varepsilon(y,z) &= \frac{N}{EA} + \frac{Ne_y y}{EI_{yy}} + \frac{Ne_z z}{EI_{zz}} \\ &= \frac{N}{EA}\left(1 + \frac{EA}{EI_{yy}}\,e_y y + \frac{EA}{EI_{zz}}\,e_z z\right). \end{aligned} \tag{9.71}$$

The *radius of inertia* r is used to relate the second moment of area I to the cross-sectional area A, according to[2]

$$I = A \cdot r^2.$$

For inhomogeneous cross-sections, we use the *radius of inertia* to relate the bending stiffness EI to the axial stiffness EA:

$$r_y^2 = \frac{EI_{yy}}{EA} \text{ and } r_z^2 = \frac{EI_{zz}}{EA}\,. \tag{9.72}$$

1 See Section 9.4.
2 See also Section 3.2.1.

Since y and z are principal coordinate axes, r_y and r_z are the principal radii of inertia. They have the dimension of a length.

Comment: Although the notation of the radii of inertia r_y and r_z suggest that they are components of a vector, this is not the case. Upon rotation of the coordinate system they do not transform like the components of a vector.

Using (9.7.2) we can further simplify expression (9.71) for the strain distribution:

$$\varepsilon(y, z) = \frac{N}{EA}\left(1 + \frac{e_y y}{r_y^2} + \frac{e_z z}{r_z^2}\right). \tag{9.73}$$

This expression shows the strain ε at point (y, z) for a normal force N with its point of application at (e_y, e_z). As an experiment of mind we can think of a force N acting at (y, z) and observing the strain ε at (e_y, e_z). It appears to result in exactly the same strain. This is due to the equivalence of e_y and y in the strain formula, and of e_z and z.

We can summarise this phenomenon as follows.

The strain ε at P due to a force N at Q is equal to the strain ε at Q due to a force N at P (see Figure 9.79).

This is also known as *Maxwell's reciprocal theorem* and is general applicable to linear elastic systems for which the superposition theorem holds. We will make use of this theorem in the next section on the core of a cross-section.

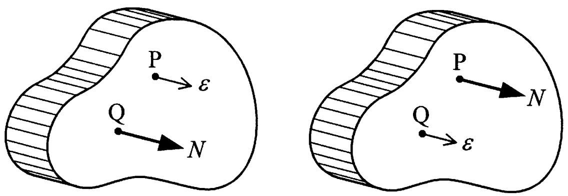

Figure 9.79 *Maxwell's reciprocal theorem*: the strain ε at P due to a force N at Q is equal to the strain ε at Q due to a force N at P.

9.15 Core of a cross-section

When the neutral axis intersects the cross-section, both tensile and compressive zones will occur on either side of the neutral axis. Some materials can hardly sustain tensile stresses, e.g. brick walls and unreinforced concrete. For these materials, the cross-section should be loaded in such a way that only compression occurs. The neutral axis should then be outside the cross-section or just at its boundary. With this requirement we can determine the area in which the centre of force should be positioned in order to prevent sign changes in the stress distribution. This area is called the *core* or *kern* of the cross-section. In other words: the core of a cross-section is the set of centres of force for which the neutral axis is outside the cross-section.

In Section 4.9 the core was introduced for a rectangular cross-section with dimensions $b \times h$, as shown in Figure 9.80. The core appeared to be a diamond with the corner points on the y and z axis with a distance to the NC of $b/6$ and $h/6$ respectively.

After discussing some properties of the core in Section 9.15.1, we will in Section 9.15.2 outline a general method to find the core of (in)homogeneous and unsymmetrical cross-sections. Some examples are given in Section 9.15.3.

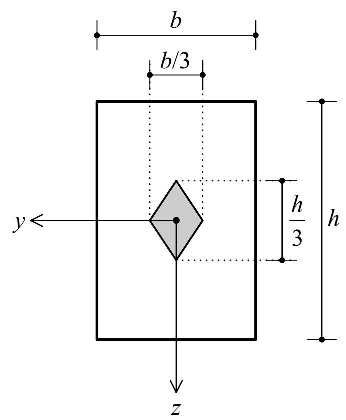

Figure 9.80 The core of a homogeneous rectangular cross-section.

9.15.1 Properties of the core

In the following we will make use of two important properties:

- For a neutral axis tangent to the cross-section, the associated centre of force is located on the edge of the core.
- Cross-sections for which all valid boundary positions of the neutral axis form a polygon, also have a polygon as core. The number of

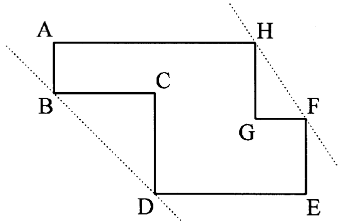

Figure 9.81 There are six valid boundary positions of the neutral axis: AB, AH, HF, FE, ED and DB. For each of the six valid boundary positions of the neutral axis, the associated centre of force is a *corner* of the core.

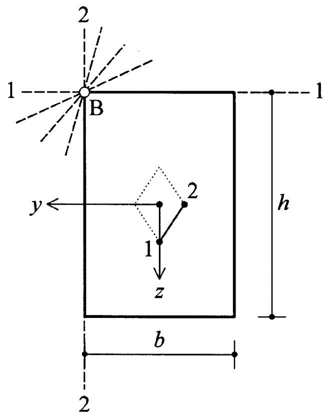

Figure 9.82 Each neutral axis passing through B and not intersecting the cross-section, relates to a centre of force on the straight line between the points 1 and 2. The valid boundary positions 1-1 and 2-2 of the neutral axis correspond with the centres of force 1 and 2 respectively; they are corners of the core, and are called *core points*.

corners of the core is equal to the number of valid boundary positions of the neutral axis.

A centre of force within the core corresponds with a stress distribution in the cross-section that does not exhibit a change in sign. So the total cross-section is in tension or in compression, which implies that the neutral axis is outside the cross-section.

For the cross-section in Figure 9.81 there are six valid boundary positions of the neutral axis: AB, AH, HF, FE, ED and DB. Note that a neutral axis which coincides with boundary BC cannot be valid since the neutral axis then intersects the cross-section. For each of the six valid boundary positions of the neutral axis, the associated centre of force is a corner of the core, and is called *core point*.

The second property states that when all valid boundary positions of the neutral axis form a polygon, the core has straight edges and is also a polygon. We will explain this for the simple rectangular cross-section in Figure 9.82. From the two boundary lines 1-1 and 2-2 the associated centres of force (core points) are denoted as 1 and 2. The quest now is to determine the boundary of the core between these points. Therefore we use Maxwell's reciprocal theorem: the strain ε at P due to a force N at Q is equal to the strain ε at Q due to a force N at P (see Figure 9.79).

When the centre of force is chosen at core point 1, then the neutral axis is the line 1-1 along the upper edge of the cross-section. So there is a zero strain at B. In reverse, according to Maxwell's reciprocal theorem: when the centre of force is chosen at B, there will be zero strain at core point 1. When the centre of force is chosen at core point 2, then the neutral axis is the line 2-2 along the left edge of the cross-section. Again there is a zero strain at B. In reverse, according to Maxwell's reciprocal theorem: when the centre of force is chosen at B, there will be zero strain at core point 2.

Therefore the neutral axis associated with the centre of force at B is a (straight) line which passes through the core points 1 and 2.

Conclusion:

- *For a force at B there is a zero strain at all points on the straight line through the core points 1 and 2.*

Maxwell's reciprocal theorem implies the following:

- *For all centres of force on the straight line between the core points 1 and 2 there is a zero strain at B.*

Each neutral axis passing through B and not intersecting the cross-section, relates to a centre of force on the straight line between the points 1 and 2 (see Figure 9.82). The valid boundary positions 1-1 and 2-2 of the neutral axis correspond with the centres of force 1 and 2; they are corners of the core and are called *core points*.

This proves that the boundary of the core for cross-sections with straight edges is built up by straight lines.

Figure 9.83 shows two cross-sections for which not all edges are straight. In Figure 9.83a the four valid boundary positions of the neutral axis form a polygon, so the core of the cross-section is also a polygon, with four sides.

Figure 9.83b shows a cross-section for which edge AB is not straight. The core points (centres of force) 2 and 3 are associated with the neutral axes 2-2 through A and 3-3 through B respectively. Any neutral axis tangent to the curved edge AB is associated with a centre of force on the boundary of the core between the points 2 and 3. This results in a *curved boundary of the core* between these points. The determination of this part of the core is quite laborious.

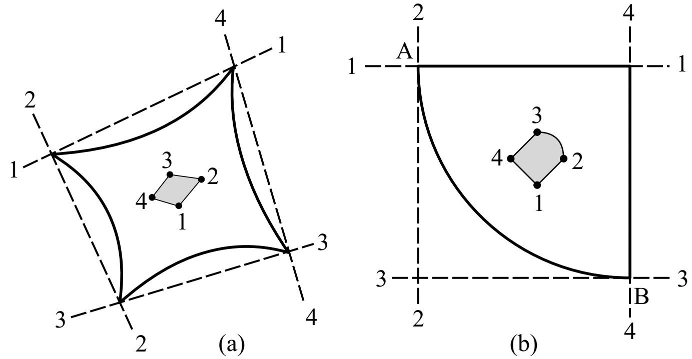

Figure 9.83 Cross-sections with curved edges. (a) The four valid boundary positions of the neutral axis form a polygon, so the core of the cross-section is also a polygon, with four sides. (b) Any neutral axis tangent to the curved edge AB is associated with a centre of force on the boundary of the core. This results in a *curved boundary of the core* between the points 2 and 3.

9.15.2 General method to find the core

The neutral axis in a cross-section is defined by

$$\varepsilon(y, z) = \varepsilon + \kappa_y \cdot y + \kappa_z \cdot z = 0.$$

When the yz coordinate system is chosen in such a way that its origin coincides with the normal centre NC, then extension and bending can be treated separately. The strain ε at the origin of the coordinate system is caused by extension and the curvatures κ_y and κ_z are caused by bending. The cross-sectional constitutive relationships are

$$N = EA\varepsilon \qquad \text{(extension)}, \qquad (9.26)$$

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \qquad \text{(bending)}. \qquad (9.40)$$

For a cross-section with non-zero normal force (and therefore $\varepsilon \neq 0$), the equation for the neutral axis can be written as

$$1 + \frac{\kappa_y}{\varepsilon} y + \frac{\kappa_z}{\varepsilon} z = 0. \qquad (9.71)$$

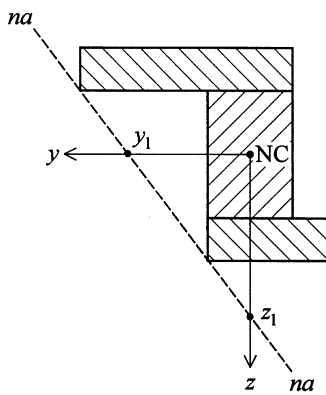

Figure 9.84 A neutral axis *na* which is bounding the cross-section. This neutral axis intersects the y and z coordinate axes in the points $(y_1, 0)$ and $(0, z_1)$ respectively.

Consider a neutral axis *na* which is bounding the cross-section (see Figure 9.84). Assume this neutral axis intersects the y and z coordinate axes in the points $(y_1, 0)$ and $(0, z_1)$ respectively. Using (9.71) we can relate these points of intersection to the three cross-sectional deformation quantities:

$$y_1 = -\frac{\varepsilon}{\kappa_y} \quad \text{and} \quad z_1 = -\frac{\varepsilon}{\kappa_z}. \qquad (9.72)$$

Between the components M_y and M_z of the bending moment and the components e_y and e_z of the eccentricity of the normal force N, there is the following relationship:[1]

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = N \begin{bmatrix} e_y \\ e_z \end{bmatrix}. \tag{9.18}$$

Substitute in (9.18) the constitutive relationship for extension:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = EA\varepsilon \begin{bmatrix} e_y \\ e_z \end{bmatrix}. \tag{9.73}$$

Substitute (9.73) in the constitutive relationship for bending,

$$EA\varepsilon \begin{bmatrix} e_y \\ e_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix},$$

and use (9.72) to find the coordinates e_y and e_z of the centre of force:

$$\begin{aligned} \begin{bmatrix} e_y \\ e_z \end{bmatrix} &= \frac{1}{EA} \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y/\varepsilon \\ \kappa_z/\varepsilon \end{bmatrix} \\ &= -\frac{1}{EA} \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} 1/y_1 \\ 1/z_1 \end{bmatrix}. \end{aligned} \tag{9.74}$$

1 See Section 9.4.

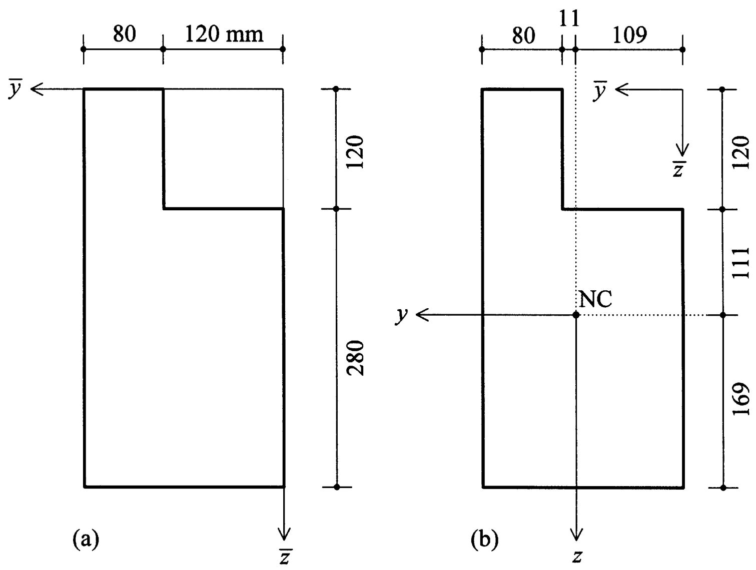

Figure 9.85 (a) A homogeneous and unsymmetrical cross section. (b) The location of the normal centre NC.

For a neutral axis which is tangent to the cross-section we now have found the associated centre of force (e_y, e_z), which is a point on the boundary of the core.

For a homogeneous cross-section the location is independent of the modulus of elasticity E, since this quantity can be cancelled in (9.74):

$$\begin{bmatrix} e_y \\ e_z \end{bmatrix} = -\frac{1}{A} \begin{bmatrix} I_{yy} & I_{yz} \\ I_{zy} & A_{zz} \end{bmatrix} \begin{bmatrix} 1/y_1 \\ 1/z_1 \end{bmatrix}. \tag{9.75}$$

Reminder: Do not forget the minus sign in the formulae (9.74) and (9.75)!

Finding the core points has reduced to a straightforward procedure in which all valid boundary positions of the neutral axis (tangent to the cross-section) are considered. Each neutral axis is defined by the two points of intersection with the coordinate axes. With formula (9.74) – or (9.75) for homogeneous cross-sections – the location of the associated core point can be found. This procedure will be illustrated by three examples.

9.15.3 Examples

Example 1: Core of a homogeneous unsymmetrical cross-section

Question:
Find the core of the homogeneous and unsymmetrical cross-section as shown in Figure 9.85a.

Solution (units mm):
If the neutral axis coincides with a straight edge of the cross-section, without intersecting the cross-section, the corresponding centre of force is a corner of the core or *core point*. Four of such neutral axes can be drawn. A fifth neutral axis, not coinciding with a straight edge, is bounding the

cross-section at two corners (see Figure 9.86).[1] The five neutral axes will result in five core points; so the core is a five-sided polygon.

To find the core points of a homogeneous cross-section we need to know

- the cross-sectional area A;
- the location of the normal centre NC;
- the centroidal moments of inertia I_{yy}, $I_{yz} = I_{zy}$ and I_{zz}.

The cross-sectional area is

$$A = 400 \times 200 - 120^2 = 65.6 \times 10^3 \text{ mm}^2.$$

The position of the normal center NC can be found with the method explained before.[2] In a $\overline{yz}$ coordinate system with the $\bar{y}$ axis along the upper side of the cross-section, and the $\bar{z}$ axis along the right side of the cross-section (see Figure 9.85a),

$$\bar{y}_{\text{NC}} = \frac{ES_{\bar{y}}}{EA} = \frac{S_{\bar{y}}}{A} \frac{400 \times 200 \times 100 - (120)^2 \times 60}{65.6 \times 10^3} = 109 \text{ mm},$$

$$\bar{z}_{\text{NC}} = \frac{ES_{\bar{z}}}{EA} = \frac{S_{\bar{z}}}{A} \frac{400 \times 200 \times 200 - (120)^2 \times 60}{65.6 \times 10^3} = 231 \text{ mm}.$$

The location of the NC is shown in Figure 9.85b.

Note that in a homogeneous cross-section the location of the normal centre NC is independent of the modulus of elasticity E.

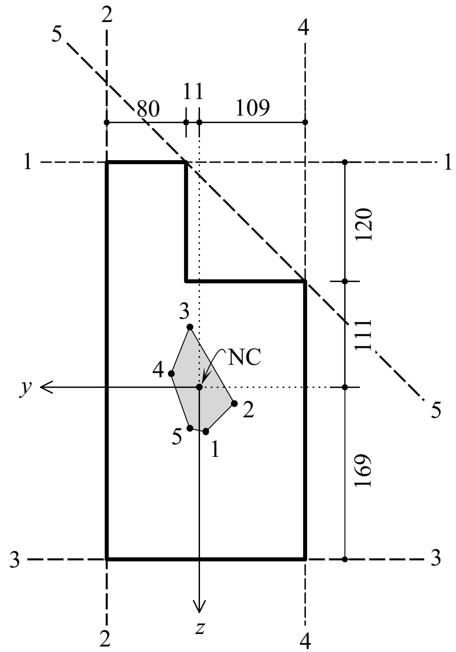

Figure 9.86 The core of the cross-section.

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1 See Section 7.1.1.

2 See Section 3.1.3 for homogeneous cross-sections, and Section 9.4 for homogeneous and inhomogeneous cross-sections.

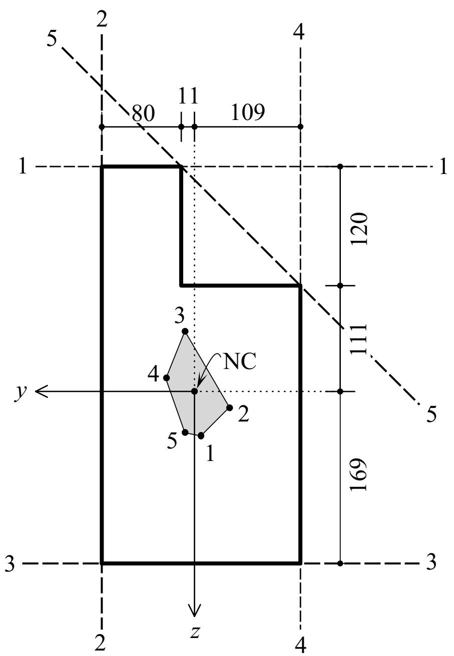

Figure 9.86 The core of the cross-section.

The centroidal moments of inertia are

$$I_{yy} = \tfrac{1}{12} \times 400 \times 200^3 + 400 \times 200 \times 9^2 - \tfrac{1}{12} \times 120^4 - 120^2 \times 49^2$$
$$= 221.3 \times 10^6 \text{ mm}^4,$$
$$I_{yz} = 400 \times 200 \times 9 \times 31 - 120^2 \times (-49) \times (-171)$$
$$= -98.3 \times 10^6 \text{ mm}^4,$$
$$I_{zz} = \tfrac{1}{12} \times 200 \times 400^3 + 400 \times 200 \times 31^2 - \tfrac{1}{12} \times 120^4 - 120^2 \times 171^2$$
$$= 705.2 \times 10^6 \text{ mm}^4.$$

In order to find all core points we will tabulate the calculation and use a systematic numbering of the five neutral axes tangent to the cross-section, and their associated centres of force (see Figure 9.86). All calculus can be done with a spreadsheet, for instance in Excel. For each assumed position of the neutral axis the points of intersection $(y_1, 0)$ and $(0, z_1)$ with the coordinate axes are determined first. The position of the associated centre of force can then be found with (9.75):

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = -\frac{1}{A} \begin{bmatrix} I_{yy} & I_{yz} \\ I_{zy} & E_{zz} \end{bmatrix} \begin{bmatrix} 1/y_1 \\ 1/z_1 \end{bmatrix},$$

or, after substituting the numerical values of A, I_{yy}, $I_{yz} = I_{zy}$ and I_{zz},

$$e_y = -\frac{3.373 \times 10^3 \text{ mm}^2}{y_1} + \frac{1.498 \times 10^3 \text{ mm}^2}{z_1},$$

$$e_z = +\frac{1.498 \times 10^3 \text{ mm}^2}{y_1} - \frac{10.750 \times 10^3 \text{ mm}^2}{z_1}.$$

The results can be found in Table 9.10 and the graph of Figure 9.86.

If the normal force (the resultant of all normal stresses) has its point of application within the core, there will be no change of sign in the stress distribution, irrespective of the magnitude of the normal force!

Particularly for prestressed concrete beams this can be important. If the prestressing tendons are within the core of the cross-section no tensile stresses due to the prestressing will occur. We mention again that this is independent of the magnitude of the prestressing force!

Example 2: Core of a composite steel-concrete column

The composite steel-concrete column AB in Figure 9.87 is fixed at A and free at B. The column is loaded by an eccentric compressive force F at the free end B. The steel I-section is not exactly in the centre of the column, as can be seen from the cross-sectional measurements in the figure.

For the column a linear elastic behaviour is assumed. The moduli of elasticity are $E_a = 210 \times 10^3$ N/mm^2 for the steel I-section and $E_c = 20 \times 10^3$ N/mm^2 for the concrete. Furthermore, the following cross-sectional properties are given for the steel I-section in its centroidal $\overline{yz}$ coordinate system (see Figure 9.87b):

$$A_a = 10 \times 10^3 \text{ mm}^2,$$

$$I_{\overline{yy};a} = 40 \times 10^6 \text{ mm}^4,$$

$$I_{\overline{yz};a} = I_{\overline{zy};a} = 0 \quad \text{(symmetrical cross-section)},$$

$$I_{\overline{zz};a} = 112.5 \times 10^6 \text{ mm}^4.$$

Table 9.10 Calculation results.

neutral axis	y_1 (mm)	z_1 (mm)	core point	e_y (mm)	e_z (mm)
1-1	$\pm\infty$	-231	1	-6.5	$+46.5$
2-2	$+91$	$\pm\infty$	2	-36.7	$+16.3$
3-3	$\pm\infty$	$+169$	3	$+8.9$	-63.6
4-4	-109	$\pm\infty$	4	$+30.9$	-13.7
5-5	-220	-220	5	$+8.5$	$+42.1$

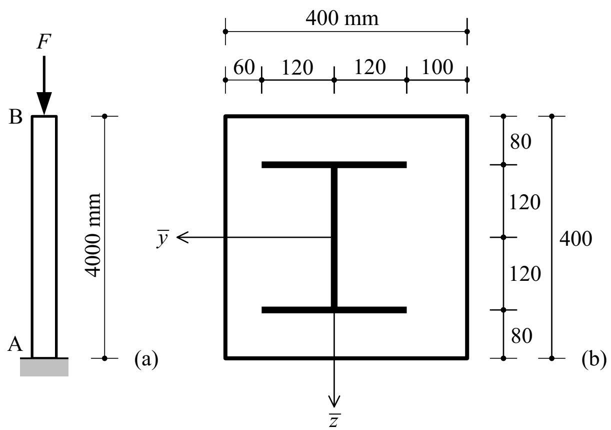

Figure 9.87 Composite steel-concrete column.

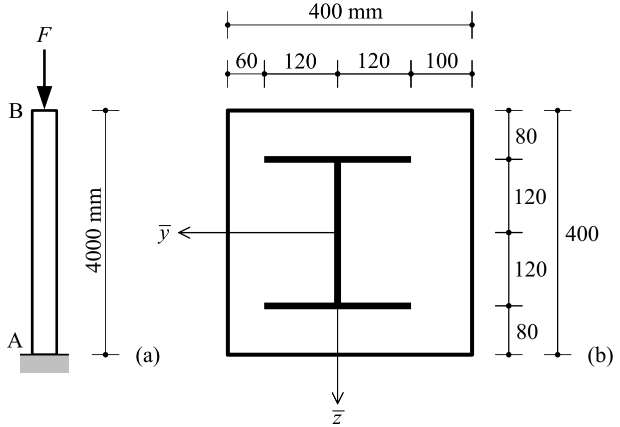

Figure 9.87 Composite steel-concrete column.

Questions:

a. Locate the normal centre NC.
b. Calculate for the composite column the cross-sectional stiffness quantities in a yz coordinate system with its origin at the normal centre NC of the cross-section.
c. Which points of application are allowed for the compressive force to prevent tensile stresses in the column.

Solution (units N and mm):

a. We start to work in the $\overline{yz}$ coordinate system with its origin at the centre of the steel I-section (see Figure 9.87b). The $\bar{y}$ axis is a line of symmetry for the composite column, so the normal centre NC is on the $\bar{y}$ axis:

$$\bar{z}_{\text{NC}} = 0.$$

The $\bar{y}$ coordinate of NC follows from

$$\bar{y}_{\text{NC}} = \frac{ES_{\bar{y}}}{EA}.$$

So we have to find EA and $ES_{\bar{y}}$. We distinguish two contributions: the steel section, and the full concrete cross-section reduced with the recess for the I-section (applied as a negative area of concrete).

$$\begin{aligned} EA &= \underbrace{(210 \times 10^3) \times (10 \times 10^3)}_{(EA)_\text{a}} + \underbrace{(20 \times 10^3)(400^2 - \overbrace{10 \times 10^3}^{\text{recess in the concrete}})}_{(EA)_\text{c}} \\ &= 5.1 \times 10^9 \text{ N}, \\ ES_{\bar{y}} &= (ES_{\bar{y}})_\text{c} \\ &= (20 \times 10^3) \times 400^2 \times (-20) = -6.4 \times 10^9 \text{ Nmm}. \end{aligned}$$

Note that in the $\overline{yz}$ coordinate system $S_{\bar{y}} = 0$ for both the steel section and the recess in the concrete.

We now find:

$$\bar{y}_{\mathrm{NC}} = \frac{ES_{\bar{y}}}{EA} = \frac{-6.4 \times 10^9 \text{ Nmm}}{5.1 \times 10^9 \text{ N}} = -12.55 \text{ mm}.$$

b. Next we work in the yz coordinate system with its origin at the normal centre NC, as shown in Figure 9.88. Since the y axis is a line of symmetry for the cross-section, $EI_{yz} = EI_{zy} = 0$. The yz coordinate system is a principal coordinate system, and EI_{yy} and EI_{zz} are the principal bending stiffness values.

$$EI_{yy} = \underbrace{(210 \times 10^3)}_{E_\mathrm{a}} \times [(40 \times 10^6) + (10 \times 10^3) \times 12.55]^2$$

$$+ \underbrace{(20 \times 10^3)}_{E_\mathrm{c}} \times \big[\tfrac{1}{12} \times 400^4 + 400^2 \times 7.45^2$$

$$- \underbrace{\{(40 \times 10^6) + (10 \times 10^3) \times 12.55^2\}}_{\text{recess in the concrete}}\big]$$

$$= 50.774 \times 10^{12} \text{ Nmm}^2,$$

$$EI_{zz} = \underbrace{(210 \times 10^3)}_{E_\mathrm{a}} \times 112.5 \times 10^6$$

$$+ \underbrace{(20 \times 10^3)}_{E_\mathrm{c}} \times \big\{\tfrac{1}{12} \times 400^4 - \underbrace{112.5 \times 10^6}_{\text{recess in the concrete}}\big\}$$

$$= 64.042 \times 10^{12} \text{ Nmm}^2.$$

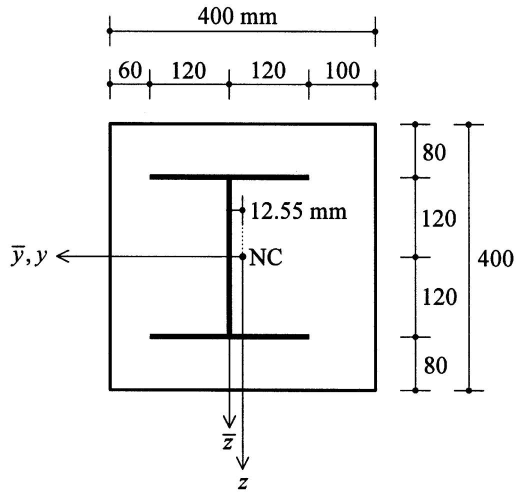

Figure 9.88 The normal centre NC of the composite cross-section.

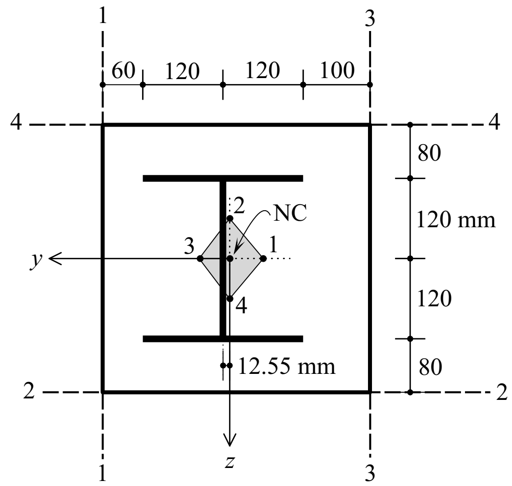

Figure 9.89 The core of the composite cross-section.

Table 9.11 Calculation results.

neutral axis	y_1 (mm)	z_1 (mm)	core point	e_y (mm)	e_z (mm)
1-1	+192.45	$\pm\infty$	1	−51.7	0
2-2	$\pm\infty$	+200	2	0	−62.8
3-3	−207.45	$\pm\infty$	3	+48.0	0
4-4	$\pm\infty$	−200	4	0	+62.8

The axial stiffness EA was found earlier:

$$EA = 5.1 \times 10^9 \text{ N}.$$

c. Due to the eccentric load F at B, the normal force and bending moment are constant along the column, and therefore the stress distribution is the same for all cross-sections. If no tensile stresses are allowed, the point of application of the compressive force F (the centre of force) must be within the core of the cross-section. To find the core we have to consider four bounding neutral axes, for which the associated centres of force are the core points (see Figure 9.89). The core is a four sided polygon. The core points can be found with the procedure explained in Section 9.15.2, by using formula (9.74), appropriate for inhomogeneous cross-sections:

$$\begin{bmatrix} e_y \\ e_z \end{bmatrix} = -\frac{1}{EA}\begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix}\begin{bmatrix} 1/y_1 \\ 1/z_1 \end{bmatrix}. \tag{9.74}$$

Substituting the numerical values of EA, EI_{yy}, $EI_{yz} = EI_{zy}$ (= 0) and EI_{zz}, we find

$$e_y = -\frac{EI_{yy}}{EA}\frac{1}{y_1} = -\frac{50.774 \times 10^{12} \text{ Nmm}^2}{5.1 \times 10^9 \text{ N}}\frac{1}{y_1} = -\frac{9.956 \times 10^3 \text{ mm}}{y_1},$$

$$e_z = -\frac{EI_{zz}}{EA}\frac{1}{z_1} = -\frac{64.042 \times 10^{12} \text{ Nmm}^2}{5.1 \times 10^9 \text{ N}}\frac{1}{z_1} = -\frac{12.557 \times 10^3 \text{ mm}}{z_1}.$$

The results can be found in Table 9.11 and the graph of Figure 9.89.

If the point of application of the compressive force F is within the core, the neutral axis is outside the cross-section, and only compressive stresses occur in the cross-section. With the point of application on the boundary of the core, the neutral axis is bounding the cross-section, and there are zero stresses in one or more points on the edge of the cross-section.

Example 3: Core of a part of an inhomogeneous cross-section
The third and last example concerns the composite cross-section in Figure 9.90, built up by four squares of different materials. For this cross-section, the normal centre NC and axial stiffness EA were determined in Section 9.7, Example 2, and the bending stiffness quantities were calculated in Section 9.9.3:

$$EA = 624 \text{ MN},$$

$$EI_{yy} = 260.43 \times 10^9 \text{ Nmm}^2,$$

$$EI_{yz} = EI_{zz} = 59.07 \times 10^9 \text{ Nmm}^2,$$

$$EI_{zz} = 324.76 \times 10^9 \text{ Nmm}^2.$$

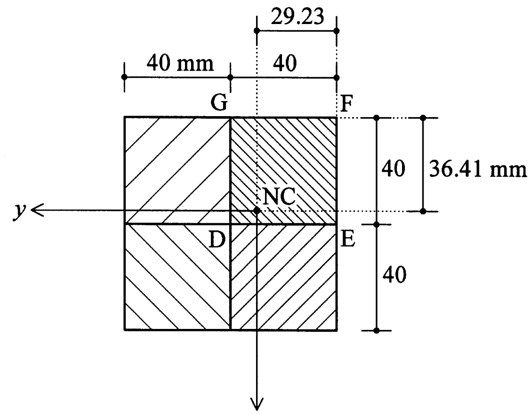

Figure 9.90 A composite cross-section, built up out of four squares of different materials, and the location of the normal centre NC.

The cross-section is loaded by an eccentric compressive force.

Question:
Which points of application are possible for the compressive force to prevent tensile stresses in the material DEFG?

Solution:
For this problem we find a way out with the core of part DEFG of the cross-section. For a compressive force with its point of application (centre of force) within this core, the neutral axis will not intersect DEFG, so there are only compressive stresses within DEFG. With the point of application on the boundary of the core, the neutral axis is bounding the edge of part DEFG, and there are zero stresses in one or more points on edge DEFG.

To calculate the core points associated with the four neutral axes bounding

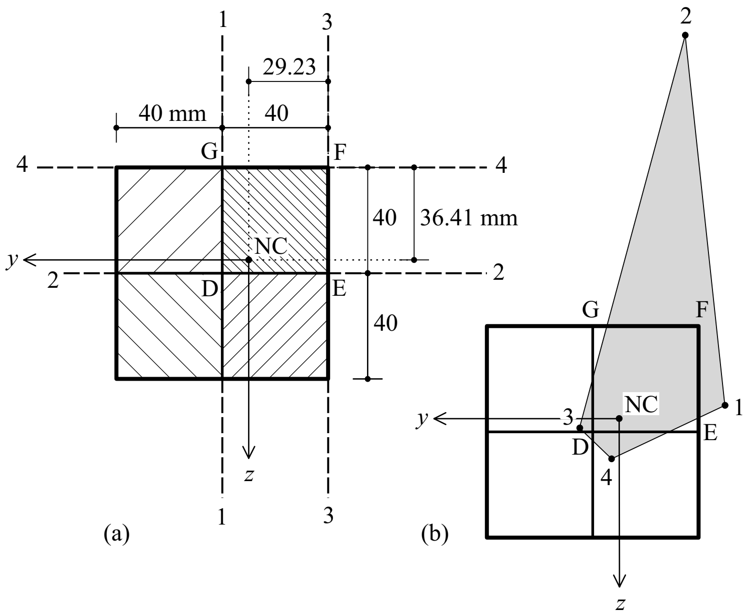

Figure 9.91 (a) The location of the normal centre NC. (b) If the cross-section is subject to a compressive force, and no tensile stresses are allowed in the material DEFG, then the points of application of the compressive force have to be within the shaded area shown.

DEFG, as shown in Figure 9.91a, we use again (9.75):

$$\begin{bmatrix} e_y \\ e_z \end{bmatrix} = -\frac{1}{EA}\begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix}\begin{bmatrix} 1/y_1 \\ 1/z_1 \end{bmatrix}$$

$$= -\frac{10^9 \text{ Nmm}^2}{624 \times 10^6 \text{ N}} \times \begin{bmatrix} 260.43 & 59.07 \\ 59.07 & 324.76 \end{bmatrix}\begin{bmatrix} 1/y_1 \\ 1/z_1 \end{bmatrix}$$

$$= -(1 \text{ mm}^2) \times \begin{bmatrix} 417.36 & 94.66 \\ 94.66 & 520.45 \end{bmatrix}\begin{bmatrix} 1/y_1 \\ 1/z_1 \end{bmatrix},$$

or

$$e_y = -\frac{417.36 \text{ mm}^2}{y_1} - \frac{94.66 \text{ mm}^2}{z_1},$$

$$e_z = -\frac{94.66 \text{ mm}^2}{y_1} - \frac{520.45 \text{ mm}^2}{z_1}.$$

The calculation is performed in Table 9.12. The core for DEFG is sketched in Figure 9.91b.

Table 9.12 Calculation results.

neutral axis	y_1 (mm)	z_1 (mm)	core point	e_y (mm)	e_z (mm)
1-1	+10.77	±∞	1	−38.8	−8.8
2-2	±∞	+3.59	2	−26.4	−145.0
3-3	−29.23	±∞	3	+14.3	+3.2
4-4	±∞	−36.41	4	+2.6	+14.3

9.16 Thermal effects

Constrained or restricted deformations due to thermal effects may lead to considerable stresses in structures. These stresses can even be larger than stresses due to normal loading conditions. Temperature loads can therefore not be neglected. In Section 4.12 the influence of a linear temperature distribution over the depth of a cross-section was discussed for a homogeneous symmetrical cross-section. In this section, we will consider inhomogeneous and/or unsymmetrical cross-sections, and a temperature distribution which can be an arbitrary function of y and z.

In Section 9.16.1 we investigate the effect of a change in temperature on the cross-sectional constitutive relationships for extension and bending. Next the results are applied to a statically determinate beam in Section 9.16.2 and a statically indeterminate beam in Section 9.16.3. In these sections we will look at the strain and stress distribution in the cross-section and the deflection of the beam.

9.16.1 The constitutive relationships

The modulus of elasticity E as well as the *coefficient of thermal expansion* α and the change in temperature T are functions depending on y and z:

$$E = E(y, z),$$

$$\alpha = \alpha(y, z),$$

$$T = T(y, z).$$

To simplify the expressions we introduce the following two- and three-letter symbols as functions of y and z:

$$\alpha(y,z) \cdot T(y,z) = \alpha T(y,z),$$

$$\alpha(y,z) \cdot E(y,z) \cdot T(y,z) = \alpha E T(y,z).$$

The extension of the fibre model subject to a change in temperature results in a fibre strain which is the combination of a strain due to the stresses and a strain due to the thermal effect. In order to clearly distinct between these two components we use a superscript T or σ:

ε^T – strain due to the thermal effect;

ε^σ – strain due to the stresses.

The strain in a fibre due to a change in temperature, denoted with the temperature distribution function $T(y,z)$, can be expressed as

$$\varepsilon^T(y,z) = \alpha T(y,z). \tag{9.76}$$

The strain due to the stress in a fibre follows from the constitutive relationship, e.g. Hooke's law:

$$\varepsilon^\sigma(y,z) = \frac{\sigma(y,z)}{E(y,z)}. \tag{9.77}$$

The total strain definition thus becomes

$$\varepsilon(y,z) = \varepsilon^T(y,z) + \varepsilon^\sigma(y,z) = \alpha T(y,z) + \frac{\sigma(y,z)}{E(y,z)}. \tag{9.78}$$

The assumptions as presented in Section 9.1 will also hold in this section. Since plane cross-sections remain plane, the cross-sectional strain distribution is linear in y and z, and can be found with the earlier derived kinematic relationship (9.5):

$$\varepsilon(y,z) = \varepsilon + \kappa_y y + \kappa_z z. \tag{9.5}$$

When we combine the latter two expressions, we obtain

$$\begin{aligned}\sigma(y,z) &= E(y,z)\varepsilon^{\sigma}(y,z)\\ &= E(y,z)\{\varepsilon(y,z) - \varepsilon^{T}(y,z)\}\\ &= E(y,z)\{\underbrace{\varepsilon + \kappa_y y + \kappa_z z}_{\varepsilon(y,z)} - \underbrace{\alpha T(y,z)}_{\varepsilon^{T}(y,z)}\}.\end{aligned} \tag{9.79}$$

With this expression the section forces N, M_y and M_z can be determined as described in Section 1.4 using the well-known *double letter symbols*:

Normal force:

$$\begin{aligned}N &= \int_A \sigma(y,z)\,\mathrm{d}A = \int_A E(y,z)\{\varepsilon + \kappa_y y + \kappa_z z - \alpha T(y,z)\}\,\mathrm{d}A\\ &= EA\varepsilon + ES_y\kappa_y + ES_z\kappa_z - \int_A \alpha E T(y,z)\,\mathrm{d}A.\end{aligned}$$

Bending moment:

$$\begin{aligned}M_y &= \int_A y\sigma(y,z)\,\mathrm{d}A\\ &= ES_y\varepsilon + EI_{yy}\kappa_y + EI_{yz}\kappa_z - \int_A y\cdot\alpha E T(y,z)\,\mathrm{d}A,\\ M_z &= \int_A z\sigma(y,z)\,\mathrm{d}A\\ &= ES_z\varepsilon + EI_{zy}\kappa_y + EI_{zz}\kappa_z - \int_A z\cdot\alpha E T(y,z)\,\mathrm{d}A.\end{aligned}$$

With the origin of the yz coordinate system chosen at the *normal centre* NC of the cross-section (then $ES_y = ES_z = 0$), the expressions for the section forces simplify to

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \kappa_y \\ \kappa_z \end{bmatrix} - \begin{bmatrix} \int_A \alpha ET(y,z)\,\mathrm{d}A \\ \int_A y \cdot \alpha ET(y,z)\,\mathrm{d}A \\ \int_A z \cdot \alpha ET(y,z)\,\mathrm{d}A \end{bmatrix}. \quad (9.80)$$

If the beam is unconstrained (can deform freely) and is not loaded by forces, the section forces are zero. So the deformations which may occur are the result of the thermal effect only. The cross-sectional deformation quantities due to a change in temperature are denoted with the superscript T, and can be found from (9.80):

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \varepsilon^T \\ \kappa_y^T \\ \kappa_z^T \end{bmatrix} - \begin{bmatrix} \int_A \alpha ET(y,z)\,\mathrm{d}A \\ \int_A y \cdot \alpha ET(y,z)\,\mathrm{d}A \\ \int_A z \cdot \alpha ET(y,z)\,\mathrm{d}A \end{bmatrix}. \quad (9.81)$$

This system of equations can be solved with the expression for the inverse stiffness tensor:

$$\begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{zy} & EI_{zz} \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{EA} & 0 & 0 \\ 0 & \dfrac{EI_{zz}}{\mathrm{Det}(EI)} & -\dfrac{EI_{zy}}{\mathrm{Det}(EI)} \\ 0 & -\dfrac{EI_{yz}}{\mathrm{Det}(EI)} & \dfrac{EI_{yy}}{\mathrm{Det}(EI)} \end{bmatrix}$$

in which $\mathrm{Det}(EI)$ is the determinant of the bending stiffness matrix:

$$\mathrm{Det}(EI) = EI_{yy}EI_{zz} - EI_{yz}^2.$$

For the cross-sectional deformation quantities due to a change in temperature we find in an arbitrary yz coordinate system, with its origin at the normal centre NC,

$$\varepsilon^T = \frac{1}{EA}\int_A \alpha ET(y,z)\,\mathrm{d}A, \tag{9.82}$$

$$\kappa_y^T = \frac{1}{\mathrm{Det}(EI)}\left\{EI_{zz}\int_A y\cdot\alpha ET(y,z)\,\mathrm{d}A - EI_{zy}\int_A z\cdot\alpha ET(y,z)\,\mathrm{d}A\right\},$$

$$\kappa_z^T = \frac{1}{\mathrm{Det}(EI)}\left\{-EI_{yz}\int_A y\cdot\alpha ET(y,z)\,\mathrm{d}A + EI_{yy}\int_A z\cdot\alpha ET(y,z)\,\mathrm{d}A\right\}.$$

With these expressions, formula (9.80) can be simplified to

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{zy} & EI_{zz} \end{bmatrix}\begin{bmatrix} \varepsilon - \varepsilon^T \\ \kappa_y - \kappa_y^T \\ \kappa_z - \kappa_z^T \end{bmatrix}. \tag{9.83}$$

For the special case in which the coordinate system coincides with the principal directions of the cross-section, the expressions will further simplify. Since the bending in the xy and the xz plane then are uncoupled ($EI_{yz} = EI_{zy} = 0$), the constitutive relationships become

$$N = EA(\varepsilon - \varepsilon^T), \quad \text{in which} \quad \varepsilon^T = \int_A \frac{\alpha ET(y,z)\,\mathrm{d}A}{EA}, \tag{9.84a}$$

$$M_y = EI_{yy}(\kappa_y - \kappa_y^T), \quad \text{in which} \quad \kappa_y^T = \int_A \frac{y \cdot \alpha ET(y,z)}{EI_{yy}}\,\mathrm{d}A, \tag{9.84b}$$

$$M_z = EI_{zz}(\kappa_z - \kappa_z^T), \quad \text{in which} \quad \kappa_z^T = \int_A \frac{z \cdot \alpha ET(y,z)}{EI_{zz}}\,\mathrm{d}A. \tag{9.84c}$$

On cross-sectional level, only the constitutive relationships have to be modified to obtain strains and stresses when introducing thermal effects. However on structural level there will arise some problems, and we have to distinct between statically determinate and statically indeterminate structures.

Statically determinate structures
In a statically determinate structure the structural elements are unconstrained and can deform freely. The displacements and rotations due to thermal effects can be determined directly from the obtained cross-sectional temperature deformations ε^T, κ_y^T and κ_z^T. A change in temperature results only in additional deformations which may occur freely; it does not effect the section forces N, M_y and M_z. The section forces are independent of the temperature and depend only upon the loading by forces. The section forces can be found directly from the equilibrium conditions. An example is given in Section 9.16.2.

Statically indeterminate structures

In statically indeterminate structures the deformations cannot occur freely but are constrained, and the situation becomes more complex. The section forces can not be determined with the equilibrium conditions only; the kinematic and constitutive relationships are also required. Therefore the section forces in a statically indeterminate structure are influenced by a change in temperature. Using the compatibility or deformation conditions for the deformation due to the loading by forces and the thermal effects, we can determine the force distribution in the structure. In a cross-section the deformation quantities ε, κ_y and κ_z can now be found from the section forces N, M_y and M_z with help of the constitutive relationship (9.83):

$$\begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 \\ 0 & EI_{yy} & EI_{yz} \\ 0 & EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \varepsilon - \varepsilon^T \\ \kappa_y - \kappa_y^T \\ \kappa_z - \kappa_z^T \end{bmatrix}.$$

Subsequently the stress distribution in the cross-section can be found from (9.79):

$$\sigma(y,z) = E(y,z)\{\underbrace{\varepsilon + \kappa_y y + \kappa_z z}_{\varepsilon(y,z)} - \underbrace{\alpha T(y,z)}_{\varepsilon^T(y,z)}\}.$$

This approach for statically indeterminate structures will be illustrated by an example in Section 9.16.3.

9.16.2 Statically determinate beam subject to a temperature load

The inhomogeneous prismatic cantilever beam AB in Figure 9.92, with a T-shaped cross-section, is constructed with different materials for flange and web. Both materials behave linear elastic. The web of the T-section has a

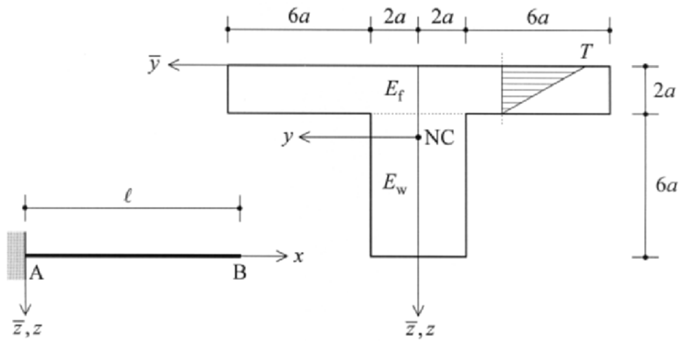

Figure 9.92 A prismatic cantilever T-beam with inhomogeneous cross-section. In the flange the temperature is increased. The increase is linear over the depth of the flange, and constant over the width of the flange and the length of the beam.

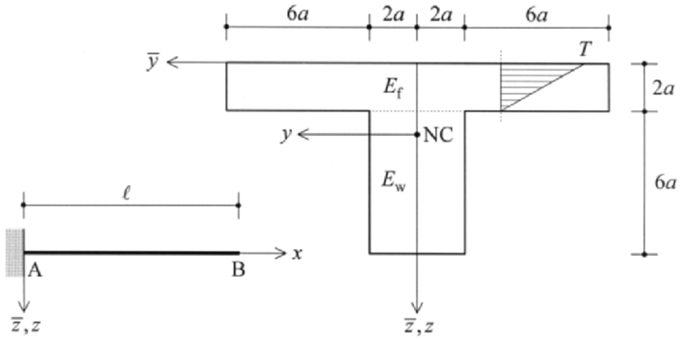

Figure 9.92 A prismatic cantilever T-beam with inhomogeneous cross-section. In the flange the temperature is increased. The increase is linear over the depth of the flange, and constant over the width of the flange and the length of the beam.

modulus of elasticity $E_\mathrm{w} = E$ and the flange has a modulus of elasticity of $E_\mathrm{f} = 0.75E$. The temperature distribution over the depth of the flange is linear, as can be observed from the figure, and is constant over the width of the flange and the length of the beam. The *coefficient of thermal expansion* of the flange is α. The web of the beam remains under constant temperature conditions. There are no loading forces.

Questions:

a. Find the force distribution in the structure.
b. Calculate the strain and stress distribution for the cross-section at fixed end A.
c. Draw a sketch of the deformed beam.
d. Determine the maximum vertical displacement of this cantilever beam.

Solution:
Since the cross-sectional dimensions of the beam and temperature distribution are mirror symmetrical, there are only section forces, deformations and displacements in the (vertical) plane of mirror symmetry, the xz plane or $\overline{xz}$ plane (see Figure 9.92).

a. The cantilever beam is statically determinate and can deform unconstrained. The force distribution can be obtained directly from the equilibrium conditions and is independent of the temperature load. Since there are no loading forces, all section forces are zero: $N = 0$ and $M_z = 0$.

b. Since all section forces are zero, the stresses in the beam are the result of the temperature load only. To find the stress distribution in cross-section A, we first have to determine the cross-sectional deformation quantities due to the temperature load: ε^T and κ_z^T. For this purpose the location of the normal centre and the stiffness quantities are required.

Normal centre NC
The location of the normal centre NC must be on the vertical line of sym-

metry. Therefore only the vertical position of the NC has to be found. In the $\overline{yz}$ coordinate system in Figure 9.92, we find

$$\bar{z}_{NC} = \frac{S_{\bar{z}}}{EA} = \frac{a \cdot EA_f + 5a \cdot EA_w}{EA}$$
$$= \frac{a \cdot \left(\frac{3}{4} E \cdot 2a \cdot 16a\right) + 5a \cdot (E \cdot 6a \cdot 4a)}{\frac{3}{4} E \cdot 2a \cdot 16a + E \cdot 6a \cdot 4a} = 3a.$$

The location of the NC is shown in Figure 9.93. Here we choose a yz coordinate system, with the z axis coinciding with the line of mirror symmetry. Since the z axis is a line of symmetry, the yz coordinate system is a principal coordinate system, which means that $EI_{yz} = EI_{zy} = 0$.

Axial stiffness
The axial stiffness of the cross-section can be found as:

$$EA = EA_f + EA_w = \tfrac{3}{4} E \cdot 2a \cdot 16a + E \cdot 6a \cdot 4a = 48EA^2.$$

Bending stiffness
Due to the mirror symmetry the beam will curve in the xz plane. Thus we need to determine only the bending stiffness EI_{zz}:

$$EI_{zz} = \underbrace{\tfrac{3}{4} E \times \left\{\tfrac{1}{12} \cdot 16a \cdot (2a)^3 + 16a \cdot 2a \cdot (2a)^2\right\}}_{\text{flange}}$$
$$+ \underbrace{E \times \left\{\tfrac{1}{12} \cdot 4a \cdot (6a)^3 + 4a \cdot 6a \cdot (2a)^2\right\}}_{\text{web}}$$
$$= 272Ea^4.$$

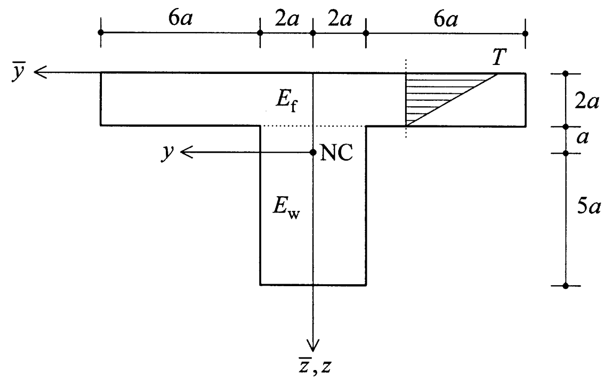

Figure 9.93 The location of the normal centre NC.

Table 9.13 Strain and stress distribution due to the temperature load for the (unconstrained) cantilever beam.

	z	$\varepsilon \times \dfrac{272}{\alpha T}$	$\sigma \times \dfrac{272}{E\alpha T}$
top flange	$-3a$	+152	−90
bottom flange	$-a$	+96	+72
top web	$-a$	+96	+96
normal centre	0	+68	+68
bottom web	$+5a$	−72	−72

The temperature function for the flange can be expressed as

$$T(y,z) = -\tfrac{1}{2}\,T\left(1+\frac{z}{a}\right) \quad \text{with} \quad -2a \le z \le -1.$$

With the basic formulae (9.84a) and (9.84c) we can now find the sectional deformation quantities ε^T and κ_z^T due to the temperature load:[1]

$$\begin{aligned}
\varepsilon^T &= \int_A \frac{\alpha E T(y,z)\,\mathrm{d}A}{EA} \\
&= \int_{-3a}^{-a} \frac{1}{EA}\cdot\alpha\cdot\tfrac{3}{4}\,E\cdot\left\{-\tfrac{1}{2}\,T\left(1+\frac{z}{A}\right)\right\}\cdot 16a\cdot\mathrm{d}z \\
&= \tfrac{1}{4}\,\alpha T, \\
\kappa_z^T &= \int_A \frac{z\cdot\alpha E T(y,z)}{EI_{zz}}\,\mathrm{d}A \\
&= \int_{-3a}^{-a} \frac{1}{EI_{zz}}\cdot z\cdot\alpha\cdot\tfrac{3}{4}\,E\cdot\left\{-\tfrac{1}{2}\,T\left(1+\frac{z}{A}\right)\right\}\cdot 16a\cdot\mathrm{d}z \\
&= -\tfrac{7}{68}\,\frac{\alpha T}{a}\,.
\end{aligned}$$

For the unconstrained beam without loading forces, there are only strains due to the temperature load. Therefore the strain distribution in the cross-section is

[1] Remember $\kappa_y = 0$.

$$\varepsilon(y,z) = \varepsilon^T + \kappa_z^T z = \tfrac{1}{4}\alpha T - \tfrac{7}{68}\frac{\alpha T}{a} z$$
$$= \frac{\alpha T}{272}\left(68 - 28\frac{z}{a}\right).$$

According to (9.79) the stress distribution is

$$\sigma(y,z) = E(y,z)\varepsilon^\sigma(y,z)$$
$$= E(y,z)\{\varepsilon(y,z) - \varepsilon^T(y,z)\}$$
$$= E(y,z)\{\underbrace{\varepsilon^T + \kappa_z^T z}_{\varepsilon(y,z)} - \underbrace{\alpha T(y,z)}_{\varepsilon^T(y,z)}\}.$$

If we elaborate this expression for the flange we find

$$\sigma(y,z) = \tfrac{3}{4}E \times \left\{\frac{\alpha T}{272}\left(68 - 28\frac{z}{a}\right) + \tfrac{1}{2}\alpha T\left(1 + \frac{z}{a}\right)\right\}$$
$$= \frac{E\alpha T}{272}\left(153 + 81\frac{z}{a}\right).$$

For the web, $\varepsilon^T(y,z) = \alpha T(y,z) = 0$, so the stress distribution is

$$\sigma(y,z) = E \times \frac{\alpha T}{272}\left(68 - 28\frac{z}{a}\right) = \frac{E\alpha T}{272}\left(68 - 28\frac{z}{a}\right).$$

For a number of key points the strain and stress values are calculated in Table 9.13. The total strain and stress distribution for the cross-section are shown in Figure 9.94.

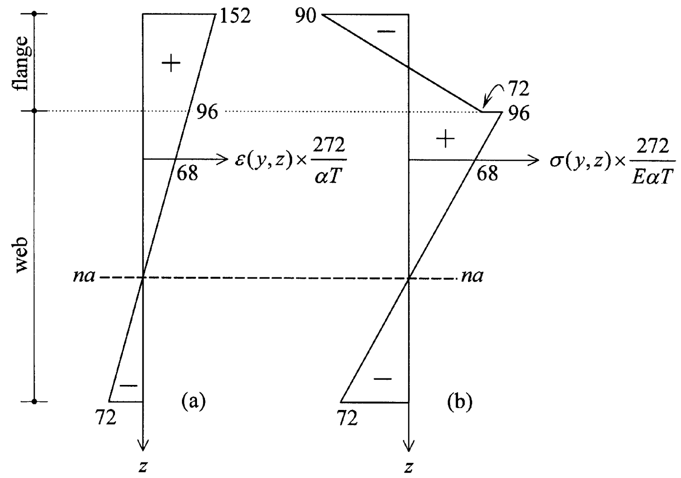

Figure 9.94 (a) Strain diagram and (b) stress diagram in case of an unconstrained deformation. Note that the stress distribution is not similar to the strain distribution. Even more noticeable is the double root in the stress distribution. One of the zero stress points coincides with the position of the *neutral axis na*, but the other zero stress point does not! Remember that the neutral axis is defined as the (straight) line in the cross-section with *zero strain*.

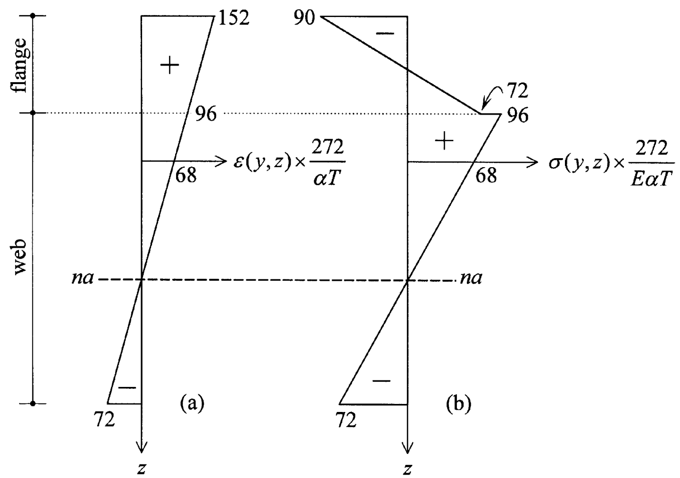

Figure 9.94 (a) Strain diagram and (b) stress diagram in case of an unconstrained deformation. Note that the stress distribution is not similar to the strain distribution. Even more noticeable is the double root in the stress distribution. One of the zero stress points coincides with the position of the *neutral axis na*, but the other zero stress point does not! Remember that the neutral axis is defined as the (straight) line in the cross-section with *zero strain*.

It is remarkable to notice that the stress distribution is not similar to the strain distribution. Even more noticeable is the double root in the stress distribution. One of the zero stress points coincides with the position of the *neutral axis na*, but the other zero stress point does not! The fact that we observe a double change of sign in the stress distribution is a direct result of the elongation of the fibres due to the temperature gradient and the requirement that the section forces are zero. The elongation of fibres must be such that the resulting stress distribution forms an equilibrium system ($\sum N = 0$; $\sum M = 0$. *The beam will deform unconstrained without section forces but with non-zero normal stresses!*

The neutral axis is defined as the (straight) line in the cross-section with zero strain.[1] The neutral axis divides the cross-section into two parts: one with positive straining and one with negative straining. This example shows clearly that the neutral axis can not be defined unambiguously as the (straight) line in the cross-section with zero normal stresses, since there are two of these lines here.

As a result of the temperature load and the unconstrained deformation of the beam, the strain distribution will be constant along the beam axis. Therefore all cross-sections will exhibit the same strain and stress distribution as shown in Figure 9.94.

c. The strain ε at the normal centre NC and the curvature κ_z are constant along the length ℓ of the beam:

$$\varepsilon(x) = \varepsilon^T = \tfrac{1}{4}\alpha T,$$

$$\kappa_z(x) = \kappa_z^T = -\tfrac{7}{68}\frac{\alpha T}{a}.$$

---

1 See Sections 4.2 and 9.6.

In Figure 9.95 this is visualised in the ε and κ_z diagram for the beam. Also a sketch is given of the beam deformation in the xz plane.

The elongation u of the beam equal to the area of the ε diagram:[1]

$$u = \varepsilon^T \ell = \tfrac{1}{4}\,\alpha T \ell.$$

Using the moment-area formulae we can find the vertical displacement w_B at B from the κ_z diagram:[2]

$$\theta = \kappa_z^T \ell = \tfrac{7}{68}\,\alpha T \frac{\ell}{a}\,,$$

$$w_B = \theta \times \tfrac{1}{2}\,\ell = \tfrac{7}{136}\,\alpha T \frac{l^2}{a}\,.$$

d. The maximum deflection occurs at B.

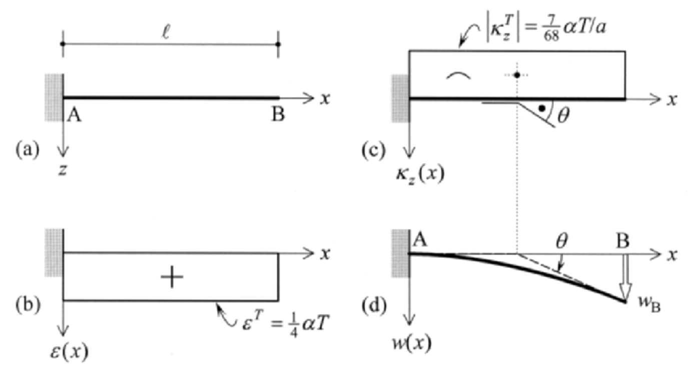

Figure 9.95 (a) Cantilever beam, (b) ε diagram, (c) κ diagram, and (d) the curved beam in case of unconstrained deformation.

9.16.3 Statically indeterminate beam subject to a temperature load

For the statically indeterminate beam AB in Figure 9.96 the same cross-section and temperature load will be used as in the previous example. The beam is fixed at A and roller supported at B.

Questions:

a. Find the force distribution in the structure.

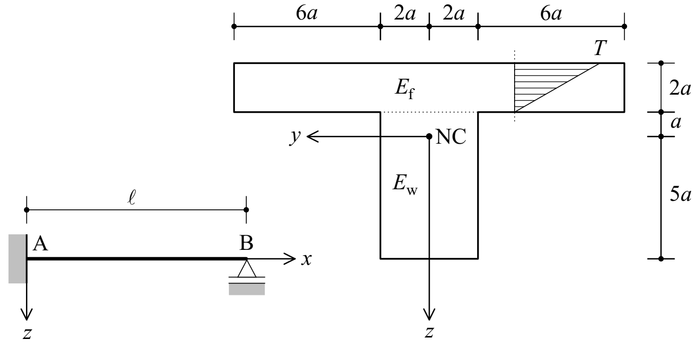

Figure 9.96 A statically indeterminate T-beam with inhomogeneous cross-section. In the flange the temperature is increased. The increase is linear over the depth of the flange, and constant over the width of the flange and the length of the beam.

1 See Section 2.6.1.

2 See Sections 8.4 and 9.13. The angle θ is equal to the area of the κ_z diagram, and is located at the centroid of the diagram.

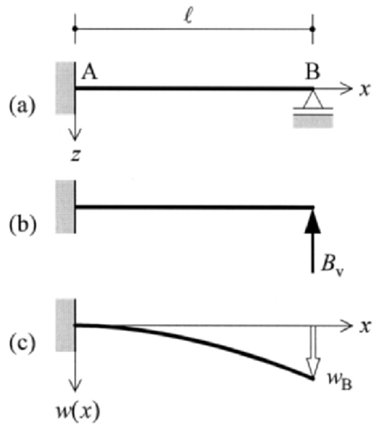

Figure 9.97 (a) For the statically indeterminate beam the roller support is selected as the *redundant constraint*, and the reaction force B_v as the *static redundant*. (b) By removing the roller support at B we obtain a cantilever beam. This structure is called the *released structure*. The as yet unknown static redundant B_v is applied as a load on the released structure. (c) The displacement associated with the static redundant B_v, i.e. the vertical displacement w_B at B, due to both the static redundant and the actual load must be zero. From this compatibility condition the unknown static redundant B_v can be solved.

b. Calculate the strain and stress distribution for the cross-section at fixed end A.
c. Draw a sketch of the deformed beam.
d. Determine the maximum vertical displacement of this cantilever beam.

Solution:
a. The beam is statically indeterminate to the first degree.[1] Since a statically indeterminate structure cannot deform freely, the deformation is constrained. To find the force distribution we have to take into account the deformation behaviour of the beam.

Applying the force method, we first change the statically indeterminate beam in such a way that it becomes statically determinate. For this purpose we change the structure in a cantilever beam by removing the roller support at B (see Figure 9.97b). The cantilever beam is called the *statically determinate released structure* or, in short, *released structure*. Here we selected the roller support as the *redundant constraint*, and the reaction force B_v, as the *static redundant*.

The as yet unknown static redundant B_v is now applied as a load on the released structure, acting together with the actual load – here the temperature load only.

Next the displacement associated with the static redundant B_v is calculated, i.e. the vertical displacement w_B at B, due to the static redundant and the actual load (see Figure 9.97c). Subsequently we formulate the deformation or compatibility condition for the displacement w_B associated with the static redundant B_v. The roller support as redundant constraint requires that the vertical displacement w_B, due to both the actual load and the unknown

1 See *Engineering Mechanics* – Volume 1, Section 4.5.

static redundant B_v, must be zero. In the end B_v can be solved from this compatibility condition.

The unconstrained deformation of the cantilever beam, due to a temperature load, was solved in the previous example in Section 9.16.2. Using $EI_{zz} = 272Ea^4$, we find the vertical support reaction B_v at B from the compatibility condition $w_B = 0$:

$$w_B = \underbrace{\tfrac{7}{136}\alpha T \frac{\ell^2}{a}}_{\text{due to } T} - \underbrace{\frac{B_v \ell^3}{3EI_{zz}}}_{\text{due to } B_v} = 0 \Rightarrow B_v = \tfrac{21}{136}\alpha T \frac{EI_{zz}}{a\ell} = -42E\alpha T \frac{a^3}{\ell}.$$

Since the beam is not subject to force loads, the bending moment and shear force in the beam are the result of the support reaction B_v only. They can directly be calculated from the equilibrium of the statically determinate released structure (the cantilever beam) (see Figure 9.97b). The bending moment and shear force diagrams are shown in Figure 9.98.

b. The bending moment at A is

$$M_z = 42E\alpha T a^3.$$

Since the yz coordinate system is a principal coordinate system the curvature κ_z in the xz plane can be obtained directly from the bending moment M_z:

$$\kappa_z^\sigma = \frac{M_z}{EI_{zz}} = \tfrac{42}{272}\frac{E\alpha T a^3}{Ea^4} = \tfrac{42}{272}\frac{\alpha T}{a}.$$

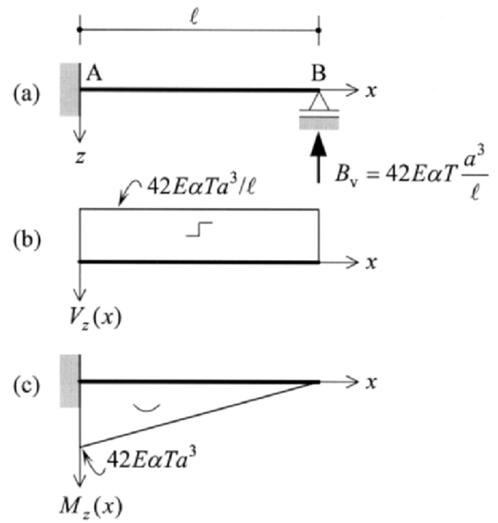

Figure 9.98 (a) Support reaction at B, (b) shear force diagram, and (c) bending moment diagram.

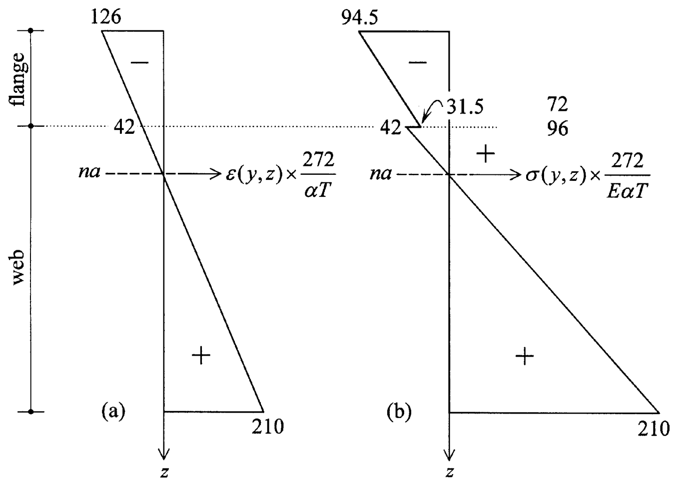

Figure 9.99 (a) ε diagram and (b) σ diagram at the fixed end A due to only the bending moment $M_z = 42E\alpha Ta^3$.

Table 9.14 Strain and stress distribution at A, due to the section forces.

	z	$\varepsilon \times \frac{272}{\alpha T}$	$\sigma \times \frac{272}{E\alpha T}$
top flange	$-3a$	-126	-94.5
bottom flange	$-a$	-42	-31.5
top web	$-a$	-42	-42
normal centre	0	0	0
bottom web	$+5a$	$+210$	$+210$

The associated strain and stress distribution are

$$\varepsilon^\sigma(y,z) = \kappa_z^\sigma z = \tfrac{42}{272}\,\alpha T\,\frac{z}{a}\,,$$

$$\sigma(y,z) = E\varepsilon^\sigma(y,z) = E(y,z)\times\tfrac{42}{272}\,\alpha T\,\frac{z}{a}\,.$$

For a number of key points, the strain and stress values at A due to the section forces are computed in Table 9.14. In Figure 9.99 the strain and stress distribution at A is shown in diagrams.

The actual strain and stress distribution are found by superposition of the influence of the section forces (see Figure 9.99) and the influence of the temperature load for the statically determinate released structure with zero section forces as found in the previous example (see Figure 9.94). The result is shown in Figure 9.100.

Note that the zero stress point in the cross-section is not on the neutral axis *na*, which is outside the cross-section.

c. The actual curvature in Figure 9.101d is also a summation of the earlier found curvature of the unconstrained beam due to the temperature load in Figure 9.101b and the curvature due to the bending moment in Figure 9.101c. The values of the curvature are expressed in terms of k:

$$k = \tfrac{14}{272}\,\frac{\alpha T}{a}\,.$$

d. The maximum vertical displacement will occur at the location were the rotation φ is zero. From the curvature diagram in Figure 9.101d we observe a zero rotation at C ($x = \frac{2}{3}\,\ell$). This can also be derived from the moment-area formula for the rotation:

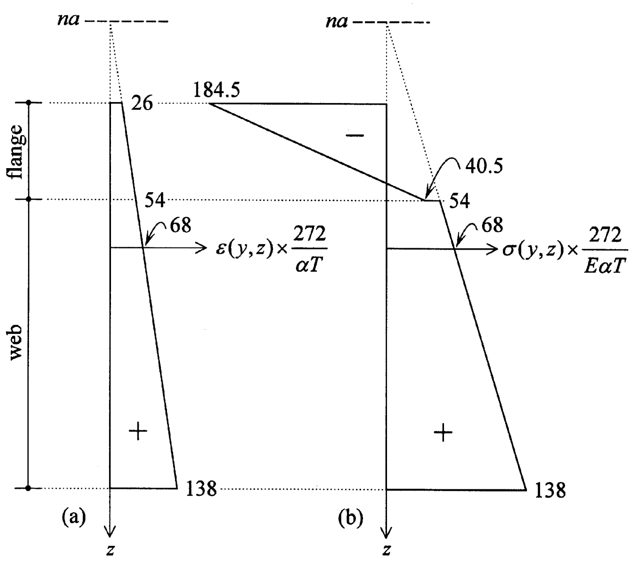

Figure 9.100 The actual (a) strain and (b) stress distribution is found by superposition of the influence of the bending moment (see Figure 9.99) and the influence of the temperature load for the released structure with zero section forces (see Figure 9.94). Note that the zero stress point in the cross-section is not on the neutral axis *na*, which is outside the cross-section.

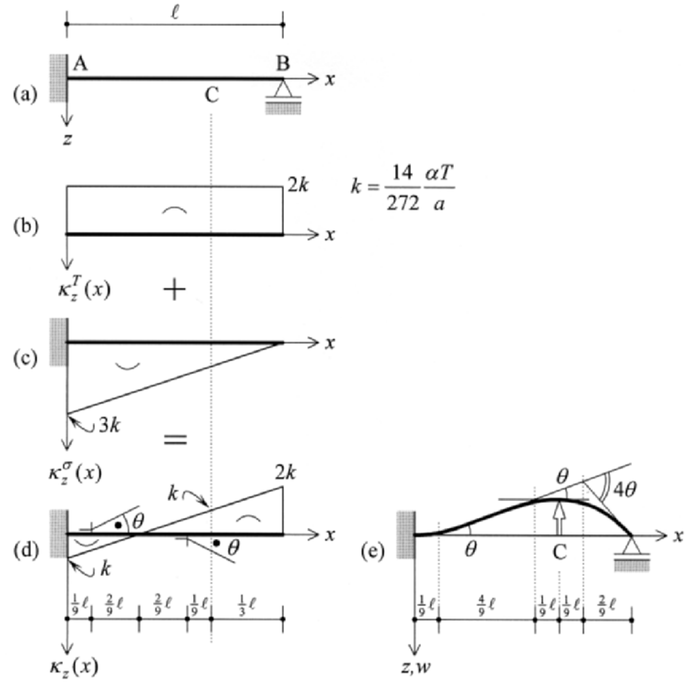

Figure 9.101 (a) A statically indeterminate beam. (b) The released structure with the κ diagram due to the temperature load, and (c) the released structure with the κ diagram due to the bending moments. (d) The resultant κ diagram. (e) The deformed beam with the maximum deflection at C.

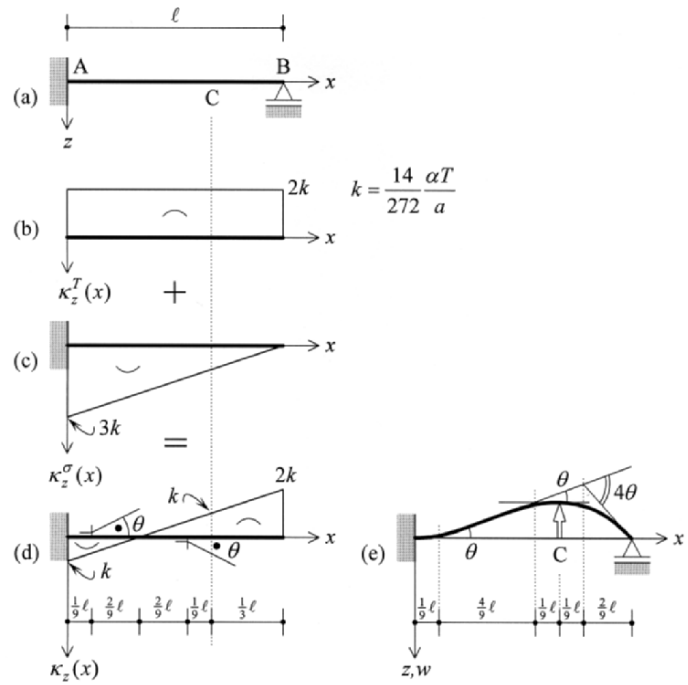

Figure 9.101 (a) A statically indeterminate beam. (b) The released structure with the κ diagram due to the temperature load, and (c) the released structure with the κ diagram due to the bending moments. (d) The resultant κ diagram. (e) The deformed beam with the maximum deflection at C.

$$\varphi_C = \varphi_A + \int_A^C \kappa_z \, dx$$
$$= \underbrace{0}_{\varphi_A} + \underbrace{(\text{area of the } \kappa_z \text{ diagram between A and C})}_{\text{integral}} = 0.$$

So the maximum displacement occurs at C:

$$x_C = \tfrac{2}{3}\,\ell.$$

The vertical displacement at C can be found using the rotations θ shown in Figure 9.101d:

$$\theta = \tfrac{1}{2} \times \tfrac{1}{3}\,\ell \times k = \tfrac{1}{6}\,kl,$$

and

$$w_C = -\tfrac{5}{9}\,\ell\theta + \tfrac{1}{9}\,\ell\theta = -\tfrac{4}{54}\,k\ell^2 = -\tfrac{4}{54} \times \tfrac{14}{272}\frac{\alpha T \ell^2}{a} \approx -\tfrac{1}{272}\frac{\alpha T \ell^2}{a}.$$

A sketch of the deformed beam is given in Figure 9.101e.

Comment: For both examples the section forces N and M at B are zero. The strain and stress distribution at B should therefore be the distributions found for the unconstrained deformation due to the temperature load, as shown in Figure 9.94. However this stress distribution is not very realistic since the end cross-section of the beam at B is a free edge where the normal stresses should all be zero. Over a small distance from the free edge at B the presented theory will therefore predict a wrong strain and stress distribution. According to Saint Venant's principle[1] this will only be the case for a distance equal to about the depth of the beam. For the rest we can make use of this theory without any hesitation, but with this remark in mind.

9.17 Shear flow and shear stresses in arbitrary cross-sections – shear centre

Apart from normal stresses there are also shear stresses in cross-sections. In Chapter 5, a systematic method is outlined for homogeneous cross-sections with at least one line of symmetry. In this section we will extend this approach to unsymmetrical and or inhomogeneous cross-sections.

We discuss the shear flow in a longitudinal section plane, following the traditional approach in Section 9.17.1 and an alternative approach in Section 9.17.2. Shear stresses in longitudinal and cross-sectional planes can be derived from the shear flow. For details we refer to Section 5.3. One of the applications in Section 9.17.2 is the shear stress distribution in an inhomogeneous rectangular cross-section, which is derived from the bending stress distribution. At the end, in Section 9.17.3, we will pay attention to the location of the *shear* (*force*) *centre* SC of open thin-walled cross-sections.

9.17.1 Shear flow in longitudinal direction (traditional approach)

From the force equilibrium of a small sliding element we found the following formula for s_x^{a}, the shear force per length (or *shear flow*) on the

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[1] Named after Barré de Saint Venant (1797–1886), French civil engineer who contributed to the development of the theory of elasticity.

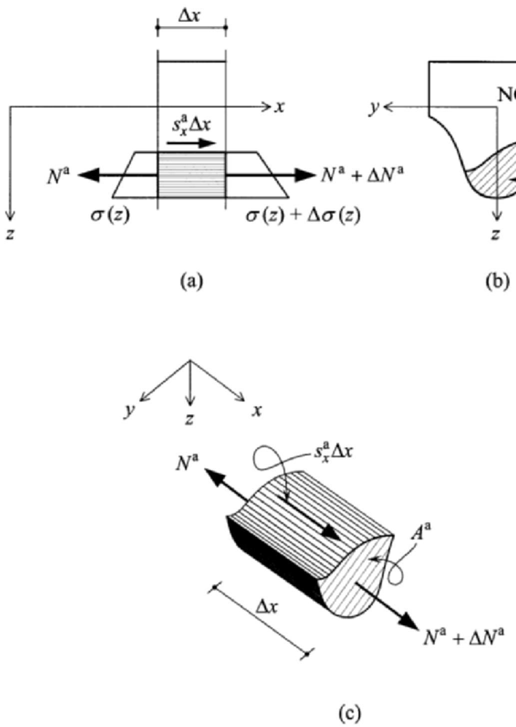


Figure 9.102 The lower part of the beam segment, wit small length Δx, has been isolated and is called the *sliding element*. (b) The cross-sectional area of the sliding element is A^a. (c) Spatial representation of the sliding element with all the forces acting on it. Since the resultant of all normal stresses on the front and back of the sliding element are not equal, a longitudinal shear force must act on the longitudinal section plane.

longitudinal section plane[1] (see Figure 9.102):

$$s_x^a = -\frac{dN^a}{dx}. \tag{9.85}$$

Here, N^a is the resultant force due to all normal stresses on the cross-sectional area A^a of the sliding element:

$$N^a = \int_{A^a} \sigma(y, z)\, dx. \tag{9.86}$$

So

$$s_x^a = -\frac{dN^a}{dx} = -\frac{d}{dx}\left(\int_{A^a} \sigma(y, z)\, dx\right) = -\int_{A^a} \frac{d\sigma(y, z)}{dx}\, dx. \tag{9.87}$$

Differentiating with respect to the longitudinal direction and integrating with respect to the cross-sectional area of the sliding element are two independent operations that may be interchanged.

The normal stress distribution in an inhomogeneous cross-section follows from the constitutive relationship:

$$\sigma(y, z) = E(y, z)\varepsilon(y, z), \tag{9.88}$$

in which $\varepsilon(y, z)$ is the linear strain distribution:

$$\varepsilon(y, z) = \varepsilon + \kappa_y y + \kappa_z z. \tag{9.89}$$

---

[1] See Section 5.1.2.

The cross-sectional deformation quantities ε, κ_y and κ_z follow from the cross-sectional constitutive relationships as introduced in Section 9.5. If the cross-sectional yz coordinate system coincides with the principal coordinate system the cross-sectional constitutive relationships are uncoupled:

$$\varepsilon = \frac{N}{EA}, \quad \kappa_y = \frac{M_y}{EI_{yy}} \text{ and } \kappa_z = \frac{M_z}{EI_{zz}}. \tag{9.90}$$

Combining the expressions (9.88), (9.89) and (9.90) gives

$$\sigma(y,z) = E(y,z) \times \left[\frac{N}{EA} + \frac{M_y y}{EI_{yy}} + \frac{M_z z}{EI_{zz}} \right]. \tag{9.91}$$

For a prismatic beam[1]

$$\frac{\mathrm{d}\sigma(y,z)}{\mathrm{d}x} = E(y,z) \times \left[\frac{1}{EA}\frac{\mathrm{d}N}{\mathrm{d}x} + \frac{y}{EI_{yy}}\frac{\mathrm{d}M_y}{\mathrm{d}x} + \frac{z}{EI_{zz}}\frac{\mathrm{d}M_z}{\mathrm{d}x} \right].$$

Substituting this result in (9.87), we find for the shear flow

$$\begin{aligned} s_x^{\mathrm{a}} &= -\frac{\mathrm{d}N^{\mathrm{a}}}{\mathrm{d}x} = -\int_{A^{\mathrm{a}}} \frac{\mathrm{d}\sigma(y,z)}{\mathrm{d}x}\,\mathrm{d}x \\ &= -\int_{A^{\mathrm{a}}} E(y,z) \times \left[\frac{1}{EA}\frac{\mathrm{d}N}{\mathrm{d}x} + \frac{y}{EI_{yy}}\frac{\mathrm{d}M_y}{\mathrm{d}x} + \frac{z}{EI_{zz}}\frac{\mathrm{d}M_z}{\mathrm{d}x} \right] \mathrm{d}A. \end{aligned} \tag{9.92}$$

---

[1] Note that in a prismatic beam the stiffness quantities, denoted with the *double letter symbols* EA, EI_{yy} and EI_{zz}, are constant and independent of x.

We assume a constant normal force N for the beam segment, so

$$\frac{\mathrm{d}N}{\mathrm{d}x} = 0.$$

Furthermore, there are the following static relationships between the shear forces and the bending moments:

$$V_y = \frac{\mathrm{d}M_y}{\mathrm{d}x} \quad \text{and} \quad V_z = \frac{\mathrm{d}M_z}{\mathrm{d}x}.$$

Equation (9.92) now changes into

$$\begin{aligned} s_x^{\mathrm{a}} &= -\int_{A^{\mathrm{a}}} E(y,z) \times \left(+ \frac{V_y y}{EI_{yy}} + \frac{V_z z}{EI_{zz}} \right) \mathrm{d}A \\ &= -\frac{V_y}{EI_{yy}} \int_{A^{\mathrm{a}}} E(y,z) \cdot y \,\mathrm{d}A - \frac{V_z}{EI_{zz}} \int_{A^{\mathrm{a}}} E(y,z) \cdot z \,\mathrm{d}A. \end{aligned} \tag{9.93}$$

Since the bending stiffness quantities EI_{yy} and EI_{zz}, and the shear forces V_y and V_z are independent of $\mathrm{d}A = \mathrm{d}y\,\mathrm{d}z$, they can be put in front of the integrals. By introducing the double letter symbols

$$ES_y^{\mathrm{a}} = \int_{A^{\mathrm{a}}} E(y,z) \cdot y \,\mathrm{d}A \quad \text{and} \quad ES_z^{\mathrm{a}} = \int_{A^{\mathrm{a}}} E(y,z) \cdot z \,\mathrm{d}A,$$

expression (9.93) simplifies to the following formula for the shear flow:

$$s_x^{\mathrm{a}} = -\frac{V_y ES_y^{\mathrm{a}}}{EI_{yy}} - \frac{V_z ES_z^{\mathrm{a}}}{EI_{zz}}. \tag{9.94}$$

For a homogeneous cross-section the moduli of elasticity in the numerator and denominator cancel each other out, and we have

$$s_x^{\mathrm{a}} = -\frac{V_y S_y^{\mathrm{a}}}{I_{yy}} - \frac{V_z S_z^{\mathrm{a}}}{I_{zz}} \,. \tag{9.95}$$

This expression is almost identical to the expression found in Section 5.1.2. New element here is the contribution of the shear forces in two directions.

For a yz coordinate system with its origin at the normal centre NC, but *not coinciding with the principal directions*, the constitutive relationships for bending are coupled:

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}$$

and

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{1}{\mathrm{Det}(EI)} \begin{bmatrix} EI_{zz} & -EI_{zy} \\ -EI_{yz} & EI_{yy} \end{bmatrix} \begin{bmatrix} M_y \\ M_z \end{bmatrix},$$

in which $\mathrm{Det}(EI)$ is the determinant of the bending stiffness matrix:

$$\mathrm{Det}(EI) = EI_{yy} EI_{zz} - EI_{yz}^2 .$$

Following the same procedure as before, we find for the shear flow

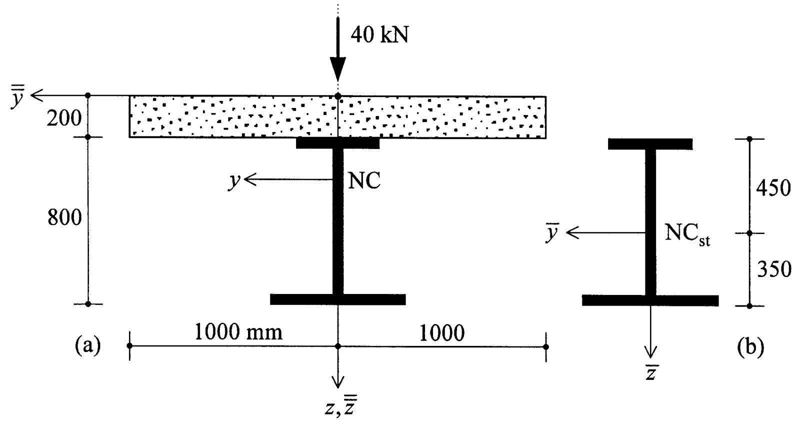


Figure 9.103 (a) The cross-section of a concrete-steel composite beam subject to bending. The cross-section has to transmit the shear force $V_z = 40$ kN. (b) The location of the normal centre (centroid) NC_{st} of the homogeneous steel section.

$$s_x^a = -\frac{V_y}{\text{Det}(EI)}(ES_y^a \cdot EI_{zz} - ES_z^a \cdot EI_{zy})$$
$$-\frac{V_z}{\text{Det}(EI)}(ES_z^a \cdot EI_{yy} - ES_y^a \cdot EI_{yz}). \quad (9.96)$$

For homogeneous cross-sections this changes into

$$s_x^a = -\frac{V_y}{\text{Det}(I)}(S_y^a \cdot I_{zz} - S_z^a \cdot I_{zy}) - \frac{V_z}{\text{Det}(I)}(S_z^a \cdot I_{yy} - S_y^a \cdot I_{yz}). \quad (9.97)$$

The expressions (9.96) and (9.97) are far from attractive. Therefore, for general situations we will derive an alternative method in Section 9.17.3.

We will illustrate the traditional approach with two examples.

Example 1: Shear flow in a symmetric inhomogeneous cross-section
Figure 9.103a shows the cross-section of a concrete-steel composite beam subject to bending. The cross-section has to transmit the shear force $V_z = 40$ kN as is shown in the figure. The yz coordinate system is chosen at the normal centre NC of the composite cross-section. The concrete flange is without slip firmly attached to the steel section. Therefore we can assume both parts to act together as one piece.

The location of the normal centre (centroid) NC_{st} of the homogeneous steel section is given in Figure 9.103b. More structural data:

- Area of steel: $A_{st} = 32 \times 10^3$ mm^2,
- Second moment of area of steel: $I_{\overline{zz};st} = 432 \times 10^6$ mm^4,
- Modulus of elasticity of steel: $E_{st} = 210 \times 10^3$ N/mm^2,
- Modulus of elasticity of concrete: $E_c = 14 \times 10^3$ N/mm^2.

Question:
Compute the longitudinal *shear flow* in the interface between steel and concrete.

Solution (units in N and mm):
The composite cross-section has one of line symmetry, on which the normal centre NC is located. In the $\bar{\bar{y}}\bar{\bar{z}}$ coordinate system the vertical location of the normal centre NC is (see Figure 9.104)

$$\bar{\bar{z}}_{\mathrm{NC}} = \frac{E_{\mathrm{c}} \times (200 \times 2000) \times 100 + E_{\mathrm{st}} \times (32000) \times 650}{E_{\mathrm{c}} \times (200 \times 2000) + E_{\mathrm{st}} \times (32000)} = 400 \text{ mm.}$$

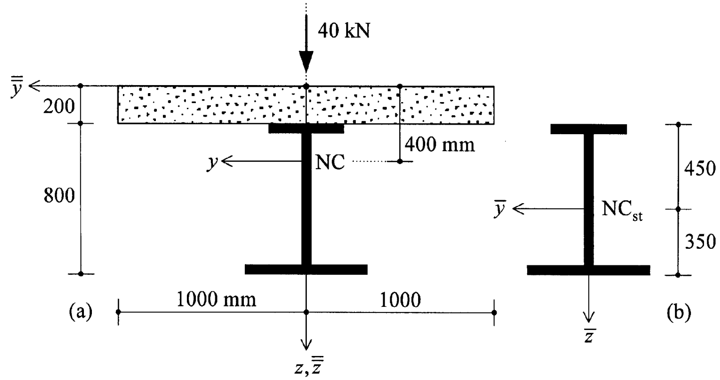


Figure 9.104 (a) The location of the normal centre NC of the composite cross-section. (b) The location of the normal centre (centroid) NC_{st} of the homogeneous steel section.

The yz coordinate system is a principal coordinate system, so we can use formula (9.94) to find the shear flow:

$$s_x^{\mathrm{a}} = -\cancel{\frac{V_y E S_y^{\mathrm{a}}}{E I_{yy}}} - \frac{V_z E S_z^{\mathrm{a}}}{E I_{zz}} .$$

Since the shear force in the y direction is zero we only have to consider the quantities in the z direction. The bending stiffness in the z direction is

$$\begin{aligned} E I_{zz} &= E_{\mathrm{c}} \times \left[\tfrac{1}{12} \times 2000 \times 200^3 + (200 \times 2000) \times (300)^2\right] \\ &\quad + E_{\mathrm{st}} \times [432 \times 10^6 + 32 \times 10^3 \times (250)^2] \\ &= 1033.4 \times 10^{12} \text{ Nmm}^2. \end{aligned}$$

If we choose the concrete flange as the sliding element,

$$E S_z^{\mathrm{a}} = E_{\mathrm{c}} \times (2000 \times 200) \times (-300) = -1.68 \times 10^{12} \text{ Nmm.}$$

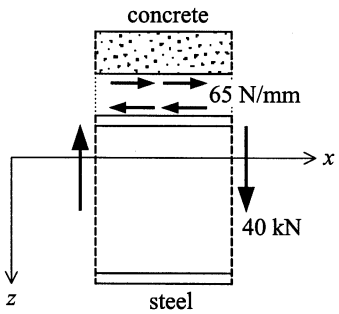


Figure 9.105 The shear flow (shear force per length) between the concrete flange and the steel section.

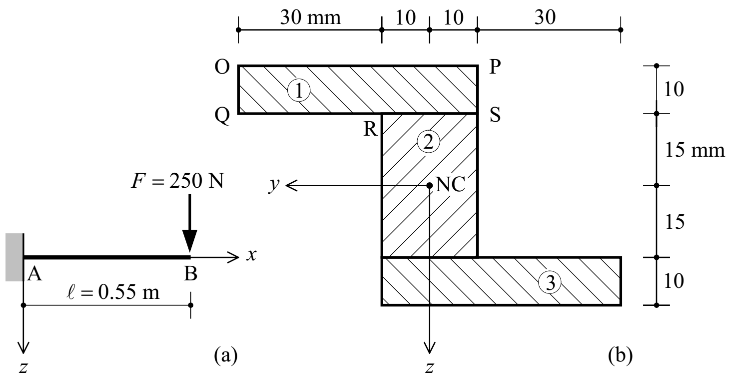


Figure 9.106 Cantilever beam with an unsymmetrical and inhomogeneous cross-section.

For the shear flow (shear force per length) in the interface between the concrete flange and the steel section we find

$$s_x^{\mathrm{a}} = -\frac{V_z E S_z^{\mathrm{a}}}{E I_{zz}} = -\frac{(40 \times 10^3 \text{ N}) \times (-1.68 \times 10^{12} \text{ Nmm})}{1033.4 \times 10^{12} \text{ Nmm}^2}$$
$$= +65 \text{ N/mm}.$$

Since the shear flow is positive, it acts in the positive x direction on the flange (the sliding element), and in the negative x direction on the steel section (see Figure 9.105).

Example 2: Shear flow in an unsymmetrical and inhomogeneous cross-section

The cantilever beam in Figure 9.106, with an inhomogeneous cross-section and subject to a point load of 250 N at B, was discussed in Section 9.9.2, Example 2. The beam is constructed of three parts, numbered by 1 to 3, which are firmly glued together. Different materials are used for flanges and web. The moduli of elasticity are $E_1 = E_3 = 12000\,\text{N/mm}^2$ and $E_2 = 6000\,\text{N/mm}^2$.

Question:
Determine the shear flow at the glue line RS.

Solution:
Here we can use formula (9.96). Since $V_y = 0$, this slightly simplifies the formula:

$$s_x^{\mathrm{a}} = -\frac{V_z}{\mathrm{Det}(EI)}\left(ES_z^{\mathrm{a}} \cdot EI_{yy} - ES_y^{\mathrm{a}} \cdot EI_{zy}\right)$$
$$= -\frac{V_z \cdot ES_z^{\mathrm{a}} \cdot EI_{yy}}{\mathrm{Det}(EI)} + \frac{V_z \cdot ES_y^{\mathrm{a}} \cdot EI_{yz}}{\mathrm{Det}(EI)}.$$

In Section 9.9.2, Example 2, we derived

$$EI_{yy} = 5.32 \times 10^9 \text{ Nmm}^2,$$

$$EI_{yz} = EI_{zy} = -3.6 \times 10^9 \text{ Nmm}^2,$$

$$EI_{zz} = 5.17 \times 10^9 \text{ Nmm}^2,$$

from which we find

$$\text{Det}(EI) = EI_{yy}EI_{zz} - EI_{yz}^2 = 14.54 \times 10^{18} \text{ N}^2\text{mm}^4.$$

Consider the upper flange OPQS as the sliding element. The cross-sectional area A^{A} of the sliding element is (see Figure 9.107)

$$A^{\text{a}} = (50 \text{ mm})(10 \text{ mm}) = 500 \text{ mm}^2.$$

Furthermore,

$$EA^{\text{a}} = E_1 \cdot A^{\text{a}} = (12 \times 10^3 \text{ N/mm}^2)(500 \text{ mm}^2) = 6 \times 10^6 \text{ N},$$

$$ES_y^{\text{a}} = E_1 A^{\text{a}} \cdot y_{\text{NC}}^{\text{a}} = (6 \times 10^6 \text{ N})(+15 \text{ mm}) = +90 \times 10^6 \text{ Nmm},$$

$$ES_z^{\text{a}} = E_1 A^{\text{a}} \cdot z_{\text{NC}}^{\text{a}} = (6 \times 10^6 \text{ N})(-20 \text{ mm}) = -120 \times 10^6 \text{ Nmm}.$$

For $V_z = +250$ N the shear flow is

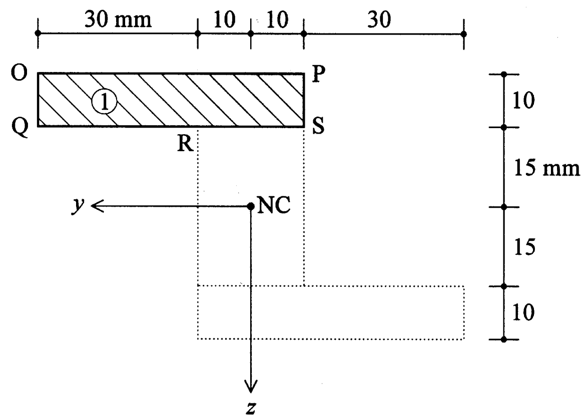


Figure 9.107 The upper flange OPQS is chosen as the sliding element.

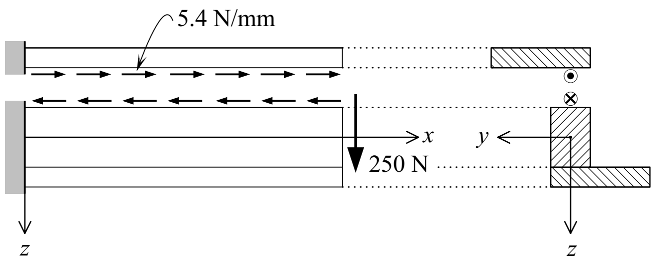


Figure 9.108 The shear flow between upper flange and web. Since the shear force V_z is constant along the length of the beam, the shear flow will also be constant.

$$s_x^a = -\frac{V_z \cdot ES_z^a \cdot EI_{yy}}{\text{Det}(EI)} + \frac{V_z \cdot ES_y^a \cdot EI_{yz}}{\text{Det}(EI)}$$

$$= \frac{(250\text{ N})(-120 \times 10^6\text{ Nmm})(5.32 \times 10^9\text{ Nmm}^2)}{14.54 \times 10^{18}\ (\text{Nmm}^2)^2}$$

$$+ \frac{(250\text{ N})(+90 \times 10^6\text{ Nmm})(-3.6 \times 10^9\text{ Nmm}^2)}{14.54 \times 10^{18}\ (\text{Nmm}^2)^2}$$

$$= 5.4\text{ N/mm}.$$

Since the shear force V_z is constant along the length of the beam, the shear flow between the upper flange and web will also be constant. The direction of the shear flow is shown in Figure 9.108. Since the shear flow is positive, it acts in the positive x direction on the flange (the sliding element), and in the negative x direction on the web.

The glued joint should be designed in such a way that it can resist this shear flow. If so, the glued joint can be seen as an interface without slip, and the composite cross-section acts as one piece.

9.17.2 Shear flow in longitudinal direction (alternative approach)

If the yz coordinate system does not coincide with the principal directions of the cross-section we cannot use the rather simple formula (9.94), based on the uncoupled bending terms of the constitutive relationship,

$$s_x^a = -\frac{V_y ES_y^a}{EI_{yy}} - \frac{V_z ES_z^a}{EI_{zz}}, \qquad (9.94)$$

but we have to use the far more complicated formula (9.96)

$$s_x^{\mathrm{a}} = -\frac{V_y}{\mathrm{Det}(EI)}\,(ES_y^{\mathrm{a}} \cdot EI_{zz} - ES_z^{\mathrm{a}} \cdot EI_{zy})$$
$$\qquad - \frac{V_z}{\mathrm{Det}(EI)}\,(ES_z^{\mathrm{a}} \cdot EI_{yy} - ES_y^{\mathrm{a}} \cdot EI_{yz}). \qquad (9.96)$$

Below we will derive an alternative formula for general situations. We start with the result of the horizontal equilibrium of the sliding element as shown in Figure 9.109a. The shear flow (shear force per length in longitudinal direction) is

$$s_x^{\mathrm{a}} = -\frac{\mathrm{d}N^{\mathrm{a}}}{\mathrm{d}x}\,, \qquad (9.85)$$

in which

$$N^{\mathrm{a}} = \int_{A^{\mathrm{a}}} \sigma(y,z)\,\mathrm{d}A.$$

Since the normal stresses $\sigma(y,z)$ on the cross-sectional area A^{a} of the sliding element are proportional to the normal force N and the bending moment M, we can split the resultant N^{a} of these stresses on A^{a} into the contributions $N_N^{\mathrm{a}} = c_1 N$ due to N and $N_M^{\mathrm{a}} = c_2 M$ due to M:

$$N^{\mathrm{a}} = N_N^{\mathrm{a}} + N_M^{\mathrm{a}} = c_1 N + c_2 M, \qquad (9.98)$$

in which

$$M = \sqrt{M_y^2 + M_z^2}.$$

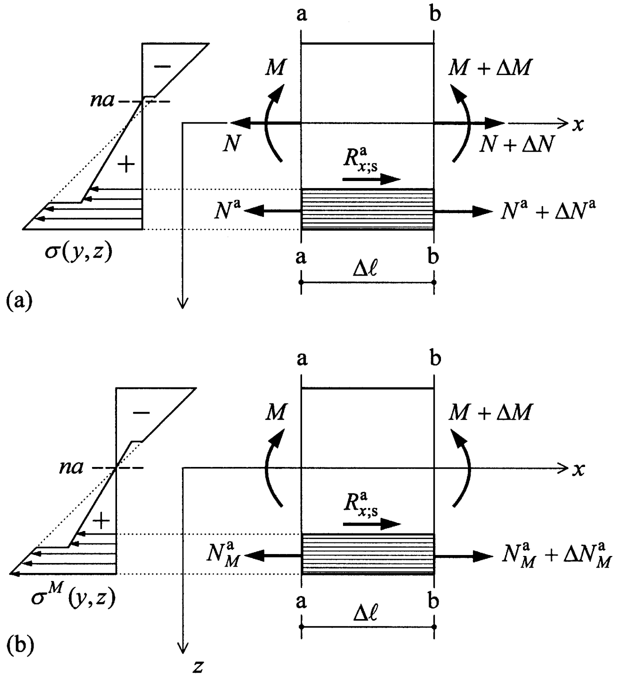


Figure 9.109 (a) The shear force $R^{\mathrm{a}}_{x;s}$ in longitudinal direction follows from the horizontal equilibrium of the sliding element. (b) If N_M^{a} is the resultant of only the bending stresses $\sigma^M(y,z)$ on the sliding element, then N_M^{a} is proportional to the bending moment M: $N_M^{\mathrm{a}} = c_2 M$. c_2 is a proportionality factor determined by the cross-sectional shape of the sliding element.

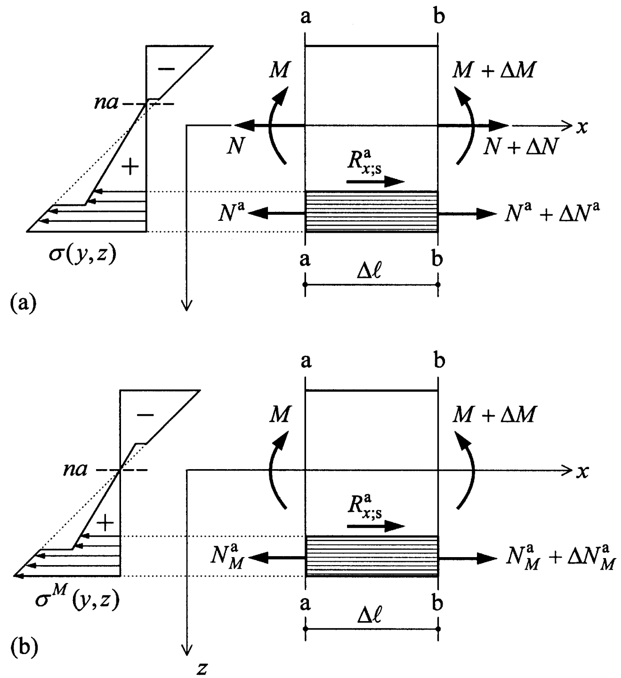


Figure 9.109 (a) The shear force $R^{\mathrm{a}}_{x;s}$ in longitudinal direction follows from the horizontal equilibrium of the sliding element. (b) If N^{a}_M is the resultant of only the bending stresses $\sigma^M(y, z)$ on the sliding element, then N^{a}_M is proportional to the bending moment M: $N^{\mathrm{a}}_M = c_2 M$. c_2 is a proportionality factor determined by the cross-sectional shape of the sliding element.

For a prismatic beam, c_1 and c_2 are constant; they can be found as

$$c_1 = \frac{N^{\mathrm{a}}_N}{N} \text{ and } c_2 = \frac{N^{\mathrm{a}}_M}{M}. \tag{9.99}$$

Substitute (9.98) in (9.99):

$$s^{\mathrm{a}}_x = -\frac{\mathrm{d}N^{\mathrm{a}}}{\mathrm{d}x} = -\frac{\mathrm{d}(c_1 N + c_2 M)}{\mathrm{d}x} = -c_1\frac{\mathrm{d}N}{\mathrm{d}x} - c_2\frac{\mathrm{d}M}{\mathrm{d}M}.$$

Assuming a constant normal force N, we have $\mathrm{d}N/\mathrm{d}x = 0$ and find for the shear flow

$$s^{\mathrm{a}}_x = -c_2\frac{\mathrm{d}M}{\mathrm{d}x} = -c_2 V = -\frac{N^{\mathrm{a}}_M}{M}\,V, \tag{9.100}$$

in which

$$V = \sqrt{V_y^2 + V_z^2}.$$

In this expression V is the resultant shear force in the cross-section.

In words, formula (9.100) states that the shear flow (shear force per length) s^{a}_x is equal to the *resultant shear force* V multiplied with a scaling factor c_2. This scaling factor is the resultant N^{a}_M of the *bending stresses*[1] on the sliding element only, divided by the resultant bending moment M in the cross-section (see Figure 9.109b).

[1] Bending stresses are normal stresses due to bending only.

Note that the scaling factor c_2 is independent of the magnitude of the bending moment M. Important in the application of this method is however that both the shear force V and the bending moment M act in the *same plane of loading* since we made use of the equilibrium condition

$$V = \frac{dM}{dx} .$$

Moreover, we have to take into account that in this equilibrium the positive directions of V and M condition are related as shown in Figure 9.110.

If in a cross-section only the shear force is known, or the bending moment is zero, we can apply a non-zero *dummy moment* to calculate the scaling factor c_2. This will be illustrated in the second example below.

Example 1: Shear flow in an unsymmetrical and inhomogeneous cross-section

In Section 9.9.2, Example 2, the bending stress distribution in Figure 9.111 was found for the inhomogeneous cross-section subject to a bending moment M with components $M_y = 0$ and $M_z = -137.5 \times 10^3$ Nmm. The cross-section has to transmit a shear force V with components $V_y = 0$ and $V_z = +250$ N.

Question:
Determine the shear flow in the glue line RS.

Solution (units in N and mm):
Since we know the normal stress distribution due to bending only, we can use the alternative formula for the shear flow:

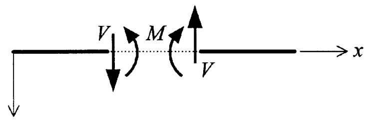


Figure 9.110 The equilibrium equation $V = dM/dx$ holds only when the positive directions of V and M are related as shown in the figure.

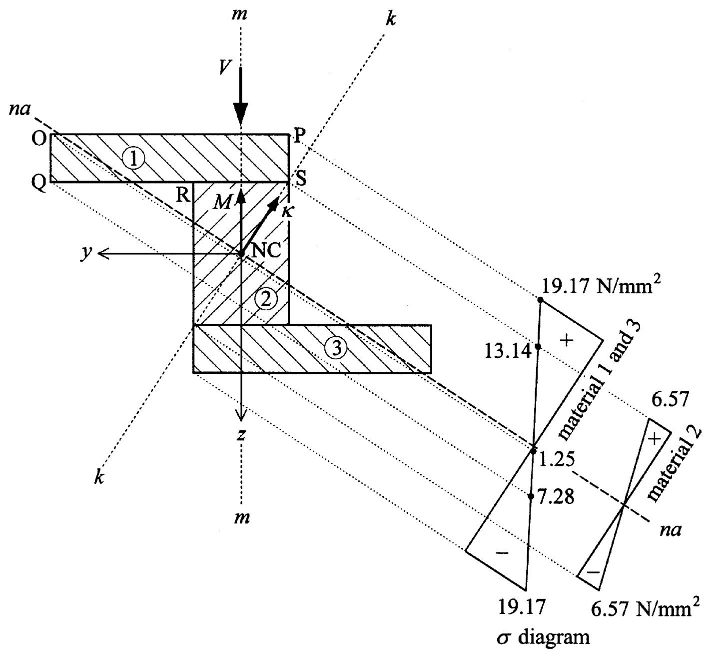


Figure 9.111 Distribution of the bending stresses for an unsymmetrical and inhomogeneous cross-section.

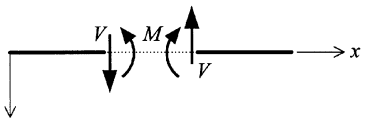


Figure 9.110 The equilibrium equation $V = \mathrm{d}M/\mathrm{d}x$ holds only when the positive directions of V and M are related as shown in the figure.

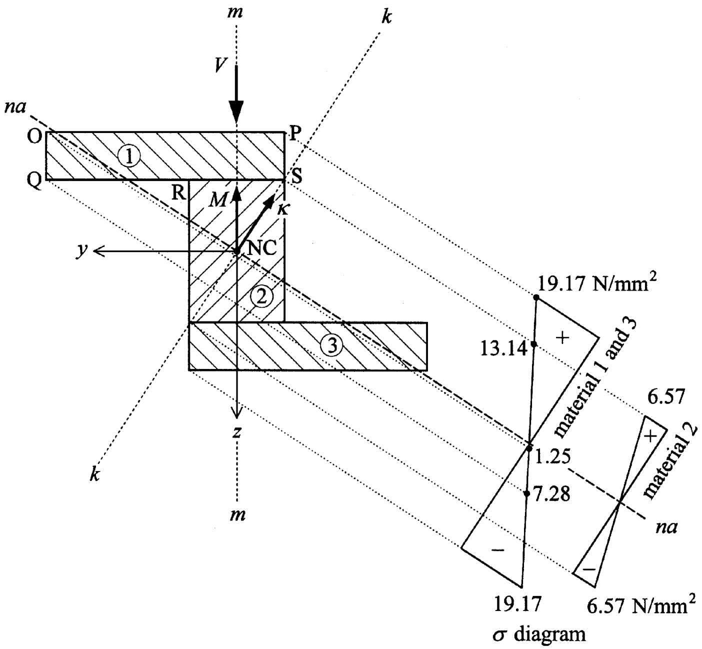


Figure 9.111 Distribution of the bending stresses for an unsymmetrical and inhomogeneous cross-section.

$$s_x^{\mathrm{a}} = -c_2 \frac{\mathrm{d}M}{\mathrm{d}x} = -c_2 V = -\frac{N_M^{\mathrm{a}}}{M} V, \qquad (9.100)$$

The xz plane is the plane of loading, in which both the bending moment M and the shear force V act. As mentioned before, the signs of V and M are related to the positive directions as shown in Figure 9.110. So, in the xz coordinate system, M is negative and V is positive:

$$M = M_z = -137.5 \times 10^3 \text{ Nmm},$$

$$V == V_z = +250 \text{ N}.$$

To determine the shear flow in the glue line RS we consider the flange OPQS as *sliding element*. The normal stresses at the four corners of this flange are given in Table 9.15.[1]

Since the normal stress distribution over the rectangle OPQS is linear, the resultant normal force on the sliding element can be found by multiplying the *average normal stress* with the cross-sectional *area of the sliding element*:

$$N_M^{\mathrm{a}} = \frac{(-1.25 + 19.17 - 7.28 + 13.14)\ \text{N/mm}^2}{4} \times (50 \times 10\ \text{mm}^2)$$

$$= +2972.5 \text{ N}.$$

[1] See Section 9.9.2, Example 2, Table 9.5.

For the shear flow in the glue line RS we now find

$$s_x^{\mathrm{a}} = -\frac{N_M^{\mathrm{a}}}{M} V = -\frac{+2972.5 \text{ N}}{-137.5 \times 10^3 \text{ Nmm}} \times (+250 \text{ N}) = +5.4 \text{ N/mm}.$$

The shear flow (shear force per length) on the upper flange acts in the positive x direction, that on the web in the negative x direction. The same result was found in Section 9.17.1, Example 2.

Example 2: Shear stresses in an inhomogeneous cross-section
The rectangular inhomogeneous cross-section in Figure 9.112a is built up out of three parts which are firmly glued together. The cross-section has to transmit a vertical shear force $V = 40$ kN as shown in Figure 9.112b.The moduli of elasticity are

- part 1: $E_1 = 40 \times 10^3$ N/mm^2,
- part 2: $E_2 = 160 \times 10^3$ N/mm^2,
- part 3: $E_3 = 120 \times 10^3$ N/mm^2.

The location of the normal centre NC is given in the figure. Furthermore, it is given that $EI_{zz} = 200 \times 10^{12}$ Nmm^2 in the yz coordinate system shown.[1]

Question:
Determine the shear stress distribution, assuming the shear stresses are constant across the width of the cross-section.

[1] It is left to the reader to verify the location of the normal centre NC and the mentioned value of EI_{zz}, the bending stiffness in the xz plane.

Table 9.15

point	y (mm)	z (mm)	Stress σ (N/mm^2)
O	+40	−25	−1.25
P	−10	−25	+19.17
Q	+40	−15	−7.28
S	−10	−15	+13.14

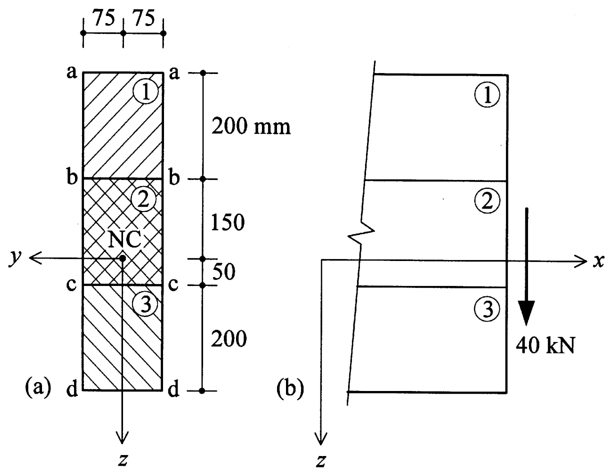


Figure 9.112 (a) A rectangular inhomogeneous cross-section, built up out of three parts which are firmly glued together. (b) The cross-section has to transmit a vertical shear force $V = 40$ kN.

Solution:

Section 5 3.2 shows how the shear stresses in the cross-section can be found from the shear flow (see Figure 9.113):

$$\sigma_{xm} = \sigma_{mx} = \frac{s_x^{\mathrm{a}}}{b^{\mathrm{a}}}\,. \tag{9.101}$$

For an unambiguous notation of the shear stress, a temporarily m axis is introduced, normal to the cut and with its positive direction pointing out of the (hatched) material of the sliding element.

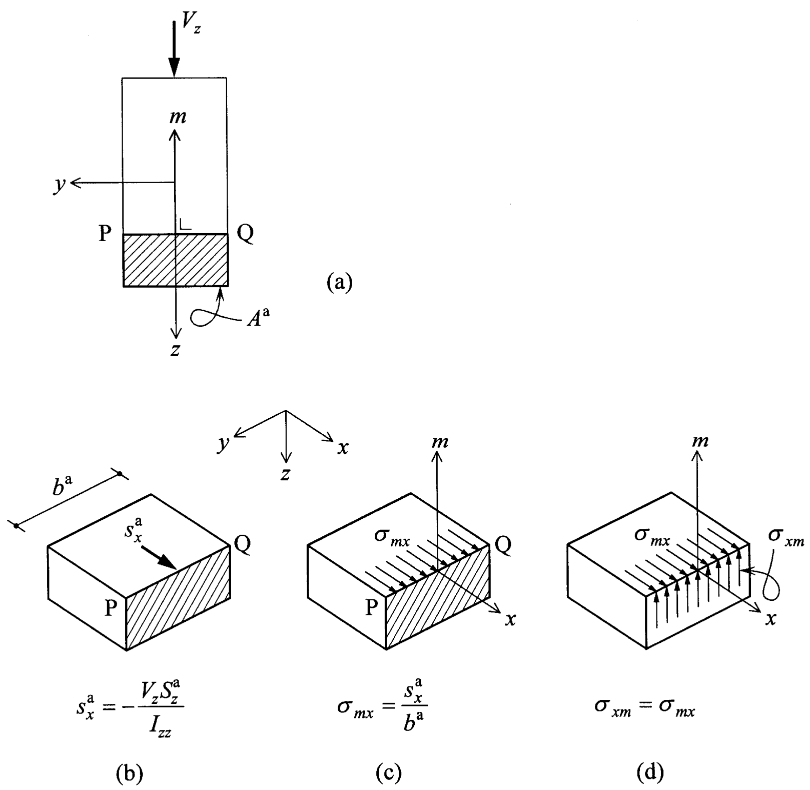


Figure 9.113

Figure 9.113 (a) A rectangular cross-section subject to a shear force V_z. To ensure clarity in the figure, the shear force is placed outside the cross-section. The sliding element of the cross-section is hatched. Plane cut PQ has a width b^{a} and is normal to the edges of the cross-section. In order to label the shear stresses, the m axis has been introduced, perpendicular to the cut PQ and in such a way that the positive direction, indicated by the arrowhead, points out of the (hatched) material of the sliding part of the cross-section. (b) The sliding element with the longitudinal shear force s_x^{a} (force per length) on the longitudinal section plane. (c) Smearing the shear force s_x^{a} (force per length) uniformly over width b^{a} leads to the longitudinal shear stress σ_{mx} (force per area). (d) In the cut PQ, the longitudinal section plane and the cross-sectional plane are perpendicular to one another. Since the shear stresses on two perpendicular planes are equal, $\sigma_{xm} = \sigma_{mx}$.

Comment: Formula (9.101) is based on the assumption that shear stresses are uniformly distributed over the width of the longitudinal cut.[1] We also use the property that the shear stresses in two perpendicular planes are of the same magnitude, thus $\sigma_{xm} = \sigma_{mx}$.[2]

To find the shear flow s_x^{a}, we use the alternative approach:

$$s_x^{\mathrm{a}} = -\frac{N_M^{\mathrm{a}}}{M} V. \tag{9.100}$$

The shear force is $V = V_z = 40$ kN. The bending moment M is not given, so we apply a *dummy moment*. By choosing $M = M_z = 40$ kNm (see Figure 9.114a), we have

$$s_x^{\mathrm{a}} = -\frac{N_M^{\mathrm{a}}}{M} V = -\frac{N_M^{\mathrm{a}}}{40\text{ kNm}} (40\text{ kN}) = -\frac{N_M^{\mathrm{a}}}{10^3\text{ mm}}, \tag{9.101}$$

and for the shear stress

$$\sigma_{xm} = \frac{s_x^{\mathrm{a}}}{b^{\mathrm{a}}} = -\frac{N_M^{\mathrm{a}}}{b^{\mathrm{a}} \times (10^3\text{ mm})}, \tag{9.102}$$

in which N_M^{a} is the resultant of all bending stresses on the sliding element, due to $M = 40$ kNm.

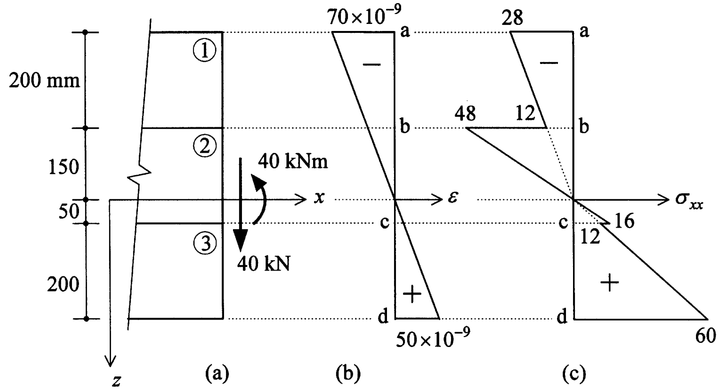


Figure 9.114 (a) The shear force is given, $V = V_z = 40$ kN, but not the bending moment M. Therefore a *dummy moment* is applied, for which is chosen $M = M_z = 40$ kNm. Diagrams for (b) the strain $\varepsilon(z)$ and (c) bending stress $\sigma_{xx}(z)$, both due to the dummy moment $M = M_z = 40$ kNm.

[1] For restrictions and special cases, see Sections 5.3.3, 5.3.4 and 5.4.
[2] See the proof in Section 5.3.1.

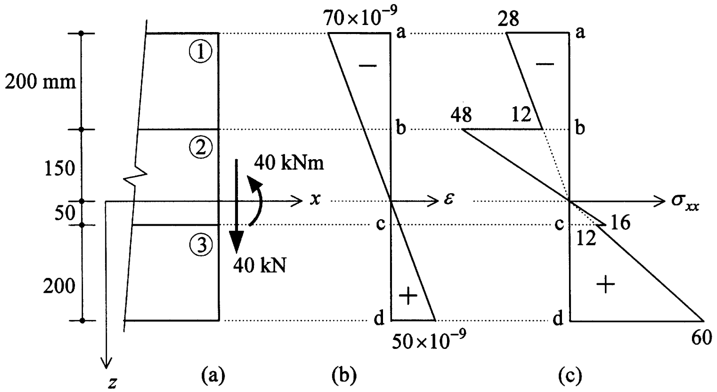


Figure 9.114 (a) The shear force is given, $V = V_z = 40$ kN, but not the bending moment M. Therefore a *dummy moment* is applied, for which is chosen $M = M_z = 40$ kNm. Diagrams for (b) the strain $\varepsilon(z)$ and (c) bending stress $\sigma_{xx}(z)$, both due to the dummy moment $M = M_z = 40$ kNm.

Table 9.16

part i	cut	z (mm)	E_i (N/mm^2)	$\varepsilon(z)$	$\sigma_{xx}(z)$ (N/mm^2)
1	a-a	-350	40×10^3	-70×10^{-9}	-2.8
1	b-b	-150	40×10^3	-30×10^{-9}	-1.4
2	b-b	-150	160×10^3	-30×10^{-9}	-4.8
2	c-c	$+50$	160×10^3	$+10 \times 10^{-9}$	$+1.6$
3	c-c	$+50$	120×10^3	$+10 \times 10^{-9}$	$+1.2$
3	d-d	$+250$	120×10^3	$+50 \times 10^{-9}$	$+6.0$

Since the yz coordinate system is a principal coordinate system, the bending stresses can easily be found. The curvature in the xz plane is

$$\kappa_z = \frac{M_z}{EI_{zz}} = \frac{40 \times 10^6 \text{ Nmm}}{200 \times 10^{12} \text{ Nmm}^2} = 0.2 \times 10^{-9} \text{ mm}^{-1}.$$

The strain, stress and modulus of elasticity are functions of z only. The linear strain distribution is

$$\varepsilon(z) = \kappa_z z = (0.2 \times 10^{-9} \text{ mm}^{-1}) \times z.$$

The bending stresses follow from[1]

$$\sigma_{xx}(z) = E(z) \cdot \varepsilon(z).$$

The strains and stresses are calculated in Table 9.16 and their distributions are shown in Figure 9.114.

The resultant of all bending stresses on the sliding element is

$$N_M^{\text{a}} = b^{\text{a}} \times (\text{area of the bending stress diagram}).$$

Hence

$$\sigma_{xm} = -\frac{N_M^{\text{a}}}{b^{\text{a}} \times (10^3 \text{ mm})} = -\frac{\text{area of the bending stress diagram}}{10^3 \text{ mm}}. \quad (9.103)$$

[1] Since the *double index notation* (tensor notation) is used for the shear stress σ_{xm}, we will use it also for the bending stress σ.

Procedure to derive the shear stress distribution from the bending stress distribution:

- Choose an m coordinate normal to the cut, and pointing out of the material of the sliding element considered.
- Find the bending stress, denoted as σ_{xx}, as a function of m.
- Integrate σ_{xx} to find the area of the bending stress diagram, as a function m.
- With formula (9.103) we have the shear stress σ_{xm} as a function of m.

Below the units are omitted in the intermediate calculations; we use mm and N.

Part 1 (see Figure 9.115):

$$\sigma_{xx} = -2.8 + \frac{1.6}{200} m_1 = -2.8 + 0.008 m_1,$$

$$\text{area} = \int_0^{m_1} \sigma_{xx} \, \mathrm{d}m_1 = -2.8 m_1 + 0.004 m_1^2,$$

$$\text{area1} = (-2.8 m_1 + 0.004 m_1^2)\Big|_0^{200} = -400,$$

$$\sigma_{xm} = -\frac{-2.8m + 0.004m^2}{10^3}. \qquad (9.104)$$

Note that area1 is the area of the bending stress diagram with respect to part 1.

Part 2 (see Figure 9.116):

$$\sigma_{xx} = -4.8 + \frac{6.4}{200} m_2 = -4.8 + 0.032 m_2,$$

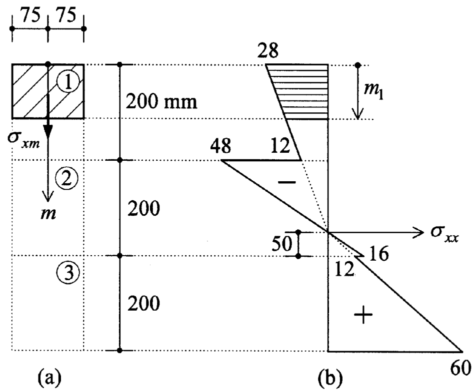


Figure 9.115 (a) The sliding element with the m coordinate axis. (b) For part 1, the associated area of the bending stress diagram can be calculated as function of m_1.

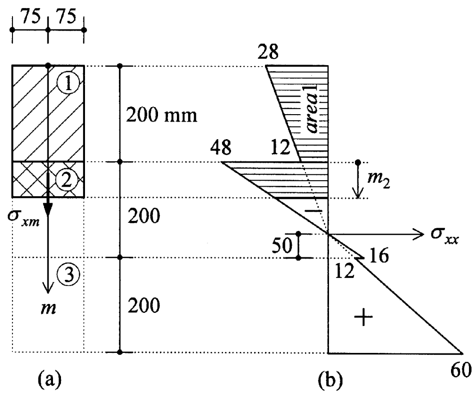


Figure 9.116 (a) The sliding element with the m coordinate axis. (b) For part 2, the associated area of the bending stress diagram can be calculated as function of m_2.

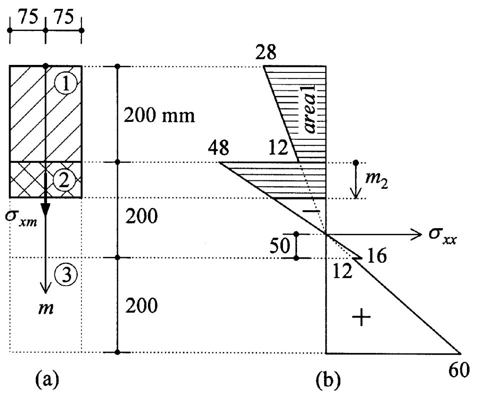


Figure 9.116 (a) The sliding element with the m coordinate axis. (b) For part 2, the associated area of the bending stress diagram can be calculated as function of m_2.

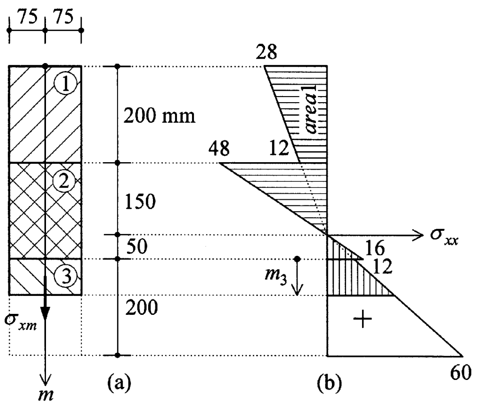


Figure 9.117 (a) The sliding element with the m coordinate axis. (b) For part 3, the associated area of the bending stress diagram can be calculated as function of m_3.

$$\text{area} = \text{area1} + \int_0^{m_2} \sigma_{xx}\, \mathrm{d}m_2 = -400 - 4.8m_2 + 0.016m_2^2,$$

$$\text{area1} + \text{area2} = (-400 - 4.8m_2 + 0.016m_2^2)\big|_0^{200} = -720,$$

$$\sigma_{xm} = -\frac{-400 - 4.8m_2 + 0.016m_2^2}{10^3}. \tag{9.105}$$

Part 3 (see Figure 9.117):

$$\sigma_{xx} = +1.2 + \frac{4.8}{200} m_3 = +1.2 + 0.024m_3,$$

$$\text{area} = \text{area1} + \text{area2} + \int_0^{m_3} \sigma_{xx}\, \mathrm{d}m_3 = -720 + 1.2m_3 + 0.012m_3^2,$$

$$\text{area1} + \text{area2} + \text{area3} = (-720 + 1.2m_3 + 0.012m_3^2)\big|_0^{200} = 0,$$

$$\sigma_{xm} = -\frac{-720 + 1.2m_3 + 0.012m_3^2}{10^3}. \tag{9.106}$$

Note: area1 + area2 + area3 = 0 we can use as a check. Since we consider bending stresses only, there is no resultant force and the total area of the bending stress diagram has to be zero.

For each homogeneous part the vertical shear stress is quadratic in m.

The parabolic shear stress distribution is in agreement with the following rule, derived in Section 5.3.3: *if the bending stress is linear, the shear stress must be parabolic*. This and the other rules, summarised in Section 5.7.2, apply also to the homogeneous parts within an inhomogeneous cross-section.

At P and S, $\sigma_{xm} = 0$. At the glue lines we find

$$\text{Q:} \quad \sigma_{xm} = -\frac{\text{area1}}{10^3} = -\frac{-400}{10^3} = +0.4\ \text{N/mm}^2,$$

$$\text{R:} \quad \sigma_{xm} = -\frac{\text{area1} + \text{area2}}{10^3} = -\frac{-720}{10^3} = +0.72\ \text{N/mm}^2.$$

The shear stress diagram is sketched in Figure 9.118. All shear stresses σ_{xm} are positive, which means that they act in the positive m direction. This is in accordance with the direction of the shear force.

The shear stress is a maximum at the cut through the normal centre NC. For $m_2 = 150$ mm we find from (9.105)

$$\sigma_{xm} = -\frac{(-400 - 4.8m_2 + 0.016m_2^2)\big|_{m_2=150}}{10^3} = +0.76\ \text{N/mm}^2.$$

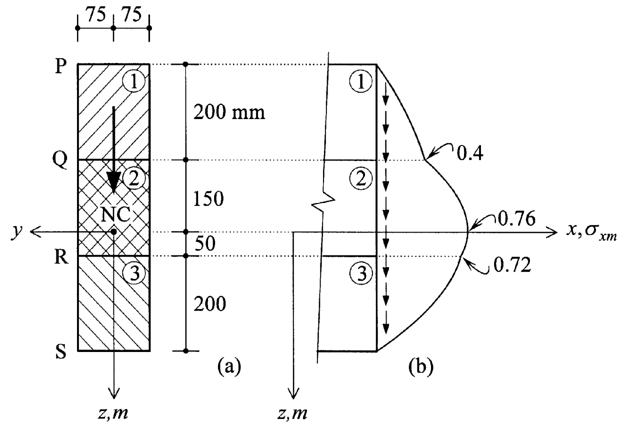


Figure 9.118 (a) Cross-section with the shear force. (b) The vertical shear stress distribution. For each homogeneous part the shear stress is parabolic. All shear stresses σ_{xm} are positive, which means that they act in the positive m direction. This agrees with the direction of the shear force.

9.17.3 Shear centre for unsymmetrical thin-walled cross-sections

If there is no torsion, the resultant force of all shear stresses in a cross-section is equal to the shear force in that cross-section. However its line of action will in most cases not pass through the normal centre NC, but through the so-called *shear* (*force*) *centre* SC. The shear centre is defined as follows:

The shear centre SC is that point in the cross-sectional plane through which the line of action of the shear force must pass so that there will be no torsion.

For rotation symmetrical cross-sections the shear centre SC coincides with

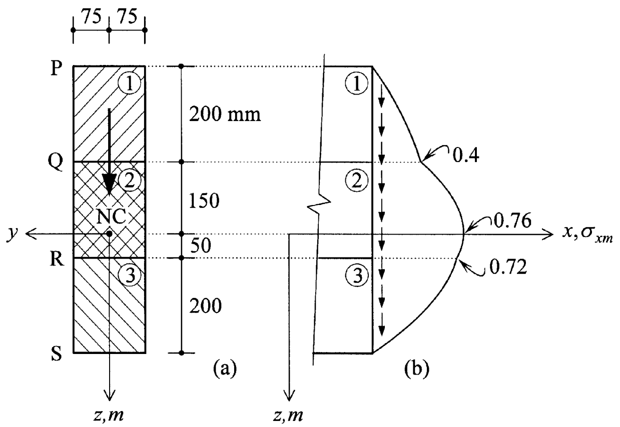


Figure 9.118 (a) Cross-section with the shear force. (b) The vertical shear stress distribution. For each homogeneous part the shear stress is parabolic. All shear stresses σ_{xm} are positive, which means that they act in the positive m direction. This agrees with the direction of the shear force.

the normal centre NC. If the cross-section has a line of symmetry, the shear centre SC will be located on this axis. For a number of thin-walled sections this is illustrated in Section 5.5.

In this section we will show how to find the location of the shear centre SC for unsymmetrical thin-walled cross-sections. The necessary steps in the procedure to find the shear centre are:

- Step 1: For an arbitrary shear force V_1, find the shear stress distribution in the cross-section and determine the line of action of the resultant force due to the shear stresses. This is the line of action of V_1 for shear without torsion. So the shear centre SC is located on this line.
- Step 2: Repeat step 1 for an arbitrary shear force V_2 in a direction other than V_1. The shear centre SC is located on the line of action of V_2, found from the shear stress distribution.
- Step 3: The shear centre is the point of intersection of the two lines of action of V_1 and V_2.

Since we try to find the shear centre SC as the point of intersection of two lines of action, it is essential to choose two shear forces V_1 and V_2 not parallel to one another.

The term *arbitrary* means that we are free to choose the directions and the magnitudes of the set of shear forces as long as these shear forces are not parallel.

The procedure is illustrated in an example.

Example: Shear centre of an unsymmetrical thin-walled cross-section
The thin-walled homogeneous cross-section PQRS in Figure 9.119a has a constant wall-thickness $t = 30$ mm. Use for the longitudinal measurement $a = 180$ mm. The modulus of elasticity is $E = 200 \times 10^3$ Nmm^2.

Questions:

a. Determine the normal centre NC of the cross-section and the central bending stiffness values.[1]

b. Determine the shear centre SC of the cross-section.

Solution:

a. In the $\overline{yz}$ coordinate system, shown in Figure 9.119b, we find for the coordinates of the normal centre NC

$$\bar{y}_{NC} = \frac{S_{\bar{y}}}{A} = \frac{t \cdot a \cdot 2a + t \cdot 2a \cdot a}{t \cdot (a + 2a + 3a)} = \tfrac{2}{3}\,a = 120 \text{ mm},$$

$$\bar{z}_{NC} = \frac{S_{\bar{z}}}{A} = \frac{t \cdot a \cdot \frac{1}{2}\,a + t \cdot 3a \cdot \frac{3}{2}\,a}{t \cdot (a + 2a + 3a)} = \tfrac{5}{6}\,a = 150 \text{ mm}.$$

In the yz coordinate system with its origin at the normal centre NC the components of the bending stiffness tensor are

$$EI_{yy} = E \times \left[\underbrace{ta \cdot (\tfrac{4}{3}\,a)^2}_{\text{PQ}} + \underbrace{\tfrac{1}{12}\,t \cdot (2a)^3 + 2ta \cdot (\tfrac{1}{3}\,a)^2}_{\text{QR}} + \underbrace{3ta \cdot (\tfrac{2}{3}\,a)^2}_{\text{RS}} \right]$$

$$= 4Eta^3 = 139.968 \times 10^{12} \text{ Nmm}^2,$$

$$EI_{yz} = E \times \left[\underbrace{ta \cdot (\tfrac{4}{3}\,a) \cdot (-\tfrac{1}{3}\,a)}_{\text{PQ}} + \underbrace{2at \cdot (\tfrac{1}{3}\,a) \cdot (-\tfrac{5}{6}\,a)}_{\text{QR}} + \underbrace{3at \cdot (-\tfrac{2}{3}\,a) \cdot (\tfrac{2}{3}\,a)}_{\text{RS}} \right]$$

$$= -\tfrac{7}{3}\,Eta^3 = -81.648 \times 10^{12} \text{ Nmm}^2,$$

[1] These are the values in a central coordinate system, which is a coordinate system with its origin at the normal centre NC of the cross-section.

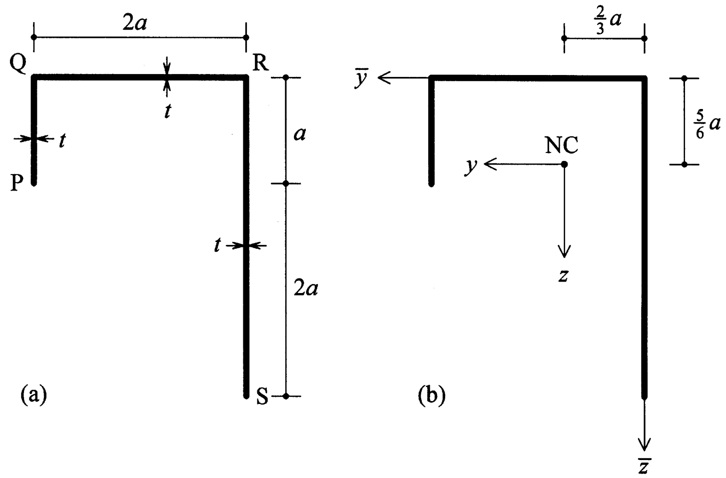


Figure 9.119 (a) A homogeneous and unsymmetrical thin-walled cross-section. (b) The location of the normal centre NC.

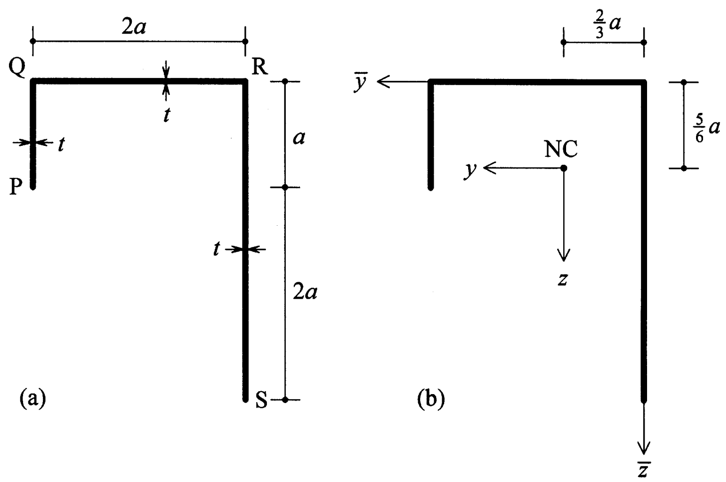


Figure 9.119 (a) A homogeneous and unsymmetrical thin-walled cross-section. (b) The location of the normal centre NC.

$$EI_{zz} = E \times \left[\underbrace{\tfrac{1}{12}ta^3 \cdot ta \cdot (\tfrac{1}{3}a)^2}_{\text{PQ}} + \underbrace{2ta \cdot (\tfrac{5}{6}a)^2}_{\text{QR}} + \underbrace{\tfrac{1}{12}t \cdot (3a)^3 + 3ta \cdot (\tfrac{2}{3}a)^2}_{\text{RS}} \right]$$

$$= \tfrac{31}{6}\,Eta^3 = 180.792 \times 10^{12}\ \text{Nmm}^2.$$

In order to find the shear centre SC, we need these values.

b. The shear stress distribution due to an arbitrary shear force V is determined with the alternative approach, using a dummy moment M:

$$s_x^{\text{a}} = -\frac{N_M^{\text{a}}}{M}V \quad \text{and} \quad \sigma_{xm} = \frac{s_x^{\text{a}}}{b^{\text{a}}} = -\frac{N_M^{\text{a}}}{b^{\text{a}}}\frac{V}{M}.$$

N_M^{a} is the resultant of all bending stresses on the sliding element due to the dummy moment:

$$N_M^{\text{a}} = b^{\text{a}} \times (\text{area of the bending stress diagram}).$$

So

$$\sigma_{xm} = -\frac{N_M^{\text{a}}}{b^{\text{a}}}\frac{V}{M} = -(\text{area of the bending stress diagram}) \times \frac{V}{M}.$$

V and M have to be chosen in the same plane and in the correct directions with respect to one another. Using a unity shear force for V and a unity bending moment for M, we can simplify the calculation considerably:

$$\sigma_{xm} = -(\text{area of the bending stress diagram}) \times \dim\left(\frac{V}{M}\right).$$

The numerical value of the shear stress is, with opposite sign, equal to the area of the bending stress diagram for the sliding element. Since V/M is not dimensionless, we may not forget to pay attention to the units in which we work.

To find the location of the shear centre we have to compute the shear stress distributions due to two shear forces, for which we choose shear forces in the y and z direction. So the first distribution is the result of a unity shear force V_y and the associated (dummy) unity moment M_y, and the second is the result of a unity shear force V_z with the associated (dummy) unity moment M_z:

$$\text{Loading case 1:} \quad \begin{bmatrix} V_y \\ V_z \end{bmatrix} = \begin{bmatrix} 1\text{ N} \\ 0 \end{bmatrix}; \quad \begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 1\text{ Nmm} \\ 0 \end{bmatrix},$$

$$\text{Loading case 2:} \quad \begin{bmatrix} V_y \\ V_z \end{bmatrix} = \begin{bmatrix} 0 \\ 1\text{ N} \end{bmatrix}; \quad \begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 \\ 1\text{ Nmm} \end{bmatrix}.$$

In order to find the bending stress distribution we use the constitutive relationship

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix}$$

in the inverse form:

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{1}{\text{Det}(EI)} \begin{bmatrix} EI_{zz} & -EI_{zy} \\ -EI_{yz} & EI_{yy} \end{bmatrix} \begin{bmatrix} M_y \\ M_z \end{bmatrix}.$$

Table 9.17

point	y **(mm)**	z **(mm)**	$\sigma(y,z)\times 10^9$ **(N/mm²)**
P	+240	+30	+491.9
Q	+240	−150	+334.2
R	−120	−150	−364.2
S	−120	+390	+108.8

Here

$$\text{Det}(EI) = EI_{yy}EI_{zz} - (EI_{yz})^2.$$

After substitution of the bending stiffness values found before, we find

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{1}{10^{15}\ \text{Nmm}^2} \begin{bmatrix} 9.70 & 4.38 \\ 4.38 & 7.51 \end{bmatrix} \begin{bmatrix} M_y \\ M_z \end{bmatrix}.$$

Loading case 1

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 100 \times 10^6\ \text{Nmm} \\ 0.0 \end{bmatrix},$$

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{1}{10^{15}\ \text{Nmm}^2} \times \begin{bmatrix} 9.70 & 4.38 \\ 4.38 & 7.51 \end{bmatrix} \begin{bmatrix} 1\ \text{Nmm} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9.70 \\ 4.38 \end{bmatrix} \times (10^{-15}\ \text{mm}^{-1}).$$

The strain distribution due to bending:

$$\varepsilon(y,z) = \kappa_y y + \kappa_z z$$
$$= (9.70 \times y + 4.38 \times z) \times (10^{-15}\ \text{mm}^{-1}).$$

The normal stress distribution due to bending:

$$\sigma(y,z) = E \cdot \varepsilon(y,z) = E\kappa_y y + E\kappa_z z$$
$$= (1.940 \times y + 0.876 \times z) \times (10^{-9}\ \text{N/mm}^{-3}).$$

The bending stresses at the key points P, Q, R and S are calculated in Table 9.17. Figure 9.120 shows the bending stress diagram. When the picture is drawn to scale, we have the check that the neutral axis *na* must pass through the normal centre NC.

Next we will derive the shear stress distribution from the bending stress diagram. Below, all quantities will be expressed in the units N and mm. These units are omitted in the calculations. Since $\dim(V/M) = 1\ \text{mm}^{-1}$, this term does not influence the numerical results, but provides only the correct dimension.

- PQ – loading case 1: $V_y = 1$ N (see Figure 9.120)

The bending stress σ_{xx} as a function of the auxiliary coordinate m_1:

$$\sigma_{xx} = \left(491.9 - \frac{491.9 - 334.2}{180} m_1\right) \times 10^{-9}$$

$$= (491.9 - 0.876 m_1) \times 10^{-9}.$$

Area of the bending stress diagram from P to the cut at m_1:

$$\text{area} = \int_0^{m_1} \sigma_{xx}\, \mathrm{d}m_1 = \left(491.9 m_1 - \tfrac{1}{2} \times 0.8761 m_1^2\right) \times 10^{-9},$$

$$\text{area(PQ)} = \int_0^{180} \sigma_{xx}\, \mathrm{d}m_1 = 74349 \times 10^{-9}.$$

Shear stress σ_{xm} at the cut m_1 (see Figure 9.121a):

$$\sigma_{xm} = -\text{area} = \left(-491.9 m_1 + \tfrac{1}{2} \times 0.8761 m_1^2\right) \times 10^{-9}.$$

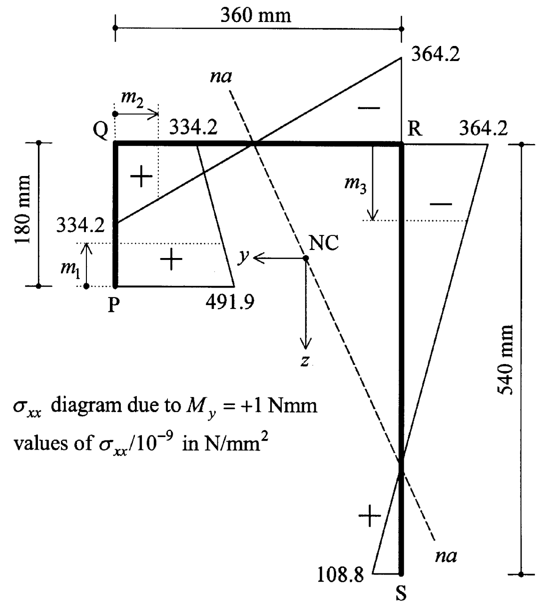


Figure 9.120 The bending stress distribution due to $M_y = +1$ Nmm.

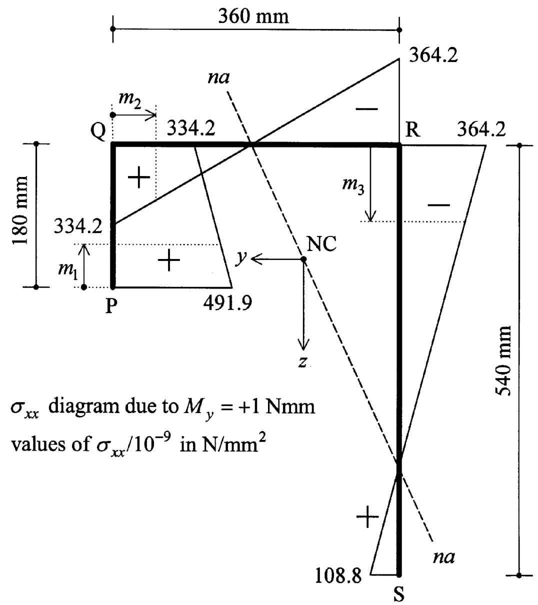


Figure 9.120 The bending stress distribution due to $M_y = +1$ Nmm.

Resultant shear force in PQ (see Figure 9.121b):

$$R_{PQ} = t \times \int_0^{180} \sigma_{xm}\, dm_1$$
$$= 30 \times \left(-\tfrac{1}{2} \times 491.9 m_1^2 + \tfrac{1}{6} \times 0.8761 m_1^3\right) \times 10^{-9} \Big|_0^{180}$$
$$= -0.21352 \text{ N}.$$

- QR – loading case 1: $V_y = 1$ N (see Figure 9.120)

The bending stress σ_{xx} as a function of the auxiliary coordinate m_2:

$$\sigma_{xx} = \left(334.2 - \frac{334.2 + 364.2}{360} m_2\right) \times 10^{-9}$$
$$= (334.2 - 1.94 m_2) \times 10^{-9}.$$

Area of the bending stress diagram from P to the cut at m_2:

$$\text{area} = \text{area(PQ)} + \int_0^{m_2} \sigma_{xx}\, dm_2$$
$$= \left(74349 + 334.2 m_2 - \tfrac{1}{2} \times 1.94 m_2^2\right) \times 10^{-9},$$
$$\text{area(PQR)} = \text{area(PQ)} + \int_0^{360} \sigma_{xx}\, dm_2 = 68949 \times 10^{-9}.$$

Shear stress σ_{xm} at the cut m_2 (see Figure 9.121a):

$$\sigma_{xm} = -\text{area} = \left(-74349 - 334.2 m_2 + \tfrac{1}{2} \times 1.94 m_2^2\right) \times 10^{-9}.$$

Resultant shear force in QR (see Figure 9.121b):

$$R_{QR} = t \times \int_0^{360} \sigma_{xm}\, dm_2$$
$$= 30 \times \left(-74349 m_2 - \tfrac{1}{2} \times 334.2 m_2^2 + \tfrac{1}{6} \times 0.194 m_2^3\right) \times 10^{-9}\Big|_0^{360}$$
$$= -1.00000\ \text{N}.$$

- RS – loading case 1: $V_y = 1$ N (see Figure 9.120)

The bending stress σ_{xx} as a function of the auxiliary coordinate m_3:

$$\sigma_{xx} = \left(-364.2 + \frac{364.2 + 108.8}{540} m_3\right) \times 10^{-9}$$
$$= (-364.2 + 0.8759 m_3) \times 10^{-9}.$$

Area of the bending stress diagram from P to the cut at m_3:

$$\text{area} = \text{area(PQR)} + \int_0^{m_3} \sigma_{xx}\, dm_3$$
$$= \left(68949 - 364.2 m_3 + \tfrac{1}{2} \times 0.8759 m_3^2\right) \times 10^{-9}.$$

Since there is no normal force, the total area of the normal stress diagram must be zero. This condition is satisfied:

$$\text{area(PQRS)} = \text{area(PQR)} + \int_0^{540} \sigma_{xx}\, dm_3 = -13 \times 10^{-9} \approx 0.$$

Shear stress σ_{xx} at the cut m_3 (see Figure 9.121a):

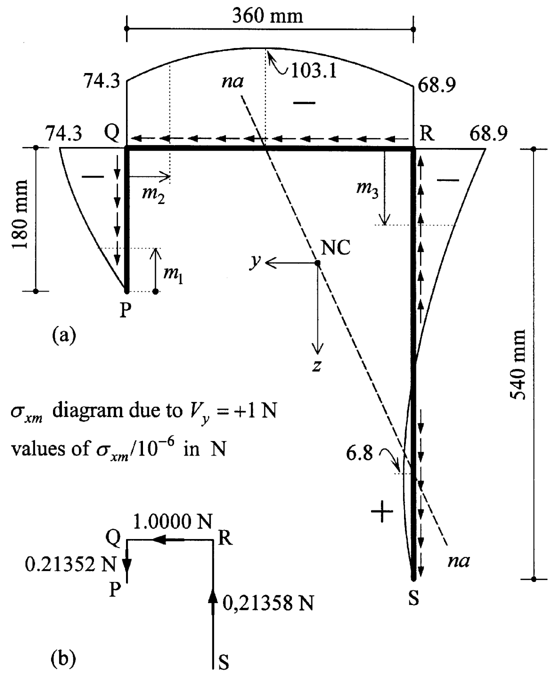


Figure 9.121 (a) The shear stress distribution due to $V_y = +1$ N. (b) The resultants of the shear stresses in the flange and webs.

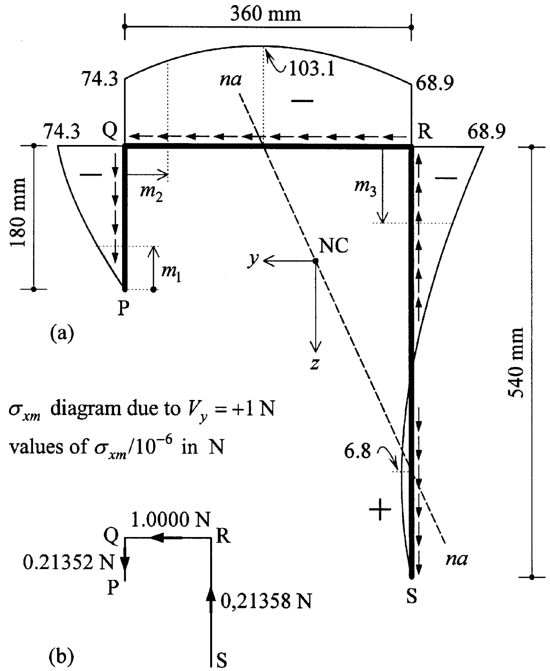


Figure 9.121 (a) The shear stress distribution due to $V_y = +1$ N. (b) The resultants of the shear stresses in the flange and webs.

$$\sigma_{xm} = -\text{area} = \left(-68949 + 364.2m_3 - \tfrac{1}{2} \times 0.8759m_3^2\right) \times 10^{-9}.$$

Resultant shear force in RS (see Figure 9.121b):

$$R_{RS} = t \times \int_0^{540} \sigma_{xm}\, dm_3$$
$$= 30 \times \left(-68949m_3 + \tfrac{1}{2} \times 364.2m_3^2 - \tfrac{1}{6} \times 0.8759m_3^3\right) \times 10^{-9}\Big|_0^{540}$$
$$= -0.21358 \text{ N}.$$

The shear stress distribution is shown in Figure 9.121a. The resultants of the shear stresses in the segments PQ, QR an RS are shown Figure 9.121b. The top of the parabolic shear stress distribution appears where the bending stress is zero, that is where the neutral axis *na* intersects the thin-walled cross-section. The plus and minus signs of the shear stresses and their resultants are related to the *m* coordinate axis for the specific segment. With arrows the actual directions are indicated.

Figure 9.121 shows that the horizontal shear force $V_y = 1$ N is transmitted by the flange QR. In the webs PQ and RS there are two equal and opposite forces. The difference between the shear forces in PQ and RS are caused by rounding off some values in the manual calculation, and is negligible.

Loading case 2

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \text{ Nmm} \end{bmatrix},$$

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{1}{10^{15} \text{ Nmm}^2} \times \begin{bmatrix} 9.70 & 4.38 \\ 4.38 & 7.51 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \text{ Nmm} \end{bmatrix}$$
$$= \begin{bmatrix} 4.38 \\ 7.51 \end{bmatrix} \times (10^{-15} \text{ mm}^{-1}).$$

Strain distribution due to bending:

$$\varepsilon(y,z) = \kappa_y y + \kappa_z z$$
$$= (4.38 \times y + 7.51 \times z) \times (10^{-15}\ \text{mm}^{-1}).$$

Normal stress distribution due to bending:

$$\sigma(y,z) = E \cdot \varepsilon(y,z) = E\kappa_y y + E\kappa_z z$$
$$= (0.876 \times y + 1.502 \times z) \times (10^{-9}\ \text{N/mm}^3).$$

The bending stresses at the key points P, Q, R and S are calculated in Table 9.18. Figure 9.122 shows the bending stress diagram.

Table 9.18

point	y (mm)	z (mm)	$\sigma(y,z) \times 10^9$ (N/mm^2)
P	+240	+30	+255.3
Q	+240	−150	−15.1
R	−120	−150	−330.4
S	−120	+390	+480.7

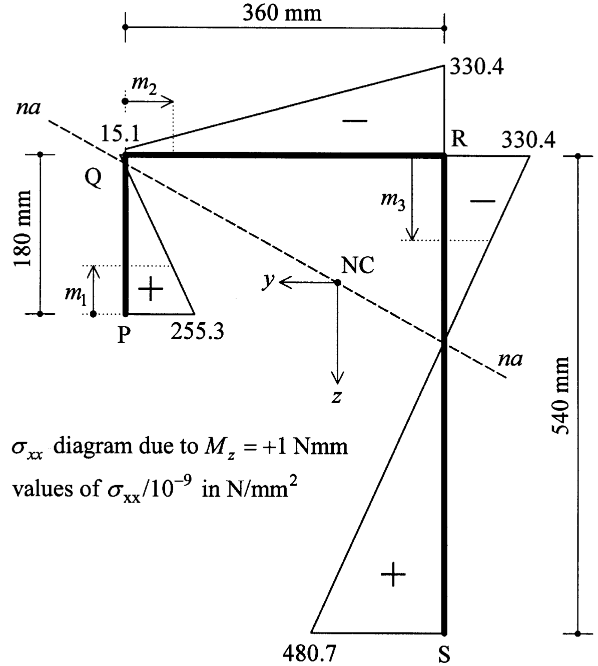


Figure 9.122 The bending stress distribution due to $M_z = +1$ Nmm.

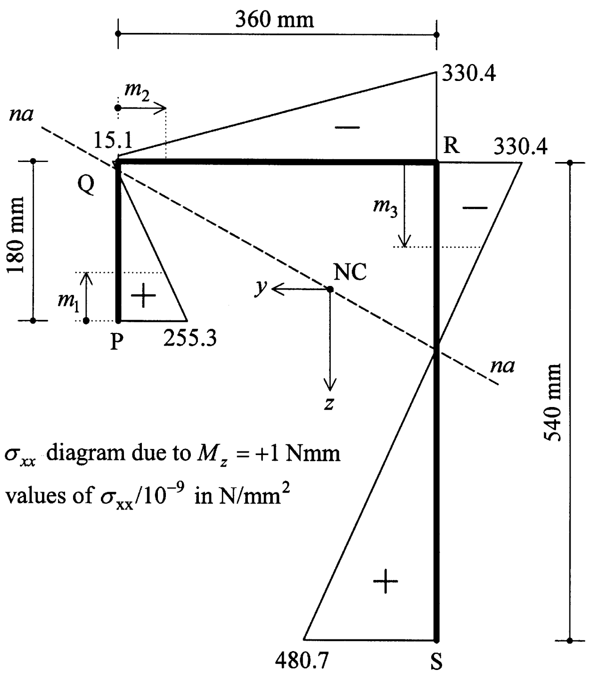


Figure 9.122 The bending stress distribution due to $M_z = +1$ Nmm.

- PQ – loading case 2: $V_z = 1$ N (see Figure 9.122)

The bending stress σ_{xx} as a function of the auxiliary coordinate m_1:

$$\sigma_{xx} = \left(255.3 - \frac{255.3 + 15.1}{180} m_1\right) \times 10^{-9}$$
$$= (255.3 - 1.5022 m_1) \times 10^{-9}.$$

Area of the bending stress diagram from P to the cut at m_1:

$$\text{area} = \int_0^{m_1} \sigma_{xx}\, \mathrm{d}m_1 = \left(255.3 m_1 - \tfrac{1}{2} \times 1.5022 m_1^2\right) \times 10^{-9}$$

$$\text{area(PQ)} = \int_0^{180} \sigma_{xx}\, \mathrm{d}m_1 = 21618 \times 10^{-9}.$$

Shear stress σ_{xm} at the cut m_1 (see Figure 9.123a):

$$\sigma_{xm} = -\text{area} = \left(-255.3 m_1 + \tfrac{1}{2} \times 1.5022 m_1^2\right) \times 10^{-9}.$$

Resultant shear force in PQ (see Figure 9.123b):

$$R_{\mathrm{PQ}} = t \times \int_0^{180} \sigma_{xm}\, \mathrm{d}m_1$$
$$= 30 \times \left(-\tfrac{1}{2} \times 255.3 m_1^2 + \tfrac{1}{6} \times 1.5022 m_1^3\right) \times 10^{-9}\Big|_0^{180}$$
$$= -0.08027 \text{ N}.$$

- QR – loading case 2: $V_z = 1$ N (see Figure 9.122)

The bending stress σ_{xx} as a function of the auxiliary coordinate m_2:

$$\sigma_{xx} = \left(-15.1 - \frac{330.4 - 15.1}{360} m_2\right) \times 10^{-9}$$

$$= (-15.1 - 0.8758 m_2) \times 10^{-9}.$$

Area of the bending stress diagram from P to the cut at m_2:

$$\text{area} = \text{area(PQ)} + \int_0^{m_2} \sigma_{xx}\, \mathrm{d}m_2$$

$$= \left(21618 - 15.1 m_2 - \tfrac{1}{2} \times 0.8757 m_2^2\right) \times 10^{-9}$$

$$\text{area(PQR)} = \text{area(PQ)} + \int_0^{360} \sigma_{xx}\, \mathrm{d}m_2 = -40572 \times 10^{-9}.$$

Shear stress σ_{xm} at the cut m_2 (see Figure 9.123a):

$$\sigma_{xm} = -\text{area} = \left(-21618 + 15.1 m_2 + \tfrac{1}{2} \times 0.8758 m_2^2\right) \times 10^{-9}.$$

Resultant shear force in QR (see Figure 9.123b):

$$R_{\text{QR}} = t \times \int_0^{360} \sigma_{xm}\, \mathrm{d}m_2$$

$$= 30 \times \left(-21618 m_2 + \tfrac{1}{2} \times 15.1 m_2^2 + \tfrac{1}{6} \times 0.8758 m_2^3\right) \times 10^{-9} \Big|_0^{360}$$

$$= 0.190 \times 10^{-12} \approx 0.$$

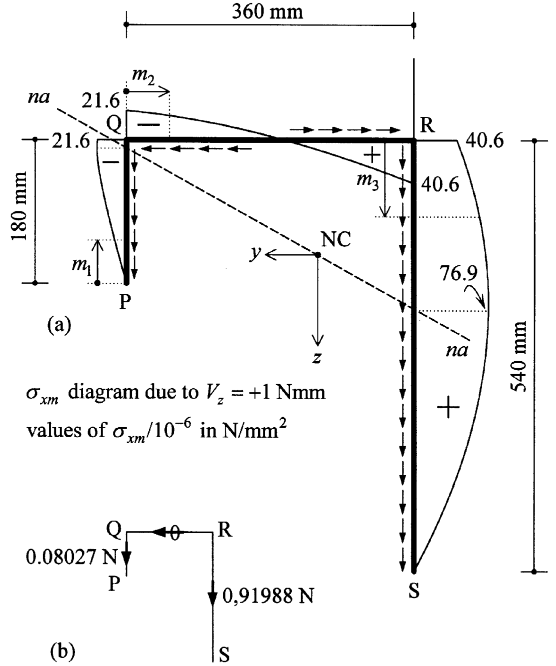


Figure 9.123 (a) The shear stress distribution due to $V_z = 1$ N. (b) The resultants of the shear stresses in the flange and webs.

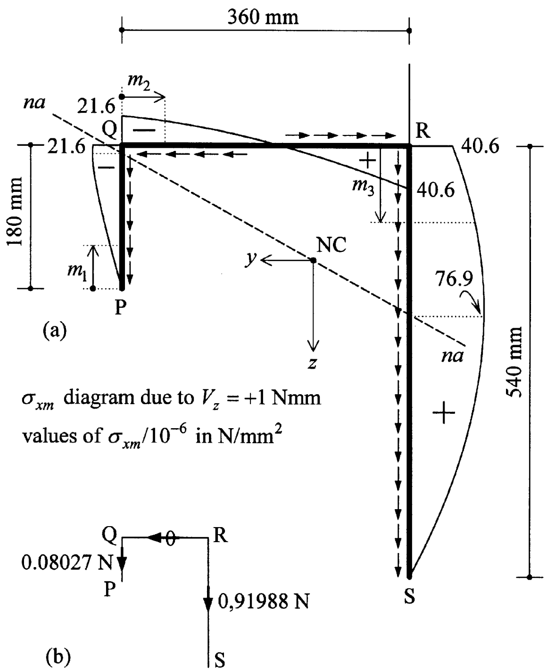


Figure 9.123 (a) The shear stress distribution due to $V_z = 1$ N. (b) The resultants of the shear stresses in the flange and webs.

- RS – loading case 2: $V_z = 1$ N (see Figure 9.122)

The bending stress σ_{xx} as a function of the auxiliary coordinate m_3:

$$\sigma_{xx} = \left(-330.4 + \frac{330.4 + 480.7}{540} m_3\right) \times 10^{-9}$$
$$= (-330.4 + 1.502 m_3) \times 10^{-9}.$$

Area of the bending stress diagram from P to the cut at m_3:

$$\text{area} = \text{area(PQR)} + \int_0^{m_3} \sigma_{xx}\, \mathrm{d}m_3$$
$$= \left(-40572 - 330.4 m_3 + \tfrac{1}{2} \times 1.502 m_3^2\right) \times 10^{-9}.$$

Since there is no normal force, the total area of the normal stress diagram must be zero. This condition is satisfied:

$$\text{area(PQRS)} = \text{area(PQR)} + \int_0^{540} \sigma_{xx}\, \mathrm{d}m_3 = -4 \times 10^{-9} \approx 0.$$

Shear stress σ_{xm} at the cut m_3 (see Figure 9.123a):

$$\sigma_{xm} = -\text{area} = \left(+40554 + 330.4 m_3 - \tfrac{1}{2} \times 1.5020 m_3^2\right) \times 10^{-9}.$$

Resultant shear force in RS (see Figure 9.123b):

$$R_{\text{RS}} = t \times \int_0^{540} \sigma_{xm}\, \mathrm{d}m_3$$
$$= 30 \times \left(+40572 m_3 + \tfrac{1}{2} \times 330.4 m_3^2 - \tfrac{1}{6} \times 1.502 m_3^3\right) \times 10^{-9}\Big|_0^{540}$$
$$= +0.91988\ \text{N}.$$

The shear stress distribution and resulting shear forces per segment are shown in Figure 9.123.

The vertical shear force $V_z = 1$ kN is transmitted by the webs PQ and RS. The resultant of the shear stresses in flange QR is zero. The resultant of the shear stresses in both webs must be equal to V_z:

$$V_z = R_{\mathrm{PQ}} + R_{\mathrm{RS}} = (0.08027\ \mathrm{N}) + (0.91988\ \mathrm{N}) = 1.00015\ \mathrm{N}.$$

The small difference is caused by rounding off some values in the manual calculation, and is negligible.[1]

To find the shear centre SC we summarise both loading cases (see Figure 9.124).

Loading case 1: $V_y = 1$ N
The shear forces in PQ and RS are statically equivalent with a moment in the plane of the cross-section. Taking the mean value of R_{PQ} and R_{RS}, we find

$$(0.21355\ \mathrm{N})(360\ \mathrm{mm}) = 76.88\ \mathrm{Nmm}\ (\circlearrowleft).$$

The resultant of this moment and the shear force in QR is a horizontal force with its line of action 76.88 mm above the centre line of the flange QR.

Loading case 2: $V_z = 1$ N
The resultant of $R_{\mathrm{PQ}} = 0.08026$ and $R_{\mathrm{RS}} = 0.91974$ is a vertical force with

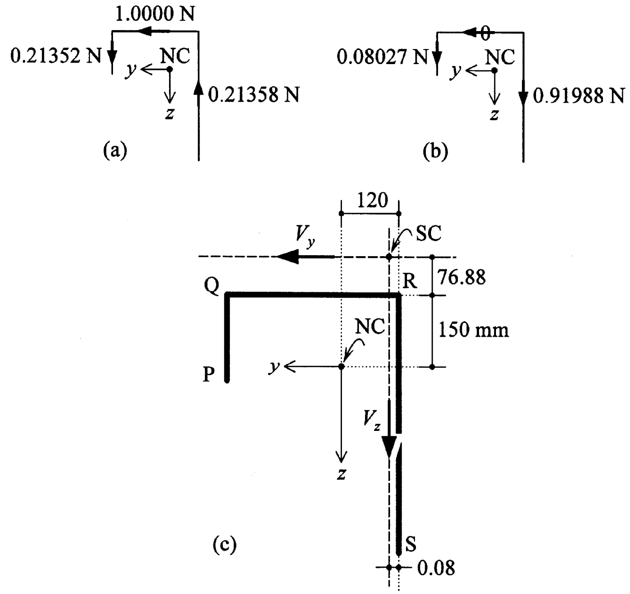


Figure 9.124 (a) The resultant of al stresses is a horizontal force with its line of action 76.88 mm above the centre line of the flange QR. (b) The resultant of all stresses is a vertical force with its line of action 0.08 mm to the left of the centre line of web RS. (c) The point of intersection of both lines of action is the *shear centre* SC of the cross-section. The shear centre SC is that point in the cross-sectional plane through which the line of action of the shear force must pass so that there will be no torsion.

[1] We can correct the values by multiplying them by 1/1.00015, but this is not necessary to find the correct line of action of the resultant of all shear stresses in the cross-section.

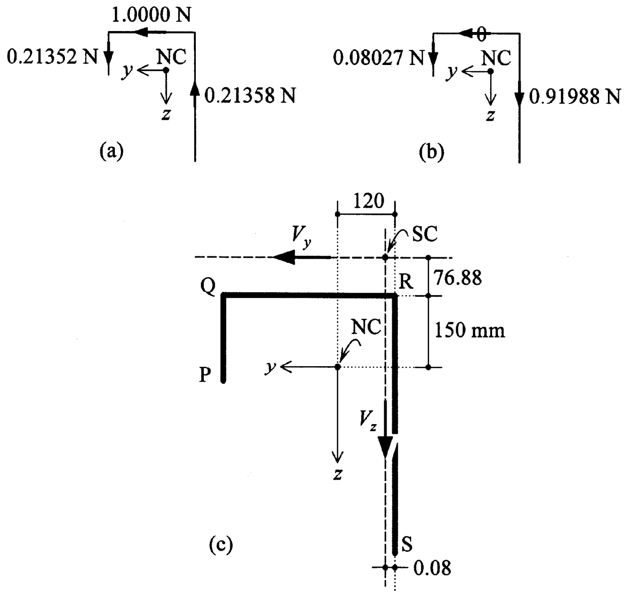


Figure 9.124 (a) The resultant of al stresses is a horizontal force with its line of action 76.88 mm above the centre line of the flange QR. (b) The resultant of all stresses is a vertical force with its line of action 0.08 mm to the left of the centre line of web RS. (c) The point of intersection of both lines of action is the *shear centre* SC of the cross-section. The shear centre SC is that point in the cross-sectional plane through which the line of action of the shear force must pass so that there will be no torsion.

its line of action at the following distance to the left of the centre line of web RS:

$$\frac{0.08026}{0.08026 + 0.91974}\ \text{mm} = 0.08\ \text{mm}.$$

The intersection of the two lines of action is the shear centre SC. For this thin-walled cross-section the shear centre SC and normal centre NC do not coincide.

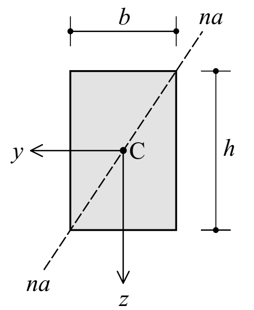
Only when the line of action of a shear force passes through the shear centre SC, there is no torsion, and the shear stress distribution can be calculated by using the formulae for the shear stresses due to a shear force. When the line of action of the shear force does not pass through the shear centre SC, the cross-section is also subject to torsion, generating additional shear stresses.

9.18 Problems

- Unless mentioned otherwise, all cross-sections are homogeneous.
- A *centroidal* coordinate system is a coordinate system with its origin at the centroid of the (homogeneous) cross-section.
- A *central* coordinate system is a coordinate system with its origin at the normal centre of the cross-section.

Plane of loading and plane of curvature (Section 9.6)

9.1 For a rectangular cross-section, the location of the neutral axis *na* is known and shown in the figure.



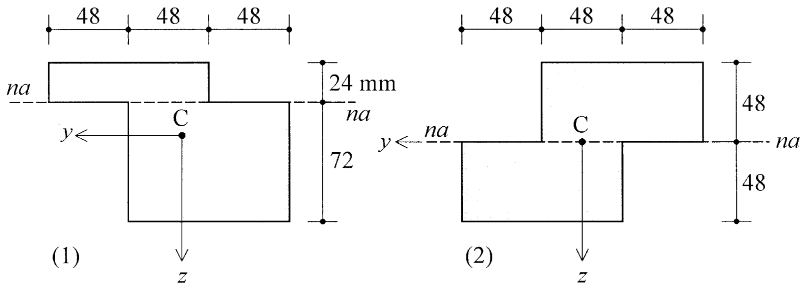
Questions:

a. Find the associated plane of loading m.
b. Is it possible to find the plane of loading m directly from the stress distribution?

9.2: 1–2 For two cross-sections the neutral axis *na* is given. In both cross-sections the maximum normal stress is a tensile stress of 144 N/mm^2.

Questions:

a. Draw the σ diagram.
b. Find the resultant of all tensile stresses and its point of application.
c. Find the resultant of all compressive stresses and its point of application.
d. Find the normal force N and bending moment M.
e. Draw in the cross-section the plane of loading m and plane of curvature k.
f. If a centre of force exists, find its location (e_y, e_z).

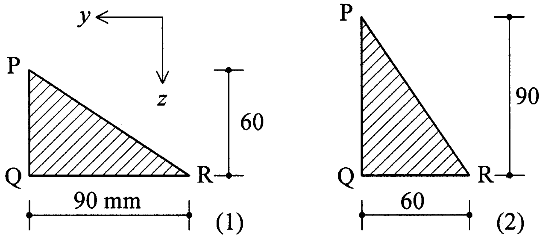


9.3: 1–2 For the figure and data, see problem 9.2.

Questions:

a. Draw the plane of curvature k.
b. Calculate the centroidal moments of inertia in the yz coordinate system.
c. Without calculating the resultants of the tensile and compressive stresses, find the normal force N and bending moment M.
d. Draw in the cross-section the plane of loading m and plane of curvature k.

9.4 The triangular cross-sections are subject to bending without extension. The same normal stresses occur at P and Q: $\sigma_P = \sigma_Q = +10$ MPa.

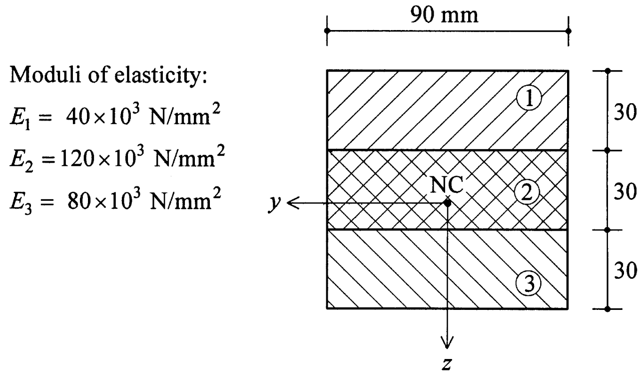


Questions:

a. Find the stress at R.
b. Calculate the magnitude and direction of the bending moment. Use a centroidal yz coordinate system.
c. Draw in the cross-section the neutral axis na, the plane of curvature k and the plane of loading m.

The normal centre NC (Section 9.7) ***and the central bending stiffness values*** (Section 9.8) ***for inhomogeneous cross-sections***

9.5 The cross-section is constructed of three materials which are firmly glued together. The moduli of elasticity are given in the figure.



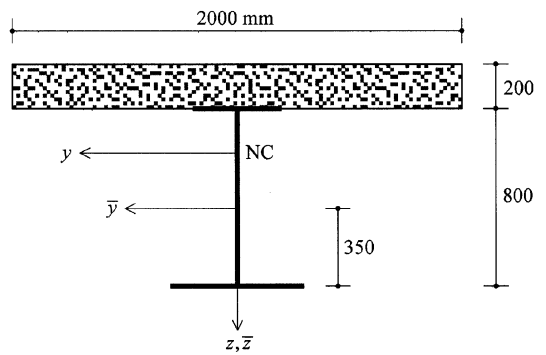
Questions:

a. Find the location of the normal centre NC.
b. Calculate the central bending stiffness values EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz}.

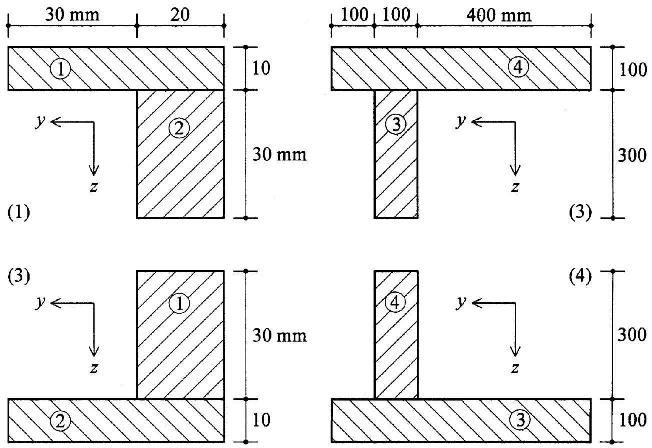
9.6 The composite concrete-steel cross-section of a bridge girder is constructed of a concrete flange and a steel I-section. The cross-sectional area of the steel section is $A_s = 32 \times 10^3$ mm^2. The centroidal bending stiffness of the steel section in the vertical plane of symmetry is $EI_{\overline{zz}} = 432 \times 10^6$ mm^4. The moduli of elasticity of steel and concrete are $E_s = 210 \times 10^3$ N/mm^2 and $E_c = 14 \times 10^3$ N/mm^2 respectively.

Questions:

a. Find the location of the normal centre NC.
b. Calculate the central bending stiffness value EI_{zz}.



9.7: 1–4 You are given four unsymmetrical and inhomogeneous cross-sections, with the following moduli of elasticity:
$E_1 = 12$ GPa, $E_2 = 6$ GPa, $E_3 = 40$ GPa and $E_4 = 80$ GPa.

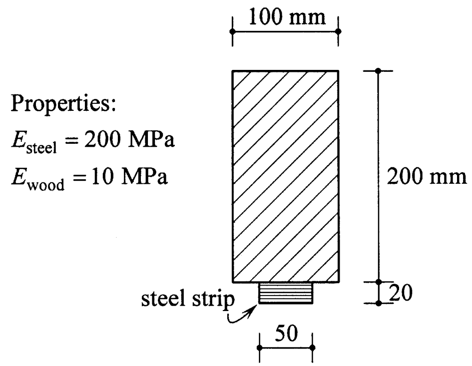


Questions:

a. Find the location of the normal centre NC.
b. Calculate the central bending stiffness values EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz}.

Stresses due to extension and bending – straightforward method (Sections 9.8 and 9.9)

9.8 A steel strip of 20×50 mm^2 is firmly fixed along the entire length of a wooden beam. Beam and strip behave as one piece. The composite beam is loaded by a tensile force in such a way that there is no bending.



Question:
Find the centre of force (point of application of the tensile force) for an arbitrary cross-section.

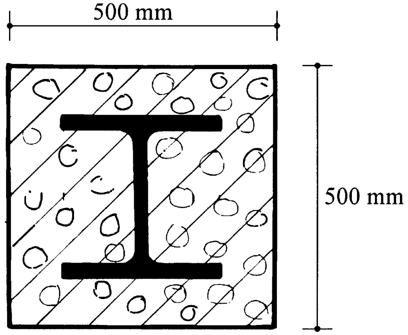
9.9 The steel-concrete column is loaded by an axial compressive force, in such a way that there is no bending but only extension. The compressive stress in the concrete is 6 N/mm^2.

Properties:

$A_{steel} = 25 \times 10^3$ mm^2

$E_{steel} = 210$ GPa

$E_{concrete} = 10$ GPa



Question:
Find the compressive force in the column.

9.10 The steel-concrete column in problem 9.9 is subject to extension by a compressive force of 5800 kN.

Questions:

a. Calculate the stresses in the concrete and in the steel I-section.
b. Which part of the compressive force is transmitted by the concrete and which part by the steel I-section?

9.11 A tensile bar is constructed of three materials. The bar is loaded by a tensile force of 440 kN.

Questions:

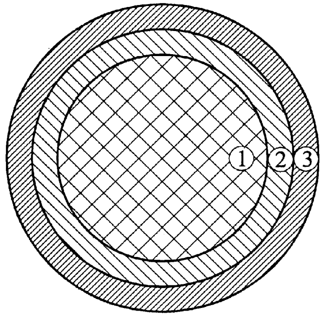
a. Calculate the axial stiffness of the bar.
b. Find the tensile stress in each of the three materials.

Properties:

$A_1 = 4000$ mm^2, $E_1 = 200$ GPa,

$A_2 = 2000$ mm^2, $E_2 = 100$ GPa,

$A_3 = 2000$ mm^2, $E_3 = 50$ GPa.

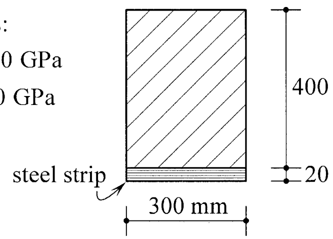


9.12 A steel strip is firmly fixed at the lower side of a wooden beam. Beam and strip behave as one piece. The composite beam is axially loaded by a central tensile force. The normal stress may not exceed the limiting values $\overline{\sigma}_{steel} = 150$ N/mm^2 and $\overline{\sigma}_{wood} = 10$ N/mm^2.

Properties:

$E_{steel} = 200$ GPa

$E_{wood} = 10$ GPa



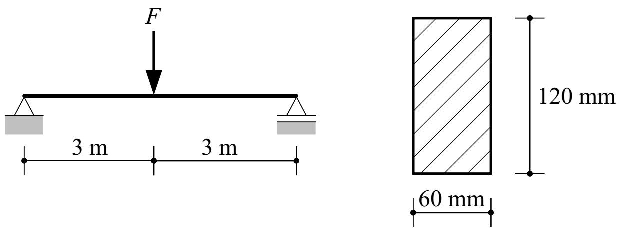
Questions:

a. Find the point of application of the central tensile force.
b. Find the maximum tensile force which can be transmitted by the composite cross-section.

9.13: 1–3 A simply supported wooden beam is loaded by a force F at midspan. The beam can be strengthened by steel strips in three different ways:

1. Two steel strips of $120 \times 10\ \text{mm}^2$ at both sides.
2. One steel strip of $60 \times 10\ \text{mm}^2$ at the upper side of the beam.
3. Two strips of $60 \times 10\ \text{mm}^2$, one at the upper side of the beam and the other at the lower side.

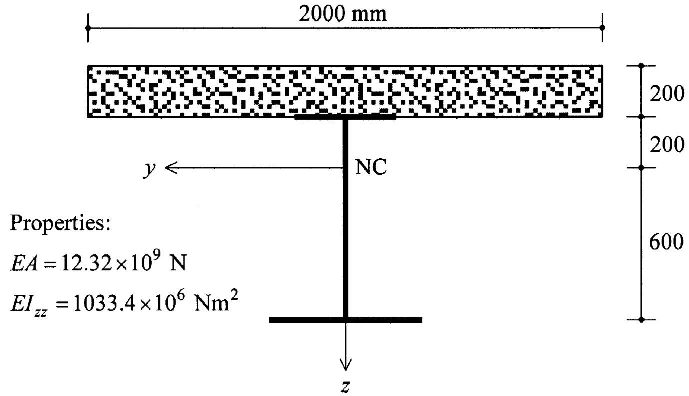
The moduli of elasticity are $E_w = 15$ GPa and $E_s = 210$ GPa for wood and steel respectively. The bending stresses may not exceed the limiting values $\overline{\sigma}_w = 7$ MPa and $\overline{\sigma}_s = 140$ MPa.



Questions:

a. Find the maximum force F the wooden beam can carry without steel strips.
b. Plot for the beam without steel strip(s) the bending stress diagram for the cross-section at midspan.
c. Find the maximum force F the wooden beam can carry with the steel strip(s).
d. For the beam with steel strip(s), draw the bending stress diagram for the cross-section at midspan.

9.14 The composite concrete-steel cross-section of a bridge girder is constructed of an steel I-section and a concrete flange. The location of the normal centre NC is shown in the figure. The central bending stiffness in the vertical plane of symmetry is $EI_{zz} = 1033.4 \times 10^6\ \text{Nm}^2$. The axial stiffness is $EA = 12.32 \times 10^9$ N. The cross-section is loaded in the plane of symmetry. At the upper side of the flange occurs a compressive stress of $7\ \text{N/mm}^2$. At the lower side of the steel section occurs a tensile stress of $105\ \text{N/mm}^2$.

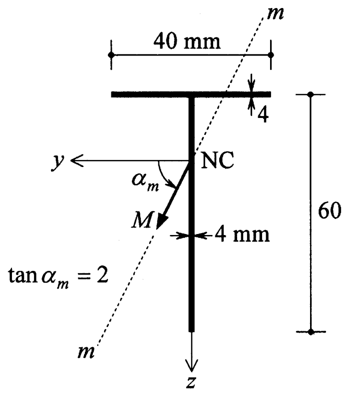


Questions:

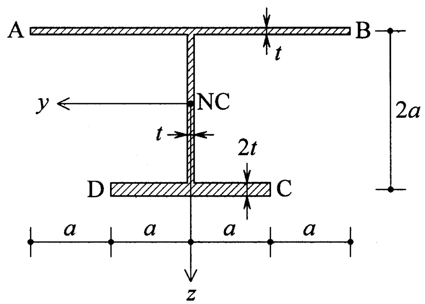
a. Draw the diagrams for the strain and stress distribution in the composite cross-section.
b. Find the cross-sectional deformation quantities ε and κ.
c. Find the normal force and bending moment in the cross-section.

9.15 The thin-walled cross-section is subject to a bending moment $M = 100$ kNm. The plane of loading m and the direction of M are shown in the figure: $\tan \alpha_m = 2$.

Questions:
a. Find the normal centre NC.
b. Draw in the cross-section the neutral axis na.
c. Draw the normal stress diagram.
d. Draw in the cross-section the plane of curvature k.



9.16 The thin-walled cross-section has to transmit the bending moments $M_y = +80\sigma a^2 t$ and $M_z = +52\sigma a^2 t$.

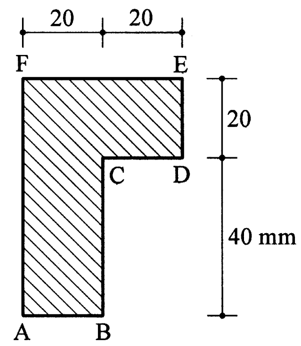


Questions:
a. Find the stresses at the four corners A to D, and plot the normal stress diagram.
b. Determine the equation for the neutral axis in y and z.

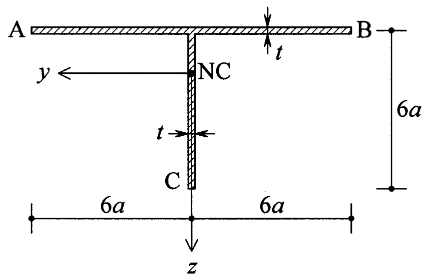
c. Draw in the cross-section the neutral axis na, the plane of loading m, and the plane of curvature k.

9.17 The cross-section shown is subject to bending without extension. The largest stress is a tensile stress of 70 MPa. The neutral axis is horizontal.

Questions:
a. Draw the σ diagram.
b. Find the magnitude and direction of the bending moment in the cross-section.
c. Draw in the cross-section the plane of loading m and the plane of curvature k.



9.18 In the thin-walled T-section the normal stresses are $+3\sigma$, -3σ and -4σ at A, B and C respectively.



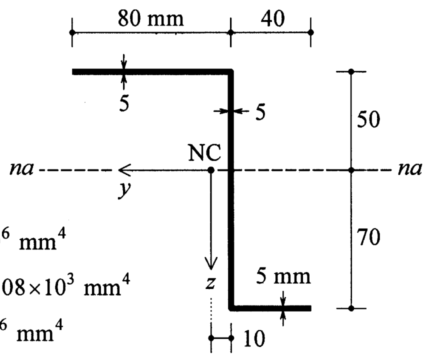
Questions:
a. Draw the normal stress diagram.

b. Find the resultant of all tensile stresses, and its point of application.
c. Find the resultant of all compressive stresses, and its point of application.
d. Find from b and c the normal force N, and the bending moments M_y and M_z.
e. Verify the stress formula

$$\sigma = \frac{N}{A} + \frac{M_y y}{I_{yy}} + \frac{M_z z}{I_{zz}}.$$

f. Draw in the cross-section the neutral axis *na*, the plane of curvature k, and the plane of loading m.

9.19 The homogeneous thin-walled cross-section is loaded by bending in such a way that the neutral axis *na* is horizontal. The maximum bending stress is a tensile stress of 140 N/mm^2.



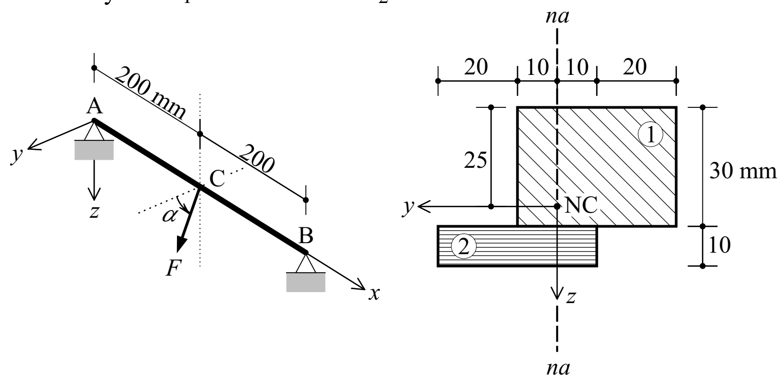
$I_{yy} = 0.84 \times 10^6$ mm^4

$I_{yz} = I_{zy} = -1.08 \times 10^3$ mm^4

$I_{zz} = 2.76 \times 10^6$ mm^4

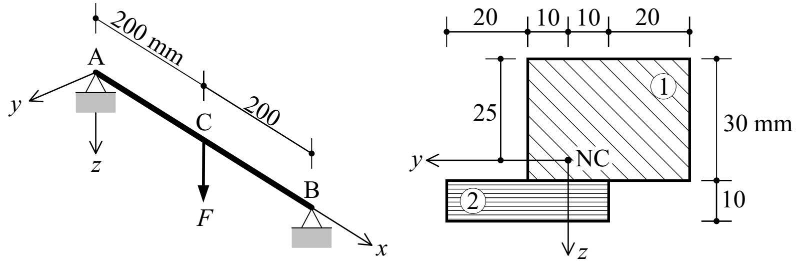
Questions:
a. Verify the location of the normal centre NC and the values of the centroidal moments of inertia.
b. Draw the stress diagram.
c. Find the components M_y and M_z of the bending moment M in the cross-section.
e. Draw in the cross-section the plane of curvature k and the plane of loading m.

9.20 The simply supported beam AB, with inhomogeneous and unsymmetrical cross-section, is loaded by a force F at midspan C. The force F is applied perpendicular to the beam axis, in such a way that the neutral axes *na* in the cross-section at C is vertical. In this cross-section the maximum bending stress is a compressive stress of 90 N/mm^2. The moduli of elasticity are $E_1 = 70$ GPa and $E_2 = 210$ GPa.



Questions:
a. Verify the location of the normal centre NC.
b. Draw the strain diagram and bending stress diagram.
c. Verify the central bending stiffness values: $EI_{yy} = 39.2 \times 10^9$ Nmm^2, $EI_{yz} = EI_{zy} = 16.8 \times 10^9$ Nmm^2, and $EI_{zz} = 23.8 \times 10^9$ Nmm^2.
d. Find the bending moment in the cross-section at C.
e. Draw in the cross-section the plane of curvature k and the plane of loading m.
f. Find the magnitude and direction α of the force F.

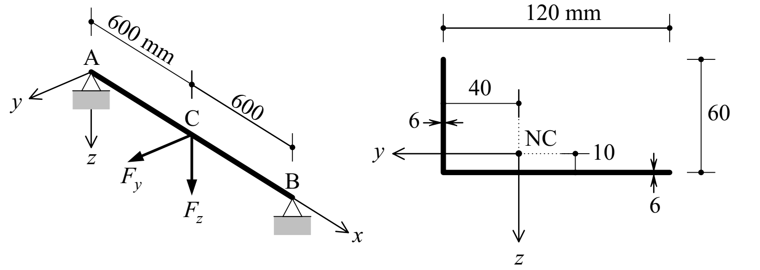
9.21 The simply supported beam AB, with inhomogeneous and unsymmetrical cross-section, is loaded by a vertical force $F = 680$ N at midspan C. The moduli of elasticity are $E_1 = 70$ GPa and $E_2 = 210$ GPa.



Questions:
a. Verify the location of the normal centre NC.
b. Verify the central bending stiffness values: $EI_{yy} = 39.2 \times 10^9$ Nmm^2, $EI_{yz} = EI_{zy} = 16.8 \times 10^9$ Nmm^2, and $EI_{zz} = 23.8 \times 10^9$ Nmm^2.
c. Find the bending moment in the cross-section at C.
d. Draw the strain diagram and bending stress diagram.
e. Draw in the cross-section the plane of loading m, the plane of curvature k, and the neutral axis na.

9.22 The simply supported beam AB is loaded at midspan C by a horizontal force $F_y = 2880$ kN and a vertical force $F_z = 720$ kN. The dimensions of the thin-walled cross-section are shown in the figure. The modulus of elasticity is $E = 200$ GPa.

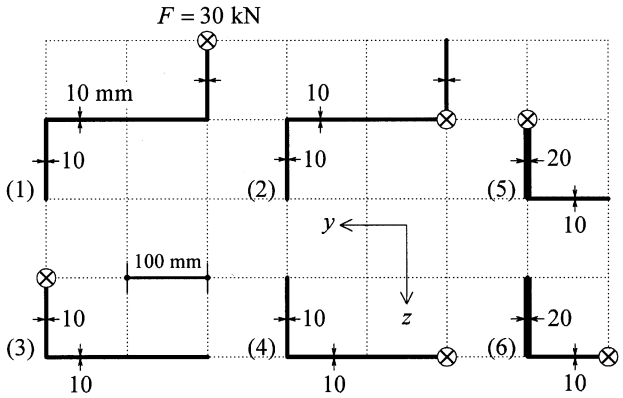
Questions:
a. Verify the location of the normal centre NC.
b. Verify the central bending stiffness values: $EI_{yy} = 345.6 \times 10^9$ Nmm^2, $EI_{yz} = EI_{zy} = -86.4 \times 10^9$ Nmm^2, and $EI_{zz} = 64.8 \times 10^9$ Nmm^2.



c. For the cross-section at midspan C, find the components of the curvature κ.
d. Draw in this cross-section the plane of loading m, the plane of curvature k, and the neutral axis na. Also draw the vectors (straight single-pointed arrows) for the bending moment M and curvature κ.
d. For the cross-section at midspan C, draw the strain diagram and stress diagram.

9.23: 1–6 You are given six thin-walled cross-sections loaded by an eccentric compressive force $F = 30$ kN, as shown in the figure. The distance between two grid lines is 100 mm.

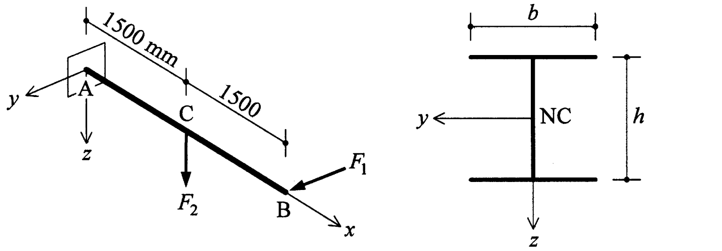
Questions:
a. Find the (components of the) bending moment in the cross-section.
b. Calculate the centroidal moments of inertia I_{yy}, $I_{yz} = I_{zy}$ and I_{zz}.
c. Draw the stress diagram due to extension.
d. Draw the stress diagram due to bending.
e. Draw the stress diagram due to the combination of extension and bending.



9.24 A cantilever beam with an I-section is loaded by the two forces F_1 (in the y direction) and F_2 (in the z direction), as shown in the figure. The length of the beam is 3 m. For the I-section: $b = 160$ mm, $h = 152$ mm, $I_{yy} = 6.12 \times 10^6$ mm^4 and $I_{zz} = 16.73 \times 10^6$ mm^4. The normal stress in the cross-section is limited to $\overline{\sigma} = 100$ MPa.

Questions:

a. Find the maximum values for F_1 and F_2 if $F_1 = F_2$.
b. For the critical cross-section, plot the σ diagram.
c. For the critical cross-section, draw the neutral axis na, the plane of curvature k, and the plane of loading m. Also draw the direction of the resultant shear force.



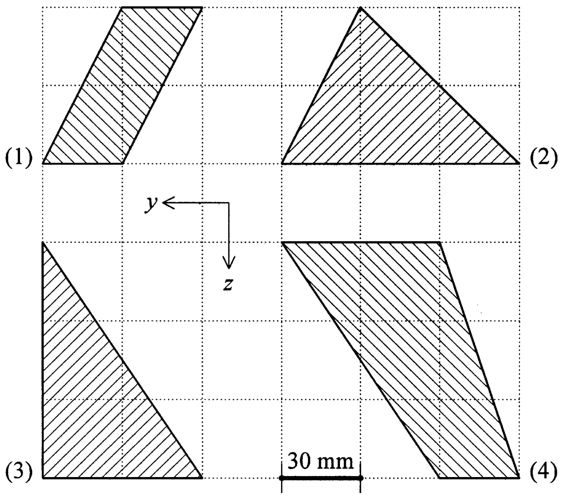
9.25 As problem 9.24, but now with a different I-section: $b = h = 160$ mm, $I_{yy} = 9.58 \times 10^6$ mm^4 and $I_{zz} = 26.34 \times 10^4$ mm^4. The normal stress in the cross-section is limited to $\overline{\sigma} = 140$ MPa.

Questions:

a. Find the maximum values for F_1 and F_2 if $F_1 = 0.5F_2$.
b. For the critical cross-section, plot the σ diagram.
c. For the critical cross-section, draw the neutral axis na, the plane of curvature k, and the plane of loading m. Also draw the direction of the resultant shear force.

Mohr's circle for the moments of inertia and the bending stiffness tensor (Section 9.11)

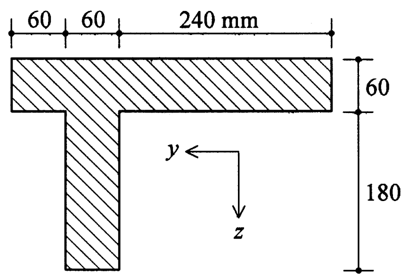
9.26: 1–4 You are given four cross-sections. The distance between two grid lines is 30 mm.



Questions:

a. Find the normal centre NC of the cross-section.
b. Calculate the moments of inertia in the central yz coordinate system.
c. Draw Mohr's circle for the moments of inertia.
d. Find from Mohr's circle the principal directions and the principal values of the moments of inertia I_1 and I_2.

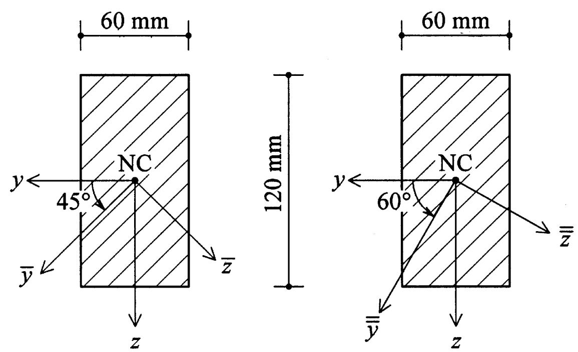
9.27 For the cross-section the modulus of elasticity is $E = 15$ GPa.



Questions:

a. Find the normal centre NC of the cross-section.
b. Calculate bending stiffness values EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz} in the centroidal yz coordinate system.
c. Draw Mohr's circle for the bending stiffness tensor.
d. Find from Mohr's circle the principal directions and the principal bending stiffness values EI_1 and EI_2.

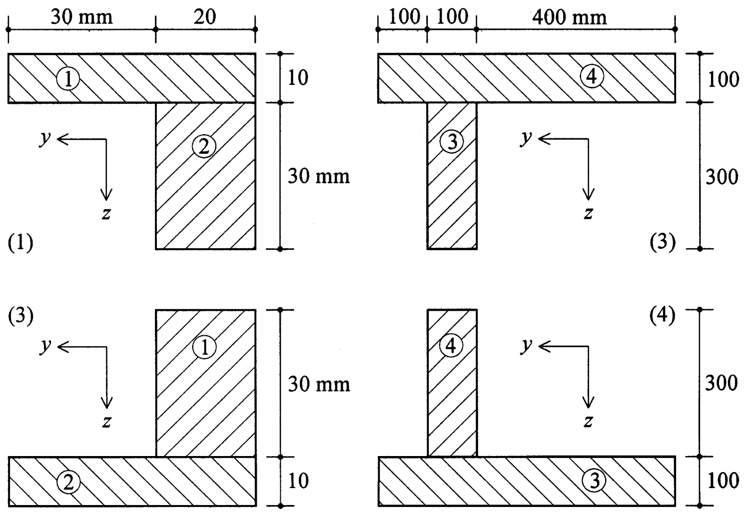
9.28 You are given a rectangular cross-section.



Questions:

a. Calculate the centroidal moments of inertia I_{yy}, $I_{yz} = I_{zy}$ and I_{zz}, and draw Mohr's circle for the moments of inertia.
b. Find from Mohr's circle the moments of inertia in the rotated $\overline{yz}$ coordinate system.
c. Find from Mohr's circle the moments of inertia in the rotated $\overline{\overline{yz}}$ coordinate system.

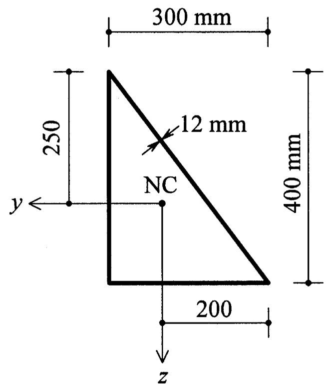
9.29: 1–4 You are given four unsymmetrical and inhomogeneous cross-sections, with the following moduli of elasticity:
$E_1 = 12$ GPa, $E_2 = 6$ GPa, $E_3 = 40$ GPa and $E_4 = 80$ GPa.



Questions:

a. Find the normal centre NC.
b. Calculate the central bending stiffness values EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz}.
c. Draw Mohr's circle for the bending stiffness tensor.
d. Find from Mohr's circle the principal directions and the principal bending stiffness values.

9.30 The wall thickness of the thin-walled triangular cross-section is $t = 12$ mm. The modulus of elasticity is $E = 2000$ MPa.

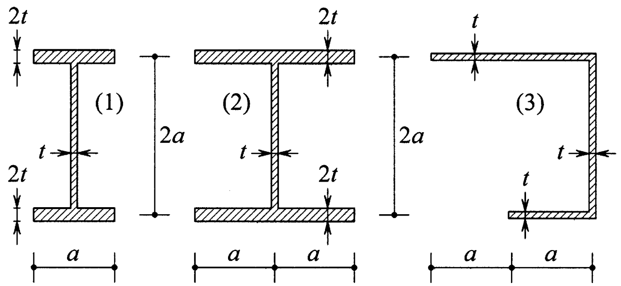


Questions:

a. Verify the location of the normal centre NC.
b. Calculate bending stiffness values EI_{yy}, $EI_{yz} = EI_{zy}$ and EI_{zz} in the centroidal yz coordinate system.
c. Draw Mohr's circle for the bending stiffness tensor.
d. Find from Mohr's circle the principal directions and the principal bending stiffness values EI_1 and EI_2.

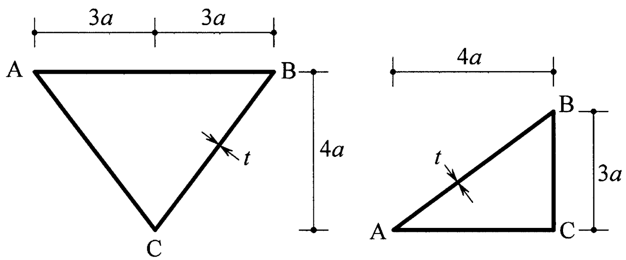
Core of the cross-section (Section 9.15)

9.31: 1–3 You are given three thin-walled cross-sections.



Question:
Find the core of the cross-section.

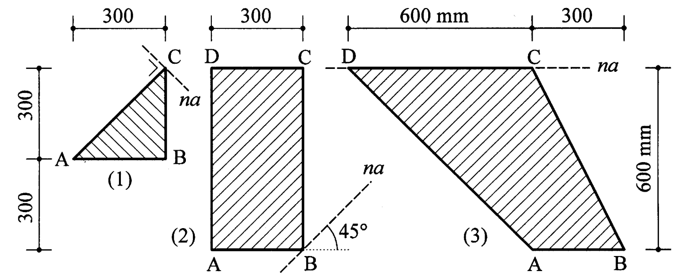
9.32: 1–2 All sides of the two thin-walled hollow cross-sections have the same thickness t.



Questions:

a. Find the core of the cross-section.
b. Find the stress distribution when the cross-section is subject to an eccentric tensile force F applied at A.

9.33: 1–3 The three cross-sections transmit an eccentric compressive force of 540 kN. For each cross-section the neutral axis *na* is shown in the figure.



Questions:

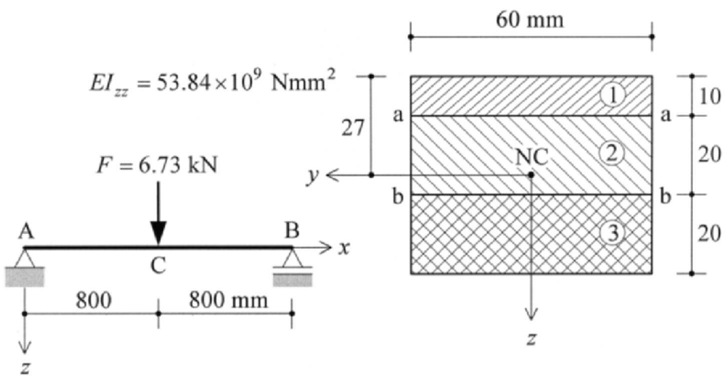
a. Find the core of the cross-section.
b. Find the centre of force associated with the neutral axis given.
c. Draw the σ diagram due to the eccentric compressive force, and calculate the stresses at the corners of the cross-section.

Displacements due to bending (Section 9.13)***, normal stresses and shear stresses*** (Section 9.17)

9.34 The simply supported beam AB, with span $\ell = 1.6$ m, is loaded at midspan C by a vertical force $F = 6.73$ kN. The cross-section is constructed of three materials, firmly glued to one another, with the following moduli of elasticity:
$E_1 = 80$ GPa, $E_2 = 60$ GPa and $E_3 = 100$ GPa.

Questions:

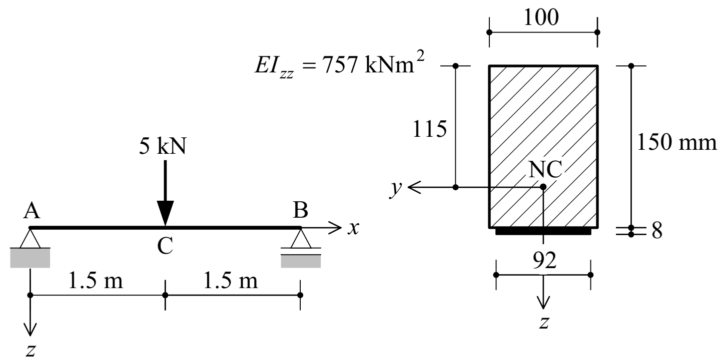
a. Verify the location of the normal centre NC and the bending stiffness EI_{zz} in the xz plane.
b. Find the deflection at C.
c. Plot the strain and normal stress diagram for the cross-section at C.



d. For the cross-section immediately to the right of C, calculate in a number of relevant points the value of the shear stress, and sketch the shear stress diagram.
e. Where in cross-section C is the shear stress a maximum and what is its value?
f. Which of the glue lines, a or b, has to transmit the larger (longitudinal) shear stress?

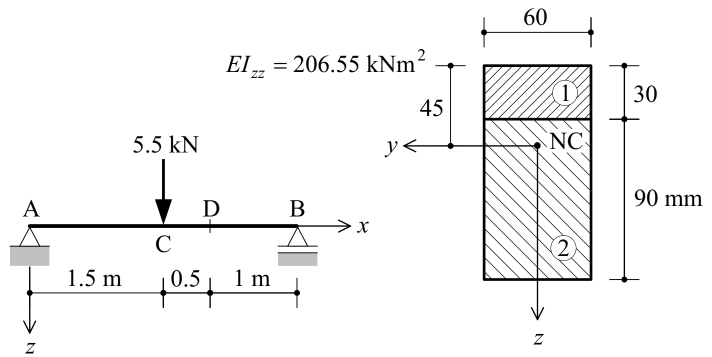
9.35 The simply supported beam AB, with span $\ell = 3$ m, is loaded at midspan C by a vertical force $F = 5$ kN. The cross-section is constructed of a wooden beam with a steel strip firmly bond to its lower side. The moduli of elasticity are $E_{\text{wood}} = 10$ GPa and $E_{\text{steel}} = 210$ GPa.

Questions:
a. Verify the location of the normal centre NC.
b. Show $EI_{zz} = 757$ kNm^2.
c. Plot the strain and normal stress distribution for the cross-section at C in diagrams.



d. Find the deflection at C.
e. Find the maximum shear flow in the connection between the wooden beam and the steel strip.

9.36 The simply supported beam AB, with span $\ell = 3$ m, is loaded at midspan C by a vertical force of 5.5 kN. The cross-section is constructed of two materials, firmly glued to one another, with the following moduli of elasticity: $E_1 = 45$ GPa and $E_2 = 15$ GPa. The dimensions of the cross-section and the location of the normal centre NC are shown in the figure.

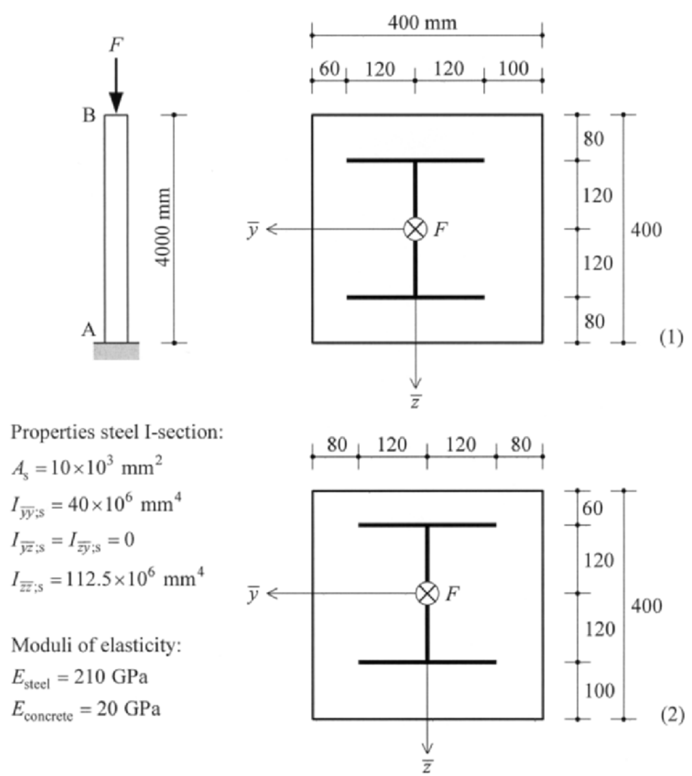


Questions:

a. Verify the location of the normal centre NC.
b. Show $EI_{zz} = 206.55\ \text{kNm}^2$.
c. Plot the strain and normal stress diagrams for the cross-section at midspan C.
d. Find the deflection at C.
e. Find the maximum shear flow in the glue line between both materials.
f. For the cross-section at D, find the shear stress distribution as a function of z.
g. Where is the shear stress a maximum in the cross-section at D and what is its value?

Mixed problems: stresses due to bending and extension, normal centre, stresses due to shear and torsion, shear centre, displacements due to bending and extension

9.37: 1–2 The composite steel-concrete column AB is fixed at A and free at B. The column is loaded by an eccentric compressive force $F = 2500$ kN at the free end B. The point of application coincides with the centroid of the steel I-section. The steel I-section is not exactly at the centre of the column, as can be seen from the cross-sectional measurements in the figures. For the column a linear elastic behaviour is assumed. The moduli of elasticity and the cross-sectional properties for the steel I-section in its centroidal $\overline{yz}$ coordinate system are given in the figure.

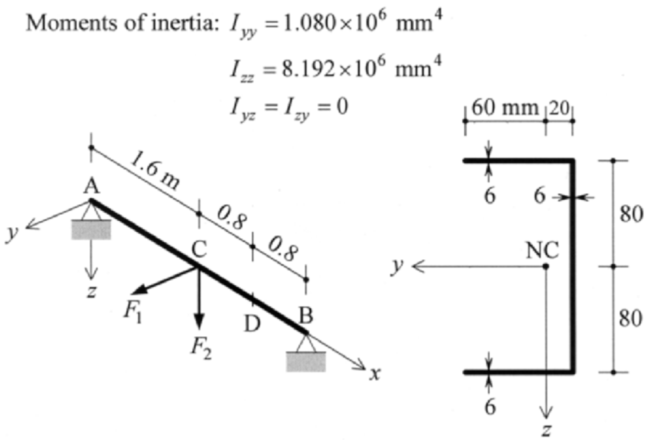


Questions:

a. Find the normal centre NC.
b. Calculate for the composite column the cross-sectional stiffness quantities in a yz coordinate system with its origin at the normal centre NC of the composite cross-section.
c. Draw for a cross-section the strain diagram and normal stress diagram.
d. Find the shortening of the column axis.
e. Find the horizontal displacement of the beam axis at the free end B.

9.38 The simply supported beam AB is loaded at midspan C by a horizontal force $F_1 = 1.6$ kN and a vertical force $F_2 = 5.12$ kN. The beam is loaded in such a way that there is no torsion. The length of the beam is

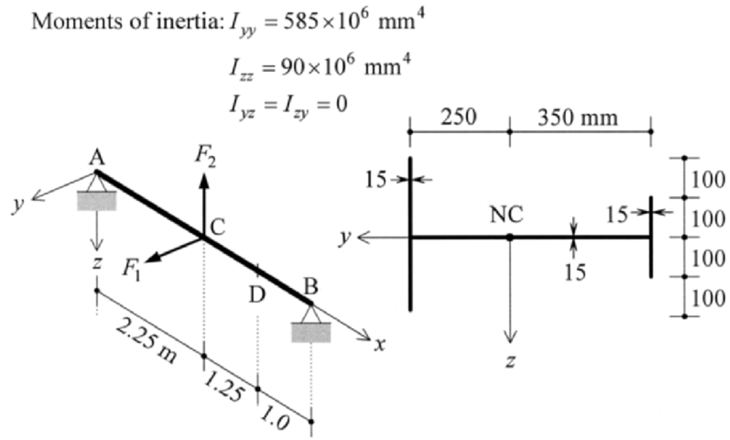
$\ell = 3.2$ m. The dimensions of the thin-walled cross-section, the location of the normal centre NC, and the moments of inertia are shown in the figure. The modulus of elasticity is $E = 200$ GPa.



Questions:

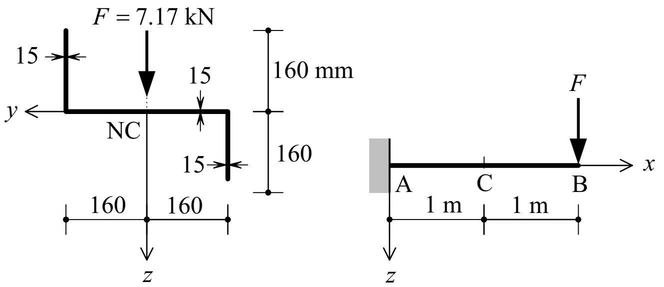
a. Find the bending moment at C.
b. Find the radius of curvature of the deflected beam at C.
c. For the cross-section at C find the equation (as a function of y and z) of the neutral axis.
d. Plot the σ diagram for the cross-section at C.
e. Find the horizontal and vertical component of the deflection at C.
f. For the cross-section at D, calculate the value of the shear stress τ in a number of relevant points, and sketch the τ diagram.
g. Where in the cross-section at D is the shear stress τ a maximum and what is its value?

9.39 The simply supported beam AB is loaded at midspan C by a horizontal force $F_1 = 156$ kN and a vertical force $F_2 = 48$ kN. The beam is loaded in such a way that there is no torsion. The length of the beam is $\ell = 4.5$ m. The dimensions of the thin-walled cross-section, the location of the normal centre NC, and the moments of inertia are shown in the figure. The modulus of elasticity is $E = 210$ GPa.



Questions:
The same questions as in problem 9.38.

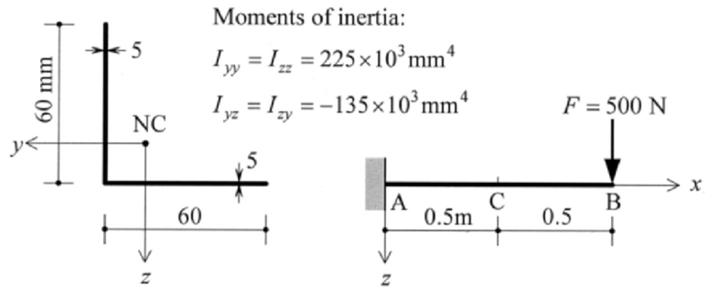
9.40 Cantilever beam AB, with a length $\ell = 2$ m, carries a force $F = 7.17$ kN at its free end B. The dimensions of the thin-walled cross-section are shown in the figure. The moments of inertia are the following: $I_{yy} = 163.84 \times 10^6$ mm^4, $I_{yz} = I_{zy} = -61.44 \times 10^6$ mm^4 and $I_{zz} = 40.96 \times 10^6$ mm^4. The modulus of elasticity is $E = 200$ GPa.



Questions:

a. For the cross-section at A, show that $\kappa_y = -1.5 \times 10^{-6}$ mm^{-1} and $\kappa_y = -4.0 \times 10^{-6}$ mm^{-1}. Find the radius of curvature at A.
b. For the cross-section at A, draw the vectors for the bending moment M and the curvature κ. Also draw the neutral axis *na*.
c. Plot the σ diagram for the cross-section at A, with the correct signs for tension and compression.
d. Plot the distribution of κ_y and κ_z along the beam. Find the displacement of the beam at the free end B.
e. At which locations in flanges and/or web are the shear stresses a maximum or minimum for the cross-section at C. Calculate these shear stresses.
f. Find in this cross-section the shear stresses at Q and R. Make a reasonable sketch of the shear stress distribution in the cross-section.

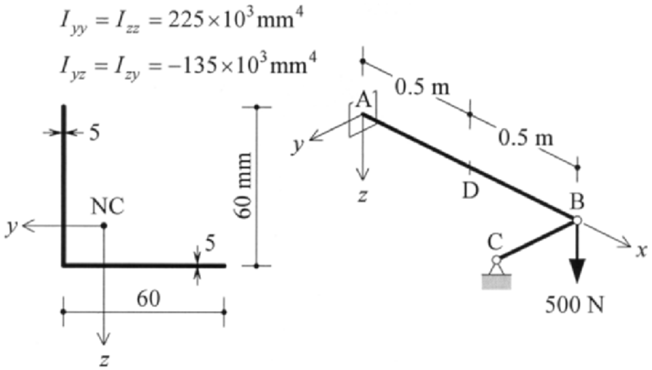
9.41 Cantilever beam AB, with a length $\ell = 1$ m, carries a vertical force $F = 500$ N at its free end B. The line of action of the force passes through the shear centre SC. The dimensions of the thin-walled L-section and the centroidal moments of inertia are shown in the figure. The modulus of elasticity is $E = 200$ GPa.



Questions:

a. Find the shear centre SC of the cross-section.
b. For the cross-section at A, draw the vectors for the bending moment M and the curvature κ. Also draw the neutral axis *na*.
c. Plot the σ diagram for the cross-section at A, with the correct signs for tension and compression.
d. Plot the distribution of κ_y and κ_z along the beam. Find the displacement of the beam at the free end B.
e. At which locations in flange and/or web are the shear stresses a maximum or minimum for the cross-section at C. Calculate these shear stresses.
f. Sketch the shear stress distribution in this cross-section. Calculate the values at relevant points.

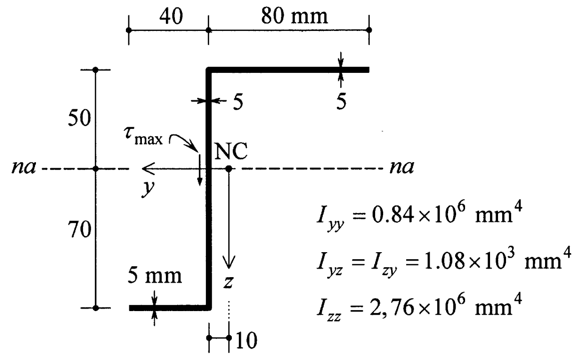
9.42 Beam AB from problem 9.41 is connected at B at the two-force member BC in such a way that B can move only in the vertical direction. The hinged connection at B is realised at the shear centre SC of the cross-section. A vertical force $F = 500$ N is applied at B; the line of action of the force passes through the shear centre SC. The dimensions of the thin-walled L-section and the centroidal moments of inertia are shown in the figure. The modulus of elasticity is $E = 200$ GPa.



Questions:

a. For the cross-section at A, draw the vectors for the bending moment M and the curvature κ. Also draw the neutral axis *na*.
b. Plot the σ diagram for the cross-section at A, with the correct signs for tension and compression.
d. Plot the distribution of κ_y and κ_z along the beam. Find the displacement of the beam at B.
e. At which locations in flange and/or web are the shear stresses a maximum or minimum for the cross-section at D. Calculate these shear stresses.
f. Sketch the shear stress distribution in this cross-section. Calculate the values at relevant points.
g. Find the normal force in two-force member BC.

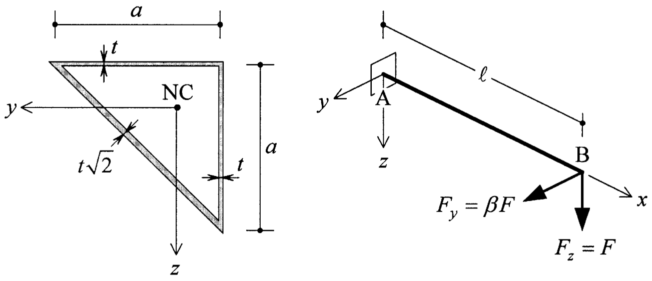
9.43 The thin-walled cross-section transmits a bending moment M and shear force V in such a way that the neutral axis *na* coincides with the y axis. The torsional moment in the cross-section is zero. Furthermore is given that the maximum bending stress is a compressive stress of 140 N/mm^2, and that the maximum shear stress of 3.8 N/mm^2 occurs in the web and is directed downwards. The cross-sectional dimensions are shown in the figure.



Questions:

a. Plot the bending stress diagram.
b. Find the magnitude and direction of the bending moment M.
c. Plot the shear stress diagram. Calculate the values at relevant points.
d. Find the magnitude and direction of the shear force V.
e. On which line in the cross-section is the shear centre SC is located? Draw this line.

9.44 Cantilever beam AB is fixed at A and loaded by the forces F_y and F_z at the free end B, in such a way that there occurs no torsion. The beam has a thin-walled triangular cross-section. All dimensions are shown in the figure. The modulus of elasticity is E.



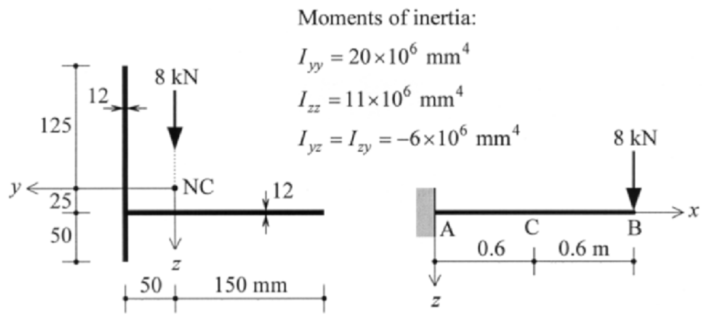
Questions:

a. Find the normal centre NC.
b. Find the principal directions and principal moments of inertia. Draw Mohr's circle, and locate the direction centre DC. Also find the moments of inertia in the yz coordinate system.
c. For $F_z = F$ and $F_y = \beta F$ the deflection of the beam at B is vertically. Find β, and find the vertical deflection at B.
d. Draw the normal stress diagram for the cross-section at A.

9.45 Cantilever beam AB, with a length $\ell = 1.2$ m, carries a force $F = 8$ kN at its free end B. The line of action of the force passes through the normal centre NC. The dimensions of the thin-walled cross-section and the moments of inertia are shown in the figure. The modulus of elasticity is $E = 210$ GPa.

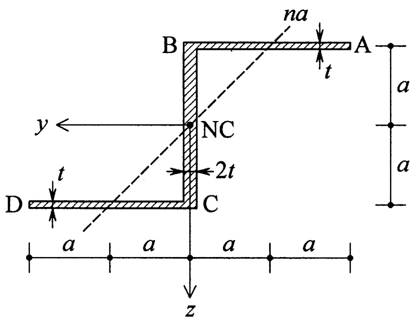
Questions:

a. Verify the correctness of the location of the normal centre NC.



b. Verify the correctness of values of the moments of inertia, including the signs. Find from Mohr's circle the principal directions and principal values.
c. Find the curvature κ of the beam at A. Use the yz coordinate system given. Compute the radius of curvature at A.
d. For the cross-section at A, draw the vectors for the bending moment M and the curvature κ. Also draw the neutral axis *na*.
e. Plot the σ diagram for the cross-section at A, with the correct signs for tension and compression.
f. Plot the distribution of κ_y and κ_z along the beam. Find the displacement of the beam axis at the free end B.
g. Sketch for the cross-section at C the τ diagram for the shear stresses due to shear without torsion. Compute the values at relevant places. Where in the cross-section is the shear stress a maximum or minimum and compute these values.
h. Find the shear centre SC of the cross-section. How large is the torsional moment in cross-section C. For this cross-section, find the shear stresses due to torsion.
i. For the cross-section at C, find the maximum shear stress due to both shear and torsion, and the location where it occurs.

9.46 A thin-walled beam, with the cross-section shown, is loaded by a shear force V in such a way that the neutral axis *na* in every cross-section is located as shown in the figure.

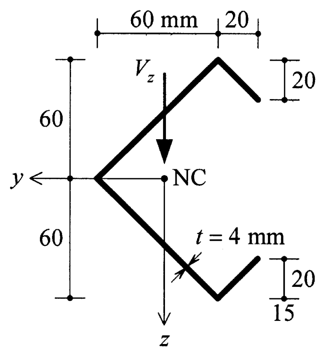


Question:
Find the maximum shear stress in the cross-section, expressed in terms of V, a and t.

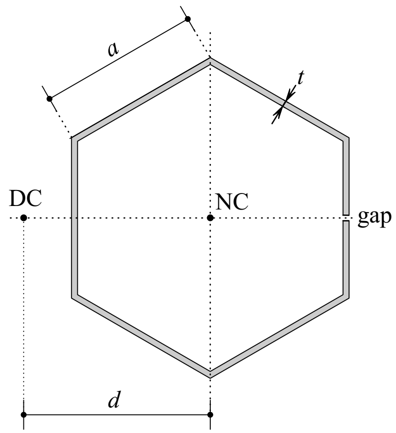
9.47 The thin-walled cross-section with a constant wall thickness t = 4 mm is subject to a shear force $V_z = 6400\sqrt{2}$ N. The line of action of V_z passes through the normal centre NC.

Questions:

a. Find the normal centre NC.
b. Draw the shear stress diagram for the cross-section due to the shear force V_z, if there is no torsion. Show the direction of the shear stresses, and write relevant values in the diagram.
c. Find the shear centre SC.
d. Find the torsional moment and the shear stresses due to torsion.
e. Find the maximum shear stress in the cross-section, due to both shear and torsion, and the location where it occurs.



9.48 The thin-walled hexagon has a gap at A. NC is the normal centre of the cross-section and SC the shear centre.



Question:
Find the distance d between NC and SC.

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Homogeneous and symmetrical cross-sections

Definition cross-sectional quantities

area	static moments	moments of inertia	polar moment of inertia
$A = \int_A \mathrm{d}A$	$S_y = \int_A y\,\mathrm{d}A$	$I_{yy} = \int_A y^2\,\mathrm{d}A$	$I_\mathrm{p} = \int_A r^2\,\mathrm{d}A = I_{yy} + I_{zz}$
	$S_z = \int_A z\,\mathrm{d}A$	$I_{yz} = I_{zy} = \int_A yz\,\mathrm{d}A$	
		$I_{zz} = \int_A z^2\,\mathrm{d}A$	

Coordinates of the normal centre NC (or centroid C) of an homogeneous cross-section

$$\bar{y}_\mathrm{NC} = \bar{y}_\mathrm{C} = \frac{S_{\bar{y}}}{A} \quad \text{and} \quad \bar{z}_\mathrm{NC} = \bar{z}_\mathrm{C} = \frac{S_{\bar{z}}}{A}$$

Steiner's parallel axis theorem

$$I_{\overline{yy}} = I_{yy(\mathrm{centr})} + \bar{y}_\mathrm{C}^2 A$$
$$I_{\overline{yz}} = I_{\overline{yz}} = I_{yz(\mathrm{centr})} + \bar{y}_\mathrm{C}\bar{z}_\mathrm{C} A$$
$$I_{\overline{zz}} = I_{zz(\mathrm{centr})} + \bar{z}_\mathrm{C}^2 A$$

Basic relationships for prismatic members subject to extension and bending[1]

	kinematic relationships	constitutive relationships	static relationships	differential equations
extension	$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}$	$N = EA\varepsilon$	$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0$	$EA\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + q_x = 0$
bending	$\varphi = -\frac{\mathrm{d}w}{\mathrm{d}x}$, $\kappa = \frac{\mathrm{d}\varphi}{\mathrm{d}x}$ → $\kappa = -\frac{\mathrm{d}^2w}{\mathrm{d}x^2}$	$M = EI\kappa$	$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0$, $\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0$ → $\frac{\mathrm{d}^2M}{\mathrm{d}x^2} + q_z = 0$	$-EI\frac{\mathrm{d}^4w}{\mathrm{d}x^4} + q_z = 0$

Strain formula for bending and extension

$$\varepsilon(z) = \varepsilon + \kappa_z z$$

Normal stress formulae for bending and extension

- extension: $\sigma^N = \dfrac{N}{A}$
- bending (bending stresses): $\sigma^M = \dfrac{M_z z}{I_{zz}}$
- combined extension and bending: $\sigma(z) = \sigma^N + \sigma^M = \dfrac{N}{A} + \dfrac{M_z z}{I_{zz}}$

Maximum bending stresses

These stresses occur in the outermost fibre layers $z = -e_\mathrm{t}$ and $z = +e_\mathrm{b}$.

- non-symmetrical cross-section:

$$\sigma_\mathrm{t}^M = -\frac{M_z e_\mathrm{t}}{I_{zz}} = -\frac{M_z}{W_{z;\mathrm{t}}} \quad \text{in which} \quad W_{z;\mathrm{t}} = \frac{I_{zz}}{e_\mathrm{t}} \quad \text{(section modulus)}$$
$$\sigma_\mathrm{b}^M = +\frac{M_z e_\mathrm{b}}{I_{zz}} = +\frac{M_z}{W_{z;\mathrm{b}}} \quad \text{in which} \quad W_{z;\mathrm{b}} = \frac{I_{zz}}{e_\mathrm{b}} \quad \text{(section modulus)}$$

- symmetrical cross-section:

$$\sigma_\mathrm{max}^M = \pm\frac{M}{W}$$

General normal stress formula for combined bending and extension

$$\sigma(y, z) = \frac{N}{A} + \frac{M_y y}{I_{yy}} + \frac{M_z z}{I_{zz}}$$

Coordinates of the centre of force in the cross-section

$$e_y = \frac{M_y}{N} \quad \text{and} \quad e_z = \frac{M_z}{N}$$

---

1 All formulas related to bending are derived for the principal directions. For bending in the xz plane the z axis is therefore a principal axis of the cross-section.

Core of the cross-section

- non-symmetrical cross-section:

$$k_{\text{b}} = \frac{W_{z;\text{t}}}{A} \text{ (lower core radius)}$$

$$k_{\text{t}} = \frac{W_{z;\text{b}}}{A} \text{ (upper core radius)}$$

- symmetrical cross-section:

$$k = \frac{W}{A}$$

Shear flow (shear force per length) in the longitudinal direction

$$s_x^{\text{a}} = -\frac{V_z S_z^{\text{a}}}{I_{zz}} \qquad \text{(traditional approach)}$$

$$s_x^{\text{a}} = -V_z \cdot \left[\frac{N_M^{\text{a}} \text{ (due to } M_z^*)}{M_z^*} \right] \quad \text{(alternative approach)}$$

Resultant shear force in the longitudinal direction

$$R_{x;\text{s}}^{\text{a}} = -\frac{\Delta M_z S_z^{\text{a}}}{I_{zz}}$$

Shear stresses due to shear force

$$\sigma_{xm} = -\frac{V_z S_z^{\text{a}}}{b^{\text{a}} I_{zz}} \text{ (traditional approach)}$$

$$\sigma_{xm} = -\frac{V_z}{b^{\text{a}}} \left[\frac{N_M^{\text{a}} \text{ (due to } M_z^*)}{M_z^*} \right] \text{ (alternative approach)}$$

Basic relationships for members subject torsion

- Constitutive relationship: $M_{\text{t}} = G I_{\text{t}} \chi$
- Kinematic relationship: $\chi = \dfrac{\mathrm{d}\varphi_x}{\mathrm{d}x}$

Shear stresses due to torsion

- Thin-walled circular cross-section:

$$\tau = \frac{M_{\text{t}} R}{I_{\text{t}}} \text{ in which } I_{\text{t}} = I_{\text{p}} = 2\pi R^3 t \text{ (torsion constant)}$$

- Thick-walled circular cross-section:

$$\tau = \frac{M_{\text{t}} r}{I_{\text{t}}} \text{ in which } I_{\text{t}} = I_{\text{p}} = \tfrac{1}{2}\pi (R_{\text{e}}^4 - R_{\text{i}}^4)$$

- Solid circular cross-section:

$$\tau = \frac{M_{\text{t}} r}{I_{\text{t}}} \text{ in which } I_{\text{t}} = I_{\text{p}} = \tfrac{1}{2}\pi R^4$$

- Closed thin-walled cross-sections (general formula):

$$s = \tau t = \text{constant (the shear flow is constant)}$$

$$\tau = \frac{M_{\text{t}}}{2A_{\text{m}} t}$$

- Thin-walled strip:

$$\tau = \frac{M_{\text{t}} e_{\text{m}}}{\frac{1}{2} I_{\text{t}}} \text{ in which } I_{\text{t}} = \tfrac{1}{3} h t^3$$

- Open thin-walled cross-sections:

$$\tau = \frac{M_{\text{t}} e_{\text{m}}}{\frac{1}{2} I_{\text{t}}} \text{ with } I_{\text{t}} = \sum \tfrac{1}{3} h t^3$$

Change in rotation due to torsion

- General formula:

$$\Delta\varphi_x = \int_\ell \chi \cdot \mathrm{d}x = \int_\ell \frac{M_{\text{t}}}{G I_{\text{t}}} \mathrm{d}x$$

- Prismatic member with constant torsional moment:

$$\Delta\varphi_x = \frac{M_{\text{t}} \ell}{G I_{\text{t}}}$$

Torsion constant

- a closed thin-walled cross-section: $I_t = \dfrac{4A_m^2}{\oint \frac{1}{t}\,\mathrm{d}m}$
- other cross-sectional shapes: see the shear stress formulas.

Change in length due to extension

- General formula:

$$\Delta\ell = \int_\ell \varepsilon\,\mathrm{d}x = \int_\ell \frac{N}{EA}\,\mathrm{d}x$$

- Prismatic member with constant normal force:

$$\Delta\ell = \frac{N\ell}{EA}$$

Deformation due to bending

- First moment-area theorem:

$$\Delta\varphi = \varphi_B - \varphi_A = \underbrace{\int_A^B \frac{M}{EI}\,\mathrm{d}x}_{\text{area } M/EI \text{ diagram}}$$

- Second moment-area theorem:

$$\Delta w = w_B - w_A = -\varphi_A(x_B - x_A) - \underbrace{\int_A^B \frac{M}{EI}(x_B - x)\,\mathrm{d}x}_{\text{static moment } M/EI \text{ diagram wrt B}}$$

Inhomogeneous and/or non-symmetrical cross-sections

Definition cross-sectional quantities

axial stiffness	weighted static moments	bending stiffness quantities
$EA=\int_A E(y,z)\cdot \mathrm{d}A$	$ES_y=\int_A E(y,z)\cdot y\,\mathrm{d}A$	$EI_{yy}=\int_A (y,z)\cdot y^2\,\mathrm{d}A$
	$ES_z=\int_A E(y,z)\cdot z\,\mathrm{d}A$	$EI_{yz}=EI_{zy}=\int_A E(y,z)\cdot yz\,\mathrm{d}A$
		$EI_{zz}=\int_A E(y,z)\cdot z^2\,\mathrm{d}A$

Coordinates of the normal centre NC of the cross-section

$$\bar{y}_{NC} = \frac{ES_{\bar{y}}}{EA} \quad\text{and}\quad \bar{z}_{NC} = \frac{ES_{\bar{z}}}{EA}$$

Strain formula for combined bending and extension

$$\varepsilon(y,z) = \varepsilon + \kappa_y y + \kappa_z z$$

Normal stresses due to combined bending and extension

$$\sigma(y,z) = E(y,z)\cdot\varepsilon(y,z) = E(y,z)\cdot(\varepsilon + \kappa_y y + \kappa_z z)$$

Constitutive relationships[1]

$$N = EA\varepsilon \qquad \text{(extension)}$$

$$\begin{bmatrix} M_y \\ M_z \end{bmatrix} = \begin{bmatrix} EI_{yy} & EI_{yz} \\ EI_{zy} & EI_{zz} \end{bmatrix} = \begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} \quad \text{(bending)}$$

In inverse form:

$$\varepsilon = \frac{N}{EA}$$

$$\begin{bmatrix} \kappa_y \\ \kappa_z \end{bmatrix} = \frac{1}{\mathrm{Det}(EI)} \begin{bmatrix} EI_{zz} & -EI_{yz} \\ -EI_{zy} & EI_{yy} \end{bmatrix} = \begin{bmatrix} M_y \\ M_z \end{bmatrix}$$

in which $\mathrm{Det}(EI)$ is the determinant of the bending stiffness matrix:

$$\mathrm{Det}(EI) = EI_{yy}EI_{zz} - EI_{yz}^2$$

1 The origin of the yz coordinate system coincides with the normal centre NC of the cross-section.

Transformation rules for the bending stiffness tensor

$$EI_{\overline{yy}} = +EI_{yy}\cos^2\alpha + 2EI_{yz}\sin\alpha\cos\alpha + EI_{zz}\sin^2\alpha$$

$$EI_{\overline{yz}} = EI_{zy} = -(EI_{yy} - EI_{zz})\sin\alpha\cos\alpha + EI_{yz}(\cos^2\alpha - \sin^2\alpha)$$

$$EI_{\overline{zz}} = +EI_{yy}\sin^2\alpha - 2EI_{yz}\sin\alpha\cos\alpha + EI_{zz}\cos^2\alpha$$

Principal directions

$$\tan 2\alpha = \frac{EI_{yz}}{\frac{1}{2}(EI_{yy} - EI_{zz})}$$

Shear flow (shear force per length) in the longitudinal direction

- traditional approach (in a principal yz coordinate system)

$$s_x^{\mathrm{a}} = -\frac{V_y E S_y^{\mathrm{a}}}{EI_{yy}} - \frac{V_z E S_z^{\mathrm{a}}}{EI_{zz}}$$

- alternative approach

$$s_x^{\mathrm{a}} = -V \cdot \left[\frac{N_M^{\mathrm{a}}\ (\text{due to } M^*)}{M^*}\right]$$

The dummy moment M^* acts in the same plane as the resultant shear force V.

Shear stresses due to shear force

$$\sigma_{xm} = \frac{s_x^{\mathrm{a}}}{b^{\mathrm{a}}}$$

Temperature effects[1]

Strain

$$\varepsilon(y,z) = \varepsilon^{\sigma}(y,z) + \varepsilon^{T}(y,z) = \frac{\sigma(y,z)}{E(y,z)} + \alpha T(y,z)$$

Strain distribution

$$\varepsilon(y,z) = \varepsilon + \kappa_y y + \kappa_z z$$

Stress distribution

$$\sigma(y,z) = E(y,z)\varepsilon^{\sigma}(y,z) = E(y,z)\{\underbrace{\varepsilon + \kappa_y y + \kappa_z z}_{\varepsilon(y,z)} - \underbrace{\alpha T(y,z)}_{\varepsilon^T(y,z)}\}$$

Unconstrained temperature deformations

$$\varepsilon^T = \int_A \frac{\alpha E T(y,z)\,\mathrm{d}A}{EA}$$

$$\kappa_y^T = \int_A \frac{y \cdot \alpha E T(y,z)}{EI_{yy}},\mathrm{d}A$$

$$\kappa_z^T = \int_A \frac{z \cdot \alpha E T(y,z)}{EI_{zz}},\mathrm{d}A$$

Constitutive relationships

$$N = EA(\varepsilon - \varepsilon^T)$$

$$M_y = EI_{yy}(\kappa_y - \kappa_y^T)$$

$$M_z = EI_{zz}(\kappa_z - \kappa_z^T)$$

1 The yz coordinate system is a principal coordinate system.

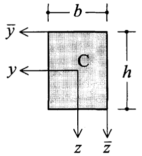
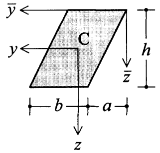
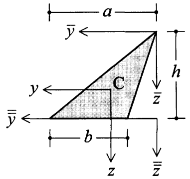
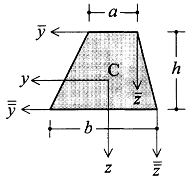
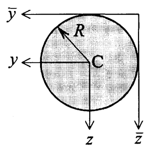
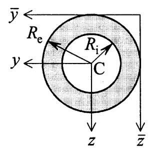
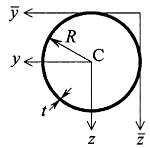
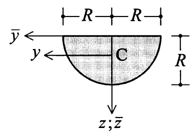
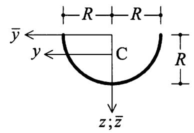
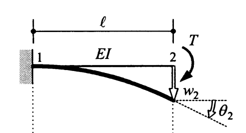
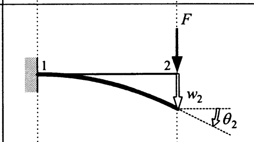
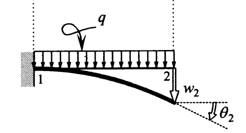
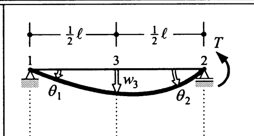
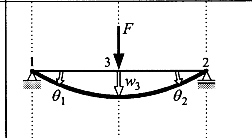
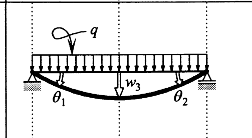
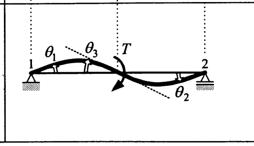
Figure	Area, coordinates centroid C	Second moments of area	
		centroidal	other
	Rectangle $A = bh$ $\bar{y}_C = \frac{1}{2}b$ $\bar{z}_C = \frac{1}{2}h$	$I_{yy} = \frac{1}{12}b^3h$ $I_{zz} = \frac{1}{12}bh^3$ $I_{yz} = 0$	$I_{\bar{y}\bar{y}} = \frac{1}{3}b^3h$ $I_{\bar{z}\bar{z}} = \frac{1}{3}bh^3$ $I_{\bar{y}\bar{z}} = \frac{1}{4}b^2h^2$
	Parallelogram $A = bh$ $\bar{y}_C = \frac{1}{2}(a+b)$ $\bar{z}_C = \frac{1}{2}h$	$I_{yy} = \frac{1}{12}(a^2+b^2)bh$ $I_{zz} = \frac{1}{12}bh^3$ $I_{yz} = \frac{1}{12}abh^2$	$I_{\bar{z}\bar{z}} = \frac{1}{3}bh^3$
	Triangle $A = \frac{1}{2}bh$ $\bar{y}_C = \frac{1}{3}(2a-b)$ $\bar{z}_C = \frac{2}{3}h$	$I_{yy} = \frac{1}{36}(a^2-ab+b^2)bh$ $I_{zz} = \frac{1}{36}bh^3$ $I_{yz} = \frac{1}{72}(2a-b)bh^2$	$I_{\bar{z}\bar{z}} = \frac{1}{4}bh^3$ $I_{\bar{y}\bar{z}} = \frac{1}{8}(2a-b)bh^2$ $I_{\bar{\bar{z}}\bar{\bar{z}}} = \frac{1}{12}bh^3$
	Trapezium $A = \frac{1}{2}(a+b)h$ $\bar{z}_C = \frac{1}{3}\frac{a+2b}{a+b}h$	$I_{zz} = \frac{1}{36}\frac{a^2+4ab+b^2}{a+b}h^3$	$I_{\bar{z}\bar{z}} = \frac{1}{12}(a+3b)h^3$ $I_{\bar{\bar{z}}\bar{\bar{z}}} = \frac{1}{12}(3a+b)h^3$
	Circle $A = \pi R^2$	$I_{yy} = I_{zz} = \frac{1}{4}\pi R^4$ $I_{yz} = 0$ $I_p = \frac{1}{2}\pi R^4$	$I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = \frac{5}{4}\pi R^4$ $I_{\bar{y}\bar{z}} = \pi R^4$

Figure	Area, coordinates centroid C	Second moments of area	
		centroidal	other
	Thick-walled ring $A = \pi(R_e^2 - R_i^2)$	$I_{yy} = I_{zz} = \frac{1}{4}\pi(R_e^4 - R_i^4)$ $I_{yz} = 0$ $I_p = \frac{1}{2}\pi(R_e^4 - R_i^4)$	
	Thin-walled ring $A = 2\pi Rt$	$I_{yy} = I_{zz} = \pi R^3 t$ $I_{yz} = 0$ $I_p = 2\pi R^3 t$	$I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = 3\pi R^3 t$
	Semicircle $A = \frac{1}{2}\pi R^2$ $\bar{y}_C = 0$ $\bar{z}_C = \frac{4}{3\pi}R$ $= 0.424R$	$I_{yy} = \frac{1}{8}\pi R^4 = 0.393R^4$ $I_{zz} = (\frac{\pi}{8} - \frac{8}{9\pi})R^4$ $= 0.110R^4$ $I_{yz} = 0$	$I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = \frac{1}{8}\pi R^4$ $I_{\bar{y}\bar{z}} = 0$
	Semicircular ring $A = \pi Rt$ $\bar{y}_C = 0$ $\bar{z}_C = \frac{2}{\pi}R$ $= 0.637R$	$I_{yy} = \frac{1}{2}\pi R^3 t$ $I_{zz} = (\frac{\pi}{2} - \frac{4}{\pi})R^3 t$ $= 0.298R^3 t$ $I_{yz} = 0$	$I_{\bar{y}\bar{y}} = I_{\bar{z}\bar{z}} = \frac{1}{2}\pi R^3 t$ $I_{\bar{y}\bar{z}} = 0$

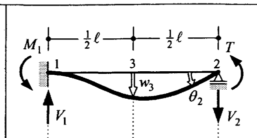
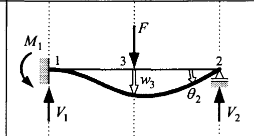
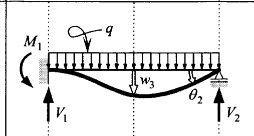
forget-me-nots

(1)		$\theta_2 = \dfrac{T\ell}{EI};\quad w_2 = \dfrac{T\ell^2}{2EI}$
(2)		$\theta_2 = \dfrac{F\ell^2}{2EI};\quad w_2 = \dfrac{F\ell^3}{3EI}$
(3)		$\theta_2 = \dfrac{q\ell^3}{6EI};\quad w_2 = \dfrac{q\ell^4}{8EI}$

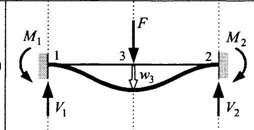
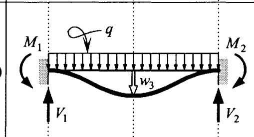
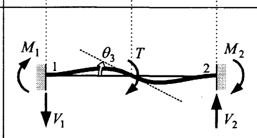
simply supported beam (statically determinate)

(4)		$\theta_1 = \dfrac{1}{6}\dfrac{T\ell}{EI};\quad \theta_2 = \dfrac{1}{3}\dfrac{T\ell}{EI};\quad w_3 = \dfrac{1}{16}\dfrac{T\ell^2}{EI}$
(5)		$\theta_1 = \theta_2 = \dfrac{1}{16}\dfrac{F\ell^2}{EI};\quad w_3 = \dfrac{1}{48}\dfrac{F\ell^3}{EI}$
(6)		$\theta_1 = \theta_2 = \dfrac{1}{24}\dfrac{q\ell^3}{EI};\quad w_3 = \dfrac{5}{384}\dfrac{q\ell^4}{EI}$
(a)		$\theta_1 = \theta_2 = \dfrac{1}{24}\dfrac{T\ell}{EI};\quad \theta_3 = \dfrac{1}{12}\dfrac{T\ell}{EI};\quad w_3 = 0$

statically indeterminate beam (one fixed end)

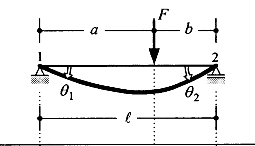
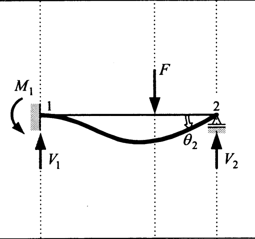
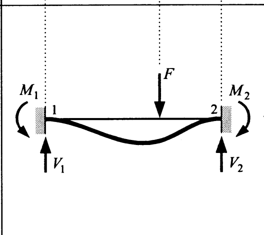
(7)		$\theta_2 = \dfrac{1}{4}\dfrac{T\ell}{EI};\quad w_3 = \dfrac{1}{32}\dfrac{T\ell^2}{EI}$ $M_1 = \dfrac{1}{2}T;\quad V_1 = V_2 = \dfrac{3}{2}\dfrac{T}{\ell}$
(8)		$\theta_2 = \dfrac{1}{32}\dfrac{F\ell^2}{EI};\quad w_3 = \dfrac{7}{768}\dfrac{F\ell^3}{EI}$ $M_1 = \dfrac{3}{16}F\ell;\quad V_1 = \dfrac{11}{16}F;\quad V_2 = \dfrac{5}{16}F$
(9)		$\theta_2 = \dfrac{1}{48}\dfrac{q\ell^3}{EI};\quad w_3 = \dfrac{1}{192}\dfrac{q\ell^4}{EI}$ $M_1 = \dfrac{1}{8}q\ell^2;\quad V_1 = \dfrac{5}{8}q\ell;\quad V_2 = \dfrac{3}{8}q\ell$

statically indeterminate beam (two fixed ends)

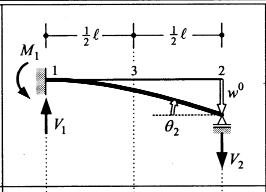
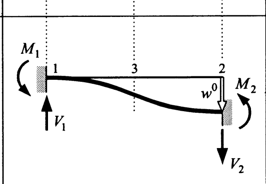
(10)		$w_3 = \dfrac{1}{192}\dfrac{F\ell^3}{EI}$ $M_1 = M_2 = \dfrac{1}{8}F\ell;\quad V_1 = V_2 = \dfrac{1}{2}F$
(11)		$w_3 = \dfrac{1}{384}\dfrac{q\ell^4}{EI}$ $M_1 = M_2 = \dfrac{1}{12}q\ell^2;\quad V_1 = V_2 = \dfrac{1}{2}q\ell$
(b)		$\theta_3 = \dfrac{1}{16}\dfrac{T\ell}{EI};\quad w_3 = 0$ $M_1 = M_2 = \dfrac{1}{4}T;\quad V_1 = V_2 = \dfrac{3}{2}\dfrac{T}{\ell}$

Some formulae for prismatic beams with bending stiffness EI.
T, F and q represent the load by a couple, force and uniformly distributed load respectively.
M_i and V_i represent the bending moment and shear force on the end i of the beam, due to the support reactions.

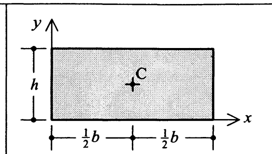
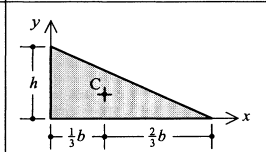
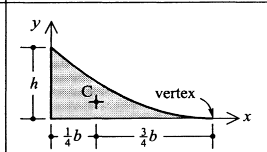
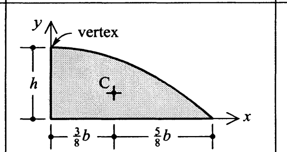
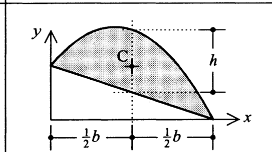
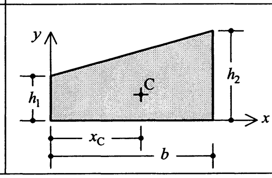
support reactions and rotations at the beam ends

(c)		$\theta_1 = \dfrac{Fab(\ell+b)}{6EI\ell} = \dfrac{F\ell^2}{6EI}\left(2\dfrac{a}{\ell} - 3\dfrac{a^2}{\ell^2} + \dfrac{a^3}{\ell^3}\right)$ $\theta_2 = \dfrac{Fab(\ell+a)}{6EI\ell} = \dfrac{F\ell^2}{6EI}\left(\dfrac{a}{\ell} - \dfrac{a^3}{\ell^3}\right)$
(d)		$M_1 = \dfrac{Fb(\ell^2-b^2)}{2\ell^2} = F\ell\left(\dfrac{a}{\ell} - \dfrac{3}{2}\dfrac{a^2}{\ell^2} + \dfrac{1}{2}\dfrac{a^3}{\ell^3}\right)$ $V_1 = \dfrac{Fb(3\ell^2-b^2)}{2\ell^3} = F\left(1 - \dfrac{3}{2}\dfrac{a^2}{\ell^2} + \dfrac{1}{2}\dfrac{a^3}{\ell^3}\right)$ $V_2 = \dfrac{Fa^2(3\ell-a)}{2\ell^3} = F\left(\dfrac{3}{2}\dfrac{a^2}{\ell^2} - \dfrac{1}{2}\dfrac{a^3}{\ell^3}\right)$ $\theta_2 = \dfrac{Fa^2b}{4EI\ell} = \dfrac{F\ell^2}{4EI}\left(\dfrac{a^2}{\ell^2} - \dfrac{a^3}{\ell^3}\right)$
(e)		$M_1 = \dfrac{Fab^2}{\ell^2} = F\ell\left(\dfrac{a}{\ell} - 2\dfrac{a^2}{\ell^2} + \dfrac{a^3}{\ell^3}\right)$ $V_1 = \dfrac{Fb^2(\ell+2a)}{\ell^3} = F\left(1 - 3\dfrac{a^2}{\ell^2} + 2\dfrac{a^3}{\ell^3}\right)$ $M_2 = \dfrac{Fa^2b}{\ell^2} = F\ell\left(\dfrac{a^2}{\ell^2} - \dfrac{a^3}{\ell^3}\right)$ $V_2 = \dfrac{Fa^2(\ell+2b)}{\ell^3} = F\ell\left(3\dfrac{a^2}{\ell^2} - 2\dfrac{a^3}{\ell^3}\right)$

settlements

(f)		$M_1 = \dfrac{3EI}{\ell^2}w^0;\quad V_1 = V_2 = \dfrac{3EI}{\ell^3}w^0$ $\theta_2 = \dfrac{3}{2}\dfrac{w^0}{\ell}$ $\theta_3 = \dfrac{9}{8}\dfrac{w^0}{\ell};\quad w_3 = \dfrac{5}{16}w^0$
(g)		$M_1 = M_2 = \dfrac{6EI}{\ell^2}w^0;\quad V_1 = V_2 = \dfrac{12EI}{\ell^3}w^0$ $\theta_3 = \dfrac{3}{2}\dfrac{w^0}{\ell};\quad w_3 = \dfrac{1}{2}w^0$

properties of plane figures to be used for the moment-area theorems

(1)		rectangle: $y = h$ $A = bh$ $x_C = \frac{1}{2}b$
(2)		triangle: $y = h\left\{1 - \dfrac{x}{b}\right\}$ $A = \frac{1}{2}bh$ $x_C = \frac{1}{3}b$
(3)		parabola: $y = h\left\{1 - \dfrac{x}{b}\right\}^2$ $A = \frac{1}{3}bh$ $x_C = \frac{1}{4}b$
(4)		parabola: $y = h\left\{1 - \left(\dfrac{x}{b}\right)^2\right\}$ $A = \frac{2}{3}bh$ $x_C = \frac{3}{8}b$
(5)		parabola: $A = \frac{2}{3}bh$ $x_C = \frac{1}{2}b$
(6)		trapezium: $y = h_1 + (h_2 - h_1)\dfrac{x}{b}$ $A = \frac{1}{2}b(h_1 + h_2)$ $x_C = \frac{1}{3}b\dfrac{h_1 + 2h_2}{h_1 + h_2}$

Latin capitals

Quantity		SI unit
Symbol	Name	Symbol[1]
A^{a}	cross-sectional area of the sliding element	m^2
A_{m}	area enclosed by the centre line of a closed thin-walled cross-section	m^2
E_{v}	modulus of elasticity	$\mathrm{N/m}^2$
EA	axial stiffness	N
EI	bending stiffness	Nm^2
EI_{yy}	bending stiffness in the xy plane	Nm^2
$EI_{yz} = EI_{zy}$	bending stiffness quantity	Nm^2
EI_{zz}	bending stiffness in the xz plane	Nm^2
G	shear modulus	Nm^2
GI_{t}	torsional stiffness	Nm^2
I	second moment of area, moment of inertia	m^4
I_{yy}	moment of inertia in the xy plane	m^4
$I_{yz} = I_{zy}$	product of inertia	m^4
I_{zz}	moment of inertia in the xz plane	m^4
I_{p}	polar moment of inertia	m^4
I_{t}	torsion constant	m^4
M^*	dummy bending moment	Nm

Quantity		SI unit
Symbol	Name	Symbol[1]
N^{a}	resultant of all normal stresses on the cross-sectional area of the sliding element	N
N^{a}_{M}	resultant of all normal stresses on the cross-sectional area of the sliding element, due to bending	N
R	radius, radius of curvature	m
R_{e}	outside radius of a circular tube	m
R_{i}	inside radius of a circular tube	m
$R^{\mathrm{a}}_{x;\mathrm{s}}$	resultant shear force in the longitudinal direction, acting on the sliding element	N
S	first moment of area, static moment	m^3
S_y	static moment in the xy plane	m^3
S_z	static moment in the xz plane	m^3
S^{a}_{z}	static moment in the xz plane of the cross-sectional area of the sliding element	m^3
T	temperature; increase in temperature	K
W	section modulus	m^3
W_{t}	section modulus with respect to the upper fibre layer	m^3
W_{b}	section modulus with respect to the lower fibre layer	m^3
W_z	section modulus in the xz plane	m^3

Latin lower case

Quantity		SI unit
Symbol	Name	Symbol[1]
b^{a}	width of the sliding segment, width of a cut	m
e_{b}	distance of the lower fibre layer to the member axis	m
e_m	distance to the centre line m (for thin-walled parts) of the cross-section	m
e_{t}	distance of the upper fibre layer to the member axis	m
e_y, e_z	coordinates of the centre of force in the yz plane	m
$f_{0.2}$	0.2% offset yield strength	$\mathrm{N/m}^2$
f_{t}	tensile strength	$\mathrm{N/m}^2$
f_{y}	yield point	$\mathrm{N/m}^2$
k	factor of proportionality	1
k	core radius	m
k_{b}	lower core radius	m
k_{t}	upper core radius	m
m	coordinate axis along the centre line (of a thin-walled part) of the cross-section	m
q_y	distributed load acting in the y direction	N/m
q_z	distributed load acting in the z direction	N/m

[1] Expressed in the basic units.

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
r	radius of inertia, radius of gyration	m
r_y	radius of inertia in the y direction	m
r_z	radius of inertia in the z direction	m
s	shear flow (shear force per length)	N/m
s_{in}	flow-in, shear flow towards the joint	N/m
s_m	shear flow in the m direction	N/m
s_{out}	flow-out, shear flow from the joint	N/m
s_x^{a}	shear flow in the longitudinal direction, acting on the sliding element	N/m
t	wall thickness	m
t_{f}	web thickness	m
$y_{\mathrm{C}}, z_{\mathrm{C}}$	coordinates of the centroid	m
$y_{\mathrm{NC}}, z_{\mathrm{NC}}$	coordinates of the normal centre of a cross-section	m
$y_{\mathrm{SC}}, z_{\mathrm{SC}}$	coordinates of the shear centre of a cross-section	m
$y_{\mathrm{C}}^{\mathrm{a}}, z_{\mathrm{C}}^{\mathrm{a}}$	coordinates of the centroid of the sliding part of the cross-section	m

Greek letters

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
α	coefficient of thermal expansion	1/K
ε	strain	1
ε^T	unconstrained strain due to temperature effects	1
$\varepsilon_{\mathrm{pl}}$	strain at the end of the yield stage	1
ε_{t}	strain associated with the tensile strength f_{t}	1
ε_{u}	strain at failure	1
ε_{y}	yield strain, strain at the start of the yield stage	1
χ	distortion, torsional strain	rad/m
κ	curvature	1/m
κ^T	unconstrained curvature due to temperature effects	1/m
κ_y	curvature in the xy plane	1/m
κ_z	curvature in the xz plane	1/m
γ	load factor	1
γ	change of the right angle due to shear, shear strain	rad
γ_{g}	load factor related to the permanent load	1
γ_{q}	load factor related to the variable load	1

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
γ_{y}	shear strain associated with the yield strain stress τ_{y}	1
θ	angle, angle of rotation	rad
$\bar{\sigma}$	limiting value of the normal stress, admissible value	N/m^2
σ^M	normal stress due to bending, bending stress	N/m^2
σ^N	normal stress due to extension	N/m^2
σ_{b}	normal stress in the lower fibre layer	N/m^2
σ_{t}	normal stress in the upper fibre layer	N/m^2
σ_{upset}	upsetting stress	N/m^2
σ_{mx}	shear stress in the longitudinal direction	N/m^2
σ_{xm}	shear stress in the plane of the cross-section	N/m^2
$\bar{\tau}$	limiting value of the shear stress, admissible value	N/M^2
τ_{y}	yield shear stress	N/m^2

Other symbols

$(\cdots)'$	$\frac{\mathrm{d}(\cdots)}{\mathrm{d}x}$, derivative with respect to x

[1] Expressed in basic units.