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Dynamics of Synchronising Systems

Translated by Alexander K. Belyaev

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Introduction

Current thinking suggests that the closed-form integrability of the equations of motion of a particular mechanical system, or a system which is close to this system, is a special "physical" property which provides the system with special degenerating properties. For instance, the motion of an "integrable" conservative system with a positive definite Hamilton function is nearly always of a quasi-periodic (in particular, periodic) character, whereas small conservative perturbations result in motions of a more complex form. Quasi-conservative (that is nearly conservative) problems are important in modern technical and celestial mechanics, as well as in physics (quantum mechanics) and electrical engineering. The reasons for this are as follows. Firstly, the overwhelming majority of integrable problems belong to conservative systems. It is sufficient to mention the classical example of the equations of motion of a rigid body with a fixed point which are integrable by quadratures. Moreover, relatively general conditions of integrability are obtained only for conservative systems. Secondly, while investigating quasi-conservative problems one can successfully apply the analytical methods of small parameters, primarily the averaging method, which has been developed over recent years.

The efficient application of the methods of small parameters has proved to be feasible for a broader class of perturbed problems, which are referred to as locally integrable in the first chapter of the present book. These are the systems whose equations of motion have a particular periodic solution or a particular family of solutions, the variational equations in the vicinity of this family being integrable by quadratures. A distinctive "physical" peculiarity of locally integrable problems is their relative simplicity in the vicinity of the closed-form particular solution. Examples of this include quasi-linear problems and the problems which are reducible to essentially nonlinear piecewise-linear equations in the original approximation. Other examples of locally integrable systems are an autonomous dynamic system in the plane, provided that its particular integral can be constructed, some cases of the partial integration in rigid body dynamics, etc. The analysis of such systems is developed in the first chapter and is subsequently adapted to the investigation of more complex systems.

The second chapter is of an auxiliary character and contains a brief description of existing ways of describing mechanical and electromechanical systems which can be considered as being conservative to some extent.

The third chapter investigates conservative systems, where attention is given to analysing the dependence of the characteristics of the periodic motions of librational and rotational types on the "action-angle" variables. While analysing the dependence of the energy constant on frequency, one can classify these problem by means of the degree of anisochronism. Further analysis of integrable conservative systems with several degrees of freedom is also considered using action-angle variables. The most rational ways of introducing these variables, the corresponding variables "phase-frequency", and the harmonic canonical variables are studied in detail. The degeneracy of similar problems caused either by reducing the number of "true" phases or by the appearance of fixed isochronous frequencies is analysed.

This allows one to suggest a more general statement of the perturbed problem in the fifth chapter. A set of special slow variables which have the meaning of anisochronous frequencies is separated from the whole set of slow variables. Further multi-frequency averaging of the obtained system with a multi-dimensional rapidly rotating phase is carried out by a modified averaging procedure with accuracy up to third order. These modifications allows one to use other variables (the averaged anisochronous frequencies) instead of the traditional frequency detunings and, thus, write down the final result in a compact form which is convenient for further "physical" analysis. Considerable attention is given to the study of stationary solutions of the averaged equations and construction of the explicit relationships for determining the different stability criteria of these solutions. The case of essentially nonlinear equations for non-critical fast variables close to the quasi-static ones is further pursued. In this case, instead of the "action-angle" variables it is convenient to use the harmonic quasi-static variables and locally integrable systems which are piecewise-continuous in the generating approximation. In all these cases, the Lyapunov-Poincaré local method of small parameter is used instead of the averaging method.

The seventh, eighth and tenth chapters are devoted to the problem of weak interaction of quasi-conservative dynamic objects. The seventh chapter deals with the classification of the interaction types, the criteria of weakness of the interaction and the general statement of the problem. Synchronous regimes in a system of anisochronous single-degree-of-freedom objects are treated in detail. The resulting relationships, enabling determination of the stable phasing of synchronous motions, are expressed in terms of the parameters having the meaning of the dynamic influence coefficients. It is essential that these coefficients can be expressed both theoretically and experimentally. It is shown that in the case of the quasi-conservative mechanism of interaction between the objects, a stable synchronous regime possesses certain extremum properties. Namely, a stable synchronous phasing renders an extremum to the Hamilton action of the interaction elements, the extremum character being determined by the type of anisochronism of the objects and the peculiarities of the weak interaction. When modified, this result can be generalised to a more complex problem of the weak interaction of dynamical objects with several degrees of freedom. The chapter is concluded with a non-quasiconservative theory of synchronisation of inertial vibration exciters, the similarity of and the difference between the quasi-conservative theory being discussed. From this perspective it is clearly important to perform a preliminary analysis of the order of smallness of the factors affecting the weak interaction of the objects in the system.

The present book is based upon the monograph [74] published in Russian by Nauka, St. Petersburg in 1996. However the present book contains an increased number of applications and this is why the sixth, ninth and eleventh chapters have been added. The sixth chapter, written by A.G. Chirkov, is concerned with a description of the averaging procedure by means of the methods of non-relativistic quantum mechanics. Particularly, this chapter is important since non-rigorous approaches of the intuitive character are still used to analyse some physical problems and they may lead to inaccuracies and, possibly, gross error. The ninth chapter is devoted to the analysis of the phenomenon of self-synchronisation of the inertial vibration exciters which are frequently used to drive modern vibrational facilities. The basis for this chapter is the material of monograph [76] published by Mashinostroenie, St. Petersburg in 1990. The last, eleventh, chapter of the present book is written by D.Yu. Skubov. Its origin is in the earlier publications by K.Sh. Khodzhaev and is also aimed at applications in the field of vibrational technology. It is worthwhile noting the peculiarity of the problems of synchronisation in electromechanical systems. The mechanism of weak interaction of electromechanical objects is of an essentially non-conservative character. Nevertheless, stable synchronous regimes can also have close extremum properties in these problems.

The present book is based on "physical" reasoning. The author does not suggest principle improvements to the modern methods of the theory of nonlinear oscillations. The main objective is to find a rational means of obtaining sufficiently general averaged equations of motion which have a clear physical interpretation and are valid for a broad class of weak interaction problems of a mechanical (or other) nature. Using these equations allows one to avoid, at least to the first approximation, the necessity for cumbersome derivations. This results in a justified prediction of the character of the stable, stationary motion of the system without constructing the original equations of motion. The author hopes that the results obtained, as well as the proposed style, provide a degree of interest in terms of both science and teaching.

It is assumed that the reader has a basic knowledge of analytical mechanics, rigid body dynamics, theory of nonlinear oscillations, as well as quantum mechanics and electrical engineering. The book is written primarily for researchers in Mechanics and Physics with a university or equivalent education, mathematicians specialising in the field of the theory of ordinary differential equations, as well as graduate and post-graduate students. Nevertheless, the author has tried to make the book understandable for a broad range of readers. For this reason, detailed mathematically rigorous proofs and substantiations are omitted and the special terminology related to these proofs is not included.

The present book is the result of many years of activity by the author in the corresponding field of research. It brings together and, to a great extent, completes his previously published works in various journals and proceedings. The works by I.I. Blekhman published in 1953-1960 years gave an initial impetus to the present research. The following colleagues and coworkers of the author: I.Kh. Akhmetshin, P.S. Goldman, V.V. Guzev, A.A. Danilin, F.F.Fazullin, K.Sh. Khodzhaev and S.D. Shatalov contributed to the research at different times. In particular, the results of Sections 1.5, 2.4, 4.6, 5.4 and 5.6 were obtained by the author together with P.S. Goldman. Many useful suggestions about the manuscript were made by A.K. Belyaev and S. McWilliam.

Contents

1

Locally integrable dynamical systems

1.1 Concept of local integrability

Nowadays, a number of analytical methods based on the theory of small parameters are widely used in the theory of nonlinear oscillation. It is worth mentioning the local method of analytical continuation with respect to the parameters suggested by Lyapunov and Poincaré [61] and [84], the method of averaging [19] and [98], the methods based on Lie transformations [103], [38] and others. Consistent applications of these methods allows one to reveal a series of general laws for the behaviour of mechanical systems, as well as to solve a number of theoretical and applied problems of technical and celestial mechanics. Application of the method of small parameters has gained importance in solving nonlinear problems of physics, electronics, biology and chemistry, though these fields of knowledge have been based on linear models until recent times.

One of the principal problems of the theory of nonlinear oscillations is that of constructing periodic solutions to the system of differential equations

$$\dot{x} = X\left(x, t, \varepsilon\right) \tag{1.1}$$

in the form of a power series in a sufficiently small positive parameter ε. In eq. (1.1) x denotes an $n \times 1$ vector-row and the vector-function X is assumed to be analytic with respect to parameter ε and the components of x in the considered region of phase space of the system. In addition to

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this, X is assumed to be periodic with respect to the time t, which appears implicitly in the latter equation.

It can be stated that construction of successive approximations to the exact periodic solution of system (1.1) can be performed successfully by means of a finite number of operations provided that:

1) the generating system obtained from (1.1) for $\varepsilon = 0$ admits construction of a particular $T-$periodic solution $x_0(t)$;

2) the variational system of equations about a given periodic solution of the generating system (i.e. for $\varepsilon = 0$)

$$\dot{y} = \left(\frac{\partial X}{\partial x}\right) y \quad (y = \delta x) \tag{1.2}$$

has a general integral. Here $\frac{\partial X}{\partial x}$ is an $n \times n$ Jacobi matrix, and throughout this book parentheses imply that the value in the parentheses is calculated on the generating solution.

Let us show that under these conditions the original system (1.1) can be reduced to the so-called standard form in some local vicinity of the generating trajectory. This is important from the perspective of the further application of the averaging method. To this end, we first make the substitution

$$x = x_0 + \varepsilon\left(x_1 + x_*\right), \tag{1.3}$$

where x_1 is a $T-$ periodic function of t and is determined from the following linear inhomogeneous system with periodic coefficients

$$\dot{x}_1 = \left(\frac{\partial X}{\partial x}\right) x_1 + \left(\frac{\partial X}{\partial \varepsilon}\right). \tag{1.4}$$

The homogeneous part of system (1.4) is coincident with (1.2). For this reason, the very feasibility of solving system (1.4) by quadratures is beyond question.

In order to determine the conditions for the existence of solutions of eq. (1.4) which are periodic with respect to t, it is necessary to introduce the system of equations which is the conjugate to (1.2)

$$\dot{z} = -z\left(\frac{\partial X}{\partial x}\right), \tag{1.5}$$

where z is a $1 \times n$ vector-row. It is easy to see, cf. [61], that the solutions of systems (1.2) and (1.5) satisfy the relationship of the form $zy = \text{const}$. The characteristic exponents of systems (1.2) and (1.5) differ only in their sign, namely, if system (1.2) has a $l-$fold exponent α corresponding to m sets of solutions, then system (1.5) has a $l-$fold exponent $(-\alpha)$ corresponding to m sets of solutions of the same type. Let us assume that system (1.2) has a multiple zero exponent corresponding to m sets of solutions. Then system (1.5) possesses m periodic solutions $z_1, \dots, z_m$ with period T. We differentiate the scalar product $z_i x_1$, take into account eqs. (1.4) and (1.5), and average the obtained expression over the period T

$$\int_0^T z_i \left(\frac{\partial X}{\partial \varepsilon} \right) dt = 0 \quad (i = 1, \dots, m) . \tag{1.6}$$

As shown in the theory of small parameters [61] the fulfillment of these relationships is necessary not only to ensure the periodicity of x_1 but also the existence of the periodic solution of the original system in the local vicinity of the point $\varepsilon = 0$.

It is essential for the forthcoming analysis that the periodic solutions of the perturbed system is unstable if system (1.2) or (1.5) has zero exponents with the elementary divisor of the power higher than two and no special degeneracy is assumed. For this reason, in what follows we assume that system (1.2) or (1.5) has m sets of the solutions of the following form

$$\begin{aligned} z &= z_i, \quad z = z_i t + \theta_i \quad (i = 1, \dots, m_1) , \\ z &= z_i, \quad (i = m_1 + 1, \dots, m) , \end{aligned} \tag{1.7}$$

where $\theta_i, \dots, \theta_m$ are also T−periodic with respect to t. In other words, it is assumed that the zero characteristic exponents have either simple or square elementary divisors. As for the other characteristic exponents $(n > m_1 + m)$, the same assumption implies that they must have non-positive real parts.

Let us write down the system obtained by substituting eq. (1.3) into eq. (1.1)

$$\dot{x}_* = \left(\frac{\partial X}{\partial x} \right) x_* + \varepsilon X_* \left(x_*, t \right) + \varepsilon^2 \dots , \tag{1.8}$$

where X_* is a well-defined function of its arguments which is obtained by means of the quadratic terms after expanding the right hand side of eq. (1.1) as a series in terms of ε. This function is T−periodic with respect to time t.

Since the structure of the characteristic determinant of the homogeneous part of system (1.8) has been determined, it can be expressed in the following form

$$\begin{aligned} \dot{p}_1 &= \varepsilon P_1 + \varepsilon^2 \dots , \quad \dot{q}_1 = p_1 + \varepsilon Q_1 + \varepsilon^2 \dots , \\ \dot{p}_2 &= \varepsilon P_2 + \varepsilon^2 \dots , \quad \dot{r} = A r + \varepsilon R + \varepsilon^2 \dots \end{aligned} \tag{1.9}$$

by means of a non-singular linear replacement of the coordinates with the periodic coefficients $x_* \to p_1, q_1, p_2, r$. The components of the $m_1 \times 1$ vectors

p_1 and q_1, $(m - m_1) \times 1$ vector p_2 and $(n - m - m_1) \times 1$ vector r are the new variables of the problem. The characteristic exponents of the matrix A have non-positive real parts and are not equal to zero whereas the vector-functions P_1, Q_1, P_2, R are $T-$ periodic with respect to explicit time t and are analytic with respect to the components of the vectors p_1, q_1, p_2, r. Let us stress that, by virtue of the integrability of the systems (1.2) and (1.5), the above substitution is determined with the help of a finite number of operations and its matrix is formed by means of the expressions for the periodic solutions of the conjugate system.

By means of the new substitution we can express system (1.9) in the standard form

$$\begin{aligned} \dot{p}_1' &= \sqrt{\varepsilon} P_1 + \varepsilon \dots, & \dot{q}_1 &= \sqrt{\varepsilon} p_1' + \varepsilon \dots, \\ \dot{p}_2 &= \varepsilon P_2 + \varepsilon^{3/2} \dots, & \dot{r} &= Ar + \varepsilon R + \varepsilon^{3/2} \dots. \end{aligned} \tag{1.10}$$

It is essential that, under the above two conditions, the functions on the right hand side of the system can be determined with any accuracy with respect to ε by means of a finite number of operations. Then the generalised averaging method [98] can be applied directly to system (1.10). As a result, the averaged equations can be obtained with any accuracy, and their solutions will approximate the solutions of the exact equations (1.1) or (1.10) in a finite time interval $O\left(\varepsilon^{-1/2}\right)$. The averaged equations are more convenient than the original ones since they do not depend explicitly on the argument t.

By analogy, one can prove the feasibility of constructing any approximation with the help of any other method of small parameters, say, the Poincaré-Lyapunov method or Hori's method by a finite number of operations. Under certain restrictions of the principal character, a similar statement is also valid for the problems of determining the quasi-periodic solutions for systems with small parameters. This gives grounds to classify the system under consideration as a special class. Clearly, the characteristic property of these systems is that their integration for $\varepsilon = 0$ can be reduced to quadratures in the local (linear) neighbourhood of the generating solution. From this perspective, these systems can be naturally referred to as being locally integrable. It is the local integrability of the generating system that is the necessary and sufficient condition for the applicability of the analytical methods of small parameters to the analysis of the perturbed system.

It is expedient to start by considering various classes of dynamical system with those problems admitting isolated generating solutions which are strongly stable in the sense of Lyapunov [61]. In other words, arbitrary small perturbations of the initial conditions of these solutions completely decay away either in a finite time or asymptotically. The theorem [61], as applied to this case, guarantees the existence of a strongly stable unique periodic solution of the perturbed problem, this solution being coincident with the generating one as $\varepsilon \to 0$.

The system can have an isolated periodic generating solution only if it is non-autonomous (i.e. it depends on time explicitly) when $\varepsilon = 0$. This solution can be obtained efficiently for the following systems which are locally integrable:

1) the linear inhomogeneous system

$$\dot{x} = Ax + f(t), \tag{1.11}$$

where f denotes a periodic vector-function of time and the eigenvalues of the $n \times n$ matrix A with the constant coefficients have finite negative real parts;

2) non-autonomous piecewise continuous systems which are integrable by quadratures within the regions of continuity.

More often than not, in the latter case one has to deal with the so-called piecewise linear systems which are described by equations of the form of (1.11) within the regions of continuity. It is clear that the above examples do not exhaust the entire class of the problem which are locally integrable and admit the construction of an isolated generating solution. However it is hardly possible to formulate the necessary and sufficient conditions for the complete class. We restrict ourselves to a characteristic example which demonstrates that the possibility of efficiently constructing a stable isolated solution of the generating systems does not imply that the system is locally integrable.

The example considers the forced oscillation of the oscillator governed by the equation

$$m\ddot{x} + \beta\dot{x} + cx = P\sin\omega t, \tag{1.12}$$

where m, P and ω are positive constants whilst the damping factor β and the rigidity c vary and depend upon the "oscillator energy"

$$e = \frac{m}{2}\left(\dot{x}^2 + \omega^2 x^2\right). \tag{1.13}$$

It is easy to see that eq. (1.12) has an isolated harmonic solution

$$x = a\cos(\omega t - \vartheta), \tag{1.14}$$

whose amplitude a and phase ϑ are given by

$$\begin{gathered} a = \frac{1}{\omega}\sqrt{\frac{2e}{m}}, \quad \tan\vartheta = \frac{\beta\omega}{c - m\omega^2}, \\ f(e) \equiv \frac{2e}{m\omega^2}\left[\left(c - m\omega^2\right)^2 + \beta^2\omega^2\right] = P^2. \end{gathered} \tag{1.15}$$

If β and c are positive and increase monotonically as e increases, then this solution is unique and asymptotically stable. At $f' = 0$ the corresponding variational equation

$$m\ddot{y} + \beta\dot{y} + cy + \left(\beta'\dot{x} + c'x\right)m\left(\dot{x}\dot{y} + \omega^2 xy\right) = 0 \tag{1.16}$$

admits a particular periodic solution $y = \cos(\omega t - \vartheta - \gamma)$ where γ is constant and a prime denotes differentiation with respect to e. Hence, the appearance of the double root of the latter equation in (1.15) means that solution (1.14) reaches the boundary of its region. However, in general, the linear homogeneous equation (1.16) of Hill's type is not integrable by quadratures. Therefore, the equation which takes the form of eq. (1.12) at $\varepsilon = 0$ does not admit construction of a periodic solution using the analytical methods of small parameters.

1.2 Linear heterogeneous systems

For the sake of generality we assume that the homogeneous system

$$\dot{y} = Ay \tag{1.17}$$

corresponding to the inhomogeneous system (1.11) has $2r$ zero characteristic exponents with squared elementary divisors and s zero exponents with simple elementary divisors $(2r + s \leq n)$. The remaining characteristic exponents are assumed to have negative real parts. Then, neglecting the terms which decrease exponentially as $t \to \infty$ we cast the general integral of system (1.17) in the form

$$y = Y_1C_1 + (Y_1t + \Theta)\, C_2 + Y_2C_3. \tag{1.18}$$

Here the $n \times r$ matrices Y_1 and Θ as well as the $n \times s$ matrix Y_2 with mutually independent rows are $T-$periodic with respect to time t, whereas the coefficients of the $r \times 1$ vectors C_1 and C_2 as well as the $s \times 1$ vectors C_3 are the integration constants.

It is important for further study that the components of the $n \times r$ matrix Θ are determined from the equation

$$\dot{\Theta} = A\Theta - Y_1. \tag{1.19}$$

The homogeneous linear system

$$\dot{z} = -zA \tag{1.20}$$

aimed at determining the components of the conjugate $1 \times n$ vector z has, by virtue of the above, the following general integral

$$z = D_1Z_1 + D_2Z_2 + \dots, \tag{1.21}$$

where the components of the $r \times n$ matrix Z_1 and the $s \times n$ matrix Z_2 are also $T-$periodic whereas the $1 \times r$ vector D_1 and the $1 \times s$ vector D_2 are constant. The components increasing linearly and decreasing exponentially in time are omitted in expression (1.21). Since the rectangular matrix Y_1

corresponds to the zeroth roots with the squared elementary divisors, the following matrix identities

$$Z_i Y_1 = 0 \quad (i = 1, 2) \tag{1.22}$$

always hold true. With regard to the components of the matrices Θ and Y_2, their choice is subject to the following conditions of orthogonality and normalisation

$$Z_2 Y_2 = E_s, \quad Z_1 \Theta = E_r, \tag{1.23}$$

where E_s denotes the $s \times s$ unit matrix. While proving equalities (1.22) and (1.23) and in what follows it is taken into account that the solutions of the original (1.17) and the conjugate (1.22) systems satisfy the relationship of the form $zy =$ const. The equalities of this type can be considered as the first integrals of system (1.17) provided that the independent particular solutions of the conjugate system are taken.

We next introduce into consideration the so-called impulse $T-$periodic function

$$\Phi(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \tag{1.24}$$

where $\delta(t)$ is the Dirac delta-function. According to the definition of the latter function the following identity

$$f(t) = \int_0^T \Phi(\xi) f(\tau) \, d\tau \quad (\xi = t - \tau) \tag{1.25}$$

holds for any $T-$periodic function f for any values of t. Let us seek a particular $T-$periodic solution of system (1.11) in the form of the convolution integral [89]

$$x(t) = \int_0^T K(t, \tau) f(\tau) \, d\tau, \tag{1.26}$$

where the $n \times n$ matrix K is referred to as the impulse-frequency characteristic of the original system. It is essential that eq. (1.26) is valid for any values of t. Inserting expressions (1.25) and (1.26) into eq. (1.11) leads immediately to the inhomogeneous matrix equation for the components of K

$$\dot{K} = AK + \Phi(\xi) E_n, \tag{1.27}$$

where τ should be considered as being a constant value. The term on the right hand side of eq. (1.27) is evidently not orthogonal to the $T-$periodic and linearly increasing particular solutions of eq. (1.17). Hence in the general case under consideration the components of the impulse-frequency matrix characteristic are not periodic and must contain components with the secular terms t and t^2. Let us show that, despite this general property, function $x(t)$, see eq. (1.26), can be T−periodic with respect to t provided that K is properly chosen. Let us look for the corresponding solution of system (1.27) in the form

$$K = U\xi^2 + V\xi + W, \tag{1.28}$$

where components of the $n \times n$ matrices U, V and W are T−periodic with respect to $\xi = t - \tau$. Substituting eq. (1.28) into eq. (1.27) and equating the coefficients for all powers of ξ yields

$$\begin{aligned} \dot{U} &= AU, \quad \dot{V} = AV - 2U, \\ \dot{W} &= AW - V + \Phi(\xi) E_n. \end{aligned} \tag{1.29}$$

While deriving this result we took into account that τ is a parameter in eq. (1.27). Thus differentiating with respect to ξ is equivalent to differentiating with respect to t. Due to eqs. (1.18) and (1.19) the general T−periodic solutions of the two first equations in (1.29) have the form

$$U = Y_1 R_1, \quad V = 2\Theta R_1 + Y_1 R_2 + Y_2 R_3, \tag{1.30}$$

where the components of the $r \times n$ matrices R_1 and R_2 as well as the $s \times n$ matrix R_3 do not depend upon t.

Let us write down the conditions for the existence of a T−periodic solution of the third equation in (1.29). The corresponding equalities are determined in terms of the known T−periodic solutions of the conjugate system (1.20) and, by virtue of eq. (1.22), have the form

$$\int_0^T Z_i \left[\Phi(\xi) E_n - V\right] dt = 0 \quad (i = 1, 2). \tag{1.31}$$

Taking into account the expressions for V and relationships (1.22) and (1.23) we can cast these matrix equalities in the form

$$\frac{1}{T} Z_1(\tau) = 2R_1 + PR_3, \quad \frac{1}{T} Z_2(\tau) = 2QR_1 + R_3. \tag{1.32}$$

Here the rectangular $r \times s$ matrix P and the $s \times n$ matrix Q with constant coefficients are as follows

$$P = Z_1(0) Y_2(0), \quad Q = Z_2(0) \Theta(0). \tag{1.33}$$

It follows from eq. (1.32) that R_1 and R_3 are functions of parameter τ and are given respectively by

$$R_1 = \frac{1}{2T}(E_r - PQ)^{-1}[Z_1(\tau) - PZ_2(\tau)],$$
$$R_3 = \frac{1}{2T}(E_s - QP)^{-1}[Z_2(\tau) - QZ_1(\tau)]. \tag{1.34}$$

The components of the matrix R_2 are still undetermined, see eq. (1.30). Let us try to determine them from the condition (1.26) of periodicity of the considered solution, i.e. $x(t) = x(t+T)$. Due to eq. (1.28) this equality takes the form

$$\int_0^T [U(2\xi + T) + V] f(\tau)\, d\tau = 0. \tag{1.35}$$

Substituting expression (1.30) into this equation and allowing for eqs. (1.34), (1.22) and (1.23), we arrive at the following result

$$Y_1(t)\int_0^T (2\tau R_1 - R_2) f(\tau)\, d\tau = 0. \tag{1.36}$$

These equalities are satisfied for any values of t only if

$$\int_0^T (2\tau R_1 - R_2) f(\tau)\, d\tau = 0. \tag{1.37}$$

The impulse-frequency characteristic by its nature (and in turn the $r \times n$ matrix R_2) does not depend upon the external excitation $f(\tau)$. Thus, it follows from eq. (2.21) that

$$R_2 = 2\tau R_1. \tag{1.38}$$

Let us note that, by virtue of eq. (1.38), matrix R_2 is not periodic with respect to τ in contrast to R_1 and R_3. Therefore only the "secular" part of the impulse-frequency characteristic of the original system is determined. As for the purely periodic component of W, its existence is proved and its expression is obtained after integration of the third equation in (1.29). The final expression for the impulse-frequency characteristic has the form

$$K = \frac{(t-\tau)^2}{2T} Y_1(t)(E_r - PQ)^{-1}[Z_1(\tau) - PZ_2(\tau)] +$$
$$\frac{(t-\tau)}{T}\left\{2[\Theta(t) + \tau Y_1(t)](E_r - PQ)^{-1}[Z_1(\tau) - PZ_2(\tau)] +\right.$$
$$\left. Y_2(t) Z_2(\tau)\right\} + W(t,\tau). \tag{1.39}$$

If matrix A has only a multiple characteristic exponent with the simple elementary divisors ($r = 0$, $Y_1 = \Theta = 0$, $Z_1 = 0$) then, instead of eq. (1.29), we arrive at the expression which is quadratic in time

$$K = \frac{(t-\tau)}{T} Y_2(t)\, Z_2(\tau) + W(t, \tau). \tag{1.40}$$

In the case of a non-singular matrix A ($r = s = 0$) the matrix impulse-frequency characteristic of the system is purely periodic (i.e. $K = W$) and hence can be expanded using Fourier series

$$K = \sum_{i=-\infty}^{\infty} K_i \exp\left(\sqrt{-1} i\omega\xi\right) \quad \left(\xi = t - \tau, \omega = \frac{2\pi}{T}\right). \tag{1.41}$$

Inserting this series into eq. (1.27) and taking into account that

$$\Phi(\xi) = \frac{1}{T} \sum_{i=-\infty}^{\infty} \exp\left(\sqrt{-1} i\omega\xi\right) \tag{1.42}$$

leads to the expression for the matrix Fourier coefficients

$$K_i = \frac{1}{T}\left(\sqrt{-1} i\omega E_n - A\right)^{-1}. \tag{1.43}$$

Thus, in the non-singular case, the matrix impulse-frequency characteristic of the system is a function of the difference between the main arguments, i.e. $\xi = t - \tau$. In contrast to this, in the critical case studied above the expression for K is a function of both arguments t and τ, cf. eqs. (1.39) and (1.40).

To conclude this section, we notice that the case in which matrix A is a T−periodic function of t and the general integral of system (1.17) is known reduces to the above case with the help of analytical methods, see [96].

1.3 Piecewise-continuous systems

Let us consider an infinite ordered sequence of dynamic systems

$$\dot{x}_i = X_i(x_i, t), \quad i = \dots, -1, 0, 1, \dots, \tag{1.44}$$

where x_i is a $k_i \times 1$ vector. Let us assume that if x_i belongs to a certain region G_i of its phase space, then the vector-function X_i is regular with respect to all its arguments and the integral trajectories due to eq. (1.44) intersect the hypersurface

$$g_{i+1}(x_i, t) = 0 \tag{1.45}$$

first at the time instant $t = t_{i+1}$. We presume a rigorous correspondence between the dynamical states of the $i-th$ and $(i+1)-th$ systems at this time instant

$$x_{i+1} = \Phi_{i+1}(x_i, t), \tag{1.46}$$

where $x_{i+1} \in G_{i+1}$. The scalar function and the vector-function are regular in G_i with respect to all their arguments.

Finally, we assume that there exists a positive natural number n ensuring that the following equalities

$$\begin{aligned} k_i = k_{i+n}, \quad X_i(x_i, t) = X_{i+n}(x_i, t+T), \\ g_{i+1}(x_i, t) = g_{i+n+1}(x_i, t+T), \\ \Phi_{i+1}(x_i, t) = \Phi_{i+n+1}(x_i, t+T) \end{aligned} \tag{1.47}$$

hold for some positive constant T.

The sequence of solutions of the continuous systems (1.44) subject to conditions (1.45) and (1.46) enables us to judge the qualitative motion of the corresponding piecewise-continuous system of the varying structure with n essentially different switches. This statement assumes that the motions analysed are characterised by a strictly prescribed order of transition through the discontinuity surfaces and are accompanied by a finite number of switches within finite time intervals. This allows us to exclude from consideration the impact processes of finite duration having an infinite number of collisions [75]. Vector x_i, determining the position of the piecewise-continuous system under the motion studied, has the number of components which depends on the number i of the continuity interval and is equal to

$$x = x_i, \quad t_i < t < t_{i+1}. \tag{1.48}$$

The question of correspondence of components of the vectors x_i and x_{i+1} is easily solved by means of physical reasoning.

The advantage of the present approach over the traditional one [77] is that, in contrast to vector x, vector x_i experiences no jumps at time instants t_i and t_{i+1} and, in general, is continuous for any real t.

Let k_0 be the smallest of the numbers $k_0, k_1, \ldots, k_{n-1}$, that is $k_0 \leq k_i$. Then the statement of the boundary condition

$$x_0\big|_{t=t_*} = a \quad (a \in G_0), \tag{1.49}$$

where time instant t_* may occur beyond the interval (t_0, t_1) allows one to determine a unique piecewise-continuous solution x for any real t. Prescribing a certain vector x_j at the initial instant t_* such that $k_j > k_0$ does allow one to continue x in the direction of decreasing t (eq. (1.46) can not be resolved for x_i) beyond $t_l < t_j$ for which $k_{l-1} < k_l$ at the first time.

The family of solutions of eq. (1.44) extended in both directions of t and satisfying the initial conditions (1.49) include all $T-$periodic solutions for which

$$x_i(t) = x_{i+n}(t+T) \tag{1.50}$$

due to eq. (1.48). Let us notice that all remaining solutions transform into the family of solutions extended in both directions within a finite time interval as t increases. If any of the dynamical systems from the sequence (1.44) is integrable by quadratures, then it proves possible to find, by means of the switching conditions (1.45) and (1.46), the closed form relationships relating the values of x_i and x_{i+n} through n switches. These relationships must cover only the family of the solutions continued in both directions, and thus they ascribe a point mapping of k_0-dimensional space into itself. Imposing the periodicity conditions ($x_i = x_{i+n}$) on these relationships we reduce the problem of finding periodic solutions to the analysis of closed relationships. In other words, the problem of determining the $T-$periodic solution is reduced to the search of the fixed point of the above-mentioned transformation [77].

In what follows, we use the theory of generalised functions [31] and write down the equations of motion for successive dynamical systems in the form

$$\dot{x}_i = F_i(x_i, x_{i-1}, t), \quad F_i = X_i\sigma(g_i) + \Phi_i\dot{\sigma}(g_i). \tag{1.51}$$

Here $\sigma(g_i)$ denotes the unit step function due to the formula

$$\begin{aligned} \sigma(g_i) &= \begin{cases} 0, & g_i < 0 \\ 1, & g_i > 0 \end{cases} \\ \dot{\sigma}(g_i) &= \dot{g}_i\delta(g_i) = \delta(t-t_i). \end{aligned} \tag{1.52}$$

Without loss of generality we assume that $g_i(x_{i-1}, t) < 0$ if $t < t_i$ and $g_i(x_{i-1}, t) > 0$ if $t > t_i$. The solution of system (1.51) should be obtained under the assumption that $x_{i-1} = 0$ for $t < t_i$. Let us assume now that system (1.51) admits a solution which is $T-$periodic in the sense of eq. (1.50). It is required here to obtain the variational system for the $T-$periodic solutions and to study its property. In accordance with eq. (3.8) we have

$$\begin{aligned} \dot{y}_i = \left(\frac{\partial F_i}{\partial x_i}\right) y_i + \left(\frac{\partial F_i}{\partial x_{i-1}}\right) y_{i-1}, \quad \frac{\partial F_i}{\partial x_i} = \frac{\partial X_i}{\partial x_i}\sigma(g_i), \\ \frac{\partial F_i}{\partial x_{i-1}} = X_i\delta(g_i)\frac{\partial g_i}{\partial x_{i-1}} + \frac{\partial \Phi_i}{\partial x_{i-1}}\dot{\sigma}(g_i) + \Phi_i\frac{d}{dt}\left[\delta(g_i)\frac{\partial g_i}{\partial x_{i-1}}\right]. \end{aligned} \tag{1.53}$$

Here and throughout the present book, the expressions in the parentheses are calculated on the generating solutions. For further analysis it is expedient to prove the following statement: given the first-order equations of the following type

$$\dot{x} = X + U\frac{d}{dt}[V\delta(W)], \tag{1.54}$$

where X is a generalised function whilst U, V, W denote functions of time which are continuous along with their first derivatives, then $U\frac{d}{dt}\left[V\delta\left(W\right)\right]$ can be replaced by $-\dot{U}V\delta\left(W\right)$ as it causes the same jump in the variable x. Indeed, let $W=0$, $\dot{W}\neq 0$ at $t=t'$. Then integrating eq. (1.54) within the interval $(t'-0, t'+0)$ yields

$$
\begin{aligned}
x|_{t'-0}^{t'+0} &= \int\limits_{t'-0}^{t'+0} X dt + \int\limits_{t'-0}^{t'+0} U\frac{d}{dt}\left[V\delta\left(W\right)\right] dt \\
&= \int\limits_{t'-0}^{t'+0} X dt + UV\delta\left(W\right)|_{t'-0}^{t'+0} - \int\limits_{t'-0}^{t'+0} \dot{U}V\delta\left(W\right) dt. \qquad (1.55)
\end{aligned}
$$

The next to last term in this expression is equal to zero which proves the statement.

Let us notice that quantities $x_{i-1}, g_i, \frac{\partial g_i}{\partial x_{i-1}}, \Phi_i$ and their first derivatives with respect to time do not experience jumps at instant t_i when $g_i=0$. Hence the rule formulated above can be directly applied to system (1.53). As a result, taking into account eq. (1.52) we arrive at the following infinite sequence of equations

$$
\begin{aligned}
&\dot{y}_i = A_i y_i \sigma\left(t-t_i\right) + B_i y_{i-1}\delta\left(t-t_i\right), \quad i=\ldots,-1,0,1,\ldots, \\
&A_i = \left(\frac{\partial X_i}{\partial x_i}\right), \quad B_i = \left(\frac{\partial \Phi_i}{\partial x_{i-1}}\right) + \left(X_i - \dot{\Phi}_i\right)\left(\frac{\partial g_i}{\partial x_{i-1}}\dot{g}_i^{-1}\right). \qquad (1.56)
\end{aligned}
$$

System (1.56) can be obtained from the original system (1.44) by direct differentiation of the latter with respect to an arbitrary integration constant which does appear explicitly in this system. For this reason, equations (1.56) comprise the variational system for the original system (1.44) in the vicinity of the $T-$periodic solution under consideration. The solutions of the piecewise-continuous system corresponding to (1.56) satisfy the superposition principle and thus this system is linear in contrast to the piecewise-linear systems.

Generally speaking the correct solution of (1.56) can be extended only in the direction of increasing argument t. At the same time, the solution which exists for any real time instant t and is thus determined by the initial condition

$$y_0|_{t=t_*} = \beta \qquad (1.57)$$

can be cast in the form

$$y_i = U_i\left(t, t_*\right)\beta. \qquad (1.58)$$

Here U_i denotes the $k_i \times k_0$ matrix solution of eq. (1.56) satisfying the initial condition

$$U_0(t_*, t_*) = E_{k_0}, \tag{1.59}$$

where E_{k_0} is the $k_0 \times k_0$ unity matrix.

Since the original solution is $T-$periodic in the sense of eq. (1.50) then the matrix coefficients A_i and B_i are $T-$periodic in the same sense. This means that the $k_i \times k_0$ matrix $U_n(t+T, t_*)$ satisfies system (1.56) and moreover it belongs to the family (1.58). Hence we can write

$$U_{i+n}(t+T, t_*) = U_i(t, t_*)\, U_n(t_* + T, t_*). \tag{1.60}$$

Just as in the theory of continuous linear equations with periodic coefficients, relationship (1.60) results in the characteristic equation

$$\left| U_n(t_* + T, t_*) - e^{\lambda T} E_{k0} \right| = 0, \tag{1.61}$$

where λ denotes the characteristic exponent. For any root λ of eq. (1.61) there is a particular solution of eq. (1.56) belonging to the family (1.58)

$$y_i = e^{\lambda t} v_i(t), \tag{1.62}$$

where $v_i(t)$ is a $T-$periodic function of time in the sense of (1.50). If all of the k_0 characteristic exponents of system (1.56) are different or have simple elementary divisors, then there exist k_0 independent particular solutions of the type (1.62). Superposition of these solutions yields the general form of family (1.58) of solutions of eq. (1.56) continued in both directions of t.

Let us introduce into consideration an ordered sequence of systems

$$\dot{z}_i = -z_i A_i \left[1 - \sigma(t - t_{i+1})\right] - z_{i+1} B_{i+1} \delta(t - t_{i+1}). \tag{1.63}$$

Each equation in (1.63) serves to determine vectors z_i of dimension $1 \times k_i$, however, unlike eqs. (1.51) and (1.56) it is necessary to take $z_i = 0$ for $t > t_{i+1}$. Generally speaking, an arbitrary solution of the linear piecewise-continuous system can be continued in the direction of decreasing t. There exists a certain correspondence between the solutions of eqs. (1.63) and (1.56) which can be continued into both directions of t. Indeed,

$$\frac{d}{dt} \sum_{i=-\infty}^{\infty} z_i y_i = \sum_{i=-\infty}^{\infty} \{-z_i A_i y_i \left[1 - \sigma(t - t_{i+1})\right] - z_{i+1} B_{i+1} y_i \delta(t - t_{i+1}) + z_i A_i y_i \sigma(t - t_{i+1}) + z_i B_i y_{i-1} \delta(t - t_i)\} = 0. \tag{1.64}$$

Let us integrate this relationship over t between the limits t_* and t ($t_0 < t_* < t_1$). As $y_i = 0$ for $t < t_i$ and $z_i = 0$ for $t > t_{i+1}$ we have

$$z_i y_i = z_0 y_0 \big|_{t=t_*} \quad t_i < t < t_{i+1}. \tag{1.65}$$

Relationship (1.65) allows us to speak about system (1.63) as being the conjugate to system (1.56).

Inserting the particular solution of eq. (1.56) in the form (1.62) into eq. (1.65) we obtain k_0 mutually independent first integrals. Resolving these integrals results in construction of a family of solutions of eq. (1.63) continued into both directions of t. This family can be represented as superposition of the particular solutions

$$z_i = e^{-\lambda t} w_i (t), \tag{1.66}$$

where w_i is a $T-$periodic function of t. It means that for any exponent λ of system (1.56) there is the exponent $(-\lambda)$ of system (1.63). In particular the number of the zeroth characteristic exponents is coincident for these systems and thus the number of the periodic solutions also coincides. Determination of the roots $\rho = e^{\lambda T}$ of the determinant (1.61) is equivalent to determining the characteristic values of the immovable point of the above transformation of the k_0-dimensional space into itself.

When the set of characteristic exponents is completely determined the general integral of the variational system (more precisely, the complete family of solutions continued into both directions) can be obtained in closed form as a result of a finite number of operations.

By virtue of the above, one can state that the considered piecewise-continuous dynamical systems are locally integrable in the vicinity of the periodic solutions. In general, the global structure of their phase space is very complex, especially if one takes into account that the order and the character of switches may vary for various motions. With this in view, the global integrability of such systems is not studied.

1.4 Homogeneous Lyapunov systems

If the generating system for problem (1.1) is autonomous, i.e. it does not depend on time explicitly, then the generating solution can not be isolated. Indeed, if this system has a particular $T-$periodic solution $x(\omega t)$, where $\omega = 2\pi/T$ denotes the circular frequency, then along with this solution there exists a one-parametric family $x(\varphi)$ of the same form, where $\varphi = \omega t + \alpha$ is the rotating phase and the initial phase $\alpha = \varphi|_{t=0}$ depends upon the initial conditions. For any α the variational system of equations (1.2) has a $T-$periodic solution $\dfrac{\partial x}{\partial \alpha} = \dfrac{\dot{x}}{\omega}$, and the "critical" zeroth characteristic exponent corresponds to this case. Let us assume that the other characteristic exponents of system (1.2) have negative real parts. Then the family of solutions $x(\varphi)$ under consideration are orbitally stable and the corresponding single closed trajectory in the phase space $(x_1, \dots, x_n)$ is isolated. It is essential that the existence of an orbitally stable generating solution ensures that the autonomous perturbed system possesses the family of solutions of the same type. However it is worth mentioning that the period of these solutions is not known in advance and differs from T in the value of order ε.

The situation changes drastically when the terms of order ε on the right hand side of system (1.1) are periodic with respect to explicit time t. Clearly, the period of perturbation must be equal to the period of the generating solution $x(\varphi)$ or differs from it by a factor which is not a very large natural number. This is ordinarily achieved by means of introducing so-called small detuning, i.e. a change in the generating system of values of order ε. The isolated asymptotically stable periodic solution of the perturbed motion, if it exists, can transformed into only one solution of the family $x(\varphi)$ at $\varepsilon = 0$. Hence, the problem of determining the generating value for the initial phase α becomes non-trivial. In addition to this, since system (1.2) has the critical zeroth characteristic exponent at $\varepsilon = 0$, the criteria of stability of the required solution is also non-trivial.

An orbitally stable generating solution, as well as the general integral of the generating variational system of equations, can be obtained, as before, by standard methods, provided that we consider the piecewise-continuous autonomous system to be integrable by quadratures within the intervals of continuity for $\varepsilon = 0$. The case in which the generating system is a Lyapunov system is more challenging. Let us recall that the systems of first order differential equations which, in the vicinity of the stable equilibrium, can be reduced by means of a non-singular change of variables to the following form

$$\begin{aligned} \dot{x} &= -\lambda y + X(x, y, x_1, \dots, x_n), \\ \dot{y} &= \lambda x + Y(x, y, x_1, \dots, x_n), \\ \dot{x}_i &= \sum_{j=1}^{n} a_{ij} x_j + X_i(x, y, x_1, \dots, x_n) \quad (i = 1, \dots, n) \end{aligned} \tag{1.67}$$

and additionally admit an analytical first integral

$$H = x^2 + y^2 + W + S = \text{const} = h \tag{1.68}$$

are referred to as Lyapunov systems. Here $X, Y, X_1, \dots, X_n$ are analytical functions of their arguments whose expansions in terms of power series begin with the terms of order not lower than second, W denotes a positive definite quadratic form of the variables $x_1, \dots, x_n$, and S is an analytical function of $x, y, x_1, \dots, x_n$ whose expansion in terms of power series begins with the terms of order not lower than third.

By Lyapunov's theorem [80] such systems have two-parametric family of periodic solutions

$$\begin{aligned} x &= x(\varphi, h), \ y = y(\varphi, h), \ x_i = x_i(\varphi, h) \quad i = 1, \dots, n \\ \varphi &= \omega(h) t + \alpha, \end{aligned} \tag{1.69}$$

depending upon the constants $h > 0$ and α and being analytic with respect to $\sqrt{h}$ in the vicinity of zero. The variational system of equations admit two particular solutions which are obtained if eq. (1.69) is differentiated with respect to α and h

$$\begin{aligned} &\frac{\partial x}{\partial \alpha}, \frac{\partial y}{\partial \alpha}, \frac{\partial x_i}{\partial \alpha}; \\ &\frac{\partial x}{\partial h} + \frac{\partial x}{\partial \alpha}\frac{d\omega}{dh}t, \ \frac{\partial y}{\partial h} + \frac{\partial y}{\partial \alpha}\frac{d\omega}{dh}t, \ \frac{\partial x_i}{\partial h} + \frac{\partial x_i}{\partial \alpha}\frac{d\omega}{dh}t. \end{aligned} \tag{1.70}$$

The periodic solution and the linearly increasing solution (1.70) form a pair [61]. Thus we can argue that the characteristic determinant of solution (1.69) has a double zeroth root with quadratic elementary divisor [103]. However it is not possible to obtain the general integral of the variational equations in the general case because of the complexity of the structure of the small vicinity of family (1.69).

Let us consider an example of a Lyapunov system which admits construction of a closed form solution of type (1.70) and is locally integrable in the vicinity of this solution. We can cast the equations of motion of this system in the form

$$m_i \ddot{x}_i + \frac{\partial \Pi}{\partial x_i} = 0 \quad (i = 1, \ldots, n). \tag{1.71}$$

Here m_i are constant positive masses and the potential energy $\Pi(x_1, \ldots, x_n)$ is a positive definite form of power $k = 4, 6, 8, \ldots$. The "normal" solution of this nonlinear system is sought in the form [88]

$$x_i = c_i x, \tag{1.72}$$

where $c_1, \ldots, c_n$ are constants. Inserting eq. (1.72) into eq. (1.71) yields

$$m_i c_i \ddot{x} + \frac{\partial \Pi}{\partial c_i} x^{k-1} = 0, \tag{1.73}$$

where here, and in what follows, $\Pi = \Pi(c_1, \ldots, c_n)$. All n equations (1.73) are identical to each other if the n constants satisfy the following $(n-1)$ relationships

$$m_1 c_1 \frac{\partial \Pi}{\partial c_i} = m_i c_i \frac{\partial \Pi}{\partial c_1}. \tag{1.74}$$

Additionally, we assume that the normalisation condition

$$\sum_{i=1}^{n} m_i c_i^2 = 1 \tag{1.75}$$

holds. Then, taking into account the identity

$$\sum_{i=1}^{n} c_i \frac{\partial \Pi}{\partial c_i} = k\Pi \tag{1.76}$$

we can write

$$\frac{1}{m_i c_i} \frac{\partial \Pi}{\partial c_i} = k\Pi \tag{1.77}$$

instead of eqs. (1.74) and (1.75). Hence, n equations (1.73) are satisfied if the unknown variable x is governed by the following second-order differential equation

$$\ddot{x} + k\Pi x^{k-1} = 0. \tag{1.78}$$

This equation has the following first integral

$$\frac{\dot{x}^2}{2} + \Pi x^k = h, \tag{1.79}$$

which, due to eq. (1.77), is coincident with the first integral of the original system (1.71) for the considered normal solution (1.72). On the other hand, the variables in relationship (1.79) are separated, and thus the definition of the general integral (1.78) is reduced to quadratures. Without formulating these quadratures we just notice that the general solution $x = x(\varphi, h)$ is periodic with respect to phase $\varphi = \omega t + \alpha$ for $\Pi > 0$ and an even k.

Let us vary the original system of equations (1.71) about the obtained two-parametric family

$$x_i = c_i x + \frac{y_i}{\sqrt{m_i}}. \tag{1.80}$$

The resulting variational system is then written as follows

$$\ddot{y}_i + x^{k-2} \sum_{j=1}^{n} a_{ij} y_j = 0, \tag{1.81}$$

where the constant coefficients

$$a_{ij} = \frac{1}{\sqrt{m_i m_j}} \frac{\partial^2 \Pi}{\partial c_i \partial c_j} \tag{1.82}$$

form a symmetric positive definite matrix. Systems of type (1.81) were first studied in [62].

Let us introduce into consideration the eigenvalues κ_s $(s = 1, \dots, n)$ and the corresponding orthogonal and normalised eigenvectors ξ_{js} of matrix a_{ij} by means of the following relationships

$$\sum_{j=1}^{n} a_{ij}\xi_{js} = \kappa_s \xi_{is}, \quad \sum_{i=1}^{n} \xi_{is}\xi_{ir} = \delta_{sr}. \tag{1.83}$$

The eigenvalues κ_s are real-valued and positive whilst the eigenvectors ξ_{is} $(s = 1, \ldots, n)$ are linearly independent, even in the case of the multiple eigenvalues. One of the solutions of system (1.81), say for $s = 1$, has the form

$$\xi_{i1} = \sqrt{m_i}c_i, \quad \kappa_1 = k\,(k-1)\,\Pi. \tag{1.84}$$

Let us perform a non-singular change of variables in system (1.81) by using the formula

$$y_i = \sum_{i=1}^{n} \xi_{is} z_s. \tag{1.85}$$

Taking into account eq. (1.83) we can split the variational system into n second-order differential equations

$$\ddot{z}_s + \kappa_s x^{k-2} z_s = 0. \tag{1.86}$$

The general integral of eq. (1.86) for $s = 1$ is given by, see eq. (1.70)

$$z_1 = A_1 \frac{\partial x}{\partial \alpha} + A_2 \left(\frac{\partial x}{\partial h} + \frac{\partial x}{\partial \alpha}\frac{d\omega}{dh} t \right), \tag{1.87}$$

where A_1 and A_2 are constant values. This can be proved easily by directly differentiating eq. (1.78) with respect to parameters of the generating solution α and h. To solve eq. (1.86) for $s = 2, \ldots, n$ we use the new non-dimensional argument

$$u = \frac{\Pi}{h} x^k. \tag{1.88}$$

We will additionally take into account that, by virtue of eq. (4.13) $0 \leq u \leq 1$ and $\dot{x}^2 = 2h\,(1-u)$. As a result we arrive at the following hypergeometric equation of Gauss [39]

$$u\,(1-u)\,\frac{d^2 z_s}{du^2} - \left(\frac{3k-2}{2k} u - \frac{k-1}{k} \right) \frac{dz_s}{du} + \frac{\kappa_s}{2k^2 \Pi} z_s = 0. \tag{1.89}$$

The general integral of this equation which is valid for the half-open intervals $0 \leq u < 1$ and $0 < u \leq 1$, respectively, is represented in the form, see [39]

$$z_s = A_1 F\left(\alpha_s, \beta_s, \frac{k-1}{k}, u\right) + A_2 x F\left(\alpha_s + \frac{1}{k}, \beta_s + \frac{1}{k}, \frac{k+1}{k}, u\right),$$

$$z_s = A^{(1)} F\left(\alpha_s, \beta_s, \frac{1}{2}, 1-u\right) + A^{(2)} x F\left(\alpha_s + \frac{1}{2}, \beta_s + \frac{1}{2}, \frac{3}{2}, u\right), \tag{1.90}$$

where $A_1, A_2, A^{(1)}, A^{(2)}$ are constants, values α_s and β_s are determined as follows

$$\alpha_s, \beta_s = \frac{k-2}{4k} \pm \frac{1}{4k}\left[(k-2)^2 + \frac{8\kappa_s}{\Pi}\right]^{1/2}, \tag{1.91}$$

and F denotes the hypergeometric series

$$F(\alpha_s, \beta_s, \gamma, u) = 1 + \sum_{l=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+l-1)\,\beta(\beta+1)\cdots(\beta+l-1)}{l!\gamma(\gamma+1)\cdots(\gamma+l-1)} u^l. \tag{1.92}$$

If the negative value β_s is an integer , i.e. $\beta_s = -m$ $(m = 1, 2, 3, \dots)$ or equivalently

$$\kappa_s = (2mk + k - 2)\, mk\Pi, \tag{1.93}$$

then series of the type (1.92) is a finite series and, due to eq. (1.90), there exist particular periodic solutions with half of the period. In accordance with the Floquet-Lyapunov theory this means that the obtained normal solution reaches the border of stability. Alternatively, the periodic solutions of eq. (1.86) with the principal period $2\pi/\omega$ exists if $\beta_s = -m - 1/k$, $\beta_s = -m - 1/2$ $(m = 1, 2, 3, \dots)$. These equalities can be rewritten in the following form

$$\kappa_s = k(mk+1)\Pi, \quad \kappa_s = k\left[((m+1)k - 1)\right](2m+1)\Pi \tag{1.94}$$

and describe the stability border of another type.

1.5 On local integrability of the equations of motion of Hess's gyro

For practical purposes, local integrability of the following equations

$$\dot{x} = X(x, x_1, \dots, x_n), \quad \dot{x}_i = X_i(x, x_1, \dots, x_n), \quad (i = 1, \dots, n) \tag{1.95}$$

is of importance. Here $X(0, x_1, \dots, x_n) \equiv 0$ and the n last equations in (1.95) admit periodic solutions $x_i = x_i(t)$, the system of linear equations

$$\dot{y}_{i0} = \sum_{j=1}^{n} \left(\frac{\partial X_i}{\partial x_j}\right) y_{j0} \tag{1.96}$$

being integrable by quadratures. In eq. (1.96) and in the following, parentheses imply that the corresponding values are calculated on the periodic solution of the original system (1.95) $x = 0, x_i = x_i(t)$. The variational system of equations about the given solution has the form

$$\dot{y} = \left(\frac{\partial X_i}{\partial x}\right) y, \quad \dot{y}_i = \sum_{j=1}^{n} \left(\frac{\partial X_i}{\partial x_j}\right) y_j + \left(\frac{\partial X_i}{\partial x}\right) y. \tag{1.97}$$

As the first equation in (1.97) can be considered separately, n mutually independent particular solutions of this system are characterised by the fact that $y = 0$ and the unknown variables coincide with the corresponding solutions of the homogeneous system (1.96). For the remaining $(n+1)-th$ solution

$$y = \exp \int_0^t \left(\frac{\partial X_i}{\partial x}\right) dt, \tag{1.98}$$

and the feasibility of determining $y_1, \dots, y_n$ in closed form is guaranteed through the integrability of eq. (1.96). This justifies the validity of the initial statement completely.

This is the situation which one faces while investigating the so-called fast gyroscope of Hess [7]. Motion of a heavy rigid body about a fixed point is known to be governed by the vectorial Euler-Poisson equations

$$\frac{d'L}{dt} + \omega \times L = \rho \times P, \quad \frac{d'\nu}{dt} + \omega \times \nu = 0. \tag{1.99}$$

Here ω is the angular velocity of the body, $L = J \cdot \omega$ denotes its kinetic moment, J is the tensor of inertia at the immovable point, P is the gravity force, ρ is the radius vector of the centre of mass, ν is the unit vector of the vertical axis, and a prime means that the time-derivative is taken in the moving frame which is rigidly bounded to the body. Hess's gyroscope is characterised by the fact that in the system of principal axes of inertia, the coordinates x_c, y_c, z_c of the centre of mass satisfy the conditions

$$y_c = 0, \quad x_c\sqrt{A(B-C)} + z_c\sqrt{C(A-B)} = 0, \tag{1.100}$$

where A, B, C are the principal moments of inertia $(A > B > C)$.

It is convenient to project the equations in (1.99) on axes of the special (not principal) moving system $O\eta_1\eta_2\eta_3$ [7]. Axis $O\eta_1$ passes through the fixed point O and the centre of mass C, whereas axes $O\eta_2$ and $O\eta_3$ are chosen in such a way that the components of the gyration tensor $l = J^{-1}$ have the form

$$l = \begin{vmatrix} a & b_1 & 0 \\ b_1 & a_1 & 0 \\ 0 & 0 & a_1 \end{vmatrix}. \tag{1.101}$$

In what follows, let us take $b_1 < 0$. As the main non-dimensional components of the problem we choose the direction cosines ν_1, ν_2, ν_3 of the vertical relative to the basis $O\eta_1\eta_2\eta_3$, as well as quantities x, r and α. The kinetic moment in terms of these variables is given by

$$L = \sqrt{-\frac{P\rho}{b_1}}\left[xe_1 + r\left(e_2\cos\alpha + e_3\sin\alpha\right)\right]. \tag{1.102}$$

Here e_i denotes the unit vector of axis $O\eta_i$ $(i = 1, 2, 3)$. Projecting eq. (1.99) on axes $O\eta_1\eta_2\eta_3$ yields

$$\begin{aligned}
&\frac{dx}{d\tau} = rx\sin\alpha, \quad \frac{dr}{d\tau} = \nu_3\cos\alpha - \left(\nu_2 + x^2\right)\sin\alpha,\\
&\frac{d\alpha}{d\tau} = 2\frac{d-c}{cd}x + r\cos\alpha - \frac{1}{r}\left[\left(\nu_2 + x^2\right)\cos\alpha + \nu_3\sin\alpha\right],\\
&\frac{d\nu_1}{d\tau} = \frac{2r}{c}\left(\nu_2\sin\alpha - \nu_3\cos\alpha\right) + x\nu_3,\\
&\frac{d\nu_2}{d\tau} = \frac{2}{d}x\nu_3 - r\left(\nu_3\cos\alpha + \frac{2}{c}\nu_1\sin\alpha\right),\\
&\frac{d\nu_3}{d\tau} = -\left(\nu_1 + \frac{2}{d}\nu_2\right)x + r\left(\nu_2\cos\alpha + \frac{2}{c}\nu_1\sin\alpha\right),
\end{aligned} \tag{1.103}$$

where $\tau = \sqrt{-P\rho b_1}t$ denotes non-dimensional time, $c = -2b_1/a_1$ and $d = -2b_1/a$. The first equation in (1.103) is satisfied at $x = 0$ (see eq. (1.95)). This equality characterises the so-called particular Hess's integral. In accordance with the above, it is necessary to prove that the closed system obtained from eq. (1.103) at $x = 0$ is locally integrable. One must also take into account the energy integral, the area integral and the geometric relationship between the direction cosines which are set as follows

$$\begin{aligned}
\frac{x^2}{d} + \frac{r^2}{c} - \nu_1 - rx\cos a &= k_1, \quad x\nu_1 + r\left(\nu_2\cos\alpha + \nu_3\sin\alpha\right) = k_2,\\
\nu_1^2 + \nu_2^2 + \nu_3^2 &= 1
\end{aligned} \tag{1.104}$$

in the system $O\eta_1\eta_2\eta_3$. Here k_1 and k_2 are two arbitrary non-dimensional constants. If $x = 0$ then the two integrals of eq. (1.104) are

$$r^2 = c\left(\nu_1 + k_1\right), \quad r\left(\nu_2\cos\alpha + \nu_3\sin\alpha\right) = k_2. \tag{1.105}$$

Then, the third and fourth equations in (1.103) are rewritten as follows

$$\frac{d\alpha}{d\tau} = r\cos\alpha - \frac{k_2}{r^2}, \quad c\frac{d\nu_1}{d\tau} = \pm\sqrt{c\left(\nu_1 + k_1\right)\left(1 + \nu_1^2\right) - k_2^2}. \tag{1.106}$$

The second equation in (1.106) allows us to determine ν_1 in terms of the elliptic functions. In the simplest case, i.e. for $k_2 = 0$, we have the periodic solution

$$\nu_1 = 1 - 2\,\mathrm{sn}^2 \left[\left(1 + \frac{1}{k_1} \right) \sqrt{\frac{k_1}{c}} \left(\tau + \mathrm{const} \right) \right], \tag{1.107}$$

where sn denotes the elliptic sine-function with modulus $k = \sqrt{2/(1+k_1)}$. The non-dimensional period of function (1.107) is equal to

$$T = \frac{4\sqrt{c}K(k)}{\sqrt{1+k_1}}. \tag{1.108}$$

Now we proceed to integration of the first equation in (1.106) accounting for eq. (1.107). This equation can not be integrated by quadratures. However in the case when the area integral vanishes, i.e. $k_2 = 0$, there is a particular solution $\alpha = \pi/2$ corresponding to rotation of Hess's gyro about the vertical axis.

In order to prove that the variational system is integrable by quadratures in the vicinity of the constructed periodic solution ($k_2 = \nu_3 = 0, \alpha = \pi/2$) it is sufficient to vary the first integrals (1.104)

$$\frac{2r}{c}\delta r = \delta\nu_1 + \delta k_1, \quad r\left(\delta\nu_3 - \nu_2\delta\alpha\right) = \delta k_2, \quad \nu_1\delta\nu_1 + \nu_2\delta\nu_2 = 0 \tag{1.109}$$

and equations in (1.106)

$$\frac{d\delta\nu_1}{d\tau} = F_1\delta\nu_1 + F_2\delta k_1, \quad \frac{d\delta\alpha}{d\tau} = -r\delta\alpha - \frac{\delta k_2}{r^2}. \tag{1.110}$$

Here variations of the integration constants k_1 and k_2 are also constant and additionally the notation

$$F_1 = \pm\frac{1}{2}\frac{1 - 2k_1\nu_1 - 3\nu_1^2}{\sqrt{c\left(\nu_1 + k_1\right)\left(1 - \nu_1^2\right)}}, \quad F_1 = \pm\frac{1}{2}\sqrt{\frac{1-\nu_1^2}{c\left(\nu_1 + k_1\right)}} \tag{1.111}$$

is introduced.

The general integrals of the first-order linear inhomogeneous equations (1.110) are found by means of conventional methods. Thereafter variations of the variables r, ν_1, ν_2 are obtained with the help of eq. (1.109).

2

Conservative dynamical systems

2.1 Introductory remarks

A number of locally integrable dynamical systems were considered in the previous chapter. It is evident that other classes of example of such systems will also exist. The problems for which the generating system is not only locally integrable in the vicinity of a certain periodic motion but are completely integrable by quadratures are much more important in non-linear mechanics and perturbation theory.

Currently it is not possible to formulate the necessary and sufficient conditions for integrability by quadratures in its general form. Nevertheless, the generally accepted viewpoint is that integrability by quadratures is a special "physical" property of the mechanical system which provides the system with special degenerating properties. The dynamical systems exhibit quasi-periodic (or conditionally-periodic) motions only if they: (i) are integrable by quadratures, (ii) are subject to some restrictions of a rather general character, and (iii) the initial conditions densely fill certain regions. To be rigorous, this statement has not been proved. Yet any counterexample is absent. Sufficiently general conditions for integrability by quadratures are formulated only for conservative problems. Correspondingly, the overwhelming majority of mechanical problems with available solutions by quadratures fall into the category of problems involving the motion of conservative systems. Moreover, integrable non-mechanical systems may be treated, as a rule, as being conservative. Finally, let us note that the general conditions for non-integrability by quadratures, as well as the theorems on the non-existence of the integrals (autonomous, single-valued and analytical), were formulated by Poincaré [84] for conservative systems which are close to integrable ones.

Hence, the importance of integrable conservative problems is determined to a great extent by the fact that it is these systems which are usually considered as a generating approximation of more difficult perturbed problems.

2.2 Conservative mechanical systems

Dynamical systems whose equations of motion admit autonomous single-valued first integral (the energy integral) are referred to as conservative systems. Alternatively, the total mechanical energy of the conservative system

$$H = K + \Pi, \tag{2.1}$$

with K and Π denoting respectively the kinetic and potential energy, remain unaltered over time

$$H = h \quad (h = \mathrm{const})\,. \tag{2.2}$$

Let the current configuration of a mechanical system be characterised by the generalised (Lagrange's) coordinates $q_1, \ldots, q_n$ which are mechanically meaningful. In general, this system is non-holonomic, and the equations for the non-holonomic (non-integrable) constraints

$$A_{ij}\dot{q}_j + A_i = 0 \quad (i = 1, \ldots, m,\ m < n) \tag{2.3}$$

are linear with respect to the generalised velocities $\dot{q}_1, \ldots, \dot{q}_n$. Here and in what follows the sign of the summation over the repeated subscripts from 1 to n is omitted, i.e. $A_{ij}\dot{q}_j = \sum\limits_{j=1}^{n} A_{ij}\dot{q}_j$, and coefficients A_{ij} and A_i are functions of the generalised coordinates $q_1, \ldots, q_n$ and time t.

The equations of motion for the system can be expressed as follows

$$\mathcal{E}_i(K) = Q_i + \sum_{k=1}^{m} A_{ki}\lambda_k \quad (i = 1, \ldots, n)\,, \tag{2.4}$$

where $Q_i = Q_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ denotes the generalised force corresponding to coordinate q_i, then $\lambda_1, \ldots, \lambda_m$ are undetermined multipliers and $\mathcal{E}_i$ is the Eulerian operator

$$\mathcal{E}_i(K) = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i}. \tag{2.5}$$

Relationships (2.3) and (2.4) form a closed system of ordinary differential equations with $n+m$ unknown variables $q_1, \dots, q_n, \lambda_1, \dots, \lambda_m$. The system under consideration admits a first integral which can be interpreted as the energy integral provided that

1) coefficients $A_1, \dots, A_m$ are equal to zero and thus equations (2.3) for non-holonomic constraints are homogeneous with respect to the generalised velocities (the so-called catastatic system, see [83]),

2) the kinetic energy does not depend explicitly on time, and

3) the generalised forces $Q_1, \dots, Q_n$ satisfy the identity

$$Q_i \dot{q}_i \equiv -\frac{d\Pi}{dt}, \tag{2.6}$$

where $\Pi = \Pi(q_1, \dots, q_n)$ designates the potential energy determined up to an additive constant. Identity (2.6) is always satisfied for all functions $q_1(t), \dots, q_n(t)$ admissible by constraints if the generalised forces $Q_1, \dots, Q_n$ can be represented as a sum of two components

$$Q_i = \Gamma_i + \left(-\frac{\partial \Pi}{\partial q_i}\right), \tag{2.7}$$

where quantities $\Gamma_1, \dots, \Gamma_n$ are linear homogeneous forms of the generalised velocities with the skew-symmetric matrix of the coefficients

$$\Gamma_i = b_{ij}(q_1, \dots, q_n)\, \dot{q}_i, \quad b_{ij} = -b_{ji}. \tag{2.8}$$

The first and second terms in eq. (2.7) are called the generalised gyroscopic force and the generalised potential force, respectively. It is easy to see that work of the potential forces in an arbitrary finite displacement of the system does not depend upon the form of the path and is equal to the difference in the potential energy in the initial and final positions, respectively. At the same time, the work of the gyroscopic forces in any admissible displacement is equal to zero. Gravitational and electrical forces are examples of the potential forces. The forces due to small deformations of ideally elastic bodies admit the introduction of a potential. Magnetic forces are examples of the gyroscopic forces given by eq. (2.8). What is more, magnetic forces are described by the generalised potential depending on the generalised coordinates, see next section for details.

2.3 Generalised Jacobi integral

The autonomous single-valued first integral of the above mechanical system can be obtained in the following way. One multiplies the $i-th$ equation in (2.4) by $\dot{q}_i$ and sums up the result over i from 1 to n. By virtue of eq. (2.3) the coefficients of multipliers $\lambda_1, \dots, \lambda_m$ vanish and the resulting equation, due to eq. (2.6), is cast in the form

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\dot{q}_i - K + \Pi\right) = 0. \tag{2.9}$$

Jacobi's integral is obtained by integrating eq. (2.9) and has the form

$$\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \text{const}, \tag{2.10}$$

where

$$L = K - \Pi \tag{2.11}$$

denotes Lagrange's function or the kinetic potential of the system.

In the most general case, the kinetic energy is a sum of three terms

$$K = K_2 + K_1 + K_0, \tag{2.12}$$

where K_2 and K_1 are respectively homogeneous quadratic and linear forms of the generalised velocities

$$K_2 = \frac{1}{2}a_{ij}\left(q_1, \dots, q_n\right)\dot{q}_i\dot{q}_j, \quad K_2 = b_i\left(q_1, \dots, q_n\right)\dot{q}_i, \tag{2.13}$$

whereas K_0 depends only on $q_1, \dots, q_n$. Jacobi's integral is then recast as follows

$$K_2 + \Pi - K_0 = \text{const}. \tag{2.14}$$

If the constraints imposed on the system are stationary, that is they do not depend on time explicitly, then $K_1 = K_0 = 0$, $K = K_2$ and Jacobi's integral is coincident with the energy integral (2.2). In the case of non-stationary constraints the quantities

$$K^* = K_2, \quad \Pi^* = \Pi - K_0 \tag{2.15}$$

can be treated as modified kinetic and potential energies of the system. Jacobi's integral has the sense of the integral of modified energy $H^* = K^* + \Pi^*$. When a mechanical system is considered in a non-inertial (rotating) coordinate system, then K_2 is the relative kinetic energy and $(-K_0)$ is the "potential energy" of the centrifugal forces. The linear part of the kinetic energy of the system K_1 does not affect the form of Jacobi's integral and leads to the appearance of Coriolis forces of inertia in the non-inertial coordinate system. Their contribution to Lagrange's equations (2.4) is naturally included in the modified (or total) gyroscopic forces

$$\Gamma_i^* = \Gamma_i + \left(\frac{\partial b_j}{\partial q_i} - \frac{\partial b_i}{\partial q_j}\right)\dot{q}_j. \tag{2.16}$$

Hence, Lagrange's equations of motion for a conservative system with holonomic non-stationary constraints have the form

$$\mathcal{E}_i\left(L^*\right) = \Gamma_i^* \quad \left(L^* = K^* - \Pi^*\right). \tag{2.17}$$

Along with Lagrange's equations the equations of motion in Hamilton's form

$$\dot{q}_i = \frac{\partial H^*}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H^*}{\partial q_i} + \Gamma_i^* \tag{2.18}$$

are frequently used in analytical mechanics. Here quantities

$$p_i = -\frac{\partial K^*}{\partial \dot{q}_i} = a_{ij}\dot{q}_j \tag{2.19}$$

are generalised momenta and H^* denotes Hamiltonian function coinciding with the modified energy of the system. The generalised velocities in the expressions for H^* and Γ_i^* are expressed in terms of the momenta by means of eq. (2.19). Let us notice in passing that superscript * can be omitted in the case of stationary constraints.

2.4 Conservative system in the presence of the generalised gyroscopic forces

Let the conservative system be subjected only to holonomic stationary constraints. Then its kinetic energy is a quadratic form of the generalised velocities $(K = K_2)$, and due to eqs. (2.7) and (2.8) Lagrange's equations can be represented in the form

$$\mathcal{E}_i\left(K - \Pi\right) = b_{ij}\dot{q}_j. \tag{2.20}$$

Let us determine under which conditions there exists such a generalised potential linearly depending on the generalised velocities

$$\Pi^* = \Pi + F_i\left(q_1, \dots, q_n\right)\dot{q}_i \tag{2.21}$$

that eq. (2.20) takes the form

$$\mathcal{E}_i\left(K - \Pi^*\right) = 0 \tag{2.22}$$

or equivalently

$$\mathcal{E}_i\left(F_i\dot{q}_i\right) = -b_{ij}\dot{q}_j. \tag{2.23}$$

Relationships (2.23) are equivalent to the following ones

$$\frac{\partial F_i}{\partial q_j} = \frac{\partial F_j}{\partial q_i} + b_{ij}. \tag{2.24}$$

It is clear that the components $F_1, \dots, F_n$ of vector F can not be found for an arbitrary skew-symmetric matrix b_{ij}. In order to determine the corresponding conditions we rewrite eq. (2.24) by changing the subscripts in the following way

$$\frac{\partial F_i}{\partial q_l} = \frac{\partial F_l}{\partial q_i} + b_{il}. \tag{2.25}$$

Differentiating eqs. (2.24) and (2.25) with respect to q_i and q_j, respectively, equating the right hand sides and rearranging the result yields

$$\frac{\partial}{\partial q_i}\left(\frac{\partial F_j}{\partial q_l} - \frac{\partial F_l}{\partial q_j}\right) = \frac{\partial b_{il}}{\partial q_j} - \frac{\partial b_{ij}}{\partial q_l}. \tag{2.26}$$

Taking into account eq. (2.24) we finally obtain

$$\frac{\partial b_{jl}}{\partial q_i} + \frac{\partial b_{ij}}{\partial q_l} = \frac{\partial b_{il}}{\partial q_j}. \tag{2.27}$$

Thus, if eq. (2.27) holds identically for any i, j and l, then there exists a vector F satisfying eq. (2.24). Then, according to eq. (2.6), the components of vector F do not affect the expression for Jacobi's integral

$$H = K + \Pi = h, \tag{2.28}$$

but appear in the expression for the generalised momenta

$$p_i = \frac{\partial}{\partial \dot{q}_i}(K - \Pi^*) = a_{ij}\dot{q}_i + F_i. \tag{2.29}$$

Expressing the generalised velocities $\dot{q}_1, \dots, \dot{q}_n$ in terms of the generalised momenta $p_1, \dots, p_n$ by means of eq. (2.29) and substituting them into the corresponding formulae in (2.28) results in the Hamiltonian of the system

$$\begin{aligned} H &= H_2 + H_1 + H_0, \quad H_2 = \frac{1}{2}a_{ij}^{-1}p_i p_j, \\ H_2 &= a_{ij}^{-1}F_i p_j, \quad H_0 = \frac{1}{2}a_{ij}^{-1}F_i F_j + \Pi, \end{aligned} \tag{2.30}$$

where a_{ij}^{-1} are components of the matrix inverse of matrix a_{ij}. The equations of motion in Hamilton's form are written as follows

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2.31}$$

Let us consider a charged particle of mass m in a stationary electromagnetic field as an example of the conservative system with generalised gyroscopic forces. It is known [82] that the vector of force Q acting on the mass can be expressed in terms of its charge e, the scalar potential of the electric field $U(r)$ and the magnetic intensity A by means of the following formula

$$Q = -e \operatorname{grad} U + \frac{e}{c}\dot{r} \times \operatorname{rot} A. \tag{2.32}$$

Here c and r denote respectively the velocity of light and radius vector of the particle. It is easy to see that the component of the generalised force, which is linear in velocity, satisfies eq. (2.27). Vector F is easily expressed in terms of A

$$F = -\frac{e}{c}A. \tag{2.33}$$

Lagrange's function of the system is then set in the form

$$L = \frac{m}{2}\dot{r}^2 - \Pi^*, \quad \Pi^* = eU - \frac{e}{c}A \cdot \dot{r}. \tag{2.34}$$

If x, y, z denote Cartesian coordinates, then the corresponding generalised momenta are written as follows

$$p_x = m\dot{x} + \frac{e}{c}A_x,\ p_y = m\dot{y} + \frac{e}{c}A_y,\ p_z = m\dot{z} + \frac{e}{c}A_z, \tag{2.35}$$

whereas the Hamiltonian function of the system, due to eq. (2.30), takes the form

$$H = \frac{1}{2m}\left[\left(m\dot{x} + \frac{e}{c}A_x\right)^2 + \left(m\dot{y} + \frac{e}{c}A_y\right)^2 + \left(m\dot{z} + \frac{e}{c}A_z\right)^2\right] + eU(x, y, z). \tag{2.36}$$

In particular, let $U = -eEx$ and $A = (\gamma x, 0, 0)$. Then the Hamiltonian function (2.36) becomes

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) - \frac{e\gamma}{mc}xp_x + \frac{e^2\gamma^2}{2mc^2}x^2 - eEx. \tag{2.37}$$

As can be easily shown, linear differential equations in the canonical form obtained by means of (2.37) have the following general integral

$$\begin{aligned} p_z &= \text{const}, \quad z = \frac{p_z}{m}t + z_0, \quad p_x = -\frac{e\gamma}{c}R\sin\varphi, \\ x &= R\cos\varphi + x_0, \quad p_y = \frac{mEe}{\gamma} + \frac{e\gamma}{c}R\cos\varphi, \\ y &= R\cos\varphi + \frac{Ee}{\gamma}t + y_0, \end{aligned} \tag{2.38}$$

where the fast phase $\varphi = \omega t + \alpha$, $\omega = \dfrac{e\gamma}{mc}$ and

$$\tan\varphi = \frac{v_{x_0}}{\dfrac{Ee}{\gamma} - v_{y_0}}, \quad R = \frac{1}{\omega}\left[\left(v_{y_0} - \frac{Ee}{\gamma}\right)\cos\alpha - v_{x_0}\sin\alpha\right], \tag{2.39}$$

with $v_{x_0} = \dot{x}(0), v_{y_0} = \dot{y}(0)$. Notice that for $E = 0$ equations (2.38) describe a helix whose axis is coincident with axis x.

2.5 Electromechanical systems

In the present section we basically follow [49]. We refer to systems in which mechanical and electromagnetic processes are intimately linked with each other as electromechanical systems. In mechanics, the state of the system is considered to be given if its generalised coordinates and velocities are prescribed. However, for electromechanical systems they define only part of its character and the quantities describing the electromagnetic processes comprise another part. In order to characterise the electromagnetic field it is necessary to prescribe the vector of magnetic induction B and the vector of electric intensity E. These quantities are known to satisfy the system of partial differential equations of Maxwell [82] which differ drastically from the equations of mechanics. However, while considering a wide class of applied electromechanical problems, the field variables B and E can be expressed in terms of a finite number of other variables which are equivalent to the generalised coordinates and momenta to some extent. To meet this requirement, the conditions of quasi-stationarity enabling electromagnetic waves to be neglected must hold. Besides, the transverse dimensions of conductors are assumed to be much smaller than their longitudinal dimensions, with the exception of capacitor plates. Such conductors and currents in them are referred to as being linear. Linear conductors connected to each other, capacitor plates and external power sources form an electric circuit. Let an electromechanical system under consideration consist of l parallel circuits connected with each other by means of inductors. Each parallel circuit has nodes where more than two conductors and branches, i.e. non-parallel subcircuits between the nodes consisting of linear conductors, capacitors and power sources, are connected. The current in any cross-section of the linear conductor in the branch remains constant. It is assumed that the circuit does not change topologically under mechanical motion.

Let z_c and y_c denote respectively the number of branches and nodes in the $s-th$ circuit. Let us arbitrarily choose the positive direction of the current in any branch and designate these currents as $i_1, \dots, i_N$ $(N = z_1 + \ldots + z_l)$. According to Kirchhoff's first law these currents are related as follows

$$\sum_{j=1}^{N} \gamma_{jk} i_j = 0, \quad k = 1, \dots, \sum_{s=1}^{l} y_s. \tag{2.40}$$

Here $\gamma_{ik} = 0$ if the $j-th$ branch is not adjacent to the $k-th$ node and $\gamma_{ik} = \pm 1$ if this branch is adjacent to the $k-th$ node, with a positive sign corresponding to the current entering the branch and a negative sign for the circuit leaving the branch.

It is easy to see that the number of independent equations in (2.40) is equal to $\sum_{s=1}^{l} y_s - l$, thus we can introduce m $\left(m = N + l - \sum_{s=1}^{l} y_s\right)$ independent currents in such a way that the other currents can be expressed in terms of these independent circuits.

As will be explained below, the currents play the role of the generalised velocities, that is $i_j = \dot{g}_j$ where g_j denotes the charge transferred by current i_j in the corresponding branch. This means that relationships (2.40) after integration plays the role of the holonomic stationary constraints

$$\sum_{j=1}^{N} \gamma_{jk} (g_j - g_{j0}) = 0. \tag{2.41}$$

Currents $i_1, \ldots, i_m$ cause a magnetic field to develop in the surrounding. Vector B describing this field can be viewed as a function of the corresponding point in the space and these currents. Let the space be filled by a medium in which B is a linear function of currents $i_1, \ldots, i_m$. Then

$$B = \sum_{s=1}^{m} B_s (x, y, z) i_s, \tag{2.42}$$

where x, y, z are Cartesian coordinates of the point at which vector B is determined. Electrical field is considered only in the space between the capacitor plates, E being a linear function of the capacitor charge g_j

$$E = g_j E_j (x, y, z). \tag{2.43}$$

Here charge g_j is related to current i_j which flows through the branch with the capacitor.

The energy of the magnetic field is given by

$$W = \frac{1}{2} \int \frac{B^2}{\mu} dV, \tag{2.44}$$

where the integral is evaluated over the whole space and $\mu = \mu(x, y, z)$ denotes the magnetic permeability of the medium. In accordance with eq. (2.42) we have

$$W = \frac{1}{2} \sum_{s,r=1}^{m} L_{rs} i_r i_s, \quad L_{rs} = \int \frac{B_r B_s}{\mu} dV, \tag{2.45}$$

that is, W is a homogeneous quadratic form of the currents $i_1, \ldots, i_m$ which are understood as being analogous to the generalised velocities. With this in view, W is an analogue of the kinetic energy for the electromagnetic field.

The energy of the electric field is given by

$$V = \frac{1}{2} \sum_{j=1}^{N} \int_{\Omega_j} \varepsilon E_j^2 dV, \tag{2.46}$$

where $\varepsilon = \varepsilon(x, y, z)$ denotes dielectric permittivity of the medium and the integral is evaluated over the space between the capacitor plates. Substituting eq. (2.43) in eq. (2.46) we obtain

$$V = \frac{1}{2}\sum_{j=1}^{N}\frac{g_j^2}{c_j}, \quad c_j^{-1} = \int\limits_{\Omega_j} \varepsilon E_j^2 dV. \tag{2.47}$$

Here c_j denotes capacitance of the capacitors in the branch corresponding to i_j. For simplicity, we assume that each branch has not more than one capacitor. In the case of no capacitor in the branch it is necessary to take $c_j = \infty$. By virtue of relationships (2.41), function V can be expressed in terms of m independent charges $g_1, \ldots, g_m$ and their initial values $g_{10}, \ldots, g_{m0}$

$$V = \frac{1}{2}\sum_{s,r=1}^{m}\frac{g_r g_s}{c_{rs}} + \sum_{r=1}^{m} b_r g_r + V_0. \tag{2.48}$$

The values $1/c_{rs}$ for $r = s$ and for $r \neq s$ are called the inverse capacitance and mutual inverse capacitance, respectively, and are expressed in terms of the capacitance of the capacitors. Values b_r and V_0 are expressed in terms of the initial values of the charges and the capacitance of the capacitors. As follows from eq. (2.48), V is a function of charges $g_1, \ldots, g_m$ treated as the generalised coordinates, therefore, V can be understood to be the potential energy of the field.

Another characteristic of circuits is active resistances R_j of the branches. Let $R_j = \text{const}$. Then we can enter an analogue of Rayleigh's dissipation function

$$\Psi = \frac{1}{2}\sum_{j=1}^{N} R_j i_j^2. \tag{2.49}$$

Function Ψ, referred to as the electric dissipation function, is related to the power P of the generated heat by the formula $P = 2\Psi$. Expressing, as before, N currents in terms of $i_1, \ldots, i_m$ we obtain

$$\Psi = \frac{1}{2}\sum_{r,s=1}^{m} R_{rs} i_r i_s. \tag{2.50}$$

The mechanical motion of the system can be described by the generalised coordinates $q_1, \ldots, q_n$. The induction coefficients L_{rs} and the inverse capacitances c_{rs} depend on these coordinates. If $K(q, \dot{q})$ and $\Pi(q)$ denote the kinetic and potential mechanical energies of the system, respectively, then the system of equations governing mechanical and electrical processes have the following general form

$$\begin{aligned}\left(\frac{d}{dt}\frac{\partial}{\partial i_r}-\frac{\partial}{\partial g_r}\right)(W-V) &= E_r, \quad r=1,\dots,m,\\ \mathcal{E}_i\,(K+W-\Pi-V) &= Q_i, \quad i=1,\dots,n. \end{aligned} \tag{2.51}$$

Here Lagrange's electric generalised forces take into account the influence of the external power sources and the energy dissipation in the active resistances. Let us notice that, in general, the values of the inductances L_{rs} and the capacitances c_{rs}, see eqs. (2.45) and (2.48), change under mechanical displacements. Therefore, magnetic W and electric E energies of the electromechanical system of the general form are functions of the mechanical coordinates. At the same time, mechanical energies K and Π do not explicitly depend on the electrical coordinates $g_1,\dots,g_m$ and their velocities.

When the generalised forces on the right hand side of eq. (2.51) are absent, the system is conservative and admits the first integral

$$K+W+\Pi+V=\text{const}, \tag{2.52}$$

which can be considered as integral of the total energy. It is needless to say that the validity of eq. (2.52) requires that system (2.51), for $E_r=Q_i=0$, is autonomous and the constraints imposed on the mechanical part are stationary.

In passing we notice that electromechanical systems in the above sense are frequently encountered in the theory of superconductivity [82]. The problems, in which the conservative terms on the right hand side of (2.51) are small in a certain sense, are actual for a number of applied problems. While solving these problems in the generating approximation one has to deal with the conservative electromechanical system.

2.6 Planar systems which admit the first integral

Various problems of chemistry, biology, sociology etc. are often reduced to integration of differential equations. These equations have no "regular" structure and do not allow one to obtain such general characteristics as kinetic and potential energies. Hence, even if integration of these equations can be reduced to quadratures, the very interpretation of the considered system as being conservative is made difficult. In this sense a natural solution of the problem can be obtained only for non-mechanical systems reducible to the system of two first-order differential equations

$$\dot q = Q\,(q,p)\,, \quad \dot p = P\,(q,p)\,. \tag{2.53}$$

It is known that integration of this system can be reduced to quadratures if it admits a single-valued autonomous first integral

$$H(q,p) = \text{const}. \tag{2.54}$$

This results in the identity

$$\frac{\partial H}{\partial q}Q + \frac{\partial H}{\partial q}P = 0. \tag{2.55}$$

By virtue of the latter it is possible to introduce function $M(q,p)$ with the help of the relationship

$$\frac{\partial F(H)}{\partial p}P = -\frac{\partial F(H)}{\partial q}Q = MPQ, \tag{2.56}$$

where F is, in general, an arbitrary function of H. The following differential identity

$$\frac{\partial(MQ)}{\partial q} + \frac{\partial(MP)}{\partial q} = 0, \tag{2.57}$$

which does not contain H, follows immediately from eq. (2.56).

One can consider eq. (2.57) as a partial differential equation for M. In the general case, this equation does not admit single-valued analytical solutions of arguments q and p. However the existence of such a solution enables one to find the first integral (2.54) and eventually integrate the original system. For this reason, M is referred to as Jacobi's last multiplier. Let us substitute expressions (2.56) for Q and P into the right hand side of eq. (2.53).

The resulting expressions form Pfaff's system of equations

$$\dot{q} = \frac{1}{M}\frac{\partial F}{\partial p}, \quad \dot{p} = -\frac{1}{M}\frac{\partial F}{\partial q}. \tag{2.58}$$

Due to eqs. (2.57) and (2.58) velocity $v(\dot{q},\dot{p})$ of the representative point in the phase plane (q,p) satisfies the scalar equation

$$\operatorname{div}(Mv) = 0. \tag{2.59}$$

Hence, motion of the point (q,p) describing the instantaneous dynamical state of Pfaff's system are similar to the motion of a particle of the ideal compressible fluid with density M [57].

Introducing new time by means of the non-linear substitution

$$dt = M d\tau, \tag{2.60}$$

we obtain, instead of eq. (2.45), the following canonical system

$$\frac{dq}{d\tau} = \frac{\partial F}{\partial p}, \quad \frac{dp}{d\tau} = -\frac{\partial F}{\partial q}. \tag{2.61}$$

If after integrating eq. (2.61) we obtain dependences $q(\tau)$ and $p(\tau)$ then the relationship between the arguments t and τ is obtained from eq. (2.60) by means of quadratures.

Equations (2.61) have the form which is often encountered in mechanics. The location of the integral lines on the phase plane (q, p) can be made similar to the flow of an incompressible fluid with unit mass density. While analysing eq. (2.61) it is expedient to interpret function $F(H)$ as a Hamiltonian and to choose the particular form of dependence $F = F(H)$ from the perspective of simplicity, symmetry etc. in accordance with mechanical analogy. Partial differential equations (2.57) for the unknown variable $M(q, p)$ can be replaced by the integral equation

$$U = V, \tag{2.62}$$

where

$$U = \int MQdp, \quad U = -\int MPdq. \tag{2.63}$$

Thus, if the original system admits a single-valued analytical integral and, for this reason, can be reduced to quadratures, then one can always find such a function M that quantities U and V are equal to each other by virtue of eq. (2.63).

3

Dynamical systems in a plane

3.1 Conservative systems in a plane

Let us consider a single-degree-of-freedom system. In this case the gyroscopic forces, according to definition (2.8), are absent whereas the term in expression (2.13) for the kinetic energy, which is linear in the generalised velocity $\dot{q}$, does not affect the motion. This means that, in general, it is sufficient to know expressions (2.15) for the modified kinetic and potential energies

$$K = \frac{1}{2} m(q) \dot{q}^2, \quad \Pi = \Pi(q). \tag{3.1}$$

Here and in what follows the sign "*" is omitted. Let us recall that the expression for Π contains the translational kinetic motion ("potential energy of the centrifugal forces"). Lagrange's equations of motion are written in the following form

$$m\ddot{q} + \frac{1}{2}\frac{dm}{dq}\dot{q}^2 + \frac{d\Pi}{dq} = 0. \tag{3.2}$$

We assume that the motion governed by eq. (3.2) continues with respect to t in both directions for as long as is required. Normally, this is ensured by positiveness of the inertial coefficient in the expression for K, i.e. $m(q) > 0$, and boundedness of Π for finite values of q.

Models which do not possess this property are sometimes used in mechanics. An example that has a vanishing inertial coefficient m is the problem of ring rolling without slippage on the horizontal plane. The ring of radius r is massless and carries an attached mass m. Due to eq. (3.1) the total mechanical energy of the system can be represented in the form

$$h = mr^2 (1 + \cos q) \left(\dot{q}^2 + \frac{g}{r}\right), \tag{3.3}$$

where g is the free fall acceleration and q denotes the angle of turn of the radius vector of mass m from the ring centre relative to the vertical axis. The law of motion of this system can be obtained by integrating eq. (3.3), and for $h > 2mgr$ has the form

$$\sin\frac{q}{2} = \sqrt{\frac{h}{2mgr} - 1}\sinh\left(\sqrt{\frac{g}{r}}\frac{t - t_0}{2}\right). \tag{3.4}$$

It is assumed here that $q|_{t=t_0} = 0$, $\dot{q}|_{t=t_0} > 0$. As follows from eq. (3.4), the motion can not be continued beyond the time instant

$$t_* = t_0 + 2\sqrt{\frac{r}{g}}\left[\ln\left(\sqrt{\frac{h}{2mgr}} + 1\right) - \ln\left(\sqrt{\frac{h}{2mgr} - 1}\right)\right], \tag{3.5}$$

when the mass m is in the extreme low position ($q = \pi, \dot{q} \to \infty$) and the inertial coefficient $m(q) = 2mr^2(1 + \cos q)$ is equal to zero, see eqs. (3.1) and (3.3).

An example in which the potential energy Π tends to infinity is the motion of a particle m along a straight line under the action of a force that is inversely proportional to the square of the distance r to the attracting centre. The total mechanical energy is cast as follows

$$h = \frac{m}{2}\dot{r}^2 - \frac{f}{r}. \tag{3.6}$$

Here f is a constant and value h is determined by the initial state of the system. For the sake of simplicity, we assume that $\dot{r} = 0$, $r = R = -f/h$ at $t = 0$. Direct integration yields the following law of motion

$$\sqrt{\frac{2f}{m}}t = R^{3/2}\left[\frac{\pi}{2} - \arctan\sqrt{\frac{r}{R - r}} + \frac{1}{R}\sqrt{r(R - r)}\right]. \tag{3.7}$$

Now it is easy to find time instant t^* when the particle reaches the attracting centre $r = 0$

$$t_* = \frac{\pi}{2}\sqrt{\frac{m}{2f}R^3}. \tag{3.8}$$

Therefore, the particle reaches point $r = 0$ within the finite time t^*. The potential energy is infinite at this point and thus the trajectory can not be continued in the direction of increasing time. Clearly, all rectilinear motions in this problem possess this property for $h < 0$ $(R > 0)$.

It is evident that the fact that certain motions can not be continued through t is caused by imperfect modelling and that it does not occur in real systems. For example, in order to obtain the motion of the ring for any t it is sufficient to take account of the distributed inertial properties of the ring. With this in view, we will consider those perfect models for which $m(q) > 0$, $\Pi(q) < \infty$ for values of q under consideration.

By virtue of eq. (3.2), the equilibrium position for a conservative system in plane $\dot{q} = 0$ is determined from the condition of extremum of the potential energy

$$\frac{d\Pi}{dq} = 0. \tag{3.9}$$

The minimum ($\frac{d^2\Pi}{dq^2} > 0$, singular point referred to as a centre) corresponds to the stable equilibrium, whereas the maximum ($\frac{d^2\Pi}{dq^2} < 0$, singular point referred to as a saddle) corresponds to unstable equilibrium. The degenerate case where $\frac{d^2\Pi}{dq^2} = 0$ is beyond the scope of the present book.

Equations of motion for conservative systems with one degree of freedom are always integrablc by quadratures. Indeed, the first integral of motion corresponding to eq. (3.2)

$$\frac{m}{2}\dot{q}^2 + \Pi = h, \tag{3.10}$$

see eq. (2.14), is a first-order differential equation with separable variables. Thus, motion of the system is governed by the following differential equation

$$\pm\sqrt{\frac{m}{2(h-\Pi)}}\,dq = dt. \tag{3.11}$$

Analysis of eq. (3.10) shows that four types of motion, namely, libration, rotational, escaping and limitation motions [80], [83] can exist in the conservative system with one degree of freedom depending upon the particular form of the relationships $m(q)$ and $\Pi(q)$. Libration (or oscillatory) motions are periodic and exist if

1) equation

$$\Pi(q) = h \tag{3.12}$$

has two sequential simple roots $q_1(h)$ and $q_2(h)$ ($d\Pi/dq \neq 0$, $q_1 < q_2$) and

2) inequality

$$\Pi(q) < h \tag{3.13}$$

holds in the interval $q_1 < q < q_2$.

Rotational motions exist if

1) the generalised coordinate q is periodic such that

$$m(q) = m(q + 2\pi), \quad \Pi(q) = \Pi(q + 2\pi),$$

2) inequality $h > \max \Pi$ is met.

Non-periodic motions of the escaping type exist if the potential energy Π is not periodic with respect to q, and either it is limited by absolute value or $\Pi \to -\infty$ as $q \to \infty$. They are distinctive in that the absolute value of q tends to infinity as $t \to \pm\infty$.

Motions of the special limitation type are also non-periodic and exist only for discrete values of the energy constant h coinciding with the values of h at points $q = q^*$ of positions of unstable equilibria

$$\left.\frac{d\Pi}{dq}\right|_{q=q_*} = 0, \quad \left.\frac{d^2\Pi}{dq^2}\right|_{q=q_*} < 0. \tag{3.14}$$

Limitation motions are characterised by the fact that $q \to q_*$ and $\dot{q} \to 0$ as $t \to \pm\infty$. One speaks of the corresponding phase trajectories as separatrices.

The phase plane $(q, p = m\dot{q})$ of the conservative system with one degree of freedom is symmetric about axis q. In the general case, separatrices divide the phase plane into a finite or countable number of regions which are completely filled by phase trajectories of libration, rotational or escaping motions. Hence, the separatrices separate the regions of motions of essentially different types.

This theory is applicable to a wide class of conservative systems with one degree of freedom for which functions m and Π are continuous with respect to q, whilst their derivatives in the region of q may have only a finite number of discontinuities [31]. However, only points at which derivative $\dfrac{d\Pi}{dq}$ changes its sign stepwise correspond to the equilibria.

Conservative systems with one degree of freedom of the impact-oscillatory type deserve special consideration. These systems are characterised by the fact that the motion takes place within bounded (two side stops) or half-bounded (one side stop) regions. If the system reaches a stop, the generalised velocity changes sign, however the energy constant h remains unaltered. Motion of a conservative systems with two side stops are always of the periodic libration type. Notice that in the case of two side stops, the form of the potential energy within the intervals of continuity does not affect the presence of librations even if the potential energy has no smooth extrema in the equilibrium positions.

3.2 Libration in the conservative system with a single degree of freedom

Under libration motion the generalised coordinate q executes $T-$periodic oscillations within the interval (q_1, q_2). By virtue of eq. (3.11), the duration of the forward ($\dot{q} > 0$) and the backward motion ($\dot{q} < 0$) coincide and are equal to the half-period. Hence, the libration period is equal

$$T(h) = \int_{q_1}^{q_2} \sqrt{\frac{2m}{h - \Pi}} dq \tag{3.15}$$

and, generally speaking, depends upon the energy constant. The half-swing of the oscillation is as follows

$$a(h) = \frac{q_1 - q_2}{2} \tag{3.16}$$

and can be understood to be the amplitude of oscillation. The equation for the phase trajectory on plane (q, p) directly stems from eq. (1.26)

$$\frac{p^2}{2m} + \Pi = h, \tag{3.17}$$

where $p = m\dot{q}$ is the generalised momentum.

The phase trajectory is symmetric about axis q and bounds an area which is equal to

$$S(h) = \oint p dq = 2 \int_{q_1}^{q_2} \sqrt{2m(h - \Pi)} dq. \tag{3.18}$$

It is easy to see that this area is equal to the increase in Lagrange's action W of the system over the period of libration [1], [33]

$$S = W|_0^T, \quad W = \int_0^t p\dot{q} dt = \int_0^t 2K dt. \tag{3.19}$$

For this reason, value $S(h)$ is referred to as the modulus of periodicity of Lagrange's action or simply the integral of action.

It is essential that, by virtue of eqs. (3.15) and (3.18), the period of libration motion is equal to the derivative of the action with respect to energy

$$\frac{dS}{dh} = T. \tag{3.20}$$

Because $T > 0$ the action is a monotonically increasing function of energy. The arbitrary constant in the expression for Π therewith can be chosen in such a way that $S|_{h=0} = 0$. From the above consideration we can state that the energy is a function of the action of the same type, that is $h = h(S)$.

Let us introduce the phase of motion

$$\Phi = \Omega t + \mathcal{A}, \tag{3.21}$$

which is measured in revolutions. Here $\Omega = 1/T$ is the cyclic frequency measured in Hz, and $\mathcal{A}$ is an initial phase determined by the initial conditions at $t = 0$. The general periodic solution, depending upon the two constants S and $\mathcal{A}$, can be cast in the form

$$q = q(\Phi, S), \quad p = p(\Phi, S), \tag{3.22}$$

where $q(\Phi) = q(\Phi + 1), p(\Phi) = p(\Phi + 1)$. Relationships (3.22) can be understood to be equations for the transformation of variables $q, p \rightarrow \Phi, S$. As the equations of motion in the new variables have the simple canonical structure

$$\dot{\Phi} = \frac{\partial h}{\partial S}, \quad \dot{S} = 0, \quad \left(h = h(S), \Omega = \frac{dh}{dS}\right), \tag{3.23}$$

the new "action-angle" variables [33], [83] are canonical.

The circular phase

$$\varphi = 2\pi\Phi = \omega t + \alpha \tag{3.24}$$

and the corresponding action $s = S/2\pi$ are often used in applications. Here $\omega = 2\pi/T = dh/ds$ is the circular frequency measured in radian per second, and $\alpha = 2\pi\mathcal{A}$ denotes the initial circular phase. Obviously, the motions under consideration are $2\pi-$periodic with respect to φ.

For the purposes of further analysis it is expedient to introduce Hamilton's action and its modulus of periodicity

$$V = \int_0^t L dt, \quad \Lambda = V|_0^T \quad (L = K - \Pi). \tag{3.25}$$

By virtue of eq. (3.10), Hamilton's action and Lagrange's action are related as follows

$$V = W - ht. \tag{3.26}$$

Correspondingly, the moduli of periodicity of these actions are related by the formula

$$\Lambda = S - hT. \tag{3.27}$$

Differentiating eq. (3.27) with respect to h and taking into account eq. (3.20), we obtain

$$\frac{d\Lambda}{dh} = -h\frac{dT}{dh} = -h\frac{d^2S}{dh^2}. \tag{3.28}$$

Libration of the considered type disappears if one of the roots q_1, q_2 of eq. (3.12) disappears with increasing h or inequality (3.13) no longer holds with decreasing h.

Solving quadratures of eq. (3.11) presents a challenging mathematical problem. Closed form expressions for the periodic solutions of the equation for the physical pendulum and the Duffing equation are obtained in terms of the elliptic functions [39]. This question is studied in detail in [2] where it is shown that the periodic motion of the conservative system with one degree of freedom can be represented using Fourier series whose coefficients are expressed in terms of Bessel functions of several variables. Let us consider construction of the solution in the form of a series in terms of small parameters in the case of libration about a stable equilibrium.

Without loss of generality we assume that both the potential energy and the generalised coordinate become zero at the stable equilibrium

$$\Pi(0) = \left.\frac{d\Pi}{dq}\right|_{q=0} = 0, \quad c_0 = \left.\frac{d^2\Pi}{dq^2}\right|_{q=0} > 0. \tag{3.29}$$

Besides, the potential energy and the inertial coefficient are assumed to be represented in the form of a Taylor series in the vicinity of zero, i.e.

$$\Pi = \frac{c_0}{2}q^2 + \frac{c_1}{6}q^3 + \frac{c_2}{24}q^4 + \ldots, \quad m = m_0 + m_1 q + \frac{m_2}{2}q^2 + \ldots \tag{3.30}$$

We proceed from the energy integral which after replacing argument $t \to \varphi$ by means of eq. (3.24) can be written in the following form

$$\frac{m}{2}\left(\frac{dh}{ds}\frac{dq}{d\varphi}\right)^2 + \Pi = h. \tag{3.31}$$

This differential equation is invariant under a change of sign of the circular phase φ. Hence, the general integral describing libration can be represented by an even $2\pi-$periodic function of φ. Alternatively, its Fourier series contains only the even harmonics $\cos k\varphi$ $(k = 1, 2 \ldots)$ and such a solution is sought in what follows. It is also essential that in the linear approximation ($c_1 = c_2 = \cdots = 0$, $m_1 = m_2 = \cdots = 0$) the required solution has the form

$$q = q_0\sqrt{s}, \quad q_0 = \sqrt{\frac{2\omega_0}{c_0}}\cos\varphi, \quad h = \omega_0 s, \tag{3.32}$$

where $\omega_0 = \sqrt{c_0/m_0}$ is the linearised natural frequency of the system. It is expected that in the neighbourhood of the equilibrium where $s \to 0$, q and h are analytical functions of $\sqrt{s}$ and s, respectively, and we have

$$q = q_0\sqrt{s} + q_1 s + q_2 s^{3/2} + s^2 \dots, \qquad h = \omega_0 s + \frac{h_1}{2}s^2 + s^3 \dots \tag{3.33}$$

Let us now substitute series (3.30) and (3.33) into eq. (3.31) and equate terms of the same powers of s. The balance of terms of $s^{3/2}$ yields the equation of the first approximation

$$-\sin\varphi q_1' + \cos\varphi q_1 = -\frac{1}{\sqrt{c_0 m_0}}\left(\frac{m_1}{m_0}\cos\varphi\sin^2\varphi + \frac{c_1}{3c_0}\cos^3\varphi\right), \tag{3.34}$$

where, here and throughout the book, a prime denotes a derivative with respect to φ.

The even solution which is $2\pi-$periodic over φ is set in the form of the quadratic form of harmonics $\sin\varphi$ and $\cos\varphi$

$$q_1 = -\frac{1}{\sqrt{c_0 m_0}}\left[\left(\frac{2c_1}{3c_0} - \frac{m_1}{m_0}\right)\sin^2\varphi + \frac{c_1}{3c_0}\cos^2\varphi\right]. \tag{3.35}$$

The homogeneous part of the equation for the next approximation q_2, as well as for $q_3, q_4 \dots$ coincides with the homogeneous part of eq. (3.34). At the same time, the right hand side of this equations can be represented in the form of a homogeneous even function of harmonics of the fourth power. To this aim, it is sufficient to multiply the quadratic terms having the constant factor h_1, see eq. (3.33), with $\sin^2\varphi + \cos^2\varphi = 1$. In connection with this, a general question is whether the equation

$$-\sin\varphi x' + \cos\varphi x = \sum_{k=0}^{n} A_k \cos^k\varphi \sin^{n-k}\varphi, \tag{3.36}$$

possesses a bounded solution which is $2\pi-$periodic over φ and can be cast as a homogeneous form of power $n-1$

$$x = \sum_{k=0}^{n-1} a_k \cos^k\varphi \sin^{n-1-k}\varphi. \tag{3.37}$$

Equations for determining the required constants $a_0, a_1, \dots, a_{n-1}$ are obtained immediately after substituting eq. (3.37) into eq. (3.36) and equating the coefficients in front of the same powers of harmonics. Then we immediately obtain

$$a_1 = A_0, \quad a_{n-1} = A_n \tag{3.38}$$

and also $A_{n-1} = 0$. The latter imposes a certain restriction on the form of the right hand side of eq. (3.36) and its fulfillment is the necessary condition for existence of the solution (3.37) bounded with respect to φ. As for the other coefficients, they are determined from the system

$$(k+1)\,a_{k+1} - (n-k-1)\,a_{k-1} = A_k \quad (k = 1, 2, \dots, n-2)\,. \tag{3.39}$$

This system is always split into two subsystems which are independent of each other. If n is an odd number, then these subsystems are closed (if eq. (3.38) is taken into account) and enable one to determine uniquely and consequently constants $a_{n-3}, a_{n-5}, \dots, a_2, a_0$ from one subsystem and $a_3, a_5, \dots, a_{n-4}, a_{n-2}$ from the other. Therefore, for an odd n the solution (3.37) of eq. (3.36) always exists and is unique under the condition $A_{n-1} = 0$. If n is even, then the subsystem for the unknown variables $a_0, a_2, \dots, a_{n-4}, a_{n-2}$ is indeterminate, i.e. the number of equations is less than the number of unknown variables by one. The apparent arbitrariness is due to the fact that one can always add the expression $C \sin\varphi = C\left(\sin^2\varphi + \cos^2\varphi\right)^{n/2-1} \sin\varphi$ to solution (3.37). This expression satisfies the homogeneous differential equation (3.36) and is determined up to an arbitrary factor C. Alternatively, for the other subsystem the number of unknown variables $a_3, a_5, \dots, a_{n-5}, a_{n-3}$ is less than the number of equations by one. Hence, a certain constraint should be imposed on coefficients $A_0, A_2, \dots, A_{n-2}, A_n$. For $n = 4$ and $n = 6$ these constraints have the form

$$A_0 + A_2 = 3A_4, \quad 3A_0 + A_2 + 3A_4 = 15A_6. \tag{3.40}$$

Equations for determining the successive approximations $q_2, q_3, \dots$ are given by eq. (3.36). However, the terms which are even with respect to φ are absent on the right hand side of this equation ($A_{n-1} = A_{n-3} = \dots = 0$) otherwise the evenness of the required general integral with respect to φ is violated. On the other hand, relationships (3.40) appearing when equations for approximations of the even orders of $q_2, q_4, \dots$ are considered, serve to determine the correction terms $h_1, h_2, \dots$ in the expansion of h. Retaining the terms of order $s^{3/2}$ in the expressions for q and the terms of order s^2 in the expressions for h, we can write the resulting series in the form

$$\begin{aligned} q &= \sqrt{\frac{\omega_0}{c_0}}\xi - \frac{\omega_0}{2c_0}\left[\left(\frac{2c_1}{3c_0} - \frac{m_1}{m_0}\right)\eta^2 + \frac{c_1}{3c_0}\xi^2\right] + \\ &\quad \frac{\xi}{32m_0\sqrt{c_0\omega_0}}\left[\left(\frac{13c_1^2}{18c_0^2} + \frac{23m_1c_1}{3m_0c_0} - \frac{21m_1^2}{2m_0^2} + 3\frac{m_2}{m_0} - \frac{c_2}{2c_0}\right)\eta^2 + \right. \\ &\quad \left.\left(\frac{25c_1^2}{18c_0^2} + \frac{m_1c_1}{m_0c_0} + \frac{m_1^2}{2m_0^2} - \frac{m_2}{m_0} - \frac{5c_2}{6c_0}\right)\xi^2\right] + \dots, \\ h &= \omega_0 s + \left(\frac{m_1^2}{m_0^2} - \frac{5c_1^2}{3c_0^2} + \frac{2m_1c_1}{m_0c_0} - \frac{2m_2}{m_0} + \frac{c_2}{c_0}\right)\frac{s^2}{16m_0} + \dots, \end{aligned} \tag{3.41}$$

where

$$\xi = \sqrt{2s}\cos\varphi, \quad \eta = -\sqrt{2s}\sin\varphi. \tag{3.42}$$

Hence, in the neighbourhood of the stable equilibrium, the generalised coordinate is represented as an analytical function of variables ξ and η and is an even function of φ. The generalised momentum $p = m\dfrac{dh}{ds}\dfrac{dq}{d\varphi}$ is an analytical function of variables ξ and η and is an odd function of φ. The convergence of these series for sufficiently small s is proved in much the same fashion as convergence of the series in Lyapunov's method [80] is proved.

Relationships $q = q(\xi, \eta)$ and $p = p(\xi, \eta)$ can be viewed as prescribing the change of variables $q, p \to \xi, \eta$. This change is canonical, with the Hamiltonian function in terms of the new variables being a function of $s = \frac{1}{2}\left(\xi^2 + \eta^2\right)$. The canonical equations of motion in terms of the new variables

$$\dot{\xi} = \frac{\partial h}{\partial \eta} = \omega(s)\,\eta, \quad \dot{\eta} = -\frac{\partial h}{\partial \xi} = -\omega(s)\,\xi \tag{3.43}$$

are equivalent to the equations of motion of the customary harmonic oscillator. Here

$$\omega = \frac{dh}{ds} = \omega_0 + \left(\frac{m_1^2}{m_0^2} + \frac{2m_1c_1}{m_0c_0} - \frac{5c_1^2}{3c_0^2} - \frac{2m_2}{m_0} + \frac{c_2}{c_0}\right)\frac{s}{8m_0} + \ldots$$

is the circular frequency of libration. However, one must bear in mind that this frequency is dependent on the energy (or the action), and this oscillator is actually anisochronous. In view of eq. (3.42), the canonical variables ξ and η are referred to as the harmonic variables.

3.3 Rotational motion of the conservative system with one degree of freedom

Let the conditions for existence of rotational motion introduced in Sec. 3.1 be satisfied. In this case it is natural to identify values of q and $q + 2\pi$, and consider the cylindrical phase surface (p, q) rather than the phase plane. In this case, trajectory (3.17) is closed and the rotational motion is periodic. Due to eq. (3.11) the period of rotation is equal to

$$T = \int_0^{2\pi} \sqrt{\frac{m}{2(h - \Pi)}}\,dq. \tag{3.44}$$

By using eq. (3.19) we can also enter the integral of action S. However, instead of eq. (3.18) we should take

$$S = \int_0^{2\pi} \sqrt{2m(h - \Pi)}\,dq. \tag{3.45}$$

It is clear that relationships (3.20)-(3.22) are valid in the case of rotation, and additionally $p(\Phi) = p(\Phi+1)$, $q(\Phi+1) = q(\Phi) + 2\pi$. Assuming the direction of rotation as positive and making use of the circular phase (3.24), we can write relationships (3.22) in the following form

$$p = p(\varphi, s), \quad q = \varphi + q_1(\varphi, s), \tag{3.46}$$

where $p(\varphi + 2\pi) = p(\varphi)$, $q_1(\varphi + 2\pi) = q_1(\varphi)$.

The "action-angle" variables can also be applied to the non-periodic rotational motions provided that quantities m and Π are bounded $2\pi-$periodic motions of n partial angles $\theta_i = k_i q$ $(i = 1, \ldots, n)$. It is assumed that, due to the properties of the considered system, the numbers $k_1, \ldots, k_n$ are mutually incommensurable. It is also essential that the generalised coordinate q can always be chosen such that $k_1 = 1$. For this reason, partial angle $\theta_1 = q$ can be called the supporting angle. Indeed, for $h > \max \Pi$ and $0 < m < \infty$ the general solution of equation (3.2) governing rotation in the positive direction is written as following

$$t - t_0 = \int_0^q \sqrt{\frac{m}{2(h-\Pi)}} dq \quad (t_0 = \text{const}). \tag{3.47}$$

In accordance with eq. (3.47) both argument t and function q have the same rate of tending to infinity. Hence we can speak of a constant average velocity of rotation for the supporting angle

$$\omega = \lim_{t\to\infty} \frac{q}{t} = \lim_{q\to\infty} q \left[\int_0^q \sqrt{\frac{m}{2(h-\Pi)}} dq \right]^{-1}. \tag{3.48}$$

It is essential that for calculation of ω in the case in which coefficients $k_1, \ldots k_n$ are mutually incommensurable one can replace averaging over q by averaging over all partial angles [19], [98]

$$\omega = \left\langle \sqrt{\frac{m}{2(h-\Pi)}} \right\rangle^{-1}. \tag{3.49}$$

Here and in what follows $\langle f(\theta_1 \ldots, \theta_n) \rangle$ implies the following averaging

$$\langle f \rangle = \frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} f d\theta_1 \ldots d\theta_n. \tag{3.50}$$

By analogy we can introduce the action variable

$$s = \lim_{q\to\infty} \frac{1}{q} \int_0^q \sqrt{2m(h-\Pi)} dq = \left\langle \sqrt{2m(h-\Pi)} \right\rangle. \tag{3.51}$$

Here, as before, the following relationship

$$\frac{dh}{ds} = \omega \tag{3.52}$$

holds, which results in the energy constant being a monotonically increasing function of the action.

In the case under consideration the original canonical variables q and p are quasi-periodical functions of the phase determined by means of the following implicit dependence

$$\varphi = \omega \int_0^q \sqrt{\frac{m}{2\,(h - \Pi)}} dq. \tag{3.53}$$

Notice that in this particular case choice of the "action-angle" variables depends on choice of the supporting angle. The resulting rotations with respect to this angle are represented as a superposition of stationary rotation with frequency ω and oscillations of the quasi-periodic type.

As an example, let us consider a system consisting of two unbalanced rotors with masses m_1 and m_2, eccentricities e_1 and e_2 and rotating in the gravitational field. The angles of rotation are related with each other by means of a mechanical gearing with an irrational gear ratio k. The total energy of this system up to constant order can be cast in the form

$$h = \frac{1}{2}\left[\dot{q}^2 + a_1\,(1 - \cos q) + a_2\,(1 - \cos kq)\right], \tag{3.54}$$

where $a_1 = \dfrac{2m_1e_1g}{m_1e_1^2 + m_2e_2^2k^2}$ and $a_2 = \dfrac{2m_2e_2g}{m_1e_1^2 + m_2e_2^2k^2}$.

By virtue of eqs. (3.49) and (3.51), the average frequency and action constant are determined by quadratures

$$\omega = 4\pi^2 \left[\int_0^{2\pi}\int_0^{2\pi} \frac{d\theta_1 d\theta_2}{\sqrt{2h - a_1\,(1 - \cos\theta_1) - a_2\,(1 - \cos\theta_n)}}\right]^{-1},$$

$$s = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \sqrt{2h - a_1\,(1 - \cos\theta_1) - a_2\,(1 - \cos\theta_n)}d\theta_1 d\theta_2, \tag{3.55}$$

which, generally speaking, can not be expressed in terms of the elementary functions. For rotations to exist, it is necessary that inequality $2h > a_1 + a_2$ holds.

As for the libration, solving quadratures (3.53), which is necessary for constructing dependence $q\,(t)$, presents considerable difficulties.

In the particular case of rotation at ultra-high energy this problem is solvable using power series in small parameters. Considering this particular case, we assume that $h, s >> 1$ and look for the solution in the form of the following series

$$q = q_0 + \frac{1}{s^2}q_1 + \frac{1}{s^4}q_2 + \dots,$$
$$h = \frac{1}{2}h_0 s^2 + h_1 + \frac{1}{s^4}q_2 + \dots. \quad (3.56)$$

Inserting expression (3.56) for h into eq. (3.51) and expanding the right hand side as a series in powers of s, we obtain

$$\left\langle \sqrt{mh_0}\left[1 + \frac{h_1 - \Pi}{h_0 s^2} + \left(\frac{h_2}{h_0} - \frac{(h_1 - \Pi)^2}{2h_0^2}\right)\frac{1}{s^4} + \dots\right]\right\rangle = 1. \quad (3.57)$$

Equating the terms of the same order of small parameter $1/s^2$, we obtain

$$h_0 = \left\langle\sqrt{m}\right\rangle^{-2}, \quad h_1 = \frac{1}{\left\langle\sqrt{m}\right\rangle}\left\langle\sqrt{m}\Pi\right\rangle,$$
$$h_2 = \frac{1}{2}\left\langle\sqrt{m}\right\rangle\left\langle\sqrt{m}\Pi^2\right\rangle - \frac{1}{2}\left\langle\sqrt{m}\Pi\right\rangle^2, \quad (3.58)$$

Next it is necessary to substitute series (3.56) in equality (3.53) and perform the corresponding expansions on the right hand side of this equality.

The resulting expression is given by

$$\varphi = \sqrt{h_0}\int_0^{q_0}\sqrt{m}dq + \left[h_0\sqrt{m}q_1 + \int_0^{q_0}\sqrt{m}\left(h_1 - \Pi\right)dq\right]\frac{1}{\sqrt{h_0}s^2} +$$
$$\left[\sqrt{m}h_0 q_2 + \sqrt{m}q_1\left(h_1 - \Pi\right) - \frac{3}{2}\int_0^{q_0}\sqrt{m}\left(h_2 - \frac{(h_1 - \Pi)^2}{h_0}\right)dq + \right.$$
$$\left.\frac{1}{4\sqrt{m}}\frac{dm}{dq}q_1^2\right]\frac{1}{\sqrt{h_0}s^4} + \dots \quad (3.59)$$

Here all values out of the integrals, except for the correction terms $q_1, q_2, \dots$, are calculated at $q = q_0$. Let us equate the coefficients of the same powers of $1/s^2$. This leads to the following expressions for the first three approximations

$$\varphi = \frac{1}{\left\langle\sqrt{m}\right\rangle}\int_0^{q_0}\sqrt{m}dq, \; q_1 = \frac{\left\langle\sqrt{m}\right\rangle}{\sqrt{m}}\left[\left\langle\sqrt{m}\Pi\right\rangle\left\{\sqrt{m}\right\} - \left\langle\sqrt{m}\right\rangle\left\{\sqrt{m}\Pi\right\}\right],$$
$$q_2 = \left\langle\sqrt{m}\right\rangle\left[\left\langle\sqrt{m}\Pi\right\rangle - \left\langle\sqrt{m}\right\rangle\Pi\right]q_1 - \frac{1}{4m}\frac{dm}{dq}q_1^2 -$$
$$3\left\langle\sqrt{m}\Pi\right\rangle\frac{\left\langle\sqrt{m}\right\rangle^2}{\sqrt{m}}\left[\left\langle\sqrt{m}\Pi\right\rangle\left\{\sqrt{m}\right\} - \left\langle\sqrt{m}\right\rangle\left\{\sqrt{m}\Pi\right\}\right] -$$
$$\frac{3}{2}\frac{\left\langle\sqrt{m}\right\rangle^3}{\sqrt{m}}\left[\left\langle\sqrt{m}\right\rangle\left\{\sqrt{m}\Pi^2\right\} - \left\langle\sqrt{m}\Pi^2\right\rangle\left\{\sqrt{m}\right\}\right], \quad (3.60)$$

where

$$\{f(q)\} = \int_0^{q_0} [f(q) - \langle f \rangle]\, dq. \tag{3.61}$$

If function f is 2π–periodic over q, then the value of $\{f\}$, due to eq. (3.61), is its periodic antiderivative. The first equality in eq. (3.60) determines $q_0(\varphi)$. As follows from this equality, at ultra-high energies rotation can be stationary ($q \approx \varphi$) only in the case $m = \text{const}$. According to eqs. (3.58) and (3.60) the resulting expansions take an especially simple form at $m = \text{const}$

$$\begin{aligned} h &= \frac{s^2}{2m} + \langle \Pi \rangle + \frac{m}{2s^2}\left(\langle \Pi^2 \rangle - \langle \Pi \rangle^2\right) + \frac{1}{s^4}\dots, \\ q &= \varphi - \{\Pi\}\frac{m}{s^2} + \left[\Pi\{\Pi\} + 2\langle \Pi \rangle \{\Pi\} - \frac{3}{2}\{\Pi^2\}\right]\frac{m^2}{2s^4} + \frac{1}{s^6}\dots. \end{aligned} \tag{3.62}$$

If we make use of formulae (3.62) for describing fast rotations of system (3.54) then we can obtain the following expressions

$$h = \frac{s^2}{2} + \frac{a_1 + a_2}{2} + \frac{a_1^2 + a_2^2}{16s^2}\dots, \tag{3.63}$$

$$\begin{aligned} q = \varphi - \frac{1}{2s^2}\left(a_1 \sin\varphi + \frac{a_2}{k}\sin k\varphi\right) + \frac{1}{4s^4}\left\{\frac{7}{8}\left(a_1^2 \sin 2\varphi + \frac{a_2^2}{k}\sin 2k\varphi\right) + \right. \\ \left. \frac{a_1 a_2}{k}\left[\frac{k^2 + 5k + 1}{k}\sin(k+1)\varphi + \frac{5k - 1 - k^2}{k}\sin(k-1)\varphi\right]\right\}\dots. \end{aligned}$$

Let us notice an essential difference between these expansions and those obtained above for the librations about the position of stable equilibrium, see eqs. (3.37)-(3.39). In the end, this difference causes the inefficiency of introducing the harmonic variables ξ and η (3.36) for the analysis of rotation. Indeed, due to eqs. (3.59) and (3.61), the generalised coordinate q is a non-analytical many-valued function of these variables since, for example, $\varphi = -\operatorname{Arctan}(\eta/\xi)$.

3.4 Backbone curve and its steepness coefficient

Periodic (libration or rotational) motions as well as quasi-periodic rotations of the conservative system with one degree of freedom are usually characterised by the type of dependence of the energy constant h on frequency ω. This dependence determines the backbone curve which can be given implicitly in the following form

$$h = h(s), \quad \omega = \frac{dh}{ds} = \omega(s). \tag{3.64}$$

In practical applications, the dependence of the oscillation amplitude a or amplitude of the first harmonic of the Fourier expansion upon the frequency is often understood as the backbone curve. Such definitions are sometimes more illustrative however when they have a particular character and are usually used for the description of libration of continuous systems. In the latter case, the amplitude (3.16) is a monotonically increasing function of energy and, for this reason, dependences $a(\omega)$ and $h(\omega)$ are qualitatively coincident. However, in general, the concept "amplitude" has no sense for rotational motion.

In the case of isochronism, when the frequency is constant and depends only on the system parameters (rather than initial conditions), the backbone curve degenerates. Oscillations of the linear harmonic oscillator are isochronous. Small librations ($h, s \to 0$) of the system with distributed parameters about the equilibrium are approximately isochronous. In this case, due to eqs. (3.41) and (3.42), we have

$$q = \sqrt{\frac{2\omega_0 s}{c_0}} \cos\varphi, \qquad h = \omega_0 s \tag{3.65}$$

up to higher order terms.

It is necessary to mention that isochronism is not a property which is inherent in linear conservative systems. For instance, it is shown that dependence $\Pi = \Pi(q)$ for $\left.\frac{d\Pi}{dq}\right|_{q=0} = 0$, $\left.\frac{d^2\Pi}{dq^2}\right|_{q=0} > 0$ and $m = \text{const}$ prescribed for positive q can be uniquely continued into the negative direction $q < 0$ in such a way that the resulting oscillations are isochronous, see [43].

An example of a non-linear isochronous object is an oscillator with a piecewise linear restoring force. The governing equation is

$$m\ddot{q} + (c_1 + c_2 \operatorname{sign} q)\, q = 0 \qquad (c_1 > c_2 > 0)\,. \tag{3.66}$$

The general solution has a constant period which is independent of the energy

$$T = \pi \left(\sqrt{\frac{m}{c_1 + c_2}} + \sqrt{\frac{m}{c_1 - c_2}} \right). \tag{3.67}$$

In the general non-linear case, the isochronism of a system is characterised by the proportionality between the energy and the action

$$h = \omega s \qquad (\omega = \text{const})\,. \tag{3.68}$$

Let us notice that by virtue of eqs. (3.68) and (3.27), the modulus of periodicity of Hamilton's action, see eq. (3.25), vanishes for the isochronous systems, i.e. $\Lambda = 0$. This follows directly from equality (3.27). In other words, the values of the kinetic and potential energies, averaged over the period, coincide for the isochronous systems.

Whereas isochronism is a manifestation of peculiar degeneration, anisochronism, that is the ability of the frequency to change with changing initial conditions in a finite or infinite range $\left(\omega^{(1)}, \omega^{(2)}\right)$ is an inherent property of the conservative systems with one degree of freedom. The main characteristic of the backbone curve of an anisochronous object is the steepness coefficient of the backbone curve

$$e = \frac{d^2h}{ds^2} = \omega\frac{d\omega}{dh}. \tag{3.69}$$

Provided that the steepness coefficient is positive (negative) in a certain region of the energy constant, and thus energy increases (decreases) as frequency grows, then the considered conservative object is referred to as hard (soft) anisochronous and the backbone curve is called hard (soft). The sign of the derivative of the libration amplitude with respect to frequency $\frac{da}{d\omega}$ coincides with the sign of the steepness coefficient. According to eq. (3.28), the steepness coefficient of the backbone curve is proportional to the derivative of the periodicity modulus of Hamilton's action with respect to energy

$$e = \frac{\omega^3}{2\pi h}\frac{d\Lambda}{dh}. \tag{3.70}$$

Thus, for hard anisochronous systems, action Λ is positive and increases with the growth in energy, whereas $\Lambda < 0$ for soft anisochronous systems. By virtue of this, the value of Hamilton's action Λ characterises the degree of anisochronism of the system. One can also express this fact by saying that the greater the difference between the values of the kinetic and potential energies averaged over the period, the stronger the anisochronism.

Rotational motions are always hard anisochronous. Indeed, the rotation frequency (3.49) increases as h increases. Due to eqs. (3.56) and (3.58), at ultra-high energies ($h, s \to \infty$) the steepness coefficient of the backbone curve of rotation is constant up to the values of order $1/s^4$

$$e = \left\langle\sqrt{m}\right\rangle^{-2} > 0. \tag{3.71}$$

The character of the librations in the small neighbourhood of the equilibrium position is determined by the sign of the coefficient of s^2 in expansion (3.41). Hence, these librations are soft anisochronous ($e < 0$), if

$$\frac{5c_1^2}{3c_0^2} + \frac{2m_2}{m_0} > \frac{m_1^2}{m_0^2} + \frac{2m_1c_1}{m_0c_0} + \frac{c_2}{c_0}. \tag{3.72}$$

An intersecting separatrix is associated with the passage over the minimum value of ω ($\omega = 0$, $T = \infty$) and is always characterised by a change in the type of anisochronism. More precisely, we speak of the separatrices

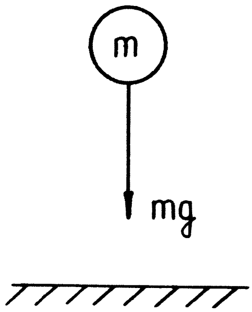

FIGURE 3.1.

which separate the regions of different periodic (quasi-periodic) motions, for example, librations and rotations. The type of anisochronism is conditioned by the single-valued dependence $\omega(h)$, see eqs. (3.15) and (3.49). For instance, a physical pendulum [4] is soft anisochronous ($e < 0$) in the region of libration and hard anisochronous ($e > 0$) in the region of rotation. However a change in the type of anisochronism can occur at other values of the energy when $e \sim \dfrac{d\omega}{dh}$ (which does not correspond to any separatrix) and is not caused by a qualitative change in the character of the motion. In the vicinity of these values, the system behaves approximately as an isochronous one. Therefore, in various regions of the energy, the same conservative system can expose soft and hard isochronism or even be isochronous.

A simple example of a hard isochronous object is an ideal conservative rotor, for which

$$K = \frac{1}{2}J\dot{q}^2, \quad \Pi = 0. \tag{3.73}$$

Here q and J denote an angle and a moment of inertia about the rotation axis. Taking into account eqs. (3.45), (3.53) and (3.69) we can describe the rotational motion of system (3.73) by means of the following relationships

$$q = \varphi, \quad h = \frac{s^2}{2J} = \frac{J\omega^2}{2}, \; e = \frac{1}{J}. \tag{3.74}$$

From the perspective of anisochronism, the impact-oscillatory object shown in Fig. 3.1 has the opposite properties. This is the case of soft isochronous oscillation (libration) for a ball moving vertically over a one-sided "conservative" stop. It is easy to see that in this case

$$h = \sqrt[3]{\frac{3m}{8}}\,(\pi g s)^{2/3} = \frac{\pi m g^2}{2\omega^2}. \tag{3.75}$$

As an example of a conservative object with limited frequency range, we consider an oscillator with a symmetric piecewise-linear force characteristic, Fig 3.2. Librations of the oscillator with low intensity, i.e. when $0 < h < c_1 l^2/2$, are isochronous with frequency $\omega = \omega^{(1)} = \sqrt{c_1/m}$. When $h > c_1 l^2/2$, the additional spring is active and oscillations gain hard anisochronous character. The upper limit of the frequency range is

$$\omega^{(2)} = \lim_{h \to \infty} \omega = \sqrt{\frac{c_1 + c_2}{m}}. \tag{3.76}$$

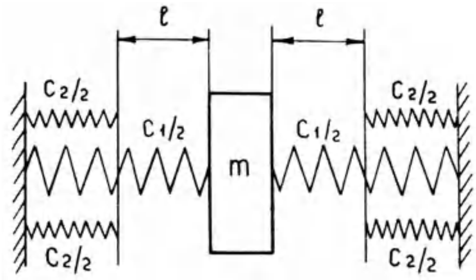

FIGURE 3.2.

3.5 Dynamical system with an invariant relationship

It is known that the motion of an autonomous mechanical or electromechanical second-order system can be described by Lagrange's differential equation, see eqs. (2.4) and (2.51),

$$\left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}\right) K = Q. \tag{3.77}$$

Here q and K denote the generalised coordinate of the system and its kinetic energy, whilst Q designates the generalised force which, in mechanics, is usually represented as the sum of two components

$$Q = Q_1 + Q_2. \tag{3.78}$$

Force Q_1, referred to as a potential force, can be determined by the formula

$$Q_1 = -\frac{d\Pi}{dq},$$

where $\Pi(q)$ is the generalised potential energy of the system, see eq. (2.15).

If $Q_2 = 0$, eq. (3.77) admits an energy integral (3.10). As already mentioned, in this case the gyroscopic forces are absent and the system is conservative. The non-conservative correction term Q_2 characterises the supply of external energy into the system.

Separation of the potential generalised forces is usually carried out by means of pure physical reasoning. However, from a general perspective, it is necessary to announce the presence of the total arbitrariness to the choice of generalised potential energy of the system. We can liquidate this arbitrariness in such a way that the non-potential parts of the generalised forces gain certain specific properties.

Let us imagine the situation in which all the singular trajectories of the non-conservative systems comprise a subset of the set of integral curves of the conservative part of the system. We refer to the non-potential generalised forces resulting in such a transformation of the phase space of the system as natural ones. Introducing natural non-potential forces leads to stabilising motions on a finite (or countable) number of energetic levels, the character of the motion of the system at each level being unchanged.

The dissipative forces of the type $Q_2 = f(\dot{q})$ $(\dot{q}f(\dot{q}) < 0,\ f(0) = 0)$ are natural since the only singular motions are the "conservative" equilibria. Examples of self-excited systems with natural non-potential forces are less trivial. The simplest among them is the "non-conservative" mechanical rotor whose motion is governed by the equation

$$J\ddot{q} = M(\dot{q}). \tag{3.79}$$

Here the natural non-potential force M stabilises the "conservative" uniform rotation of the rotor $q = \nu t + \alpha$ with frequency ν, the latter being the root of equation $M(\nu) = 0$. However, for the overwhelming majority of self-excited (and other) systems, the "physical" non-potential forces are not in general natural and the question of their separation becomes non-trivial.

Let us consider an arbitrary autonomous second-order system

$$\dot{q} = Q(q,p), \quad \dot{p} = P(q,p). \tag{3.80}$$

Let us assume that functions Q and P are regular with respect to q and p in the considered region of the phase plane. Let system (3.80) admit a periodic particular solution

$$q = a(\varphi), \quad p = b(\varphi), \tag{3.81}$$

where a and b are continuous 2π−periodic functions of the fast phase $\varphi = \omega t + \alpha$. The following invariant relationship [57]

$$H(q,p) = h, \tag{3.82}$$

describing a closed trajectory in the phase plane, corresponds to this solution. We assume that function $H(q,p)$ is regular in the neighbourhood of the given trajectory and that h is positive. We represent functions Q and P as sums of two regular components

$$Q = Q_1 + Q_2, \quad P = P_1 + P_2, \tag{3.83}$$

where Q_1 and P_1 fulfill the following identity

$$\frac{\partial H}{\partial q} Q_1 + \frac{\partial H}{\partial p} P_1 = 0. \tag{3.84}$$

Next we change the variables $q, p \to \varphi, f$ in the original system and require fulfillment of the following identities

$$\begin{aligned} \frac{\partial p}{\partial \varphi} \nu(f) &= P_1, \quad \frac{\partial q}{\partial \varphi} \nu(f) = Q_1, \ \nu(0) = \omega, \\ q|_{f=0} &= a(\varphi), \quad p|_{f=0} = b(\varphi). \end{aligned} \tag{3.85}$$

In what follows, functions $q(\varphi, f)$ and $p(\varphi, f)$ are assumed to be 2π–periodic with respect to φ and regular with respect to f about point $f = 0$. Inserting the new variables into the original system (3.80) yields

$$\frac{\partial q}{\partial \varphi} \dot{\varphi} + \frac{\partial q}{\partial f} \dot{f} = Q_1 + Q_2, \quad \frac{\partial p}{\partial \varphi} \dot{\varphi} + \frac{\partial p}{\partial f} \dot{f} = P_1 + P_2. \tag{3.86}$$

The determinant of this system, due to eq. (3.85), is

$$\Delta = \frac{1}{\nu} \left(Q_1 \frac{\partial p}{\partial f} - P_1 \frac{\partial q}{\partial f} \right). \tag{3.87}$$

The standard form of the equations under consideration in the new variables takes the form

$$\dot{f} = \frac{1}{\Delta} \left(P_2 \frac{\partial q}{\partial \varphi} - Q_2 \frac{\partial p}{\partial \varphi} \right), \quad \dot{\varphi} = \nu(f) + \frac{1}{\Delta} \left(Q_2 \frac{\partial p}{\partial f} - P_2 \frac{\partial q}{\partial f} \right). \tag{3.88}$$

If the non-conservative part is absent, i.e. $Q_2 = P_2 = 0$, this system has a first integral $f = 0$ which, in the case $f = 0$ (see eq. (3.85)), must degenerate into the invariant relationship (3.82). Thus, in the general case we have

$$f = F(H), \tag{3.89}$$

where

$$F(h) = 0. \tag{3.90}$$

In the simplest case we can take

$$F = H - h. \tag{3.91}$$

Besides, since the right hand sides of system (3.88) are analytical about point $f = 0$ for the solution $f = 0$, $\varphi = \omega t + \alpha$ (which is identical to solution (3.81)) it is necessary and sufficient that the relationships

$$Q_2 = F(H)\lambda_1(q,p), \; P_2 = F(H)\lambda_2(q,p) \tag{3.92}$$

hold. Hence, the considered system, admitting the invariant relationship (3.82), can be represented in terms of the new variables in the following form

$$\dot{f} = f\Psi(f,\varphi), \quad \dot{\varphi} = \nu(f) + f\Phi(f,\varphi), \tag{3.93}$$

where

$$\Psi = \frac{1}{\Delta}\left(\lambda_1\frac{\partial p}{\partial \varphi} - \lambda_2\frac{\partial q}{\partial \varphi}\right), \quad \Phi = \frac{1}{\Delta}\left(\lambda_1\frac{\partial p}{\partial f} - \lambda_2\frac{\partial q}{\partial f}\right). \tag{3.94}$$

In the small neighbourhood of point $f = 0$, this system can be treated as a system with fast rotating scalar phase α. It is not easy to show that this system is locally integrable about the solution $f = 0$, $\varphi = \omega t + \alpha$, see Chapter 1. Indeed, the variational system of equations

$$\delta\dot{f} = \Psi(0,\varphi)\,\delta f, \quad \delta\dot{f} = \left(\frac{d\nu}{df}\right)\delta f + \Phi(0,\varphi)\,\delta f, \tag{3.95}$$

where $\left(\frac{d\nu}{df}\right) = \left.\frac{d\nu}{df}\right|_{f=0}$, admits two independent solutions, one of which is periodic

$$\delta f^{(1)} = 0, \quad \delta\varphi^{(1)} = 1, \tag{3.96}$$

whilst the second one depends exponentially on time

$$\begin{aligned} \delta f^{(2)} &= u(\varphi), \quad \delta\varphi^{(2)} = \frac{1}{\omega}\int_0^{\varphi}\left[\left(\frac{d\nu}{df}\right) + \Phi(0,\varphi)\right]u(\varphi)\,d\varphi + C, \\ u &= \exp\frac{1}{\omega}\int_0^{\varphi}\Psi(0,\eta)\,d\eta, \end{aligned} \tag{3.97}$$

with the integration constant C being determined below. The orbital stability of the solution under consideration is ensured by negativeness of the characteristic exponent

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi}\Psi(0,\eta)\,d\eta < 0, \tag{3.98}$$

see eq. (3.97). The periodic solution $\delta\bar{f}, \delta\bar{\varphi}$ of the system conjugated to (3.95) is determined from the conditions of normalisation and orthogonality

$$\delta\bar{f}\delta f^{(i)} + \delta\bar{\varphi}\delta\varphi^{(i)} = \delta_{1i} \quad (i = 1, 2) \tag{3.99}$$

and is given by

$$\delta\bar{f} = -\frac{\delta\varphi^{(2)}}{u}, \quad \delta\bar{\varphi} = 1. \tag{3.100}$$

By virtue of the periodicity condition $\delta\bar{f}\big|_0^{2\pi} = 0$, the integration constant C is equal to

$$C = -\frac{1}{1 - \exp(2\pi\lambda/\omega)} \int_0^{2\pi} \left[\left(\frac{d\nu}{df}\right) + \Phi(0, \varphi)\right] u(\varphi)\, d\varphi. \tag{3.101}$$

Due to eqs. (3.82) and (3.84), relationships (3.85) determining the change of variables $q, p \to \varphi, f$ are consistent. At the same time, this change admits a certain arbitrariness. We can reduce this arbitrariness by choosing functions Q_1 and P_1 with the help of eq. (3.84) such that the conservative part of this system has the canonical structure. In this case it is natural that variables φ, f gain the character of the "action-angle" variables ($f = s$). To this end, it is necessary to adopt the following additional constraint

$$s = \frac{1}{2\pi\nu} \int_0^{2\pi} pQ_1 d\varphi. \tag{3.102}$$

The above arbitrariness, however, is not completely removed. It is sufficient to say that the choice of dependence $\nu(s)$ remains arbitrary. It is shown in the next section that splitting (3.83) can be carried out rationally in a certain vicinity of the singular point of eq. (3.80). Therefore, this system uniquely reduces to a typical form in terms of the "action-angle" variables. Let us notice that the above analysis can be generalised on a more general case in which the particular solution (3.81) is not periodic and correspondingly a non-closed phase trajectory corresponds to the invariant relationship (3.82). Variable φ no longer has the meaning of fast phase and we can not speak of variable t as an action.

The usefulness of the studied procedure, despite its arbitrariness, is demonstrated through the modified Van-der-Pol equation

$$m\ddot{x} + cx = -\beta\left(\frac{m\dot{x}^2}{2} + \frac{cx^2}{2} - h\right)\dot{x}, \tag{3.103}$$

where m, c, β and h are positive constants. We can reduce the problem to the following non-dimensional system

$$q' = p, \quad p' = -q - \gamma\left(\frac{p^2 + q^2}{2} - 1\right)p. \tag{3.104}$$

Here $q = \sqrt{c/h}$, $\gamma = \beta h/\sqrt{mc}$, and a prime denotes differentiation with respect to the non-dimensional time $\tau = \sqrt{c/m}t$.

System (3.104) admits the following family of periodic solutions

$$q = \sqrt{2}\cos\varphi, \quad p = -\sqrt{2}\sin\varphi, \quad \varphi = \tau + \alpha. \tag{3.105}$$

In accordance with eq. (3.91), we enter the following new variable

$$f = \frac{1}{2}\left(p^2 + q^2\right) - 1, \tag{3.106}$$

so that the required substitution is as follows

$$q = \sqrt{2\left(f+1\right)}\cos\varphi, \quad p = -\sqrt{2\left(f+1\right)}\sin\varphi. \tag{3.107}$$

Then the equations of motion take the following form

$$f' = -2\gamma f\left(f+1\right)\sin^2\varphi, \; \varphi' = -1 - \gamma f \sin\varphi\cos\varphi. \tag{3.108}$$

In the vicinity of the solution $f = 0$, $\varphi = \tau + \alpha$, this variational system of equations has, is addition to periodic solution (3.96), the following exponential solution, see eq. (3.97)

$$\begin{aligned}
\delta f &= \exp\left(-\gamma\left[\varphi - \frac{1}{2}\sin 2\varphi\right]\right), \\
\delta\varphi &= -\gamma\int_0^{\varphi}\sin\varphi\cos\varphi\exp\left(-\gamma\left[\varphi - \frac{1}{2}\sin 2\varphi\right]\right)d\varphi + \\
&\quad \frac{\gamma}{1-\exp\left(-\pi\gamma\right)}\int_0^{\pi}\sin\varphi\cos\varphi\exp\left(-\gamma\left[\varphi - \frac{1}{2}\sin 2\varphi\right]\right)d\varphi.
\end{aligned} \tag{3.109}$$

By virtue of eq. (3.100) the periodic solution of the system conjugated to the variational equations is cast in the form

$$\begin{aligned}
\delta\bar{f} &= \gamma\exp\left(\gamma\left[\varphi - \frac{1}{2}\sin 2\varphi\right]\right)\left[\int_0^{\varphi}\sin\varphi\cos\varphi\exp\left(-\gamma\left[\varphi - \frac{1}{2}\sin 2\varphi\right]\right)d\varphi\right. \\
&\quad \left.-\frac{1}{1-\exp\left(-\pi\gamma\right)}\int_0^{\pi}\sin\varphi\cos\varphi\exp\left(-\gamma\left[\varphi - \frac{1}{2}\sin 2\varphi\right]\right)\right]d\varphi, \\
\delta\bar{\varphi} &= 1.
\end{aligned} \tag{3.110}$$

3.6 Canonisation of a system about the equilibrium position

As shown in Sec. 2.6, the truncated conservative system, which is obtained from eq. (3.80) by setting $Q_2 = P_2 = 0$, is a system of Pfaff's equations

$$\dot{q} = \frac{1}{M}\frac{\partial F_1(H)}{\partial p}, \quad \dot{p} = -\frac{1}{M}\frac{\partial F_1(H)}{\partial q}, \tag{3.111}$$

where F_1 is function of variable $H = H(q,p)$ of the same type as

$$F_2(H) = \int F(H)\,dH, \tag{3.112}$$

whereas factor M satisfies eq. (2.57) in which Q_1 and P_1 are substituted for Q and P respectively.

Then, taking into account eqs. (2.56), (3.83), (3.92) and (3.112) we can write for system (3.80) that

$$MQ = \frac{\partial F_1}{\partial p} + \frac{dF_2}{dH}\mu_1, \; MP = -\frac{\partial F_1}{\partial q} + \frac{dF_2}{dH}\mu_2, \tag{3.113}$$

where $\mu_1 = M\lambda_1, \mu_2 = M\lambda_2$.

We refer to the process of selection of the true non-potential forces in system (3.80), determined by eq. (3.92), as canonisation.

As already mentioned above, the corresponding procedure is not unique. However we will try to make it precise in the small vicinity of the equilibrium. Without loss of generality, we assume that $Q(0,0) = P(0,0) = 0$.

As functions Q and P are regular about zero, the following power series expansions

$$Q = a_{11}q + a_{12}p + \sum_{i=2}^{\infty} Q_i, \quad P = a_{21}q + a_{22}p + \sum_{i=2}^{\infty} P_i \tag{3.114}$$

are valid, where a_{jk} $(j,k = 1,2)$ are real constants, and Q_i and P_i $(i = 1,2\ldots)$ are homogeneous forms of the $i - th$ degree in variables q and p.

The eigenvalues of the linearised system (3.80) are defined as the roots of the quadratic equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{3.115}$$

Let us assume that $\lambda_1 \neq \lambda_2$ and $\lambda_1\lambda_1 \neq 0$ and thus

$$(a_{11} - a_{22})^2 + 4a_{12}a_{21} \neq 0, \; a_{11}a_{22} - a_{12}a_{21} \neq 0. \tag{3.116}$$

Then after a non-singular change of variables $q,p \to x,y$, the original system (3.80) can be brought to the form

$$\dot{x} = X(x,y), \quad \dot{y} = Y(x,y), \tag{3.117}$$

where in the considered vicinity of the zero

$$X = \lambda_1 x + \sum_{i=2}^{\infty} X_i, \quad Y = \lambda_2 y + \sum_{i=2}^{\infty} Y_i. \tag{3.118}$$

The new variables x and y as well as the coefficients of expansion (3.118) are, in general, complex-valued. We assume additionally that function H, which determines the invariant relationship (3.82), is regular in the vicinity of zero.

To begin with, we assume that only linear terms $X = \lambda_1 x$, $Y = \lambda_2 y$ are kept in expansions (3.118). Then it is easy to see that relationships (3.113) are satisfied if

$$\begin{aligned} M &= 1, \quad H = xy, \quad F_1 = \frac{\lambda_1 - \lambda_2}{2} H, \quad F_2 = \frac{\lambda_1 + \lambda_2}{2} H, \\ \mu_1 &= \frac{\partial H}{\partial y}, \quad \mu_2 = \frac{\partial H}{\partial x}. \end{aligned} \tag{3.119}$$

In order to prove this, it is sufficient to substitute X, Y and x, y instead of Q, P and q, p, respectively.

Let us admit now that the two latter relationships (3.119) are valid in the general case, i.e. for all the consequent terms of expansion. Then these equalities can be rewritten as follows

$$MX = \frac{\partial F_1}{\partial y} + \frac{dF_2}{dy}, \quad MY = -\frac{\partial F_1}{\partial x} + \frac{dF_2}{dx} \tag{3.120}$$

or

$$MX = \frac{\partial U}{\partial y}, \quad MY = -\frac{\partial V}{\partial x}, \tag{3.121}$$

where $U(H) = F_1 + F_2$ and $V(H) = F_1 - F_2$.

Assuming the existence of functions U and V is equivalent to the following statement. If functions X and Y satisfy expansions (3.118), then there exists such function $M(x, y)$ that the quantities

$$U = \int^y MXdy, \quad V = -\int^x MYdx \tag{3.122}$$

are independent of each other. In other words, there exists an analytical function M which satisfies the following non-linear integro-differential equation

$$J\left(\frac{U, V}{x, y}\right) = 0. \tag{3.123}$$

Interestingly, for the conservative system ($Q_2 = P_2 = 0$) we have $F_2 = 0$ and thus $U = V$, see also eq. (2.62). In other words, for a conservative system, the quantity M is determined from the equality $U = V$, see Sec. 2.6. In non-conservative systems, the integral invariant of Poincaré [84] is absent and equality $U = V$ can not be attained. However, as it is shown below, it is possible to achieve a mutual dependence of these functions by means of a special choice of M in a certain vicinity of the equilibrium.

We proceed now to the proof of the existence of the canonisation process based on equalities (3.121) or (3.123). The solutions are sought in the following series form

$$M = 1 + \sum_{i=1}^{\infty} M_i, \quad H = xy + \sum_{i=3}^{\infty} H_i,$$
$$U = \lambda_1 \left(H + \sum_{i=2}^{\infty} u_i H^i \right), \quad V = -\lambda_2 \left(H + \sum_{i=2}^{\infty} v_i H^i \right), \tag{3.124}$$

where M_i and H_i are homogeneous forms of $i - th$ degree in x and y. Let us insert these series in relationships (3.121). Then the corresponding multiplication yields the following equations of balance of the forms of even degree $2i$

$$\lambda_1 x M_{2i-1} - \lambda_1 \frac{\partial H_{2i+1}}{\partial y} = A_{2i}, \quad \lambda_2 y M_{2i-1} - \lambda_2 \frac{\partial H_{2i+1}}{\partial x} = B_{2i} \tag{3.125}$$

and the forms of odd degree $2i + 1$

$$\begin{aligned} \lambda_1 x M_{2i} - \lambda_1 \frac{\partial H_{2i+2}}{\partial y} - \lambda_1 (i+1) u_{i+1} x^{i+1} y^i &= A_{2i+1}, \\ \lambda_2 y M_{2i} - \lambda_2 \frac{\partial H_{2i+2}}{\partial x} - \lambda_2 (i+1) v_{i+1} x^i y^{i+1} &= B_{2i+1}. \end{aligned} \tag{3.126}$$

In eqs. (3.125) and (3.126), $i = 1, 2, 3, \ldots$, A_i and B_i are homogeneous forms of the $i - th$ degree depending on coefficients of the forms $M_1, \ldots, M_{i-2}$; $H_2, \ldots, H_i$; $X_2, \ldots, X_i$; $Y_2, \ldots, Y_i$ as well as on values $u_2, \ldots, u_{i'}$; $v_2, \ldots, v_{i'}$ where i' denotes the integer part of $\dfrac{i+1}{2}$.

Simple analysis of relationships (3.125) shows that the coefficients of the forms of odd degree M_{2i-1} and H_{2i+1} are always uniquely determined, regardless of the forms A_{2i} and B_{2i} determined earlier. As for the forms of the even degree M_{2i} and H_{2i+2}, all of their coefficients are uniquely determined from eq. (3.126) except those of $(xy)^i$ and $(xy)^{i+1}$ (denoted by c and d respectively). Equating coefficients of $x^{i+1}y^i$ in the first equation in (3.126) and of $x^{i+1}y^{i+1}$ in the second equation in (3.126) leads to the following two equations relating the four unknown variables c, d, u_{i+1} and v_{i+1}

$$c - (i+1)\, d - (i+1)\, u_{i+1} = \frac{a}{\lambda_1}, \quad c - (i+1)\, d - (i+1)\, v_{i+1} = \frac{b}{\lambda_2}, \tag{3.127}$$

where a and b are the corresponding coefficients of forms A_{2i+1} and B_{2i+1}.

The structure of relationships (3.127) is such that the following restrictions

$$u_{i+1} - v_{i+1} = \frac{1}{i+1}\left(\frac{b}{\lambda_2} - \frac{a}{\lambda_1}\right) \tag{3.128}$$

are imposed on the differences of the coefficients u_{i+1} and v_{i+1}. Thus, the coefficients of the expansion for functions $U(H)$ and $V(H)$ can not be taken arbitrary, for example, they can not be set to zero or be coincident.

On the other hand, expansions of the quantities in eq. (3.121) are determined up to arbitrariness in the coefficients of $(xy)^i$ in the expressions for form M_{2i} and H_{2i}. The question of convergence of the obtained formal series is rather difficult and is beyond the scope of the present book. It is clear that any choice of the coefficients of $(xy)^i$ affects the convergence.

Now let eq. (3.115) have a double non-zero root λ with square elementary divisors, that is

$$a_{11}a_{22} - a_{12}a_{21} \neq 0, \quad (a_{11} - a_{22})^2 + 4a_{12}a_{21} = 0, \tag{3.129}$$

and besides a_{12} and a_{21} do not vanish simultaneously.

Then, the original system can be expressed in form (3.117) by a non-singular linear transformation of variables, where X and Y have the following form

$$X = \lambda x + y + \sum_{i=2}^{\infty} X_i, \quad Y = \lambda y + \sum_{i=2}^{\infty} Y_i \tag{3.130}$$

in the vicinity of zero.

Let us find such expressions for H, M, F_1 and F_2 that relationships (3.120) or (3.121) are satisfied by linear approximation.

It is easy to see that in the case under consideration they are satisfied if we take

$$M = 1, \quad H = \lambda xy + \frac{y^2}{2}, \quad F_1 = 0, \quad F_2 = H. \tag{3.131}$$

As before, we will try to obtain formal series for M, H, U and V which satisfy equalities (3.121). The construction is carried out by analogy with eq. (3.124) and is based upon the following general expressions

$$\begin{aligned} M &= 1 + \sum_{i=1}^{\infty} M_i, \quad H = \lambda xy + \frac{y^2}{2} + \sum_{i=3}^{\infty} H_i, \\ U &= H + \sum_{i=2}^{\infty} u_i H^i, \quad V = H + \sum_{i=2}^{\infty} v_i H^i. \end{aligned} \tag{3.132}$$

Inserting these relationships into (3.121), rearranging the resulting expression and equating the forms of even degrees $2i$ yields the following equations

$$(\lambda x + y)\, M_{2i-1} - \frac{\partial H_{2i+1}}{\partial y} = A_{2i}, \quad \lambda y M_{2i-1} - \frac{\partial H_{2i+1}}{\partial x} = B_{2i}. \quad (3.133)$$

Balance of the forms of degree $2i + 1$ results in the equations

$$(\lambda x + y)\, M_{2i} - \frac{\partial H_{2i+2}}{\partial y} - (i+1)\, u_{i+1}\, (y + \lambda x) \left(\lambda xy + \frac{y^2}{2} \right)^i = A_{2i+1},$$

$$\lambda y M_{2i} - \frac{\partial H_{2i+2}}{\partial x} - (i+1)\, v_{i+1} \lambda y \left(\lambda xy + \frac{y^2}{2} \right)^i = B_{2i+1}. \quad (3.134)$$

Here A_i and B_i depend on the same values which appear in eqs. (3.125) and (3.126).

It is easy to show that the coefficients of the forms of odd degrees are uniquely determined from eq. (3.133) for any A_{2i} and B_{2i}. The forms of even degrees, M_{2i} and H_{2i+2} are determined up to the coefficients of the combinations $\left(\lambda xy + y^2/2\right)^i$ and $\left(\lambda xy + y^2/2\right)^i$.

Denoting these coefficients, as above, by c and d we obtain the equations which are analogous to (3.127)

$$c - (i+1)\, d - (i+1)\, u_{i+1} = a, \quad c - (i+1)\, d - (i+1)\, v_{i+1} = b. \quad (3.135)$$

This equation confirms the above made conclusions, see eq. (3.127), that coefficients u_{i+1} and v_{i+1} can not be taken arbitrarily.

Therefore, also in the case of a double eigenvalue we have succeeded in obtaining formal series (3.132) up to arbitrary coefficients of $\left(\lambda xy + y^2/2\right)^i$ which satisfy eq. (3.121).

If $\lambda_1 \neq \lambda_2$, then the coefficients of $(xy)^i$ are arbitrary, whilst if $\lambda_1 = \lambda_2$ then the coefficients of $\left(\lambda xy + y^2/2\right)^i$ are arbitrary. This arbitrariness associated with constructing the solutions of eq. (3.121), gives rise to the question of whether it is necessary to change expansion (3.124) (or (3.132)) for a change in the arbitrary coefficients.

Let us denote expansions (3.124) or (3.132) which do not contain terms with undetermined coefficients by M_*, H_*, U_* and V_*. Another solution corresponding to a certain choice of the coefficients in the expansions for M and H can be obtained in two stages.

First, a new multiplier is introduced

$$M = M_* \left(1 + \alpha_1 H_* + \alpha_2 H_*^2 + \ldots\right), \quad (3.136)$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots$ are chosen subsequently from the condition that the coefficients of terms $(xy)^i$ are equal to given values. The new functions

$U(H_*)$ and $V(H_*)$ are introduced by the formulae

$$U = \int^{H_*} \frac{\partial U_*}{\partial H_*}(1+\alpha_1 H_* + \dots)\, dH_*,$$
$$V = \int^{H_*} \frac{\partial V_*}{\partial H_*}(1+\alpha_1 H_* + \dots)\, dH_*. \qquad (3.137)$$

Then, we introduce the new argument of functions U and V

$$H = H_* + \beta_1 H_*^2 + \beta_2 H_*^3 + \dots, \qquad (3.138)$$

where values $\beta_1, \beta_2, \dots$ are obtained from the condition that the undetermined coefficients are equal to the given values.

The formal solution constructed in such a way satisfies, as before, system (3.121). However it does not lead to any change in the geometric properties of the canonised dynamical system. Hence the mentioned arbitrariness is not essential, however it determines the convergence of the obtained series.

3.7 Canonised form of the equations of motion

Let us derive the real-valued form of the completely canonised equations of motion for the system under consideration. To this aim, it is sufficient to return to the original physical variables q and p of the problem. For the case of $\lambda_1 \neq \lambda_2$ we finally arrive at the following equations

$$\frac{dq}{d\tau} = \frac{\partial R}{\partial p} + \left[(a_{11}-a_{22})\frac{\partial S}{\partial p} - 2a_{12}\frac{\partial S}{\partial p}\right], \quad \left(d\tau = M^{-1}dt\right)$$
$$\frac{dp}{d\tau} = -\frac{\partial R}{\partial q} + \left[(a_{11}-a_{22})\frac{\partial S}{\partial q} + 2a_{21}\frac{\partial S}{\partial p}\right], \qquad (3.139)$$

where

$$R(E) = E + \sum_{i=2}^{\infty} r_i E^i, \quad S(E) = \frac{(a_{11}+a_{22})\,E}{(a_{11}-a_{22})^2 + 4a_{12}a_{21}} + \sum_{i=2}^{\infty} s_i E^i,$$
$$E(q,p) = \frac{1}{2}\left[a_{12}p^2 + (a_{11}-a_{22})\,qp - aq^2\right] + \sum_{i=3}^{\infty} E_i, \qquad (3.140)$$

r_i and s_i are some real constants, and E_i are homogeneous forms of power i in q and p.

In the case of $\lambda_1 = \lambda_2$ we obtain the following canonised equations in real-valued form

$$\frac{dq}{d\tau} = \frac{\partial R}{\partial p} + \left[2\frac{\partial S}{\partial p} + \left(1 + \frac{a_{22} - a_{11}}{a_{12}}\right)\frac{\partial S}{\partial p}\right], \tag{3.141}$$

$$\frac{dp}{d\tau} = -\frac{\partial R}{\partial q} + \left[\left(1 + \frac{a_{22} - a_{11}}{a_{12}}\right)\frac{\partial S}{\partial q} + \frac{1}{2}\left(1 + \left(1 + \frac{a_{22} - a_{11}}{a_{12}}\right)^2\right)\frac{\partial S}{\partial p}\right],$$

where

$$R = E + \sum_{i=2}^{\infty} r_i E^i, \quad S = E + \sum_{i=2}^{\infty} s_i E^i, \quad E = \frac{p^2}{2}\left(1 + \frac{a_{11} + a_{22}}{2}\right) +$$

$$\frac{pq}{2a_{12}}\left[(a_{11} + a_{22})(a_{22} - a_{11} + a_{12}) + a_{11} - a_{22}\right] + \tag{3.142}$$

$$\frac{q^2}{8a_{12}^2}\left[(a_{11} - a_{22})^2 (1 - a_{11} - a_{22}) - 2a_{12}(a_{11} + a_{22})\right] + \sum_{i=3}^{\infty} E_i.$$

The components in square brackets in eqs. (3.139) and (3.141) can be treated as components of the natural non-potential force acting on the system, whereas function E can be understood as its natural energy.

The canonised form of the equations of motion (3.139) and (3.141) allows us to immediately determine the singular solutions of the system under consideration.

Clearly, they satisfy the conservative (and thus integrable) part of the non-canonised equations. The values of the constant of the natural energy $E = e$, corresponding to these solutions, are determined from the condition of stationarity of function S, that is

$$\left.\frac{dS}{dE}\right|_{E=e} = 0. \tag{3.143}$$

Generally speaking, eq. (3.143) may have several solutions in the considered vicinity of the zero singular point. Among them, there may occur solutions corresponding to other equilibria of the truncated conservative subsystem, as well as to certain libration, escaping or limitation trajectories. In the latter two cases, the corresponding motion gains a special character in non-conservative systems.

In a rather general form, the canonised system can be written down as follows

$$\dot{q} = \frac{1}{M}\left[\frac{\partial R}{\partial p} + F(E)\, X(q,p)\right],$$

$$\dot{p} = \frac{1}{M}\left[-\frac{\partial R}{\partial q} + F(E)\, Y(q,p)\right], \tag{3.144}$$

where functions $M(q,p)$ and $R(E)$ are considered as being positive in a certain vicinity of zero. Introducing new time by means of eq. (2.60), we can cast the conservative part of system (3.144) in the canonical form

$$\frac{dq}{d\tau} = \frac{\partial R}{\partial p}, \quad \frac{dp}{d\tau} = -\frac{\partial R}{\partial q}. \tag{3.145}$$

This system, see Secs. 3.2 and 3.3, admits a periodic general integral

$$\begin{aligned} q &= q(\psi, s), \quad p = p(\psi, s), \quad \psi = \nu(s)\tau + \alpha, \\ s &= \frac{1}{2\pi}\int_0^{2\pi} p\frac{\partial q}{\partial \psi}d\psi, \end{aligned} \tag{3.146}$$

where p and q are 2π−periodic functions of ψ. The original system (3.144) in the "action-angle" variables (s, ψ) has the form

$$\begin{aligned} \frac{ds}{d\tau} &= \left(\frac{\partial q}{\partial \psi}X - \frac{\partial p}{\partial \psi}Y\right)F, \\ \frac{d\psi}{d\tau} &= \nu + \left(\frac{\partial p}{\partial s}X - \frac{\partial q}{\partial s}Y\right)F. \end{aligned} \tag{3.147}$$

Relationship (2.60) is equivalent to the following one

$$t = \int_0^{\tau} M d\tau = \frac{1}{\nu}\int_0^{\psi} M d\psi. \tag{3.148}$$

By virtue of eq. (3.148) the following equality

$$\frac{2\pi}{\omega} = \frac{1}{\nu}\int_0^{2\pi} M d\psi \tag{3.149}$$

holds true, where ω denotes the frequency of the fast phase $\varphi = \omega(s)t + \alpha$ corresponding to the "old " time. It is easy to obtain the expression for ω

$$\omega = \frac{\nu}{\langle M\rangle} \quad \left(\langle M\rangle = \frac{1}{2\pi}\int_0^{2\pi} M d\psi\right). \tag{3.150}$$

Then we have

$$d\varphi = \omega dt = \nu\frac{M}{\langle M\rangle}d\tau = \frac{M}{\langle M\rangle}d\psi. \tag{3.151}$$

Inserting eqs. (3.151) and (2.60) into (3.147) we finally arrive at the equations of motion in the form

$$\dot{s} = F_*(s)U(\varphi, s), \quad \dot{\varphi} = \omega + F_*(s)V(\varphi, s), \tag{3.152}$$

where it is taken into account that "energy" E is function of action s and the following notation

$$\begin{aligned} F_* &= F\left[E\left(s\right)\right], \quad U = \frac{1}{\langle M \rangle}\left(\frac{\partial q}{\partial \varphi}Y - \frac{\partial q}{\partial \varphi}X\right), \\ U &= \frac{1}{\langle M \rangle}\left(\frac{\partial p}{\partial s}X - \frac{\partial q}{\partial s}Y\right) \end{aligned} \tag{3.153}$$

is introduced.

The periodic solution of system (3.152) is characterised by the relationships

$$s, a = \text{const}, \quad F_*\left(s\right) = 0, \quad \varphi = \omega\left(s\right)t + \alpha. \tag{3.154}$$

The variational equation about this solution is identical to eq. (3.95) and hence admits the single periodic solution

$$\delta s = 0, \quad \delta\varphi = 1. \tag{3.155}$$

The corresponding periodic solution to the conjugated system $\delta\bar{s}, \delta\bar{\varphi}$ is determined by means of eq. (3.100)

$$\delta\bar{s} = -\frac{1}{\Delta}\left[\frac{1}{\omega}\int\limits_0^{\varphi}\left(\frac{d\omega}{ds} + \frac{dF_*\left(s\right)}{ds}V\right)\Delta d\varphi + c\right], \quad \delta\bar{\varphi} = 1, \tag{3.156}$$

where

$$\begin{aligned} \Delta\left(\varphi, s\right) &= \exp\left(\frac{1}{\omega}\frac{dF_*\left(s\right)}{ds}\int\limits_0^{\varphi}Ud\varphi\right), \qquad (3.157) \\ c\left(s\right) &= \frac{1}{\exp\left(2\pi\lambda/\omega\right) - 1}\left(\frac{1}{\omega}\int\limits_0^{2\pi}\left(\frac{d\varphi}{ds} + \frac{dF_*\left(s\right)}{ds}V\right)\Delta d\varphi\right) \end{aligned}$$

and λ denotes a characteristic exponent given by the formula

$$\lambda = \frac{1}{2\pi}\frac{dF_*\left(s\right)}{ds}\int\limits_0^{2\pi}Ud\varphi. \tag{3.158}$$

Due to eq. (3.98), the condition for the orbital stability of the periodic solution (3.154) is $\lambda < 0$.

4

Conservative systems with many degrees of freedom

4.1 Action-angle variables

We consider the canonical equations of motion for conservative system with n degrees of freedom $(q_1, , \dots, q_n)$ which is described by the Hamiltonian function $H = H(q_1, \dots, q_n, p_1, \dots, p_n)$. Integration of such systems can be carried out rationally by means of the autonomous canonical univalent change of variables $q_i, p_i \to \Phi_i, S_i$, [1], [60]. This transformation does not affect the value of Hamiltonian H and is usually given in an implicit form

$$p_i = \frac{\partial W}{\partial q_i}, \quad \Phi_i = \frac{\partial W}{\partial S_i}, \tag{4.1}$$

where W denotes the generating function of the transformation

$$W = W(q_1, \dots, q_n, S_1, \dots, S_n), \tag{4.2}$$

depending upon the "old" coordinates and the "new" momenta.

Let us assume that Hamiltonian h of the system under consideration depend only on the new momenta $S_1, \dots, S_n$. Then taking into account the first set of relationships in eq. (4.1) and condition $H = h$, we arrive at the autonomous Hamilton-Jacobi equation

$$H\left(q_1, \dots, q_n, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_n}\right) = h(S_1, \dots, S_n). \tag{4.3}$$

This partial differential equation serves to determine the generating function W of the canonical transformation introduced above.

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The canonical equations of motion in the new variables are given by

$$\dot{S}_i = -\frac{\partial h}{\partial \Phi_i} = 0, \quad \dot{\Phi}_i = \frac{\partial h}{\partial S_i}. \tag{4.4}$$

Thus, the new momenta are constant, i.e. they are the integrals of motion, whereas the new coordinates vary as linear functions of time

$$\Phi_i = \frac{\partial h}{\partial S_i} t + A_i, \ (A_i = \text{const}). \tag{4.5}$$

Hence, it is necessary to determine the total integral (rather than the general one) of the Hamilton-Jacobi equation. When this integral depending on m arbitrary constants $S_1, \ldots, S_n$ is determined, then the general integral of the original system depending upon $2n$ constants $S_1, \ldots, S_n, A_1, \ldots, A_n$ is obtained by resolving eq. (4.1)

$$q_i = q_i(\Phi_1, \ldots, \Phi_n, S_1, \ldots, S_n), \ p_i = p_i(\Phi_1, \ldots, \Phi_n, S_1, \ldots, S_n). \tag{4.6}$$

We also notice that the generating function W is Lagrange's action for the system under consideration

$$W = \int p_i \dot{q}_i dt. \tag{4.7}$$

This can be proved easily by differentiating W, eq. (4.2), with respect to t and taking into account eqs. (4.1) and (4.4)

$$\frac{dW}{dt} = \frac{\partial W}{\partial q_i}\dot{q}_i + \frac{\partial W}{\partial S_i}\dot{S}_i = p_i \dot{q}_i. \tag{4.8}$$

As suggested by Stäckel, see [60], for a rather wide class of conservative mechanical systems with stationary holonomic constraints ($K = K_2 = \frac{1}{2}p_i\dot{q}_i$, eq. (2.15)) the total integral of the Hamilton-Jacobi equation is determined with the help of the method of separation of variables

$$W = \sum_{i=1}^{n} W_i(q_i, S_1, \ldots, S_n). \tag{4.9}$$

It is worthwhile mentioning that usually, while solving these problems, the separation of variables turns out to be possible only by an appropriate choice of the coordinates.

In conservative systems, more general indications of the separability of variables in the Hamilton-Jacobi equation are formulated in [100] and [37]. There also exist examples of integrability of the conservative systems of the impact-oscillatory type, see Sec. 4.4. In all these cases, the general integral of the system in non-small finite or infinite regions of the phase space has quasi-periodic (or conditionally periodic) character with not more than n

periods [5]. The new momenta can always be taken such that the new coordinates $\Phi_1, \ldots, \Phi_n$ gain the meaning of the partial rotating phases of the quasi-periodic functions q_i and p_i. In other words, functions q_i and p_i can be taken to be periodic with respect to $\Phi_1, \ldots, \Phi_n$ with unit period

$$q_i = \sum_{j_1,\ldots,j_n=-\infty}^{\infty} a_{j_1,\ldots,j_n}(S_1,\ldots,S_n)\exp\left(2\pi\sqrt{-1}\left[j_1\Phi_1+\ldots+j_n\Phi_n\right]\right),$$

$$p_i = \sum_{j_1,\ldots,j_n=-\infty}^{\infty} b_{j_1,\ldots,j_n}(S_1,\ldots,S_n)\exp\left(2\pi\sqrt{-1}\left[j_1\Phi_1+\ldots+j_n\Phi_n\right]\right). \tag{4.10}$$

These relationships characterise the quasi-periodic motions of libration (oscillatory) type with respect to all phases. On the other hand, if coordinate q_i has the meaning of an angle and for this reason Hamiltonian H is $2\pi-$periodic with respect to q_i, then rotation with respect to any of these phases, say Φ_j, is observed. Instead of the first relationship in eq. (4.10) we should write

$$q_i = 2\pi\Phi_j + \sum_{j_1,\ldots,j_n=-\infty}^{\infty} a_{j_1,\ldots,j_n}(S_1,\ldots,S_n)\exp\left(2\pi\sqrt{-1}\left[j_1\Phi_1+\ldots+j_n\Phi_n\right]\right). \tag{4.11}$$

The partial frequencies of the considered quasi-periodic motions are dependent on the new momenta and, by virtue of eq. (4.5), are equal to

$$\Omega_i = \frac{\partial h}{\partial S_i}. \tag{4.12}$$

Finally, we notice that relationships (4.10) and (4.11) for fixed values of $S_1, \ldots, S_n$ determine a hypersurface in the $2n-$dimensional phase space of the system, which is Cartesian and cylindrical with respect to libration and rotational coordinates, respectively. This hypersurface is topologically equivalent to a $2n-$dimensional torus, with phases $\Phi_1, \ldots, \Phi_n$ being angular coordinates of this torus. With this in view, the phase space of the integrable conservative system is said to be entirely filled by tori of the quasi-periodic solutions.

Let us proceed now to determine the physical meaning of the new momenta $S_1, \ldots, S_n$ which are conjugated to the phases of the quasi-periodic general integral. Since the transformation $q_i, p_i \to \Phi_i, S_i$ is canonical, we consider the following equations

$$[\Phi_i, S_j] = \delta_{ij}, \tag{4.13}$$

where δ_{ij} denotes the Kronecker symbol and

$$[\Phi_i, S_j] = \frac{\partial q_l}{\partial \Phi_i}\frac{\partial p_l}{\partial S_j} - \frac{\partial p_l}{\partial \Phi_i}\frac{\partial q_l}{\partial S_j} \tag{4.14}$$

denotes Lagrange's brackets, [60]. Let us rewrite this relationship as follows

$$\frac{\partial}{\partial S_j}\left(p_l \frac{\partial q_l}{\partial \Phi_i}\right) - \frac{\partial}{\partial \Phi_i}\left(p_l \frac{\partial q_l}{\partial S_j}\right) = \delta_{ij} \tag{4.15}$$

and integrate it over $\Phi_1, \dots, \Phi_n$ from 0 to 1. Then, both in the case of libration and rotation, we obtain

$$\frac{\partial}{\partial S_j} \int_0^1 \cdots \int_0^1 p_l \frac{\partial q_l}{\partial \Phi_i} d\Phi_1 \cdots d\Phi_n = \delta_{ij}. \tag{4.16}$$

The result is the required formulae

$$S_i = \int_0^1 \cdots \int_0^1 p_j \frac{\partial q_j}{\partial \Phi_i} d\Phi_1 \cdots d\Phi_n. \tag{4.17}$$

Assume that Lagrange's action of the system (4.7) is expressed in terms of the new coordinates and momenta, that is $W = W(q(\Phi, S), S)$. By virtue of eq. (3.111) the "total" partial derivative of W with respect to S_i is equal to

$$\frac{\partial' W}{\partial S_i} = \frac{\partial W}{\partial S_i} + \frac{\partial W}{\partial q_j}\frac{\partial q_j}{\partial S_i} = \Phi_i + \frac{\partial q_j}{\partial S_i} p_j. \tag{4.18}$$

The quantities p_j and $\dfrac{\partial q_j}{\partial S_i}$ can be represented as generalised Fourier series both for libration and rotation, see eqs. (4.10) and (4.11). Direct integration of the latter equation yields

$$W = S_i \Phi_i + \Delta W(\Phi_1, \dots, \Phi_n, S_1, \dots, S_n), \tag{4.19}$$

where function ΔW can also be represented by Fourier series. As follows from eq. (4.19) the new momentum S_i is equal to the increment in Lagrange's action of the system during one revolution of the phase Φ_i under unaltered values of the other phases

$$S_i = W|_{\Phi_i}^{\Phi_i + 1}. \tag{4.20}$$

This explains why the new momenta conjugated to the phases are referred to as the partial moduli of periodicity of Lagrange's action or partial integrals of action.

In what follows, along with the introduced "action-angle" variables S_i, Φ_i we use their circular analogues, cf. (3.24)

$$\varphi_i = 2\pi\Phi_i = \omega_i t + \alpha_i, \quad s_i = \frac{S_i}{2\pi} = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} p_j \frac{\partial q_j}{\partial \varphi_i} d\varphi_1 \cdots d\varphi_n,$$

$$\omega_i(s_1, \dots, s_n) = 2\pi\Omega_i = \frac{\partial h}{\partial s_i}, \quad \alpha_i = 2\pi A_i. \tag{4.21}$$

Evidently, the original canonical variables are $2\pi-$periodic with respect to the circular phases $\varphi_1, \dots, \varphi_n$.

It is worthwhile mentioning another characteristic equality

$$s_i\omega_i = \lim_{t\to\infty} \frac{W}{t}, \tag{4.22}$$

which follows directly from eq. (4.19).

Transformation to the "action-angle" variables is simplified in the presence of the cyclic coordinates. For instance, let the conservative system be described by n positional generalised coordinates $q_1, \dots, q_n$ and a single cyclic coordinate q, the latter having the meaning of the angle. Then its Hamiltonian is written as follows

$$H = H\left(q_1, \dots, q_n, p, p_1, \dots, p_n\right), \tag{4.23}$$

and the cyclic momentum p is an integral of motion. Transformation to the "action-angle" variables $p, p_1, \dots, p_n, q, q_1, \dots, q_n \rightarrow s, s_1, \dots, s_n, \varphi, \varphi_1, \dots, \varphi_n$ is performed by means of the generating function

$$W = sq + W_1\left(q_1, \dots, q_n, s_1, \dots, s_n\right). \tag{4.24}$$

The positional coordinates are independent of the phase φ

$$q_i = q_i\left(\varphi_1, \dots, \varphi_n, s, s_1, \dots, s_n\right), \quad p_i = p_i\left(\varphi_1, \dots, \varphi_n, s, s_1, \dots, s_n\right). \tag{4.25}$$

At the same time, we have for the cyclic variables

$$p = \frac{\partial W}{\partial q} = s, \quad \varphi = q + \frac{\partial W_1}{\partial s}. \tag{4.26}$$

Therefore, the cyclic momentum is the corresponding partial action, whilst the cyclic coordinate differs from the phase conjugated to this action in a quasi-periodic function of the "positional" phases $\varphi_1, \dots, \varphi_n$. Let us find the partial derivative of the identity $H = h$ with respect to s and account for eq. (4.25) and the equality $p = s$. We also take into account that for the canonical transformation

$$\left[\varphi_j, s\right] = \frac{\partial q_i}{\partial \varphi_j}\frac{\partial p_i}{\partial s} - \frac{\partial p_i}{\partial \varphi_j}\frac{\partial q_i}{\partial s} + \frac{\partial q}{\partial \varphi_j} = 0 \tag{4.27}$$

for any $j = 1 \dots, n$. The result is

$$\frac{\partial h}{\partial s} = \frac{\partial H}{\partial p} + \left(\frac{\partial q_i}{\partial \varphi_j}\frac{\partial p_i}{\partial s} - \frac{\partial p_i}{\partial \varphi_j}\frac{\partial q_i}{\partial s}\right)\dot{\varphi}_j = \frac{\partial H}{\partial p} - \frac{\partial q}{\partial \varphi_j}\dot{\varphi}_j. \tag{4.28}$$

Direct averaging of this expression leads to the following formula for the cyclic frequency

$$\omega = \frac{\partial h}{\partial s} = \frac{1}{(2\pi)^n}\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\partial H}{\partial p} d\varphi_1 \cdots d\varphi_n. \tag{4.29}$$

4.2 Conservative systems moving by inertia

Integrable conservative systems with stationary constraints which move by inertia ($\Pi = 0, K = \frac{1}{2}p_i\dot{q}_i = h, W = 2ht$) are rather specific. In this case, equality (4.22) takes the form

$$h = \frac{1}{2}s_i\omega_i. \tag{4.30}$$

Clearly, the following differential identity holds true

$$s_i\frac{\partial h}{\partial s_i} = 2h. \tag{4.31}$$

Direct proof convinces us that function h of variables $s_1, \ldots, s_n$ satisfying eq. (4.31) is given by

$$h = \frac{s_n^2}{2D}, \tag{4.32}$$

where D is a function of the action ratios, i.e. $D = D\left(\frac{s_1}{s_n}, \ldots, \frac{s_{n-1}}{s_n}\right)$.

In the case of the system with one degree of freedom, i.e. $n = 1, s_n = s$, the value of D does not depend on the action and is equal to the value which is the inverse of the steepness coefficient of the backbone curve, see eq. (3.71),

$$D = \left\langle\sqrt{m}\right\rangle^2 = \frac{1}{e}. \tag{4.33}$$

In accordance with eq. (4.33), for a "conservative" crank mechanism shown in Fig. 4.1 we have

$$\begin{aligned} K &= \frac{m(q)}{2}\dot{q}^2, \quad m(q) = J + mr^2\sin^2 q, \\ D &= \frac{4}{\pi^2}E^2(k)\left(J + mr^2\right), \quad k = \sqrt{\frac{mr^2}{J + mr^2}}, \end{aligned} \tag{4.34}$$

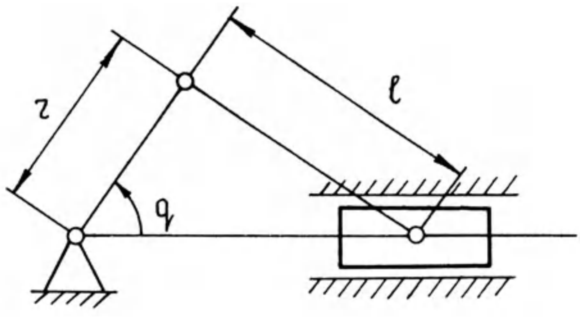

FIGURE 4.1.

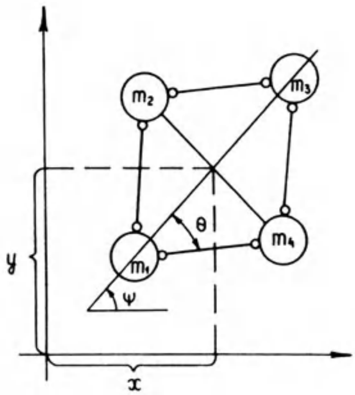

FIGURE 4.2.

where m and J denote respectively the mass of the slide block and the moment of inertia of the balanced crank (the mass of the connecting rod as well as the values of order r/l are neglected). Furthermore, $E(k)$ denotes the complete elliptic integral of the second kind

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \alpha} d\alpha. \tag{4.35}$$

The problems of this class, having several degrees of freedom, are much more complex. For example, let us consider the free planar motion of a rhombus consisting of bars connected by joints, see Fig. 4.2. The masses are assumed to be concentrated at the joints. The generalised coordinates are angles ϑ and ψ as well as the coordinates x and y of the centre of the rhombus. The kinetic energy of the rhombus does not depend upon x and y and, up to the terms linear in $\dot{x}$ and $\dot{y}$, is equal to

$$\begin{aligned} K &= \frac{1}{2}(m_1 + m_2 + m_3 + m_4)\left(\dot{x}^2 + \dot{y}^2\right) + \frac{m_1 + m_3}{2} l^2 \left(\dot{\vartheta}^2 \sin^2 \vartheta + \right. \\ & \left. \dot{\psi}^2 \cos^2 \vartheta\right) + \frac{m_1 + m_2}{2} l^2 \left(\dot{\vartheta}^2 \cos^2 \vartheta + \dot{\psi}^2 \sin^2 \vartheta\right) + \ldots, \end{aligned} \tag{4.36}$$

with l denoting the length of the rhombus side. The cyclic momenta $p_x = \dfrac{\partial K}{\partial \dot{x}}$ and $p_x = \dfrac{\partial K}{\partial \dot{y}}$ are constant. Besides, as K, eq. (4.36), is independent of ψ, the Routhian function of the system [60] is also independent of ψ

$$K - p_x \dot{x} - p_y \dot{y} = (m_1 + m_2 + m_3 + m_4) l^2 R,$$

$$R = \frac{a}{2}\left(\dot{\vartheta}^2 \sin^2 \vartheta + \dot{\psi}^2 \cos^2 \vartheta\right) + \frac{b}{2}\left(\dot{\vartheta}^2 \cos^2 \vartheta + \dot{\psi}^2 \sin^2 \vartheta\right) - c\dot{\vartheta}\dot{\psi}. \tag{4.37}$$

Here the following notation is introduced

$$a = \mu_1 + \mu_3 - (\mu_1 - \mu_3)^2, \quad b = \mu_2 + \mu_4 - (\mu_2 - \mu_4)^2,$$
$$c = (\mu_1 - \mu_3)(\mu_2 - \mu_4), \quad \mu_i = m_i / \sum_{j=1}^{4} m_j \quad (i = 1,2,3,4). \tag{4.38}$$

It is easy to see that the Routhian function (4.37) is equal to the kinetic energy of the rhombus in the inertial system moving together with its center of mass. Then the "new" cyclic integral

$$\frac{\partial R}{\partial \dot{\psi}} = \left(a\cos^2\vartheta + b\sin^2\vartheta\right)\dot{\psi} - c\dot{\vartheta} = p_\psi \tag{4.39}$$

ensures that the kinetic moment about the centre of mass is constant. In addition to this, the "relative" kinetic energy of the rhombus is constant at any time instant, that is $R = h$.

It follows from eq. (4.39) that ψ rotates with frequency ω_ψ

$$\psi = \varphi_\psi + \Delta\psi, \quad \varphi_\psi = \omega_\psi t + \alpha_\psi, \quad \omega_\psi = \left\langle \frac{p_\psi + c\dot{\vartheta}}{a\cos^2\vartheta + b\sin^2\vartheta} \right\rangle, \tag{4.40}$$

the variables $\Delta\psi$ and ϑ being independent of phase φ_ψ. Here $\alpha_\psi = \text{const}$ and $\langle\ \rangle$ designates time averaging. Then the partial action conjugated to φ_ψ is equal to the corresponding cyclic momentum, i.e. $s_\psi = p_\psi$, see Sec. 4.1 for the proof of this statement in the general case. Additionally we take into account equality (4.32), i.e.

$$h = \frac{s_\psi^2}{2D}. \tag{4.41}$$

As a result, the integrals of the relative kinetic moment (4.39) and the energy $R = h$ can be cast as follows

$$\dot{\psi} = \frac{2}{u}\left(s_\psi + c\dot{\vartheta}\right), \quad \dot{\vartheta}^2 = \frac{s_\psi^2}{2D}\frac{u - 2D}{v}, \tag{4.42}$$

where we introduced the notation

$$u = a + b + (a - b)\cos 2\vartheta, \quad v = ab - c^2 + \frac{(a-b)^2}{4}\sin^2 2\vartheta. \tag{4.43}$$

Let us notice that the value

$$ab - c^2 = 4\mu_1\mu_3(\mu_2 + \mu_4) + 4\mu_2\mu_4(\mu_1 + \mu_3) \tag{4.44}$$

is always positive and thus, during the motion, velocities $\dot{\vartheta}$ and $\dot{\psi}$ are always bounded.

Angle ϑ, determined by eq. (4.42), can change according to the law of a periodic libration $\vartheta(t) = -\vartheta(t + T_\vartheta/2)$ with amplitude ϑ_* and circular frequency $\omega_\vartheta = 2\pi/T_\vartheta$ given by the formulae

$$\cos 2\vartheta_* = \frac{2D - a - b}{a - b}, \quad \omega_\vartheta = \frac{\pi s_\psi}{2\sqrt{2D}} \left[\int_0^{\vartheta_*} \sqrt{\frac{v}{u - 2D}} d\vartheta \right]^{-1}. \tag{4.45}$$

The forthcoming analysis depends essentially on the sign of the difference $a - b$, which, because $\sum_{i=1}^{4} \mu_i = 1$, can be represented in the form

$$a - b = 4(\mu_1\mu_3 - \mu_2\mu_4). \tag{4.46}$$

If $a > b$ and thus $m_1 m_3 > m_2 m_4$ then by virtue of eq. (4.45), for the periodic libration to exist, it is necessary to require fulfillment of the inequality $b < D < a$. Conversely, if $0 < D < b$ then $\dot{\vartheta}^2 > 0$ and the rhombus sides perform rotational motions relative to each other. Value $D = a$ is the maximum and corresponds to a stable quasi-stationary solution $\vartheta = 0$.

When $a < b$, i.e. $m_1/m_2 < m_4/m_3$, then the ranges of libration and rotation of angle ϑ has the form $a < D < b$ and $0 < D < a$, respectively. The maximum value $D = b$ corresponds to the quasi-stationary solution $\vartheta = \pi/2$.

In the case of libration, the explicit expression for the partial frequency of rotation of angle ψ, see eq. (4.40), is given by

$$\omega_\psi = \frac{\omega_\vartheta}{2\pi} \int_0^{2\pi/\omega_\vartheta} \frac{2}{u} \left(s_\psi + c\dot{\vartheta} \right) dt = \frac{4\omega_\vartheta}{\pi} \int_0^{\vartheta_*} \left(\sqrt{\frac{2Dv}{u - 2D}} + c \right) \frac{d\vartheta}{u}. \tag{4.47}$$

We insert expressions (4.45) and (4.47) for the partial frequencies of the considered two-frequency general integral into the following equation

$$\frac{s_\psi^2}{2D} = \frac{1}{2}(s_\psi\omega_\psi + s_\vartheta\omega_\vartheta), \tag{4.48}$$

which is a direct sequence of eqs. (4.30) and (4.32). Then we arrive at the equation which enables us to determine D as function of the ratio of partial actions

$$\frac{\pi}{4}\frac{s_\vartheta}{s_\psi} = \frac{1}{\sqrt{2D}} \int_0^{\vartheta_*} \sqrt{v(u - 2D)} \frac{d\vartheta}{u} - \frac{c}{2\sqrt{ab}} \arctan\sqrt{\frac{b(a - D)}{a(D - b)}}$$
$$(a > b). \tag{4.49}$$

Due to eq. (4.49), D is a monotonically decreasing function of the ratio s_ϑ/s_ψ, with the maximum value $D = a$ $(\vartheta_* = 0)$ corresponding to the minimum of the positional action $s_\vartheta = 0$.

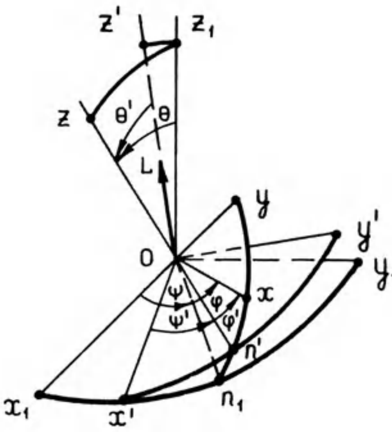

FIGURE 4.3.

Therefore, the required expression for the Hamiltonian of the problem as a function of the partial actions s_ϑ and s_ψ is given parametrically by means of eqs. (4.41) and (4.49).

It is important for the forthcoming analysis that the quasi-static motion

$$\begin{aligned} \vartheta &= 0, \quad \psi = \frac{s_\psi}{a}t + \alpha_\psi, \quad m_1 m_3 > m_2 m_4, \\ \vartheta &= \frac{\pi}{2}, \quad \psi = \frac{s_\psi}{b}t + \alpha_\psi, \quad m_1 m_3 < m_2 m_4 \end{aligned} \tag{4.50}$$

corresponds to the minimum value $s_\vartheta = W|_{\varphi_\vartheta}^{\varphi_\vartheta+2\pi}$, see eq. (4.20). For the considered class $(\Pi = 0, W = 2V)$ we can speak both of Lagrange's action W, cf. eq. (3.19), and Hamilton's action V, cf. eq. (3.25).

4.3 The problem of spherical motion of a free rigid body (Euler's case)

The problem of free motion $(\Pi = 0)$ of a rigid body having a fixed point, see eq. (1.99), is ideologically close to the above problem. However in the present problem two cyclic coordinates are not directly selected and thus the fast phases do not correspond to the original coordinates. At the same time, solving the problem by quadratures is caused by the fact that, in addition to the energy integral $K = h$, the kinetic moment of the body about point O has constant absolute value and direction.

Proceeding to determining the dependence of the Hamiltonian of the body on the partial actions in form (4.32), we introduce three Cartesian coordinate systems with the same origin at the immovable point O, see Fig. 4.3:

1) the moving system $Oxyz$ of the principal axes of the body;

2) fixed system $Ox_1y_1z_1$;

3) special fixed system $Ox'y'z'$ whose axis Oz' is directed along the kinetic moment L, whereas axis Ox' lies in plane Ox_1y_1.

The position of system $Oxyz$ relative to systems $Ox_1y_1z_1$ and $Ox'y'z'$ is determined by Euler's angles which are denoted by ψ, ϑ, φ and $\psi', \vartheta', \varphi'$, respectively. The projections of the angular velocity vector ω on the moving axes are given in terms on the auxiliary Euler angles by the formulae [7]

$$\begin{aligned} p &= \dot{\psi}' \sin\vartheta' \sin\varphi' + \dot{\vartheta}' \cos\varphi', \quad q = \dot{\psi}' \sin\vartheta' \cos\varphi' - \dot{\vartheta}' \sin\varphi', \\ r &= \dot{\psi}' \cos\vartheta' + \dot{\varphi}'. \end{aligned} \tag{4.51}$$

Similar formulae prescribe the dependence of p, q, r on Euler's angles ψ, ϑ, φ.

Without loss of generality we assume that $A > B > C$ where A, B, C are the moments of inertia about the principal axes $Oxyz$, see eq. (1.99). The relationships

$$L \sin\vartheta' \sin\varphi' = Ap, \quad L \sin\vartheta' \cos\varphi' = Bq, \quad L \cos\varphi' = Cr \tag{4.52}$$

are obtained by projecting the kinetic moment L on the moving axis. By virtue of eqs. (4.51) and (4.52) and equality $K = \frac{1}{2}\left(Ap^2 + Bq^2 + Cr^2\right)$ one obtains the following equations for the momenta corresponding to the auxiliary Euler's angles

$$p_{\psi'} = \frac{\partial K}{\partial \dot{\psi}'} = L, \quad p_{\psi'} = \frac{\partial K}{\partial \dot{\vartheta}'} = 0, \quad p_{\psi'} = \frac{\partial K}{\partial \dot{\varphi}'} = Cr. \tag{4.53}$$

The spherical motion of the body is governed by the first set of equations in (1.99) for projections of the angular velocity on the moving axis $Oxyz$

$$A\dot{p} - (B - C)\, qr = 0, \quad B\dot{q} - (C - A)\, pr = 0, \quad C\dot{r} - (A - B)\, pq = 0. \tag{4.54}$$

The method of integration of system (4.54) is well known, [7]. First, by means of the first integrals $K = h$ and $A^2p^2 + B^2q^2 + C^2r^2 = L^2$ one removes p and r from the above system. The result is a first-order differential equation with separated variables whose general integral is expressed in terms of Jacobi's elliptic sine-function. Next p,r and then the auxiliary angles of nutation ϑ' and spin φ', see (4.51), are expressed in terms of the elliptic functions. All of these quantities are $2\pi-$periodic functions of the first phase $\varphi_1 = \omega_1 t + \alpha$.

The auxiliary angle of precession ψ' is determined by quadratures from the last equation in (4.51) and is a superposition of uniform rotation with the second partial frequency ω_2 and a $2\pi-$periodic function of φ_1

$$\psi' = \varphi_2 + \Delta\psi\,(\varphi_1) \quad (\varphi_2 = \omega_2 t + \alpha_2)\,. \tag{4.55}$$

According to eqs. (4.23) and (4.53), the partial action corresponding to phase φ_2 is equal to the modulus of the kinetic moment

$$\begin{aligned} s_2 &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left(L\frac{\partial\psi'}{\partial\varphi_2}+cr\frac{\partial\varphi'}{\partial\varphi_2}\right)d\varphi_1 d\varphi_2 = L, \\ &\left(\frac{\partial\psi'}{\partial\varphi_2}=1,\quad \frac{\partial\varphi'}{\partial\varphi_2}=0\right). \end{aligned} \tag{4.56}$$

Thus, instead of (4.32) we can write, see eq. (4.41), that

$$L^2 = 2hD\left(\frac{s_1}{s_2}\right). \tag{4.57}$$

In order to expand the equality

$$\frac{s_2^2}{D} = s_1\omega_1 + s_2\omega_2, \tag{4.58}$$

which is analogous to eq. (4.48), we recall the formulae for the partial frequencies of the two-frequency integral under consideration

$$\omega_1 = \frac{\pi}{2K(k)}\sqrt{\frac{2h(A-D)(B-C)}{ABC}}, \quad \omega_2 = \frac{\sqrt{2hD}}{C}\left[1-\frac{A-C}{A}\frac{P(n,k)}{K(k)}\right], \tag{4.59}$$

where the complete elliptic integral of the third kind is denotes as

$$\begin{aligned} P(n,k) &= \int_0^{\pi/2}\frac{d\alpha}{(1+n\cos^2\alpha)\sqrt{1-k^2\sin^2\alpha}}, \\ k^2 &= \frac{(D-C)(A-B)}{(A-D)(B-C)}, \quad n = \frac{C(A-B)}{A(B-C)}, \end{aligned} \tag{4.60}$$

[39], and it is adopted that $C < D < B$. As a result we arrive at the equation for the constant D as a function of the ratio of the partial actions

$$(C-D)K(k) + \frac{D}{A}(A-C)P(n,k) = \frac{\pi s_1}{2s_2}\sqrt{\frac{CD}{AB}(A-D)(B-C)}. \tag{4.61}$$

In the case of $B < D < A$ such equation is constructed by analogy, with the modulus of the elliptic function being equal to $1/k$, eq. (4.60). If $D = B$ the motion is an aperiodic "separatrix".

Hence, the considered two-frequency solution depends on four constants s_1, s_2, α_1 and α_2 and completely determines the spherical motion of the body relative to the special fixed system $Ox'y'z'$. The general integral of the problem depending on six constants can be constructed if the position of the special system $Ox'y'z'$ is determined relative to the basic system $Ox_1y_1z_1$. To this aim, it is expedient to enter the canonical pair of Andooyer [7], namely angle φ_3 between the fixed basic axis Ox_1 and fixed special axis Ox' and the cyclic momentum $s_3 = p_3 = \dfrac{\partial K}{\partial \dot{\psi}}$ corresponding to the true angle of precession ψ. Let us notice that the introduced quantities φ_3 and s_3 are canonically conjugate but they are not true "action-angle" variables.

It is easy to show that s_3 is equal to the projection of the kinetic moment L on fixed axis Oz_1. Thus, the transition from system $Ox_1y_1z_1$ to system $Ox'y'z'$ is carried out by the following two turns: the first turn is about axis Oz_1 through angle φ_3 and the second turn is about axis Ox' through angle $\arccos \dfrac{s_3}{s_2}$. Then $\psi = \psi' + \varphi_3$.

It is essential that, due to its physical meaning, $s_2 = L$ is always greater than the absolute values of s_1 and s_2. The formula for the true angle of nutation $\vartheta = \angle(z, z_1)$ is given by

$$\cos\vartheta = \frac{s_3}{s_2}\cos\vartheta' + \sqrt{1 - \frac{s_3^2}{s_2^2}}\sin\vartheta'\cos\psi'. \tag{4.62}$$

Here ϑ is seen to be dependent on $s_1, s_3, \alpha_1, \alpha_3$. In addition to this, ψ and φ depend on φ_3.

4.4 On degeneration of integrable conservative systems

As the above example show, the total number of pairs of "true" action-angle variables is often less than the number of degrees of freedom in the conservative system under consideration. The cause is that the partial frequencies of the general quasi-periodic integral (4.6) are mutually commensurable. For example, let the relationships

$$\sum_{i=1}^{n} n_{ji}\omega_i = 0 \quad (j = 1, \dots, l,\ l < n) \tag{4.63}$$

with mutually simple integers $n_{j1}, \dots, n_{jn}$ be identically satisfied for $s_1, \dots,$ s_n. In this case, the motion has $m = n - l$ true partial periods and the system is referred to as l times degenerate. Correspondingly, the number of the true partial frequencies as well as the true pairs of the action-angle variables is equal to m. Indeed, let us carry out the canonical transformation of the variables $s_i, \varphi_i \to s'_i, \varphi'_i$ with the following generating function

$$F = \sum_{i=1}^{n}\sum_{j=1}^{l} n_{ji} s'_j \varphi_i + \sum_{j=l+1}^{n} s'_j \varphi_j. \qquad (4.64)$$

Due to eq. (4.64), the new variables are equal to

$$\begin{aligned} \varphi'_j &= \sum_{i=1}^{n} n_{ji}\varphi_i = \sum_{i=1}^{n} n_{ji}\alpha_i \quad j = 1, \dots, l, \\ \varphi'_j &= \varphi_i = \omega_j t + \alpha_j, \quad j = l+1, \dots, n. \end{aligned} \qquad (4.65)$$

Thus, the l first new frequencies vanish ($\omega'_j = 0$, $j = 1, \dots, l$) whilst the remaining $m = n - l$ frequencies do not change ($\omega'_j = \omega_j$, $j = l+1, \dots, n$). Then the true action-angle variables are only $s'_{l+1}, \varphi'_{l+1}, \dots, s'_n, \varphi'_n$. As $\omega_j = \dfrac{\partial h}{\partial s'_j}$, the Hamiltonian of the system is only a function of the true actions

$$h = h\left(s'_{l+1}, \dots, s'_n\right). \qquad (4.66)$$

Let us notice that the suggested way of transition to the new action-angle variables is not unique.

In what follows, while considering degenerated problems, we use only true action-angle variables and keep the previous notation $s_1, \varphi_1, \dots, s_m, \varphi_m$ ($m = n - l$) for the l times degenerate system.

As before, see Sec. 3.4, the true phase is called isochronous if the corresponding partial phase is constant and it does not depend on the initial conditions. If all phases of the l times degenerate system are isochronous, then the energy constant is a linear homogenous form of the true action constants

$$h = \omega_1 s_1 + \ldots + \omega_m s_m, \qquad (4.67)$$

where values $\omega_1, \dots, \omega_m$ must be mutually incommensurable. Notice that expression (4.67) contradicts eq. (4.30) even for $m = n$. Thus, the conservative system moving by inertia can not be isochronous.

The linear conservative system for which

$$K = \frac{1}{2} a_{ij}\dot{q}_j\dot{q}_i, \quad \Pi = \frac{1}{2} c_{ij} q_j q_i, \quad (a_{ij}, c_{ij} = \text{const}) \qquad (4.68)$$

admits the following general quasi-periodic solution of libration type

$$q_i = x_{ij}\xi_j. \qquad (4.69)$$

Here the quantities

$$\xi_1 = \sqrt{2s_1}\cos\varphi_1, \dots, \xi_n = \sqrt{2s_n}\cos\varphi_n \qquad (4.70)$$

are harmonic variables, see eq. (3.42) and the partial frequencies $\omega_1, \dots, \omega_n$ are roots of the determinant of the homogeneous system

$$\left(c_{ij} - \omega_l^2 a_{ij}\right) x_{jl} = 0, \quad l = 1, \dots, n. \tag{4.71}$$

The values $x_{1l}, \dots, x_{nl}$ comprise particular solutions corresponding to the root ω_l and satisfy the following condition of orthogonality

$$a_{ij} x_{is} x_{jr} = 0 \quad s \neq r\,. \tag{4.72}$$

The action-angle variables in eq. (4.69) are true variables and the system as a whole is not degenerate ($l = 0$) provided that the eigenfrequencies $\omega_1, \dots, \omega_n$ are mutually incommensurable.

In a more general case, the canonical pair is referred to as anisochronous if the corresponding partial frequency is an essential function of $s_1, , \dots, s_m$ (for l times degenerate system). A completely anisochronous conservative system can be characterised by the form of its $m-$dimensional backbone hypersurface whose equation is given, for example, in the form

$$h = h\left(s_1, , \dots, s_m\right), \; \omega_i = \omega_i\left(s_1, , \dots, s_m\right) \quad (i = 1, \dots, m)\,. \tag{4.73}$$

One can speak of a hard (soft) anisochronism with respect to all phases, see eq. (3.69), if the matrix coefficient of steepness

$$e_{ij} = \frac{\partial^2 h}{\partial s_i \partial s_j} \quad (i, j = 1, \dots, m) \tag{4.74}$$

is positive (negative) definite. The "true" character of m pairs $s_1, \varphi_1, \dots, s_m, \varphi_m$ and thus the presence of only l integer relationships (4.63) guarantees only non-degenerate character of the $m \times m$ matrix e_{ij}.

Now let us assume that the conservative system moves by inertia, i.e. $\Pi = 0$. Differentiating eq. (4.31) with respect to s_j we immediately obtain

$$\omega_i = e_{ij} s_j. \tag{4.75}$$

Inserting this expression into eq. (4.30) yields

$$h = \frac{1}{2} e_{ij} s_i s_j = \frac{1}{2} e_{ij}^{-1} \omega_i \omega_j > 0, \tag{4.76}$$

where e_{ij}^{-1} denotes the coefficients of the matrix inverse of e_{ij}. As directly follows from eq. (4.76), the quasi-conservative systems moving by inertia are hard anisochronous. This conclusion is valid regardless of the fact that systems with many degrees of freedom, in contrast to systems with a single degree of freedom, may have limitation motions and separatrices at $\Pi = 0$, and thus quasi-periodic motions may have qualitatively different character in various regions of the phase space.

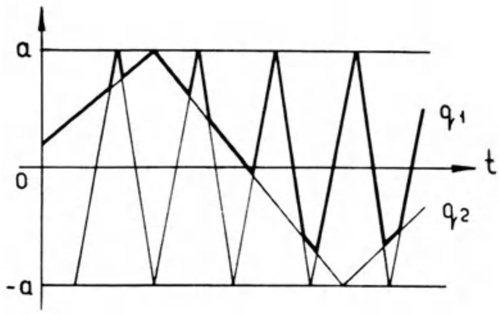

FIGURE 4.4.

The two-frequency spherical motion of a rigid body with three degrees of freedom by inertia, Sec. 4.3, is single degenerate and hard anisochronous.

Let us demonstrate an example of a hard anisochronous with respect to both phases and non-degenerate ($l = 0$) impact-oscillatory problem with two degrees of freedom. We consider the motion of two identical balls along a straight line within a two-sided limiter of length $2a$. It is assumed that the balls move by inertia between the absolutely elastic collisions of each other or the limiter. According to the stereometric theory of impact [75], the balls exchange their velocities. Thus, the resulting two frequency motion can be easily reconstructed graphically, cf. Fig. 4.4. The closed form expressions for the motion of the balls are given by

$$q_1 = \frac{1}{2}\left(x_1 + x_2 + |x_1 - x_2|\right), \quad q_2 = \frac{1}{2}\left(x_1 + x_2 - |x_1 - x_2|\right), \qquad (4.77)$$

where x_1 and x_2 denote the actual displacements of the balls when there is no interaction between them

$$x_i = \frac{2a}{\pi} \arcsin \sin \varphi_i \quad (\varphi_i = \omega_i t + \alpha_i \; i = 1, 2). \qquad (4.78)$$

The partial frequencies ω_1 and ω_2 are linearly proportional to the absolute values of the velocities ($\omega_i = \pi v_i / 2a$).

The Fourier expansions of the dependences (4.77) as well as the expression for the energy constant in terms of the partial actions s_1 and s_2 are given by

$$
\begin{aligned}
q_1 &= \frac{a}{3} + \frac{4a}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} \left[\cos i \left(\frac{\pi}{2} - \varphi_1 \right) + \cos i \left(\frac{\pi}{2} - \varphi_2 \right) - \right. \\
&\qquad \left. \cos i \left(\frac{\pi}{2} - \varphi_1 \right) \cos i \left(\frac{\pi}{2} - \varphi_2 \right) \right], \\
q_1 &= -\frac{a}{3} - \frac{4a}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} \left[\cos i \left(\frac{\pi}{2} + \varphi_1 \right) + \cos i \left(\frac{\pi}{2} + \varphi_2 \right) - \right. \\
&\qquad \left. \cos i \left(\frac{\pi}{2} + \varphi_1 \right) \cos i \left(\frac{\pi}{2} + \varphi_2 \right) \right], \\
h &= \frac{\pi^2}{8ma^2} \left(s_1^2 + s_2^2 \right).
\end{aligned}
\tag{4.79}
$$

The peculiarity of the obtained solution is the absence of the explicit dependence of q_1 and q_2 on the partial actions which is typical for the impact-oscillatory systems in general.

4.5 Conservative systems with a single positional coordinate

Conservative systems having one positional coordinate while the remaining coordinates are cyclic are an important class of system integrable by quadratures. Contrary to the above examples, the generalised coordinates of these conservative systems can be chosen from the very beginning in such a way that the kinetic and potential energies are represented as follows

$$
K = \frac{1}{2} m_{ij}(q)\, \dot{q}_i \dot{q}_j + m_i(q)\, \dot{q}_i \dot{q} + \frac{1}{2} m(q)\, \dot{q}^2, \quad \Pi = \Pi(q). \tag{4.80}
$$

It is assumed that the conservative system has $n+1$ generalised coordinates $q, q_1, \ldots, q_n$, the positional coordinate being q. The system is integrable by quadratures since there exist n first cyclic integrals

$$
\frac{\partial K}{\partial \dot{q}_i} = m_{ij} \dot{q}_j + m_i \dot{q} = p_i, \tag{4.81}
$$

where $p_1, \ldots, p_n$ are constant cyclic momenta. Resolving the n equations (4.81) for the cyclic velocities $\dot{q}_1, \ldots, \dot{q}_n$, we obtain

$$
\dot{q}_i = m_{ij}^{-1} \left(p_i - m_i \dot{q} \right), \tag{4.82}
$$

where values $m_{ij}^{-1} = m_{ji}^{-1}$ are components of the matrix inverse of m_{ij}

$$
m_{ij} m_{il}^{-1} = \delta_{jl}. \tag{4.83}
$$

Next we express the kinetic energy of the system in terms of the cyclic momenta $p_1, \ldots, p_n$ by means of eq. (4.82). The corresponding expression leads to the form

$$K = \frac{1}{2} M(q) \dot{q}^2 + \frac{1}{2} m_{ij}^{-1} p_i p_j, \tag{4.84}$$

where

$$M = m + m_{ij}^{-1} m_i m_j. \tag{4.85}$$

The first energy integral of the system can be cast as follows

$$K + \Pi = \frac{1}{2} M \dot{q}^2 + U(q) = h, \tag{4.86}$$

where

$$U(q) = \Pi + \frac{1}{2} m_{ij}^{-1} p_i p_j. \tag{4.87}$$

It is useful to treat $M\dot{q}^2$ and U respectively as the modified kinetic and potential energies of the equivalent conservative system with a single degree of freedom which is obtained as a result of excluding cyclic velocities.

The first-order differential equation (4.86) can admit a constant solution $q = q_*$, where the constant q_* depends upon the cyclic momenta $p_1, \ldots, p_n$ and is determined from the following equation

$$\left(\frac{dU}{dq}\right)_* = \left(\frac{1}{2}\frac{m_{ij}^{-1}}{dq} p_i p_j + \frac{d\Pi}{dq}\right)_* = 0. \tag{4.88}$$

Here and in what follows, subscript $*$ denotes that the value in parentheses is calculated for $q = q_*$. By virtue of eq. (4.82), the following values of the cyclic velocities

$$\dot{q}_i = \omega_{i*}, \quad \omega_{i*} = \left(m_{ij}^{-1}\right)_* p_j \tag{4.89}$$

correspond to the value $q = q_*$.

It is assumed in the forthcoming analysis that the cyclic coordinates are the angles of rotation measured in radians. Then the cyclic momenta are the angular momentum whereas the equalities in eq. (4.89) characterise the uniform rotation of the system with respect to the coordinates $q_1, \ldots, q_n$ with the angular velocities $\omega_{1*}, \ldots, \omega_{n*}$. For this reason, this solution can be called quasi-static.

Equation (4.88) for q_* can be rewritten in a somewhat different form in terms of $\omega_{1*}, \ldots, \omega_{n*}$. To this end, the expressions

$$p_j = \left(m_{ij}\omega_j\right)_*, \tag{4.90}$$

which follow directly from (4.89) are substituted into eq. (4.88). Taking into account the identity

$$\frac{dm_{ij}}{dq}m_{il}^{-1} + m_{ij}\frac{dm_{il}^{-1}}{dq} = 0, \tag{4.91}$$

which is a consequence of eq. (4.83), we obtain

$$\left(\frac{1}{2}\frac{dm_{ij}}{dq}\omega_i\omega_j - \frac{d\Pi}{dq}\right)_* = 0. \tag{4.92}$$

It is reasonable to determine the value of q_* from (4.92) as a function of $\omega_{1*}, \dots, \omega_{n*}$. This enable us to obtain the cyclic momenta due to eq. (4.90). In this case, eq. (4.92) describes the extremum of the kinetic potential $L = K - \Pi$ of the system with respect to the positional coordinate under independent cyclic velocities

$$\left(\frac{dL}{dq}\right)_* = 0. \tag{4.93}$$

According to Routh's theorem, for stability of the considered quasi-static solution it is necessary and sufficient that for independent cyclic momenta $p_1, \dots, p_n$ and under the condition $q = q_*$ the modified potential energy U has a minimum

$$c_0 = \left(\frac{d^2U}{dq^2}\right)_* = \left(\frac{1}{2}\frac{d^2m_{ij}^{-1}}{dq^2}p_ip_j + \frac{d^2\Pi}{dq^2}\right)_* > 0. \tag{4.94}$$

Differential equation (4.86) in the vicinity of the obtained stable quasi-static solution is similar to eq. (3.10) for the case of a single degree of freedom. Therefore we can state that, in the considered vicinity of the quasi-static solution, the positional coordinate is equal to

$$q = q_* + \delta q, \tag{4.95}$$

where the dynamic components δq and $h - U(q_*)$ are given in the form of the following series

$$\delta q = \sqrt{\frac{\omega_0}{c_0}}\xi - \frac{\omega_0}{2c_0}\left[\left(\frac{2}{3}\nu_1 - \mu_1\right)\eta^2 + \frac{\nu_1}{3}\xi^2\right] +$$
$$\frac{\xi}{32m_0\sqrt{c_0\omega_0}}\left[\left(\frac{13}{18}\nu_1^2 + \frac{23}{3}\mu_1\nu_1 - \frac{21}{2}\mu_1^2 + 3\mu_2 - \frac{\nu_2}{2}\right)\eta^2 +\right.$$
$$\left.\left(\frac{25}{18}\nu_1^2 + \mu_1\nu_1 + \frac{1}{2}\mu_1^2 - \mu_2 - \frac{5}{6}\nu_2\right)\xi^2\right] + \dots,$$
$$h - U(q_*) = \omega_0 s + \left(\mu_1^2 - \frac{5}{3}\nu_1^2 + 2\mu_1\nu_1 - 2\mu_2 + \nu_2\right)\frac{s^2}{16m_0} + \dots, \tag{4.96}$$

which coincide with those in eq. (3.41). Here ξ and η are harmonic canonical variables, see eq. (3.31), and, additionally, the following notation

$$\begin{aligned} m_0 &= (M)_*, \quad \omega_0 = \sqrt{\frac{c_0}{m_0}}, \quad \mu_i = \left(\frac{1}{M}\frac{d^i M}{dq^i}\right)_*, \\ \nu_1 &= \frac{1}{c_0}\left(\frac{d^{i+2}U}{dq^{i+2}}\right)_*, \quad (i = 1, 2, \dots) \end{aligned} \tag{4.97}$$

is introduced. All these constants depend on the cyclic momenta, and the positional frequency is $\omega = dh/ds$. The cyclic coordinates $\omega_1, \dots, \omega_n$ can be determined by averaging expression (4.82) over the positional phase $\varphi = \omega t + \alpha$

$$\omega_i = \langle m_{ij}^{-1} \rangle p_j = \omega_{i*} - \frac{\omega_0}{2c_0}\left[\left(\frac{dm_{ij}^{-1}}{dq}\right)_* (\nu_1 - \mu_1) - \left(\frac{d^2 m_{ij}^{-1}}{dq^2}\right)_*\right] s + \dots. \tag{4.98}$$

Because of eq. (4.82), the cyclic coordinates $q_1, \dots, q_n$ under oscillation of q is a superposition of the uniform rotation with frequency ω_i and periodic oscillation of frequency ω

$$q_i = \varphi_i + \delta q_i(\varphi) \quad (\varphi_i = \omega_i t + \alpha_i), \tag{4.99}$$

where the additional terms are analytical with respect to ξ and η

$$\begin{aligned} \delta q_i = -\frac{\eta}{\omega_0}\Bigg\{ &\left(\frac{dm_{ij}^{-1}}{dq}\right)_* \left[\sqrt{\frac{\omega_0}{c_0}} + \frac{\omega_0}{4c_0}\left(\frac{\nu_1}{3} - \mu_1\right)\xi + \dots\right] + \\ &\left(\frac{d^2 m_{ij}^{-1}}{dq^2}\right)_* \frac{\omega_0}{2c_0}\xi + \dots\Bigg\} p_j - \left(m_{ij}^{-1} m_j\right)_* \left[\sqrt{\frac{\omega_0}{c_0}}\xi + \right. \\ &\left.\frac{\omega_0}{4c_0}\left(\frac{\nu_1}{3} - \mu_1\right)(\xi^2 - \eta^2) + \dots\right] - \frac{\omega_0}{4c_0}\left(\frac{dm_{ij}^{-1} m_j}{dq}\right)_* (\xi^2 - \eta^2) - \dots. \end{aligned} \tag{4.100}$$

Here the terms of order $s^{3/2}$ and higher are not included.

In accordance with formulae (4.21), the positional action is the ratio of the modulus of periodicity of Lagrange's action of the equivalent conservative system with a single degree of freedom to the full revolution of the phase 2π

$$\begin{aligned} s &= \frac{1}{(2\pi)^{n+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \left(p_i \frac{\partial p_i}{\partial \varphi} + p\frac{\partial p}{\partial \varphi}\right) d\varphi_1 \cdots d\varphi_n d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi/\omega} M\dot{q}^2 dt. \end{aligned} \tag{4.101}$$

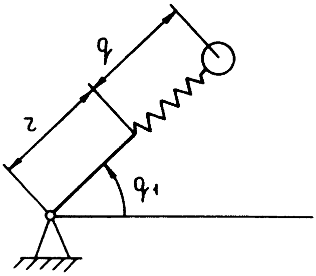

FIGURE 4.5.

Since $\frac{\partial q_i}{\partial \varphi_j} = \delta_{ij}, \frac{\partial q}{\partial \varphi_j} = 0$ and $p_i =$ const it is also essential that the cyclic constants of action are equal to the cyclic momenta, i.e. $s_i = p_i$. In other words, these actions are equal to the first integrals of the angular momentum. This is in full agreement with the result obtained at the end of Sec. 4.1. Let us recall that the same situation arises in the problem of the spherical motion of the rigid body by inertia, see Sec. 4.3.

4.6 Motion of an elastically mounted, unbalanced rotor

As an example we consider the motion of an elastically mounted, unbalanced rotor, Fig. 4.5. The system has two generalised coordinates, namely the positional coordinate q describing the deformation of the elastic spring and cyclic coordinate q_1 which is equal to the eccentricity turn. Assuming the inertia of the rotor to be concentrated in the particle m at the end of the spring with rigidity c, we cast the expressions for the kinetic and potential energies in the form

$$K = \frac{m}{2}\left[\dot{q}^2 + (r+q)^2 \dot{q}_1^2\right], \quad \Pi = \frac{c}{2}q^2.$$

Using the notation of the previous section, see (4.80) and (4.85), we have $n = 1, m_{11} = m(r+q)^2, m_1 = 0$ and $M = m$. The cyclic constant of action is the kinetic moment of the rotor about the centre of rotation

$$p_1 = s_1 = m(r+q)^2 q_1, \tag{4.102}$$

whereas the modified kinetic energy, due to eq. (4.87), is equal to

$$U = \frac{cq^2}{2} + \frac{s_1^2}{2m(r+q)^2}. \tag{4.103}$$

Equation (4.87) for the constant value of the positional coordinate in the quasi-static regime has the form

$$\frac{s_1^2}{m\,(r+q_*)^3} = cq_*. \tag{4.104}$$

This equation admits two roots, both having physical meaning. One root is positive, whilst the other is negative and less than $-r$. In the neighbourhood of $s_1 = 0$ the first and second roots are analytical with respect to s_1^2 and $s_1^{3/2}$, i.e.

$$\begin{aligned} (q_*)_1 &= r\vartheta\left(1 - 3\vartheta + 15\vartheta^2 + \vartheta^3 \ldots\right), \quad \vartheta = \frac{s_1^2}{mcr^2}, \\ (q_*)_2 &= -r\left(1 + \vartheta^{1/3} - \frac{1}{3}\vartheta^{2/3} + \frac{1}{3}\vartheta + \vartheta^{4/3} \ldots\right). \end{aligned} \tag{4.105}$$

On the other hand, for $s_1 \to \infty$ the following expansion

$$(q_*)_{1,2} = r\left(\pm\vartheta^{1/4} - \frac{3}{4} \mp \frac{27}{16}\vartheta^{-1/4} + \vartheta^{-1/2} \ldots\right) \tag{4.106}$$

holds, where the upper and lower signs correspond respectively to the first and second roots. The cyclic angular velocity of the rotor in quasi-static regime, see eq. (4.89), is expressed in terms of q_* by means of eq. (4.104)

$$\omega_{1*}^2 = \frac{c}{m}\frac{q_*}{r+q_*}. \tag{4.107}$$

Hence, the system has two principally different quasi-static regimes corresponding to the uniform rotation under unaltered deformation of the spring. The quasi-static velocity of the first regime ($q_* > 0$) is less than the eigenfrequency of the mass m attached to the spring, i.e. $\omega_{1*} < c/m$, whereas under the second regime ($q_* < -r$) we have $\omega_{1*} > c/m$. For this reason, the first and second regimes are referred to as pre-resonant and post-resonant, respectively. It is also important that both regimes are stable since value c_0, eq. (4.94), expressed in terms of q_* is equal to

$$c_0 = \frac{r+4q_*}{r+q_*}c \tag{4.108}$$

and thus is always possible.

We express all quantities of the problem in terms of the quasi-static deformation q_* of the spring which is a function of the cyclic action s_1, see eq. (4.104). The final expression for the energy constant, see eq. (4.96), takes the form

$$h = \frac{cq_*}{2}(r+2q_*) + \sqrt{\frac{c}{m}\frac{r+4q_*}{r+q_*}}\,s + \frac{15q_*r}{(r+q_*)^2(r+4q_*)^2}\frac{s^2}{4m} + s^3 \ldots. \tag{4.109}$$

If $s_1 \to 0$ $(\vartheta \to 0)$ then, due to expansion (4.105) for $(q_*)_1$, the energy constant (4.109) is an analytical function of s and s_1^2

$$(h)_1 = \frac{cr^2}{2}\vartheta\left(1-\vartheta+3\vartheta^2+\ldots\right)+\sqrt{\frac{c}{m}}\left(1+\frac{3}{2}\vartheta+\ldots\right)s+\frac{15\vartheta}{mr^2}s^2+\ldots. \tag{4.110}$$

The dependence of the energy constant on action s_1 for the second solution $(q_*)_2$ is more complex and has a non-analytical character

$$\begin{aligned}(h)_2 &= \frac{cr^2}{2}\left(1+3\vartheta^{1/3}-\vartheta^{2/3}+\ldots\right)+\\ &\quad\sqrt{\frac{3c}{m}}\vartheta^{-1/6}s-\frac{5}{12mr^2}\vartheta^{-2/3}s^2+\ldots.\end{aligned} \tag{4.111}$$

When $s_1 \to \infty$ then, by virtue of eq. (4.106), we can write instead of (4.109) that

$$\begin{aligned}h &= cr^2\vartheta^{1/2}\left(1\mp\vartheta^{-1/4}-\frac{57}{16}\vartheta^{-1/2}+\ldots\right)+\\ &\quad 2\sqrt{\frac{c}{m}}\left(1\pm\frac{1}{6}\vartheta^{-1/4}+\ldots\right)\pm\frac{15}{64mr^2}\vartheta^{-3/4}s^2+\ldots.\end{aligned} \tag{4.112}$$

Comparing eqs. (4.112) and (3.56) shows that in the considered system with two degrees of freedom dependence of the energy on "rotational action" s_1 is of a character which can not be realised in a system with one degree of freedom. If $s_1 = O(s) \to 0$, then an approximation $h \approx \sqrt{c_1/m}\,(s_1+2s)$ follows from eq. (4.112) which demonstrates that the isochronous harmonic oscillations of frequency $\sqrt{c/m}$ are observed in the system. In this case, distance q changes with a double frequency $\partial h/\partial s \cong 2\sqrt{c_1/m}$.

Finally, we present the expressions for q and q_1 obtained in accordance with eqs. (4.96) and (4.100) in the form of the analytical functions of variables ξ and η

$$\begin{aligned}q &= q_* + (mc_0)^{-1/4}\xi + \frac{2q_*(mc_0)^{-1/2}}{(r+q_*)(r+4q_*)}\left(\xi^2+2\eta^2\right) - \\ &\quad \frac{q_*(mc_0)^{-3/4}\xi}{16(r+q_*)^2(r+4q_*)^2}\left[25r\xi^2-(45r+232q_*)\eta^2\right]+\ldots,\\ q_1 &= \varphi_1 + \frac{2(mc_0)^{-1/4}\eta}{(r+q_*)}\sqrt{\frac{q_*}{r+4q_*}}\times\\ &\quad\left\{1-\frac{(mc_0)^{-1/4}}{3}\frac{(r+7q_*)\xi}{(r+q_*)(r+4q_*)}+\frac{(mc_0)^{-1/2}}{48}\times\right.\\ &\quad\left.\frac{3\left[32\left(r^2+4q_*^2\right)-75q_*r\right]\xi^2+\left[32\left(r^2-4q_*^2\right)-51q_*r\right]\eta^2}{(r+q_*)(r+4q_*)}+\ldots\right\}.\end{aligned} \tag{4.113}$$

4.7 Spherical motion of an axisymmetric heavy top

An axisymmetric heavy top is a classical example of a conservative system with one positional and two cyclic coordinates. Indeed, let us take the true angles of precession ψ and spin φ as well as variable $u = \cos\vartheta$ (ϑ denotes the angle of nutation, see (4.62)) as the generalised coordinates. Unlike the problem studied in Sec. 4.3, there is no need to introduce a special fixed coordinate system since the present problem has a selected direction and the kinetic moment about the fixed point is not constant. The coordinates introduced above are the most natural if the fixed axis Oz_1 is vertical and the moving axis Oz is coincident with the symmetric axis of the body. Then, due to equality of the equatorial moments ($A = B$), the kinetic energy of the body is written as follows

$$K = \frac{A}{2}\left[\left(1-u^2\right)\dot{\psi}^2 + \frac{\dot{u}^2}{1-u^2}\right] + \frac{C}{2}\left(u\dot{\psi} + \dot{\varphi}\right)^2. \tag{4.114}$$

The potential energy depends only on u

$$\Pi = P\rho u, \tag{4.115}$$

where P and ρ denote the weight and the eccentricity (the distance between the centre of mass and the fixed point) of the heavy top. Hence, coordinates ψ and φ are cyclic whilst u is the positional coordinate.

Using eq. (4.114) we can write expressions for the cyclic actions s_φ and s_ψ in the form

$$s_\varphi = p_\varphi = C\left(u\dot{\psi} + \dot{\varphi}\right), \quad s_\psi = p_\psi = A\left(1-u^2\right)\dot{\psi} + s_\varphi u. \tag{4.116}$$

The energy integral of the heavy top in the form of eq. (4.86) equals

$$\frac{A\dot{u}^2}{2\left(1-u^2\right)} + U(u) = h, \tag{4.117}$$

where the modified potential energy is given by

$$U = P\rho u + \frac{\left(s_\psi - s_\varphi u\right)^2}{2A\left(1-u^2\right)} + \frac{s_\varphi^2}{2C}. \tag{4.118}$$

It is convenient to set equation (4.88) for the quasi-static solution in the following non-dimensional form

$$\left(\sigma_\psi^2 + \sigma_\varphi^2\right)u_* - \sigma_\psi\sigma_\varphi\left(1+u_*^2\right) + \left(1-u_*^2\right)^2 = 0, \tag{4.119}$$

where σ_ψ and σ_φ denote non-dimensional analogues of the cyclic actions

$$\sigma_\psi = \frac{s_\psi}{\sqrt{AP\rho}}, \quad \sigma_\varphi = \frac{s_\varphi}{\sqrt{AP\rho}}. \tag{4.120}$$

According to eq. (4.119), quantity u_* is a symmetric function of σ_ψ and σ_φ. Thus, the structure of isolines in the plane $(\sigma_\psi, \sigma_\varphi)$ is symmetric about bisectrices $\sigma_\psi = \sigma_\varphi$ and $\sigma_\psi = -\sigma_\varphi$. It is also important that for all solutions of eq. (4.119) the value

$$c_0 = \left(\frac{d^2U}{du^2}\right)_* = \frac{\left(s_\psi^2 + s_\varphi^2\right)\left(1 + 3u_*^2\right) - s_\psi s_\varphi u_* \left(3 + u_*^2\right)}{A\left(1 - u_*^2\right)^3} \tag{4.121}$$

is positive and thus the quasi-static solution is stable.

The first-order differential equation (4.117) is actually coincident with the equation which is used for construction of the general three-frequency integral of the problem of pseudoregular precession of the heavy top. Closed form expressions characterising this solution are constructed in accordance with the scheme of Sec. 3.2 and expressed in terms of Jacobi's elliptic functions. Using the formulae of Sec. 4.5 it is also possible to obtain an expression for the energy constant h as power series in terms of the positional action s as well as expansions for the generalised coordinates in terms of powers of the harmonic canonical variables ξ and η. The latter expansions are not convenient if eccentricity $\rho \to 0$. Indeed, if $\rho = 0$ then the advantageous direction of the vertical $0z_1$ no longer exists and there appears a necessity to introduce a special fixed system whose axis $0x'$ is directed along the fixed kinetic moment of the heavy top, see Sec. 4.3. Correspondingly, the efficient solution for $\rho = 0$ is obtained from the solution of the problem of the spherical motion of the body by inertia.

Let us proceed to the solution of this problem. Assuming $A = B$ $(C < D < A)$ we have $k = n = 0$, $K = P = \pi/2$ and thus, instead of eq. (4.61) we obtain

$$\frac{1}{D} = \frac{1}{A}\left(1 + \frac{s_1^2}{s_2^2}\frac{A - C}{C}\right). \tag{4.122}$$

Expression (4.57) takes the following form

$$h = \frac{1}{2}\frac{A - C}{AC}s_1^2 + \frac{s_2^2}{2A}. \tag{4.123}$$

It follows from this equation that

$$\omega_1 = \frac{A - C}{AC}s_1, \quad \omega_2 = \frac{s_2}{A}. \tag{4.124}$$

In this case the elliptic functions reduce to circular functions, so that the general integral of the problem is given by

$$p = \frac{\sigma_1}{A}\sqrt{s_2^2 - s_1^2}\cos\varphi_1, \quad q = \frac{\sigma_2}{A}\sqrt{s_2^2 - s_1^2}\sin\varphi_1, \quad r = \frac{s_1}{C},$$
$$\cos\vartheta' = \frac{s_1}{s_2}, \quad \varphi' = \varphi - \frac{\pi}{2}, \quad \psi = \varphi_2. \tag{4.125}$$

This solution describes the regular precession of the heavy top [7]. Since $r = \text{const}$ we immediately obtain that the partial action s_1 is equal to the projection of the kinetic moment on the symmetry axis Oz_1 ($s_1 = Cr$) and thus coincides with the generalised momentum $p_\varphi = \dfrac{\partial K}{\partial \dot{\varphi}}$. Hence $s_1 = s_\varphi$, see eq. (4.116).

Using relationships (4.125) we write down expression for u, eq. (4.62),

$$u = \frac{1}{s_2^2}\left[s_\psi s_\varphi + \sqrt{\left(s_2^2 - s_\varphi^2\right)\left(s_2^2 - s_\psi^2\right)}\cos\varphi_2\right], \tag{4.126}$$

where $s_3 = p_\psi = s_\psi$ is taken into account. Expression (4.126) is a general integral of equation (4.117) for $\rho = 0$. The quasi-static solution ($u = \text{const}$) which is obtained from (4.126) has the form

$$u_* = \begin{cases} s_\psi/s_\varphi & s_2 = s_\varphi > s_\psi, \\ s_\varphi/s_\psi & s_2 = s_\psi > s_\varphi. \end{cases} \tag{4.127}$$

It is easy to see that it is the solution which is obtained from eq. (4.119) for $\rho = 0$. It is also essential that in the first case ($s_\varphi > s_\psi$) the kinetic moment is directed along the symmetric axis of the heavy top, $h = \dfrac{s_\varphi^2}{2C}$ and the motion gains the character of uniform rotation. In the second case ($s_\psi > s_\varphi$), the basic $Ox_1y_1z_1$ and the auxiliary coordinate systems coincide.

Phase φ_2 is positional for solution (4.126). Therefore, the positional action s introduced in Sec. 4.5 should be additive with respect to $s_2 = L$. On the other hand, $s = 0$ for the quasi-static solution (3.18). Therefore, $s_2 = s_\varphi + s$ for $s_\varphi > s_\psi$ and $s_2 = s_\psi + s$ for $s_\psi > s_\varphi$. Now it is easy to represent expression (4.126) for u in the form of a series in terms of harmonic variables ξ and η by using equality $s = \frac{1}{2}\left(\xi^2 + \eta^2\right)$.

Closing the study of the free spherical motion of the symmetric heavy top, we notice that expression (4.126), equations for ψ, φ and $p_u = \dfrac{\partial K}{\partial \dot{u}}$ (not shown here), as well as identities $p_\varphi = s_\varphi = s_1$, $p_\psi = s_\psi = s_3$ define the transition from the original canonical variables to the new canonical variables s_i, φ_i ($i = 1, 2$) which are referred to as Andooyer's variables [7]. This replacement is especially efficient for solving problems by the method of small parameters provided that their generating approximation deals with a free symmetric heavy top. For example, in the case of the heavy top the Hamiltonian in terms of Andooyer's variables is given by

$$h = \frac{1}{2}\frac{A - C}{AC}s_1^2 + \frac{s_2^2}{2A} + \frac{P\rho}{s_2^2}\left[s_1 s_3 + \sqrt{\left(s_2^2 - s_1^2\right)\left(s_2^2 - s_3^2\right)}\cos\varphi_2\right], \tag{4.128}$$

see eqs. (4.115), (4.123) and (4.126).

Let us assume that the eccentricity is small. Then the generating Hamiltonian does not depend on the positional coordinate, which is phase φ_2, as well as the cyclic momentum s_3. The same problem in terms of the original canonical variables is much more difficult.

4.8 Selecting the canonical action-angle variables

As shown in Sec. 4.7, the spherical motion of a heavy symmetric top is a three-frequency motion and the corresponding system is thus non-degenerate. We denote the true phases, actions and frequencies characterising this motion by φ_i', s_i' and ω_i' respectively in order to distinguish them from the original Andooyer variables s_i, φ_i. We list here some properties of the considered solution.

1. The cyclic momenta s_1 and s_3 coincide with the corresponding true actions s_1' and s_3', see Sec. 4.1.

2. The partial frequencies ω_1' and ω_3' corresponding to these actions are respectively equal to the values of $\dfrac{\partial h}{\partial s_1}$ and $\dfrac{\partial h}{\partial s_3}$ averaged over phase φ_2 within the period of 2π.

3. Positional variables s_2 and φ_2 as well as the nutation angle $\vartheta = \arccos u$ are periodic with respect to t and thus are $2\pi-$periodic with respect to the true phase φ_2'.

It is important that while constructing quasi-periodic general integrals of the integrable system with several (more than one) degrees of freedom we face an arbitrariness associated with the choice of the true phases and thus the actions. Indeed, instead of the adopted true phases φ_1', φ_2' and φ_3', we can enter the new phases φ_1'', φ_2'' and φ_3'' by means of the relationships

$$\varphi_i' = n_{i1}\varphi_1'' + n_{i2}\varphi_2'' + n_{i3}\varphi_3'', \tag{4.129}$$

where integers n_{ij} $(i, j = 1, 2, 3)$ form a non-zero determinant. Nevertheless, in the problem under consideration there are grounds to choose one set of phases among an infinite number of such sets. This set can be taken as being the most reasonable and natural one if the physically meaningful cases of the additional degeneration of the problem are characterised by eliminating one or several partial frequencies.

In the problem of the spherical motion of a symmetric heavy top, the first degeneration occurs under vanishing perturbation when $\rho \to 0$. The suggested way of introducing true phases is rational since the third partial frequency $\omega_3' = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{\partial h}{\partial s_3} d\varphi_2'$ vanishes at $\rho = 0$. The second additional degeneration takes place at $A = C$, i.e. when the body possesses a complete physical symmetry. In this case, the angular velocity of the regular precession is zero ($\omega_1 = 0$, see eq. (4.124)), the phase φ_1 is constant ($\varphi_1 = \alpha_1$)

and the energy is independent of the corresponding action, see eq. (4.123). Besides, at $A = C$ $(\rho = 0)$ the angular velocity vector is constant and its constant absolute value coincides with the second partial frequency which is the angular spin velocity ω_2. For this reason, the motion of the body has the character of uniform rotation about a fixed axis.

In what follows we speak of an integrable conservative system with n degrees of freedom as completely degenerate or recurrent [53] if the degree of its degeneration is $l = n - 1$, see Sec. 4.4. The recurrent system has a single rigorously defined pair of true "action-angle" variables, namely s, φ with the Hamiltonian $h = h(s)$. Correspondingly, like the system with one degree of freedom, see Sec. 3.4, the single frequency $\omega = \dfrac{dh}{ds}$ and the matrix coefficient of steepness is degenerated into a value $e = \dfrac{d^2h}{ds^2} = \omega \dfrac{d\omega}{dh}$. Motion of the recurrent systems in non-small regions of the $2n$–dimensional phase space has periodic (libration or rotational) character.

Hence, a recurrent conservative system corresponds to a completely symmetric heavy top $(A = B = C,\ \rho = 0)$ being moved by inertia. Another typical example of the recurrent system is the classical Kepler's problem of motion of a particle in Newton's central field, cf. [33], [60]. We do not dwell on the well-known construction of the general periodic integral of this problem. Our aim is to find the relationship between the energy and the action by means of elementary reasoning. To this end, we first assume that the particle moves on a circular orbit with a constant velocity $v = \omega r$. The orbit radius is easy to determine by equating the gravitational and centrifugal forces

$$m\omega^2 r = \frac{f}{r^2}, \tag{4.130}$$

where m denotes the mass of the particle and f is the gravitational constant.

Inserting the obtained value of the radius into the expression for the energy constant

$$h = \frac{m\omega^2 r^2}{2} - \frac{f}{r} \tag{4.131}$$

we can find the relation between the energy and frequency

$$h = -\frac{1}{2} m^{1/3} f^{2/3} \omega^{2/3} \quad (h < 0)\,. \tag{4.132}$$

Now it is easy to determine the expression for the action in terms of the energy

$$s = \int \frac{dh}{\omega} = f\sqrt{\frac{m}{2(-h)}}. \tag{4.133}$$

Since the considered system is recurrent, we can state that eqs. (4.132) and (4.133) are also valid for the more general case of an elliptic orbit.

Let us demonstrate an example of an isochronous recurrent system with an infinite numbers of degrees of freedom. We consider the free vibration of a homogeneous string governed by the wave equation

$$N\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} = 0, \quad u|_{x=0,l} = 0, \tag{4.134}$$

where N, ρ and l denote tension, mass per unit length and length respectively. The frequency spectrum of such a string is equidistant, i.e. the eigenfrequencies are $\frac{\pi}{l}\sqrt{\frac{N}{\rho}}, \frac{2\pi}{l}\sqrt{\frac{N}{\rho}}, \ldots$. The frequency of the periodic general integral of the problem is equal to the fundamental frequency. The dependence of the energy on the action is of the linear character

$$h = \sqrt{\frac{N}{\rho}}\frac{\pi}{l}s, \tag{4.135}$$

which is typical for isochronous systems.

4.9 Nearly recurrent conservative systems

Speaking of conservative systems integrable by quadratures it is necessary to mention another class of systems whose order can be reduced by two by means of a special asymptotic change of variables. The class of system is the nearly recurrent conservative system with one fast, rapidly rotating phase. In accordance with the above, the Hamiltonian of such a system with $n+1$ degrees of freedom can be represented in the form

$$H = H_0(p) + \varepsilon H_1(q, p, q_1, p_1, \ldots, q_n, p_n) + \varepsilon^2 \ldots, \tag{4.136}$$

where ε is a small parameter characterising the proximity of the considered system to the recurrent one. The consequent corrections H_1, H_2 are $2\pi-$periodic with respect to the fast phase q related the "slow" action p whereas q_i, p_i $(i = 1, \ldots, n)$ denote other slow canonical variables. Let us notice that q, q_i, p_i are constant and phase q rotates uniformly if $\varepsilon = 0$. While solving a more complex perturbed problem which yields this nearly recurrent problem in the generating approximation, it is necessary to check that parameter ε is not related to the perturbation parameter and, generally speaking, that it considerably exceeds the latter.

Following the averaging method [19], we seek a nearly identical univalent transformation of variables $q, p, q_i, p_i \to u, v, u_i, v_i$ with the following generating function

$$F = qv + q_i v_i + \varepsilon F_1(q, p, q_1, p_1, \ldots, q_n, p_n) + \varepsilon^2 \ldots \tag{4.137}$$

such that the new Hamiltonian $\bar{H}$ does not depend on u

$$\bar{H} = H_0(v) + \varepsilon \bar{H}_1(v, u_1, v_1, \dots, u_n, v_n) + \varepsilon^2 \dots . \tag{4.138}$$

Functions $F_1, F_2, \dots$ in the expression for F must be $2\pi-$periodic with respect to the fast phase q. We equate the values of the old and new Hamiltonians, i.e. $H = \bar{H}$, and exclude the old momenta and the new coordinates by means of the following formulae

$$p = \frac{\partial F}{\partial q}, \quad p_i = \frac{\partial F}{\partial q_i}, \quad u = \frac{\partial F}{\partial v}, \quad u_i = \frac{\partial F}{\partial v_i}. \tag{4.139}$$

Next, equating the terms in front of the coinciding powers of ε, we obtain

$$\omega_0(v) \frac{\partial F_2}{\partial q} + H_1(q, v, q_1, v_1, \dots, q_n, v_n) = \bar{H}_1(v, q_1, v_1, \dots, q_n, v_n),$$

$$\omega_0(v) \frac{\partial F_2}{\partial q} + \frac{e_0(v)}{2} \left(\frac{\partial F_1}{\partial q} \right)^2 + \frac{\partial H_1}{\partial v} \frac{\partial F_1}{\partial q} +$$
$$\frac{\partial H_1}{\partial v_i} \frac{\partial F_1}{\partial q_i} + H_2 = \frac{\partial \bar{H}_1}{\partial q_i} \frac{\partial F_1}{\partial v_i} + \bar{H}_2, \dots , \tag{4.140}$$

where

$$\omega_0 = \frac{dH_0}{dv}, \quad e_0 = \frac{d^2 H_0}{dv^2}. \tag{4.141}$$

As the averaging method suggests [98], we average the obtained equalities with respect to q over one revolution 2π and integrate them over q. As a result, the consequent corrections to the generating functions of the required transformation that are determined are correct to an arbitrary function of variables $v, q_1, v_1, \dots, q_n, v_n$. The most rational way is to find these functions from the conditions $\langle F_1 \rangle = 0$ $(i = 1, 2, \dots)$ where $\langle\,\rangle$ denotes the operation of averaging over q, i.e. $\langle \cdot \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cdot \, dq$, see Sec. 3.3. Thus, we finally obtain

$$\bar{H}_1 = \langle H_1 \rangle, \quad F_1 = \frac{1}{\omega_0} \{H_1\},$$
$$\bar{H}_2 = \langle H_2 \rangle + \frac{1}{2\omega_0} \left(\frac{\partial}{\partial v} - \frac{e_0}{\omega_0} \right) \left(\langle H_1 \rangle^2 - \langle H_1^2 \rangle \right) + \left\langle \frac{\partial H_1}{\partial v_i} \frac{\partial F_1}{\partial q_i} \right\rangle. \tag{4.142}$$

Here and, in what follows, $\{\,\}$ denotes the operation of determining the antiderivative of the dummy variable q provided that this antiderivative has zero mean value, see eq. (3.61).

The canonical equations of motion of the system in terms of the new variables have the form

$$\dot{v} = 0. \quad \dot{u} = \omega_0 + \varepsilon \frac{\partial \bar{H}_1}{\partial v} + \varepsilon^2 \dots ,$$
$$\dot{v}_i = -\varepsilon \frac{\partial \bar{H}_1}{\partial u_i} - \varepsilon^2 \dots , \quad \dot{u}_i = -\varepsilon \frac{\partial \bar{H}_1}{\partial v_i} + \varepsilon^2 \dots , \tag{4.143}$$

where, due to eq. (4.142), the new variables should be substituted into the expressions for $\bar{H}_1, \bar{H}_2$. Hence, the new momentum v is constant and the equality $v =$ const is a new first integral of motion which is essentially different from the energy integral. The action p corresponding to the fast phase q is close to a constant value and is referred to as the adiabatic invariant of the nearly recurrent system under consideration. The canonical variables u_i and v_i are determined separately from the truncated system of order $2n$ and are functions of slow time $\tau = \varepsilon t$. The new fast phase u is found by quadratures from the second equation (4.143).

Hence, the order of the system is reduced by two. If the original system has two degrees of freedom, then the selected subsystem of slow motions has second order and is thus integrable by quadratures. In this regard, a nearly recurrent conservative system with two degrees of freedom is integrable. Such a problem is the planar Kepler's problem under small conservative perturbations. However it is necessary to bear in mind that the asymptotic series (4.137) are actually divergent [19] and hence one can speak of the presence of a quasi-periodic (two-frequency) general integral only in an asymptotic sense (within a finite time interval of order $1/\varepsilon$).

The problem of motion of a canonical system with n degrees of freedom under periodic perturbation of high frequency $\nu \gg \nu_0$, where ν_0 implies a scaling value of frequency, is reduced to the above problem. The Hamiltonian of the system is written as

$$H_* = H_* \left(\nu t, q_1, p_1, \dots, q_n, p_n\right). \tag{4.144}$$

Let us introduce the non-dimensional time $q = \nu t$ and the corresponding "energy" p whereas $\varepsilon = \nu_0/\nu$ is a small parameter. Then the motion of the system in the enlarged phase space $(q, p, q_1, p_1, \dots, q_n, p_n)$ is completely defined by the Hamiltonian

$$H = p + \frac{\varepsilon}{\nu_0} H_* \left(q, q_1, p_1, \dots, q_n, p_n\right). \tag{4.145}$$

Here the independent variable is the non-dimensional time q. According to eqs. (4.137), (4.138) and (4.142) the generating function of the asymptotic transformation under consideration and the Hamiltonian are given by

$$\begin{aligned} F &= qv + q_i v_i - \frac{1}{\nu}\{H_*\} + \frac{1}{\nu^2}\cdots, \\ H &= v + \frac{1}{\nu}\{H_*\} - \frac{1}{\nu^2}\left\langle \frac{\partial H_*}{\partial v_i}\frac{\partial}{\partial u_i}\{H_*\}\right\rangle + \frac{1}{\nu^3}\cdots. \end{aligned} \tag{4.146}$$

While substituting the arguments in H_* it is necessary to keep in mind that F depends on the old coordinates and new momenta. It is easy to see that the corrections of order $\varepsilon, \varepsilon^2, \dots$ in the expressions for F and H are independent of v. Hence, see eq. (4.139), $u = q = \nu t$ and the equations for the slow motions do not depend upon the modified "energy", i.e. $v =$ const. With this in view it is reasonable to return to the physical time and consider the non-autonomous canonical transformation $q_i, p_i \to u_i, v_i$. The generating function F_* of this transformation and the new Hamiltonian $\bar{H}_* \neq H_*$ are equal to

$$\begin{aligned} F_* &= q_i v_i - \frac{1}{\nu}\{H_*\} + \frac{1}{\nu^2}\dots, \\ \bar{H}_* &= \langle H_* \rangle - \frac{1}{\nu}\left\langle \frac{\partial H_*}{\partial v_i}\frac{\partial}{\partial u_i}\{H_*\}\right\rangle + \frac{1}{\nu^2}\dots, \end{aligned} \tag{4.147}$$

where integration is carried out with respect to $q = \nu t$.

The considered approach can be easily generalised to the case of the nearly recurrent canonical systems, whose Hamiltonian depends upon slow time $\tau = \varepsilon t$

$$H = H_0(p, \tau) + \varepsilon H_1(q, p, q_1, p_1, \dots, q_n, p_n) + \varepsilon^2 \dots . \tag{4.148}$$

In this case the generating function of the asymptotic change of variables and the new Hamiltonian are also dependent on τ, i.e.

$$F = qv + q_i v_i + \varepsilon F_1(q, v, q_1, v_1, \dots, q_n, v_n, \tau) + \varepsilon^2 \dots,$$

$$\bar{H} = H_0(v, \tau) + \varepsilon \bar{H}_1(v, u_1, v_1, \dots, u_n, v_n, \tau) + \varepsilon^2 \dots . \tag{4.149}$$

The considered canonical transformation is, generally speaking, non-autonomous. Thus $H \neq \bar{H}$ and it is necessary to write

$$\bar{H} = H + \frac{\partial F}{\partial t} = H + \varepsilon \frac{\partial F}{\partial \tau}. \tag{4.150}$$

Otherwise the above procedure of constructing successive asymptotic approximations is fully preserved. In particular, as before, it is rational to remove the arbitrariness in determining $F_1, F_2, \dots$ in such a way that $\langle F_i \rangle = 0$ for $i = 1, 2, 3 \dots$. Then formulae (4.142) for $\bar{H}_1, F_1$ and $\bar{H}_2$ hold. However, now they explicitly depend on τ. It is essential that in this case the formula for F_2 changes.

The canonical equations of motion in the new variables (4.143) also admit the first integral $v =$ const. Therefore, the adiabatic invariance of the action corresponding to the fast phase is retained under slow perturbations. The system of equations for slow motions serving to determine u_i and v_i becomes isolated, as above. This system is non-autonomous and thus can not be integrated by quadratures even for the original system with two degrees of freedom ($n = 1$).

To conclude, the canonical system with one degree of freedom under slow perturbations can be integrated in an asymptotic sense. For example, this is the case of the oscillation of a pendulum with slowly varying length.

5

Resonant solutions for systems integrable in generating approximation

5.1 Introductory remarks

As pointed out above, the considered classes, as well as examples of systems, that are locally and globally (by quadratures) integrable are not only of interest by themselves, but also because they can be considered in the generating approximation of more complex perturbed problems. Analytical methods of investigation of such problems are well developed and are based upon expansions with respect to small parameters.

The most important among them are the local method of analytic continuation with respect to small parameter (the Lyapunov-Poincaré method) and the asymptotic averaging method whose fundamentals were formulated in the classical papers by Krylov and Bogolyubov [19], [98]. Lie's expansions [58] are successfully applied for the latter method, which were first suggested by Deprit and Hori in [24], [38].

The detailed presentation of these methods is beyond the scope of the present book. A number of treatises, for example [19], [103], [61], are devoted to this aim. For this reason, before we proceed to make direct use of these methods we restrict ourselves to some remarks.

First of all, we notice that the smallness of parameter ε is usually based on the reasoning of purely physical nature. For instance, in the problem of weak interaction of several mechanical objects this parameter characterises the smallness (in a certain sense) of action of one object on another. From this perspective, using expansions which are typical for the Lyapunov-Poincaré method give rise to a certain logical inconsistency, [18]. Indeed, the substantiation of this method indicates only that the corresponding series converge for a sufficiently small value of the parameter and, moreover, that reliable estimates for convergence radii are absent for these series. Under the circumstances, in the forthcoming analysis we will prefer the averaging method which allows us to approximate the solution with any degree of accuracy within a finite but sufficiently large time interval.

More often than not, investigations with the help of the methods of small parameters is related, in one way or another, to the basic assumption that the phase space of the system under consideration is bounded. In order to illuminate this statement let us consider two simple problems of the perturbed motion of the harmonic oscillator

$$\begin{aligned} x + k^2 x + \varepsilon \sin x &= 0, \\ x + k^2 x + \varepsilon \left(\dot{x}^2 - 1\right) \dot{x} &= 0 \quad (k = \text{const}) \ . \end{aligned} \tag{5.1}$$

Both equations can be studied by means of the averaging method with relative ease. In the first case, the obtained averaged equation of the first order completely describes, in a certain sense, the dynamical system under consideration. For small ε, this is a direct result of the inequality $|\sin x| \leq 1$ for any real values of x. This statement is, however, not valid for the second equation in (5.1) known as Rayleigh's equation. In other words, the solution of the corresponding averaged equation makes sense only under those initial conditions for which $\dot{x} = O(1)$. In what follows, we always assume that the studied motions, also periodic, belong to the regions of the indicated type. Let us notice that the size of these regions can be small (of the order of ε or $\sqrt{\varepsilon}$) in some directions. This is above all relevant to the regions where one can apply the asymptotic transformations of the variables to averaging the systems with many frequencies in the case of the internal resonance.

It is important that any generating approximation to the dynamical system with a small parameter is somewhat uncertain from the mathematical perspective. Indeed, any group of terms with the order of ε can either be treated as being a part of the generating system or be included into the small perturbation. Hence, if by virtue of physical reasoning one has to consider a nearly conservative system, then a small non-conservative term should be considered as a perturbation of order ε. This means that there exists no nearly conservative generating system. For the same reason, while considering the problem of weak interaction of nearly identical dynamic objects their equations should coincide in the generating approximation.

As mentioned above, the aim of the forthcoming analysis is not a further development and improvement of the approaches of non-linear analysis. The aim is to apply them to determine the general physical properties and to obtain the simplified equations of the first approximation for rather general classes of mechanical and other systems. The classes of dynamical systems whose equations of motion have a rather general mechanical structure are investigated in what follows. While formulating such problems it is often not reasonable to bring them to the non-dimensional form and in particular to separate a small parameter explicitly. With this in view, for solving the physical problems we will use the concept of the formal indicator of smallness μ which is equal to unity. In order to explain the concept, let us consider a system of equations with small parameter

$$\dot{x} = X(x, t, \varepsilon), \tag{5.2}$$

where x denotes an $n \times 1$ vector, by means of one of the asymptotical methods. Alternatively, the same method can be applied to the system

$$\dot{x} = X(x, t, \mu\varepsilon) \quad (\mu = 1). \tag{5.3}$$

One can "forget" that parameter ε is small and use an expansion in terms of μ instead of an expansion in terms of ε. Let us stress, that before any power series in terms of μ is constructed one must carry out a thorough analysis of the smallness of the components of the analysed equations and clarify the physical sense of the true small parameter ε of the problem.

5.2 On transition to the angle-action variables

Let us consider a dynamical system whose motion is described by the following system of the differential equations

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} + \varepsilon X_i(q_1, \dots, q_n, p_1, \dots, p_n, \tau_1, \dots, \tau_r, \varepsilon), \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + \varepsilon Y_i(q_1, \dots, q_n, p_1, \dots, p_n, \tau_1, \dots, \tau_r, \varepsilon), \\ & (i = 1, \dots, n). \end{aligned} \tag{5.4}$$

Here $0 < \varepsilon \ll 1$ and $H = H(q_1, \dots, q_n, p_1, \dots, p_n)$ denotes the Hamiltonian of the system, whilst X_i and Y_i characterise small non-conservative perturbations which are 2π-periodic with respect to the phases $\tau_1 = \nu_1 t, \dots, \tau_r = \nu_r t$ of the external excitation. In addition to this, we assume that equations (5.4) are integrable in the generating conservative approximation ($\varepsilon = 0$), that is, we can construct a transformation to the "action-angle" variables

$$\begin{aligned} q_i &= q_i(\varphi_1, \dots, \varphi_n, s_1, \dots, s_n), \quad p_i = p_i(\varphi_1, \dots, \varphi_n, s_1, \dots, s_n), \\ H &= H(s_1, \dots, s_n), \quad \dot{\varphi}_i = \frac{\partial H}{\partial s_i} = \omega_i, \quad \dot{s}_i = 0. \end{aligned} \tag{5.5}$$

Generally speaking, this assumes the degeneracy of the conservative generating system in the sense that the matrix coefficient of steepness $e_{ij} = \dfrac{\partial^2 H}{\partial s_i \partial s_j}$ has the rank $f \le n$. The deficiency of matrix e_{ij} is equal to $n - f$ and can be caused by the degeneration of the type described in Sec. 4.4 and a linear dependence of the Hamiltonian on some of the partial actions. If the variables are rationally taken, in the first and second cases, some of the partial frequencies turn respectively either to zero or to constant values independent of the actions.

Let us carry out the transformation to the "action-angle" variables in the perturbed system (5.4) by means of eq. (5.5). We take into account the following identities

$$\frac{\partial H}{\partial p_i} = \frac{\partial q_i}{\partial \varphi_j}\omega_j, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial p_i}{\partial \varphi_j}\omega_j \tag{5.6}$$

that are valid due to relationships (5.5). On the other hand, direct differentiation yields

$$\dot{q}_i = \frac{\partial q_i}{\partial \varphi_j}\dot{\varphi}_j + \frac{\partial q_i}{\partial s_j}\dot{s}_j, \quad \dot{p}_i = \frac{\partial p_i}{\partial \varphi_j}\dot{\varphi}_j + \frac{\partial p_i}{\partial s_j}\dot{s}_j. \tag{5.7}$$

Inserting eqs. (5.6) and (5.7) into (5.4) allows us to write the equations of motion in the following form

$$\begin{aligned} \frac{\partial q_i}{\partial \varphi_j}\left(\dot{\varphi}_j - \omega_j\right) + \frac{\partial q_i}{\partial s_j}\dot{s}_j &= \varepsilon X_i, \\ \frac{\partial p_i}{\partial \varphi_j}\left(\dot{\varphi}_j - \omega_j\right) + \frac{\partial p_i}{\partial s_j}\dot{s}_j &= \varepsilon Y_i. \end{aligned} \tag{5.8}$$

We first multiply the first equation in (5.8) with $\frac{\partial p_i}{\partial s_l}$ and subtract the second equation multiplied with $\frac{\partial q_i}{\partial s_l}$ and sum over i from 1 to n. Then we multiply the second equation in (5.8) with $\frac{\partial q_i}{\partial \varphi_l}$ and subtract the first equation multiplied with $\frac{\partial p_i}{\partial \varphi_l}$ and sum up i from 1 to n. Accounting for eq. (4.3) we obtain

$$\begin{aligned} \dot{\varphi}_i - \omega_i &= \varepsilon\left(\frac{\partial p_j}{\partial s_i}X_j - \frac{\partial q_j}{\partial s_i}Y_j\right), \\ \dot{s}_i &= \varepsilon\left(\frac{\partial q_j}{\partial \varphi_i}X_j - \frac{\partial p_j}{\partial \varphi_i}Y_j\right). \end{aligned} \tag{5.9}$$

As the conservative generating system is assumed to have degeneration of the two types mentioned above, we can take that the generating Hamiltonian can be represented as a sum of two components, $H = H_1 + H_2$. The first component H_1 is a non-degenerated function of the f first actions

$s_1, \dots, s_f$ in the sense that the matrix

$$e_{ij} = \left(\frac{\partial^2 H_1}{\partial s_i \partial s_j} \right)_{i,j=1,\dots,f} \tag{5.10}$$

has rank f. The second component is represented in the form of a linear homogeneous form of the remaining g actions

$$H_2 = \sum_{i=f+1}^{f+g} \omega_i s_i \quad (f+g \le n), \tag{5.11}$$

the constants $\omega_{f+1}, \dots, \omega_{f+g}$ being mutually incommensurable. Thus, the partial frequencies (and the corresponding phases and actions) are split into three groups. The frequencies of the first group are referred to as anisochronous and are essential functions of the corresponding anisochronous actions

$$\omega_i = \frac{\partial H_1}{\partial s_i} = \omega_i (s_1, \dots, s_f) \quad (i = 1, \dots, f). \tag{5.12}$$

It is natural to call the frequencies $\omega_{f+1}, \dots, \omega_{f+g}$ of the second group isochronous. As for the third group, its frequencies vanish identically, i.e. $\omega_{f+g+1} = \dots = \omega_n = 0$. For this reason, the corresponding "phases" $\varphi_{f+g+1}, \dots, \varphi_n$ are slow ones.

In further analysis, it is often efficient to perform a transformation to the non-canonical anisochronous "phase-frequency" variables φ_i, ω_i $(i = 1, \dots, f)$ by means of eq. (5.12). In this case, instead of the first equations in (5.9) we have

$$\begin{aligned} \dot{\varphi}_i - \omega_i &= \varepsilon \sum_{k=1}^{f} e_{ik} \sum_{j=1}^{n} \left(\frac{\partial p_j}{\partial \omega_k} X_j - \frac{\partial q_j}{\partial \omega_k} Y_j \right), \\ \dot{\omega}_i &= \varepsilon \sum_{k=1}^{f} e_{ik} \sum_{j=1}^{n} \left(\frac{\partial q_j}{\partial \omega_k} Y_j - \frac{\partial p_j}{\partial \omega_k} X_j \right). \end{aligned} \tag{5.13}$$

This transformation must conserve the order of smallness of the right hand sides of the equations in (5.13) in the considered region of the phase space of the system.

Further analysis assumes a certain commutation between the partial isochronous frequencies $\omega_{f+1}, \dots, \omega_{f+g}$ and the frequencies of external excitation $\nu_1, \dots, \nu_r$. The corresponding combination resonance is characterised by the equalities

$$\omega_i = \sum_{k=1}^{f} a_{ij} \nu_j + \varepsilon \gamma_i \quad (i = f+1, \dots, f+g), \tag{5.14}$$

where a_{ij} is a $g \times r$ matrix with an integer coefficients whilst the isochronous detunings $\gamma_{f+1}, \ldots, \gamma_{f+g}$ are prescribed and are of the order of unity. If we transform from the isochronous phases to the isochronous phase shifts $\vartheta_{f+1}, \ldots, \vartheta_{f+g}$ with the help of the following formulae

$$\varphi_i = \sum_{k=1}^{f} a_{ij}\tau_j + \vartheta_i, \tag{5.15}$$

then the second group of equations in (5.9) is recast as follows

$$\begin{aligned} \dot{\vartheta}_i &= \varepsilon\left[\gamma_i + \frac{\partial p_j}{\partial s_i}X_j - \frac{\partial q_j}{\partial s_i}Y_j\right], \\ \dot{s}_i &= \varepsilon\left(\frac{\partial q_j}{\partial \vartheta_i}Y_j - \frac{\partial p_j}{\partial \vartheta_i}X_j\right) \quad (i = f+1, \ldots, f+g). \end{aligned} \tag{5.16}$$

It is important from the perspective of further analysis that both isochronous actions and phase shifts in these equations are slow. Finally, all of the variables in the third group of the equations in (5.9) are slow

$$\begin{aligned} \dot{\varphi}_i &= \varepsilon\left(\frac{\partial p_j}{\partial s_i}X_j - \frac{\partial q_j}{\partial s_i}Y_j\right), \\ \dot{s}_i &= \varepsilon\left(\frac{\partial q_j}{\partial \varphi_i}Y_j - \frac{\partial p_j}{\partial \varphi_i}X_j\right) \quad (i = f+g+1, \ldots, n). \end{aligned} \tag{5.17}$$

Equations (5.13), (5.16) and (5.17) form the basic system for further analysis. Their right hand sides are $2\pi-$periodic with respect to $\tau_1, \ldots, \tau_r$, $\varphi_1, \ldots, \varphi_f, \vartheta_f \ldots, \vartheta_{f+g}$. The slow "phases" $\varphi_{f+g+1}, \cdots, \varphi_n$ need not satisfy this condition. It is important that all the variables of the system, except the anisochronous phases $\varphi_1, \ldots, \varphi_f$, are slow. Among them, anisochronous frequencies $\omega_1, \ldots, \omega_f$ determining the velocity of change of anisochronous phases plays a special role.

Therefore, while describing the general scheme of averaging the obtained system, it is expedient to introduce uniform notation for the "non-singular" slow variables of the problem

$$x_k = \begin{cases} \vartheta_{f+k} & k = 1, \ldots, g \\ s_{f-g+k} & k = g+1, \ldots, 2g \\ \varphi_{f-g+k} & k = 2g+1, \ldots, n-f+g \\ s_{2f-n+k} & k = n-f+g+1, \ldots, 2(n-f) \end{cases} . \tag{5.18}$$

As a result, instead of eqs. (5.13), (5.16) and (5.17) we obtain the following system with $f-$dimensional fast rotating phase

$$\begin{aligned} \dot{\varphi}_i - \omega_i &= \varepsilon U_i^{(1)} + \varepsilon^2 U_i^{(2)} + \varepsilon^2 \ldots, \\ \dot{\omega}_i &= \varepsilon V_i^{(1)} + \varepsilon^2 V_i^{(2)} + \varepsilon^2 \ldots, \\ \dot{x}_k &= \varepsilon W_k^{(1)} + \varepsilon^2 W_k^{(2)} + \varepsilon^2 \ldots, \\ & (i = 1, \ldots, f,\ k = 1, \ldots, m). \end{aligned} \tag{5.19}$$

Here $m = 2(n - f)$ and functions $U_i^{(j)}, V_i^{(j)}, W_k^{(j)}$ are obtained by expanding the right hand sides of eqs. (5.13), (5.16) and (5.17) in power series in terms of the explicit small parameter ε, and satisfy the above listed properties.

5.3 Excluding non-critical fast variables

In what follows we study equations which are somewhat more general than eq. (5.4) and differ from the latter in that functions X_i and Y_i depend not only on the canonical variables $q_1, p_1, \ldots, q_n, p_n$ and phases of the external excitation $\tau_1, \ldots, \tau_r$ but also on the components of the entries of the $k \times 1$ vector y which is not known in advance. This vector is assumed to be introduced by means of a special k−dimensional differential equation. For $\varepsilon = 0$, the latter admits a quasi-periodic solution to be constructed which has the frequency basis $\omega_1, \ldots, \omega_f, \nu_1, \ldots, \nu_r$ and is globally stable. This means that eq. (5.4) should be supplemented with the following equation

$$\dot{y} = Ay + F + \varepsilon \ldots , \tag{5.20}$$

which is linear in y. Here the $k \times 1$ vector F depends on $q_1, p_1, \ldots, q_n, p_n$ and is 2π−periodic with respect to $\tau_1, \ldots, \tau_r$. After performing the transformation described in the previous section, the components of this vector depend upon $\omega_1, \ldots, \omega_f, x_1, \ldots, x_m$ and are 2π−periodic with respect to $\varphi_1, \ldots, \varphi_f$ and $\tau_1, \ldots, \tau_r$. Further analysis is carried out for the case $r = 1$ ($\tau_1 = \tau = \nu t$). The generalisation to the case for $r > 1$ is obvious.

Let us introduce the following two assumptions for the $k \times k$ matrix A:

1) after transformation to the final equations (5.19) the components of this matrix depend only on the slow variables $\omega_1, \ldots, \omega_f, x_1, \ldots, x_m$, and

2) the eigenvalues of the matrix have non-small negative real parts for arbitrary fixed values of $\omega_1, \ldots, \omega_f, x_1, \ldots, x_m$.

It seems likely that it is not possible to consider sufficiently general cases that admit construction of a generating approximation of the stable quasi-periodic solution.

In accordance with the adopted nomenclature, the components of vector y are referred to as the non-critical fast variables. Let us demonstrate that these variables can be removed by means of the following asymptotic series in terms of the powers of ε

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 \ldots , \tag{5.21}$$

where the coefficients are 2π−periodic with respect to $\varphi_1, \ldots, \varphi_f$ and depend upon $\omega_1, \ldots, \omega_f, x_1, \ldots, x_m$.

This means that only a small vicinity of the generating quasi-periodic family is considered, and the transient motions under arbitrary initial conditions for y are not investigated.

Substituting series (5.21) into eq. (5.20) and accounting for eq. (5.19) in the original approximation leads to the linear partial differential equation for y_0

$$\sum_{k=1}^{f} \omega_i \frac{\partial y_0}{\partial \varphi_i} + \nu \frac{\partial y_0}{\partial \tau} = A y_0 + F. \tag{5.22}$$

Clearly, the fast phases $\varphi_1, \dots, \varphi_f, \tau$ should be understood as the independent arguments of this equation whereas the slow variables $\omega_1, \dots, \omega_f$, $x_1, \dots, x_m$ can be taken to be constant.

A general investigation of eq. (5.22) is very difficult. However its stable quasi-periodic solution of the required type can be constructed easily in closed form with the help of the ideas of Sec. 1.2. To this aim, we introduce into consideration the generalised impulse-frequency characteristic of the system defined as the $(f+1)$ −frequency solution of the following matrix equation

$$\sum_{k=1}^{f} \omega_i \frac{\partial K}{\partial \theta_i} + \nu \frac{\partial y_0}{\partial \theta} = AK + E_k \Phi(\theta) \prod_{i=1}^{f} \Phi(\theta_i), \tag{5.23}$$

where, as in Sec. 1.2 Φ designates the impulse-frequency function

$$\Phi(\theta) = \sum_{i=-\infty}^{\infty} \delta(\theta - 2\pi i), \tag{5.24}$$

and E_k is the $k \times k$ unity matrix. It is easy to see that solution of eq. (5.23) of the required type is cast in the form of the $(f+1)$ −fold Fourier series

$$K = \sum_{j_1, \dots, j_f, j=-\infty}^{\infty} K_{j_1, \dots, j_f, j} \exp\left[\sqrt{-1}\left(\sum_{k=1}^{f} j_i \theta_i + j\theta\right)\right], \tag{5.25}$$

where

$$K_{j_1, \dots, j_f, j} = \frac{1}{(2\pi)^{f+1}} \left[\sqrt{-1}\left(\sum_{k=1}^{f} j_i \theta_i + j\nu\right) E_k - A\right]^{-1}. \tag{5.26}$$

Notice that the Fourier coefficients (5.26) are bounded since the real parts of the eigenvalues of matrix A are not small for any integer $j_1, \dots, j_f, j$.

Now it is not difficult to show that the original approximation of the required type is determined in closed form by means of the $(f+1)$ −fold convolution integral

$$y_0 = \int_0^{2\pi} \cdots \int_0^{2\pi} KF \Big|_{\varphi_i = \xi_i} d\xi_1 \cdots d\xi_f d\xi, \tag{5.27}$$

where

$$\theta_i = \varphi_i - \xi_i \quad (i = 1, \dots, f) \quad \theta = r - \xi, \tag{5.28}$$

and the dummy variables $\xi_1, \dots, \xi_f, \xi$ are substituted into vector function F instead of $\varphi_1, \dots, \varphi_f, \tau$, respectively.

It is necessary to mention that one can separate from series (5.25) an arbitrary number of infinite subsequences with coefficients which do not decrease at infinity. The summation indices $j_1, \dots, j_f, j$ corresponding to each of these sequences are characterised by the fact that the integer combinations $\sum_{k=1}^{f} j_i\theta_i + j\nu$ become as close to any a priori given real number as is wished. This is indicative of the fact that the structure of the generalised impulse-frequency characteristic is much more complex than the structure of the single-frequency one and series (5.25) should be understood in the generalised sense. However, the character of function y_0, given by eq. (5.27) or by the corresponding Fourier expansion, is not more complex than that of function F. In particular, the Fourier coefficients for y_0 at infinity decrease at least not slower than those for F.

Equations for the higher approximations have a similar structure. This testifies that the coefficients of the formal series (5.21) can be determined up to any power of the small parameter. Inserting the obtained expansions into the original system leads to the isolated system of type (5.19). The right hand sides of this system, see eqs. (5.13), (5.16) and (5.17), are subject to a certain transformation related to excluding the non-critical fast variables. In what follows, we assume that the corresponding change is carried out and retain the notation.

5.4 Averaging equations of motion in the vicinity of the chosen torus

Let us turn our attention to averaging system (5.19). We assume that the right hand sides of equations (5.19) after removing the non-critical fast variables are explicit functions of the slow variables $\omega_1, \dots, \omega_f, x_1, \dots, x_m$ and are 2π−periodic with respect to the fast phases $\varphi_1, \dots, \varphi_f$ and the phases of external excitation $\tau_1, \dots, \tau_r$. The frequencies of the external excitation $\nu_1, \dots, \nu_r$ are taken to be essentially incommensurable. This means, see [36], that for any integer r−dimensional vector l with a non-zero norm there exists such a value L that the inequality

$$|l \cdot v| > L\,|l|^{-r+1} \tag{5.29}$$

holds, where $|l| = \sum_{i=1}^{r} |l_i|$, $l \cdot \nu = \sum_{i=1}^{r} l_i \cdot \nu_i$ and $l_1, \dots, l_r$ denote components of vector l.

Let us choose a $f \times r$ matrix a_{ij} having integer components of the order of unity and carry out the following change of variables

$$\varphi_i = \sum_{k=1}^{r} a_{ij}\tau_i + \vartheta_i, \quad \omega_i = \sum_{k=1}^{f} a_{ij}\nu_i + \sqrt{\varepsilon}\gamma_i \quad (i = 1, \dots, f). \tag{5.30}$$

Let us assume that anisochronous phase shifts $\vartheta_1, \dots, \vartheta_f$ and the corresponding detunings of the anisochronous frequencies $\gamma_1, \dots, \gamma_f$ have order not lower than unity, see (5.14). Then replacement (5.30) is correct only in a certain $\sqrt{\varepsilon}$ neighbourhood of the selected torus with the frequency basis $\nu_1, \dots, \nu_r$. Direct substitution of eq. (5.30) into (5.19) leads to the system of equations in the standard form

$$\begin{aligned} \dot{\vartheta}_i &= \sqrt{\varepsilon}\gamma_i + \varepsilon U_i^{(1)} + \varepsilon^2 U_i^{(2)} + \varepsilon^3 \dots, \\ \dot{\gamma}_i &= \sqrt{\varepsilon} V_i^{(1)} + \varepsilon^{3/2} V_i^{(2)} + \varepsilon^{5/2} \dots, \quad (i = 1, \dots, f) \\ \dot{x}_k &= \varepsilon W_k^{(1)} + \varepsilon^2 W_k^{(2)} + \varepsilon^3 \dots, \quad (k = 1, \dots, m), \end{aligned} \tag{5.31}$$

the right hand sides being 2π–periodic with respect to the phases of the external excitation $\tau_1 = \nu_1 t, \dots, \tau_r = \nu_r t$. The time rates of the modified anisochronous variables ϑ_i and γ_i have the lower order of smallness $\sqrt{\varepsilon}$ than the others. This means that constructing only the first approximation to the solutions of system (1.44) by the averaging method is not sufficient. With this in view, we first determine three asymptotic approximations to the solutions of the system of equations in the standard form

$$\dot{z}_i = \sum_{k=1}^{\infty} \mu^k Z_i^{(k)} (z_1, \dots, z_n, \tau), \quad (i = 1, \dots, f), \tag{5.32}$$

where μ denotes a small parameter, and, for the sake of simplicity, functions $Z_i^{(k)}$ are assumed to be 2π–periodic with respect to the single phase $\tau = \nu t$. Acting formally, we use the nearly identical change of variables

$$z_i = \zeta_i + \sum_{k=1}^{\infty} \mu^k u_i^{(k)} (\zeta_1, \dots, \zeta_n, \tau). \tag{5.33}$$

Here the coefficients are 2π–periodic with respect to τ and are not known in advance. They are chosen in such a way that the equations of motion in the new variables are autonomous, i.e.

$$\dot{\zeta}_i = \sum_{k=1}^{\infty} \mu^k \Xi_i^{(k)} (\zeta_1, \dots, \zeta_n) \quad (i = 1, \dots, n). \tag{5.34}$$

Then, as usual, cf. [19], [98], we differentiate eq. (5.33) with respect to t, remove derivatives $\dot{z}_i$ and $\dot{\zeta}_i$ by means of eqs. (5.32) and (5.34), exclude the old variables with the help of eq. (5.33) and equate the coefficients of the same powers of μ. The balance of the coefficients of μ yields

$$\Xi_i^{(1)} + \nu \frac{\partial u_i^{(1)}}{\partial \tau} = Z_i^{(1)}, \tag{5.35}$$

where, here and in what follows, $\zeta_1, \ldots, \zeta_n$ are inserted in functions $Z_i^{(k)}$ $(k = 1, 2, \ldots)$ instead of $z_1, \ldots, z_n$. Averaging eq. (5.35) with respect to explicit τ, we obtain, due to the 2π–periodicity of $u_i^{(1)}$ that

$$\Xi_i^{(1)} = \langle Z_i \rangle, \tag{5.36}$$

where $\langle \cdot \rangle = \int_0^{2\pi} \cdot \, d\tau$. Function $u_i^{(1)}$ is then determined by quadratures up to an arbitrary function of $\zeta_1, \ldots, \zeta_n$. Let us agree to remove this arbitrariness in such a way that function $u_i^{(1)}$ (as well as $u_i^{(2)}, u_i^{(3)}, \ldots$) is 2π–periodic with respect to τ and has a zero mean value. Thus,

$$u_i^{(1)} = \left\{ Z_i^{(1)} \right\}_1, \tag{5.37}$$

where here and further on, in accordance with the above notation, cf. (3.61), $\{\ \}_k$ with $k = 1, 2, 3, \ldots$ denotes the antiderivative of the corresponding function of order k which is periodic with respect to t and has a zero mean value.

Equations for the higher approximations are identical to eq. (5.35) and their solution is obtained by analogy. As a result, equating coefficients of μ^2 and μ^3, we finally obtain

$$\begin{aligned}
\Xi_i^{(2)} &= \left\langle Z_i^{(2)} + \frac{\partial Z_i^{(1)}}{\partial \zeta_j} u_j^{(1)} \right\rangle, \ u_i^{(2)} = \left\{ Z_i^{(2)} + \frac{\partial Z_i^{(1)}}{\partial \zeta_j} u_j^{(1)} - \frac{\partial u_i^{(1)}}{\partial \zeta_j} \Xi_j^{(1)} \right\}_1, \\
\Xi_i^{(3)} &= \left\langle Z_i^{(3)} + \frac{\partial Z_i^{(2)}}{\partial \zeta_j} u_j^{(1)} + \frac{\partial Z_i^{(1)}}{\partial \zeta_j} u_j^{(2)} + \frac{1}{2} \frac{\partial^2 Z_i^{(1)}}{\partial \zeta_j \partial \zeta_l} u_j^{(1)} u_l^{(1)} \right\rangle, \\
u_i^{(3)} &= \left\{ Z_i^{(3)} + \frac{\partial Z_i^{(2)}}{\partial \zeta_j} u_j^{(1)} + \frac{\partial Z_i^{(1)}}{\partial \zeta_j} u_j^{(2)} - \frac{\partial u_i^{(1)}}{\partial \zeta_j} \Xi_j^{(2)} - \frac{\partial u_i^{(2)}}{\partial \zeta_j} \Xi_j^{(1)} + \right. \\
&\quad \left. \frac{1}{2} \frac{\partial^2 Z_i^{(1)}}{\partial \zeta_j \partial \zeta_l} u_j^{(1)} u_l^{(1)} \right\}_1,
\end{aligned} \tag{5.38}$$

where summation over repeated subscripts j and l from 1 to n is implied.

The formulae obtained allow us to average the original system (5.29) in the case of a single frequency excitation

$$r = 1, \ \tau_1 = \tau = \nu t, \quad a_{i1} = 1, 0 \tag{5.39}$$

up to quantities of order $\varepsilon^{3/2}$ included. For this purpose, it is sufficient to put $\mu = \sqrt{\varepsilon}, z = (\vartheta_i, \gamma_i, x_k)$. It is important that if a natural number

$a_{i1} = 0$ for some i, then the corresponding frequency ω_i can not be made to be coincident with the excitation frequency, and phase φ_i is a slow variable.

Summarising, by virtue of eqs. (5.37) and (5.38), the required asymptotic replacement of variables $\vartheta_i \to \alpha_i$, $\gamma_i \to \eta_i$, $x_k \to \sigma_k$ up to terms of order $\varepsilon^{3/2}$ is represented in the form

$$\begin{aligned}
\vartheta_i &= \alpha_i + \varepsilon\left(\left\{U_i^{(1)}\right\}_1 + \left\{V_i^{(1)}\right\}_2\right) - \varepsilon^{3/2}\frac{\partial}{\partial\alpha_j}\left\{V_i^{(1)}\right\}_2 \eta_j + \varepsilon^2 \dots, \\
\gamma_i &= \eta_i + \sqrt{\varepsilon}\left\{V_i^{(1)}\right\}_1 - \varepsilon\frac{\partial}{\partial\alpha_j}\left\{V_i^{(1)}\right\}_2 \eta_j + \varepsilon^{3/2}\left\{V_i^{(2)} + \right. \\
&\quad \frac{\partial V_i^{(1)}}{\partial\alpha_j}\left(\left\{U_j^{(1)}\right\}_1 + \left\{V_j^{(1)}\right\}_2\right) + \frac{\partial V_i^{(1)}}{\partial\Omega_j}\left\{V_j^{(1)}\right\}_1 + \frac{\partial V_i^{(1)}}{\partial\sigma_k}\left\{W_k^{(1)}\right\}_1 \\
&\quad - \left(\left\langle U_j^{(1)}\right\rangle \frac{\partial}{\partial\alpha_j} + \left\langle V_j^{(1)}\right\rangle \frac{\partial}{\partial\Omega_j} + \left\langle W_k^{(1)}\right\rangle \frac{\partial}{\partial\sigma_k}\right)\left\{V_i^{(1)}\right\}_1 + \\
&\quad \left. \eta_j\eta_l\frac{\partial^2}{\partial\alpha_j\partial\alpha_l}\left\{V_i^{(1)}\right\}_1\right\}_1 + \varepsilon^2 \dots, \\
x_k &= \sigma_k + \varepsilon\left\{W_k^{(1)}\right\}_1 - \varepsilon^{3/2}\frac{\partial}{\partial\alpha_j}\left\{W_k^{(1)}\right\}_2 \eta_j + \varepsilon^2 \dots. \qquad (5.40)
\end{aligned}$$

Here the repeated subscripts j and l imply summation from 1 to n whilst the repeated subscript k means summation from 1 to m. The original arguments ϑ_i, x_k, ω_i in the expressions for functions $U_i^{(s)}, V_i^{(s)}, W_k^{(s)}$ $(s = 1, 2, \dots)$ should be replaced respectively by α_i, σ_k and

$$\Omega_i = a_{i1}\nu + \sqrt{\varepsilon}\eta_i \quad (i = 1, \dots, f), \qquad (5.41)$$

where the latter three variables are naturally referred to as the averaged anisochronous frequencies. In general, they differ from the original frequencies in values of order ε, i.e. $\omega_i - \Omega_i = O(\varepsilon)$. This also means that while constructing asymptotic approximations we "forgot" for the time being the explicit dependence of the averaged frequencies Ω_i on ε. We used the fact that all functions on the right hand sides of original system (5.31), i.e. $U_i^{(1)}, V_i^{(1)}, W_k^{(1)}, U_i^{(2)}, \dots$ depend on variables γ_i only in terms of anisochronous frequencies $\omega_i = \alpha_{ij}\nu + \sqrt{\varepsilon}\gamma_i$. By virtue of eq. (5.41) all of the terms in series (5.40) can be, in turn, represented as power series in terms of $\sqrt{\varepsilon}$ about point $\Omega_i = \alpha_{ij}\nu$.

When we write down the averaged equations of motion, it is more convenient to enter frequencies $\Omega_1, \dots, \Omega_f$ instead of averaged detunings $\eta_1, \dots,$ η_f. With the help of eqs. (5.36) and (5.38) we obtain

$$\begin{aligned}
\dot{\alpha}_i &= \Omega_i - a_{i1}\nu + \varepsilon R_i + \varepsilon \dots, \quad \dot{\Omega}_i = \varepsilon P_i + \varepsilon^2 P_i^{(1)} + \varepsilon^{5/2} \dots, \\
\dot{\sigma}_k &= \varepsilon Q_k + \varepsilon^2 \dots. \qquad (5.42)
\end{aligned}$$

These equations are very convenient since, up to the achieved accuracy, all functions on the right hand sides are dependent only on $\alpha_1, \dots, \alpha_f$,

$\Omega_1, \dots, \Omega_f, \sigma_1, \dots, \sigma_m$

$$\begin{aligned} R_i &= \left\langle U_i^{(1)} \right\rangle, \quad P_i = \left\langle V_i^{(1)} \right\rangle, \quad Q_k = \left\langle W_k^{(1)} \right\rangle, \\ P_i^{(1)} &= \left\langle V_i^{(2)} + \frac{\partial V_i^{(1)}}{\partial \alpha_j} \left(\left\{ U_j^{(1)} \right\}_1 + \left\{ V_j^{(1)} \right\}_2 \right) + \frac{\partial}{\partial \Omega_j} \left\{ V_j^{(1)} \right\}_1 + \right. \\ & \left. \frac{\partial}{\partial \sigma_k} \left\{ W_k^{(1)} \right\}_1 \right\rangle . \end{aligned} \tag{5.43}$$

It is important that differentiating with respect to Ω_i, in contrast to η_i, does not affect the order of smallness of the corresponding term, cf. $\dfrac{\partial}{\partial \eta_i} = \sqrt{\varepsilon} \dfrac{\partial}{\partial \Omega_i}$. With accuracy of values of order $\varepsilon^{3/2}$ the equations in (5.42) are obtained by formally averaging the right hand sides of the original system (5.19) if we set $\varphi_i = \alpha_{ij} \nu t + \alpha_i, \omega_i = \Omega_i, x_k = \sigma_k$. Taking this into account and eq. (5.13) we have

$$\begin{aligned} R_i &= \sum_{k=1}^{f} e_{ik} \sum_{j=1}^{n} \left\langle \frac{\partial p_j}{\partial \omega_k} X_j - \frac{\partial q_j}{\partial \omega_k} Y_j \right\rangle , \\ P_i &= \sum_{k=1}^{f} e_{ik} \sum_{j=1}^{n} \left\langle \frac{\partial q_j}{\partial \varphi_k} Y_j - \frac{\partial p_j}{\partial \varphi_k} X_j \right\rangle . \end{aligned} \tag{5.44}$$

Due to eq. (5.30) anisochronous fast phases φ_i are additive with respect to phase shifts ϑ_i. By virtue of eq. (5.40) they are additive with respect to the averaged phase shifts α_i up to non-small terms. This substantiates the replacement $\dfrac{\partial}{\partial \varphi_k} \to \dfrac{\partial}{\partial \alpha_k}$ (and clearly $\dfrac{\partial}{\partial \omega_k} \to \dfrac{\partial}{\partial \Omega_k}$) in expression (5.44). Variables $\alpha_1, \dots, \alpha_f$ in system (5.42) are formally not slow. However, it is necessary to remember that this system is valid only in the considered $\sqrt{\varepsilon}$ vicinity of the single-frequency regime. Correspondingly, analysis of eq. (5.42) requires the initial conditions for $\Omega_1, \dots, \Omega_f$, see also (5.41).

Like (5.31), averaging the multi-frequency system

$$\dot{z}_i = \sum_{k=1}^{\infty} \mu^k Z_i^{(k)} \left(z_1, \dots, z_n, \tau_1, \dots, \tau_n \right) \tag{5.45}$$

is carried out using a similar scheme. Asymptotic transformation of variables is, as before, represented in the form of series (5.43) with coefficients $u_i^{(1)}, u_i^{(2)}, \dots$ which are quasi-periodic functions of explicit time and have zero mean values. This transformation results in autonomous system (5.34). Due to the formal scheme of averaging [19], [98], equating of terms of order

μ^k leads to the equations

$$\sum_{j=1}^{r} \nu_j \frac{\partial u_i^{(k)}}{\partial \tau_j} + \Xi_i^{(k)} = \Gamma_i^{(k)} (\zeta_1, \dots, \zeta_n, \tau_1, \dots, \tau_r)$$

$$\left(k = 1, 2, \dots, \ \Gamma_i^{(1)} = Z_i^{(1)}\right), \tag{5.46}$$

where quasi-periodic functions of time are determined at the previous step of the iteration process and are represented by the generalised Fourier series

$$\Gamma_i^{(k)} = \sum_l \Gamma_{il}^{(k)} \exp\left[\sqrt{-1}\,(l \cdot \tau)\right]. \tag{5.47}$$

Here l is an integer $r-$dimensional vector, $\Gamma_{il}^{(k)}(\zeta_1, \dots, \zeta_n)$ denotes the Fourier coefficients and $\tau = (\tau_1, \dots, \tau_r)$ is a $r \times 1$ vector. It follows from eq. (5.46) that

$$\Xi_i^{(k)} = \left\langle \Gamma_i^{(k)} \right\rangle = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \Gamma_i^{(k)} d\tau_1 \cdots d\tau_r,$$

$$u_i^{(k)} = \left\{ \Gamma_i^{(k)} \right\} = \sum_{|l| \neq 0} \frac{\Gamma_{il}^{(k)}}{\sqrt{-1}\,(l \cdot \nu)} \exp\left[\sqrt{-1}\,(l \cdot \nu)\right], \tag{5.48}$$

where $(l \cdot \nu) = \sum_{i=1}^{r} l_i \nu_i$. Inequality (5.29) holds because of essential rational incommensurability of frequencies $\nu_1, \dots, \nu_r$. Let us additionally assume that function $\Gamma_i^{(k)}$ is bounded (i.e. $\left|\Gamma_i^{(k)}\right| \leq M$) in the region $|\mathrm{Im}\, \tau_j| \leq R$. This substantiates the following estimate for the Fourier coefficients

$$\left|\Gamma_{il}^{(k)}\right| \leq M \exp\left(-R\,|l|\right). \tag{5.49}$$

The above said ensures convergence of series (5.47), see [19], [98]. However, due to the existence of an infinite number of small denominators of the sort $(l \cdot \nu)^{-1}$, the constructed function $u_i^{(k)}$ is a discontinuous function of frequencies $\nu_1, \dots, \nu_r$. A number of papers devoted to improving the convergence of the method, see e.g. [20], [36], substantiate the basic idea of overcoming this difficulty which prevents direct application of the formal averaging scheme. In accordance with this idea, function $\Gamma_i^{(k)}$ given by series (5.49) is represented as a sum of two components

$$\left(\Gamma_i^{(k)}\right)_N = \sum_{|l| \leq N} \Gamma_{il}^{(k)} \exp\left[\sqrt{-1}\,(l \cdot \tau)\right],$$

$$R_N \Gamma_i^{(k)} = \sum_{|l| > N} \Gamma_{il}^{(k)} \exp\left[\sqrt{-1}\,(l \cdot \tau)\right], \tag{5.50}$$

with N being a natural number. It is known that the following inequality

$$R_N\Gamma_i^{(k)} \leq \left(\frac{2r}{|l|}\right)^r \frac{M}{\delta^{r+1}} \exp\left(-N\gamma\right) \tag{5.51}$$

holds in the narrowed region $|\mathrm{Im}\,\tau_j| = R - \gamma - \delta$, where $4\delta \leq 2\gamma \leq R < 1$.

Let us choose number N such that the remainder of series $R_N\Gamma_i^{(k)}$ has order of smallness $\mu^{k'}$ where k' is a natural number. By virtue of eq. (5.50) we immediately have

$$N \sim -\frac{1}{\gamma} \ln \left[\mu^{k'} \left(\frac{e}{2r}\right)^r \frac{\delta^{r+1}}{M}\right]. \tag{5.52}$$

The final estimate for N is obtained if we drop all the values of zero order in eq. (5.51) and has the form

$$N = \frac{k'}{\gamma} \ln \frac{1}{\mu}. \tag{5.53}$$

Let us notice that this estimate does not depend on the particular form of function $\Gamma_i^{(k)}$.

Hence, due to eq. (5.53), the remainder of series $R_N\Gamma_i^{(k)}$ can be taken into account by means of the next approximation of order $\mu^{k+k'}$ whereas the right hand side of eq. (5.46) contains a finite trigonometric sum $\left(\Gamma_i^{(k)}\right)_N$. In turn, the expression for the $k-th$ correction $u_i^{(k)}$, eq. (5.48), can be represented as a finite trigonometric sum. As the norm of vector l in this sum does not exceed N, there exists the following restriction

$$|l \cdot \nu| \geq LN^{-r+1} \gg \mu \tag{5.54}$$

for the small denominators, cf. eq. (5.29).

A similar correction is needed for each step of the many frequency averaging and it is often convenient to take $k' = 1$. This means that the series remainder $R_N\Gamma_i^{(k)}$ should be taken into account at the next $(k+1)-th$ approximation. However, for solving a number of practical problems, the equations for determining several first approximations are characterised by the fact that functions $\Gamma_i^{(1)}, \Gamma_i^{(2)}, \ldots$ are automatically represented as a sum of a small number of harmonics. Therefore, the very necessity of this correction for constructing the approximations vanishes.

Taking this into account we consider expressions (5.36)-(5.38), (5.40) and (5.43) to be valid for multi-frequency averaging. The operations of averaging and determination of the quasi-periodic antiderivative with zero mean value should be carried out in accordance with (5.48). Finally, the averaged system is written in the form, cf. (5.42)

$$\begin{aligned}\dot{\alpha}_i &= \Omega_i - \sum_{j=1}^{r} a_{ij}\nu_j + \varepsilon R_i + \varepsilon^2 \ldots , \quad \dot{\Omega}_i = \varepsilon P_i + \varepsilon^2 P_i^{(1)} + \varepsilon^{5/2} \ldots , \\ \dot{\sigma}_k &= \varepsilon Q_k + \varepsilon^2 \ldots . \end{aligned} \tag{5.55}$$

As for the existence of infinite series of the averaging method (of the type of (5.33) or (5.40)) it is understood in the asymptotic sense and is determined by the existing theorems [19].

5.5 Existence and stability of stationary solution of the averaged system

In accordance with [19], the existence of an asymptotically stable stationary (constant) solution of system (5.55) guarantees, in the general case, existence and stability of the quasi-periodic solution of the original system (5.31) and (5.19) with the frequency basis $\nu_1, \ldots , \nu_r$. For sufficiently small values of ε this stationary solution is analytic with respect to $\sqrt{\varepsilon}$ and is represented in the form of the series

$$\begin{aligned}\alpha_i &= \alpha_i^{(0)} + \varepsilon\alpha_i^{(1)} + \varepsilon^{3/2} \ldots , \quad \Omega_i = \sum_{j=1}^{r} a_{ij}\nu_j + \varepsilon\Omega_i^{(1)} + \varepsilon^{3/2} \ldots , \\ \sigma_k &= \sigma_k^{(0)} + \varepsilon\sigma_i^{(1)} + \varepsilon^{3/2} \ldots . \end{aligned} \tag{5.56}$$

Inserting eq. (5.56) into (5.52) we obtain in the original approximation

$$(P_i) = 0, \quad (Q_k) = 0, \quad (i = 1, \ldots , f,\ k = 1, \ldots , m), \tag{5.57}$$

where, here and in what follows, parentheses mean that the corresponding quantity is calculated for $\alpha_i = \alpha_i^{(0)}, \sigma_k = \sigma_k^{(0)}$ and $\Omega_i = \sum_{j=1}^{r} a_{ij}\nu_j$ (in the single frequency case $\Omega_i = a_{i1}\nu$). Thus, in the general case, equalities (5.57) form a system of transcendental equations for parameters of the "generating" solution $\alpha_1^{(0)}, \ldots , \alpha_f^{(0)}, \sigma_1^{(0)}, \ldots , \sigma_m^{(0)}$. Let us assume that this system has a simple solution. Then all of the successive approximations to the stationary solution will be determined from the linear heterogeneous systems with non-trivial determinant

$$\Delta = \begin{vmatrix} \left(\dfrac{\partial P}{\partial \alpha}\right) & \left(\dfrac{\partial P}{\partial \sigma}\right) \\ \left(\dfrac{\partial Q}{\partial \alpha}\right) & \left(\dfrac{\partial Q}{\partial \sigma}\right) \end{vmatrix} \neq 0. \tag{5.58}$$

For this reason, inequality (5.57) is a sufficient condition for the existence of the required stationary solution (5.56) to system (5.55) for sufficiently small ε.

The local stability of the constructed stationary solution is established from the linear homogeneous system with constant coefficients

$$\begin{aligned}
\delta\dot{\alpha}_i &= \delta\Omega_i + \varepsilon\left[\left(\frac{\partial R_i}{\partial \alpha_j}\right)\delta\alpha_j + \left(\frac{\partial R_i}{\partial \Omega_j}\right)\delta\Omega_j + \left(\frac{\partial R_i}{\partial \sigma_k}\right)\delta\sigma_k\right] + \varepsilon^2 \ldots , \\
\delta\dot{\Omega}_i &= \varepsilon\left[\left(\frac{\partial P_i}{\partial \alpha_j}\right)\delta\alpha_j + \left(\frac{\partial P_i}{\partial \Omega_j}\right)\delta\Omega_j + \left(\frac{\partial P_i}{\partial \sigma_k}\right)\delta\sigma_k\right] + \varepsilon^2 \ldots , \\
\delta\dot{\sigma}_k &= \varepsilon\left[\left(\frac{\partial Q_k}{\partial \alpha_j}\right)\delta\alpha_j + \left(\frac{\partial Q_k}{\partial \Omega_j}\right)\delta\alpha_j + \left(\frac{\partial Q_k}{\partial \sigma_l}\right)\delta\sigma_l\right] + \varepsilon^2 \ldots , \quad (5.59)
\end{aligned}$$

obtained by means of variation of system (5.42) about the stationary point. Here and in what follows, the repeated subscript j implies summation from 1 to f, and subscripts k and l from 1 to m, see eq. (5.40).

In the generating approximation, system (5.59) admits $f+m$ independent constant solutions

$$\begin{aligned}
\delta\alpha_i &= \delta_{ij}, \quad \delta\Omega_i = \delta\sigma_k = 0 \quad (j = 1, \ldots , f), \\
\delta\alpha_i &= \delta\Omega_i = 0, \quad \delta\sigma_k = \delta_{kl} \quad (l = 1, \ldots , m) \qquad (5.60)
\end{aligned}$$

and f solutions which increase linearly in time

$$\delta\Omega_i = \delta_{ij}, \quad \delta\alpha_i = \delta_{ij}t, \quad \delta\sigma_k = 0 \quad (j = 1, \ldots , f) . \qquad (5.61)$$

Thus, characteristic equation of system (5.59) at $\varepsilon = 0$ has a $m-$fold zero root with simple elementary divisors and a $2f-$fold zero root with square elementary divisors. Investigation of the local asymptotic stability is based upon constructing mutually independent particular solutions of system (5.59) in the form

$$\delta\alpha_i = a_i e^{\lambda t}, \quad \delta\Omega_i = b_i e^{\lambda t}, \quad \delta\sigma_k = c_k e^{\lambda t}, \qquad (5.62)$$

which coincide with (5.60) at $\varepsilon = 0$. Here constant factors a_i, b_i, c_k and the characteristic exponent λ depend on ε.

The right hand sides of the equations in system (5.59) are analytic with respect to $\sqrt{\varepsilon}$. However, the terms of the lowest order of smallness are proportional to ε and ε^2. Then, according to the known theorem of Malkin [61] for a $m-$fold zero root of the generating solution, there exist m particular solutions (5.62) for which

$$\begin{aligned}
\lambda &= \lambda_1\varepsilon + \lambda_2\varepsilon^2 + \varepsilon^{5/2} \ldots , \quad a_i = a_i^{(0)} + \varepsilon a_i^{(1)} + \varepsilon^2 \ldots , \\
b_i &= b_i^{(0)} + \varepsilon b_i^{(1)} + \varepsilon^2 \ldots , \quad c_k = c_k^{(0)} + \varepsilon c_k^{(1)} + \varepsilon^2 \ldots . \qquad (5.63)
\end{aligned}$$

If we now substitute eqs. (5.62) and (5.63) into (5.59) and equate the non-small terms, we obtain $b_1^{(0)} = \cdots = b_f^{(0)} = 0$. Equating terms of the first order in ε yields the following linear homogeneous system of equations

$$\begin{aligned} \left(\frac{\partial P_i}{\partial \alpha_j}\right) a_j^{(0)} + \left(\frac{\partial P_i}{\partial \sigma_l}\right) c_l^{(0)} &= 0, \quad (i = 1, \dots, f), \\ \left(\frac{\partial Q_k}{\partial \alpha_j}\right) a_j^{(0)} + \left(\frac{\partial Q_k}{\partial \sigma_l}\right) c_l^{(0)} &= \lambda_1 c_k^{(0)}, \quad (k = 1, \dots, m). \end{aligned} \tag{5.64}$$

Equating the determinant of this system to zero we arrive at the equation of $m - th$ degree in λ_1. The following m inequalities

$$\operatorname{Re} \lambda_1 < 0 \tag{5.65}$$

are the necessary conditions for the asymptotic stability of the regime and form the isochronous criteria of its stability. Let us notice that $\lambda_1 \neq 0$ due to (5.58).

Conforming with the above theorem [61], the following solution of system (5.59)

$$\begin{aligned} \lambda &= \lambda_1 \sqrt{\varepsilon} + \lambda_2 \varepsilon + \varepsilon^{3/2} \dots, \qquad a_i = a_i^{(0)} + a_i^{(1)} \sqrt{\varepsilon} + \varepsilon \dots, \\ b_i &= b_i^{(0)} + b_i^{(1)} \sqrt{\varepsilon} + \varepsilon \dots, \qquad c_k = c_k^{(0)} + c_k^{(1)} \sqrt{\varepsilon} + \varepsilon \dots \end{aligned} \tag{5.66}$$

is analytical with respect to ε and corresponds to $2f$–fold zero root.

Substituting eqs. (5.62) and (5.66) into (5.59) yields in the zeroth approximation that $b_1^{(0)} = \cdots = b_f^{(0)} = 0$. Equating terms of order of $\sqrt{\varepsilon}$ leads to the equations

$$b_i^{(1)} = \lambda_1 a_i^{(1)}, \quad c_k^{(0)} \lambda_1 = 0. \tag{5.67}$$

Since $\lambda_1 \neq 0$ we have $c_1^{(0)} = \cdots = c_m^{(0)} = 0$. Equating terms of order ε and accounting for eq. (5.67) results in the following

$$\begin{aligned} \left(\frac{\partial P_i}{\partial \alpha_j}\right) a_j^{(0)} &= \lambda_1^2 a_i^{(0)}, . \\ \lambda_2 a_i^{(0)} + \lambda_1 a_i^{(1)} &= b_i^{(2)} + \left(\frac{\partial R_i}{\partial \alpha_j}\right) a_j^{(0)}, \quad (i = 1, \dots, f), \\ \lambda_1 c_k^{(1)} &= \left(\frac{\partial Q_k}{\partial \alpha_j}\right) a_j^{(0)} \quad (k = 1, \dots, m). \end{aligned} \tag{5.68}$$

The first set of these equalities forms a separate system of f homogeneous linear equations in $a_1^{(0)}, \dots, a_f^{(0)}$. The condition for existence of non-trivial solutions of this system leads to the equation of degree f for λ_1^2

$$\det\left(\left(\frac{\partial P_i}{\partial \alpha_j}\right) - \lambda_1^2 \delta_{ij}\right) = 0. \tag{5.69}$$

Let us assume that this equation has a complex-valued solution λ_1^2. Then

$$\operatorname{Re}\lambda_1 = \pm\sqrt{\frac{1}{2}\left(\operatorname{Re}\lambda_1^2 + |\lambda_1^2|\right)}, \tag{5.70}$$

and one of these λ_1 is bound to have a positive real part. Thus for the solution to be stable it is necessary that $\operatorname{Re}\lambda_1 = 0$ or equivalently

$$\lambda_1^2 < 0. \tag{5.71}$$

Hence, all f roots of determinant (5.69) must be real and negative. The corresponding inequalities (5.71) form a set of anisochronous criteria of stability of the first kind. It is important that the pairs of λ_1 corresponding to these roots are purely imaginary and form a complex conjugate. Hence, the conditions for the asymptotic stability ($\operatorname{Re}\lambda < 0$) are determined by the values of the next correction λ_2 of order ε, see eq. (5.66). The value of this correction is obtained by balancing terms of order $\varepsilon^{3/2}$ after substituting eqs. (5.62) and (5.66) into the second set of equations in (5.59). Accounting for eq. (5.68) we have

$$\left(\frac{\partial P_i}{\partial \alpha_j}\right) a_j^{(1)} - \lambda_1^2 a_i^{(1)} = \lambda_1 \left[2\lambda_2\delta_{ij} - \left(\frac{\partial P_i}{\partial \Omega_j}\right) - \left(\frac{\partial R_i}{\partial \alpha_j}\right) - \frac{1}{\lambda_1^2}\left(\frac{\partial P_i}{\partial \sigma_k}\frac{\partial Q_k}{\partial \alpha_j}\right)\right] a_j^{(0)} \quad (i = 1, \dots, f). \tag{5.72}$$

Equalities (5.72) form a system of linear heterogenous equations for unknown values $a_1^{(1)}, \dots, a_f^{(1)}$ with zero determinant, see (5.69). In order to find the solvability condition we introduce a linear homogeneous system

$$\left(\frac{\partial P_j}{\partial \alpha_i}\right) a_j^* - \lambda_1^2 a_i^* = 0, \tag{5.73}$$

which is the conjugate to (5.68). Let us subject the solution of this system to the following condition of normalisation

$$a_j^{(0)} a_j^* = 1. \tag{5.74}$$

Let us multiply eq. (5.72) with a_i^* and sum up over i from 1 to f. Taking into account eqs. (5.73) and (5.74) we arrive at the formula for determining the required correction

$$\lambda_2 = \frac{1}{2}\left[\left(\frac{\partial P_i}{\partial \Omega_j}\right) + \left(\frac{\partial R_i}{\partial \alpha_j}\right) + \frac{1}{\lambda_1^2}\left(\frac{\partial P_i}{\partial \sigma_k}\frac{\partial Q_k}{\partial \alpha_j}\right)\right] a_i^* a_j^{(0)}. \tag{5.75}$$

Provided that inequalities (5.71) hold for all f values, λ_2, due to eq. (5.75), are real. Hence, the necessary condition for stability is fulfillment of the following inequalities

$$\lambda_2 < 0. \tag{5.76}$$

These inequalities are referred to as the anisochronous criteria of stability of the second kind. Thus, for the local asymptotic stability of the stationary quasi-periodic (in a particular periodic) solution of the original nearly conservative system, it is sufficient to satisfy three sets of different criteria of stability, namely isochronous criteria (5.65) and anisochronous criteria of the first (5.71) and second (5.76) kind. From the perspective of both constructing the regime and determining the sufficient conditions of its stability, it is necessary to find only expressions for P_i, R_i, Q_i $(i = 1, \dots, f,\ k = 1, \dots, m)$ in the form of explicit expressions of functions of the variables $\alpha_i, \Omega_i, \sigma_k$ of the averaged equations of motion.

5.6 Existence and stability of "partially-autonomous" tori

The "partially-autonomous" case is characterised by the fact that the frequency spectrum $\nu_1, \dots, \nu_r$ of the considered solution is broader than the frequency spectrum of the external excitation $\nu_{p+1}, \dots, \nu_r$ $(p < \nu)$. Then the right hand sides of the original system (5.4) or (5.9) are explicit functions only of $r - p$ external phases, whereas the autonomous (or internal) frequencies $\nu_1, \dots, \nu_p$ appear during the studied problem and thus are not known in advance. It is important that the autonomous frequencies as well as the frequencies of the external excitation are mutually incommensurable. The number of autonomous frequencies does not exceed the total number of isochronous and anisochronous phases $(d < f + g)$.

In general, transformations (5.14), (5.15) and (5.30) are applicable for determination of the $r-$frequency motion which results in the situation that the problem reduces to investigation of the system in the standard form (5.31). It is essential for the forthcoming analysis that, due to the mutual incommensurability of the autonomous frequencies, the $(f + g) \times p$ matrix a_{ij} $(i = 1, \dots, f + g,\ j = 1, \dots, p)$ with integer coefficients has rank p.

The process of averaging system (5.31) follows the method described in Sec. 5.4. The resulting averaged system is put in the form (5.42). However the right hand sides of this system of equations have certain peculiarities. This is because all functions on the right hand sides of (5.31) depend on autonomous phases $\tau_1 = \nu_1 t, \dots, \tau_p = \nu_p t$ only by means of anisochronous and isochronous phases

$$\varphi_i = \sum_{j=1}^{r} a_{ij}\tau_j + \vartheta_i \quad (i = 1, \dots, f + g), \tag{5.77}$$

cf. eqs. (5.15) and (5.30). For this reason, we have, for example

$$\frac{\partial V_i^{(1)}}{\partial \tau_j} = \sum_{l=1}^{f+g} a_{ij} \frac{\partial V_i^{(1)}}{\partial \vartheta_l} \quad (j = 1, \dots, p). \tag{5.78}$$

Multi-frequency averaging, as above, is performed in the sense of eq. (5.40), the original variables being replaced by their averaged counterparts (in particular $\vartheta_i \to \alpha_i, i = 1, \dots, f$). By virtue of eq. (5.18), the notation $\vartheta_{f+k} = x_k$ $(k = 1, \dots, g)$ is introduced for the isochronous phase shifts. This suggests entering averaged isochronous phase shifts $\alpha_{f+k} = \sigma_k$ $(k = 1, \dots, g)$, see eq. (5.40). Considering this, we carry out averaging of expression (5.78) and use notation (5.43). Since $V_i^{(1)}$ is $2\pi-$periodic with respect to τ_j, we arrive at the following result

$$\sum_{l=1}^{f+g} a_{lj} \frac{\partial P_i}{\partial \alpha_l} = 0 \quad (j = 1, \dots, p). \tag{5.79}$$

Equalities (5.79) imply in fact that functions P_i depend only on $f+g-p$ independent linear combinations of phase shifts with integer coefficients. Particularly, it follows from eq. (5.79) that

$$P_i \equiv P_i \left(a_{1j}\alpha_2 - a_{2j}\alpha_1, \dots, a_{1j}\alpha_{f+g} - a_{f+gj}\alpha_1 \right). \tag{5.80}$$

The other functions on the right hand sides of the averaged system (5.42) for all approximations and, in particular R_i, $P_i^{(1)}$, Q_k, possess similar properties. For this reason, we can write, for instance, that

$$\sum_{l=1}^{f+g} a_{lj} \frac{\partial Q_k}{\partial \alpha_l} = 0 \quad (j = 1, \dots, p). \tag{5.81}$$

The stationary solution of the truncated system represented in the form of series (5.56) with constant coefficients has certain peculiarities. This solution is a $p-$parametric manifold and the characteristic equation of the variational system about this solution has a $p-$fold zero root for any approximation. Clearly, this root does not affect the stability. Thus, the process of constructing successive approximations to the stationary solution can be extended as far as is wished, regardless of the fact that the rank of determinant (5.58) is equal to $f + g - p$.

Therefore, phase shifts $\alpha_1, \dots, \alpha_{f+g}$ are determined from the stationarity conditions up to p arbitrary additive constants. This means that not only constants $\sigma_{g+1}, \dots, \alpha_m$ but also the autonomous frequencies $\nu_1, \dots, \nu_p$ are determined with any degree of accuracy. Obviously, the latter should be sought in the form of series

$$\nu_i = \nu_i^{(0)} + \varepsilon \nu_i^{(1)} + \varepsilon^2 \dots. \tag{5.82}$$

Investigations of the stability of the stationary regime in the partially-autonomous case reveal a number of features and can be split into two essentially different cases.

1. Number of autonomous frequencies does not exceed the number of isochronous phases, i.e. $p \leq g$. The isochronous criteria of stability of the stationary solution are convenient to study on the basis of the system conjugated to (5.64) (with a transposed determinant)

$$\begin{aligned} \left(\frac{\partial P_j}{\partial \alpha_i}\right) a_j^* + \left(\frac{\partial Q_l}{\partial \alpha_i}\right) c_l^* &= 0 \quad (i = 1, \dots, f), \\ \left(\frac{\partial P_i}{\partial \sigma_k}\right) a_j^* + \left(\frac{\partial Q_l}{\partial \sigma_k}\right) c_l^* &= \lambda_1 c_k^* \quad (k = 1, \dots, m). \end{aligned} \tag{5.83}$$

Here, the repeated subscripts j and l imply summation from 1 to f and from 1 to m, respectively. Since $\sigma_k = \alpha_{f+k}$ the first g equations of the second set can be cast as follows

$$\left(\frac{\partial P_j}{\partial \alpha_i}\right) a_j^* + \left(\frac{\partial Q_l}{\partial \alpha_i}\right) c_l^* = \lambda_1 c_{i-f}^* \quad (i = f+1, \dots, f+g). \tag{5.84}$$

Let us consider the first f equations in (5.83) and the consequent g equations in (5.84). Multiplying the $i-th$ equation with a_{ij}, summing up the result over i from 1 to $f+g$ and taking into account eqs. (5.79) and (5.81) we obtain

$$\lambda_1 \sum_{i=f+1}^{f+g} a_{ij} c_{i-f}^* = 0 \quad (j = 1, \dots, p). \tag{5.85}$$

According to eq. (5.85) there is a $p-$fold zero root ($\lambda_1 = 0$) which, as mentioned above, is equal to zero for any approximation and thus does not affect the stability. For non-trivial roots ($\lambda_1 \neq 0$) we have

$$\sum_{i=f+1}^{f+g} a_{ij} c_{i-f}^* = 0 \quad (j = 1, \dots, p). \tag{5.86}$$

Thus, the isochronous criteria of stability are proved for the linear system consisting of equations (5.83) for $i = 1, \dots, f$ and $k = g+1, \dots, m$ as well as eq. (5.86). The simplest case of performing this investigation is $p = g$ when, due to the non-degeneracy of the $g \times g$ matrix a_{ij} ($i = f+1, \dots, f+g, j = 1, \dots, g$), closed system (5.86) admits only non-trivial solution $c_1^* = \dots = c_g^* = 0$. The determinant of the above system (for $p \leq g$) yields an algebraic equation of order $m-p$ for determination of the non-trivial values of λ_1 ($\lambda_1 \neq 0$). The anisochronous criteria of stability are based upon eqs. (5.69) and (5.75). Clearly, $\lambda_1^2 \neq 0$, because the $f \times f$ matrix $\dfrac{\partial P_i}{\partial \alpha_j}$ ($i, j = 1, \dots, f$) is not, generally speaking, degenerate.

2. The number of autonomous frequencies exceeds the number of isochronous phases, i.e. $p > g$. The peculiarity of this study is caused by the fact that rank of matrix $\frac{\partial P_i}{\partial \alpha_j}$ $(i, j = 1, \ldots, f)$ is equal to $f + g - p < f$. Indeed, let us resolve the last g equations in (5.79) for the derivatives with respect to the isochronous phase shifts

$$\frac{\partial P_i}{\partial \alpha_l} = -\sum_{k=1}^{f} \sum_{j=p-g+1}^{p} a_{kj} a_{lj}^{-1} \frac{\partial P_i}{\partial \alpha_k} \quad (l = f + 1, \ldots, f + g), \tag{5.87}$$

where

$$\sum_{j=p-g+1}^{p} a_{kj} a_{lj}^{-1} = \delta_{kl} \quad (k, l = f + 1, \ldots, f + g). \tag{5.88}$$

Inserting expressions (5.87) into the first $p - g$ equations (5.79), we arrive at the relationships relating the derivatives of P_i with respect only to anisochronous phase shifts

$$\sum_{k=1}^{f} b_{kj} \frac{\partial P_i}{\partial \alpha_k} = 0 \quad (j = 1, \ldots, p - g), \tag{5.89}$$

where

$$b_{kj} = a_{kj} - \sum_{i=f+1}^{f+g} \sum_{l=p-g+1}^{p} a_{ij} a_{il}^{-1} a_{kl}. \tag{5.90}$$

All $f\,(p - g)$ numbers b_{kj} are rational. Equalities (5.89) indicates the above mentioned degeneracy of matrix $\frac{\partial P_i}{\partial \alpha_j}$ $(i, j = 1, \ldots, f)$. Similar equalities are valid, as before, for functions $R_i, P_i^{(1)}, Q_k \ldots$.

Let us proceed to direct analysis of the isochronous criteria of stability. To this end, in system (5.64) we introduce the new variables $l_1, \ldots, l_{p-g}$, $m_{p-g-1}, \ldots, m_p$, $n_{p-g+1}, \ldots, n_f$ by means of the formulae

$$\begin{aligned}
a_i^{(0)} &= \sum_{j=1}^{p-g} b_{ij} l_j + \sum_{j=p-g+1}^{p} a_{ij} m_j, \quad (i = 1, \ldots, p - g) \\
a_i^{(0)} &= \sum_{j=1}^{p-g} b_{ij} l_j + \sum_{j=p-g+1}^{p} a_{ij} m_j + n_j, \quad (i = p - g + 1, \ldots, f) \\
c_k^{(0)} &= \sum_{j=p-g+1}^{p} a_{f+k,j} m_j \quad (k = 1, \ldots, g).
\end{aligned} \tag{5.91}$$

Substituting eq. (5.91) into eq. (5.64) and taking into account eqs. (5.79), (5.81) and (5.89) yields

$$\sum_{j=p-g+1}^{f}\left(\frac{\partial P_i}{\partial \alpha_j}\right)n_j+\sum_{k=g+1}^{m}\left(\frac{\partial P_i}{\partial \sigma_k}\right)c_k^{(0)}=0, \quad (i=1,\dots,f)$$

$$\sum_{j=p-g+1}^{f}\left(\frac{\partial Q_k}{\partial \alpha_j}\right)n_j+\sum_{l=g+1}^{m}\left(\frac{\partial Q_k}{\partial \sigma_l}\right)c_l^{(0)}=\lambda_1\sum_{j=p-g+1}^{f}a_{f+k,j}m_j,$$

$$(k=1,\dots,g) \tag{5.92}$$

$$\sum_{j=p-g+1}^{f}\left(\frac{\partial Q_k}{\partial \alpha_j}\right)n_j+\sum_{l=g+1}^{m}\left(\frac{\partial Q_k}{\partial \sigma_l}\right)c_l^{(0)}=\lambda_1 c_k^{(0)}, \quad (k=g+1,\dots,m).$$

One can see that variables $l_1,\dots,l_{p-g}$ do not appear in eq. (5.92). Moreover, the system consisting of the first f equations and the last $m-g$ equations can be considered separately. As the total number of equations of this subsystem exceeds the number of variables $(m-g+f>m-p+f)$ this subsystem has the unique trivial solution

$$n_{f-g+1}=\dots=n_f=c_{g+1}^{(0)}=\dots=c_m^{(0)}=0. \tag{5.93}$$

Then we obtain from the second set of equations in (5.92)

$$\lambda_1\sum_{j=p-g+1}^{p}a_{f+k,j}m_j=0 \quad (k=1,\dots,g). \tag{5.94}$$

If $\lambda_1=0$, then g unknown variables $m_{p-g+1},\dots,m_p$ can be taken arbitrarily. Then the determinant of system (5.92) and in turn (5.64) have a $g-$fold root which is zero for any approximation. If $\lambda_1\neq 0$ then

$$\sum_{j=p-g+1}^{p}a_{f+k,j}m_j=0 \quad (k=1,\dots,g). \tag{5.95}$$

The determinant of this system of g equations with the same number of variables is not equal to zero. Hence, it follows from eq. (5.95) that

$$m_{p-g+1}=\dots=m_p=0. \tag{5.96}$$

Equalities (5.93) and (5.96) are satisfied simultaneously for $\lambda_1\neq 0$. Hence, the unknown parameters $m_1,\dots,m_{p-g}$ can be chosen arbitrarily. According to eq. (5.91), the original system (5.64) admits the following non-trivial family of solutions depending on $p-g$ arbitrary constants

$$a_i^{(0)}=\sum_{j=1}^{p-g}b_{ij}l_j=0, \quad c_k^{(0)}=0$$

$$(i=1,\dots,f; \quad k=1,\dots,m). \tag{5.97}$$

Therefore, the isochronous criteria of stability in the first approximation (i.e. of order ε) can not be determined and the higher approximation (i.e. of order ε^2) should be considered. To this aim, we first note that in the first approximation from the first set of equations in (5.59) with the help of eqs. (5.97) and (5.90) we obtain, cf. (5.63),

$$b_i^{(1)} = \sum_{j=1}^{p-g} b_{ij} d_j = 0, \quad d_j = \lambda_1 l_j. \tag{5.98}$$

Inserting eqs. (5.62) and (5.63) into eq. (5.59) we can balance the terms of order ε^2, to obtain

$$\begin{gathered}
\left(\frac{\partial P_i}{\partial \alpha_j}\right) a_j^{(1)} + \left(\frac{\partial P_i}{\partial \Omega_j}\right) b_j^{(1)} + \left(\frac{\partial P_i}{\partial \sigma_l}\right) c_l^{(1)} = \lambda_1 b_i^{(1)}, \\
\left(\frac{\partial Q_k}{\partial \alpha_j}\right) a_j^{(1)} + \left(\frac{\partial Q_k}{\partial \Omega_j}\right) b_j^{(1)} + \left(\frac{\partial Q_k}{\partial \sigma_l}\right) c_l^{(1)} = \lambda_1 c_k^{(1)} \\
(i = 1, \dots, f \quad k = 1, \dots, m) .
\end{gathered} \tag{5.99}$$

These equations are a consequence of only the second and third sets of system (5.59). Carrying out a replacement similar to eq. (5.91) in eq. (5.100) we have

$$\begin{aligned}
a_i^{(1)} &= \sum_{j=1}^{p-g} b_{ij} l_j^{(1)} + \sum_{j=p-g+1}^{p} a_{ij} m_j^{(1)}, \quad (i = 1, \dots, p-g) \\
a_i^{(1)} &= \sum_{j=1}^{p-g} b_{ij} l_j^{(1)} + \sum_{j=p-g+1}^{p} a_{ij} m_j^{(1)} + n_i^{(1)}, \quad (i = p-g+1, \dots, f) \\
c_k^{(1)} &= \sum_{j=p-g+1}^{p} a_{f+k,j} m_j^{(1)} \quad (k = 1, \dots, g) .
\end{aligned} \tag{5.100}$$

Taking into account eq. (5.98) we obtain

$$\begin{gathered}
\sum_{j=p-g+1}^{f} \left(\frac{\partial P_i}{\partial \alpha_j}\right) n_j^{(1)} + \sum_{l=g+1}^{m} \left(\frac{\partial P_i}{\partial \sigma_l}\right) c_l^{(1)} + \\
\sum_{l=1}^{p-g} \sum_{j=1}^{f} \left[\left(\frac{\partial P_i}{\partial \Omega_j}\right) - \delta_{ij} \lambda_1 \right] b_{jl} d_l = 0, \quad (i = 1, \dots, f)
\end{gathered}$$

$$\sum_{j=p-g+1}^{f}\left(\frac{\partial Q_k}{\partial \alpha_j}\right)n_j^{(1)}+\sum_{l=g+1}^{m}\left(\frac{\partial Q_k}{\partial \sigma_l}\right)c_l^{(1)}+\sum_{l=1}^{p-g}\sum_{j=1}^{f}\left(\frac{\partial Q_k}{\partial \Omega_j}\right)b_{jl}d_l$$
$$=\lambda_1\sum_{j=p-g+1}^{f}a_{f+k,j}m_j^{(1)},\quad (k=1,\dots,g)$$
$$\sum_{j=p-g+1}^{f}\left(\frac{\partial Q_k}{\partial \alpha_j}\right)n_j^{(1)}+\sum_{l=g+1}^{m}\left[\left(\frac{\partial Q_k}{\partial \sigma_l}\right)-\delta_{kl}\lambda_1\right]c_l^{(1)}+$$
$$\sum_{l=1}^{p-g}\sum_{j=1}^{f}\left(\frac{\partial Q_k}{\partial \Omega_j}\right)b_{jl}d_l=0\quad (k=g+1,\dots,m)\,. \tag{5.101}$$

The unknown parameters $l_1^{(1)},\dots,l_{p-g}^{(1)}$ do not appear in eq. (5.101). However the constant parameters $d_1,\dots,d_{p-g}$ can be considered as being additional unknown parameters. For this reason, the first f and last $m-g$ equations in (5.101) form a closed subsystem of $f+m-g$ equations with unknown variables $n_{p-g-1}^{(1)},\dots,n_f^{(1)}$, $c_{g+1}^{(1)},\dots,c_m^{(1)}$, $d_1,\dots,d_{p-g}$. The determinant of this system provides us with an equation of degree $m+p-2g$ for determining the isochronous criteria of stability of the stationary solution. After resolving the subsystem, the values of $m_{p-g-1}^{(1)},\dots,m_p^{(1)}$ are determined from the second set of equations in (5.101).

In the problem under consideration ($p>g$) the total number of anisochronous criteria of stability of the first and second kind is equal to $f+g-p$. Indeed, we multiply the $i-th$ equation of the conjugated system (5.73) by b_{ij} $(j=1,\dots,p-g)$ and sum up over i from 1 to f. Taking into account eq. (5.89) we obtain

$$\lambda_1^2\sum_{i=1}^{f}b_{ij}a_i^*=0\quad (j=1,\dots,p-g)\,. \tag{5.102}$$

If $\lambda_1^2\neq 0$, then

$$\sum_{i=1}^{f}b_{ij}a_i^*=0\quad (j=1,\dots,p-g)\,. \tag{5.103}$$

In this situation it is reasonable to consider the closed system consisting of $f+g-p$ equations (5.73) and $p-g$ equations (5.103). The determinant of this system is a polynomial of degree $f+g-p$ in λ_1^2. Correspondingly, for stability it is necessary to satisfy $f+g-p$ inequalities (5.71) and (5.76), the second corrections λ_2 being determined by formula (5.75). We can state that determinant (5.69) has a $(p-g)$ −fold root $\lambda_1^2=0$, that is $2\,(p-g)$ −fold root $\lambda_1=0$. Only the $p-g$ characteristic exponents are equal to zero in all approximations, the remaining exponents are non-trivial and have order ε, the first corrections being determined from system (5.101).

To close the section we consider some typical particular cases.

1. The case of pure isochronism in which the anisochronous pairs "phase-frequency" are completely absent $(f=0)$. The averaged equations have the form, cf. eq. (5.42)

$$\dot{\sigma}_k = \varepsilon Q_k(\sigma_1, \dots, \sigma_m) + \varepsilon^2 \dots \quad (k=1,\dots,m), \tag{5.104}$$

the right hand sides of this system being expanded in series in terms of integer degrees of ε. Correspondingly, solutions of system (5.104) are close to the solution of the original problem in time intervals of order $1/\varepsilon$. The number of autonomous frequencies is always less than the number of synchronous phases $(p<g)$. Hence, only synchronous criteria of stability are essential. They are proved while solving the following equation, see (5.64),

$$\det\left[\left(\frac{\partial Q_k}{\partial \sigma_l}\right) - \delta_{kl}\lambda_1\right] = 0. \tag{5.105}$$

2. The case of pure anisochronism in which the slow variables which are not conjugate to anisochronous pairs are completely absent $(m=g=0)$. Then the last m "isochronous" equations in averaged equations (5.42) drop out. The isochronous criteria of stability, see eqs. (5.64) and (5.65), are absent and the anisochronous criteria are, as before, proved by means of eqs. (5.69) and (5.75). In this case

$$\lambda_2 = \frac{1}{2}\left[\left(\frac{\partial P_i}{\partial \Omega_j}\right) + \left(\frac{\partial R_i}{\partial \alpha_j}\right)\right] a_i^* a_j^{(0)}. \tag{5.106}$$

Inequality $g<p\le f$ holds automatically in partially-autonomous problems. Thus, determinant (5.69) has $f-p$ values λ_1 which are conjugated to zero values and are determined, due to eq. (5.101) from the system

$$\sum_{j=p+1}^{f}\left(\frac{\partial P_i}{\partial \alpha_j}\right) n_j^{(1)} + \sum_{l=1}^{p}\sum_{j=1}^{f}\left[\left(\frac{\partial P_i}{\partial \Omega_j}\right) - \delta_{ij}\lambda_1\right] a_{jl} d_l = 0,$$
$$(i=1,\dots,f), \tag{5.107}$$

where it is taken into account that, by virtue of eq. (5.90) $b_{jl}=a_{jl}$ for $g=0$. In this case, transformation (5.100) takes the form

$$\begin{aligned} a_i^{(1)} &= \sum_{j=1}^{p} a_{ij} l_j^{(1)}, \quad (i=1,\dots,p), \\ a_i^{(1)} &= \sum_{j=1}^{p} a_{ij} l_j^{(1)} + n_i^{(1)}, \quad (i=p+1,\dots,f). \end{aligned} \tag{5.108}$$

We multiply the $i-th$ equation in (5.108) with $\dfrac{\partial P_l}{\partial \alpha_i}$ and sum up over i from 1 to f. Accounting for eq. (5.79) we obtain

$$\sum_{j=p+1}^{f}\left(\frac{\partial P_i}{\partial \alpha_j}\right) n_j^{(1)} = \sum_{j=1}^{f}\left(\frac{\partial P_i}{\partial \alpha_j}\right) a_j^{(1)}. \qquad (5.109)$$

Let us introduce into consideration the particular solutions a_{jl}^* $(l = 1,\dots,p)$ of the conjugated system

$$\left(\frac{\partial P_i}{\partial \alpha_j}\right) a_{jl}^* = 0, \qquad (5.110)$$

corresponding to the zero root and satisfying the conditions of orthogonality and normalisation

$$a_{ij}a_{il}^* = \delta_{jl} \quad (j, l = 1,\dots,p). \qquad (5.111)$$

In the above manner, see eqs. (5.72)-(5.75), starting with system (5.107) and accounting for (5.110) we arrive at the system of p equations with p unknown parameters

$$\sum_{k=1}^{p}\sum_{i,j=1}^{f}\left(\frac{\partial P_i}{\partial \Omega_j}\right) a_{ij}a_{il}^* d_k = \lambda_1 d_l \quad (l = 1,\dots,p). \qquad (5.112)$$

For the single-frequency stationary solution of the pure autonomous problem $(p = r = 1)$, we obtain

$$\lambda_1 = \sum_{i,j=1}^{f}\left(\frac{\partial P_i}{\partial \Omega_j}\right) a_{j1}a_{i1}^*. \qquad (5.113)$$

The absolute value of this expression is twice what is obtained due to eq. (5.106) for the zero root, [73]. Thus, all anisochronous criteria of stability of the latter problem can be uniformly written down.

5.7 Anisochronous and quasi-static criteria of stability of a single-frequency regime

Transformation to the "action-angle" variables in the original conservative system (5.4) with the help of eq. (5.5) is not always efficient. Indeed, let us assume for simplicity that all fast phases are anisochronous, i.e. $f = n$. Then the original canonical variables q_i, p_i are analytical functions of the canonically conjugated harmonic variables, see Sec. 4.5,

$$\xi_i = \sqrt{2s_i}\cos\varphi_i, \quad \eta_i = -\sqrt{2s_i}\sin\varphi_i \quad (i = m+1,\dots,n). \qquad (5.114)$$

It is clear, that introducing harmonic variables for the other "action-angle" variables $\varphi_1, s_1, \dots, \varphi_m, s_m$ is not efficient. The same situation is observed in the case of libration of the second order system about the equilibrium position, see Sec. 3.2, as well as in the case of motion of the system with one positional and several cyclic coordinates in the neighbourhood of the quasi-static solution, Sec. 4.5. In what follows, in the perturbed problem, we are also interested in quasi-static solutions, for which $s_i \to 0$ when $\varepsilon \to 0$ $(i = m+1, \dots, n)$. Variables (5.114), as well as their complex harmonic counterparts

$$z_i = \xi_i + \sqrt{-1}\eta_i = \sqrt{2s_i}\exp\left(-\sqrt{-1}\varphi_i\right),$$
$$z_i^* = \xi_i - \sqrt{-1}\eta_i = \sqrt{2s_i}\exp\left(\sqrt{-1}\varphi_i\right), \quad (i = m+1, \dots, n), \tag{5.115}$$

are referred to as quasi-static.

Constructing perturbed trajectories based on eq. (5.9) is not efficient in this case since, for example,

$$\frac{\partial p_j}{\partial s_i} = \frac{1}{2s_i}\left(z_i\frac{\partial p_j}{\partial z_i} + z_i^*\frac{\partial p_j}{\partial z_i^*}\right) \quad (i = m+1, \dots, n). \tag{5.116}$$

Here and in what follows it is assumed that the old variables q_i, p_i are analytical functions of $z_{m+1}, z_{m+1}^*, \dots, z_n, z_n^*$.

Thus, the right hand sides of system (5.9) are no longer analytic with respect to $z_{m+1}, z_{m+1}^*, \dots, z_n, z_n^*$. With this in view, it is more rational to use mixed variables $\varphi_1, \omega_1, \dots, \varphi_m, \omega_m, z_{m+1}, z_{m+1}^*, \dots, z_n, z_n^*$ where the matrix

$$e_{ij} = \left(\frac{\partial^2 H}{\partial s_i \partial s_j}\right)_{i,j=1,\dots,m} \tag{5.117}$$

is assumed to be non-singular. As the result we obtain, instead of eq. (5.9)

$$\begin{aligned} \dot\varphi_i - \omega_i &= \varepsilon U_i, \quad \dot\omega_i = \varepsilon V_i, \quad (i = 1, \dots, m) \\ \dot z_i + \sqrt{-1}\omega_i z_i &= \varepsilon M_i, \quad \dot z_i^* - \sqrt{-1}\omega_i z_i^* = \varepsilon N_i, \quad (i = m+1, \dots, n), \end{aligned} \tag{5.118}$$

where

$$U_i = \sum_{k=1}^m e_{ik}\sum_{j=1}^n\left(\frac{\partial p_j}{\partial \omega_k}X_j - \frac{\partial q_j}{\partial \omega_k}Y_j\right),$$
$$V_i = \sum_{k=1}^m e_{ik}\sum_{j=1}^n\left(\frac{\partial q_j}{\partial \varphi_k}Y_j - \frac{\partial p_j}{\partial \varphi_k}X_j\right), \quad (i = 1, \dots, m)$$
$$M_i = 2\sqrt{-1}\sum_{j=1}^n\left(\frac{\partial q_j}{\partial z_i^*}Y_j - \frac{\partial p_j}{\partial z_i^*}X_j\right),$$
$$N_i = -2\sqrt{-1}\sum_{j=1}^n\left(\frac{\partial q_j}{\partial z_i}Y_j - \frac{\partial p_j}{\partial z_i}X_j\right), \quad (i = m+1, \dots, n) \tag{5.119}$$

and the quasi-static frequencies $\omega_{m+1}, \dots, \omega_n$ are functions of $\omega_1, \dots, \omega_m$, $s_{m+1} = \frac{1}{2} z_{m+1} z_{m+1}^*, \dots, s_n = \frac{1}{2} z_n z_n^*$. It is essential that these frequencies are analytical functions of $s_{m+1}, \dots, s_n$ and they are not equal to zero for $s_{m+1}, \dots, s_n = 0$.

Directly applying the averaging method to system (5.118) is not possible. In order to construct quasi-periodic solutions of such systems one can use Poisson's series based on the method of generalised normalisation by Bryuno [22]. The derivations accompanying this method are extremely cumbersome. For this reason, we restrict our consideration to the analysis of the existence and local stability of the periodic solutions of this system constructed by the Lyapunov-Poincaré method [61]. We assume that functions X_i, Y_i are $2\pi-$periodic with respect to a single phase $\tau = \nu t$ of the external excitation.

We seek $2\pi/\nu-$periodic solution of system (5.118) for sufficiently small ε in the form of the following series

$$\begin{aligned} \varphi_i &= \varphi_{i0} + \varepsilon \varphi_{i1} + \varepsilon^2 \dots, \quad \omega_i = \omega_{i0} + \varepsilon \omega_{i1} + \varepsilon^2 \dots, \quad (i = 1, \dots, m) \\ z_i &= \varepsilon z_{i1} + \varepsilon^2 \dots, \quad z_i^* = \varepsilon z_{i1}^* + \varepsilon^2 \dots, \quad (i = m+1, \dots, n). \quad (5.120) \end{aligned}$$

The anisochronous phases, Sec. 4.8, can always be chosen such that the generating approximation has the form

$$\omega_{i0} = \nu, \quad \varphi_{i0} = \tau + \alpha_i \quad (i = 1, \dots, m), \quad (5.121)$$

where the generating phase shifts $\alpha_1, \dots, \alpha_m$ are constant. The generating values for the quasi-static frequencies

$$\Omega_i = \omega_i \big|_{\omega_1 = \dots = \omega_m = \nu, \; s_{m+1} = \dots = s_n = 0} \quad (i = m+1, \dots, n) \quad (5.122)$$

are incommensurable with the frequency of the external excitation ν.

While determining only periodic solutions, it is possible to generalise the analysis by assuming that functions X_i, Y_i also depend on the non-critical fast variables $y_1, \dots, y_k$ which are introduced by means of the following essentially non-linear differential equations, cf. (5.20),

$$\dot{y}_s = F_s \left(y_1, \dots, y_k, q_1, p_1, \dots, q_n, p_n, \tau \right) \quad (s = 1, \dots, k), \quad (5.123)$$

functions F_s being $2\pi-$periodic with respect to $\tau = \nu t$. Clearly, one can not speak of removing variables $y_1, \dots, y_k$ in this particular case. However we assume that functions F_s are such that the generating system

$$\dot{y}_s^{(0)} = f_s \left(y_1^{(0)}, \dots, y_k^{(0)}, \tau + \alpha_1, \dots, \tau + \alpha_m, \nu_1, \dots, \nu_m, \tau \right) \quad (5.124)$$

is locally integrable. Here f_s is the result of substitutions $\varphi_i = \tau + \alpha_i, \omega_i = \nu_i \; (i = m+1, \dots, n), \; z_i = z_i^* = 0 \; (i = m+1, \dots, n)$. In other words, this system has $2\pi/\nu-$periodic solution

$$y_s^{(0)} = y_s^{(0)} \left(\tau, \alpha_1, \dots, \alpha_m, \nu_1, \dots, \nu_n, \nu \right), \quad (5.125)$$

depending upon the constants $\alpha_1, \dots, \alpha_m, \nu_1, \dots, \nu_n, \nu$, whilst the variational system

$$\delta \dot{y}_s = \sum_{r=1}^{k} \left(\frac{\partial f_s}{\partial y_r} \right) \delta y_r \tag{5.126}$$

admits construction of the general integral in closed form. Moreover, it is assumed that solution (5.125) is asymptotically stable in the sense that all the characteristic exponents of linear system (5.126) with periodic coefficients have non-small negative real parts. An explicit dependence $y_s^{(0)}$ on excitation frequency ν is caused by the fact that this frequency is contained explicitly in the following system

$$\nu \frac{dy_s^{(0)}}{d\tau} = f_s, \tag{5.127}$$

which is equivalent to system (5.124).

Derivative $\dfrac{\partial y_s^{(0)}}{\partial \nu}$ equals the periodic solution of the following linear system

$$\frac{d}{dt} \left(\frac{\partial y_s^{(0)}}{\partial \nu} \right) = \sum_{r=1}^{k} \left(\frac{\partial f_s}{\partial y_r} \right) \frac{\partial y_r^{(0)}}{\partial \nu} - \frac{1}{\nu} \dot{y}_s^{(0)}. \tag{5.128}$$

This can be proved easily by means of the partial differentiation of (5.127) with respect to ν. The above mentioned local integrability takes place, first of all, in the problems of piecewise-linear problems or in problems which are integrable within the continuity region, see Sec. 1.3.

System (5.124) of essentially non-linear equations under the action of periodic excitation with period $2\pi/\nu$ can admit subperiodic solutions of periods $2\pi/j\nu$, where $j = 2, 3, \dots$ denotes multiplicity of the regime. In this case the consideration does not change if we make the replacement $\varphi_i \to \varphi_i j$ in eq. (5.9) from the very beginning.

A periodic solution of problems (5.118) and (5.123) having period $2\pi/\nu$ is sought in the form of series (5.120) and

$$y_s = y_{s0} + \varepsilon y_{s1} + \varepsilon^2 \dots \quad (s = 1, \dots, k), \tag{5.129}$$

where $y_{s0} = y_s^{(0)} \Big|_{\nu_1 = \dots = \nu_m = \nu}$.

As is easy to see, the first periodic approximation (of order ε) can be constructed provided that there exists a periodic solution of the system

$$\dot{\omega}_i^{(1)} = (V_i) \quad (i = 1, \dots, m). \tag{5.130}$$

One must substitute $\varphi_i = \tau + \alpha_i$, $\omega_i = \nu_i$ $(i = 1, \dots, m)$, $z_i = z_i^* = 0$ $(i = m+1, \dots, n)$, $y_s = y_s^{(0)}$, so that the true first correction is $\omega_{i1} =$

$\omega_i^{(1)}\Big|_{\nu_1=\cdots=\nu_m=\nu}$. The periodicity conditions reduce to the following equations

$$P_i\left(\alpha_1,\ldots,\alpha_m,\nu_1,\ldots,\nu_m,\nu\right)=0$$
$$\left(\nu_1=\ldots=\nu_m=\nu,\ i=1,\ldots,m\right), \tag{5.131}$$

where, see eq. (5.43)

$$P_i=\frac{1}{2\pi}\int\limits_0^{2\pi}\left(V_i\right)d\tau. \tag{5.132}$$

As proved in the Lyapunov-Poincaré theory of small parameters [61], the existence of a solution of system (5.131), which is simple with respect to $\alpha_1,\ldots,\alpha_m$, is the sufficient condition for the existence of the periodic solution of the original problem for sufficiently small ε. This periodic solution can be represented in the form of series (5.120) and (5.129).

Proceeding to an investigation of the local stability of the considered periodic solution, let us write down the following variational system of equations

$$\delta\dot{\varphi}_i-\delta\omega_i=\varepsilon\left[\sum_{j=1}^{m}\left(\frac{\partial U_i}{\partial\varphi_j}\right)\delta\varphi_j+\sum_{r=1}^{k}\left(\frac{\partial U_i}{\partial y_r}\right)\delta y_r+\ldots\right]+\varepsilon^2\ldots,$$
$$\delta\dot{\omega}_i=\varepsilon\left\{\sum_{j=1}^{n}\left[\left(\frac{\partial V_i}{\partial\varphi_j}\right)\delta\varphi_j+\left(\frac{\partial V_i}{\partial\omega_j}\right)\delta\omega_j\right]+\right.$$
$$\left.\sum_{r=1}^{k}\left(\frac{\partial V_i}{\partial y_r}\right)\delta y_r+\ldots\right\}+\varepsilon^2\ldots,\quad(i=1,\ldots,m)$$
$$\delta\dot{z}_i+\sqrt{-1}\Omega_i\delta z_i=\varepsilon\left\{\sum_{j=m+1}^{n}\left[\left(\frac{\partial M_i}{\partial z_j}\right)\delta z_j+\left(\frac{\partial M_i}{\partial z_j^*}\right)\delta z_j^*\right]+\right.$$
$$\left.\sum_{r=1}^{k}\left(\frac{\partial M_i}{\partial y_r}\right)\delta y_r+\ldots\right\}+\varepsilon^2\ldots,$$
$$\delta\dot{z}_i^*-\sqrt{-1}\Omega_i\delta z_i^*=\varepsilon\left\{\sum_{j=m+1}^{n}\left[\left(\frac{\partial N_i}{\partial z_j}\right)\delta z_j+\left(\frac{\partial N_i}{\partial z_j^*}\right)\delta z_j^*\right]+\right.$$
$$\left.\sum_{r=1}^{k}\left(\frac{\partial N_i}{\partial y_r}\right)\delta y_r+\ldots\right\}+\varepsilon^2\ldots,\quad(i=m+1,\ldots,n) \tag{5.133}$$

$$\delta\dot{y}_s = \sum_{r=1}^{k}\left(\frac{\partial f_s}{\partial y_r}\right)\delta y_r + \sum_{j=1}^{m}\left[\left(\frac{\partial f_s}{\partial \varphi_j}\right)\delta\varphi_j + \left(\frac{\partial f_s}{\partial \omega_i}\right)\delta\omega_j\right] +$$

$$\sum_{j=m+1}^{n}\left[\left(\frac{\partial f_s}{\partial z_j}\right)\delta z_j + \left(\frac{\partial f_s}{\partial z_j^*}\right)\delta z_j^* + \dots\right] + \varepsilon\dots \qquad (s = 1,\dots,k).$$

Here we omitted those terms among the terms of order ε in the first $2n$ equations which are not needed for constructing the required criteria of stability. In the case of $\varepsilon = 0$, the characteristic equation for system (5.133) has a $2m$–fold zero root with square elementary divisors and $n-m$ pairs of pure imaginary roots $\pm\sqrt{-1}\Omega_i$. All these roots are non-critical ones since the remaining k roots have non-small negative real parts. The stability criteria corresponding to the zero root and pure imaginary roots are called, as above, anisochronous and quasi-static criteria respectively.

The coefficients of the linear homogeneous system (5.133) are periodic, in contrast to system (5.59) with constant coefficients. However, the search for the anisochronous criteria of stability is very similar to the described one.

Following Floquet's theory, the particular solutions of (5.133) are sought in the form

$$\begin{aligned} \delta\varphi_i &= a_i e^{\lambda t}, \quad \delta\omega_i = b_i e^{\lambda t}, \quad (i = 1,\dots,m) \\ \delta z_i &= c_i e^{\lambda t}, \quad \delta z_i^* = d_i e^{\lambda t}, \quad (i = m+1,\dots,n) \\ \delta y_s &= e_s e^{\lambda t}, \quad (s = 1,\dots,k), \end{aligned} \qquad (5.134)$$

values a_i, b_i, c_i, d_i, e_s being $2\pi/\nu$-periodic with respect to t. As above, cf. (5.66), the following solutions

$$\begin{aligned} \lambda &= \lambda_1\sqrt{\varepsilon} + \lambda_2\varepsilon + \varepsilon^{3/2}\dots, \quad a_i = a_i^{(0)} + \sqrt{\varepsilon}a_i^{(1)} + \varepsilon\dots, \\ b_i &= b_i^{(0)} + \sqrt{\varepsilon}b_i^{(1)} + \varepsilon\dots, \quad c_i = c_i^{(0)} + \sqrt{\varepsilon}c_i^{(1)} + \varepsilon\dots, \\ d_i &= d_i^{(0)} + \sqrt{\varepsilon}d_i^{(1)} + \varepsilon\dots, \quad e_s = e_s^{(0)} + \sqrt{\varepsilon}e_s^{(1)} + \varepsilon\dots, \end{aligned} \qquad (5.135)$$

which are analytic with respect to $\sqrt{\varepsilon}$ correspond to the $2m$–fold zero root. In the original approximation we obtain

$$\begin{aligned} a_i^{(0)} &= \text{const}, \quad b_1^{(0)} = \dots = b_m^{(0)} = 0, \\ c_{m+1}^{(0)} &= d_{m+1}^{(0)} = \dots = c_n^{(0)} = d_n^{(0)} = 0, \\ \dot{e}_s^{(0)} &= \sum_{r=1}^{m}\left(\frac{\partial f_s}{\partial y_r}\right)e_r^{(0)} + \sum_{j=1}^{m}\left(\frac{\partial f_s}{\partial \alpha_j}\right)a_j^{(0)}. \end{aligned} \qquad (5.136)$$

Comparing the latter equation with eq. (5.124) yields

$$e_s^{(0)} = \sum_{j=1}^{m}\left(\frac{\partial y_s^{(0)}}{\partial \alpha_j}\right)a_j^{(0)}. \qquad (5.137)$$

Equating terms of order $\sqrt{\varepsilon}$ and taking into account eqs. (5.136) and (5.137) we obtain

$$b_i^{(1)} = \lambda_1 a_i^{(0)},\ a_i^{(1)} = \text{const},\ c_{m+1}^{(1)} = d_{m+1}^{(1)} = \cdots = c_n^{(1)} = d_n^{(1)} = 0,$$
$$\dot{e}_s^{(1)} = \sum_{r=1}^{k} \left(\frac{\partial f_s}{\partial y_r}\right) e_r^{(1)} + \sum_{j=1}^{m} \left[\frac{\partial f_s}{\partial \alpha_j} a_j^{(1)} + \lambda_1 \frac{\partial f_s}{\partial \nu_j} a_j^{(0)} - \lambda_1 \frac{\partial y_s^{(0)}}{\partial \alpha_j} a_j^{(0)}\right]. \tag{5.138}$$

By using eq. (5.124) the periodic solution of the latter system of equations can be set in the following form

$$e_s^{(1)} = \sum_{j=1}^{m} \left[\frac{\partial f_s}{\partial \alpha_j} a_j^{(1)} + \lambda_1 \left(\frac{\partial y_s^{(0)}}{\partial \nu_j} + y_{sj}\right) a_j^{(0)}\right], \tag{5.139}$$

where periodic functions of time y_{sj} are obtained from integration of the system

$$\dot{y}_{sj} = \sum_{r=1}^{k} \left(\frac{\partial f_s}{\partial y_r}\right) y_{rj} - \frac{\partial y_s^{(0)}}{\partial \alpha_j}. \tag{5.140}$$

The feasibility of determining all of these functions by means of a finite number of operations is beyond question due to the integrability of the variational system.

Equating terms of order ε we include only the relationships that follow from the first and second sets of equations in (5.133). We also take into account equalities (5.136)-(5.138) and the formula for the "total" partial differentiation with respect to α_j

$$\frac{\partial}{\partial \alpha_j} = \left(\frac{\partial}{\partial \varphi_j}\right) + \sum_{r=1}^{k} \frac{\partial y_r^{(0)}}{\partial \alpha_j} \left(\frac{\partial}{\partial y_r}\right). \tag{5.141}$$

The result is as follows

$$\begin{aligned} \dot{a}_i^{(2)} + \lambda_1 a_i^{(1)} + \lambda_2 a_i^{(0)} &= b_i^{(2)} + \sum_{j=1}^{m} \frac{\partial (U_i)}{\partial \alpha_j} a_j^{(0)}, \\ \dot{b}_i^{(2)} + \lambda_1^2 a_i^{(0)} &= \sum_{j=1}^{m} \frac{\partial (V_i)}{\partial \alpha_j} a_j^{(0)}. \end{aligned} \tag{5.142}$$

The periodic solution of the second set of equations (5.142) exists only if

$$\sum_{j=1}^{m} \frac{\partial P_i}{\partial \alpha_j} a_j^{(0)} = \lambda_1^2 a_i^{(0)} \quad (i = 1, \dots, m) \tag{5.143}$$

and is given by

$$b_i^{(2)} = f_i + \sum_{j=1}^{m} \frac{\partial \{V_i\}}{\partial \alpha_j} a_j^{(0)}. \tag{5.144}$$

Here f_i = const and $\{\ \}$ denotes the periodic antiderivative with zero mean value, see Sec. 5.4. Now from the condition of the existence of periodic quantities $a_i^{(2)}$, we obtain from the first set of equations in (5.142) that

$$f_i = \lambda_1 a_i^{(1)} + \lambda_2 a_i^{(0)} - \sum_{j=1}^{m} \frac{\partial R_i}{\partial \alpha_j} a_j^{(0)}, \tag{5.145}$$

where, see (5.132)

$$R_i = \frac{1}{2\pi} \int_0^{2\pi} (U_i)\, d\tau. \tag{5.146}$$

The linear homogeneous system (5.143) is identical to (5.68) and serves to determine the anisochronous criteria of stability of the first kind $\left(\lambda_1^2 < 0\right)$. Omitting the determination of non-trivial periodic functions $c_{m+1}^{(2)}, d_{m+1}^{(2)}, c_n^{(2)}, d_n^{(2)}, e_1^{(2)}, \ldots, e_k^{(2)}$ we obtain the relationships which are obtained from the second set of eq. (5.133) by equating the terms of order $\varepsilon^{3/2}$ and accounting for eqs. (5.138), (5.139) and (5.143)-(5.145)

$$\dot b_i^{(3)} + \lambda_1^2 a_i^{(1)} - \sum_{j=1}^{m} \frac{\partial (V_i)}{\partial \alpha_j} a_j^{(1)} = \lambda_1 \sum_{j=1}^{m} \left[\frac{\partial R_i}{\partial \alpha_j} + \frac{\partial (V_i)}{\partial \nu_j} + \sum_{r=1}^{m} \left(\frac{\partial V_i}{\partial y_r} \right) y_{rj} \right] a_j^{(0)} - 2\lambda_1\lambda_2 a_i^{(0)} - \sum_{j=1}^{m} \lambda_1 \frac{\partial \{V_i\}}{\partial \alpha_j} a_j^{(0)}. \tag{5.147}$$

The anisochronous criteria of stability of the second kind are obtained with the help of eq. (5.147) by analogy with Sec. 5.5. To this end, we introduce the linear homogeneous system

$$\sum_{j=1}^{m} \frac{\partial P_i}{\partial \alpha_j} a_j^* = \lambda_1^2 a_j^*, \tag{5.148}$$

which is conjugate with respect to (5.143), see (5.73).

The solutions of this system are subject to the normalisation condition

$$\sum_{j=1}^{m} a_j^{(0)} a_j^* = 1. \tag{5.149}$$

We average eq. (5.147) with respect to t, multiply it by a_i^* and sum up over i from 1 to n. Taking into account eqs. (5.132), (5.148) and (5.149) we obtain

$$\lambda_2 = \frac{1}{2} \sum_{i,j=1}^{m} \left[\frac{\partial P_i}{\partial \nu_j} + \frac{\partial R_i}{\partial \alpha_j} + P_{ij} \right] a_j^{(0)} a_i^*, \tag{5.150}$$

where

$$P_{ij} = \frac{1}{2\pi} \sum_{r=1}^{k} \int_0^{2\pi} \left(\frac{\partial V_i}{\partial y_r} \right) y_{rj} d\tau. \tag{5.151}$$

Determining quantities λ_2, due to eq. (5.150), is complicated by the necessity of finding periodic solutions of m linear systems (5.140) with periodic coefficients.

Let us assume that the system of equations

$$\begin{aligned} \dot{y}_s^{(0)} &= f_s \left(y_1^{(0)}, \ldots, y_k^{(0)}, \tau_1, \ldots, \tau_m, \nu_1, \ldots, \nu_m, \tau \right) \\ \tau_1 &= \nu_i' t + \alpha_i \end{aligned} \tag{5.152}$$

has a stable $(m+1)-$ frequency solution

$$y_s^{(0)} = y_s^{(0)} \left(\tau_1, \ldots, \tau_m, \nu_1', \ldots, \nu_m', \nu_1, \ldots, \nu_m, \tau \right), \tag{5.153}$$

which coincides with eq. (5.125) when $\nu_1', \ldots, \nu_m'$ (appearing in $\tau_1, \ldots, \tau_m$) are equal to ν. Then, the system of partial differential equations

$$\sum_{i=1}^{m} \nu_i' \frac{\partial y_s^{(0)}}{\partial \tau_i} + \nu \frac{\partial y_s^{(0)}}{\partial \tau} = f_s \left(y_1^{(0)}, \ldots, y_k^{(0)}, \tau_1, \ldots, \tau_m, \nu_1, \ldots, \nu_m, \tau \right) \tag{5.154}$$

admits quasi-periodic solution (5.153) which can be cast in the form of a generalised Fourier series. Let us differentiate eq. (5.154) with respect to ν_j'. Taking into account eq. (5.140) we obtain

$$\dot{y}_{sj} = \left. \frac{\partial y_s^{(0)}}{\partial \nu_j'} \right|_{\nu_1' = \cdots = \nu_m' = \nu} . \tag{5.155}$$

By virtue of eqs. (5.132), (5.150) and (5.151) we have

$$\frac{\partial P_i}{\partial \nu_j} + P_{ij} = \frac{\partial' P_i}{\partial \nu_j}, \tag{5.156}$$

where a prime denotes that, while calculating functions P_i, we substitute expression (5.153) for $\nu_i' = \nu_i$ and all frequencies ν_i' in τ_i are taken to be equal to ν. Thus, expression (5.150) is coincident with (5.106). The above said has no principal importance inasmuch as it is apparently not possible to indicate sufficiently broad classes of non-linear systems of type (5.152).

In the autonomous case, functions P_i and R_i depend only on the differences in the generating phase shifts $\alpha_i - \alpha_j$, that is

$$\sum_{i=1}^{m} \frac{\partial P_i}{\partial \alpha_j} = \sum_{i=1}^{m} \frac{\partial R_i}{\partial \alpha_j} = 0. \tag{5.157}$$

The synchronous frequency should be determined as series $\nu = \nu^{(0)} + \varepsilon \nu^{(1)} + \varepsilon^2 \ldots$, with the generating values being obtained from eq. (5.131). The determinant of system (5.143) has a zero root $\lambda_1^2 = 0$ with the corresponding non-trivial solution $a_i^{(0)} = 1$, see (5.157). One of the corresponding characteristic exponents is zero in all approximations. The other exponent is non-trivial and analytic with respect to ε, i.e. $\lambda = \lambda_2 \varepsilon^{(1)} + \varepsilon^2 \ldots$. The first correction is given by, see eq. (5.113)

$$\lambda_2 = \sum_{i,j=1}^{n} \left[\frac{\partial P_i}{\partial \nu_j} + \frac{\partial R_i}{\partial \alpha_j} + P_{ij} \right] a_j^{(0)} a_i^*. \tag{5.158}$$

Let us expand this expression, take into account eq. (5.157) and the following equality

$$\sum_{j=1}^{m} y_{rj} = \frac{\partial y_r^{(0)}}{\partial \nu}, \tag{5.159}$$

which follows from eqs. (5.128) and (5.140).

The resulting condition for stability takes the following typical form, see eq. (5.113)

$$\lambda_2 = \sum_{i=1}^{m} \frac{\partial' P_i}{\partial \nu} a_i^* < 0, \tag{5.160}$$

where $\sum\limits_{i=1}^{m} a_i^* = 1$ and, additionally, we introduced the following notation for the "total" partial differentiation with respect to synchronous frequency

$$\frac{\partial'}{\partial \nu} = \left(\sum_{i=1}^{m} \frac{\partial}{\partial \nu_i} + \frac{\partial}{\partial \nu} \right)_{\nu_1 = \cdots = \nu_m = \nu} . \tag{5.161}$$

Let us proceed to determine the quasi-static criteria of stability. For generality, we assume the presence of the l-fold quasi-static frequency $\Omega_{m+1} = \cdots = \Omega_{m+l} = \Omega$ $(m + l \leq n)$. The particular solutions of system

(5.133) corresponding to this value are determined by eq. (5.134). However, contrary to (5.135), these solutions are analytic with respect to ε

$$\lambda = -\sqrt{-1}\Omega + \varepsilon\lambda_1 + \varepsilon^2 \ldots, \quad a_i = \varepsilon a_i^{(1)} + \varepsilon^2 \ldots, \quad b_i = \varepsilon b_i^{(1)} + \varepsilon^2 \ldots,$$
$$c_i = c_i^{(0)} + \varepsilon c_i^{(1)} + \varepsilon^2 \ldots, \quad d_i = \varepsilon d_i^{(1)} + \varepsilon^2 \ldots, \quad e_s = e_s^{(0)} + \varepsilon e_s^{(1)} + \varepsilon^2 \ldots. \tag{5.162}$$

Inserting eq. (5.134) into the latter equations yields in the original approximation

$$c_i^{(0)} = \text{const} \ (i = m+1, \ldots, m+l), \quad c_i^{(0)} = 0 \ (i = m+l+1, \ldots, n),$$
$$\dot{e}_s^{(0)} - \sqrt{-1}\Omega e_s^{(0)} = \sum_{r=1}^{k} \left(\frac{\partial f_s}{\partial y_r}\right) e_r^{(0)} + \sum_{j=m+1}^{m+l} \left(\frac{\partial F_s}{\partial z_j}\right) c_j^{(0)}$$
$$(s = 1, \ldots, k). \tag{5.163}$$

The general integral of the homogeneous part of the latter system is related to the general integral of the variational system of equations (5.126) by the equality

$$e_s^{(0)} = \exp\left(\sqrt{-1}\Omega t\right) \delta y_s. \tag{5.164}$$

With this in view, the periodic solution of this system can be constructed by means of a finite number of operations. Let us represent this solution in the form

$$e_s^{(0)} = \sum_{j=m+1}^{m+l} e_{sj} c_j^{(0)}, \tag{5.165}$$

where the periodic functions $e_{1j}, \ldots, e_{kj}$ are determined from the system

$$\dot{e}_{sj} - \sqrt{-1}\Omega e_{sj} = \sum_{r=1}^{k} \left(\frac{\partial f_s}{\partial y_r}\right) e_{rj} + \left(\frac{\partial F_s}{\partial z_j}\right). \tag{5.166}$$

Equating terms of order ε in the first f equations (5.133) of the third set we obtain

$$\dot{c}_i^{(1)} + \lambda_1 c_i^{(0)} = \sum_{j=m+1}^{m+l} \left[\left(\frac{\partial M_i}{\partial z_j}\right) + \sum_{r=1}^{k} \left(\frac{\partial M_i}{\partial y_r}\right) e_{rj}\right] c_j^{(0)}$$
$$(i = m+1, \ldots, m+l). \tag{5.167}$$

Because of the periodicity of the first correction $c_i^{(1)}$ with respect to t we immediately obtain from eq. (5.167) the following linear homogeneous system for constant values $c_{m+1}^{(0)}, \dots , c_{m+l}^{(0)}$

$$\sum_{j=m+1}^{m+l} \left\langle \left(\frac{\partial M_i}{\partial z_j} \right) + \sum_{r=1}^{k} \left(\frac{\partial M_i}{\partial y_r} \right) e_{rj} \right\rangle c_j^{(0)} = \lambda_1 c_i^{(0)}$$
$$(i = m+1, \dots , m+l) . \tag{5.168}$$

Expanding the determinant of the obtained system we arrive at an equation of degree l for λ_1. For stability it is necessary that all of its roots have negative real parts, i.e. $\operatorname{Re} \lambda_1 < 0$.

5.8 Periodic solutions of the piecewise continuous systems

As shown in Sec. 1.3, the piecewise-continuous systems integrable within the continuity intervals and in particular the piecewise-linear systems are locally integrable. Thus methods of small parameter can be successfully applied to solving perturbed problems which belong to the above class in the generating approximation ($\varepsilon = 0$). However it is not possible to find criteria of integrability of piecewise-continuous systems by quadratures which are rather general and actual for practical applications. From this perspective, while investigating periodic solutions of perturbed piecewise-continuous systems we make use of the local method of Lyapunov-Poincaré.

Let us begin with an infinite sequence of the system of continuous equations

$$\dot{x}_i = F_i, \quad F_i = X_i \sigma (g_i) + \Phi_i \dot{\sigma} (g_i)$$
$$(i = - \dots 1, 0, 1, \dots) , \tag{5.169}$$

cf. (1.51), which corresponds to the piecewise-continuous problem under consideration.

Let us assume that the $k_i \times 1$ vector-functions X_i depend on components x_i, whilst the $k_i \times 1$ vector-functions Φ_i and the scalar functions g_i depend on x_{i-1}, see eqs. (1.44)-(1.46). Besides, all of these functions are $T-$periodic with respect to t in the sense of (1.47) and, as opposed to the previous analysis, are analytic with respect to small parameter ε. The other assumptions of Sec. 1.3, for instance, (1.52) remain valid.

Let us assume that for $\varepsilon = 0$ the successive systems (5.169) admit efficient construction of a family of $T-$periodic, in the sense of (1.50), solutions $x_i = x_i^{(0)}$ which is dependent on s arbitrary parameters $h_1, \dots , h_s$ $(s < k_0)$. The main goal of the previous investigations consists of determining conditions under which the perturbed problem admits an isolated $T-$periodic solution coinciding with one of the solutions of the generating family for $\varepsilon \to 0$.

To begin with, we first differentiate the equations in (5.169) with respect to ε. We take into account that quantities

$$\Phi_i \frac{d}{dt}\left[\delta\left(g_i\right)\frac{\partial' g_i}{\partial\varepsilon}\right], \quad -\dot{\Phi}_i\delta\left(g_i\right)\frac{\partial' g_i}{\partial\varepsilon}$$

have an equal jump in the components of vector $x_i^{(1)} = \dfrac{\partial x_i}{\partial\varepsilon}$ at the time instant of the "switch" $t - t_i$, see eq. (1.55). A prime designates the "total" partial derivative with respect to ε

$$\frac{\partial' g_i}{\partial\varepsilon} = \frac{\partial' g_i}{\partial x_{i-1}} x_{i-1}^{(1)} + \frac{\partial g_i}{\partial\varepsilon}\,. \tag{5.170}$$

The final result is as follows

$$\dot{x}_i^{(1)} = A_i x_i^{(1)}\sigma\left(g_i\right) + B_i x_{i-1}^{(1)}\dot{\sigma}\left(g_i\right) + \frac{\partial X_i}{\partial\varepsilon}\sigma\left(g_i\right) + Y_i\dot{\sigma}\left(g_i\right), \tag{5.171}$$

where coefficients $k_i \times k_i$ matrix A_i and $k_i \times k_{i-1}$ matrix B_i are determined due to eq. (1.56) and the notation

$$Y_i = \frac{\partial\Phi_i}{\partial\varepsilon} - \left(\dot{\Phi}_i - X_i\right)\frac{\partial g_i}{\partial\varepsilon}\left(\dot{g}_i\right)^{-1} \tag{5.172}$$

is introduced. For solving eq. (5.171) it is necessary to adopt, as in Sec. 1.3, that $x_i^{(1)} = 0$ for $t < t_i$.

The second differentiation of eq. (5.171) with respect to ε, carried out in accordance with the above scheme, results in an equation for the second derivative $\dfrac{\partial^2 x_i}{\partial\varepsilon^2}$. Repeating the process of differentiation we obtain an infinite system of linear piecewise-continuous equations whose homogeneous parts coincide with the variational equations (1.56). As the generating system is locally integrable, these equations can also be integrated in closed form for $\varepsilon = 0$. The only restriction imposed on this strategy is the following inequality

$$\dot{g}_i\big|_{t=t_i,\,\varepsilon=0} \neq 0. \tag{5.173}$$

For sufficiently small ε the solution of the perturbed problem can be cast as follows

$$x_i\left(t,\varepsilon\right) = x_i\left(t,0\right) + \left(\frac{\partial x_i}{\partial\varepsilon}\right)\varepsilon + \frac{1}{2}\left(\frac{\partial^2 x_i}{\partial\varepsilon^2}\right)\varepsilon^2 + \varepsilon^3\ldots\,. \tag{5.174}$$

The time instants of the switch $t_i\left(\varepsilon\right)$, needed for determination of the piecewise-continuous solution (1.48), are given by the series

$$t_i\left(\varepsilon\right) = t_i\left(0\right) + \left(\frac{\partial t_i}{\partial\varepsilon}\right)\varepsilon + \frac{1}{2}\left(\frac{\partial^2 t_i}{\partial\varepsilon^2}\right)\varepsilon^2 + \varepsilon^3\ldots\,. \tag{5.175}$$

To determine its coefficients it is necessary to differentiate the following expression

$$g_i\left(x_{i-1}, t, \varepsilon\right)\Big|_{t=t_i(\varepsilon)} = 0, \tag{5.176}$$

and then put $\varepsilon = 0$. In particular, at the first step we have

$$\left(\frac{\partial t_i}{\partial \varepsilon}\right) = -\left.\frac{\dfrac{\partial' g_i}{\partial \varepsilon}}{\dot{g}_i}\right|_{t=t_i(\varepsilon),\, \varepsilon=0}. \tag{5.177}$$

It follows from this equation that correction $\left(\dfrac{\partial^k t_i}{\partial \varepsilon^k}\right)$ of order ε^k ($k = 1, 2, \ldots$) is determined after determining functions $\left(\dfrac{\partial^k x_i}{\partial t^k}\right)$.

Let us consider system (5.171) as well as the consequent systems for determining higher order approximations for $T-$ periodic generating solution $x_i = x_i^{(0)}$ depending on parameters $h_1, \ldots, h_s$. Variational equations of the generating system admit s periodic solutions $\dfrac{\partial x_i^{(0)}}{\partial h_1}, \ldots, \dfrac{\partial x_i^{(0)}}{\partial h_s}$. For these solutions there exist s $T-$periodic solutions of the conjugate system $z_i^{(1)}, \ldots, z_i^{(s)}$, see (1.63). Assume that these piecewise-continuous systems with periodic coefficients have no other $T-$ periodic solutions. In order to find the conditions for the existence of a $T-$ periodic solution of the equations of first approximation, we differentiate the value

$$\sum_{i=-\infty}^{\infty} z_i^{(r)} x_i^{(0)} \quad (r = 1, \ldots, s) \tag{5.178}$$

with respect to t. In a similar fashion to eq. (1.64) we obtain

$$\frac{d}{dt}\sum_{i=-\infty}^{\infty} z_i^{(r)} x_i^{(0)} = \sum_{i=-\infty}^{\infty}\left[z_i^{(r)}\left(\frac{\partial X_i}{\partial \varepsilon}\right)\sigma\left(t - t_i\right) + z_i^{(r)}\left(Y_i\right)\delta\left(t - t_i\right)\right]. \tag{5.179}$$

Let us recall that the parentheses imply the necessity to determine the corresponding value for $\varepsilon = 0, x_i = x_i^{(0)}$. We integrate equality (5.179) with respect to t from t_0 to $t_n = t_0 + T$. Since notations (5.171) and (1.63) suggest that $x_i^{(1)} = 0$ for $t < t_i$ and $z_i^{(1)} = 0$ for $t > t_i$, respectively, the final conditions for the periodicity are represented in the form

$$P_r\left(h_1, \ldots, h_s\right) \equiv \sum_{i=1}^{n}\left[\int_{t_{i-1}}^{t_i} z_i^{(r)}\left(\frac{\partial X_i}{\partial \varepsilon}\right) dt + z_i^{(r)}\left(t_{i-1}\right)\left(Y_i\right)_{t=t_{i-1}}\right] = 0. \tag{5.180}$$

Similar to the continuous case, it is shown that the necessary and sufficient condition for the existence of T–periodic solutions of the considered problem, which can be represented by series (5.174) for sufficiently small ε, is that system (5.180) has a simple solution for which

$$J\left(\frac{P_1,\dots,P_s}{h_1,\dots,h_s}\right) \neq 0. \tag{5.181}$$

A study of the local stability is based upon the variational system (1.56) for $\varepsilon \neq 0$. The particular solutions should be sought in form (1.62). Let us assume that the characteristic equation of this system, see eq. (1.61), for $\varepsilon = 0$ has a s–fold zero root with simple elementary divisors, whereas the remaining "non-critical" roots have non-small negative real parts. Then the "critical" solutions of the variational equations are analytic with respect to ε, and

$$v_i = \sum_{r=1}^{s} \frac{\partial x_i^{(0)}}{\partial h_r} a_r + \varepsilon v_i^{(1)} + \varepsilon^2 \dots, \qquad \lambda = \lambda_1 \varepsilon + \varepsilon^2 \dots, \tag{5.182}$$

cf. eq. (1.62).

Finally, the condition for T–periodicity of the first correction $v_i^{(1)}$ in the sense of (1.47) leads to the system of linear homogeneous equations for the constants $a_1, \dots, a_s$

$$\sum_{r=1}^{s} \frac{\partial P_j}{\partial h_r} a_r = \lambda_1 a_j \quad (j = 1, \dots, s). \tag{5.183}$$

The determinant of this system enables us to formulate s criteria of stability of the type $\operatorname{Re}\lambda_1 < 0$. All these criteria can be characterised as isochronous, see for example eq. (5.64) for $f = 0$. As for the situations in which the anisochronous criteria of the first and second kind are essential, they are less actual in the piecewise-continuous case. This is caused by the fact that there exists no sufficiently general class of conservative piecewise-continuous systems with several degrees of freedom, which admits efficient construction of a general integral of the quasi-periodic type. It appears that the only exception is the problem of weak interaction of the conservative piecewise-continuous systems, each having a single degree of freedom. In this problem any of the isolated conservative piecewise-continuous systems has a periodic general integral, see Sec. 10.1. Provided that the generalised coordinate q and momentum p are continuous with respect to t, the transformation to the "action-angle" variables is feasible. As a result we arrive at the system with a multi-dimensional fast phase of the type (5.9). The right hand sides of the equations of this system can have several discontinuities with respect to the canonical variables $\varphi_1, \dots, \varphi_n, s_1, \dots, s_n$. Further multi-frequency averaging of the system is quite similar to that described in Sec. 5.4.

Transformation to the "action-angle" variables according to the general scheme of Sec. 5.2 is not feasible in the problem of the weak interaction of the dynamical objects of the impact-oscillatory type for which the generalised momentum p (and the generalised velocity $\dot{q}$) experiences a discontinuity at certain time instants. Not touching upon this question, we nevertheless indicate that this is one case which is important for a series of practical applications. In this case formulae (5.131), (5.132), (5.143), (5.146), (5.150) and (5.151) for determining the criteria of stability of the first and second kind for a single-frequency regime are valid in a certain sense. This is the problem of determining the non-critical fast variables studied in the previous section. Instead of continuous system (5.123), we considered a system as being piecewise-continuous, linear or integrable by quadratures within the continuity intervals. In addition to this, we assume that the original variables "phase-frequency" are continuous at the "switch", see Sec. 10.1 for the physical substantiation of this assumption. To this aim it is sufficient to

1) replace the integrals over period by the sum of integrals over the continuity intervals within the period,

2) write a piecewise-continuous analogue for system (5.154), differentiate it with respect to ν_j' according to the rules developed in Sec. 1.3 and then put $\nu_1' = \cdots = \nu_m' = \nu$, see eq. (5.155). The result is a linear piecewise-continuous system for determining $T-$periodic functions which are piecewise-continuous analogues for quantities y_{sj}. It is important that inequality (5.160) is also valid in the autonomous case.

6

Canonical averaging of the equations of quantum mechanics

6.1 Introductory remarks

The very possibility of applying the modern methods of the classical theory of non-linear oscillations to quantum mechanics is based upon the representation of the non-stationary Schrödinger's equation as a classical Hamiltonian system. From this perspective it is quite natural to construct a special asymptotic perturbation theory which utilises the advantages of the Hamiltonian formalism, that is to apply canonical transformations. Special and sufficiently efficient approaches [60], [35] and [29] were developed by mathematicians for canonical systems. However, the generality of these approaches makes them very cumbersome whereas the first two non-trivial approximations are ordinarily sufficient for practical application.

Traditionally, mathematicians use the methods of spectral analysis of operators for constructing perturbation theory in non-relativistic quantum mechanics, see [28], [42]. However, it is necessary to take into account that, in practice, physicists do not distinguish between the concepts of self-adjoint and symmetric operators. This gives rise to two unpleasant things: firstly, the domain of definition of the operator in Hilbert space remains unclear which does not allow one to apply the methods of spectral theory, and secondly, the domain of definition of the operator includes all functions for which analytical operations are meaningful regardless of the fact whether these functions (and the result of applying an operator to them) belong to Hilbert space. A rigorous consideration of the latter case requires the introduction of an equipped Hilbert space, see [11], [14].

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For this reason it is not possible to prove even the conditions for applicability of the regular perturbation theory developed by Kato-Relih [42], [87] providing us with the criterion for the Rayleigh-Schrödinger formal series to have a non-zero radius of convergence.

It is well known that orthodox perturbation theory is not applicable to many cases since the corresponding series diverge. The asymptotic character of the series used in perturbation theory was first proved by Titchmarsh [95]. A rigorous proof of the divergence of this series for an anharmonic oscillator ($V \sim x^4$) is given by Bender and Woo [13].

In addition to the associated complexity, the above-mentioned methods possess another shortcoming, namely they do not allow one to obtain the wave function which plays an important role in the investigation of physical systems.

In the present chapter we suggest another approach which is based upon the representation of the non-stationary Schrödinger's equation as a classical Hamiltonian system. This representation enables one to make use of the powerful, modern, rigorously substantiated methods of the classical theory of non-linear oscillations (asymptotic perturbation theory) and indicate simple conditions for justifying the applicability of the results obtained.

An important part for the transition from the hypotheses of Planck and Einstein to quantum mechanics was played by the adiabatic hypothesis by Ehrenfest. Born and Fock showed in 1928, [21], that Ehrenfest's hypothesis is a consequence of the postulates of quantum mechanics. A rigorous mathematical proof of the adiabatic theory was given by Kato in 1949, [41]. Later on, the adiabatic Landau-Dykhne approximation, [54], [26], [27], was built on the analogy between the adiabatic and quasi-classical approximations.

The Born-Fock adiabatic approximation is actually not an approximation since all of the terms of the adiabatic Born-Fock series have the same order of smallness, [27], [23], which, in turn, does not allow us to construct a post-adiabatic approximation. The Born-Fock condition, which implies real-valued wave functions, does not allow us to use this approximation in problems involving magnetic fields.

The results of the Landau-Dykhne adiabatic approximation relate to the results of non-stationary perturbation theory only approximately. In addition to this, both approximations yield an incorrect factor in front of the exponential function, see [23].

The problem of the time interval, within which the difference between the approximated and exact solutions is small, plays an important part in the non-stationary case. In the above works, this problem is not discussed at all.

In the present chapter, the adiabatic and the post-adiabatic theories, as well as the adiabatic perturbation theory and the post-adiabatic approximation, are constructed by means of the method of canonical averaging (phase perturbation theory). The basic assumptions of the Born-Fock theory are not satisfied. The obtained approximations are compared with exact solutions of the non-stationary Schrödinger's equation for a harmonic oscillator in a homogeneous time-dependent field and with the approximations obtained by traditional formulae [54]. One can see from this comparison that the standard non-stationary approximations [54] are valid only within a non-dimensional time intervals $t \sim 1$, whereas the approximations of the present chapter are valid within asymptotically longer time intervals $t \sim 1/\varepsilon$.

6.2 Stationary Schrödinger's equation as a classical Hamiltonian system

In this section the classical canonical perturbation theory is applied to constructing asymptotic solutions of Schrödinger's equation with a discrete spectrum. The main subject of analysis of the non-relativistic quantum theory is Schrödinger's equation, [54],

$$i\hbar \frac{\partial \Psi(q,t)}{\partial t} = \hat{H}\Psi(q,t), \tag{6.1}$$

where $i^2 = -1, \hbar = 1.054 \cdot 10^{-34}\,\mathrm{J\,s}$ denotes Planck's constant, $q = (q_1, q_2, ..., q_n)$ denotes a point of the configuration space of the corresponding classical system, t is time and $\Psi(q,t)$ denotes a complex-valued function with integrable square of the absolute value. Further, $\hat{H}$ denotes a self-adjoint (symmetric) operator in Hilbert space, which in Cartesian coordinates, in Schrödinger's representation for one particle, has the form, [54],

$$\hat{H} = \hat{T} + \hat{V} = -\frac{\hbar^2}{2m}\Delta + \hat{V}(x,y,z), \tag{6.2}$$

where $\hat{T}$ and $\hat{V}$ designate operators of the kinetic and potential energies, respectively, $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$, and m is mass of the particle. Schrödinger's equation (6.1) is subject to an initial condition $\Psi(q,0) = \Psi_0(q)$ and some boundary conditions.

The case studied in the framework of perturbation theory appears when operator $\hat{H}$ can be cast as the sum

$$\hat{H} = \hat{H}_0 + \varepsilon\hat{V}, \quad 0 < \varepsilon << 1, \tag{6.3}$$

of two self-adjoint operators, the corresponding problem (6.1) for operator $\hat{H}_0$ being assumed to have an exact solution and the second operator (perturbation) being small in some sense, [42], [87], [54].

The majority of the physically interesting problems turn out to be mathematically incorrect, since the perturbation operators are usually not bounded and not even self-adjoint. The latter is related to the fact that physicists never distinguish between the concepts of self-adjoint and symmetric operators. This leads to the operator space in Hilbert space being unclear, which in turn does not allow one to apply the methods of spectral theory [14], [87]. For this reason, it is difficult to indicate the conditions for the applicability of perturbation theory and estimate the discrepancy between the exact and an approximate solution for practical problems.

However, it is possible to reduce Schrödinger's equation to a form of classical Hamiltonian system, which is well-developed in non-linear mechanics. This allows one to apply the methods of classical dynamics which are rigorously substantiated and simpler from the perspective of application.

Let us consider eq. (6.1) with Schrödinger's operators (6.3), i.e. the problem

$$\begin{aligned} i\hbar\frac{\partial\Psi(q,t)}{\partial t} &= (\hat{H}_0+\varepsilon\hat{V})\Psi(q,t), \\ \Psi(q,0) &= \Psi_0(q), \end{aligned} \tag{6.4}$$

where operator $\hat{H}_0$ does not depend on time and ε is a formal small parameter. The question of choosing the small parameters is discussed below.

Along with the problem we consider, the generating approximation, which is obtained from eq. (6.4) at $\varepsilon = 0$

$$i\hbar\frac{\partial\Psi^0(q,t)}{\partial t} = \hat{H}_0\Psi^0(q,t), \qquad \Psi^0(q,0) = \Psi_0(q)\,. \tag{6.5}$$

Assuming the spectrum to be discrete, we can apply Fourier's method and set the general solution of problem (6.5) in the form

$$\begin{aligned} \Psi^0(q,t) &= \sum_{n=0}^{\infty} c_n^0\psi_n^0(q)\exp(-i\omega_n^0 t), \\ c_n^0 &= \int \Psi_0(q)\psi_n^{0*}(q)dq, \quad \omega_n^0 = E_n^0/\hbar, \end{aligned} \tag{6.6}$$

where $\psi_n^0(q)$ and E_n^0 denote respectively the eigenfunctions and eigenvalues of the following problem

$$\hat{H}_0\psi_n^0(q) = E_n^0\psi_n^0(q) \tag{6.7}$$

and an asterisk denotes the complex conjugate.

The self-adjoint character of operator $\hat{H}_0$ means that for $\Psi(q,t)\in L^2$ the following expansion is valid

$$\Psi(q,t) = \sum_{n=0}^{\infty} c_n(t)\psi_n^0(q)\exp(-i\omega_n^0 t)\,. \tag{6.8}$$

Inserting expansion (6.8) into eq. (6.4) yields the following equations for the coefficients of the expansion $c_n(t), c_n^*(t)$

$$\begin{aligned} \dot{c}_n(t) &= -i\varepsilon \sum_{m=0}^{\infty} v_{nm} c_m(t) \exp(-i\omega_{mn}^0 t), \\ \dot{c}_n^*(t) &= i\varepsilon \sum_{m=0}^{\infty} v_{mn} c_m^*(t) \exp(i\omega_{mn}^0 t), \end{aligned} \tag{6.9}$$

$$v_{mn}(t) = \frac{1}{\hbar} \int \psi_m^{0*}(q) V(q,t) \psi_n^0(q) dq \,, \tag{6.10}$$

where a dot implies time derivative and $\omega_{mn}^0 = \omega_m^0 - \omega_n^0$.

The system of equations (6.9) is Hamiltonian (in the classical sense) with the following Hamilton function

$$\varepsilon H_1(c, c^*, t) = -i\varepsilon \sum_{n,m=0}^{\infty} v_{nm} c_n^* c_m \exp(i\omega_{nm}^0 t) \tag{6.11}$$

and describes a classical distributed system with an infinite number of internal resonances. The system is Hamiltonian as matrix v_{nm} is Hermitian, that is the perturbation operator is self-adjoint. By separating the principal resonance $\omega_n^0 = \omega_m^0$, we can cast Hamilton's function (6.11) as follows

$$\varepsilon H_1(c, c^*, t) = -i\varepsilon v_{nn} c_n c_n^* - i\varepsilon \sum_{n,m=0}^{\infty}{}' v_{nm} c_n^* c_m \exp(i\omega_{nm}^0 t) \,, \tag{6.12}$$

where a prime denotes the sum without the term with $n = m$. Transformation of variables c_n, c_n^* to the real-valued "action-angle" variables I_n, ψ_n by means of the formulae

$$\begin{aligned} c_n &= \sqrt{I_n} \exp(-i\psi_n), \quad I_n = c_n c_n^*, \\ c_n^* &= \sqrt{I_n} \exp(i\psi_n), \quad \psi_n = -\arctan \frac{(c_n - c_n^*)}{i(c_n + c_n^*)}, \end{aligned} \tag{6.13}$$

we can set system (6.9) in the form

$$\begin{aligned} \dot{I}_n &= 2\varepsilon \sum_{m=0}^{\infty}{}' \sqrt{I_n I_m} \, \mathrm{Im}\{v_{nm} \exp[-i(\psi_m - \psi_n + \omega_{mn}^0 t)]\}, \\ \dot{\psi}_n &= \varepsilon v_{nn} + \varepsilon \sum_{m=0}^{\infty}{}' \sqrt{\frac{I_n}{I_m}} \, \mathrm{Re}\{v_{nm} \exp[-i(\psi_m - \psi_n + \omega_{mn}^0 t)]\} \end{aligned} \tag{6.14}$$

and Hamilton's function (6.12) as follows

$$\varepsilon H_1(I, \psi, t) = \varepsilon \sum_{m,n=0}^{\infty} v_{nm} \sqrt{I_n I_m} \exp[-i(\psi_m - \psi_n + \omega_{mn}^0 t)] \,. \tag{6.15}$$

Systems (6.9) and (6.14) do not contain Planck's constant explicitly and are the classical Hamiltonian systems with an infinite number of internal resonances. Estimates of the norm of discrepancy between the exact and approximate solutions as well as the conditions for applicability of the averaging method for these systems are given by the Los theorem, see [65] and [59], which is a generalisation of Bogolyubov theorem for the case of an infinite-dimensional coordinate Hilbert space.

The canonical form of systems (6.9) and (6.14) allows us to consider the evolutionary equations by operating only with Hamilton's functions (6.12) and (6.15), i.e. by calculating an averaged Hamilton's function. For example, for eqs. (6.11) and (6.15) the second approximation $\bar{H}^{(2)}$ for the averaged Hamilton's function is constructed with the help of the following formulae

$$\bar{H}^{(2)}(\bar{c},\bar{c}^*) = \varepsilon \bar{H}_1(\bar{c},\bar{c}^*) + \varepsilon^2 \bar{H}_2(\bar{c},\bar{c}^*),$$
$$\bar{H}_1 = \langle H_1 \rangle, \quad \bar{H}_2 = -\left\langle \frac{\partial \tilde{H}_1}{\partial \bar{c}^*} \frac{\partial \{H_1\}}{\partial \bar{c}} \right\rangle, \tag{6.16}$$

where $\bar{c}_n$ and $\bar{c}_n^*$ denote the evolutionary components of variables c_n and c_n^*. In the latter equation the following notation is used

$$\langle f \rangle = \lim_{T\to\infty} \frac{1}{T} \int\limits_{t_0}^{t_0+T} f(\bar{c},\, \bar{c}^*, t)dt, \quad \tilde{f}(\bar{c},\, \bar{c}^*, t) = f(\bar{c},\, \bar{c}^*, t) - \langle f \rangle,$$
$$\{f\} = \int \tilde{f}(\bar{c},\, \bar{c}^*, t)dt, \tag{6.17}$$

the arbitrary function of slow variables $\bar{c}$ and $\bar{c}^*$ being set to zero while estimating the last integral.

Let us notice that $\bar{H}$ is an integral of the averaged equations of motion, i.e. an adiabatic invariant [6], [90].

The first approximation $\bar{c}_n^{(1)}$ to expansion coefficient c_n is given by the formula $c_n^{(1)} = \bar{c}_n$, where $\bar{c}_n$ satisfies the following equation

$$\dot{\bar{c}}_n = \varepsilon \frac{\partial \bar{H}_1}{\partial \bar{c}_n^*}\ . \tag{6.18}$$

The second approximation $c_n^{(2)}$ to expansion coefficient c_n is given by

$$\bar{c}_n^{(2)} = \bar{c}_n + \varepsilon \frac{\partial \{H_1\}}{\partial \bar{c}_n^*}, \tag{6.19}$$

where the second approximation to evolutionary component $\bar{c}_n$ is obtained from the equation

$$\dot{\bar{c}}_n = \varepsilon \frac{\partial \bar{H}_1}{\partial \bar{c}_n^*} + \varepsilon^2 \frac{\partial \bar{H}_2}{\partial \bar{c}_n^*}\ . \tag{6.20}$$

6.3 General properties of the canonical form of Schrödinger's equation

Representation of Schrödinger's equation in canonical form (6.9) or (6.14) allows us to draw a number of conclusions without performing any calculations.

1. The original formulation of the problem of perturbation theory for the non-stationary Schrödinger's equation results in formulae enabling us to study all cases: stationary (non-degenerate and degenerate), non-stationary, resonant, adiabatic etc.

2. In the stationary case, for which matrix elements v_{mn} are independent of time, in the absence of degeneracy and the internal resonances ($\omega_{mn}^0 \neq O(\varepsilon)$, $m \neq n$) except for the selected principal resonance ($m = n$), systems (6.9) or (6.14) admit direct averaging. The equation for the evolutionary components of phase $\bar{\psi}_n$ has the form $\dot{\bar{\psi}}_n = \varepsilon v_{nn}$ (the mean values of the sums in eqs. (6.9) or (6.14) are zero) and is easily integrated, to give $\bar{\psi}_n = \varepsilon v_{nn} t + \psi_{n0}$. In turn, this means that the first correction to the energy appears due to the principal internal resonance. This conclusion remains valid for any order of the perturbation theory since the corrections to the eigenvalues of the unperturbed operator $\hat{H}_0$ are basically determined by the principal internal resonance.

On the other hand, it is evident that the coefficients of the diagonal terms in the averaged Hamilton function are the corrections to the energy.

Let us notice that direct averaging without introducing the resonant terms would lead to the appearance of divergent terms which are proportional to $1/(E_m - E_n)$ $(m \to n)$. Thus, separation of the resonant terms eliminates the divergent terms in the perturbation theory series.

3. It is obvious that the first-order contribution to the averaged Hamilton function or to the above equations from the non-resonant sums is zero for a stationary perturbation and non-zero for a non-stationary perturbation. This is the only difference between the stationary and non-stationary cases. In both cases, calculations are carried out by means of formula (6.16).

4. Contributions from these sums are not zero if the perturbation is stationary and the unperturbed levels contain close levels for which $\omega_{mn}^0 = O(\varepsilon)$. Thus, the problem of close levels should be solved only in a non-stationary form. It is evident that the problem for degenerate levels is a particular case of the previous one for which, along with the principal quantum number n, there is a multiindex α characterising the unperturbed eigenvalue, so that the relationship $\omega_{n\alpha,n\beta} = O(\varepsilon)$ holds. All these cases are manifestations of the resonances additional to the principal internal resonance.

5. Generally speaking, a general analysis is not applicable for time-dependent perturbations because it is necessary to know the spectrum of the perturbation so as to take correct account of the possible resonances.

Only general analysis of periodic (in particular, single-frequency) perturbation is feasible.

6. For the sake of simplicity, let us consider the case of single-frequency perturbation ($v_{mn} \sim \exp(\pm i\omega t)$). Clearly, the problem in this case is reduced to the stationary one, for which the expansion is performed for wave functions of the stationary states with the new frequencies $\omega_{mn} = \omega_m - \omega_n$ such that $\omega_m = \omega_m^0 - \omega/2$, $\omega_n = \omega_n^0 + \omega/2$ (for $v \sim \exp(i\omega t)$) and $\omega_m = \omega_m^0 + \omega/2$, $\omega_n = \omega_n^0 - \omega/2$ (for $v \sim \exp(-i\omega t)$).

Thus, the corresponding quantities $E_{m,n} = \hbar\omega_{m,n}$ are energies, that is the energy of the system: unperturbed system (atom) and a field. Their interaction is absent, the original interaction being included into the definition of the quasi-energy. This gives rise to the concept of the system dressed by a field (dressed atom, [90]).

Traditionally, this conclusion is obtained in a rather sophisticated way by means of Floquet theorem and the conclusion on the level splitting is obtained only in the resonant approximation, see [27], [6]. Let us notice that the results of this point are valid for any value of parameter ε, that is regardless of the perturbation theory.

7. The exact eigenfrequencies of the perturbed system are given by the relationship $\omega_n = \omega_n^0 + \dot{\psi}_n$. Thus, the second equations in (6.14) determine corrections to the eigenfrequencies caused by perturbation. In principle, these corrections can be removed by a canonical transformation of the phase which can serve as a distinctive procedure of renormalisation which allows one to remove the secular terms from the series of the perturbation theory. It is obvious that in the non-stationary case (even for $v_{nn} = 0$) there exists a non-zero contribution of the first order stemming from the sum. The presence of this contribution is not possible using orthodox perturbation theory. The coherent interaction with the external field is realised under the condition of constant phase difference (the condition of equality of the original frequencies is only a necessary condition) and has the form $\dot{\varphi}_2 - \dot{\varphi}_1 - \omega = \omega_2^0 + \dot{\psi}_2 - \omega_1^0 + \dot{\psi}_1 = \omega_{21}^0 - \omega + \dot{\psi}_2 - \dot{\psi}_1 = 0$, where ω denotes the frequency of the external field satisfying the condition that $\omega_{21}^0 - \omega = O(\varepsilon)$.

One usually uses the condition that the transition frequency is close to that of the external field. It follows from the form of the equations for $\dot{\psi}_n$ that the type of problem for resonant interaction coincides with the type considered in point 4 above.

The exact frequencies ω_n in the system are anisochronous which is a characteristic of the non-linear classical system and leads to bounded solutions at resonance even in the case of no damping, in spite of the linearity of Schrödinger's equation.

Thus, Schrödinger's equation is equivalent to some classical non-linear distributed system whereas representations (6.9) or (6.14) are expansions in terms of normal forms of the unperturbed system.

To some extent, it is the picture to which Schrödinger tended and which is most close to the classical one. "There is no need to explain that the representation of the energy transformation from one oscillatory form to another under a quantum energy transition is much more satisfactory than that of electron jump"[1], [91].

8. It is evident that no specific "quantum-mechanical" properties of the generating operator $\hat{H}_0$, but being self-adjoint, is used. Nevertheless, this technique allows us to analyse other self-adjoint problems of mathematical physics with a discrete spectrum. Taking into account the particular structure of the generating operator $\hat{H}_0$ we can construct a more efficient theory [66].

An attempt to apply the averaging method to quantum mechanics was undertaken in [93]. However, as follows from the above, the absence of resonances (the main assumption of the authors) is not correct. Besides, despite the title of this article, the authors did not succeed in a proof of the theorem of convergence of the constructed perturbation theory.

9. All formulae remain valid for the case of adiabatic perturbation, i.e. under the additional dependence of perturbation on the slow time $\tau = \varepsilon t$ ($\varepsilon V = \varepsilon V(\vec{r}, t, \tau)$). In this case, equations for the evolutionary components become non-stationary and require more sophisticated integration methods.

10. Small parameter ε is introduced in systems (6.9) and (6.14) in a formal way. Generally speaking, the question of a rigorous introduction into equations should be considered individually for each particular problem. Let us point out some general ideas.

Let us introduce some characteristic values $[E]$ and $[V]$ for the eigenvalues E and matrix elements V_{mn} respectively. Then $[E]/\hbar = \omega_0$ and $[V]/\hbar = \Omega_0$ can be referred to as a characteristic eigenfrequency and the generalised Rabi frequency (for a dipole interaction $V \sim \vec{d}\vec{E}_0$ and $V/\hbar \sim \Omega$ is called the Rabi frequency). Further study depends on the relationship between frequencies ω_0 and Ω_0. Let $\omega_<$ and $\omega_>$ denote respectively the smaller and the larger of frequencies ω_0 and Ω_0. Entering a non-dimensional time $t_n = \omega_> t$ into dimensional systems (6.9) and (6.14) we obtain the following value $\varepsilon = \omega_< / \omega_>$.

Three cases are possible:

a) the case of a weak field $\omega_0 >> \Omega_0$, $t_n = \omega_0 t$, $\varepsilon = \Omega_0/\omega_0$,

b) the case of a strong field $\omega_0 << \Omega_0$, $t_n = \Omega_0 t$, $\varepsilon = \omega_0/\Omega_0$,

c) the case in which the frequencies are of the same order, that is, $\omega_0 \sim \Omega_0$. In this case an additional resonance occurs in the system and the small parameter is absent. This situation requires special consideration.

Clearly, both $[E]$ and $[V]$ are, in general, functions of n and m which should be taken into account while carrying out estimates.

[1] Translator's note: translation from Russian

11. In problems with initial conditions, the values of the coefficients $c_n|_{t=0}$ in the expansions of the initial functions have order of unity, whilst those which do not appear ($c_n|_{t=0} = 0$) are of order of ε. As follows from Parseval's equality $\left(\sum_n |c_n|^2 = 1\right)$ coefficient c_n must rapidly decreases with the growth of n, thus, the first order approximation in (6.8) contains a finite sum with terms having non-zero coefficients c_n of the expansion of the initial function.

It becomes clear from the above that an accurate account of all possible internal and external resonances in the system, i.e. the analysis of the phase relationships, plays a crucial part for obtaining a correct result. For this reason, it is natural to refer to this perturbation theory as the phase perturbation theory.

6.4 Stationary perturbation of a non-degenerate level of the discrete spectrum

Let us consider a perturbation of a non-degenerate level of the discrete spectrum, i.e. the case $\hat{V} = \hat{V}(\vec{r})$. The canonical procedure of averaging is carried out by using Hamilton's function (6.12) under the condition $\omega^0_{mn} \neq O(\varepsilon)$ implying no degeneracy and no close energy levels. Simple calculation by means of eq. (6.16) yields

$$\begin{aligned} \bar{H}_1 &= -i\sum_k v_{kk}\bar{c}_k\bar{c}_k^*, \ \{H_1\} = \sum_{n,m} \frac{v_{nm}}{\omega^0_{mn}}\bar{c}_m\bar{c}_n^* \exp(-i\omega^0_{mn}t), \\ \bar{H}_2 &= -i\sum_{l\neq k}\sum_k \frac{|v_{kl}|^2}{\omega^0_{kl}}\bar{c}_k\bar{c}_k^* . \end{aligned} \tag{6.21}$$

The second approximation to the averaged Hamilton's function (6.12) is given by

$$\begin{aligned} \bar{H}^{(2)} &= \varepsilon\bar{H}_1 + \varepsilon^2\bar{H}_2 = -i\sum_k \Delta\omega_k\bar{c}_k\bar{c}_k^* , \\ \Delta\omega_k &= \varepsilon v_{kk} + \varepsilon^2{\sum_l}' \frac{|v_{kl}|^2}{\omega^0_{kl}} \end{aligned} \tag{6.22}$$

Both first and second terms on the averaged Hamilton function can be renormalised by the phase (frequency) renormalisation in the original expansion (6.8), i.e. by replacing ω^0_k by $\Omega^0_k = \omega^0_k + \Delta\omega_k$. In addition to this, $\bar{H}^{(2)} \equiv 0$. This procedure can be performed in any order of calculations. This means in turn that, instead of a standard time interval $\Delta t \sim 1/\varepsilon$, this approximation is valid for exponentially large time intervals which is in full agreement with the general theorems of mechanics on the behaviour of Hamiltonian systems close to integrable systems [6].

The Hamilton function has a diagonal form and the coefficients of the quadratic form are corrections to the phase (energy) of the unperturbed wave function.

Equation (6.20) of the second approximation for the evolutionary component $\bar{c}_k^{(2)}$ has the form

$$\dot{\bar{c}}_k = \frac{\partial \bar{H}^{(2)}}{\partial \bar{c}_k^*} = -i\Delta\omega_k \bar{c}_k \,. \tag{6.23}$$

Then we easily obtain

$$\bar{c}_k^{(2)} = A_k \exp(-i\Delta\omega_k t)\,. \tag{6.24}$$

Coefficients A_k are determined from the initial conditions.

The second approximation to coefficients c_k in expansion (6.8), obtained by means of formula (6.19), is as follows

$$c_k^{(2)} = A_k^{(2)} \exp(-i\Delta\omega_k t) + \varepsilon \sum_m \frac{v_{km}}{\omega_{mk}^0} A_m^{(1)} \exp(-i\Omega_{mk} t), \tag{6.25}$$

where $\Omega_{mk} = \Omega_m - \Omega_k, \Omega_k = \omega_k^0 + \Delta\omega_k$, and $A_k^{(1)}, A_k^{(2)}$ denote the first and the second approximations to coefficients A_k (it is sufficient to substitute only the first approximation to A_k into the second term in eq. (6.25)).

The second approximation to wave function $\Psi^{(2)}$ is constructed with the help of coefficients $c_k^{(2)}$

$$\Psi^{(2)} = \sum_k{}' \left[A_k^{(2)} - \varepsilon \sum_m \frac{v_{km}}{\omega_{km}^0} A_m^{(1)} \exp(-i\Omega_{mk} t) \right] \Psi_k^0 \exp(-i\Omega_k t), \tag{6.26}$$

where $\Omega_k = \omega_k^0 + \Delta\omega_k$, i.e. it is sufficient to restrict the consideration by the first correction to the eigenfrequency.

Provided that the system is in the $n-th$ stationary state of the discrete spectrum, then $A_k = \delta_{kn}$ and we obtain from eq. (6.26) that

$$\Psi_n^{(2)} = \left[\Psi_n^0 + \varepsilon \sum_k{}' \frac{v_{kn}}{\omega_{nk}^0} \Psi_k^0 \right] \exp(-i\Omega_n t)\,. \tag{6.27}$$

In the case of Cauchy's problem, the system at the initial time instant is in a certain stationary state of the discrete spectrum, in the $s-th$ state say, that is $\Psi(q,t)|_{t=0} = \Psi_s^0$ and $c_n|_{t=0} = \delta_{ns}$. Coefficients $A_n^{(1)}, A_n^{(2)}$ are obtained from the relationship

$$\delta_{ns} = A_n^{(2)} + \varepsilon \sum_m \frac{v_{nm}}{\omega_{mn}^0} A_m^{(1)}\,. \tag{6.28}$$

From this relationship we obtain $A_n^{(1)} = \delta_{ns}$ and $A_n^{(0)} = \delta_{ns} - \varepsilon \frac{v_{ns}}{\omega_{sn}^0}$. The second approximation to the wave function has the form

$$\Psi^{(2)} = \left(\Psi_s^{(0)} + \varepsilon {\sum_k}' \frac{v_{ks}}{\omega_{sk}^0} \Psi_k^0 \right) \exp(-i\Omega_s t) + \varepsilon {\sum_k}' \frac{v_{ks}}{\omega_{sk}^0} \Psi_k^0 \exp(-i\Omega_k t) \,. \tag{6.29}$$

This formula is absent in the standard textbooks on quantum mechanics. It is important to mention that it is adopted in courses on quantum mechanics that the probability of transition to this problem is determined by the square of the absolute value of the first correction to the expansion coefficients c_n, that is, by the second term in eq. (6.25), see for example [54]. The relationships in eq. (6.26) show that it is not correct. This term determines the correction to the unperturbed wave function of the initial state. The transition probability is determined by the second terms in $A_n^{(2)}$ which is completely absent in the standard perturbation theory. This is due to the fact that initial condition $c_n|_{t=0} = \delta_{ns}$ is not satisfied. The whole coefficient $c_n^{(2)}$ rather than a part of it, as in the standard theory, must satisfy this initial condition. In the case under consideration, coincidence is occasional because the coefficients are independent of time. But these coefficients are different in the non-stationary theory.

As an example of applying formula (6.28), we consider the problem of the excitation of a charged oscillator by an abruptly applied homogeneous electric field $\vec{\varepsilon}$, directed along the oscillation axis, [54], [30].

In this case it is necessary to solve the problem

$$\begin{aligned} i\hbar \frac{\partial \Psi(x,t)}{\partial t} &= (\hat{H}_0 + \hat{V})\Psi(q,t) = (\frac{\hat{p}^2}{2m} + \frac{kx^2}{2} - e\varepsilon x)\Psi, \\ \Psi|_{t=0} &= \Psi_0^0, \quad \Psi|_{x\to\pm\infty} - \text{bounded}, \end{aligned} \tag{6.30}$$

where $\hat{p}^2 = -\hbar^2\Delta$, m denotes the oscillator mass, k is the rigidity coefficient, e is the electron charge and $\hat{V} = -e\varepsilon x$.

Let us introduce into eq. (6.30) a non-dimensional variable $\xi = x/a$, $(a = \sqrt{\hbar/m\omega_0})$ and the eigenfrequency $\omega = \sqrt{k/m}$. The eigenfrequency ω $(\omega = \omega_0)$ is taken as a characteristic frequency ω_0 and the generalised Rabi frequency is $\Omega_0 = e\varepsilon a/\hbar$. Assuming the external field to be weak, we enter a small parameter ε by the relationship $\varepsilon = \Omega_0/\omega = e\varepsilon a/\hbar\omega = e\varepsilon/ka \ll 1$.

An exact solution of the problem of eigenfunctions and eigenvalues in terms of the non-dimensional units is given by

$$\begin{aligned} \Psi_n(x,t) &= (2^n\sqrt{\pi}an!)^{-1/2} \exp\left[-(\xi-\varepsilon)^2/2\right] H_n(\xi-\varepsilon) \times \\ &\quad \exp\left[-i\left(n + 1/2 - \varepsilon^2/2\right)\right], \\ \omega_n &= \omega_0 - \varepsilon^2/2 = n + 1/2 - \varepsilon^2/2\,, \end{aligned} \tag{6.31}$$

where $H_n(z)$ denotes Hermite polynomials, [54], [92].

The general solution of problem (6.30), constructed by means of eigenfunctions (6.31), has the form

$$\Psi(x,t)=\sum_{n=0}^{\infty} c_n \Psi_n(x,t)\,. \tag{6.32}$$

Using the initial condition we obtain the following expansion coefficients c_n in eq. (6.32) $c_n=\varepsilon^n\left(2^n n!\right)^{-1/2}\exp(-\varepsilon^2/4)$, so that the final result is as follows

$$\begin{aligned}\Psi(x,t) \;&=\; \sum_{n=0}^{\infty}\frac{\varepsilon^n\exp(-\varepsilon^2/4)}{\sqrt{2^n n!}}\left(2^n\sqrt{\pi}an!\right)^{-1/2}\exp\left[-(\xi-\varepsilon)^2/2\right]\times\\ &\quad H_n(\xi-\varepsilon)\exp\left[-i\left(n+1/2-\varepsilon^2/2\right)\right]\,.\end{aligned} \tag{6.33}$$

Performing expansion with accuracy up to terms of order of ε^2 yields

$$\Psi^{(2)}(x,t)=\left(\Psi_0^0+\varepsilon\frac{1}{\sqrt{2}}\Psi_1^0+\varepsilon\frac{1}{\sqrt{2}}\Psi_1^0\exp(-it)\right)\exp\left(-it/2\right)\,, \tag{6.34}$$

where Ψ_n^0 denotes the eigenfunctions of the unperturbed problem.

Let us now construct the solution of problem (6.30) by means of the canonical theory of perturbation. The matrix elements of the perturbation operator are $\varepsilon v_{mn}=-\varepsilon\left[(n/2)^{-1/2}\delta_{m,n-1}+((n+1)/2)^{-1/2}\delta_{m,n+1}\right]$, the corrections to the eigenfrequencies are $\Delta\omega_k=-\varepsilon^2/2$, the eigenfrequencies of the unperturbed system are $\omega_0^0=1/2$, $\omega_1^0=3/2$, and eq. (6.29) takes the form

$$\Psi^{(2)}(x,t)=\left(\Psi_0^0+\varepsilon\frac{1}{\sqrt{2}}\Psi_1^0+\varepsilon\frac{1}{\sqrt{2}}\Psi_1^0\exp(-it)\right)\exp\left(-it/2\right)\,.$$

As expected, the solution constructed by the perturbation theory coincides with the series expansion of the exact solution. Let us notice that obtaining solution (6.34) is simpler than expanding the exact solution in the series.

The probability of transition to the first excitation state is the square of the absolute value of the coefficient of $\Psi_1^0\exp(-i3t/2)$ and is equal to $\varepsilon^n/2$.

6.5 Stationary excitation of two close levels

In the case of stationary perturbation, it is expedient to take into account the principal resonance by renormalisation of the frequency in the original expansion, i.e. to present expansion (6.8) as follows

$$\Psi(q,t)=\sum_n c_n(t)\psi_n^0(q)\exp(-i\Omega_n^0 t), \tag{6.35}$$

where $\Omega_n^0 = \omega_n^0 + \varepsilon v_{nm}$. This transformation removes the term with εv_{nm} from the equations and the effective Hamilton function, the latter taking the form

$$\varepsilon H_1(c, c^*, t) = -i\varepsilon \sum_{n,m}{}' v_{nm} c_m c_n^* \exp(i\Omega_{nm}^0 t)\,. \tag{6.36}$$

The presence of two close levels in the unperturbed system, for example with indices α and β ($\omega_\alpha^0 - \omega_\beta^0 = \varepsilon\delta_0$) means that $\Omega_{\alpha\beta}^0 = \Omega_\alpha^0 - \Omega_\beta^0 = \varepsilon\delta$ ($\delta = \delta_0 + v_{\alpha\alpha} - v_{\beta\beta}$) and leads to the necessity to take into account the dependence of sum (6.36) on slow time $\tau = \varepsilon t$ in the process of averaging

$$\begin{aligned} \varepsilon H_1 &= -i\varepsilon v_{\alpha\beta} c_\alpha^* c_\beta \exp(i\delta\tau) - i\varepsilon v_{\beta\alpha} c_\alpha c_\beta^* \exp(-i\delta\tau) - \\ & i\varepsilon \sum_{n,m}{}'' v_{nm} c_m c_n^* \exp(-i\Omega_{nm}^0 t), \end{aligned} \tag{6.37}$$

where notation $\sum''$ implies that this sum does not contain the diagonal elements ($m = n$) and the elements with $m = \alpha, n = \beta$ and $m = \beta, n = \alpha$.

The calculation of the first-order approximation yields the following averaged Hamilton function

$$\varepsilon \bar{H}_1 = -i\varepsilon v_{\alpha\beta} \bar{c}_\alpha^* \bar{c}_\beta \exp(i\delta\tau) - i\varepsilon v_{\beta\alpha} \bar{c}_\beta^* \bar{c}_\alpha \exp(-i\delta\tau)\,. \tag{6.38}$$

The Hamilton equations for the evolutionary components $\bar{c}_\alpha, \bar{c}_\beta$ have the form

$$\begin{aligned} \dot{\bar{c}}_\alpha &= \varepsilon \frac{\partial \bar{H}_1}{\partial \bar{c}_\alpha^*} = -i\varepsilon v_{\alpha\beta} \bar{c}_\beta \exp(i\delta\tau)\,, \\ \dot{\bar{c}}_\beta &= \varepsilon \frac{\partial \bar{H}_1}{\partial \bar{c}_\beta^*} = -i\varepsilon v_{\beta\alpha} \bar{c}_\alpha \exp(-i\delta\tau)\,. \end{aligned} \tag{6.39}$$

Thus, the averaging procedure "cuts" a two-level system with close levels α and β from the whole spectrum. It is evident that the case of two-fold degeneracy is a particular case of this problem for $\delta_0 = 0$. The same solution has the problem of resonant interaction with an external field with frequency ω ($\omega - \omega_{\alpha\beta}^0 = \varepsilon\delta_0$), the difference being only in the resonance type. In the problem of the close levels there is an internal resonance.

The solution of problem (6.39) subject to the initial condition $\bar{c}_\alpha|_{t=0} = \delta_{n\alpha}$ is as follows

$$\begin{aligned} \bar{c}_\alpha &= \frac{1}{\Delta}\left[\Omega_1 \exp(i\Omega_2\tau) - \Omega_2 \exp(i\Omega_1\tau)\right], \\ \bar{c}_\beta &= \frac{g^*}{\Delta}\left[\exp(-i\Omega_1\tau) - \exp(-i\Omega_2\tau)\right], \\ \bar{c}_n &= 0, \quad n \neq \alpha, \beta\,, \end{aligned} \tag{6.40}$$

where $\Omega_{1,2} = \frac{\delta}{2} \pm \frac{\Delta}{2}$, $\Delta = \sqrt{\delta^2 + 4|g|^2}$, $g = v_{\alpha\beta}$.

Representing the first equation in eq. (6.40) in the form

$$\bar{c}_\alpha = \frac{\exp(i\Omega_1\tau)}{\Delta}\left\{\Delta + \Omega_1[\exp(-i\Delta\tau) - 1]\right\}, \tag{6.41}$$

it is easy to find that the probability of being in the state become unity ($w_\alpha = 1$) after the time interval $\tau^* = 2\pi/\Delta$. Thus, the oscillations between the levels α and β have period $T = 2\pi/\Delta$ or frequency $\omega = \Delta$.

Substituting coefficients (6.40) into expansion (6.35) we obtain that the wave function is a superposition of two stationary states with

$$\omega_{\alpha,\beta} = \frac{\Omega_\alpha^0 + \Omega_\beta^0}{2} \pm \frac{1}{2}\sqrt{\left(\Omega_\alpha^0 - \Omega_\beta^0\right)^2 + 4|g|^2}. \tag{6.42}$$

In the case of a degenerate level $\delta_0 = 0$ and from eq. (6.42) we obtain the correction $\Delta\omega$ to the frequency of the stationary state $\omega_\alpha^0 = \omega_\beta^0 = \omega^0$

$$\Delta\omega = \varepsilon\frac{v_{\alpha\alpha} + v_{\beta\beta}}{2} \pm \frac{1}{2}\sqrt{(v_{\alpha\alpha} - v_{\beta\beta})^2 + 4|v_{\alpha\beta}|^2}. \tag{6.43}$$

Therefore, all three problems, namely the problems on close levels, two-fold degenerate level and resonant interaction with an external single-frequency field, are all solved in the framework of the same approach and yield the results coinciding with the traditional one with first order accuracy.

While solving the problem, we determine the conditions under which the quantum system with a discrete spectrum can be modelled, in the first approximation, by a two-level system. The main point of this procedure is the possibility of averaging Hamilton's function (6.38). The condition of weakness of the external field is needed for this. Then, the non-trivial initial conditions are required, at least for one of the coefficients c_α, c_β, otherwise the solutions of the homogeneous equations in (6.39) are trivial.

Nowadays, the procedure of solving these problems is performed backwards. A two-level systems is first taken, then a so-called resonant approximation (rotating wave approximation), [54], [23], is applied to it. At this stage it is incorrectly assumed that the resonant approximation is also applicable in the cases where the perturbation theory is invalid, see for example [23].

6.6 Non-stationary Schrödinger's equation as a Hamiltonian system

The situation studied in the non-stationary perturbation theory occurs when operator $\hat{H}$ can be represented as sum $\hat{H} = \hat{H}_0 + \varepsilon\hat{V}(q,t)$ ($0 < \varepsilon << 1$) of two self-adjoint operators. In the adiabatic approximation the perturbation operator $\hat{V}(q,t)$ is not small and depends on slow time $\tau = \varepsilon t$

such that $\hat{V}(q,t) = \hat{V}(q,\tau)$. The solution should be constructed within the asymptotically large time interval $\tau \sim 1/\varepsilon$ when change in the perturbation operator is large. In this case splitting the total Schrödinger's operator $\hat{H}$ into two operators, namely the generating (unperturbed) operator and a perturbation operator makes no sense. In order to embrace both possibilities we consider problem (6.4) with the time-dependent Schrödinger operator $\hat{H} = \hat{H}(q,t)$.

Let us assume that the stationary problem corresponding to (6.4) is solvable for a parametric dependence of Schrödinger's operator on time and has a discrete spectrum. This means that the eigenfunctions and the eigenvalues of the problem are given by

$$\hat{H}\,(t)\,\psi_n(q,t) = E_n(t)\psi_n(q,t), \tag{6.44}$$

with time t being fixed. The eigenfunctions are assumed to be orthonormalised as follows

$$\int\limits_{-\infty}^{\infty} \bar{\psi}_m(q,t)\psi_n(q,t)dq = \delta_{mn} \tag{6.45}$$

where a bar denotes the complex conjugate.

The existing approximations of Born-Fock [92] and Landau-Dykhne [27], [23] suggest that the eigenfunctions can be chosen as being real-valued (i.e. no magnetic field is assumed) which essentially reduces the applicability of the method. In the present study this assumption is not needed.

In the case of weak fields the results of the adiabatic approximation of Landau-Dykhne do not coincide with the results of perturbation theory [23]. The approximation of Born-Fock is actually not an approximation at all since all higher approximations turn out to be of the order of the first approximation [23]. In addition to this, both approximations yield an incorrect factor in front of the exponential function [23].

Let us look for the solution of the exact problem in the form

$$\Psi(q,t) = \sum_n c_n(t)\psi_n(q,t)\exp\left\{-i\int\limits_0^t \Omega_n(z)dz\right\}, \tag{6.46}$$

where

$$\Omega_n(t) = \omega_n(t) + v_{nn}(t),\ \omega_n(t) = E_n(t),\ v_{nn} = -i\int\limits_{-\infty}^{\infty} \bar{\psi}_n \frac{\partial\psi_n}{\partial t}dq.$$

The meaning of this choice of the phase becomes clear in what follows.

Inserting eq. (6.46) into eq. (6.4) yields the following equation for the expansion coefficients $c_m(t)$

$$\dot{c}_m(t) = -i\sum_{n,m}{}' v_{mn} c_n \exp\left\{-i\int\limits_0^t \Omega_{mn}(z)dz\right\}, \tag{6.47}$$

where a dot denotes a total time derivative and a prime at the summation sign denotes the absence of a diagonal components with $m = n$. The matrix of coefficients v_{mn} has the form

$$v_{mn} = -i\int\limits_{-\infty}^{\infty} \bar{\psi}_m(q,t)\frac{\partial\psi_n(q,t)}{\partial t}dq \tag{6.48}$$

and is Hermitian, i.e. $v_{mn} = \bar{v}_{nm}$.

The choice of phase indicated in eq. (6.46) ensures that the sum has no diagonal component which is responsible for the principal resonance. If eq. (6.46) had this diagonal component, the sum (6.47) would have a small resonant denominator.

Indeed, differentiating eq. (6.44) with respect to time and taking into account that Schrödinger's operator is self-adjoint, we obtain

$$v_{mn} = \left(\frac{i}{\hbar\omega_{mn}}\right)\left(\frac{\partial\hat{H}}{\partial t}\right)_{mn}, \quad m \neq n\,. \tag{6.49}$$

It is clear that in the case of the real-valued eigenfunctions $\bar{\psi}_n = \psi_n$, that is, the diagonal elements $v_{nn} = 0$. It is this fact that is the reason for the real-valued normalisation in the Born-Fock approximation.

Similar actions in the case when $m = n$ leads to the relationship $\left(\frac{\partial\hat{H}}{\partial t}\right)_{nn} = \frac{\partial E_n}{\partial t}$ and do not determine the diagonal matrix elements. In the case in which Schrödinger's operator depends on time τ in terms of the set of functions $\xi_i(\tau)$ $(i = 1, 2..N)$, elements v_{nn} determine the topological adiabatic Berry phase [97] whose value does not depend on the evolution time and is determined only by a closed contour in the parameter space.

Equation (6.49) indicates three cases allowing the development of the perturbation theory. In the adiabatic case $\hat{H} = \hat{H}(\xi(\tau))$, so that $\frac{\partial\hat{H}}{\partial t} = \varepsilon\left(\frac{\partial\hat{H}}{\partial\xi}\right)\left(\frac{\partial\xi}{\partial\tau}\right)$. In the case of the non-stationary perturbation theory, Schrödinger's operator has the form $\hat{H} = \hat{H}_0 + \varepsilon\hat{V}(q,t)$. Finally, in the case of the adiabatic perturbation theory $\hat{H} = \hat{H}_0 + \varepsilon\hat{V}(q,\tau)$.

The system of equations (6.47), along with the complex conjugate one, is Hamiltonian (in the classical sense) having the following Hamilton function

$$H(c, c^*, t) = -i\sum_{n,m}{}' v_{mn}(t) c_n c_m^* \exp\left\{-i\int_0^t \Omega_{mn}(z)dz\right\} \qquad (6.50)$$

which describes the classical distributed system. The matrix of coefficients v_{mn} is Hermitian which ensures that this system is Hamiltonian and in turn enables one to apply the phase perturbation theory.

6.7 Adiabatic approximation

We assume that Schrödinger's operator has the form $\hat{H} = \hat{H}_0 + \hat{V}(q, \xi(\tau))$, where $\tau = \varepsilon t$ denotes slow time. Then the matrix elements $v_{mn} \sim \varepsilon\xi'$ and Hamilton's function (6.50) can be cast in the form

$$\varepsilon H_1(c, c^*, t, \tau) = -i\varepsilon\sum_{n,m}{}' v_{mn}(\tau) c_n c_m^* \exp\left\{i\int_0^t \Omega_{mn}(\tau)dt\right\}, \qquad (6.51)$$

where $\dot{\tau} = \varepsilon$ and $\Omega_{mn} = \Omega_m - \Omega_n$.

The canonical form allows us to convert the evolutionary equations by means of the formulae

$$\bar{H}^{(2)}(\bar{c}, \bar{c}^*, \tau) = \varepsilon\bar{H}_1(\bar{c}, \bar{c}^*, \tau) + \varepsilon^2\bar{H}_2(\bar{c}, \bar{c}^*, \tau),$$
$$\bar{H}_1 = \langle H_1\rangle,\ \bar{H}_2 = -\left\langle\left(\frac{\partial\tilde{H}_1}{\partial\bar{c}^*}\right)\left(\frac{\partial\{H_1\}}{\partial\bar{c}}\right)\right\rangle, \qquad (6.52)$$

where $\bar{H}^{(2)}$ denotes the second approximation to the averaged Hamilton's function, whilst $\bar{c} = (\bar{c}_1, \bar{c}_2, ..)$ and $\bar{c}^* = (\bar{c}_1^*, \bar{c}_2^*, ..)$ are the evolutionary components of variables c and c^*.

Averaging expression (6.51) along the generating solution (c_k = const, τ = const), we obtain $\bar{H}_1 = \langle H_1\rangle = 0$ which in turn implies that $\dot{\bar{c}}_k = 0$. The latter result is the adiabatic theorem of Kato [41] which is obtained in fact without calculations, cf. ([64]) for the proof. In the classical sense, the evolutionary components $\bar{c}_k$ of the original variables c_k are the adiabatic invariants, see [87], [6], i.e. they retain the initial values for the asymptotic time interval $t \sim 1/\varepsilon$. For deriving this result it is necessary to assume that $\Omega_{mn}(\tau) \neq O(\varepsilon)$, i.e. the system has no degeneracy, there are no close levels and the levels do not intersect during the evolution time.

In the adiabatic (first) approximation, the solution of Schrödinger's equation has the form

$$\Psi^{(1)}(q,t) = \sum_{n=0}^{\infty} c_n^{(1)} \psi_n(q,t) \exp\left\{ -i \int_0^t \Omega_{mn}(z) dz \right\} . \tag{6.53}$$

Under rather general assumptions, Los's theorem [59] renders estimates for the difference $\left| \Psi(q,t) - \Psi^{(1)}(q) \right| < C\varepsilon$, where C is a constant independent of ε for time interval $t \sim 1/\varepsilon$.

6.8 Post-adiabatic approximation

In order to construct the second (post-adiabatic) approximation we make use of relationships in eq. (6.52). Simple calculation yields

$$\bar{H}_2 = -i \sum_k \Delta\Omega_k(\tau) \bar{c}_k \bar{c}_k^*, \ \Delta\Omega_k = \sum_l{}' \frac{|v_{kl}|^2}{\Omega_{kl}} , \tag{6.54}$$

so that the second approximation to the averaged Hamilton's function $\bar{H}^{(2)}$ has the form

$$\bar{H}^{(2)} = -i\varepsilon^2 \sum_k \Delta\Omega_k(\tau) \bar{c}_k \bar{c}_k^* . \tag{6.55}$$

Hamiltonian equations with Hamilton's function (6.55) for the evolutionary components $\bar{c}_k$ are integrated easily, to give

$$\bar{c}_k = A_k \exp\left\{ -i\varepsilon^2 \int_0^\tau \Delta\Omega_k(z) dz \right\} = A_k \exp\left(-i\alpha_k\right) . \tag{6.56}$$

Integration constants A_k are determined by means of the initial conditions.

Let us notice that the phase of coefficients $\bar{c}_k$ could be included into the original expansion (6.46), then we would obtain $\bar{H}_2 = 0$, $\bar{H}^{(2)} = 0$.

The second approximation to the expansion coefficients in eq. (6.46) is constructed by means of formulae (6.16)

$$c_k^{(2)} = A_k^{(2)} \exp(-i\alpha_k) - \varepsilon \sum_m{}' \frac{v_{km}}{\Omega_{km}} A_m \exp(-i\alpha_m + i\Omega_{km} t), \tag{6.57}$$

With the help of coefficients $c_k^{(2)}$ we obtain the second approximation $\Psi^{(2)}(q,t)$ to the solution of Schrödinger's equation

$$\Psi^{(2)}(q,t) = \sum_k [A_k \exp(-i\alpha_k) - \tag{6.58}$$

$$\left. \varepsilon \sum_m{}' \frac{v_{km}}{\Omega_{km}} A_m \exp(-i\alpha_m - i\Omega_{km} t) \right] \Psi_k \exp\left[-i \int_0^t \Omega_k(z) dz \right] .$$

It is necessary to mention that in eq. (6.57) we can limit our consideration to the terms in the sum by the first approximation $A_m^{(1)}$ with respect to ε.

When Cauchy's problem is studied, the system at the initial time instant is at a certain stationary state, say $s-th$, of the discrete spectrum of the unperturbed problem with Schrödinger's operator $\hat{H}_0$, i.e. $\Psi(q,t)|_{t=0}=\Psi_s^0$. In contrast to the stationary case we can not take that $c_n|_{t=0}=\delta_{ns}$, since the expansion is carried out in terms of the eigenfunctions of the perturbed problem $\psi_n(q,t)=\psi_n(q,\xi(t))$, where $\xi(t)$ denotes parameters determining the dependence of the perturbation on time. With this in view, we additionally assume that the switch fulfills the conditions $\xi(0)=\dot{\xi}(0)=0$. The problem can be solved under other conditions which implies an instantaneous switching of the perturbation followed by its adiabatic change.

Therefore, we take that the equation for coefficients A_k has the following form

$$c_k|_{t=0}=\delta_{ks}=A_k-\varepsilon\sum_m{}' \left[\frac{v_{km}(\tau)}{\Omega_{km}(\tau)}\right]_{t=0} A_m\,. \tag{6.59}$$

Obviously the matrix elements $v_{km}\sim\dot{\xi}(\tau)$, thus under the adopted conditions we obtain $A_{ks}=\delta_{ks}$, that is

$$c_{ks}^{(2)}=\delta_{ks}\exp(-i\alpha_k)-\varepsilon\frac{v_{ks}}{\Omega_{ks}}\exp(-i\alpha_s+i\Omega_{ks}t)\,. \tag{6.60}$$

Finally, the second (post-adiabatic) approximation $\Psi^{(2)}$ for the wave function is as follows

$$\Psi^{(2)}(q,t)=\psi_s\exp\left[-i\int_0^t\Omega_s(z)dz\right]-$$
$$\sum_k{}'\frac{v_{ks}}{\Omega_{ks}}\psi_k\exp\left[i\Omega_{ks}t-i\alpha_s-i\int_0^t\Omega_k(z)dz\right]\,. \tag{6.61}$$

It can be proved, see [59], that the estimate $\left|\Psi(q,t)-\Psi^{(2)}(q,t)\right|<B\varepsilon^2$ is valid for the time interval $t\sim 1/\varepsilon$. Papers by physicists do not take into account the boundedness of the time interval in which an approximate solution approximates the exact solution. As one can see from the forthcoming examples, this interval, in general, can not be enlarged.

The problem of constructing the adiabatic approximation and taking account of the transition through the virtual levels was posed by Dykhne: "In order to obtain the correct factor of the exponential function it would be necessary to take into account all higher approximations of the perturbation theory, all yielding results of the same order. In practice, this is, of course, not feasible. The obtained formulae give answers to the question of calculating the probability of transition of a quantum system to an "adjacent" level. As for transitions to more remote levels, then the transitions through virtual levels may compete with the considered process of the "direct" transition. However, this question needs an additional investigation", [27]. Nevertheless, this problem has not been solved so far by the existing methods of perturbation theory.

6.9 Quantum linear oscillator in a variable homogeneous field

In order to compare the obtained results with the known ones, let us study the following problem having an exact solution. The situation considered is the motion of a particle in the field of a parabolic potential subjected to a variable external force, i.e.

$$i\hbar\frac{\partial\Psi(q,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{m\omega^2 q^2}{2} - e\varepsilon_0 f(\nu t)q\right]\Psi(q,t) \tag{6.62}$$

$$\Psi(q,t)|_{t=0} = \pi^{-1/4}a^{1/2}\exp\left[-\left(q/a\sqrt{2}\right)^2\right], \quad \Psi(q,t)|_{q\to\pm\infty} - \text{bounded},$$

where m and e denote the mass and charge of the oscillator, ε_0 is amplitude of the electric field, ν^{-1} denotes a characteristic time constant of the field, $a = (\hbar/m\omega)^{1/2}$ denotes a characteristic length scale. Clearly, the initial state is the main state of the free harmonic oscillator. Let us introduce the non-dimensional time $t_n = \omega t$, the non-dimensional coordinate $x = q/a$ and the non-dimensional force amplitude $\varepsilon_1 = e\varepsilon_0 a/\hbar\omega = \Omega/\omega$. Problem (6.62) in the terms of non-dimensional variables takes the form

$$\begin{aligned} i\frac{\partial\Psi(x,t)}{\partial t} &= \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2 - \xi(\varepsilon t)x\right]\Psi(x,t), \\ \Psi(x,t)|_{t=0} &= \pi^{-1/4}\exp\left(-x^2/2\right) \quad , \quad \Psi(x,t)|_{x\to\pm\infty} - \text{bounded}, \end{aligned} \tag{6.63}$$

where $\varepsilon = \nu/\omega$, $\xi(\varepsilon t) = \varepsilon_1 f(\varepsilon t)$ and the non-dimensional time is denoted by t.

The exact solution of this problem is given by, cf. [25],

$$\begin{aligned} \Psi_0(x,t) &= \pi^{-1/4}\exp\left\{-\frac{it}{2} + i\int_0^t dz\delta^2(z)\exp(-2iz) - \frac{x^2}{2} - \right. \\ &\quad \left. \sqrt{2}x\delta(t)\exp(-it)\right\}, \end{aligned} \tag{6.64}$$

$$\delta(t) = -\frac{i}{\sqrt{2}}\int_0^t \xi(\varepsilon z)\exp(iz)dz \tag{6.65}$$

for arbitrary values of the parameters ε and ε_1.

In order to construct expansion (6.64) in the adiabatic case ($0 < \varepsilon << 1$) we integrate eq. (6.65) by parts three times and take into account that $\xi(0) = \dot{\xi}(0) = \ddot{\xi}(0) = 0$. The result is

$$\delta(t) = -\frac{1}{\sqrt{2}}\left[\xi + i\xi - \dot{\xi}\right]\exp(it) - \frac{i}{2}\int_0^t \exp(iz)\dddot{\xi}(z)dz\,. \tag{6.66}$$

Taking $M = \max\left|\ddot{\xi}(t)\right|$ in time interval $[0, T]$ we obtain the following estimate

$$\left|\int_0^T \exp(iz)\dddot{\xi}(z)dz\right| \leqslant \int_0^T \left|\dddot{\xi}(z)\right| dz \leqslant MT \sim \varepsilon^3 T\,. \tag{6.67}$$

It is clear from this equation that one can neglect the latter term in eq. (6.66) within time interval $\Delta t \sim T \sim 1/\varepsilon$. For this reason, it is necessary to keep the second order term $\ddot{\xi}$ in eq. (6.66) for evaluating integral (6.64) and omit it by substituting without integration.

With the same accuracy we evaluate the following integral

$$i\int_0^t dz\exp(-2iz)\delta^2(z) = i\int_0^t \frac{\xi^2}{2}dz - \frac{\xi^2}{2} - i\xi\dot{\xi} + i\int_0^t \frac{\dot{\xi}^2}{2}dz \tag{6.68}$$

and the expansion of the exact solution (6.64) as a series in time interval $t \sim 1/\varepsilon$

$$\begin{aligned}\Psi_0(x,t) \quad = \quad & \left[\Psi_0(x-\xi) + i\dot{\xi}\Psi_1(x-\xi)\right]\exp\left\{-\frac{it}{2} + \right. \\ & \left. i\int_0^t \frac{\xi^2}{2}dz + i\int_0^t \frac{\dot{\xi}^2}{2}dz\right\} + O\left(\varepsilon^2\right)\,.\end{aligned} \tag{6.69}$$

Next, we construct the expansion of the exact solution (6.64) in the case of the harmonic non-resonant perturbation ($\xi(t) = \varepsilon_1 \sin\nu t, 0 < \varepsilon_1 << 1,\ \nu \neq O(\varepsilon_1),\ \nu \neq 1$). In this case

$$\delta(t) = -\varepsilon_1\frac{i}{\sqrt{2}(\nu^2-1)}\left[(i\sin\nu t - \nu\cos\nu t)\exp(it) + \nu\right] \tag{6.70}$$

$$i\int_0^t dz\delta^2(z)\exp(-2it) = -i\frac{\varepsilon_1^2}{4(\nu^2-1)}t + O\left(\varepsilon_1^2\right) \tag{6.71}$$

The terms omitted in eq. (6.71) are uniformly bound such that approximation (6.71) is valid at any time instant. Inserting eqs. (6.70) and eq. (6.71) into eq. (6.64) yields the following expansion of the exact solution

$$\Psi_0(x,t) = \pi^{-1/4}\exp\left\{-\frac{it}{2} - i\frac{\varepsilon_1^2}{4(\nu^2-1)}t - \frac{x^2}{2} + x\frac{i\varepsilon_1}{\nu^2-1}(i\sin\nu t - \nu\cos\nu t) + \exp(-it)\frac{i\varepsilon_1\nu}{\nu^2-1}x\right\} = \left\{\left[\psi_0 + \frac{i\varepsilon_1}{\sqrt{2}(\nu^2-1)}\psi_1(i\sin\nu t - \nu\cos\nu t)\exp\left(-\frac{it}{2}\right) + \frac{i\varepsilon_1\nu}{\sqrt{2}(\nu^2-1)}\psi_1\exp\left(-i\frac{3t}{2}\right)\right]\right\}\exp\left[-i\frac{\varepsilon_1^2}{4(\nu^2-1)}t\right], \tag{6.72}$$

where ψ_0 and ψ_1 are the eigenfunctions of the unperturbed Schrödinger's operator ($\xi(t) = \varepsilon_1\cos t$, $0 < \varepsilon_1 << 1$). In this case

$$\delta(t) = -\frac{i\varepsilon_1}{2\sqrt{2}}(\exp(it)\sin t + t), \tag{6.73}$$

$$i\int_0^t \delta^2(z)\exp(-2iz)dz = -i\frac{\varepsilon_1^2}{8}\left[\frac{1}{2}\left(t - \frac{\sin 2t}{2}\right) + t^2\exp(-it)\sin t\right], \tag{6.74}$$

so that within non-dimensional time interval $t \sim 1/\varepsilon$ the expansion of the exact solution with accuracy up to the first order of smallness has the form

$$\Psi_0(x,t) = \pi^{-1/4}\exp\left\{-\frac{it}{2} - \frac{x^2}{2} - \left(\frac{\varepsilon_1 t}{4}\right)^2 + ix\frac{\varepsilon_1 t}{2}\exp(-it) + \left(\frac{\varepsilon_1 t}{4}\right)^2\exp(-2it)\right\}. \tag{6.75}$$

6.10 Charged linear oscillator in an adiabatic homogeneous field

In order to demonstrate application of the suggested theory, let us solve the problem of motion of a particle in the field of a parabolic potential subjected to a variable external force. Let $\varepsilon = \nu/\omega$ be a small parameter ($0 < \varepsilon << 1$) in the adiabatic case. Let us also take that parameter ε_1 is of order of unity. The perturbation operator $\xi(\varepsilon t)x$ can not be taken to be small in any way, cf. [87], that is, the problem of the perturbation theory can not principally be solved by the methods of spectral analysis of operators.

To make calculations more transparent, it is worthwhile carrying out the calculations from the very beginning rather than to use the resulting formula (6.61). For a fixed time instant t problem (6.44) is written in the form

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{2}x^2-\xi(\varepsilon t)x\right]\Psi_n(x,t)=E_n(t)\Psi_n(x,t) \tag{6.76}$$

and has the following solution

$$\begin{aligned}\Psi_n(x,t) &= (2^n\sqrt{\pi}n!)^{-1/2}\exp\left[-(x-\xi(\varepsilon t))^2/2\right]H_n\left(x-\xi(\varepsilon t)\right),\\ \omega_n &= n+1/2-\xi^2/2,\end{aligned} \tag{6.77}$$

where $H_n(z)$ denote Hermite polynomials [54], [92].

Using the recurrent relationships for Hermite polynomials it is easy to calculate the matrix elements $v_{mn}(t)$ for $m\neq n$

$$v_{mn}(t)=-\left(i\dot{\xi}/2\right)\left[\sqrt{n+1}\delta_{m,n+1}-\sqrt{n}\delta_{m,n-1}\right] \tag{6.78}$$

where a dot denotes the total derivative with respect to time, so that $\dot{\xi}\sim\varepsilon$ and $\delta_{m,n}$ denotes Kronecker's delta. Since the eigenfunctions (6.77) are real-valued, the diagonal matrix elements $v_{nn}=0$, i.e. $\Omega_n=\omega_n=n+1/2-\xi^2/2$.

With the help of eq. (6.78) we can easily calculate values $\Delta\Omega_k=\Delta\omega_k=-\dot{\xi}/2$ and coefficients $\bar{c}_k$, see eq. (6.56)

$$\bar{c}_k=A_k\exp\left\{i\int\left[\dot{\xi}^2(\varepsilon z)/2\right]dz\right\}. \tag{6.79}$$

In the case under consideration $\langle H_1\rangle=0$ and $\tilde{H}_1=H_1$, then we find $\{H_1\}$ and $\dfrac{\partial\{H_1\}}{\partial\bar{c}_k^*}$

$$\begin{aligned}\{H_1\} &= -i\sum_{n,m}{}'\frac{v_{mn}(t)}{\omega_{mn}(t)}\bar{c}_n\bar{c}_m^*\exp\left[i\omega_{mn}(\tau)t\right],\\ \frac{\partial\{H_1\}}{\partial\bar{c}_k^*} &= -\sum_n{}'\frac{v_{kn}(t)}{\omega_{kn}(t)}\bar{c}_n\exp\left[i\omega_{kn}(\tau)t\right].\end{aligned} \tag{6.80}$$

Making use of relationships (6.80) we obtain

$$c_k^{(2)}=A_k^{(2)}\exp\left[i\int_0^t\frac{\dot{\xi}^2}{2}dz\right]-\sum_n{}'\frac{v_{kn}(t)}{\omega_{kn}(t)}A_n\exp\left[i\int_0^t\frac{\dot{\xi}^2}{2}dz+i\omega_{kn}t\right]. \tag{6.81}$$

The equations for determining the integration constants have the form

$$c_k^{(2)}\Big|_{t=0} = \delta_{k0} = A_k - \sum_n{}' \left(\frac{v_{kn}(t)}{\omega_{kn}(t)}\right)_{t=0} A_n \,. \tag{6.82}$$

Assuming, as above, that $\dot{\xi}(0) = v_{kn}(0) = 0$, we obtain $A_k = \delta_{k0}$ and

$$c_k^{(2)} = \delta_{k0} \exp\left[i\int_0^t \frac{\dot{\xi}^2}{2} dz\right] - \frac{v_{k0}}{\omega_{k0}} \exp\left[i\int_0^t \frac{\dot{\xi}^2}{2} dz + i\omega_{k0} t\right] \,. \tag{6.83}$$

As follows from eq. (6.78) $v_{k0} = -i\dot{\xi}/\sqrt{2}$, thus

$$c_k^{(2)} = \left[\delta_{k0} + i\frac{\dot{\xi}}{\sqrt{2}}\delta_{k1}\exp(it)\right] \exp\left[i\int_0^t \frac{\dot{\xi}^2(z)}{2} dz\right] \,. \tag{6.84}$$

Inserting these equalities into expansion (6.46) yields the post-adiabatic approximation

$$\Psi^{(2)}(x,t) = \left[\Psi_0(x-\xi) + \frac{i\dot{\xi}}{\sqrt{2}}\Psi_1(x-\xi)\right] \exp\left\{-\frac{it}{2} + i\int_0^t \frac{\xi^2}{2} dz + i\int_0^t \frac{\dot{\xi}^2}{2} dz\right\} \tag{6.85}$$

which coincides with the expansion of the exact solution (6.69) within time interval $\Delta t \sim 1/\varepsilon$, approximation (6.85) not being applicable for increased times.

As follows from eq. (6.85), the probability of the oscillator excitation is equal to $\dot{\xi}^2/2$, i.e. it is zero at extreme points of function $\xi(t)$. For a Gaussian distribution, $\xi(t) \sim \exp(-\tau^2)$ and the excitation probability has its only maximum at $\tau = 1/\sqrt{2}$.

In the traditional Born-Fock approximation, [54], [97], the expansion coefficient corresponding to eq. (6.81) is given by

$$c_{kn} = \delta_{kn} + \int_0^t \frac{1}{\omega_{kn}(t')}\left(\frac{\partial V}{\partial t}\right)_{kn} \exp\left[i\int_0^t \omega_{kn}(t'')dt''\right] dt', \tag{6.86}$$

where $V(x,t) = -\xi(t)x$. Carrying out simple manipulations we obtain

$$c_{k0} = \delta_{k0} - \frac{\delta_{k1}}{\sqrt{2}} \int_0^t \dot{\xi}\exp(it')dt' \,. \tag{6.87}$$

Comparing the latter equation with eq. (6.84) indicates that the factor $\exp\left[i\int_0^t \left(\dot{\xi}^2(z)/2\right) dz\right]$ is absent in eq. (6.87). Equations (6.87) and (6.84) coincide at time instant $t \sim 1$, when a change in $\dot{\xi}(\varepsilon t)$ in the integrand can be neglected and there are no conditions $\xi(0) = \dot{\xi}(0) = 0$ under which approximation (6.84) is valid.

6.11 Adiabatic perturbation theory

In the case of the adiabatic perturbation theory, Schrödinger's operator is as follows $\hat{H} = \hat{H}_0 + \varepsilon \hat{V}(q, \tau)$ which allows us to carry out calculations by using the results of Sec.6.6. However, as mentioned in Sec. 6.2 it is more convenient to apply the formalism of the stationary phase perturbation theory.

Indeed, by casting problem (6.62) in terms of the eigenfunctions of unperturbed Schrödinger's operator $\hat{H}_0$ we find the effective Hamilton function

$$\varepsilon H_1(c, c^*, t, \tau) = -i\varepsilon \sum_{n,m}{}' v_{mn}(\tau) c_n c_m^* \exp\left(i\Omega_{mn}^0 t\right), \tag{6.88}$$

where $v_{mn}(\tau)$ denote matrix elements of the perturbation operator which are calculated by means of the unperturbed eigenfunctions, further $\Omega_n^0 = \omega_n^0 + \varepsilon \int_0^t v_{nn}(\tau) d\tau$ and $\Omega_{mn}^0 = \Omega_m^0 - \Omega_n^0$, ω_n^0 denote the eigenvalues of the unperturbed problem.

In this case formulae (6.52) remain valid since the dependence of the effective Hamilton function on slow time τ is observed only in terms of the third order of smallness.

For problem (6.62) we have

$$\begin{aligned} \Psi_n^0(x) &= \left(2^n \sqrt{\pi} n!\right)^{-1/2} \exp\left[-x^2/2\right] H_n(x), \quad \omega_n^0 = n + 1/2, \\ v_{mn}(\tau) &= -\frac{\varepsilon_1 \xi(\tau)}{\sqrt{2}} \left[\sqrt{\frac{n+1}{2}} \delta_{m,n+1} + \sqrt{\frac{n}{2}} \delta_{m,n-1}\right], \\ v_{nn}(\tau) &= 0, \quad \Omega_n^0 = \omega_n^0 = n + 1/2, \quad \Delta\omega_k = -\frac{1}{2} \varepsilon_1^2 \xi^2(\tau), \end{aligned} \tag{6.89}$$

with parameter ε_1 being a small value of order of ε.

By virtue of the latter relationship in eq. (6.89) we easily find the second approximation to the expansion coefficients $c_n^{(2)}$ (under the conditions $\xi(0) = \dot{\xi}(0) = 0$)

$$c_n^{(2)} = \exp\left[i\varepsilon_1^2 \int_0^t \frac{\xi^2}{2} dz\right] \delta_{k0} \tag{6.90}$$

and the second approximation to the wave function

$$\Psi^{(2)} = \psi_0(x) \exp\left[-\frac{it}{2} + i\int_0^t \frac{\xi^2}{2} dz\right] , \tag{6.91}$$

which coincides with adiabatic expansion (6.85) when we take into account a small factor at $\dot{\xi}$ due to parameter ε_1.

6.12 Harmonic excitation of a charged oscillator. Non-resonant case

To demonstrate the way of constructing solutions in the case of non-stationary perturbation theory, we consider the case of a harmonic external field which is frequently encountered in practical applications. In this case function $\xi(t)$ in eq. (6.62) takes the form $\xi(t) = \varepsilon_1 \sin \nu t$ and we can adopt that $\nu \neq 1$ and $\nu \neq O(\varepsilon)$. In other words, we consider the non-resonant case and parameter ε_1 is taken as having the order of smallness of parameter ε.

Let us carry out the corresponding calculations in three ways: first, stationary phase perturbation theory, second, non-stationary phase perturbation theory developed here and finally, traditional method of [54].

In the first case the matrix elements of the perturbation operator, calculated by means of the unperturbed eigenfunctions, have the form

$$\begin{aligned} v_{mn}(t) &= \varepsilon_1 (v^0_{mn} \exp(i\nu t) - v^0_{mn} \exp(-i\nu t)) , \\ v^0_{mn} &= \frac{i}{2\sqrt{2}} \left[\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right] . \end{aligned} \tag{6.92}$$

The effective Hamilton function is given by

$$\begin{aligned} \varepsilon_1 H_1(c, c^*, t) &= -i\varepsilon_1 \sum_{n,m}^{\infty}{}' v^0_{mn} \exp\left[i(\omega^0_{mn} + \nu)t\right] c_n c^*_m + \\ & \quad i\sum_{n,m}{}' v^0_{mn} \exp\left[i(\omega^0_{mn} - \nu)t\right] c_n c^*_m . \end{aligned} \tag{6.93}$$

In this case $\bar{H}_1 = \langle H_1 \rangle = 0$, so that $H_1 = \tilde{H}_1$. The correction of the second order $\bar{H}_2$ to the averaged Hamilton function is as follows

$$\bar{H}_2 = -i\frac{\varepsilon_1^2}{4(\nu^2-1)} \sum_n \bar{c}_n \bar{c}^*_n . \tag{6.94}$$

Next we find the second approximation for variables $c_n^{(2)}$

$$c_n^{(2)} = \delta_{n0} - i\frac{\varepsilon_1 \nu}{\sqrt{2}(1-\nu^2)}\delta_{n1} - i\frac{\varepsilon_1}{\sqrt{2}}\delta_{n1}\left[i \sin \nu t - \nu \cos \nu t\right] \exp(it) . \tag{6.95}$$

Taking into account these relationships we obtain the second approximation $\Psi^{(2)}(x,t)$ to the solution of the non-stationary problem (6.62)

$$\Psi^{(2)}(x,t) = \left\{\left[\Psi_0 + \frac{i\varepsilon_1}{\sqrt{2}(\nu^2-1)}\Psi_1(i\sin\nu t - \nu\cos\nu t)\right]\exp\left(-\frac{it}{2}\right) + \frac{i\varepsilon\nu}{\sqrt{2}(\nu^2-1)}\Psi_1\exp\left(-i\frac{3t}{2}\right)\right\}\exp\left[-i\frac{\varepsilon_1^2}{4(\nu^2-1)}t\right]. \quad (6.96)$$

This result suggests that the spectrum remains equidistant and is only subjected to a common shift $\Delta\omega = \varepsilon_1^2/4\left(\nu^2-1\right)$. In this case $\Delta\omega < 0$ for $\nu < 1$ ($\omega < \omega_0$, ω being the frequency of excitation force), and $\Delta\omega > 0$ for $\nu > 1$ ($\omega > \omega_0$). The shift $\Delta\omega$ at $\omega \to 0$ corresponds to effective elevation of the bottom of the potential well. In the case of a high frequency external field ($\nu >> 1$) $\Delta\omega = \varepsilon_1^2/4\nu^2$ that coincides with the effective potential energy. Hence, the constructed expansion (6.96) coincides with the expansion of the exact solution (6.72).

Let us now construct a solution by using formulae of the non-stationary phase perturbation theory. In this case the matrix elements (6.48) calculated by means of eigenfunctions (6.77) of the instantaneous Schrödinger's operator have the form

$$v_{mn}(\tau) = -\frac{i\dot{\xi}}{\sqrt{2}}\left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right) \quad (6.97)$$

or

$$\begin{aligned} v_{mn}(t) &= v_{mn}^0\exp(i\nu t) + v_{mn}^0\exp(-i\nu t), \\ v_{mn}^0 &= -i\frac{\varepsilon_1\nu}{2\sqrt{2}}\left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right). \end{aligned} \quad (6.98)$$

As the eigenfunctions are real, then $v_{nn} = 0$ and $\Omega_n = \omega_n = n + \frac{1}{2} - \frac{\xi^2}{2}$.

The effective Hamilton function is as follows

$$\varepsilon_1 H_1(c,c^*,t) = -i\varepsilon_1\sum_{n,m}{}' \left[v_{mn}^0\exp\left[i(\omega_{mn}^0+\nu)t\right] + v_{mn}^0\exp\left[i(\omega_{mn}^0-\nu)t\right]\right]c_n c_m^*. \quad (6.99)$$

It is evident that $\bar{H}_1 = \langle H_1\rangle = 0$, so that $H_1 = \tilde{H}_1$. The correction $\bar{H}_2$ to the averaged Hamilton function, calculated by formulae (6.52) with the help of eq. (6.98), is given by

$$\bar{H}_2 = -i\frac{\nu^2}{4(\nu^2-1)}\sum_k \bar{c}_k\bar{c}_k^*. \quad (6.100)$$

The second approximation $\bar{c}_k$ for the evolutionary components of variables c_k is found from Hamilton's equations with Hamilton's function $\bar{H}^{(2)} =$

$\varepsilon_1 \bar{H}_1 + \varepsilon_1^2 \bar{H}_2$ and has the form

$$\bar{c}_k = A_k \exp\left[-i\frac{\varepsilon_1^2\nu^2}{4(\nu^2-1)}t\right] . \tag{6.101}$$

Constants A_k should be obtained from the initial conditions.

The second approximation $c_k^{(2)}$ to the original variables c_k is as follows

$$c_k^{(2)} = \left\{A_k - \varepsilon_1 \sum_n v_{kn}^0 A_n \left[\frac{\exp\left[i\left(\omega_{kn}+\nu\right)t\right]}{\omega_{kn}+\nu} + \frac{\exp\left[i\left(\omega_{kn}-\nu\right)t\right]}{\omega_{kn}-\nu}\right]\right\} \exp\left[-i\frac{\varepsilon_1^2\nu^2}{4(\nu^2-1)}t\right] . \tag{6.102}$$

For determining the integration constants A_k from eq. (6.102) we obtain the following equation

$$c_k\big|_{t=0} = \delta_{k0} = A_k^{(2)} - \varepsilon_1 \sum_n v_{kn}^0 A_n^{(1)} \left[\left(\omega_{kn}+\nu\right)^{-1} + \left(\omega_{kn}-\nu\right)^{-1}\right] , \tag{6.103}$$

where $A_k^{(1)}$ and $A_k^{(2)}$ are the first and the second approximations to coefficients A_k with respect to parameter ε_1. As the first approximation we can take $A_k^{(1)} = \delta_{k0}$, then

$$A_k^{(2)} = \delta_{k0} - i\frac{\varepsilon_1\nu}{\sqrt{2}(1-\nu^2)}\delta_{k1} , \tag{6.104}$$

where it is taken into account that $v_{k0}^0 = -i\varepsilon_1\nu\delta_{k1}/2\sqrt{2}$ and the final expression for $c_k^{(2)}$ takes the form

$$c_k^{(2)} = \left\{\delta_{k0} + i\frac{\varepsilon_1\nu}{\sqrt{2}(\nu^2-1)}\delta_{k1} + i\frac{\varepsilon_1\nu\exp(it)}{\sqrt{2}(\nu^2-1)}\delta_{k1}[\cos\nu t - i\nu\sin\nu t]\right\} \exp\left[-i\frac{\varepsilon_1^2\nu^2}{4(\nu^2-1)}t\right] . \tag{6.105}$$

Then we find the expression for phases in expansion (6.46)

$$-i\int_0^t \omega_n(t)dt = i\frac{\varepsilon^2}{4}t - i(n+\frac{1}{2})t + O(\varepsilon^2). \tag{6.106}$$

Inserting eqs. (6.105) and (6.106) into expansion (6.46) we finally obtain

$$\Psi^{(2)}(x,t) = \left\{\left[\Psi_0 + \frac{i\varepsilon_1}{\sqrt{2}(\nu^2-1)}\Psi_1(i\sin\nu t - \nu\cos\nu t)\right]\exp\left(-\frac{it}{2}\right) + \frac{i\varepsilon_1\nu}{\sqrt{2}(\nu^2-1)}\Psi_1\exp\left(-i\frac{3t}{2}\right)\right\}\exp\left[-i\frac{\varepsilon_1^2}{4(\nu^2-1)}t\right] \tag{6.107}$$

coinciding with eqs. (6.96) and (6.72).

Finally, taking into account that the second approximation of the present analysis coincides with the first approximation of the traditional approach, we obtain, for the case $a_k = c_k$, by means of the standard formulae of the non-stationary perturbation theory, that

$$a_k = a_k^{(0)} + a_k^{(1)} = \delta_{k0} + i\frac{\varepsilon_1 \delta_{k1}}{\sqrt{2}(\nu^2 - 1)} (i \sin \nu t - \nu \cos \nu t) \exp(it) . \quad (6.108)$$

Comparison of eqs. (6.105) and (6.108) allows us to indicate a number of inaccuracies in the standard courses. Firstly, a phase multiplier $\exp[-i\varepsilon_1^2 \nu^2 t/4(\nu^2 - 1)]$, which is of crucial importance for investigation of the coherent processes, is absent in expression (6.108). It can be neglected within a non-dimensional time interval $t \sim 1$, but not within asymptotical intervals $t \sim 1/\varepsilon$. Thus, approximation (6.108) and in turn the whole solution is valid only within this small time interval.

Secondly, only the first approximation $a_k^{(1)} = \delta_{k0}$ rather than a_k is subject to the initial condition (6.103). This explains the absence of the term $i\varepsilon_1 \nu/\sqrt{2}(\nu^2 - 1)$. Indeed, by assuming $a_k^{(0)} = \delta_{k0} + \varepsilon_1 \tilde{a}_{k0}^{(0)}$ and subjecting the whole coefficient a_k to the initial condition $a_k|_{t=0} = \delta_{k0}$ we find

$$\tilde{a}_{k0}^{(0)} = i\frac{\nu}{\sqrt{2}(\nu^2 - 1)}\delta_{k1} . \quad (6.109)$$

Finally, as follows from solution (6.107) or (6.72) it is this absent correction (rather than $a_k^{(1)}$) that determines the probability of transition to the excited state.

6.13 Harmonic excitation of an oscillator. Transition through a resonance

Let us consider excitation of an oscillator by a weak resonant harmonic field $V(x, t) = -\varepsilon_1 x \cos \nu t$. In this case $1 - \nu = \varepsilon$ $(0 \ll \varepsilon < 1)$ and parameter ε_1 is a small value.

The effective Hamilton function constructed by means of the eigenfunctions of the unperturbed Schrödinger's operator has the form

$$\varepsilon H_1(c, c^*, t) = i\frac{\varepsilon_1}{2\sqrt{2}} \sum_n \left\{ \sqrt{n} c_n c_{n-1}^* \exp[-i\varepsilon t] + \sqrt{n+1} c_n c_{n+1}^* \exp[i\varepsilon t] \right\} +$$
$$i\frac{\varepsilon_1}{2\sqrt{2}} \sum_n \left\{ \sqrt{n} c_n c_{n-1}^* \exp[-i(\nu + 1)t] + \sqrt{n+1} c_n c_{n+1}^* \exp[i(\nu + 1)t] \right\} . \quad (6.110)$$

In the first approximation we obtain the averaged Hamilton function

$$\varepsilon \bar{H}_1 = i\frac{\varepsilon_1}{2\sqrt{2}} \sum_n \left\{ \sqrt{n}\bar{c}_n \bar{c}^*_{n-1} \exp\left[-i\varepsilon t\right] + \sqrt{n+1}\bar{c}_n \bar{c}^*_{n+1} \exp\left[i\varepsilon t\right] \right\} \quad (6.111)$$

and the equation of first approximation for the evolutionary components $\bar{c}_k$ of the original variables c_k takes the form

$$\dot{\bar{c}}_k = i\frac{\varepsilon_1}{2\sqrt{2}} \left[\sqrt{k+1}\bar{c}_{k+1} \exp\left(-i\varepsilon t\right) + \sqrt{k}\bar{c}_{k-1} \exp\left(i\varepsilon t\right) \right] . \quad (6.112)$$

It is easy to prove by direct differentiation that the solution of the equation

$$\dot{c}_k = i\frac{\xi(t)}{\sqrt{2}} \left[\sqrt{k+1}c_{k+1} \exp\left(-i\varepsilon t\right) + \sqrt{k} \exp\left(i\varepsilon t\right) c_{k-1} \right]$$

has the following form

$$\begin{aligned} c_k(t) &= \frac{\left[-\delta(t)\right]^k}{\sqrt{k!}} c_0(t) , \\ c_0(t) &= \exp\left\{ i\varepsilon \int_0^t dz \delta^2(z) \exp(-2i\varepsilon z) + \frac{1}{2}\delta^2(t) \exp(-2i\varepsilon t) \right\} , \\ \delta(t) &= -\frac{i}{\sqrt{2}} \int_0^t \xi(z) \exp(i\varepsilon z) dz . \end{aligned} \quad (6.113)$$

Using this solution one can investigate the transition of the system through the resonance.

We restrict our further investigation to the case of exact resonance ($\varepsilon = 0$). Using relationship (6.113) we find coefficients $\bar{c}_k$

$$\bar{c}_k(t) = \frac{1}{\sqrt{k!}} \left(\frac{i\varepsilon_1 t}{2\sqrt{2}} \right)^k \exp\left[-\left(\frac{\varepsilon_1 t}{4} \right)^2 \right]$$

and solution $\Psi(x,t)$ in the first approximation

$$\begin{aligned} \Psi(x,t) &= \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} \left(\frac{i\varepsilon_1 t}{2\sqrt{2}} \right)^k \exp\left(-\frac{\varepsilon_1 t}{4} \right)^2 (2^k \sqrt{\pi} k!)^{-1/2} \times \\ & \exp\left(-\frac{x^2}{2} \right) H_k(x) \exp\left[-i\left(k + \frac{1}{2} \right) t \right] . \end{aligned} \quad (6.114)$$

Carrying out summation in eq. (6.114) by means of the generating function for Hermite polynomials [92]

$$\exp\left(2xz - z^2\right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} H_k(x), \quad (6.115)$$

we obtain the final solution for the case of exact resonance

$$\Psi(x,t) = \pi^{-1/4}\exp\left\{-\frac{it}{2}-\frac{x^2}{2}-\left(\frac{\varepsilon_1 t}{4}\right)^2+ ix\frac{\varepsilon_1 t}{2}\exp(-it)+\left(\frac{\varepsilon_1 t}{4}\right)^2\exp(-2it)\right\}. \qquad (6.116)$$

This solution coincides with the expansion of the exact solution (6.75).

The probability of excitation of the oscillator has the form of a Poisson distribution

$$w_n(t) = |c_n(t)|^2 = \frac{(\bar{n})^n}{n!}\exp(-\bar{n}), \qquad (6.117)$$

where $\bar{n} = \left(\varepsilon_1 t/2\sqrt{2}\right)^2$.

7

The problem of weak interaction of dynamical objects

7.1 The types of conservative interaction and criteria of their weakness

As shown in the previous chapter, the problem of the motion of dynamical systems which are conservative in the generating approximation and integrable by quadratures can be reduced to the investigation of equations with multi-dimensional fast rotating phase. The corresponding problem can then be treated as the problem of weak interaction of several fast phases, complicated by the presence of some slow variables. In this regard, it is desirable to select the class of problems which, by means of pure physical reasoning, can be deemed to be problems of the weak interaction of real dynamical objects from the very beginning. Indeed, only such problems can be reduced to the analysis of equations with a sufficiently large number of fast phases. When the weak physical interaction is absent, one has to consider in the generating approximation either the Keplerian problem of the motion of a particle about an attracting centre or a rigid body with an immovable point or some other integrable problem. Of course, all of these problems are of crucial importance. However their total number, as well as the order of the corresponding equations, are relatively low.

Besides, it is important that in the problems of weak physical interaction of an arbitrary number of relatively simple objects, one often succeeds in suggesting a rather general physical interpretation of the obtained results applicable to entire classes of problems of such types. Similar interpretations are most natural in cases in which the presence of the weak interaction

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between the objects does not eliminate the general conservative (or nearly conservative) nature of the dynamical system.

Taking into account the importance of the problems of weak interaction of nearly conservative objects for practical purposes, let us proceed to a rather general description of such objects.

Let there be given a system of n dynamical objects weakly interacting with each other. The motion of a generic $i-th$ object in the system is characterised by $l_i \times 1$ vector-column of the "own" generalised coordinates $q_i = (q_{i1}, \dots, q_{il_i})$. The way of introducing the own coordinates is assumed to be independent of the character of the interaction between the objects and is not absolutely arbitrary. Thus the own coordinates must keep their physical meaning even in the case of no interaction. Hence, the character of the dependence of the dynamic characteristics of motion of the object determined by means of their own generalised coordinates and velocities on these values is invariant with respect to the interaction type. For example, the kinetic and potential energies of the $i-th$ object

$$K_i = \frac{1}{2}\dot{q}_i' A_i(q_i)\,\dot{q}_i, \quad \Pi_i = \Pi_i(q_i) \tag{7.1}$$

have the form of the corresponding counterparts in the case of no interaction. Here A_i denotes a $l_i \times l_i$ symmetric matrix of the inertial coefficients and a prime denotes the transpose operation.

When the interactions in the system are conservative in the above sense, the objects in the system gain a certain additional mobility such that a $l \times 1$ vector of additional generalised coordinates $x = (x_1, \dots, x_l)$ is needed to describe their motion. In this case, the total kinetic and potential energies of the system are given by

$$K^* = \sum_{i=1}^{n} K_i + \Delta K, \quad \Pi^* = \sum_{i=1}^{n} \Pi_i + \Delta\Pi\,. \tag{7.2}$$

Let us expand the expressions for the additional kinetic and potential energies of the objects as series in terms of x and $\dot{x}$, then in the linear approximation we obtain

$$\begin{aligned} \Delta K &= \sum_{i=1}^{n} \dot{q}_i' A_{li}(q_1, \dots, q_n)\,\dot{x} + \dots, \\ \Delta\Pi &= x'C(q_1, \dots, q_n, t) + \dots, \end{aligned} \tag{7.3}$$

where A_{li} and C denote the rectangular $l_i \times l$ matrix and $l \times 1$ vector respectively. An explicit dependence of the component of C on time characterises a certain mechanism of exerting an external perturbation, caused by perturbation, on the objects.

In what follows, we relate a set of auxiliary generalised coordinates $x_1, \dots, x_l$ to the concept of a carrying body or a carrying system of bodies with l degrees of freedom, oscillatory nature being assumed. Then, the interactions due to the carrying system provide it with a certain additional mobility. Interactions of this type are referred to as carrying interactions or interactions of the first kind.

The kinetic and potential energies of the carrying system are cast up to terms quadratic in the components of x and $\dot{x}$ as follows

$$K^{(1)} = \frac{1}{2}\dot{x}'M\dot{x} + \dots, \quad \Pi^{(1)} = \frac{1}{2}x'Cx + \dots. \tag{7.4}$$

Here M and C are positive definite $l \times l$ matrices with constant coefficients.

In a number of problems, the carrying system has distributed parameters and thus has an infinite number of degrees of freedom. We do not dwell on all of the difficulties associated with this fact and we will assume that it is possible to introduce normal coordinates for describing small "own" oscillations of the carrying system. In other words, for any l (in particular $l = \infty$) the coordinates $x_1, \dots, x_l$ can be taken in such a way that matrices M and C become diagonal. This enables us to make the forthcoming analysis independent of the number of degrees of freedom of the carrying system.

Carried interactions or interactions of the second kind are of a different nature. Their presence does not increase mobility of the objects. In order to describe the motion of the inertial elements of the corresponding carried system in the general case it is necessary to ascribe a $m \times 1$ vector $y = (y_1, \dots, y_m)$ in addition to the introduced quantities $x, q_1, \dots, q_l$. The kinetic and generalised potential energies of the carried system depend on all the generalised coordinates and velocities of the system of interacting objects. In what follows we use $K^{(2)}(q, \dot{q}, y, \dot{y})$ and $\Pi^{(2)}(q, \dot{q}, y, \dot{y})$ which correspond the immovable carrying system $x = \dot{x} = 0$. Using these expressions allows us to describe how the external perturbation acts on the objects via elements of the carried system.

Let us assume that, due to the weakness of interaction, we can introduce a certain small parameter ε such that the interaction is absent for $\varepsilon = 0$. The scales of the physical parameters for all objects are assumed to coincide. Then the dynamical and kinematical characteristics of the motion for any object have the same order which determines the scales of displacement, velocity, mass, rigidity etc. ($q, \dot{q}, A, \Pi_i, C = O(1)$, see eqs. (7.1)-(7.3)). In other words, the mass of any object, the rigidity of any of its elastic elements and the characteristic velocity are of the order of the scales of mass, rigidity and velocity.

The smallness of the parameter of interaction implies that for the class of motions under consideration the general dynamical characteristics change by values of the order ε when the interaction is considered. Thus, the kinetic and potential energies become

$$K = K^* + K^{(1)} + K^{(2)} = \sum_{i=1}^{n} K_i + O(\varepsilon),$$
$$\Pi = \Pi^* + \Pi^{(1)} + \Pi^{(2)} = \sum_{i=1}^{n} \Pi_i + O(\varepsilon). \tag{7.5}$$

We list the more natural conditions for estimates (7.5) by using the idea of the formal indicator μ of smallness of order ε, see Sec. 5.1.

1. Since the components of the $l_i \times l$ matrix A_{li} and the $l_i \times 1$ vector C, see eq. (7.3), characterise the inertial and elastic properties of the $i-th$ object and thus have the order of unity, the estimates $\Delta K, \Pi = O(\varepsilon)$ are valid only if the values of the coordinates of the carrying system under motion are small, i.e. $x = \mu x$. This, in particular, determines the higher order of smallness of the omitted terms in expansions (7.3) and (7.4).

2. For the dynamic characteristics of the motion of the carrying system to be small, it is sufficient that

$$M = \frac{M}{\mu}, \quad C = \frac{C}{\mu}. \tag{7.6}$$

3. The inertial and force characteristics of the carried system are small, whereas the components of the $m \times 1$ vector y are, in general, of the order of unity.

Thus, the weakness of the interactions of the first kind is caused by the smallness of oscillations of the large carrying system. On the other hand, the weakness of the interactions of the second kind is due to the smallness of the generalised momenta of the carried system.

7.2 Examples of interactions of carrying and carried types

An illustrative example of the classification of the types of interaction introduced in the previous section is the system depicted in Fig. 7.1. The weak interactions of the potential character between two dynamical objects (harmonic oscillators) appear either due to mobility of the large carrying body ($m/M = O(\varepsilon)$) or due to weak force ($c_0/c = O(\varepsilon)$), inertial ($m_0/m = O(\varepsilon)$) or non-inertial interactions. Non-inertial interactions in the considered system result in increasing the total number of degrees of freedom. For example, this is the case for a finite rigidity of the inertialess hinged rods which determine the position of mass m_0.

In view of the above, we notice that in the forthcoming study the generalised coordinates are absolute coordinates, the coordinates of the interacting objects describe motion relative to the "frozen" carrying system,

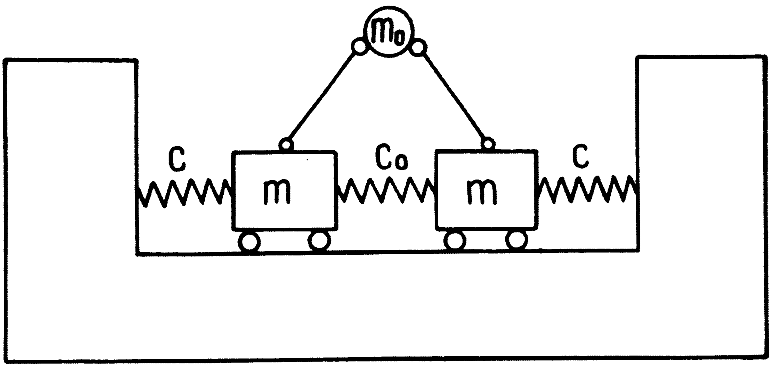

FIGURE 7.1.

and the auxiliary generalised coordinates of the inertial elements of the carried system describe motion relative to an arbitrary configuration of the carrying system and objects in space.

Interaction of both types manifests itself in the celebrated problem of motion of many bodies under gravitational attraction [33]. Assume that n particles ("planets") with masses $m_1, \ldots, m_n$ rotate about a "sun" of mass M. The position of the "sun" relative to the centre of mass of the whole system is prescribed by the radius vector r_0 whereas the positions of planets with respect to the "sun" are given by the radius vectors $r_1, \ldots, r_n$ (the barycentric system of coordinates). Without loss of generality, we can assume that the centre of mass does not move, hence the kinetic and potential energies of the system are written in the form

$$\begin{aligned} K &= \frac{1}{2}\left(M + \sum_{i=1}^{n} m_i\right)\dot{r}_0^2 + \sum_{i=1}^{n} m_i \dot{r}_i \cdot \dot{r}_0 + \frac{1}{2}\sum_{i=1}^{n} m_i \dot{r}_i^2, \\ \Pi &= -\frac{\gamma}{2}\left(M \sum_{i=1}^{n} \frac{m_i}{|r_i|} + \sum_{i,j=1}^{n} \frac{m_i m_j}{|r_i - r_j|}\right). \end{aligned} \tag{7.7}$$

Here γ denotes the gravitational constant which is equal to $0,296 \cdot 10^{-3}$ in the system of fundamental astronomic unities. Let us assume that $M \gg m_i$ $(i = 1, \ldots, n)$. Then, according to the results of the previous section

$$M = \frac{M}{\mu}, \quad m_i = O(1), \quad r_0 = \mu r_0, \quad r_i = O(1).$$

Additionally, the adopted system of unities allows us to take $\gamma = O(\varepsilon)$ $(\gamma = \mu\gamma)$. Then it is natural to take the "sun" as being the carrying body with kinetic energy

$$K^{(1)} = \mu \frac{M}{2} \dot{r}_0^2 \quad \left(\Pi^{(1)} = 0\right). \tag{7.8}$$

The proper kinetic and potential energies of the $i - th$ planet are as follows

$$K_i = \frac{m_i}{2}\dot{r}_i^2, \quad \Pi_i = -\frac{\gamma}{2}M\frac{m_i}{|r_i|}, \tag{7.9}$$

whereas the additional kinetic energy of the "planets" due to mobility of the "sun" takes the form

$$\Delta K = \mu \sum_{i=1}^{n} m_i \dot{r}_i \cdot \dot{r}_0, \quad (\Delta\Pi = 0)\,. \tag{7.10}$$

The specific feature of the considered simplified statement of the problem is that

$$M r_0 + \sum_{i=1}^{n} m_i r_i = 0, \tag{7.11}$$

and thus $2K^{(1)} + \Delta K = 0$, see Sec. 8.5.

Along with the carrying interactions, the carried interactions caused by the mutual interaction of the "planets" are considerable. Since the inertial elements are absent, the carried interactions are of pure force character and do not increase the total number of degrees of freedom. Their potential energy is given by

$$\Pi^{(2)} = -\mu\frac{\gamma}{2}\sum_{i,j=1}^{n}\frac{m_i m_j}{|r_i - r_j|}\,. \tag{7.12}$$

The generating motions of the "planets" are easily determined by integration of Kepler's equations. Determination of the consequences of weak interactions of the first and second kinds is one of the principal problems of celestial mechanics [5], [12].

To conclude we notice that in a system of unbalanced rotors (inertial vibration exciters) driven by independent asynchronous electric motors and mounted on a massive body or a system of elastically connected bodies, an internal resonance (synchronisation) may appear due to the weak carrying interactions between these bodies. Nowadays this phenomenon is extensively used in vibration technology [16].

7.3 Equations of motion in Routh's form

The previous sections are devoted to the interactions of pure conservative nature. However, in the general case the weak interactions between the objects can also occur as a result of the action of the generalised non-potential forces.

Let any dynamical object be subjected to a $l_i \times 1$ vectorial generalised force. Its dependence on the generalised coordinates and velocities of the system and possibly on the dynamical reaction forces is specified below when the particular problems are considered, see for instance Sec. 8.1. The order of smallness of the components of this generalised force is determined in what follows. Anyway, the weak interaction between the objects can also occur by means of these components. Additionally, we assume that small oscillations in the system are accompanied by the non-potential forces vanishing at $\dot{x} = 0$. With accuracy up to the terms of higher order, the generalised force of dimension $l \times 1$ is given by

$$Q_x = -B\dot{x} + \varepsilon \ldots , \tag{7.13}$$

the components of $l \times l$ matrix B being constant values of order $1/\varepsilon$. Let us assume that the quadratic form

$$\Phi = \frac{1}{2}\dot{x}' B_s \dot{x}, \tag{7.14}$$

where $B_s = \frac{1}{2}(B + B')$ denotes the symmetric part of matrix B, is positive definite and thus can be understood as Rayleigh's dissipation function. The skew-symmetric part $B_a = \frac{1}{2}(B - B')$ accounts for the existence of gyroscopic forces under oscillations of the carrying system.

In the case of weak interactions, the $m \times 1$ vectorial non-potential force corresponding to the coordinates of the carried system should be small ($Q_y = \mu Q_y$) and does not depend on the coordinates and velocities of the carrying system up to small values of higher order.

The above relationships are sufficient for the description of weak non-conservative interaction of carrying and carried types between the dynamical objects in continuous systems. A description of the piecewise-continuous systems is studied in what follows, see Sec. 8.6.

The equations of motion for the system of weakly interacting objects are conveniently cast in Routh's form which is usually introduced in the case of cyclic coordinates [1], [60]. To this end, we first introduce into consideration the vector-rows of the generalised momenta corresponding to the proper coordinates of the objects

$$p_i = \frac{\partial K}{\partial \dot{q}_i} = \dot{q}_i' A_i + \mu \ldots . \tag{7.15}$$

With accuracy up to the values of order ε included, the required Routhian function is written down in the form

$$R = K - \sum_{i=1}^{n} p_i \dot{q}_i = -\frac{1}{2}\sum_{i=1}^{n} p_i A_i^{-1} p_i' + \mu\left(\Delta K + K^{(1)} + K^{(2)}\right) + \mu^2 \ldots , \tag{7.16}$$

where the "proper" generalised coordinates in the equations for ΔK and $K^{(2)}$ are expressed in terms of the corresponding generalised momenta by eq. (7.15).

Equation (7.16) yields the following formula for Routh's kinetic potential

$$L_R = R - \Pi = -\sum_{i=1}^{n} H_i + \mu L_0 + \mu^2 \dots . \tag{7.17}$$

Here

$$H_i = \frac{1}{2} p_i A_i^{-1} p_i' + \Pi_i \tag{7.18}$$

denotes the "proper" Hamiltonian of the $i-th$ object, furthermore

$$L_0 = \Delta L + L^{(1)} + L^{(2)} \tag{7.19}$$

denotes the total kinetic potential of the weak interaction, $\Delta L = \Delta K - \Delta\Pi$ designates the additional kinetic potential of the objects caused by small oscillations of the carrying system, then

$$L^{(1)} = \frac{1}{2}\dot{x}' M \dot{x} - \frac{1}{2} x' C x \tag{7.20}$$

is the kinetic potential of the carrying system and $L^{(2)} = K^{(2)} - \Pi^{(2)}$ is the kinetic potential of the carried system.

Now it is easy to write down the general equations of weak interaction in Routh's form

$$\begin{aligned}
\dot{q}_i - \frac{\partial H_i}{\partial p_i} &= -\mu \frac{\partial}{\partial p_i}\left(\Delta L + L^{(2)}\right) + \mu^2 \dots , \\
\dot{p}_i + \frac{\partial H_i}{\partial q_i} &= \mu \frac{\partial}{\partial p_i}\left(\Delta L + L^{(2)}\right) + Q_i + \mu^2 \dots , \\
M x + B\dot{x} + Cx + \left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}'} - \frac{\partial}{\partial x'}\right)\Delta L + \mu \dots &= 0 , \\
\mu \left[\left(\frac{d}{dt}\frac{\partial}{\partial \dot{y}'} - \frac{\partial}{\partial y'}\right) L^{(2)} - Q_y\right] + \mu^2 \dots &= 0 .
\end{aligned} \tag{7.21}$$

These equations of motion for the objects are written with accuracy up to the terms of order ε included whereas the equations of motion for the carrying and carried systems contain only the terms of the lower order of smallness.

8

Synchronisation of anisochronous objects with a single degree of freedom

8.1 Eliminating coordinates of the carrying system

In this chapter we restrict our consideration to the weak interaction of the carrying and carried types of dynamic objects with one degree of freedom, that is $l_1 = \cdots = l_n = 1$. The generalised forces corresponding to the proper coordinates of the objects have the order of the small parameter of interaction ($Q_i = \mu Q_i$). This assumption conforms to the quasi-conservative concept and is opposed to the non-conservative concept for which the assumption $Q_i = O(1)$ is typical, see Sec. 10.3.

The generating system for the considered problem, see (7.21), is split into n second order, independent conservative subsystems

$$\dot{q}_i^{(0)} = \frac{\partial H_i}{\partial p_i^{(0)}}, \quad \dot{p}_i^{(0)} = -\frac{\partial H_i}{\partial q_i^{(0)}}. \tag{8.1}$$

Each of these subsystems admits construction of the general integral of libration or rotational type in the considered region of the phase space, see Secs. 3.2 and 3.3,

$$\begin{aligned} q_i^{(0)} &= q_i(\varphi_i, s_i), \quad p_i^{(0)} = p_i(\varphi_i, s_i), \quad H_i = h_i(s_i), \\ \varphi_i &= \omega_i t + \alpha_i, \quad \omega_i = \frac{dh_i}{ds_i}, \end{aligned} \tag{8.2}$$

where φ_i and s_i are the "action-angle" variables, and ω_i denotes the corresponding circular frequency. Instead of the first $2n$ equations (7.21) we can write

$$\begin{aligned}\dot{s}_i &= \mu\left[\frac{\partial q_i}{\partial \varphi_i}Q_i + \frac{\partial}{\partial \varphi_i}\left(\Delta L + L^{(2)}\right)\right] + \mu^2\dots, \\ \dot{\varphi}_i - \omega_i &= -\mu\left[\frac{\partial q_i}{\partial s_i}Q_i + \frac{\partial}{\partial s_i}\left(\Delta L + L^{(2)}\right)\right] + \mu^2\dots,\end{aligned} \qquad (8.3)$$

$\Delta L, L^{(2)}$ and Q_i being expressed in terms of $\varphi_1, s_1, \dots, \varphi_n, s_n$ by means of eq. (8.2). Additionally, we assume a sufficiently smooth character of dependence of these quantities and the "action-angle" variables on $x, \dot{x}, y, \dot{y}$.

As mentioned above, the presence of a carrying system causes the appearance of additional mobility of the interacting objects. It is natural to assume that for an exact description of an arbitrary $i-th$ object we need to prescribe both its proper coordinate and the components of a certain $m_i \times 1$ vector x_i whose physical meaning is completely determined by the nature of the object. These components usually have a clear geometric meaning and can be referred to as the quasi-coordinates of its interaction with the carrying system (the above remains valid for objects with several degrees of freedom also). For instance, in the case of an unbalanced rotor these coordinates are the components of displacement of the centre of rotation in the three mutually orthogonal directions. The coordinates of the carrying system (the components of vector x) may be of a rather abstract character, for example these can be the amplitudes of the normal modes of small oscillation.

The time dependence of the quasi-coordinates of the interaction is determined from the change of the absolute coordinates of the carrying system, so that

$$x_i = F_{m_i}(x). \qquad (8.4)$$

Assuming additionally that $F_{m_i}(0) = 0$ we obtain, due to the smallness of x, that

$$x_i = \mu F_{m_i l} x + \mu^2\dots, \quad \left(F_{m_i l} = \left.\frac{dF_{m_i}}{dx}\right|_{x=0}\right), \qquad (8.5)$$

where $F_{m_i l}$ denotes a $m_i \times l$ matrix with constant coefficients, describing the orientation of the object relative to the carrying system. The additional kinetic potential of all objects is obtained with the help of the superposition principle

$$\Delta L = \sum_{i=1}^{n} \Delta L_i. \qquad (8.6)$$

Here component ΔL_i characterises the change of state of the $i-th$ object under the interaction and, up to higher order terms, is linear in components of x_i and $\dot{x}_i$

$$\Delta L_i = \mu \left(\dot{x}_i' B_i^{(1)} + x_i' B_i^{(2)} \right) + \mu^2 \ldots . \tag{8.7}$$

By virtue of the above, the components of the $m_i \times 1$ vectors $B_i^{(1)}$ and $B_i^{(2)}$ are functions of the partial derivatives of the "action-angle" variables φ_i, s_i. Then, the differential equations describing the small oscillations of the carrying system can be presented in the following form, see (7.21),

$$M x + B \dot{x} + C x = \sum_{i=1}^{n} F'_{m_i l} G_i, \tag{8.8}$$

where the $m_i \times 1$ vector $G_i(\varphi_i, s_i)$ given by the equality

$$G_i = \left(\frac{\partial}{\partial x_i'} - \frac{d}{dt} \frac{\partial}{\partial \dot{x}_i'} \right) \Delta L_i = B_i^{(2)} - \omega_i \frac{\partial B_i^{(1)}}{\partial \varphi_i} + \mu \ldots \tag{8.9}$$

can be understood as the $m_i \times 1$ dimensional "force" exerted on the carrying system from the $i-th$ object, corresponding to the quasi-coordinates of interaction. If the vector of the quasi-coordinates of interaction has three mutually orthogonal spatial coordinates, then G_i is a force in the conventional meaning of the word. If the quasi-coordinates are rotations, then G_i is a mechanical moment. Thus, the dynamical objects under consideration play the part of exciters of finite-dimensional forces.

The procedure of removing the non-critical fast variables of the carrying system explained in Sec. 5.3 is used below. Then, in the generating approximation, we obtain

$$x = \sum_{i=1}^{n} x^{(i)}(\varphi_i, s_i), \tag{8.10}$$

where each component $x^{(i)}$ is obtained by analogy with eq. (5.27) by means of the convolution integral

$$x^{(i)} = \int_0^{2\pi} K(\varphi_i - \xi) F'_{m_i l} G_i(\xi, s_i) \, d\xi, \tag{8.11}$$

K denoting the $l \times l$ matrix impulse-periodic function of the carrying system determined by eq. (5.25) for $\nu = 0, f = 1$. The vector of quasi-coordinates of interaction of the $i - th$ object in the generating approximation has, by virtue of eq. (8.5), the following form

$$x_i = \sum_{j=1}^{n} \int_0^{2\pi} K_{ij}(\xi, \omega_j) G_j(\varphi_j - \xi, s_j) \, d\xi. \tag{8.12}$$

Here it is taken into account that K and G_i are $2\pi-$ periodic with respect to their fast changing argument, and the following $m_i \times m_j$ matrix

$$K_{ij} = F_{m_i l} K\left(\xi, \omega_j\right) F'_{m_j l} \tag{8.13}$$

can be referred to as the dynamic influence matrix of the $j-th$ object on the $i-th$ object through the carrying system. Indeed, component $k_{ij}^{(p,q)}$ $(p = 1, \ldots, m_i, q = 1, \ldots, m_j)$ of matrix K_{ij} is equal to the time-dependence of the $p-th$ quasi-coordinate of the $i-th$ object subjected to an impulse $2\pi/\omega_j-$periodic perturbation through the $q-th$ quasi-coordinate of the $j-th$ object.

Hence, the coordinates of the carrying system on the right hand sides of equations (8.3) can be excluded, and formulae (8.12) can be used in the first approximation. In what follows, we assume that a similar removal is also feasible for the coordinates of the carried system. To this aim it is necessary that the equations of the oscillations of the carried system, see eq. (7.21), are linear in the components of the vector y and have a structure close to eq. (8.8). Additionally, an important particular case is that in which the presence of non-inertial carried interactions does not result in the appearance of additional degrees of freedom.

The above is related to the problem of internal synchronisation for the case of a totally autonomous system. For practical applications, one is also interested in the problem of external synchronisation when an external perturbation is exerted on the system through the elements of the carrying and carried systems. Let us assume that this perturbation is quasi-periodic with given, mutually independent frequencies $\omega_{n+1}, \ldots, \omega_{n+n'}$. In particular, let the carrying system be subjected to given $2\pi/\omega_j-$periodic forces $G_i\left(\psi_i\right)$, where $\psi_i = \omega_i t, j = n+1, \ldots, n+n'$. Let the matrices of the dynamical influence of forces $K_{i,n+1}, \ldots, K_{i,n+n'}$ on the time rate of change of the coordinates describing the interaction of the carrying system with the $i-th$ object $(i = 1, \ldots, n)$ be given also. As a result, formula (8.12) in the problem of external synchronisation remains valid provided that the summation over subscript j is carried out from 1 to $n + n'$.

8.2 The principal resonance in the system with weak carrying interactions

Let us assume that the inertialess interactions are absent, and the carrying system is subjected to a single $2\pi/\nu-$periodic perturbation $(n' = 1, \omega_{n+1} = \nu)$. All the objects in the system are assumed to be anisochronous. Then, the dependence of the partial frequencies on the actions, i.e. $\omega_i = \omega_i\left(s_i\right)$ is essential, and the steepness coefficients of the corresponding backbone curves are $e_i\left(\omega_i\right) = \dfrac{d\omega_i}{ds_i} = O\left(1\right)$.

Let us transform eq. (5.22) to the new "anisochronous" variables "phase-frequency" φ_i, ω_i. We take into account that only component ΔL_i of the additional Lagrange's function ΔL in eq. (8.3) depends explicitly on φ_i, ε_i. The result is as follows

$$\begin{aligned} \dot{\omega}_i &= \mu e_i \left[\frac{\partial q_i}{\partial \varphi_i} Q_i + \frac{\partial \Delta L_i}{\partial \varphi_i} \right] + \mu^2 \dots , \\ \dot{\varphi}_i - \omega &= -\mu e_i \left[\frac{\partial q_i}{\partial \omega_i} Q_i + \frac{\partial \Delta L_i}{\partial \omega_i} \right] + \mu^2 \dots . \end{aligned} \tag{8.14}$$

This system with the multidimensional fast rotating phase admits use of the procedure of averaging within $\sqrt{\varepsilon}-$vicinity of the principal resonance explained in Sec. 5.4. In the first approximation we obtain

$$\begin{aligned} \dot{\Omega}_i &= \mu e_i (\Omega_i) P_i (\alpha_1, \dots , \alpha_n, \Omega_1, , \dots , \Omega_n) + \mu^2 \dots , \qquad (8.15) \\ \dot{\alpha}_i &= \Omega_i - \nu + \mu e_i (\Omega_i) R_i (\alpha_1, \dots , \alpha_n, \Omega_1, , \dots , \Omega_n) + \mu^2 \dots , \end{aligned}$$

where, in accordance with eqs. (31.12), (31.14) and (31.15), the averaged variables α_i and Ω_i differ from the old variables $\varphi_i = \nu t$ and ω_i by small values of order ε and hence have the meaning of the averaged phase shifts and frequencies whereas functions P_i and R_i are obtained by substituting $\omega_i = \Omega_i$ $\varphi_i = \varphi + \alpha$ in the terms of the order of ε into the right hand sides of eq. (8.14) and averaging over $\varphi = \nu t$. While averaging, it is necessary to keep in mind that differentiating the first equation in (8.14) is carried out only with respect to explicit φ_i (not in terms of x_i) and, with accuracy up to non-small terms, $\dfrac{d}{dt} = \nu \dfrac{d}{d\varphi}$ and $\Omega_i = \nu$. Thus, the equalities following from eqs. (8.7) and (8.9)

$$\begin{aligned} \frac{\partial \Delta L_i}{\partial \varphi_i} &= \frac{\partial}{\partial \varphi_i} \left(x_i' B_i^{(2)} \right) - \frac{\partial x_i'}{\partial \varphi_i} G_i, \\ \frac{\partial \Delta L_i}{\partial \Omega_i} &= \nu \frac{\partial}{\partial \varphi_i} \left(x_i' \frac{\partial B_i^{(1)}}{\partial \Omega_i} \right) + x_i' \frac{\partial G_i}{\partial \Omega_i} \end{aligned} \tag{8.16}$$

hold. Then we have

$$P_i \equiv M_i - \sum_{j=1}^{n+1} W_{ij}, \quad R_i \equiv N_i + \sum_{j=1}^{n+1} U_{ij}, \tag{8.17}$$

where the quantity

$$M_i = \left\langle \frac{\partial q_i'}{\partial \varphi_i} Q_i \right\rangle = \frac{1}{\nu} \left\langle \dot{q}_i' Q_i \right\rangle \tag{8.18}$$

is equal to the work of the proper non-potential forces of the objects averaged over the period. Moreover,

$$N_i = - \left\langle \frac{\partial q_i'}{\partial \omega_i} Q_i \right\rangle , \tag{8.19}$$

and, by virtue of eqs. (8.12) and (8.16),

$$W_{ij} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} G_i'(\varphi+\alpha_i,\Omega_i)\,K_{ij}^*(\varphi-\psi,\Omega_i)\,G_j(\psi+\alpha_j,\Omega_j)\,d\varphi d\psi,$$

$$U_{ij} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\partial}{\partial\Omega_i}G_i'(\varphi+\alpha_i,\Omega_i)\,K_{ij}(\varphi-\psi,\Omega_j)\,G_j(\psi+\alpha_j,\Omega_j)\,d\varphi d\psi, \tag{8.20}$$

where $K_{ij}^* = \dfrac{\partial K_{ij}}{\partial\varphi}$.

Let us recall that the $(n+1)-th$ component in the sums in (8.17) appears because the problem under consideration is non-autonomous. All objects of the system are tuned to the frequency of "force" $G_{n+1}(\varphi)$ in eq. (8.20), $(\alpha_{n+1}=0)$.

Quantities W_{ij} determined by eq. (8.20) as double convolution integrals are referred to as the particular vibrational moments. They characterise an averaged interaction of the $i-th$ and $j-th$ objects through the carrying system and are functions of the averaged partial frequencies Ω_i and Ω_j, as well as the difference $\alpha_i-\alpha_j$ of the phase shifts $(i,j=1,\dots,n)$. It follows from eq. (8.20) that

$$\frac{\partial U_{ij}}{\partial\alpha_i} = -\frac{\partial W_{ij}}{\partial\Omega_i}. \tag{8.21}$$

The quantity

$$W_j = \sum_{j=1}^{n+1} W_{ij} \tag{8.22}$$

describes an averaged action of the carrying system on the $i-th$ object and is called the total vibrational moment.

Let us average expression (8.7) in the first approximation and estimate the integral of the first terms by parts. By virtue of eq. (8.9) the obtained expression has the form

$$\Delta\Lambda_i = \frac{1}{2\pi}\int_0^{2\pi}\Delta L_i d\varphi = \frac{1}{2\pi}\int_0^{2\pi} G_i'\dot{x}_i d\varphi \tag{8.23}$$

and is equal to the average work which is done by the force excited by the $i-th$ object in its quasi-coordinates. Quantity $\Delta\Lambda_i$ can be referred to as the additional Hamilton action of the $i-th$ object. Inserting eq. (8.12) into eq. (8.23) yields

$$\Delta\Lambda_i = \sum_{j=1}^{n+1} \Delta\Lambda_{ij}, \tag{8.24}$$

$$\Delta\Lambda_{ij} = \frac{1}{2\pi} \int_0^{2\pi}\int_0^{2\pi} G_i'(\varphi + \alpha_i, \Omega_i)\, K_{ij}(\varphi - \psi, \Omega_j)\, G_j(\psi + \alpha_j, \Omega_j)\, d\varphi d\psi.$$

It immediately follows from this equation that

$$W_{ij} = -\frac{\partial \Delta\Lambda_{ij}}{\partial \alpha_i}, \quad U_{ij} = \frac{\partial \Delta\Lambda_{ij}}{\partial \Omega_i}. \tag{8.25}$$

The above can be generalised easily to the case of the internal synchronisation when the external synchronising action is absent and thus $G_{n+1} = 0$. In this case all of the above formulae remain valid. However, the synchronous frequency ν is not known in advance, $W_{ij} = U_{ij} = 0$, and the right hand sides of the averaged equations (8.15) depend only on the differences of phase shifts $\alpha_i - \alpha_j$.

Nearly trivial is the generalisation to the case of so-called multiple synchronisation, when the frequencies of the motion in the generating approximation are bound by certain integer relationships. Then it is sufficient to perform some integer transformation of the "phase-frequency" variables in the original equations (8.14) using Sec. 4.8. Certain lower harmonics may be absent in the finite-dimensional forces G_i. This may result in the absence of frequency commutation between these forces and identical vanishing W_{ij} and U_{ij}. Such a degenerating situation requiring consideration of the higher approximations appears, for example, in the problem of multiple synchronisation of the exciters of harmonic forces [10].

The obtained results allow us to judge the peculiarities of the incomplete synchronisation in the system under consideration. For example, let the interacting objects be split into n' sets each of which is tuned to one of the mutually independent frequencies $\omega_{n+1}, \dots, \omega_{n+n'}$. In the first approximation it is of no concern whether they are prescribed frequencies of the external excitations or the frequencies of internal incomplete perturbations. In this case, the process of averaging system (8.14) is understood in the generalised sense and strictly follows the approach of Sec. 5.4. The absence of any frequency communication between the forces exerted on the carrying system by the objects of the different sets leads to the averaged system, which is split into n' isolated subsystems (8.15). Each of these subsystems describes the weak interaction of the averaged frequencies and phase shifts of the corresponding set whilst its right hand sides are determined according to eqs. (8.14)-(8.20). This allows us to formulate the following independence principle, namely, that the weak interaction between objects of the same set in the vicinity of a certain (internal or external) resonance does not depend on the actions exerted by the other objects.

Keeping in mind this possibility of the above generalisation we restrict our consideration, for brevity, to the analysis of only complete internal synchronisation of anisochronous objects $(i, j = 1, \ldots, n)$.

The single-frequency stationary regime in such a system (complete synchronisation) is characterised, due to (8.15), by the fact that the constant values of the averaged values α_i and Ω_i, of the order of unity, are determined from the following equations

$$\Omega_i = \nu, \quad P_i (\alpha_1, \ldots, \alpha_n, \Omega_1, , \ldots, \Omega_n) = 0. \tag{8.26}$$

By virtue of eqs. (8.17) and (8.22), the second equations in (8.26) take the form

$$P_i \equiv M_i - W_i = 0. \tag{8.27}$$

Actually, these equations serve to determine the original approximations to the differences of the phase shifts $\alpha_2 - \alpha_1, \ldots, \alpha_n - \alpha_1$ and to the synchronous frequency ν. With this in view, these transcendental equations can be referred to as the equations of the problem of autophasing of the synchronous motions of the objects.

The asymptotic local stability is determined according to the asynchronous criteria of stability of the first and second sets derived in Sec. 5.5. In particular, the criteria of stability of the first set are satisfied if the $n-1$ roots λ_1^2 of the determinant of the linear system

$$e_i \sum_{j=1}^{n} \frac{\partial P_i}{\partial \alpha_j} a_j = \lambda_1^2 a_j \quad (e_i = e_i(\nu)) \tag{8.28}$$

are negative, the $n - th$ root being equal to zero due to autonomy of the problem. The anisochronous criteria of the second set are determined by eq. (5.75). The corresponding inequalities take a very specific form in the important particular case in which the non-potential forces for the proper coordinates of the objects are either of the partial character $(Q_i = Q_i(q_i, p_i))$ or depend on some other parameters which usually have no mechanical meaning and can be expressed in terms of q_i and p_i (or φ_i and ω_i) by means of an infinite procedure analogous to that used for excluding the coordinated of the carrying system, see Sec. 9.1. In both cases, values M_i and N_i do not depend on the phase shifts and are functions of Ω_i. Then we can perform the replacement $P_i \to -W_i$ in eq. (8.28) so that we finally have

$$e_i \sum_{j=1}^{n} W_{ij}^{*} (a_i - a_j) + \lambda_1^2 a_i = 0. \tag{8.29}$$

Here an asterisk denotes differentiation of the partial vibrational moment $W_{ij}(\alpha_i - \alpha_j, \Omega_i, \Omega_j)$ with respect to its first argument.

In view of the absence of the slow isochronous variables in the analysed equation (8.14), the anisochronous criteria of the second set require fulfillment of the following inequalities, cf. (5.75)

$$\sum_{i,j=1}^{n} e_i \left(\frac{\partial P_i}{\partial \Omega_j} + \frac{\partial R_i}{\partial \alpha_j} \right) a_j a_i^* < 0. \tag{8.30}$$

In the case under consideration, instead of eq. (8.30) we can write

$$\sum_{i=1}^{n} \left[\frac{dM_i}{d\nu} a_i - \sum_{j=1}^{n} a_j \left(\frac{\partial}{\partial \Omega_i} + \frac{\partial}{\partial \Omega_j} \right) W_{ij} \right] e_i a_i^* < 0, \tag{8.31}$$

where identity (8.21) is taken into account. Let us introduce $b_i = e_i a_i^*$, which by virtue of eq. (5.73) are determined from the following equation

$$e_i \sum_{j=1}^{n} \left(W_{ij}^* b_i - W_{ji}^* b_j \right) + \lambda_1^2 b_i = 0 \tag{8.32}$$

and satisfy the normalisation condition $\sum\limits_{i=1}^{n} \frac{1}{e_i} a_i b_i = 1$ with the weight function $\frac{1}{e_i}$. In addition to this, we take into account that one can take $\frac{\partial}{\partial \Omega_i} + \frac{\partial}{\partial \Omega_j} = \frac{\partial}{\partial \nu}$ if no difference is made between the partial frequencies for calculation of the partial vibrational moments (8.20). Then we can finally write down

$$\sum_{i=1}^{n} \left[\frac{dM_i}{d\nu} a_i - \sum_{j=1}^{n} \frac{\partial W_{ij}}{\partial \nu} a_j \right] b_i < 0. \tag{8.33}$$

Equations replacing eqs. (8.28) and (8.32) are obtained for the above non-autonomous case provided that we replace n by $n+1$, then omit the $(n+1) - th$ equation and put $a_{n+1} = a_{n+1} = b_{n+1} = 0$. The form of inequalities (8.33) does not change.

It is easy to see that for $\lambda_1^2 = 0$ we have $a_i = 1$ and $\sum\limits_{i=1}^{n} \frac{b_i}{e_i}{}_i = 1$. Then the inequality ensuring stability for the second set corresponding to this root has the form

$$\sum_{i=1}^{n} b_i \frac{\partial}{\partial \nu} (M - W_i) < 0. \tag{8.34}$$

In this regard it is worthwhile finding the first approximation to the synchronous frequency from the following equation

$$\sum_{i=1}^{n} (M_i - W_i) = 0, \tag{8.35}$$

which is obtained by summing up eq. (8.17) over i and can be treated as the equation of energy balance. Thus, inequality (8.34) has a standard meaning of the condition for self-excited oscillations. In the particular case of a single object ($n = 1$) the problem of autophasing does not exist at all, and, instead of eqs. (8.35) and (8.34) we have

$$M = W, \quad e\frac{d}{d\nu}(M - W) < 0. \tag{8.36}$$

Two additional solutions of the first equation in (8.36), namely under-resonant $\left(\frac{dW}{d\nu} > 0\right)$ and over-resonant $\left(\frac{dW}{d\nu} < 0\right)$ ones, may appear near the resonance with the coordinates of the carrying system. If the object is hard anisochronous ($e > 0$) then the over-resonant solution is unstable due to eq. (8.36). This obstacle determines specific features of the quasi-stationary transition of the system through the resonance, jumps accompanying this transition, the Sommerfeld effect and so on [52]. In the case of several interacting objects ($n \geq 2$) the stability inequalities (8.33) corresponding to the non-trivial values of λ_1^2 have a much more complicated nature. However, they are most essential near to the resonances of the coordinates of the carrying system.

In the particular case in which the proper non-potential forces are of partial character and are explicit functions of the proper coordinates q_i, p_i (and possibly also of a external synchronous phase φ) the following relationship

$$\frac{dM_i}{d\nu} = e_i \left\langle \frac{\partial Q_i}{\partial p_i} \right\rangle \tag{8.37}$$

is valid. This can be proved easily if we calculate the partial derivative of the original expression (8.18) with respect to Ω_i and estimate the integral with $\frac{\partial^2 q_i}{\partial \varphi_i \partial \omega_i}$ by parts. It should be taken into account that the quantity $\frac{\partial q_i}{\partial \omega_i}$ is 2π–periodic with respect to φ_i, and the determinant of the transformation to the "phase-frequency" variables is equal to the corresponding steepness coefficient of the backbone curve

$$\frac{\partial q_i}{\partial \varphi_i}\frac{\partial p_i}{\partial \omega_i} - \frac{\partial q_i}{\partial \omega_i}\frac{\partial p_i}{\partial \varphi_i} = e_i. \tag{8.38}$$

8.3 Dynamic matrix and harmonic influence coefficients of the carrying system

If small oscillations of the carrying system are not accompanied by the gyroscopic forces, i.e. $B = B'$, see eq. (8.8), then the matrix impulse-frequency characteristic of the carrying system $K(\xi, \omega)$ is also symmetric.

The following relationships of dynamic reciprocity

$$K_{ij} = K'_{ji} \tag{8.39}$$

are fulfilled due to eq. (8.13).

The matrix coefficients of the Fourier expansion

$$K_{ij} = \frac{1}{2\pi} K_{ij}^{(0)} + \frac{1}{\pi} \sum_{\rho=1}^{\infty} \left(K_{ij}^{(\rho)} \cos \rho\xi + \bar{K}_{ij}^{(\rho)} \sin \rho\xi \right) \tag{8.40}$$

are equal to the harmonic influence coefficients of the m–dimensional force on the m_j–dimensional amplitude of the corresponding harmonics. Under the harmonic influence coefficients one understands the reaction on the harmonic perturbation of unit amplitude. If $B = 0$, then $\bar{K}_{ij}^{(\rho)} = 0$ and the dynamic influence matrix is an even function of the synchronous phase

$$K_{ij} (\xi, \omega) = K_{ij} (-\xi, \omega) . \tag{8.41}$$

Then, due to eq. (8.20), the $n \times n$ matrix of the partial vibrational moments is skew-symmetric in the sense that

$$W_{ij} (\alpha_i - \alpha_j, \Omega_i, \Omega_j) = -W_{ij} (\alpha_j - \alpha_i, \Omega_j, \Omega_i) . \tag{8.42}$$

This circumstance determines the potential character of the averaged equations of motion, see Sec. 8.5.

Assume now that the carrying system is an elastic continuum with prescribed properties. This means that for any two points M and N of the carrying system there exists a unique tensor of second rank $K (M, N) = K (N, M)$ such that the displacement u of point M under the action of force Q applied at point N is equal to

$$u = K (M, N) Q. \tag{8.43}$$

The Fredholm equation for small oscillations of the carrying system under harmonic force $e_j \cos \nu t$ of the unit amplitude ($|e_j| = 1$) applied at point N_j is as follows

$$\begin{aligned} u (M, t) \quad = \quad & - \int_{(V)} K (M, N) \left[\rho (N) \frac{\partial^2 u (N, t)}{\partial t^2} + R \right] dV_N + \\ & K (M, N_j) e_j \cos \nu t. \end{aligned} \tag{8.44}$$

Here $u (M, t)$ denotes the displacement vector of point M of the carrying system, whilst $\rho (N)$ and R denote the mass density and the vectorial density of the dissipation force respectively. Integration is performed over the whole volume of the carrying system.

Let us assume that the influence tensor of the carrying system with l degrees of freedom is represented as the following bilinear expansion

$$K(M,N) = \sum_{\rho=1}^{l} \frac{\Theta_\rho(M)\,\Theta_\rho(N)}{\lambda_\rho^2}, \tag{8.45}$$

where λ_ρ and Θ_ρ denote respectively the eigenvalues and the vectorial eigenfunctions which satisfy the following conditions of orthogonality and normalisation

$$\int\limits_{(V)} \rho(N)\,\Theta'_\rho(M)\,\Theta_\rho(N)\,dV_N = \delta_{\rho\sigma} \tag{8.46}$$

and are determined from the following homogeneous equation

$$\Theta_\rho(M) = \lambda_\rho^2 \int\limits_{(V)} \rho(N)\,K(M,N)\,\Theta_\rho(N)\,dV_N. \tag{8.47}$$

The solution of eq. (8.44) is sought as the series

$$u(M,t) = \sum_{\rho=1}^{l} \Theta_\rho(N)\,u_\rho(t), \tag{8.48}$$

where $u_1, \ldots, u_l$ are *a priori* unknown scalar amplitudes of the normal forms. We use the following expression for the density of the energy dissipation in the carrying system

$$R = \sum_{\rho=1}^{l} \beta_\rho \Theta_\rho(N)\,\dot{u}_\rho, \tag{8.49}$$

where $\beta_1, \ldots, \beta_l$ are positive constants. The hypothesis behind the latter relationship, known as Voigt's hypothesis [8], implies, in a certain sense, a proportionality of the forces of internal resistance to the strain rate.

Inserting eq. (8.49) into (8.44) and rearranging the result by means of eqs. (8.45), (8.46) and (8.49) we arrive at the system of ordinary differential equations

$$\ddot{u}_\rho + \beta_\rho \dot{u}_\rho + \lambda_\rho^2 u_\rho = \Theta_\rho(N_j)\cdot e_j \cos\nu t. \tag{8.50}$$

The periodic solution of this equation has the form

$$u_\rho = \Theta_\rho(N_j)\cdot e_j \frac{\left(\lambda_\rho^2 - \nu^2\right)\cos\nu t + \nu\beta_\rho \sin\nu t}{\left(\lambda_\rho^2 - \nu^2\right)^2 + \nu^2\beta_\rho^2}. \tag{8.51}$$

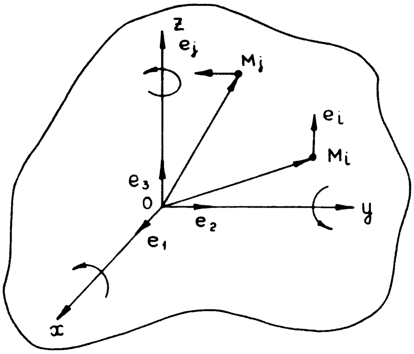

FIGURE 8.1.

Let us substitute the latter equation into eq. (8.48), adopt $M = M_i$ and project the obtained displacement on the direction of vector e_j. The resulting scalar harmonic coefficient of influence of the force on the displacement in the corresponding direction is given by

$$k_{ij} = \sum_{\rho=1}^{l} \Theta_\rho (M_i) \cdot e_i \Theta_\rho (N_j) \cdot e_j \frac{\left(\lambda_\rho^2 - \nu^2\right) \cos \nu t + \nu \beta_\rho \sin \nu t}{\left(\lambda_\rho^2 - \nu^2\right)^2 + \nu^2 \beta_\rho^2} . \tag{8.52}$$

This general expression also remains valid in the degenerate case when a few first eigenvalues λ_ρ are equal to zero and formula (8.45) loses its meaning. We can use this expression for the case of the carrying system with distributed parameters ($l = \infty$) and a discrete spectrum of eigenvalues.

Let us consider several particular examples of determining the harmonic influence coefficients.

1. A free rigid body with six degrees of freedom, Fig. 8.1. Let us neglect the rigidity of the springs which fix the equilibrium position of the body and the influence of the energy dissipation at its small oscillation. The eigenvalues of the system are zero and the orthogonal and normalised modes are determined by the formulae

$$\begin{aligned} \Theta_\rho &= \frac{e_\rho}{\sqrt{M}} \quad (\rho = 1, 2, 3), \quad \Theta_4 = \frac{-ye_2 + ze_3}{\sqrt{J_x}}, \\ \Theta_4 &= \frac{xe_1 - ze_3}{\sqrt{J_x}}, \quad \Theta_4 = \frac{-xe_1 + ye_2}{\sqrt{J_x}}. \end{aligned} \tag{8.53}$$

Here e_1, e_2, e_3 are the unit vectors of the principal axes of inertia of the body Ox, Oy, Oz, whereas M and J_x, J_y, J_z denote the mass and the principal moments of inertia of the body, respectively. Substituting eq. (8.53) into

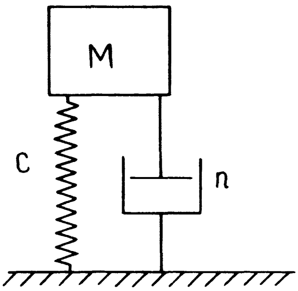

FIGURE 8.2.

eq. (8.52) and taking into account that $l = 6$ and $\beta_\rho = \lambda_\rho = 0$ yields

$$k_{ij} = -\left\{\frac{1}{M}\sum_{\rho=1}^{3}\kappa_{i\rho}\kappa_{j\rho} + \frac{1}{J_x}\left(y_i\kappa_{i2} - z_i\kappa_{i3}\right)\left(y_j\kappa_{j2} - z_j\kappa_{j3}\right) + \frac{1}{J_y}\left(x_i\kappa_{i1} - z_i\kappa_{i3}\right)\left(x_j\kappa_{j1} - z_j\kappa_{j3}\right) + \frac{1}{J_z}\left(x_i\kappa_{i1} - y_i\kappa_{i2}\right)\left(x_j\kappa_{j1} - y_j\kappa_{j2}\right)\right\}\frac{\cos\nu t}{\nu^2}. \qquad (8.54)$$

where x_i, y_i, z_i and x_j, y_j, z_j are the coordinates of the points M_i and M_j respectively, and $\kappa_{i\rho} = e_i \cdot e_j$ $(\rho = 1, 2, 3)$ denote the direction cosines of the considered directions.

2. A rigid body of mass M with a single degree of freedom $(l = 1)$ mounted on a spring of rigidity c, Fig. 8.2. As the only normalised mode in this case is $\Theta = 1/\sqrt{M}$, the harmonic coefficient of influence of the force in direction x on the displacement in this direction is given by

$$k = \frac{\cos(\xi - \zeta)}{M\sqrt{\left(\lambda^2 - \nu^2\right)^2 + \nu^2 n^2/M^2}} \quad (\xi = \nu t)\,, \qquad (8.55)$$

cf. eq. (8.52). Here $\lambda = \sqrt{c/M}$ denotes the natural frequency of oscillation of the system, n denotes the coefficient of viscous resistance and ζ denotes the harmonic coefficient of influence of the force on the oscillation phase

$$\tan\zeta = \frac{1}{M}\frac{\nu n}{\lambda^2 - \nu^2} \quad (0 < \zeta < \pi)\,. \qquad (8.56)$$

According to eq. (8.55) the maximum value of k is naturally considered to be the harmonic coefficient of the force on the oscillation amplitude.

3. A rigid platform with two degrees of freedom mounted on two springs, Fig. 8.3. The mass and the moment of inertia are designated by M and J respectively, and the centre of mass is assumed to lie at the midspan.

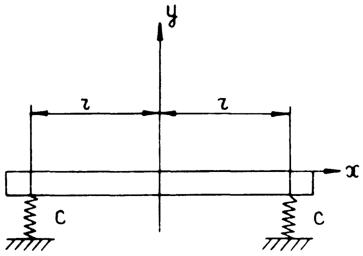

FIGURE 8.3.

The system has two natural frequencies $\lambda_1 = \sqrt{2c/M}$ and $\lambda_2 = \sqrt{2cr^2/J}$ and correspondingly two orthonormalised modes $\Theta_1 = 1/\sqrt{M}$ and $\Theta_2 = x/\sqrt{J}$. Denoting the coefficient of viscous resistance in any spring by n, the harmonic coefficient of influence of the transverse force at point x_1 on the displacement of point x_2 is, due to eq. (8.52), given by

$$k_{12} = \frac{1}{M}\frac{\cos(\xi - \zeta_1)}{\sqrt{\left(\lambda_1^2 - \nu^2\right)^2 + 4\nu^2 n^2/M^2}} + \frac{x_1 x_2}{J}\frac{\cos(\xi - \zeta_2)}{\sqrt{\left(\lambda_2^2 - \nu^2\right)^2 + 4\nu^2 n^2 r^2/J^2}}, \tag{8.57}$$

where

$$\tan\zeta_1 = \frac{1}{M}\frac{2\nu n}{\lambda_1^2 - \nu^2}, \quad \tan\zeta_2 = \frac{r^2}{J}\frac{2\nu n}{\lambda_2^2 - \nu^2}. \tag{8.58}$$

4. Homogeneous beam. The natural frequencies of the beam [8] are conveniently expressed in terms of the non-dimensional eigenvalues p_ρ ($\rho = 1, 2, \ldots \infty$) in the following form

$$\lambda_\rho = \sqrt{\frac{G}{\rho}\frac{p_\rho^2}{l^2}}, \tag{8.59}$$

where G, ρ and l denote the beam rigidity, mass per length unit and the total length respectively. Thus, the harmonic coefficient of influence of the transverse force on the displacement is given by

$$k_{12} = \frac{\rho l^4}{G}\sum_{\rho=1}^{\infty}\Theta_\rho(x_1)\,\Theta_\rho(x_2)\,\frac{\cos\xi}{p_\rho^4 - p^4}, \tag{8.60}$$

where $p = \sqrt[4]{\rho\nu^2/Gl}$ denotes the non-dimensional frequency parameter and $\Theta_\rho(x)$ denotes the orthogonal and normalised modes of the beam. In the case of a simply supported beam $p_\rho = \pi\rho$, and

$$\Theta_\rho = \sqrt{\frac{2}{M}}\sin\pi\rho\frac{x}{l} \quad (0 < x < l), \tag{8.61}$$

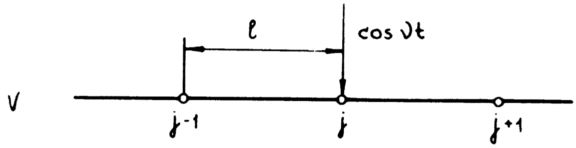

FIGURE 8.4.

where M is the beam mass. In the case of a free-free beam (actually a beam on an elastic suspension of negligibly small rigidity) $\lambda_1 = \lambda_2 = 0$, $\Theta_1 = 1/\sqrt{M}$ and $\Theta_2 = x/\sqrt{J}$ one can separate a "rigid" part from the series

$$k_{12} = -\left\{ \frac{1}{\nu^2}\left(\frac{1}{M} + \frac{x_1 x_2}{J}\right) - \frac{\rho l^4}{G}\sum_{\rho=3}^{\infty} \frac{\Theta_\rho(x_1)\,\Theta_\rho(x_2)}{p_\rho^4 - p^4} \right\}\cos\xi. \tag{8.62}$$

Here it is assumed that J denotes the moment of inertia of the beam and the centre mass $x = 0$ is assumed to lie at the midspan of the beam.

The non-trivial eigenvalues of the free-free beam are determined from the equation $\cosh p_\rho \cos p_\rho = 1$ and the corresponding normal modes are equal to, see [8],

$$\begin{aligned}
\Theta_\rho &= \frac{1}{\sqrt{M}}\left(\frac{\cosh p_\rho \frac{x}{l}}{\cosh \frac{p_\rho}{2}} + \frac{\cos p_\rho \frac{x}{l}}{\cos \frac{p_\rho}{2}}\right), \quad \rho = 3, 5, 7, \ldots \\
\Theta_\rho &= \frac{1}{\sqrt{M}}\left(\frac{\sinh p_\rho \frac{x}{l}}{\sinh \frac{p_\rho}{2}} + \frac{\sin p_\rho \frac{x}{l}}{\sin \frac{p_\rho}{2}}\right), \quad \rho = 4, 6, 8, \ldots .
\end{aligned} \tag{8.63}$$

Let us notice that the harmonic coefficient of influence of the force on the angle of rotation of the cross-section x_1 is equal to $\dfrac{\partial k_{12}}{\partial x_1}$, that of the moment on the displacement $\dfrac{\partial k_{12}}{\partial x_2}$, and that of the moment on the angle of rotation $\dfrac{\partial^2 k_{12}}{\partial x_1 \partial x_2}$.

This list can be continued. However we restrict our consideration to a unbounded system for which applying formula (8.52) is not possible. We consider a chain-like one-dimensional system of identical rigid hinged rods, Fig. 8.4. The system is assumed to be unbounded in both directions, the particles of mass M being attached at the hinges. The rigidities of the elastic elements fixing the straight equilibrium position of the system are assumed, as above, to be negligibly small. The equation for the forced oscillations of the chain subjected to a unit force applied at the $i-th$ hinge is as follows

$$\left(M + \frac{2}{3}\rho l\right) \ddot{k}_{ij} + \frac{\rho l}{6} \left(\ddot{k}_{i,j+1} + \ddot{k}_{i,j-1}\right) = \delta_{ij} \cos\xi, \qquad (8.64)$$

where k_{ij} denotes the displacement of the $j-th$ hinge. While constructing the latter equation we assume that the rods have a constant mass per unit length ρ and their central moment of inertia is thus $\rho l^3/12$. The pure harmonic forced oscillation governed by eq. (8.64) must satisfy the condition that $k_{ij} \to 0$ when $j \to \infty$ and can be constructed by means of the analytical method of finite differences [68]. The corresponding expression has the form

$$k_{ij} = -\frac{3}{\rho l \nu^2} \frac{(-1)^{i+j}}{\sinh q} \exp\left(-q|i-j|\right) \cos\xi, \quad \xi = \nu t, \qquad (8.65)$$

where the positive parameter q is determined from the equation

$$\sinh q = 2 + 3\frac{M}{\rho l}. \qquad (8.66)$$

Thus, the harmonic relation between the hinges decays exponentially with increasing distance.

8.4 Synchronisation of the force exciters of the simplest type

Construction and analysis of the averaged equations of motion of the first approximation in the problem of weak interaction of anisochronous dynamical objects with a single degree of freedom by means of a "linear" carrying system consists of a several steps

1) determination of the values of the forces G_i exerted by the generating conservative objects on the immovable carrying system,

2) theoretical or experimental determination of the components of the dynamical influence matrices,

3) constructing expressions for the partial vibrational moments and average power of the non-potential forces of the proper coordinates of the objects, and

4) determination of the synchronous phasings and criteria of their existence and stability.

Solution of the latter problem is considerably simplified in the important particular case in which the dissipative forces are small under oscillations of the carrying system $(B = 0)$. Indeed, due to skew-symmetry of the matrix of partial vibrational moments, see eq. (8.39), the equation of the energy balance (8.35) typically takes the following form

$$\sum_{i=1}^{n} M_i = 0 \tag{8.67}$$

and serves to determine the synchronous frequency ν. Matrix W'_{ij} is, in contrast, symmetric. One can easily symmetrise the determinant of system (8.29), thus, all of its roots are real. The direct and conjugate systems (8.29) and (8.32) coincide $\left(b_i = \sum_{j=1}^{n} \frac{a_j^2}{e_j} a_i\right)$ and the stability inequalities of the second set in eq. (8.33) take the form

$$\left(\sum_{j=1}^{n} \frac{a_j^2}{e_j}\right) \sum_{i=1}^{n} \frac{dM_i}{d\nu} a_i^2 < 0.$$

Assume now that all of the objects coincide in the sense that $\operatorname{sign} e_1 = \ldots = \operatorname{sign} e_n$ and $M_1 = \ldots = M_n$. Then, as the numbers $a_1, \ldots, a_n$ are real-valued, we obtain, instead of eqs. (8.33) and (8.67), that

$$W_i = 0, \quad M = 0, \quad \frac{dM}{d\nu} \operatorname{sign} e_i < 0. \tag{8.68}$$

Thus, synchronous phasing in the system of identical objects results in the total vibrational moments vanishing, whereas the appearance of the interactions do not shift the synchronous frequency in the first approximation. The latter inequality in (8.68) means that the stationary motion of each objects with the given frequency must be stable under no interaction.

Let us notice that, due to eq. (8.37), the latter inequality (8.68) can be recast in the form

$$\left\langle \frac{dQ_i}{dp_i} \right\rangle < 0.$$

In the particular case of a single object, which is a nearly conservative non-autonomous system with a single degree of freedom, the only necessary and sufficient condition coincides with that obtained by Kats [61].

Let us study first the most simple case of synchronisation of scalar forces or moments (i.e. forces or moments of a fixed direction), then series (8.40) can be written as follows

$$K_{ij} = \frac{1}{2\pi} k_{ij}^{(0)} + \frac{1}{\pi} \sum_{\rho=1}^{\infty} k_{ij}^{(\rho)} \cos\left(\rho\xi - \psi_{ij}^{(\rho)}\right) \quad (\xi = \nu t), \tag{8.69}$$

where $k_{ij}^{(\rho)}$ and $\psi_{ij}^{(\rho)}$ are the coefficients of influence of a unit harmonic force of frequency $\rho\nu$ excited by the $j-th$ object on the amplitude and phase shift of the single quasi-coordinate of interaction of the $i-th$ object respectively. The scalar force, excited by the $i-th$ object, is assumed to be anharmonic

$$G_i = G_i^{(0)} + \sum_{\rho=1}^{\infty} G_i^{(\rho)} \cos\left(\rho\varphi_i - \vartheta_i^{(\rho)}\right) \quad (\varphi_i = \nu t + \alpha_i). \tag{8.70}$$

By an appropriate choice of the proper coordinates of the objects we can always show that the first harmonics G_i is a maximum $\left(\vartheta_i^{(1)} = 0\right)$ at $\varphi_i = 0$.

Inserting series (8.69) and (8.68) into eq. (8.16) leads to the following general expression for the partial vibrational moment

$$W_{ij} = \frac{1}{2}\sum_{\rho=1}^{\infty} G_i^{(\rho)} G_j^{(\rho)} k_{ij}^{(\rho)} \sin\left[\rho\left(\alpha_i - \alpha_j\right) + \psi_{ij}^{(\rho)} - \left(\vartheta_i^{(\rho)} - \vartheta_j^{(\rho)}\right)\right]. \tag{8.71}$$

If all of the objects excite forces, each component of which has the same phase, i.e. $\vartheta_i^{(\rho)} = \vartheta^{(\rho)}$, then, by virtue of eq. (8.71), the known values of $\vartheta^{(\rho)}$ $(\rho = 2, 3, \dots)$ do not affect the values of W_{ij} and the character of the synchronous phasing. If, moreover, the friction in the carrying system is small, i.e. $\psi_{ij}^{(\rho)} = 0$ and equalities (8.68) are valid, then the equations of the problem of autophasing take the form

$$W_i = \frac{1}{2}\sum_{j=1}^{n}\sum_{\rho=1}^{\infty} G_i^{(\rho)} G_j^{(\rho)} k_{ij}^{(\rho)} \sin\rho\left(\alpha_i - \alpha_j\right) = 0. \tag{8.72}$$

This system always has a family of solutions $\alpha_i = \pi\sigma_i$, where σ_i is either zero or unity. The character of such phasing, referred to as the phasing of the first kind can be described by $n-$digital binary number $\sigma_1\sigma_2\dots\sigma_n$. Among the phasings of the first kind, the synchronous-synphase regime is of fundamental importance. The regime of two oscillations for which the phases are coincident is referred to as the synphase regime. Clearly, the other possible phasings depend on the properties of the carrying system and the objects and are referred to as the phasing of the second kind. Correspondingly, these phasings exist only in a bounded region of the space of the problem's parameters.

Let us make the problem more precise by assuming that the forces excited by the objects are monoharmonic, i.e. $G_i^{(\rho)} = 0$ $(\rho = 2, 3, \dots)$. Then the equations in (8.72) can be recast in the form

$$W_i = \sum_{j=1}^{n} p_{ij} \sin\rho\left(\alpha_i - \alpha_j\right) = 0, \quad p_{ij} = \frac{1}{2} G_i^{(1)} G_j^{(1)} k_{ij}^{(1)} = p_{ji}. \tag{8.73}$$

In addition to this, we assume for definiteness, that all of the objects in the system are hard anisochronous, i.e. $e > 0$. Then the direct analysis of

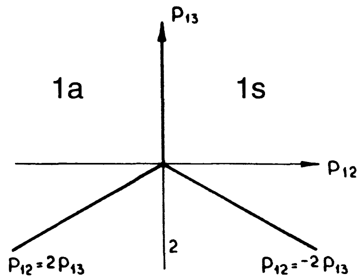

FIGURE 8.5.

the latter equation and the stability criteria for the synchronous phasings, following from eq. (8.29), leads to the following results.

In the system of two exciters ($n = 2$) only a synphase ($\alpha_1 - \alpha_2 = 0$) and antiphase phasing ($\alpha_1 - \alpha_2 = \pi$) of the first kind are possible, the synphase being stable if $p_{12} > 0$.

In a symmetric system of three exciters ($n = 3, p_{12} = p_{23}$), there exist the following phasings:

a) $1s$ which is a symmetric phasing of the first kind (synphase), $\alpha_1 = \alpha_2 = \alpha_3 = 0$. It is stable if $p_{12} > 0$ and $p_{12} + 2p_{23} > 0$;

b) $1a$ which is an antisymmetric phasing of the first kind (antiphase), $\alpha_1 = \pi, \alpha_2 = 0, \alpha_3 = -\pi$. It is stable if $p_{12} > 0$ and $p_{12} + 2p_{23} > 0$;

c) 2 which is an antisymmetric phasing of the second kind, for which

$$\cos\alpha_1 = -\frac{p_{12}}{2p_{13}}, \quad \alpha_2 = 0, \ \alpha_3 = -\alpha_1. \tag{8.74}$$

The condition for the existence and stability has the form $p_{13} < -|p_{12}|/2$.

The regions for the existence and stability of these phasings do not intersect each other and cover the entire plane, see Fig. 8.5. Typically, both phasings of the first kind coincide with the phasing of the second kind, and a jump is observed on their intersection border ($p_{12} = 0, p_{13} > 0$). In the particular case of exciters of equal harmonical forces of the permanent direction mounted on the free body, symmetric along its central axis, see Fig. 8.6, the coefficients p_{12} and p_{13} differ only by the same positive factor from the corresponding harmonic influence coefficients, see eq. (8.73)

$$k_{12} = -\frac{1}{M\nu^2}, \quad k_{13} = -\frac{1}{\nu^2}\left(\frac{1}{M} - \frac{r^2}{J}\right). \tag{8.75}$$

As a result, the synphase is always unstable, the antiphase is stable if $J < Mr^2$ whereas the phasing of the second kind is stable if $J > Mr^2$.

Not going into detail for the symmetric system of four exciters ($n = 4$, $p_{12} = p_{34}, p_{13} = p_{24}$), we notice only that the antisymmetric phasings of the second kind are determined in closed form and are characterised by the

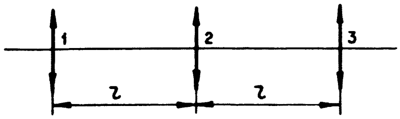

FIGURE 8.6.

equalities

$$\cos(\alpha_1-\alpha_2) = -\frac{p_{13}}{2}\left[\frac{1}{p_{14}}+\frac{1}{p_{23}}\pm\left(\frac{1}{p_{14}}-\frac{1}{p_{23}}\right)\left(\frac{1-p_{14}p_{23}/p_{13}^2}{1-p_{14}p_{23}/p_{12}^2}\right)^{1/2}\right],$$
$$\cos(\alpha_1+\alpha_2) = -\frac{p_{12}}{2}\left[\frac{1}{p_{14}}+\frac{1}{p_{23}}\pm\left(\frac{1}{p_{14}}-\frac{1}{p_{23}}\right)\left(\frac{1-p_{14}p_{23}/p_{12}^2}{1-p_{14}p_{23}/p_{13}^2}\right)^{1/2}\right],$$
$$\alpha_4 = -\alpha_1,\ \alpha_3 = -\alpha_2. \tag{8.76}$$

The above said is equally valid for synchronisation of the exciters of both forces and moments of the permanent direction. In the latter case, the values G_i and k_{ij} are the mechanical moments and harmonic coefficients of influence of the moments on the angle of rotation, respectively.

Let us consider the problem of a weak interaction of the exciters of two-dimensional harmonic forces

$$G_i = \begin{vmatrix} G_i^{(1)}\cos\varphi_i \\ G_i^{(2)}\cos(\varphi_i-\gamma_i) \end{vmatrix} \quad (\varphi_i = \nu t + \alpha_i), \tag{8.77}$$

where the amplitudes of the components may be interpreted as forces and moments, whilst γ_i denotes a constant phase shift of the second harmonic relative to the first one. The dynamic matrix of influence of the $j-th$ exciter on the $i-th$ one, or, strictly speaking, its pure harmonic part, can be presented in the following form

$$K_{ij}(\xi,\nu) = \frac{1}{\pi}\begin{vmatrix} k_{ij}^{(1,1)}\cos\left(\xi-\psi_{ij}^{(1,1)}\right), & k_{ij}^{(1,2)}\cos\left(\xi-\psi_{ij}^{(1,2)}\right) \\ k_{ij}^{(2,1)}\cos\left(\xi-\psi_{ij}^{(2,1)}\right), & k_{ij}^{(2,2)}\cos\left(\xi-\psi_{ij}^{(2,2)}\right) \end{vmatrix}, \tag{8.78}$$

where $k_{ij}^{(p,q)}$ and $\psi_{ij}^{(p,q)}$ $(p,q=1,2)$ are the harmonic coefficients of influence of the force on the amplitude and phase of the displacement of the corresponding quasi-coordinate of the interaction. By virtue of the reciprocity condition we have $k_{ij}^{(1,2)} = k_{ji}^{(2,1)}$ and $\psi_{ij}^{(1,2)} = \psi_{ji}^{(2,1)}$. Inserting expressions (8.77) and (8.78) into eq. (8.20) we finally arrive at the following general formula for the partial vibrational moment

$$W_{ij} = \frac{1}{2}\left[G_i^{(1)}G_j^{(1)}k_{ij}^{(1,1)}\sin\left(\alpha_i - \alpha_j + \psi_{ij}^{(1,1)}\right) + \right. \tag{8.79}$$
$$G_i^{(1)}G_j^{(2)}k_{ij}^{(1,2)}\sin\left(\alpha_i + \gamma_j - \alpha_j + \psi_{ij}^{(1,2)}\right) + G_i^{(2)}G_j^{(1)}k_{ij}^{(2,1)}\sin\left(\alpha_i - \alpha_j - \right.$$
$$\left.\left.\gamma_i + \psi_{ij}^{(2,1)}\right) + G_i^{(2)}G_j^{(2)}k_{ij}^{(2,2)}\sin\left(\alpha_i - \alpha_j - \gamma_i + \gamma_j + \psi_{ij}^{(1,2)}\right)\right].$$

It follows from the latter expression that, generally speaking, in the system of exciters of two-dimensional harmonic forces there exists no phasing of the first kind. In particular, a synchronous-synphase regime does not exist even in the case of (8.68) when $\psi_{ij}^{(p,q)} = 0$. For their existence, the presence of the synphase component of the force is sufficient, i.e. $\gamma_i = 0, \pi$, see eq. (8.77).

8.5 Extremum properties of stationary synchronous motions

Historically, the averaged equations of the problem of weak interaction of anisochronous dynamical objects with one degree of freedom were cast in a form distinct from that of Sec. 8.2. This form does not use the concept of the dynamic influence matrix and thus is less convenient for the study of particular problems. On the other hand, by using this form we can obtain certain qualitative conclusions about stationary synchronous motions in systems of a more general structure in the presence of the interaction of both carrying and carried types.

Let us begin with the following equations for weak interaction of the dynamical objects of the considered type in terms of the "phase-frequency" variables, see for example eq. (8.14)

$$\begin{aligned} \dot{\omega}_i &= \mu e_i\left[\frac{\partial q_i}{\partial \varphi_i}Q_i + \frac{\partial}{\partial \varphi_i}\left(\Delta L + L^{(2)}\right)\right] + \mu^2\ldots, \\ \dot{\varphi}_i - \omega_i &= -\mu e_i\left[\frac{\partial q_i}{\partial \omega_i}Q_i + \frac{\partial}{\partial \omega_i}\left(\Delta L + L^{(2)}\right)\right] + \mu^2\ldots. \end{aligned} \tag{8.80}$$

Here, as before, the coordinates of the carrying and carried systems are presented as implicit functions of these variables. However, the partial derivatives with respect to φ_i and ω_i are taken without accounting for the dependence of quantities ΔL and $L^{(2)}$ on these coordinates. It is immaterial for the forthcoming analysis that the additional kinetic potential of the objects is represented in the form (8.6).

The equations for the carried system in the generating approximation, see eq. (7.21), are assumed to be linear in components of y and thus admit a quasi-periodic family of solutions for arbitrary $\omega_1, \dots, \omega_n$. This result enables the kinetic potential $L^{(2)}$ also to be represented as a function of the "phase-frequency" variables.

The averaged equations of motion in the considered $\sqrt{\varepsilon}$ vicinity of the principal resonance have, as before, the form (8.15), where

$$P_i = M_i - W_i, \quad R_i = N_i + U_i \tag{8.81}$$

and quantities M_i and N_i are given by formulae (8.18) and (8.19), whilst W_i and W_i^*, eq. (8.20), are obtained when the corresponding partial derivatives of the "total" additional kinetic potential of the system

$$L_0 = L^{(1)} + \Delta L + L^{(2)} \tag{8.82}$$

with respect to φ_i, ω_i is averaged over the synchronous phase $\varphi = \nu t$.

The averaged expression can be set in the form

$$\begin{aligned} \frac{\partial L_0}{\partial \varphi_i} &= \frac{\partial L_0}{\partial \alpha_i} - \left(\frac{\partial}{\partial \dot{x}'}\frac{\partial \dot{x}}{\partial \alpha_i} + \frac{\partial}{\partial x'}\frac{\partial x}{\partial \alpha_i}\right)\left(L^{(1)} + \Delta L\right) - \\ &\left(\frac{\partial}{\partial \dot{y}'}\frac{\partial \dot{y}}{\partial \alpha_i} + \frac{\partial}{\partial y'}\frac{\partial y}{\partial \alpha_i}\right) L^{(2)}, \end{aligned} \tag{8.83}$$

where it is taken into account that, with accuracy up to higher order of smallness, $L^{(1)}$ and $L^{(2)}$ do not depend explicitly on q_i, p_i and x respectively. Accounting for the equations of motion for the carrying and carried systems, the integrals with the vectorial factors $\frac{\partial \dot{x}}{\partial \alpha_i}$ and $\frac{\partial y}{\partial \alpha_i}$ are integrated by parts to yield

$$\begin{aligned} M\ddot{x} + B\dot{x} + Cx + \left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}'}\right)\Delta L + \mu \dots &= 0, \\ \left(\frac{d}{dt}\frac{\partial}{\partial \dot{y}'} - \frac{\partial}{\partial y'}\right) L^{(2)} - Q_y' + \mu^2 \dots &= 0. \end{aligned} \tag{8.84}$$

Matrix B is assumed to be not symmetric, i.e. $B \neq B'$. Averaging (8.83) leads to the following typical expression for the total vibrational moment

$$W_i = -\frac{\partial \Lambda}{\partial \alpha_i} + \left\langle \dot{x}' B' \frac{\partial x}{\partial \alpha_i} \right\rangle - \left\langle Q_y' \frac{\partial y}{\partial \alpha_i} \right\rangle, \tag{8.85}$$

where

$$\Lambda = \langle L_0 \rangle . \tag{8.86}$$

Quantity Λ is referred to as the averaged Hamilton action of the characteristics of the weak interaction in the system. Notice that in eq. (8.83) and in what follows we imply only the "total" partial differentiation with respect to the averaged variables α_i, Ω_i, whereas the averaging is carried out as above in Sec. 8.2. Thus, all averaged quantities are functions of $\alpha_1, \ldots, \alpha_n, \Omega_1, \ldots, \Omega_n$.

The analogous transformations yield

$$U_i = -\frac{\partial \Lambda}{\partial \Omega_i} + \left\langle \dot{x}' B' \frac{\partial x}{\partial \Omega_i} \right\rangle - \left\langle Q_y' \frac{\partial y}{\partial \Omega_i} \right\rangle . \tag{8.87}$$

Here

$$\Lambda = \Lambda^{(1)} + \Lambda^{(2)} + \Delta\Lambda, \tag{8.88}$$

where $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are the averaged Hamilton action of the carrying and carried systems respectively, whereas $\Delta\Lambda$ means the additional action of the objects caused by small oscillations of the carrying system

$$\Lambda^{(1)} = \left\langle L^{(1)} \right\rangle, \quad \Lambda^{(2)} = \left\langle L^{(2)} \right\rangle, \quad \Delta\Lambda = \langle \Delta L \rangle . \tag{8.89}$$

There are also two possible modified expressions for the total vibrational moment (8.85) and (8.87). In order to obtain them, we perform the virial transformation [83] of the equation for small oscillations of the carrying system (8.84). In other words, we calculate the scalar product of this equation and the vector-row x' and average the result over φ. Then, as $L^{(1)}$ and ΔL are quadratic and linear forms of components of x and $\dot{x}$, see eqs. (7.20) and (8.7), we obtain, after integration by parts, that

$$2\Lambda^{(1)} + \Delta\Lambda = \Gamma, \tag{8.90}$$

where

$$\Gamma = \langle x' B_a \dot{x} \rangle \tag{8.91}$$

is the averaged virial of the gyroscopic forces in the carrying system, where $B_a = \frac{1}{2}(B - B')$. Using identity (8.90) and eq. (8.88) we obtain, instead of eq. (8.85) that

$$\begin{aligned} W_i &= -\frac{\partial}{\partial \alpha_i} \left\langle \Lambda^{(2)} - \Lambda^{(1)} \right\rangle + \left\langle \dot{x}' B \frac{\partial x}{\partial \alpha_i} \right\rangle - \left\langle Q_y' \frac{\partial y}{\partial \alpha_i} \right\rangle , \\ W_i &= -\frac{\partial}{\partial \alpha_i} \left\langle \Lambda^{(2)} + \frac{1}{2}\Delta\Lambda \right\rangle + \left\langle \dot{x}' B_s \frac{\partial x}{\partial \alpha_i} \right\rangle - \left\langle Q_y' \frac{\partial y}{\partial \alpha_i} \right\rangle , \end{aligned} \tag{8.92}$$

where $B_s = \frac{1}{2}(B + B')$.

Further analysis is restricted to the most simple particular case in which the stationary synchronous regime in the system is characterised by vanishing the total vibrational moments, see eq. (8.68). To this end, as shown in Sec. 8.4, it is necessary that the objects are identical in a certain sense ($\operatorname{sign} e_i = \operatorname{sign} e$, $M_i = M$) and the non-potential forces of the carrying and carried system have a higher order of smallness, i.e. $B_s = 0, Q_y = 0$. Then, by virtue of the second equality in (8.92), the equations of the problem of autophasing take the form of conditions for stationarity of the potential function $D = \Lambda^{(2)} + \frac{1}{2}\Delta\Lambda$ for the generating phase shifts $\alpha_1, \dots, \alpha_n$

$$\frac{\partial D}{\partial \alpha_i} = 0. \tag{8.93}$$

Moreover, the criteria for stability of the synchronous regime of the first set, see for example eq. (8.28), which are not only necessary but also sufficient in this case (cf. eq. (8.68)), are obtained from the equation

$$e_i \sum_{j=1}^{n} d_{ij} a_j = \lambda_1^2 a_i, \tag{8.94}$$

where

$$d_{ij} = d_{ji} = \frac{\partial^2 D}{\partial \alpha_i \partial \alpha_j}. \tag{8.95}$$

Since the signs of the steepness coefficients of the backbone curve coincide for the objects ($\operatorname{sign} e_i = \operatorname{sign} e$) all n roots λ_1^2 of the determinant of (8.94) are real. In order to prove this, it is sufficient to rewrite this determinant in the following "symmetrised" form

$$\left| d_{ij}\sqrt{e_i e_j} - \lambda_1^2 \delta_{ij} \operatorname{sign} e \right| = 0. \tag{8.96}$$

All n roots of the determinant of system (8.94) have the same sign provided that the homogeneous quadratic form

$$D_2 = \frac{1}{2} \sum_{i,j=1}^{n} d_{ij} a_i a_j \tag{8.97}$$

is positive definite (positive semi-definite in the autonomous case) in the vicinity of zero. It follows from the equality that

$$\lambda_1^2 = 2D_2 \left(\sum_{i=1}^{n} \frac{a_i^2}{e_i} \right)^{-1}. \tag{8.98}$$

It is important that the real values $a_1, \dots, a_n$ in expression (8.97) can be understood to be the deviations of the phase shifts from their stationary values a_i^* in accordance with eq. (8.93). In this sense, form D_2 is a result of the expansion of the potential function in power series in terms of deviations $a_i = \alpha_i - \alpha_i^*$.

The positive-definiteness (or positive semi-definiteness in the autonomous case) of form D_2 is known, cf. [60], to be ensured if all n roots $\kappa_1, \dots, \kappa_n$ of the equation

$$|d_{ij} - \kappa \delta_{ij}| = 0 \tag{8.99}$$

have the same sign (but one zero root in the autonomous case). Therefore, we arrive at the following statement: for a stable synchronous phasing the potential function $D = \Lambda^{(2)} + \frac{1}{2}\Delta\Lambda$ has a maximum for $e_i > 0$ (minimum for $e_i < 0$) which is determined by quadratic terms in the power expansion of this function in the vicinity of a stationary point. In the general case $(e_i \neq e_j)$ the values of $\kappa_1, \dots, \kappa_n$ are not proportional to the squares of the first approximations to the characteristic exponents and thus do not allow us to judge the degree of stability of the regime.

The presence of the above extremum property enables us to write down the system of explicit inequalities which is equivalent to the stability conditions following from eq. (8.93). To this end, it is sufficient to apply the special linear transformation $a_i \to b_i$ with a triangular matrix [33] in order to represent quadratic form (8.97) in the following form

$$D_2 = \frac{1}{2}\left(d_1 b_1^2 + \frac{d_2}{d_1} b_2^2 + \dots + \frac{d_n}{d_{n-1}} b_n^2\right), \tag{8.100}$$

where $d_1, \dots, d_n$ are the sequential principal minor determinants of matrix d_{ij}

$$d_p = |d_{ij}|_{i,j=1,\dots,p} \quad (p = 1, \dots, n). \tag{8.101}$$

Representation (8.100) yields the necessary and sufficient conditions for stability of the synchronous regime in the form of Sylvester's inequalities

$$d_1 \operatorname{sign} e < 0, \dots, d_n \operatorname{sign} e < 0. \tag{8.102}$$

In the problem of internal synchronisation when $d_n = |d_{ij}| = 0$, the last inequality in (8.102) no longer exists. Assume now that some rigid kinematic constraints stabilising the phasing $\alpha_{j+1}^*, \dots, \alpha_n^*$ are imposed on the synchronous motion of the objects with numbers $j+1, \dots, n$. Simple analysis shows that in this case, due to eq. (8.102) the stable phasing of the objects with the numbers $1, \dots, j$ is not affected. Thus, imposing rigid kinematic constraints does not eliminate both phasing and synchronism. The situation changes drastically when such rigid constraints are imposed for example on the displacement of the elements of the carrying system.

We demonstrate now some particular expressions for the potential function which exist provided that, in addition to the above, small vibrations are not accompanied by gyroscopic forces $(B_a = 0)$

$$D = \Lambda, \quad D = \Lambda^{(2)} - \Lambda^{(1)}. \tag{8.103}$$

These expressions are a direct consequence of eq. (8.85) and the first equation in (8.92). The possibility of using Λ, eq. (8.88), as the potential function testifies that a stable synchronous regime is described by an extremum (minimum for $e < 0$ and maximum for $e > 0$) of the averaged

Hamilton action of the total system consisting of the objects of the carrying and carried systems, provided that the terms of order of ε included are considered. Indeed, inasmuch as the averaged values of the proper dynamic characteristics of motion in the generating approximation do not depend on the phase shifts, we can write

$$\frac{\partial \Lambda}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i} \langle L \rangle , \tag{8.104}$$

where L denotes the total kinetic potential of the system, see eq. (7.5).

The second representation in eq. (8.103) shows that the difference of the averaged actions of the carrying and carried systems can be taken as a potential function. This results in the typical situation illustrated in the following table.

	hard anisochronism ($e > 0$)	soft anisochronism ($e < 0$)
carrying system	min	max
carried system	max	min

This table allows us to determine the character of the extremum of $\Lambda^{(2)}$ or $\Lambda^{(1)}$ due to a stable phasing in the system of objects under weak interaction of the solely carrying or carried type. In particular, in a system of rigid anisochronous objects with weak interactions of the carrying body, action $\Lambda^{(1)}$ has a minimum under a stable synchronous phasing. Another proof of the considered extremum property was first suggested by Blekhman [16] in the problem of synchronisation in the system of objects with "uniform fast rotations" of a particular sort. There, this extremum property was referred to as the integral stability criterion of synchronous motion. In particular, the so-called "principle of minimum of the kinetic energy of the carrying (oscillatory) system" follows immediately from this integral criterion. For a class of simple vibrational facilities, see next chapter for inertial vibration exciters, the rigid carrying body has a soft shock-absorbing suspension and thus we can take $L^{(1)} = K^{(1)}, \Pi^{(1)} = 0$ with accuracy up to the values of higher orders of smallness. For this reason, we can assert from the very beginning that the antiphase rotation regime of identical vibration exciters in the four-vibrator-machine, depicted in Fig. 8.7, is stable. Indeed, in this case the carrying body is immovable in the first approximation and hence $\Lambda^{(1)} = \langle K^{(1)} \rangle = 0$.

The hard anisochronous objects are a peculiar kind of absorber of oscillations for the carrying system of pure inertial type. Quite the reverse, the same objects in the presence of a carried system of inertial type lead to maximum increase in amplitude. In the case of weak interaction of the soft anisochronous objects, the inherent tendency to stable single-frequency synchronous regime is of diametrically opposite character. In the case of

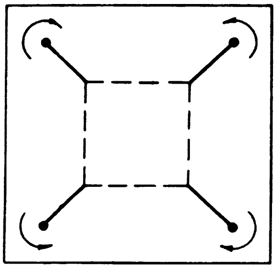

FIGURE 8.7.

interactions of two types their synchronising influence on the objects can partly or totally suppress each other. As a result, the stability margin decreases which results in an increasing probability of dephasing the synchronous motions of the objects, caused by non-potential random imperfections. In the limiting case, the system completely loses the synchronism.

As an example, let us consider the problem of synchronisation of two identical unbalanced rotors rotating about the same axis in the same direction. The rotors are driven by two identical electric motor which do not interact with each other. The centres of rotation of the rotors are coincident with the centre of mass of a massive free carrying body of mass M, see Fig. 8.8. The carried interactions between the rotors are performed by means of a particle m_0 attached at the vertex of a rhombus with side l. The sides of the rhombus are non-inertial rods connected by joints. The pure conservative model of this problem is studied in Sec. 4.2.

The small parameter of the problem is $\varepsilon = 0\,(m/M) = 0\,(m_0/m)$ where m denotes the mass of the rotor. In the generating approximation, the rotors rotate uniformly obeying the law

$$q_1 = \nu t + \alpha_1, \quad q_2 = \nu t + \alpha_2, \tag{8.105}$$

the synchronous frequency ν being equal, in the first approximation, to the partial frequency of rotation without interaction. The kinetic energies of the carrying and carried bodies, up to values of the order of ε^2, do not depend on time and are equal to the corresponding averaged actions

$$\begin{aligned} K^{(1)} &= \Lambda^{(1)} = \frac{m^2 l^2 \nu^2}{M}\left[1 + \cos\left(\alpha_1 - \alpha_2\right)\right], \\ K^{(2)} &= \Lambda^{(2)} = m_0 l^2 \nu^2 \left[1 + \cos\left(\alpha_1 - \alpha_2\right)\right]. \end{aligned} \tag{8.106}$$

The potential function introduced earlier is given by

$$D = \Lambda^{(2)} - \Lambda^{(1)} = m l^2 \nu^2 \left(\frac{m_0}{m} - \frac{m}{M}\right)\left[1 + \cos\left(\alpha_1 - \alpha_2\right)\right]. \tag{8.107}$$

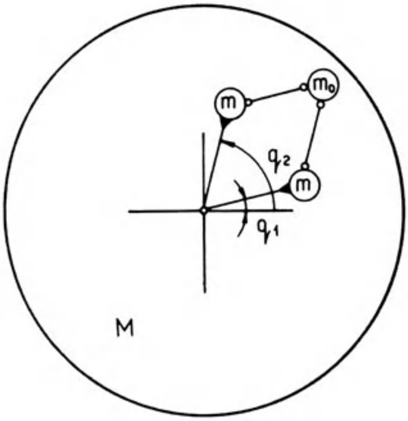

FIGURE 8.8.

Since the problem is autonomous in the stable synchronous regime, function D must have a maximum with respect to the difference of the phase shifts $\alpha_1 - \alpha_2$. Simple analysis leads to the following conclusions.

1. A stable antiphase regime of synchronous rotation of the rotors, under which the kinetic energies of the carrying and carried bodies vanish, exists in the case of prevailing carrying interactions: $m/M > m_0/m$.

2. A stable synphase regime under which the kinetic energies of the carrying and carried bodies achieve a maximum, is realised in the case of prevailing carried interactions: $m_0/m > m/M$.

3. When $m/M \approx m_0/m$ (an approximate compensation of the synchronising influences of interactions of different types) the system may lose the synchronism with a considerable probability.

Let us notice that the mentioned synchronous regimes in the pure conservative system of Sec. 4.2 describe a singular point of the type centered in the phase plane $\left(\dot{\vartheta}/\sqrt{h}, \vartheta\right)$, for which the first integral σ takes the maximum value equal to $2b$ for $m/M > m_0/m$ and $2a$ for $m/M < m_0/m$.

8.6 Synchronisation in a piecewise continuous system

The general theory developed in the previous Sections is generalised to the case of weak interaction of dynamic objects with discontinuities by means of the continuous carrying and carried systems. All the general results and formulae obtained above are fully applicable provided that in the original generating approximation all the proper coordinates and velocities (or momenta) q_i and $\dot{q}_i$ remain continuous. This means, in turn, that the proper Hamiltonian of the object H_i, as function of q_i, p_i, should be continuous

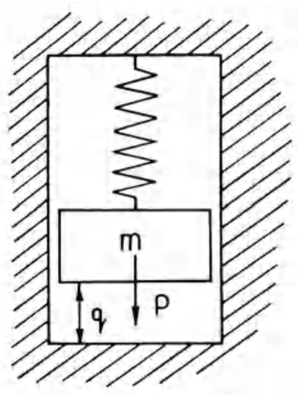

FIGURE 8.9.

whereas its derivatives $\frac{\partial H_i}{\partial q_i}$ and $\frac{\partial H_i}{\partial p_i}$ contain only discontinuities of the first kind.

A maximum non-linear case of the weak interaction of objects of the impact-oscillatory type when the proper velocities (or momenta) undergo finite jumps has certain peculiar features. Let us illustrate this for the example of the problem of weak interaction of exciters of one-dimensional impulse periodic actions on a continuous carrying system, see Fig. 8.9. Let us assume that the spring acts on the mass with a constant force P. If $q = 0$, then according to the stereomechanical theory of impact [75] the mass experiences a jump of the proper velocity

$$\dot{q}_+ = -R\dot{q}_-, \tag{8.108}$$

where subscripts $+$ and $-$ denote the corresponding quantity calculated just for and after the impact, respectively, and R denotes the coefficient of restitution. It is also assumed that the continuous non-potential force F is of the order of ε with respect to coordinate q and moreover $R = 1 - O(\varepsilon)$. The latter implies that the impact is nearly fully elastic. Adopting that the impact takes place at $t = 0$ we write down the equation of motion for the fixed carrying system in the interval of time between the impacts

$$m\ddot{q} + 2m\dot{q}_-\delta(t) = -P + \mu\left[F + m(1-R)\,\dot{q}_-\delta(t)\right]. \tag{8.109}$$

Here the number of the object (i) is omitted for the time being. This equation is conservative for $\varepsilon = 0$ and has a continuum of discontinuous solutions of libration type

$$q = \frac{P\varphi}{m\omega^2}\left(\pi - \frac{\varphi}{2}\right) \quad (\varphi = \omega t,\ +0 < t < T - 0). \tag{8.110}$$

For these motions

$$h = \frac{\pi^2 P^2}{2m\omega^2}, \quad e = -\frac{m\omega^4}{\pi^2 P^2} < 0, \tag{8.111}$$

and thus the considered objects are soft anisochronous ($e < 0$). The one-dimensional force

$$G = P\left[1 - 2\pi\delta\left(\varphi\right)\right] \tag{8.112}$$

generated by the object is impulse-periodic and has a zero mean value. Due to eq. (8.109), the generalised non-potential force corresponding to the proper coordinate is given by

$$Q = F - \pi\left(1 - R\right) P\delta\left(\varphi\right). \tag{8.113}$$

Assume that force F is an odd function of the proper velocity and that it can be represented using the series

$$F = \sum_{k=1,3,5,\ldots}^{\infty} \beta_k \dot{q}^k. \tag{8.114}$$

Inserting expressions (8.112) and (8.113) into formulae (8.18), (8.19) and (8.20) for the coefficients of the averaged equations (8.15), taking into account eq. (8.110) and integrating the result yields

$$\begin{aligned} M_i &= \frac{P}{m\Omega_i^2} f\left(\Omega_i\right) - \left(1 - R\right) \frac{\pi P^2}{2m\Omega_i^2}, \\ W_{ij} &= \pi P^2 K'_{ij}\left(\alpha_j - \alpha_i, \Omega_j\right), \\ N_i &= U_{ij} = 0. \end{aligned} \tag{8.115}$$

Here, according to Sec. 8.2, the dynamic matrix of influence of the $j-th$ object on the $i-th$ object becomes a scalar. Further, the following notation

$$f\left(\omega\right) = \left\langle\left(\pi - \varphi\right) F\right\rangle = \sum_{k=1,3,5,\ldots}^{\infty} \frac{\pi\beta_k}{k+2}\left(\frac{\pi P}{m\omega}\right)^k \tag{8.116}$$

is introduced.

All of the objects in the system are assumed to be identical. Let us now write down the averaged equation of motion of the separate object when the carrying system is fixed. Accounting for the expression for the steepness coefficient of the backbone curve, eq. (8.111), we obtain

$$\dot{\Omega} = \frac{1-R}{2\pi}\Omega^2 - \frac{\Omega^2}{\pi^2 P} f\left(\Omega\right). \tag{8.117}$$

A stationary solution of this equation $\Omega = \nu$ corresponds to periodic self-excited oscillations and satisfies the following condition

$$f(\nu) = \frac{\pi}{2}(1-R)P. \tag{8.118}$$

The stability condition for this solution is obtained, as usual, by varying eq. (8.117) and has the form $\frac{df}{d\nu} > 0$. In the case of a cubic characteristic ($\beta_5 = \beta_7 = \ldots = 0$) for existence and stability of the regime, it is necessary that the following inequalities

$$\beta_1 > 0, \quad \beta_3 < -\frac{80}{729}\frac{\beta_1^3}{(1-R)^2 P^2} \tag{8.119}$$

hold. Using the averaged equations for the problem of synchronisation of n identical objects of the considered type, mounted on a continuous carrying system, see eq. (8.15), we obtain

$$\begin{aligned} \dot{\Omega}_i &= \frac{1-R}{2\pi}\Omega_i^2 - \frac{\Omega_i^2}{\pi^2 P} f(\Omega_i) + \frac{2m\Omega_i^4}{\pi}\sum_{j=1}^{n} K_{ij}^*(\alpha_j - \alpha_i, \Omega_j), \\ \dot{\alpha}_i &= \Omega_i - \nu \quad (i = 1, \ldots, n). \end{aligned} \tag{8.120}$$

It is important that the influence function $K_{ij}(\varphi, \omega)$ is continuous with respect to both arguments. Its derivative $K_{ij}^* = \frac{\partial K_{ij}}{\partial \varphi}$ has discontinuities of the first kind at $\varphi = 0, 2\pi, 4\pi, \ldots$, see (8.13). For this reason, the right hand sides of the equations in (8.120) have finite jumps when $\alpha_i = \alpha_j$ and, in particular for the synchronous-synphase regime, when $\Omega_1 = \ldots = \Omega_n, \alpha_1 = \ldots = \alpha_n$. This is a typical feature of all problems of synchronisation of the dynamical objects of the impact-oscillatory type using a continuous carrying system.

Let us consider a simple problem of synchronisation of identical vibration exciters of the considered type which are symmetrically mounted on a free rigid body with three degrees of freedom. Let us assume that an impulse-periodic force Φ, eq. (5.24) acts along the direction y (i.e. the quasi-coordinate of the interaction of the right object) whereas oscillations of the carrying body in the directions of y and φ are controlled by a linear and torsional spring respectively. Oscillations of the carrying body are governed by the following equations

$$M\ddot{y} + c_y y = \Phi, \quad J\ddot{\varphi} + c_\varphi \varphi = r\Phi, \tag{8.121}$$

where M and J denote the mass and the central moment of inertia of the carrying body, respectively, c_y and c_φ denote rigidities of the corresponding springs, and the centre of mass lies in the middle of the straight line between the points where the forces are applied at the distance r for each point.

Assume that these rigidities are related to each other $c_y = \frac{M}{J} c_\varphi$. Then, the dynamic function of influence of the right object on the left one $K_{12} = y - r\varphi$ is determined as the $2\pi/\nu-$periodic solution of the following equation

$$K_{12}^{**} + \lambda^2 K_{12} = \frac{J - Mr^2}{JM\nu^2}, \tag{8.122}$$

where $\lambda = \frac{1}{\nu}\sqrt{\frac{c_y}{M}}$, and an asterisk denotes differentiation with respect to $\psi = \nu t$. Simple analysis shows that

$$K_{12} = \frac{J - Mr^2}{2JM\nu^2} \frac{\cos\lambda\left(\pi - \psi\right)}{\lambda \sin \pi\lambda} \quad \left(0 < \psi < 2\pi\right). \tag{8.123}$$

By virtue of eq. (8.95)

$$K_{12}^{*}|_{\lambda=0} = \frac{J - Mr^2}{JM\nu^2} u\left(\psi\right), \tag{8.124}$$

where u is a $2\pi-$periodic function of phase ψ with a zero mean value which, for $0 < \psi < 2\pi$, is given by

$$u = \frac{1}{2}\left(1 - \frac{\psi}{\pi}\right). \tag{8.125}$$

At the points of discontinuity $\psi = 0, 2\pi, 4\pi, \ldots$ one can put $u = 0$. Let us notice that one can not pass to the limit $\lambda \rightarrow 0$ by using eq. (8.121). Formula (8.124) is obtained if one integrates the original equations in eq. (8.121) for $c_y = c_\varphi = 0$ and we omit the integration constant and the linear increasing component of the obtained discontinuous solution.

By virtue of the above, the equations in (8.120) admit a stationary synphase ($\alpha_1 - \alpha_2 = 0$) and antiphase ($\alpha_1 - \alpha_2 = \pi$) solutions whose averaged frequency coincides with the frequency of oscillation without interaction ($\Omega_1 = \Omega_2 = \nu$, see eq. (8.118)). The antiphase solution is ordinary and for its stability the fulfillment of the anisochronous criteria of the first and the second sets is necessary, see eqs. (8.29) and (8.33)

$$\frac{df}{d\nu} > 0, \quad \frac{Mr^2}{J} > 1. \tag{8.126}$$

Strictly speaking, the method of investigating local stability is not applicable for the synphase, essentially discontinuous solution and thus the direct varying eq. (8.120) is not correct. In the vicinity of the point $\alpha_1 - \alpha_2, \Omega_1 = \Omega_2 = \nu$, equations (8.120) can be reduced to the essentially nonlinear "variational equation"

$$\ddot{\alpha} + \frac{\nu^2}{\pi^2 P}\frac{df}{d\nu}\dot{\alpha} + \frac{2m\nu^2}{\pi}\frac{J - Mr^2}{JM} \operatorname{sign}\alpha = 0, \tag{8.127}$$

where $\alpha = \alpha_1 - \alpha_2$. Simple investigation of this equation leads to the inequalities

$$\frac{df}{d\nu} > 0, \quad \frac{Mr^2}{J} < 1. \tag{8.128}$$

Since the energy dissipation under oscillations of the carrying body is not considered in this example, the selected synchronous motions satisfy the extremum property formulated in the previous section. In other words, for stable motion the kinetic energy of the carrying body averaged over the period achieves a maximum with respect to the phase shift α. Thus, for a stable synchronous regime the soft anisochronous objects, unlike hard anisochronous ones of the type of inertial vibration exciters, result in the maximum build-up of oscillations of the free carrying body. In the stable synchronous-synphase regime this maximum is of non-analytical character since the derivative $\dfrac{\partial L^{(1)}}{\partial \alpha} \sim \operatorname{sign} \alpha$ experiences a jump.

The problems of synchronisation of asynchronous dynamic objects influenced by a massive carrying system of the piecewise-continuous (discontinuous) type are of crucial importance from the perspective of practical applications. Such problems applied to the analysis of synchronisation of inertial vibration exciters for vibration facilities are considered in the next chapter. The equations of motion of the objects can be cast in the form of eq. (8.80). However, the procedure of removing the non-critical fast variables of the carrying system is not applicable here. For this reason, one should use the local method of small parameters developed by Lyapunov and Poincaré, whose details are given in Secs. 5.7 and 5.8 for the systems under consideration.

Let us assume that the equations for the discontinuity surfaces depend only on the coordinates and velocities of the carrying system. The original physical coordinates are smooth, whereas the velocities are either smooth or experience jumps of the order of unity. In the first case, small oscillations of the carrying system are accompanied by action of piecewise-continuous elastic or dissipative forces (forces of dry friction), whilst in the second case one has to deal with the carrying system of the vibroimpact type. In the latter case, the proper generalised velocities of the objects experience jumps of the order of ε. Applying the approaches developed in Secs. 5.7 and 5.8 allows us to split the procedure of investigation of the periodic (synchronous) solution into the following steps.

1. Construction of the generating family of periodic solutions. As above, cf. eq. (8.80), in the original approximation $\omega_i = \nu, \varphi_i = \nu t + \alpha_i$. The periodic solution of the equation for oscillation of the carrying system is constructed by exact semi-inverse methods of stepwise integration under arbitrary values of $\alpha_1, \dots, \alpha_n$ with the conditions of discontinuity and periodicity being taken into account. The conditions for the existence and local stability of the considered solution depending upon $\alpha_1, \dots, \alpha_n$ are

established. This is a principal difference of a piecewise-continuous (or in general an essentially non-linear) system from a continuous one. Another essential feature of the piecewise-continuous carrying system for all vibro-impact ones is the existence of subharmonic oscillations whose period is related to the averaged period of the proper fast phases by whole numbers.

2. Equations for determining the generating phasing of the synchronous motions of the objects, see eq. (5.131),

$$P_i\left(\alpha_1,\dots,\alpha_n,\nu\right)=0 \quad (i=1,\dots,n)\,, \tag{8.129}$$

as well as those for values R_i, eq. (5.146), are obtained by formally averaging the right hand sides of equations (8.80) over the period (or a subperiod). In the vibroimpact case, the integration is understood in the generalised sense since, due to eq. (8.7),

$$\nu\int\limits_{t_1-0}^{t_1+0}\frac{\partial\Delta L_i}{\partial\varphi_i}dt=\Delta\dot{x}_i'B_i^{(1)}\,, \tag{8.130}$$

where t_1 denotes the time instant of the discontinuity, $\varphi_i=\nu t+\alpha_i$ and $\Delta\dot{x}_i$ denotes a jump of the vectorial quasi-velocity of the $i-th$ object at the jump instant.

3. Determination of the anisochronous criteria of stability of the first and second sets due to eqs. (5.143) and (5.150). According to eq. (5.143) there remains thc inequality of the first set

$$\frac{\partial\left(P_1-P_2\right)}{\partial\alpha_1}<0 \tag{8.131}$$

in the case of two objects. Then the stability condition (5.150) takes the following form

$$\frac{\partial P_1}{\partial\alpha_1}\frac{\partial' P_2}{\partial\nu}+\frac{\partial P_2}{\partial\alpha_2}\frac{\partial' P_1}{\partial\nu}>0. \tag{8.132}$$

Let us consider, as an example, a planar problem of synchronisation of two identical exciters mounted on a rigid body, see Fig. 8.10. Its vertical oscillations are damped by a dry friction force. In the original approximation the vibration exciters rotate uniformly, that is $\varphi_i=\nu t+\alpha_i$ $(i=1,2)$. In this approximation, the equations for small oscillations of the carrying body are given by

$$\begin{aligned}M\ddot{x}&=me\nu^2\left[\cos\left(\nu t+\alpha_1\right)+\cos\left(\nu t+\alpha_2\right)\right],\\ M\ddot{y}+F\,\mathrm{sign}\,\dot{y}&=me\nu^2\left[\sin\left(\nu t+\alpha_1\right)+\sin\left(\nu t+\alpha_2\right)\right],\\ J\ddot{\varphi}&=-mer\nu^2\left[\sin\left(\nu t+\alpha_1\right)-\sin\left(\nu t+\alpha_2\right)\right],\end{aligned} \tag{8.133}$$

where M and J are the mass and the central moment of inertia of the carrying body, F denotes the absolute value of the dry friction force, m

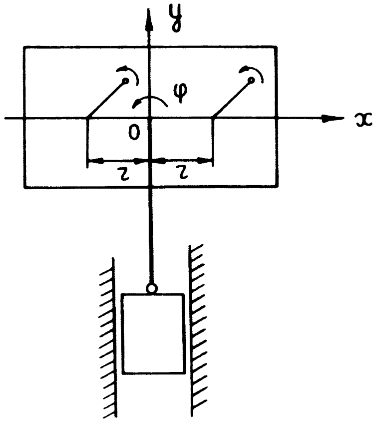

FIGURE 8.10.

and e are the mass and the eccentricity of the exciters and the spring rigidity is neglected. The second, essentially non-linear equation in (8.133) can be recast in the following non-dimensional form

$$u'' + f \operatorname{sign} u' = \cos\alpha \sin\tau, \tag{8.134}$$

where

$$\tau = \nu t + \frac{\alpha_1 + \alpha_2}{2}, \quad u = \frac{My}{2me}, \quad \alpha = \frac{\alpha_1 - \alpha_2}{2}, \quad f = \frac{F}{2me\nu^2}. \tag{8.135}$$

A solution of this equation having two instantaneous changes in the sign of the velocity and satisfying the periodicity condition $u(\tau) = -u(\tau + \pi)$ exists and is stable if

$$\cos\alpha > \sqrt{1 + \frac{\pi^2}{4} f}. \tag{8.136}$$

At the time instant

$$\tau = \gamma = \arccos\frac{\pi f}{2\cos\alpha} \tag{8.137}$$

the sign of u' changes from minus to plus and the non-dimensional coordinate achieves its minimum $u = -a$ where

$$a = \sqrt{\cos^2\alpha - \frac{\pi^2 f^2}{4}}. \tag{8.138}$$

Within the next half-period $\gamma < \tau < \pi + \gamma$ we have

$$u' = f\left(\frac{\pi}{2} + \gamma - \tau\right) - \cos\alpha\cos\tau. \tag{8.139}$$

In the problems of synchronisation of the inertial vibration exciters one can obtain equations for determining the possible autophasing of their rotation if the Lagrange's equations for rotation of the exciters are averaged over the period along the trajectories of the generating solution. Then, these equations take the form

$$I\ddot{\varphi}_i = L(\dot{\varphi}_i) - D(\dot{\varphi}_i) - me\left[\ddot{x}_i \sin\varphi_i - \ddot{y}_i \cos\varphi_i\right], \quad (i = 1, 2), \tag{8.140}$$

where I is the moment of inertia of the exciters, $x_{1,2} = x, y_{1,2} = y \pm r\varphi$ are the quasi-coordinates of interaction. The driving torque L and the anti-torque moment of resistance in the bearings D are assumed to be dependent only on the proper angular velocities, see Sec. 10.2. Taking into account eqs. (8.133) and (8.139) and averaging the above equations yields

$$P_{1,2} = \frac{1}{I}\left[L(\nu) - D(\nu) \pm \pi\nu^2 m^2 e^2 \left(\frac{2}{M} - \frac{r^2}{J}\right)\sin 2\alpha - \frac{8m^2e^2\nu^2}{M} f \sin(\gamma \pm \alpha)\right] = 0. \tag{8.141}$$

System (8.141) admits a solution which describes a synchronous-synphase rotation of the exciters ($\alpha = 0$). The first approximation to the synchronous frequency of this motion is determined from the equation

$$L(\nu) = D(\nu) + \frac{8m^2e^2\nu^2}{M} f\sqrt{1 - \frac{\pi^2 f^2}{4}}, \tag{8.142}$$

which represents energy balance. The stability condition of the first set, eq. (8.131), is as follows

$$\frac{Mr^2}{J} > 2\left(1 - f^2\right). \tag{8.143}$$

Here it is taken into account that, due to eq. (8.137), $\left.\dfrac{d\gamma}{d\alpha}\right|_{\alpha=0} = 0$. The stability condition of the second set, eq. (8.132), is obtained by directly differentiating eq. (8.142) with respect to ν

$$\frac{dL}{d\nu} < \frac{dD}{d\nu} + \frac{4m^2e^2\nu\pi^2}{M} \frac{f^3}{\sqrt{1 - \dfrac{\pi^2 f^2}{4}}}. \tag{8.144}$$

In addition to this, it is necessary to bear in mind the following condition for the existence and stability of the generating synchronous-synphase solution

$$f < \frac{1}{\sqrt{1 + \pi^2/4}}, \tag{8.145}$$

which follows from eq. (8.136) for $\alpha = 0$.

9
Synchronisation of inertial vibration exciters

9.1 Inertial vibration exciter generated by rotational forces

The most simple and natural harmonic exciter is an unbalanced rotor whose axis of rotation is rigidly bound to one of the bodies of the carrying system, see Fig. 9.1. The exciter has two linear quasi-coordinates of interaction x_1 and x_2 which are measured in the plane of rotation in two orthogonal directions. The kinetic energy of the exciter has the form

$$K = \frac{J}{2}\dot{q}^2 - m\left[\mu r\left(\dot{x}_1 \sin q - \dot{x}_2 \cos q\right) - \frac{\mu^2}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)\right]. \tag{9.1}$$

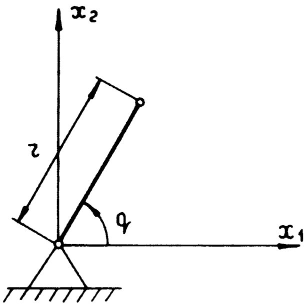

FIGURE 9.1.

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Here $\mu = 1$ is the formal indicator of smallness of order ε. In addition, m, J, r denote the mass, moment of inertia and eccentricity of the rotor, respectively, and q is the proper angular coordinate. The subscript i, which is the number of the exciter, is omitted here and up to eq. (9.6). In the case of small spatial oscillations, the expression for K gains an additional term which has no influence on the character of the averaged equations of the first approximation.

Since the potential energy of the rotor is equal to zero, the exact expressions for the components of the two-dimensional force exerted by the rotor on the carrying system are as follows

$$\begin{aligned} G_1 &= -\frac{d}{dt}\frac{\partial K}{\partial \dot{x}_1} = -mr\left(r\cos q\right)^{\cdot\cdot} - \mu m \ddot{x}_1, \\ G_2 &= -\frac{d}{dt}\frac{\partial K}{\partial \dot{x}_2} = -mr\left(r\sin q\right)^{\cdot\cdot} - \mu m \ddot{x}_2. \end{aligned} \tag{9.2}$$

The second terms on the right hand sides of the equations in (9.2) characterise the translational force of inertia and are of the order of ε.

The generalised force for the proper angle coordinate of the exciter is usually represented in the form $Q = L - D$, where L denotes the torque moment of the electric motor and D denotes the moments of the frictional forces in the bearing of the axis. The moment of an asynchronous electric motor is expressed in terms of the electric variables (currents and magnetic-flux linkage) which are determined from the special equations constructed for the windings of the rotor and stator in accordance with the laws of electrical engineering [49]. Under some rather general assumptions, these inhomogeneous equations are linear in the electric variables and can be made to be autonomous by certain replacements accounting for the axial symmetry of the motor. It is important that the coefficients of these equations depend on the instantaneous angular velocity $\omega = \dot{\varphi}$ which is a slow mechanical variable. Thus, while solving the problem of weak interaction of the considered objects in the neighbourhood of an internal resonance, we can apply an infinite process of removing electric variables as described in general in Sec. 5.3. As a result we obtain $L = L(\omega)$, this dependence being referred to (in electrical engineering) as the static characteristic of the electric motor.

The moment of resistance of the rotation of the rotor is ordinarily caused by the forces of dry friction in the bearing. If the forces of the viscous damping hinder the rotation of the rotor, then their total moment is dependent on the instantaneous angular velocity $\omega = \dot{\varphi}$ and thus can be deemed to be part of the torque moment $L(\omega)$. Thus, the reactive torque due to dry friction is proportional to the absolute value of force G transmitted by the exciter to the carrying system. By virtue of eq. (9.2) up to values of the higher order of smallness, we obtain

$$D = -fr\sqrt{G_1^2 + G_2^2}\,\mathrm{sign}\,\omega = -\left(D_0 + \mu\Delta D\right)\mathrm{sign}\,\omega, \tag{9.3}$$

where the coefficient of dry friction in the bearing f is, in general, a function of ω. Furthermore,

$$D_0 = fmr^2 \left(\omega^4 + \dot{\omega}^2\right)^{1/2}, \quad \Delta D = fmr \frac{\ddot{x}_1 (\cos q)^{\cdot\cdot} + \ddot{x}_2 (\sin q)^{\cdot\cdot}}{\sqrt{\omega^4 + \dot{\omega}^2}}. \tag{9.4}$$

The quasi-conservative concept in the synchronisation theory of inertial vibration exciters involves the assumption of an approximate balance of the electric motor torque and the anti-torque moment in the considered vicinity of the resonance, so that

$$L(\omega) - D_0 \operatorname{sign} \omega = O(\varepsilon). \tag{9.5}$$

In other words, the non-potential generalised force is a small quantity of order ε.

In the generating approximation ($\varepsilon \to 0$) the carrying system is "immovable" and the non-potential forces are "absent". The angular coordinate has a meaning of an uniformly rotating fast phase $q = \varphi = \omega t + \alpha$. The proper energy of the vibration exciter is $h = \frac{1}{2} J\omega^2$ and thus the positive steepness coefficient of the backbone curve is $e = 1/J$. In this approximation the vibration exciters generate two-dimensional harmonic (uniformly rotating) forces, which, due to eq. (9.2), are given by

$$G_i = g_i \begin{vmatrix} \cos \varphi_i \\ \sin \varphi_i \end{vmatrix} \quad (\varphi_i = \nu t + \alpha_i), \tag{9.6}$$

where the force amplitudes are denoted by $g_i = m_i r_i \nu^2$. The partial vibrational moment in the problem of synchronisation of inertial exciters is obtained from eq. (8.79) if one sets $G_i^{(1)} = G_i^{(2)} = g_i, \gamma_i = \pi/2$

$$W_{ij} = \frac{g_i g_j}{2} \left[k_{ij}^{(1.1)} \sin\left(\alpha_i - \alpha_j + \psi_{ij}^{(1.1)}\right) + k_{ij}^{(1.2)} \cos\left(\alpha_i - \alpha_j + \psi_{ij}^{(1.2)}\right) - k_{ij}^{(2.1)} \cos\left(\alpha_i - \alpha_j + \psi_{ij}^{(2.1)}\right) + k_{ij}^{(2.2)} \sin\left(\alpha_i - \alpha_j + \psi_{ij}^{(2.2)}\right)\right]. \tag{9.7}$$

The work of the proper non-potential forces averaged over the period, due to eq. (8.18) and $\left.\dfrac{\partial q_i}{\partial \varphi_i}\right|_{\varepsilon=0} = 1$ are equal to

$$M_i = L_i - D_i - \Delta D_i, \tag{9.8}$$

where $L_i(\nu)$ denotes static characteristic of the $i-th$ electric motor, see eq. (9.4)

$$D_i = m_i f_i r_i^2 \nu^2, \quad \Delta D_i = -\frac{f_i}{\nu^2} \langle \ddot{x}_i' G_i \rangle. \tag{9.9}$$

Here $\langle\ \rangle$ denotes averaging over $\xi = \nu t$, and x_i is a two-dimensional vector of quasi-coordinates of the $i - th$ exciter determined by eq. (8.12). This results in the additional averaged anti-torque moment

$$\Delta D_i = \sum_{j=1}^{n} \Delta D_{ij}\left(\alpha_i - \alpha_j, \Omega_i, \Omega_i\right), \tag{9.10}$$

where ΔD_{ij} describes the average non-potential anti-torque moment of the $i - th$ exciter due to rotation of the $j - th$ exciter and is equal to

$$\Delta D_{ij} = -\frac{f_i}{2\pi} \int_0^{2\pi}\int_0^{2\pi} G_i'\left(\varphi + \alpha_i, \Omega_i\right) K_{ij}^{**}\left(\varphi - \psi, \Omega_j\right) G_i\left(\psi + \alpha_j, \Omega_j\right) d\varphi d\psi. \tag{9.11}$$

Here a double asterisk denotes the second derivative with respect to φ, see eq. (8.20). Evaluating the integrals in the latter equation and taking into account eqs. (8.78) and (9.6) we obtain

$$\Delta D_{ij} = \frac{1}{2} f_i g_i g_j \left[k_{ij}^{(1.1)} \cos\left(\alpha_i - \alpha_j + \psi_{ij}^{(1.1)}\right) - k_{ij}^{(1.2)} \sin\left(\alpha_i - \alpha_j + \psi_{ij}^{(1.2)}\right)\right.$$
$$\left. + k_{ij}^{(2.1)} \sin\left(\alpha_i - \alpha_j + \psi_{ij}^{(2.1)}\right) + k_{ij}^{(2.2)} \cos\left(\alpha_i - \alpha_j + \psi_{ij}^{(2.2)}\right)\right]. \tag{9.12}$$

The quantities P_i on the right hand sides of the averaged equations for Ω_i are cast as follows

$$P_i = L_i - D_i - \sum_{j=1}^{n} V_{ij}, \tag{9.13}$$

where the difference $L_i\left(\Omega_i\right) - D_i\left(\Omega_i\right)$ characterises the excessive "static" torque of the $i - th$ exciter. When the carrying system does not move, then $L_i = D_i$, and this equation determines the partial frequency of the rotation of the rotor in the case of no interaction. Quantities V_{ij} determine the "complete" dynamic action of the $j - th$ exciter on the $i - th$ exciter and is equal to

$$V_{ij} = \frac{g_i g_j}{2\cos\rho_i} \left[k_{ij}^{(1.1)} \sin\left(\alpha_i - \alpha_j + \rho_i + \psi_{ij}^{(1.1)}\right) + \right. \tag{9.14}$$
$$k_{ij}^{(1.2)} \cos\left(\alpha_i - \alpha_j + \rho_i + \psi_{ij}^{(1.2)}\right) -$$
$$\left. k_{ij}^{(2.1)} \cos\left(\alpha_i - \alpha_j + \rho_i + \psi_{ij}^{(2.1)}\right) + k_{ij}^{(2.2)} \sin\left(\alpha_i - \alpha_j + \rho_i + \psi_{ij}^{(2.2)}\right)\right],$$

where $\rho_i = \arctan f_i$ is the angle of friction and lies in the interval $(0, \pi/2)$. The stability criteria of the stationary synchronous regime are obtained by means of relationships (8.29), (8.33) and (8.34), provided that the replacement $W_{ij} \to V_{ij}, M_i \to L_i \to D_i$ is performed.

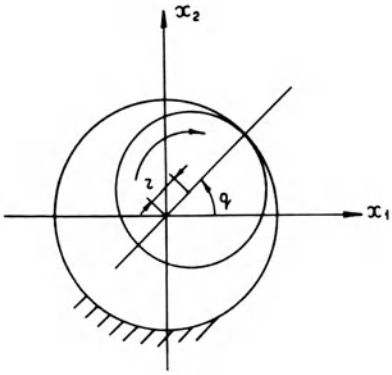

FIGURE 9.2.

In the case of an inertial vibration exciter, the mechanism of interaction has a non-conservative nature even if the friction in the carrying system is absent $\left(\psi_{ij}^{(p.q)} = 0\right)$. In practice, the values of the angles of friction ρ_i are rather small for conventional misbalances (active vibration exciters). Hence one can put $V_{ij} = W_{ij}, M_i = L_i - D_i$ with an acceptable degree of accuracy. This is what is accepted in the simple examples of the previous chapter, see Sec. 8.4. An exception is the passive planetary vibration exciter shown in Fig. 9.2 which can be studied in the framework of the suggested theory under assumptions different from the above, see Sec. 7.1.

Let us consider a massive planetary vibration exciter which has a small radial clearance. The mass m, the central moment of inertia J and the eccentricity r have orders of $1/\varepsilon, 1/\varepsilon^2$ and ε respectively. In contrast to eqs. (9.1) and (9.2), the total kinetic energy and the components of the two-dimensional force are given by

$$\begin{aligned} K &= \frac{Jr^2}{2R^2}\dot{q}^2 + \mu\left[\frac{mr^2}{2}\dot{q}^2 - mr\dot{q}\left(\dot{x}_1\sin q - \dot{x}_2\cos q\right) + \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right)\right], \\ G_1 &= mr\left(\cos q\right)^{\cdot\cdot} + m\ddot{x}_1, \quad G_2 = mr\left(\sin q\right)^{\cdot\cdot} + m\ddot{x}_2. \end{aligned} \tag{9.15}$$

Thus, both components in the expressions for G_1 and G_2 have the order of unity. This does not affect the final results since the moment of the rolling resistance is proportional to the normal component of force G rather than the absolute value of G, that is

$$D = f'RN, \quad N = G_1\cos q + G_2\sin q. \tag{9.16}$$

Here the radius of the roller $R = 0\left(1\right)$ and the coefficient of the rolling friction is assumed to take a value of the order of ε. Then, instead of eq. (9.3), we arrive at the formula

$$D = -\mu\left(D_0 + \Delta D\right)\operatorname{sign}\omega, \tag{9.17}$$

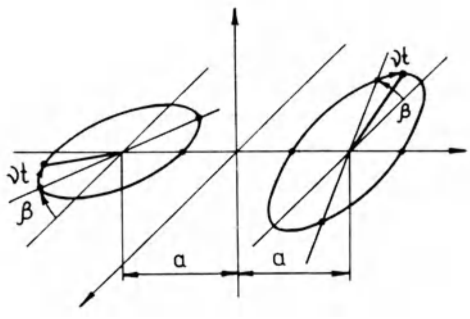

FIGURE 9.3.

the formulae for D_0 and ΔD at $\varepsilon = 0$ being identical to eq. (9.4) provided that the replacement $f \to f'R/r$ is performed. As D_0 is small, the quasi-conservative statement of the problem is correct despite the absence of the compensating torque of the electric motor ($L = 0$). It is also essential that the equivalent angle of friction for the planetary vibration exciter $\rho_i = \arctan \dfrac{f_i R}{r}$ is not small. For this reason, the corresponding components in eq. (9.14) can not be neglected.

Due to eqs. (8.9) and (9.15) the two-dimensional force excited by an arbitrary $i-th$ planetary vibration exciter is

$$G_i = G_i^{(0)} - m_i F_{m,l} \ddot{x},$$

where $G_i^{(0)}$ is given by eq. (9.6). The equation for small oscillation of the carrying system (8.8) should be sought in the form

$$\left(M + \sum_{i=1}^{n} m_i F'_{m,l} F_{m,l} \right) \ddot{x} + B\dot{x} + Cx = \sum_{i=1}^{n} F'_{m,l} G_i^{(0)}.$$

The dynamic matrix should also be sought under this condition.

This is also valid under the following assumptions $J = 0\,(1/\varepsilon)\,, m = 0\,(1/\varepsilon)\,, r = 0\,(\varepsilon)\,, R = 0\,(\sqrt{\varepsilon})\,, f' = 0\,(\sqrt{\varepsilon})$, different from those made above. In both cases the proper part of the kinetic energy of the vibration exciter at $\varepsilon = 0$ is value of the order of unity.

The harmonic vibration exciters of the different types can be obtained by utilising two or more unbalanced rotors which are rigidly coupled either kinematically or electrically. For example, a paired rotor with two identical misbalances whose rotation is synchronised and synphase in the way shown in Fig. 9.3 transmits the following two-dimensional harmonic rotational force

$$G = 2mr\nu^2 \begin{vmatrix} \cos\beta \\ a\sin\beta \end{vmatrix} \cos\xi \quad (\xi = \nu t)\,, \tag{9.18}$$

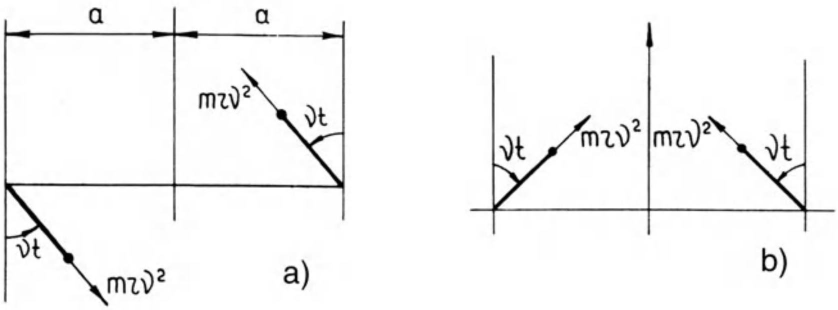

FIGURE 9.4.

to the carrying system. Here, $2a$ and β denote the distance between the axes of the misbalances and the angle between the plane perpendicular to the exciter axis and the planes of the rotation of the misbalances, respectively. Clearly, the axes of rotation of both misbalances are fixed to a rigid body which is an element of the carrying system. In the particular cases when $\beta = 0, \pi/2$, i.e. when both misbalances rotate in the same plane, cf. Fig. 9.4, the exciter under consideration generates only a moment or only a force of constant direction.

Thus, in general, this facility is one of the examples of excitation of a harmonic synphase rotational force. The exciter of a rotational force of another, antiphase type

$$G = 2mr\nu^2 \left| \begin{array}{c} \cos\xi \\ a\sin\xi \end{array} \right| \tag{9.19}$$

is a paired rotor shown in Fig. 9.5. An exciter of a rotating harmonic

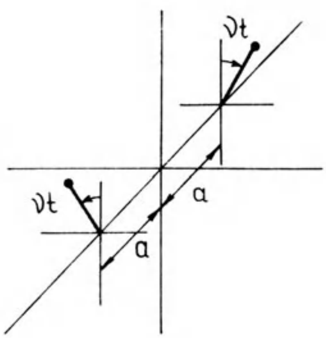

FIGURE 9.5.

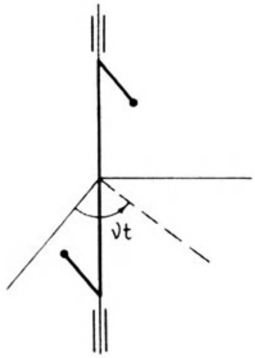

FIGURE 9.6.

moment

$$G = 2mra\nu^2 \begin{vmatrix} \cos\xi \\ \sin\xi \end{vmatrix} \tag{9.20}$$

is obtained if equal misbalances are attached to the common axis of rotation in the opposite directions, see Fig. 9.6.

A rotating mechanical moment excited by a two-frame gyroscope with an axisymmetric rotor [76] is of another physical nature. The massive rotor and inertialess frame are assumed to be driven independently by electric motors. The suggested theory is applicable to the considered facility without modifications if we assume that the electric motor has unbounded power. Thus its own relative angular velocity n is constant and considerably exceeds the angular velocity of the frame, i.e. $n \gg \nu$. The resulting gyroscopic moment written in projections on the quasi-coordinates of the interaction is equal to

$$G = An\nu \begin{vmatrix} \cos\xi \\ \sin\xi \end{vmatrix}, \tag{9.21}$$

where A denotes the axial moment of inertia of the rotor. The main technical advantage of the gyroscopic vibration exciter lies in the ease of adjusting the amplitude of the moment by changing the angular velocity n for a fixed value of ν, see Fig. 9.7.

9.2 The case of a single vibration exciter mounted on a carrying system

As indicated above, the averaged equations of the first approximation, see eq. (8.15), allow us to determine the motion with accuracy up to values

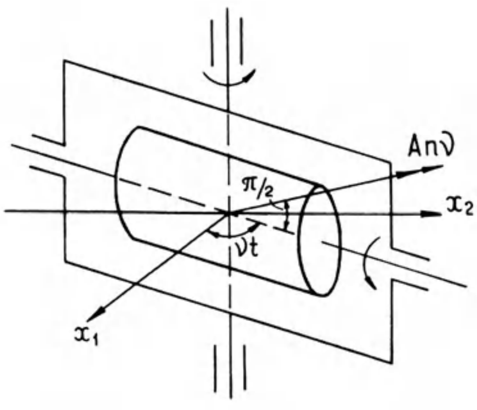

FIGURE 9.7.

of order $\sqrt{\varepsilon}$ in the $\sqrt{\varepsilon}$–vicinity of the considered resonance within a time interval of order $1/\sqrt{\varepsilon}$. Thus, using these equations provides us with the possibility to construct a stationary solution (synchronous regime) with an appropriate accuracy and to study its stability. However, in the simplest case the averaged equation also turns out to be valid for the analysis of the transient processes. The situation considered is that of the interaction of a single dynamical object (vibration exciter with a limited supply) with a massive carrying system of the oscillatory type.

Let us assume, as above, that dry friction in the bearings of the single vibration exciter is small, so that, see eq. (8.20),

$$V = W = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} G'(\varphi, \Omega) \frac{\partial K(\varphi - \psi, \Omega)}{\partial \varphi} G(\psi, \Omega)\, d\varphi d\psi. \tag{9.22}$$

It is clear that this value characterises an averaged dynamical action of the exciter on itself through the carrying system. Because in this particular case $W = W(\omega)$, with the adopted accuracy we can consider the first equation in (8.15) separately

$$J\dot{\Omega} = L - D - W, \tag{9.23}$$

$L(\Omega)$ describing the static characteristic of the driving motor and the reaction torque of the viscous friction in the bearings and $D = mf^2r^2\Omega^2$. The variables in this first-order equation are separated. Thus, assuming $t = 0, \Omega = \omega_0$ we obtain the equation for the angular velocity in the following implicit form

$$\int_{\omega_0}^{\Omega} \frac{J d\Omega}{L - D - W} = t. \tag{9.24}$$

This expression can be used to investigate the process of speed-up of the rotor of the exciter for zero initial velocity $\omega_0 = 0$. Indeed, at the beginning of this process the driving torque considerably exceeds the reaction torque, so that the necessary condition for slow change in the angular velocity of the rotor $\left(\dot{\Omega} = O(\varepsilon)\right)$ is not satisfied. At the same time, investigation of the process of run-down, that is when the motor is switched-off ($L = 0$) at the initial time instant, proves to be very efficient. The equation for run-down has the form

$$J\int_{\Omega}^{\omega_0} \frac{d\Omega}{D+W} = t. \tag{9.25}$$

The main focus comprises the process of transition of the carrying system through the resonance which is accompanied with an abrupt increase in the oscillation intensity. Let us consider, for example, the problem of the interaction of the exciter of the harmonic force of constant direction, Fig. 9.4b, with the simplest carrying system which consists of a mass mounted on a linear viscoelastic suspension, Fig. 8.2, which is idealised by a single degree of freedom. In this case, see eq. (8.55),

$$\begin{aligned} G &= 2mr\Omega^2\cos\varphi, \quad K(\varphi,\omega) = \frac{\cos(\varphi-\xi)}{\sqrt{\left(c-M\Omega^2\right)^2+n^2\Omega^2}}, \\ \tan\xi &= \frac{\Omega n}{c-M\Omega^2}, \end{aligned} \tag{9.26}$$

where m, r denote respectively the mass and the eccentricity of the misbalance, M is the mass of the carrying system, and c, n denote the coefficients of the rigidity and viscosity of the suspension respectively. Inserting these expression into eq. (9.22) yields

$$W = \frac{2m^2r^2n\Omega^5}{\left(c-M\Omega^2\right)^2+n^2\Omega^2}. \tag{9.27}$$

Let us take into account that, for the paired misbalance, Fig. 9.4b, $J = 2I$, where I denotes the axial moment of inertia of each misbalance. Let us introduce the non-dimensional variables and parameters

$$z = \frac{\omega_0^2}{\omega^2}, \ \tau = \frac{m^2r^2n}{IM^2}t, \ \beta = \frac{n}{M\omega_0}, \ \nu = \omega_0\sqrt{\frac{M}{C}}. \tag{9.28}$$

In addition to this, we neglect the reaction torque of the forces of dry friction in the bearing, i.e. $f \sim D = 0$. Direct integration of eq. (9.25) yields

$$\tau = \frac{z^2-1}{4\nu^2} - \left(\frac{1}{\nu} - \frac{\beta^2}{2}\right)(z-1) + \frac{1}{2}\ln z. \tag{9.29}$$

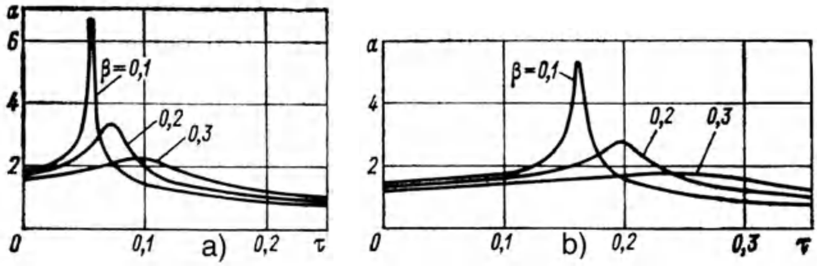

FIGURE 9.8.

This is an implicit dependence of the angular velocity on time in the process of the system run-down. Instead of the averaged amplitude A of mass M, we introduce its non-dimensional counterpart

$$a = \frac{MA}{2mr} = \left[\left(\frac{z}{\nu^2} - 1\right)^2 + \beta^2 z\right]^{-1/2}. \tag{9.30}$$

Figure 9.8 displays the non-dimensional amplitude versus time for $\nu = 1.5$ (Fig. 9.8a) and $\nu = 1.9$ (Fig. 9.8b) which are plotted using eqs. (9.29) and (9.30). One can observe from this Figure that the maximum (resonant) value of the amplitude decreases as the frequency of the proper oscillation of the carrying body decreases (parameter ν increases). However, the duration of the time interval with intensive oscillation increases.

9.3 Stability of the synchronous-synphase regime

Oscillations of the working element of the vibration facility providing the correct transport-technological process correspond, in the majority, to the synchronous-synphase regime of rotor rotation of synchronising vibration exciters, for which $\alpha_i = \alpha$. Let us assume that all of the exciters are identical or nearly identical and that the influence of dry friction in the bearings on the rotation of the rotor can be neglected. Then, a stable synchronous-synphase regime can be very frequently realised in the system. First of all, this corresponds to the case in which the damping in the carrying system is small $(B = 0)$.

The above is valid for systems with harmonic vibration exciters of constant direction (Fig. 9.4b) which are frequently used to drive vibrational screens and horizontal conveyors [76]. In this case the vibrational moments vanish, due to eq. (8.73). The equations for the problem of autophasing in the system of vibrations excites the synphase rotational forces, see eq. (9.18) and Fig. 9.3, which are usually used for driving vibration hoists of considerable height [76]. To this end, it is sufficient to take, see eqs. (8.79) and (8.73), that

$$p_{ij} = 2m_i m_j r_i^2 r_j^2 \omega_i^2 \omega_j^2 \left(k_{ij}^{(1.1)} \sin \beta_i \sin \beta_j + a_i a_j k_{ij}^{(2.2)} \cos \beta_i \cos \beta_j \right). \tag{9.31}$$

We consider here the most important, from the perspective of practical application, case in which the coefficients of influence of force on the rotation $\left(k_{ij}^{(1,2)}\right)$ and the moment on the displacement $\left(k_{ij}^{(2,1)}\right)$ vanish. Let us recall that in the system of exciters of two-dimensional harmonic forces (8.77) with the synphase components ($\gamma_1 = 0, \pi$) the total vibrational moments also reduce to the form

$$W_i = \sum_{i=1}^{n} p_{ij} \sin(\alpha_i - \alpha_j) \quad (p_{ij} = p_{ji}). \tag{9.32}$$

In real facilities, an ideal synchronous-synphase regime is usually slightly violated due to the inevitable influence of frictional forces and parameter dispersion which sometimes considerably decreases the efficiency of the transport-technological process. Among the factors which mostly affect the quality of the synchronous-synphase regime, the most important is the dispersion of the partial frequencies of rotation of the rotors of vibrational exciters $\nu_1, \nu_2, \dots, \nu_n$, that is, the frequencies of their rotation under a fixed carrying system ($M_i(\nu_i) = 0$, see eq. (8.68)). Assume now that this dispersion is not considerable, i.e. the synchronous frequency ν is close to the partial one. Then, with sufficient accuracy we can set

$$M_i(\nu) = z(\nu - \nu_i), \tag{9.33}$$

where coefficient $z = \dfrac{dM_i}{d\nu}$, being negative for the majority of motors, is taken to be equal for all motors. Thus, we begin with the following equations for the problem of autophasing

$$P_i \equiv z(\nu - \nu_i) - \sum_{i=1}^{n} p_{ij} \sin(\alpha_i - \alpha_j) = 0, \tag{9.34}$$

where $\omega_i = \nu$ should be substituted into the expressions for p_{ij}. By virtue of eq. (8.67)

$$\nu = \frac{1}{n} \sum_{i=1}^{n} \nu_i. \tag{9.35}$$

Let us assume now that near the synchronous-synphase regime ($\alpha_i = 0$)

$$\nu_i = \nu + \delta\nu_i, \quad \alpha_i = \delta\alpha_i, \tag{9.36}$$

where variations $\delta\nu_i$ and $\delta\alpha_i$ are small quantities of the same order. In the first approximation, instead of eq. (9.34) we can write the following linear inhomogeneous system for determination of $\delta\alpha_1, \dots, \delta\alpha_n$

$$\sum_{j=1}^{n} p_{ij}\left(\delta\alpha_i - \delta\alpha_j\right) = z\left(\frac{1}{n}\sum_{j=1}^{n}\delta\nu_j - \delta\nu_i\right). \tag{9.37}$$

The synchronous-synphase regime in the ideal system ($\nu_i = \nu$) is stable. The necessary and sufficient condition for its stability, see eq. (8.68), coincide with the criteria of stability for the first set given by the following equations

$$\sum_{j=1}^{n} p_{ij}\left(a_i - a_j\right) = -e\nu^2\kappa a_i, \quad \left(\kappa = \frac{\lambda_1^2}{\nu^2}\right), \tag{9.38}$$

where the steepness coefficient of the backbone curve is equal to $e = 1/J$, whereas the effective moment of inertia of the rotor J is taken, for simplicity, to be equal for all exciters. The stability criteria for the second set are satisfied automatically.

Let us place the non-dimensional stability coefficients $\kappa_1, \kappa_2, \dots, \kappa_n$ in descending order, i.e. $\kappa_1 = 0 > \kappa_2 > \dots > \kappa_n$. Let us subject the values $a_i^{(s)}$ corresponding to κ_s to the following conditions of orthogonality and normalisation

$$\sum_{i=1}^{n} a_i^{(s)} a_i^{(r)} = \delta_{sr} \quad (s, r = 1, \dots, n). \tag{9.39}$$

For the zero root $\kappa_1 = 0$, we can write $a_i^{(s)} = 1/\sqrt{n}$.

The form of the homogeneous part of eq. (9.37) coincides with that of system (9.38). Thus, it is natural to look for the solution of (9.37) in the form of the superposition of discrete mutually orthogonal vectors $a_i^{(1)}, \dots, a_i^{(n)}$

$$\delta\alpha_i = \sum_{s=1}^{n} C_s a_i^{(s)}, \tag{9.40}$$

where $C_1, \dots, C_n$ denote the coefficients of the mode which should be determined. The term corresponding to subscript 1 can be omitted in this sum, i.e. $C_1 = 0$. Indeed, this term does not depend on i whereas all of the solutions of the original system (9.37) are determined up to an arbitrary additive constant. We now insert eq. (9.40) into eq. (9.37), multiply the result with $a_i^{(r)}$ and sum up the products over i from 1 to n. Taking into account eq. (9.39) we obtain

$$C_r = \frac{z}{e\nu^2 z_r}\sum_{i=1}^{n} a_i^{(r)}\delta\nu_i \quad (r = 2, \dots, n). \tag{9.41}$$

Thus, explicit expressions for the deviations of the phase shifts in a non-ideal machine are as follows

$$\delta\alpha_i = \frac{z}{e\nu^2}\sum_{j=1}^{n} A_{ij}\delta\nu_i, \tag{9.42}$$

where

$$A_{ij} = \sum_{s=2}^{n} \frac{a_i^{(s)}a_j^{(s)}}{\kappa_s}. \tag{9.43}$$

Let us take that deviations $\delta\nu_i$ of the partial frequencies from the nominal values are mutually independent random values obeying a Gaussian distribution with a zero mean value and dispersion σ_ω^2. According to eq. (9.42) the mean value $\frac{1}{n}\sum_{l=1}^{n}\delta\alpha_i = 0$.

The dispersion of the deviation of the phase shifts of the $i-th$ and the $j-th$ exciters $\delta\alpha_i - \delta\alpha_j$ is determined by the formula

$$\sigma_{ij}^2 = \frac{z^2\sigma_\omega^2}{e^2\nu^4}\sum_{l=1}^{n}\left(A_{li} - A_{lj}\right)^2. \tag{9.44}$$

By virtue of eqs. (9.39) and (9.43) we obtain

$$\sigma_{ij}^2 = \frac{z^2\sigma_\omega^2}{e^2\nu^4}\sum_{s=2}^{n}\frac{\left(a_i^{(s)} - a_j^{(s)}\right)^2}{\kappa_s^2}. \tag{9.45}$$

The quantitative measure of stability of the synchronous-synphase regime in the considered case of random partial frequencies of the exciters is understood as the relative value of the maximum dephasing of the exciters

$$S = \max_{i,j}\frac{\sigma_{ij}^2}{\sigma_\omega^2} = \frac{z^2}{e^2\nu^4}\max_{i,j}\sum_{s=2}^{n}\frac{\left(a_i^{(s)} - a_j^{(s)}\right)^2}{\kappa_s^2}. \tag{9.46}$$

Alternatively, the lower the value of S due to eq. (9.46), the more stable is the regime. Let us notice that increasing stability can be achieved by decreasing the maximum non-trivial stability coefficient $\kappa_2 < 0$.

In the case of two exciters ($n = 2$) the only solution of system (9.38) satisfying condition $\kappa < 0$ has the form (see eq. (8.73))

$$\kappa_2 = -\frac{2p_{12}}{e\nu^2}, \quad a_1^{(2)} = -a_2^{(1)} = \frac{1}{\sqrt{2}}. \tag{9.47}$$

Therefore, in this case the stability degree is equal to

$$S = \frac{z^2}{2p_{12}}. \tag{9.48}$$

The mean root square of the phase shifts is given by

$$\sigma_{12} = \frac{z}{\sqrt{2}p_{12}}\sigma_\omega \,. \tag{9.49}$$

The value of p_{12} is frequently called the absolute value of the vibrational moment. Thus, one can assert that the probabilistic detuning of the phase shifts is inversely proportional to the absolute value of the vibrational moment.

The situation becomes more difficult even for three exciters ($n = 3$). Let us consider for example a system of three exciters which is symmetric in the sense that $p_{12} = p_{23}$, eq. (8.74). The non-trivial stability coefficients for the synchronous-synphase regime and the normalised solutions of system (9.38) have the form

$$\begin{aligned} \kappa^{(1)} &= -\frac{p_{12} + 2p_{13}}{e\nu^2}, \quad a_1^{(1)} = -a_3^{(1)} = \frac{1}{\sqrt{2}}, \quad a_2^{(1)} = 0, \\ \kappa^{(2)} &= -\frac{3p_{13}}{e\nu^2}, \quad a_1^{(2)} = a_3^{(2)} = \frac{1}{\sqrt{6}}, \quad a_2^{(2)} = -\frac{2}{\sqrt{6}}. \end{aligned} \tag{9.50}$$

Here we use another notation for the stability coefficients $\kappa^{(1,2)}$ because it is unclear, in the general case, what coefficient of them has the smaller absolute value.

The region of existence and stability of the synchronous-synphase regime is shown in Fig. 8.5 and is marked by $1s$. It is easy to see that in the subregion $-p_{12}/2 < p_{13} < 0$ the stability margin determines the coefficients $\kappa^{(1)}$ $\left(\kappa_2 = \kappa^{(1)} > \kappa_3 = \kappa^{(2)}\right)$ whereas in the subregion $p_{13} > 0$ it determines the coefficient $\kappa^{(2)}$ $\left(\kappa_2 = \kappa^{(2)} > \kappa_3 = \kappa^{(1)}\right)$.

By virtue of eq. (9.45), the dispersion of deviation of the phase shifts of the exciters is given by

$$\begin{aligned} \sigma_{12}^2 &= \sigma_{23}^2 = \frac{z\sigma_\omega^2}{2}\left[\frac{1}{(p_{12} + 2p_{13})^2} + \frac{1}{3p_{12}^2}\right], \\ \sigma_{13}^2 &= \frac{2z\sigma_\omega^2}{(p_{12} + 2p_{13})^2}. \end{aligned} \tag{9.51}$$

It is easy to see that $\sigma_{12} < \sigma_{13}$ for $p_{13} > 2p_{12}\left(\sqrt{3} - 1\right)$. In this case an inadmissibly large detuning between the synchronous rotation of the extreme left and right exciters (the first and the third ones) is most likely. Contrary to this, if $p_{13} < 2p_{12}\left(\sqrt{3} - 1\right)$, then large phase shifts are probable between the middle and the extreme exciters. Due to eq. (9.51), the phase discrepancies between the exciters increase up to infinity as the boundary of the region for the existence and stability of the synchronous-synphase regime is approached. This would not occur if the original non-linear system (9.34) was not linearised, see eq. (9.37).

The results obtained remain valid if the determinant of system (9.38) has multiple non-zero roots. Indeed, in this case, a multiple root has simple elementary divisors. Thus, to it correspond some solutions which can be made orthogonal and normalised with respect to each other as well as to the other solutions.

9.4 Self-synchronisation of vibration exciters of anharmonic forces of the constant direction

As an example of such a vibration exciter let us consider the crank mechanism depicted in Fig. 4.1. The mechanism is mounted on a moving carrying rigid body. Let us assume, as above, that the crank is balanced and the ratio r/l is negligibly small. The mechanism is driven by a torque applied to the crank. This torque, the anti-torque moment and the resistance force of the slider are assumed to be small. Then relationships (4.34) are valid in the generating "conservative" approximation. According to eq. (4.33), the energy constant and the frequency of motion are equal to $h = \frac{s^2}{2D}$ and $\omega = \frac{dh}{ds} = \frac{s}{D}$ respectively. Let us remove the energy constant from the latter equations which allows us to express the frequency in terms of the energy

$$\omega = \sqrt{\frac{2h}{D}} = \frac{\pi}{E(k)}\sqrt{\frac{h}{2(J+mr^2)}}, \tag{9.52}$$

with the previous notation being conserved. If we integrate the energy equality $\frac{1}{2}m(q)\dot{q}^2 = h$ we obtain the law of motion of the mechanism

$$E(q,k) = \frac{2}{\pi}E(k)\varphi \quad (\varphi = \omega t + \alpha), \tag{9.53}$$

where $E(q,k)$ denotes the incomplete elliptic integral of the second kind

$$E(q,k) = \int_0^q \sqrt{1-k^2\sin^2\alpha d\alpha}. \tag{9.54}$$

As the implicit dependence (9.53) suggests, variable q is a function of the phase φ and the energy constant h (and correspondingly action s). For simplicity, we assume that the inertia of the flywheel bounded to the crank is much greater than the inertia of the slider $\left(J \gg mr^2, 0 < k \ll 1\right)$ and thus the rotation angle q can be determined as a power series in terms of k^2. As a result, instead of eq. (9.53) we obtain up to terms of order k^4

$$q = \varphi - \frac{k^2}{8}\sin 2\varphi - \frac{k^4}{16}\left(\sin 2\varphi - \frac{5}{16}\sin 4\varphi\right) + k^6 \ldots . \tag{9.55}$$

Let us proceed to determining the force which the mechanism transmits to the fixed carrying body in the generating conservative approximation. Under the adopted assumptions this force is directed along the axis of the mechanism and is equal to

$$G = -m\omega^2 r \frac{d^2 \sin q}{d\varphi^2}. \tag{9.56}$$

Let us insert series (9.55) into eq. (9.56). Then with the adopted accuracy we have

$$\begin{aligned} G(\omega,\varphi) &= m\omega^2 r \left[\sin\varphi - \frac{k^2}{16}(9\sin 3\varphi + \sin\varphi) + \right. \\ & \left. \frac{k^4}{256}(75\sin 5\varphi - 54\sin 3\varphi - 9\sin\varphi) + k^6 \ldots \right]. \end{aligned} \tag{9.57}$$

Thus, only sine harmonics of odd order are observed in the expressions for the force.

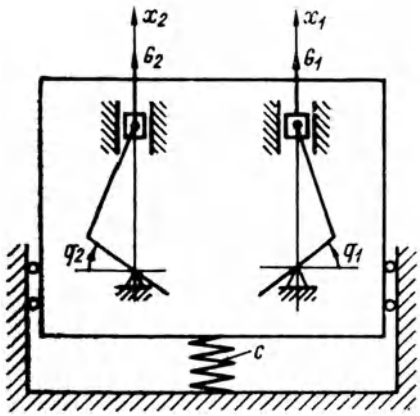

FIGURE 9.9.

Let us consider, as an example, the problem on synchronisation of two identical anharmonic exciters of the same type attached to a carrying body of mass M with one degree of freedom which is mounted on a viscoelastic suspension, Fig. 9.9. The partial vibrational moments are easy to determine by using eqs. (8.71). Comparing eqs. (8.70) and (9.57) we see that $G_i^{(0)} = 0, \vartheta_i^{(\rho)} = \pi/2, \rho = 1,3,5,\ldots$ $(\rho = 1,2)$. Besides, due to eqs. (8.55) and (8.56) we have

$$\begin{aligned} k_{ij}^{(\rho)} \cos\psi_{ij}^{(\rho)} &= \frac{c - M\rho^2\nu^2}{(c - M\rho^2\nu^2)^2 + n^2\nu^2}, \\ k_{ij}^{(\rho)} \sin\psi_{ij}^{(\rho)} &= \frac{n\rho\nu}{(c - M\rho^2\nu^2)^2 + n^2\nu^2}. \end{aligned} \tag{9.58}$$

Let us recall that ν denotes the synchronous frequency. Using eq. (9.57) the replacement $\omega \longrightarrow \nu, \varphi = \nu t + \alpha_i$ should be carried out in the expressions for G_1 and G_2. A special feature of the problem is that the harmonic influence coefficients (9.58) do not depend upon subscripts $i, j = 1, 2$.

Directly substituting the expressions obtained into formula (8.71) yields the following result

$$W_{11} = W_{22} = \frac{m^2 r^2 n \nu^5}{2} \left[\frac{1 - \frac{k^2}{8} - \frac{17k^4}{256} + k^6 \ldots}{\left(c - M\nu^2\right)^2 + n^2\nu^2} + \frac{\frac{243k^4}{256} + k^6 \ldots}{\left(c - 9M\nu^2\right)^2 + 9n^2\nu^2} \right], \tag{9.59}$$

$$W_{12} = \frac{m^2 r^2 \nu^4}{2} \left[\left(1 - \frac{k^2}{8} - \frac{17k^4}{256} + k^6 \ldots \right) \times \frac{\left(c - M\nu^2\right) \sin\left(\alpha_1 - \alpha_2\right) + n\nu \cos\left(\alpha_1 - \alpha_2\right)}{\left(c - M\nu^2\right)^2 + n^2\nu^2} + \left(\frac{81k^4}{256} + k^6 \ldots \right) \frac{\left(c - 9M\nu^2\right) \sin 3\left(\alpha_1 - \alpha_2\right) + 3n\nu \cos 3\left(\alpha_1 - \alpha_2\right)}{\left(c - 9M\nu^2\right)^2 + 9n^2\nu^2} \right].$$

An expression for W_{21} is obtained if the inversion $\alpha_1 \longleftrightarrow \alpha_2$ is performed in the formula for W_{12}. Hence, the influence of the harmonics of the triple frequency can only be taken into account in the expressions for the vibrational moments.

Let us write down the equations for the problem on autophasing, see eq. (8.27)

$$\begin{aligned} P_1 &\equiv M_1 - W_{11} - W_{12} = 0, \\ P_2 &\equiv M_2 - W_{22} - W_{21} = 0. \end{aligned} \tag{9.60}$$

The influence of the dry friction force in the elements of the mechanism is neglected. Then, as shown in Sec. 9.1, the possibility of non-conservative interaction between the exciters is excluded and we can take that

$$M_1 = M_2 = M(\nu), \tag{9.61}$$

where $M(\nu)$ includes the driving moment, as well as the averaged moments of the forces of viscous damping acting on the crankshaft. It is easy to see that the equations in (9.61) admit a solution $\alpha_1 = \alpha_2$ which corresponds to the synchronous–synphase regime. Indeed, substituting these values into eq. (9.61) we arrive at the following equations for the energy balance in the system

$$M(\nu) = m^2r^2n\nu^5 \left[\frac{1 - \frac{k^2}{8} - \frac{17k^4}{256} + k^6 \ldots}{(c - M\nu^2)^2 + n^2\nu^2} + \frac{\frac{243k^4}{256} + k^6 \ldots}{(c - 9M\nu^2)^2 + 9n^2\nu^2} + k^6 \ldots \right]. \tag{9.62}$$

This equation also serves to determine the first approximation to the synchronous frequency ν. The steepness coefficient of the backbone curve in the considered case is positive

$$e = \frac{1}{D} = \frac{\pi^2}{4} E^{-2}(k) \left(J + mr^2\right)^{-1}. \tag{9.63}$$

The only condition for stability of the synchronous-synphase regime of the first set has the form $\left.\frac{\partial W_{12}}{\partial \alpha_1}\right|_{\alpha_1=\alpha_2} < 0$. An expanded form of this inequality is as follows

$$\frac{\left(1 - \frac{k^2}{8} - \frac{17k^4}{256} + k^6 \ldots\right)(c - M\nu^2)}{(c - M\nu^2)^2 + n^2\nu^2} + \frac{\left(\frac{243k^4}{256} + k^6 \ldots\right)(c - 9M\nu^2)}{(c - 9M\nu^2)^2 + 9n^2\nu^2} + k^6 \ldots < 0. \tag{9.64}$$

If $n \to 0$, then this condition is certainly satisfied in the over-resonant region $\nu > \sqrt{C/M}$. The structure of the under-resonant region $\left(\nu < \sqrt{C/M}\right)$ is more difficult. For example, for $n \to 0$ the character of the stable phasing changes while passing the resonances of the third $\left(\nu = \frac{1}{3}\sqrt{C/M}\right)$, fifth $\left(\nu = \frac{1}{5}\sqrt{C/M}\right)$ and higher odd orders. The stability condition of the second set is reduced to the following inequality

$$\frac{dM}{d\nu} < \frac{\partial}{\partial \nu} \left(W_{11} + W_{12}\right)_{\alpha_1=\alpha_2}. \tag{9.65}$$

An explicit form of this inequality is very cumbersome and is omitted. Let us only notice that its violation is possible about both principal and higher order resonances.

The regimes of the multiple synchronisation are also possible in the considered system. Let us assume for example, that under a synchronous regime the right (second) exciter rotates with an angular velocity which is three times as large as that of the left exciter. The parameters of the exciter are assumed to be different, however the values

$$k_1 = \sqrt{\frac{m_1 r_1^2}{J_1 + m_1 r_1^2}}, \quad k_2 = \sqrt{\frac{m_2 r_2^2}{J_2 + m_2 r_2^2}} \tag{9.66}$$

are assumed to have the same order of smallness. Using eq. (9.36) we should write

$$G_1 = m_1 r_1 \nu^2 \left[\left(1 - \frac{k_1^2}{16} - \frac{9k_1^4}{256} + k_1^6 \ldots \right) \sin\varphi_1 - \right.$$
$$\left. \left(\frac{9k_1^2}{8} - \frac{27k_1^4}{128} + k_1^6 \ldots \right) \sin 3\varphi_1 + \left(\frac{75k_1^4}{256} + k_1^6 \ldots \right) \sin 5\varphi_1 + k_1^6 \ldots \right],$$
$$G_2 = 9 m_2 r_2 \nu^2 \left[\left(1 - \frac{k_2^2}{16} + k_2^4 \ldots \right) \sin 3\varphi_2 - \right.$$
$$\left. \left(\frac{9k_2^2}{16} + k_2^4 \ldots \right) \sin 9\varphi_2 + k_2^4 \ldots \right] \quad (\varphi_i = \nu t + \alpha_i, \ i = 1, 2). \tag{9.67}$$

It is easy to see from these equations that only harmonics with numbers 3,9,15,21,... can commute.

Further calculations based upon eq. (8.71) are completely analogous to those which were carried out earlier. With accuracy up to values of the order of $k_{1,2}^4$ included, we obtain

$$W_{12} = -\frac{81}{32} m_1 m_2 r_1 r_2 \nu^4 k_1^2 \left(1 + \frac{3k_1^2}{8} - \frac{k_2^2}{16} + k_{1,2}^4 \ldots \right) \times$$
$$\frac{\left(c - 9M\nu^2\right) \sin 3(\alpha_1 - \alpha_2) + 3n\nu \cos 3(\alpha_1 - \alpha_2)}{\left(c - 9M\nu^2\right)^2 + 9n^2\nu^2} + k_{1,2}^4 \ldots . \tag{9.68}$$

The components appearing due to commutation of harmonics of frequency 9ν have the order of $k_{1,2}^{10}$. This provides evidence that the multiple synchronous regime (of type 1/3), if it exists, has a very small stability margin.

It is important for practical applications that the vibration machine tuned to a stable multiple synchronous regime can be used, under certain circumstances, as a vibrational reduction gear.

9.5 Stabilisation of the working synchronous regime

The vibrational machines driven by several synchronising inertial vibration exciters have considerable promise nowadays. The working vibrational regime of such machines can be characterised, in a certain sense, as synchronous-synphase. As shown in Sec. 8.4, the conditions for existence of the synchronous-synphase regime in the case of two or three exciters of scalar harmonic forces are reduced to satisfying some inequalities which are rather rough and easily reachable in practice. For this reason, a number of highly-effective machines with two vibration exciters have been developed and successfully function now. At the same time the necessity for creating heavy vibrational machines of large size with an extended working element indicates a problem of stabilisation of synchronous-synphase regime in systems with more than three exciters. In particular, the most natural scheme of a long horizontal vibrational conveyer involves a working element in the form of a beam mounted on a system of shock-absorbing springs of negligibly small rigidity and n equal exciters of harmonical forces of constant direction which are uniformly distributed over the beam length and have the same angle β to the beam axis.

Let us place the origin of the longitudinal axis x at the midspan of the beam. Then the abscissa of the point of attachment of the $i-th$ vibration exciter is $x_i = \dfrac{l}{2n}(2i-n-1)$, where $i = 1, 2, \dots, n$. The beam is assumed to be homogeneous and free. The influence of the forces of energy dissipation under oscillation is neglected, and the longitudinal rigidity is taken to be infinite. This is correct if the synchronous frequency is much less than the fundamental frequency of longitudinal oscillation, the latter being higher than some natural frequencies of bending oscillation of the beam. The stability analysis of the synchronous-synphase regime ($\alpha_i = 0$) reduces only to determining the criteria of stability of the first set, see eq. (8.68). Thus, the analysis is based upon relationships (8.29) and (8.73). Instead of eq. (8.29) we obtain

$$\frac{G^2}{2J}\sum_{j=1}^{n}\left(k_{ij}\sin^2\beta - \frac{\cos^2\beta}{M\nu^2}\right)(a_j - a_i) = \lambda_1^2 a_i, \tag{9.69}$$

where J denotes the moment of inertia of the rotor of the exciter, G denotes the amplitude of the excited force and k_{ij} denotes the harmonic coefficient of influence of the shear force on the bending amplitude which is determined by means of eq. (8.62). Summing eq. (9.69) over i we obtain $\lambda_1^2\sum\limits_{i=1}^{n} a_i = 0$. Let us take into account that $\sum\limits_{i=1}^{n} a_i = 0$ for the non-trivial roots $\left(\lambda_1^2 \neq 0\right)$ and introduce the non-dimensional coefficient of stability

$$\kappa = \frac{2JM}{nG^2}\nu^2\lambda_1^2. \tag{9.70}$$

Equation (9.69) can be rewritten in the following form

$$\frac{M\nu^2}{n}\sum_{j=1}^{n} k_{ij}\left(a_j - a_i\right) = \kappa_0 a_i, \tag{9.71}$$

where $\kappa = \kappa_0 \sin^2\beta + \cos^2\beta$. It is important that the value of κ_0 can be used as a coefficient of stability for the regime $a_i = 0$ in the case of pure transverse forces $(\beta = \pi/2)$. Thus, it is sufficient to solve the problem only in this typical case.

A numerical analysis of the latter problem shows that the working synchronous-synphase regime is stable in the frequency region $p_{2n} < p < p_{2n+1}$ which is preceded by the critical resonance whose mode has $2n$ nodes [76]. Here p denotes the non-dimensional frequency parameter and p_p is the eigenvalue of the free beam, see eqs. (8.59) and (8.60).

On the other side, it is necessary to take into account that the nodeless form of the forced vibration of the beam, which is needed to provide continuous uninterrupted transportation of material along the conveyer is completely lost if for $n = 3, 4, 5, 6, 7$ the frequency parameter p does not exceed the values equal to 10, 15.5, 21, 22, 27.5. Since the critical number p_{2n} for $n = 3, 4, 5, 6, 7$ takes the values 17.3, 23.6, 28.9, 36.1, 42.4 respectively, the frequency domains of the transportability and stability does not intersect for $n \geq 3$. Hence the conveyer under consideration for $n \geq 3$ can be realised only under additional synchronising facilities [76].

The necessary condition for the applicability of these facilities, which are referred to as the stabilisers of the synchronous-synphase regime, is that the carrying system must possess the so-called longitudinal axis of rigidity. For this reason, a single harmonic force of unit amplitude acting along the axis of rigidity causes the same harmonic displacements in the same direction and amplitude k at all points within this body. For the horizontal vibrational conveyor $k = -\dfrac{1}{M\nu^2}$.

There exist active and reactive stabilisers. An active stabiliser is an additional zeroth exciter of the harmonic force of amplitude G_0 directed along the longitudinal axis of rigidity. Contrary to eq. (9.69), we use the following equations

$$\begin{gathered}(-1)^{\sigma}\frac{GG_0}{2J}k\cos\beta\left(a_0 - a_i\right) + \\ \frac{G^2}{2J}\sum_{j=1}^{n}\left(k_{ij}\sin^2\beta + k\cos^2\beta\right)\left(a_j - a_i\right) = \lambda_1^2 a_i, \quad (i = 1, \dots, n) \\ (-1)^{\sigma}\frac{GG_0}{2J}k\cos\beta\sum_{j=1}^{n}\left(a_j - a_0\right) = \lambda_1^2 a_0,\end{gathered} \tag{9.72}$$

where $\sigma = 0, 1$ and J denotes the moment of inertia of the stabiliser, to prove the stability of phasing of the first kind

$$\alpha_0 = \pi\sigma + \alpha, \quad \alpha_i = \alpha \quad (i = 1, \dots, n) . \tag{9.73}$$

Summing up the first n equations in (9.72) over i we arrive at the system of two equations with two unknown values a_0 and $\bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i$

$$\begin{aligned} (-1)^\sigma \frac{GG_0}{2J} k \cos\beta\,(a_0 - \bar{a}) &= \lambda_1^2 \bar{a}, \\ (-1)^\sigma \frac{GG_0}{2J} k \cos\beta\,(\bar{a} - a_0) &= \frac{\lambda_1^2}{n} a_0. \end{aligned} \tag{9.74}$$

The determinant of this system has two roots

$$\left(\lambda_1^2\right)_1 = 0, \quad \left(\lambda_1^2\right)_2 = -\left(-1\right)^\sigma \frac{GG_0}{2} k \cos\beta \left(\frac{1}{J} + \frac{n}{J_0}\right) . \tag{9.75}$$

It follows from this equation that for $k > -\frac{1}{M\nu^2}$ it is necessary for stability that $\sigma = 1$. Hence, the stabiliser works, in a certain sense, in opposition to the other exciters.

The remaining $n - 1$ non-trivial criteria of stability are obtained from eq. (9.72) for $a_0 = \bar{a} = 0$ and thus are determined from system (9.71). However, the stability coefficient κ, eq. (9.70), as distinct from the above, is determined by the following equation

$$\kappa = \kappa_0 \sin^2\beta + \cos^2\beta - \frac{G_0}{nG}\cos\beta. \tag{9.76}$$

Clearly, all of these values can not be made negative. For this, it is necessary that amplitude G_0 of the force excited by the stabiliser has the same order as the summed amplitude value of nG forces of the other exciters. Since using such powerful exciters is not expedient from the perspective of strength and technology, active stabilisers are not widespread.

From this viewpoint, vibrational machines with reactive stabilisers of synchronous-synphase regime are more feasible. In the simplest case this stabiliser is a mass which is attached to the carrying body by a spring oriented along the axis of rigidity of the latter, see Fig. 9.10. While studying stability we also use system (9.71). However the coefficients κ and κ_0, see eq. (9.70), are related as follows

$$\kappa = \kappa_0 \sin^2\beta + \frac{\lambda_0^2 - \nu^2}{\lambda^2 - \nu^2}\cos^2\beta, \tag{9.77}$$

where $\lambda = \sqrt{\frac{c\,(m+M)}{mM}}$ and $\lambda_0 = \sqrt{\frac{c}{m}}$, $(\lambda_0 < \lambda)$ denote the natural frequency of the longitudinal oscillation of the two-mass carrying system and

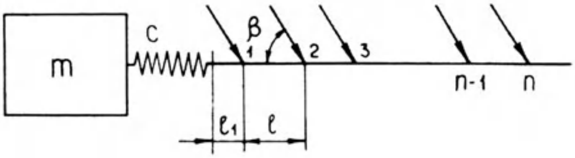

FIGURE 9.10.

the partial frequency of the stabiliser respectively. It is also easy to see that in this case the negative coefficient of stability can be ensured in the frequency domain between the partial and natural frequencies $(\lambda_0 < \nu < \lambda)$. It is usually advantageous to use stabilisers as the working element and tune the body carrying vibration exciters to the antiresonant regime, see [76] for detail.

A radical change in stable phasing always occurs when the synchronous frequency passes through the resonances of the carrying system. That is why the existence of the synchronous-synphase regime, stable in a sufficiently broad frequency domain, is expected in facilities of the principal other, non-inertial type. The sectional vibrational conveyor schematically depicted in Fig. 9.11 is of interest, [76]. The conveyor consists of n relatively short sections which can be considered as being absolutely rigid. Special attachments ensuring hermetic sealing can be idealised by standard slipping joints. Thus, the longitudinal oscillations of the sections are not related to each other. The rigidity of the supporting shock-absorbing springs are negligibly low, so that the facility is non-inertial in a sufficiently broad frequency domain. The section parameters and the vibration exciters of directed action are chosen such that, in the synchronous-synphase regime, the facility moves in the axial direction as a rigid body. For a particular facility this condition was satisfied by a special matching of the parameters of the extreme left and extreme right sections, the remaining sections being absolutely identical. It turned out necessary to place the extreme right exciter at a certain distance from the end of the corresponding section, see Fig. 9.11.

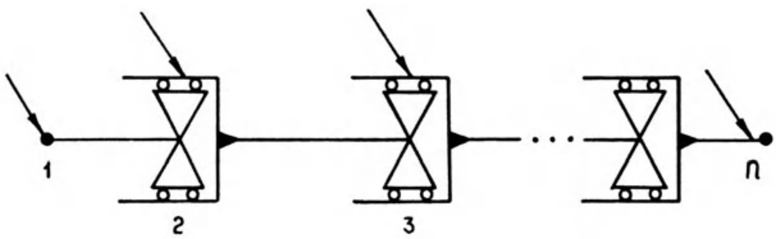

FIGURE 9.11.

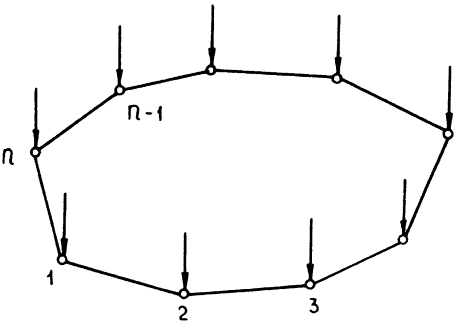

FIGURE 9.12.

The solution of the problem in the framework of the above theory and, further, the numerical work showed that the working regime is stable for $n = 2, 3, \ldots, 7$. However, the degree of stability, i.e. the smallest absolutely value of the characteristic exponent $|\lambda_1|$, monotonically decreases with growth of the number of the sections. This fact of stability was experimentally confirmed in tests.

A qualitative explanation of this effect can be obtained through the example of a system of n identical exciters of vertical harmonic force which interact by means of a closed system of horizontal rods connected via joints, see Fig. 9.12. According to eq. (8.64), the harmonic coefficients of influence of force on the amplitude of vertical displacement is determined from the solution of the inhomogeneous finite-difference system

$$\nu^2 \left(M + \frac{2\rho l}{3} \right) k_{ij} + \nu^2 \frac{\rho l}{6} \left(k_{ij+1} + k_{ij-1} \right) = -\delta_{ij} \tag{9.78}$$

subjected to the boundary conditions of the periodicity

$$k_{i0} = k_{in}, \quad k_{i1} = k_{in+1}. \tag{9.79}$$

Equations for the stability criteria of the synchronous-synphase regime of the first kind have the form

$$\frac{G^2}{2J} \sum_{j=1}^{n} k_{ij} \left(a_j - a_i \right) = \lambda_1^2 a_i. \tag{9.80}$$

Let us sum up equations in (9.78) over j from 1 to n and take into account eq. (9.79)

$$\sum_{j=1}^{n} k_{ij} = -\frac{1}{(M + \rho l)\, \nu^2}. \tag{9.81}$$

Using the latter equation we can recast eq. (9.80) as follows

$$\sum_{j=1}^{n} k_{ij} a_j = \left[\frac{2J\lambda_1^2}{G^2} - \frac{1}{(M+\rho l)\,\nu^2} \right] a_i. \tag{9.82}$$

Now we multiply the $i-th$ equation in (9.82) with $\nu^2\,(M + 2\rho l/3)$, the $(i+1)-th$ equation and the $(i-1)-th$ equation with $\rho l/6$ and sum up the obtained expressions. As $a_{n+1} = a_1$ and $a_n = a_0$ and taking into account eq. (9.78) and the reciprocity condition $k_{ij} = k_{ji}$, we obtain

$$-a_i = \left[\frac{2J\nu^2}{G^2}\lambda_1^2 - \frac{1}{M+\rho l} \right] \left[\left(M + \frac{2\rho l}{3} \right) a_i + \frac{\rho l}{6}\,(a_{i+1} + a_{i-1}) \right]. \tag{9.83}$$

The finite-difference equations (9.83) admits n different particular solutions

$$a_i = C_1 \cos \frac{2\pi s}{n} i + C_2 \sin \frac{2\pi s}{n} i \quad (s = 0, 1, \ldots, n-1), \tag{9.84}$$

satisfying the periodicity conditions and depending upon two arbitrary constants C_1 and C_2. Inserting eq. (9.84) into eq. (9.83) yields the following formula for n different values of λ_1^2

$$\lambda_1^2 = -\frac{G^2}{2J\nu^2\,(M+\rho l)} \frac{\rho l \left(1 - \cos \dfrac{2\pi s}{n} \right)}{3M + 2\rho l + \rho l \sin \dfrac{2\pi s}{n}}, \tag{9.85}$$

where $\lambda_1^2\big|_{s=0} = 0$. One can see from the latter equation that $\lambda_1^2\big|_{s=1} = O\left(1/n^2\right)$. Thus, the stability margin monotonically tends to zero with increasing numbers of exciters.

9.6 Two vibration exciters mounted on the carrying system of vibroimpact type

In Sec. 8.6, we considered the problem of synchronisation of two identical inertial vibration exciters by means of a carrying rigid body whose small oscillations are damped by dry friction forces. The case in which oscillations of the carrying body (or system of carrying bodies) are accompanied by repeated collisions is of even more importance in practice. This non-linearity, which is an essential non-linearity of a discontinuous type imposes a number of specific restrictions on the process of the solution to the problem which are formulated in the most general form in Sec. 8.6.

Let us consider, for example, the problem of synchronising two identical vibration exciters mounted on a carrying system of vibroimpact type, see

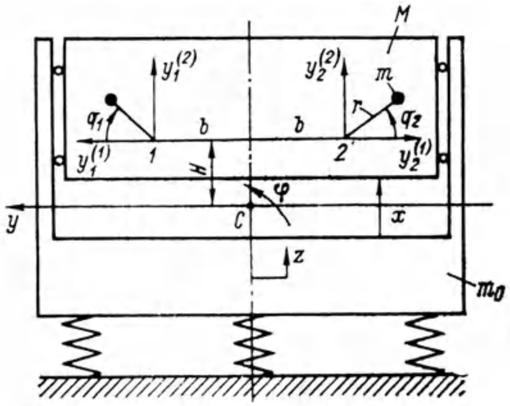

FIGURE 9.13.

Fig. 9.13. This system presents a simple dynamical model of the modern vibrational hammer crusher. It is assumed that the exciters are mounted on the hammer symmetrically about the vertical axis and rotate in the opposite directions. The anvil is attached to the immovable base by means of a system of shock-absorbers of negligibly small rigidity whereas the hammer has the only "vertical' degree of freedom. The centres of mass of the hammer and the anvil as well as the eccentricities of the rotors of the exciters lie in one vertical plane, so the problem under consideration is actually planar.

In order to describe small oscillations of the carrying system, we introduce the following generalised coordinates: the distance between the hammer and the anvil x, the horizontal y and vertical z displacements of that point of the anvil which coincides with the common centre of mass in the position of static equilibrium and the total rotation φ. Rotations of the exciters' rotors in the generating approximation are assumed, as above (see eq. (8.133)) to be uniform, that is $\varphi_i = \nu t + \alpha_i$ $(i = 1, 2)$. The differential equations for the horizontal and rotational oscillations of the carrying system are linear in the generating approximation and are given by

$$
\begin{aligned}
(M + m_0)\,\ddot{y} &= me\nu^2 \left[\cos(\nu t + \alpha_1) - \cos(\nu t + \alpha_2)\right], \\
J\ddot{\varphi} &= me\nu^2 h \left[\cos(\nu t + \alpha_1) - \cos(\nu t + \alpha_2)\right] - \\
& \quad me\nu^2 b \left[\sin(\nu t + \alpha_1) - \sin(\nu t + \alpha_2)\right]. \qquad (9.86)
\end{aligned}
$$

Here M and m_0 denote the masses of the hammer and anvil, respectively, J is the total central moment of inertia of the carrying system of two bodies, h denotes the distance from the common centre of mass to the horizontal line between the centres of rotation of the exciters and b designates the half-distance between these centres. Besides, it is necessary to take into account the following generating equations for the vertical oscillations of the hammer and the anvil between the impacts

$$\begin{aligned} M(\ddot{x}+\ddot{z}) &= -Mg + me\nu^2\left[\sin(\nu t+\alpha_1)+\sin(\nu t+\alpha_2)\right], \\ m_0\ddot{z} &= Mg \quad (x>0) \end{aligned} \tag{9.87}$$

According to the stereomechanic theory of impact [34], the collisions of the hammer and the anvil are governed by the equations

$$\dot{x}_+ = -R\dot{x}_-, \quad \dot{z}_+ - \dot{z}_- = -\frac{M}{M+m_0}(\dot{x}_+ - \dot{x}_-), \quad x=0, \tag{9.88}$$

where $\dot{x}_\pm$ and $\dot{z}_\pm$ denote the values of the corresponding velocities just after and before the collisions, respectively, whilst R is the so-called coefficient of restitution $(0 \le R \le 1)$.

First we eliminate acceleration $\ddot{z}$ from eq. (9.88) to get

$$M\ddot{x} = -\frac{M}{m_0}(M+m_0)g + me\nu^2\left[\sin(\nu t+\alpha_1)+\sin(\nu t+\alpha_2)\right]. \tag{9.89}$$

Then, we introduce the following non-dimensional variables and parameters, cf. eq. (8.135)

$$\begin{aligned} \tau &= \nu t + \frac{\alpha_1+\alpha_2}{2}, \quad u = \frac{Mx}{2me\cos\alpha}, \\ w &= \frac{m_0}{M+m_0}\frac{2me\nu^2}{Mg}, \quad \alpha = \frac{\alpha_1-\alpha_2}{2}. \end{aligned} \tag{9.90}$$

Instead of eq. (9.89) and the first relationship in eq. (9.88) we obtain

$$\begin{aligned} u'' &= \sin\tau - \frac{1}{w\cos\alpha}, \quad (u>0), \\ u'_+ &= -Ru'_- \quad (u=0), \end{aligned} \tag{9.91}$$

where a prime denotes differentiation with respect to τ.

In contrast to eq. (8.134), this essentially non-linear problem has an infinite number of stable periodic, subperiodic and stochastic solutions [75]. Let us consider in more detail construction of the simplest single-impact i–fold periodic solution than was developed in Sec. 8.6. The corresponding regime is characterised by the occurrence of only one collision within one period, which is j times as large as the perturbation period $2\pi/\nu$. Let, in a certain time instant τ_i immediately after the collision, the relative non-dimensional velocity be equal to $u'_+ = v_i$ $(i=1,2)$. Since at this instant $u=0$, direct integration yields the following law of motion in the time interval between the impacts $\tau_i < \tau < \tau_{i+1}$

$$\begin{aligned} u' &= v_i + \cos\tau_i - \cos\tau - \frac{\tau-\tau_i}{w\cos\alpha}, \\ u &= (v_i+\cos\tau_i)(\tau-\tau_i) + \sin\tau_i - \sin\tau - \frac{(\tau-\tau_i)^2}{2w\cos\alpha}. \end{aligned} \tag{9.92}$$

Just before the next impact $\tau = \tau_{i+1}$ $u = 0, u' = -v_{i+1}/R$. Then, by virtue of eq. (9.92)

$$-\frac{v_{i+1}}{R} = v_i + \cos\tau_i - \cos\tau_{i+1} - \frac{\tau_{i+1}-\tau_i}{w\cos\alpha},$$
$$(v_i + \cos\tau_i)(\tau_{i+1}-\tau_i) + \sin\tau_i - \sin\tau_{i+1} - \frac{(\tau_{i+1}-\tau_i)^2}{2w\cos\alpha} = 0. \qquad (9.93)$$

The non-linear finite-difference equations (9.93) describe the point mapping of the plane of impact interactions into itself. The solution of these equations, corresponding to the single-impact j–fold regime, has the form

$$\tau_i = 2\pi j i + \psi, \quad v_i = v, \qquad (9.94)$$

where $\psi = \tau_0$ denotes the phase of the initial impact. Substituting eq. (9.94) into eq. (9.93) we obtain

$$v = \frac{R}{1+R}\frac{2\pi j}{w\cos\alpha}, \quad \cos\psi = \frac{1-R}{1+R}\frac{\pi j}{w\cos\alpha}. \qquad (9.95)$$

The second equation in (9.95) admits a unique solution having a physical meaning only if

$$w > \frac{1-R}{1+R}\pi j. \qquad (9.96)$$

Here and in what follows while studying the stability of the generating solution we assume that the rotors of the exciters rotate synchronous and are synphase ($\alpha = 0$).

Contrary to the problem investigated in Sec. 8.6, in the present vibroimpact problem the analysis of the asymptotic stability of the generating single-impact regime is important. To this aim, it is necessary to vary the recurrent relationships (9.93) in the vicinity of the expressions (9.94) (which evidently satisfy eq. (9.93)). Rearranging the resulting system of linear homogeneous variational equations, we obtain

$$\frac{\delta v_{i+1}}{R} + \delta v_i + \left(\sin\psi - \frac{1}{w}\right)(\delta\tau_{i+1} - \delta\tau_i) = 0,$$
$$\delta v_i - \sin\psi\delta\tau_i - \frac{1}{w(1+R)}(\delta\tau_{i+1} - \delta\tau_i) = 0. \qquad (9.97)$$

According to the analytical theory of the finite-difference equations [15] the particular solution of this system is sought in the form

$$\delta v_i = C_1 h^i, \quad \delta\tau_i = C_2 h^i, \qquad (9.98)$$

where constant values C_1 and C_2 do not depend on i, and h denotes the characteristic number (the multiplier) which is required to be determined.

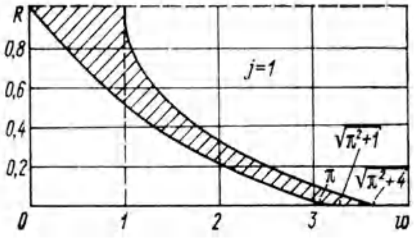

FIGURE 9.14.

Let us insert eq. (9.98) into eq. (9.97) and cancel factor h^i. Equating the determinant of the obtained linear homogeneous system for C_1 and C_2 to zero we arrive at the following characteristic equation

$$h^2 - \left[1 + R^2 - (1+R)^2\, w \sin\psi\right] h + R^2 = 0. \tag{9.99}$$

For stability of the vibroimpact regime, it is necessary and sufficient that the absolute value of each root of this equation is less than unity, i.e. $|h_{1,2}| < 1$. This means that $0 < \psi < \pi/2$ and, due to the second relationship in (9.95),

$$\psi = \arccos\left(\frac{1-R}{1+R}\frac{\pi j}{w}\right). \tag{9.100}$$

Additionally, it is necessary that inequality (9.96) ($h < 1$) and the following inequality

$$w < \frac{1}{(1+R)^2}\sqrt{\pi^2 j^2 \left(1-R^2\right)^2 + 4\,(1+R)^2} \quad (h > -1)\,. \tag{9.101}$$

hold.

In addition, there exists an additional border of the existence region which ensures the absence of an extra impact of the hammer on the anvil within one period of motion [71]. This border is characterised by the fact that at a certain time instant $\tau = \psi + \vartheta$ from the interval $(\psi, \psi + 2\pi j)$ the hammer touches the anvil ($u = u' = 0, u'' > 0$). The region for the existence and stability of the single-impact principal ($j = 1$) regime in the plane (w, R) is shown in Fig. 9.14. This region is basically bounded by inequalities (9.96) and (9.101). The above mentioned additional border cuts a small subregion which is adjacent to the interval $\left(\pi, \sqrt{\pi^2+1}\right)$ on axis w.

Outside the region shown in Fig. 9.14 there exist stable periodic, subperiodic and stochastic motions of principally another nature. For taken values of w and R there may exist several motions of such a type rather than a single motion. Moreover, within the constructed region there may exist other stable stationary motions. The above said readily illustrate the considerable complexity of the vibroimpact system compared with the system with dry friction, see Sec. 8.6. Nevertheless, there are good grounds to believe that within the region shown in Fig. 9.14 a single-impact principal regime possesses an attraction domain which considerably exceeds the corresponding regions of the stable motions of the other types. One can say that the stability margin of this regime is much higher. This is a very important fact since the normal functioning of the crusher is normally based upon this regime of motion.

We proceed now to investigate the stability of the synchronous-synphase rotation of the exciters under periodic single-impact oscillation of the carrying system. We first take into account that, due to eqs. (9.88), (9.90) and (9.92), the jumps in the vertical components of the velocities of the hammer and the anvil are given by the formulae

$$\Delta\dot{x} = \frac{2\pi j}{\nu}\frac{M+m_0}{m_0}g, \quad \Delta\dot{z} = -\frac{2\pi j}{\nu}\frac{M}{m_0}g. \tag{9.102}$$

Within one period $\psi < \tau < 2\pi j + \psi$, because of eqs. (9.86), (9.87), (9.89) and (9.90), the generalised accelerations corresponding to the carrying system are given by

$$\ddot{x} = \frac{M+m_0}{m_0}g + \frac{2me\nu^2}{M}\cos\alpha\sin\tau, \quad \ddot{z} = \frac{M}{m_0}g, \tag{9.103}$$
$$\ddot{y} = -\frac{2me\nu^2}{M+m_0}\sin\alpha\sin\tau, \quad \ddot{\varphi} = -\frac{2me\nu^2}{J}\sin\alpha\,(h\sin\tau + b\cos\tau)\,.$$

Constructing equations of the problem of autophasing should be carried out by means of the basic approach developed in Sec. 8.6, see eqs. (8.129)-(8.132). It is necessary to keep in mind that the equations for rotation of the rotors of the exciters in the first approximation have the form

$$\begin{aligned} I\ddot{\varphi}_1 &= L(\nu) - D(\nu) + me\left[(\ddot{y} + h\ddot{\varphi})\sin(\nu t + \alpha_1) - \right.\\ &\quad \left.(\ddot{x} + \ddot{z} - b\ddot{\varphi})\cos(\nu t + \alpha_1)\right], \\ I\ddot{\varphi}_2 &= L(\nu) - D(\nu) + me\left[(\ddot{y} - h\ddot{\varphi})\sin(\nu t + \alpha_2) - \right.\\ &\quad \left.(\ddot{x} + \ddot{z} + b\ddot{\varphi})\cos(\nu t + \alpha_2)\right]. \end{aligned} \tag{9.104}$$

Here I denotes the moment of inertia of the rotor of the exciters, the driving torque of the motor is determined with the help of its static characteristic and the influence of dry friction in the bearing of the rotor can be neglected, i.e. $D = D(\nu)$, see Sec. 9.1. Let us recall that averaging the right hand sides of the latter equations along the trajectories of the generating solutions should be understood in the generalised sense. This implies, for example, that

$$\int_{t_1-0}^{t_1+0} \ddot{x}\cos(\nu t + \alpha_1)\,dt = \Delta\dot{x}\cos(\nu t + \alpha_1)\,, \tag{9.105}$$

where $t_1 = \frac{1}{\nu}\left(\psi - \frac{\alpha_1 + \alpha_2}{2}\right)$ denotes the time instant of the impact, and $\nu t + \alpha_1 = \psi + \alpha$. Finally, the required equations are written in the following form

$$P_{1,2} = L(\nu) - D(\nu) - meg\left[\frac{1-R}{1+R}\frac{\pi j}{w} \mp \sin\alpha\left(\sin\psi + \frac{w}{2}\frac{M+m_0}{m_0}\cos\alpha\right)\right] \mp m^2e^2\nu^2\left(\frac{1}{M+m_0} + \frac{h^2+b^2}{J}\right)\sin\alpha\cos\alpha. \tag{9.106}$$

Transcendental equations in (9.106) admit solutions corresponding to the synchronous-synphase regime ($\alpha = 0$). The equation of energy balance in the system which is simultaneously an equations for the first approximation to the synchronous frequency ν takes the form

$$L(\nu) = D(\nu) + \frac{(M+m_0)\,Mg^2}{m_0\nu^2}\frac{1-R}{1+R}\pi j. \tag{9.107}$$

The second term on the right hand side of (9.107) describes the average (over the period) energetic loss due to the collision of the hammer with the anvil.

The stability criterion of the first set, see eq. (8.131), is obtained by accounting for $\left.\frac{\partial\psi}{\partial\alpha}\right|_{\alpha=0} = 0$, see eq. (9.95). An expanded form of the corresponding inequality is as follows

$$\frac{w}{2} + \sqrt{1 - \left(\frac{1-R}{1+R}\right)^2\frac{\pi^2 j^2}{w^2}} < \frac{(M+m_0)}{2m_0}\frac{M\left(h^2+b^2\right)}{J}. \tag{9.108}$$

Thus, a rotational mobility of the facility ($J < \infty$) is needed for stability of the synchronous-synphase regime. However, the lateral and rotational oscillations are absent ($y = \psi = 0$). With growth of the multiplicity j of the regime, the restrictions imposed by inequality (9.108) become more rigid. The stability conditions of the second set, see eq. (8.132), are reduced to the form $\left.\frac{\partial' P_1}{\partial\nu}\right|_{\alpha=0} = \left.\frac{\partial' P_2}{\partial\nu}\right|_{\alpha=0} < 0$. Hence, the required inequality is obtained by means of directly differentiating the equations of the energy balance (9.107) with respect to ν.

Finally, we notice that the j-fold synchronous-synphase regime is characterised by the average angular velocity of rotation of the rotors of the exciters being j times as large as the main frequency of the oscillations of the carrying system. Subperiodic motions of this type are not feasible in a "linear" carrying system as well as in the case of a carrying system with dry friction.

10

Synchronisation of dynamical objects of the general type

10.1 Weak interaction of anisochronous and isochronous objects

Let us proceed to investigate the weak interaction of quasi-conservative dynamical objects with one degree of freedom whose equations of motion in terms of the "action-angle" variables are written in form (8.3). For the sake of simplicity, the carried system is excluded from our consideration, i.e. $L^{(2)} = 0$. Let us assume that the first m objects are essentially anisochronous whereas the remaining $n - m$ objects are isochronous. The system is assumed to be tuned to a single frequency of external perturbation. In the autonomous case it is tuned to the only synchronous frequency which is not known in advance. The isochronous phases $\varphi_{m+1}, \dots, \varphi_n$ can be taken so that the corresponding isochronous detunings are small, i.e. $\omega_i - \nu = \gamma_i, \gamma_i = O(\varepsilon)$ $(i = m+1, \dots, n)$, see eq. (5.14). Introducing the "phase-frequency" variables, the proper equations of motion for the objects, eq. (8.3) are written as follows

$$\begin{aligned}
\dot{\omega}_i &= \mu e_i \left[\frac{\partial q_i}{\partial \varphi_i} Q_i + \frac{\partial \Delta L}{\partial \varphi_i} \right] + \mu^2 \dots , \\
\dot{\varphi}_i - \omega_i &= -\mu e_i \left[\frac{\partial q_i}{\partial \omega_i} Q_i + \frac{\partial \Delta L}{\partial \omega_i} \right] + \mu^2 \dots , \\
& (i = 1, \dots , m)
\end{aligned}$$

$$\begin{aligned}
\dot{s}_i &= \mu\left[\frac{\partial q_i}{\partial \vartheta_i}Q_i + \frac{\partial \Delta L}{\partial \vartheta_i}\right] + \mu^2\dots, \\
\dot{\vartheta}_i &= -\mu\left[\frac{\partial q_i}{\partial s_i}Q_i - \gamma_i + \frac{\partial \Delta L}{\partial s_i}\right] + \mu^2\dots. \\
&\quad (i = m+1,\dots,n). \qquad (10.1)
\end{aligned}$$

Here the isochronous phase shifts $\vartheta_i = \varphi_i - \tau \; (i = m+1,\dots,n), \tau = \nu t$ are introduced, cf. eq. (5.15). Removing the coordinates of the carrying system from the right hand sides of system (10.1) is carried out according to the approach of Sec. 8.1 by introducing the dynamic influence matrix K_{ij}, see eq. (8.13). The further averaging system (10.1) in the $\sqrt{\varepsilon}-$vicinity of the principal resonance is performed due to the basic rules formulated in Sec. 5.4. Similar to Sec. 8.2 the averaged equations, with accuracy up to the terms of order $\varepsilon^{3/2}$ included, are cast in the following form

$$\begin{aligned}
\dot{\Omega}_i &= e_i(\Omega_i)\, P_i(\alpha_1,\dots,\alpha_n,\Omega_1,\dots,\Omega_m,\sigma_{m+1},\dots,\sigma_n), \\
\dot{\alpha}_i &= \Omega_i - \nu + e_i(\Omega_i)\, R_i(\alpha_1,\dots,\alpha_n,\Omega_1,\dots,\Omega_m,\sigma_{m+1},\dots,\sigma_n), \\
&\quad (i = 1,\dots,m) \\
\dot{\sigma}_i &= P_i(\alpha_1,\dots,\alpha_n,\Omega_1,\dots,\Omega_m,\sigma_{m+1},\dots,\sigma_n), \\
\dot{\alpha}_i &= \gamma_i + e_i(\Omega_i)\, R_i(\alpha_1,\dots,\alpha_n,\Omega_1,\dots,\Omega_m,\sigma_{m+1},\dots,\sigma_n), \\
&\quad (i = m+1,\dots,n). \qquad (10.2)
\end{aligned}$$

Here $\alpha_1,\dots,\alpha_n,\Omega_1,\dots,\Omega_m,\sigma_{m+1},\dots,\sigma_n$ denote the averaged values of the phase shifts, anisochronous frequencies and isochronous action constants respectively, and the following notation is introduced

$$\begin{aligned}
P_i &= M_i - \sum_{j=1}^{n+1} W_{ij}, \quad R_i = N_i + \sum_{j=1}^{n+1} U_{ij}, \quad M_i = \left\langle \frac{\partial q_i}{\partial \varphi_i} Q_i \right\rangle, \\
W_{ij} &= \frac{1}{2\pi}\int_0^{2\pi}\!\!\int_0^{2\pi} G_i'(\varphi+\alpha_i)\, K_{ij}^*(\varphi-\psi)\, G_j(\psi+\alpha_j)\, d\varphi d\psi, \\
N_i &= -\left\langle \frac{\partial q_i}{\partial \omega_i} Q_i \right\rangle, \quad (i = 1,\dots,n) \\
U_{ij} &= \frac{1}{2\pi}\int_0^{2\pi}\!\!\int_0^{2\pi} \frac{\partial G_i'(\varphi+\alpha_i)}{\partial \Omega_i} K_{ij}(\varphi-\psi)\, G_j(\psi+\alpha_j)\, d\varphi d\psi, \\
&\quad (i = 1,\dots,m), \\
U_{ij} &= \frac{1}{2\pi}\int_0^{2\pi}\!\!\int_0^{2\pi} \frac{\partial G_i'(\varphi+\alpha_i)}{\partial s_i} K_{ij}(\varphi-\psi)\, G_j(\psi+\alpha_j)\, d\varphi d\psi, \\
N_i &= -\left\langle \frac{\partial q_i}{\partial s_i} Q_i \right\rangle, \quad (i = m+1,\dots,n). \qquad (10.3)
\end{aligned}$$

Let us recall that G_{n+1} denotes an external $2\pi/\nu$–periodic force transmitted to the carrying system. The other forces G_j as well as matrix K_{ij} for $j = 1, \dots, m$ depend, in addition to the argument shown in the above equations, on Ω_j and for $j = m+1, \dots, n$ on σ_j.

Similar to Sec. 8.2, we limit our consideration only to internal synchronisation when any external force is absent $(G_{n+1} = 0)$. The asynchronous criteria of stability for the first set $\left(\lambda_1^2 < 0\right)$ of the stationary solutions of system (10.2) ($\alpha_i = \text{const}, i = 1, \dots, n$, $\Omega_i = \nu, i = 1, \dots, m$, $\sigma_i = \text{const}, i = m+1, \dots, n$) due to eqs. (5.68) and (8.29) are determined from the following system

$$e_i \sum_{j=1}^{m} W'_{ij}\left(a_i - a_j\right) + \lambda_1^2 a_i = 0 \quad (i = 1, \dots, m), \tag{10.4}$$

where a prime denotes differentiation of $W_{ij}\left(\alpha_i - \alpha_j, \Omega_i, \Omega_j\right)$ with respect to its first argument. While determining the anisochronous criteria of stability of the second set it is necessary to make use of the general expressions (5.75). As a result, we obtain

$$\lambda_2 = \frac{1}{2} \sum_{i,j=1}^{m} \left[\frac{\partial P_i}{\partial \Omega_j} + \frac{\partial R_i}{\partial \alpha_j} + \frac{1}{\lambda_1^2} \sum_{l=m+1}^{n} \left(\frac{\partial P_i}{\partial \sigma_l}\frac{\partial P_l}{\partial \alpha_j} + \frac{\partial P_i}{\partial \alpha_l}\frac{\partial R_l}{\partial \alpha_j}\right)\right] a_j b_i, \tag{10.5}$$

where the conjugate values $b_1, \dots, b_m$ are determined from the system

$$e_i \sum_{j=1}^{m} \left(W'_{ij} b_i - W'_{ji} b_j\right) + \lambda_1^2 b_i = 0 \tag{10.6}$$

and satisfy the normalisation condition $\sum_{i,=1}^{m} a_i b_i / e_i = 1$, cf. eq. (8.33). Identity (8.22) is valid in this case only for $i = 1, \dots, m$. On the other hand, by virtue of eq. (10.3), we have

$$\frac{\partial U_{ij}}{\partial \alpha_j} = -\frac{\partial W_{ij}}{\partial \sigma_i} \quad (i = m+1, \dots, n). \tag{10.7}$$

Due to eqs. (10.3), (8.22) and (10.7), the inequalities which follow from eq. (10.5) take the form

$$\sum_{i=1}^{m} \left\{\frac{dM_i}{d\nu} a_j - \sum_{j=1}^{n} \left[\frac{\partial W_{ij}}{\partial \nu} - \frac{1}{\lambda_1^2} \sum_{l=m+1}^{n} \left(\frac{\partial W_{il}}{\partial \sigma_l}\frac{\partial W_{lj}}{\partial \alpha_l} + \frac{\partial W_{lj}}{\partial \sigma_l}\frac{\partial W_{il}}{\partial \alpha_l}\right)\right] a_j \right\} b_i < 0. \tag{10.8}$$

According to eq. (5.64) the isochronous criteria of stability ($\lambda_1 < 0$) are determined by analysing the following linear homogeneous system of $2n-m$ equations with $2n-m$ unknown parameters $a_1, \dots, a_n, b_{m+1}, \dots, b_n$

$$\begin{aligned}
\sum_{j=1}^{m} \frac{\partial P_i}{\partial \alpha_j} a_j + \sum_{j=m+1}^{n} \frac{\partial P_i}{\partial \sigma_j} b_j &= \lambda_1 \kappa_i a_i, \\
&\quad (i = 1, \dots, n) \\
\sum_{j=1}^{m} \frac{\partial R_i}{\partial \alpha_j} a_j + \sum_{j=m+1}^{n} \frac{\partial R_i}{\partial \sigma_j} b_j &= \lambda_1 b_i, \\
&\quad (i = m+1, \dots, n) \qquad (10.9)
\end{aligned}$$

where $\kappa_i = 0$ if $i = 1, \dots, m$ and $\kappa_i = 1$ if $i = m+1, \dots, n$. An explicit form of the equations in (10.9) is as follows

$$\begin{gathered}
\sum_{j=1}^{n} W'_{ij} (a_i - a_j) + \sum_{j=m+1}^{n} \frac{\partial W_{ij}}{\partial \sigma_j} b_j = 0, \\
(i = 1, \dots, m) \\
\sum_{j=1}^{n} W'_{ij} (a_i - a_j) + \sum_{j=m+1}^{n} \frac{\partial W_{ij}}{\partial \sigma_j} b_j - \frac{dM_i}{d\sigma_i} b_i + \lambda_1 a_i = 0, \\
\sum_{j=1}^{n} \frac{\partial W_{ij}}{\partial \sigma_i} (a_i - a_j) + \sum_{j=m+1}^{n} \frac{\partial U_{ij}}{\partial \sigma_j} b_j + \left(\frac{dN_i}{d\sigma_i} - \lambda_1 \right) b_i = 0, \\
(i = m+1, \dots, n) . \qquad (10.10)
\end{gathered}$$

The zero root $\lambda_1 = 0$, caused by the autonomy of the considered system, is obtained directly from the latter system according to the conclusions of Sec. 5.6. Thus, all m anisochronous stability criteria of the first set, being a sequence of eq. (10.4), are non-trivial.

Let us consider a carrying system which is pure conservative in the original approximation ($B = 0$). Similar to Sec. 8.5 one can prove the following identities

$$\begin{aligned}
W_i &= \sum_{j=1}^{n} W_{ij} = -\frac{\partial \Lambda}{\partial \alpha_i}, \quad (i = 1, \dots, n) \\
\Lambda &= \Lambda^{(1)} + \Delta\Lambda = -\Lambda^{(1)} = \frac{1}{2} \Delta\Lambda, \\
U_i &= \sum_{j=1}^{n} U_{ij} = -\frac{\partial \Lambda}{\partial \Omega_i}, \quad (i = 1, \dots, m) \\
U_i &= -\frac{\partial \Lambda}{\partial \sigma_i}, \quad (i = m+1, \dots, n) . \qquad (10.11)
\end{aligned}$$

Here the averaged Hamilton action of the characteristics of the weak interaction of the objects in system Λ, see eq. (8.86), is a function of $\alpha_1, \dots, \alpha_n$, $\Omega_1, \dots, \Omega_m, \sigma_{m+1}, \dots, \sigma_n$. Let us assume that the non-potential forces corresponding to the proper coordinates of the isochronous objects vanish, i.e. $M_i = N_i = 0$, $i = m+1, \dots, n$. Differentiating the second set of equation in (10.2) with respect to time and accounting for eqs. (10.3) and (10.11) we obtain, up to values of higher orders of smallness

$$\begin{aligned} \ddot{\alpha}_i &= e_i\left(M_i + \frac{\partial \Lambda}{\partial \alpha_i}\right), \quad (i = 1, \dots, m), \\ \dot{\alpha}_i &= -\frac{\partial \Lambda}{\partial \sigma_i} + \gamma_i, \quad \dot{\sigma}_i = \frac{\partial \Lambda}{\partial \alpha_i}, \quad (i = m+1, \dots, n). \end{aligned} \qquad (10.12)$$

Without loss of accuracy, we can carry out the transformation $\Omega_i \to \nu$ $(i = 1, \dots, m)$ on the right hand sides of these equations.

Let us introduce the notation

$$\begin{aligned} D &= -\Lambda + \sum_{i=1}^{m}\left(\frac{e_i}{2}\sigma_i^2 - M_i\alpha_i\right) + \sum_{i=m+1}^{n} \gamma_i\sigma_i, \\ \sigma_i &= \frac{\dot{\alpha}_i}{e_i}, \quad (i = 1, \dots, m). \end{aligned} \qquad (10.13)$$

Then eq. (10.12) can be rewritten in the following compact canonical form

$$\dot{\alpha}_i = \frac{\partial D}{\partial \sigma_i}, \quad \dot{\sigma}_i = -\frac{\partial D}{\partial \alpha_i}, \quad (i = 1, \dots, n). \qquad (10.14)$$

The first approximations to the true values of $\nu, \sigma_{m+1}, \dots, \sigma_n$ are determined uniquely from the stationarity conditions for the potential function D, whilst the values of the phase shifts $\alpha_1, \dots, \alpha_n$ are determined up to an arbitrary additive constant and additionally $\sigma_1 = \dots = \sigma_m = 0$. To this solution there corresponds a stable single-frequency regime of motion of the original system provided that function D has, at the stationary point, an approximate extremum (minimum or maximum) which is established by the quadratic terms in the expansion of D [84]. If the first m objects are hard anisochronous ($e_i > 0$) then deviations from the stationary point lead to an increase in the corresponding set of components (containing $\sigma_1, \dots, \sigma_m$) in the expression for D. For this reason, the synchronous regime in the problem of the interaction of hard anisochronous and isochronous objects is stable if function D has a minimum. Conversely, if there exist m soft anisochronous objects ($e_i < 0$) it is necessary to find a maximum of function D. Since the averaged action Λ appears in the formula for D with a minus sign, the above corresponds completely to the conclusions of Sec. 8.5 for systems of solely anisochronous objects.

It is worthwhile mentioning that the presence of extremum of the required type is sufficient for the fulfillment of isochronous stability criteria and anisochronous criteria of the first set, cf. eqs. (10.4) and (10.10), however, in general, it is not necessary. This particular extremum property of the synchronous motions of the considered "mixed" system differs drastically from that obtained in Sec. 8.2 for the "pure anisochronous systems". Nonetheless, the anisochronous conditions of stability for the second group, see eq. (10.8), are satisfied automatically in this case ($B = 0, M_i = N_i = 0$, $i = m+1, \dots, n$) if the corresponding motions of the isolated anisochronous objects are stable, see eq. (8.68). It is important that system (10.12) in which the terms of order $\varepsilon^{3/2}$ are omitted is not appropriate for obtaining these conditions (unlike system (10.2)).

In the pure anisochronous case ($m = 0$), the equations averaged by the described method have special importance. Their solutions are in the ε−vicinity of the exact solutions within a longer time interval (of order of $1/\varepsilon$).

10.2 Synchronisation of the quasi-conservative objects with several degrees of freedom

The problems of synchronisation in the process of weak interaction of recurrent quasi-conservative objects, cf. Sec. 4.8, are ideologically close to those considered in the previous section.

In the generating approximation, there exist n mutually independent conservative subsystems, see eq. (8.1), for which the conjugate canonical variables $q_i^{(0)}, p_i^{(0)}$ have vectorial meaning. In the considered region of the "proper phase space", each of these subsystems admits a periodic general integral

$$q_i^{(0)} = q_i^{(0)}(\varphi_i, s_i, \vartheta_i, \kappa_i), \quad p_i^{(0)} = p_i^{(0)}(\varphi_i, s_i, \vartheta_i, \kappa_i). \tag{10.15}$$

Here $\varphi_i = \omega_i t + \alpha, s_i$ are the scalar "action-angle" variables, whereas the constant vectors ϑ_i, κ_i are taken to be canonically conjugate to each other. The "proper" energy is a function only of the action, i.e. $H_i = h_i(s_i)$, so for the frequency and the steepness coefficient of the backbone curve we have

$$\omega_i = \frac{dh}{ds_i}, \quad e_i = \frac{d^2h}{ds_i^2}. \tag{10.16}$$

The original equations of the weak interaction are written, as above, in Routh's form (7.21). Let us assume that the carrying system is smooth and its coordinates can be eliminated according to the approach of Sec. 8.1. For simplicity, the presence of the carried system is assumed not to be associated with additional degrees of freedom, so that $L^{(2)} = L^{(2)}(q_1, \dots, q_n$, $p_1, \dots, p_n)$ in the first approximation.

Let us perform the transformation $q_i, p_i \to \varphi_i, \omega_i, \vartheta_i, \kappa_i$. Instead of eq. (38.9) we obtain, see eq. (8.14)

$$\begin{aligned}
\dot{\omega}_i &= \mu e_i \left[\frac{\partial q_i}{\partial \varphi_i} Q_i + \frac{\partial}{\partial \varphi_i} \left(\Delta L + L^{(2)} \right) \right] + \mu^2 \ldots , \\
\dot{\varphi}_i - \omega_i &= -\mu e_i \left[\frac{\partial q_i}{\partial \omega_i} Q_i + \frac{\partial}{\partial \omega_i} \left(\Delta L + L^{(2)} \right) \right] + \mu^2 \ldots , \\
\dot{\kappa}_i &= \mu \left[\frac{\partial q_i}{\partial \vartheta_i} Q_i + \frac{\partial}{\partial \vartheta_i} \left(\Delta L + L^{(2)} \right) \right] + \mu^2 \ldots , \\
\dot{\vartheta}_i &= -\mu \left[\frac{\partial q_i}{\partial \kappa_i} Q_i + \frac{\partial}{\partial \kappa_i} \left(\Delta L + L^{(2)} \right) \right] + \mu^2 \ldots . \qquad (10.17)
\end{aligned}$$

Here the vectorial non-potential force corresponding to the proper coordinates of the $i - th$ object is taken to be small, as above.

Therefore, the problem is reduced to the analysis of the system with multi-dimensional fast rotating phase which is similar to that in eq. (10.1). Instead of isochronous actions and phase shifts, one deals with the slow components of mutually conjugate vectors ϑ_i, κ_i. For this reason, further averaging of system (10.17) in the $\sqrt{\varepsilon}$ vicinity of the principal resonance is carried out using a similar scheme. To ensure stability of the stationary solutions of the averaged system it is necessary to prove fulfillment of isochronous and anisochronous stability criteria of the first and second sets constructed according to the conclusions of Sec. 5.5.

Let us restrict out consideration to the pure conservative model of interaction. The result of averaging up to the values of order $\varepsilon^{3/2}$ is as follows, see eq. (5.42),

$$\begin{aligned}
\dot{\Omega}_i &= e_i \frac{\partial \Lambda}{\partial \alpha_i}, \quad \dot{\alpha}_i = \Omega_i - \nu - e_i \frac{\partial \Lambda}{\partial \Omega_i}, \\
\dot{\xi}_i &= \frac{\partial \Lambda}{\partial \eta_i}, \quad \dot{\eta}_i = -\frac{\partial \Lambda}{\partial \xi_i}, \quad (i = 1, \ldots , n) , \qquad (10.18)
\end{aligned}$$

where ξ_i, η_i are averaged values of variables κ_i, ϑ_i, whilst the quantity

$$\Lambda = \Lambda^{(2)} - \Lambda^{(1)} \qquad (10.19)$$

is an averaged Hamilton's action of the characteristics of the weak interaction in the system, see eq. (8.86), whereas $\Lambda^{(1)}$ and $\Lambda^{(2)}$ denote the averaged actions of the carrying and carried systems, respectively, see eq. (8.89). Let us differentiate the second equation in (10.18) with respect to time. All of the variables of this system are slow, then by accounting for the first equation we obtain, with accuracy up to the higher order of smallness, that

$$\ddot{\alpha}_i = e_i \frac{\partial \Lambda}{\partial \alpha_i}. \qquad (10.20)$$

The system consisting of eq. (10.20) and the last set of vectorial equations (10.18) can be considered separately. We can substitute $\Omega_i = \nu$ into this system and expressions for Λ and e_i, which implies that values $e_1, \dots, e_n$ are constant. As above, cf. eq. (10.13), we can introduce the notation

$$\sigma_i = \frac{\dot{\alpha}_i}{e_i} \quad (i = 1, \dots, n), \quad D = \frac{1}{2}\sum_{i=1}^{m} e_i \sigma_i^2 - \Lambda. \tag{10.21}$$

As a result, the above system is cast in the canonical form

$$\begin{aligned} \dot{\alpha}_i &= \frac{\partial D}{\partial \sigma_i}, \quad \dot{\sigma}_i = -\frac{\partial D}{\partial \alpha_i}, \\ \dot{\xi}_i &= -\frac{\partial D}{\partial \sigma_i}, \quad \dot{\eta}_i = \frac{\partial D}{\partial \alpha_i}. \end{aligned} \tag{10.22}$$

What remains is to repeat the derivation of the previous section. The constant solutions of system (10.22) correspond to the stationary points of the Hamiltonian D and are characterised by $\sigma_1 = \cdots = \sigma_m = 0$. Let us assume that all of the objects in the system are soft anisochronous ($e_i < 0$). Then deviation of values $\sigma_1, \dots, \sigma_m$ from zero results in a decrease in D. On the other hand, following Poincaré [84], for stability it is sufficient (but not necessary) that function D at the stationary point has an isolated maximum or minimum. Hence, for $e_i < 0$ a sufficient condition of stability of the considered solution is a maximum of Hamiltonian D and correspondingly the minimum of action Λ.

The situation described above is observed in the celebrated problem of n bodies in celestial mechanics, see Sec. 7.2. All objects (the planets) in the system are soft anisochronous, see eq. (4.133). According to eqs. (7.8)-(7.11) the action Λ is represented in the form, see eq. (10.19)

$$\Lambda = -\left\langle K^{(1)} \right\rangle - \left\langle \Pi^{(2)} \right\rangle. \tag{10.23}$$

Thus, one can speak of the maximum of the kinetic energy of the interaction averaged over the synchronous period of interaction, i.e. $\langle K^{(1)} + \Pi^{(2)} \rangle$. If we neglect the kinetic energy of the carrying body (the sun) then the maximum of the averaged potential energy of the gravitational interaction between the planets is achieved, see eq. (7.12),

$$\left\langle \Pi^{(2)} \right\rangle = -\frac{\gamma}{2} \sum_{i,j=1}^{n} m_i m_j \left\langle \frac{1}{\rho_{ij}} \right\rangle \quad \left(\rho_{ij} = |r_i - r_j|\right). \tag{10.24}$$

This value is considered as a function of the phase shifts $\alpha_1, \dots, \alpha_n$ as well as the other canonically conjugated elements of the perturbed (Keplerian) orbits of the planets.

The above corresponds to the well-known principle of a least interaction by Ovenden [81]: the system of n satellites moving under the force of gravitational attraction is mostly in the configuration for which the time-average of the power function of perturbations is minimum and this configuration is resonant, i.e. there exist certain rational relationships between the averaged motions (and in turn between the frequencies of the revolutions). Let us recall that the potential power function U and the potential energy Π are related by expression $U = -\Pi$, [60]. The indication of a finite time in this assertion is caused by the fact that asymptotically stable synchronous motions are not feasible in a pure conservative system. Similar statements can be found in [12], [17] and [79].

It is well known that the period of rotation of the moon about the earth is coincident with the period of its revolution about its own axis. This problem and similar problems of celestial mechanics do not fit in the adopted general scheme of Chapter 7 since we considered the synchronisation of various components of motions of the same object in the process of weak interaction rather than various physical objects. Nevertheless one can assert from the very beginning that the synchronous solutions of such problems do not possess the extremum property proved above since rotation of the moon about the earth (Kepler's problem) is soft anisochronous (see eq. (4.133)) whilst revolution of the moon about its own axis is hard anisochronous.

Let us consider now a slightly more complex problem of the weak interaction of dynamical objects with two degrees of freedom. One of the proper generalised coordinates q_{1i} is cyclic whereas the second one q_{2i} is positional, that is $H_i = H_i\left(q_{2i}, p_{1i}, p_{2i}\right)$. In this case the cyclic momentum, due to eq. (4.26), coincides with the corresponding action, $p_1 = s_1$ and the equations of motion in the generating approximation are given by

$$\begin{aligned} q_1 &= \varphi + \frac{1}{\omega}\int\limits_0^{\varphi_1}\left(\frac{\partial H}{\partial p_1} - \omega\right) d\varphi_1, \quad q_2 = q_2\left(\varphi_1, s_1, s\right), \\ p_2 &= p_2\left(\varphi_1, s_1, s\right), \quad H = h\left(s_1, s\right), \\ \omega &= \frac{\partial h}{\partial s} = \frac{1}{2\pi}\int\limits_0^{2\pi}\frac{\partial H}{\partial p_1} d\varphi_1, \quad \omega_1 = \frac{\partial h}{\partial s_1}. \end{aligned} \tag{10.25}$$

It is important that in the vicinity of the quasi-static solution ($s_i \to 0$) the dependence of solution (10.25) on the positional "action-angle" variables φ_1, s_1 is realised in the series (4.96), (4.100) in terms of the harmonic canonical variables $\xi = \sqrt{2s_1}\cos\varphi_1, \eta_1 = -\sqrt{2s_1}\sin\varphi_1$, see eq. (3.42).

Let us recall that subscript i, which is the number of the object, has been omitted up to this point.

It is shown above, that in this problem introducing the new positional coordinates φ_i, s_i is not efficient for constructing the solution of the perturbed motion, see eq. (5.116). For this reason, proceeding to the problem of weak interaction we use the transformation $q_1, p_1, q_2, p_2 \to \varphi, s, \xi, \eta$, see eq. (5.118). However, unlike Sec. 5.7, the real-valued harmonic variables are used in the present analysis. As a result, the general equations of motions of the objects (7.21) for the considered problem are transformed to the form

$$\dot{s}_i = \mu \left[Q_i^{(1)} + \frac{\partial \Delta L}{\partial \varphi_i} \right] + \mu^2 \dots ,$$
$$\dot{\varphi}_i - \omega_i = -\mu \left[\frac{\partial q_{1i}}{\partial s_i} Q_i^{(1)} + \frac{\partial q_{2i}}{\partial s_i} Q_i^{(2)} + \frac{\partial \Delta L}{\partial s_i} \right] + \mu^2 \dots ,$$
$$\dot{\xi}_i - \omega_{1i}\eta_i = \mu \left[\frac{\partial q_{1i}}{\partial \eta_i} Q_i^{(1)} + \frac{\partial q_{2i}}{\partial \eta_i} Q_i^{(2)} + \frac{\partial \Delta L}{\partial \eta_i} \right] + \mu^2 \dots ,$$
$$\dot{\eta}_i + \omega_{1i}\xi_i = -\mu \left[\frac{\partial q_{1i}}{\partial \xi_i} Q_i^{(1)} + \frac{\partial q_{2i}}{\partial \xi_i} Q_i^{(2)} + \frac{\partial \Delta L}{\partial \xi_i} \right] + \mu^2 \dots , \qquad (10.26)$$

where $Q_i^{(1)}$ and $Q_i^{(2)}$ denote the generalised non-potential forces corresponding to coordinates q_{i1} and q_{i2}. For simplicity, interactions of the carried type are excluded from consideration. In the considered problem, similar to Sec. 8.1 we can introduce the quasi-coordinates of interactions of the objects with the carrying system and the corresponding forces transmitted by the objects to the carrying system. In the original approximations these forces are functions of all proper coordinates of the corresponding object $G_i(\varphi_i, s_i, \xi_i, \eta_i)$. The equations for small oscillations of the carrying system have, as above, the form of eq. (8.8).

The analysed single-frequency regime in the generating approximation corresponds to the quasi-statical solutions of the corresponding canonical subsystems of fourth order isolated from each other

$$\varphi_i = \omega t + \alpha_i, \quad \omega_1 = \dots = \omega_n = \nu, \quad \xi_i = \eta_i = 0. \qquad (10.27)$$

The conditions of its existence and local stability can be determined with the help of the Lyapunov-Poincaré method by using the approach of Sec. 5.7 in the particular case in which the equations are linear with respect to non-critical fast variables, see eq. (5.123). Equations (8.23) for determining the generating phasing of the synchronous rotations $\alpha_1, \dots, \alpha_n$ by means of the cyclic coordinates remain valid. Then, see eq. (8.18)

$$M_i = \left\langle Q_i^{(1)} \right\rangle , \qquad (10.28)$$

where $\xi_i = \eta_i = 0$ should be substituted into this formula as well as into the expression for the particular vibrational moments (8.20). If we take into account only this fact, then the relations for determination of anisochronous stability criteria of the first and second sets, see eqs. (8.29) and (8.33), as well as the extremum properties of the synchronous solution formulated in Sec. 8.5 remain valid. Thus, one can assert that in the case of absence of considerable damping in the carrying system, the stable phasing results in the minimum of the averaged action $\Lambda^{(1)}$, see eq. (8.89), if $e_i = \left.\frac{\partial \omega_i}{\partial s_i}\right|_{\omega_i = \nu, s_{i1} = 0} > 0$.

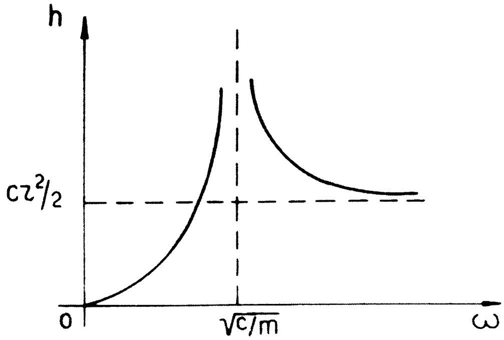

FIGURE 10.1.

A new feature (compared to Sec. 8.2) is the necessity to prove the quasi-static criteria of stability. Their role can be very considerable if the positional frequencies, in the generating approximation, are equal to each other, i.e. $\omega_i|_{\omega_i=\nu, s_{i1}=0} = \Omega$. The corresponding inequalities are obtained in the framework of the general approach of Sec. 5.7 and are based upon the equations in (5.168). Not going into the detail of these equations, we briefly consider the problem of weak interaction of vibration exciters with an elastic eccentricity, cf. Sec. 4.6. We first take into account that the proper energy of the exciter under uniform quasi-static rotation is, due to eq. (4.109), equal to

$$h = \frac{cq_*}{2}(r + 2q_*). \tag{10.29}$$

If we remove the quasi-static positional coordinate q_* from eqs. (5.120) and (10.29), then we arrive at the equation of the backbone curve in the quasi-static approximation

$$h = \frac{1}{2}cmr^2\omega^2\frac{c + m\omega^2}{(c - m\omega^2)^2}. \tag{10.30}$$

This backbone curve versus frequency is shown in Fig. 10.1.

Therefore, the exciter is soft anisochronous $\left(e = \omega\frac{d\omega}{dh} < 0\right)$ in the over-resonant regime $\left(\omega > \sqrt{c/m}\right)$. A stable synchronous regime corresponds to the maximum of the averaged action of the carrying system. In the case of a softly decaying rigid body its kinetic energy (for $\omega > \sqrt{c/m}$) must be maximum. Hence, a regime of oscillation of higher intensity is excited, which is ordinarily desirable for an efficient vibrational facility.

10.3 Non-quasiconservative theory of synchronisation

Contrary to the case of Sec. 8.1, it is assumed that the generalised forces for the proper coordinates of the interaction objects $Q_1, \dots, Q_n$ are of the order of unity and thus we have to consider n isolated non-conservative systems in the generating approximation. The local integrability of each subsystem (cf. Chapter 1) is the necessary and sufficient condition for applying the analytical methods of small parameters for constructing synchronous solutions. In the particular case of weak interaction of the objects with one degree of freedom it is sufficient that, in the generating approximation, equations of each subsystem can be set in the canonised form (3.139) or even (3.144). The generating equations must admit particular solutions which are $2\pi-$periodic with respect to the proper phases $\varphi_i = \nu t + \alpha_i$ $(i = 1, \dots, n)$

$$q_i = q_i(\varphi_i), \quad p_i = p_i(\varphi_i). \tag{10.31}$$

Frequency ν of the generating solution coincides with the frequency of the external perturbation and is given with accuracy up to non-small terms in the autonomous case. For this reason, the non-quasiconservative problem assumes determining a stable phasing $\alpha_1, \dots, \alpha_n$ of the synchronous motions of the objects. In this case, by using eq. (3.156), one succeeds in finding an explicit form for the periodic solutions of the system which is conjugated with the variational equations about solution (10.31). Using for example eq. (1.6) one can also write down the equations of the problem of the autophasing in an explicit form

$$P_i(\alpha_1, \dots, \alpha_n) = 0 \quad (i = 1, \dots, n). \tag{10.32}$$

In the autonomous case these transcendental equations yield the differences of the phase shifts $\alpha_2 - \alpha_1, \dots, \alpha_n - \alpha_1$ and the first approximation of order ε to the synchronous frequency ν. On the other hand, if system (10.32) has a simple solution, this guarantees the existence of a synchronous solution of the problem which is analytical with respect to ε, [61]. For the local stability of this solution, it is sufficient to fulfill criteria similar to the isochronous ones, see (5.64). In other words, the solution is stable if all n roots of the determinant

$$\left| \frac{\partial P_i}{\partial \alpha_j} - \delta_{ij}\lambda \right| = 0 \tag{10.33}$$

have negative real parts $(\mathrm{Re}\,\lambda < 0)$. (Recall that in the autonomous case one of the roots is zero, $\lambda = 0$, see Sec. 5.6.) By analogy we can construct the synchronous solution of the problem of weak interaction of the modified Van-der-Pol oscillators, see eq. (3.103), when the equations in (10.32) are constructed by means of expressions (3.110).

In what follows we limit ourselves to the particular case which is important from the practical perspective and in which q_i, p_i in the general equations for the weak interaction (7.21) are scalar quantities and additionally

$$H_i = \frac{p_i^2}{2J_i}, \quad Q_i = Q_i(p_i). \tag{10.34}$$

Therefore, the situation is that of a weak interaction of the unbalanced rotors in the cases in which the torque of the electric motor is due to its static characteristic and the angle of dry friction in the bearings is sufficiently small, cf. Sec. 9.1. This is the problem which has been studied in detail by Blekhman [16]. Instead of the equations of motion (7.21) we should write

$$\begin{aligned} \dot{q}_i - \frac{p_i}{J_i} &= -\mu\frac{\partial \Delta L}{\partial p_i} + \mu^2 \ldots, \\ \dot{p}_i - Q_i &= \mu\frac{\partial \Delta L}{\partial q_i} + \mu^2 \ldots. \end{aligned} \tag{10.35}$$

Here the right hand sides are $2\pi-$periodic with respect to the proper angles of the rotor rotation $q_1, \ldots, q_n$, whereas $J_1, \ldots, J_n$ are their axial moments of inertia. The interactions of the carried type are neglected, i.e. $L^{(2)} = 0$.

The root of equation $Q_i(p_i) = 0$ differs from $p_i = J_i\nu$ by a small value of the order of ε. Then, rewriting the second equation in (10.35) in the form

$$\dot{p}_i - Q_i + M_i = \mu\left(M_i + \frac{\partial \Delta L}{\partial q_i}\right) + \mu^2 \ldots, \tag{10.36}$$

where $M_i = Q_i(J_i\nu) = O(\varepsilon)$, we arrive at the problem which admits, in the generating approximation, a family of periodic solutions depending upon the constants $\alpha_1, \ldots, \alpha_n$

$$p_i = J_i\nu, \quad q_i = \nu t + \alpha_i. \tag{10.37}$$

Equations conjugated to the variational equations of the generating system have the form

$$\delta \dot{\bar{q}}_i = 0, \quad \delta \dot{\bar{p}}_i = \frac{k_i}{J_i}\delta \bar{p}_i - \frac{1}{J_i}\delta \bar{q}_i, \tag{10.38}$$

where the parameter

$$k_i = -\frac{dM_i}{d\nu} \tag{10.39}$$

is referred to as the integrated damping coefficient as suggested in [16]. Equations (10.38) admit n periodic solutions which are constant values in this particular case

$$\delta \bar{q}_i = \delta_{ij}, \quad \delta \bar{p}_i = \frac{1}{k_i}\delta_{ij} \quad (j = 1, \ldots, n). \tag{10.40}$$

Inertial vibration exciters, as in a quasi-conservative case, excite two-dimensional harmonic (i.e. uniformly rotating) forces, see eq. (9.6). As above, see Sec. 8.1, the dynamic matrix and the harmonic coefficients of influence of the objects on each other by means of the carrying system can be introduced. The conditions for the existence of a synchronous solution in the considered non-quasiconservative problem are based upon eq. (1.6) with eq. (10.40) being taken into account and are determined by averaging the right hand sides of equations (10.36) over the period along the trajectories of the generating solution (10.37). The corresponding derivations are fully analogous to those in Sec. 8.2 and lead to the system of equations (8.27), the particular vibrational moments W_{ij} being determined by formulae (9.7). There exists another notation of this equations which follows from eq. (8.85) and has the form

$$P_i = M_i + \frac{\partial \Lambda}{\partial \alpha_i} - \left\langle \dot{x}' B' \frac{\partial x}{\partial \alpha_i} \right\rangle = 0, \tag{10.41}$$

where Λ denotes an averaged Hamilton's action of the characteristics of the weak interactions in the system. Let us notice in passing that equations (10.41) are also valid in the case of the carried interactions of a pure conservative type. For the considered synchronous solution to be locally stable, it is sufficient that the roots of the equation

$$\left| \frac{1}{k_i} \frac{\partial P_i}{\partial \alpha_j} - \delta_{ij} \lambda_1 \right| = 0 \tag{10.42}$$

have negative real parts, i.e. $\operatorname{Re} \lambda_1 < 0$. Comparing these criteria with the quasi-conservative stability criteria, see eqs. (8.28), (8.29) and (8.33), shows that the non-quasiconservative theory of synchronisation of the inertial vibration exciters differs from their quasi-conservative variant as follows:

1. The role of the steepness coefficients of the backbone curve ($e_i = 1/J_i$, eq. (9.6)) is played by values which are inversely proportional to the integrated damping coefficients (10.39).

2. All coefficients $k_1, \dots, k_n$ must be positive. The appearance of a negative damping coefficient ($k_i < 0$) leads to instability of the generating solution (10.37) even if all of the roots of determinant (10.42) are negative.

3. In the non-quasiconservative theory the requirement that all roots of the following equation

$$\left| \frac{\partial P_i}{\partial \alpha_j} - \delta_{ij} \kappa \right| = 0 \tag{10.43}$$

are negative is replaced by a weaker requirement that their real parts must be negative.

4. The necessity to prove the stability conditions of the second set no longer exists.

It is important that if the extremum property of the synchronous solution ($B_s = 0$, see Sec. 8.5) is valid, then the stability conditions of the second set are satisfied automatically in the quasi-conservative problem, whereas for the inertial vibration exciters $e_i = 1/J_i$ is always positive. For this reason, the various assumptions about the order of the values of the proper non-potential forces have no influence on the final result. However, in the general case, the more stringent "quasi-conservative" stability conditions may not be satisfied in contrast to the less stringent "non-quasiconservative" conditions.

10.4 On the influence of friction in the carrying system on the stability of synchronous motion

Let us assume that small oscillations of the continuous carrying system are accompanied by small forces of energy dissipation. We characterise this "smallness" by inserting a formal indicator of smallness $\beta = 1$ in front of the symmetric matrix B of coefficients of linearised damping in the generating equation of oscillations of the carrying system, see eq. (8.8)

$$M\ddot{x} + \beta B\dot{x} + Cx = \sum_{i=1}^{n} F'_{m,l} G_i. \tag{10.44}$$

The pure forced solution of the above weakly damped system is sought in the form of the following series

$$x = x_0 + \beta x_1 + \beta^2 \dots . \tag{10.45}$$

The original approximation x_0 corresponds to the synchronous oscillations of the carrying system without damping ($B = 0$) whilst the next approximation is determined from the system

$$M\ddot{x}_1 + Cx_1 = -B\dot{x}_0. \tag{10.46}$$

The kinetic potential of the carrying system $L^{(1)} = K^{(1)} - \Pi^{(1)}$, see eq. (7.4), in the generating approximation is also represented in the form of the series

$$L^{(1)} = \frac{1}{2}\dot{x}'M\dot{x} - \frac{1}{2}x'Cx = L_0^{(1)} + \beta L_1^{(1)} + \beta^2 \dots , \tag{10.47}$$

where, due to symmetry of M and C, we have

$$L_1^{(1)} = \dot{x}_0' M\dot{x}_1 - x_0' C x_1. \tag{10.48}$$

Let us premultiply eq. (10.46) with x_0'. Then, by virtue of eq. (10.48) we obtain

$$L_1^{(1)} = \frac{d}{dt}\left(\frac{1}{2}x_0' B x_0 + x_0' M \dot{x}_1\right). \tag{10.49}$$

The averaged Hamilton action of the carrying system, see eq. (8.89), due to eq. (10.47) is equal to

$$\Lambda^{(1)} = \Lambda_0^{(1)} + \beta^2 \ldots, \quad \Lambda_0^{(1)} = \left\langle L_0^{(1)} \right\rangle. \tag{10.50}$$

Now, as $\Lambda = -\Lambda^{(1)}$, see eq. (8.103), we can write down the equations of the problem of autophasing with accuracy up to the values of the first order of smallness included

$$P_i = M_i - \frac{\partial \Lambda_0^{(1)}}{\partial \alpha_i} - \beta \left\langle \dot{x}_0' B \frac{\partial x_0}{\partial \alpha_i} \right\rangle + \beta^2 \ldots. \tag{10.51}$$

Let us recall that in the case of inertial vibration exciters these equations are equally applicable to both quasiconservative and non-quasiconservative theories, see Sec. 10.3. Equations for the quasiconservative theory for determination of stability criteria of the first set, see eq. (8.28) are written by means of eq. (10.51) in the "symmetrised" form, see eq. (8.96), $e_i > 0$,

$$\sum_{j=1}^{n} \left(l_{ij} + \beta f_{ij} + \beta^2 \ldots\right) a_j = \lambda_1^2 a_i. \tag{10.52}$$

Here we introduce the notation

$$\begin{aligned} l_{ij} &= -\sqrt{e_i e_j} \frac{\partial^2 \Lambda_0^{(1)}}{\partial \alpha_i \partial \alpha_j} = l_{ji}, \\ f_{ij} &= -\sqrt{e_i e_j} \left\langle \frac{\partial^2 x_0^{(j)}}{\partial^2 \alpha_j} B \frac{\partial x_0^{(i)}}{\partial \alpha_i} \right\rangle = -f_{ji}, \end{aligned} \tag{10.53}$$

where $x_0^{(i)} = x_0^{(i)}(\nu t + \alpha_i, \nu)$ are determined by formula (8.11) for the case $B = 0$. Let us emphasise that the values l_{ij} and f_{ij} form symmetric and skew-symmetric $n \times n$ matrices respectively. If the roots of the determinant of the system (10.52) are negative and different for $B = 0$, then the roots of the slightly perturbed determinant are also negative and different. Let us assume that for $B = 0$ there exists a l-fold negative root κ_0 $(l < n)$. The appearance of multiple roots often occurs in problems with weak interaction, provided that these problems are symmetric from a physical perspective. This also take place if a certain symmetry is observed in the position of several objects relative to the carrying system. Thus, this case is of a principal and practical importance.

Simple elementary divisors correspond to the root $\lambda_1^2 = \kappa_0$ due to the symmetry of matrix l_{ij}. Thus, at $B = 0$ system (10.52) admits a family of solutions depending on the arbitrary constants $C_1, \dots, C_l$

$$a_{i0} = \sum_{k=1}^{l} C_k a_i^{(k)}, \tag{10.54}$$

where $a_i^{(k)}$ $(k = 1, \dots, l)$ denote partial independent particular solutions of the system

$$\sum_{j=1}^{n} l_{ij} a_j^{(k)} = \kappa_0 a_i^{(k)}, \tag{10.55}$$

which satisfy conditions of orthogonality and normalisation

$$\sum_{i=1}^{n} a_i^{(k)} a_i^{(s)} = \delta_{ks} \quad (k, s = 1, \dots, l). \tag{10.56}$$

Let us seek the particular solution of the full system (10.52) which belongs to the family (10.54) for $B = 0$ in the form of the following series

$$a_i = a_{i0} + \beta a_{i1} + \beta^2 \dots, \quad \lambda_1^2 = \kappa_0 + \beta \kappa_1 + \beta^2 \dots. \tag{10.57}$$

The equations of the first approximation are cast as follows

$$\sum_{j=1}^{n} (l_{ij} a_{j1} + f_{ij} a_{j0}) = \kappa_0 a_{i1} + \kappa_1 a_{i0}. \tag{10.58}$$

We multiply this equation by $a_i^{(k)}$ and subtract equation (10.55) multiplied by a_{i1}. The result of summing up this relationship over i from 1 to n, due to eqs. (10.54) and (10.56), is given by

$$\sum_{s=1}^{l} f^{(s,k)} C_s = \kappa_1 C_k, \tag{10.59}$$

where the factors

$$f^{(s,k)} = \sum_{i,j=1}^{n} f_{ij} a_i^{(s)} a_j^{(k)} \tag{10.60}$$

are skew-symmetric by virtue of eq. (10.53), i.e. $f^{(s,k)} = -f^{(k,s)}$. Let us now multiply eq. (10.59) with C_k and sum up this relationship over k from 1 to l. Since the quadratic form $\sum_{k,s=1}^{l} f^{(s,k)} C_k C_s \equiv 0$, we obtain

$$\kappa_1 \sum_{s=1}^{l} C_k^2 = 0. \tag{10.61}$$

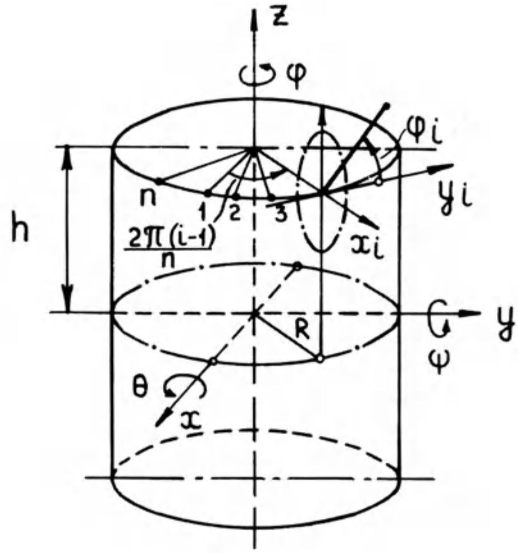

FIGURE 10.2.

If determinant $|f_{ij}| \neq 0$, then the correction $\kappa_1 \neq 0$. Thus, in the general case

$$\sum_{s=1}^{l} C_k^2 = 0, \tag{10.62}$$

and the values of C_k and κ_1 are complex-valued. In other words, the roots of the skew-symmetric secular determinant can not be real in the case of non-degeneracy. This fact causes the appearance of complex coefficients of stability λ_1^2 and, as a consequence (see eq. (5.70)) a destabilising influence of the resistance force under small oscillations of the carrying system on the stability of the synchronous regime.

In the non-quasiconservative theory of synchronisation of the inertial vibration exciters, the above derivation is also valid under the replacements $e_i \to 1/k_i, \lambda_1^2 \to \lambda_1$. The characteristic exponents of the synchronous regime are then also complex. However, this does not lead to a destabilising regime if weaker "non-quasiconservative" stability conditions $\operatorname{Re}\lambda_1 < 0$ are satisfied.

The example [32] considered below shows that the discovered effect is not necessarily related to the original assumption of the smallness of the oscillations of the carrying system. We consider the weak interaction of n identical vibration exciters of rotating centrifugal force $mr\nu^2$ mounted periodically on an axisymmetric rigid body with six degrees of freedom, see Fig. 10.2. Let us assume that the translational and rotational oscillations of the carrying body about its principal central axes $Oxyz$ are independent of each other. In other words, the matrices of rigidity and viscous resistance corresponding to the coordinates $x, y, z, \psi, \vartheta, \varphi$ are diagonal. The adjacent exciters are linked by the carried force constraint responding on the difference of their rotation angles. The linear rigidity of this quasi-elastic synchronising moment c has the order of the small parameter of interaction ε. Let us notice that the considered system models the vibration pile driver. The pile is assumed to execute harmonic, in the original approximation, "screw" oscillations about axis Oz due to synchronising inertial vibration exciters. The stabilisation of the needed synchronous-synphase regime is provided by the proper choice of the value of c.

The investigation carried out in accordance with the approach of Chapter 8 shows that the synchronous-synphase regime does exist in the system, whereas the stability criteria of the first set are obtained as the result of studying the following linear homogeneous system

$$
\begin{aligned}
&ca_{i+1} - (2c + np + \kappa)\, a_i + ca_{i-1} = -p\bar{a} - \\
&(qa_c + sa_s)\cos\frac{2\pi i}{n} - (qa_s - sa_c)\sin\frac{2\pi i}{n} \\
&i = 1, \ldots, n, \quad a_0 = a_n, \quad a_1 = a_{n+1},
\end{aligned}
\qquad (10.63)
$$

where

$$
\begin{aligned}
\bar{a} &= \sum_{j=1}^{n} a_j, \quad a_c = \sum_{j=1}^{n} a_j \cos\frac{2\pi j}{n}, \quad a_s = \sum_{j=1}^{n} a_j \sin\frac{2\pi j}{n}, \\
p &= \frac{m^2 r^2 \nu^4}{2}\left[\frac{c_z - M\nu^2}{(c_z - M\nu^2)^2 + b_z^2\nu^2} + R^2 \frac{c_\varphi - J_z\nu^2}{(c_\varphi - J_z\nu^2)^2 + b_\varphi^2\nu^2}\right], \\
q &= \frac{m^2 r^2 \nu^4}{2}\left[\frac{c_x - M\nu^2}{(c_x - M\nu^2)^2 + b_x^2\nu^2} + \left(h^2 + R^2\right)\frac{c_\psi - J_x\nu^2}{(c_\psi - J_x\nu^2)^2 + b_\psi^2\nu^2}\right] \\
s &= m^2 r^2 \nu^2 h R \frac{b_\psi \nu}{(c_\psi - J_x\nu^2)^2 + b_\psi^2\nu^2},
\end{aligned}
\qquad (10.64)
$$

and $M, J_x = J_y, J_z$ denote the mass and the central principal moments of inertia of the shell, $c_x = c_y, c_z, c_\psi = c_\vartheta, c_\varphi$ denote the coefficients of rigidity of the corresponding translational and rotational displacement of the shell, and $b_x = b_y, b_z, b_\psi = b_\vartheta, b_\varphi$ denote the coefficients of the viscous damping corresponding to these displacements. Let us notice that the stability coefficient κ in the quasiconservative and non-quasiconservative cases is proportional to λ_1^2 and λ_1 respectively.

Analysis of system (10.63) for $n > 2$ is carried out in the following fashion. Summing up the equations over i from 1 to n we obtain $\kappa\bar{a} = 0$. This ensures the existence of a zero root $\kappa_1 = 0$. For non-trivial roots $\bar{a} = 0$. Taking this into account, we multiply eq. (10.63) first with $\sin 2\pi i/n$, then with $\cos 2\pi i/n$ and sum up over i from 1 to n. Then we arrive at the following homogeneous equations for a_c and a_s

$$\begin{aligned} \left[2c\left(1-\cos\frac{2\pi}{n}\right)+n\left(p-\frac{q}{2}\right)+\kappa\right]a_c-\frac{ns}{2}a_s &= 0, \\ \frac{ns}{2}a_c+\left[2c\left(1-\cos\frac{2\pi}{n}\right)+n\left(p-\frac{q}{2}\right)+\kappa\right]a_s &= 0. \end{aligned} \qquad (10.65)$$

The following stability coefficients

$$\kappa_{2,3}=-2c\left(1-\cos\frac{2\pi}{n}\right)-n\left(p-\frac{q}{2}\right)\pm\frac{ns}{2}\sqrt{-1} \qquad (10.66)$$

correspond to the non-trivial values of a_c and a_s. The remaining coefficients are determined from the linear homogeneous finite-difference equation which is obtained from (10.63) at $\bar{a}=a_c=a_s=0$. The result is

$$\kappa_{i+2}=-2c\left(1-\cos\frac{2\pi i}{n}\right)-np \quad (i=2,\dots,n-2). \qquad (10.67)$$

The real values of the stability coefficients $\kappa_2,\dots,\kappa_n$ are negative if the value of the stabilising rigidity satisfies the following inequalities

$$c>-\frac{n}{2}\frac{p-\dfrac{q}{2}}{1-\cos\dfrac{2\pi}{n}}, \qquad c>-\frac{np}{2\left(1-\cos\dfrac{4\pi}{n}\right)}. \qquad (10.68)$$

Fulfillment of these inequalities guarantees local stability of the regime in the non-quasiconservative system. Since coefficients κ_2 and κ_3 are complex-valued, stabilisation of the regime is not feasible under the quasiconservative idealisation. The system loses stabilisation because of the destabilising action of damping under the rotational oscillations of the shell. The only exception is the case when $h=0$, where the centres of rotations of the vibration exciters lie in the plane passing through the centre of mass of the shell. Indeed, in this case, by virtue of eq. (10.64), $s=0$ and κ_2 and κ_3 are real. However, even in this problem, in addition to eq. (10.68), it is necessary to prove the fulfillment of stability criteria of the second set.

This confirms once again the importance of the preliminary analysis of the order of all parameters of the original equations referred to the small parameter of interaction ε.

11

Periodic solutions in problems of excitation of mechanical oscillations

11.1 Special form of notation for equations of motion and their solutions

To begin with, the systems under consideration are naturally split into two subsystems: a mechanical oscillating system and a vibration exciter (or exciters). These subsystems are prescribed to some extent arbitrarily, in particular, the same exciter can be related to several oscillatory systems. However, in each particular case the processes in the exciter and the motion of the oscillatory system influence each other. The forces created by the exciter affect the oscillatory system and there always exist some elements of the system on which these forces act directly. Alternatively, the reaction forces of the element comprise a part of the exciter and their motion affects the processes in the exciter. If the motion of the mentioned elements is known then the processes in the exciter are prescribed and the motion of the remaining elements of the oscillatory system is no longer needed.

Taking this into account we cast the equations for the values describing the processes in the exciter in such a way that they contain not the coordinates of the oscillatory system but auxiliary quantities (denoted by $\xi_1, \dots, \xi_k$) which are the displacements or the rotation angles corresponding to the reaction forces in the elements. Let us also assume that the oscillatory system is linear, i.e. its motion is governed by linear differential equations with time-independent coefficients. If we denote the set of generalised coordinates describing the configuration of the oscillatory system by v, then v is either a vector of finite dimension or an element of a

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Hilbert space G. Let us denote the "proper" coordinates of the exciter by $q = (q_1, \dots, q_m)$ and write down the kinetic potential of the system in the form

$$L = L_1(q, \dot{q}, \xi, \dot{\xi}) + L_2(v, \dot{v}). \tag{11.1}$$

Provided that dependence $\xi(t)$ is given, the equations of motion must provide us with the possibility to determine $q(t)$ regardless of the dimension of vector v and the time-history of its separate components. Thus, the form of $L_1(q, \dot{q}, \xi, \dot{\xi})$ must be independent of the way of introducing the coordinates v, their number etc. In other words, function L_1 must be invariant to the particular form of the oscillatory system. But if the oscillatory system is taken then $\xi_1, \dots, \xi_k$ should be expressed in terms of the components of vector v. If the system is linear, then ξ_i must be linear functionals of v and thus be represented by the scalar products of the form

$$\xi_i = (v, v_i), \quad i = 1, \dots, k, \tag{11.2}$$

where v_i denote constant vectors or elements of the same space as v (however it may also be a larger space). The values of ξ_i describing the "backward" influence of the oscillations on the motion of the exciter do not necessarily belong to the generalised coordinates of the system. For this reason, they are referred to as the "functionals (or parameters) of the backward influence".

The Lagrange's equations are as follows

$$\begin{aligned} \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}} - \frac{\partial L_1}{\partial q} &= R(q, \dot{q}) + E(t), \\ M\ddot{v} + B\dot{v} + Cv &= \sum_{i=1}^{k} Q_i v_i \,. \end{aligned} \tag{11.3}$$

Here

$$Q_i = -\left(\frac{d}{dt}\frac{\partial L_1}{\partial \dot{\xi}_i} - \frac{\partial L_1}{\partial \xi_i}\right),$$

R and E describe non-potential forces in the exciter and the non-mechanical actions, respectively, M, B, C denote quadratic matrices or linear operators in space G. Guided by the physical meaning of the problem we take that M, B, C are symmetric and in addition to this M, C are positive-definite matrices whereas B is positive semi-definite.

One can see from eq. (11.3) that vectors v_i describe the distribution of the exciter's forces over the oscillatory system whereas coefficients Q_i in front of v_i determine the value of these forces. Additionally, the condition that v_i are constant values implies that the loading is referred to the undeformed oscillatory system. In contrast to quantities ξ_i whose mechanical meaning is solely determined by the exciter, vectors v_i will be determined only when the oscillatory system is prescribed.

The assumption that the oscillatory system is linear is related to the assumption of the smallness of its displacements compared with a certain typical dimension. However this does not mean that quantities Q_i and the first six equations in (11.3) can be linearised with respect to ξ, as the displacements can be compared with typical dimensions of the exciters. For instance, the linear elastic electromechanical systems possess this property. If the displacements are small in comparison with the latter dimensions, the problem, nonetheless, may remain non-linear (with a small parameter proportional to the ratio of the displacement to a typical size). In particular, this is the case for the oscillations excited by rotation due to an unbalanced rotor.

Eliminating ξ_i from eq. (11.3) by means of eq. (11.2) for any particular oscillatory system we can obtain the equations of motion in terms of the coordinates. However, there exists a number of cases in which one can study a class of system by using eq. (11.3). For example, let the equations in (11.3) be linear, then they are described by the Lagrange function

$$L = L_1 + L_2 \,,$$

where

$$\begin{aligned} L_1 &= \frac{1}{2}\sum_{r,s}^{m}(m_{rs}\dot{q}_r\dot{q}_s - c_{rs}q_rq_s) + \sum_{r}^{m}\sum_{i}^{k}(d_{ri}\dot{q}_r\dot{\xi}_i - h_{ri}q_r\xi_i), \\ L_2 &= \frac{1}{2}(M\dot{v},\dot{v}) - \frac{1}{2}(Cv,v)\,. \end{aligned} \tag{11.4}$$

Assuming for simplicity that the prescribed forces are harmonic and taking into account the damping we arrive at the following equations of motion

$$\begin{aligned} \sum_{s}^{m}(m_{rs}\ddot{q}_s + b_{rs}\dot{q}_s + c_{rs}q_s) &= U_r\sin\omega t + V_r, (r = 1,\dots,m), \\ M\ddot{v} + B\dot{v} + Cv &= \sum_{i}^{k} Q_i v_i\,. \end{aligned} \tag{11.5}$$

Here

$$V_r = -\sum_{i}^{k}(d_{ri}\ddot{\xi}_i + h_{ri}\xi_i), \quad Q_i = -\sum_{r}^{m}(d_{ri}\ddot{q}_r + h_{ri}q_r)\,. \tag{11.6}$$

The quantities V_r can also be introduced in the general case by recasting the first m equations (11.3) in the following form

$$\frac{d}{dt}\frac{\partial L_{10}}{\partial \dot{q}} - \frac{\partial L_{10}}{\partial q} = R(q,\dot{q}) + E(t) + V\,, \tag{11.7}$$

where

$$L_{10} = L_1(q, \dot{q}, \xi \equiv 0, \dot{\xi} \equiv 0), \quad V = \frac{d}{dt}\frac{\partial(L_{10} - L_1)}{\partial \dot{q}} - \frac{\partial(L_{10} - L_1)}{\partial q}.$$

Putting $V = 0$ in eq. (11.7) we obtain equations of motion for the oscillatory system fixed in the equilibrium position. Thus, values of V characterise, to some extent, the effect of the backward influence of oscillations on the exciters. Quantities V can also be considered as the generalised forces of the same physical nature as the prescribed forces $E(t)$. With this in view, we refer to these forces as the "total generalised vibrational forces". Such quantities are encountered, for instance, in the problems of the dynamics of systems with mechanical exciters in which the values averaged over a period are called vibrational moments, [16].

Let us consider periodic solutions of system (11.5) having period $2\pi/\omega$. Looking for the solution in the form $q_r = a_r \sin(\omega t - \varepsilon_r)$ and inserting this into eq. (11.6), we can find Q_i as a function of $a = (a_1, \dots, a_m)$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, i.e.

$$Q_i = Q_{i1}(a, \varepsilon) \sin(\omega t - \vartheta_i(a, \varepsilon)). \tag{11.8}$$

Let us introduce into consideration the components $k_1^{(ij)} \geq 0$ and $\psi_1^{(ij)}$ of the square $k \times k$ matrices K_1 and Ψ_1, determined as follows. Let a $2\pi/\omega$-periodic solution of equation

$$M\ddot{v}^{(i)} + B\dot{v}^{(i)} + Cv^{(i)} = v_i \sin \omega t \tag{11.9}$$

be given. Let us find the scalar products $(v^{(i)}, v_j)$. Their amplitude and phase angles are denoted in the following $k_1^{(ij)}, \psi_1^{(ij)}$

$$(v^{(i)}, v_j) = k_1^{(ij)} \sin(\omega t - \psi_1^{(ij)}). \tag{11.10}$$

Quantities $k_1^{(ij)}$ and $\psi_1^{(ij)}$ are determined from the solution of the problem of the forced oscillations of the oscillatory system subjected to prescribed harmonic forces and have the following physical meaning. The system is assumed to be subjected to a harmonic load of frequency ω with a unit amplitude ($Q_i = 1$) which is distributed over the oscillatory system as well as the part of the exciter's load corresponding to vector v_i. Let us determine the time-dependence of the $j - th$ functional of the backward influence ξ_j under forced harmonic oscillations caused by the mentioned load. The amplitude of ξ_j and the phase shift between ξ_j and the load are equal to $k_1^{(ij)}$ and $\psi_1^{(ij)}$ respectively.

If a system without damping is considered, then $k_1^{(ij)}$ are the influence coefficients. The reciprocity property $k_1^{(ij)} = k_1^{(ji)}$ as well as the property $\psi_1^{(ij)} = \psi_1^{(ji)}$ is also valid for the system with friction.

Having eq. (11.8) we can express the functional of the backward influence in terms of the amplitude a, angles ε and values $k_1^{(ij)}, \psi_1^{(ij)}$ as follows

$$\xi_j = \sum_i^k Q_{i1}(a,\varepsilon) k_1^{(ij)} \sin(\omega t - \vartheta_i(a,\varepsilon) - \psi_1^{(ij)}) . \tag{11.11}$$

The amplitudes and phases of the total vibrational forces can be expressed from eq. (11.6) in terms of the above values

$$V_r = V_{r1}(a,\varepsilon,K_1,\Psi_1) \sin(\omega t - \gamma_r(a,\varepsilon,K_1,\Psi_1)) . \tag{11.12}$$

Let us insert eq. (11.6) into the first m equations in (11.5). This allows us, in principle, to find the dependences $q_r = q_r(t,a,\varepsilon,K_1,\Psi_1)$. Determining the amplitudes and phases of q_r we arrive at the system for $a_1,\ldots,a_m$, $\varepsilon_1,\ldots,\varepsilon_m$ which allows us to determine these values as functions of the components of the matrices K_1 and Ψ_1. Making use of eq. (11.8), we finally obtain

$$a = a(K_1,\Psi_1),\ \varepsilon = \varepsilon(K_1,\Psi_1),\ Q_{i1} = Q_{i1}(K_1,\Psi_1),\ \vartheta_i = \vartheta_i(K_1,\Psi_1) . \tag{11.13}$$

The latter set of these relationships allows us to find the forces in terms of matrices K_1 and Ψ_1. Thus, if relationships (11.13) are constructed for an exciter, then the problem of the forced oscillation reduces to the determination of matrices K_1 and Ψ_1 as well as to using these relationships. In order to construct relationships (11.13) it is necessary to know only that part of Lagrange's function L_1, which is written in terms of the parameters of the backward influence.

Instead of amplitudes a and phases ε one can determine the amplitudes of the sine and cosine components. The corresponding matrices of these amplitudes should also be introduced for the oscillatory system.

Let us consider now the case in which the equations of motion contain a small parameter μ and are reduced to the form

$$\begin{aligned} \dot{x} &= X(x,t) + \mu Y(x,\xi,\dot{\xi},t,\mu), \\ M\ddot{v} + B\dot{v} + Cv &= \sum_i^k Q_i(x,\dot{x}) v_i + \mu \ldots . \end{aligned} \tag{11.14}$$

Here $x = (x_1,\ldots,x_p)$ are unknown variables which should be introduced instead of $q_1,\ldots,q_m$ in order to cast the equations in the form of (11.14).

The equations of this particular type are obtained in the case when the displacements in the oscillatory system can be understood as being small of order of μ in comparison with a typical dimension of the exciter. Particularly, this is the case for systems with mechanical vibration exciters [52] and synchronising systems [16]. In what follows we consider the case when the equations in (11.14) are Routh's equations, this case covering the problems of oscillations under action of electromagnets.

Let functions X and Y be $2\pi/\omega-$periodic with respect to t. Let us also assume that for any continuous $2\pi/\omega-$periodic function $\varphi(t)$ each of the equations

$$M\ddot{v} + B\dot{v} + Cv = \varphi(t)v_i \tag{11.15}$$

admits a unique $2\pi/\omega-$periodic solution $v_\varphi^{(i)}(t)$ such that $\max_t |(v_\varphi^{(i)}(t), v_i)|/\max_t |\varphi(t)| < h^{(i)}$, where a constant $h^{(i)}$ is coincident for all φ and $h^{(i)} = O(1)$. This assumption corresponds to the so-called "non-resonant" case, see Sec. 11.4 for the resonant case in which $h^{(i)} = O(1/\mu)$.

The generating solutions of eq. (11.14) are considered to be non-isolated. In the case of an isolated generating solution the influence of the oscillations results only in small corrections to variables x obtained without vibrational forces. Hence, the backward influence of the oscillations on the exciter's motion can not lead to qualitatively new effects and is of little interest from this perspective. For this reason, let us consider the case in which the system

$$\dot{x} = X(x,t) \tag{11.16}$$

admits a family of $2\pi/\omega-$periodic solutions $x = x_0(t,\alpha), \alpha = (\alpha_1, \dots, \alpha_n)$ depending upon n arbitrary parameters. Let functions $x_0(t,\alpha)$ be given, then functions $Q_i(x_0, \dot{x}_0)$ are also given functions of t, α. Therefore, the coefficients of the Fourier series

$$Q_i = \sum_\nu Q_{i\nu}(\alpha)\cos(\nu\omega t - \vartheta_{i\nu}(\alpha)) \tag{11.17}$$

can be also treated as given functions of α. Let us introduce, as above, the matrices of the harmonic influence coefficients and phase shifts. These matrices should be introduced for all frequencies appearing in eq. (11.17). Let us denote these matrices and their components as K_ν, Ψ_ν and $k_\nu^{ij}, \psi_\nu^{ij}$, respectively. They are defined by the relationships

$$(v_\nu^i, v_j) = k_\nu^{(ij)}\cos(\nu\omega t - \psi_\nu^{(ij)}), \quad k_\nu^{(ij)} \geq 0\,, \tag{11.18}$$

where $v_\nu^{(i)}$ denotes a $2\pi/\nu\omega-$ periodic solution of the equation

$$M\ddot{v}_\nu^{(i)} + B\dot{v}_\nu^{(i)} + Cv_\nu^{(i)} = v_i\cos\nu\omega t\,. \tag{11.19}$$

The physical meaning of matrices K_ν, Ψ_ν for $\nu \neq 1$ is coincident with that of matrices K_1, Ψ_1, with the only difference being that the oscillatory system is subjected the unit load of frequency $\nu\omega$ rather than ω.

Now we can write down the expressions for the functionals of the backward influence in the generating approximation which contain the components of matrices K_ν, Ψ_ν and values $\alpha_1, \dots, \alpha_n$ as parameters, i.e.

$$\xi_{j0} = \sum_i^k \sum_\nu Q_{i\nu}(\alpha) \cos(\nu\omega t - \vartheta_{i\nu}(\alpha) - \psi_\nu^{(ij)}) k_\nu^{(ij)} . \tag{11.20}$$

Let us also assume that we have $2\pi/\omega$−periodic solutions $z_1, \dots, z_n$ of the system of linear equations with periodic coefficients which is referred to as the system conjugate to the variational system of equations for the generating solution

$$\dot{z} + Zz = 0, \qquad Z = \| (\partial X_s/\partial x_r)_0 \| . \tag{11.21}$$

Here z denotes a vector with the same dimension as x and the zero subscript implies that the derivatives are taken at $x = x_0$. Then the equations for determination of $\alpha_1, \dots, \alpha_n$ can be written such that they contain the components K_ν, Ψ_ν as parameters. They have the form

$$P_r(\alpha_1, \dots, \alpha_n, K, \Psi) = \left\langle \sum_s^n z_{rs} Y_{s0}(t, \alpha, K, \Psi) \right\rangle = 0, \quad r = 1, \dots, n . \tag{11.22}$$

Here $Y_{s0}(t, \alpha, K, \Psi) = Y_s(x_0, \xi_0, \dot{\xi}_0, t, 0)$ denotes the components of vector Y, whereas symbols K, Ψ denote the entire set of matrices K_ν, Ψ_ν.

Provided that solutions $\alpha = \alpha(K, \Psi)$ of system (11.22) are found as functions of K, Ψ and are substituted into the above relationships, then all of the required variables in the generating approximation are also found as functions of K, Ψ. As a result, determination of the oscillations in any linear oscillatory system excited by an exciter with given expression for Q_i reduces to the solution of the problem of the forced oscillation and using derived expressions. The elements of the determinant $|\partial P_r/\partial \alpha_s|$ and thus the stability conditions can be expressed in a similar form in terms of the components of matrices K, Ψ.

Relationships containing matrices K, Ψ are also useful in that they can be utilised when the equations of motion for the oscillating system are not yet derived, whereas these matrices are obtained, for example, experimentally.

The solution can be cast in this form when the system (11.14) is autonomous. In this case it is necessary to know the dependence $K(\nu\omega), \Psi(\nu\omega)$ in some intervals of the values of ω. Additionally, if system (11.14) is nearly conservative, then the stability conditions [45] contain not only $K(\nu, \omega)$, $\Psi(\nu\omega)$ but also their derivatives with respect to ω.

Khodzhaev suggested solutions for a number of problems on oscillations under the action of electromagnets in [44], [46], [48] and solutions for the problems on synchronisation of mechanical vibrations in [45]. The same approach was suggested by Sperling [94] however the stability conditions for the second set (derived in [45]) are absent in this paper.

11.2 Integral stability criterion for periodic motions of electromechanical systems and systems with quasi-cyclic coordinates

Let us construct the equations governing the quasi-stationary oscillations of electromechanical systems with closed linear currents without assuming small displacements

$$\begin{gathered}\frac{d}{dt}\frac{\partial W}{\partial i_s}+\frac{\partial F}{\partial i_s}=E_s(t), \quad s=1,\dots,m,\\ \frac{d}{dt}\frac{\partial L_M}{\partial \dot q_{m+s}}-\frac{\partial L_M}{\partial q_{m+s}}-\frac{\partial W}{\partial q_{m+s}}=Q_{m+s}, \quad s=1,\dots,n-m.\end{gathered} \tag{11.23}$$

Here $i_1,\dots,i_m$ denote currents, F is the "electric dissipation function", $E_1,\dots,E_m$ are prescribed $2\pi/\omega$-periodic electromotive forces (emf), q_{m+1}, $\dots,q_n$ are the generalised coordinates, L_M is the kinetic potential, and $Q_{m+1},\dots,Q_n$ are the non-potential generalised forces.

For inductively-connected non-branched loops we have

$$F=\frac{1}{2}\sum_s^m R_s i_s^2\,. \tag{11.24}$$

Let us adopt that F has the same form in the general case which can be achieved by a linear transformation of the original currents.

Let us express the field energy W in terms of the inductances and currents

$$W=\frac{1}{2}\sum_{r,s}^m L_{rs}(q_{m+1},\dots,q_n)i_s i_r\,. \tag{11.25}$$

Let E_*, L_* and R_* denote the characteristic values of the emf, inductance and resistance, respectively. Let us introduce characteristic current $i_*=E_*/\omega L_*$ and take $\tau=\omega t$. Then we can recast the first m equations in the following non-dimensional form

$$\frac{d}{d\tau}\sum_r^s l_{rs}\eta_r+\rho\beta_s\eta_s=e_s, \quad s=1,\dots,m\,. \tag{11.26}$$

Here $l_{rs}=L_{rs}/L_*,\eta_r=i_r/i_*,\beta_s=R_s/R_*,e_s=E_s/(\omega L_* i_*),\rho=R_*/\omega L_*$. Assuming the electric dissipation to be small, that is the characteristic inductive (ωL_*) and active (R_*) resistances fulfill the relationship $\omega L_*\gg R_*$, we deem ρ to be a small parameter. Systems with spatial conductors are considered under analogous assumptions. Besides, in what follows we study a peculiar, but interesting from the technological perspective, case in which it is sufficient to assume that only few of the R_s are small. Thus, eq. (11.26) contains equations with small parameter ρ. Let η_r, q_{m+s} be obtained in the form of a power series in terms of ρ. If we return to the original dimensional variables, then the result must coincide with that obtained if the solution of eq. (11.23) is sought in the form of a power series in terms of R_*, provided that the value of R_* is conditionally taken as being small in comparison with the other values of any dimension. With this in view, we do not introduce the small parameter ρ explicitly and manipulate directly the equations in (11.23).

Averaging the first m equations in (11.23) over the period and denoting the mean values by $\langle\ \rangle$ we obtain

$$R_s\langle i_s\rangle = \langle E_s\rangle, \quad \langle i_s\rangle = \frac{\omega}{2\pi}\int\limits_0^{2\pi/\omega} i_s(t)dt\,. \tag{11.27}$$

It follows that the constant components $\langle E_s\rangle$ of the given emf should be small. Indeed, if $\langle E_s\rangle \sim E_*$, then $\langle i_s\rangle/i_* \sim 1/\rho$. In practice, these currents are approximately equal to the short circuit current under the nominal voltage. For the formal viewpoint, another procedure of Poincaré's method is needed. It is natural to take $\langle E_s\rangle$ as being small of the order of R_s, then $\langle i_s\rangle \sim i_*$. This corresponds to the nominal working regimes of the electromechanical facilities.

Let us express the ponderomotive force $\partial W/\partial q_{m+r}$ in terms of the magnetic fluxes $\Phi_r = \partial W/\partial i_r$. Then equations (11.23) can be integrated by means of the following scheme. Let us represent E_s in the form $E_s = U_s + \langle E_s\rangle, \langle U_s\rangle = 0$. Omitting the small terms in eq. (11.23) and denoting the generating approximation by a zero subscript we obtain

$$\Phi_{s0} = \alpha_s + V_s(t), \quad \dot{V}_s = U_s, \quad \langle V_s\rangle = 0\,. \tag{11.28}$$

Here α_s denote arbitrary constant components of yet undetermined magnetic fluxes. Let us substitute eq. (11.28) into the second set of equations in (11.23). Then we arrive at the equations for the oscillations of a mechanical system under the actions of forces which are prescribed functions of time, mechanical coordinates and parameters α_s. They allow us, in principle, to determine q_{m+r} as a function of t and α_s. This means that the family of generating solutions depending on m is obtained. Among these solutions, we should take those which are equal to the periodic solutions of the original system (with small terms) for $\rho = 0$. Only certain values of $\alpha_1, \dots, \alpha_m$ correspond to these generating solutions. In this particular case these values are the solutions of the following system of equations

$$P_s(\alpha_1, \dots, \alpha_m) \equiv \langle i_{s0}(\alpha_1, \dots, \alpha_m)\rangle - i_{cs} = 0, \quad s = 1, \dots, m\,. \tag{11.29}$$

Here $i_s = \partial W/\partial\Phi_s$ denote the currents which are considered as functions of the magnetic fluxes and the mechanical coordinates, i.e. $i_{cs} = \langle E_s\rangle/R_s$. For sufficiently small ρ, for each solution of system (11.29) there is a periodic solution of system (11.23). Taking one of them and inserting the values $\alpha_1, \dots, \alpha_m$ into eq. (11.28) and into the expressions for $q_{m+s0}(t, \alpha_1, \dots, \alpha_m)$, $i_{s0}(t, \alpha_1, \dots, \alpha_m)$ we find the required variables with accuracy up to the order of ρ. The higher order terms can be found, if required. Obtaining the derivatives $\partial P_s / \partial \alpha_r$ one can analyse the stability of the regime under consideration, [61].

However, application of the described procedure is made difficult by the fact that the equations for determining $q_{m+s0}(t, \alpha_1, \dots, \alpha_m)$ are in general non-linear. Provided that they are piecewise-linear, their solution can be obtained by the matching method. In general, one should use either approximate methods, or numerical integration. But this strategy, especially the numerical approaches, does not allow one to determine explicit dependences of q_{m+s0} on $\alpha_1, \dots, \alpha_m$. Therefore, what remains is to prescribe various values of $\alpha_1, \dots, \alpha_m$, determine solutions $q_{m+s0}(t)$ corresponding to each set of $\alpha_1, \dots, \alpha_m$ and calculate the values of P_s aiming at approaching the solution of eq. (11.29) by changing $\alpha_1, \dots, \alpha_m$.

The above calculations are simplified if it is known that the required values of $\alpha_1, \dots, \alpha_m$ renders an extremum to a certain function $\Lambda(\alpha_1, \dots, \alpha_m)$. Indeed, in this case, the sets of $\alpha_1, \dots, \alpha_m$ can be taken by using the conventional search methods for an extremum. In the present chapter we indicate the cases in which such a function Λ exists and establish its physical meaning. Since equations (11.23) comprise a particular case of the equations of mechanics with quasi-cyclic coordinates we will consider the latter, more general equations.

The relations between the stable periodic solutions of the system with small parameter and the points of minima of a certain scalar function Λ was first established by Blekhman for synchronisation problems [16]. He referred to this principle as the integral stability criterion. This result was generalised in [74] for the class of systems called quasiconservative synchronising systems. Particular cases of the problem of synchronisation were studied by Lavrov in [55] by means of the integral criterion. These authors are of the opinion that the integral criterion is of interest itself (regardless of the usefulness for calculations) and link this with physical clarity and the fact that Λ can be constructed without detailed equations of motion.

It is interesting to clarify what systems, along with the above mentioned, obey the integral criterion. As follows from its derivation, see [74], [47], it is necessary that the equations with small parameter have the structure of mechanical equations and arbitrary constants appear in the generating approximation in a "natural" way. In mechanics, there exist two classes of systems whose periodic motions form families depending upon arbitrary constants. These are the conservative systems (for which the arbitrary parameters are initial phases and the energy constant or period) and the systems with cyclic coordinates (the parameters are cyclic momenta). The system with quasi-cyclic coordinates which is considered in what follows belongs to the second most natural class of the systems for which the integral criterion holds.

Let us consider a system with holonomic stationary constraints described by m quasi-cyclic $(q_1, q_2, \dots, q_m)$ and $n-m$ positional $(q_{m+1}, \dots, q_n)$ coordinates. It is assumed that the quasi-cyclic coordinates correspond to the generalised coordinates of two sorts, namely small forces of viscous damping and forces depending only on time, with the latter being assumed to be $2\pi/\omega$–periodic. Lagrange's equations for such a system has the form

$$\begin{aligned} &\dot{p}_s + \mu b_s \dot{q}_s = U_s(t) + \mu U_{cs}, \quad s = 1, \dots, m, \\ &\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{m+s}} - \frac{\partial L}{\partial q_{m+s}} = N_{m+s}, \quad s = 1, \dots, n-m, \\ &L = T(q_{m+1}, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) - \Pi(q_{m+1}, \dots, q_n). \end{aligned} \tag{11.30}$$

Here p_s denote the quasi-cyclic momenta, $\langle U_s \rangle = 0, U_{cs} = \text{const}, N_{m+s}$ are the non-potential generalised forces corresponding to the positional coordinates, L is the kinetic potential of the system, and μ is a small parameter. One can adopt that $\mu > 0$, then $b_s > 0$. A more general case in which the "quasi-cyclic" dissipative forces are given by the dissipation function of the form

$$F = \frac{1}{2}\sum_{r,s}^{m} b_{rs}\dot{q}_r\dot{q}_s$$

reduces to the previous case by a linear transformation of only quasi-cyclic coordinates.

Let us consider Routh's equations corresponding to eq. (11.30)

$$\begin{aligned} &\dot{p}_s - \mu b_s \frac{\partial L_R}{\partial p_s} = U_s(t) + \mu U_{cs}, \quad s = 1, \dots, m, \\ &\frac{d}{dt}\frac{\partial L_R}{\partial \dot{q}_{m+s}} - \frac{\partial L_R}{\partial q_{m+s}} = N_{m+s}, \quad s = 1, \dots, n-m. \end{aligned} \tag{11.31}$$

Here L_R denote Routh's kinetic potential related to Lagrange's function as follows

$$L_R = L - \sum_{s}^{m} p_s \dot{q}_s\,. \tag{11.32}$$

L_R is inserted into eq. (11.31) as a function of $p_1, \dots, p_m$, $\dot{q}_{m+1}, \dots, \dot{q}_n$, $q_{m+1}, \dots, q_n$. If the original Lagrange's function L is prescribed as a function of $\dot{q}_1, \dots, \dot{q}_m$ and the positional coordinates and velocities, then it is necessary to express the quasi-cyclic velocities in terms of momenta and $\dot{q}_{m+1}, \dots \dot{q}_n, q_{m+1}, \dots, q_n$ from the linear equations for $\dot{q}_1, \dots, \dot{q}_m$

$$\frac{\partial T}{\partial \dot{q}_s} = p_s, \quad s = 1, \dots, m\,. \tag{11.33}$$

For $\mu = 0$, the following system of m equations can be separated from eq. (11.31)

$$\dot{p}_{s0} = U_s(t), \quad s = 1, \dots, m, \tag{11.34}$$

which yields, up to the constants $\alpha_1, \dots, \alpha_m$, the quasi-cyclic momenta in the generating approximation

$$p_{s0} = \alpha_s + V_s(t), \quad s = 1, \dots, m. \tag{11.35}$$

Here and in what follows, the antiderivatives $V_s, \dot{V}_s = U_s$ are taken such that $\langle V_s \rangle = 0$.

Let us insert eq. (11.35) into the last $n - m$ equations in (11.31). Then we obtain equations for q_{m+s0}, in which $\alpha_1, \dots, \alpha_m$ appear as parameters. Let us assume that these equations admit a stable $2\pi/\omega-$periodic isolated solution (i.e. such a solution which has no new constants) q_{m+s0} for any $\alpha_1, \dots, \alpha_m$ from a certain region A. The set of functions $p_{s0}(t, \alpha_s)$, $q_{m+s0}(t, \alpha_1, \dots, \alpha_m)$ comprises a family of the generating solutions. Inserting p_{s0}, q_{m+s0} into the small terms of the first m equations in eq. (11.31) we obtain the equations for the constants $\alpha_1, \dots, \alpha_m$ from the conditions of periodicity of the first approximation (p_{s1}) to the quasi-cyclic momenta. In this case, the periodicity conditions reduce to the requirement that the small terms do not contain constant components after substitution of $p_s = p_{s0}, q_{m+s} = q_{m+s0}$. Then we arrive at equations

$$P_s(\alpha_1, \dots, \alpha_m) = -\left\langle \frac{\partial L_R}{\partial p_s} \right\rangle_0 - i_{cs} = 0, s = 1, \dots, m. \tag{11.36}$$

Here $i_{cs} = U_{cs}/b_s$. Let system (11.36) admits the solution $\alpha_s = \alpha_{s*}, s = 1, \dots, m$ belonging to A. Then the periodic regime corresponding to this solution is stable if roots $\lambda'_1, \dots, \lambda'_m$ of the equation

$$\det \left\| \left(\frac{\partial P_r}{\partial \alpha_s} \right)_* + \lambda' \kappa_s \delta_{rs} \right\| = 0 \tag{11.37}$$

have negative real parts and is unstable if $\operatorname{Re} \lambda'_i > 0$. The case of zero and pure imaginary roots is beyond the scope of the present book. In eq. (11.37) $\kappa_s = 1/b_s$, δ_{rs} being the Kronecker delta.

Let us clarify when the conditions

$$P_s = -\frac{\partial}{\partial \alpha_s} \left(\langle L_R \rangle_0 + \sum_{s=1}^{m} i_{cs} \alpha_s \right), \quad s = 1, \dots, m \tag{11.38}$$

are satisfied. Integrating by parts we obtain

$$\begin{aligned} \left\langle \frac{\partial L_R}{\partial p_s} \right\rangle_0 &= \frac{\partial}{\partial \alpha_s} \langle L_R \rangle_0 - \sum_{r=1}^{n-m} \left\langle \frac{\partial L_{R0}}{\partial \dot{q}_{m+r0}} \frac{\partial \dot{q}_{m+r0}}{\partial \alpha_s} + \frac{\partial L_R}{\partial q_{m+r0}} \frac{\partial q_{m+r0}}{\partial \alpha_s} \right\rangle \\ &= \frac{\partial}{\partial \alpha_s} \langle L_R \rangle_0 + \sum_{r=1}^{n-m} \left\langle \left[\frac{d}{dt} \frac{\partial L_R}{\partial \dot{q}_{m+r}} - \frac{\partial L_R}{\partial q_{m+r}} \right]_0 \frac{\partial q_{m+r0}}{\partial \alpha_s} \right\rangle \end{aligned} \tag{11.39}$$

It follows from Routh's equations that relationships (11.38) are valid provided that

$$\sum_{r=1}^{n-m} \left\langle N_{m+r0} \frac{\partial q_{m+r0}}{\partial \alpha_s} \right\rangle = 0, \quad s = 1, \dots, m. \tag{11.40}$$

Point $(\alpha_{1*}, \dots, \alpha_{m*})$ is a stationary point of function

$$\Lambda(\alpha_1, \dots, \alpha_m) = -\langle L_R \rangle_0 - \sum_{s=1}^{m} i_{cs} \alpha_s . \tag{11.41}$$

In addition to this, if conditions (11.40) are satisfied then matrix $\partial P_r / \alpha_s$ is symmetric. In this case the eigenvalues λ_i' are real-valued and their signs (under the condition that all $\kappa_s > 0$) do not depend on the values of κ_s. Thus, the stability can be judged by signs of the roots $\lambda_1, \dots, \lambda_m$ of the equation

$$\det \left\| \left(\frac{\partial P_r}{\partial \alpha_s} \right)_* + \lambda \delta_{rs} \right\| = 0 . \tag{11.42}$$

Finally, we conclude that values $\alpha_1 = \alpha_{1*}, \dots, \alpha_m = \alpha_{m*}$ corresponding to the stable solution delivers a minimum for function $\Lambda(\alpha_1, \dots, \alpha_m)$.

The latter assertion is a formulation of the integral criterion of stability for systems of the considered class. The criterion is obviously valid if all of the forces corresponding to the positional coordinates are potential. However, it can also be valid in the case of non-potential forces N_{m+s}.

For example, let N_{m+s} be represented as linear forms in the generalised velocities

$$N_{m+s} = \sum_{r}^{n-m} \beta_{rs} \dot{q}_{m+r} , \tag{11.43}$$

whereas q_{m+s} can be cast in the form of a series

$$q_{m+s0} = \sum_{\nu} q_{m+s0}^{(\nu)}(\alpha_1, \dots, \alpha_m) \cos(\nu\omega t - \varphi_\nu) ,$$

where the phase shifts $\varphi_{m+s0}^{(\nu)} = \varphi_\nu, \nu = 1, \dots$ of harmonics (i.e. components) of q_{m+s0} do not depend on $\alpha_1, \dots, \alpha_m$ and are equal to each other for all q_{m+s0}, $s = 1, \dots, n-m$. Then

$$\sum_{s}^{n-m} \left\langle N_{m+s0} \frac{\partial q_{m+s0}}{\partial \alpha_r} \right\rangle = \sum_{s,k}^{n-m} \beta_{ks} \left\langle -\sum_{\nu} \nu\omega q_{m+k0}^{(\nu)} \sin(\nu\omega t - \varphi_\nu) \times \right.$$
$$\left. \sum_{\nu} \frac{\partial q_{m+s0}^{(\nu)}}{\partial \alpha_r} \cos(\nu\omega t - \varphi_\nu) \right\rangle = 0 . \tag{11.44}$$

We refer to functions with equal phase shifts of the components in the Fourier expansion as component-synphase, whilst the conditions $\varphi_{m+10}^{(\nu)} = \dots = \varphi_{n0}^{(\nu)} = \varphi_\nu$ are called the component-synphase conditions. The above means that for the integral criterion to exist in the case where N_{m+s} are linear forms of $\dot q_{m+1}, \dots, \dot q_n$, it is sufficient that the phase shifts in the expansion of the positional coordinates determined in the generating approximation do not depend on $\alpha_1, \dots, \alpha_m$, whereas these coordinates satisfy the component-synphase conditions. Since no condition is imposed on the properties of coefficients β_{rs}, this criterion is valid in this case, in particular, when N_{m+s} are viscous damping forces.

This case has no analogy in the problems of periodic motion of quasi-conservative systems since their constants $\alpha_1, \dots, \alpha_m$ are the phase shifts of the object coordinates [16], [74] which appear in the solution in the combinations $\omega t + \alpha_s$, values $\varphi_{m+s0}^{(\nu)}$ depend necessarily on $\alpha_1, \dots, \alpha_m$ and the integral criterion holds only for a carrying (oscillatory) system without dissipation, [74].

The integral criterion holds also in the case when the sums in eq. (11.40) are derivatives of some function with respect to α_s, and in particular when they do not depend on $\alpha_1, \dots, \alpha_m$. Such a case is studied in Sec. 11.5.

Let us write down the expression for function Λ in detail. The kinetic energy has the following form: $T = T_1 + U + T_2$, where T_1 and T_2 are respectively the quadratic forms of the quasi-cyclic and positional generalised coordinates, and U is their bilinear form. Using this notation, the expression for L_R is cast in the form: $L_R = T_2 - \Pi - T_1 = L_2 - T_1$. Hence, $\Lambda = \langle T_1 \rangle_0 - \langle L_2 \rangle_0 - W_c$, where

$$W_c = \sum_r^m i_{cr} \alpha_r \,. \tag{11.45}$$

Function Λ also retains this form in the case in which the sums in eq. (11.45) are values i'_{cr} which are independent of $\alpha_1, \dots, \alpha_m$. What is required is to replace i_{cr} by $i_{cr} - i'_{cr}$.

The generating approximation and the form of function Λ do not change under adding terms of the form $\mu()^\bullet$ into the first m equations in eq. (11.30) and any terms of order of μ into the last $n-m$ equations. Finally the dependence $\langle L_R \rangle_0$ on $\alpha_1, \dots, \alpha_m$ does not change if the system gains such k additional degrees of freedom corresponding to the coordinates $q_{n+1}, \dots, q_{n+k}$ that the expression for T is written down as follows

$$\begin{aligned} T &= T_1 + U + T_2 + T_3, \qquad T_3 = T_3(q_{n+1}, \dots, q_{n+k}, \dot q_{n+1}, \dots, \dot q_{n+k}) \\ T_1 &= \frac{1}{2} \sum_{r,s}^m a_{rs} \dot f_r \dot f_s, \qquad U = \sum_r^m \sum_s^{n-m} a_{rm+s} \dot f_r \dot q_{m+s} \,. \end{aligned} \tag{11.46}$$

Here, as above, $a_{rs}, a_{r,m+s}$ depend only on $q_{m+1}, \dots, q_n$. Moreover, T_2 has the previous form and

$$\dot{f}_r = \dot{q}_r + \sum_{j=1}^{k} n_{rj}\dot{q}_{n+j}, \quad n_{rj} = \text{const} . \tag{11.47}$$

Let us find the form of Routh's equations. Denoting the sum in eq. (11.47) as $\dot{g}_r$ we have $\partial T/\partial \dot{q}_r = \partial T/\partial \dot{f}_r$ and

$$\dot{q}_r = \dot{q}_r^{(0)}(p_1, \dots, p_m, \dot{q}_{m+1}, \dots, \dot{q}_n, q_{m+1}, \dots, q_n) - \dot{g}_r , \tag{11.48}$$

where $\dot{q}_r^{(0)}$ denotes the quasi-cyclic velocity in the system without additional arguments (i.e. $\dot{q}_r^{(0)}$ also depend on their arguments as $\dot{q}_r$ in the former system). The expression for Routh's kinetic potential is given by

$$L_R = -T_1(\dot{q}_r) + T_1(\dot{g}_r) + T_2(\dot{q}_{m+r}) + U(\dot{g}_r, \dot{q}_{m+s}) + T_3 - \Pi . \tag{11.49}$$

Here only dependences on the generalised velocities are shown (in an abbreviated form). Removing $\dot{q}_r$ with the help of eq. (11.48) from (11.49) and applying the identity

$$\sum_{r,s} a_{rs}\dot{q}_r^{(0)}\dot{g}_s + \sum_{s}^{m}\sum_{r}^{n-m} a_{sm+r}\dot{q}_{m+r}\dot{g}_s = \sum_{s}^{m} p_s\dot{g}_s \tag{11.50}$$

we obtain

$$L_R = L_R^{(0)}(p_1, \dots, p_m, \dot{q}_{m+1}, \dots, \dot{q}_n, q_{m+1}, \dots, q_n) + \sum_{s}^{m} p_s\dot{g}_s + T_3 . \tag{11.51}$$

Here $L_R^{(0)}$ denotes Routh's kinetic potential for the system without additional coordinates. Thus, the generating equations for $p_1, \dots, p_m, q_{m+1}, \dots,$ q_n have the same form and the same solutions as those for the system without additional coordinates. The equations for the additional coordinates are as follows

$$\frac{d}{dt}\frac{\partial T_3}{\partial \dot{q}_{n+s}} - \frac{\partial T_3}{\partial q_{n+s}} = -\sum_{r}^{m} n_{rs}\dot{p}_r + N_{n+s}, \quad s = 1, \dots, k . \tag{11.52}$$

System (11.52) in the generating approximation does not contain $\alpha_1, \dots,$ α_m. Let us assume that, for $\dot{p}_r = U_r$, the system admits an isolated stable solution for which $\dot{q}_{n+10}, \dots, \dot{q}_{n+k0}$ are $2\pi/\omega$–periodic functions of time. Then equations for determining $\alpha_1, \dots, \alpha_m$, the stability conditions and function Λ differ from those of the case without additional coordinates only in that i_{cr} is replaced by

$$i_{cr} + \sum_{j=1}^{k} n_{rj}\langle \dot{q}_{n+j0}\rangle .$$

The above results are also generalised to the rotational motions, that is, to the motions with some coordinates obeying the law $q_{m+s} = \omega t + \psi(t)$, where $\psi(t)$ is a $2\pi/\omega-$periodic function of time. Functions T, Π and N_{m+s} should be $2\pi-$periodic with respect to the corresponding coordinates q_{m+s} or contain only their differences $q_{m+s} - q_{m+r}$.

In the case of a linear positional coordinate the integral criterion can be modified. Routh's equations corresponding to the positional coordinates are given by

$$M\ddot{v} + Cv = Q + N, \qquad Q = \frac{d}{dt}\frac{\partial T_1}{\partial \dot{v}} - \frac{\partial T_1}{\partial v}. \tag{11.53}$$

Here $v = (q_{m+1}, \dots, q_n)$, $N = (N_{m+1}, \dots, N_n)$, M, C are symmetric $(n - m) \times (n - m)$ matrices with constant entries. Denoting the scalar products by parentheses we obtain

$$L_2 = \frac{1}{2}(M\dot{v}, \dot{v}) - \frac{1}{2}(Cv, v). \tag{11.54}$$

Let us take equation (11.53) in the generating approximation, multiply both its parts by $\partial v_0/\partial \alpha_r$ and average over the period. Under the condition (11.40) we obtain

$$\left\langle \left(M\ddot{v}_0, \frac{\partial v_0}{\partial \alpha_r} \right) \right\rangle + \left\langle \left(Cv_0, \frac{\partial v_0}{\partial \alpha_r} \right) \right\rangle = \left\langle \left(Q_0, \frac{\partial v_0}{\partial \alpha_r} \right) \right\rangle. \tag{11.55}$$

Differentiating eq. (11.53) with respect to α_r and calculating the scalar product of the result and v_0 yields

$$\left\langle \left(M\frac{\partial \ddot{v}_0}{\partial \alpha_r}, v_0 \right) \right\rangle + \left\langle \left(C\frac{\partial v_0}{\partial \alpha_r}, v_0 \right) \right\rangle = \left\langle \left(\frac{\partial N_0}{\partial \alpha_r}, v_0 \right) \right\rangle + \left\langle \left(\frac{\partial Q_0}{\partial \alpha_r}, v_0 \right) \right\rangle. \tag{11.56}$$

We sum up relationships (11.55) and (11.56) term by term and integrate the terms with $\ddot{v}_0$ by parts. The result is

$$\frac{\partial}{\partial \alpha_r}\langle L_2 \rangle_0 = \frac{\partial}{\partial \alpha_r}(V_N + V_Q)_0. \tag{11.57}$$

Let us refer to quantities $V_Q = -1/2\langle (Q, v) \rangle$ and $V_N = -1/2\langle (N, v) \rangle$ as the virial of the forces of action of the quasi-cyclic subsystem on the positional subsystem and the virial of the non-potential generalised forces respectively. Using eq. (11.57) we can remove $\langle L_2 \rangle_0$ from the expression for and express Λ in terms of the following virials

$$\Lambda = \langle T_1 \rangle_0 - V_{Q0} - V_{N0} - W_c. \tag{11.58}$$

If $N_m + 1, \dots, N_n$ are linear forms of $\dot{q}_{m+1}, \dots, \dot{q} + n$, whilst $q_{m+10}, \dots, q_{n0}$ are component synphase with phases independent of $\alpha_1, \dots, \alpha_m$, then $V_{N0} = 0$, eq. (11.40) is fulfilled and

$$\Lambda = \langle T_1 \rangle_0 - V_{Q0} - W_c.$$

This representation is useful due to the following fact. In a number of problems of vibration excitations (see the next section) it is convenient to express T_1 such that it contains some linear functionals $\xi_i = (v, v_i), v_i =$ const rather than the coordinates v themselves. The dependence of $T_1(p, \xi)$ will be "invariant" to the form of the positional subsystem. If additionally $V_{N0} = 0$, then the form of the averaged functions in eq. (11.58) as functions of p, ξ is immaterial to the details of the positional subsystem. However, the form of $\xi(t, \alpha_1, \ldots, \alpha_m)$ depends essentially on this positional subsystem.

11.3 Energy relationships for oscillations of current conductors

Given a system of bodies including m linear conductors subjected to external $2\pi/\omega$−periodic electromotive forces let us consider the case where the magnetic field can be taken to be quasi-stationary for frequencies $\omega, \ldots, \nu_*\omega$, where ν_* is sufficiently high. In general, the forthcoming analysis is valid with accuracy up to the high-frequency "tails" of the required functions effective from an $(\nu_* + 1) - th$ harmonic. This is due to the fact that the dynamic effects in the material are neglected. The relations between **B** and **H** in the material is considered to be linear and the active resistances of the conductors are assumed to be small in comparison with the inductive resistances at frequency ω. The Lagrange-Maxwell equations for the system under consideration are given by

$$\begin{aligned} \dot{\Phi}_r + \mu R_r i_r &= U_r(t) + \mu U_{cr}, \quad r = 1, \ldots, m \\ \frac{d}{dt}\frac{\partial L_2}{\partial \dot{q}_{m+r}} - \frac{\partial L_2}{\partial q_{m+r}} &= N_{m+r} + Q_{m+r}, \quad r = 1, \ldots, n-m, \end{aligned} \tag{11.59}$$

where $i_r = \dot{q}_r, r = 1, \ldots, m$ denote currents in the conductors, $q_{m+1}, \ldots, q_n$ are the mechanical generalised coordinates, with the coordinates (charges) q_r being quasi-cyclic. We introduce the notation for the ponderomotive forces

$$Q_{m+r} = \frac{\partial}{\partial q_{m+r}} W(i_1, \ldots, i_m, q_{m+1}, \ldots, q_n) \tag{11.60}$$

and the magnetic fluxes through the conductor loops

$$\Phi_r = \frac{\partial}{\partial i_r} W(i_1, \ldots, i_m, q_{m+1}, \ldots, q_n), \tag{11.61}$$

with W denoting the energy of the magnetic field.

In eq. (11.59) μR_r denotes active resistance of the conductors, whereas U_r and μU_{rc} denote variable and constant parts of the external emf, respectively. The latter are assumed to be small, so that no currents $i = O(1/\mu)$ are observed in a stationary regime.

The parameters of the generating solution $\alpha_1, \dots, \alpha_m$ are the constant components of the magnetic fluxes calculated with accuracy up to small terms

$$\Phi_{r0} = \alpha_r + V_r(t), \quad r = 1, \dots, m. \tag{11.62}$$

The value

$$W_c = \sum_{r=1}^{m} i_{cr}\alpha_r, \quad i_{cr} = \frac{U_{cr}}{R_r} \tag{11.63}$$

is referred to as the bias energy and have the following physical meaning. Let all $U_r = 0$, and the $r-th$ loop be subjected to a constant flux α_r. Then the current in the $r-th$ loop is $i_r = i_{cr}$, and the energy of the system of direct currents (bias currents) is equal to W_c (the field is considered as being external for the currents).

Relationship (11.41) allows us to formulate the following statement: if a system has no non-potential forces or these forces fulfill condition (11.40), then, under a stable periodic motion, the constant components of the magnetic fluxes (up to small values) render a minimum to the function of these components which is equal to the energy of the magnetic field minus the mechanical kinetic potential and bias energy, all energies (the mechanical kinetic potential included) being averaged over the period.

In the case where L_2 corresponds to a linear oscillating system, the mechanical kinetic potential can be replaced by a sum of the virials of the non-potential mechanical and ponderomotive forces due to eq. (11.58).

Let us assume that other linear conductors are located near the mentioned conductors, so that if a magnetic induction line encloses the "original" conductor then it also encloses the whole adjacent set of the additional conductors. The resistances of the additional conductors are assumed not to be small, otherwise the currents are $i = O(1/\mu)$). Then the charges transmitted along the additional conductors are additional coordinates, see Sec. 3.2. The quantities f_r are as follows

$$f_r = q_r + \sum_{j=1}^{l_r} w_{rj} q_j^{(r)} . \tag{11.64}$$

Here l_r denotes the number of additional conductors located near the $r-th$ original conductor, $l_1 + \dots + l_m = l, q_j^{(r)}$ is the charge in $j-th$ conductor of the $r-th$ set. The rational numbers w_{rj} are defined as follows. Let the loop of the $r-th$ original conductor follow a certain line $n_1^{(r)}$ times, whereas the loop of $j^{(r)}-th$ conductor follows a line $n_j^{(r)}$ times (in fact $n_j^{(r)}$ is the number of turns). Then $w_{jr} = n_j^{(r)}/n_1^{(r)}$.

The influence of additional conductors on the motion of the system, regardless of the types of emf, connections and connected elements (coils, capacitors, rectifiers etc.) is taken into account by replacing i_{cr} in the expression for W_c by

$$i'_{cr} = i_{cr} + \sum_{j=1}^{l_r} w_{rj} \langle i_{j0}^{(r)} \rangle \,. \tag{11.65}$$

As there always exist induction lines enclosing the original loop and not enclosing the additional loops, then the present derivation is valid only under the condition that this "difference" can be described only by terms of the order of μ in the expression for W. The first m equations in eq. (11.59) gain terms of the type $\mu()^\bullet$, and the generating solution does not change with any addition into the equations for the mechanical coordinates.

11.4 On the relationship between the resonant and non-resonant solutions

Let us consider a finite-dimensional system

$$\begin{gathered} \dot{x} = X(x,t) + \mu Y(x,\xi,\dot{\xi},t,\mu), \\ M\ddot{v} + B\dot{v} + Cv = f\left[\sum_i^k Q_i(x)v_i + \mu \ldots\right], \end{gathered} \tag{11.66}$$

which differs from eq. (11.14) by the scalar multiplier f on the right hand side of the equations of motion of the oscillatory system. In the non-resonant case $f = O(1)$ and, as above, this multiplier can be included in Q_i.

While studying systems (11.66), in addition to the assumptions of Sec. 11.1, one uses the assumptions corresponding to the resonant case. Their essence is as follows. Let us take $M = M_0 + \delta M_1$, $C = C_0 + \sigma C_1$, $B = B_0 + \gamma B_1$ and consider the equation

$$M_0\ddot{v} + C_0 v = 0\,. \tag{11.67}$$

Its eigenvalues, i.e. the roots of the polynomial $|C_0 - \lambda^2 M_0|$ are denoted as λ_ρ^2. The resonant case assumes that one of these values is $\lambda_\rho = \omega$, whilst the other eigenvalues differ from $\nu\omega$ (ν is an integer) in non-small values and $\delta, \sigma, \gamma, f$ are small values of the order μ, with δ, σ, γ not being equal to zero simultaneously.

We describe now the procedure of determining the periodic solutions in the resonant case. Let us take $\delta = \mu\delta_1, \sigma = \mu\sigma_1$ and so on. Equation (11.66) takes the form

$$\dot{x} = X(x,t) + \mu Y(x,\xi,\dot{\xi},t,\mu), \tag{11.68}$$

$$M_0\ddot{v} + C_0 v = \mu\left[-\delta_1 M_1 \ddot{v} - \gamma_1 B_1 \dot{v} - \sigma C_1 v + f_1 \sum_i^k Q_i(x) v_i + \mu \ldots\right].$$

In the generating approximation x, v are determined independently. Let us assume for simplicity that only one eigenvector corresponds to the eigenvalue $\lambda = \omega$. We denote this eigenvector as $v^{(1)}$ however ω is not necessarily the first eigenvalue. The generating solution is given by

$$\begin{gathered} x = x_0(t,\alpha_1,\ldots,\alpha_n), \\ v = v_0(t,A_1,A_2) = (A_1\cos\omega t + A_2 \sin\omega t)\, v^{(1)}\,. \end{gathered} \tag{11.69}$$

It contains not n, but $n+2$ constants $\alpha_1,\ldots,\alpha_n, A_1, A_2$. The equations for their determination are

$$\begin{gathered} P_r(\alpha_1,\ldots,\alpha_n,A_1,A_2) = \left\langle \sum_s^p z_{rs} Y_{s0} \right\rangle = 0, \quad r = 1,\ldots,n, \\ P_{n+1}(\alpha_1,\ldots,\alpha_n,A_1,A_2) = A_1\left((\sigma_1 C_1 - \omega^2\delta_1 M_1)v^{(1)}, v^{(1)}\right) + \\ \gamma_1 A_2 (B_1 v^{(1)}, v^{(1)}) - f_1 \sum_i^k Q_{i1}\cos\vartheta_{i1}(v_i, v^{(1)}) = 0, \\ P_{n+2}(\alpha_1,\ldots,\alpha_n,A_1,A_2) = -\gamma_1 A_1 (B_1 v^{(1)}, v^{(1)}) + \\ A_2((\sigma_1 C_1 - \omega^2\delta_1 M_1)v^{(1)}, v^{(1)}) - f_1 \sum Q_{i1}\sin\vartheta_{i1}(v_i, v^{(1)}) = 0, \end{gathered} \tag{11.70}$$

Here z_{rs} are the same functions as in eq. (11.21), $Y_{s0} = Y_s(x_0, \xi_0.\dot{\xi}_0, t, 0)$, $\xi_{i0} = (v_0, v_i), Q_{i1}, \vartheta_{i1}$ denote the amplitude and phase of the first harmonic $Q_i(t)$ (see eq. (11.16)). The latter two equations in eq. (11.70) have the following meaning. Inserting the generating approximation into the terms of the order of μ in the second equation in (11.68) and expanding the result as a Fourier series yields vectors v_c and v_s which are the coefficients of $\cos\omega t$ and $\sin\omega t$. The equations under consideration express the condition that v_c and v_s are orthogonal to $v^{(1)}$.

Provided that the equations in (11.70) are resolvable, one can find, in principle, that $\alpha_1,\ldots,\alpha_n, A_1, A_2$, which allows one to determine the generating solution, analyse its stability etc. Under the analogous assumptions one can also seek the periodic solutions for autonomous systems.

Generally speaking, both the non-resonant and resonant cases should be studied for each exciter. Both cases are considered in particular in the problems of the oscillations caused by mechanical exciters [16], [52]. It can be shown that the non-resonant solution is more general and the resonant solution can be obtained from the non-resonant one.

Let us refer to the region in the parameter space as resonant (non-resonant) if the assumptions of the resonant (non-resonant) case are valid there. In the non-resonant region we have the following equations

$$P_r(\alpha_1, \dots, \alpha_n, fK_0, fK_1, \Psi_1, \dots) = 0, \quad r = 1, \dots, n \tag{11.71}$$

for determining $\alpha_1, \dots, \alpha_n$. It is easy to see that matrices K_ν and parameter f appear in these equations only in the form of a product fK_ν. Matrices K_ν exist also at points of the resonant region however $K_1 = O(1/\mu), fK_\nu = O(\mu), \nu \neq 1$ here. In both regions $fK_1 = O(1)$. Therefore, equations (11.71) can also be written for the resonant region. Let us take a point in this region and construct equations (11.71). Alternatively, let us construct equations (11.70) for this point, express A_1, A_2 in terms of $\alpha_1, \dots, \alpha_n$ from the last two equations and insert the results into the first n equations. Let us show that the obtained equations for $\alpha_1, \dots, \alpha_n$ coincide with the equations in (11.71) up to the values of order μ.

To this end, we consider the equation for v_0 in the non-resonant case

$$M\ddot{v}_0 + B\dot{v}_0 + Cv_0 = f\sum_i^k Q_i(x_0(t,\alpha))v_i \,. \tag{11.72}$$

Solving these equations for v_0, inserting it into $Y(x_0, \xi_0, \dot{\xi}_0, t, 0)$ and averaging the result yields eq. (11.71) corresponding to the given values of M, B and C.

In general, matrices C_0, M_0, as well as matrices C, M should be taken as being positive definite. For this reason, the eigenvectors $v^{(\rho)}$ of equation $(C - \lambda^2 M)v = 0$ form a basis in the space of configurations of the oscillatory system. This allows one to seek the solution of eq. (11.72) in the form

$$v_0 = \sum_\nu \sum_\rho (C_{\nu\rho} \cos\nu\omega t + D_{\nu\rho} \sin\nu\omega t)v^{(\rho)} \,. \tag{11.73}$$

The equations for $C_{\nu,\rho}, D_{\nu,\rho}$ are as follows

$$\begin{aligned}
&\sum_\rho ((C - \nu^2\omega^2 M)v^{(\rho)}, v^\kappa)C_{\nu,\rho} + \nu\omega(Bv^{(\rho)}, v^\kappa)D_{\nu,\rho} \\
&= f\sum_i^k Q_{i\nu} \cos\vartheta_{i\nu}(v_i, v^{(\kappa)}), \\
&\sum_\rho -\nu\omega(Bv^{(\rho)}, v^{(\kappa)})C_{\nu,\rho} + ((C - \nu^2\omega^2 M)v^{(\rho)}, v^{(\kappa)})D_{\nu,\rho} \\
&= f\sum_i^k Q_{i\nu} \sin\vartheta_{i\nu}(v_i, v^{(\kappa)}), \quad \kappa = 1, 2, \dots .
\end{aligned} \tag{11.74}$$

Because $\delta_1, \sigma_1, \gamma_1$ do not vanish simultaneously, equations (11.74) are resolvable in the resonant region. Let us elucidate the form of the solutions.

Let $\nu \neq 1$. Assuming $v^{(\rho)}$ to be orthonormalised, i.e. $\left(M_0 v^{(\rho)}, v^{(\kappa)}\right) = \delta_{\rho.\kappa}$, we obtain

$$
\begin{aligned}
&(\lambda_\kappa^2 - \nu^2\omega^2)C_{\nu\kappa} + \mu \sum_\rho ((\sigma_1 C_1 - \nu^2\omega^2\delta_1 M_1)v^{(\rho)}, v^{(\kappa)})C_{\nu.\rho} + \\
&\nu\omega\gamma_1(Bv^{(\rho)}, v^{(\kappa)})D_{\nu\rho} = \mu f_1 \sum_i^k Q_{i\nu}\cos\vartheta_{i\nu}(v_i, v^{(\kappa)}), \\
&(\lambda_\kappa^2 - \nu^2\omega^2)D_{\nu\kappa} + \mu \ldots = \mu f_1 \sum_i^k Q_{i\nu}\sin\vartheta_{i\nu}(v_i, v^{(\kappa)})\,.
\end{aligned}
\tag{11.75}
$$

One can see that all $C_{\nu\kappa}, D_{\nu\kappa} = O(\mu), \nu \neq 1$. For $\nu = 1$ we have

$$
\begin{aligned}
&(\lambda_\kappa^2 - \omega^2)C_{1\kappa} + \mu \sum_\rho ((\sigma C_1 - \omega^2\delta_1 M_1)v^{(\rho)}, v^{(\kappa)})C_{1\rho} + \\
&\mu\omega\gamma_1(Bv^{(\rho)}, v^{(\kappa)})D_{1\rho} = \mu f_1 \sum_i^k Q_{i1}\cos\vartheta_{i1}(v_i, v^{(\kappa)}), \\
&(\lambda_\kappa^2 - \omega^2)D_{1\kappa} + \mu \sum_\rho -\omega\gamma_1(Bv^{(\rho)}, v^{(\kappa)})C_{1\rho} + \\
&((\sigma_1 C_1 - \omega^2\delta_1 M_1)v^{(\rho)}, v^{(\kappa)})D_{1\rho} = \mu f_1 \sum_i^k Q_{i1}\sin\vartheta_{i1}(v_i, v^{(\kappa)})\,.
\end{aligned}
\tag{11.76}
$$

Let us write down the equation corresponding to $\kappa = 1$

$$
\begin{aligned}
&\sum_\rho ((\sigma_1 C_1 - \omega^2\delta_1 M_1)v^{(1)}, v^{(1)})C_{11} + \omega\gamma_1(B_1 v^{(1)}, v^{(1)})D_{11} - \\
&f_1 \sum_i^k Q_{i1}\cos\vartheta_{i1}(v_i, v^{(1)}) + \mu \ldots = 0\,.
\end{aligned}
\tag{11.77}
$$

If we denote here $C_{11} = A_1, D_{11} = A_2$, then eq. (11.77) coincides with the $(n+1) - th$ equation in (11.70) with accuracy up to the terms of order μ. Similarly, the second equation in (11.76) for $\kappa = 1$ coincides with the $(n+2) - th$ equation in eq. (11.70). Thus, dependences $C_{11}(\alpha_1, \ldots, \alpha_n)$, $D_{11}(\alpha_1, \ldots, \alpha_n)$, obtained from eq. (11.74) are coincident with dependences $A_1(\alpha_1, \ldots, \alpha_n)$, $D_1(\alpha_1, \ldots, \alpha_n)$ obtained from the last two equations in (11.70) with accuracy up to the values of the order $O(\mu)$. Functions $P_r(\alpha_1, \ldots, \alpha_n)$ obtained by substituting the solution of eq. (11.73) for the taken point of the region into Y_r and further averaging (i.e. functions P_r obtained for the resonant region by the non-resonant procedure) coincide with the above accuracy with functions $P_r(\alpha_1, \ldots, \alpha_n)$ obtained for the same point after removing A_1, A_2 from (11.70). If $|\partial P_r/\partial \alpha_s| \neq 0$, then $\alpha_1, \ldots, \alpha_n$, determined from eqs. (11.71) and (11.70) differ by values of the order μ. However, the latter means that the resonant generating solution can be obtained from the non-resonant one and the non-resonant solution can be utilised in the resonant region.

This conclusion is also valid in the cases of multiple roots as well as in the cases of the roots $\lambda_\rho = \nu\omega, \nu \neq 1$ etc.

Thus, if only the generating solution is of interest, the special consideration of the resonant case is unnecessary.

This can not be extended to the stability conditions. However, in the problem of oscillations excited by a rotating unbalanced body when the vibrator is considered as a nearly conservative object, the stability conditions in the non-resonant case [74] coincide with those in the resonant case, the latter being obtained via an asymptotical method by Kononenko in [52].

In contrast to this, if the resonant solution is used in the non-resonant region, the result is coincident with that which is obtained by omitting all of the harmonics in eq. (11.73) except the first, and all the modes except $v^{(1)}$. This strategy is often admissible. In principle, the resonant assumptions allows us to solve the problem in the cases in which eq. (11.72) is non-linear, i.e. when the oscillatory system is non-linear or Q_i depend on ξ. In this case, utilising the resonant solution in the non-resonant region is equivalent to applying the method of harmonic balance for solving eq. (11.72).

11.5 Routh's equations which are linear in the positional coordinates

Even if the kinetic potential L_2 corresponds to a linear system, the equations for the positional coordinates are, generally speaking, non-linear by virtue of the dependence of actions Q on the positional coordinates. There exist however two cases which result in linear equations. Under stationary constraints, the structure of Routh's function $R = L + \Pi$ is as follows, see [60],

$$R = \frac{1}{2}\sum_{r,s=1}^{n-m}(A_{m+rm+s} + D_{m+rm+s})\dot{q}_{m+r}\dot{q}_{m+s} + \sum_{r=1}^{n-m}\sum_{s=1}^{m} D_{m+rs}p_s\dot{q}_{m+r} - \frac{1}{2}\sum_{r,s=1}^{m} A^{(rs)}p_rp_s, \quad (\| A^{(rs)} \| = \| A_{rs} \|^{-1}). \tag{11.78}$$

Expressions for the forces of action of a quasi-cyclic subsystem on a positional one are given by

$$Q_{m+r} = -\frac{d}{dt}\sum_{s=1}^{n-m} D_{m+rm+s}\dot{q}_{m+s} + \frac{1}{2}\sum_{i,s=1}^{n-m}\frac{\partial D_{m+im+s}}{\partial q_{m+r}}\dot{q}_{m+i}\dot{q}_{m+s} - \sum_{s=1}^{m} p_s \sum_{i+1}^{n-m}\dot{q}_{m+1}\left(\frac{\partial D_{m+rs}}{\partial q_{m+i}} - \frac{\partial D_{m+is}}{\partial q_{m+r}}\right) - \sum_{s=1}^{m} D_{m+rs}\dot{p}_s - \frac{1}{2}\sum_{i,s=1}^{m}\frac{\partial A^{(is)}}{\partial q_{m+r}}p_ip_s, \quad (r = 1, \dots, n-m). \tag{11.79}$$

To obtain linear equations in the generating approximation for the positional coordinates, we take that the non-potential forces corresponding to the positional coordinates are linear forms in $\dot{q}_{m+1}, \dots, \dot{q}_n$ with constant coefficients. As follows from eq. (11.79), two cases are possible.

1. *Quasi-harmonic generating system.* Quasi-harmonic equations, i.e. the linear equations with periodic coefficients, are obtained if $D_{m+rm+s} =$ const $(r, s = 1, \dots, n-m)$, $D_{m+rs}(r = 1, \dots, n-m, s = 1, \dots, m)$ are sums of the constant values and linear forms of the positional coordinates, whereas $A^{(rs)}(r, s = 1, \dots, m)$ are sums of constant values, linear and quadratic forms of the positional coordinates. The constant terms in $A^{(rs)}$ do not affect the form of Q_r, whilst the second term on the right hand side of eq. (11.79) vanishes. The system of generating equations is inhomogeneous if at least one of two conditions is satisfied: 1) $D_{m+r,s}$ contains constant terms and 2) $A^{(rs)}$ contains linear terms. This system is homogeneous if all $D_{m+r,s}$ are linear forms and $A^{(rs)}$ is a sum of a constant value and a quadratic form.

Let us notice one specific case. Let the products of the quasi-cyclic and positional coordinates be absent in the expression for the kinetic energy, i.e. $U = 0$. Then all of $D_{m+rs} = 0, D_{m+rm+s} = 0$. Let also $A^{(rs)}$ contain no linear terms. Splitting T_1 into the energy of quasi-cyclic subset for the fixed positional subsystem, T_1^*, and an "additional" energy, ΔT_1,

$$T_1 = T_1^* + \Delta T_1, \; T_1^* = \frac{1}{2}\sum_{r,s=1}^{m} A_*^{(rs)} p_r p_s, \; \Delta T_1 = \frac{1}{2}\sum_{r,s=1}^{m} \Delta A^{(rs)} p_r p_s, \tag{11.80}$$

where $\Delta A^{(rs)}$ denote the quadratic forms of q_{m+r}, we obtain

$$V_Q = \Delta T_1 . \tag{11.81}$$

Function Λ has the following form

$$\Lambda = \langle T_1^* \rangle - V_Q - W_c . \tag{11.82}$$

If additionally $V_{Q0} = 0$, then function Λ has the form corresponding to the case of $q_{m+1}, \dots, q_n = 0$, i.e. under the fixed positional subsystem. Hence, in this case the positional subsystem does not affect the motion of the cyclic one with accuracy up to small terms (however motion of the positional subsystem essentially depends upon the motion of the quasi-cyclic one). In the problems of the vibration excitation this implies that the backward influence of oscillations on the exciter is not essential despite the presence of the family of generating solutions and the importance of small terms depending on the positional coordinates.

If $T_1^* = 0$ and at least one $U_{cs} \neq 0$, then the solutions of the considered type do not exist at all. If $U_{cs} = 0$, then we arrive at the special case of the method of small parameters $(P_r \equiv 0)$ which requires consideration of the terms of the order μ^2 in the solutions sought.

2. *Generating system with constant coefficients.* Equations with constant coefficients are obtained if $D_{m+rs} = \text{const}, D_{m+rm+s} = \text{const}$, whilst $A^{(rs)}$ are linear form of the positional coordinates. The positional coordinates in the generating approximation are determined from the solution of the problem of the forced oscillations of a linear system subjected to forces which are prescribed functions of time and parameters $\alpha_1, \dots, \alpha_m$.

Let the expression for the kinetic energy of the system with quasi-cyclic coordinates contain no products of the quasi-cyclic and positional velocities, i.e. their bilinear form $U = 0$. If the expression for T_1 can be written such that it contains the parameters of the backward influence (i.e. in the form invariant to the particular form of the oscillatory system), then equations for the parameters of the generating solution and the stability conditions can be cast in the form containing the harmonic influence coefficients for the oscillatory system as parameters.

Let us represent the expression for Routh's kinetic potential L_R in terms of the functional of the backward influence

$$L_R = L_2(v, \dot{v}) - \frac{1}{2} \sum_{r,s=1}^{m} \left(A_{rs} + \sum_{j=1}^{k} \Delta A_{rs}^{(i)} \xi_i \right) p_r p_s = L_2 - T_1^* - \Delta T_1 \,. \tag{11.83}$$

Keeping the assumptions of Sec. 11.1, we write down Routh's equations

$$\begin{aligned} &\dot{p}_s - \mu \beta_s \frac{\partial L_r}{\partial p_s} = U_s(t) + \mu U_{cs}, \quad s = 1, \dots, n, \\ &M\ddot{v} + B\dot{v} + Cv = \sum_{i}^{k} Q_i v_i \,. \end{aligned} \tag{11.84}$$

The generalised forces Q_i are given by the relationships

$$Q_i = -\frac{1}{2} \sum_{r,s+1}^{m} \Delta A_{rs}^{(i)} p_r p_s \,. \tag{11.85}$$

In the generating approximation we have

$$\begin{aligned} &p_{s0} = \alpha_s + V_s(t), \quad \dot{V}_s = U_s, \quad \langle V_s \rangle = 0, \\ &Q_{i0} = -\frac{1}{2} \sum_{r,s=1}^{m} \Delta A_{rs}^{(i)} (\alpha_r \alpha_s + 2\alpha_r V_s + V_r V_s) \,. \end{aligned} \tag{11.86}$$

Given Q_{i0}, we can write the following expressions for ξ_{j0}

$$\xi_{j0} = -\tfrac{1}{2} \sum_{i}^{k} \sum_{r,s}^{m} \Delta A_{rs}^{(i)} \left[k_0^{(ij)} (\alpha_r \alpha_s + V_{rs}^{(0)}) + \right.$$

$$\left. 2 \sum_{\nu, \nu \neq 0} k_\nu^{(ij)} \alpha_r V_{s\nu} \cos(\nu \omega t - \vartheta_{s\nu} - \psi_\nu^{(ij)}) \right] + \xi_{*j}, \quad j = 1, \dots, k \,. \tag{11.87}$$

The adopted notation corresponds to the equalities

$$V_s = \sum_{\nu,\nu\neq 0} V_{s\nu}\cos(\nu\omega t - \vartheta_{s\nu}),$$

$$V_rV_s = V_{rs}^{(0)} + \sum_{\nu,\nu\neq 0} V_{rs}^{(\nu)}\cos(\nu\omega t - \vartheta_{rs}^{(\nu)}), \tag{11.88}$$

$$\xi_{*j} = -\frac{1}{2}\sum_i^k \sum_{\nu,\nu\neq 0} k_\nu^{(ij)} \sum_{r,s}^m \Delta A_{rs}^{(i)} V_{rs}^{(\nu)} \cos(\nu\omega t - \vartheta_{rs}^{(\nu)} - \psi_\nu^{(ij)}).$$

Thus ξ_{*j} are the parts of ξ_j which are independent of $\alpha_1, \dots, \alpha_m$ and $\langle\xi_{*j}\rangle = 0$.

Inserting ξ_{j0} from eq. (11.87) into the relationships

$$\dot{q}_r = -\frac{\partial L_R}{\partial p_r} = \sum_{s=1}^m \left[A_{rs} + \sum_i^k \Delta A_{rs}^{(i)}\xi_i\right] p_s \tag{11.89}$$

and averaging the result we obtain equations for $\alpha_1, \dots, \alpha_m$

$$P_r(\alpha_1, \dots, \alpha_m) \equiv \sum_{s,u,z=1}^m a_{rsuz}\alpha_s\alpha_u\alpha_z + \sum_{s=1}^m a_{rs}\alpha_s - e_r = 0, \; r = 1, \dots, m. \tag{11.90}$$

Here

$$a_{rsuz} = -\frac{1}{2}\sum_{i,j}^k \Delta A_{rz}^{(j)} \Delta A_{su}^{(i)} k_{(0)}^{(ij)},$$

$$a_{rs} = A_{rs} + \sum_{u,z=1}^m a_{rsuz}V_{uz}^{(0)} -$$

$$\frac{1}{2}\sum_{i,j}^k \sum_{u,z}^m \sum_{\nu,\nu\neq 0} \Delta A_{ru}^{(j)} \Delta A_{sz}^{(i)} V_{u\nu}V_{z\nu}K_\nu^{(ij)} \cos(\vartheta_{u\nu} - \vartheta_{z\nu} - \psi_{(\nu)}^{(ij)}), \tag{11.91}$$

$$e_r = i_{cr} - \sum_s^m \sum_j^k \Delta A_{rs}^{(j)} \langle\xi_{*j}V_s\rangle, \quad i_{cr} = U_{cr}/\beta_r.$$

The dependences $\alpha_r(K, \Psi)$ can be found and analysed sufficiently simply only in particular cases, for example, for $k, m = 1, 2$. Generally speaking, $\alpha_1, \dots, \alpha_m$ should be determined from equations in eq. (11.90), the particular values of K, Ψ having been substituted into these equations.

The further calculation of $\alpha_1, \dots, \alpha_m$ is simplified if the integral criterion holds. Let us find the sufficient conditions of its existence. Let us construct the derivatives

$$\frac{\partial P_r}{\partial \alpha_s} = \sum_{u,z}^m (a_{rsuz} + a_{rusz} + a_{ruzs})\alpha_u\alpha_z + a_{rs}. \tag{11.92}$$

It follows from the relationships $k_0^{ij} = k_0^{ji}$ and evident equalities $\Delta A_{rs}^{(i)} = \Delta A_{sr}^{(i)}$ that the coefficients a_{rsuz} do not change if we interchange the extreme (left and right) subscripts, as well as the middle subscripts, and simultaneously interchange the subscripts in the first and second pairs. Indeed,

$$a_{rusz} = -\frac{1}{2}\sum_{i,j}^{k} \Delta A_{rz}^{(i)} \Delta A_{su}^{(j)} k_0^{(ij)} = -\frac{1}{2}\sum_{i,j}^{k} \Delta A_{su}^{(i)} \Delta A_{rz}^{(j)} k_0^{(ij)} = a_{srzu} \,. \tag{11.93}$$

Hence, $a_{rsuz} = a_{rusz}$, that is, the first two coefficients in the sum in eq. (11.92) are equal. Let us interchange the subscripts r and s in eq. (11.92). The third coefficient in the sum does not change. In addition to this,

$$\sum_{u,z} a_{sruz}\alpha_u\alpha_z = \sum_{u,z} a_{aszu}\alpha_u\alpha_z = \sum_{u,z} a_{rsuz}\alpha_u\alpha_z \,. \tag{11.94}$$

Hence, the sum in eq. (11.92) itself does not change. The first two terms in the expression for a_{rs} possess the same property. Using the property of the reciprocity of the harmonic influence coefficients and phases $\psi_\nu^{(ij)}$ and interchanging the subscripts i, j, u, z yields

$$\begin{aligned} a_{rs} - a_{sr} = -\tfrac{1}{2}\sum_{i,j}^{k}\sum_{u,z}^{m}\sum_{\nu,\nu\neq 0} \Delta A_{ru}^{(j)} \Delta A_{sz}^{(i)} V_{u\nu} V_{z\nu} \times \\ k_\nu^{(ij)} \left[\cos(\vartheta_{u\nu} - \vartheta_{z\nu} - \psi_\nu^{(ij)}) - \cos(\vartheta_{z\nu} - \vartheta_{u,\nu} - \psi_\nu^{(ij)})\right] \,. \end{aligned} \tag{11.95}$$

Hence, $a_{rs} = a_{sr}$, if $\vartheta_{u\nu} = \vartheta_{z\nu}$ where $u, z = 1, \ldots, m$. Thus the equalities $\partial P_r/\partial\alpha_s = \partial P_s/\partial\alpha_r$ hold in the case when the generalised forces corresponding to the quasi-cyclic coordinates are component-synphase. Then, $P_r = \partial\Lambda/\partial\alpha_r$, that is, the condition of the component-synphase coordinates is the sufficient condition for the existence of the integral criterion. Another sufficient condition is $\psi_\nu^{(ij)} = 0$ which is satisfied when all of the generalised forces corresponding to the positional coordinates are potential forces. The forces $U_s(t)$ are not necessarily component-synphase.

Under component-synphase $U_s(t)$ the integral criterion is valid because the non-potential forces in the generating approximation satisfy the relationships

$$\left\langle \sum_{s}^{n-m} N_{m+s0}\frac{\partial q_{m+s0}}{\partial\alpha_r} \right\rangle \equiv -\left\langle \left(B\dot{v}, \frac{\partial v_0}{\partial\alpha_r} \right) \right\rangle = i'_{cr} \,, \tag{11.96}$$

where i'_{cr} are not zero but do not depend on $\alpha_1, \ldots, \alpha_m$. Indeed, the external forces

$$\sum_{i}^{k} Q_{i0}v_i = -\frac{1}{2}\sum_{i}^{k}\sum_{r,s}^{m} \Delta A_{rs}^{(i)}[\alpha_r\alpha_s + 2\alpha_r \sum_{\nu,\nu\neq 0} V_{s\nu}\cos(\nu\omega t - \vartheta_\nu) + V_r V_s]v_i \tag{11.97}$$

cause oscillations of the form $v_0 = v_c + v_g + v_*$, where v_c is time-independent, v_g is represented by the following expansion

$$v_g = \sum_{\nu,\nu\neq 0} v_\nu^{(1)} \cos(\nu\omega t - \vartheta_\nu) + v_\nu^{(2)} \sin(\nu\omega t - \vartheta_\nu)\,, \tag{11.98}$$

with $v_\nu^{(1)}, v_\nu^{(2)}$ being linear forms of $\alpha_1, \dots, \alpha_m$ of the type $v_\nu^{(1)} = L_\nu(\alpha) v_{\nu *}^{(1)}$, $v_\nu^{(2)} = L_\nu(\alpha) v_{\nu *}^{(2)}$, whilst v_* does not depend on $\alpha_1, \dots, \alpha_m$. Therefore,

$$\left\langle\left(B\dot{v}_0, \frac{\partial v_0}{\partial \alpha_r}\right)\right\rangle = \left\langle\left(B(v_g + v_*)^\bullet, \frac{\partial v_c}{\partial \alpha_r} + \frac{\partial v_g}{\partial \alpha_r}\right)\right\rangle\,. \tag{11.99}$$

Since the derivatives do not contain a constant term, and v_ν is independent of $\alpha_1, \dots, \alpha_m$, then

$$\left\langle\left(B(v_g + v_*)^\bullet, \frac{\partial v_c}{\partial \alpha_c}\right)\right\rangle, \qquad \left\langle\left(B\dot{v}_g, \frac{\partial v_g}{\partial \alpha_r}\right)\right\rangle = 0\,.$$

Finally we have

$$\left\langle\left(B\dot{v}_0, \frac{\partial v_0}{\partial \alpha_r}\right)\right\rangle = \left\langle\left(B\dot{v}_*, \frac{\partial v_g}{\partial \alpha_r}\right)\right\rangle\,. \tag{11.100}$$

Clearly, the value on the right hand side of eq. (11.100) does not depend on $\alpha_1, \dots, \alpha_m$. Let us indicate the particular form of the integral criterion corresponding to the case under consideration. Let us present the kinetic energy of the exciter in the form $T_1 = T_e + \Delta T$. Here T_e denotes the energy for $\xi_i \equiv 0$, i.e. for an oscillatory system which is immovable in the undeformed state. ΔT denotes an "additional" energy

$$T_e = \frac{1}{2}\sum_{r,s}^{m} A_{rs} p_r p_s\,, \quad \Delta T = \frac{1}{2}\sum_{r,s}^{m}\sum_{i}^{k} \Delta A_{rs}^{(i)} \xi_i p_r p_s\,. \tag{11.101}$$

Since ΔT is linear in ξ_i, the virial of the external forces is related to the additional energy of the exciter by the relationship

$$V_Q = \frac{1}{2}\langle \Delta T\rangle\,. \tag{11.102}$$

Integrating by parts we obtain

$$\langle (B\dot{v}_0, v_0)\rangle = -\langle (Bv_0, \dot{v}_0)\rangle\,. \tag{11.103}$$

Matrix (or operator) B is symmetric, thus

$$-\langle (Bv_0, \dot{v}_0)\rangle = -\langle (B\dot{v}_0, v_0)\rangle, \qquad \langle (B\dot{v}_0, v_0)\rangle = 0\,, \tag{11.104}$$

that is, the virial V_N of the non-potential forces in the oscillatory system is zero if these forces are those of viscous damping. By using eq. (11.51) we arrive at the following expression for Λ

$$\Lambda = \langle T_e \rangle_0 + \frac{1}{2}\langle \Delta T \rangle_0 - W_c, \quad W_c = \sum_r^m (i_{cr} - i'_{cr})\alpha_r \,. \tag{11.105}$$

Entering the total kinetic energy T_1, we have

$$\Lambda = \langle T_1 \rangle_0 - \frac{1}{2}\langle \Delta T \rangle_0 - W_c \,. \tag{11.106}$$

Along with the more general representation of function Λ in terms of the averaged Routh's kinetic potential or the kinetic potential of the oscillatory system

$$\Lambda = \langle T_1 \rangle_0 - \langle L_2 \rangle_0 - W_c \tag{11.107}$$

expressions (11.105) and (11.106) provide us with three forms of the integral criterion, each utilising two of four functions $T_1, T_e, \Delta T, L_2$.

It follows from eqs. (11.106) and (11.107) that $\langle L_2 \rangle_0 = 1/2\langle \Delta T \rangle_0$. Calculating the scalar product of the equation of motion for the oscillatory system and v, and averaging over the period we can obtain a more general relationship $\langle L_2 \rangle = 1/2\langle \Delta T \rangle$. Let us derive an explicit expression for function Λ. It is presented by a sum of a form of fourth degree, a quadratic and a linear form in $\alpha_1, \dots, \alpha_m$, and it can also contain an arbitrary term Λ_c which does not depend on $\alpha_1, \dots, \alpha_m$. If Λ is defined according to eq. (11.105)-(11.107), then Λ_c is not zero. Then we have

$$\Lambda = \frac{1}{4} \sum_{r,s,u,z} a_{rsuz}\alpha_r\alpha_s\alpha_u\alpha_z + \frac{1}{2}\sum_{r,s}^m a_r s\alpha_r\alpha_s - \sum_r^m (i_{cr} - i'_{cr})\alpha_r + \Lambda_c \,. \tag{11.108}$$

In order to calculate coefficients a_{rsuz}, a_{rs} there is no need to use their representation in terms of the Fourier coefficients. We can use, for example, the following notation. Let us introduce the matrix impulse-frequency characteristic of the oscillatory system $K(t) = \| K_{ij}(t) \|, i, j = 1, \dots, k$, which is defined as follows. Let a single $2\pi/\omega$−periodic load $f(t)v_j$ act on the oscillatory system. Then for pure forced oscillations, the dependence of the functional ξ_i on time can be written in the form

$$\xi_i(t) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} K_{ij}(t-\tau) f(\tau) d\tau \,, \tag{11.109}$$

where K_{ij} does not depend upon the particular form of the $2\pi/\omega$−periodic function $f(t)$. This enables us to rewrite eq. (11.87) in the following form

$$\xi_{j0} = \frac{\omega}{2\pi}\sum_{i}^{k}\int_{0}^{2\pi/\omega} K_{ij}(t-\tau)Q_{i0}(\tau)d\tau \tag{11.110}$$

and the expression for $\langle\Delta T\rangle_0$ as follows

$$\langle\Delta T\rangle_0 = -\frac{\omega^2}{4\pi^2}\sum_{i,j}^{k}\int_{0}^{2\pi/\omega} dt \int_{0}^{2\pi/\omega} d\tau Q_{j0}(t)K_{ij}(t-\tau)Q_{i0}(\tau)\,. \tag{11.111}$$

The above expressions are obtained from this equation with the help of the following relationship

$$K_{ij}(t) = k_0^{(ij)} + 2\sum_{\nu,\nu\neq 0} k_\nu^{(ij)}\cos(\nu\omega t - \psi_\nu^{(ij)})\,. \tag{11.112}$$

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