CISM COURSES AND LECTURES NO. 340
INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

DYNAMIC MOTION
CHAOTIC AND STOCHASTIC BEHAVIOUR

EDITED BY
F. CASCIATI

SPRINGER-VERLAG WIEN GMBH

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DYNAMIC MOTION: CHAOTIC AND STOCHASTIC BEHAVIOUR

EDITED BY

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PREFACE

This book is a collection of the main results of the research project on Nonlinear Mechanical Systems under Uncertainty, which was funded by the Italian Ministry of the University and of the Scientific and Technological Research (MURST) over a period of three years (from 1991 to 1993).

The partners which were coordinated by the Editor are academic research teams working in six different Italian universities: University of Florence, University of Messina, University of Naples, University of Pavia, University of Palermo and Technical University of Milan. The single researchers involved in the project provided the single chapters, but, in a joint effort, the original material has been re-arranged and re-ordered in order to avoid overlapping and to give a systematic overview of the topic.

Dynamic systems are commonly encountered in several fields of Science and Technology. Nonlinear dynamic systems are generally regarded as a topic which still requires phenomenological understanding and tools of analysis. Chaotic behaviour can occur when dealing with these nonlinear systems, i.e. the behaviour can become unpredictable even under a deterministic environment. The presence of a stochastic excitation (noise) and/or of a stochastic distribution in space of the mechanical properties of the system are other sources of unpredictability.

The chapters of this book were written by structural engineers. The approach, therefore, is not aiming toward a scientific modelling of the response but to the definition of engineering procedures for detecting and avoiding undesired phenomena. In this sense chaotic and stochastic behaviour can be tackled in a similar manner. This aspect is illustrated in Chapter 1 which covers the first lectures of the course "Dynamic Motion: Chaotic and Stochastic Behaviour", held in Udine in September 1993, under the DESEG

program of CISM. The kernel of the course is illustrated in the remaining six chapters. In particular, Chapters 2 and 3 are entirely devoted to Stochastic Dynamics and cover single-degree-of-freedom systems and impact problems, respectively. Chapter 4 provides details on the numerical tools necessary for evaluating the main indexes useful for the classification of the motion and for estimating the response probability density function. The randomness of the material characteristics and the relevant stochastic models are considered in Chapter 5. Chapter 6 gives an overview of random vibration methods for linear and nonlinear multi-degree-of-freedom systems. Chapter 7, eventually, deals with large engineering systems under stochastic excitation and allows for the stochastic nature of the mechanical and geometrical properties.

Thankings are due to several persons: professor G. Sacchi Landriani, who sponsored the course as responsible of the CISM-DESEG program, professor C. Tasso, who made possible this editorial format in the Springer-Verlag/CISM series, the researchers and the Ph.D students who assisted the authors in their research activities. But a special thank is due to MURST, without the funding policy of which this book would not have been possible.

F. Casciati

CONTENTS

Chapter 1

STOCHASTIC AND CHAOTIC MOTION IN ENGINEERING SYSTEMS

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1.1 INTRODUCTION.

This Chapter deals with the definition of a nonlinear dynamical system, the characterizations of its properties, the qualitative and quantitative tools for detecting them, the governing mathematical equations (independently of their solvability) and the consequent engineering pitfall. It is conceived as a bird's eye introduction of Nonlinear Dynamics topics.

Physical phenomena will be studied by their equivalent, in some sense, mathematical models, i.e. by solving mathematical problems. A mathematical problem is made by the equation which governs the evolution of the phenomenon and its initial and boundary conditions:

$$\mathcal{Q}(\mathbf{x}, \mathbf{y}, t) = 0, \qquad \mathbf{x} \in \Omega \tag{1}$$

$$\mathbf{y}(\mathbf{x}, t = 0) = \mathbf{b}(\mathbf{x}, t = 0), \qquad \mathbf{x} \in \Gamma \cup \Omega \tag{2}$$

$$\mathbf{y}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x}, t), \qquad \mathbf{x} \in \Gamma,\ t > 0 \tag{3}$$

Eq. (1) states a relation between the response $\mathbf{y}(t)$, the space coordinates $\mathbf{x}$ and the time t (generally $\mathbf{y}$ is a vector with n components y_j and n finite or infinite, vectors and matrices being printed in bold-face). The region in the space where $\mathbf{y}(\mathbf{x}, t)$ is defined is Ω and Γ is its boundary.

[1] Professor of Structural Mechanics

In particular a dynamical system is governed by the evolutionary problem

$$\dot{\mathbf{y}} = \mathbf{Q}(\mathbf{y}, t) \tag{4}$$

with the initial conditions

$$\mathbf{y}(t = 0) = \mathbf{b} \tag{5}$$

The number of degrees of freedom of the system is n, i.e. it is equal to the number of the components of the vector $\mathbf{y}$. When the time t does not explicitly appear in Eq.(4) the system is called **autonomous**; otherwise it will be **non-autonomous**.

Mathematical problems can also be classified according to their linear and nonlinear behaviour. The difference is based on the applicability or not of the **principle of superposition** . A mathematical problem is **well-posed** when

1. the solution of the problem is unique;
2. the solution depends continuously on the initial and boundary conditions.

It follows that linear problems are well-posed as well as many problems treated within classical analysis; only recently, one has focused attention on ill-posed problems.

Special care will be devoted to the second order nonlinear differential equation

$$\mathbf{m} \cdot \ddot{\mathbf{y}}(t) + \mathbf{c} \cdot \dot{\mathbf{y}}(t) + \mathbf{f}(\mathbf{y}(t)) = \mathbf{F}(t) \tag{6}$$

with mass $\mathbf{m}$, damping $\mathbf{c}$, nonlinear restoring force $\mathbf{f}(\mathbf{y})$ and external excitation $\mathbf{F}(t)$. Eq. (6) is a system of $\frac{n}{2}$ equations and is a special form of Eq. (4) with n equations. The number of physical degrees of freedom is therefore one half of the mathematical degrees of freedom. (Some illustrative figures of this Chapter consider a special form of Eq.(4): $f(t)$ is a cubic polynomial (Duffing oscillator) of the scalar displacement y).

The presence of known parameters in the governing equation makes it a **deterministic equation**. The presence of some random terms (additive or multiplicative) defines a **stochastic equation** (a stochastic process is a temporal sequence of correlated or uncorrelated random variables). Furthermore, the mathematical problem can be either sensitive or insensitive to the initial and boundary conditions.

The combinations of these cases gives the following classification of the equivalent mathematical problems [1, 2, 3] (see Table I):

- **deterministic problems** , governed by a deterministic equation and insensitive to initial and boundary conditions,
- **chaotic problems** , governed by a deterministic or stochastic equation but manifesting sensitivity to initial and boundary conditions,
- **stochastic problems** , governed by a stochastic equation.

Table I - Classification of dynamical systems.

	presence of random aspects	sensitivity to initial conditions
deterministic problems	no	no
chaotic deterministic problems	no	yes
chaotic stochastic problems	yes	yes
stocastic problems	yes	no

From an engineer point of view a source of noise (i.e. a random term) is always present in any real system, and then the corresponding mathematical problem should take into account random terms in the governing equation. In this sense this book only studies chaotic and stochastic motion of nonlinear structural systems.

1.2 QUALITATIVE INVESTIGATION

1.2.1 ANALYSIS IN TIME DOMAIN.

A natural way of studying a dynamical system is through the observation of its time histories, i.e. the plot of each component of the vector $\mathbf{y}$ as a function of the time t. Reference is made, throughout this paper, to the Duffing oscillator with unit mass:

$$\ddot{u} + d\dot{u} + (u^3 - u) = a\cos t + \sigma_0 w(t) \tag{7}$$

with $w(t)$ a white noise of unit strength. This equation is numerically integrated in time with time step $\Delta t = \frac{2\pi}{M}$, with M ranging between 20 and 400. Since it consists of a second order differential equation, the length of the associated vector $\mathbf{y}$ is 2.
The time history of Figure 1 leads to a static equilibrium. To distinguish between a transient and a stationary motion does not offer difficulties. This also occurs in Figures 2 and 3 where the stationary behaviour is of a periodic and a chaotic nature, respectively.

The characteristics of the system can be different due to different values of some system parameters, i.e. a system can show different behaviour according to different values of some internal parameters [4]. The passage from one kind of motion to the other, is characterized by a **bifurcation** or in general by a **catastrophe** . The classification of the system behaviour only by inspection of its time history is not simple. In particular a system with many degrees of freedom, even under a harmonic excitation, shows almost chaotic features just due to the superposition of the different components.

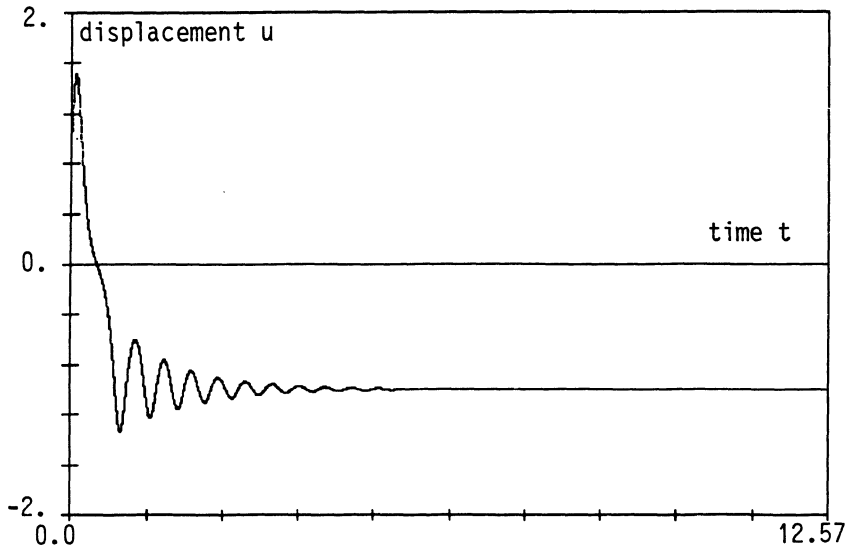

Figure 1 - Duffing oscillator with damping parameter d=0.185, no periodic excitation and no noise input: time history of the response leading to a static equilibrium.

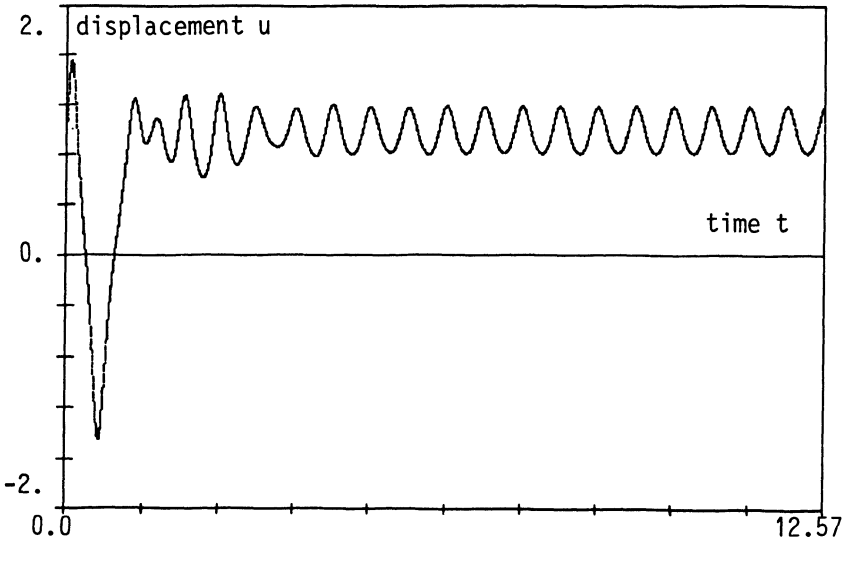

Figure 2 - Duffing oscillator with damping parameter d=0.185, intensity of the periodic excitation a=0.176 and no noise input: time history of the response leading to a periodic motion.

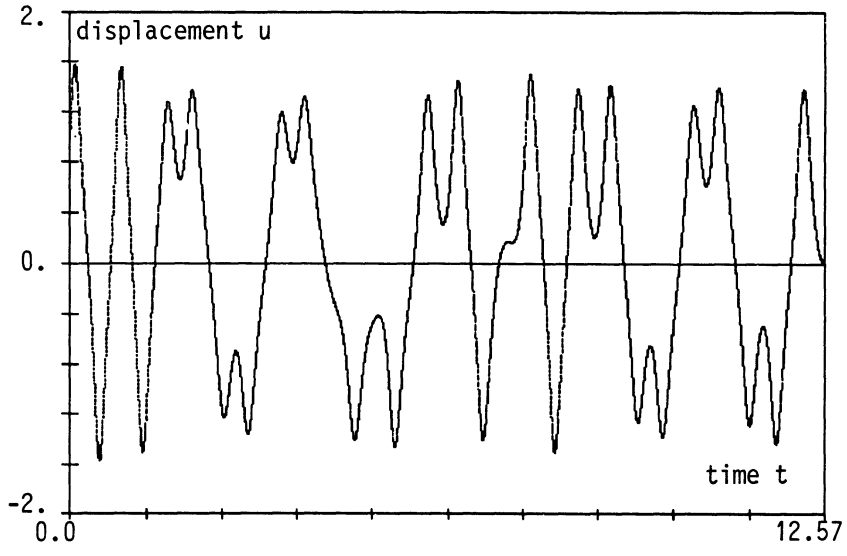

Figure 3 - Duffing oscillator with damping parameter d=0.185, intensity of the periodic excitation α=0.3 and no noise input: time history of the chaotic motion.

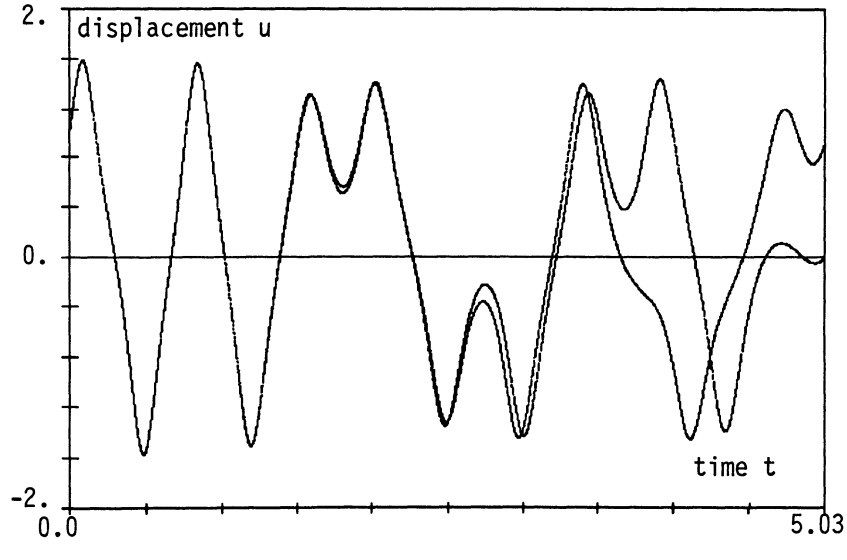

Figure 4 - Duffing oscillator of figure 3: sensitivity to the initial conditions for initially close time histories.

By drawing in the same plot time histories of systems initially close one to each other (Figure 4) [5, 6] one can appreciate their divergence. One can alternatively think to the same system that makes different realizations from initially near positions and with close boundary conditions.

The time evolution of the dynamical system of equation (4) is represented by $\mathbf{y}(t)$ that can be regarded as a time series $\mathbf{y}_h$, when the motion of the system is sampled at regular instants of time t_h [7].

1.2.2 ANALYSIS IN FREQUENCY DOMAIN.

If the single component $y(t)$ is periodic with period T, we can represent it through the **Fourier series**, i.e. as a superposition of periodic components

$$y(t) = \sum_k A_k \cdot e^{ik\omega_0 t} \tag{8}$$

where $\omega_0 = (2\pi)/T$ is the basic frequency and i is the imaginary unit. The **spectral analysis** of $y(t)$ is the determination of the amplitude of each component A_k:

$$A_k = \frac{\omega_0}{2\pi} \cdot \int_{-\pi/\omega_0}^{\pi/\omega_0} y(t) \cdot e^{-ik\omega_0 t} \cdot dt \tag{9}$$

If $y(t)$ is not periodic (that is in the limit case for $T \to \infty$), the spectrum must be expressed in terms of oscillations with continuous frequencies. This spectral representation (that is an extension of the one of equation (7)) is called the **Fourier transform** of $y(t)$

$$\hat{y}(\omega) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} y(t) \cdot e^{-i\omega t} \cdot dt \tag{10}$$

In general it is a complex function. A real function that can be extracted from the Fourier series or from the Fourier transform is its modulus (Figure 5). For a chaotic/stochastic motion, this function is continuous, i.e the time history $y(t)$ has components with a continuous spectrum of frequencies.

1.2.3 REPRESENTATION IN THE PHASE SPACE.

An endochronic approach to the description of the dynamical system (4) makes use of a representative point in a space with dimension equal to the number n of variables necessary to define in an univocal way the dynamical status of the system (Figure 6). This is the **phase space** where equation (4) can then be regarded as the motion of a fluid. This interpretation is the key for concepts as the Lyapunov exponent, the equations of Liouville and Fokker-Planck, and the idea of Gibbs set.

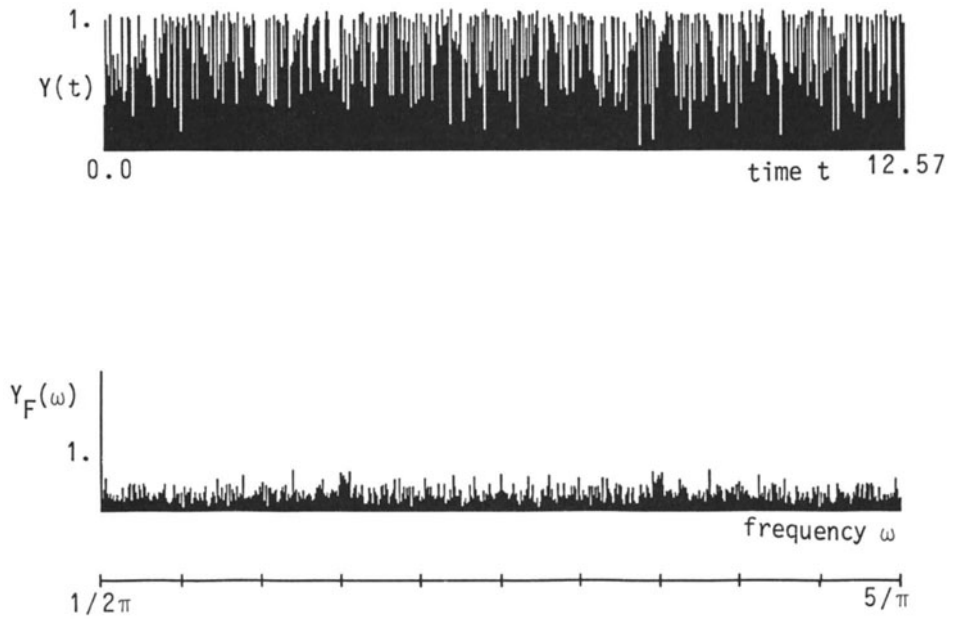

Figure 5 - Chaotic time history of the Duffing oscillator, without noise: a) squared response over its squared maximum, *Y(t)*; b) the same ratio for the modulus of its Fourier transform, $Y_F(\omega)$.

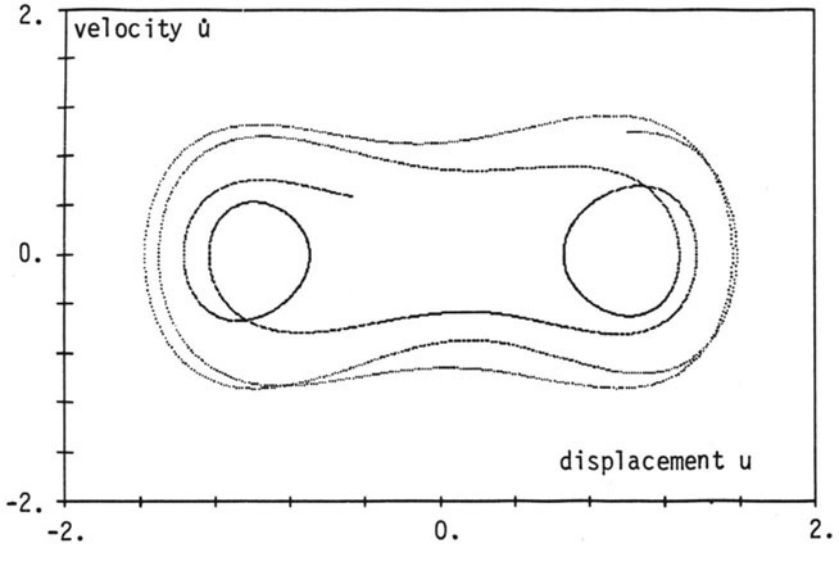

Figure 6 - Duffing oscillator with damping parameter d=0.185, intensity of the periodic excitation a=0.3 and no noise input: representation in the phase space.

In order to detect the kind of behaviour of the system between static equilibrium, periodicity and chaoticity/stochasticity, one must study the asymptotic location of the system. The set of points, where the system remains after a transient, is called **attractor** of the system [5]. When the system is nonlinear, the actractor is not necessarily unique. Figure 7 shows the presence of two attractors (the bottom of the two basins) and the sensitivity to initial conditions for the Duffing oscillator.

In the presence of sensitivity to the initial conditions or in the presence of noise in the equation of motion of the system, the asymptotic status of the system can be found only with statistical methods [1, 8, 9].

1.2.4 POINCARE' SECTIONS.

The observation of the system by its time histories and by its behaviour in the phase space is continuous. Alternatively, the status of the system can be detected only at discrete instants of time. In this way one obtains the Poincare' section (Figure 8) of the behaviour of the system, which depends on the selection of the instants of observation.
When the transient is disregarded, the aspect of the Poincare' section (in particular the number of points that appear) is representative of the behaviour of the system.

1.2.5. BIFURCATION DIAGRAMS.

Consider the dynamics of a system over the range of one **internal parameter.** In this range, the behaviour of the system can be very different. For example, the dynamics for the oscillator with trilinear restoring force can be different for different amplitudes of the harmonic excitation [10]. The transition between one kind of motion to the other occurs through a **bifurcation** . Figure 9 a) presents a diagram of bifurcation for the Duffing oscillator under a harmonic excitation.
Such a diagram, is costructed by:

- integrating the motion of the system for any given value of the parameter the influence of which must be studied (in this case the amplitude of the harmonic excitation);
- representing at some istant t after some transient, (in particular at each period of the exciting force, when it is periodic) the point $y(t)$ in a diagram where the abscissa is the internal parameter under investigation;
- repeating the previous two steps for another value of the internal parameter.

In particular, different kinds of behaviour with periodic or chaotic/stochastic cases can be immediately distinguished. Bifurcations occur where the behaviour of the system changes.

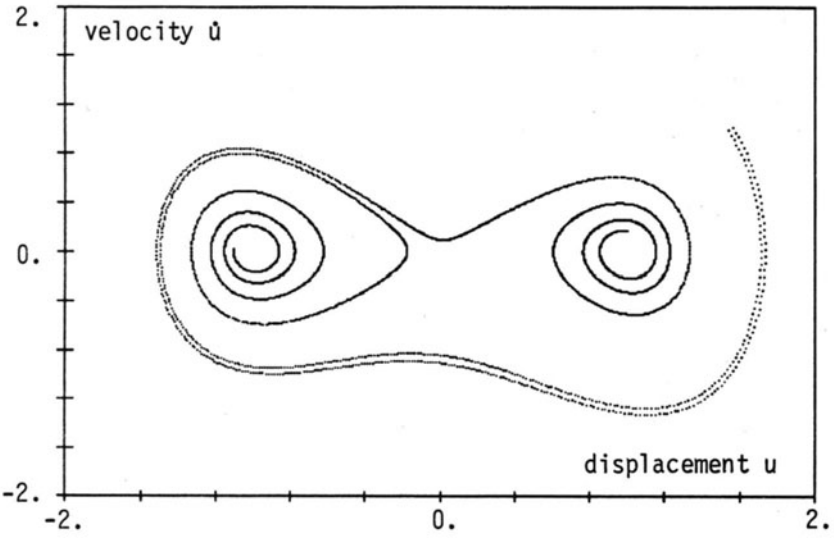

Figure 7 - Duffing oscillator with damping parameter d=0.185, no periodic excitation and no noise input: example of multiple attractor and sensitivity to the initial conditions.

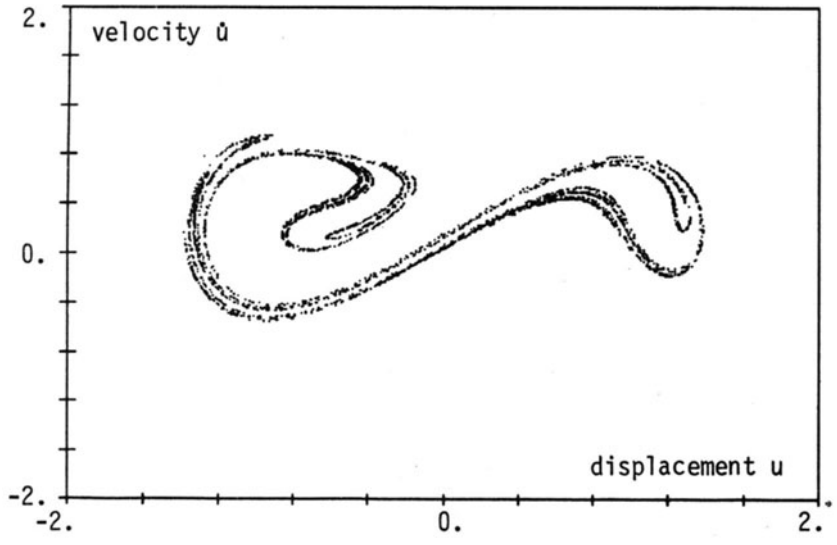

Figure 8 - Duffing oscillator with damping parameter d=0.185, intensity of the periodic excitation a=0.3 and no noise input: Poincare' section.

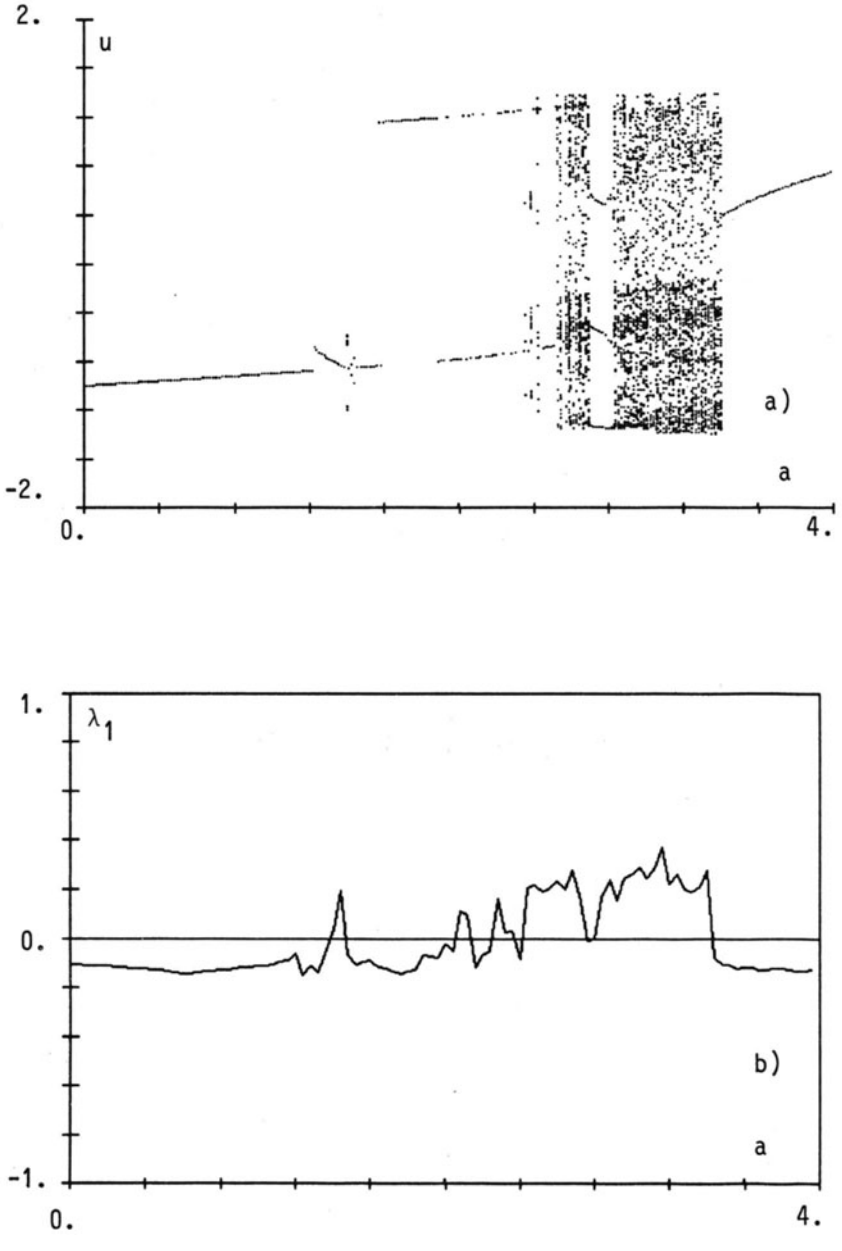

Figure 9 - Duffing oscillator with damping parameter d=0.185 and no noise input: a) bifurcation diagram for the intensity of the periodic excitation a ; b) maximum Lyapunov exponent λ_1 versus the intensity of the periodic excitation a.

1.2.6. ROUTES TO CHAOS.

Three main models of prechaotic behaviour have been observed [11]. These three scenarios are characterized by the name of the scientists who detected them:

1. **Period doubling (Feigenbaum)**: one starts with a system with a fundamental periodic motion. As some system parameter varies, the system starts an infinite sequence of bifurcations to periodic motions with twice the period of the previous oscillation. This route to chaos occurs, for instance, in the analysis of a piece-wise linear oscillator [12];
2. **Quasi-periodic route (Ruelle-Takens)**: one starts with a quasi-periodic motion the actractor of which is a torus. Tori with three or more dimensions are not robust in view of structural stability: they disappear for a small perturbation and give rise to strange attractors. This path has been encountered in [12] when studying a truss structure with bi-linear constitutive law.
3. **Intermittancy (Pomeau-Mannesville)**: long periods of periodic motion are interrupted by bursts of chaos. A multi-body model can show this route to chaos [12], with frequency of the burst of chaos higher as the number of mutual interactions increases.

1.2.7. CAUGHEY'S REMARK.

In reference [13] the integration of the Duffing oscillator

$$\ddot{x} + c\dot{x} + \omega^2(x + \alpha x^3) = F(t) \tag{11}$$

with $c = 0, \omega = 1, \alpha = 1$, is pursued by:

- the Euler numerical integration scheme with integration step $\Delta t = 0.5$;
- the Euler numerical integration scheme with integration step $\Delta t = 0.1$;
- the Euler numerical integration scheme with integration step $\Delta t = 0.01$;
- writing the exact solution through the adoption of elliptic functions.

The numerical results achieved by the three numerical schemes shows forms of chaotic/stochastic behaviour which do no affect the exact solution. The chaotic aspects are therefore the result of the numerical discretization of the problem.

This does not mean that any chaotic situation results from the discretization: numerical errors spread around the representative points, but these do not show a structure as chaotic situations do. Nevertheless, the check of the intrinsic physical meaning of chaos remains an aspect of primary importance.

3. QUANTITATIVE INVESTIGATIONS

1.3.1. PROBABILITY DENSITY FUNCTION AND DISCRETIZATION OF THE PHASE SPACE.

Connected with the attractor and the shape of the potential energy of the system there is the possibility of introducing the probability density function $p(\mathbf{y},t)$ that the system is in some part of the phase space at a given instant of time when starting with initial conditions specified by a given function $p(\mathbf{y},0)$ [8, 10, 14, 15]. Let dV the infinitesimal hypervolume in the phase space, then $p(\mathbf{y},t)dV$ expresses the probability that the system belongs to dV when centered at $\mathbf{y}$.

Analytical models (the Fokker-Planck differential equation) govern its behaviour and will be discussed in Section 1.4. It is to stress that for chaotic or stochastic systems one has probability density functions that are regular functions. In the stochastic case, in particular, they are smoothed due to the diffusion process introduced by the noise (parabolic contribution to the Fokker-Planck equation). In the case of regular deterministic systems one has singular distributions (like Dirac δ-function) which move with inalterate shape as the motion proceeds (hyperbolic contribution to the Fokker-Planck equation). The presence of extreme gradients in the $p(\mathbf{y},t)$ is responsible of the numerical difficulties that are met in solving the analytical governing equation as the intensity of the noise tends to zero.

However a frequentist interpretation of the concept of probability permits one to achieve an estimation of $p(\mathbf{y},t)$ by assessing the frequencies of finding the system, during a numerical simulation, in a subset of the phase space. Quantitative estimates of the behaviour of a dynamical system are in particular obtained by discretization of the phase space through a grid of boxes of characteristic size ϵ, which is indicative of our capacity to measure the status of the system in equation (4) (the union of these boxes is the original phase space and all their intersections are void). The frequencies extracted through the partition of the phase space are only approximated, and this approximation is increased with a higher number of boxes (and then with smaller ϵ). The number of boxes used is clearly related to the computational effort. Also the region of the phase space considered for the partitioning is related with the same computational considerations. In particular, it is necessary to extend the partition well beyond the points of the phase space visited from the system during its motion.
The first elementary measure is the number of boxes $N_{box}(t,\epsilon)$ which are visited. A more significant information is the relative frequency of points belonging to the i-th box which is an estimate of the probability mass function $p_i(t,\epsilon)$ of finding the status of the system in the i-th box. By definition one has:

$$p_i(t,\epsilon) = \int_{V_i} p(\mathbf{y},t) \cdot dV_i \tag{12}$$

The situation of Figure 10 was found for the Duffing oscillator (see also equation (10)). In

this way one emphasizes the presence of many attractors, the effect of noise sources or the sensitivity to initial conditions. As for $p(\mathbf{y}, t)$, also for the mass probability $p_i(t, \epsilon)$ singular functions characterize regular motions, while chaotic and stochastic ones are characterized by regular functions. It follows that the class of chaotic systems, even if perfectly deterministic, can be managed as stochastic from a computational point of view.

1.3.2. METRIC ENTROPY

Related to the form of the probability mass function $p_i(t, \epsilon)$ there is a quantity that measures the disorder connected with the motion of the system: the **metric entropy** of the system $I(t, \epsilon)$ [1, 14]. This quantity is a function of the time since the probability mass function $p_i(t, \epsilon)$ is continuously varying in time.
The entropy can be computed as

$$I(t, \epsilon) = -\sum_i p_i(t, \epsilon) \cdot \lg_2(p_i(t, \epsilon)) \tag{13}$$

where the summation covers all the visited boxes. $I(t)$ is zero when the motion of the system is perfectly ordered (i.e. $p_i(t, \epsilon)$ is 1 in one box and zero in any other box) and reaches a positive upper-bound for a uniform occupation of all the cells.

Note that the entropy defined in (11) is dependent on the discretization of the phase space. The need for an invariant definition of entropy, more satisfactory from a theoretical point of view, leads one to the definition of the Kolmogorov entropy [1, 16], the price being a high mathematical sophistication.

1.3.3. KOLMOGOROV ENTROPY.

The computed function $I(t, \epsilon)$ is not smooth (Figure 11), due to the discretization of the phase space. Then, numerically, it is impossible to evaluate the numerical derivative of the metric entropy. With reference to Figure 11, one sees a first interval of time during which the entropy is linearly increasing and a subsequent stationary status.

A simplified definition of the Kolmogorov entropy can be reached by introducing a linear interpolation $I'(t, \epsilon)$ for $I(t, \epsilon)$ in the range between the origin and the stationary status [17]:

$$I'(t, \epsilon) = I'_o(\epsilon) + K(\epsilon)t = I(t, \epsilon) - \eta(t, \epsilon) \tag{14}$$

$\eta(t, \epsilon)$ being a zero mean fluctuation from the interpolating straight line. The velocity of variation of the entropy $I'(t, \epsilon)$, i.e. its time derivative, for ϵ sufficiently small is known as the **Kolmogorov entropy**[2, 4],

$$K = \frac{\partial I'(t, \epsilon)}{\partial t} \qquad \epsilon \text{ small} \tag{15}$$

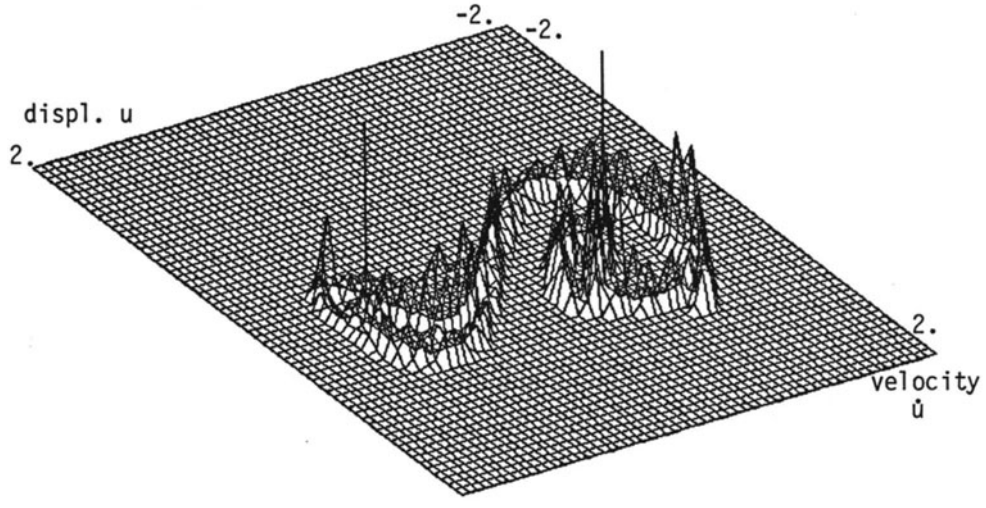

Figure 10 - Duffing oscillator with damping parameter d=0.185, intensity of the periodic excitation a =0.3 and no noise input: probability density function in the partitioned phase space. The spikes denote the initial condition.

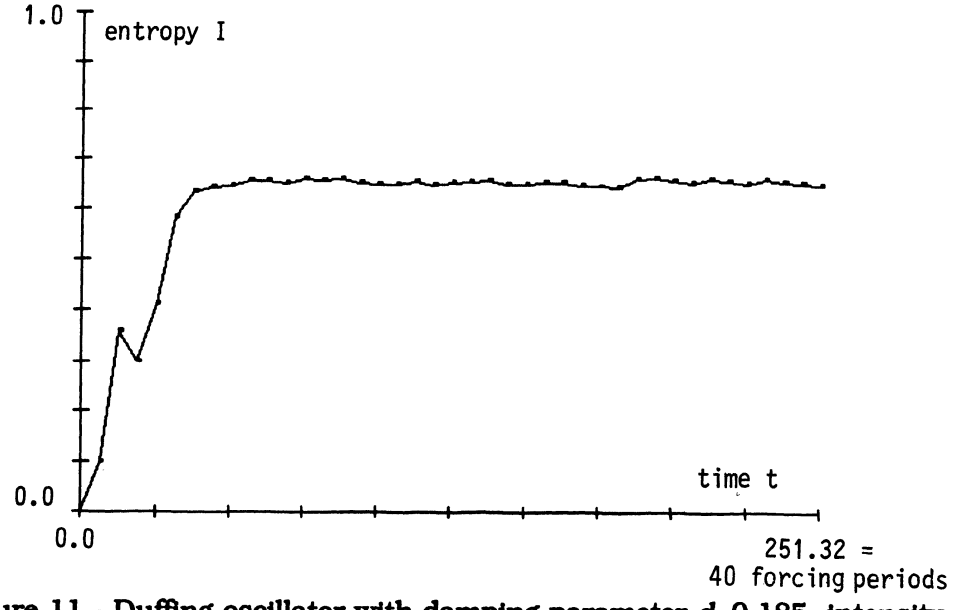

Figure 11 - Duffing oscillator with damping parameter d=0.185, intensity of the periodic excitation a =0.3 and no noise input: entropy versus time.

In words, the Kolmogorov entropy is the constant value of the slope of the interpolating straight line. Since, for ϵ sufficiently small, it turns out that K is not sensitive to the partition grid, such an operative definition can be adopted instead of the rigorous, but useless expression:

$$K = \lim_{\epsilon \to 0} \lim_{t \to T_s} \left(\frac{I(t, \epsilon)}{t} \right) \tag{16}$$

where T_s is the period necessary to $I'(t, \epsilon)$ for reaching the stationary status. (The presence of I'_o in Eq. (13) accounts for initial conditions of nonzero metric entropy, but Eq. (15) does not).

Of course, the numerical use of Eq. (14) fails when the stationary status is reached just after a single step Δt of time integration. A second analysis making use of a shorter Δt should be conducted in this case. However, when this event occurs for a map (f.i. the random number generator map [12]), it means an infinite value of K.
A classification of the behaviour of the system can then be based on the value of the Kolmogorov entropy:

- $K = 0$: the motion is regular,
- $K > 0$: the motion is chaotic or stochastic,
- $K = \infty$: the motion is entirelly random.

The third case appears when the representation of the system takes place at intervals of time which give the system the chance to cover any point of the phase space in two subsequent observed situations. This is the case toward which the random number generators tend.

While physically the entropy is the information needed to locate the system in time, the Kolmogorov entropy is the velocity with which one loses the information about the status of the system. Other quantities that measure the same effect are the Lyapunov exponents, and it will be shown that the Kolmogorov entropy is related with them.

1.3.4 LYAPUNOV EXPONENTS.

Classical Lyapunov stability approach makes use of the eigenvalues of the matrix **A** in the linearized equation:

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} \tag{17}$$

which can be derived from Eq. (4) [18]. An extensions to stochastic stability was provided by Khasminskii [19] and a significant generalization to initial condition dependency was reached with the ergodicity theorem given by Osdelec [20]. Both these developments introduced, as a substitute of the eigenvalues, the Lyapunov exponents. They are defined as the limit as t tends to ∞ of a norm of the vector, in the phase space, representative for the system under investigation.

Given a continuous dynamical system in a n-dimensional space, consider the points of the phase space included in a sphere of radius ϵ_0. This sphere is deformed in time to become an ellipsoid of principal axes

$$\epsilon_i = \epsilon_0 \cdot 2^{\lambda_i t},\ i = 1,..n \tag{18}$$

from which it turns out the definition of the i-th Lyapunov exponent

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \cdot \lg_2[\frac{\epsilon_t}{\epsilon_0}] \tag{19}$$

In Eq. (18) 2 is used as basis for the logarithm: this choice is convenient but not necessary.

These Lyapunov exponents are the average exponential rates of divergence or convergence of nearby orbits, which correspond to nearly identical states. Therefore, exponential orbital divergence means that close systems will soon behave quite differently and any predictive ability is lost.

It often occurs in physics, in the presence of necessary and sufficient conditions, that part of the literature misses some logical links. For chaotic motion, in particular, the previously stated link between chaos and Lyapunov exponents it often replaced by a definition such as: *any system containing at least one positive Lyapunov exponent is defined to be chaotic with the magnitude of the exponent reflecting the time scale on which system dynamics becomes unpredictable* [21] (see Figure 9 b)).

Since these exponents are strictly related to the principal components of the strain tensor in continuum mechanics, it is easily understood that the sum of the Lyapunov exponents (i.e. the Lyapunov exponent of order n) governs the evolution of the volume of the sphere. For a dissipative system one expects some contraction and then

$$\Sigma_i \lambda_i < 0 \tag{20}$$

Consider a dynamical point without a fixed point. Since there is a slow change along to the axis tangent to the flow, in the phase space, one Lyapunov exponent must be zero. Eq. (18) then requires a minimal dimension $n = 3$ for having chaotic phenomena. In the last case one has:

- $\{\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0\}$: chaotic motion with strange attractor;
- $\{\lambda_1 = 0, \lambda_2 = 0, \lambda_3 < 0\}$: quasi-periodic motion with torus of dimension 2;
- $\{\lambda_1 = 0, \lambda_2 < 0, \lambda_3 < 0\}$: periodic motion with limit cycle;
- $\{\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0\}$: equilibrium with fixed point.

Numerically, the Lyapunov exponents can be obtained with standard techniques [21]. They will be discussed in Chapter 4 of this book.

1.3.5. DIMENSION OF THE ATTRACTOR.

An attractor can be characterized by a number that is a measure of the variables needed to specify the position of the system over the actractor itself. No supplementary variable is required in order to locate a system in its equilibrium point in the phase space. An arbitrary abscissa locates a system over its limit cycle. To locate a system over the corresponding torus in a three dimensional phase space, one needs two coordinates. So, the equilibrium point is an attractor of dimension 0, the limit cycle is an attractor of dimension 1 and the torus is an attractor of dimension 2. From a matematical point of view one can expect attractors of noninteger dimension. These are **strange attractors** and are related to stochastic and chaotic motion. Geometrically the strange attractor are **fractal objects**[5, 6].
One can compute several dimensions, that are summarized in the following equation [4], which assumes a preliminary partitioning of the phase space

$$d^{(q)} = \frac{1}{q-1} \cdot \lim_{\epsilon \to 0} \left(\frac{\lg_2(\sum p_i^q)}{\lg_2(\epsilon)} \right) \tag{21}$$

where q is a positive integer and ϵ denotes the size of the boxes in the partitioning. For $q = 0$ one produces the capacity dimension, $q = 1$ the information dimension and $q = 2$ the correlation dimension, with:

$$d^{(0)} \geq d^{(1)} \geq d^{(2)}$$

It is worth noting that the presence of noise produces an attractor more smeared. An increase of dimension for the attractor can be detected as the noise intensity is increased.

1.3.6. LYAPUNOV EXPONENTS VS. ACTRACTOR DIMENSION AND KOLMOGOROV ENTROPY

The following relation holds [2]

$$K(t) \leq \Sigma_i \lambda_i \tag{22}$$

where the summation is extended over the ordered Lyapunov exponents of positive sum.

Moreover the Lyapunov exponents are related with a special form of the fractal dimension of the strange attractor:

$$d_L = j - \frac{\sum_{i=1,j} \lambda_i}{\lambda_{j+1}} \tag{23}$$

where j is the index of the last positive exponent among the ordered (in a decreasing way) Lyapunov exponent.

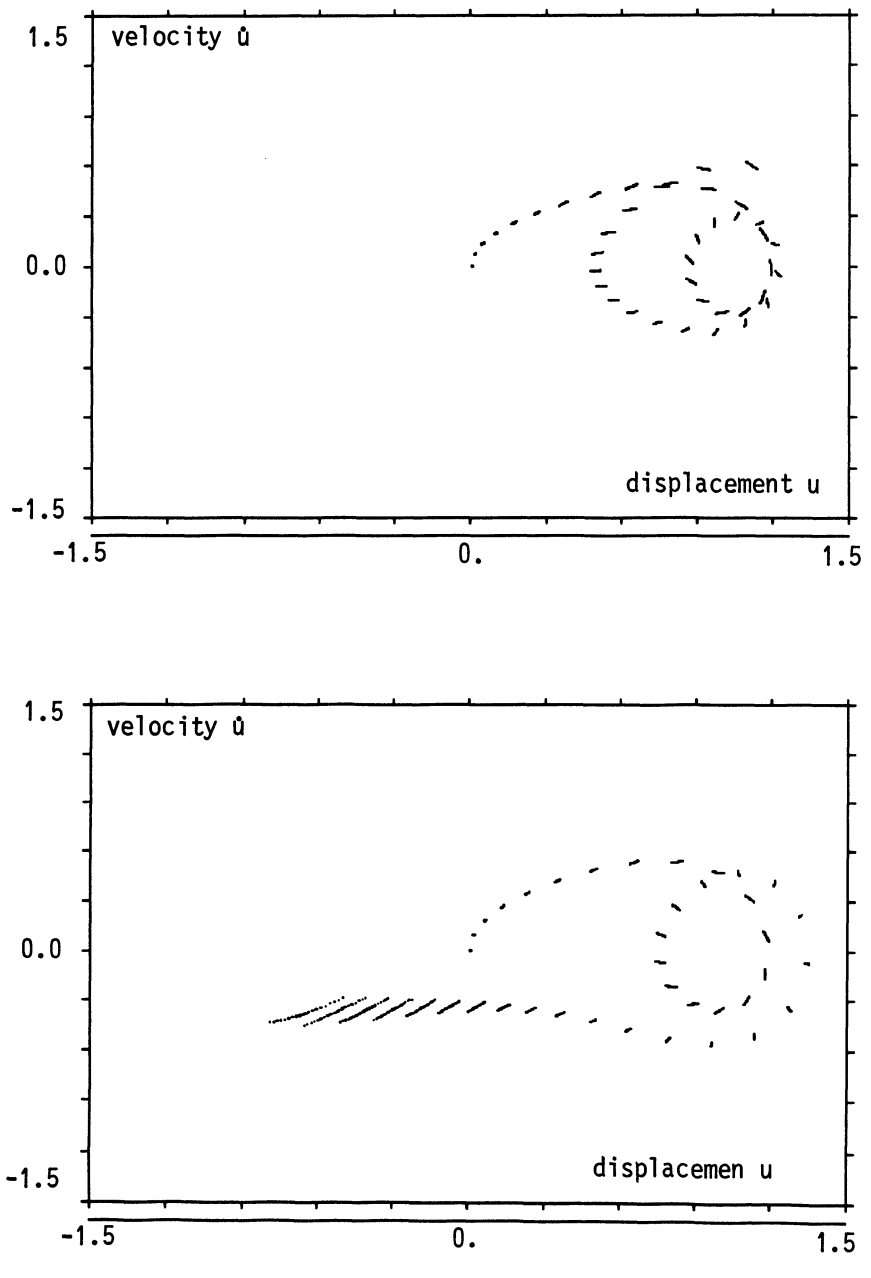

Figure 12 - Duffing oscillator with damping parameter d=0.185, periodic excitation and noise input: a) evolution of a Gibbs set in the regular case; b) evolution of the same Gibbs set in the chaotic case.

1.3.7 STUDY OF MANY TRAJECTORIES : THE GIBBS SET.

In the presence of stochasticity or chaoticity the evolution of certain average quantities must be determined, rather than the trajectory corresponding to a given set of initial conditions. Such formulation in terms of average quantities is also the basis of the well codified Statistical Mechanics [22].

Consider a collection of N dynamical systems identical to the one represented by equation (4)

$$\dot{\mathbf{y}}^{(k)} = \mathbf{Q}(\mathbf{y}^{(k)}, t), \quad k = 1, ..N \tag{24}$$

with slightly different initial conditions (5). This collection is called a **Gibbs set** . One will observe the evolution in time of this collection of systems, that start nearly at time t_0, each of them exhibiting a motion that is not affected by the presence of the other systems. In Figure 12, one sees the initial disposition of the Gibbs set and how this configuration has changed at a generic instant of time t.
With this set one simulates his uncertainty about the effective initial status of the system, i.e. the uncertainty about the initial conditions. These systems reproduce then the small phase space volume considered inside the ball introduced for computing the Lyapunov exponents. For a problem of classical kind, the size of the Gibbs set remains substantially the same (the uncertainty about the location of the system remains constant). In cases of chaotic or stochastic dynamics, the size of the collection grows in time (information about the true status of the system is lost). The idea is then to use the Gibbs set and its motion to compute statistics of the system behaviour.

1.4 THE LIOUVILLE AND THE FOKKER-PLANCK EQUATIONS.

This section deals with an alternative approach by which the evolution of the dynamical system of equation (4) can be followed. However, while the simulation of the system requires the solution of ordinary differential equations, this approach requires the solution of partial differential equations: as a consequence it is more synthetic but its numerical solution is much more complicated.

1.4.1. LIOUVILLE EQUATION

Equation (24) can be regarded as governing the components of the vector of the velocity of a fluid, with mass density $p(\mathbf{y}, t)$ (normalized to have integral equal to 1), which moves in the phase space of the dynamical system (4) [16]. For the generic portion of this fluid in the subset D of the phase space, we can establish an integral equation of continuity as

$$\frac{d}{dt}\int_D p(\mathbf{y},t)\cdot dD = 0 \tag{25}$$

Equation (24) is equivalent to the differential continuity equation (**Liouville equation**)

$$\frac{\partial p}{\partial t} + \sum_k \frac{\partial}{\partial y_k}[\dot{y}_k p] = 0 \tag{26}$$

where the terms under the sum allow the convection equation (25) to belong to the class of hyperbolic partial differential equation in the unknown quantity $p(\mathbf{y},t)$. With the appropriate initial conditions, Eq. (25) solves the dynamical problem in this special case.

1.4.2. FOKKER-PLANK EQUATION

In the presence of noise, the continuity equation is altered due to the different meaning of the flow drift and the presence of diffusive. In particular, one assumes the following limits exist [23]:

$$b_k(y_k,t) = \lim_{\tau\to 0}\frac{E[\underline{y}_k(t+\tau) - y_k(t)]}{\tau} \tag{27}$$

$$c_{ij}(y_i,y_j,t) = \lim_{\tau\to 0}\frac{E[(\underline{y}_i(t+\tau) - y_i(t))(\underline{y}_j(t+\tau) - y_j(t))]}{\tau} \tag{28}$$

where $E[\ ]$ denotes the ensemble average and underlined letters distinguish the stochastic process from its realization. Terms of order higher than the one in equation (27) may be nonzero. They vanish for a Gaussian noise. Then, the governing equation becomes (**Fokker-Planck equation**)

$$\frac{\partial p}{\partial t} + \sum_k \frac{\partial}{\partial y_k}[b_k\cdot p] - \sum_i\sum_j \frac{\partial^2}{\partial y_i \partial y_j}[c_{ij}\cdot p] = 0 \tag{29}$$

which is of the parabolic-hyperbolic kind: b_k is responsible of the hyperbolic character and c_{ij} of the parabolic one.

The Fokker-Planck equation, with the appropriate initial condition, solves fully the problem of determining the aspects of the motion of a real system with noise: from the solution $p(y,t)$ of the equation (28), one knows all the properties of the system. For its rigorous derivation, the reader is referred to Chapter 2 of this book.

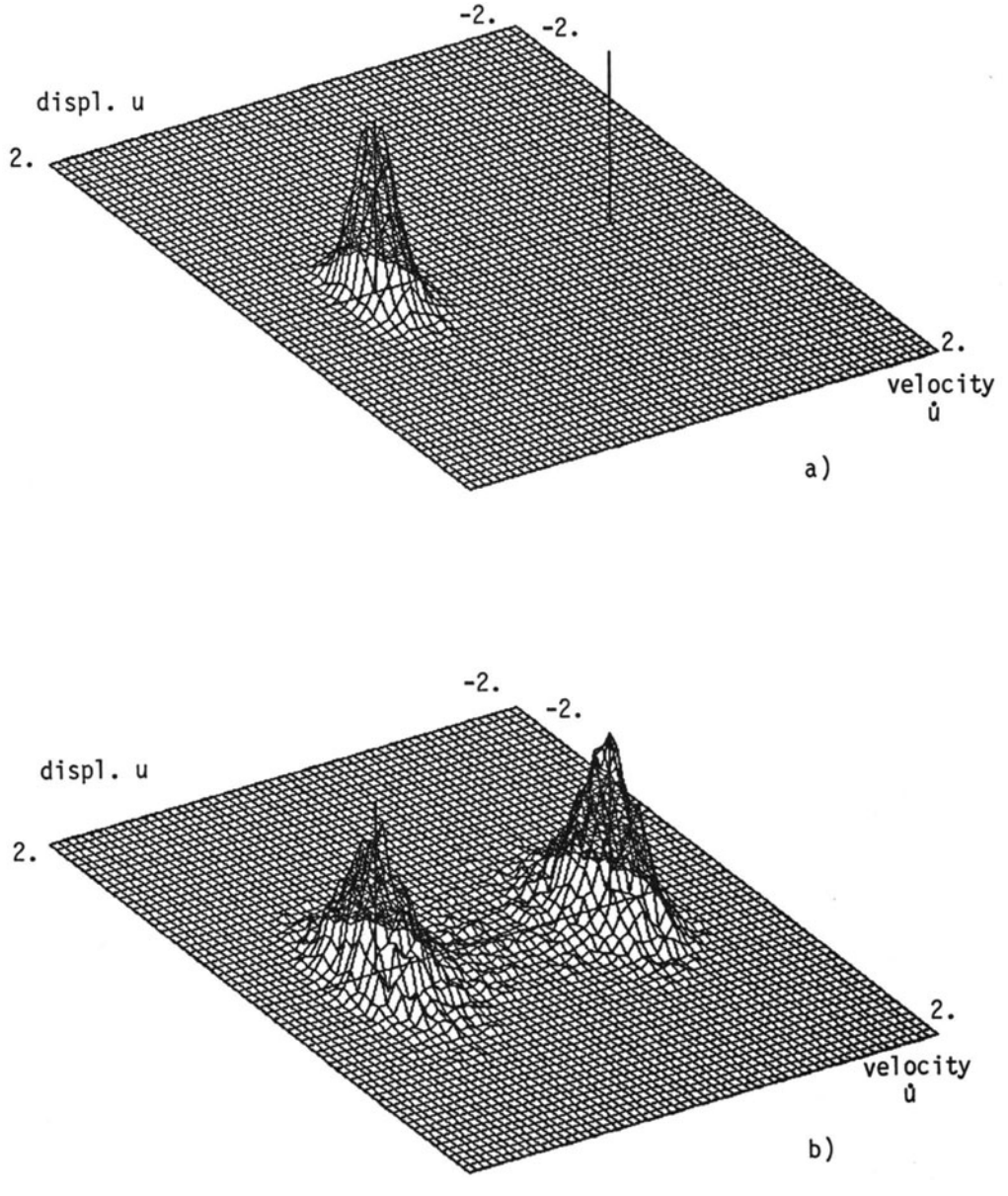

Figure 13 - The concept of stochastic bifurcation for the Duffing oscillator with damping parameter d=0.185, no periodic excitation and noise input of intensity σ_0: a) σ_0 =0.15; b) σ_0 =0.3.

1.4.3. EFFECTS OF RANDOM PERTURBATIONS ON DYNAMICAL SYSTEMS

The fluctuations due to the presence of noise allow a dynamical system to visit in an almost-continuous way the phase-space. In particular, when several actractors exist, the response of the system can be shifted from the neighbourhood of the first actractor to the second and vice versa.

It follows the concept of stochastic bifurcation illustrated in Figure 13. A unimodal probability density function of the response, which characterizes the dynamical behaviour, around a single actractor is drawn in the top of Figure 13. In the presence of two actractors the system visits both the regions and a bi-modal probability density function is found (bottom). The passage from the first situation to the second one defines the critical value of stochastic bifurcation for the parameter under investigation. It corresponds to a flat central part of the probability density function.

If one considers a simple elasto-plastic oscillator under a cyclic stochastic excitation, a phenomenon of stochastic bifurcation can be emphasized by just increasing the intensity of the excitation. At the beginning, the elastic response locates around the zero, but for large intensities a bimodal probability density function shows that the more likely states are now the positive and negative plastic branch.

When several actractors exist, multimodal probability density functions may occur, but other forms of information may also be collected: the temporal sequence of the changes of actractors, the time spent around each actractor and so on.

1.4.4. EFFECTS OF RANDOM PERTURBATIONS ON CHAOTIC SYSTEMS

If a chaotic state is considered, something of similar to stochastic bifurcation may occur for the probability density function. The main difference results in the patterns of transfer from one attractor to the other, which are very narrow, and in the inter-transfer times which are governed by the interaction between system and excitation properties. Adding noise to such phenomena means to increase the scatter of the transfer paths and to modify the rate of changes.

1.5 SAFETY AND PREDICTABILITY.

The evolution of the concept of safety which characterized the last two decades is the main responsible of the increased interest in dealing with extreme events. The old approach to this problem was just requiring to locate the design in a region of stable equilibrium with

a sufficiently redundant carrying capacity. This was not always possible in aerospace technology (due to the required weight limitations) but was appropriate in civil and mechanical engineering.

1.5.1. SAFETY

The modern understanding of structural safety [7, 15, 23, 24] requires the study of extreme situations with the goal of assessing the probability of any undesired behaviour. The last concept is often translate in a zero-one (Boolean) logic by introducing in the space of the design variables the safe region and the unsafe one. The boundary which separates these two regions is the geometrical representation of the limit state.

Figures 9 b) gives an idea of this situation when the limit state of chaotic motion is introduced. It is defined operatively by the positiveness of the larger Lyapunov exponent. The Kolmogorov entropy could also be adopted.

When the interaction of 2 (or more) parameters is significant in view of the chaotic behaviour, one obtains Figure 14 for the maximum Lyapunov exponent. Two remarks turn out from this picture:

1. one cannot rely on a continuous unsafe region. This means that the space of the possible design situations must be tested by a sufficiently dense grid;
2. the location of the optimal design is not necessarily very far from a chaotic region, since too far may involve sensitivity to another chaotic region.

Given the probabilistic description of the design variables of interest and the whole set of surfaces denoting the transition to chaos, the integral of the joint probability density function over the unsafe region will provide the probability of failure, which is the complement to 1 of the structural reliability.

1.5.2. PREDICTABILITY

Safety is a first-level measure of the structural performance. A second-level investigates the damages arising as consequence of the fact that an unsafe region has been reached. For the limit state of the previous subsection this second level means to study what occurs when the system becomes chaotic.

The main step required by such a task are:

- to select an appropriate (i.e. significant) subset of the phase space;
- to associate to each point of this subset a stress/strain description of the dynamic system;

- to identify the limit states arising from limitation on stress and/or strain functions;
- to evaluate the relevant probability density function of the response;

As a result of the whole analysis one has the probability that the system will be in a given chaotic state (safety analysis) and for each point of the chaotic state regions the associate probability of reaching a given limit state. Of course, out of the chaotic regions, in the absence of noise, the system would be fully predictable and the design could be classified as safe or unsafe. Both chaoticity and stochasticity, alone or together, makes the adoption of a probabilistic approach unquestionable.

1.6 CONCLUSIONS.

In this chapter the tools by which a dynamical system can be investigated have been introduced and discussed. Emphasis has been put on the possibility of using the same tools for studying chaoticity and stochasticity.

The basic steps of the study of an engineering dynamical system turn out to be:

1. identification of the chaocity region: this can be conveniently performed, even for multi-degree-of-freedom systems, by computing the Kolmogorov entropy (see Chapter 5);
2. probabilistic description of the response associated with each point of this chaotic region. This could be performed by solving the Fokker-Plank equation (see Chapter 2) which is the governing equation of stochastic dynamics. Unfortunately for complex engineering systems a solution of that equation is not available and approximate algorithms should be adopted (Chapter 3 to 4).

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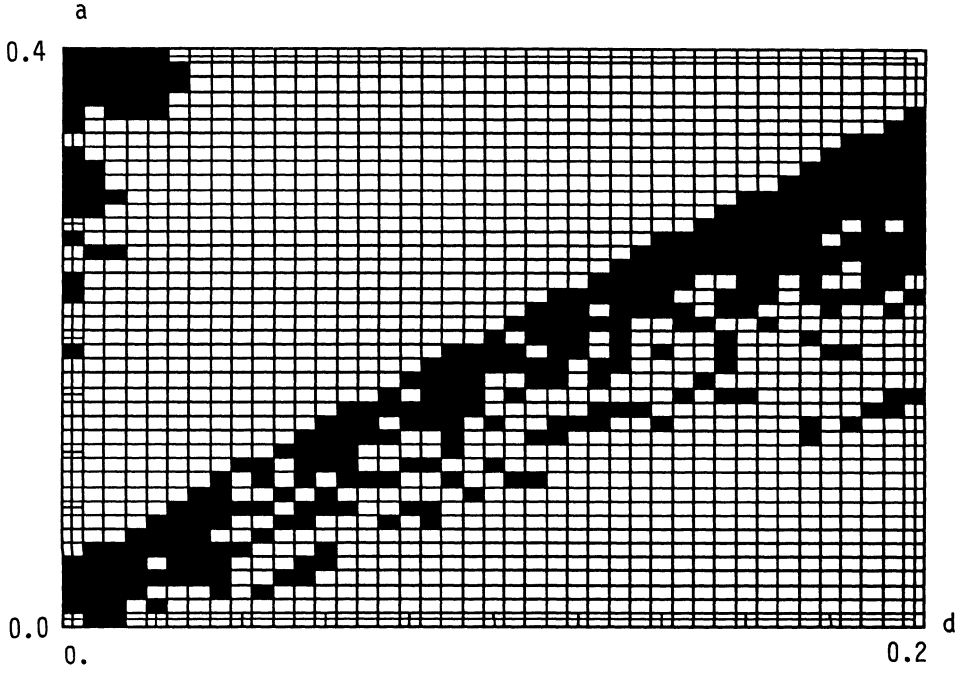

Figure 14 - Duffing oscillator with damping parameter *d*, intesity of the periodic excitation *a* and no noise input: sign of the maximum Lyapunov exponent λ_1 versus *d* and *a*. A black cell denotes a positive value.

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Chapter 2

STOCHASTIC DIFFERENTIAL CALCULUS

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1 INTRODUCTION.

In many cases of engineering interest it has become quite common to use stochastic processes to model loadings resulting from earthquake, turbulent winds or ocean waves. In these circumstances the structural response needs to be adequately described in a probabilistic sense, by evaluating the cumulants or the moments of any order of the response (see e.g. [1, 2]). In particular, for linear systems excited by normal input, the response process is normal too and the moments or the cumulants up to the second order fully characterize the probability density function of both input and output processes. Many practical problems involve processes which are approximately normal and the effect of the non-normality can often be regarded as negligible. This explains the popularity of second order analyses.

'or linear or non-linear systems driven by normal delta-correlated processes the Ito stochastic ifferential calculus [3, 5] represents the main tool for evaluating the response in terms of statistical ıoments. A paper by Srinvasan [6] reviews the evolution of the researches on the stochastic ıtegrals and on the problems connected to their applications. By using the Ito stochastic integral nd the corresponding Ito differential calculus, the analysis of the probabilistic response of non-ıear systems under external and/or parametric normal excitations becomes simple and immediate. ılthough, for the Ito integral, the usual rules of integration of the ordinary differential calculus fail, ı the applications it is preferred respect to another stochastic integral, called Stratonovich integral 7], which remains the fundamental tools of the ordinary integral calculus. This preference is due to he computational benefits related to the statistical independence between the increments of the Wiener process and the response process, evaluated at the same time; this independence can be ;vocated only if the Itô integral is applied, while, if the Stratonovich integral is used, the analysis ıecomes more and more complicated. The common strategy for evaluating the Stratonovich ntegral, which, as said before, satisfies the fundamental tools of the ordinary differential calculus, :onsists of modifying the drift coefficients, in the differential equations, by adding some corrective erms ([8] or Stratonovich correction terms), and then, of applying the Itô integration procedure. These corrective terms are such that the Ito integral of the modified differential equation coincides with the Stratonovich integral of the original differential equation. The presence of the Wong-Zakai ır Stratonovich correction terms is usually related to the local irregularities of the white noise ıroces s, that is, to the fact that a normal white noise process exhibit unbounded variations in nfinitesimal time intervals. In the last decade, a common justification of their presence has been :onnected to the passage from an ideal white noise to a physical white noise, in order to treat real ıroblems [2]. But, as pointed out in a recent work [9], this explanation is not satisfactory.

Although the assumption of Gaussianity is adequate in many cases, there may be excitation processes for which this assumption is not justified by the experimental data. Relevant examples of non Gaussian excitation are event-type represented as train of pulses occurring at random times such as highway traffic load on bridges [10], buffeting on airplan tail [11] ground motion acceleration due to strong earthquakes or shock waves [12, 13]. For these input processes the linear case was treated [14, 15, 2, 6] by obtaining moments and cumulants of higher order in order to evidence the departure from the normality of the response. Non linear case excited by external excitation has also been treated by Feller [17], Roberts [18]. Iwankievicz et al. [19, 20]. Only recently the case of parametric-type delta-correlated process has been treated by Di Paola and Falsone (Di Paola and Falsone [21]) obtaining an extension of the stochastic differential calculus for such types of excitation.

In this chapter the main results of the classical stochastic calculus will first be treated. An extension of the Fokker-Plank equation and a moment equation approach for parametric type delta-correlated input process will be also presented together with numerical examples and simulation procedures.

2 PRELIMINARY CONCEPTS AND DEFINITIONS.

In this section some well known concepts on the probabilistic characterization of stochastic processes are outlined, the scalar case is treated but extension to multidimensional processes is straightforward.

Let X(t) be a stochastic process; its probabilistic description at a fixed time t can be obtained by means of the knowledge of the *probability density function* $p_X(x; t)$ or by its Fourier transform that is the so-called *characteristic function*, defined as

$$M_X(\vartheta; t) = E[\exp(-i\,\vartheta\,X)] = \int_{-\infty}^{\infty} p_X(x; t)\, e^{-i\,\vartheta\,x}\, dx \tag{1}$$

where ϑ is a real parameter, i is the imaginary unit, and $E[\cdot]$ means stochastic average.

A Taylor expansion of the characteristic function gives

$$M_X(\vartheta; t) = \sum_{j=0}^{\infty} \frac{(-i)^j}{j!}\, m_j[X]\, \vartheta^j \tag{2}$$

where $m_j[X]$ is the so-called *moment* of order j of X(t), that is

$$m_j[X] = E[X^j] = \int_{-\infty}^{\infty} p_X(x; t)\, x^j\, dx = \frac{1}{(-i)^j} \left[\frac{d^j M_X(\vartheta; t)}{d\vartheta^j} \right]_{\vartheta = 0} \tag{3}$$

A different representation of the characteristic function can be made by introducing the so-called *cumulants* $k_j[X]$, see e.g. Lin [2] in the form

$$M_X(\vartheta; t) = \exp\left(\sum_{j=1}^{\infty} \frac{(-i)^j}{j!}\, k_j[X]\, \vartheta^j \right) \tag{4}$$

where $k_j[X]$ is the j-th coefficient of the Taylor expansion of the *log-characteristic function*

$$k_j[X] = \frac{1}{(-i)^j} \left[\frac{d^j \ln M_X(\vartheta; t)}{d\vartheta^j} \right]_{\vartheta = 0} \tag{5}$$

moments and cumulants are related to each other by means of non-linear algebraic recurrence relationship (with argument omitted),

$$\begin{aligned} m_1 &= k_1 \\ m_2 &= k_2 + k_1 m_1 \\ m_3 &= k_3 + 2 k_2 m_1 + k_1 m_2 \\ m_4 &= k_4 + 3 k_3 m_1 + 3 k_2 m_2 + k_1 m_3 \\ &\vdots \end{aligned} \tag{6}$$

where the coefficients in equation (6) are those of the Tartaglia triangle. From equations (1), (2) and (4), it is evident that the complete probabilistic description of the stochastic process X(t) at a fixed time t can be obtained indifferently by means of the knowledge of its probability density function or of its characteristic function or of its moments and/or cumulants of every order.
Many physical problems have small values of higher order cumulants and, for normal (Gaussian) processes they are exactly zero for order greater than two, thus in these cases the first cumulant (mean value) and the second cumulant also called *variance* fully define the stochastic process X(t) from a probabilistic point of view at a fixed time t.
A complete probabilistic description of a stochastic process X(t) can be obtained by means of the knowledge of the infinite hierarchy of the joint probability density functions

$$p_{X_1, X_2 \dots X_s}(x_1, x_2, \dots, x_s; t_1, t_2, \dots, t_s) = p_{X_s}(x_s; t_s) \quad s = 1, 2, \dots, \infty \tag{7}$$

where

$$X_s^T = [X_1 \; X_2 \; \dots \; X_s]; \quad x_s^T = [x_1 \; x_2 \; \dots \; x_s]; \quad t_s^T = [t_1 \; t_2 \; \dots \; t_s] \tag{8}$$

In equation (8) the apex T means transpose and X_k is the random variable obtained by X(t) at the time instants t_k, that is $X_k = X(t_k)$.
The stochastic process X(t) can be equivalently characterized by means of the multidimen-sional characteristic function $M_{X_s}(\vartheta_s; t_s)$ which is related to the probability density function by means of the Fourier transform operator, that is

$$M_{X_s}(\vartheta_s; t_s) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left(-i \vartheta_s^T x_s\right) p_{X_s}(x_s; t_s) \, dx_1 \, dx_2 \dots dx_s \tag{9}$$

where ϑ_s is an s-vector of real parameters

$$\vartheta_s^T = [\vartheta_1 \; \vartheta_2 \; \dots \; \vartheta_s] \tag{10}$$

A Taylor expansion of the characteristic function gives

$$M_{X_s}(\vartheta_s; t_s) = 1 + \sum_{j=1}^{s} - i\,\vartheta_j\, m_1[X_j] + \frac{1}{2!}\sum_{j=1}^{s}\sum_{k=1}^{s} (-i)^2\,\vartheta_j\,\vartheta_k\, m_2[X_j\ X_k] + \cdots \tag{11}$$

where $m_r\left[X_1^p\ X_2^q\ \ldots\ X_s^l\right]$ with $p + q + \ldots + l = r$ is the moment of order r of $X(t_1)$, $X(t_2)$, ..., $X(t_s)$, that is

$$m_r\left[X_1^p\ X_2^q \ldots X_s^l\right] = E\left[X_1^p\ X_2^q \ldots X_s^l\right] = \frac{1}{(-i)^r}\left[\frac{\partial^r}{\partial\vartheta_1^p\,\partial\vartheta_2^q \ldots \partial\vartheta_s^l} M_{X_s}(\vartheta_s; t_s)\right]_{\vartheta_s = 0} \tag{12}$$

A more compact representation of equation (11) can be obtained by means of the Kronecker algebra whose fundamentals are reported in Appendix A.

$$M_{X_s}(\vartheta_s; t_s) = \sum_{j=0}^{\infty} \frac{(-i)^j}{j!}\, \mathbf{m}_j^T[\mathbf{X}_s]\, \vartheta_s^{[j]} \tag{13}$$

in equation (13) $\mathbf{m}_j[\mathbf{X}_s]$ is the vector of order s^j collecting all possible moments of order J of the random variables $X_1, X_2, \ldots, X_s$ in the form

$$\mathbf{m}_j[\mathbf{X}_s] = E[\mathbf{X}_s \otimes \mathbf{X}_s \otimes \ldots \otimes \mathbf{X}_s] = E\left[\mathbf{X}_s^{[j]}\right] \tag{14}$$
$$\text{j - times}$$

in equations (13) and (14) the exponent into square brackets means Kronocker power and the symbol $\otimes$ means Kronecker product.
The vector of moments of order r of the vector $\mathbf{X}_s$ can be obtained by the characteristic function as follows

$$\mathbf{m}_r[\mathbf{X}_s] = \frac{1}{(-i)^r}\left[\nabla_{\vartheta_s}^{[r]}\ M_{X_s}(\vartheta_s; t_s)\right]_{\vartheta_s = 0} \tag{15}$$

where ∇_{ϑ_s} is the derivative vector defined as

$$\nabla_{\vartheta_s}^T = [\partial/\partial\vartheta_1\ \ \partial/\partial\vartheta_2\ \ \ldots\ \ \partial/\partial\vartheta_s] \tag{16}$$

A different representation of the characteristic function can be made by introducing the so-called cumulants $k_r\left[X_1^p X_2^q \dots X_s^l\right]$ with $p+q+\dots+l=r$

$$M_{X_s}(\vartheta_s; t_s) = \exp\left(\sum_{j=1}^{s} -i\,\vartheta_j\, k_1[X_j] + \frac{1}{2!}\sum_{j=1}^{s}\sum_{r=1}^{s}(-i)^2\,\vartheta_j\,\vartheta_r\, k_2[X_j X_r] + \dots\right) \tag{17}$$

where the various cumulants in equation (17) can be obtained by the following relationship

$$k_r\left[X_1^p X_2^q \dots X_s^l\right] = \frac{1}{(-i)^r}\left[\frac{\partial^r}{\partial\vartheta_1^p\,\partial\vartheta_2^q \dots \partial\vartheta_s^l} ln\, M_{X_s}(\vartheta_s; t_s)\right]_{\vartheta_s = 0} \tag{18}$$

Equation (17) can be rewritten in the form

$$M_{X_s}(\vartheta_s; t_s) = \exp\sum_{j=1}^{\infty}\frac{(-i)^j}{j!}\mathbf{k}_j^T[X_s]\,\vartheta_s^{[j]} \tag{19}$$

The entire set of cumulants of order r can be obtained in compact form as follows

$$\mathbf{k}_r[X_s] = \frac{1}{(-i)^r}\left[\nabla_{\vartheta_s}^{[r]}\, ln\, M_{X_s}(\mathbf{x}_s; t_s)\right]_{\vartheta_s = 0} \tag{20}$$

Relationships between moments and cumulants can be found in literature [22, 23, 24].
It is to be emphasized that if the stochastic average $E[X_1 X_2 \dots X_s]$ (a scalar quantity) is known for every set of instants $t_1, t_2, \dots, t_s$, then all possible moments of order s, $m_s = \left[X_1^p X_2^q \dots X_s^l\right]$ with $(p + q + l = s)$, can be constructed, that is the entire vector $\mathbf{m}_s[X_s]$ can be easily computed by means of an appropriate choice of the time instants $t_1, t_2, \dots, t_s$. For this reason it seems to be appropriate to introduce a new nomenclature for this moment and it will be called *"average at multiple times"* and denoted as $E_X^{(s)}(t_s)$. The counterpart of $E_X^{(s)}(t_s)$ in terms of cumulants is, following the nomenclature introduced by Stratonovich (1963), the *"correlation"* $R_X^{(s)}(t_s)$. If the correlation is known as a function of the various time instants $t_1, t_2, \dots, t_s$ then by means of an appropriate choice of the instants $t_1, t_2, \dots, t_s$ the entire vector $\mathbf{k}_s[X]$ can be easily constructed.
If X(t) is a strong stationary process of order s, then the average at multiple times and the correlations up to the s-th order depend on the time differences $t_1 - t_2, t_1 - t_3, \dots, t_1 - t_s$ instead of the time instants $t_1, t_2, \dots, t_s$ separately taken.
The scalar functions $E_X^{(s)}(t_s)$ and $R_X^{(s)}(t_s)$ can be obtained in the form

$$E_X^{(s)}(t_s) = \frac{1}{(-i)^s}\left[\frac{\partial^s}{\partial\vartheta_1\,\partial\vartheta_2 \dots \partial\vartheta_s}\, M_{X_s}(\vartheta_s; t_s)\right]_{\vartheta_s = 0} \tag{21}$$

$$R_X^{(s)}(t_s) = \frac{1}{(-i)^s}\left[\frac{\partial^s}{\partial\vartheta_1\,\partial\vartheta_2 \dots \partial\vartheta_s}\, ln\, M_{X_s}(\vartheta_s; t_s)\right]_{\vartheta_s = 0} \tag{22}$$

The infinite set of average at multiple times and correlations fully describe the stochastic process from a probabilistic point of view.

An important class of stochastic processes encountered in practical application are the normal ones that are defined by having all the correlations of order greater than two zero. It follows that a complete characterization of a normal process is obtained by the knowledge of

$$R_X^{(1)}(t_1) = E[X(t)] \tag{23}$$

$$R_X^{(2)}(t_1, t_2) = E[X(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)] \tag{24}$$

For stationary processes the second correlation, in the following simply called correlation function and denoted as $R_X(t_1, t_2)$ depends only from the difference $\tau = t_1 - t_2$. It follows that the complete characterization of a stationary normal process is made by means of the mean value $\mu_X = E[X]$ which takes a constant value and by the correlation of second order in the follows simply denoted as $R_X(\tau)$. The Fourier transform of the correlation function of a stationary process is the so-called *Power spectral density* (PSD) function that is

$$S_X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_X(\tau)\, e^{-i\,\omega\,\tau}\, d\tau \tag{25}$$

The inverse relationship writes

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(\omega)\, e^{+i\,\omega\,\tau}\, d\omega \tag{26}$$

The equations (25) and (26) are the Wiener-Kintchine relationships. From equation (26) we recognize that for a normal stationary processes the PSD fully characterize the stochastic process X(t) from a probabilistic point of view.

3 DELTA-CORRELATED PROCESSES.

In this section we will deal with a very important class of stochastic processes that are charecterized by correlation functions given as

$$\begin{aligned} R_W^{(1)}(t_1) &= q^{(1)}(t_1) \\ R_W^{(s)}(t_1, t_2, \dots, t_s) &= q^{(s)}(t_1)\,\delta(t_1 - t_2)\,\delta(t_1 - t_3)\dots\delta(t_1 - t_s);\quad s = 2, 3, \dots, \infty \end{aligned} \tag{27}$$

in this case the process W(t) will be called *delta correlated* (see e.g. [1, 14, 15].
In equation (27) $\delta(\cdot)$ is the Dirac's delta and $q^{(s)}(t_1)$ is an intensity coefficient. If $q^{(s)}(t_s)$, $s = 1, 2, \dots, \infty$ take a constant value, then the process W(t) is a stationary one.
A relevant example of delta-correlated process is the Poisson white noise process, defined as

$$W(t) = \sum_{k=1}^{N(t)} Y_k\,\delta(t - t_k) \tag{28}$$

This function is a train of Dirac delta impulses $\delta(t - t_1)$ occurring at Poisson-distributed random times t_k. N(t) is a counting process giving the number of impulses in the time interval [0, t] with initial condition N(0) = 0, with probability one [25]. Y_k are assumed to be mutually independent and independent of the random instants t_k. The Poisson process is completely characterized by the expected rate $\lambda(t)$ of events (impulse occurrences), that is the mean of the random impulses in the infinitesimal interval t + t + dt, that will be denoted by N(dt), is given as

$$E[N(dt)] = \lambda(t)\,dt + 0(dt) \tag{29}$$

The entire probabilistic structures of the process W(t) defined in equation (28) can be evaluated in the form

$$R_W^{(s)}(t_1, t_2, \dots, t_s) = \lambda(t)\,E[P^s]\,\delta(t_1 - t_2)\,\delta(t_1 - t_3)\dots\delta(t_1 - t_s) \tag{30}$$

If $\lambda(t)$ keeps a constant value the Poisson white noise process is a stationary one. The Poisson white noise processes have been discussed by several authors and have been used in order to model some excitations, such as earthquakes [26] and the random trains of loadings on bridges due to the vehicles traffic [10]. The Poisson process W(t) can be considered as the formal derivative of the so-called *homogeneous compound Poisson process* C(t) defined as

$$C(t) = \sum_{k=1}^{N(t)} Y_k\,U(t - t_k) \tag{31}$$

U(t) being the unit step function. It is easy to show that the correlation functions of C(t) can be expressed as

$$R_C^{(1)}(t) = \lambda\, t\, E[P] \tag{32}$$

$$R_c^{(s)}(t_1, t_2, \dots, t_s) = \lambda \min(t_1, t_2, \dots, t_s)\, E[P^s];\quad \forall s \geq 2 \tag{33}$$

Differentiating equation (33) with respect to $t_1, t_2, \dots, t_s$, we obtain

$$\frac{\partial^s R_C^{(s)}(t_1, t_2, \dots, t_s)}{\partial t_1\, \partial t_2 \dots \partial t_s} = R_W^{(s)}(t_1, t_2, \dots, t_s) \tag{34}$$

This equation shows that the correlation function of a Poisson white noise coincide with the derivative of the correlation function of the corresponding homogeneous compound Poisson process.

It can be easily seen that the correlation functions of the increments $dC(t_j) = C(t_j + dt) - C(t)$ of a homogeneous compound Poisson process are given as

$$R_{dC}^{(s)}(t_1, t_2, \dots, t_s) = \lambda E[P^s]\, \delta(t_1 - t_2)\, \delta(t_1 - t_3) \dots \delta(t_1 - t_s)\, dt_1\, dt_2 \dots dt_s \tag{35}$$

taking into account the relationships between average at multiple time and correlations, we obtain

$$\begin{aligned}
&E[dC(t_1)] = \lambda E[Y]\, dt \\
&E[dC(t_1)\, dC(t_2)] = \lambda E[Y^2]\, \delta(t_1 - t_2)\, dt_1\, dt_2 + \lambda^2 E[Y]^2\, dt_1\, dt_2 \\
&E[dC(t_1)\, dC(t_2)\, dC(t_3)] = \lambda E[Y^3]\, \delta(t_1 - t_2)\, \delta(t_1 - t_3)\, dt_1\, dt_2\, dt_3 + \\
&\quad + \lambda^2 E[Y]\, E[Y^2]\left[\delta(t_2 - t_1) + \delta(t_3 - t_1) + \delta(t_3 - t_2)\right] dt_1\, dt_2\, dt_3 + \\
&\quad + \lambda^3 E[Y]^3\, dt_1\, dt_2\, dt_3 \\
&\qquad \vdots
\end{aligned} \tag{36}$$

Taking into account that, when t_i approaches t_j, the Dirac's deltas $\delta(t_i - t_j)$ gets an infinite value of the same order of 1/dt, putting $t_1 = t_2 = \dots = t_s$ in equation (36) and neglecting higher order infinitesimal than dt, we obtain the very remarkable relationship

$$E[(dC)^r] = k_r[dC] \tag{37}$$

Equation (37) shows that cumulants and moments of the same order of an increment of the compound Poisson process coincide.
In Fig. 1 a sample function of a compound Poisson process and the corresponding white noise Poisson process are reported.

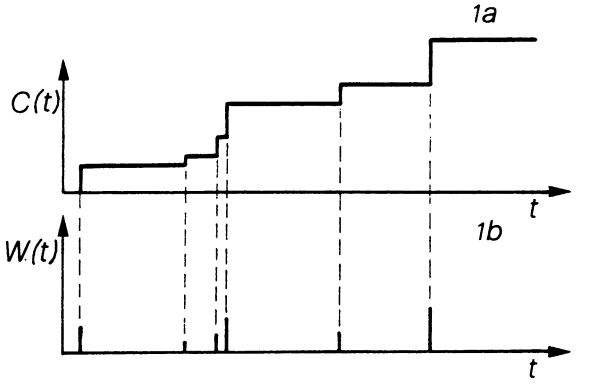

Fig. 1 - *a) Sample function of a homogeneous compound Poisson process; b) Sample function of a white noise Poisson process.*

The *normal* delta correlated process or simply *white noise* W^0 is a particular type of delta correlated process characterized by having $q^{(s)}(t) = 0$ for $s > 2$. It follows that the white noise is fully defined, from a probabilistic point of view, by means of the second correlation

$$R^{(1)}_{W^0}(t) = \mu_{W^0}(t) = q^{(1)}(t) \tag{38}$$

$$R^{(2)}_{W^0}(t_1, t_2) = R_{W^0}(t_1, t_2) = q^{(2)}(t_1)\,\delta(t_1 - t_2) \tag{39}$$

if $q^{(1)}(t)$ is zero and $q^{(2)}(t)$ is independent of t, the process W^0 is a zero mean stationary white noise. In this case the correlation function writes

$$R_{W^0}(\tau) = q^{(2)}\,\delta(\tau) \tag{40}$$

The Power spectral density function $S_{w^0}(\omega)$ in this case becomes

$$S_{w^0}(\omega) = \frac{q^{(2)}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} \delta(\tau)\, d\tau = \frac{q^{(2)}}{2\pi} \tag{41}$$

Equation (41) shows that the PSD is a constant at overall frequency range and this justifies the nomenclature of white noise; in the optic the white colour is one having constant frequency power. Like the formal derivative of an homogeneous Poisson process is the white Poisson process, the normal white noise can be obtained as the formal derivative of an other fundamental process, that is the so-called *Wiener process*, B(t). In Fig. 2 some sample functions of a Wiener process are depicted showing that all sample functions are continuous though, in accordance with a theorem of N. Wiener, nowhere differentiable functions.

Moreover, although the mean value of B(t) is zero, the mean square becomes infinite as $t \to \infty$. The extreme irregularity of the individual sample function (unbounded variation in infinitesimal intervals) account for the non differentiability of such processes.

It is to be emphasized that the white noise is totally uncorrelated, and can be obtained as the limit from a white Poisson process. As in fact the mean number of arrivals approaches to infinity and E [P] = 0, λ E [Y^2] keeps a constant value, the Poisson white noise approaches to the normal one [1, 2]. Moreover in this case the homogeneous compound Poisson process becomes a Wiener one. The departure from the normality of a Poisson white noise increases when λ decreases, for fixed E [Y^2].

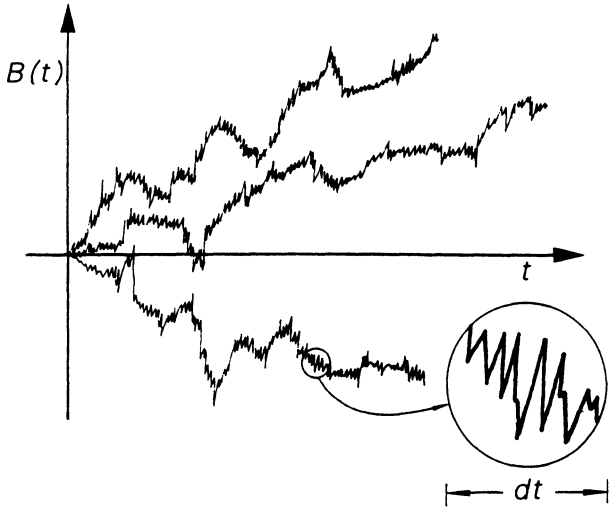

Fig. 2 - *Three sample functions of a Wiener process*

An increment of a Wiener process dB(t) is such that

$$E[dB(t_1)\, dB(t_2)] = q^{(2)}\, \delta(t_1 - t_2)\, dt_1\, dt_2 \tag{42}$$

$$E^{(s)}_{dB}[t_1, t_2, \dots, t_s] = 0; \quad \forall s > 0 \tag{43}$$

letting $t_1 = t_2$ in equation (42) we obtain

$$E[dB(t)^2] = k_2[dB] = q^{(2)}\, dt \tag{44}$$

moreover using equation (43) and the relationships between moments and cumulants we obtain

$$k_s[dB(t)] = E[(dB(t))^s] = 0 \tag{45}$$

From equation (44) we recognize that of dB(t) is an infinitesimal of order $\sqrt{dt}$.

4 OPERATION ON STOCHASTIC PROCESSES.

Many random phenomena in nature which directly interest us are expressed in terms of limiting series, derivatives, integrals. Since a stochastic process is defined in a probabilistic sense, the calculus of the random process must also be developed in the probabilistic framework. The concept of *convergence* is fundamental in the development of the differential calculus of stochastic processes. We shall confine ourselves to the notion of *mean square convergence* and state the conditions of continuity, differentiability, and integrability without proofs. For a derivation of these conditions rearchers are referred to Parzen [27], Loève [28], (see also [29, 30]).

4.1 Limit.

A sequence of random variables $X_1, X_2, \dots, X_m, \dots$ *converges in mean square* to a random variables X as $n \to \infty$ if

$$\lim_{n \to \infty} E[|X_n - X|^2] = 0 \tag{46}$$

we shall abbreviate this to

$$\underset{n \to \infty}{l.i.m.}\ X_n = X \quad \text{or} \quad X_n \overset{m.s.}{\to} X \tag{47}$$

where *l.i.m.* denotes *limit* in the *mean.* Other commonly used names are *convergence in quadratic mean* and *second order convergence.* For equation (46) to hold the necessary condition is that $E\left[X_n^2\right] < \infty$.

A fundamental theorem of the convergence in *mean square* is that for every convergent sequence X_n *l.i.m.* and mean commute that is if

$$\underset{n \to \infty}{l.i.m.}\ X_n = X \tag{48}$$

then

$$\underset{n \to \infty}{lim}\ E[X_n] = E[X] \tag{49}$$

4.2 Mean square continuity.

The concept of mean square convergence of a sequence of random variables can be extended to a stochastic process X(t) where t is a continuous parameter. We shall say that the stochastic process X(t) is continuous in mean square sense if

$$\underset{t \to t_0}{lim}\ E\left[\,|X(t) - X(t_0)|^2\right] = 0 \quad \text{or} \quad \underset{t \to t_0}{l.i.m.}\ X(t) = X(t_0) \tag{50}$$

whathever $t \to t_0$.

It can be shown that the necessary and sufficient condition for the continuity of the process X(t), $t \in T$ is that the correlation function $R_X(t_1, t_2)$ exists and is finite in T x T.

4.3 Mean square differentiation.

The concept of mean square differentiation follows naturally from the mean square continuity, and is defined as

$$\dot{X}(t) = \underset{\tau \to 0}{l.i.m.}\ \frac{X(t+\tau) - X(t)}{\tau} \tag{51}$$

The mean square derivative of the stochastic process X(t), $t \in T$ exists at t if

$$\frac{\partial^2 R_X(t_1, t_2)}{\partial t_1 \partial t_2} < \infty \tag{52}$$

and is continuous in $t_1 = t_2$.
From this condition we recognize that the Wiener and the homogeneous compound Poisson processes are not differentiable and this explain the adjective "formal" used in Sects. 3 and 4. In the next section this concept will be extensively investigated.
It can be shown that if eq. (52) holds, then

$$\frac{d}{dt} E[X(t)] = E\left[\frac{dX}{dt}\right] = E[\dot{X}(t)] \tag{53}$$

that is the operation of expectation and differentiation commute. It follows that the derivative of the cross correlation between two stochastic processes $X_1(t)$ and $X_2(t)$ can be obtained in the form

$$\frac{\partial}{\partial t_1} R_{\dot{X}_1 X_2}(t_1, t_2) = \frac{\partial}{\partial t_1} E[X_1(t_1)\, X_2(t_2)] - \frac{\partial}{\partial t_1}[E[X_1(t_1)] E[X_2(t_2)]] =$$

$$E[\dot{X}_1(t_1)\, X_2(t_2)] - E[\dot{X}_1(t_1)] E[X_2(t_2)] = R_{\dot{x}_1 x_2}(t_1, t_2) \tag{54}$$

Generalization of the result given in equation (54) are straightforward.
Let the derivative $d^n X_i(t)/dt^n$ denoted as $X_i^{(n)}$. The cross correlation $R_{x_1^{(n)} x_2^{(m)}}$ can be written in the form

$$R_{X_1^{(n)} X_2^{(m)}}(t_1, t_2) = \frac{\partial^{n+m}}{\partial t_1^n \partial t_2^m} R_{X_1 X_2}(t_1, t_2) \tag{55}$$

If $X_i(t)$ are jointly stationary processes, equation (55) reduces to

$$R_{X_1^{(n)} X_2^{(m)}}(t_1, t_2) = R_{X_1^{(n)} X_2^{(m)}}(\tau) = (-1)^n \frac{d^{n+m} R_{X_1 X_2}(\tau)}{d\tau^{n+m}}; \quad \tau = t_2 - t_1 \tag{56}$$

4.4 Mean square integration.

In order to define the *mean square Riemann integral* we first consider, as customary a collection of all finite partitions $\{p_n\}$ of an interval [a, b]. The partition p_n is defined by the subdivision points t_k, $k = 0, 1, \ldots, n$ such that

$$a = t_0 < t_1 < \ldots < t_n = b\,; \quad \Delta_n = \max_s (t_s - t_{s-1}) \tag{57}$$

and let $\bar{t}_s$ be an arbitrary point in the interval $[t_{s-1}, t_s]$. Let X(t) be a stochastic process and g(t, τ) a deterministic function. We define Riemann integral Y(t), if exists and is finite and independent of the choice of the intermediate points $\bar{t}_s$, the limit of the partial sums

$$Y(t) = \int_a^b X(\tau)\, g(t, \tau)\, d\tau = \underset{\substack{n \to \infty \\ \Delta_n \to 0}}{l.i.m.} \sum_{s=1}^{n} g(t, \bar{t}_s) X(\bar{t}_s)(t_s - t_{s-1}) \tag{58}$$

The stochastic process Y(t) exists if, and only if, the ordinary double Riemann integral

$$\int_a^b \int_a^b g(t, \tau)\, g(s, \tau)\, E[X(t)\, X(s)]\, dt\, ds \tag{59}$$

exists and is finite. In this case the usual rules of common integrals such as integration by parts, Leibniz Rule are available.
An extension of the mean square Riemann integral is the *mean square Riemann-Stieltjes* integral of the type

$$U = \int_a^b f(t)\, dX(t)\,; \quad V = \int_a^b X(t)\, df(t) \tag{60}$$

where f(t) is a deterministic function. If X(t) is mean square differentiable or f(t) is differentiable in all t belonging to (a, b), then the Riemann-Stieltjes integral reduces to a Riemann integral. In this sense we can affirm that the mean square Riemann-Stieltjes integrals are more general than the Riemann ones. If the limits of partial sums

$$U = \int_a^b f(t)\, dX(t) = \underset{\substack{n \to \infty \\ \Delta_n \to 0}}{l.i.m.} \sum_{k=1}^{n} f(\bar{t}_k)[X(t_k) - X(t_k - t_{k-1})] \tag{61}$$

$$V = \int_a^b X(t)\, df(t) = \underset{\substack{n \to \infty \\ \Delta_n \to 0}}{l.i.m.} \sum_{k=1}^{n} X(\bar{t}_k)[f(t_k) - f(t_k - t_{k-1})] \tag{62}$$

exist, are finite and are independent on the choice of the intermediate points $\bar{t}_k$, then the Riemann-Stieltjies integrals exist.

The necessary and sufficient condition for the existence of the mean square integrals (61) and (62) is that the Riemann-Stieltjes integrals

$$\int_a^b \int_a^b f(t)\, f(s)\, d^2E[X(t)\ X(s)]\,; \quad \int_a^b E[X(t)\ X(s)]\, df(t)\, df(s) \tag{63}$$

exist and are finite.

It can be shown that if U and V in equations (61) and (62) exist, then both U and V exist and are related to each other by means of the following relationship

$$\int_a^b X(t)\, df(t) = [f(t)\ X(t)]_a^b - \int_a^b f(t)\, dX(t) \tag{64}$$

Without going into details, we mention that the other properties of ordinary Riemann-Stieltjes integrals are also valid in this setting.

As an example the integral involving increments of homogeneous compound process or increments of Wiener process

$$\int_{t_0}^t f(\tau)\, dC(\tau)\,; \quad \int_{t_0}^t f(\tau)\, dB(\tau) \tag{65}$$

exists where the function $f(\tau)$ is a deterministic smooth function, since every approximation of the partial sums given in equation (61) leads to this value.

If, however, $f(\tau)$ in equation (65) is irregular like dB(t) or dC(t), itself the mean square integrals depend on the choice of the intermediate point selected. As an example

$$\int_{t_0}^{t} B(t)\, dB(t) = \underset{\substack{n \to \infty \\ \Delta_n \to 0}}{l.i.m.} \sum_{k=1}^{n} B(\bar{t}_k)[B(t_k) - B(t_{k-1})] \tag{66}$$

from equation (33) we have that

$$E[B(t_i)\, B(t_j)] = q^{(2)} \min(t_i, t_j) \tag{67}$$

it follows that

$$E\left\{\sum_{k=1}^{n} B(\bar{t}_k)[B(t_k) - B(t_{k-1})]\right\} = q^{(2)} \sum_{k=1}^{n} [\min(\bar{t}_k, t_k) - \min(\bar{t}_k, t_{k-1})]$$

$$= q^{(2)} \sum_{k=1}^{n} (\bar{t}_k - t_{k-1}) \tag{68}$$

that is the limit of partial sum depends on the choice of the intermediate points. If, for example, we choose for all k

$$\bar{t}_k = \alpha t_k + (1 - \alpha) t_{k-1} \qquad (0 < \alpha < 1) \tag{69}$$

then equation (68) write

$$E\left\{\sum_{k=1}^{n} B(\bar{t}_k)[B(t_k) - B(t_{k-1})]\right\} = q^{(2)}(t - t_0)\alpha \tag{70}$$

So that the mean value of the integral can be anything between zero and $q^{(2)}(t - t_0)$ depending on the choice of intermediate point.

In order to reach uniqueness of the solution the Ito and Stratonovich integrals will be discussed in the next section.

5 STOCHASTIC CALCULUS.

5.1 Itô integral.

In this section some fundamentals of the classical Itô stochastic differential calculus will be brievly summarized.
The analysis of stochastic dynamic systems often leads to differential equation of the form

$$\dot{Z} = a(Z(t), t) + b(Z(t), t)\, W^0(t); \quad Z(t_0) = Z_0 \tag{71}$$

where $a(\cdot)$ and $b(\cdot)$ are deterministic non-linear functions of the response process $Z(t)$, Z_0 is the deterministic or random initial condition, while $W^0(t)$ is a stationary normal white noise process. As recalled in the previous section, $W^0(t)$ is not Riemann integrable in mean square, but it can be considered as the formal derivative of a Wiener process. It follows that Eq. (71) has not a strict mathematical meaning, and can be considered equivalent to the following *stochastic differential equation*:

$$dZ = a(Z(t), t)\, dt + b(Z(t), t)\, dB(t); \quad Z(t_0) = Z_0 \tag{72}$$

in which $dB(t)$ is an increment of the Wiener process $B(t)$. An integration of equation (72), yields

$$Z(t) - Z_0 = \int_{t_0}^{t} a(Z(\tau), \tau)\, d\tau + \int_{t_0}^{t} b(Z(\tau), \tau)\, dB(\tau) \tag{73}$$

The first integral at the rigth hand side of equation (73) is a mean square Riemann integral, while the second one can not be considered in the same way because $B(t)$ is not differentiable in mean square value. Moreover, in accordance with section 3, almost all sample functions of B are of unbounded variation, it follows that the second integral cannot be interpreted as a mean square Riemann-Stialtjes integral. Let the temporal interval $[t_0, t]$ divided into n subintervals by means of such partitioning point t_j as that given in equation (57), the limit of partial sum

$$\int_{t_0}^{t} b(Z(\tau), \tau)\, dB(\tau) = \underset{\substack{n \to \infty \\ \Delta_n \to 0}}{l.i.m.} \sum_{s-1}^{n} b\left(Z(\bar{t}_s), \bar{t}_s\right)\left[B(t_s) - B(t_{s-1})\right] \tag{74}$$

takes a different value depending on the choice of the intermediate point selected (see e.g. Arnold [31], Gardiner [4], Jazwinski [5]).

If, for every interval, we choose $\bar{t}_k = t_{k-1}$ we obtain the so-called *Itô Integral* (Ito [3]), that is

$$(I)\int_{t_0}^{t} b(Z(\tau),\tau)\,dB(\tau) = \underset{\substack{n\to\infty \\ \Delta_n \to 0}}{l.i.m.} \sum_{k=1}^{n} b\left(Z(t_{k-1}), t_{k-1}\right)\left[B(t_k) - B(t_{k-1})\right] \tag{75}$$

where the (I) before the integral stands for Ito integral.
If in equation (72) the integrals are performed by this rule then the differential equation is the so-called *Itô stochastic differential equation.* Because of the importance played by this type of integral we give in the follows some fundamental properties. The proof of these properties can be found in literature [4, 30, 31].

5.1.1 Itô integral of the type $\int B(t)\,dB(t)$.

It can be shown (see Gardiner 1990) that for $q^{(2)} = 1$ (see Eq. (27)) we have

$$(I)\int_{t_0}^{t} B(\tau)\,dB(\tau) = \frac{1}{2}\left[B^2(t) - B^2(t_0) - (t - t_0)\right] \tag{76}$$

it follows that the Ito integrals do not follow the usual rules of integration by parts. From this simple example it is thus evident that the classical integration rules fail. Then making stochastic average of equation (76) we obtain

$$E\left[(I)\int_{t_0}^{t} B(\tau)\,dB(\tau)\right] = \frac{1}{2}E\left[B^2(t) - B^2(t_0)\right] - \frac{1}{2}(t - t_0) = 0 \tag{77}$$

5.1.2 Itô integrals for non-anticipating functions.

An important concept, that is the key of the Ito calculus, is that of a non-anticipating function. A deterministic or random function $\phi(t)$ is said to be non-anticipating, if for every $t_{j+1} > t_j$, $\phi(t_j)$ is independent of the increment of the Wiener process $B(t_{j+1}) - B(t_j)$.
Example of non-anticipating functions are the solution Z(t) of equation (72), or every function $\phi(Z(t), t)$ of the solution. Moreover, every deterministic function posseses this property. It can be shown that if $\phi(t)$ is a non-anticipating function then

$$(I)\int_a^b \phi(t)\,(dB(t))^2 = q^{(2)}\int_a^b \phi(t)\,dt \tag{78}$$

$$(I)\int_a^b \phi(t)\,(dB(t))^N = 0\,;\quad N>2 \tag{79}$$

From equation (78) we recognize at once that the Itô integral does not follow the integration rules because in the classical rule the integral (78) is an infinitesimal. This can be euristically explained by considering that dB(t) is an infinitesimal of order $(dt)^{1/2}$ and consequently $(dB)^2$ is an infinitesimal of first order.
It can be also shown that the following relationships hold

$$E\left[(I)\int_a^b \phi(t)\,dB(t)\right] = 0 \tag{80}$$

$$E\left[(I)\int_{T_1} \phi(t)\,dB(t)\cdot(I)\int_{T_2} \psi(t)\,dB(t)\right] = \int_{T_1\cap T_2} E[\phi(t)\,\psi(t)]\,q^{(2)}\,dt \tag{81}$$

where $\phi(t)$ and $\psi(t)$ are two arbitrary non-anticipating functions or random processes and $T_1 \cap T_2$ is the intersection of the intervals T_1 and T_2.
It is to be emphasized that the simple relationships (78-81) are correct only using the Itô integral, and this is the motivation for the preference of this type of integral, on the contrary the usual rules of the classical integration fail.

5.2 Stratonovich integral.

The applicability of the rules of classical Riemann-Stieltjes calculus was also the motivation for the definition of a stochastic integral given by Stratonovich [7] given as

$$(S)\int_{t_0}^{t} b[Z(\tau),\tau]\,dB(\tau) = \underset{\substack{n\to\infty \\ \Delta_n\to 0}}{l.i.m.} \sum_{j=1}^{n} b\left[\frac{Z(t_s)+Z(t_{s-1})}{2}, t_{s-1}\right](B(t_s)-B(t_{s-1})) \tag{82}$$

where (S) before the integral stands for Stratonovich. It can be shown that

$$(S)\int_{t_0}^{t} B(t)\,dB(t) = \frac{1}{2}\left[B^2(t) - B^2(t_0)\right] \tag{83}$$

from this equation we recognize that the Stratonovich integral obeys to the classical rules of integration. However for this integral the very useful relationships (78-81) are not valid.
In general no relationship between the Itô integral and the Stratonovich integral exists. However if the stochastic process Z(t) is the solution of equation (71) there is a connection between these two integrals given as

$$(S)\int_{t_0}^{t} \phi\,(Z(\tau),\tau)\,dB(\tau) = (I)\int \phi\,(Z(\tau),\tau)\,dB(\tau)$$

$$+\frac{1}{2}\int_{t_0}^{t} \phi\,(Z(\tau),\tau)\frac{\partial\phi\,(Z(\tau),\tau)}{\partial Z}q^{(2)}\,dt \tag{84}$$

The second integral in equation (84) is the so-called *Wong-Zakai* or *Stratonovich* (WZ or S) *correction term.*
Equation (84) can be rewritten in differential form as follows

$$(S)\,\phi(Z(t),t)\,dB(t) = (I)\,\phi(Z(t),t)\,dB(t) + \frac{1}{2}\phi(Z(t),t)\frac{\partial\phi(Z(t),t)}{\partial Z}q^{(2)}\,dt \tag{85}$$

in which the symbols (S) or (I) means Stratonovich and Ito differential respectively in the sense that the corresponding integrals are performed as in equation (82) or (75) respectively.
Letting $\phi(Z(t), t) = b(Z(t), t)$ in equation (88) we obtain

$$(S)\,b(Z(t),t)\,dB(t) = (I)\,b(Z(t),t)\,dB(t) + \frac{1}{2}b(Z(t),t)\frac{\partial b(Z(t),t)}{\partial Z}q^{(2)}\,dt \tag{86}$$

The consequence is that interpreting equation (72) as a Stratonovich differential equation we can write

$$(S)\, dZ = a(Z(t), t))\, dt + (S)\, b(Z(t), t))\, dB = a(Z(t), t))\, dt + (I)\, b(Z(t), t))\, dB(t)$$

$$+ \frac{1}{2} b(Z(t), t) \frac{\partial b(Z(t), t)}{\partial Z} q^{(2)}\, dt \tag{87}$$

Since the last term in the summation (87) is a classical differential (infinitesimal of order dt) like as the first one, we can write

$$(S)\, dZ = f(Z(t), t)\, dt + (I)\, b(Z(t), t)\, dB(t) \tag{88}$$

where f(Z(t), t) is the so-called *drift coefficient* which takes into account the WZ or S correction term that is

$$f(Z(t), t) = a(Z(t), t) + \frac{1}{2} b(Z(t), t) \frac{\partial b(Z(t), t)}{\partial Z} q^{(2)} \tag{89}$$

In the case of external noise, that is b(Z(t), t) is independent on Z(t), the WZ or S correction term is zero and the Itô and the Stratonovich integrals lead to the same result. While in the case of parametric type excitation from equation (88) we recognize that an increment of Z in the Stratonovich sense is a summation of the drift coefficient and another increment that will be evaluated in Itô sense. This means that the Stratonovich differential equation can be evaluated by inserting the WZ or S correction term in the drift coefficient and, then, performing the Ito integral of the stochastic differential equation so modified.
In this way we can use the fundamental property of the Itô integral, according to which any non-anticipating function, such as the response itself, and an increment of the Wiener process are uncorrelated.

5.3 Itô differential rule.

Let $\phi(Z, t)$ any scalar real valued function of the stochastic process Z(t) continuously differentiable on t and twice differentiable on Z, then an increment of $\phi(Z(t), t)$, with arguments omitted, writes

$$d\phi = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial Z} dZ + \frac{1}{2} \frac{\partial^2 \phi}{\partial Z^2} (dZ)^2 \tag{90}$$

that is the Taylor expansion truncated at the second term.

The last term on the rigth hand side of equation (90) accounts for the fact that $(dZ)^2$ is an infinitesimal of first (order of $dB = \sqrt{dt}$).

5.3.1 Moment equation approach.

A suitable choice of the function $\phi(Z(t), t)$ allows to obtain the probabilistic characterization of the response. Let $\phi(Z(t), t) = Z^k$ then

$$dZ^k = k\, Z^{k-1}\, dZ + \frac{k\,(k-1)}{2} Z^{k-2}\, (dZ)^2 \tag{91}$$

inserting dZ given in equation (88) we obtain

$$dZ^k = k\, Z^{k-1}\, (f\,(Z, t)\, dt + b\,(Z, t)\, dB(t)) + \frac{k\,(k-1)}{2} Z^{k-2}\, b^2(Z, t)\, (dB)^2 \tag{92}$$

making the stochastic average and dividing by dt we obtain

$$\dot{E}\left[Z^k\right] = k\, E\left[Z^{k-1}\, f(Z, t)\right] + \frac{k\,(k-1)}{2} E\left[Z^{k-2}\, b^2(Z, t)\right] q^{(2)} \tag{93}$$

In equation (93) use has been made of the independence of the increment of Wiener process and any function g(Z, t) of the response Z, that is

$$E\,[g(Z, t)\, dB] = E\,[g(Z, t)]\, E\,[dB] = 0 \tag{94}$$

$$E\,[g(Z, t)\, (dB)^2\,] = E\,[g(Z, t)]\, E\,[(dB)^2\,] = E\,[g(Z, t)]\, q^{(2)}\, dt \tag{95}$$

Equation (93) gives the evolution of the moments of every order of the response thus their evaluation completely characterizes the response from a probabilistic point of view. The numerical evaluation of equation (93) will be discussed later.

5.3.2 Fokker-Planck equation.

Letting $\phi(Z, t) = \exp(-i\,\vartheta\,Z)$, where ϑ is a real parameter. Inserting this function in equation (90), making the stochastic average and dividing by dt we obtain the differential equation of the characteristic function $M_Z(\vartheta; t)$ in the form

$$\dot{M}_Z(\vartheta; t) = -i\,\vartheta \int_{-\infty}^{\infty} \exp(-i\,\vartheta\,z)\, p_Z(z; t)\, f(z, t)\, dz$$

$$-\frac{1}{2}\vartheta^2\, q^{(2)} \int_{-\infty}^{\infty} e^{-i\,\vartheta z}\, p_Z(z; t)\, b^2(z; t)\, dz \tag{96}$$

An inverse Fourier transform gives

$$\dot{p}_Z(z, t) = -\frac{\partial}{\partial z}\left(p_Z(z; t)\, f(z, t)\right) + \frac{q^{(2)}}{2}\frac{\partial^2}{\partial z^2}\left(p_Z(z; t)\, b(z, t)\right) \tag{97}$$

This equation, which gives directly the evolution of the probability density function is known as *Fokker-Planck* (forward) equation.

5.4 Examples.

In order to familiarize readers to the stochastic differential calculus some simple examples are discussed.

5.4.1 Multiplicative noise.

Let the differential equation of motion be given in the form

$$\dot{Z} = \exp(-Z)\, W^0(t) \tag{98}$$

where $W^0(t)$ is a normal white noise process. According to equation (88) we transform the equation (98) into an Ito type stochastic differential equation

$$dZ = -\frac{q^{(2)}}{2} e^{-2Z}\, dt + e^{-Z}\, dB \tag{99}$$

equation (93) gives

$$\dot{E}\left[Z^k\right] = -\frac{q^{(2)}}{2} E\left[Z^{k-1}\, e^{-2Z}\right] + \frac{q^{(2)}\, k(k-1)}{2} E\left[Z^{k-2}\, e^{-2Z}\right] \tag{100}$$

The Fokker-Planck equation becomes

$$\dot{p}_Z(z, t) = -\frac{q^{(2)}}{2}\frac{\partial}{\partial z}\left(p_Z(z; t)\, e^{-2z}\right) + \frac{q^{(2)}}{2}\frac{\partial^2}{\partial z^2}\left(p_Z(z; t)\, e^{-z}\right) \tag{101}$$

5.4.2 Linear system.

Let the linear differential equation of motion be given in the form

$$\dot{Z} = a Z + b Z W^0(t) \tag{102}$$

where a and b take constant values and $W^0(t)$ is a normal white noise process. The Itô stochastic differential equation writes

$$dZ = \left(a Z + \frac{q^{(2)}}{2} b^2 Z\right) dt + b Z\, dB \tag{103}$$

equation (93) gives

$$\dot{E}\left[Z^k\right] = k\left[a + \frac{q^{(2)}}{2} k\, b^2\right] E\left[Z^k\right] \tag{104}$$

The Fokker-Planck equation becomes

$$\dot{p}_Z(z, t) = -\frac{q^{(2)}}{2}\frac{\partial}{\partial z}\left(p_Z(z; t)\, z\right)\left(a + \frac{q^{(2)}}{2} b^2\right) + \frac{q^{(2)} b}{2}\frac{\partial^2}{\partial z^2}\left(p_Z(z; t)\, z\right) \tag{105}$$

5.4.3 Non linear system, additive noise.

Let the equation of motion be given in the form

$$\dot{Z} = \alpha Z + \beta Z^3 + \gamma W^0 \tag{106}$$

in this case the system is non-linear einforced by external white noise process. Since the noise is additive the Stratonovich differential equation coincides with the Itô equation, it follows that the Stratonovich increment simply writes

$$dZ = \left(dZ + \beta Z^3\right) dt + \gamma \, dB \tag{107}$$

equation (93) write

$$\dot{E}\left[Z^k\right] = k\,\alpha E\left[Z^k\right] + k\,\beta E\left[Z^{k+2}\right] + \frac{\gamma^2 k(k-1)}{2} q^{(2)} E\left[Z^{k-2}\right] \tag{108}$$

that is the differential equation of the moments of order k involves higher and lower order moments than k, that is the moment equations constitute an infinite hierarchy.
In order to solve this problem closure schemes have to be adopted [31, 32].
The Fokker-Planck equation writes

$$\dot{p}_Z(z;t) = -\frac{\partial}{\partial z}\left(p_Z(z,t)\left(\alpha z + \beta z^3\right)\right) + q^{(2)} \frac{\gamma}{2} \frac{\partial^2}{\partial z^2}\left(p_Z(z,t)\right) \tag{109}$$

If $\beta = 0$ then equation (107) becomes a linear one and equation (108) becomes a linear system of ordinary differential equation. By appropriately imposing the initial conditions the moment equations can be solved obtaining the entire probability density function. If these initial conditions are deterministic or normal distributed the response is also normal and the first two equations (108), particularized for $\beta = 0$, give the complete characterization of the process Z(t).

6 NON CLASSICAL ITÔ DIFFERENTIAL RULE.

In order to extend the stochastic differential calculus already discussed for normal white noise input, in this section a different interpretation of the WZ or S correction term will be discussed [21].
Let the equation of motion be given in the form (71). Let $\phi(Z, t)$ a scalar real valued function continuously differentiable on t and twice differentiable on Z, then an increment of $\phi(Z, t)$, denoted as $\Delta\phi$, can be expanded into a Taylor series

$$\Delta\phi = d\phi + \frac{1}{2} d^2\phi \tag{110}$$

where only the first two terms of the series are retained because the remaining terms are infinitesimal of higher order. On applying the usual rules of differentiation of composite functions, neglecting infinitesimals of higher order than dt, we obtain

$$d\phi(Z(t), t) = \frac{\partial \phi(Z(t), t)}{\partial t} dt + \frac{\partial \phi(Z(t), t)}{\partial Z} dZ(t) \tag{111}$$

$$d^2\phi(Z(t), t) = \frac{\partial^2 \phi(Z(t), t)}{\partial Z^2} (dZ(t))^2 + \frac{\partial \phi(Z(t), t)}{\partial Z} d^2Z(t) \tag{112}$$

Taking into account that the dZ(t) is given by equation (72), $d^2Z(t)$ is given as

$$d^2Z(t) = \frac{\partial}{\partial Z}(dZ)\, dZ = \frac{\partial}{\partial Z}(a(Z, t)\, dt + b(Z, t)\, dB)\, dZ =$$

$$= \left(\frac{\partial a(Z, t)}{\partial Z} dt + \frac{\partial b(Z, t)}{\partial Z} dB\right)(a(Z, t)\, dt + b(Z, t)\, dB) \tag{113}$$

By neglecting nfinitesimals of higher order than dt we obtain

$$d^2Z(t) = \frac{\partial b(Z, t)}{\partial Z} b(Z, t)\, (dB)^2 \tag{114}$$

Moreover

$$(dZ(t))^2 = b^2(Z, t)\, (dB)^2 \tag{115}$$

Then the increment $\Delta\phi(Z(t), t)$ can be rewritten, with arguments omitted, in the form

$$\Delta\phi = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial Z}\left[a\, dt + b\, dB + \frac{1}{2} b \frac{\partial b}{\partial Z} (dB)^2\right] + \frac{1}{2} \frac{\partial^2 \phi}{\partial Z^2} b^2 (dB)^2 \tag{116}$$

Letting $\phi(Z(t), t) = Z(t)$ in equation (116) gives

$$\Delta Z(t) = a(Z(t), t)\, dt + b(Z, t)\, dB + \frac{1}{2} b(Z(t), t) \frac{\partial b(Z(t), t)}{\partial Z} (dB)^2 \tag{117}$$

By comparing equation (117) with equation (88) we recognize that the increment evaluated by means of the rule (110) coincides with the Stratonovich (S)dZ. The extra-term $\frac{1}{2} b \frac{\partial b}{\partial Z} (dB)^2$ compensate the difference between the Itô increments and the Stratonovich increments.
It is interesting to note that expansion (110), neglecting higher order infinitesimal, can be also written in the form

$$\Delta\phi\,(Z(t),t) = \frac{\partial\phi\,(Z(t),t)}{\partial t}dt + \frac{\partial\phi\,(Z(t),t)}{\partial Z}\Delta Z + \frac{1}{2}\frac{\partial^2\phi\,(Z(t),t)}{\partial Z^2}(\Delta Z)^2 \tag{118}$$

in which ΔZ is given in equation (117). Relationship (118) has been called *"non classical Itô differential rule"* [21]. Equation (118) in which ΔZ is given in equation (117) leads to identical results to equation (90) in which dZ is evaluated in Stratonovich sense, however expansion (118) will be used for the extension to the case of non-normal delta-correlated input how it will be discussed in the following.

6.1 Itô and Stratonovich integrals for non-normal white noise inputs.

Let us consider the SDOF system governed by the same differential equation as in (71), where, now, W(t) is a stationary non-normal delta-correlated process. Also in this case the solution can be given as in equation (73) and the second integral can be evaluated following the (I) rule or the (S) rule. If W(t) is a Poisson white noise process, then the I integral can be expressed as follows:

$$(I)\int_{t_0}^{t} b\,(Z(\tau))\,dC(\tau) = \underset{\substack{n\to\infty \\ \Delta n\to 0}}{l.i.m.} \sum_{i=1}^{n} b\left[Z\left(t_{i-1}\right)\right]\left[C\left(t_i\right) - C\left(t_{i-1}\right)\right] \tag{119}$$

where C(t) is the homogeneous Poisson compound process connected to W(t). While, generalizing what said for the normal white noise inputs, the S integral can be expressed as follows:

$$(S)\int_{t_0}^{t} b\,(Z(\tau))\,dC(\tau) = \underset{\substack{n\to\infty \\ \Delta n\to 0}}{l.i.m.} \sum_{i=1}^{n} b\left[\frac{Z\left(t_i\right)+Z\left(t_{i-1}\right)}{2}\right]\left[C\left(t_i\right) - C\left(t_{i-1}\right)\right] \tag{120}$$

It can be shown that, applying the (I) integration rule, some properties hold

$$I\int_{t_0}^{t} \phi\,(Z(\tau))\,(dC(\tau))^r = q^{(r)}\int_{t_0}^{t} \phi\,(Z(\tau))\,d\tau \quad \forall r \tag{121}$$

and:

$$R^{(r)}\left(I\int_{t_0}^{t_1} \phi_1(\tau)\,dC(\tau), I\int_{t_0}^{t_2} \phi_2(\tau)\,dC(\tau), \ldots, I\int_{t_0}^{t_r} \phi_r(\tau)\,dC(\tau)\right) =$$

$$= q^{(r)}\int_{t_0}^{\min(t_1, t_2, \ldots, t_r)} \phi_1(\tau)\,\phi_2(\tau)\ldots\phi_r(\tau)\,d\tau \tag{122}$$

where $\phi_j(\cdot)$ $(j = 1, 2, \ldots, r)$ are deterministic functions. From these equations, we can note that, applying the (I) integral, the infinitesimals $(dC(t))^r$ can be considered to be of the same order of the infinitesimal dt, for all r. Then, also when the inputs are non-normal, the (I) integration rule does not follow some fundamental properties of the ordinary differential calculus. If we want to preserve these properties, the (S) integration rule must be applied.

6.2 Extension of Itô rule.

Even in this case, if Z(t) is the stochastic response of a differential equation given in equation (72), it is possible to find an useful relationship between the (I) and the (S) integral. Considering the expression of the increment of any composite function $\phi(Z(t), t)$ a generalization of equation (110), is given as

$$\Delta\phi\,(Z(t), t) = \sum_{j=1}^{\infty} \frac{1}{j!}\,d^j\phi\,(Z(t), t) \tag{123}$$

where all terms of the series are retained because, as said before, when the (I) integration rule is used, the differentials of every order can give first order increments. Using the rules of differentiation of composite functions, the first two differentials are given in equations (111) and (112), while the other few terms are given in the form (with omitted argument):

$$d^3\phi = \frac{\partial^3\phi}{\partial Z^3}(dZ)^3 + 3\frac{\partial^2\phi}{\partial Z^2}\,dZ\,d^2Z + \frac{\partial\phi}{\partial Z}\,d^3Z \tag{124}$$

$$d^4\phi = \frac{\partial^4\phi}{\partial Z^4}(dZ)^4 + 6\frac{\partial^3\phi}{\partial Z^3}d^2Z\,(dZ)^2 + \frac{\partial^2\phi}{\partial Z^2}\left[4dZ\,d^3Z + 3\left(d^2Z\right)^2\right] + \frac{\partial\phi}{\partial Z}d^4\phi\,Z \tag{125}$$

Inserting the differentials $d^j\phi(Z(t), t)$ into equation (123), and taking into account that the differentials $(dC(t))^r$ can give first order increments, it can be shown that equation (37) can be rewritten as follows [21]

$$\Delta\phi\,(Z(t), t)) = \frac{\partial\phi}{\partial t}dt + \sum_{j=1}^{\infty}\frac{1}{j!}\frac{\partial^j\phi\,(Z(t))}{\partial Z^j}(\Delta Z(t))^j \tag{126}$$

where

$$\Delta Z(t) = \sum_{j=1}^{\infty}\frac{1}{j!}d^j Z = a\,(Z(t))\,dt + b\,(Z(t))\,dC(t) + \sum_{j=2}^{\infty}\frac{1}{j!}G^{(j)}(Z(t))\,(dC(t))^j \tag{127}$$

In this equation the functions $G^{(r)}(Z(t))$ are given by:

$$G^{(r)}(Z(t)) = \frac{\partial G^{(r-1)}}{\partial Z(t)}G\,(Z(t)); \quad G^{(1)}(Z(t)) = b\,(Z(t)) \tag{128}$$

Equation (126) is a very remarkable result, that is an increment of any scalar real valued function in the Stratonovich sense is given as the classical Taylor expansion in which the increments ΔZ of the stochastic processes have to be evaluated in the Stratonovich sense.
Setting $\phi(Z(t), t) = Z(t)$ in equation (126), where equation (127) has been considered, and applying the (I) integration rule, we obtain the following expression:

$$Z(t) - Z(t_0) = \int_{t_0}^{t} a\,(Z(\tau))\,d\tau + (I)\int_{t_0}^{t} b\,(Z(\tau))\,dC(\tau) + \sum_{j=2}^{\infty}\frac{q^{(j)}}{j!}\int_{t_0}^{t} G^{(j)}(Z(\tau))\,d\tau \tag{129}$$

In this equation the summation $\sum_{j=2}^{\infty}(q^{(j)}/j!)\,G^{(j)}$ represents the extension of the WZ or S correction terms for non-normal delta-correlated inputs. It is easy to verify that the first term in the summation is the classical WZ or S correction term.
Instead, if the (S) integration rule is applied, taking into account that it satisfies the fundamental properties of the ordinary differential calculus, the following expression is obtained:

$$Z(t) - Z(t_0) = \int_{t_0}^{t} a(Z(\tau))\,d\tau + (S)\int_{t_0}^{t} b(Z(\tau))\,dC(\tau) \tag{130}$$

A comparison between equations (129) and (130) evidences the necessity of the new correction terms.

Taking into account the way how the presence of these new correction terms (of which the WZ or S ones are the particularization for normal inputs) has been evidenced, we recognize that they are not related to the irregularities of the input process (unbounded variations in infinitesimal intervals), but only to the different choice of performing integrals. In fact, for example, in Fig. 1b, is reported a sample of a Poisson white noise which, as said before, is a particular non-normal delta-correlated process; from this figure we recognize that this sample is obviously irregular, but the irregularities are present only in a few instants and the sample does not show unbounded variations in infinitesimal intervals; nevertheless, if we consider a system subjected to a Poisson white noise input and we want to apply the (I) integration rule, the new correction terms are required.

6.2.1 Moment equation.

By using equation (126) is now possible to find the probabilistic characterization of the response process Z(t) of the non linear system driven by parametric type delta-correlated process. Setting $\phi(Z, t) = Z^k$ we obtain

$$\Delta\left(Z^k\right) = \sum_{j=1}^{k} \frac{1}{j!}\, k(k-1)\ldots(k-j+1)\, Z^{k-j}\,(\Delta Z)^j \tag{131}$$

where (ΔZ) is given in equation (127) and, by neglecting infinitesimal of higher order than dt, we can write

$$(\Delta Z)^2 = \sum_{l=1}^{\infty}\sum_{r=1}^{\infty} \frac{G^{(l)}\,G^{(r)}}{l!\,r!}\,(dC)^{l+r}$$

$$\vdots$$

$$(\Delta Z)^k = \sum_{l=1}^{\infty}\sum_{r=1}^{\infty}\cdots\sum_{s=1}^{\infty} \frac{G^{(l)}\,G^{(r)}\ldots G^{(p)}}{l!\,r!\ldots p!}\,(dC)^{(l+r+\ldots+p)} \tag{132}$$

k - times

Hence equation (131) becomes, with arguments omitted,

$$\dot{E}\left[Z^k\right] = k\,E\left[Z^{k-1}\,a\right] + k\sum_{l=1}^{\infty}\frac{1}{l!}E\left[Z^{k-1}\,G^{(l)}\right]q^{(l)} +$$

$$\frac{1}{2!}k(k-1)\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{l!\,r!}E\left[Z^{k-2}\,G^{(l)}\,G^{(r)}\right]q^{(l-r)} + \ldots +$$

$$\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}\cdots\sum_{p=1}^{\infty}\frac{1}{l!\,r!\ldots p!}E\left[G^{(l)}\,G^{(r)}\ldots G^{(p)}\right]q^{(l+r+\ldots+p)} \qquad (133)$$

k times

In this equation use has been made of the fact that, according to the Itô's stochastic calculus, the following relationship holds

$$E\left[g(Z)\,(dC)^r\right] = E\left[g(Z)\right]E\left[(dC)^r\right] = E\left[g(Z)\right]q^{(r)}\,dt \qquad (134)$$

6.2.2 Moment equation approach for external delta-correlated input.

In the case of external delta-correlated input, that is when b(Z, t) is independent of Z, from equation (127) we recognize that ΔZ coincides with dZ because all $G^{(j)}(Z, t) = 0$, j = 2, 3, ... It follows that

$$(\Delta Z)^j = (dZ)^j = b(t)^j\,(dC)^j \qquad j = 2, 3, \ldots \qquad (135)$$

Then equation (126) writes

$$\Delta\phi(Z) = \frac{\partial\phi}{\partial t}dt + \sum_{j=1}^{\infty}\frac{1}{j!}\frac{\partial^j\phi}{\partial Z^j}(dZ)^j = \frac{\partial\phi}{\partial t}dt + \sum_{j=1}^{\infty}\frac{1}{j!}\frac{\partial^j\phi}{\partial Z^j}b^j\,(dC)^j \qquad (136)$$

Equation (136) represents the Itô rule extended for external delta-correlated input.
Letting $\phi(Z) = Z^k$ we obtain the differential moment equation of the moments in the form

$$\dot{E}\left[Z^k\right] = k\,E\left[Z^{k-1}\,a\right] + \sum_{j=1}^{k} \frac{1}{j!}\,k(k-1)\ldots(k-j+1)\,E\left[Z^{k-j}\right] b^j\,q^{(j)} \tag{137}$$

This equation can be proved to be equivalent to that obtained by Iwankievicz et al [20].

6.2.3 Fokker-Planck equation for delta-correlated input process.

Let the equation of motion be given in the form

$$\dot{Z} = a(Z(t), t) + b(Z(t), t)\,W \tag{138}$$

where W(t) is a delta-correlated process. Let $\phi(Z(t), t) = \exp(-i\,\vartheta\,Z(t))$, where ϑ is a real parameter, then by using equation (126) and taking the stochastic average we obtain

$$\Delta M_Z(\vartheta) = \sum_{k=1}^{\infty} \frac{(-i\,\vartheta)^k}{k!} E\left[\exp(-i\,\vartheta\,Z(t))\,(\Delta Z)^k\right] \tag{139}$$

where ΔZ is the Stratonovich increment given in equation (127). Making the inverse Fourier transform we obtain

$$\frac{\partial p_Z(z, t)}{\partial t}\,dt = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial z^k}\left[p_Z(z)\,(\Delta z)^k\right] \tag{140}$$

where Δz is easily obtained letting z instead Z in equation (132).
In the case of external delta-correlated input, inserting equation (135) in equation (140) we obtain

$$\frac{\partial p_Z(z, t)}{\partial t} = -\frac{\partial}{\partial z}\left(p_Z(z)\,a(z)\right) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\,b^k \frac{\partial^k}{\partial z^k}\left[p_Z(z)\right] q^{(k)} \tag{141}$$

This equation is equivalent to that obtained by Roberts [18].
If $C(t) \equiv B(t)$, that is when $\lambda \to \infty$ and $\lambda\,E\,[Y^2]$ keeps a constant value. The summation in equation (140) contains only the first two terms and coincides with the classical Fokker-Planck equation.

7. SIMULATION OF DELTA-CORRELATED PROCESSES AND RESPONSE.

In order to evidence the differences between the Stratonovich integral and the Itô integral, we consider the following quasi-linear stochastic differential equation with deterministic initial condition Z(0)

$$\dot{Z}(t) = a + \alpha Z(t) W(t); \quad Z(0) = 0 \tag{142}$$

where a and α are constant and W(t) is a Poisson delta-correlated input. Equation (45) can be rewritten in the form

$$dZ(t) = a\,dt + \alpha Z(t)\,dC(t); \quad Z(0) = 0 \tag{143}$$

where C(t) is the Poisson compound process connected to W(t). If now we consider the single sample, it is easy to verify that the response is linear between two consecutive pulses; while, at the time instant t_k, in which the pulse of intensity Y_k occour, the response shows a jump which is proportional to α, to Y_k and to the value of Z in t_k. In fact, the integration of the single sample of equation (143) gives:

$$Z(t) = \int_0^t a\,d\tau + \int_0^t \alpha Z(\tau) \sum_{k=1}^{N(t)} Y_k \delta(\tau - t_k)\,d\tau = a\,t + \alpha \sum_{k=1}^{N(t)} Y_k Z(t_k) \tag{144}$$

As in t_k the response has a jump, the value $Z(t_k)$ is not univocally determined. If we indicate with $Z\left(t_k^+\right)$ and with $Z\left(t_k^-\right)$ the values of Z(t) immediately after and before the instant t_k, $Z(t_k)$ can assume any value between $Z\left(t_k^-\right)$ and $Z\left(t_k^+\right)$. Following the (I) interpretation of the integral, we set $Z(t_k) = Z\left(t_k^-\right)$ and the jump $J_k(I)$ is given by:

$$J_k(I) = Z\left(t_k^+\right) - Z\left(t_k^-\right) = \alpha Y_k Z\left(t_k^-\right) \tag{145}$$

Instead, following the (S) interpretation, $Z(t_k)$ is considered as the mean value between $Z\left(t_k^-\right)$ and $Z\left(t_k^+\right)$, and the jump $J_k(S)$ is given by:

$$J_k(S) = Z\left(t_k^+\right) - Z\left(t_k^-\right) = \alpha Y_k \frac{Z\left(t_k^+\right) + Z\left(t_k^-\right)}{2} \tag{146}$$

In Fig. 4, the single response sample, for the input sample given in Fig. 3, for $\alpha = -1$, $c = 1.$, is reported, evaluating the jumps in the (I) sense (dashed line) and in the (S) sense (solid line). The difference between the two responses is evident.
Now we want to use the (I) integration rule, taking into account the new correction terms that, in this case, have the following expression:

$$\sum_{j=2}^{\infty} \frac{1}{j!} G^{(j)} q^{(j)}; \quad G^{(j)} = Z \alpha^j \tag{147}$$

equation (127), for the case in examination, writes:

$$\Delta Z(t) = a\,dt + Z \sum_{j=2}^{\infty} \frac{1}{j!} \alpha^j (dC(t))^j + \alpha Z\, dC(t) \tag{148}$$

If the (I) integration rule for the single response sample is applied on it, we obtain the response in dotted line of Fig. 4. It is evident that the three response samples are completely different. However, quite from the single sample response we can note that the (S) and the (I) integrations of equation (143) give certainly different results even if we work in mean with a lot of samples (simulation); this is due to the fact the response evaluated by means of the (I) integration is always below that evaluated by means of the (S) integration. By contrast, the response evaluated by means of the integration of equation (148), that is introducing the corrective terms, fluctuates around the response in full line. Evaluating the mean response overal several realizations, the (S) integration of equation (143) and the (I) integration of equation (148) give exactly the same results, as it will be pointed out better in the following.
In order to show what stated before, the response moments are evaluated. At this purpose, it is important to note that, as the useful relationships (121) and (122) are valid only if the (I) integral is applied, working in mean square value, it is convenient to use this kind of integral. Obviously, the differential equations governing the first two moments of the response Z(t) of the considered system are different whethever the new correction terms are retained or they are neglected. In fact, in the former case, the first two moment equations write:

$$\dot{E}[Z] = a + E[Z] \sum_{j=1}^{\infty} \frac{1}{j!} \alpha^j q^{(j)} \tag{149}$$

$$\dot{E}\left[Z^2\right] = 2a\,E[Z] + E\left[Z^2\right]\left(2 \sum_{j=1}^{\infty} \frac{1}{j!} \alpha^j q^{(j)} + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j!\,k!} \alpha^{j+k} q^{(j+k)}\right) \tag{150}$$

Instead, if the new correction terms are neglected, the same equations write:

$$\dot{E}[Z] = a + E[Z]\,\alpha\, q^{(1)} \tag{151}$$

$$\dot{E}\left[Z^2\right] = 2a\,E[Z] + E\left[Z^2\right]\left(2\alpha\, q^{(1)} + \alpha^2\, q^{(2)}\right) \tag{152}$$

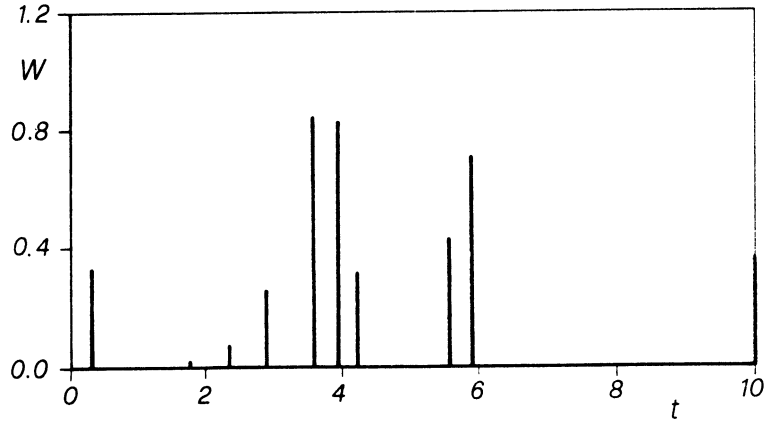

Fig. 3 - *Sample of a Poisson white noise*

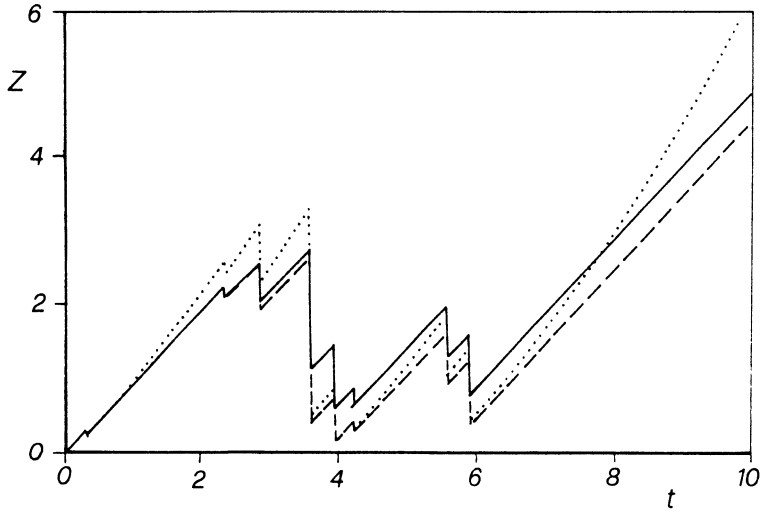

Fig. 4 - *Response sample; solid line: Stratonovich integration; dashed line: Itô integration; dotted line: Itô integration with corrective coefficients*

In Fig. 5 the solutions of the equations (149) and 151) are compared with the results of a sample of size -10000 obtained by using the integration of equation (143) in which the jumps are evaluated in (I) sense (simulation a) and in (S) sense (simulation b) and the integration of equation (148) in which the jumps are evaluated in (I) sense (simulation *c*). It is evident that the results obtained by means of the simulation *a* agree with the solution of equation (151), while the results obtained by means of the simulation *b* and *c* coincide and they agree with the solution of equation (149). In Fig. 6 the same comparison is reported for the second order moment of the response. It is important to note that, while simulations *a* and *b* are performed evaluating the mean of the square of the samples obtained by integrations in (I) and in (S) sense of equation (143), the simulation *c* must be performed by means of the following steps: 1) writing the expression for the case in examination; 2) particularizing it for $\phi(Z(t)) = Z^2(t)$; 3) evaluating the samples of $Z^2(t)$ by means of an (I) integration; 4) evaluating the mean of all the samples. From the analysis of Fig. 6 it is evident that simulation *a* agrees with the solution of equation (152), while simulations *b* and *c* coincide and they agree with the solution of equation (150).

As a conclusion, we can affirm that the presence of the new correction terms allow to obtain, using the convenient (I) integration rule, the same results obtained using the (S) integration rule.

This simple example allows to evidence the fundamental differences in the applications of the two integrals and the true significance of the corrective terms; moreover, it is important to note that it is much more difficult to do similar considerations in the case of normal white noise inputs, whose particular irregularities hide these characteristics.

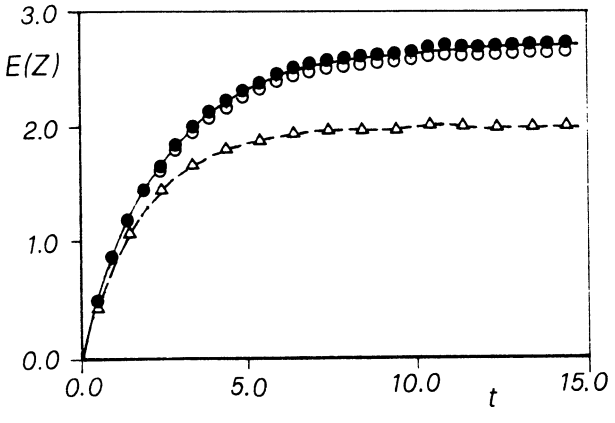

Fig. 5 - *Mean response; solid line: solution of equation (149); dashed line solution of equation (151); Δ: simulation a; ● simulation b; ○ simulation c.*

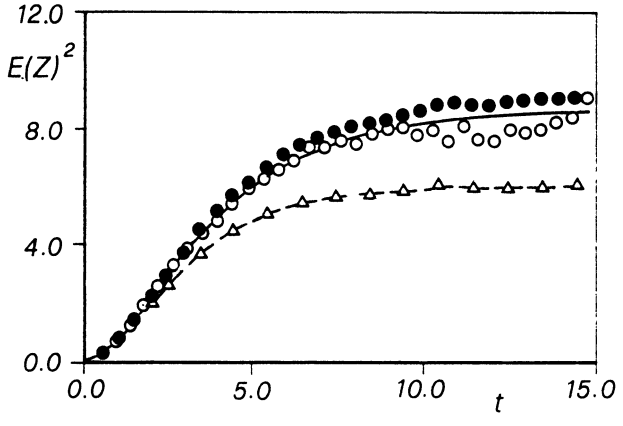

Fig. 6 - *Second order moment of the response; solid line: solution of equation (150); dashed line: solution of equation (152); Δ: simulation a; ●: simulation b; o:simulation c.*

8. MULTIDIMENSIONAL CASE.

The extension of the previous concepts to the multidimensional case is straigthforward using Kronecker algebra whose fundamentals are reported in appendix A. Let the differential equation of an n-degree of freedom model be given in the form

$$\dot{\mathbf{Z}}(t) = \mathbf{a}(\mathbf{Z}(t), t) + \mathbf{b}(\mathbf{Z}(t), t)\, \mathbf{W}(t) \tag{153}$$

where $\mathbf{a}(\mathbf{Z}(t), t)$ and $\mathbf{b}(\mathbf{Z}(t), t)$ are n x n and n x m deterministic function vector and matrix respectively, while $\mathbf{W}(t)$ is a zero mean vector of delta-correlated processes, that is the $d\mathbf{W}(t) = \mathbf{C}(t)\, dt$, were the increments of $\mathbf{C}(t)$ are given by

$$E[d\mathbf{C}(t)] = 0; \quad E\left[d\mathbf{C}^{[k]}(t)\right] = \mathbf{q}^{(k)}\, dt \tag{154}$$

where $\mathbf{q}^{(k)}$ is an m^k vector containing the strength of the white noise $\mathbf{W}(t)$. If all $\mathbf{q}^{(k)}$ $(k > 2)$ are zero then the process $\mathbf{W}(t)$ is a normal white noise.

Let the vector function $\phi(\mathbf{Z}(t), t)$, whose components are continuously differentiable on t and differentiable with respect to the components of $\mathbf{Z}(t)$, the generalization of equation (126) write [33, 34, 35]

$$\Delta[\phi(\mathbf{Z}(t), t)] = \frac{\partial \phi(\mathbf{Z}, t)}{\partial t} dt + \sum_{j=1}^{\infty} \frac{1}{j!} \left[\nabla_z^{[j]^T} \otimes \phi(\mathbf{Z}(t), t) \right] (\Delta \mathbf{Z})^{[j]} \tag{155}$$

where ∇_z is the n-vector containing the derivatives with respect to $\mathbf{Z}$, that is

$$\nabla_z^T = [\partial/\partial Z_1 \quad \partial/\partial Z_2 \quad \dots \quad \partial/\partial Z_n] \tag{156}$$

In equation (155) $\Delta\mathbf{Z}$ is given as

$$\Delta[\mathbf{Z}(t)] = \sum_{j=1}^{\infty} \frac{1}{j!} d^j \mathbf{Z} \tag{157}$$

By neglecting infinitesimal of higher order than dt

$$d^r \mathbf{Z}(t) = \mathbf{G}^{(r)}(\mathbf{Z}(t), t) (d\mathbf{C}(t))^{[r]} \tag{158}$$

where

$$\begin{aligned} \mathbf{G}^{(1)}(\mathbf{Z}(t), t) &= \mathbf{b}(\mathbf{Z}(t), t) \\ &\vdots \\ \mathbf{G}^{(r)}(\mathbf{Z}(t), t) &= \left[\nabla_z^T \otimes \mathbf{G}^{(r-1)}(\mathbf{Z}(t), t) \right] \mathbf{G}^{(1)}(\mathbf{Z}(t), t) \end{aligned} \tag{159}$$

it follows that an increment of $\mathbf{Z}(t)$ is given as

$$\Delta[\mathbf{Z}(t)] = \mathbf{a}(\mathbf{Z}(t), t)\, dt + \mathbf{b}(\mathbf{Z}(t), t)\, d\mathbf{C}(t) + \sum_{j=2}^{\infty} \frac{1}{j!} \mathbf{G}^{(j)}(\mathbf{Z}(t), t) (d\mathbf{C}(t))^{[j]} \tag{160}$$

If $\mathbf{W}(t)$ is a normal stochastic vector of white noise processes then the last summation in equation (159) contains only one term and we can write

$$\Delta[\mathbf{Z}(t)] = \mathbf{a}(\mathbf{Z}(t), t)\, dt + \mathbf{b}(\mathbf{Z}(t), t)\, d\mathbf{B}(t) + \frac{1}{2} \mathbf{G}^{(2)}(\mathbf{Z}(t), t) (d\mathbf{B}(t))^{[2]} \tag{161}$$

where the last term on the right hand side of equation (161) is the WZ or S correction term in the multidimensional case, that can to be rewritten in the explicit form [33, 34]

$$\frac{1}{2}\mathbf{G}^{(2)}(\mathbf{Z}(t), t)\,(d\mathbf{B}(t))^{[2]} = \frac{1}{2}\left[\nabla_{\mathbf{z}}^{T} \otimes \mathbf{b}(\mathbf{Z}(t), t)\right]\left(\mathbf{b}(\mathbf{Z}(t), t) \otimes \mathbf{I}_m\right)(d\mathbf{B}(t))^{[2]} \tag{162}$$

8.1 Moment equation.

A suitable choice of the vector function $\phi(\mathbf{Z}(t), t)$ allows to obtain the probabilistic charac-terization of the vector $\mathbf{Z}(t)$ from a probabilistic point of view. Setting in fact $\phi(\mathbf{Z}(t), t) = \mathbf{Z}^{[k]}(t)$, equation (155) becomes

$$\Delta\left(\mathbf{Z}^{[k]}\right) = \sum_{j=1}^{k} \frac{1}{j!}\left(\nabla_{\mathbf{z}}^{[j]^T} \otimes \mathbf{Z}^{[k]}\right)(\Delta\mathbf{Z})^{[j]} \tag{163}$$

On the other hand, it can be shown that

$$\nabla_{\mathbf{Z}}^{T} \otimes \mathbf{Z}^{[k]} = \mathbf{Q}_k\left(\mathbf{Z}^{[k-1]} \otimes \mathbf{I}_n\right) = \mathbf{Z}^{[k-1]} \otimes \mathbf{I}_n + \mathbf{Z}^{[k-2]} \otimes \mathbf{I}_n \otimes \mathbf{Z} + \ldots + \mathbf{I}_n \otimes \mathbf{Z}^{[k-1]} \tag{164}$$

where $\mathbf{Q}_k$ is the matrix (order n^k x n^k) given as

$$\mathbf{Q}_k = \sum_{j=0}^{k-1} \mathbf{E}_{n^{k-j}\, n^j} \tag{165}$$

In equation (165) $\mathbf{E}_{q,p}$ denote a permutation matrix of order (qp) x (pq) consisting of q x p arrays of p x q dimensional elementary submatrices

$$\mathbf{E}_{q,p} = \begin{bmatrix} \mathbf{E}^{11} & \mathbf{E}^{21} & \ldots & \mathbf{E}^{p1} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{E}^{1q} & \mathbf{E}^{2q} & \ldots & \mathbf{E}^{pq} \end{bmatrix} \tag{166}$$

where the p x q dimensional elementary submatrix $\mathbf{E}^{ij}$ takes the value one the (i,j)th position and zero in all other positions.
The other derivatives write

$$\nabla_{\mathbf{z}}^{[j]^T} \otimes \mathbf{Z}^{[k]} = \mathbf{Q}_k \left(\mathbf{Q}_{k-1} \otimes \mathbf{I}_n\right) \dots \left(\mathbf{Q}_{k-j+1} \otimes \mathbf{I}_n^{[j-1]}\right) \left(\mathbf{Z}^{k-j} \otimes \mathbf{I}_n^{[j]}\right) \tag{167}$$

On the other hand, $(\Delta \mathbf{Z})^{[j]}$, by neglecting infinitesimals of higher order than dt, is given in the form

$$(\Delta \mathbf{Z})^{[2]} = \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \frac{\left(\mathbf{G}^{(l)} \otimes \mathbf{G}^{(r)}\right)}{l!\, r!} (d\mathbf{C})^{[l+r]} \tag{168}$$

$$\vdots$$

$$(\Delta \mathbf{Z})^{[j]} = \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \dots \sum_{p=1}^{\infty} \frac{\mathbf{G}^{(l)} \otimes \mathbf{G}^{(r)} \otimes \dots \otimes \mathbf{G}^{(p)}}{l!\, r! + \dots + p!} (d\mathbf{C})^{[l+r+\dots+p]} \tag{169}$$

j - times

Hence equation (163) becomes

$$\Delta\left(\mathbf{Z}^{[k]}\right) = \sum_{j=1}^{k} \frac{1}{j!} \left[\mathbf{Q}_k \left(\mathbf{Q}_{k-1} \otimes \mathbf{I}_n\right) \dots \left(\mathbf{Q}_{k-j+1} \otimes \mathbf{I}_n^{[j-1]}\right)\right] \left(\mathbf{Z}^{[k-j]} \otimes \mathbf{I}_n^{[j]}\right) (\Delta \mathbf{Z})^{[j]} \tag{170}$$

Making the stochastic average and dividing by dt we obtain

$$\dot{\mathrm{E}}\left[\mathbf{Z}^{[k]}\right] = \mathbf{Q}_k \, \mathrm{E}\left[\mathbf{Z}^{[k-1]} \otimes \mathbf{a}\left(\mathbf{Z}(t), t\right)\right] + \sum_{l=1}^{\infty} \frac{1}{l!} \mathbf{Q}_k \, \mathrm{E}\left[\mathbf{Z}^{[k-1]} \otimes \mathbf{G}^{(j)}\left(\mathbf{Z}(t), t\right)\right] \mathbf{q}^{(l)} + \tag{171}$$

$$+ \frac{1}{2!} \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{l!\, r!} \mathbf{Q}_k \left(\mathbf{Q}_{k-1} \otimes \mathbf{I}_n\right) \mathrm{E}\left[\mathbf{Z}^{[k-2]} \otimes \mathbf{G}^{(l)}\left(\mathbf{Z}(t), t\right) \otimes \mathbf{G}^{(r)}\left(\mathbf{Z}(t), t\right)\right] \mathbf{q}^{(l+r)} + \dots$$

where $\mathbf{q}^{(j)}$ is the vector of order m^j collecting the strengths of the delta-correlated vector process $\mathbf{C}(t)$. If the system is driven only by external loads, then $\Delta \mathbf{Z} \equiv d\mathbf{Z}$ and then equation (171) writes

$$\dot{\mathrm{E}}\left[\mathbf{Z}^{[k]}\right] = \mathbf{Q}_k \, \mathrm{E}\left[\mathbf{Z}^{[k-1]} \otimes \mathbf{a}\left(\mathbf{Z}(t), t\right)\right] +$$

$$+ \sum_{j=1}^{k} \frac{1}{j!} \mathbf{Q}_k \left(\mathbf{Q}_{k-1} \otimes \mathbf{I}_n\right) \dots \left(\mathbf{Q}_{k-j+1} \otimes \mathbf{I}_n^{[j-1]}\right) \left[\mathrm{E}\left[\mathbf{Z}^{[k-j]}\right] \otimes \mathbf{b}^{[j]}(t)\right] \mathbf{q}^{(j)} \tag{172}$$

Moreover, neglecting higher order infinitesimal than dt, we can write

$$(\Delta \mathbf{Z})^{j} = (d\mathbf{Z})^{j} = \mathbf{b}^{[j]}(t)\,(d\mathbf{C})^{[j]}; \quad j = 2, 3, \ldots, \infty \tag{173}$$

and the non-classical Itô differential rule is given in the form

$$\Delta\phi\,(\mathbf{Z}(t), t) = \frac{\partial \phi\,(\mathbf{Z}(t), t)}{\partial t}\,dt + \sum_{j=1}^{\infty} \frac{1}{j!}\left[\nabla_{\mathbf{z}}^{[j]^{T}} \otimes \phi\,(\mathbf{Z}(t), t)\right](d\mathbf{Z}(t))^{[j]} \tag{174}$$

Letting $\phi(\mathbf{Z}, t) = \mathbf{Z}^{[k]}$, making the stochastic average and dividing by dt, we obtain

$$\dot{E}\left[\mathbf{Z}^{[k]}\right] = \mathbf{Q}_k\, E\left[\mathbf{Z}^{[k-1]} \otimes \mathbf{a}(\mathbf{Z}(t), t)\right] +$$

$$+ \sum_{j=1}^{k} \frac{1}{j!}\mathbf{Q}_k\left(\mathbf{Q}_{k-1} \otimes \mathbf{I}_n\right) \ldots \left(\mathbf{Q}_{k-j+1} \otimes \mathbf{I}_n^{[j-1]}\right)\left[E\left[\mathbf{Z}^{[k-j]} \otimes \mathbf{b}^{[j]}(t)\right]\right] \mathbf{q}^{(j)} \tag{175}$$

In the case of linear system the differential equation is given in the form

$$\Delta \mathbf{Z} = \mathbf{A}(t)\,\mathbf{Z}dt + \mathbf{G}(t)\,d\mathbf{C} \tag{176}$$

The non-classical Itô differential rule writes

$$\Delta\phi\,(\mathbf{Z}, t) = \frac{\partial \phi\,(\mathbf{Z}, t)}{\partial t}\,dt + \sum_{j=1}^{\infty} \frac{1}{j!}\left[\nabla_{\mathbf{z}}^{[j]^{T}} \otimes \phi\,(\mathbf{Z}, t)\right](d\mathbf{Z})^{[j]} \tag{177}$$

By setting $\phi(\mathbf{Z}, t) = \mathbf{Z}^{k}$, making the stochastic average and dividing by dt we obtain

$$\dot{\mathbf{m}}_k = \mathbf{A}_k(t)\,\mathbf{m}_k + \sum_{j=1}^{k} \frac{1}{j!}\mathbf{G}_{jk}(t)\,\mathbf{q}^{(j)} \tag{178}$$

where $\mathbf{A}_k(t)$ and $\mathbf{G}_{jk}(t)$ are defined as follows

$$\mathbf{A}_k(t) = \mathbf{A}(t) \otimes \mathbf{I}_n^{[k-1]} + \mathbf{I}_n \otimes \mathbf{A}(t) \otimes \mathbf{I}_n^{[k-2]} + \ldots + \mathbf{I}_n^{[k-1]} \otimes \mathbf{A}(t) =$$

$$= \mathbf{A}(t) \otimes \mathbf{I}_n^{[k-1]} + \mathbf{I}_n \otimes \mathbf{A}_{k-1}(t) \tag{179}$$

$$\mathbf{G}_{jk}(t) = \mathbf{Q}_k \left(\mathbf{Q}_{k-1} \otimes \mathbf{I}_n\right) \ldots \left(\mathbf{Q}_{k-j+1} \otimes \mathbf{I}_n^{[j-1]}\right) \left(\mathbf{m}_{k-j} \otimes \mathbf{I}_n^{[j]}\right) \mathbf{G}^{[j]}(t) \tag{180}$$

It is worth noting that equation (178) does not constitute an infinite hyerarchy and the moments of order k involved the strength of the delta-correlated process up to the k-th order.

8.2 Fokker-Planck equation, multidimensional case.

An extension of the Fokker-Planck equation for the multidimensional system of the differential equation can be easily obtained by setting $\phi(\mathbf{Z}(t), t) = \exp(-i\,\vartheta^T \mathbf{Z}(t))$, where ϑ is a vector of real parameters, inserting this expression in equation (155), taking the stochastic average and dividing by dt we obtain

$$\dot{M}_z(\vartheta, t) = \sum_{j=1}^{\infty} \frac{(-i)^j}{j!} \vartheta^{[j]^T} E\left[\exp(-i\,\vartheta^T \mathbf{Z}(t)\,(\Delta \mathbf{Z})^{[j]}\right] \tag{181}$$

taking an inverse Fourier transform we obtain

$$\frac{\partial p_z(z, t)}{\partial t} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \left[\nabla_z^{[j]^T} \left(p_z(z, t)\,(\Delta z)^{[j]}\right)\right] =$$

$$= -\nabla_z^T \left(p_z(z, t)\,\mathbf{a}(z, t)\right) - \sum_{l=1}^{\infty} \frac{1}{l!} \nabla_z^T \left(p_z(z, t)\,\mathbf{G}^{(l)}(z, t)\right) \mathbf{q}^{(l)} +$$

$$+ \frac{1}{2!} \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{l!\,r!} \nabla_z^{[2]^T} \left[p_z(z, t) \left(\mathbf{G}^{(l)}(z, t) \otimes \mathbf{G}^{(r)}(z, t)\right)\right] \mathbf{q}^{(l+r)} + \ldots \tag{182}$$

In the case of purely external loads, equation (182) reduces to

$$\frac{\partial p_z(z, t)}{\partial t} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \left\{\nabla_z^{[j]^T} \left[p_z(z, t)\,(dz)^{[j]}\right]\right\} =$$

$$= -\nabla_z^T \left[p_z(z, t)\,\mathbf{a}(z, t)\right] + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \left[\nabla_z^{[j]^T} p_z(z, t)\right] \mathbf{b}^{[j]}(t)\,\mathbf{q}^{(j)} \tag{183}$$

In the case of normal white noise input the Fokker-Planck equation can be simply obtained by neglecting all terms containing $\mathbf{q}^{(l)}$ with $l > 2$.

9 EXAMPLES.

In this section the results obtained in the previous sections are applied for a probabilistic descriptions of the response of linear and non linear systems subjected to delta-correlated input.

9.1 Linear SDOF.

Let Z_2 be a scalar real valued process satisfying the differential equation

$$dZ_2 = -B\, Z_2\, dt + \alpha\, dC\,; \quad Z_2(0) = 0 \quad \text{w.p. } 1 \tag{184}$$

where A and α are constant and $dC/dt = W(t)$ is a stationary delta-correlated input composed of train of Dirac's delta impulses $\delta(t - t_j)$ occurring at random times t_j. The random variable Y is assumed to be zero mean and having a symmetric distribution; the arrival rate $\lambda(t) = \lambda$ is kept constant.

Since the initial condition is assumed to be zero, the mean value and the odd moments of the response are always zero. The equations of moments up to fourth order, according to equation (175), are then given as:

$$\dot{E}\left[Z_2^2\right] = -2\,B\,E\left[Z_2^2\right] + \alpha^2\, q^{(2)} \tag{185a}$$

$$\dot{E}\left[Z_2^4\right] = -4\,B\,E\left[Z_2^4\right] + 6\,E\left[Z_2^2\right]\alpha^2\, q^{(2)} + \alpha^4\, q^{(4)} \tag{185b}$$

where

$$q^{(2)} = \lambda\, E\left[Y^2\right]; \quad q^{(4)} = \lambda\, E\left[Y^4\right] \tag{186}$$

The second moment coincides with the second cumulant because $E[Z_2] = 0$, and the fourth cumulant equation is simply given in the form:

$$\dot{k}_4[Z_2] = -4\,B\,k_4[Z_2] + \alpha^4\, q^{(4)} \tag{187}$$

from this equation we can observe that the fourth order cumulant is only related to the fourth order moment of the excitation. It can be easily seen that analogous results can be obtained for higher order cumulants. It follows that the equation for the 2p-th order cumulant is given in the form:

$$\dot{k}_{2p}[Z_2] = -2\,p\,B\,k_{2p}[Z_2] + \alpha^{2p}\,q^{(2p)} \tag{188}$$

The transient response for these cumulants is given as:

$$\frac{k_{2p}[Z_2]}{q^{(2p)}} = \frac{\alpha^{2p}}{2\,p\,B}(1 - \exp(-2\,p\,B\,t)) \tag{189}$$

As an example the transient excess coefficient $\gamma_e[Z_2] = k_4[Z_2] / k_2[Z_2]^2$ is given in the form:

$$\gamma_e[Z_2] = \frac{q^{(4)}}{q^{(2)^2}} B \frac{1 - \exp(-4\,B\,t)}{(1 - \exp(-2\,B\,t))^2} \tag{190}$$

9.2 Linear MDOF.

Let Z_2 be the response of the stochastic differential equation (184), representing the input on another differential equation. The complessive system will be given in the form:

$$\begin{aligned} dZ_1 &= -A\,Z_1\,dt + Z_2\,dt \\ dZ_2 &= -B\,Z_2\,dt + \alpha\,dC \end{aligned} \tag{191}$$

where $dC/dt = W(t)$ is the compound-Poisson process described in the Example 1. The analysis has been carried out up to the fourth order. The equation of second order (obtained by equation (175) particularized for $k = 2$) is:

$$\dot{E}\left[Z_1^2\right] = -2\,A\,E\left[Z_1^2\right] + 2\,E[Z_1\,Z_2] \tag{192a}$$

$$\dot{E}[Z_1\,Z_2] = -(A + B)\,E[Z_1\,Z_2] + E\left[Z_2^2\right] \tag{192b}$$

$$\dot{E}\left[Z_2^2\right] = -2\,B\,E\left[Z_2^2\right] + \alpha^2\,q^{(2)} \tag{192c}$$

The equation of fourth order moment (obtained by equation (175) particularized for $k = 4$) is:

$$\dot{E}\left[Z_1^4\right] = -4\,A\,E\left[Z_1^4\right] + 4\,E\left[Z_1^3\,Z_2\right] \tag{193a}$$

$$\dot{E}\left[Z_1^3\,Z_2\right] = -(3\,A+B)\,E\left[Z_1^3\,Z_2\right] + 3\,E\left[Z_1^2\,Z_2^2\right] \tag{193b}$$

$$\dot{E}\left[Z_1^2\,Z_2^2\right] = -2\,(A+B)\,E\left[Z_1^2\,Z_2^2\right] + 2\,E\left[Z_1\,Z_2^3\right] + E\left[Z_1^2\right]\alpha^2\,q^{(2)} \tag{193c}$$

$$\dot{E}\left[Z_1\,Z_2^3\right] = -(3\,B+A)\,E\left[Z_1\,Z_2^3\right] + E\left[Z_2^4\right] + 3\,E\left[Z_1\,Z_2\right]\alpha^2\,q^{(2)} \tag{193d}$$

$$\dot{E}\left[Z_2^4\right] = -4\,B\,E\left[Z_2^4\right] + 6\,E\left[Z_2^2\right]\alpha^2\,q^{(2)} + \alpha^4\,q^{(4)} \tag{193e}$$

The stationary solution, obtained by setting $\dot{E}\,[Z_j\;Z_k] = 0$, $\forall j,k = 1, 2$ and $\dot{E}\,[Z_j\,Z_k\,Z_r\,Z_s] = 0$, $\forall j,k,r,s = 1, 2$ gives for Z_1 and Z_2

$$E\left[Z_1^2\right] = \frac{\alpha^2\,q^{(2)}}{2\,A\,B\,(A+B)};\quad E\left[Z_2^2\right] = \frac{\alpha^2\,q^{(2)}}{2\,B} \tag{194}$$

$$E\left[Z_2^4\right] = 3E\left[Z_2^2\right]^2 + \frac{\alpha^4\,q^{(4)}}{4\,B};\quad E\left[Z_1^4\right] = 3E\left[Z_1^2\right]^2 + \frac{3}{4}\frac{\alpha^4\,q^{(4)}}{A\,B\,(A+B)\,(A+3B)\,(3A+B)} \tag{195}$$

it follows that the fourth cumulant of Z_1 and Z_2 are given in the form:

$$k_4[Z_2] = \frac{\alpha^4\,q^{(4)}}{4\,B};\quad k_4[Z_1] = \frac{3}{4}\frac{q^{(4)}}{A\,B\,(A+B)\,(A+3B)\,(3A+B)} \tag{196}$$

from these results one can observe that, even in the case of filtered processes, the fourth cumulant only depends of the fourth moment of the excitation, moreover the moments of a fixed order do not depend of the higher order terms.

9.3 Duffing oscillator.

Let the equation of motion be given in the form:

$$\ddot{X} + \nu\,\dot{X} + \left(X + \varepsilon\,X^3\right) = \sqrt{2\nu}\,W(t) \tag{197}$$

where ν is a damping factor, ε is a parameter representing the level of non-linearity and W(t) is a delta-correlated process; in particular W(t) is assumed to be the formal derivative of a compound Poisson process, in which the stochastic variables Y are uniformly distributed between -a and a. In this way, the intensities of the delta-correlated input are given by:

$$q^{(2r-1)} = 0; \quad q^{(2r)} = \lambda E\left[Y^{2r}\right] = \lambda \frac{a^{2r}}{2r+1}; \quad r = 1, 2 \ldots \tag{198}$$

If we set $a = \sqrt{3/\lambda}$, we are sure that the input process tends to a unit white noise when λ increases.
Introducing the state variables vector $\mathbf{Z}^T = [Z_1\ Z_2] = [X\ \dot{X}]$, equation (197) writes

$$\begin{aligned} dZ_1 &= \Delta Z_1 = Z_2 dt \\ dZ_2 &= \Delta Z_2 = \left(-\nu Z_2 - Z_1 - \varepsilon Z_1^3\right) dt + \sqrt{2\nu}\, dC(t) \end{aligned} \tag{199}$$

It is important to note that, since the input intensities of odd order are zeros and because of the antisymmetric characteristics of the non-linearity, the stationary odd moments of the response are zeros. So, if a six-th order cumulant neglect closure is applied, it is necessary to write the differential equations of the second, fourth and sixth order response moments only. These equations are obtained particularizing in a suitable way the vector function $\phi(Z, t)$ in eq. (171) and applying the stochastic average to both the members of equation (172). The second order moments obtained in this way are:

$$\dot{E}\left[Z_1^2\right] = 2E\left[Z_1 Z_2\right] \tag{200a}$$

$$\dot{E}\left[Z_1 Z_2\right] = E\left[Z_2^2\right] - \nu E\left[Z_1 Z_2\right] - E\left[Z_1^2\right] - \varepsilon E\left[Z_1^4\right] \tag{200b}$$

$$\dot{E}\left[Z_2^2\right] = -2\nu E\left[Z_2^2\right] - 2E\left[Z_1 Z_2\right] - 2\varepsilon E\left[Z_1^3 Z_2\right] + 2\nu q^{(2)} \tag{200c}$$

The fourth order moments are:

$$\dot{E}\left[Z_1^4\right] = 4E\left[Z_1^3 Z_2\right] \tag{201a}$$

$$\dot{E}\left[Z_1^3 Z_2\right] = 3E\left[Z_1^2 Z_2^2\right] - \nu E\left[Z_1^3 Z_2\right] - E\left[Z_1^4\right] - \varepsilon E\left[Z_1^6\right] \tag{201b}$$

$$\dot{E}\left[Z_1^2 Z_2^2\right] = 2E\left[Z_1 Z_2^3\right] - 2\nu\, E\left[Z_1^2 Z_2^2\right] - 2E\left[Z_1^3 Z_2\right] - 2\varepsilon\, E\left[Z_1^5 Z_2\right] + 2\nu\, q^{(2)} E\left[Z_1^2\right] \quad (201c)$$

$$\dot{E}\left[Z_1 Z_2^3\right] = E\left[Z_2^4\right] - 3\nu\, E\left[Z_1 Z_2^3\right] - 3E\left[Z_1^2 Z_2^2\right] - 3\varepsilon\, E\left[Z_1^4 Z_2^2\right] + 6\nu\, q^{(2)} E\left[Z_1 Z_2\right] \quad (201d)$$

$$\dot{E}\left[Z_2^4\right] = -4\nu\, E\left[Z_2^4\right] - 4E\left[Z_1 Z_2^3\right] - 4\varepsilon\, E\left[Z_1^3 Z_2^3\right] + 12\nu\, q^{(2)} E\left[Z_2^2\right] + 4\nu^2 q^{(4)} \quad (201e)$$

At last the sixth order moments are

$$\dot{E}\left[Z_1^6\right] = E\left[Z_1^5 Z_2\right] \quad (202a)$$

$$\dot{E}\left[Z_1^5 Z_2\right] = 5E\left[Z_1^4 Z_2^2\right] - \nu\, E\left[Z_1^5 Z_2\right] - E\left[Z_1^6\right] - \varepsilon\, E\left[Z_1^8\right] \quad (202b)$$

$$\dot{E}\left[Z_1^4 Z_2^2\right] = 4E\left[Z_1^3 Z_2^3\right] - 2\nu\, E\left[Z_1^4 Z_2^2\right] - 2E\left[Z_1^5 Z_2\right] - 2\varepsilon\, E\left[Z_1^7 Z_2\right] + 2\nu\, q^{(2)} E\left[Z_1^4\right] \quad (202c)$$

$$\dot{E}\left[Z_1^3 Z_2^3\right] = 3E\left[Z_1^2 Z_2^4\right] - 3\nu\, E\left[Z_1^4 Z_2^2\right] - 3\varepsilon\, E\left[Z_1^6 Z_2^2\right] + 6\nu\, q^{(2)} E\left[Z_1^3 Z_2\right] \quad (202d)$$

$$\dot{E}\left[Z_1^2 Z_2^4\right] = 2E\left[Z_1 Z_2^5\right] - 4\nu\, E\left[Z_1^2 Z_2^4\right] - 4E\left[Z_1^3 Z_2^3\right] - 4\varepsilon\, E\left[Z_1^5 Z_2^3\right] + 12\nu\, q^{(2)} E\left[Z_1^2 Z_2^2\right] + 4\nu^2 q^{(4)} E\left[Z_1^2\right] \quad (202e)$$

$$\dot{E}\left[Z_1 Z_2^5\right] = E\left[Z_2^6\right] - 5\nu\, E\left[Z_1 Z_2^5\right] - 5E\left[Z_1^2 Z_2^4\right] - 5\varepsilon\, E\left[Z_1^4 Z_2^4\right] + 20\nu\, q^{(2)} E\left[Z_1 Z_2^3\right] + 20\nu^2 q^{(4)} E\left[Z_1 Z_2\right] \quad (202f)$$

$$\dot{E}\left[Z_2^6\right] = -6\nu\, E\left[Z_2^6\right] - 6E\left[Z_1 Z_2^5\right] - 6\varepsilon\, E\left[Z_1^3 Z_2^5\right] + 30\nu\, q^{(2)} E\left[Z_2^4\right] + 60\nu^2 q^{(4)} E\left[Z_2^2\right] + 8\nu^3 q^{(6)} \quad (202g)$$

It is worth noting that, in the case of normal inputs, that is when $q^{(4)} = q^{(6)} = 0$, equations (200) and (201) coincide with those reported in [3].

For the stationary response, the first members of these equations are zeros and they reduce to algebraic equations in which the moments of greater order than six are replaced by the closure non-linear relationships (Ibrahim et al. [31]). In Fig. 7 the stationary mean square value of displacement is reported for increasing λ, compared with the exact solution in the case of normal white noise and with the results of Monte-Carlo simulations whose fundamentals are reported in

Appendix B. The parameters values choosen for the analysis are $\nu = 0.10$, $\varepsilon = 1.00$. It is clear that when λ is small, the two responses are quite different showing that the non-normality of the input strongly characterizes the response. This fact is confirmed in Fig. 8 where the kurtosis coefficients in the case of normal white noise and in the case of delta-correlated input are reported, varying λ. At last in Fig. 9 the velocity kurtosis coefficient is reported in the case of non-normal input (for normal input is zero). It is important to note that for the special case choosen the value of the stationary velocity mean square is always unitary.

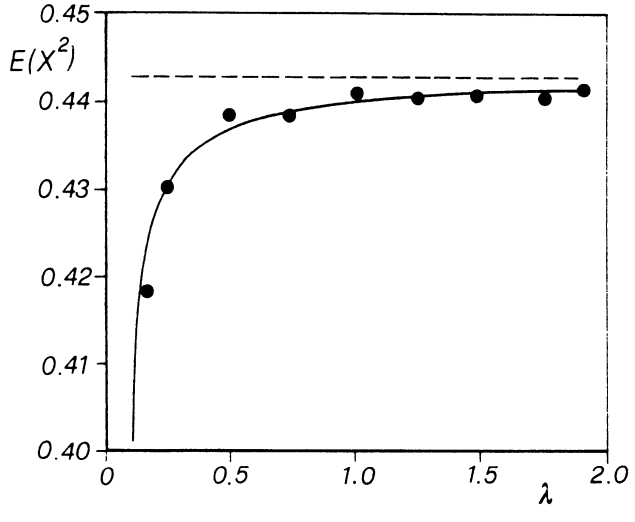

Fig. 7 - *Duffing oscillator: stationary displacement mean square value; solid line Poisson pulses input; dashed line: normal white noise input; ●: Monte-Carlo simulation for Poisson pulses.*

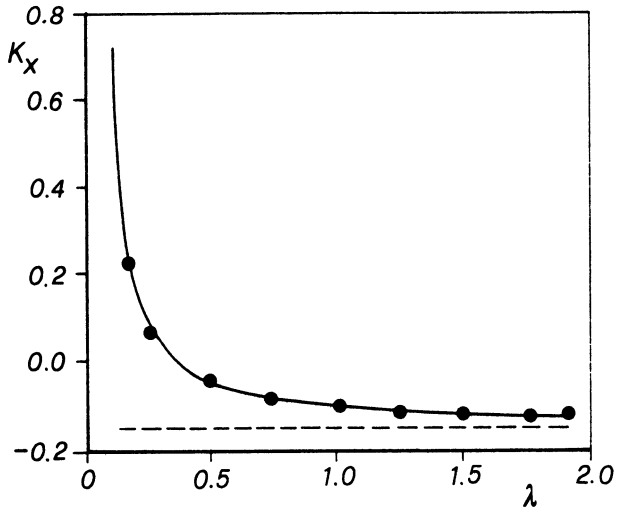

Fig. 8 - *Duffing oscillator: stationary displacement kurtosis: solid line: Poisson pulses input; dashed line: normal white noise input; ● : Monte-Carlo simulation for Poisson pulses.*

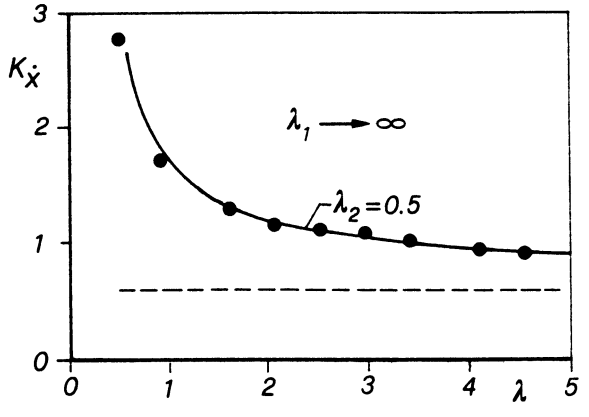

Fig. 9 - *Duffing oscillator; stationary velocity kurtosis; solid line: Poisson pulses input; dashed line: normal white noise input ●: Monte-Carlo simulation for Poisson pulses input.*

9.4 A non-linear oscillator under two parametric and one external excitations.

The last oscillator, choosen in this work, has been also studied in (Wu - Lin [32]) for normal input and it is governed by the following differential equation:

$$\ddot{X} + 2\alpha \dot{X}[1 + \nu_1 W_1(t)] + \beta \dot{X}\left(X^2 + \dot{X}^2/\Omega^2\right) + \Omega^2 X[1 + \nu_2 W_2(t)] = \nu_3 W_3(t) \quad (203)$$

Here $W_i(t)$, $i = 1, 2, 3$ are independent non-normal delta-correlated processes, whose characteristics are the same of those ones of the previous examples. It is easy to verify that the two parametric excitations $W_1(t)$ and $W_2(t)$ are such that the first one determines correction terms while the second one determines no correction term. So, in this case, differentials and increments of displacement $Z_1 = X$ and velocity $Z_2 = \dot{X}$ are different and the increments are given by:

$$\begin{aligned} \Delta Z_1 &= Z_2\, dt \\ \Delta Z_2 &= \left(-2\alpha Z_2 - \beta Z_1^2 Z_2 - \beta/\Omega^2 Z_2^3 - \Omega^2 Z_1\right) dt - 2\alpha Z_2 \nu_1\, dC_1(t) \\ &\quad - \Omega^2 Z_2 \nu_2\, dC_2(t) + \nu_3\, dC_3(t) + Z_2 \gamma_1\, dt \end{aligned} \quad (204)$$

where $\gamma_1 = (1/dt) \sum_{j=2}^{\infty} (-2\alpha\nu_1)^j [dC_1(t)]^j/j!$ and $Z_2\gamma_1$ represents the correction term: it is worth noting that the average of the first term of this summation coincide with the Wong-Zakai correction term. Even in this case the stationary odd moments are zeros, while the second moments are governed by the following equations:

$$\dot{E}\left[Z_1^2\right] = 2E[Z_1 Z_2] \quad (205a)$$

$$\dot{E}[Z_1 Z_2] = E\left[Z_2^2\right] + (E[\gamma_1] - 2\alpha) E[Z_1 Z_2] - \beta E\left[Z_1^3 Z_2\right] - \beta/\Omega^2 E\left[Z_1 Z_2^3\right] - \Omega^2 E\left[Z_1^2\right] \quad (205b)$$

$$\dot{E}\left[Z_2^2\right] = (2E[\gamma_1] - 4\alpha) E\left[Z_2^2\right] - 2\beta E\left[Z_1^2 Z_2^2\right] - 2\beta/\Omega^2 E\left[Z_2^4\right] - 2\Omega^2 E[Z_1 Z_2] + E[\gamma_2] E\left[Z_2^2\right] + \Omega^4 \nu_2^2 q_2^{(2)} E\left[Z_1^2\right] + \nu_3^2 q_3^{(2)} \quad (205c)$$

where $\gamma_i = \gamma_1^j$ and $q_j^{(i)}$, $j = 1, 2, 3$ is the strength of the j-th delta-correlated process W_j of order i. The fourth order moments are given by

$$\dot{E}\left[Z_1^4\right] = 4E\left[Z_1^3 Z_2\right] \tag{206a}$$

$$\dot{E}\left[Z_1^3 Z_2\right] = 3E\left[Z_1^2 Z_2^2\right] + \left(E[\gamma_1] - 2\alpha\right)E\left[Z_1^3 Z_2\right] - \beta\, E\left[Z_1^5 Z_2\right] - \beta/\Omega^2\, E\left[Z_1^3 Z_2^3\right] - \Omega^2\, E\left[Z_1^4\right] \tag{206b}$$

$$\dot{E}\left[Z_1^2 Z_2^2\right] = 2E\left[Z_1 Z_2^3\right] + \left(2E[\gamma_1] - 4\alpha\right)E\left[Z_1^2 Z_2^2\right] - 2\beta\, E\left[Z_1^4 Z_2^2\right] - 2\beta/\Omega^2\, E\left[Z_1^2 Z_2^4\right] - 2\Omega^2\, E\left[Z_1^3 Z_2\right] + E[\gamma_2]\, E\left[Z_1^2 Z_2^2\right] + \Omega^4 \nu_2^2 q_2^{(2)} E\left[Z_1^4\right] + \nu_3^2 q_3^{(2)} E\left[Z_1^2\right] \tag{206c}$$

$$\dot{E}\left[Z_1 Z_2^3\right] = E\left[Z_2^4\right] + \left(3E[\gamma_1] - 6\alpha\right)E\left[Z_1 Z_2^3\right] - 3\beta\, E\left[Z_1^3 Z_2^3\right] - 3\beta/\Omega^2\, E\left[Z_1 Z_2^5\right] - 3\Omega^2\, E\left[Z_1^2 Z_2^2\right] + 3E\left([\gamma_2] + E[\gamma_3]\right)E\left[Z_1 Z_2^3\right] + 3\Omega^4 \nu_2^2 q_2^{(2)} E\left[Z_1^3 Z_2\right] + 3\nu_3^2 q_3^{(2)} E\left[Z_1 Z_2\right] \tag{206d}$$

$$\dot{E}\left[Z_2^4\right] = \left(4E[\gamma_1] - 8\alpha\right)E\left[Z_2^4\right] - 4\beta\, E\left[Z_1^2 Z_2^4\right] - 4\beta/\Omega^2\, E\left[Z_2^6\right] - 4\Omega^2\, E\left[Z_1 Z_2^3\right] + \left(6E[\gamma_2] + 4E[\gamma_3] + E[\gamma_4]\right)E\left[Z_2^4\right] + 6\Omega^4 \nu_2^2 q_2^{(2)} E\left[Z_1^2 Z_2^2\right] + 6\nu_3^2 q_3^{(2)} E\left[Z_2^2\right] + \Omega^8 \nu_2^4 q_2^{(4)} E\left[Z_1^4\right] + \nu_3^4 q_3^{(4)} \tag{206e}$$

At last the sixth order moments are

$$\dot{E}\left[Z_1^6\right] = 4E\left[Z_1^5 Z_2\right] \tag{207a}$$

$$\dot{E}\left[Z_1^5 Z_2\right] = 5E\left[Z_1^4 Z_2^2\right] + \left(E[\gamma_1] - 2\alpha\right)E\left[Z_1^5 Z_2\right] - \beta\, E\left[Z_1^7 Z_2\right] - \beta/\Omega^2\, E\left[Z_1^5 Z_2^3\right] - \Omega^2\, E\left[Z_1^6\right] \tag{207b}$$

$$\dot{E}\left[Z_1^4 Z_2^2\right] = 4E\left[Z_1^3 Z_2^3\right] + \left(2E[\gamma_1] - 4\alpha\right)E\left[Z_1^4 Z_2^2\right] - 2\beta\, E\left[Z_1^6 Z_2^2\right] - 2\beta/\Omega^2\, E\left[Z_1^4 Z_2^2\right] - 2\Omega^2\, E\left[Z_1^5 Z_2\right] + E[\gamma_2]\, E\left[Z_1^4 Z_2^2\right] + \Omega^4 \nu_2^2 q_2^{(2)} E\left[Z_1^6\right] + \nu_3^2 q_3^{(2)} E\left[Z_1^4\right] \tag{207c}$$

$$\dot{E}\left[Z_1^3 Z_2^3\right] = 3E\left[Z_1^2 Z_2^4\right] + (3E[\gamma_1] - 6\alpha)E\left[Z_1^4 Z_2^2\right] - 3\beta E\left[Z_1^5 Z_2^3\right] - 3\beta/\Omega^2 E\left[Z_1^3 Z_2^5\right]$$
$$- 3\Omega^2 E\left[Z_1^4 Z_2^2\right] + 3E([\gamma_2] + E[\gamma_3])E\left[Z_1^3 Z_2^3\right] + 3\Omega^4 \nu_2^2 q_2^{(2)} E\left[Z_1^5 Z_2\right]$$
$$+ 3\nu_3^2 q_3^{(2)} E\left[Z_1^3 Z_2\right] \qquad (207d)$$

$$\dot{E}\left[Z_1^2 Z_2^4\right] = 2E\left[Z_1 Z_2^5\right] + (4E[\gamma_1] - 8\alpha)E\left[Z_1^2 Z_2^4\right] - 4\beta E\left[Z_1^4 Z_2^4\right] - \beta/\Omega^2 E\left[Z_1^2 Z_2^6\right]$$
$$- 4\Omega^2 E\left[Z_1^3 Z_2^3\right] + (6E[\gamma_2] + 4E[\gamma_3] + E[\gamma_4])E\left[Z_1^2 Z_2^4\right]$$
$$+ 6\Omega^2 \nu_2^2 q_2^{(2)} E\left[Z_1^4 Z_2^2\right] + 6\nu_3^2 q_3^{(2)} E\left[Z_1^2 Z_2^2\right] + \Omega^8 \nu_2^4 q_2^{(4)} E\left[Z_1^6\right]$$
$$+ \nu_3^4 q_3^{(4)} E\left[Z_1^2\right] \qquad (207e)$$

$$\dot{E}\left[Z_1 Z_2^5\right] = E\left[Z_2^6\right] + (5E[\gamma_1] - 10\alpha)E\left[Z_1 Z_2^5\right] - 5\beta E\left[Z_1^3 Z_2^5\right] - 5\beta/\Omega^2 E\left[Z_1 Z_2^7\right]$$
$$- 5\Omega^2 E\left[Z_1^2 Z_2^4\right] + (10E[\gamma_2] + 10E[\gamma_3] + 5E[\gamma_4] + E[\gamma_5])E\left[Z_1 Z_2^5\right]$$
$$+ 10\Omega^2 \nu_2^2 q_2^{(2)} E\left[Z_1^3 Z_2^3\right] + 10\nu_3^2 q_3^{(2)} E\left[Z_1 Z_2^3\right] + 5\Omega^8 \nu_2^4 q_2^{(4)} E\left[Z_1^5 Z_2\right]$$
$$+ 5\nu_3^2 q_3^{(4)} E[Z_1 Z_2] \qquad (207f)$$

$$\dot{E}\left[Z_2^6\right] = (6E[\gamma_1] - 12\alpha)E\left[Z_2^6\right] - 6\beta E\left[Z_1^2 Z_2^6\right] - 6\beta/\Omega^2 E\left[Z_2^8\right] - 6\Omega^2 E\left[Z_1 Z_2^5\right]$$
$$+ (15E[\gamma_2] + 20E[\gamma_3] + 15E[\gamma_4] + 6E[\gamma_5] + E[\gamma_6])E\left[Z_2^6\right]$$
$$+ 15\Omega^2 \nu_2^2 q_2^{(2)} E\left[Z_1^2 Z_2^4\right] + 15\nu_3^2 q_3^{(2)} E\left[Z_2^4\right] + 15\Omega^8 \nu_2^4 q_2^{(4)} E\left[Z_1^4 Z_2^2\right]$$
$$+ 15\nu_3^4 q_3^{(4)} E\left[Z_2^2\right] + \Omega^{16} \nu_2^6 q_2^{(6)} E\left[Z_1^6\right] + \nu_3^6 q_3^{(6)} \qquad (207g)$$

Even in this case, if the input is a normal white noise, equations (206) and (207) coincide with those ones given in (Wu and Lin [32]). The parameters choosen for the analysis are $\alpha = 0.50$, $\beta = 0.10$, $\Omega^2 = 2.00$, $\nu_1 = \nu_2 = \nu_3 = 0.10$.

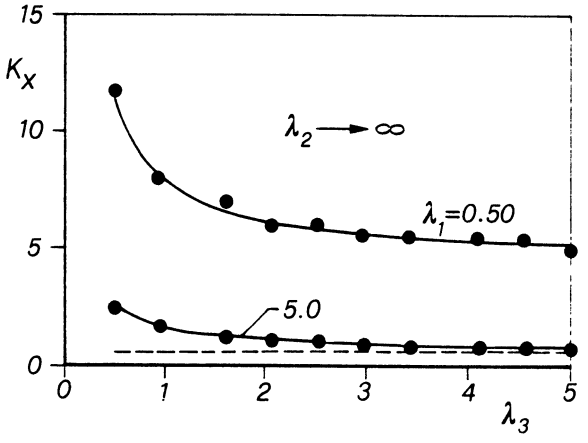

Fig. 10 - *Stationary displacement kurtosis for normal W_2: solid line: W_1, W_3 Poisson pulses; dashed line W_1, W_2: normal white noises; •: Monte-Carlo simulations for Poisson pulses ($\lambda_1 = 0.5$, $\lambda_1 = 5$).*

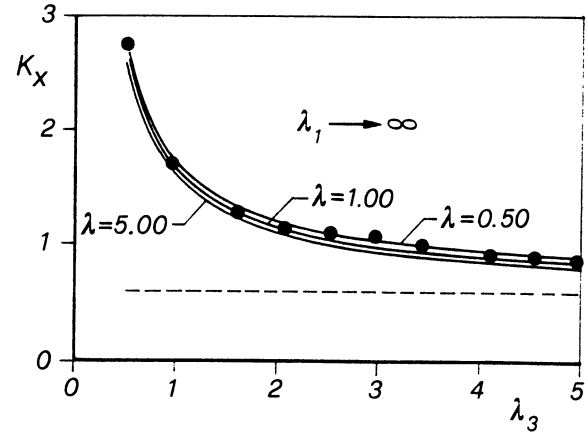

Fig. 11 - *Stationary displacement kurtosis for normal W_1: solid line: W_1, W_3 Poisson pulses; dashed line W_1, W_2: normal white noises; •: Monte-Carlo simulations for Poisson pulses ($\lambda_1 = 0.5$).*

In Fig. 10 the kurtosis coefficient of stationary displacement is reported for normal $W_2(t)$ and varying the non-normality degrees of $W_1(t)$ and $W_2(t)$, compared with that one obtained with normal input and with the results of some Monte-Carlo simulations. In Fig. 11 the same coefficient is reported for normal $W_1(t)$ and varying the non-normality degrees of the other two excitations. From these figures it is evident the great influence of the non-normality of the external excitation and of the parametric excitation $W_1(t)$ on the non-normality of the response. Moreover we can see the small influence of the parametric excitation $W_2(t)$, which does not determine corrective coefficients in the differential equations of motion.

Acknowledgements

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APPENDIX A.

In this section we shall deal with methods of Kronecker algebra in order to point out some properties which are useful in the formulation of the paper. Some of these properties are also reported in (Brewer [36]).
Let **A** and **B** be two given matrices of order (p x q) and (s x t) respectively. The Kronecker product of these two matrices, denoted by $\mathbf{A} \otimes \mathbf{B}$, is a matrix, of order (ps x qt), obtained by multiplying each element a_{ij} of **A** by the whole matrix **B**, i.e.

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots\cdots & a_{1q}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots\cdots & a_{2q}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1}\mathbf{B} & a_{p2}\mathbf{B} & \cdots\cdots & a_{pq}\mathbf{B} \end{bmatrix} \tag{A.1}$$

It is to be noted that the matrices **A** and **B** need not to be conformable to posses a Kronecker product so the above definition can be considered a generalization of matrix multiplication. The Kronecker product has the following properties:

$$\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} \tag{A.2}$$

$$(\mathbf{A} + \mathbf{B}) \otimes (\mathbf{C} + \mathbf{D}) = \mathbf{A} \otimes \mathbf{C} + \mathbf{A} \otimes \mathbf{D} + \mathbf{B} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{D} \tag{A.3}$$

$$(\mathbf{A} \otimes \mathbf{B})\,(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{A}\,\mathbf{C}) \otimes (\mathbf{B}\,\mathbf{D}) \tag{A.4}$$

$$(\mathbf{A} \otimes \mathbf{B})^{T} = \mathbf{A}^{T} \otimes \mathbf{B}^{T} \tag{A.5}$$

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \tag{A.6}$$

where **C** and **D** are two other given matrices. Equations (A.4) and (A.6) are valid, provided, the various quantities exist.
Moreover, it can be shown that if the matrix **D** is given by

$$\mathbf{D} = \mathbf{A}\,\mathbf{B}\,\mathbf{C} \tag{A.7}$$

then the following relationship holds

$$\text{Vec}(\mathbf{D}) = (\mathbf{C}^T \otimes \mathbf{A})\,\text{Vec}(\mathbf{B}) \tag{A.8}$$

where the symbol Vec (·) means vectorialized form of (·), i.e. it is a column vector formed by all columns of matrix (·) written the one below the other.
The Kronecker powers of a matrix **A** are defined recursively as

$$\begin{aligned} \mathbf{A}^{[1]} &= \mathbf{A} \\ \mathbf{A}^{[k+1]} &= \mathbf{A} \otimes \mathbf{A}^{[k]} \end{aligned} \tag{A.9}$$

They have the following properties

$$\mathbf{A}^{[k+j]} = \mathbf{A}^{[k]} \otimes \mathbf{A}^{[j]} \tag{A.10}$$

$$(\mathbf{A}\,\mathbf{B})^{[k]} = \mathbf{A}^{[k]}\,\mathbf{B}^{[k]} \tag{A.11}$$

Let $\mathbf{E}_{k,j}$ denote the permutation matrix of order (kj x kj), consisting of j x k arrays of elementary submatrices $\mathbf{E}^{mn}$ of order (k x j)

$$\mathbf{E}_{k,j} = \begin{bmatrix} \mathbf{E}^{11} & \mathbf{E}^{21} & \cdots\cdots & \mathbf{E}^{k1} \\ \mathbf{E}^{21} & \mathbf{E}^{22} & \cdots\cdots & \mathbf{E}^{k2} \\ \mathbf{E}^{1j} & \mathbf{E}^{2j} & \cdots\cdots & \mathbf{E}^{kj} \end{bmatrix} \tag{A.12}$$

where the matrix $\mathbf{E}^{mn}$ has the value one in the (m, n)th position and zero in all other positions. Then it can be shown that, for the matrices **A** and **B** previously considered, the following fundamental relationship hold:

$$\mathbf{B} \otimes \mathbf{A} = \mathbf{E}_{p,s}\,(\mathbf{A} \otimes \mathbf{B})\,\mathbf{E}_{t,q} \tag{A.13}$$

The permutation matrices have the following properties

$$\mathbf{E}_{k,1} = \mathbf{E}_{1,k} = \mathbf{I}_k \tag{A.14}$$

$$\mathbf{E}^{T}_{k,j} = \mathbf{E}^{-1}_{k,j} = \mathbf{E}_{j,k} \tag{A.15}$$

$\mathbf{I}_k$ being the identity matrix of order k.

If $\mathbf{a}_u$ and $\mathbf{b}_v$ are vectors of length u and v, respectively, particularizing equation (A.13) and taking into account equation (A.14) it can easily be shown that

$$\mathbf{B} \otimes \mathbf{a}_u = \mathbf{E}_{u,s} (\mathbf{a}_u \otimes \mathbf{B}) \tag{A.16}$$

$$\left(\mathbf{B} \otimes \mathbf{a}_u^T\right) = \left(\mathbf{a}_u^T \otimes \mathbf{B}\right) \mathbf{E}_{u,t} \tag{A.17}$$

$$\mathbf{b}_v \otimes \mathbf{a}_u = \mathbf{E}_{u,v} (\mathbf{a}_u \otimes \mathbf{b}_v) \tag{A.18}$$

$$\mathbf{b}_v \otimes \mathbf{a}_u^T = \mathbf{a}_u^T \otimes \mathbf{b}_v \tag{A.19}$$

Kronecker algebra can be advantageously applied in the differential calculus of matrices. The derivative of a matrix **A** of order (p x q), with respect to a matrix **X**, of order (j x k), is a matrix, of order (pj x qk), given by

$$\nabla_x \otimes \mathbf{A} = \begin{bmatrix} \frac{\partial}{\partial X_{11}}\mathbf{A} & \frac{\partial}{\partial X_{12}}\mathbf{A} & \cdots\cdots & \frac{\partial}{\partial X_{1k}}\mathbf{A} \\ \frac{\partial}{\partial X_{21}}\mathbf{A} & \frac{\partial}{\partial X_{22}}\mathbf{A} & \cdots\cdots & \frac{\partial}{\partial X_{2k}}\mathbf{A} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial X_{j1}}\mathbf{A} & \frac{\partial}{\partial X_{j2}}\mathbf{A} & \cdots\cdots & \frac{\partial}{\partial X_{jk}}\mathbf{A} \end{bmatrix} \tag{A.20}$$

X_{mn} being a general term of the matrix **X** while ∇_x is the differential operator containing the partial derivatives with respect to X_{mn}. If the operator ∇_x is applied to $\mathbf{X}^T$ and to **X**, we obtain

$$\nabla_x \otimes \mathbf{X}^T = \mathbf{E}_{k,j} \tag{A.21}$$

$$\nabla_x \otimes \mathbf{X} = \left(\overline{\mathbf{E}}_{k,j}\right)^T \tag{A.22}$$

where $\overline{\mathbf{E}}_{k,j}$ is a matrix of order (k^2 x j^2) built in the following manner:

$$\overline{\mathbf{E}}_{k,\,j} = \begin{bmatrix} \mathbf{E}^{11} & \mathbf{E}^{12} & \ldots\ldots & \mathbf{E}^{1j} \\ \mathbf{E}^{21} & \mathbf{E}^{22} & \ldots\ldots & \mathbf{E}^{2j} \\ \mathbf{E}^{k1} & \mathbf{E}^{k2} & \ldots\ldots & \mathbf{E}^{kj} \end{bmatrix} \tag{A.23}$$

$\mathbf{E}^{mn}$ being the elementary submatrices of order (k x j) previously defined. It can be noted that $\overline{\mathbf{E}}_{k,\,j}$ can be built from $\mathbf{E}_{k,\,j}$ transponing elementary submatrix $\mathbf{E}^{mn}$ of $\mathbf{E}_{k,\,j}$, but it is not the transpose of $\mathbf{E}_{k,\,j}$. This matrix has the following properties

$$\overline{\mathbf{E}}^{T}_{k,\,j} = \overline{\mathbf{E}}_{j,\,k} \tag{A.24}$$

$$\left(\mathbf{I}_u \otimes \mathbf{a}_u^T\right)\overline{\mathbf{E}}_{u,\,v}\left(\mathbf{I}_v \otimes \mathbf{b}_v\right) = \mathbf{b}_v^T \otimes \mathbf{a}_u \tag{A.25}$$

where $\mathbf{a}_u$ and $\mathbf{b}_v$ are the vectors previously defined.
If **A**, **B** and **C** are three given matrices such that the products **A B** and **A B C** exist, then

$$\nabla_x \otimes (\mathbf{A}\,\mathbf{B}) = \left(\nabla_x \otimes \mathbf{A}\right)\left(\mathbf{I}_k \otimes \mathbf{B}\right) + \left(\mathbf{I}_j \otimes \mathbf{A}\right)\left(\nabla_x \otimes \mathbf{B}\right) \tag{A.26}$$

$$\nabla_x \otimes (\mathbf{A}\,\mathbf{B}\,\mathbf{C}) = \left(\nabla_x \otimes \mathbf{A}\right)\left(\mathbf{I}_k \otimes \mathbf{B}\,\mathbf{C}\right) + \left(\mathbf{I}_j \otimes \mathbf{A}\right)\left(\nabla_x \otimes \mathbf{B}\right)\left(\mathbf{I}_k \otimes \mathbf{C}\right) + \left(\mathbf{I}_j \otimes \mathbf{A}\,\mathbf{B}\right)\left(\nabla_x \otimes \mathbf{C}\right) \tag{A.27}$$

while if **A** is a matrix of order (p x q) and **B** is a matrix of order (s x t), then

$$\nabla_x \otimes (\mathbf{A} \otimes \mathbf{B}) = \left(\nabla_x \otimes \mathbf{A}\right) \otimes \mathbf{B} + \left(\mathbf{I}_j \otimes \mathbf{E}_{s,\,p}\right)\left[\left(\nabla_x \otimes \mathbf{B}\right) \otimes \mathbf{A}\right]\left(\mathbf{I}_k \otimes \mathbf{E}_{q,\,t}\right) \tag{A.28}$$

Lastly, for the vector $\mathbf{a}_u$ and $\mathbf{b}_v$, it is

$$\nabla_x \otimes (\mathbf{a}_u \otimes \mathbf{b}_v) = \left(\nabla_x \otimes \mathbf{a}_u\right) \otimes \mathbf{b}_v + \left(\mathbf{I}_k \otimes \mathbf{E}_{v,\,u}\right)\left[\left(\nabla_x \otimes \mathbf{b}_v\right) \otimes \mathbf{a}_u\right] \tag{A.29}$$

If **Z** is a vector of lenght n, it is of particular interest to evaluate the derivatives of the Kronecker powers of **Z** with respect to **Z** itself; particularizing equation (A.29) for this case and using the properties of the Kronecker products previously mentioned, we obtain

$$\nabla_z^T \otimes \left(\mathbf{Z}^{[k]}\right) = \mathbf{Q}_k\left(\mathbf{Z}^{[k-1]} \otimes \mathbf{I}_n\right) = \mathbf{Z}^{[k-1]} \otimes \mathbf{I}_n + \mathbf{Z}^{[k-2]} \otimes \mathbf{I}_n + \ldots + \mathbf{I}_n \otimes \mathbf{Z}^{[k-1]} \tag{A.30}$$

where $\mathbf{Q}_k$ is a matrix of order $(n^k \times n^k)$ given by

$$\mathbf{Q}_k = \sum_{i=0}^{k-1} \mathbf{E}_{n^{k-i},\, n^i} \tag{A.31}$$

It is important to note that the sum of the elements of $\mathbf{Q}_k$ in each row is always equal to k and that $\mathbf{Q}_k$ has the following property

$$\mathbf{Q}_k \, \mathbf{c}^{[k]} = k \, \mathbf{c}^{[k]} \tag{A.32}$$

for any vector **c**. Lastly, if the second order derivatives of the Kronecker powers of **Z** are required, then it is possible to apply the differential operator $\nabla_z^{T[2]}$ to the vector $\mathbf{Z}^{[k]}$, thus obtaining:

$$\nabla_z^{T[2]} \otimes \mathbf{Z}^{[k]} = \mathbf{Q}_k \left(\mathbf{Q}_{k-1} \otimes \mathbf{I}_n\right)\left(\mathbf{Z}^{[k-2]} \otimes \mathbf{I}_n^{[2]}\right) \tag{A.33}$$

It is worth noting that the vector $E[\mathbf{Z}^{[k]}]$ contains all possible moments of k-th order. Such an example if $\mathbf{Z}^T = [Z_1 \; Z_2]$

$$E\left[\mathbf{Z}^{[2]}\right] = \begin{bmatrix} E\left[Z_1^2\right] \\ E\left[Z_1 Z_2\right] \\ E\left[Z_2 Z_1\right] \\ E\left[Z_2^2\right] \end{bmatrix} \tag{A.34}$$

APPENDIX B.

In this appendix the fundamentals of the arranged Runge-Khutta method in order to perform the integral in (S) sense of the single sample, are reported. At this purpose, it is important to note that, among the Runge-Khutta methods of various orders, when parametric Dirac's delta are present, the simplest one which better agrees with the (S) interpretation, is the second order one. Given a differential equation $\dot{z}(t) = f(t, z(t))$, the second order Runge-Khutta method gives the solution $z(t_i)$ at the instant t_i as:

$$z(t_i) = z(t_{i-1}) + a_1 s_1 + a_2 s_2 \tag{B.1}$$

where the slopes s_1 and s_2 are given by:

$$s_1 = \Delta t_i \, f\left(t_{i-1}, z(t_{i-1})\right); \quad s_2 = \Delta t_i \, f\left(t_{i-1} + \alpha \Delta t_i, z(t_{i-1}) + \beta s_1\right) \tag{B.2}$$

in which Δt_i is the temporal step choosen for the integration and the coefficients a_1, a_2, α and β are such that the following relationships must hold:

$$a_1 + a_2 = 1; \quad a_2 \alpha = \frac{1}{2}; \quad a_2 \beta = \frac{1}{2} \tag{B.3}$$

Following the (S) interpretation, it is obvious that the coefficents a_1 and a_2 must average the two contributions in the same way; so that the coefficients must have the following values:

$$a_1 = a_2 = \frac{1}{2}; \quad \alpha = \beta = 1 \tag{B.4}$$

As it can be seen froom equations (B.1) and (B.2), the second order Runge-Khutta method gives the response value $z(t_i)$, adding two quantities to the response $z(t_{i-1})$; the first one (s_1) depends on the value of $f(\cdot)$ in t_{i-1} and it is weighted by the coefficient a_1; the second one (s_2), if we set $\alpha = \beta = 1$, depends on the value of $f(\cdot)$ in t_i and it is weighted by the coefficients a_2.
Considering the integration of the single sample, if in t_k there is the presence of a parametric Dirac's delta of intensity Y_k, it is modelled as in Fig. 12, and in the time steps $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$ the second order Runge-Khutta method is used; in all the other time steps in which no Dirac's delta is present, an higher order Runge-Khutta method can be used. It is worth noting that the relationships (B.2) with the coefficients given in (B.4) are exactly conform with the (S) integral only in the quasi-linear case; while, in the case of non-linear parametric excitation, it gives an approximate result which tends to the correct one when the time step for the two intervals $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$ goes to zero. In the numerical example here reported a time step of 10^{-4} sec has been used.
Moreover, it is important to note that, if one want to perform the integration in (I) sense, when parametric Dirac's delta are present, the simplest way is using, (in the steps $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$) the first order Runge-Khutta method (i.e. the Euler method), for which:

$$z(t_i) = z(t_{i-1}) + a_1 s_1; \quad a_1 = 1 \tag{B.5}$$

That is, the response $z(t_i)$ is obtained adding the slope evaluated in t_{i-1} to the response $z(t_{i-1})$.

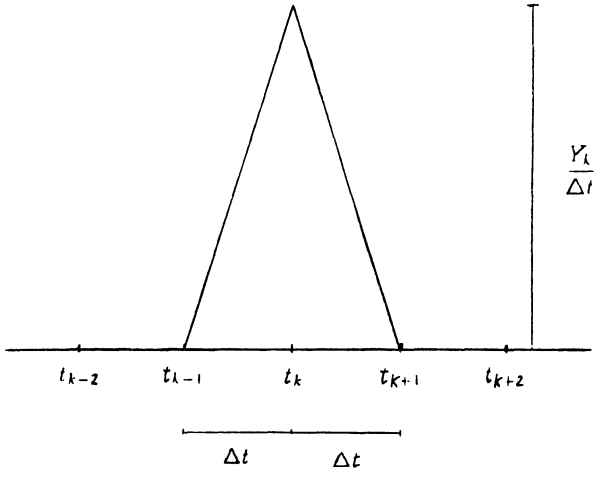

Fig. 12 - *Idealization of Dirac's delta impulse for a numerical integration scheme.*

Chapter 3

APPROXIMATE SOLUTION OF THE FOKKER-PLANCK-KOLMOGOROV EQUATION FOR DYNAMICAL SYSTEMS

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3.1) INTRODUCTION

In this chapter, the application of the Fokker-Planck-Kolmogorov (simply denoted by FPK in the sequel), to the solution of the equation of the motion of a Single-Degree-Of-Freedom (SDOF) system in the linear or non-linear range is approached. The introduction of the FPK equation as the relation governing the evolution of the instantaneous Joint-Probability-Density-Function (JPDF), identified with the transition probability function of a uni-dimensional Markov process, has been given in Chapter 2, in a very general context.
Here, the problem of the stochastic dynamics of a SDOF system will be specifically treated. Therefore, systematic reference will be made to a differential system of the type

$$\ddot{u} + g(u,\dot{u};t) = f(t) \qquad (1)$$

with initial conditions

$$u(0) = u_0$$

$$\dot{u}(0) = \dot{u}_0 \qquad (2)$$

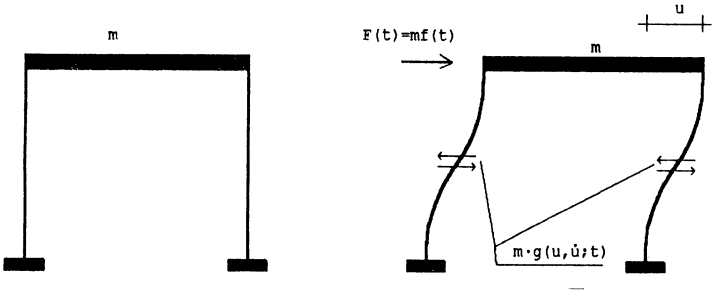

Figure 1: The model structure for a SDOF system

where u(t) denotes the displacement of the structure, superimposed dots denote time-derivative, g(u,u̇;t) is the law -possibly non linear both

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in u and û- governing the restoring force, and f(t) is the forcing function (Fig.1)

As already outlined in the previous chapters, many problems encountered in engineering practice can be efficiently treated by modeling the active force f(t) by a stochastic process.
A rather comprehensive model for f(t), able to represent, at least as a first approximation, a large number of situations in which engineers have practical interest, is a class of δ-correlated processes (see Chap.2, Sec.3) that in the simplest case are identified with zero-mean shot-noises; i.e. processes with mean value and autocorrelation function expressed by

$$E[f(t)] = 0 \tag{3}$$

$$E[f(t)f(t+\tau)] = I(t)\,\delta(\tau) \tag{4}$$

where $\delta(\bullet)$ denotes the impulsive Dirac's distribution and I(t) is the intensity function, in general allowed to vary with time.
The above relation (4) means that the process' structure is made in way that the ordinates at any two different times are uncorrelated; anyway, in general may be they remain statistically dependent. By adding the assumption that the process is Normal (i.e. the JPDF of the function values at any n times is expressed in the Gaussian form) full statistical independence is also implied by eq. (4). This means that no information can be expected on the value of the function at any point t from the knowledge of the function itself in any neighborhood of the same instant t.

A sample realization of such a process is plotted for illustration in Fig.2, with I(t) in the form

$$I(t) = I_0\, t\, \exp(-\alpha t^2) \tag{5}$$

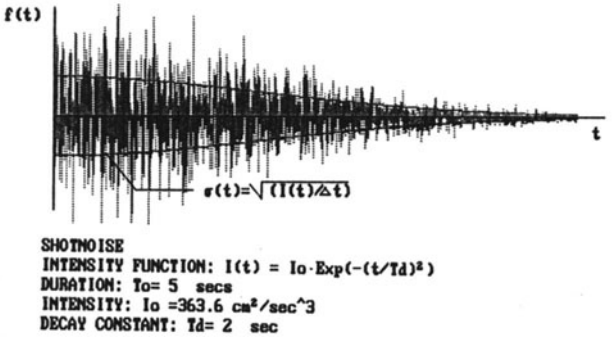

Figure 2: Sample realization of a shot noise

The shot-noise process is, in general, not stationary, unless the intensity function I(t) is independent of time and coincides with a constant value I_0. In this case, the process is named white-noise, in that it possesses a constant power spectral density, and all frequencies yield the same contribution to the total power of the process.

Under the above conditions, it is possible to understand that the conditional state of the system $\mathbf{z}(t) = [u(t) \ \dot{u}(t)]$ at time t, given the state $\mathbf{z}(\theta) = [u(\theta) \ \dot{u}(\theta)]$ at any time $\theta<t$ is physically (and therefore also statistically) independent of the state at any time prior to θ. This statement can be easily proved for a linear system (i.e. with a restoring force linearly depending on u and $\dot{u}$). In fact, in this case, the motion of the system in (θ,t) is composed by the free oscillations $z_0(\tau)$, dependent only on the initial condition $\mathbf{z}(\theta)=\mathbf{z}_0(\theta)$ and on the forced motion $z_f(\tau)$, that is not influenced by the state at times $t\leq\theta$, in that $z_f(\theta)=0$, and the active force $f(\tau)$ in (θ,t) has not any continuity link with the realization of the force for $t<\theta$. If the system is not linear, the response at time $t>\theta$ is obtained by solving eq.(1) starting from the initial state $\mathbf{z}(\theta)$. Since $f(\tau)$ in (θ,t) is independent of the values previously attained in $(0,\theta)$, it is clear that $\mathbf{z}(\tau)$ in (θ,t) has no connection with $\mathbf{z}(\tau)$ in $(0,\theta)$ other than the starting point $\mathbf{z}(\theta)$.
After the Markovian character of the state vector $\mathbf{z}(t)$ has been recognized, one concludes that the main tool for its stochastic description is the transition probability function $p_Z(\mathbf{z};t|\mathbf{z}_\theta;\theta)$, i.e. the function yielding the probability that the state of the system at time t is $\mathbf{z}$ given that its state at time θ has been observed to be $\mathbf{z}_\theta$. If the initial state $\mathbf{z}_0$ of the system is deterministically known at time t=0, then $p_Z(\mathbf{z},t) = p_Z(\mathbf{z};t|\mathbf{z}_0;0)$ is the JPDF of the components of the state of the system (i.e. the displacement u and the velocity $\dot{u}$) at time t, and this function should obey the FPK equation given by eq. (185) of Chap.2 in its general expression. In order to specify the FPK equation for the second-order dynamical system treated in this chapter, let write equation (1) under the form of the state variables. Since

$$\mathbf{z} = \begin{bmatrix} u \\ \dot{u} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{6}$$

the equation of the motion can be written in the form of eq. (155) of chap. 2

$$\dot{\mathbf{z}}(t) = \mathbf{a}[\mathbf{z}(t);t] + \mathbf{b}(t)\,\mathbf{W}(t) \tag{7}$$

where, for the case at hand, $\mathbf{a}[\mathbf{z};t]$ and $\mathbf{b}(t)$ are two-component vectors, and the load vector $\mathbf{W}(t)$ is a one-component vector (or, more simply, a scalar valued function). All these three entities are given by

$$\mathbf{a}[\mathbf{z};t] = \begin{bmatrix} z_2 \\ -g(z_1,z_2;t) \end{bmatrix} \tag{8}$$

$$\mathbf{b}(t) = \begin{bmatrix} 0 \\ \hline \sqrt{I(t)} \end{bmatrix} \tag{9}$$

$$\mathbf{W}(t) = \begin{bmatrix} W_1(t) \end{bmatrix} \tag{10}$$

where $W_1(t)$ is a zero-mean Gaussian white noise, that after modulation by the matrix $\mathbf{b}(t)$ is transformed in a non-stationary δ-correlated process of the shot-noise type.
From equations (38) and (39) of Chap.2, one can write

$$\begin{aligned} \mathbf{q}^{[k]} &= [0] \text{ for any } k \neq 2 \\ \mathbf{q}^{[2]} &= [1] \end{aligned} \tag{11}$$

Moreover the gradient operator in eq (185) can be written

$$\mathbf{V}_z = \begin{bmatrix} \dfrac{\delta}{\delta z_1} \\ \dfrac{\delta}{\delta z_2} \end{bmatrix} \tag{12}$$

By applying the rule for the Kronecker's power of matrices (see Appendix A to Chap.2), one gets the K-squares of $\mathbf{V}_z$ and $\mathbf{b}(t)$ in the transposed form (the exponent index "T" denotes transposition)

$$\left[\mathbf{V}_z^{[2]} \right]^T = \begin{bmatrix} \dfrac{\delta^2}{\delta z_1^2} & \dfrac{\delta^2}{\delta z_1 \delta z_2} & \dfrac{\delta^2}{\delta z_2 \delta z_1} & \dfrac{\delta^2}{\delta z_2^2} \end{bmatrix} \tag{13}$$

$$\left[\mathbf{b}^{[2]} \right]^T = \begin{bmatrix} 0 & 0 & 0 & I(t) \end{bmatrix} \tag{14}$$

After the above specifications, the first term on the right-hand side of eq.(185) in Chap.2 can be explicited

$$-\mathbf{V}_z^T[p_z(\mathbf{z};t)\ \mathbf{a}(\mathbf{z};t)] = \frac{\delta}{\delta z_2}[p_z(\mathbf{z};t)g(z_1,z_2;t)] - z_2 \frac{\delta p_z}{\delta z_1} \tag{15}$$

while the second term on the r.h. side, remembering eqs. (10)-(11), is reduced to

$$\frac{1}{2} \left[\mathbf{V}_z^{[2]} \right]^T \mathbf{b}^{[2]}(t)\ \mathbf{q}^{(2)} = \tag{16}$$

$$= \frac{1}{2} \begin{bmatrix} \frac{\delta^2 p_z}{\delta z_1^2} & \frac{\delta^2 p_z}{\delta z_1 \delta z_2} & \frac{\delta^2 p_z}{\delta z_2 \delta z_1} & \frac{\delta^2 p_z}{\delta z_2^2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ I(t) \end{bmatrix}$$

whence

$$\frac{1}{2} \left[\mathbf{V}_z[2] \right]^T \mathbf{b}^{[2]}(t)\, \mathbf{q}^{(2)} = \frac{1}{2} I(t) \frac{\delta^2 p_z}{\delta z_2^2} \qquad (17)$$

and the FPK equation takes the final form for a second-order dynamical system acted on by a shot-noise Gaussian force

$$\frac{\delta p(u,\dot{u};t)}{\delta t} =$$

$$\frac{\delta}{\delta \dot{u}} [p(u,\dot{u};t)\, g(u,\dot{u};t)] - \dot{u} \frac{\delta p}{\delta u} + \frac{1}{2} I(t) \frac{\delta^2 p}{\delta \dot{u}^2} \qquad (18)$$

where the components of the state vector $\mathbf{z}(t)$ have been substituted by the symbols used for the displacement and velocity $u(t)$ and $\dot{u}(t)$, and $p(u,\dot{u};t)$ has been written in place of $p_z(\mathbf{z};t)$.

3.2) THE STATIONARY SOLUTION OF THE FPK EQUATION FOR DYNAMICAL SYSTEMS

Unfortunately, no closed-form solutions for the FPK equation in the general case can be found. A technique for approximate solutions will be discussed in this and in other chapters of the present book. Numerical and finite-element type procedures will be discussed in Chap.4, showing very interesting features of the stochastic response of non-linear systems to shot-noise disturbances.

Anyway, when the excitation is stationary (i.e. it is a Gaussian white noise) and the restoring force is time-independent, in some instances at least the stationary component of the solution can be expressed in closed form. If the transient response is judged to be not significant for the scope of the analysis, the result may be considered exhaustive.

In this case, $I(t)=\text{const.} = I_0$, and $p(u,\dot{u})$ -the JPDF of $(u,\dot{u})$ in the stationary motion- and $g(u,\dot{u};t) = g(u,\dot{u})$ do not depend explicitly on time t. So, eq. (18) can be written for $f(u,\dot{u})$ in the form

$$\dot{u} \frac{\delta p}{\delta u} - \frac{\delta}{\delta \dot{u}} [p(u,\dot{u})\, g(u,\dot{u})] - \frac{1}{2} I_0 \frac{\delta^2 p}{\delta \dot{u}^2} = 0 \qquad (19)$$

Let consider the simplest case, and assume that the system is linear, i.e. that

$$g(u,\dot{u}) = 2\alpha\Omega_0\dot{u} + \Omega_\beta^2 u \tag{20}$$

Let try a Gaussian expression for $p(u,\dot{u})$, with μ_u, $\mu_{\dot{u}}$ the expected values of displacement and velocity, σ_u^2 and $\sigma_{\dot{u}}^2$ the respective variances and $\rho_{u\dot{u}}$ the coefficient of linear correlation. Because of stationarity all these five quantities are assumed to be constant in time.

If $p(u,\dot{u})$ is a Gaussian distribution, its derivatives with respect to $u,\dot{u}$ are given by

$$\frac{\delta p}{\delta u} = B(u,\dot{u})\, p(u,\dot{u}); \quad \frac{\delta p}{\delta \dot{u}} = A(u,\dot{u})\, p(u,\dot{u})$$

$$\frac{\delta^2 p}{\delta \dot{u}^2} = C(u,\dot{u})\, p(u,\dot{u}) \tag{21}$$

where

$$A(u,\dot{u}) = A_0 + A_1 u + A_2 \dot{u}$$

$$B(u,\dot{u}) = B_0 + B_1 u + B_2 \dot{u} \tag{22}$$

$$C(u,\dot{u}) = C_0 + A^2(u,\dot{u})$$

and

$$A_0 = -\frac{\sigma_{\dot{u}}\rho_{u\dot{u}}\mu_u - \sigma_u\mu_{\dot{u}}}{\sigma_{\dot{u}}^2\sigma_u(1-\rho_{u\dot{u}}^2)} \qquad A_1 = \frac{\rho_{u\dot{u}}}{\sigma_{\dot{u}}\sigma_u(1-\rho_{u\dot{u}}^2)} \qquad A_2 = -\frac{1}{\sigma_{\dot{u}}^2(1-\rho_{u\dot{u}}^2)}$$

$$B_0 = -\frac{\sigma_u\rho_{u\dot{u}}\mu_{\dot{u}} - \sigma_{\dot{u}}\mu_u}{\sigma_u^2\sigma_{\dot{u}}(1-\rho_{u\dot{u}}^2)} \qquad B_1 = -\frac{1}{\sigma_u^2(1-\rho_{u\dot{u}}^2)} \qquad A_2 = \frac{\rho_{u\dot{u}}}{\sigma_u\sigma_{\dot{u}}(1-\rho_{u\dot{u}}^2)}$$

$$C_0 = -\frac{1}{\sigma_{\dot{u}}^2(1-\rho_{u\dot{u}}^2)} \tag{23}$$

with

$$A_1 = B_2 \;\; ; \;\; A_2 = C_0 \tag{24}$$

By substituting into the stationary FPK [eq.(19)], one gets

$$(B_0+B_1u+B_2\dot{u})\dot{u} - (A_0+A_1u+A_2\dot{u})(2\alpha\Omega_0\dot{u} + \Omega_0^2u) - 2\alpha\Omega_0 +$$
$$- \tfrac{1}{2} I_0 [C_0+A_0^2+A_1^2u^2+A_2^2\dot{u}^2+2A_0A_1u+2A_0A_2\dot{u}+2A_1A_2u\dot{u}]=0 \qquad (25)$$

Since the above equation must equal zero for any values of u and $\dot{u}$, the coefficients of any product $u^n\dot{u}^m$ must be equated to 0. So one can write the following system

$$\begin{aligned}
&A_1\Omega_0^2 + \tfrac{1}{2}I_0A_1^2 = 0\\
&B_2 - 2\alpha\Omega_0A_2 - \tfrac{1}{2}I_0A_2^2 = 0\\
&B_1 - \Omega_0^2A_2 - 2\alpha\Omega_0A_1 - I_0A_1A_2 = 0\\
&A_0\Omega_0^2 + I_0A_0A_1 = 0\\
&B_0 - 2\alpha\Omega_0A_0 - I_0A_0A_2 = 0\\
&2\alpha\Omega_0 + \tfrac{1}{2}I_0C_0 + \tfrac{1}{2}I_0A_0^2 = 0
\end{aligned} \qquad (26)$$

After solution of the above system, remembering eqs. (23)-(24), one gets the following result for the parameters of the Gaussian $p(u,\dot{u})$

$$\begin{aligned}
&\mu_u = \mu_{\dot{u}} = p_{u\dot{u}} = 0\\
&\sigma_{\dot{u}}^2 = \frac{I_0}{4\alpha\Omega_0} \;;\quad \sigma_u^2 = \sigma_{\dot{u}}^2/\Omega_0^2
\end{aligned} \qquad (27)$$

yielding the final solution for the stationary response of a linear system to a white-noise disturbance.

Cases other than the simple linear one referenced in above are known, where the stationary solution can be found. In his well known book [1] Y.K. Lin yields an approach to the inverse problem, i.e. to find what class of restoring force functions are amenable to yield closed solutions.
After integration of the eq. (19) with respect to $\dot{u}$, one finds that

$$g(u,\dot{u}) = \frac{1}{p}\left[\int \frac{\delta p}{\delta u}\,\dot{u}d\dot{u} - \frac{1}{2} I_0 \frac{\delta p}{\delta \dot{u}} + P(u)\right] \qquad (28)$$

where P(u) is any function of u.
It can be seen that eq.(28) holds for a stationary JPDF of in the Caughey's form (see [1], p.266)

$$p(u,\dot{u}) = K \exp\left[-\frac{\beta}{I_0}\int_0^E v(\Phi)d\Phi\right] \qquad (29)$$

where E is the total energy of the system, i.e. the sum of the instantaneous kinetic and strain energy and $v(\Phi)$ is a positive

monotonically increasing function. This solution corresponds to a restoring force expressed by

$$g(u,\dot{u}) = \tfrac{1}{2}\,\beta\,\dot{u}v(E) + g_1(u)$$
$$E = \tfrac{1}{2}\,\dot{u}^2 + \int_0^u g_1(r)dr \qquad (30)$$

3.3) DIRECT DERIVATION OF THE FPK EQUATION FOR SECOND-ORDER DYNAMICAL SYSTEMS

In this section, the FPK equation governing the instantaneous joint probability density function (JPDF) of the phase components $u(t)$, $\dot{u}(t)$ of the motion of a single-degree-of-freedom (SDOF) structure is directly derived from the equation of the motion. The rationale is purely heuristic, but it allows autonomous consideration of the resulting equation in the context of structural dynamics.

3.3.1) Step-by-step integration of the equation of motion

Let the response of the system be described by a regular function $u(t)$, derivable and continuous with the first derivative. For random response functions one means that every sample function verifies the above properties.
Let the restoring force function $g(u,\dot{u})$ be continuous and derivable with respect to u and $\dot{u}$. Finally let the forcing function $f(t)$ be a Gaussian shot-noise stochastic process, with mean-value function and correlation function given by

$$E[f(t)] = 0$$
$$E[f(t)f(t+\tau)] = I(t)\,\delta(\tau) \qquad (31)$$

where $\delta(\bullet)$ is the Dirac distribution.

The equation of the motion is written as follows

$$\ddot{u} + g(u,\dot{u}) = f(t) \qquad (32)$$

From the above equation one infers that $\ddot{u}(t)$ is not continuous, in that every sample function of the forcing function $f(t)$ is, in general, not continuous. Every sample function of the stochastic process $f(t)$ is a distribution, and can be integrated in the sense of distributions. Consider two time instants t_1 and $t_2 > t_1$, and put

$$u_1 = u(t_1)\ ;\ \dot{u}_1 = \dot{u}(t_1)\ ;\ u_2 = u(t_2)\ ;\ \dot{u}_2 = \dot{u}(t_2) \qquad (33)$$

and let $t_2 - t_1 = \theta$ be so small that, according to usual schemes for numerical integration of the equation of the motion, one can write

$$u_2 = u_1 + \dot{u}_1\,\theta \qquad (34')$$

Since $\ddot{u}(t)$ is not a continuous function, one shall write for $\dot{u}_2$, no

matter how small is θ

$$\dot{u}_2 = \dot{u}_1 + \int_0^\theta \ddot{u}(t_1+x)dx \tag{34"}$$

Substituting from the equation of the motion

$$\ddot{u}(t_1+x) = f(t_1+x) - g[u(t_1+x),\dot{u}(t_1+x)] \tag{35}$$

one gets from eq. (34")

$$\dot{u}_2 = \dot{u}_1 - \int_0^\theta g[u(t_1+x),\dot{u}(t_1+x)]\,dx + \int_0^\theta f(t_1+x)\,dx \tag{36}$$

and remembering that u and $\dot{u}$ are supposed to be continuous

$$\dot{u}_2 = \dot{u}_1 - g[u(t^*),\dot{u}(t^*)]\,\theta + \int_0^\theta f(t_1+x)\,dx \tag{37}$$

where $t^* \in (t_1,t_1+\theta)$. By the assumed continuity of $g[u,\dot{u}]$ and of u, $\dot{u}$ with respect to their respective arguments

$$g[u(t^*),\dot{u}(t^*)] = g(u_2,\dot{u}_2) + O_1(\theta) = g(u_1,\dot{u}_1) + O_1(\theta)$$

where, here and in the sequel, $O_i(\theta)$ denotes an infinitesimal of order $\geq i$ in the variable θ.

The integral in eq. (37) is the stochastic process $F(t|\theta)$, obtained by integrating f(t) on the interval $(t,t+\theta)$

$$F(t|\theta) = \int_0^\theta f(t+x)\,dx \tag{38}$$

The stochastic characteristics of $F(t|\theta)$ are given by the usual formulas for the (mean square) integration of a stochastic process

$$E[F(t|\theta)] = \int_0^\theta E[f(t)]\,dt$$

$$C_F(t,t+\tau|\theta) = E[F(t|\theta)\,F(t+\tau|\theta)] = \int_0^\theta dx \int_0^\theta E[f(t+x)f(t+\tau+y)]dy \tag{39}$$

Remembering eqs. (31)

$$E[F(t|\theta)] = 0$$

$$C_F(t,t+\tau|\theta) = \int_0^\theta dx \int_0^\theta I(t+x)\,\delta(\tau-x+y)\,dy \tag{40}$$

After executing the integral in the expression of $C_F(t,t+\tau|\theta)$

$$C_F(t,t+\tau|\theta) = \begin{cases} \int_{\tau}^{\theta} I(t+x)\,dx & \text{for } \tau \le \theta \\ 0 & \text{for } \tau > \theta \end{cases} \tag{41}$$

and the variance function of $F(t|\theta)$ is given by

$$\sigma^2_F(t|\theta) = C_F\,(t,t|\theta) = \int_0^{\theta} I(t+x)dx = I(t^*)\,\theta \tag{42}$$

with $t^* \in (t_1, t_1+\theta)$

3.3.2) JPDF of $u,\dot{u}$ at time $t+\theta$ given the JPDF at time t

After the developments in the previous section, the phase variables at the time $t_2=t_1+\theta$ are related to displacement and velocity $u,\dot{u}$ at time $t=t_1$ by the following equations

$$\begin{aligned} u_2 &= u_1 + \dot{u}_1\,\theta \\ \dot{u}_2 &= \dot{u}_1 - g[u(t^*),\dot{u}(t^*)]\,\theta + F(t_1|\theta) \end{aligned} \tag{43}$$

or, equivalently, in the inverse form

$$\begin{aligned} u_1 &= u_2 - \dot{u}_2\theta - g[u(t^*),\dot{u}(t^*)]\,\theta^2 + F(t_1|\theta)\,\theta \\ \dot{u}_1 &= \dot{u}_2 + g[u(t^*),\dot{u}(t^*)]\,\theta - F(t_1|\theta) \end{aligned} \tag{44}$$

with $t^* \in (t_1,t_1+\theta)$ and $g[u(t*),\dot{u}(t*)]=g(u_2,\dot{u}_2)+O_1(\theta)$.
Assuming that $p_{1F}(u,\dot{u},F)$ is the JPDF of the random variables $u(t)$, $\dot{u}(t)$, $F(t|\theta)$ at time $t=t_1$, and that $p_{2F}(u,\dot{u},F)$ is the JPDF of the same variables at time $t=t_2$, the two densities are related through the usual formulas allowing for the transformation of variables

$$p_{2F}(u_2,\dot{u}_2,F) = p_{1F}[u_1(u_2,\dot{u}_2,F),\dot{u}_1(u_2,\dot{u}_2,F),F]\;|J| \tag{45}$$

where $|J|$ is the determinant of the Jacobian matrix of the transformation

$$J = \begin{vmatrix} \delta u_1/\delta u_2 & \delta u_1/\delta \dot{u}_2 & \delta u_1/\delta F \\ \delta \dot{u}_1/\delta u_2 & \delta \dot{u}_1/\delta \dot{u}_2 & \delta \dot{u}_1/\delta F \\ \delta F/\delta u_2 & \delta F/\delta \dot{u}_2 & \delta F/\delta F \end{vmatrix} = \begin{vmatrix} 1-\theta^2\frac{\delta g}{\delta u_2} & -\theta(1+\theta\frac{\delta g}{\delta \dot{u}_2}) & \theta \\ \theta\frac{\delta g}{\delta u_2} & 1+\theta\frac{\delta g}{\delta \dot{u}_2} & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

whence

$$|J| = 1 + \theta \frac{\delta g}{\delta \dot{u}_2} \tag{46}$$

Re-considering eq. (45) with the aim to calculate the JPDF of the phase variables only, taking into account that $|J|$ does not depend on F

$$p_2(u_2,\dot{u}_2) = |J| \int_{-\infty}^{+\infty} p_{1F}(u_1,\dot{u}_1,F)\, dF \tag{47}$$

with

$$p_1(u_1,\dot{u}_1) = p(u,\dot{u}|t) \quad ; \quad p_2(u_2,\dot{u}_2) = p(u,\dot{u}|t+\theta) \tag{48}$$

i.e. p_1 and p_2 are the instantaneous JPDF of the state variables at times t and $t+\theta$ respectively.

Remembering that $f(t)$ is Gaussian, $F(t_1|\theta)$ is also Gaussian, and is uncorrelated with $f(t)$ for $t<t_1$. This means that it is also statistically independent of the behaviour of $f(t)$ prior to t_1, so that it cannot depend on $u(t_1)$ and $\dot{u}(t_1)$. It is possible to conclude that

$$p_2(u_2,\dot{u}_2) = |J| \int_{-\infty}^{+\infty} p_1(u_1,\dot{u}_1)p_F(F)\, dF = |J|\, E_F[p_1(u_1,\dot{u}_1)] \tag{49}$$

where $E_F[p_1]$ means the expectation of p_1 on the random variable F and u_1 and $\dot{u}_1$ are intended to be expressed in function of u_2, $\dot{u}_2$ through the inverse relationships (44), like in eq. (45).

Consider now that

$$E_F[p_1(u_1,\dot{u}_1)] = E\{p_1[u_1(u_2,\dot{u}_2,F),\dot{u}_1(u_2,\dot{u}_2,F)]\} \tag{50}$$

and that the expected value of a function s of a random variable z with expected value m_z and variance σ^2_z can be approximated up to second order by

$$E[s(z)] = s(m_z) + \frac{1}{2} \left.\frac{\delta^2 s}{\delta z^2}\right|_{z=m_z} \sigma^2_z \tag{51}$$

By applying this formula to calculate the expected value in eq.(50), and remembering that $E[F(t|\theta)]=0$, one gets

$$E_F[p_1(u_1,\dot{u}_1)] = p_1[u_1(u_2,\dot{u}_2,0),\dot{u}_1(u_2,\dot{u}_2,0)] + \frac{1}{2} \left.\frac{\delta^2 p1}{\delta F^2}\right|_{F=0} \sigma^2_F \tag{52}$$

The 2nd order derivative in the above expression can be soon calculated:

$$\frac{\delta p1}{\delta F} = \frac{\delta p1}{\delta u}\frac{\delta u}{\delta F} + \frac{\delta p1}{\delta \dot{u}}\frac{\delta \dot{u}}{\delta F} = \frac{\delta p1}{\delta u}\theta - \frac{\delta p1}{\delta \dot{u}}$$

$$\frac{\delta^2 p1}{\delta F^2} = \frac{\delta^2 p1}{\delta u^2}\theta^2 - 2\frac{\delta^2 p1}{\delta u\,\delta u}\theta + \frac{\delta^2 p1}{\delta \dot{u}^2} = \frac{\delta^2 p1}{\delta \dot{u}^2} + O_1(\theta) \qquad (53)$$

Hence, coming back to eq. (49) and by combining with eq. (42), (46), (52) and (53) one gets

$$p_2(u_2,\dot{u}_2) =$$

$$= [1 + \frac{\delta g}{\delta \dot{u}}\theta\,]\,\{p_1[u_1(u_2,\dot{u}_2,0),\dot{u}_1(u_2,\dot{u}_2,0)] +$$

$$+ \frac{1}{2}\left[\frac{\delta^2 p1}{\delta \dot{u}^2}\bigg|_{F=0} + O_1(\theta)\right] I(t^*)\,\theta\} \qquad (54)$$

whence, by grouping all the infinitesimals of order ≥ 2, after some algebra one can write

$$p_2(u_2,\dot{u}_2) =$$

$$= O_2(\theta) + [1 + \frac{\delta g}{\delta \dot{u}}\theta]\, p_1[u_1(u_2,\dot{u}_2,0),\dot{u}_1(u_2,\dot{u}_2,0)] +$$

$$+ \frac{1}{2}\,\frac{\delta^2 p_1[u_1(u_2,\dot{u}_2,0),\dot{u}_1(u_2,\dot{u}_2,0)]}{\delta \dot{u}^2}\, I(t^*)\,\theta \qquad (55)$$

3.3.3) Derivation of the FPK equation

Let now express the JPDF p_1 in eq. (52) by the expansion in Taylor's series with initial point $(u_2,\dot{u}_2)$ and for F=0

$$p_1[u_1(u_2,\dot{u}_2,0),\dot{u}_1(u_2,\dot{u}_2,0)] =$$

$$= p_1\,\{u_2-\dot{u}_2\theta - \theta^2 g[u(t^*),\dot{u}(t^*)],\ \dot{u}_2+\theta g[u(t^*),\dot{u}(t^*)]\}$$

$$= p_1(u_2,\dot{u}_2) + \frac{\delta p_1}{\delta u}(u_1-u_2) + \frac{\delta p_1}{\delta \dot{u}}(\dot{u}_1-\dot{u}_2) + \ldots \qquad (56)$$

whence, by considering that both (u_1-u_2) and $(\dot{u}_1-\dot{u}_2)$ are infinitesimal in $\theta = t_2-t_1$

$$p_1[u_1(u_2,\dot{u}_2,0),\dot{u}_1(u_2,\dot{u}_2,0)] = \tag{57}$$

$$= p_1(u_2,\dot{u}_2) - \theta\,\dot{u}_2\,\frac{\delta p_1}{\delta u} + \theta\, g[u(t^*),\dot{u}(t^*)]\,\frac{\delta p_1}{\delta \dot{u}} + O_2(\theta)$$

Similarly

$$\theta\,\frac{\delta^2 p_1\{u_2-\dot{u}_2\theta\,-\theta^2 g[u(t^*),\dot{u}(t^*)],\ \dot{u}_2+\theta g[u(t^*),\dot{u}(t^*)]\ \}}{\delta \dot{u}^2_2} =$$

$$= \theta\,\frac{\delta^2 p_1(u_2,\dot{u}_2)}{\delta \dot{u}^2} + O_2(\theta) \tag{58}$$

By substitution of eqs. (57) and (58) in the eq. (55), one gets

$$p_2(u_2,\dot{u}_2) = p_1(u_2,\dot{u}_2) - \theta\dot{u}_2\,\frac{\delta p_1(u_2,\dot{u}_2)}{\delta u_2} +$$

$$+ \theta g[u(t^*),\dot{u}(t^*)]\,\frac{\delta p_1(u_2,\dot{u}_2)}{\delta \dot{u}_2} +$$

$$+ \theta p_1(u_2,\dot{u}_2)\,\frac{\delta g[u(t^*),\dot{u}(t^*)]}{\delta \dot{u}_1} +$$

$$+ \theta\,\frac{1}{2}\,\frac{\delta^2 p_1(u_2,\dot{u}_2)}{\delta \dot{u}^2_2}\,I(t^*) + O_2(\theta) \tag{59}$$

Remembering that p_1 and p_2 have been identified as the instantaneous JPDF of u and $\dot{u}$ at time t and at time $t+\theta$ respectively (see eqs. (48)), by considering a generic instant time t as the starting time t_1, eq.(59) can be re-written

$$\frac{p(u,\dot{u}|t+\theta) - p(u,\dot{u})|t)}{\theta} = g[u(t^*),\dot{u}(t^*)]\,\frac{\delta p_1(u_2,\dot{u}_2)}{\delta \dot{u}_2} +$$

$$+ p_1(u_2,\dot{u}_2)\,\frac{\delta g[u(t^*),\dot{u}(t^*)]}{\delta \dot{u}_1} - \dot{u}_2\,\frac{\delta p_1(u_2,\dot{u}_2)}{\delta u_2} +$$

$$+ \frac{1}{2}\,\frac{\delta^2 p_1(u_2,\dot{u}_2)}{\delta \dot{u}^2_2}\,I(t^*) + O_1(\theta) \tag{60}$$

As θ approaches 0, the first member yields the first partial derivative of the instantaneous JPDF $p(u,\dot{u}|t)$ with respect to time.

As $\theta \longrightarrow 0$, in the right-hand side $t^* \longrightarrow t_1=t$, and, by the assumed continuity, $u_2 \longrightarrow u_1=u(t)$ and $\dot{u}_2 \longrightarrow \dot{u}_1=\dot{u}(t)$; simultaneously $I(t^*) \longrightarrow I(t)$ and $O_1(\theta) \longrightarrow 0$. After this, it easy to see that eq.(60) yields

$$\frac{\delta p(u,\dot{u}|t)}{\delta t} =$$

$$\frac{\delta}{\delta \dot{u}} [p(u,\dot{u}|t)\, g(u,\dot{u})] - \dot{u} \frac{\delta p(u,\dot{u}|t)}{\delta u} + \frac{1}{2} I(t) \frac{\delta^2 p(u,\dot{u}|t)}{\delta \dot{u}^2} \qquad (61)$$

that is coincident with the Fokker-Planck-Kolmogorov equation for the analysis of dynamical second-order systems acted on by a stochastic shot-noise loading.

3.4) APPROXIMATE SOLUTION PROCEDURE FOR SMOOTH NON-LINEARITIES IN THE RESTORING FORCE

3.4.1) Basic ideas for the solution of the FPK eqn.

The basic statement assumed in the procedure is to force the solution of the FPK equation to have a given shape. In the following, the treatment will be developped assuming that the solution is forced in the form of a Gaussian joint density function (JPDF) of the state variables u, $\dot{u}$, but any other basic model (say, a Gamma, or a log-normal or else, if it is judged to be more convenient) can be chosen. The motion of the system is governed by eq.(32), where $f(t)$ is assumed to be a Gaussian shot noise obeying eqs. (31).

Considering that $u(t)$ and $\dot{u}(t)$ are the components of a time-dependent Markov vector and that $p(u,\dot{u};t)$ is the relevant instantaneous probability density at time t, the classical Fokker-Planck-Kolmogorov equation (61) holds, as well known from the theory of Markov processes, and as directly proved in the previous section with specific reference to the dynamics of a sdof system. The initial condition for $p(u,\dot{u};t)$ is assumed in the form

$$p(u,\dot{u};0) = p_0(u,\dot{u}) \qquad (62)$$

Let force the solution to have the form

$$p(u,\dot{u};t) = N(u,\dot{u};t) \qquad (63)$$

and

$$p_0(u,\dot{u}) = N(u,\dot{u};0) \qquad (64)$$

where $N(u,\dot{u};t)$ denotes the Gaussian JPDF in the random variables $u,\dot{u}$ at time t. The dependence of $N(u,\dot{u};t)$ on time is expressed by updating the parameters $\mu_u(t)$, $\mu_{\dot{u}}(t)$, $\sigma_u(t)$, $\sigma_{\dot{u}}(t)$, $\rho_{u\dot{u}}(t)$, which are the mean value functions of the displacement and of the velocity, the standard deviations of the same quantities, and the coefficient of correlation respectively.

Hence

$$N(u,\dot{u};t)= N[u,\dot{u}|\mu_u(t),\mu_{\dot{u}}(t),\sigma_u(t),\sigma_{\dot{u}}(t),p_{u\dot{u}}(t)] \tag{65}$$

Considering that

$$p(u,\dot{u};t+dt) \approx p(u,\dot{u};t) + \frac{\delta p}{\delta t}\, dt$$

with the time partial derivative given by eq. (61), the final resolving equation can be written in the form

$$N(u,\dot{u};t+dt) = N(u,\dot{u};t) + D(u,\dot{u};t)dt \tag{66}$$

where

$$D(u,\dot{u};t) = \tag{67}$$

$$= \frac{\delta N(u,\dot{u};t)}{\delta \dot{u}}\, g(u,\dot{u};t)+N(u,\dot{u};t)\, \frac{\delta g(u,\dot{u};t)}{\delta \dot{u}} - \frac{\delta N(u,\dot{u};t)}{\delta u}\, \dot{u}+ \frac{1}{2}\, I(t)\, \frac{\delta^2 N(u,\dot{u};t)}{\delta \dot{u}^2}$$

Consider that

$$\frac{\delta N}{\delta u} = B(u,\dot{u};t)\, N(u,\dot{u};t) \tag{68}$$

$$\frac{\delta N}{\delta \dot{u}} = A(u,\dot{u};t)\, N(u,\dot{u};t) \tag{69}$$

$$\frac{\delta^2 N}{\delta \dot{u}^2} = C(u,\dot{u};t)\, N(u,\dot{u};t) \tag{70}$$

where N is the bivariate Gaussian distribution and A,B,C are given by eqs (22). With these positions, Eq.(67) can be written explicitly:

$$D(u,\dot{u};t) = N(u,\dot{u};t)\, \{\, g(u,\dot{u})\cdot[a_0(t)+a_1(t)u+a_2(t)\dot{u}] + \frac{\delta g(u,\dot{u})}{\delta \dot{u}} +$$

$$- u\cdot[b_0(t)+b_1(t)u+b_2(t)\dot{u}] + \tfrac{1}{2}\cdot I(t)\cdot[c_0(t)+a_0^2(t)+a_1^2(t)u^2+a_2^2(t)\dot{u}^2+$$

$$+ 2a_0(t)a_1(t)u + 2a_0(t)a_2(t)\dot{u} + 2a_1(t)a_2(t)u\dot{u}] \,\} \tag{71}$$

By multiplying both members of eq. (66) by $u^r\cdot\dot{u}^s$ and by integrating, one obtains

$$\int_{-\infty}^{+\infty} d\dot{u} \int_{-\infty}^{+\infty} u^r\cdot\dot{u}^s\, N(u,\dot{u};t+dt)du = \int_{-\infty}^{+\infty} d\dot{u} \int_{-\infty}^{+\infty} u^r\cdot\dot{u}^s\, [N(u,\dot{u};t)+D(u,\dot{u};t)dt]du \tag{72}$$

whence

$$E[u^r \cdot \dot{u}^s | t+dt] = E[u^r \cdot \dot{u}^s | t] + E_{rs}(t)dt \tag{73}$$

with

$$E_{rs}(t) = \int_{-\infty}^{+\infty} d\dot{u} \int_{-\infty}^{+\infty} u^r \cdot \dot{u}^s \, D(u,\dot{u};t) \, du \tag{74}$$

Thus the problem is to calculate integrals in the phase-space to obtain the transient state at the instant (t+dt) of the expected displacement and velocity, and the relevant standard deviations.
By (numerical) integration of eq. (73) with respect to time, one gets the values of moments of any order at time t.

$$E[u^r \cdot \dot{u}^s; t] = E[u^r \cdot \dot{u}^s; 0] + \int_0^t E_{rs}(\tau) \, d\tau \tag{75}$$

By applying eq.(72) with respect to expectations of u and $\dot{u}$, one gets

$$\mu_u(t+dt) = \mu_u(t) + E_{10}(t)dt \tag{76}$$

$$\mu_{\dot{u}}(t+dt) = \mu_{\dot{u}}(t) + E_{01}(t)dt \tag{77}$$

By applying again eq.(72) with respect to second-order moments, it is possible to calculate updated variances and covariance respectively, as follows

$$\sigma^2_u(t+dt) = \sigma^2_u(t) + [E_{20}(t)+\mu^2_u E_{00}(t)-2\mu_u E_{10}(t)] \, dt \tag{78}$$

$$\sigma^2_{\dot{u}}(t+dt) = \sigma^2_{\dot{u}}(t) + [E_{20}(t)+\mu^2_{\dot{u}} E_{00}(t)-2\mu_{\dot{u}} E_{10}(t)] \, dt \tag{79}$$

$$K_{u\dot{u}}(t+dt)=K_{u\dot{u}}(t)+[E_{11}(t)-\mu_{\dot{u}} E_{10}(t)-\mu_u E_{01}(t)+\mu_u \mu_{\dot{u}} E_{00}(t)]dt \tag{80}$$

where μ_u and $\mu_{\dot{u}}$ denote $\mu_u(t+dt)$ and $\mu_{\dot{u}}(t+dt)$ and $K(u,\dot{u})$ denotes the covariance function of $u,\dot{u}$.

3.4.2) An example: general polynomial restoring force

Let pay attention to a restoring force $g(u,\dot{u})$ expressed by the general polynomial form

$$g(u,\dot{u}) = \sum_{i=0}^{n} \sum_{j=0}^{m} g_{ij} \cdot u^i \cdot \dot{u}^j \tag{81}$$

The derivative with respect to $\dot{u}$ is

$$\frac{\delta g}{\delta \dot{u}} = \sum_{i=0}^{n} \sum_{j=1}^{m} j \cdot u^i \cdot \dot{u}^{j-1} \tag{82}$$

where the g_{ij}'s are some suitable coefficients. Eq. (74) becomes

$$E_{rs}(t) = a_0(t)\cdot \sum_{i=0}^{n} \sum_{j=0}^{m} g_{ij}\cdot E[u^{r+i}\cdot \dot{u}^{s+j} \,|t] +$$

$$+ a_1(t)\cdot \sum_{i=0}^{n} \sum_{j=0}^{m} g_{ij}\cdot E[u^{r+i+1}\cdot \dot{u}^{s+j} \,|t] +$$

$$+ a_2(t)\cdot \sum_{i=0}^{n} \sum_{j=0}^{m} g_{ij}\cdot E[u^{r+i}\cdot \dot{u}^{s+j+1} \,|t] +$$

$$+ \sum_{i=0}^{n} \sum_{j=1}^{m} j\cdot g_{ij}\cdot E[u^{r+i}\cdot \dot{u}^{s+j-1} \,|t] -$$

$$- b_0(t)\, E[u^{r}\cdot \dot{u}^{s+1}|t] - b_1(t)\, E[u^{r+1}\cdot \dot{u}^{s+1}|t] +$$

$$- b_2(t)\, E[u^{r}\cdot \dot{u}^{s+2}|t] + \tfrac{1}{2}\, I(t)\cdot\{[c_0(t) + a_0^2(t)]\, E[u^{r}\cdot \dot{u}^{s}|t] +$$

$$+ a_1^2(t)\, E[u^{r+2}\cdot \dot{u}^{s}|t] + a_2^2(t)\, E[u^{r}\cdot \dot{u}^{s+2}|t] +$$

$$+ 2a_0(t)a_1(t)\, E[u^{r+1}\cdot \dot{u}^{s}|t] + 2a_0(t)a_2(t)\, E[u^{r}\cdot \dot{u}^{s+1}|t] +$$

$$+ 2a_1(t)a_2(t)\, E[u^{r+1}\cdot \dot{u}^{s+1}|t] \qquad (83)$$

and $E[u^k\cdot \dot{u}^h|t]$ coincides with the moments m_{kh} of the Gaussian bivariate PDF.
Since the basic distribution is forced to be Gaussian, only moments up to second order are needed to fully define the instantaneous PDF.

3.4.3) Applications and numerical results

The purpose of the applications is to show how practical results can be obtained through the approximate solution described above, for problems related to the dynamic response of a sdof non-linear and visco-elastic system under the action of a random Gaussian shot-noise type forcing function f(t).
Any kind of nonlinearity can be investigated in the transient phase giving as input only the coefficients g_{ij} of the restoring force. Of course, as the order of the nonlinearities in g(u,û) becomes larger, so increases the maximum order of the E_{rs} to be calculated. Grossly, it can be verified that if N=max(n,m) is the maximum order in g(u,û), the second member of eq.(83) contains products u·û up to (N+1)th order, and the calculation of moments up to 2nd order requires the evaluation of moments of the basic Gaussian distribution up to (N+3)th order. For explicit formulae up to 6th order see the Appendix to Ref.[10].

In order to test the performance of the procedure, one considers first the behaviour in the transient phase of a linear oscillator, and thereafter the behaviour of two classical smooth non-linear oscillators, namely the Duffing and the Van der Pol, under the action of a white-noise Gaussian force with constant intensity $I_0 = 10^3$ cm^2/sec^3.

The basic parameters of the instantaneous JPDF N(u,û;t), i.e. the

expected values $\mu_u(t)$, $\mu_{\dot{u}}(t)$, the central second-order moments $\sigma_u^2(t)$, $\sigma_{\dot{u}}^2(t)$, and the correlation coefficient $p_{u\dot{u}} = K_{u\dot{u}}(t)/[\sigma_u(t)\cdot\sigma_{\dot{u}}(t)]$, are evaluated by the present procedure, and are compared with the output of simulation made by 2,500 sample time histories of duration T=4 secs.

3.4.3.1) The linear oscillator

In this case, it is assumed n=m=1 in eq. (81) with

$$g_{00}=0 \quad ; \quad g_{10} = \Omega^2 \quad ; \quad g_{01} = 2\alpha\Omega \tag{84}$$

$$\Omega = 10 \text{ sec}^{-1} \; ; \quad \alpha = 0.05 \tag{85}$$

where α is the damping coefficient and Ω is the free undamped natural frequency.
The mean value functions are practically null on the whole duration, both for simulation and for Gaussian approximation. The results for variances of u,$\dot{u}$ and the coefficient of correlation are plotted by a continuous line in Fig.3 and are compared with the results of simulation that are quoted by small squares. The results show a quite good agreement between the analytical procedure and numerical sampling.

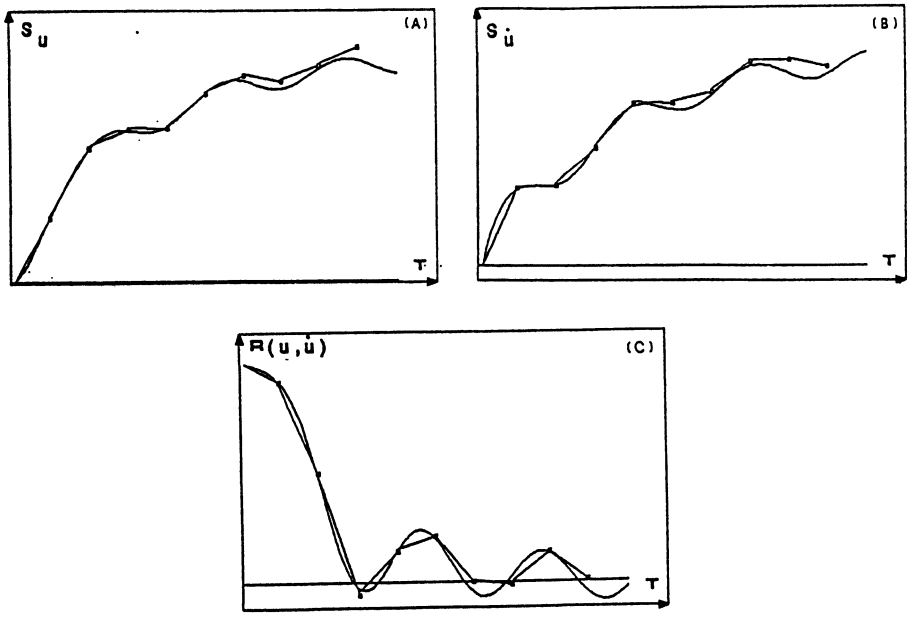

Figure 3: Linear oscillator - Simulated vs. calculated statistics
A) Standard deviation of displacement;
B) Standard deviation of velocity;
C) Coefficient of correlation of displacement and velocity

3.4.3.2) The Duffing oscillator

The restoring force in the case of the Duffing oscillator is given by an expression of the type

$$g(u,\dot{u}) = \Omega^2 u + 2\alpha\Omega\dot{u} + \beta\Omega^2\dot{u}^3 \qquad (86)$$

The nonlinearity of the 3rd order is limited to the dissipative term, and the factors of the polynomial form of $g(u,\dot{u})$, eq.(81), are all zero except

$$g_{10} = \Omega^2 \quad ; \quad g_{01} = 2\alpha\Omega \; ; \quad g_{03} = \beta\Omega^2 \qquad (87)$$

$$\Omega = 10 \text{ sec}^{-1} \; ; \quad \alpha = 0.05 \; ; \quad \beta = 0.1 \qquad (88)$$

The quadratic values of displacement, of relevant velocity andf the coefficient of correlation are plotted in Fig.4, showing good agreement, despite the fact that, as pointed in [9], the instantaneous JPDF may be well different from the assumed Gaussian.

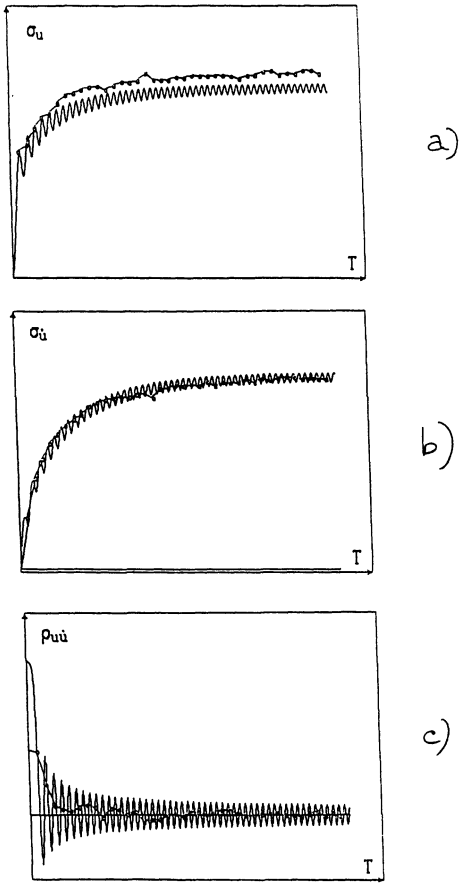

Figure 4: Duffing oscillator - Simulated vs. calculated statistics
A) Standard deviation of displacement;
B) Standard deviation of velocity;
C) Coefficient of correlation of displacement and velocity

3.4.3.3) The Van Der Pol oscillator

The restoring force in the case of Van der Pol is

$$g(u,\dot{u}) = \Omega^2 u + 2\alpha\Omega\dot{u} + \alpha\beta u^2\dot{u} \qquad (89)$$

The coefficients of the restoring force are all zero, except

$$g_{10} = \Omega^2 \quad ; \quad g_{01} = 2\alpha\Omega \; ; \quad g_{21} = \alpha\beta \qquad (90)$$

$$\Omega = 10 \text{ sec}^{-1} \; ; \quad \alpha = -0.05; \quad \beta = -1 \qquad (91)$$

The comparison with simulation is plotted in Fig.5, also showing both a good agreement and the convergence in the short period to a stationary phase.

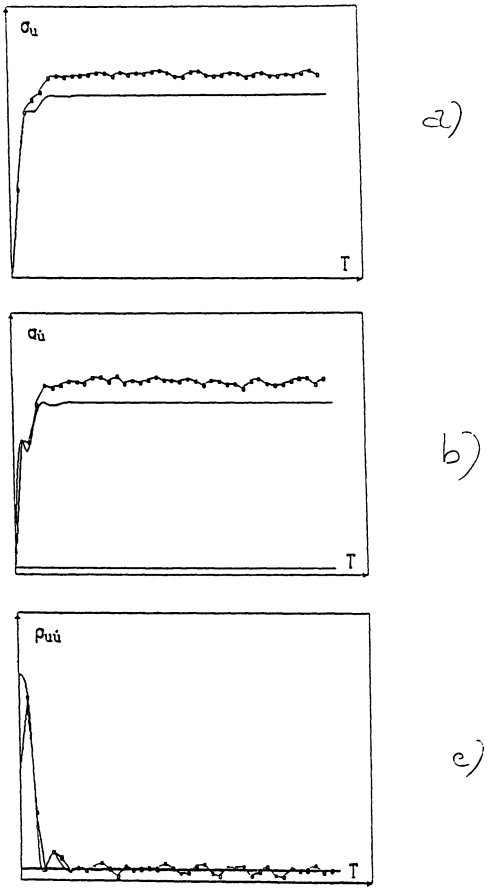

Figure 5: Van der Pol oscillator - Simulated vs. calculated statistics
A) Standard deviation of displacement;
B) Standard deviation of velocity;
C) Coefficient of correlation of displacement and velocity

4.3.4) Trajectories in the phase plane

For a more complete evaluation of the behaviour of the two nonlinear systems considered above, and for a comparison with the corresponding linear system, trajectories of the variances of u and $\dot{u}$ for the linear oscillator, for the Duffing and for the Van der Pol ones, are plotted in Fig.6 (a,b,c, respectively), and again superposed to small squares representing points resulting from simulation. Similar diagrams are quoted in Fig.7, where the instantaneous covariance is plotted versus the product of variances, thus giving an idea of the time-dependent correlation for each of the three cases considered.

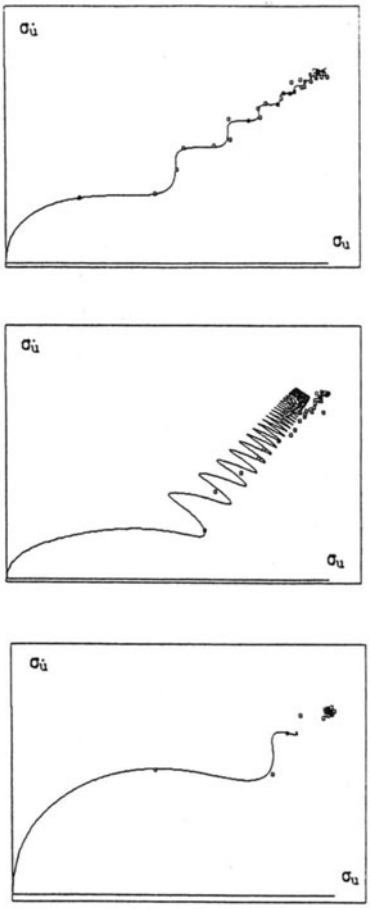

Figure 6: Trajectories of the variances of $\dot{u}$, $\dot{u}$.
Simulated vs. calculated results.
A) Linear; B) Duffing; C) Van der Pol.

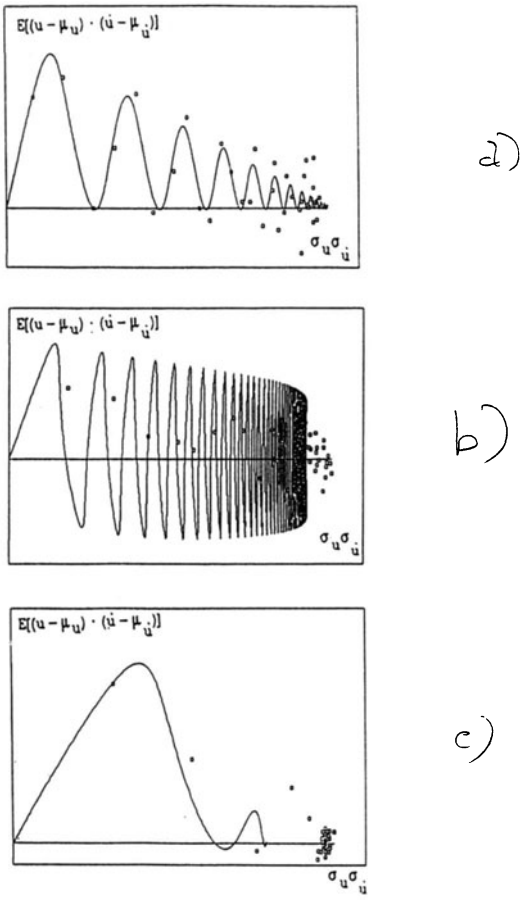

Figure 7: Instantaneous covariance vs. product of variances.
Simulated vs. calculated results.
A) Linear; B) Duffing; C) Van der Pol.

5) NONSMOOTH NONLINEARITIES: APPLICATION TO IMPACT PROBLEMS

In the previous section, the procedure has been implemented to solve problems where a smooth nonlinear restoring force is involved. Results prove to approximate to some extent the behaviour of the system, as investigated by numerical MonteCarlo simulation, and the procedure turns out to be satisfactory, at least from a qualitative point of view.
In this section an application involving a highly non-regular forcing function is attempted.

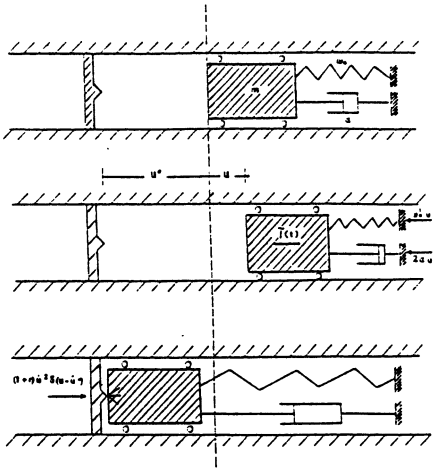

Figure 8: The oscillator under consideration

Consider the system in Fig. 8, i.e. a SDOF oscillator with a barrier on the left, located at the abscissa u^*. The active force is still a random shot-noise obeying eqs (31), and the equation of the motion is written in the form (32) with

$$g(u,\dot{u}) = \Omega_0^2 u + 2\alpha\Omega_0\dot{u} + R(u,\dot{u}) \qquad (92)$$

where $R(u,\dot{u})$ is the reaction of the obstacle, with $R(u,\dot{u})>0$ only if $u=u^*$ and $\dot{u}<0$ and $R(u,\dot{u})=0$ otherwise. If "r" is the coefficient of restitution of the obstacle (say, r=1 if the impact is purely elastic, r<1 if some dissipation of energy takes place at the impact), $R(u,\dot{u})$ can be expressed by

$$R(u,\dot{u}) = (1+r)(\dot{u}^-)\delta(u-u^*) \qquad (93)$$

This is a reactive term that introduces a strong non-linearity into the equation. It can be easily seen that, provided

$$u(0) > u^* \qquad (94)$$

the solution u(t) obeys the conditions

$$u(t) \geq u^* \qquad 0 \leq t < \infty \qquad (95)$$

$$\dot{u}^+ = -r\,\dot{u}^- \qquad \text{at the impact} \qquad (96)$$

Superscripts "+" and "-" denote values of the function on the right and on the left of the impact time, respectively.

The solutions of eq. (32) with g(u,û) specified by eq. (92) are discontinuous in the time derivative; this discontinuity causes much of the difficulty in handling the equation from an analytical point of view. The introduction of special transformations of the state variables may help in some cases to regularize the solution, as will be briefly summarized in the following section.

3.5.1)A particular case: Analytical solution

Under the set assumptions, the displacement u(t) and the relevant velocity û(t) are the components of a Markov vector, and the instantaneous joint transition probability density p(u,û;t) obeys the FPK equation

$$\frac{\delta p}{\delta t} = \frac{\delta}{\delta \hat{u}} [p(u,\hat{u};t)\, g(u,\hat{u})] - \hat{u} \frac{\delta p}{\delta u} + \frac{1}{2} I(t) \frac{\delta^2 p}{\delta \hat{u}^2} \qquad (97)$$

with initial condition

$$p(u,\hat{u};0) = p_0(u,\hat{u}) \qquad (98)$$

In the case that the initial condition of the system is deterministically known, say $u(0)=u_0$; $\hat{u}(0)=\hat{u}_0$, one gets

$$p_0(u,\hat{u}) = \delta(u-u_0,\ \hat{u}-\hat{u}_0) \qquad (99)$$

If $I(t) = I_0 = \text{const.}$ (i.e. f(t) is a stationary white noise), the response is asymptotically stationary. If, in addition, the forcing noise is a zero-mean w.n., and the impact is purely elastic (i.e. r=1), it can be proved that the stationary limit response of the system with a barrier coincides with the stationary response of the same system without barrier, after truncation and normalization to unit area of the instantaneous JPDF of u,û in the admissible range of displacements (see e.g. [3]).
In the event that the forcing process is not a zero-mean stationary w.n. and/or the impact is not elastic, or also that one is interested in the transient response p(u,û;t), or simply that the stationary JPDF for the system without barrier is unavailable, one is forced to look for approximate solutions. To this aim, it proves very useful the transformation

$$u = |x| + u^* \qquad (100)$$

first introduced by Zhuravlev (1976, see e.g. [3,6]).

If one writes the equation of the motion with reference to the transformed motion x(t), one finds that the impulsive terms is fully eliminated if the impact is purely elastic, and largely attenuated if r is not much smaller than unity (quasi-elastic impact) in comparison

with the corresponding term in the original equation. It turns out that x(t) may have much less significant discontinuities in its time-derivative. This allows normal approximation techniques for the treatment of non-linear stochastic differential equations (averaging, equivalent linearization,...) to be applied.

If none of the previous assumptions, allowing for an approximate analytical solution, holds, one is therefore urged to seek for other methods, involving a numerical solution of the differential equations.

3.5.2) Preliminary regularization of the mechanical model

The basic idea adopted here is to force the solution of the FPK equation to have the form of a Gaussian joint density function in u and $\dot{u}$, truncated for $u<u^*$.
Let $N(u,\dot{u};t)$ denote a time-dependent Gaussian JPDF, and write the solution of the problem $p(u,\dot{u};t)$ in the form

$$p(u,\dot{u};t) = h(t)\, N(u,\dot{u};t) \qquad (101)$$

where $h(t)$ denotes a time-dependent normalizing factor, and the JPDF $N(u,\dot{u};t)$ will be referred to in the following as the fundamental Gaussian. As in the smooth case, dependence on time of $N(u,\dot{u},;t)$ is assumed to be lumped into the parameters $\mu_u(t)$, $\mu_{\dot{u}}(t)$, $\sigma_u(t)$, $\sigma_{\dot{u}}(t)$, $\rho_{u\dot{u}}(t)$, which are respectively the expected values of the displacement and the velocity, the standard deviations of the same quantities, and the coefficient of correlation. In other words, it is understood that

$$N(u,\dot{u};t) = N\,[u,\dot{u}|\ \mu_u(t),\ \mu_{\dot{u}}(t),\ \sigma_u(t),\ \sigma_{\dot{u}}(t),\ \rho_{u\dot{u}}(t)] \qquad (102)$$

Let write again the FPK equation in the form

$$p(u,\dot{u};t+dt) = p(u,\dot{u};t) +$$

$$+ \left\{ \frac{\delta}{\delta u}\,[p(u,\dot{u};t)\, g(u,\dot{u})] - \dot{u}\,\frac{\delta p}{\delta u} + \frac{1}{2}\, I(t)\, \frac{\delta^2 p}{\delta \dot{u}^2} \right\} dt \qquad (103)$$

and approximate the restoring force (92) by the more amenable expression

$$g(u,\dot{u}) = 2\alpha\Omega_0\dot{u} + \Omega^2_0 u + [1-H(u-u^*)]\,[K_0(u-u^*)+C_0\dot{u}] \qquad (104)$$

where $H(.)$ is the Heaviside step function, and K_0, C_0 are parameters to be given appropriate values. It is worth noting that in expression (104) the presence of the obstacle producing the impulsive term, and the loss of kinetic energy at the impact, is simulated by the parallel coupling of a spring and a linear damper. The stiffness of the spring and the damping of the dashpot, are assumed to be zero for $u>u^*$ and to jump to values K_0, C_0, as contact with the obstacle takes place. It is clear that for $C_0 = 0$ one reproduces the case of purely elastic impact, while for $K_0 \dashrightarrow \infty$ the case of the rigid obstacle is approached.

Substitution of the expression (104) in place of (92)-(93) produces a kind of softening of the impact, but the physical behaviour is not significantly changed, as shown by the numerical experiments in the next section. Moreover, this approach is frequently encountered in the analysis of unilateral dynamics (see e.g. [7]).

The derivatives of the restoring force function, defined by eq. (104), are

$$\frac{\delta g}{\delta u} = \Omega^2_0 + [1-H(u-u^*)]\, K_0 - \delta(u-u^*)\, C_0\, \dot{u} \qquad (105)$$

$$\frac{\delta g}{\delta \dot{u}} = 2\alpha\Omega_0 + [1 - H(u - u^*)]\, C_0 \qquad (106)$$

Note that in the FPK equation only the derivative (106) enters as a factor of the unknown $p(u,\dot{u};t)$. The softening of the obstacle therefore, has the effect of eliminating impulsive terms from the equation, producing a kind of regularization of the problem.

Remembering eqs. (68)-(70), and introducing the fundamental Gaussian in the FPK equation one finds that

$$h(t+dt)N(u,\dot{u};t+dt) = h(t)N(u,\dot{u};t)[1+D(u,\dot{u};t)dt] \qquad (107)$$

with

$$D(u,\dot{u};t) = A(u,\dot{u};t)\, g(u,u) + \frac{\delta g}{\delta \dot{u}} - B(u,\dot{u};t)\dot{u} + \frac{I(t)}{2}\, C(u,\dot{u}t) \qquad (108)$$

By integrating both members of eq. (107) on R^2 with respect to $(u,\dot{u})$, one obtains

$$h(t + dt) = h(t)\, [1 + E_{00}(t)\, dt] \qquad (109)$$

where, according to eq.(74) with r=s=0

$$E_{00}(t) = \int_{-\infty}^{+\infty} d\dot{u} \int_{-\infty}^{+\infty} D(u,\dot{u};t)\, N(u,\dot{u};t)\, du \qquad (110)$$

By taking expectations of both members of eq. (107) with respect to u and $\dot{u}$, one obtains

$$\mu_u(t + dt) = \frac{\mu_u(t) + E_{10}(t)dt}{h(t)\, [1+ E_{00}(t)dt]} \qquad (111)$$

$$\mu_{\dot{u}}(t + dt) = \frac{\mu_{\dot{u}}(t) + E_{01}(t)dt}{h(t)\,[1+ E_{00}(t)dt]} \qquad (112)$$

with E_{01} and E_{10} given by eq. (74) for r=0 and s=1, and r=1 and s=0 respectively.
Finally, by taking second-order moments of both members of eq. (107) with respect to $(u - \mu_u)^2$, $(u - \mu_{\dot{u}})^2$, $(u - \mu_u)(u - \mu_{\dot{u}})$, it is possible to obtain updated variances and covariance respectively, as follows:

$$\sigma^2_u(t + dt) = \frac{\sigma^2_u(t)+[E_{20}(t)+\mu^2_u E_{00}(t)-2\mu_u E_{10}(t)]dt}{h(t)\,[1 + E_{00}(t)dt]} \qquad (113)$$

$$\sigma^2_{\dot{u}}(t + dt) = \frac{\sigma^2_u(t)+[E_{02}(t)+\mu^2_{\dot{u}} E_{00}(t)-2\mu_{\dot{u}} E_{01}(t)]dt}{h(t)\,[1 + E_{00}(t)dt]} \qquad (114)$$

$$K_{u\dot{u}}(t + dt) = \frac{K_{u\dot{u}}(t)+[E_{11}(t)+\mu_u\mu_{\dot{u}} E_{00}(t)-\mu_u E_{01}(t)-\mu_{\dot{u}} E_{10}(t)]dt}{h(t)\,[1 + E_{00}(t)dt]} \qquad (115)$$

where the meaning of the symbols E_{rs} is still referred to eq. (74).

It is clear that eqs. (110) to (115) are amenable to simple numerical integration, after the coefficients E_{ij} (i=0,1,2 - j=0,1,2) have been calculated from eq. (74). The results quoted in the present paper are obtained by direct numerical evaluations of the appropriate integrals. It can be seen, however, that the same integrals can be calculated in closed form, thanks to the particular expression, following from the introduction of the fundamental Gaussian, taken by the integrand functions. Although quite elementary in principle, the integration requires cumbersome algebraic developments, and this is the reason why, in testing the procedure, numerical quadratures have been preferred.

3.5.3) Application and numerical results

An oscillator with the following numerical characteristics is considered:

$\Omega_0 = 10\ \text{sec}^{-1}$; $\alpha = 0.01$; $u^* = -5$ cm.

In testing the procedure, first the accuracy of expression (104), as an approximation for impact, was accurately verified. In Fig. 9.a, the free oscillations for a given set of initial conditions, and purely elastic impact are shown on the time interval from 0 to 5 secs (the displacement in the plot above, the velocity below). In Fig. 9.b, the same oscillations are shown, calculated according to the softened model; this shows good agreement for $K_0 = 10{,}000\ \text{sec}^{-2}$, $C_0 = 0$. In Fig.

10.a the free oscillations, on the same time interval, are shown for the case where the impact is not elastic, the restitution factor being r=0.7. In Fig. 10.b the response with the softened obstacle is plotted, showing good agreement for $C_0 = 25$.

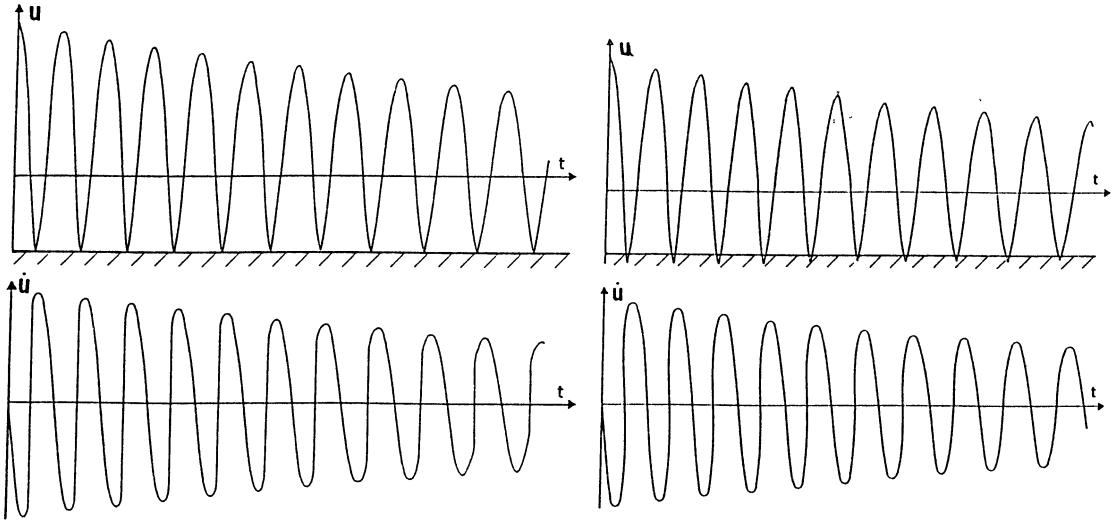

Fig. 9: Free oscillations : $u_0 = 10$ cm; $\dot{u}_0 = 0$; $r = 1$.
a) Rigid barrier;
b) Softened barrier: $K_0 = 10{,}000\ \text{sec}^{-2}$; $C_0 = 0$.

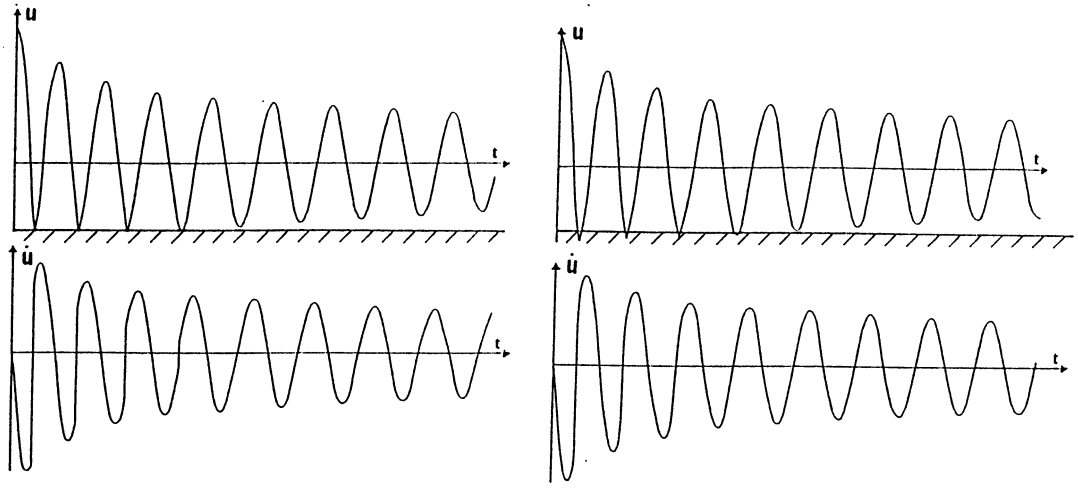

Fig.10: Free oscillations : $u_0 = 10$ cm; $\dot{u}_0 = 0$; $r = 0.7$
a) Rigid barrier;
b) Softened barrier: $K_0 = 10{,}000\ \text{sec}^{-2}$; $C_0 = 0$.

After these preliminary investigations, the analysis was concerned with the response of the oscillator with homogeneous initial conditions, acted on by a n.s.w.n. random forcing function with a zero mean-value and an intensity function given by

$$I(t) = I_0 \exp [-2*(t/T)^2] \qquad (116)$$

where $I_0 = F_0 \cdot \delta t$ is the initial intensity of the input n.s.w.n. acceleration. F_0 (10,000 cm/sec2) is the assumed initial mean square value of the forcing function and δt is the time-lag used to simulate the n.s.w.n., while T is the strong-phase duration of the excitation (1 sec.).
A sample response is plotted in Fig. 11 for $0 \leq t \leq 5$ secs.

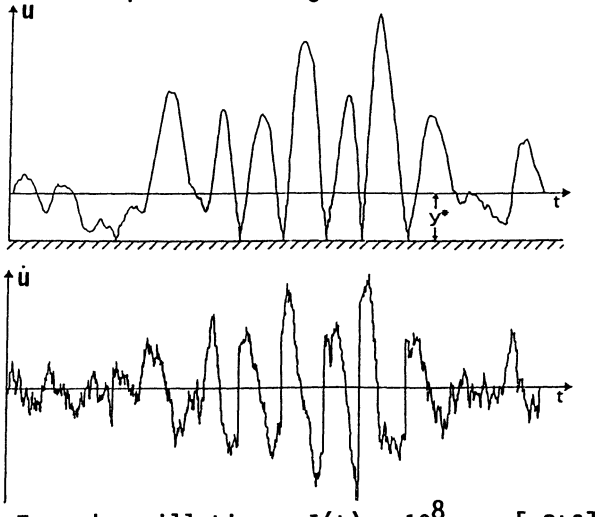

Fig. 11: Forced oscillations: $I(t) = 10^8 \exp [-2t^2]$

In the following, the time interval (0,1) is considered.
As a first step, an extended numerical simulation of the oscillator response, over the chosen time interval, was performed, by simulating 2,500 sample responses. Pairs of values (u,u̇) were stored at given times (say t=0.1, 0.2, ..., 1.0), and the instantaneous joint density functions are shown in Fig.s 12a, 12b, 12c for t= 0.1,0.3, 1.0 sec. respectively, by plotting points in the phase plane (u,u̇) corresponding to the instantaneous joint values of displacement and velocity that were recorded in the simulation. The correlation between the displacement and the velocity can be distinctly observed in the first instants, while it is clearly seen that the two quantities tend to become uncorrelated with elapsing the time.
Next, the procedure previously outlined is implemented. A problem was soon encountered, relating to the numerical integration of the equations (109),(111),(112),(113),(114) in that it is expected that the second member in eq. (107), being a probability density function, should be a positive definite function. Since it derives from a time derivative, this is not necessarily so here, and positiveness can in fact only be guaranteed if δt, the time lag in numerical integration of the differential equations, is sufficiently small (see [8]).
As it can be seen, for the larger values of δt, the updated JPDF does not remain positive. It is a lucky circumstance that for the time-lag

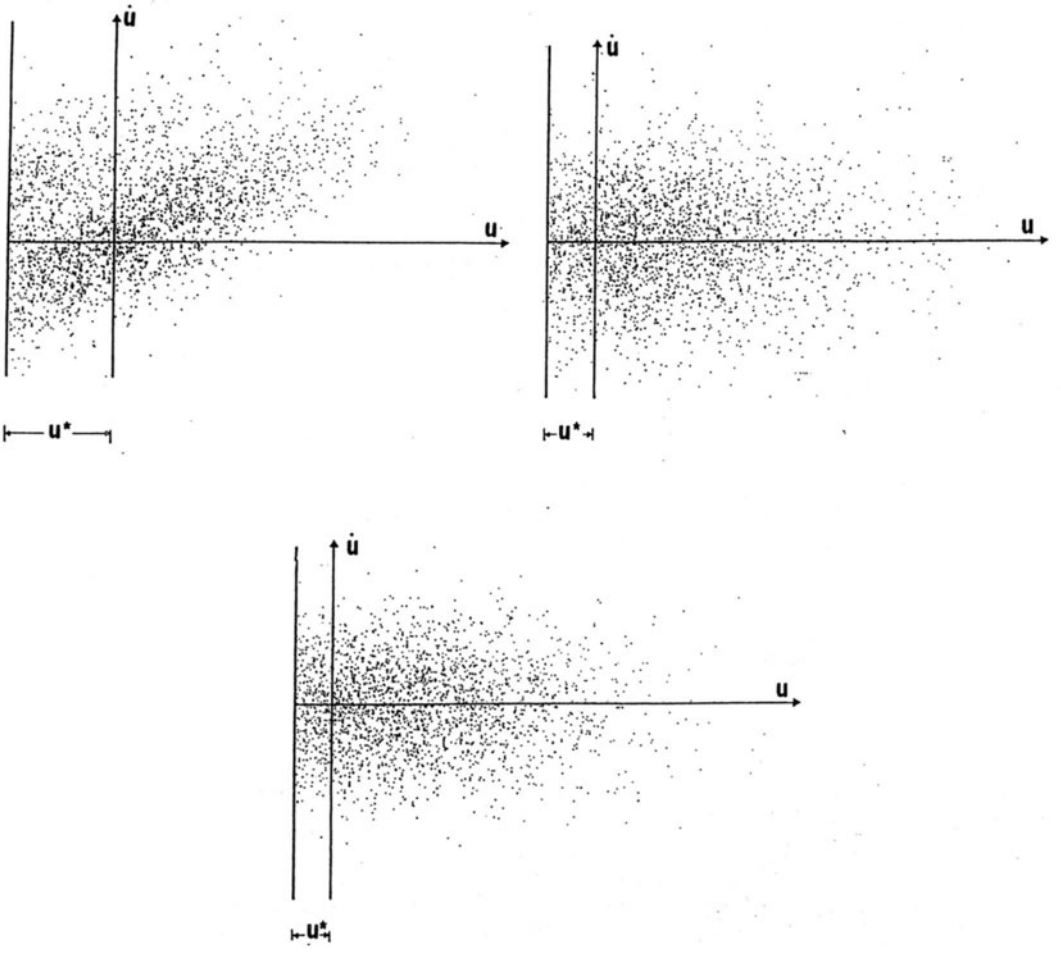

Fig.12: Instantaneous JPDF of (u,u̇) in the phase space as recorded by digital simulation.
a): t=0.1sec; b): t=0.3sec; c): t=1.0sec.

$\delta t = 0.001$ - that is of the same order of the step that has proved efficient for the deterministic integration of the equation of the motion - this problem is definitely overcome. However, in performing the numerical integrations, positiveness must be kept under control, and the time-lag reduced as the problem is encountered. It is worthwhile to note, however, that there was only a little loss in the efficiency of the procedure, in the sample studies reported here.

Finally, the evolution of the moments of the response (not to be confused with the parameters of the fundamental Gaussian μ_u, $\mu_{\dot{u}}$, σ_u, $\sigma_{\dot{u}}$, $\rho_{u\dot{u}}$), as calculated through the fundamental Gaussian after its parameters have been calculated through integration of eqs. (111) to (115), is shown by the heavy lines in Fig.s 13a,13b,13c,13d,13e for the case of elastic impact (r=1). A comparison is made with the evolution of the same quantities supplied by the simulation referred to in above, and plotted in the same diagrams as squares joined by dashed lines

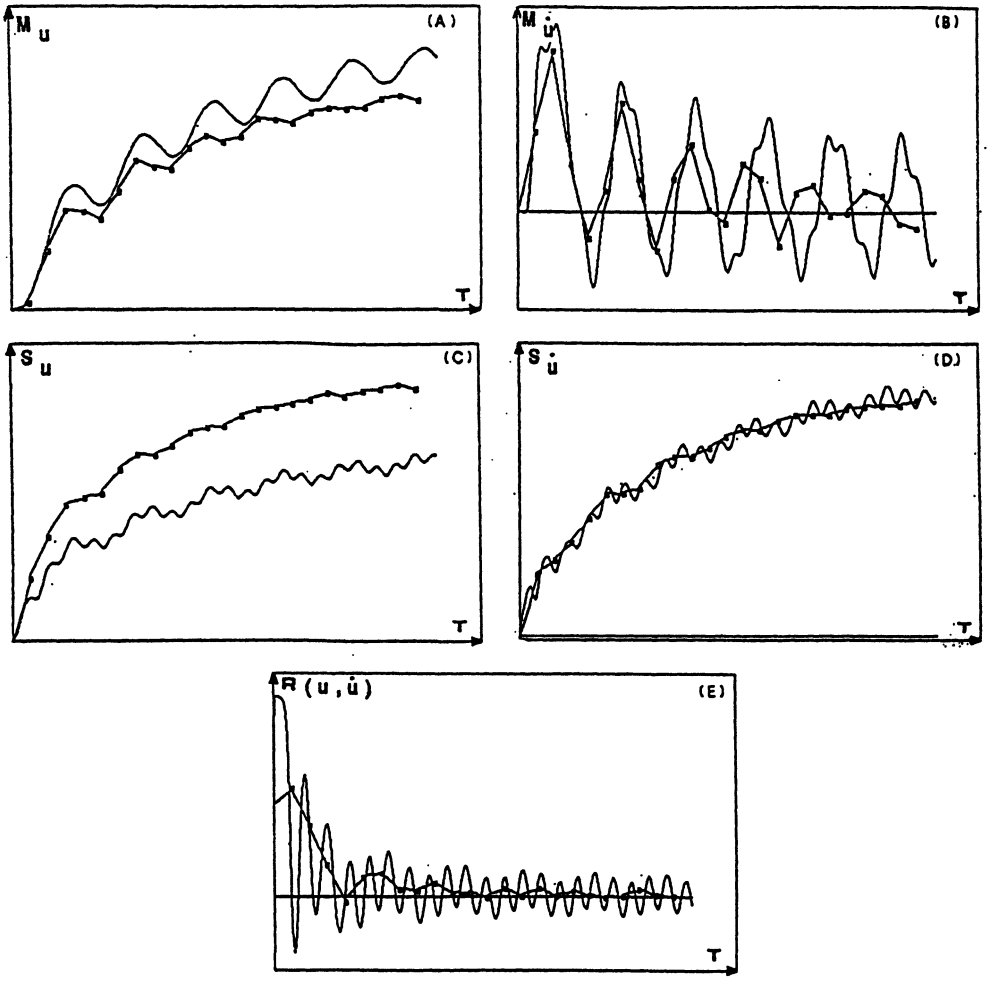

Fig.13: Montecarlo versus calculated statistics: r=1.
a): Expected displacement;
b): Expected velocity;
c): Standard deviation of displacement;
d): Standard deviation of velocity;
e): Coefficient of correlation of velocity and displacement.

$$m_u(t) = h(t) \int_{-\infty}^{+\infty} d\hat{u} \int_{u^*}^{+\infty} u\, N(u,\hat{u};t)\, du$$

$$m_{\dot{u}}(t) = h(t) \int_{-\infty}^{+\infty} \hat{u} d\hat{u} \int_{u^*}^{+\infty} N(u,\hat{u};t)\, du$$

$$s^2_u(t) = h(t) \int_{-\infty}^{+\infty} d\hat{u} \int_{u^*}^{+\infty} [u - m_u(t)]^2\, N(u,\hat{u};t)\, du \qquad (117)$$

$$s^2_{\dot{u}}(t) = h(t) \int_{-\infty}^{+\infty} [\hat{u} - m_{\dot{u}}(t)]^2\, d\hat{u} \int_{u^*}^{+\infty} N(u,\hat{u};t)\, du$$

$$C_{u\dot{u}}(t) = h(t) \int_{-\infty}^{+\infty} [\hat{u} - m_{\dot{u}}(t)]\, d\hat{u} \int_{u^*}^{+\infty} [u - m_u(t)]\, N(u,\hat{u};t)\, du$$

It should be observed that only a few points are obtained from MonteCarlo simulation; so that the oscillatory character of the curves is partially lost in the plots derived from simulation; however, it is quite evident in the results relating to the approximate solution of the FPK.

It can be seen that a reasonable agreement exists between simulation and theory. Thus, the numerical tests can be considered to be rather satisfactory, especially when one bears in mind that the procedure does not require sophisticated mathematics, and that calculations can be speedily performed, especially if analytical integrations are carried out, rather than numerical quadratures at every step of the solution procedure of the differential equations for the parameters of the fundamental Gaussian (for details, see [5]).

5) CONCLUSIONS

In this chapter, a technical approach to the FPK equation has been attempted, and an approximate procedure, nearly following the closure approach has been proposed and implemented for a number of possible non linearities in the restoring force.

The approach seems to be rather efficient and possibly open to be generalized according to the following lines:

i) Following the lines set in Sec. 3, it seems possible to set up a FPK-wise equation for a more general class of stochastic forcing functions, other than Gaussian shot-noise;

ii) The extension to multi-degree of freedom systems seems quite straightforward;

iii) Non-Gaussian closures seem possible, like partially attempted in sec. 5, thus opening the way to Galerkin-type procedures; a research effort shall be done to this aim;

iv) Implementation to higher-order nonlinearities can be pursued by the introduction of the moment generatrix function.

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Chapter 4

COMPUTATIONAL ASPECTS IN CHAOTIC AND STOCHASTIC DYNAMICS

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4.1 SOME INTRODUCTORY ASPECTS.

The computational aspects of the study of chaotic and stochastic motion can be framed in that is presently know as **numerical dynamics**. With these two words, one defines the modelling, the simulation and the understanding of an evolutionary physical system by means of the computer. It is to stress that, by means of the simulation of the physical system through a *computer experiment*, one hopes to achieve a better understanding of the unknown and forbidden structures of the system [1, 2, 3, 4, 5, 6]. In this way the study of a dynamical system through the computer is sometimes similar to the classical experimental work.
One reason of the increasing use of numerical dynamics is the *complexity* of the systems considered in engineering applications [7]. It results from

- the *nonlinearities* intrinsic to the system;
- the *sensitivity to initial conditions*;
- the *high number of dynamical degrees of freedom* that the system shows.

As a conseguence of the first item, the superposition principle is no longer valid and the use of a perturbation technique is quite inaccurate [8]. The second ground is characteristic of the chaotic systems, which are *ill-posed* in the Hadamard sense. The third ground is related to the formidable quantity of computations to be performed: it would be prohibitive to conduct them analytically.
An important contribute to the development of numerical dynamics comes from the *computer graphics* [2, 9, 11]. It permits one:

- to *synthesize* the large quantity of numerical data;
- to recognize the *spatial and temporal structures* of the system behaviour and to observe its evolution.

4.1.1. INTRODUCING NUMERICAL DYNAMICS.

One considers the evolutionary problem (see Equations (4) and (5) of Chapter 1)

$$\dot{\mathbf{y}} = \mathbf{Q}(\mathbf{y}, t) \tag{1}$$

with the initial condition

$$\mathbf{y}(t=0) = \mathbf{b} \tag{2}$$

For the mathematical definition of the problem one refers to [11, 12, 13, 14]: here one points out that Equation (1) with initial condition (2) define a **dynamical system** with n degrees of freedom , n being the number of components of the vector $\mathbf{y}$ [5, 15, 16].

In this work, one considers as example the **Duffing oscillator**, which is governed by the equation

$$\ddot{x} + d \cdot \dot{x} + (x - x^3) = A\cos(t) + w(t) \tag{3}$$

in which t denotes the time and x is the position, $\dot{x}$ the velocity, and $\ddot{x}$ the acceleration of the oscillator of mass 1 and damping d ($d = 0.185$). The term $(x - x^3)$ represents the elastic non linear restoring force, while the forcing term consists in two parts, one ($A\cos(t)$) harmonic with period $T_F = 2 \cdot \pi$ and the other ($w(t)$) random, i.e. a Gaussian white noise with intensity σ_0. With the positions $x_1 = x$ and $x_2 = \dot{x}$, one can rewrite the second order differential Equation (3) as the following nonautonomous system of first order differential equations

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -d \cdot x_2 - (x_1 - x_1^3) + A\cos(t) + w(t) \end{cases} \tag{4}$$

that has exactly the form of Equation (1), with

$$\mathbf{y} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \tag{5}$$

A natural way to represent the dynamical system of Equation (1) is to do it in the **phase space** S^n, that is an Euclidean space with number of dimension equals to the number n of the components of the vector $\mathbf{y}(t)$. In this manner, the point having as coordinates the components of the vector $\mathbf{y}$ is representative of the status of the system at instant t; the trajectory of the system is visualized by a line (t being a parameter).

One can further imagine a collection of identically systems, each of them characterized by an index k as

$$\dot{\mathbf{y}}^{(k)} = \mathbf{Q}(\mathbf{y}^{(k)}, t) \tag{6}$$

with the collection of initial conditions

$$\mathbf{y}^{(k)}(t=0) = \mathbf{b}^{(k)} \tag{7}$$

One defines these systems a **Gibbs set** [17, 18, 19] related to the system of Equation (1).
The Gibbs set can be interpreted in this way. Equation (1) defines, for different initial conditions (2), different trajectories in S^n: they can be regarded as the trajectories of the particles of a fluid during its motion in a space that coincides with S^n. In this way, Equation (6) represents the **flux** in S^n: the top of Figure 1 shows the single trajectory for the Duffing oscillator with $A = 0.3$, whereas the bottom of the same figure shows several (100) trajectories all togheter.

The set of the phase space S^n to which the trajectory of the flux tends for $t \to +\infty$, are called **attractors** [10, 15, 16, 20, 21]. From a classical point of view, the attractors can be classified as:

- **fixed point**, which corresponds to an equilibrium point;
- **limit cycle**, which corresponds to a periodic motion;
- **torus**, which corresponds to a quasi periodic motion.

A further class of attractors, can be recognized within chaotic dynamics [10, 22, 23]; it is the

- **strange attractor**.

The attractors of a system are strictly related to the stability of the system of Equation (1), because they define regions of the phase space in which the system tends to remain indefinitely, after an initial transient period.
When the system is nonlinear, it can have a multiplicity of attractors: the attractors of a dynamical system are related with the potential energy of the system. Connected with the position of the attractor and, hence of the shape of the potential energy function of the system, there is the shape of the probability density function $p(\mathbf{y}, t)$ [24, 25, 26]: this link prevents from assuming a uniform distribution even in the presence of a high intensity noise.
It is to note that the classical attractors are classical geometrical objects (like points, lines and smooth surfaces). By contrast, for the strange attractor, one has unusual geometrical objects, the *fractals* [22]: their lack of smoothness can be regarded as a requirement for numerical methods of solution, some sort of smoothness always begin implicitely requested from analytical schemes.

The starting point of numerical dynamics is the discretization in time of the problem of Equation (1). A discretized version of the dynamical system, has the form:

$$\mathbf{y}_{k+1} = \tilde{\mathbf{Q}}(\mathbf{y}_k, t_k) \tag{8}$$

in which the system is defined through a vector $\mathbf{y}_k$ at each instant $t_k, k = 1, ..K$. Equation (8) is a relationship (expressed by the approximate operator $\tilde{\mathbf{Q}}$) between the vector $\mathbf{y}_k$ and the vector $\mathbf{y}_{k+1}$ at the subsequent instant t_{k+1}. In the sense of numerical dynamics, one will no longer describe a physical system by mean of differential equations, as classically does, but by using just sets of computer instructions (or simply a *computational code*).

It is worth noting that

- a source of **noise** is always present in the real world, and hence one always has random terms in Equation (1) and in Equation (8);

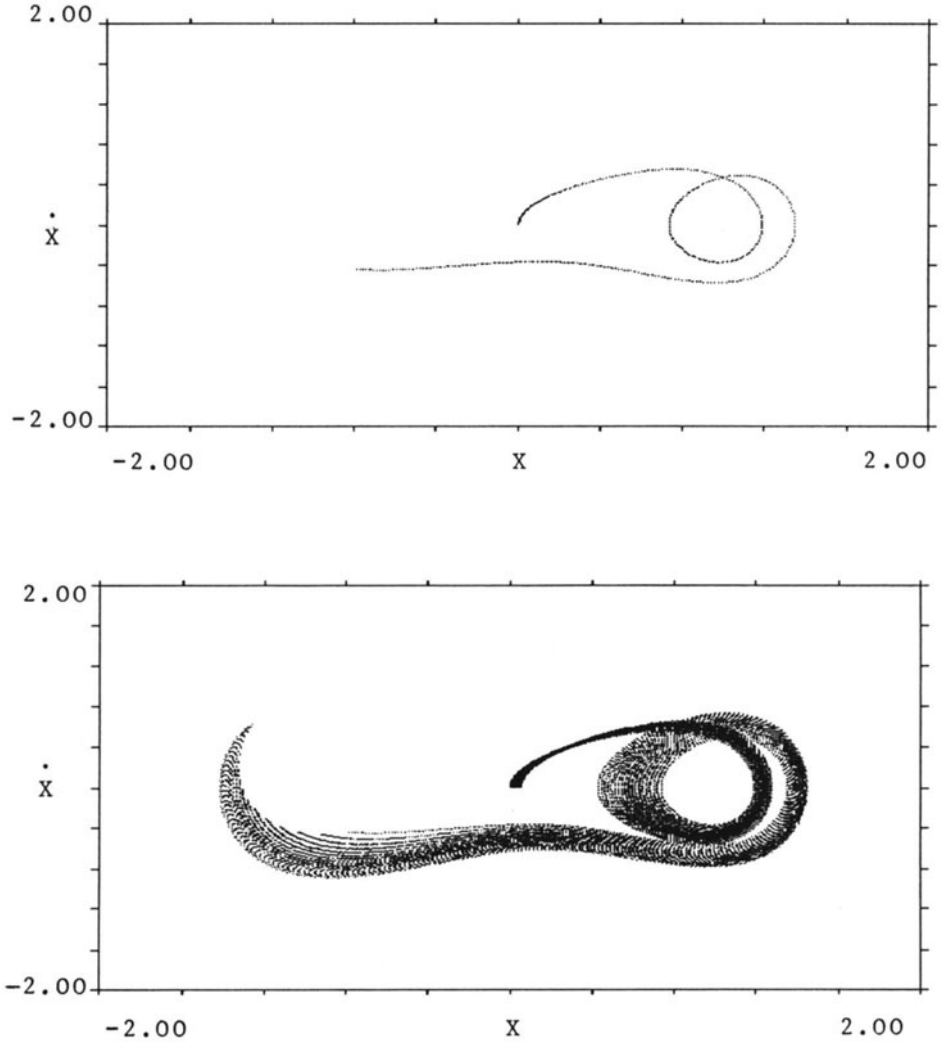

Figure 1: A single trajectory of the system in Equation (1) (top) and the corresponding flux defined by Equation (6) (bottom) (Duffing oscillator with $A = 0.3$).

- the status of a real system can be measured only with finite precision: this imply a **coarse-graining** in the space-time in which one locates the dynamical system.

For any dynamical system, one can list the following steps of analysis

qualitative analysis of the behaviour

quantitative analysis of the behaviour

construction of the probability density of the behaviour

This means:

- to bound the region of the parameters of the system in which one can have irregular motion and to known the way by which one has the transition from regular to irregular motion and vice versa;
- to give a probabilistic description of the motion of the system in the region of irregular behaviour.

4.2 BASIC TOOLS OF NUMERICAL DYNAMICS.

4.2.1 NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS.

All the numerical techniques for solving Equation (1) involve starting at the initial condition (2) and stepping along the time axis [27, 28, 29, 30, 31, 32]. Therefore one needs the incremental form of the Equation (1), that is

$$\mathbf{dy} = \dot{\mathbf{y}} \cdot dt = \mathbf{Q}(\mathbf{y}, t) \cdot dt \tag{9}$$

from which, considering a time step $\Delta t = t_{k+1} - t_k$, one obtains

$$\mathbf{y}_{k+1} - \mathbf{y}_k = \int_{t_k}^{t_{k+1}} \mathbf{dy} = \int_{t_k}^{t_{k+1}} \dot{\mathbf{y}} \cdot dt = \int_{t_k}^{t_{k+1}} \mathbf{Q}(\mathbf{y}, t) \cdot dt \tag{10}$$

Rearranging Equation (10), one obtains

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \int_{t_k}^{t_{k+1}} \mathbf{Q}(\mathbf{y}, t) \cdot dt \tag{11}$$

in which one sees that the new value $\mathbf{y}_{k+1}$ of the status of the dynamical system of Equation (1) at time t_{k+1} is obtained from the old value $\mathbf{y}_k$ at instant t_k adding a quantity resulting from a numerical integration process. This scheme is said with *one-step*, because it uses information from only one preceding point $(t_k, \mathbf{y}_k)$ to estimate the next point $(t_{k+1}, \mathbf{y}_{k+1})$.

Furthermore, comparing Equation (11) and the truncated Taylor series expansion of the solution around time t_k

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \frac{\partial \mathbf{y}_k}{\partial t} \cdot dt + \frac{\partial^2 \mathbf{y}_k}{\partial t^2} \cdot dt^2 + \dots \tag{12}$$

one can define the order of a method of integration as the highest power of the time increment dt included in this expansion.

The Runge-Kutta methods.

The general form of all Runge-Kutta methods [31, 33, 34] for advancing from time t_k to time t_{k+1} is the follows:

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \frac{\sum_{j=0}^{r-1} W_j \cdot \mathbf{K}_j}{\sum_{j=0}^{r-1} W_j} \tag{13}$$

where W_j are constant weighting coefficients and r is the order of the method. The fourth order is used in this work : one has

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \frac{1}{6} \cdot \left[\mathbf{K}_0 + 2 \cdot \mathbf{K}_1 + 2 \cdot \mathbf{K}_2 + \mathbf{K}_3 \right] \tag{14}$$

where $\mathbf{K}_0 = \mathbf{Q}(\mathbf{y}_k, t_k)$, $\mathbf{K}_1 = \mathbf{Q}(\mathbf{y}_k + \frac{1}{2} \cdot \mathbf{K}_0, t_k + \frac{1}{2} \cdot \Delta t)$, $\mathbf{K}_2 = \mathbf{Q}(\mathbf{y}_k + \frac{1}{2} \cdot \mathbf{K}_1, t_k + \frac{1}{2} \cdot \Delta t)$, and $\mathbf{K}_3 = \mathbf{Q}(\mathbf{y}_k + \mathbf{K}_2, t_k + \Delta t)$.
This kind of integration method is tipycal of physical-mathematical applications, while in engineering applications one makes a prevalent use of other methods, as the Newmark method or the Finite Difference method [28, 29, 30, 32].

4.2.2 SIMULATION OF THE RANDOMNESS.

The simulation of the real world needs a source of randomness, that is a series of random events (i.e. random numbers).
The construction of a series of this kind can be made in principle, through the observation of natural random phenomena as the decay of some isotopes. In practice, one resorts some numerical subroutines, called **random numbers generators** [6, 7, 33, 34, 35, 36], capable to supply series of pseudorandom numbers (i.e. numbers that are not truly random, due to the determinism of the computer, which is a finite machine). The series are sequences of numbers, in which the subsequent one depends on the previous number by a perfectly deterministic rule. All the numbers are therefore depending on the first one from which the generation starts: it is called the **seed**. Given the same seed, one realizes the same series of numbers. Furthermore, the series repeats itself after a period N_0 related to the word length (i.e. the capacity to represent large integer numbers) of the computer.
Among several alternatives [7], this section illustrates the **linear congruence method**, which has been adopted in all the reference examples.
Numerically it consists of the following steps [33, 37, 38]:

- one chooses two constants a and b (for example $a = 9821$ and $b = 0.211322$);
- one chooses two seeds u_0 and v_0 (for example $u_0 = 0.4$ and $v_0 = 0.6$);

- one computes two new numbers u_{k+1} and v_{k+1} for $k = 0, ..h$ using

$$\begin{aligned} u_{k+1} &= [a \cdot u_k + b] \text{mod}[1] \\ v_{k+1} &= [a \cdot v_k + b] \text{mod}[1] \end{aligned} \tag{15}$$

In this way one has a series of real number u_k (or v_k) uniformly distributed in $[0, 1]$.

To be sure that a series of pseudorandom numbers generated by some algorithm is sufficiently accurate, one must conduct some test of randomness. The dominant one has the following features:

- subdivide the interval $[0, 1]$ into N_{bin} intervals $B_j, j = 1, ..N_{bin}$, for which $\cup_{j=1,N_{bin}} B_j = [0, 1]$ and $B_i \cap B_j = \emptyset, i \neq j$;
- find the frequencies f_{B_j} that a number of the series is in the generic j-th interval;
- compute the quantity

$$e^2 = \sum_{i=1}^{N_{bin}} \left(f_{B_j} - \frac{1}{N_{bin}} \right)^2 \tag{16}$$

which is an indicator of the divergence of the distribution of the frequencies from the uniform idealization (see Figure 2).

The random number generator of Equation (15), can be regarded as a map (i.e. a transformation between points, i.e. vectors, of an Euclidean space) [9, 23, 39, 40]

$$\mathbf{u}_{k+1} = \mathbf{F}(\mathbf{u}_k) \tag{17}$$

the vector $\mathbf{u}_k$ being

$$\mathbf{u}_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix} \tag{18}$$

i.e. a special form of dynamical system.

The **Wiener process** defined in Chapter 2 has, among others, the property that for $t > 0$, $E[(\Delta W/\Delta t)^2] = \sigma_0^2/\Delta t$, $\Delta W/\Delta t$ being a zero-mean normally distributed random variable. A Gaussian signal with a constant power density spectrum, the **White Noise** $w(t)$, is obtained as the formal derivative in time of the Wiener process (whose realizations however are not differentiable [7, 36]):

$$w(t) \cdot dt = dW(t) \tag{19}$$

Numerically, an algorithm that simulates this process can therefore consist of the following steps

- consider an increment of time Δt for the evolution of the process;

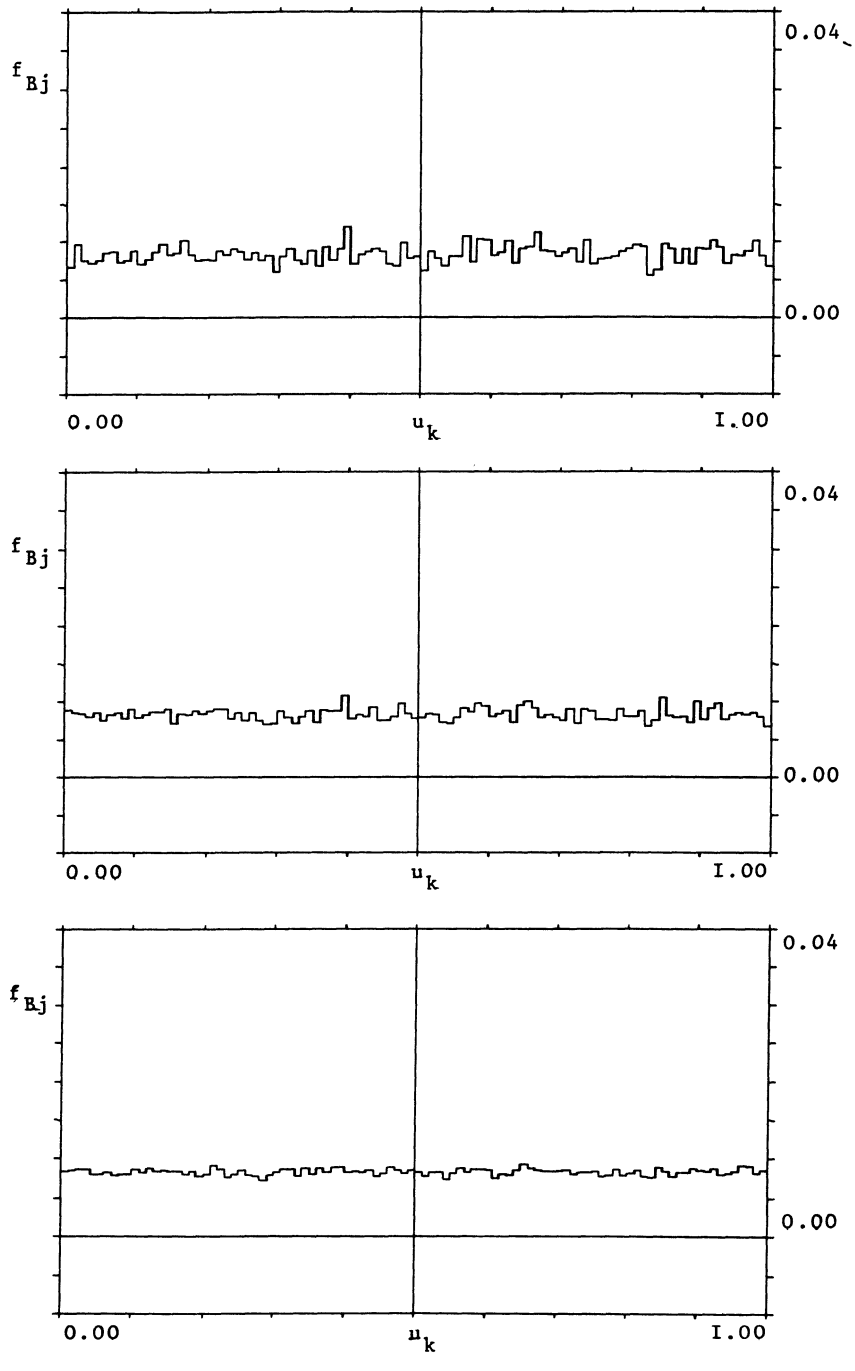

Figure 2: Test of randomness by computing the class frequencies: the frequencies f_{B_j} of 5000, 10000 and 50000 generated numbers grouped in 100 intervals over the range $[0, 1]$ are given from the top to the bottom.

- compute two new values of the random series u_{k+1} and v_{k+1} for $k = 0, ..n$ using the linear congruence generator;
- compute the random number R_{k+1} for $k = 0, ..h$ with

$$R_{k+1} = [-2 \cdot \ln u_{k+1}]^{\frac{1}{2}} \cdot \cos(2\pi v_{k+1}) \tag{20}$$

that supply a series of normally distributed numbers;

- divide each R_{k+1} by $\sqrt{\Delta t}$; the result is a sequence of independent Gaussian variables that can be regarded as Wiener process increments $\Delta W/\Delta t$, σ_0 being 1; Equation (19) shows that this sequence can also be regarded as the realization of a white noise.

In Figure 3, one sees the test for normality by grouping in 100 classes the number R_n. In Figure 4, a single realization of the Wiener process, the correspondent white noise, and the normal distribution of 1000 realization of the Wiener process are given from the top to the bottom.

4.2.3 STUDY OF MANY TRAJECTORIES: THE GIBBS SET.

Reconsider the collection of N dynamical systems of Equation (6) with the initial conditions of Equation (7). The **size** L of the Gibbs set in the phase space (which is Euclidean) can be defined either

$$L_{max}(t) = \max_{i,j=1,..N, i \neq j} \|\mathbf{y}^{(i)} - \mathbf{y}^{(j)}\| \tag{21}$$

or

$$L_{med}(t) = \frac{\sum_{i,j=1,..N; i \neq j} \|\mathbf{y}^{(i)} - \mathbf{y}^{(j)}\|}{N \cdot (N-1)} \tag{22}$$

where N is the number of system of the Gibbs set and $\mathbf{y}^{(i)}$ is the representative vector of the i-th system in the phase space.
It is to observe that through the collection of initial conditions of Equation (7), one can simulate the uncertainty about the true initial conditions of the system of Equation (1). Then, one can expect the following kinds of behaviour of the Gibbs set in time as illustrated in Figure 5. For a problem of classical kind, the size of the Gibbs set remains substantially the same. That means that the uncertainty about the location of the system remains constant. In the other cases of chaotic or stochastic dynamics, the size of the collection grows in time. That means that one loses information about the true status of the system, when the initial status is uncertain. The idea is then to use the Gibbs set and its motion to discover strangeness in the dynamics of the system and to compute the statistics of the response of the system.

4.2.4 BIFURCATION DIAGRAMS.

Consider Equation (1) rewritten as

$$\dot{\mathbf{y}} = \mathbf{Q}(\mathbf{y}, t, \mu) \tag{23}$$

in which one has put in evidence the **internal parameter** μ. The type of motion of the system can be altered as the numerical value of the parameter is changed: the passagge from one kind of motion to the other, is characterized by a singularity of the behaviour, that can be a ***bifurcation*** or, in general, a *catastrophe* [16, 21, 37, 38, 41, 42].

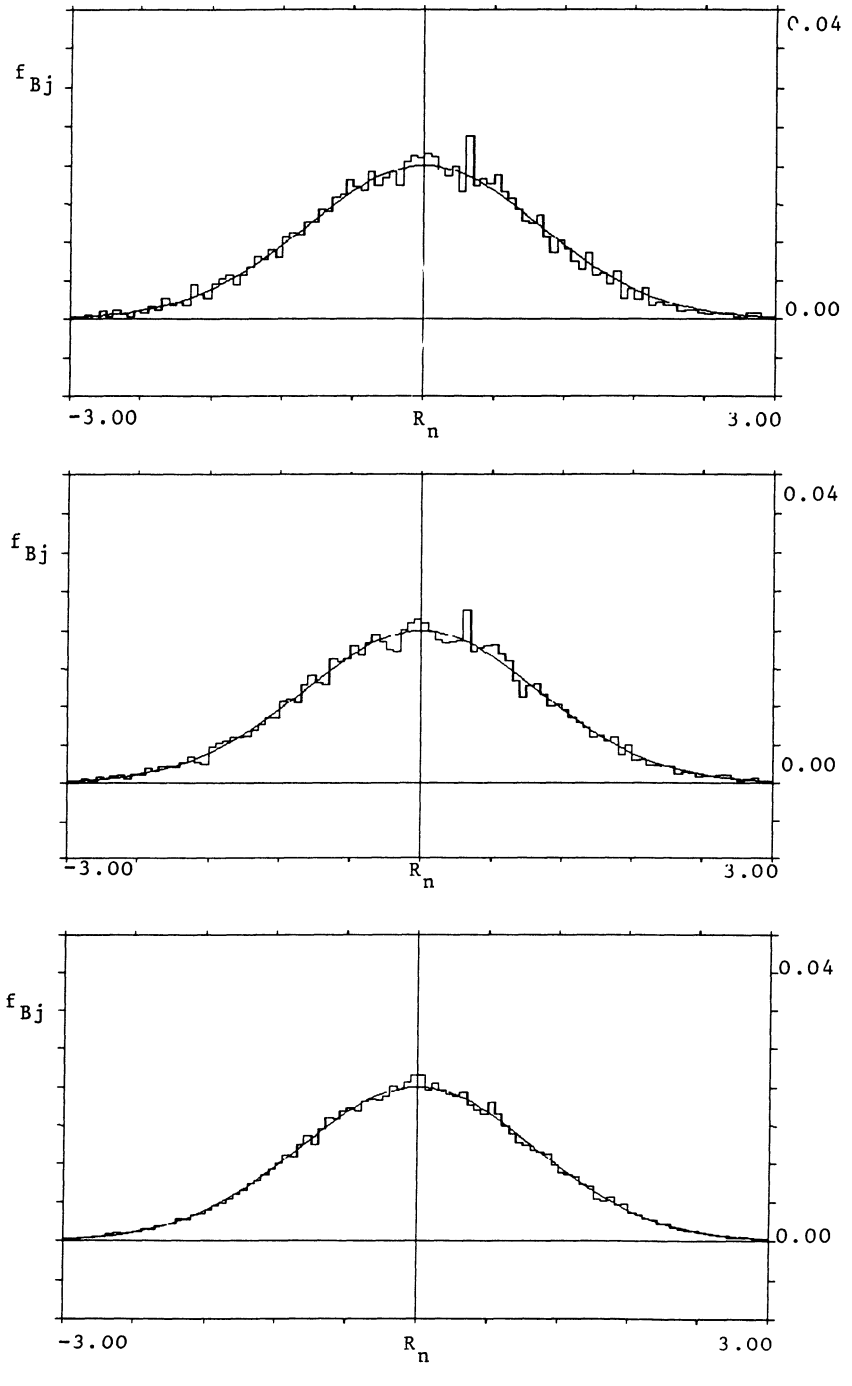

Figure 3: Test of normality for the number R_n: from the top to the bottom, 5000,10000 and 50000 numbers are generated.

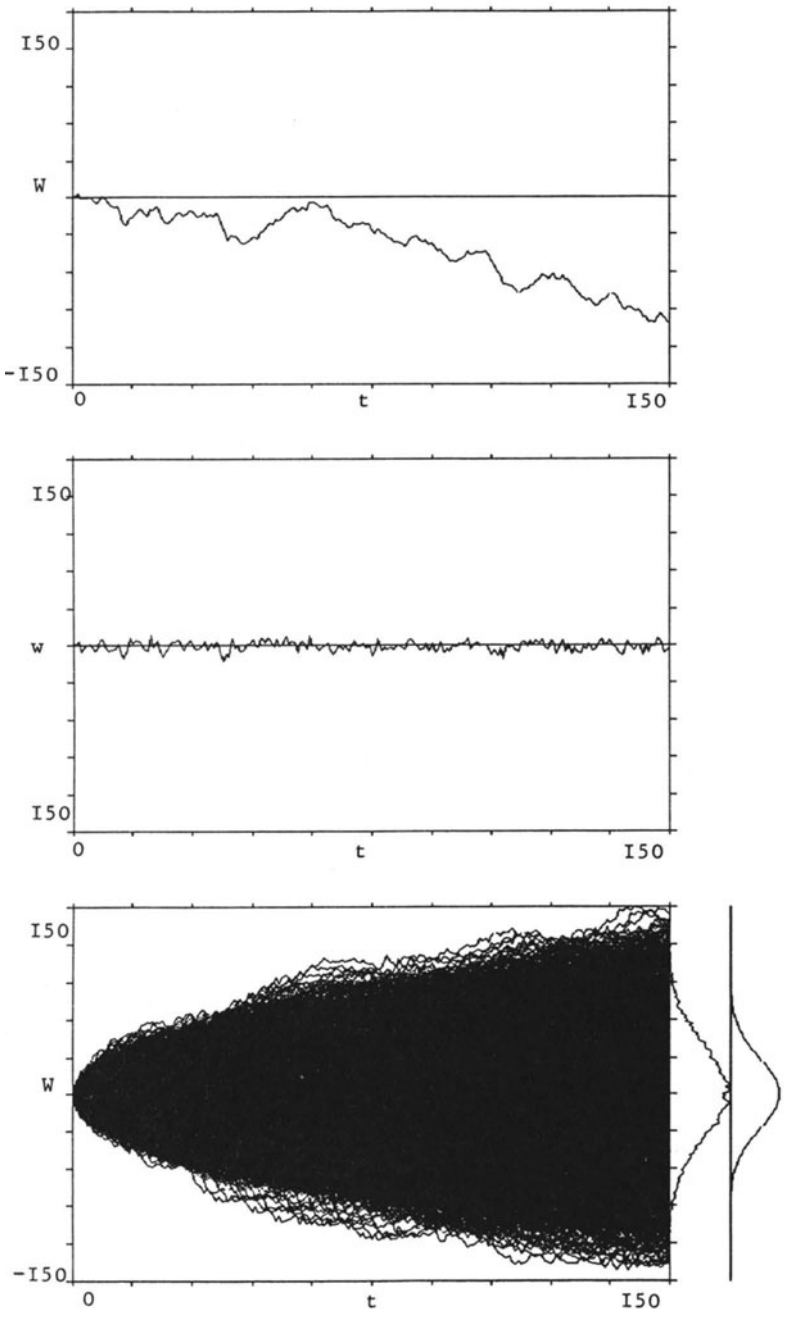

Figure 4: A single realizzation of the Wiener process and the correspondent white noise, together with the normal distribution coming from 1000 realizations.

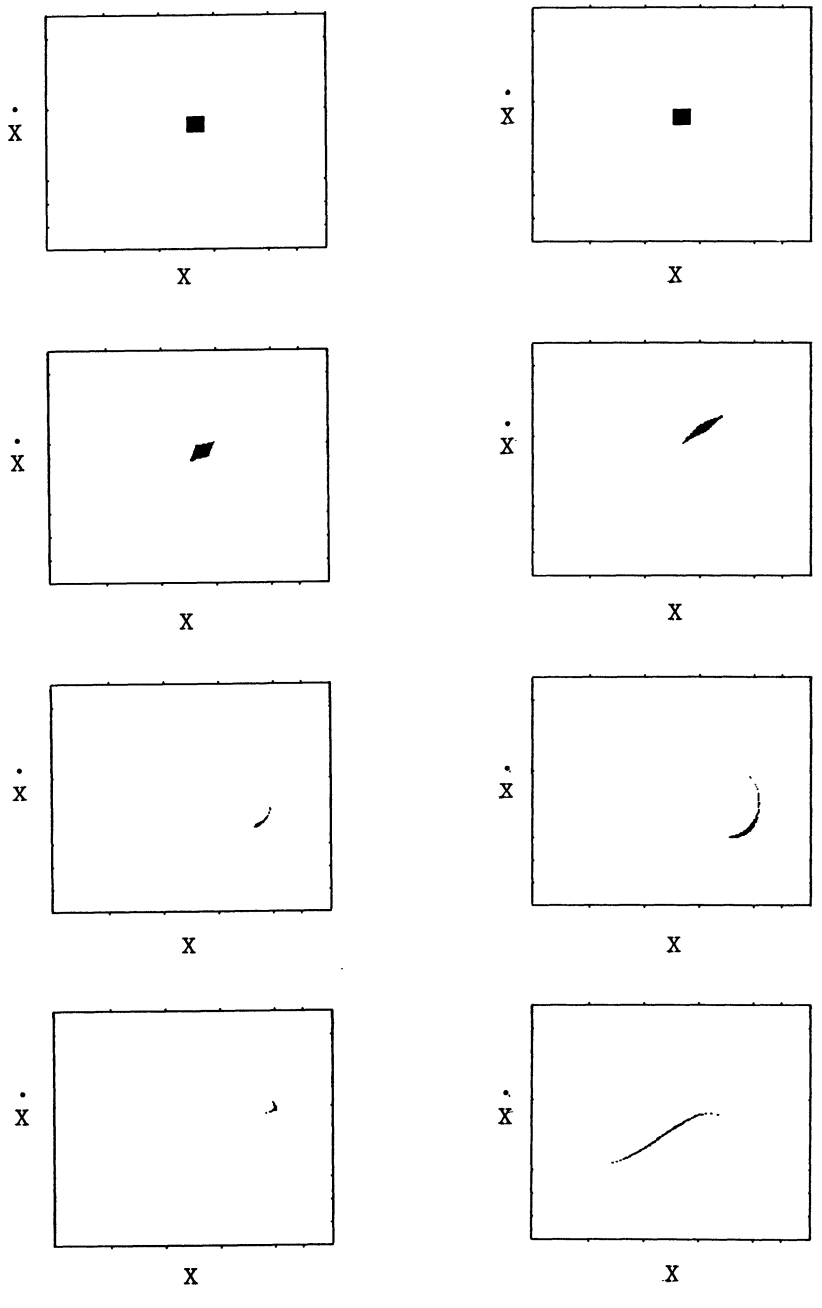

Figure 5: Different behaviours of the Gibbs set in time for the Duffing oscillator: left, regular case ($A = 0.3$); right, irregular case ($A = 0.18$).

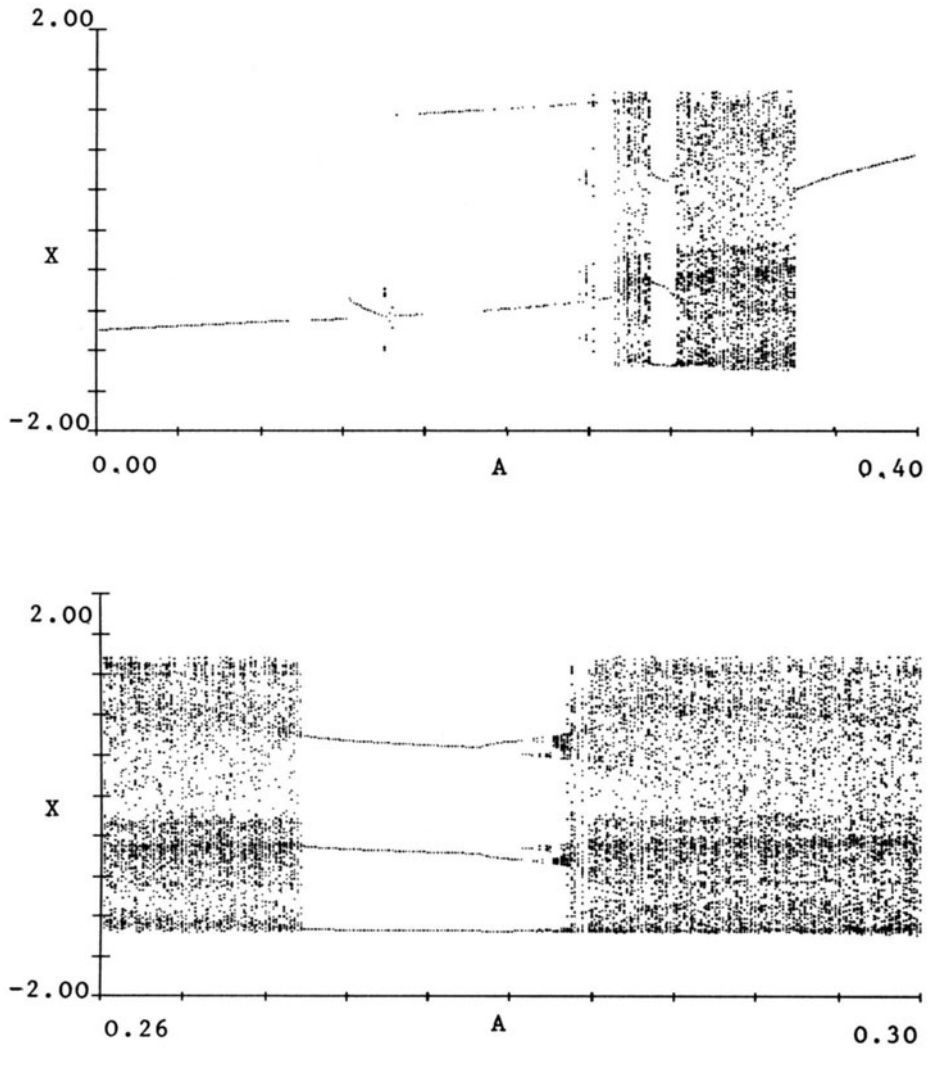

Figure 6: Bifurcation diagram of the Duffing oscillator: top, A in the range $(0.0, 0.4)$, bottom, zoom for A in the range $(0.26, 0.30)$.

In Figure 6 one studies the behaviour of the Duffing oscillator of Equation (3), the amplitude A of the harmonic force having been selected as parameter μ.
A bifurcation diagram, can be constructed in the following manner:

- choose the parameter μ, the influence of which must be studied, and a range $I_\mu = \mu_{max} - \mu_{min}$ of this parameter;
- locate $N_\mu + 1$ values of the parameter μ in this range, for example $\mu_k = \mu_{min} + k \cdot I_\mu / N_\mu, k = 0, ..N_\mu$;
- integrate for each value μ_k the equation of motion, i.e. Equation (23);
- after an initial transient period, say T_{tr}, one put $N_s + 1$ marks in a plane (μ, y_j), being y_j the j-th component of the vector $\mathbf{y}$ that defines the status of the system at time $t_n = T_{tr} + n \cdot T_{snap}, n = 0, ..N_s$. The choice of the time interval T_{snap} is critical, because it determines the appearence of the diagram. In particular, in the presence of a periodic force of period T_F, a natural choice would be $T_{snap} = T_F$; the choice of N_s, i.e. the choice of the length of the integration period, is related to the computational effort: it is always necessary to have $N_s \geq 20$ to be capable of recognizing an irregular behaviour;
- repeat the previous two steps, for all the values μ_k.

With this technique, one recognizes:

- the kind of motion of the system, for a fixed value of the parameter μ, namely:
  - a stationary or a periodic motion is revealed by a single point;
  - a quasiperiodic motion is revealed by many points, uniformly arranged;
  - a chaotic or stochastic motion (i.e. an irregular behaviour) is revealed by many points, arranged in a disordered way.
- the value of μ for which the system goes from one kind of motion to the other; this **critical value** of the parameter denotes the presence of a bifurcation or a catastrophe.

One can point out some particularities of the chaotic behaviour through the analysis of Figure 6. The top of this figure, shows the bifurcation diagram of the Duffing oscillator for $\mu = A$ in the range $(0.0, 0.4)$. One recognizes in particular

- a large region of irregular behaviour between $A = 2.5 \div 3.0$, with some periodic windows (for example around $A = 2.8$); this mixing of aperiodic and periodic motion is typical of the chaotic behaviour;
- a second region of irregularity, around $A = 1.4$; in fact also this region is chaotic, as one will see in the following by the evaluation of the Lyapunov exponents.

The bottom of the Figure 6, zooms over the range $(0.26, 0.30)$.
In Figure 7, one sees the effects of the presence of white noise (with $\sigma_0 = 0.1$) on the diagram. It reveals a more diffuse distribution of points, and, in particular, it does not reveal the fractal structure of the previous figure. This is a general law: the noise breaks down the fractal structure of the pure chaotic motion. The presence of two large branches, for $A < 1.2$, is due to the presence of the two minima of the potential energy of the oscillator, which determines the associated attractor. The bottom of Figure 7, shows how the periodic window disappears, due to the presence of noise.

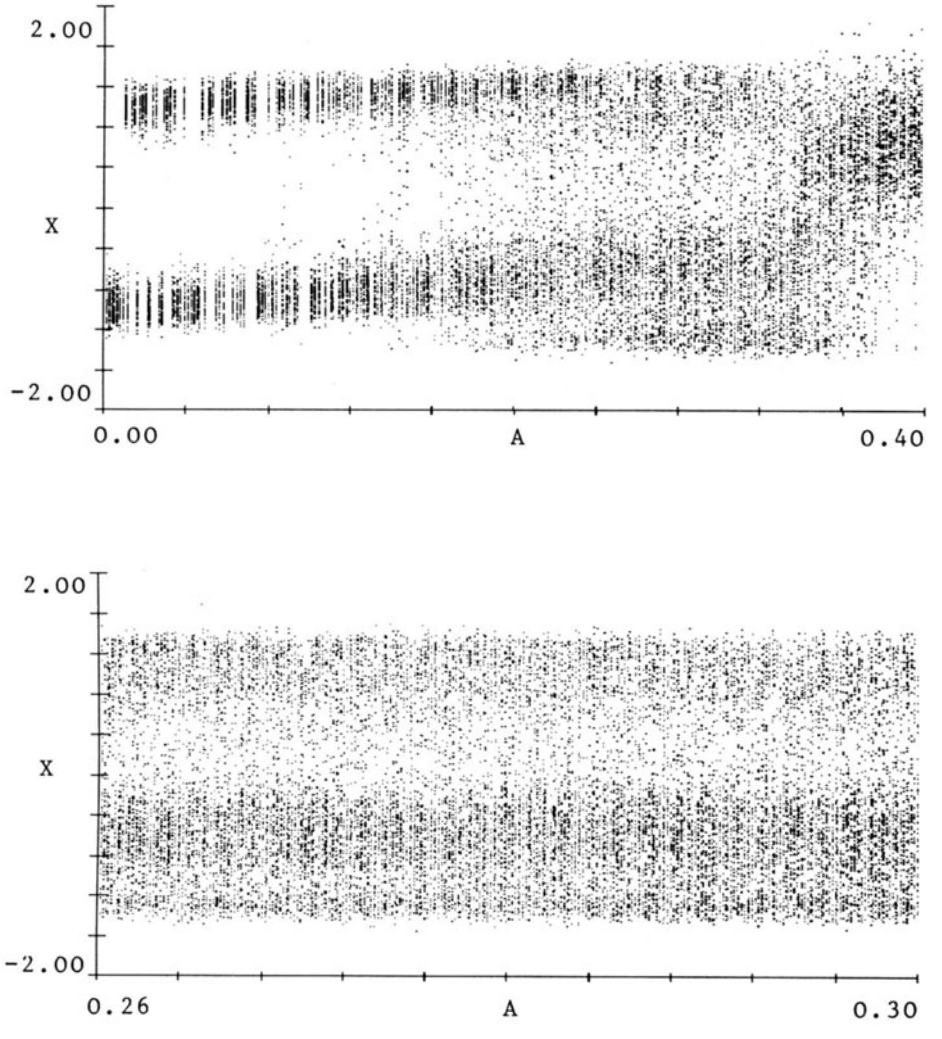

Figure 7: Bifurcation diagram of the Duffing oscillator with white noise of intensity $\sigma_0 = 0.1$: top, A in the range $(0.0, 0.4)$, bottom, zoom for A in the range $(0.26, 0.30)$.

4.2.5 THE POINCARE' SECTIONS.

The observation of the behaviour of the system in Equation (1) is made on a continuous basis through its representation in the phase space S^n. One can also observe the system in a discontinuous way at the end of time intervals of duration T_{snap}. One obtains the status of the system at discrete instants of time $t_k = T_{tr} + T_{snap} \cdot k, k = 0, ..N_s$, and the Poincare' section of the system in the phase space S^n, as show in Figure 8.
The aspect of a Poincare' section crucially depends on the choice of the instants of observation: as in the construction of the bifurcation diagram, when a periodic force is present, a convenient choice is $T_{snap} = T_F$, where T_F is the period of the force. When such a periodic term does not exist, the choice of the interval of sampling is not univocal, and the consequent look of the Poincare' section, can vary in a tremendous manner, as one can see in Figure (9).

When one has discarded the transient, the distribution of the points representative of the motion of the dynamical system in Equation (1) is indicative of the kind of dynamics.
With reference to Figure 10, one can in fact recognize that

- a finite number of points in the Poincare' section means equilibrium, or periodic (or subharmonic) oscillations (a,b);
- points arranged in a closed curve, denotes a quasiperiodic motion (c);
- points arranged along a opened line, are characteristic of a chaotic motion that can be modelled with a unidimensional map [15, 10])(d);
- points arranged in a fractal set, means a chaotic motion (e);
- points arranged in a random distribution are typical of either a chaotic motion with small dissipation (at the limit hamiltonian) or a stochastic motion (f).

4.2.6 BASINS OF ATTRACTIONS.

Another technique for the characterization of the behaviour of the dynamical system of Equation (1) is obtained by the following procedure:

- choose a subregion of the phase space in which the dynamic of the system occurs (for example, for the Duffing oscillator one choose a rectangular zone in its phase plane);
- superimpose a rectangular grid of N_{grid} points on this region: each of the points $((i, j)$ given by a pair of values $x_i, \dot{x}_j$ for the Duffing oscillator) can be thought as the initial condition for a single realization;
- integrate the equation of motion of the system for each choice of the initial conditions, until a convenient instant of time T_{int};
- mark the point of initial conditions in some way according to its status (for the Duffing oscillator, in black when $x(T_{int}) > 0$);
- repeat the previous two steps, for all the points of the grid.

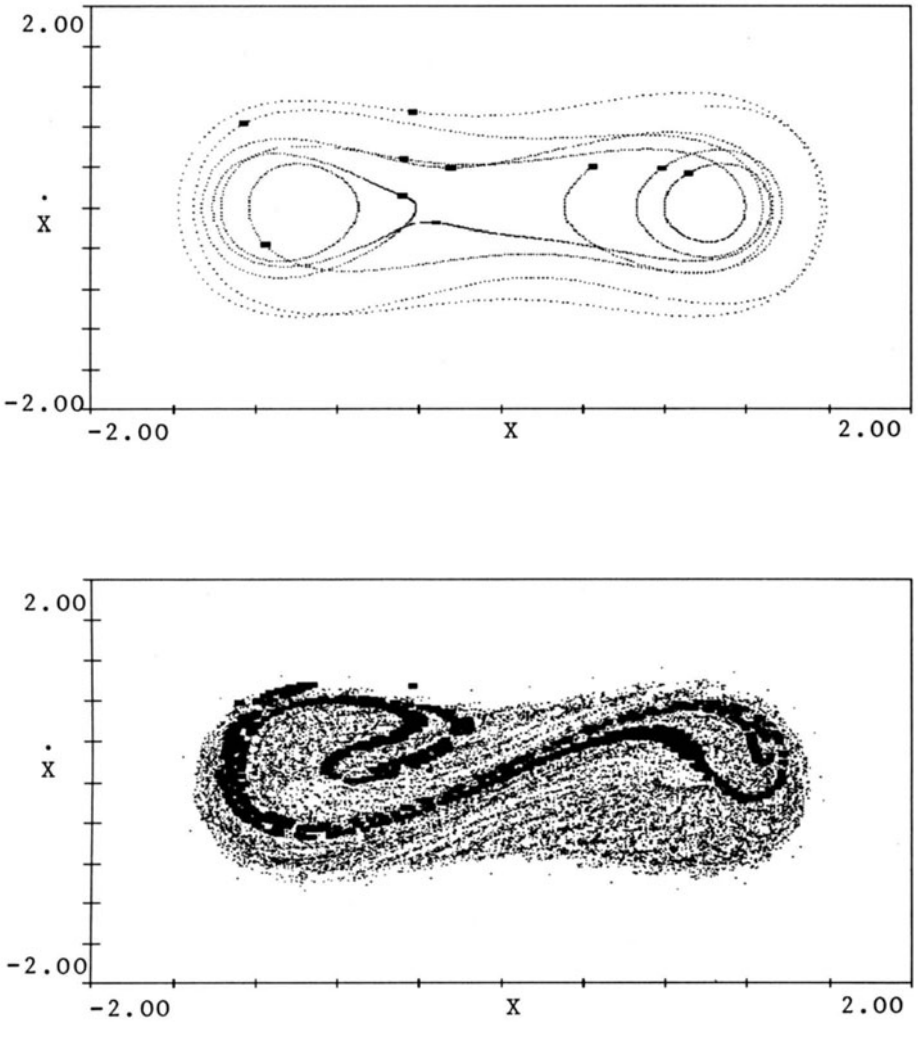

Figure 8: Observation of the dynamical system at fixed instant of time and construction of the Poincare' section (Duffing oscillator with $A = 0.3$).

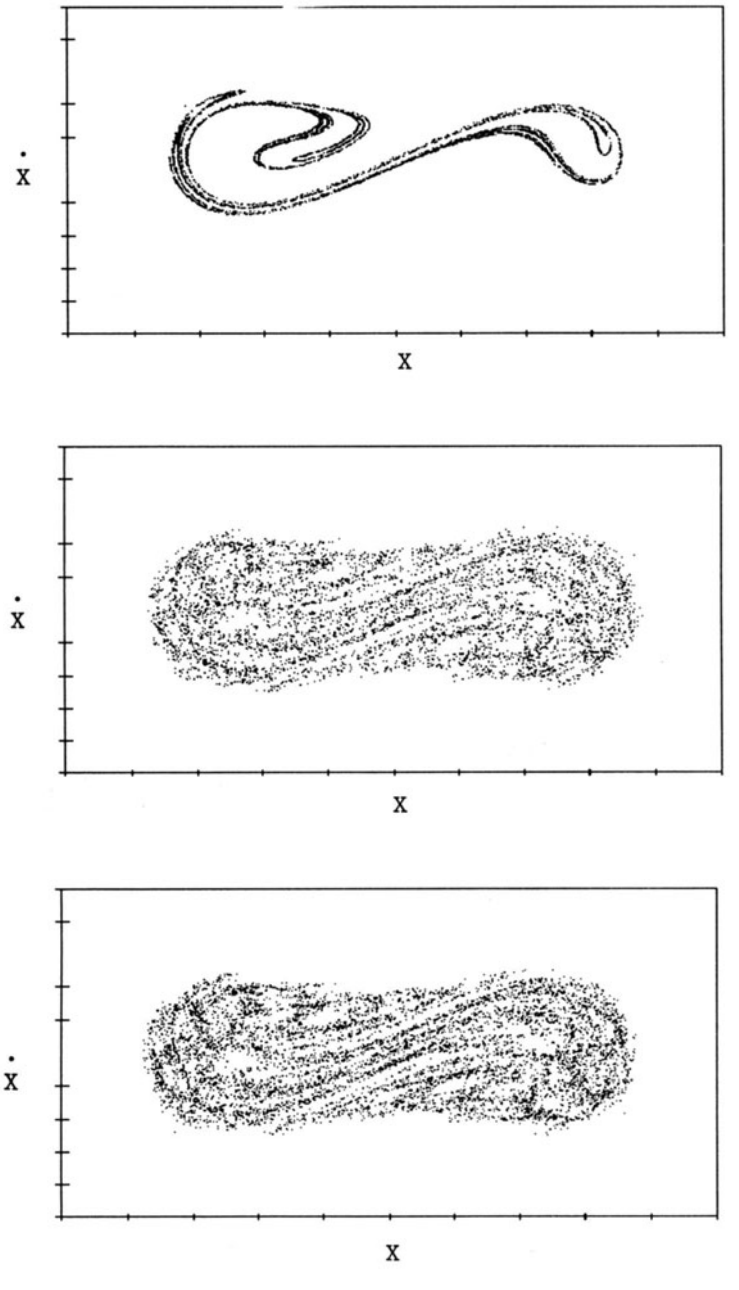

Figure 9: Aspect of the Poincare' section for different selections of T_{snap}: top, $T_{snap} = 0.7 \cdot T_F$; bottom $T_{snap} = 1.3 \cdot T_F$ (Duffing oscillator with $A = 0.3$).

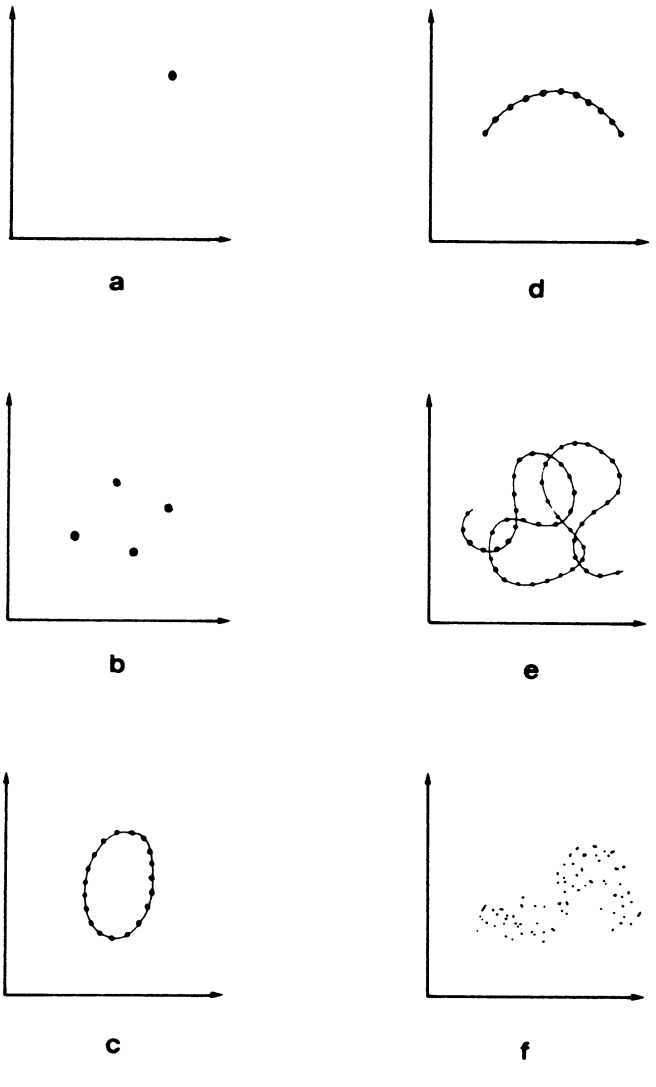

Figure 10: Different types of Poincare' sections.

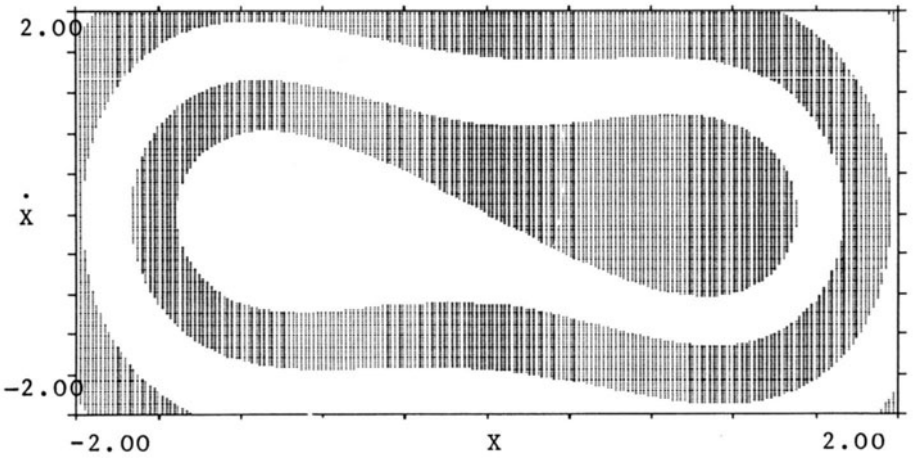

Figure 11: Basin of attraction for the unforced Duffing oscillator.

In Figure 11, one sees the basin of attraction for the square of the phase plane with both x_1 and x_2 in the range (-2, 2) and $N_{grid} = 201 \times 201$ for the unforced Duffing oscillator. In this case, one has marked with a black pixel the point representative of the initial conditions of the oscillator when it leads to a positive displacement x_1 after a period of $16 \cdot T_F$; otherwise, one has a blank. This portrait appears regular, and it is clearly related to the shape of the energy of the oscillator and, hence, with the position of the two attractors.
Introduce now the presence of a periodic excitation [16, 10, 23]. For the pure chaotic case with $A = 0.3$, one has the portrait of Figure 12: it appears very intricate, and the mixture of points that leads to a positive displacement and points that leads to a negative displacement is indicative of the sensitivity to the initial conditions. Figure 13 shows the basin of attraction in the case of a pure stochastic motion, with $\sigma_0 = 0.1$: this case reveals both the presence of the energy for the system and the sensitivity to the initial conditions.

4.2.7 RECONSTRUCTION OF THE PHASE SPACE.

Assume that a single variable is measured for the dynamical system of Equation (1) at instant of time $t_k = t_0 + k \cdot \Delta T_s$, T_s being the sampling period (for example, the time step of the numerical integration)

$$\{z_k\}, k = 1, ..K \tag{24}$$

Let n be the number of degrees of freedom of the system. One can imagine to replace the true representative vector of the status of the system $\mathbf{y}$, with another vector $\mathbf{z}$, always with n components, but obtained collecting the measured variable z_k at different instant t_k, each instant separated from the other by a time delay $T_{delay} = \Delta k \cdot T_s$. The vector $\mathbf{z}_k$ is then

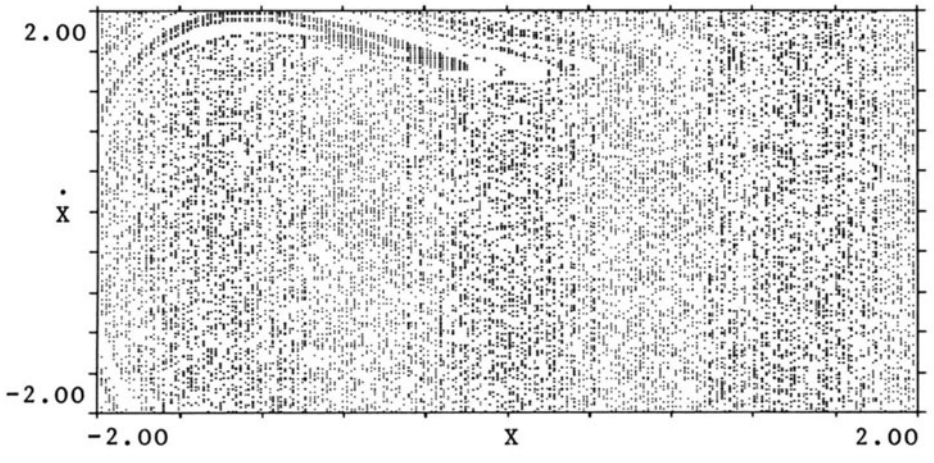

Figure 12: Basin of attraction for the forced Duffing oscillator, with $A = 0.3$.

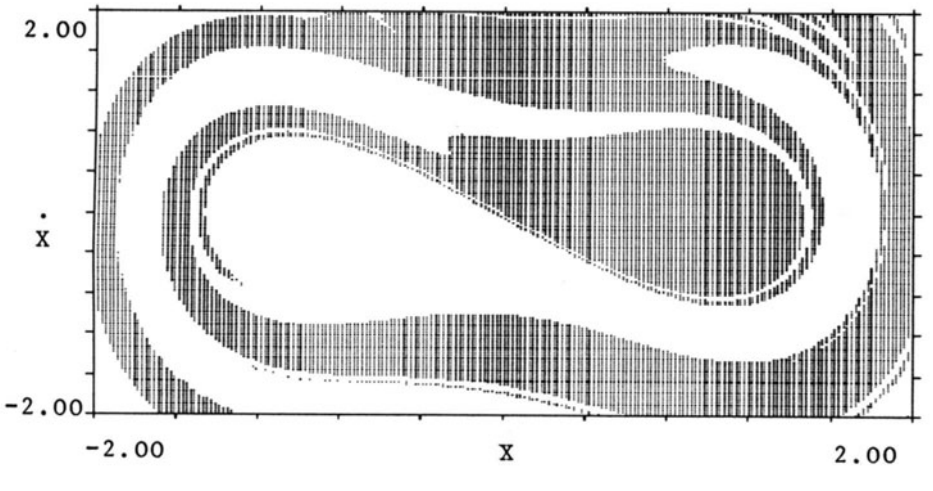

Figure 13: Basin of attraction for the forced Duffing oscillator, with $A = 0$ and noise with $\sigma_0 = 0.1$.

$$\mathbf{z}_k = \begin{pmatrix} z_k \\ z_{k+\Delta k} \\ z_{k+2\cdot\Delta k} \\ z_{k+3\cdot\Delta k} \\ \cdots \\ z_{k+(n-1)\,\Delta k} \end{pmatrix} \tag{25}$$

where the measured variable z_k appear with the delay index Δk.
The apperance of the phase plane representation for the Duffing oscillator as the delay Δk varies is represented in Figure 14. The choice of the time delay can be driven by the need of a biunivocal topological correspondence between the reconstructed phase space and the true representation [9, 23, 40].
In the reconstructed representation, one can find the tangent vector to the trajectory of the system by a finite difference scheme

$$\mathbf{v}_k = \mathbf{z}_{k+1} - \mathbf{z}_k = \begin{pmatrix} z_{k+1} - z_k \\ z_{k+\Delta k+1} - z_{k+\Delta k} \\ z_{k+2\,\Delta k+1} - z_{k+2\cdot\Delta k} \\ z_{k+3\,\Delta k+1} - z_{k+3\,\Delta k} \\ \cdots \\ z_{k+(n-1)\cdot\Delta k+1} - z_{k+(n-1)\cdot\Delta k} \end{pmatrix} \tag{26}$$

4.3. QUANTITATIVE ASPECTS.

4.3.1 THE LYAPUNOV EXPONENTS.

The Lyapunov exponents are a measure of the sensitivity to initial conditions for a dynamical system [9, 15, 10, 23, 39, 43] (see Chapter 1).
The numerical evaluation of the Lyapunov exponents differs, accordingly to the need of computing either all the exponents $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n$, or just their maximum (i.e. λ_1). For the evaluation of all the exponents, one needs the analytical knowledge of the equation of motion of the system and the related linearized equations, while the evaluation of the maximun exponent can be conducted on the basis of the knowledge of a unique time series extracted from the motion of the system (i.e. a single quantity that can be experimentally measured at a given time for a system however complex) [9, 44].

Numerical evaluation of all the Lyapunov exponents.

Considers [44] a fiduciary trajectory in a phase space.
The fiduciary trajectory obeys to the equation of motion, that for the Duffing oscillator one rewrites here in the autonomous form as

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -d \cdot x_2 - (x_1 - x_1^3) + A\cos(x_3) + w(x_3) \\ \dot{x_3} = 1 \end{cases} \tag{27}$$

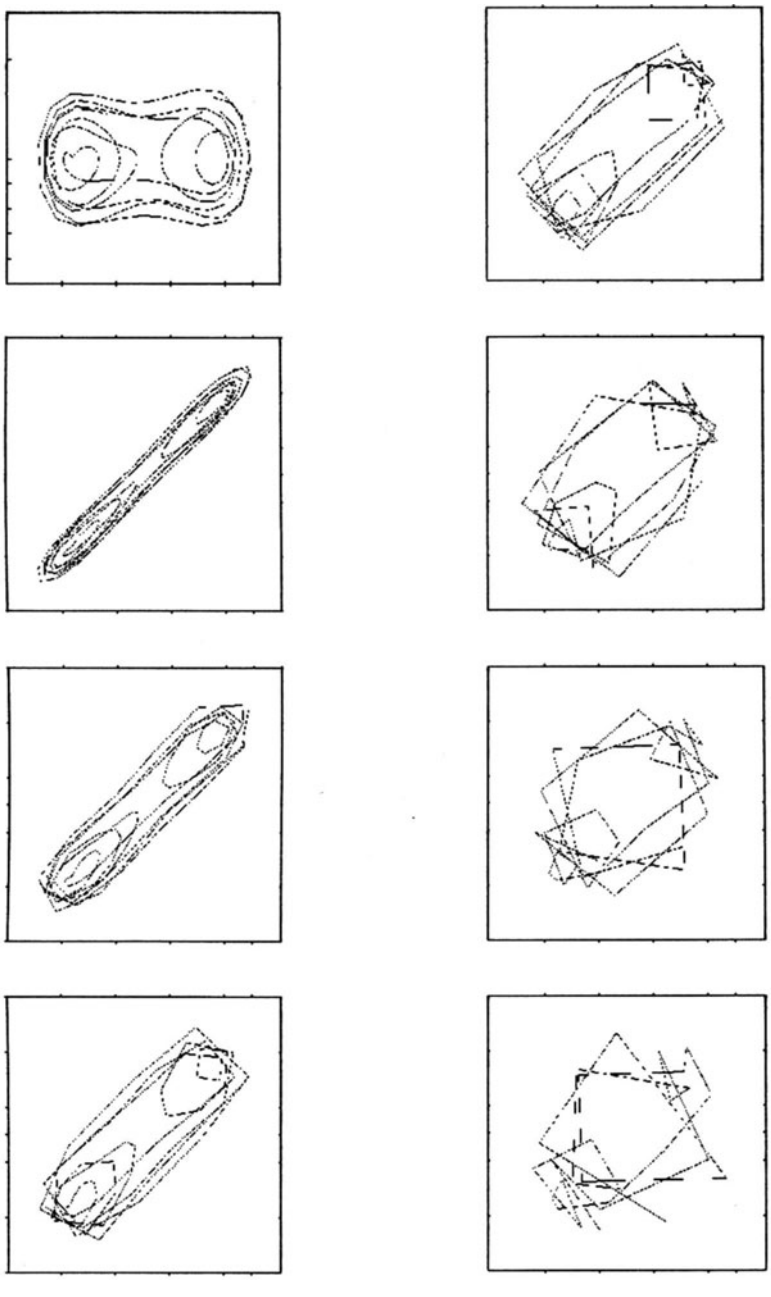

Figure 14: The appearence of the phase plane reconstruction for the Duffing oscillator, as the delay Δk varies.

in which one has made the obvious position $t = x_3$, i.e. the time is explicitly considered as a further degree of freedom.
Nearby points $\mathbf{y}+\xi$ to the fiduciary trajectory $\mathbf{y}$, obey the linearized equations of motions obtained through the Jacobian of the function $\mathbf{Q}(\mathbf{y})$:

$$\dot{\xi} = \left[\frac{\partial \mathbf{Q}}{\partial \mathbf{y}}\right] \cdot \xi \tag{28}$$

Of course, the Jacobian matrix $\left[\frac{\partial \mathbf{Q}}{\partial \mathbf{y}}\right]$ depends on the values of $\mathbf{y}$ along the fiduciary trajectory.
For the case of the Duffing oscillator (27), the Jacobian matrix has the form:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -(1-3\cdot x_1^2) & -d & -A\cdot\sin(x_3)+\dot{w}(x_3) \\ 0 & 0 & 0 \end{pmatrix} \tag{29}$$

The evolution in time of a set of $i = 1, n$ vectors ξ_i, initially orthonormal, describes the deformations of the flux in the phase space of dimension n. One must then integrate:

- the fiduciary trajectory (Eq. (27));
- the n trajectories related to the n axis of the linearized equations of motion (Eq. (28)).

The vectors ξ_i, tend as the time proceeds to collapse toward the one that grows faster, which is associated to the maximum Lyapunov exponent, as show in the top of Figure 15.
To avoid this effect, and to avoid numerical overflow, one performs at each period T_r the Gram-Schmidt renormalization [34, 44]: by this technique one pass from a set of n linearly independent vectors to a set of n orthonormally vectors. In this way, one computes the values of all the Lyapunov exponents as

$$\lambda_i = \sum_i \frac{1}{T_r} \cdot \frac{\|\xi_i(t+T_r)\|}{\|\xi_i(t)\|} \tag{30}$$

The renormalization is showed in the bottom of the Figure 15, where $T_r = T_F$.
The link between the Lyapunov exponents and the bifurcation diagram can be recognized comparing Figure 6 with Figure 16. In the latter one the Lyapunov exponent λ_1 is a function of the amplitude A of the harmonic force on the Duffing oscillator. One recognizes that for a periodic motion, λ_1 is negative, while for a chaotic situation, λ_1 is positive.

Numerical evaluation of the maximum positive Lyapunov exponent.

To compute directly the maximum Lyapunov exponent λ_1, some elementary considerations can be useful [23, 15, 44, 40]:

- one considers the time series of Equation (24) of a single variable of the dynamical system of Equation (1). This series should be sufficiently long to be representative of the stationary motion over the attractor;
- from this series, one reconstructs the phase space: so one has a vector series $\mathbf{z}_k$ and a tangent vector series $\mathbf{v}_k$ at each time t_k (see Equation (26));

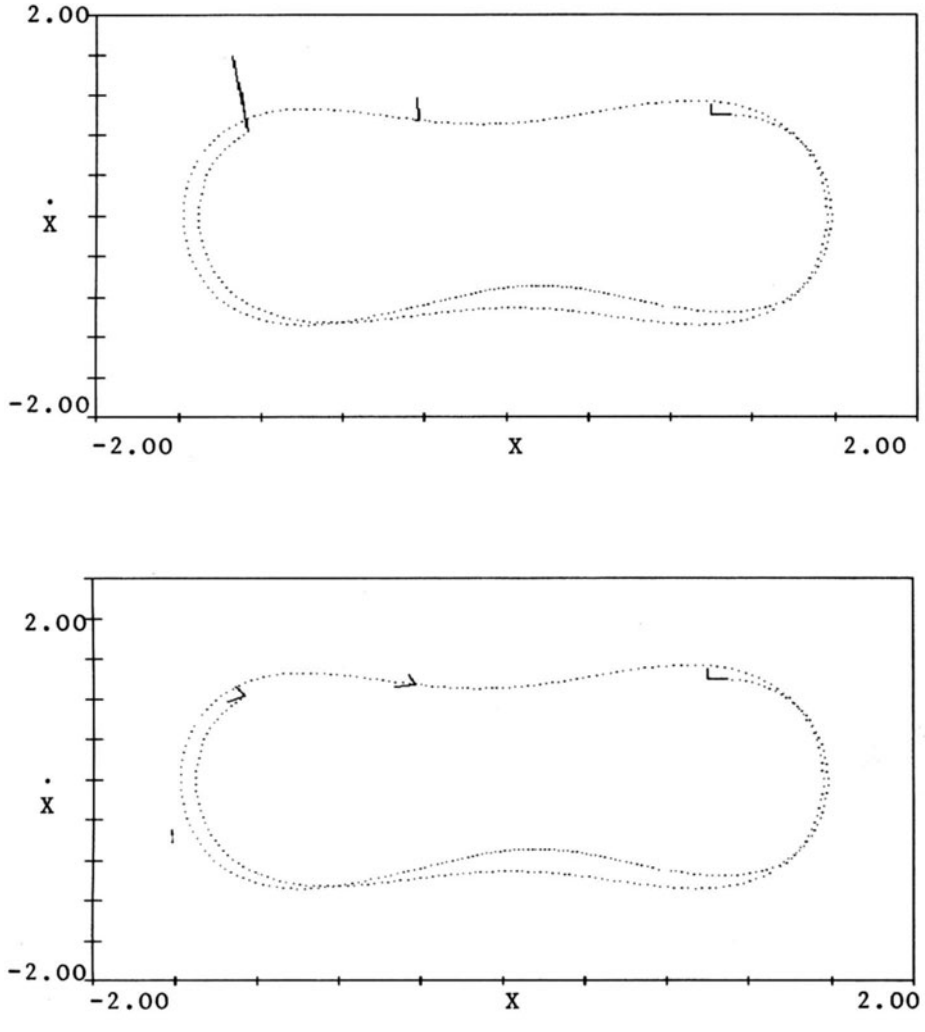

Figure 15: Evolution of the vectors ξ_i for the Duffing oscillator: top, without renormalization, bottom with renormalization.

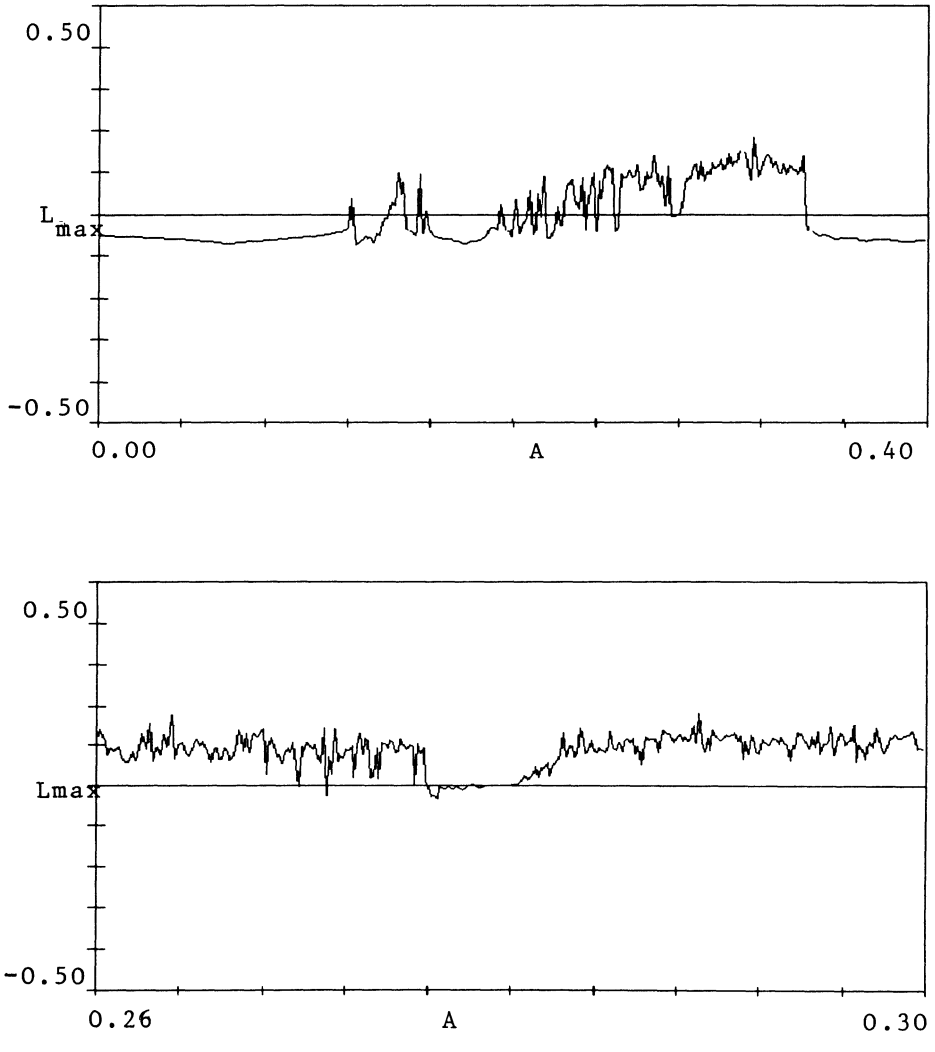

Figure 16: Lyapunov exponents λ_1 of the Duffing oscillator: top, A in the range $(0.0, 0.4)$; bottom, zoom for A in the range $(0.26, 0.30)$.

- for any index i, i.e. for any time instant t_i, one can find another index j so that

$$\epsilon_m \leq \|\mathbf{z}(t_i) - \mathbf{z}(t_j)\| \leq \epsilon_M \tag{31}$$

and that the tangent vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ to the trajectory in the phase space satisfy the relation

$$\cos\theta = \frac{(\mathbf{v}_i \cdot \mathbf{v}_j)}{\|\mathbf{v}_i\| \cdot \|\mathbf{v}_j\|} \geq \epsilon_V \tag{32}$$

that means that the points of index i and j are sufficiently near and with almost the same tangent vector, i.e. the two situation of motion are similar;

- after an interval of time τ, i.e. an increment of the index $\Delta i = \tau / T_{delay}$ (T_{delay} being, as previously defined, the delay time for the reconstructed phase space) one obtains a pair $(\mathbf{z}_{i+\Delta i}, \mathbf{z}_{j+\Delta i})$ from the initial points $(\mathbf{z}_i, \mathbf{z}_j)$; assume that

$$\|\mathbf{z}_{i+\Delta i} - \mathbf{z}_{j+\Delta i}\| = \|\mathbf{z}_i - \mathbf{z}_j\| \cdot 2^{\tilde{\lambda}_1 \cdot \tau} \tag{33}$$

then the following approximation of the exponent λ_1 is reached

$$\tilde{\lambda}_1 = \frac{1}{\tau} \cdot \log_2 \left[\frac{\|\mathbf{z}_{i+\Delta i} - \mathbf{z}_{j+\Delta i}\|}{\|\mathbf{z}_i - \mathbf{z}_j\|} \right] \tag{34}$$

- one repeats the previous steps N_λ times, so that λ_1 can be approximed by the average

$$\lambda_1 = \frac{1}{N_\lambda} \sum_1^{N_\lambda} \tilde{\lambda}_1 \tag{35}$$

The technique appears straightforward, but it is necessary to underline some points. With reference to the top of Figure 17, one sees the fiduciary trajectory $\mathbf{z}(t)$ and the nearby trajectory, $\tilde{\mathbf{z}}(t)$, at distance $d(t)$. After a period τ, this distance becomes $d(t + \tau)$, from which one obtains an approximation for λ_1:

$$\lambda_1 = \frac{1}{\tau} \cdot \log_2 \left[\frac{d(t+\tau)}{d(t)} \right] \tag{36}$$

This evaluation is correct. By contrast Equation (36) for the two points in the bottom of Figure 17, does not lead to the right result. In this case, in fact, the points belong to two acts of motion that are not initially similar. This example shows the importance to use the limit value ϵ_V.
A second remark regards the values of ϵ_m, ϵ_M, and τ. It is necessary to remember that an approximation of the maximum positive Lyapunov exponent will be obtained from the knowledge of a single time series (see Equation (24)). In particular, this series can be affected by noise, and it is necessary to choose the two points not excessively close (i.e closer than ϵ_m), to avoid that random fluctuations affect the value of λ_1. On the other hand, one will measure the departure of two close points: then it is necessary to put an upper bound ϵ_M to the initial distance d : since the motion is always limited on the attractor, one cannot measure the local rate of divergence of the trajectories when they diverge to the size of the attractor.
Generally, when one chooses the initial distance d too small, one overstimates the value of λ_1, while when d is choosen too large λ_1 is underestimated.

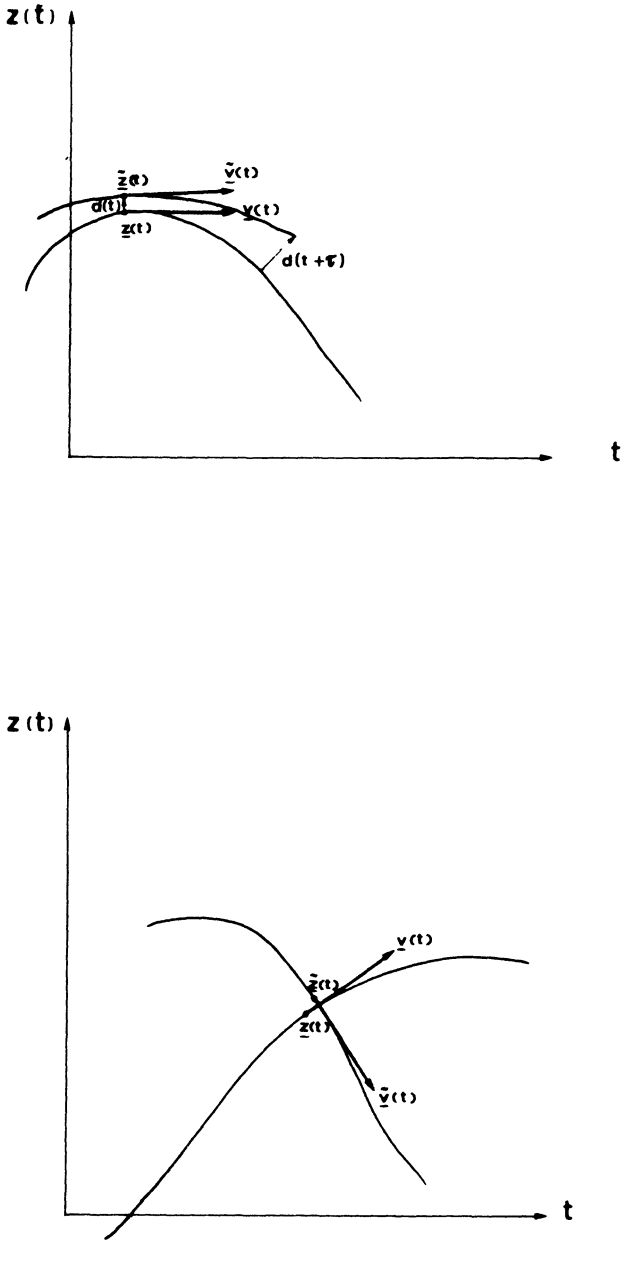

Figure 17: Choice of two nearby points to compute λ_1.

The choice of τ has a similar effect: when it is too small, one has the influence of small numerical errors (or noise) and then overstimates the value of the maximum Lyapunov exponent; when τ is too large, one understimates λ_1, since the motion is always constrained over the attractor.
One can indicate approximately the limit values of $\epsilon_m \cong 0.01 \cdot L$ and $\epsilon_M \cong 0.10 \cdot L$, where L is the size of the attractor, measured for example by the Gibbs set approach (see Equation (21)). Furthermore, when d become greater than $0.2 \cdot L$, the approximation of λ_1 can become unreliable.

4.3.2 PROBABILITY DENSITY, ENTROPY AND KOLMOGOROV ENTROPY.

Some quantitative evaluation of the behaviour of a dynamical system can be obtained partitioning the phase space: this operation is conducted through a grid of boxes $V_i, i = 1, ..N_{box}$ of characteristic size ϵ, which is indicative of our capacity to measure the status of the system in the phase space [4, 21, 45].
The numerical quantities extracted through the partition of the phase space are only approximated, and this approximation increases as the number of boxes increases. Of course the number N_{box} of boxes used is clearly related to the computational effort. The region of the phase space which is actually partitioned is related with such computational considerations. Naturally, it is necessary to extend the partition at least beyond the points of the phase space visited from the system during its motion.

The first elementary measure that one can make is the number of boxes N_{box} which are visited or not. A more significant information that one can obtain is the probability

$$p_i(t) = \int_{V_i} p(\mathbf{y}, t) \cdot dV_i \tag{37}$$

of finding the status of the system in the i-th box of volume V_i. In this way one can consider systems with many attractors, with noise and with or without sensitivity to the initial conditions. Generally one has [37, 38]:

- a *regular motion* is characterized by a δ-Dirac like functions, i.e. singular functions,
- a *chaotic/stochastic system* is characterized by a smooth probability function.

The analysis of chaotic and stochastic systems, can therefore be conducted in the same way. It is worth noting that just the presence of high gradients (in theory ∞) in the function $p(\mathbf{y}, t)$ leads to serious numerical difficulties in dealing with it.

Related to the form of the probability density function there is a quantity that measures the disorder connected with the motion of the system. This quantity is the **entropy** of the system [10, 43]. Mathematically, it is defined as

$$I(t) = -\sum_i p_i(t) \cdot \log_2(p_i(t)) \tag{38}$$

where the summation covers all the visited boxes (i.e where $p_i \neq 0$) [15, 10, 23, 39]. It is clear that numerically this expression is dependent on the discretization of the phase space. Again, this quantity is a function of the time, because the quantities $p_i(t)$ are continuously varying in time.
The average velocity of variation of the entropy, is the **Kolmogorov entropy** [15, 10, 23] (see also Chapter 1)

$$K = \overline{K(t)} = \overline{\frac{\partial I(t)}{\partial t}} \tag{39}$$

4.3.3 THE COMPUTATION OF THE KOLMOGOROV ENTROPY WITH THE RENORMALIZATION OF THE GIBBS SET.

The technique for estimating the Kolmogorov entropy is conceptually the same for both systems with few degrees of freedom and systems with many degrees, but the programming aspects are different [17, 19, 20, 46, 47].

Few degrees of freedom.

The algorithm to compute the Kolmogorov entropy consists of the following steps:

- dispose the initial conditions of Equation (7) of the systems belonging to a Gibbs set in a restricted number of boxes within the phase space, around the representative point of the first of these systems (choose as fiduciary system);
- evaluate the entropy I_0 in this initial configuration by Equation (38);
- integrate the motion equations (Equation (6));
- after a period T_r, the Gibbs set is distributed over the partitioned phase space: compute the corresponding entropy I_1;
- in the interval of time T_r, one has an increment of entropy $\Delta I = I_1 - I_0$, and then the Kolmogorov entropy is estimated as $K = \Delta I / T_r$;
- realize the renormalization of the Gibbs set, disposing the systems of the Gibbs set in a small region around the position of the first fiduciary system;
- repeat the previous five steps, for a number of times sufficiently to have an average value of the Kolmogorov entropy.

Figure 18 shows the effects of the renormalization of the Gibbs set of the Duffing oscillator, with $T_r = T_F$, i.e. at each period of the harmonic force. The choice of T_r depends on considerations similar to the ones made for the evaluation of the Lyapunov exponents: a too long T_r tends to understimates the Kolmogorov entropy, because, being the motion certainly limited over the attractor, one exceeds the stationary value of the entropy and cannot measure the diffusion of the Gibbs set.

Many degree of freedom

In cases with a high number of degrees of freedom, one has the effort to partitioning a high dimensional phase space: when one has a space of n dimension and one has to partitioning a cube of side L, with boxes of side ϵ, the number of boxes becomes

$$n_{box} \propto \left[\frac{L}{\epsilon}\right]^n \tag{40}$$

that, with the minimum choice of $L/\epsilon = 10$, leads to $N_{box} \propto 10^n$.
In this subsection a technique that is based on the renormalization of the Gibbs set, whose evolution is constructed from the knowledge of a single variable of the dynamical system, i.e. from Equation (24), is presented. One assigns a characteristic size ϵ, and proceeds as follows:

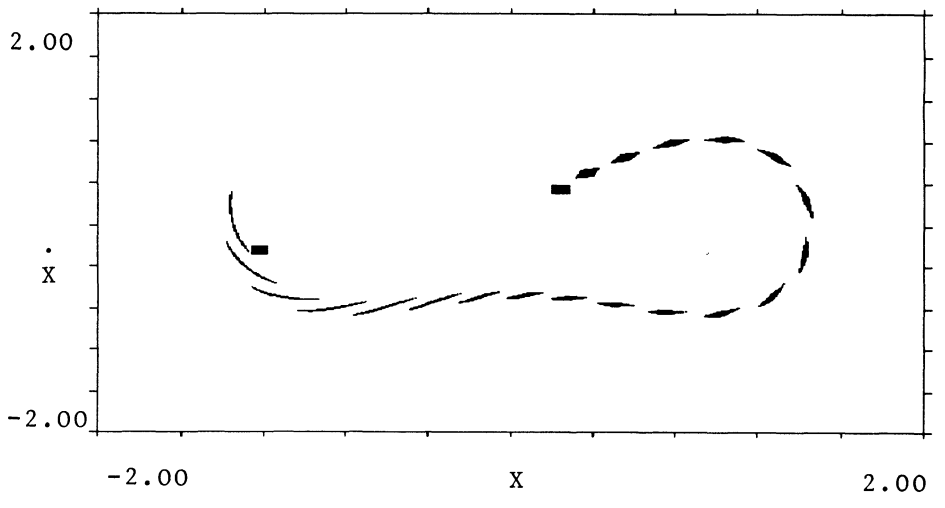

Figure 18: Renormalizations of the evolution of the Gibbs set: a set of Duffing oscillators, renormalized after a period T_F.

- considers a reference point with index i;
- at instant t_1, one considers all the points $j \neq i$ of the series in Equation (24) that are within the box of side ϵ centered around the point i; these points j should be sufficiently far in time from the point i, i.e. should be $i \gg j$ or $i \ll j$, because otherwise one considers points that are assumed during time by the reference point. The ensemble of these points forms a Gibbs set initially disposed in a unique box of side ϵ, and with zero entropy.
- one follows the evolution of the Gibbs set until an instant t_2, i.e. for a period of time $\Delta\overline{T} = t_2 - t_1$, that is for an increment of the index $\Delta i = \Delta\overline{T}/T_{delay}$ sufficient to expand the Gibbs set (but the size of the Gibbs set must remain lower than $\sim 0.1 \cdot L$, being L the size of the attractor);
- at this point, one considers a local partitioning of the phase space: the number of boxes results

$$N_{box} \propto \left[\frac{0.1 \cdot L}{\epsilon}\right]^n = \propto 2^n \tag{41}$$

(and hence, if $n = 5$ one passes from 100000 to just 32 boxes). Then, one computes $p_i(t_2)$ and $I(t_2)$ with Equation (38). The K-entropy during the interval of time $t_2 - t_1 = \Delta\overline{T}$ is

$$K_{21} = \frac{I(t_2)}{t_2 - t_1} \tag{42}$$

- repeating the previous steps, along the evolution of the reference point, one obtains an approximation of K, averaging the values K_{21}.

This algorithm is illustrated in Figure 19.

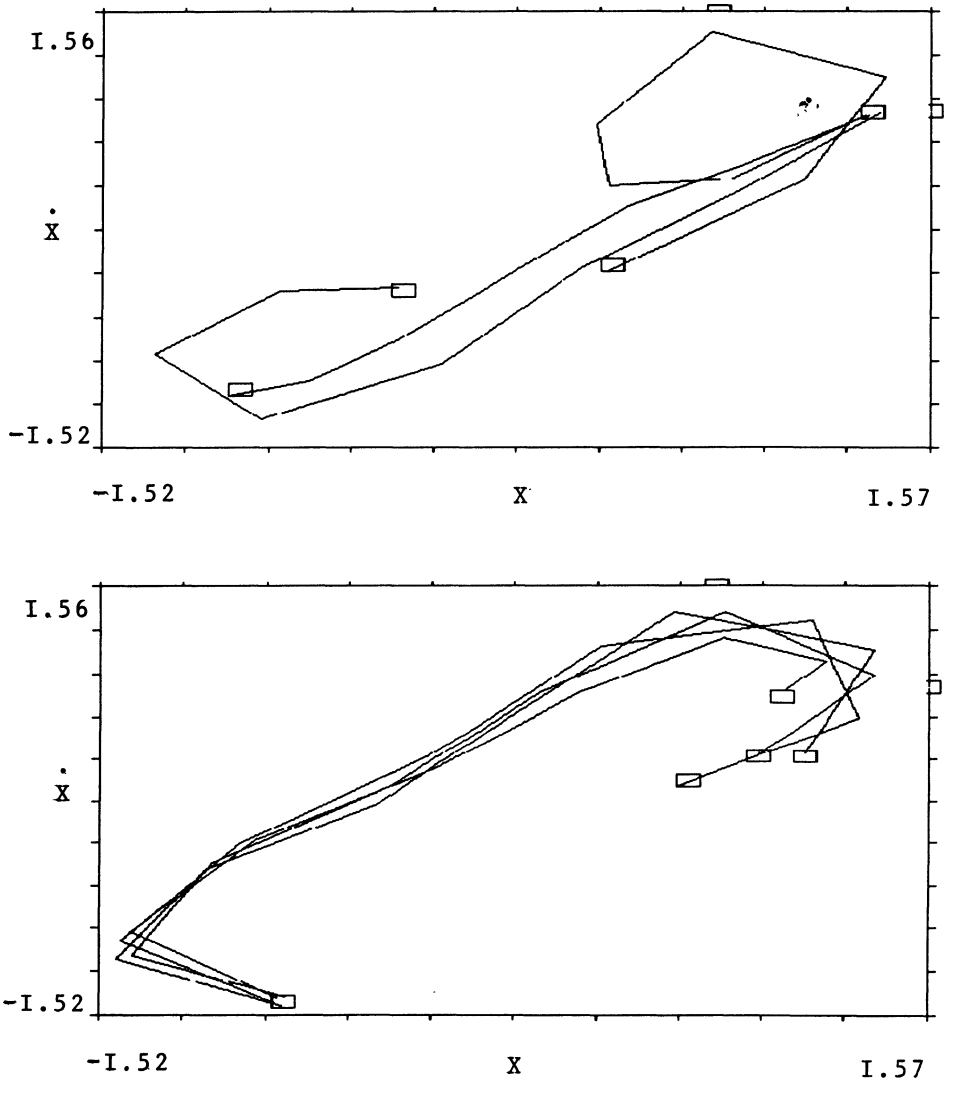

Figure 19: Renormalizations of the evolution of the Gibbs set from a single series. $T_r = 0.1 \cdot T_F$.

4.3.4 MEASURES OF THE DIMENSION OF THE ATTRACTOR.

The previous measures of chaoticity and of stochasticity are dynamics, in the sense that they must be conducted during the integration of the Equation (1) for a single system or of the Equation (6) for the Gibbs set. Other kinds of quantitative measures of the degree of irregularity of the motion are the measure of the dimension of the attractor that is underlying the motion [15, 21, 10, 22, 23, 45] (see also the Chapter 1). They are in some sense static, i.e they can be obtained separately from the integration of the motion.
Considering a partitioned phase space with boxes of size ϵ, one considers [10, 23, 45, 48]:

the **capacity dimension** d_C:

$$d_C = \lim_{\epsilon \to 0} \frac{\log_2 N(\epsilon)}{\log_2(1/\epsilon)} \tag{43}$$

where $N(\epsilon)$ is the number of non empty boxes;

the **dimension of information**:

$$d_I = \lim_{\epsilon \to 0} \frac{I(\epsilon)}{\log_2(1/\epsilon)} \tag{44}$$

where $I(\epsilon)$ is again the entropy connected with the distribution of the representative points of the dynamical system;

the **dimension of correlation**:

$$d_G = \frac{\log_2(C(R))}{\log_2(R)} \tag{45}$$

where $C(R)$ is the number of points of the trajectory of the system with distance a sampling point of the trajectory itself lower than R.

the **dimension of Lyapunov** (see Chapter 1):

$$d_\lambda = j - \frac{\sum_{i=1,j} \lambda_i}{\lambda_{j+1}} \tag{46}$$

where $\lambda_1 \geq \lambda \geq \ldots \geq \lambda_n$ are the Lyapunov exponents, and j the index of the last Lyapunov exponents λ_j for which $\sum_{i=1,j} \lambda_i \leq 0$.
For the Duffing oscillator with harmonic force ($A = 0.3$) and with white noise ($\sigma_0 = 0.01$), the Poincare' section of which is represented in Figure 20, one obtains $d_C = 1.25$, $d_I = 1.52$, $d_G = 1.53$, and $d_\lambda = 1.50$.
An important observation is related to the dimension of the phase space in which the motion develops itself. With a dissipative system of n degrees of freedom, after a transient, the motion occurs in a subspace of the n-dimensional phase space, n_e being the dimension of this subspace. This subspace form the **embedding phase space** for the motion of the dynamical system. Knowing the dimension d of the attractor for the dynamical system, one has $n_e = \text{int}(d) + 1$.

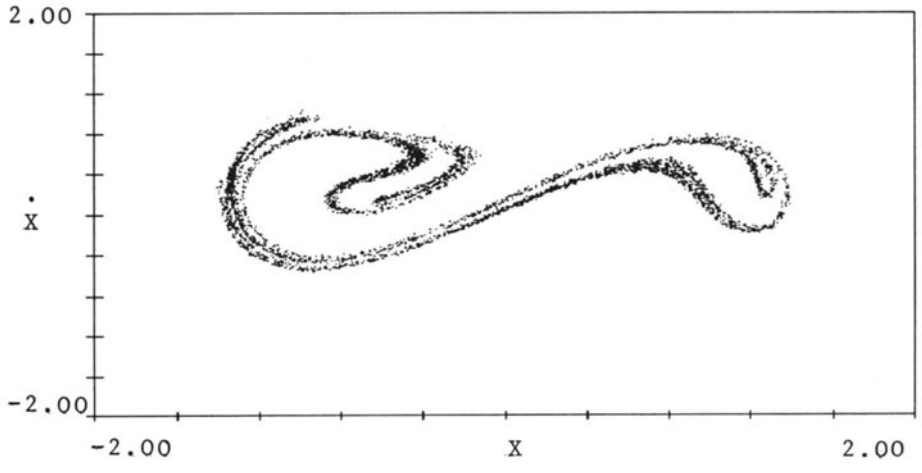

Figure 20: Attractor for $A = 0.3$ and $\sigma_0 = 0.01$

4.4 NUMERICAL SOLUTION OF THE FOKKER-PLANCK EQUATION.

For a system with two degrees of freedom, as the Duffing oscillator, the related Fokker-Planck equation can be written (see Chapter 1) [7, 24, 36, 43, 49]

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial y_i}[b_i(t, \mathbf{y})p] + 1/2 \sum_{i=1}^{2}\sum_{i=1}^{2} \frac{\partial^2}{\partial y_i \partial y_j}[c_{ij}(t, \mathbf{y})p] = 0 \tag{47}$$

The boundary conditions require that $p(y_1, y_2 \mid t)$ goes to zero as $(y_1^2 + y_2^2) \rightarrow +\infty$. Moreover the normalization condition

$$\int\int p(y_1, y_2 \mid t) \cdot dy_1 \cdot dy_2 = 1 \tag{48}$$

gives a sort of conservation equation for the probability.
The solution of the Fokker-Planck equation is difficult for the following reasons [36, 50, 51, 52, 53, 54, 55, 49]:

- when one consider a dynamical system governed by a vector $\mathbf{y}(t)$ with n components, one must solve a partial differential equation in a n dimensional space with dependence on time t. This is practically impossible for a system that has more than few degrees of freedom;
- Equation (47) has a mixed hyperbolic-parabolic character: on the r.h.s. the terms with a first order derivative are connected with the hyperbolic character (i.e. convective character), while the terms with a second order derivative are related with the parabolic one (i.e. diffusive character).

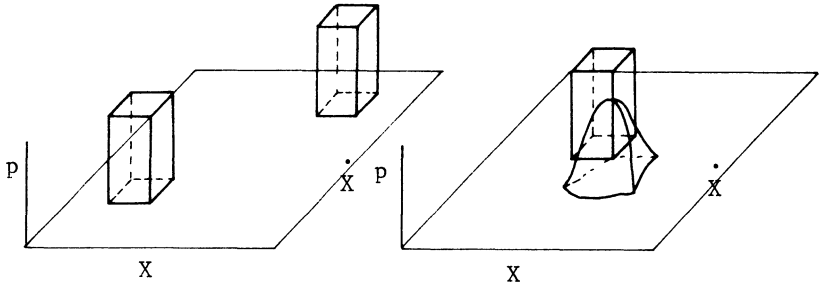

Figure 21: Hyperbolic and parabolic character of the Fokker-Planck Equation.

Figure 21 shows a probability density function and the effects on itself of the hyperbolic and parabolic character of the Fokker-Planck equation: in the first case the distribution moves with inalterated shape, while in the second, the shape is smoothed in time.

4.4.1 SOLUTION WITH LOCAL DISCRETIZATIONS.

For a bidimensional phase space, the Fokker-Planck equation can be regarded as a particular case of the general form

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y_1}(Ap) - \frac{\partial}{\partial y_2}(Bp) + \frac{\partial^2}{\partial y_1^2}(Cp) + \frac{\partial^2}{\partial y_2^2}(Dp) \tag{49}$$

with the known functions $A(y_1, y_2, t)$, $B(y_1, y_2, t)$, $C(y_1, y_2, t)$ e $D(y_1, y_2, t)$. The unknown function $p(y_1, y_2, t)$ must be estimated in the domain Ω of the phase plane (y_1, y_2) shown in Figure 22. One superimposes over the domain Ω a rectangular grid of points $(y_{1,i}, y_{2,j})$ and the time axis t is discretized through $t_k = t_0 + k \cdot \Delta t$. In this way the spatial partial derivatives of Equation (49) in the point $(y_{1,i}, y_{2,j})$ at time t_k can be approximed by the finite difference scheme

$$\begin{aligned}
\dot{p}^k_{i,j} = & -\frac{A^k_{i+1,j} \cdot p^k_{i+1,j} - A^k_{i-1,j} \cdot p^k_{i-1,j}}{2h} \\
& -\frac{B^k_{i,j+1} \cdot p^k_{i,j+1} - B^k_{i,j-1} \cdot p^k_{i,j-1}}{2h} \\
& +\frac{C^k_{i+1,j} \cdot p^k_{i+1,j} - 2 \cdot C^k_{i,j} \cdot p^k_{i,j} + C^k_{i-1,j} \cdot p^k_{i-1,j}}{h^2} \\
& +\frac{D^k_{i,j+1} \cdot p^k_{i,j+1} - 2 \cdot D^k_{i,j} \cdot p^k_{i,j} + D^k_{i,j-1} \cdot p^k_{i,j-1}}{h^2}
\end{aligned} \tag{50}$$

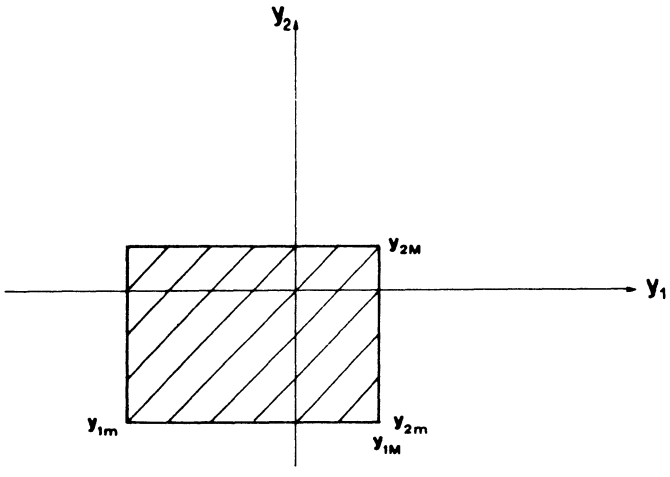

Figure 22: Domain Ω of integration in the phase plane of the Fokker-Planck Equation.

where the indexes (i,j) denote the points of the spatial grid and the apex k the time t_k. After some algebras, one writes

$$\dot{p}^k_{i,j} = \Phi_{i,j}(p^k_{i+1,j}, p^k_{i,j}, p^k_{i-1,j}, p^k_{i,j+1}, p^k_{i,j-1}, t_k) \tag{51}$$

that is a system of ordinary differential equations, in terms of the unknown functions $p^k_{i,j}$ and of the time t. The problem of Equation (51) can be discretized along the time with a finite difference scheme, or by the Runge-Kutta method of the second order. This assures good stability properties and the same levels of accuracy of the solution in time as in space. At the end, one obtains an explicit scheme, which gives the values $p^{k+1}_{i,j}$ given the $p^k_{i,j}$.
The boundary conditions are

$$p^0_{i,j} = 0 \tag{52}$$

for the grid points (i,j) belonging to the boundary of the integration domain Ω in the phase space. To have the probability conservation, i.e. Equation (48), one also imposes Equation (52) along a line of grid points inner the previous, as indicated in Figure 23, in order to have zero flux of density through the domain boundary.
Figure 24 show the evolution in time of the probability density from an initial distribution.

4.4.2 SOLUTION WITH INTEGRAL DISCRETIZATIONS.

In this case, one thinks to expand the unknown function $p(y_1, y_2, t)$ through a truncated series of trial functions $p_k(y_1, y_2)$, functions only of the spatial coordinates (y_1, y_2), multiplicated by coefficients $P_k(t)$, depending only on time t, as

$$p(y_1, y_2, t) = \sum_{k=1}^{m} P_k(t) \cdot p_k(y_1, y_2) \tag{53}$$

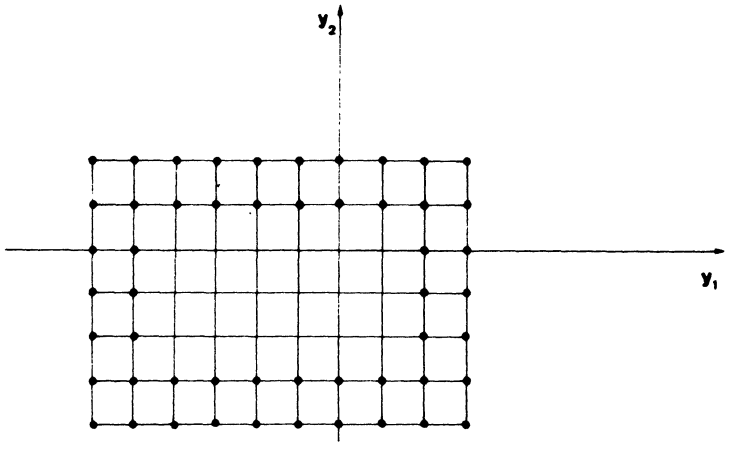

Figure 23: Points in which $p^0_{i,j}$ is set equal to zero, as boundary conditions of the discretized Fokker-Planck Equation.

where m is the number of terms of the truncated series. In this manner, one has realized a separation of the space variables from the temporal variable. One convenient choice for the trial functions, are the B-spline functions illustrated in Figure 25.
Substituting the expansion of Equation (53) in the Equation (49) one obtains

$$\sum_{k=1,m} \dot{P}_k(t) \cdot p_k(y_1, y_2) = \sum_{k=1,m} P_k(t) \cdot \Big(- \frac{\partial}{\partial y_1}(Ap_k) - \frac{\partial}{\partial y_2}(Bp_k) + \frac{\partial^2 Cp_k}{\partial y_1^2} + \frac{\partial^2 Dp_k}{\partial y_2^2} \Big) \tag{54}$$

that is not in general identically satysfied. Introducing the notation

$$< f, g >= \int_D f \cdot g \cdot d\Omega \tag{55}$$

where Ω is the domain of integration of the Fokker-Planck Equation, one can at least imposes that the m equations

$$\sum_{k=1,m} \dot{P}_k(t) \cdot < p_k, p_l >= \sum_{k=1,m} P_k(t) \cdot < -\frac{\partial}{\partial y_1}(Ap_k) - \frac{\partial}{\partial y_2}(Bp_k) + \frac{\partial^2}{\partial y_1^2}(Cp_k) + \frac{\partial^2}{\partial y_2^2}(Dp_k), p_l > \tag{56}$$

are satisfied, i.e. the integral residuals are zero. Having collected the m unknown function $P_k(t)$ in the vector $\mathbf{p} = [P_1(t), P_2(t), \ldots P_m(t)]^T$, one can recognize a *mass* matrix $\mathbf{M}$, the terms of which are

$$M_{k,l} = \int_D p_k(y_1, y_2) \cdot p_l(y_1, y_2) \cdot d\Omega \tag{57}$$

and results symmetric and time independent, and a ***stiffness matrix*** $\mathbf{K}$, the terms of which

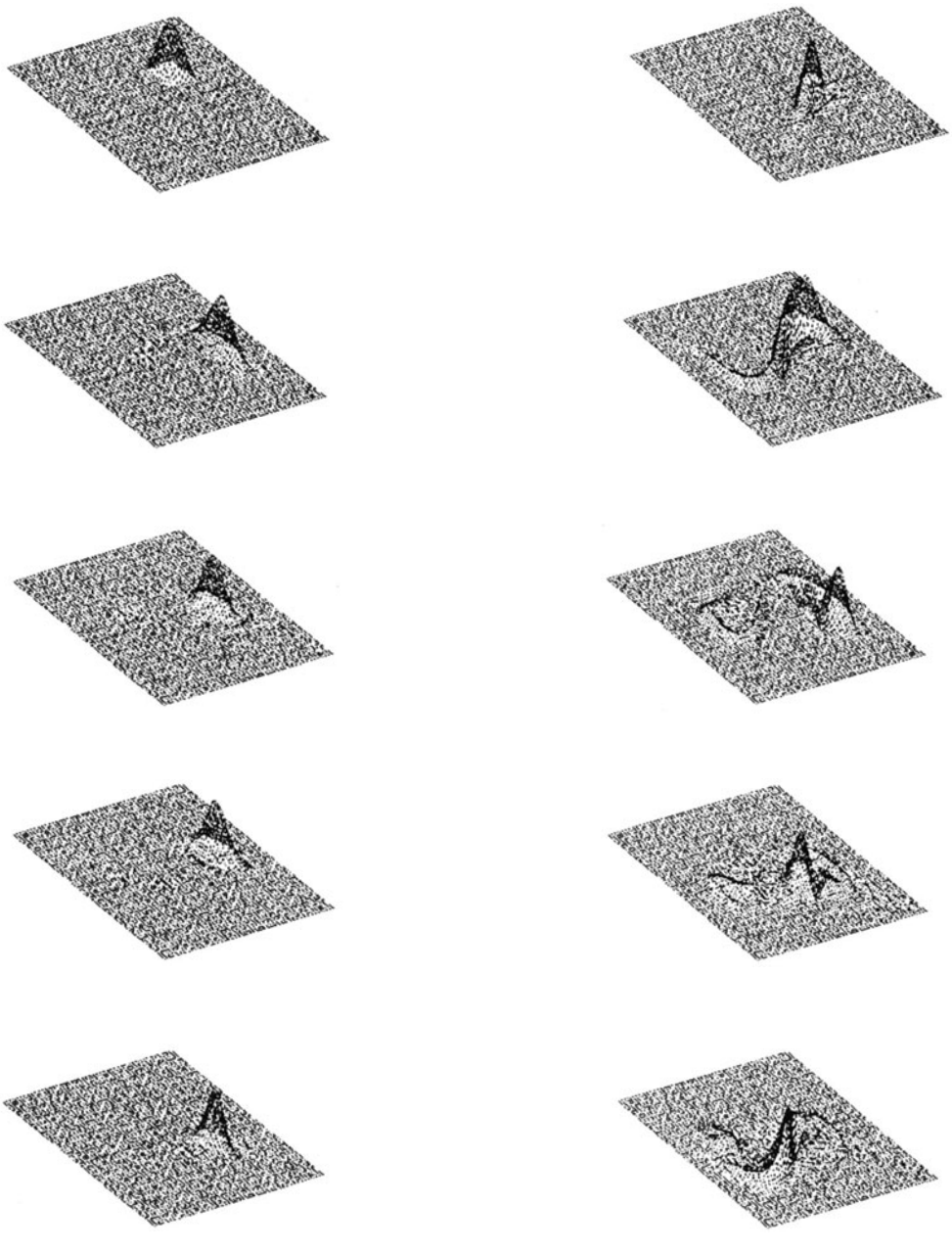

Figure 24: Evolution of the probability density function for the Duffing oscillator, with $A = 0.3$, $\sigma_0 = 0.01$.

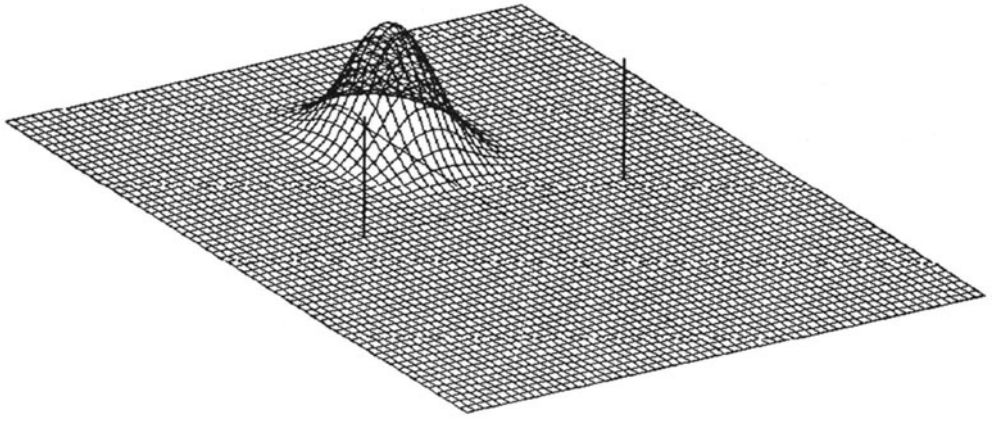

Figure 25: Generic B-spline function for the solution of the Fokker-Planck in the sense of weighted residuals.

$$K_{k,l} = K_{k,l}(t) = \int_D \Big(-\frac{\partial}{\partial y_1}(Ap_k) - \frac{\partial}{\partial y_2}(Bp_k) + \frac{\partial^2}{\partial y_1^2}(Cp_k) + \frac{\partial^2}{\partial y_2^2}(Dp_k)\Big) \cdot p_l \cdot d\Omega \qquad (58)$$

are non symmetric and time dependent. Such a discretized Fokker-Planck equation, can be regarded as the evolutive problem

$$\mathbf{M} \cdot \dot{\mathbf{p}}(t) = \mathbf{K}(t) \cdot \mathbf{p}(t) \qquad (59)$$

In this case, one obtains a solution of the same characteristics, by considering a lower number of unknowns, but one has to integrate numerically the matrix $\mathbf{M}$ and $\mathbf{K}$, operation which can result very expensive.
Figure 26 shows the solution of the Fokker-Planck Equation for the Duffing oscillator with $A = 0.3$, $\sigma_0 = 0.01$ obtained with an expansion of 25 B-splines.

ACKNOWLEDGEMENT.

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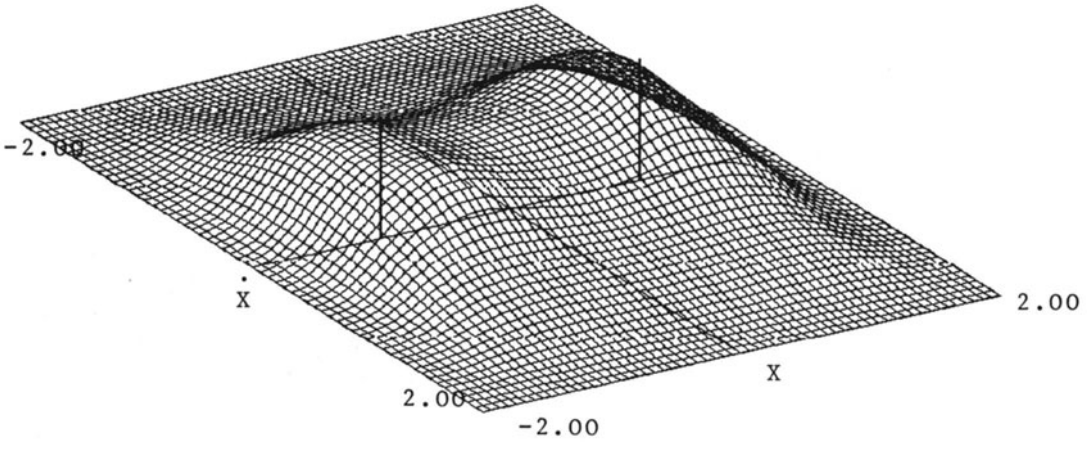

Figure 26: The probability density function for the Duffing oscillator, with $A = 0.3$, $\sigma_0 = 0.01$ obtained with an expansion of 25 B-splines.

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Chapter 5

STOCHASTIC BEHAVIOUR OF SPECIAL MATERIALS: THE COMPOSITE MATERIAL

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5.1 - INTRODUCTION

A composite material can be defined as a macroscopic combination of two or more distinct materials, having a recognizable interface between them. However, because they are usually used for their structure properties, composites are actually the materials that contain a reinforcement (such as fiber) supported by a binder material (matrix). Thus, composites typically have a discontinuous fiber that is stiffer and stronger than the continuos matrix phase.

Fiber-reinforced composites contain reinforcements having lengths much greater than their cross-sectional dimensions.

Laminar composites are fiber-reinforced materials composed of two (or more) layers with two of their dimensions being much larger than the third.

Due to their characteristics, the laminated composite materials are affected by errors in material lay-up, ply orientation and degree of matrix curing; these uncertainties can produce great variations in the mechanical material properties [1]. So, the production processes in this type of material require a remarkable technological effort to avoid defects.

These requirements become indeed more complex considering the high stresses that the fibre-reinforced composite materials need to withstand often only using very thin laminae.

Furthermore, the fibre ply of the material characterizes the behaviour of the lamina and of the structure; this can represent an additional complication in the production process in tacking care of the correct fibre alignment in parallel and in direction. Consequently, it is necessary to consider the random features of the mechanical behaviour of these materials by introducing randomness, arising from technical-productive uncertainties on some design parameters.

The random variables introduced here will take into account the dual prerogative of appointing design parameters for achieving the decided properties of the material and an indirect measure of the presence of defects.

Examples of these types of parameters are the fibre direction θ and the volume percentage of the fibres V_f in the lamina .

The basic principles of the mechanical behaviour of the laminae are first investigated so that the most important design aspects (i.e. the parameters to which the design solutions seem to be more sensitive), will be pointed out.

The defects which the technological process tries to avoid will be briefly dealt with. Afterwards, a first approach to a probability analysis will be suggested in order to characterize the mechanical behaviour of the lamina with regard to random design parameters.

5.2 - MACROMECHANICAL BEHAVIOUR OF LAMINAE

A lamina of composite material is the flat or curve assemblage of fibres (carbon fibres, boron fibres, glass fibres, etc.) which have been dipped in a matrix (epoxy resin, polyimide resin, etc.): it is the basic element which makes up laminated fibre-reinforced composites.

The knowledge of the mechanical behaviour of laminae is essential to analyze laminated fibre-reinforced composite structures. A linear elastic stress-strain relation is assumed as a basis to study this behaviour.

To keep this chapter self-contained, Hooke's laws (both for anisothropic materials and for orthotropic ones) will be briefly recalled. Some constants (called 'engineering constants'), which are of common usage in technical practice, will be introduced in the relations which describe the behaviour of the composite since they be estimated by simple tests.

These relations will refer in detail to the case of an orthotropic material under plane stress conditions: this is a typical situation for composite laminae.

The previous relations will be firstly written in the principal orthotropic axes, then the transformation in an arbitrary axes will be examined.

This transformation is useful for describing the lamina behaviour as the fibre disposition varies obtaining a general formulation for the stress-strain relation and determining the relation between the elastic constants and the fibre direction θ.

The final aim of the procedure is to show the relations of the mechanical properties (described by the engineering constants), with the disposition of the lamina itself. Once these relations are known, it will be possible (e.g. by Monte Carlo simulation) to obtain probability density functions of the elastic characteristics.

5.3 - CONSTITUTIVE RELATIONS

The generalized Hooke's law which correlates stresses and strains in the case of an elastic three-dimensional continuum, can be written as:

$$[\sigma] = [C][\varepsilon] \tag{1}$$

or:

$$\begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{vmatrix} = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{vmatrix} \begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{vmatrix} \tag{2}$$

where $\sigma_1, \sigma_2, \sigma_3, ..$ are the stress components

C_{ij} are the components of the stiffness matrix

$\varepsilon_1, \varepsilon_2, \varepsilon_3, ..$ are the strain components.

Given the symmetry of the [C] matrix (elasticity assumption) the total number of the independent elastic constants is reduced to 21.

Since the stiffness matrix is invertible, it is possible to write the strain components as:

$$\begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{vmatrix} = \begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{vmatrix} \begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{vmatrix} \tag{3}$$

where [S] is the compliance matrix which has 21 independent constants.

If two orthogonal planes of material characteristic symmetry exist, this symmetry will exist for a third plane which will be orthogonal to the previous two planes: the stress-strain relation referred to axes normal to the symmetry planes can be thus written as follows [2]:

$$\begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{vmatrix} = \begin{vmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{vmatrix} \begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{vmatrix} \tag{4}$$

which represents the constitutive relation for a linear elastic orthotropic material.

It should be noted that there are 9 independent constants and that shear strains do not cause normal stresses just as shear stresses do not cause normal strains.

In the case of a bi-dimensional lamina on an x_1-x_2 plane (Fig. 1) loaded onto this plane, the plane stress state is as known obtained:

$$\sigma_3 = 0 \; ; \; \tau_{23} = 0 \; ; \; \tau_{31} = 0 \tag{5}$$

A plane state does not in general correspond to it since

$$\varepsilon_3 = S_{13}\,\sigma_1 + S_{23}\,\sigma_2 \tag{6}$$

is generally different from zero.

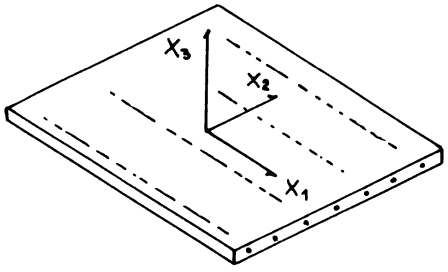


Fig. 1 - Bi-dimensional lamina on an x_1-x_2 plane.

For the lamina in linear elastic orthotropic material the strain-stress relation can be represented in this case as follows:

$$\begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{vmatrix} = \begin{vmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{vmatrix} \begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{vmatrix} \tag{7}$$

with four independent S_{ij}.

Both the stiffness matrix and the compliance matrix depend on the elastic characteristics of the material which can be expressed by several constants: Young moduli, Poisson coefficient, shear modulus and other 'engineering constants' which will be introduced in

the following. These constants can be evaluated by relatively simple tests, by applying known loading conditions and by measuring the resulting displacements and strains.

The previous constants describe the mechanical behaviour of the lamina: in the next section one considers the effects of uncertainties which result from the presence of technological random defects in the production process.

By introducing these constants, the compliance matrix for a linear elastic orthotropic lamina [2] can be written as:

$$\begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{vmatrix} = \begin{vmatrix} \frac{1}{E_1} & -\frac{\upsilon_{12}}{E_1} & 0 \\ -\frac{\upsilon_{21}}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{vmatrix} \begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{vmatrix} \tag{8}$$

where:

E_1 and E_2 are the Young moduli in the parallel and orthogonal direction to the fibres;

ν_{ij} are Poisson coefficients for the transversal deformation in the j-th direction when it is stressed in the i-th direction, i.e. :

$\nu_{ij} = - \varepsilon_j / \varepsilon_i$ when $\sigma_i = \sigma$ and all other stresses are equal to zero.

G_{12} is the shear modulus in the x_1-x_2 plane.

It should be pointed out that there are 5 constants of which only 4 are independent since the following relation must be verified:

$$\frac{\upsilon_{12}}{E_1} = \frac{\upsilon_{21}}{E_2} \tag{9}$$

because of the symmetry of the compliance matrix.

Concerning the inverted (stress-strain) relation :

$$\begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{vmatrix} = \begin{vmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{vmatrix} \begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{vmatrix} \tag{10}$$

it is possible to obtain the Q_{ij} coefficients:

$$\begin{aligned} Q_{11} &= \frac{E_1}{1-\upsilon_{12}\,\upsilon_{21}} \\ Q_{22} &= \frac{E_2}{1-\upsilon_{12}\,\upsilon_{21}} \\ Q_{12} = Q_{21} &= \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\,\upsilon_{21}} = \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\,\upsilon_{21}} \\ Q_{66} &= G_{12} \end{aligned} \tag{11}$$

In order to generalize the problem with respect to the coordinate axes, it is useful to write the previous constitutive relations with regard to generic axes x and y.

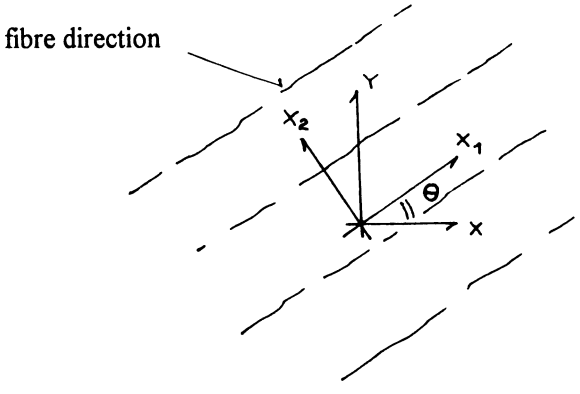


Fig. 2 - Generic reference (x-y) and principal orthotropic reference (x_1-x_2).

The transformation laws for the stress components and the strain components, for a change of reference system, is linked to a "transformation " matrix:

$$|\mathbf{L}| = \begin{vmatrix} \cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{vmatrix} \tag{12}$$

by means of which the following can be written:

$$\begin{vmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{vmatrix} = |\mathbf{L}| \begin{vmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{vmatrix} \tag{13}$$

By introducing the matrix [R]:

$$|\mathbf{R}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} \tag{14}$$

one can write:

$$\begin{vmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{vmatrix} = |\mathbf{R}||\mathbf{L}| \begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{vmatrix} \tag{15}$$

Substituting in eq. (13), one obtains:

$$\begin{vmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{vmatrix} = |\overline{\mathbf{Q}}| \begin{vmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{vmatrix} \tag{16}$$

where

$$|\overline{\mathbf{Q}}| = |\mathbf{L}||\mathbf{Q}||\mathbf{R}||\mathbf{L}|^{-1}|\mathbf{R}|^{-1} \tag{17}$$

is the stiffness matrix for linear elastic orthotropic lamina. It is referred to arbitrary axes, with components which in general will be non zero. Only four elements are mutually independent, but in general non null relations exist between normal stresses and shear strains just like between shear stresses and normal strains.

For this reason, the lamina seems to behave like an anisothropic material in the x and y directions. These laminae are defined "generally orthotropic"; the constitutive relation is similar to that of a linear elastic anisothropic lamina:

$$\begin{vmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{vmatrix} = \begin{vmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{vmatrix} \begin{vmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{vmatrix} \tag{18}$$

where, in this case, from eqs. (17) one finds:

$$\begin{aligned}
\overline{Q}_{11} &= Q_{11}\cos^4\theta + 2\,(Q_{12} + 2\,Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \\
\overline{Q}_{12} &= (Q_{11} + Q_{22} - 4\,Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta) \\
\overline{Q}_{22} &= Q_{11}\sin^4\theta + 2\,(Q_{12} + 2\,Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \\
\overline{Q}_{16} &= (Q_{11} - Q_{12} - 2\,Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} - 2\,Q_{66})\sin^3\theta\cos\theta \\
\overline{Q}_{26} &= (Q_{11} - Q_{12} - 2\,Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} - 2\,Q_{66})\sin\theta\cos^3\theta \\
\overline{Q}_{66} &= (Q_{11} + Q_{22} - 2\,Q_{12} - 2\,Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)
\end{aligned} \tag{19}$$

By inverting the eqs. (18), it turns out:

$$\begin{vmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{vmatrix} = \begin{vmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{26} \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{66} \end{vmatrix} \begin{vmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{vmatrix} \tag{20}$$

where

$$
\begin{aligned}
\overline{S}_{11} &= S_{11}\cos^4\theta + (2S_{12} + S_{66})\sin^2\theta\cos^2\theta + S_{22}\sin^4\theta \\
\overline{S}_{12} &= (S_{11} + S_{22} - S_{66})\sin^2\theta\cos^2\theta + S_{12}(\sin^4\theta + \cos^4\theta) \\
\overline{S}_{22} &= S_{11}\sin^4\theta + (2S_{12} + S_{66})\sin^2\theta\cos^2\theta + S_{22}\cos^4\theta \\
\overline{S}_{16} &= (2S_{11} - 2S_{12} - S_{66})\sin\theta\cos^3\theta - (2S_{22} - 2S_{12} - S_{66})\sin^3\theta\cos\theta \\
\overline{S}_{26} &= (2S_{11} - 2S_{12} - S_{66})\sin^3\theta\cos\theta - (2S_{22} - 2S_{12} - S_{66})\sin\theta\cos^3\theta \\
\overline{S}_{66} &= 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})\sin^2\theta\cos^2\theta + S_{66}(\sin^4\theta + \cos^4\theta)
\end{aligned}
\tag{21}
$$

Introduce now the engineering constants $\eta_{i,ji}$ and $\eta_{ij,i}$ (i=x,y; j=x,y) known as "Lekhinski's mutually influenced coefficients". They are defined as follows:

$\eta_{i,ij}$ is the coefficient of mutual influence which describes the stretching in i-th direction caused by the shear in the i-j plane, evaluated as:

$\eta_{i,ij} = \varepsilon_i / \gamma_{ij}$ when $\tau_{ij} = \tau$ and the other stresses are zero;

$\eta_{ij,i}$ is the coefficient of mutual influence, which characterizes the shearing in the i-j plane caused by a normal stress in the i-th direction, evaluated as:

$\eta_{ij,i} = \gamma_{ij} / \varepsilon_i$ when $\sigma_i = \sigma$ and all other stresses are zero.

It is possible to write:

$$
\begin{aligned}
\overline{S}_{11} &= \frac{1}{E_x} \\
\overline{S}_{12} &= -\frac{\upsilon_{xy}}{E_x} = -\frac{\upsilon_{yx}}{E_y} \\
\overline{S}_{22} &= \frac{1}{E_y} \\
\overline{S}_{16} &= \frac{\eta_{xy,x}}{E_x} = \frac{\eta_{x,y}}{G_{xy}} \\
\overline{S}_{26} &= \frac{\eta_{xy,y}}{E_y} = \frac{\eta_{y,xy}}{G_{xy}} \\
\overline{S}_{66} &= \frac{1}{G_{xy}}
\end{aligned}
\tag{22}
$$

Considering the eqs. (21), (22) and (8), it results:

$$\frac{1}{E_x} = \frac{1}{E_1}\cos^4\theta + \left[\frac{1}{G_{12}} - \frac{2\upsilon_{12}}{E_1}\right]\sin^2\theta\cos^2\theta + \frac{1}{E_2}\sin^4\theta$$

$$\upsilon_{xy} = E_x\left[\frac{\upsilon_{12}}{E_1}(\sin^4\theta + \cos^4\theta) - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta\right]$$

$$\eta_{xy,x} = E_x\left[\left(\frac{2}{E_1} + \frac{2\upsilon_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin\theta\cos^3\theta - \left(\frac{2}{E_2} + \frac{2\upsilon_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^3\theta\cos\theta\right] \qquad (23)$$

$$\frac{1}{E_y} = \frac{1}{E_1}\sin^4\theta + \left(\frac{1}{G_{12}} - \frac{2\upsilon_{12}}{E_1}\right)\sin^2\theta\cos^2\theta + \frac{1}{E_2}\cos^4\theta$$

$$\eta_{xy,y} = E_y\left[\left(\frac{2}{E_1} + \frac{2\upsilon_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^3\theta\cos\theta - \left(\frac{2}{E_2} + \frac{2\upsilon_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin\theta\cos^3\theta\right]$$

$$\frac{1}{G_{xy}} = 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\upsilon_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta + \frac{1}{G_{12}}(\sin^4\theta + \cos^4\theta)$$

where x_1 and x_2 are the principal orthotropic directions af the lamina and **x** and **y** are the arbitrary reference system (for which the lamina seems to be anisothropic).

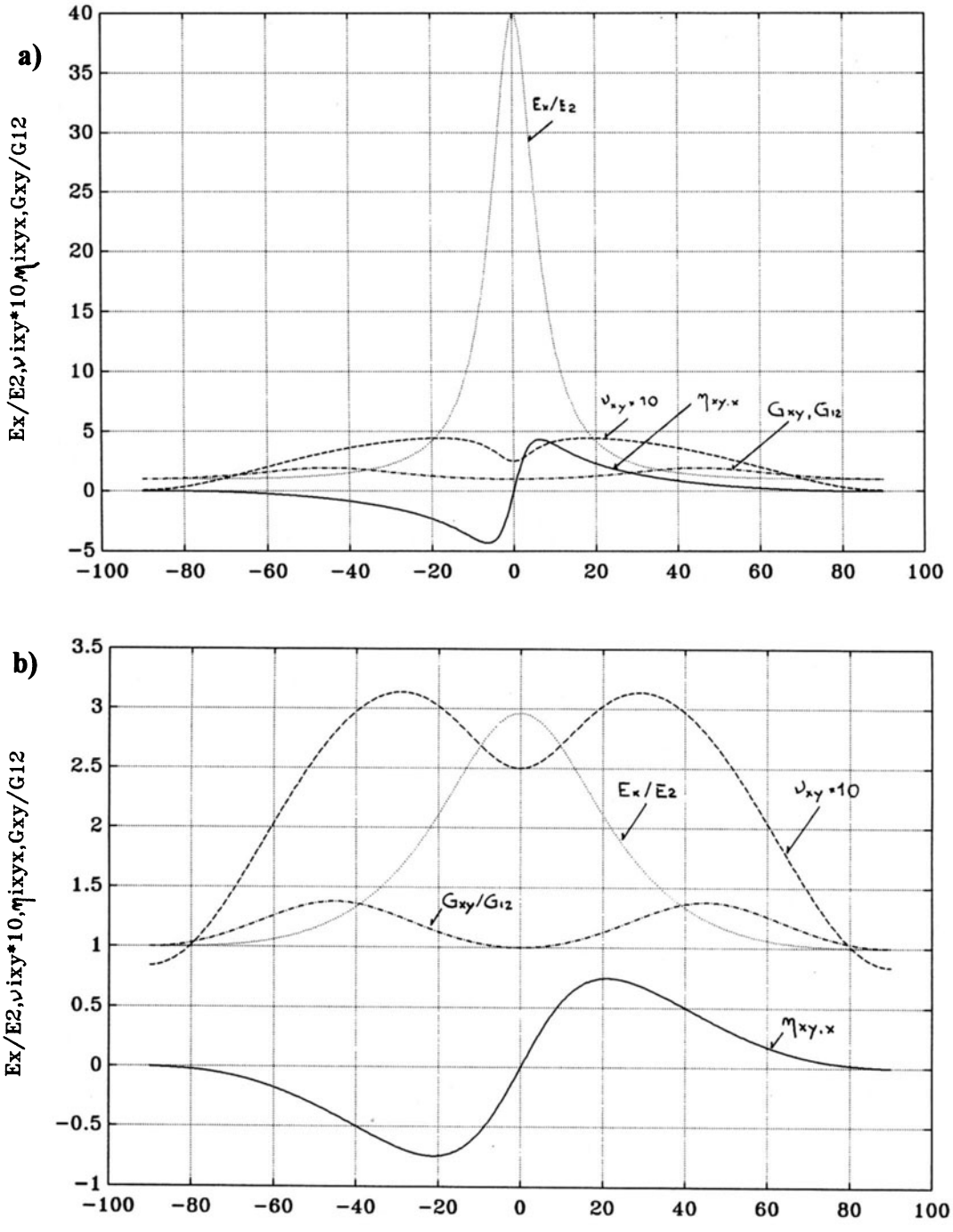
As shown, the elastic parameters in the x and y reference depend on the orthotropic elastic parameters in the x_1 and x_2 directions, and on the θ angle along which the fibres are plied. For this reason, the possible uncertainties in the fibre direction (therefore in the design parameter θ) involves uncertainties in the elastic characteristics of the material in the reference system **x** and **y**.

Figures 3a and 3b show the relations for two different types of composite laminae (graphite-epoxy and E-glass-epoxy): the maximum and minimum values of the E_x/E_2 ratio is obtained of course for $\theta=0^\circ$ and $\theta=90^\circ$ respectively; G_{xy}/G_{12} reaches the maximum value for $\theta=45^\circ$ whereas ν_{xy} takes on the maximum value for θ in the 20°-30° interval.

Indeed in eqs. (23), not only the variable θ presents a random nature but also its elastic characteristics along the principal orthotropic directions have uncertainties due to the technological defects. The elastic characteristics E_1, E_2, G_{12} and ν_{12} come directly from the characteristics of the lamina components (fibre and matrix), and from the volume percentage of these components.

To analyze these aspects, the problem has to be faced from a micromechanical point of view.

The elastic characteristics will generally depend on the material, on its components and the volume percentages in which they are present.



Figs. 3 - Relations between elastic characteristics and θ for two different types of composite laminae: a) graphite-epoxy ; b) E-glass-epoxy.

Therefore, in general, it is possible to write:

$$\begin{aligned} E_1 &= E_1 (E_f, \upsilon_f, V_f, E_m, \upsilon_m, V_m) \\ E_2 &= E_2 (E_f, \upsilon_f, V_f, E_m, \upsilon_m, V_m) \\ G_{12} &= G_{12} (E_f, \upsilon_f, V_f, E_m, \upsilon_m, V_m) \\ \upsilon_{12} &= \upsilon_{12} (E_f, \upsilon_f, V_f, E_m, \upsilon_m, V_m) \end{aligned} \tag{24}$$

where:

E_f is the Young modulus of the fibre (considered as an isothropic material);

ν_f is Poisson coefficient of the fibre (considered as an isothropic material);

V_f is the volume percentage of the fibre in the lamina;

E_m, ν_m, V_m have similar meaning regarding the matrix.

As regards these relations, the following ones are semi-empiric relations [2]:

$$\begin{aligned} E_1 &= V_f E_f + V_m E_m \\ E_2 &= \frac{E_f E_m}{V_m E_f + V_f E_m} \\ G &= \frac{G_m}{V_m + V_f \frac{G_m}{G_f}} \\ \upsilon_{12} &= \upsilon_m V_m + \upsilon_f V_f \end{aligned} \tag{25}$$

Other relations deriving from an analytical approach are in [3].

The basic assumptions are as follows: 1) the laminae under study are macroscopically homogenous, isothropic, linearly elastic, and free from initial stresses; 2) parallelism and uniform fibre distribution are assumed.

Relations (25) supplies the elastic characteristics of the lamina related to the fibre and the matrix characteristics and their volume percentage.

The volume percentage of the fibre V_f, influential on the elastic characteristics of the lamina, can be assumed as an indirect size of defectiveness, thus characterizing, as will be

shown in the next section, "undesired" aspects in the distribution of the fibre in the material.

Such a design parameter is therefore affected with uncertainties since it is linked to the inherent defectiveness in the production process; it is therefore appropriate to examine the consequences on the mechanical behaviour of the lamina.

5.4 - RANDOM CHARACTERIZATION OF LAMINATED COMPOSITE MATERIAL

The peculiar nature of the laminated composite material presents remarkable operative difficulties in the realization so that the behaviour of the laminated can be highly influenced by the technological process defects.

Current knowledge indicates that many of the defects that occur are caused by errors in manufacturing, design errors and anomalous service conditions [1].

Typically, the occurrence of defective or anomalous conditions is controlled and prevented by manufacturing controls and material inspection testing imposed during the fabrication process; however, because absolute control and inspection is generally economically infeasible and because human errors do occur, these control and inspection methods sometimes allow occasional errors.

Other errors can happen in the operation of fabrication. Fiber reinforced composites are usually fabricated by laminated together and curing multiple plies impregnated with matrix resin. Because each of the individual plies involved in a laminated composite has highly anisotropic properties, its placement and orientation can be critical in achieving the desired engineering properties. This is particularly true for composites in which each individual ply constitutes a significant percentage of the total laminate, that is, thin gage structures.

For thermosetting matrices, either improper amounts of the two resin components or the inadequate application of heat during curing can produce condition of undercure. Sometimes, these conditions can significantly degrade the properties of the matrix. Similarly, inadequate compaction during the lamination process can result in extensive porosity and reduction in material strength and durability.

Composite materials are somewhat in that both the fundamental properties of the material and the configuration of the component to be fabricated are subject to design. Correspondingly, design errors can be made at both the materials and structural levels of design. Engineering errors related to the material may include a variety of problems. The more common of these include errors in analyzing the effect of individual ply

anisotropies or the inadequate assessment of material damage and environmental sensitivities.

In fact, failures can be caused by inadequate understanding of environmental sensitivities, the effects of damage, or the fatigue sensitivity of the material used. Because the properties of composites depend upon their ply or fiber orientation and stacking sequences, the sensitivity of each design may vary significantly, posing a potential problem for design.

The engineering properties of composites can be significantly reduced by variations in temperature, foreign object impact damage, and, with some resin system, chemical attack.

Many statistical aspects and random characterization of the laminated composite strength are present in the literature. Some references are indicated in the following.

Single fiber strengths are often modeled by a two-parameter Weibull distribution [4]. An example of a Weibull distribution that accurately models the strength data for aramid filaments tested at a single gage length is presented in [5]. S. L. Phoenix and coworkers have discussed much of the work relating the probability distribution of the strength of multifiber assemblies or composites to the distribution of fiber strengths [4-9].

In the following, will be analyzed the random characterization of the elastic parameters (and of the engineering constants), previously considered, with particular regard to the single lamina.

5.5 - RANDOM CHARACTERIZATION OF THE ELASTIC PROPERTIES OF THE LAMINA

The defects related to the technological processes, for the aim of this work, can be summarized as follows:

a) interlamina voids, due to the presence of air or foreign material;
b) excess of resin;
c) porosity of the matrix;
d) incorrect orientation of the fibres;
e) damaged fibres;
f) variation in thickness.

These defects cause variations in the behaviour of the lamina and in particular how they affect the elastic and geometrical properties of the lamina itself.

Eqs. (23) show, as it has been observed, how the elastic characteristics of the lamina in x,y direction depend on the elastic characteristics in the principal orthotropic axes of the lamina and on the fibre orientation; assuming the variable θ no longer as a deterministic parameter but as a random variable, one can investigate what the consequences would be on the E_x, E_y, G_{xy}, ν_{xy}, $\eta_{xy,x}$, $\eta_{xy,y}$ characteristics.

In order to examine this problem, it is assumed that the defectiveness regarding the fibre direction be represented by a Gaussian distribution of the variable θ centered at an expected value $\bar{\theta}$ and with dispersion defined by a standard deviation σ_θ.

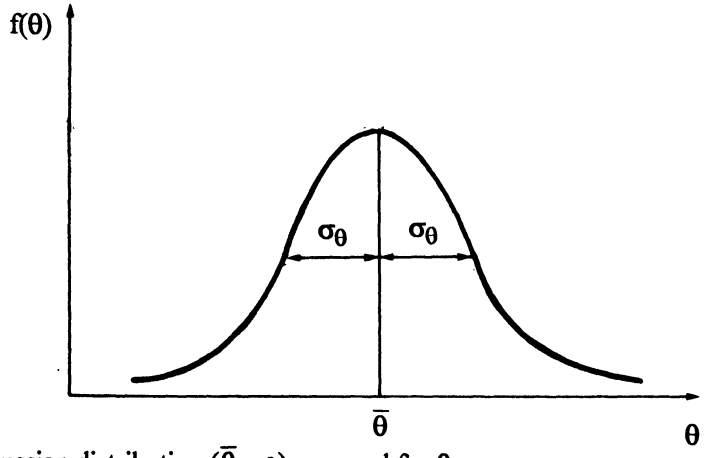


Fig.4 - Gaussian distribution $(\bar{\theta},\sigma_\theta)$ assumed for θ.

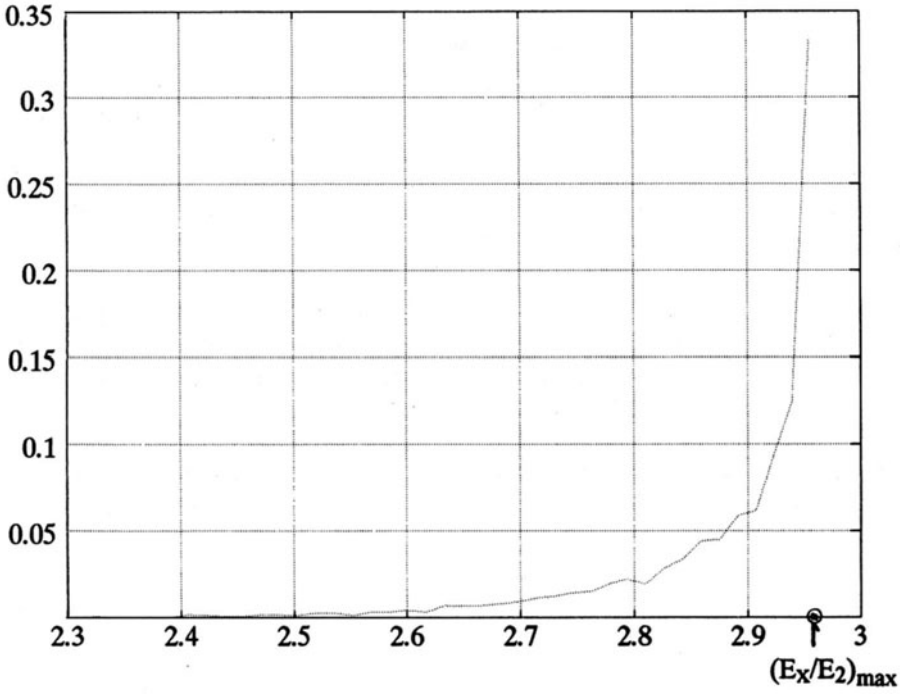
Thus known this distribution $(\bar{\theta},\sigma_\theta)$, it is possible to obtain the probability density functions of the elastic characteristics by means of Monte Carlo simulation.

The resulting type of probability density functions will depend very much on the assumed value of the expected value $\bar{\theta}$ since as previously seen the elastic characteristics vary greatly in a interval around $\theta=0^\circ$ rather than for $\theta=45^\circ$.

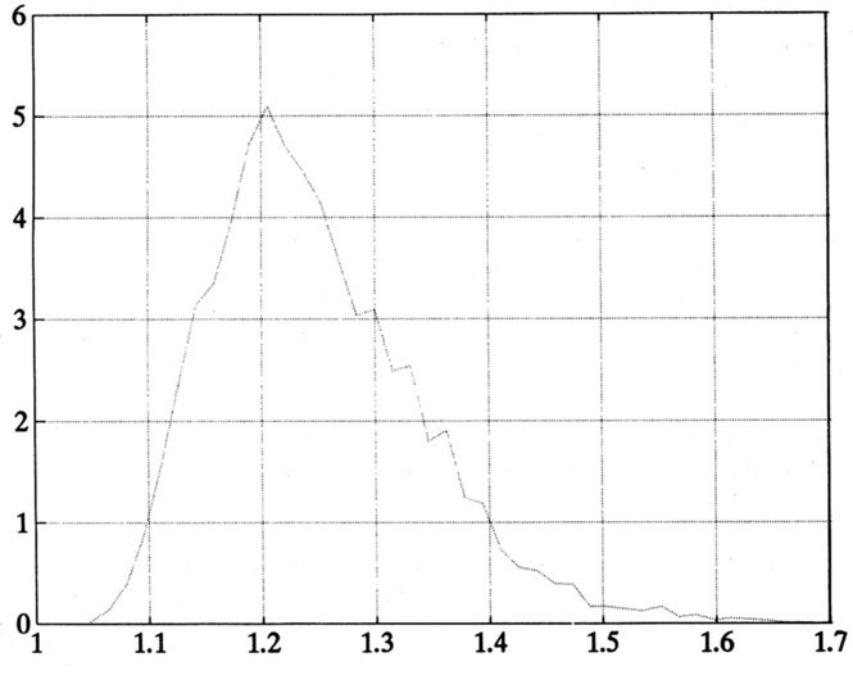
Figures 5a and 5b show the numerical results of the simulations, with the "experimental" probability density functions of E_x/E_2 for $\bar{\theta}=0^\circ$ and for $\bar{\theta}=45^\circ$ (with $\sigma_\theta=2^\circ$) in the case of E-glass-epoxy composite material. Can be noted that more dispersion of E_x/E_2 is obviously obtained in the case of higher variability ($\bar{\theta}=0^\circ$).

Furthermore, since E_x/E_2 has an absolute maximum: $(E_x/E_2)_{max}$ for $\theta=0^\circ$, the probability density function in the $\bar{\theta}=0^\circ$ case results in having non null value only for $E_x/E_2 < E_x/E_2)_{max}$.

a)



b)



Figs. 5 - Probability density functions of E_X/E_2: a) $\bar{\theta}=0°$, $\sigma_\theta=2°$; b) $\bar{\theta}=45°$, $\sigma_\theta=2°$.

Consequently the probability density function of E_X/E_2 has a mean value which is less than $(E_X/E_2)_{max}$, value of E_X/E_2 corresponding to $\theta=0^\circ$ in the deterministic case: then lower performances are expected due to the uncertainties regarding technological defects.

This observation is not obvious a priori since randomness of the design variables does not necessarily imply a lower quality of the desired structural performances; it derives from the choice of the design parameter θ corresponding to a maximum value solution.

This aspect should not be neglected since laminae in fibre-reinforced composite often represent optimized structures that are generally designed according to preferential directions aimed at improving structural performances: the structural field is often by θ values, to which stress or strain maxima (or minima) correspond.

In other words, good use of these laminae often requires placing them in an 'optimal' orientation, with θ values to which correspond the maximization of structural performances.

Similar considerations hold true for the other elastic parameters; for example, Figs. 6a and 6b show probability density functions for G_{XY}/G_{12}.

In fact, in eqs. (23) even the elastic characteristics of the lamina, as seen in eqs. (24), are random variables due to construction defects; the elastic characteristics in the principal orthotropic direction depend directly on the elastic characteristics of the comprising materials (fibre and matrix), but also on their volume percentage.

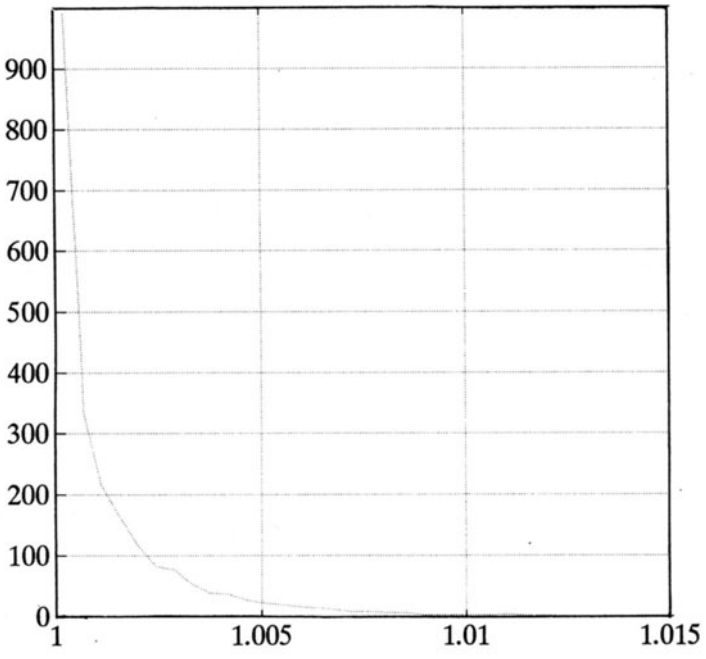
The design parameter V_f, fibre percentage in volume, is the volume ratio among the lamina components and, as clearly seen, is influenced by some of the above mentioned uncertainties (as in cases a,b,c and f).

By assuming a Gaussian distribution of V_f centered on $\overline{V}_f$ with dispersion σ_f and for θ the previously used distribution, by means eqs. (25) and Monte Carlo simulation, it is possible to obtain the distributions (Figs. 7a and 7b) of the. elastic characteristics in the principal orthotropic directions and also of the elastic characteristics according to the arbitrary **x,y** axes.

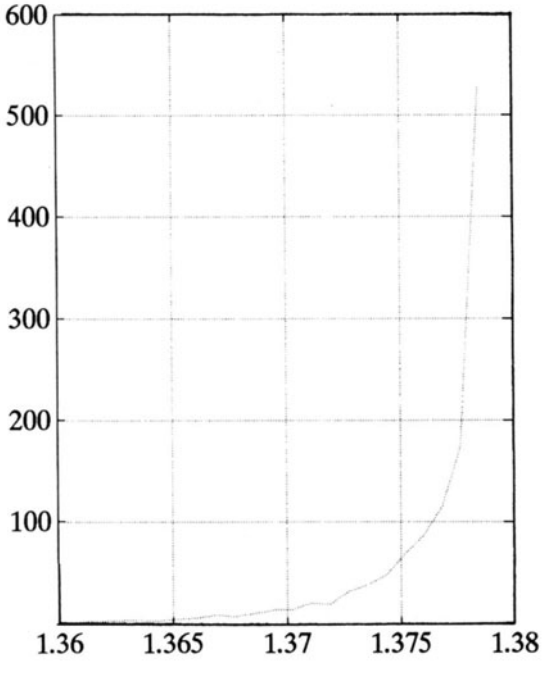
It is interesting to note (comparing Figs. 7 with the previous Figs. 5) that introducing the randomness of the parameter V_f increases what the randomness of θ had already produced.

Furthermore can be observed that this dispersion depends also on the ratio between the elastic parameters of fibre and matrix: by increasing the E_f/E_m ratio, a higher value is consequently obtained for E_1/E_2 (see eqs. (25)) and therefore a higher sensitivity of the E_X/E_2 ratio.

a)

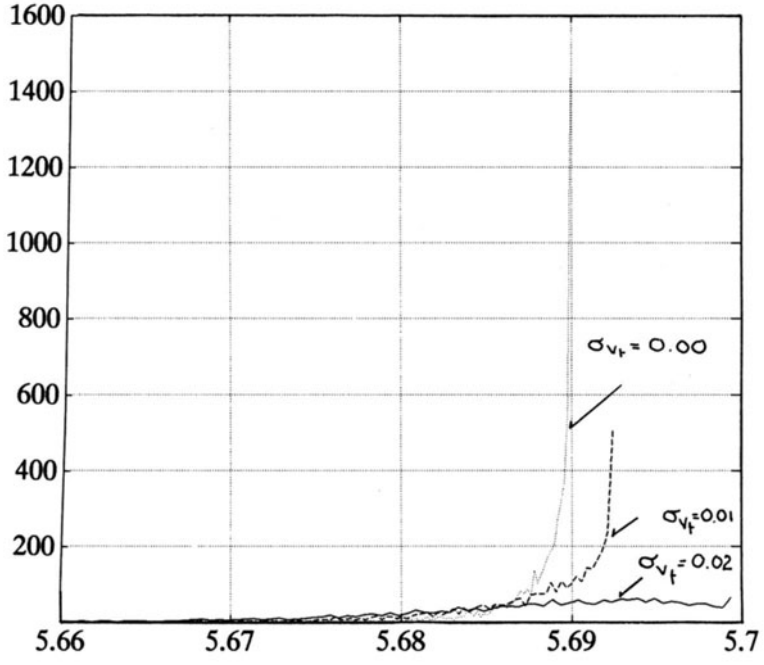


b)

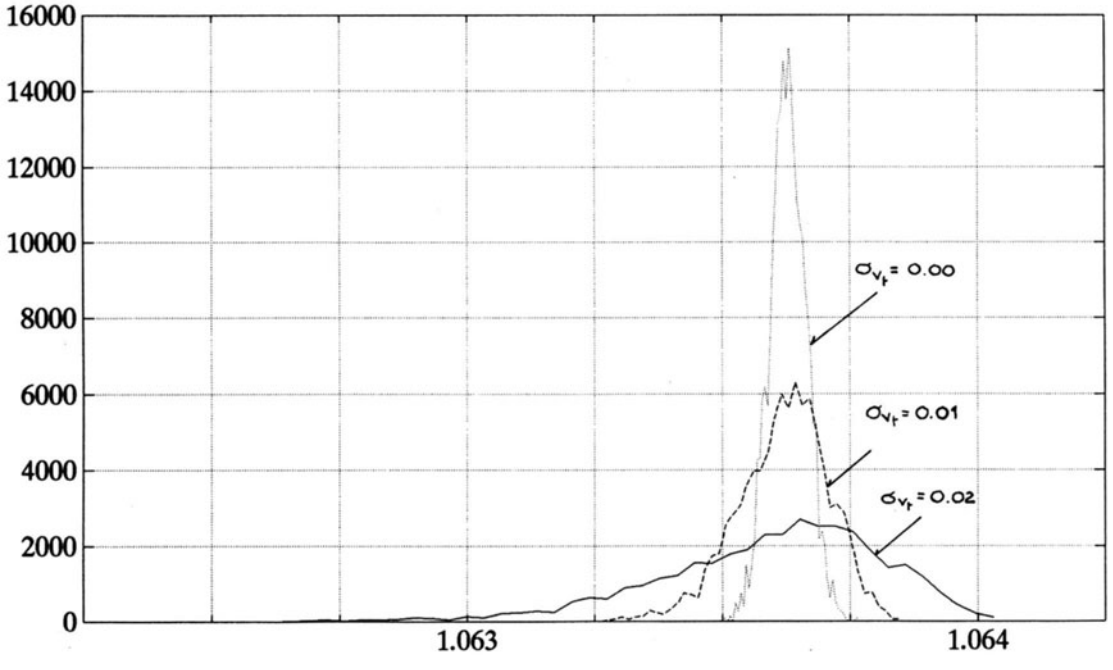


Figs. 6 - Probability density functions of G_{xy}/G_{12}: a) $\bar{\theta}=0°, \sigma_\theta=2°$; b) $\bar{\theta}=45°, \sigma_\theta=2°$.

a)



b)



Figs. 7 - Probability density functions of E_x/E_2: a) $\overline{\theta}=0°, \sigma_\theta=2°$ $\overline{V}_f=0.4$, $\sigma_{Vf}=0,0.01,0.02$; b) $\overline{\theta}=45°$, $\sigma_\theta=2°$; $\overline{V}_f=0.4, \sigma_{Vf}=0,0.01,0.02$.

The following examples quantify the influence of the uncertainties so far considered on some parameters of structural response.

5.6 - FIRST NUMERICAL EXAMPLE: A SIMPLE LAMINA

In the analysis described in the previous sections the response of an orthotropic lamina in fibre-reinforced material has been characterized with respect to the randomness of some design parameters such as the volume percentage of the fibre, V_f , and the fibre disposition θ, evidencing that such randomness result negatively in the design solution (that is, it becomes a real "defect") if the design choice was centered upon an optimum.

The principle of the analysis was to consider the elastic characteristics of the lamina to which the structural responses in terms of displacement are closely linked.

In the following example a rectangular lamina in fibre-reinforced material will be examined (Fig. 8); by introducing for V_f and θ parameters a random type of characterization the effects on the displacement state will be determined.

The analysed material is E-glass for the fibres and epoxy-resin for the matrix.

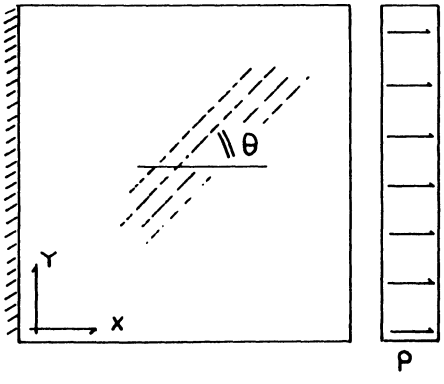


Fig.8 - Lamina loaded by uniformly pressure (E-glass-epoxy composite material).

The analyzed model is loading by an uniformly distributed pressure along one side, while the other is perfectly bound.

Fig. 9 shows the deterministic type of response in terms of the maximum displacement u_x of the free edges of the structure under study. This function, obtained by means of a series of structural analysis, shows a non increasing monotonous trend with respect to the volume percentage V_f and takes on a local minimum for $\theta=0^o$ for whatever value of V_f. It is obvious that the function trend u_x is a direct result of the E_x variation (Young modulus in x direction) as the considered parameters vary.

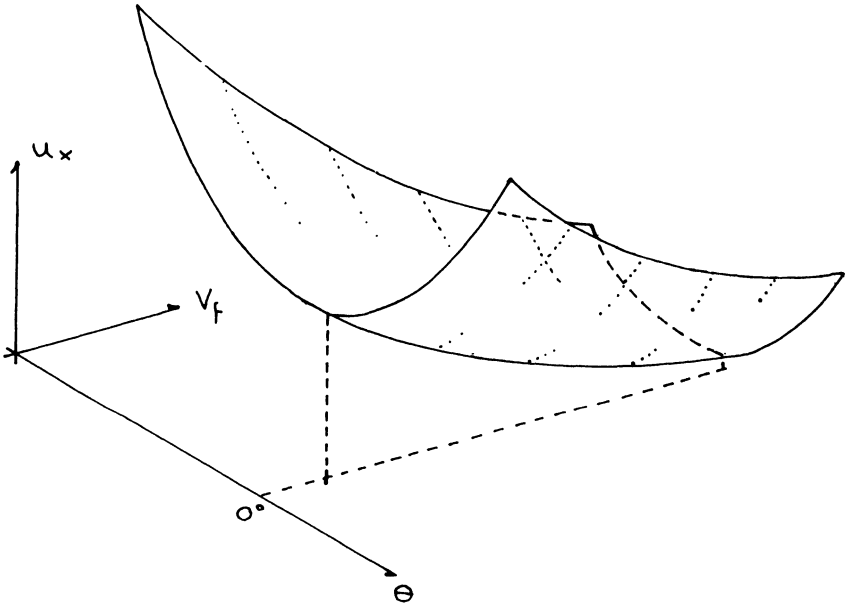


Fig. 9 - "Deterministic" relation between u_x and the design parameters θ and V_f.

As previously, assuming a random distribution for θ with expected value at $\overline{\theta}=0°$, and standard deviation $\sigma_\theta=2^o$, and for the volume percentage Vf a distribution with expected value $\overline{V}_f=0.4$ and standard deviation $\sigma_{Vf}=.01$, the probability density function of u_x (Fig. 10) is obtained by numerical simulation using a Monte Carlo technique.

The structural response is therefore again "worsened" due to the randomness that characterize the design variables since the expected value of the u_x function is greater than the deterministic value for $\theta=0^o$ and Vf=0.4 (u_x = 15.96 mm).

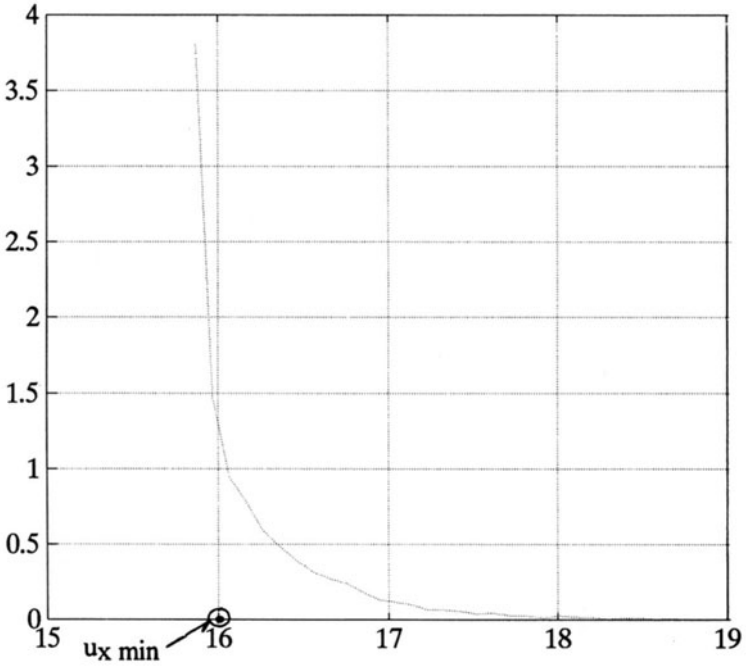


Fig. 10 - Probability density function of u_X ($\bar{\theta}=0°$, $\sigma_\theta=2°$; $\bar{V}_f=0.4$, $\sigma_{Vf}=.01$).

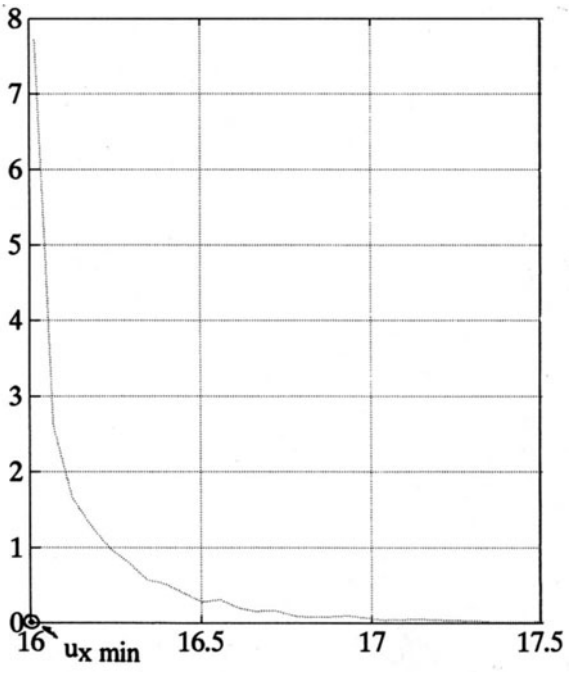
It is interesting to note the effects the randomness have on the u_X response with respect to the two different parameters V_f and θ.

If only the fibre orientation θ is considered as a random variable, it can be seen that once the value $\bar{\theta}=0°$ is chosen (optimal value for u_X in the deterministic case which corresponds $u_X = u_{X\,min} = 15.96$ mm), the probability density function of u_X is the one shown in Fig. 11a where $u_{X\,min}$ is the lower limited value. Therefore the expected value $\bar{u}_x$ will be always greater than $u_{X\,min}$.
By considering only V_f as random, a distribution of the kind reported in Fig.11b (because of the monotonic nature of the relation $u_X=u_X(V_f)$) is obtained that is the more scattered as greater the V_f dispersion will be.

The previous distributions are two limits of behaviour: the u_X distribution locates between them when both θ and V_f are random. It is possible to note that such distribution will tend to one or the other depending on the design variable dispersion: Fig. 12a shows that if there is a "small" dispersion of V_f, the probability density function of u_X will tend to a distribution closer to the one in fig.11a, and vice-versa (Fig. 12b), if the dispersion of θ is "small", a probability density function will be closer to the one in fig. 11b.

Note also that the lower limit and the expected value of u_X in Fig. 12a decrease for σ_{Vf} increasing: a "better" response is expected because of the greater uncertainties of V_f! In fact higher dispersion of V_f means higher probability values of $V_f>0.4$ which correspond lower values of u_X.

a)



b)

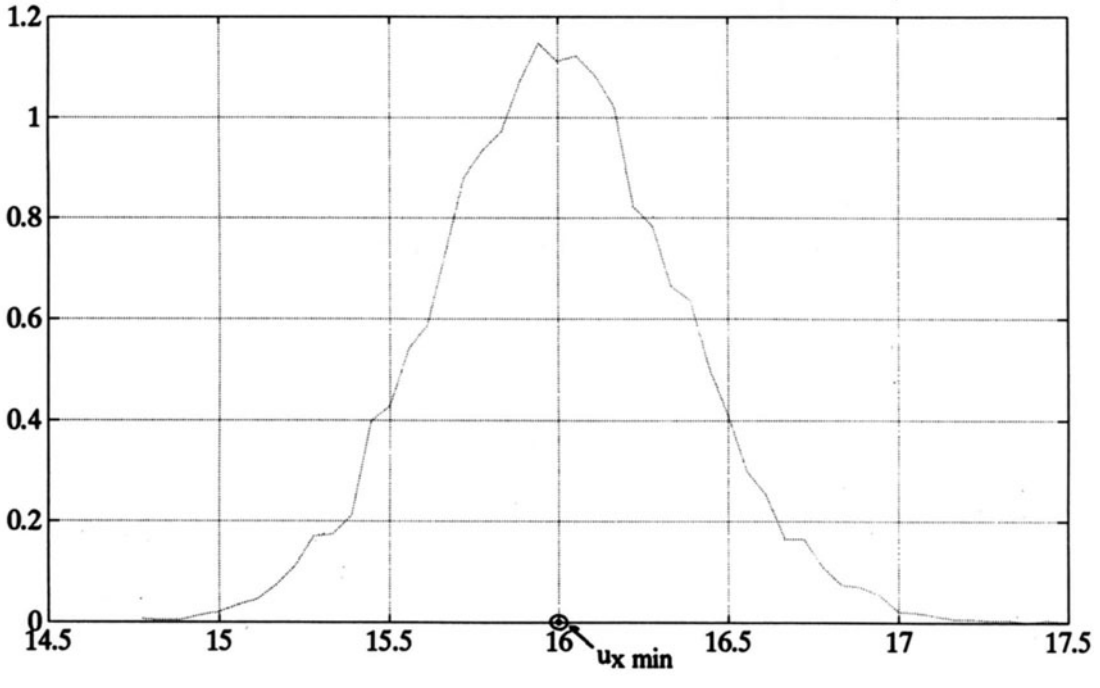
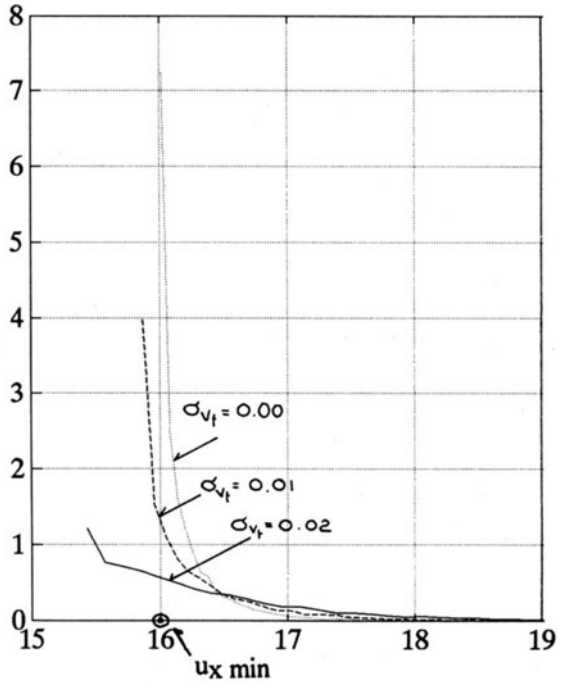
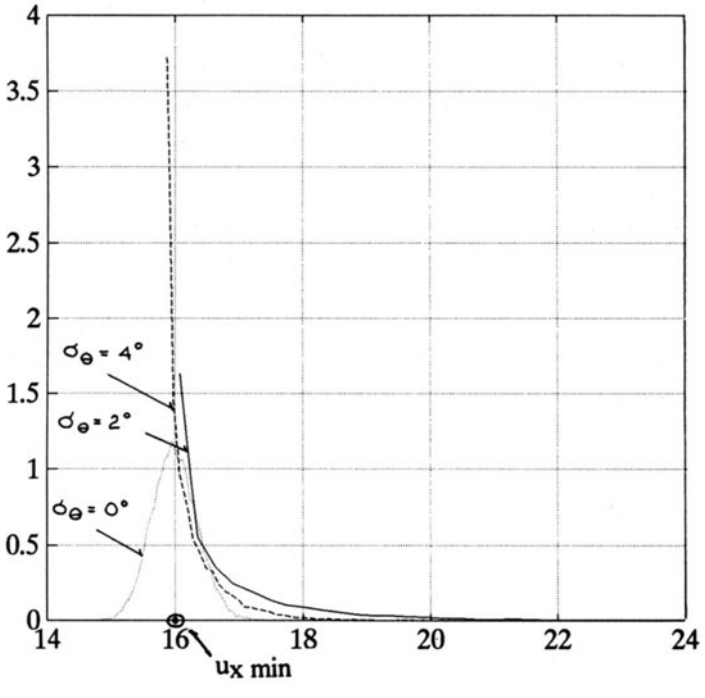


Fig. 11 - Probability density functions of u_X: a) θ random ($\bar{\theta}$=0°, σ_θ =2°) and V_f=0.4; b) V_f random ($\bar{V}_f$=0.4, σ_{Vf}=.01) and θ=0°.

a)



b)



Figs. 12 -Probability density function of u_x; influence of the dispersions of V_f and of θ:
a) $\overline{\theta}=0°, \sigma_\theta = 2°; \overline{V}_f=0.4, \sigma_{Vf}=0,0.01,0.02$; b) $\overline{V}_f=0.4, \sigma_{Vf}=0.01; \overline{\theta}=0°$, $\sigma_\theta=0°,2°,4°$.

5.7 - SECOND NUMERICAL EXAMPLE: A REINFORCED-HOLE LAMINATED STRUCTURE

In order to complete this topic, it is interesting to examine how the previous considerations hold up with respect to the response of a structure in laminated fibre-reinforced composite material.

The structure under study (Fig. 13), is characterized by the presence of a reinforced hole and subjected to a uniformly distributed pressure in the x direction.

The structure consists of five stacked fibre-reinforced laminae having equal thickness and characteristics, and assembled with cross-ply sequence $0^0/90^0/0^0/90^0/0^0$.

The package disposition is assumed to be different for the three area indicated in Fig. 13: area 1 is adjacent to the end with the load, area 2 is around the hole and area 3 is the reinforced part.

The material considered here is E-glass for the fibres and epoxy-resin for the matrix. For the symmetry, a fourth of the structure has been considered for the analysis.

The maximum displacement in x direction ,u_x of the loaded edge was taken as the parameter of the response that characterizes the structural behaviour.

As in the previous section, the link between the function u_x and θ and V_f can be obtained by means of a series of structural analyses (fig.14). This relation is of the non-increasing monotonous type with respect to V_f and has a local minimum corresponding to $\theta=0^o$ for the reinforced part and $\theta=22.5^o$ for the first and second area.

These values result to be the design choices for the production phase of the panel, in an optimal deterministic type of design.

Following the procedure used above, it is possible to introduce the uncertainties in θ and V_f, and examining their effects on u_x by Monte Carlo simulation. The resulting probability density function u_x are reported in fig. 15.

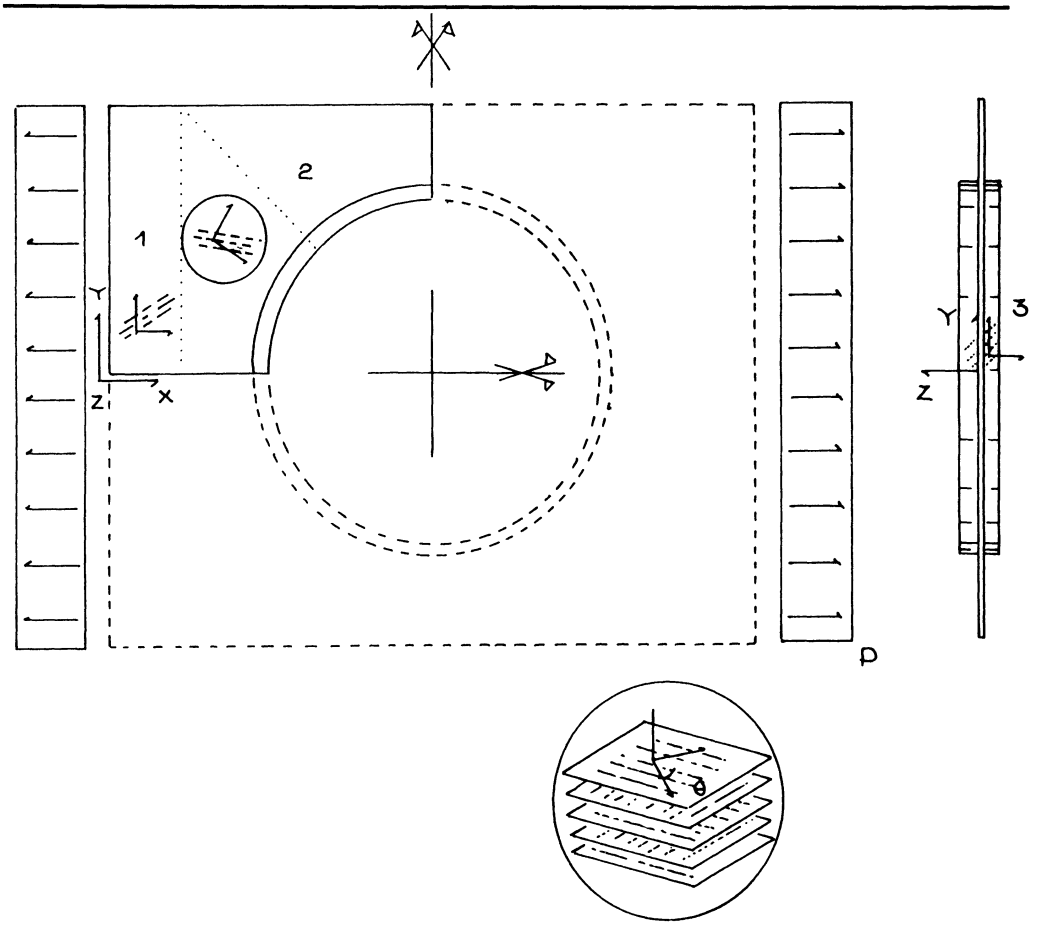


Fig. 13 - Rectangular holed shell (thickness: 4 mm, Elasticity moduli: E_f=73084 Mpa, E_m=3447 Mpa; Poisson coefficient: n_f=0.22, n_m=0.35; load: p=5 Mpa).

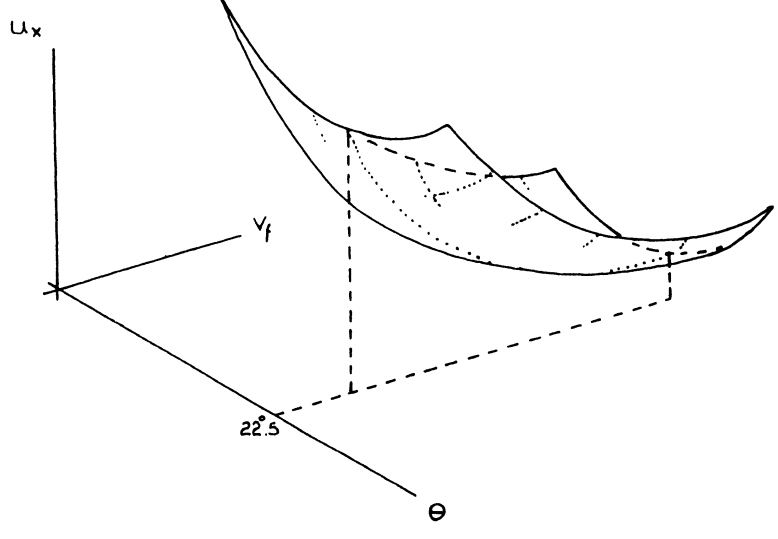


Fig. 14 - "Deterministic" linking between u_x and the design parameters θ and V_f.

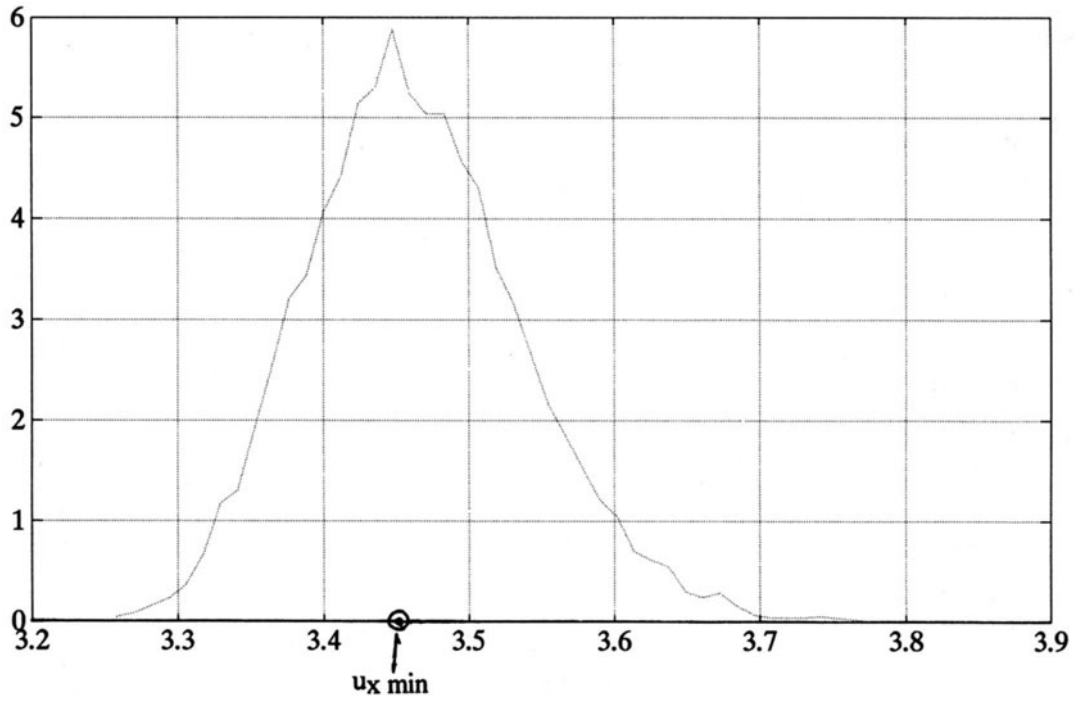


Fig. 15 - Probability density function of u_X for the structure in Fig. 13 ($\overline{\theta}$=22.5°,σ_θ=2°,$\overline{V}_f$=0.4,σ_{Vf}=.04).

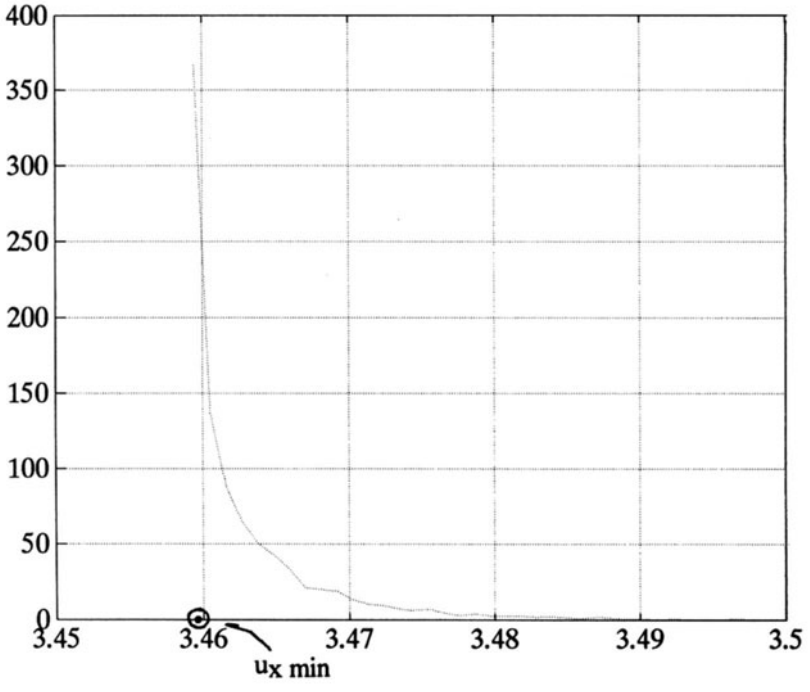
Also in this case it is interesting to note the effects the randomness have on the u_X response with respect to the two different parameters V_f and θ.

If only the fibre orientation θ is considered as a random variable, it can be seen that once the value $\overline{\theta}$=22.5° is chosen (optimal value for u_X in the deterministic case which corresponds $u_X = u_{X\,min}$), the probability density function for u_X is the one shown in Fig. 16a where $u_{X\,min}$ is the lower limited value. Therefore the expected value $\overline{u}_x$ will be always greater than $u_{X\,min}$.

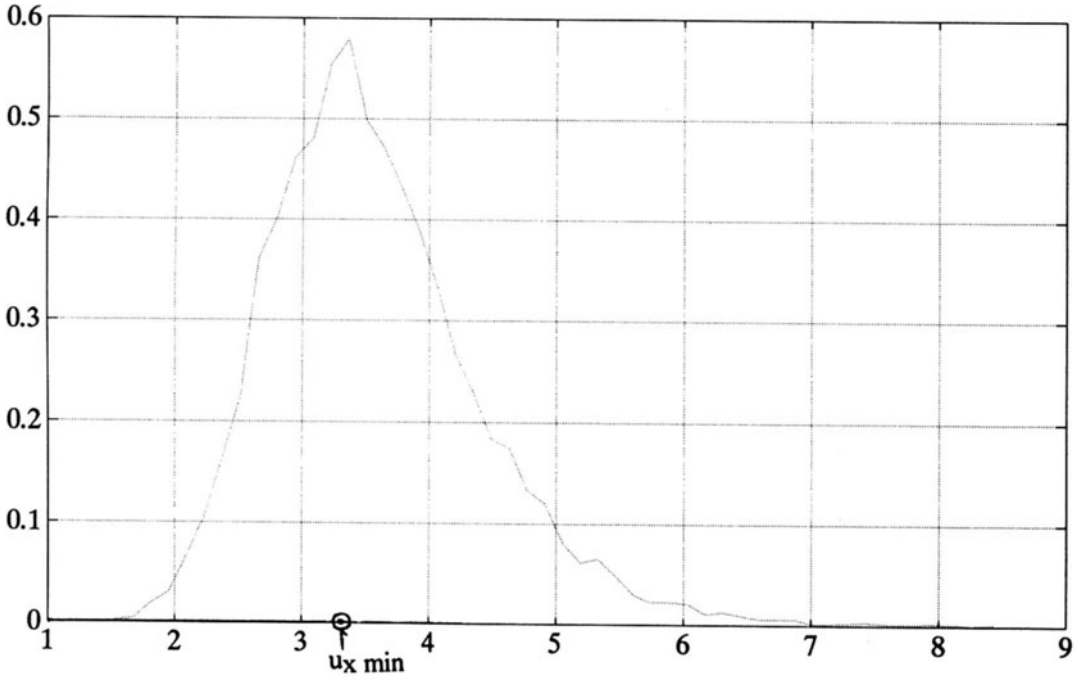
The Fig.16b shows the distribution of u_X when only V_f as random. This kind of distribution is due to the monotonic nature of the relation $u_X=u_X(V_f)$; it is the more scattered as greater the dispersion of V_f will be.

The previous distributions are the extreme limits of the u_X distribution when both θ and V_f are random. Such distribution will tend to one or the other depending on the design variable dispersion: Fig. 17a shows that when the standard deviation of V_f tends to zero, the probability density function of u_X will tend to a distribution similar to the one in fig.16a, and vice-versa (Fig. 17b), when the standard deviation of θ tends to zero, a probability density function will be similar to the one in fig. 16b.

a)

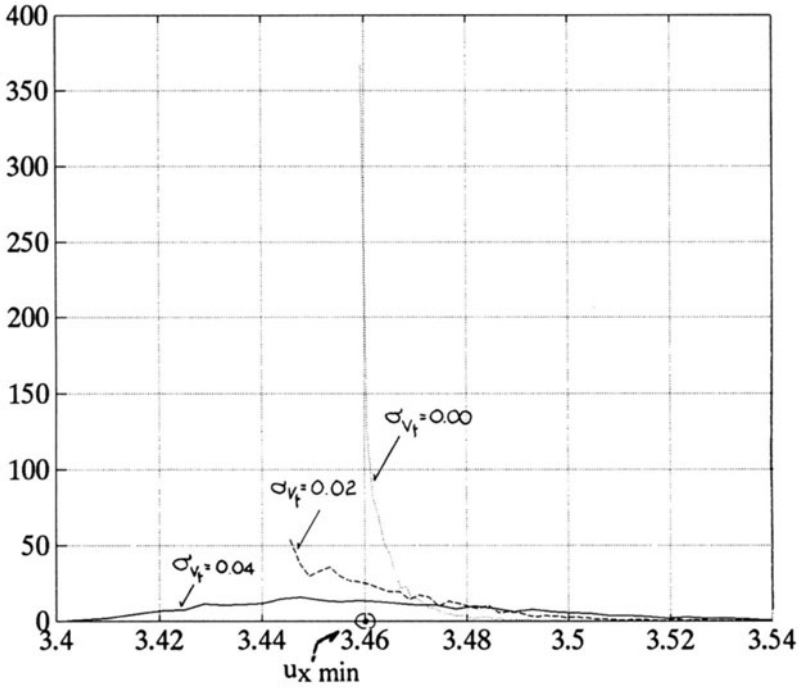


b)

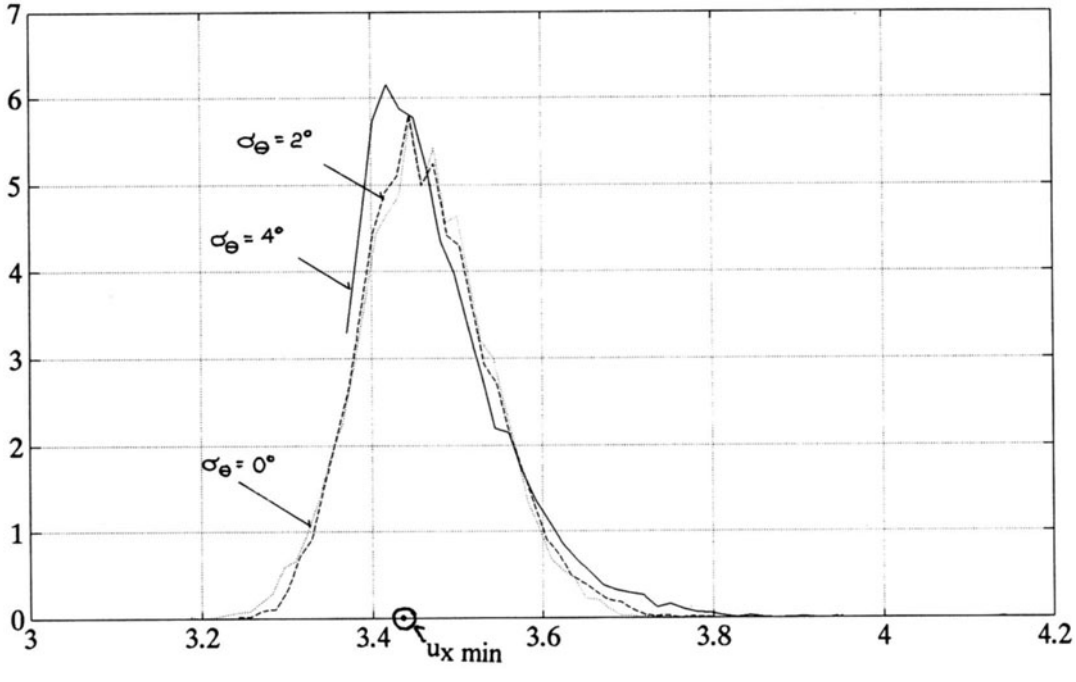


Figs. 16 - Probability density functions of u_x: a) θ random ($\bar{\theta}$=22.5°, σ_θ =2°) and V_f=0.4; b) V_f random ($\bar{V}_f$=0.4, σ_{V_f}=.01) and θ=22.5°.

a)



b)



Figs. 17 - Probability density functions of u_x; influence of the dispersions of V_f and of θ:

a) $\overline{\theta}$=22.5°, σ_θ =2° ; $\overline{V}_f$=0.4, σ_{Vf}=0,0.02,0.04;

b) $\overline{V}_f$=0.4, σ_{Vf}=0.01; $\overline{\theta}$=22.5°, σ_θ=0°,2°,4°.

From Fig. 17b it is possible to note that the effect of $\sigma\theta$ dispersion is qualitatively similar to the previous case (simple lamina) but quantitatively less significant: this is due to the fact that the lamina packaging with the $0^o/90^o/0^o/90^o/0^o$ sequence decreases the ratio between the Young modulus in the fibre direction and the Young modulus in the transversal direction.

5.8 - CONCLUSIONS

For structures in composite material characterized by fibre-reinforced laminae assembled in flat or curve shells, the structural response under deterministic design parameters often presents non monotonous relations, with local minima for certain fibre dispositions.

The optimal design of these structures can be influenced negatively by the randomness of the design variables like θ (fibre orientation) or V_f (volume percentage of fibre); the expected value of the structural response can be found in fact in a worse position from the optimal condition in the deterministic sense.

The lamina characterization regarding the V_f presents a more dispersed probability density function for the design response the greater sensitivity of the lamina to this design aspect; in general, this sensitivity increases as the volume percentage of the fibre increases.

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Chapter 6

RESPONSE OF LINEAR AND NON-LINEAR STRUCTURAL SYSTEMS UNDER GAUSSIAN OR NON-GAUSSIAN FILTERED INPUT

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1 INTRODUCTION.

Many types of loadings acting on engineering structures possess random and dynamic characteristics. Even though the study of random vibration, using the concepts of stochastic process theory, is a relatively new engineering discipline, interest in this field has grown rapidly in the last few decades. The result is a very extensive literature. However, while in the random vibration of linear structures there are now several papers which cover both theoretical and practical aspects and a number of text-books which give a good overview of the subject [1-4] is available, the study of structures which present non-linearities is more recent and exact solutions are available for few special cases only. Furthermore, while a comprehensive linear theory exists, no correspondent general theoretical framework for non-linear problems has been formulated due to the complexity of these problems.

The most commonly method used to calculate the response of weakly non-linear systems is the statistical or equivalent linearization method [5]. The main idea of this method, which actually requires relatively little numerical efforts, is to evaluate the Gaussian response properties of an equivalent linear system. However, it becomes immediately apparent that this methods gives accurate results especially for weakly non-linear systems.

An other method used for sufficiently small non-linearities is the perturbation method [6], the basic idea of this method is to expand the solution of the non-linear set of equations in terms of a small scaling parameter which characterizes the magnitude of the non-linear terms. This method is not very extensively used for the poor convergence properties and excessive computational requirements in several cases.

More general methods which give more accurate results are the methods based on the non-Gaussian closures. The basic idea of these methods is to assume the non-Gaussian probability density function of the response of non-linear systems as a series expansion with adjustable parameters of a Gaussian probability density function. The adjustable parameters depend on the statistical moments of the response which are generally governed by an infinite hierarchy of coupled differential equations. The closure approximation truncates the infinite terms series expansion in order to obtain a soluble set of equations. The simplest level of closure is the Gaussian one in which higher order statistical moments are expanded in terms of the first and second moments, as if the random processes involved were Gaussian distributed. The Gaussian closure for purely external excitations gives results identical to statistical linearization. Improvements in accuracy can be obtained by using higher order level of closure.

By no means it can be stated that the non-linear stochastic differential equations are much harder to solve than their linear counterparts and the same difficulty holds for deterministic vibration theory. Neverthless, it has been shown recently that the adoption of Kronecker algebra makes linear and non-linear differential equations governing the evolution of the statistical moments formally similar to the differential equations governing the evolution of the deterministic response [7, 8]. Moreover, the eigenproperties useful to solve both the deterministic and the moment differential equations are strictly related and a unified numerical procedure can be adopted.

In this chapter numerical procedures able to evaluate the response of linear and non-linear structures for deterministic and stochastic input are presented. The procedures described here can be applied to both classically and non-classically damped systems.

Aim of this Chapter is to show the perfect similarity of the numerical procedure to solve both linear and non-linear differential equations governing the evolution of the response for both deterministic and stochastic input. In order to do this, the response of linear systems to deterministic input and to white noise and filtered white noise input processes is evaluated first. Furthermore the linear systems subjected to delta-correlated and filtered delta-correlated input are treated.

Since readers which are not familiar with Kronecker algebra could find this algebra difficult to follow, the one-dimensional non-linear systems subjected to purely external stochastic input are treated in the first section in order to introduce concepts and solution procedures which can be easily extended to the analysis of multi-dimensional linear and non-linear structural systems.

In the stochastic analysis of non-linear systems the stochastic linearization, the cumulant and the quasi-moment neglect closures and another procedure here described and here called Hermite moment neglect closure are presented.
It will be emphasized that while the cumulant neglect closure leads always to a set of non-linear equations the quasi-moment and the Hermite moment neglect closures leads to a set of non-linear equations in the first and second statistical moments and to a set of linear equations in the central moments of order higher than two when the standardized (or the modified standardized) variable is introduced. These considerations which are not pointed out in the literature represent a fundamental step in the effective solution of non-linear systems to stochastic input.

2 ONE-DIMENSIONAL NON-LINEAR SYSTEMS

2.1 Preliminary considerations

It is well know that for linear structural systems if the input process is Gaussian, then the probability density function or the characteristic function of the response process are fully characterized by the moments or the cumulants up to second order; that is the mean, the mean square or the variance. The differential equation governing the evolution of these statistics can be easily solved for both classically or non-classically damped structures.
If the input process is not Gaussian, and the system is linear, then the response is probability characterized by the moments or the cumulants of greater order, than two. In this case the differential equations for the moments of order r will involve moments of order equal to or less than r, while the differential equations for cumulants will involve cumulants of both input and output of the same order (see e.g. Chapter two).
If the system is non-linear and the input process is or not Gaussian the response process is non Gaussian and all the moments and cumulants of the response process have to be evaluated. In this case the differential equations of order r involve also moments or alternatively cumulants or order greater than r.
If follows that an infinite hierarchy of differential equations have to be solved to fully characterize the response. This drawback is overcame by adopting the so-called closure techniques.
In this section closure techniques are presented for the study of non-linear one-dimensional systems subjected to purely external delta-correlated up to the s-th order excitations. Although this case of non-linearity is not the most general one for non-linear system, it is the most common in the structural theory. However the closure techniques here described can be also applied to non-linear systems excited by parametric excitations whose differential equations are presented in the Chapter two.
The equation of motion of a one-dimensional system with purely external delta-correlated up to s-th order excitation can be written as follows

$$dZ = \Delta Z = a(Z, t)\, dt + V(t)\, dL(t) \tag{1}$$

where dL(t) is the s-th order zero mean Levy white noise process which has the following properties

$$\begin{aligned} & m_r[dL(t)] = E[(dL(t))^r] = q_r(t)\, dt\,, \quad 2 \le r \le s \\ & k_r[dL(t)] = q_r(t)\, dt\,, \quad 2 \le r \le s\,; \quad k_{s+p}[dL(t)] = 0\,, \quad (p = 1, 2, \ldots) \\ & E[(Z(t))^k\, (dL(t))^r] = m_k[Z(t)]\, m_r[dL(t)] \\ & (dZ)^r = V^r(t)\, (dL(t))^r \quad 2 \le r \le s\,; \quad (dZ)^{s+p} = 0\,, \quad (p = 1, 2, 3, \ldots) \end{aligned} \tag{2}$$

where E [·] means stochastic average, $m_j[\cdot]$ means j-th statistical moments and $k_j[dL]$ means j-th cumulant of dL. In writing these equations we took into account the relationships between moments and cumulants (see Appendix) and we neglected infinitesimals of higher order than dt.
From Eqs. (2) it is evident that $(dL(t))^r$ and consequently $(dZ)^r$ are infinitesimals of order dt, it follows that in the series expansion of the increment ΔZ^r we have to take into account until the s-th term if $r \ge s$ and until the r-th term if $r \le s$.
Hence, for purely external excitation we can write

$$\Delta Z^r = dZ + \sum_{k=2}^{min\,(r,\,s)} \frac{1}{k!}\, d^k(Z^r) = \sum_{k=1}^{min\,(r,\,s)} \beta_{r,\,k}\, Z^{r-k}\, (dZ)^k \tag{3}$$

where

$$\beta_{r,\,k} = \frac{1}{k!}\, r\,(r-1) \ldots (r-k+1)\,; \quad q_k(t)\, dt = m_k[dL(t)] \tag{4}$$

Making the stochastic average of both sides of Eqs. (3), taking into account Eq. (1), the properties of process dL(t), given in Eqs. (2), and dividing by dt we obtain

$$\dot{m}_r[Z] = r\, E\left[a(Z, t)\, Z^{r-1}\right] + \sum_{k=2}^{min\,(r,\,s)} \beta_{r,\,k}\, m_{r-k}[Z]\, V^k(t)\, q_k(t) \tag{5}$$

We recall that the Gaussian (or normal) delta-correlated process, also called white noise process, is a particular type of delta-correlated process characterized by having $q_k(t) = 0$ for $k > 2$.
Moreover we recall that a Gaussian as well as a non-Gaussian random variable Z can be described uniquely, in a statistical sense, by its probability density function $p_Z(z, t)$, its cumulative distribution function or its characteristic function $M_Z(\vartheta; t)$.

The characteristic and the probability density function of Gaussian or non-Gaussian random variable can be written respectively as an asymptotic expansion of cumulant (also called semi-invariant) and quasi-moment functions as follows (see Appendix)

$$M_Z(\vartheta; t) = \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^j}{j!} k_j[Z] \vartheta^j \right) \tag{6a}$$

$$p_Z(z; t) = \left[1 + \sum_{j=3}^{\infty} \frac{1}{j!} \frac{b_j[Z]}{\sigma_Z^j} H_j\left(\frac{z - m_Z}{\sigma_Z} \right) \right] p_Z^0(z; t) \tag{6b}$$

where $H_j(\cdot)$ are one-dimensional Hermite polynomials (see Appendix); $k_j[Z]$ and $b_j[Z]$ are respectively the cumulant and the quasi-moments of order j of the random process Z; m_Z and σ_Z^2 are respectively the mean value and the variance of random process Z; and $p_Z^0(z; t)$ is a Gaussian density probability having $m_Z = m_1[Z]$ and σ_Z^2 as mean and variance respectively, that is

$$p_Z^0(z; t) = \frac{1}{\sqrt{2\pi}\,\sigma_Z} \exp\left[-\frac{(z - m_Z)^2}{2\sigma_Z^2} \right] \tag{7}$$

It is well known that the characteristic function is the Fourier transform of the probability density function.

Eqs. (6) fully characterize the random variable Z for Gaussian and non-Gaussian processes. As an example for Gaussian process we have $k_1[Z] = m_Z \neq 0$ and $k_2[Z] = \sigma_Z^2 \neq 0$ and $k_j[Z] = 0$ $\forall j > 2$ and $b_j[Z] = 0$ $\forall j$; for non-Gaussian processes, solution of linear differential equations forced by delta-correlated processes up the s-th order we have $k_j[Z] \neq 0$ $j \leq s$, $k_j[Z] = 0$ $\forall j > s$. The corresponding quasi-moments are obtained by the following relationship (see Eq. A.22)

$$b_j = k_j + \sum_{r=3}^{j-3} \frac{(j-1)!}{r!\,(j-r-1)!} k_{j-r}\, b_r, \quad j > 2 \tag{8}$$

For non-linear systems, even if the actual input process is idealized by a Gaussian process, the response process is not Gaussian. The most common method used to calculate the response of weakly non-linear systems is the method of equivalent or statistical linearization [5]. However, it is immediately evident that the response of actual non-linear systems to random loadings may show considerable non-Gaussian characteristics. Alternatively, some methods which allow to take into account of non-Gaussian properties of the response have been proposed. Since infinite statistics of the response cannot be evaluated the latter methods require an approximate evaluation of the characteristic function or alternatively of the probability density function. By using these methods

particular special attention has to be given to the proper choice of the probability density function of the response, because of it affects significantly the accuracy of the results.

2.2 Quasi-moment neglect closure method

The Edgeworth series (see Appendix) is a useful expansion for the non-Gaussian probability density function in terms of a Gaussian probability density function having the quasi-moments as coefficients. In this expansion setting all quasi-moments above a certain level N equal to zero we have the so-called quasi-moment neglect closure method. It follows that Eq. (6b) becomes a truncated form of the asymptotic expansion.
In order to clarify this let us consider first the case of non-linear function a(Z, t) appearing in Eq. (1) is given in the polynomial form in Z as follows

$$a(Z, t) = \sum_{i=1}^{M} a_i(t) Z^i \tag{9}$$

In this case Eq. (5) becomes

$$\dot{m}_r [Z] = r a_1(t) m_r [Z] + \sum_{i=2}^{M} a_i(t) m_{r+i-1} [Z] + \sum_{k=2}^{min (r, s)} \beta_{r,k} m_{r-k} [Z] V^k(t) q_k(t) \tag{10}$$

It follows that the differential equations for the statistics of order r will now include statistics of higher order than r. A closed set of N differential equations may be obtained by setting equal to zero the quasi-moments of order greater than N. Obviously for delta-correlated input process up the s-th order it is more convenient to choice $N \geq s$. Notice that for dynamic system with quadratic nonlinearity the differential equation of the response moments of order r will contain moments of order (r + 1). If cubic nonlinearity is included the r-th order moment equation will contain moments of order (r + 2), and so on. Thus the first-order non-Gaussian closure can be established by setting fourth-order quasi-moments equal to zero for the case of quadratic nonlinearity, or setting fourth and fifth-order quasi-moments equal to zero for the case of cubic nonlinearity, and so on. Second order non-Gaussian approximation can be obtained by generating differential equations of the response moments up to fourth-order and setting fifth-order quasi-moments and fifth and sixth-order quasi-moments to zero for quadratic and cubic nonlinearities respectively and so on. By means of this procedure the moments of higher order than N can be related to the moments of lower order by using the relationships given in Eq. (A.85) by setting $b_j = 0, j > N$ obtaining

$$m_j[Z] = -(-1)^j \sum_{\substack{k=0,2\ (j=\text{even})\\ k=3,5\ (j=\text{odd})}}^{j-1} B_{j,k}\,\sigma_Z^{j-k}\,m_k[Z]$$

$$-(-1)^j \sum_{\substack{k=0,2\ (j=\text{even})\\ k=3,5\ (j=\text{odd})}}^{j} B_{j,k}\,\sigma_Z^{j-k}\left[\sum_{r=1}^{k}(-1)^r \frac{k!}{r!\,(k-r)!}\,m_{k-r}[Z]\,m_1^r[Z]\right] \quad ;\ j > N \tag{11}$$

Hence, operating the quasi-moment neglect closure of order N, the moments of order greater than N can be related to the moments up to N, we have that the infinite hierarchy of differential equation (10) becomes a non-linear finite one.

The simplest closure is the Gaussian one in which N = 2 and all quasi-moments are zero and higher moments can be expressed in terms of the first and second moments as if the random processes involved were Gaussian distributed. Results from Gaussian closure are also known as quasi-linear approximation and coincide with that obtained from the stochastic linearization [9-10] which will be discussed later. Although the Gaussian closure is very simple to apply, unfortunately it is found to be inadeguate or unsuitable in some cases [11] and higher order-closure have to be adopted.

Moreover, since the truncated Edgeworth series can be seen as an orthogonal expansion approximation of the ratio $p_Z(z;t)/p_Z^0(z;t)$, if few terms are retained the ratio $p_Z(z;t)/p_Z^0(z;t)$ is approximated by a lower-order polynomial and the resulting behaviour of $p_Z(z;t)$ is not markedly different than that of the Gaussian model. It follows that in order to reproduce significantly a bimodal or a multimodal probability density function a big number of terms has to be retained.

It has to be noted that sometimes, since the probability density becomes very small in the region which is far from the mean value, the error caused by truncation and calculation might make an approximate density with negative values. Although it is a shortcoming of this method, it does not result in much trouble in most applications.

It is to be emphasized that the method here described can be also applied to non-linear function a(Z, t) expanded by series expansion representations as follows

$$a(Z,t) = \sum_{k=1}^{M} a_k(t)\,Z^k = \sum_{k=1}^{M} \frac{1}{k!}\left(\frac{\partial^k a(Z,t)}{\partial Z}\right)_{Z=0} Z^k \tag{12}$$

To define in alternative form the quasi-moment neglect closure let us introduce the standardized response process

$$Z^*(t) = \frac{Z(t) - m_Z(t)}{\sigma_Z(t)} \tag{13}$$

where $m_Z(t) = m_1[Z]$ and $\sigma_Z^2(t) = m_2[Z] - m_1^2[Z]$ are the mean and the variance of actual response process Z(t) respectively. The moments $m_1[Z]$ and $m_2[Z]$ can be obtained as solution of the following differential equations evaluated setting r = 1 and r = 2 into Eq. (5) where for a(Z, t) it is assumed its polynomial expansion given in Eq. (12), that is

$$\dot{m}_1[Z] = \sum_{k=1}^{M} a_k(t)\, m_k[Z] \tag{14a}$$

$$\dot{m}_2[Z] = 2 \sum_{k=1}^{M} a_k(t)\, m_{k+1}[Z] + \frac{1}{2} V^2(t)\, q_2(t) \tag{14b}$$

Since the standardized response possesses zero mean and unitary variance we have that its statistical moments coincides with its central moments, that is:

$$\mu_r[Z^*] = m_r[Z^*] = E\left[\left(\frac{Z(t) - m_Z(t)}{\sigma_Z(t)}\right)^r\right] \quad ; \quad r > 2 \tag{15}$$

In order to obtain the differential equation governing the evolution of the central moments we use Eq. (13) to write in alternative form Eq. (1) as follows

$$dZ^* = \sigma_Z^{-1}(t)\, a\,(\sigma_Z(t)\, Z^* + m_Z(t),\, t)\, dt - \sigma_Z^{-1}(t)\left[\dot{\sigma}_Z(t)\, Z^* + \dot{m}_Z(t)\right] dt + \sigma_Z^{-1}(t)\, V(t)\, dL \tag{16}$$

Since $m_Z(t)$ and $\sigma_Z(t)$ are independent of Z* we can write for purely external excitation

$$\Delta\left(Z^*\right)^r = \sum_{k=1}^{min\,(r,\,s)} \beta_{r,\,k} \left(Z^*\right)^{r-k} \sigma_Z^{-k}(t)\, (dZ^*)^k \tag{17}$$

and consequently by using Eq. (9) where $Z^i = (\sigma_Z\, Z^* + m_Z)^i$ and adopting the procedure previously described, where dZ* is given in Eq. (16), we can write

$$\begin{aligned} \dot{\mu}_r[Z^*] = r \sum_{i=1}^{M} a_i(t) \left(\sum_{j=0}^{i} \frac{i!}{j!\,(i-j)!} \sigma_Z^{i-j-1}(t)\, m_Z^j(t)\, \mu_{r+i-j-1}[Z^*] \right) \\ - r\, \sigma_Z^{-1}(t) \left(\dot{\sigma}_Z(t)\, \mu_r[Z^*] + \dot{m}_Z(t)\, \mu_{r-1}[Z^*]\right) \\ + \sum_{k=2}^{min\,(r,\,s)} \beta_{r,\,k}\, \mu_{r-k}[Z^*]\, \sigma_Z^{-k}(t)\, V^k(t)\, q_k(t) \quad , \quad r > 2 \end{aligned} \tag{18}$$

It follows that the differential equations for the central moments of order r will now include central moments of higher order than r. A closed set of N differential equations may be obtained by setting equal to zero the quasi-moments of order greater than N. By means of this procedure the central moments of higher order than N can related to the central moments of lesser order by using the relationships given in Eq. (A.59) by setting $b_j = 0$, $j > N$ obtaining the following linear relationships

$$\mu_l [Z^*] = -(-1)^l \sum_{\substack{k=0,2\ (l=\text{even}) \\ k=3,5\ (l=\text{odd})}}^{l-1} B_{l,k}\, \mu_k [Z^*] \; ; \quad l = r + i - j - 1 > N \tag{19}$$

where the coefficients $B_{l,k}$ are defined in Eq. (A.46).
So operating, the central moments of order greater than N can be related to the moments up to N by means of linear relationships and we have that the infinite hierarchy of differential Eqs. (18) becomes a set of linear differential equations in the central moments where $\sigma_Z(t)$ and $m_Z(t)$ appear in non-linear form. It follows that the numerical solution of the problem can be obtained by means of a two step iterative procedure:

(i) evaluate $m_1 [Z]$ and $m_2 [Z]$ by solving Eqs. (14) assuming a Gaussian neglect closure technique, that is assuming

$$m_k[Z] = (k-1)\left(m_2[Z] - m_1^2[Z]\right) m_{k-2}[Z] + m_1[Z]\, m_{k-1}[Z] \quad , \quad k > 2 \tag{20}$$

(ii) evaluate the central moments $\mu_r[Z^*]$ solving the set of linear differential Eqs. (18) assuming for $m_Z(t)$, $\dot{m}_Z(t)$, $\sigma_Z(t)$ and $\dot{\sigma}_Z(t)$ the functions evaluated by means of the Gaussian closure;

(iii) recalculate $m_1 [Z]$ and $m_2 [Z]$ by solving Eqs. (14) where

$$m_k[Z] = \sum_{j=0}^{k} \frac{k!}{j!\,(k-j)!}\, \sigma_Z^{k-j}(t)\, m_Z^j(t)\, \mu_{k-j}[Z^*] \tag{21}$$

with $\mu_0[Z^*] = 1$, $\mu_1[Z^*] = 0$, $\mu_2[Z^*] = 1$;

(iv) solve the linear set of differential Eqs. (18).

It is to be emphasized that introducing the standardized variable we have to solve the non-linear differential Eqs. (14) and the set of linear differential Eqs. (18) instead of a set of N non-linear differential equations.

2.3 Cumulant neglect closure method

In the cumulant neglect closure method (practically similar to the quasi-moments neglect closure method) the closure is based on cumulants. Thus, for N-th order closure, one can simply set all cumulants, k_j for which $j > N$, equal to zero. Although the basis for this method appears to be rather more arbitrary than closure based on quasi-moments, in many cases we can have very accurate results [12-14]. By using this closure technique, taking into account Eq. (A.17), we have to set in Eq. (10)

$$m_j[Z] = -\sum_{i=1}^{j-1} \frac{(j-1)!}{i!\,(j-1-i)!}(k_{j-i}[Z]\, m_i[Z]) \quad ; \quad j > N \tag{22}$$

where the cumulants $k_{j-i}[Z]$ have to be expressed in terms of moments of order equal to a lesser than $j - i$. It follows that, in order to evaluate the moments, we have to solve a set of non-linear differential equations. Once the non-linear moment differential equations are solved we have to calculate in the Edgeworth expansion the quasi-moments as a function of cumulants by means of Eq. (8) where $k_j[Z] = 0$ for $j > N$. One can easily prove that the Edgeworth expansion consequence of the N-th cumulant neglect closure is a summation of more terms than the Edgeworth expansion consequence of the N-th quasi-moments neglect closure. This fact could give difference in the approximate probability density function obtained by means of the quasi-moment and cumulant neglect closure approaches.

2.4 Hermite moment neglect closure method

If the non-linear function is of discontinuous type, or cannot be expressed as polynomials of z both the quasi-moment and the cumulant neglect closure methods seem to be not appropriate and an alternative method have to be used. Here a method called Hermite moment neglect closure is presented.
The Hermite moments have been defined by Winterstein [15] as follows

$$h_j[Z] = \frac{1}{j!} E\left[H_j\left(\frac{Z - m_Z}{\sigma_Z}\right)\right] \tag{23}$$

It follows that by using the Hermite moments the probability density function can be expanded in the Gram-Charlier expansion as follows

$$p_Z(z;t) = \left[\sum_{j=0}^{\infty} h_j[Z]\, H_j\left(\frac{z - m_Z}{\sigma_Z}\right)\right] p_Z^0(z,t) \tag{24}$$

where $p_Z^0(z; t)$ is the Gaussian probability density given in Eq. (7). In order to solve the differential Eq. (5) we have to evaluate the average $E[a(Z, t)\, Z^{r-1}]$ which by using the Gram-Charlier expansion (24) can be evaluate as follows

$$E\left[a(Z, t)\, Z^{r-1}\right] = E^0\left[a(Z, t)\, Z^{r-1}\right] + \sum_{j=3}^{\infty} E^0\left[a(Z, t)\, Z^{r-1}\, H_j\left(\frac{Z - m_Z}{\sigma_Z}\right)\right] h_j[Z] \tag{25}$$

where

$$E^0\left[a(Z, t)\, Z^{r-1}\right] = \int_{-\infty}^{\infty} a(z, t)\, z^{r-1}\, p_Z^0(z, t)\, dz$$

$$E^0\left[a(Z, t)\, Z^{r-1}\, H_j\left(\frac{Z - m_Z}{\sigma_Z}\right)\right] = \int_{-\infty}^{\infty} a(z, t)\, z^{r-1}\, H_j\left(\frac{z - m_Z}{\sigma_Z}\right) p_Z^0(z, t)\, dZ \tag{26}$$

The symbol $E^0[\cdot]$ denotes stochastic averages with normal probability density function.
Adopting an N order Hermite moments neglect closure we have to limit the summation of Eq. (25) to the first N terms. Substituting Eq. (25) into Eq. (3) and using the relationships between Hermite moments (strictly related to the quasi-moments) and statistical moments we obtain a set of N non-linear differential equations.
In a more useful form the solution of non-linear one-dimensional systems can be performed by introducing the standardized response process $Z^*(t) = [Z(t) - m_Z(t)] / \sigma_Z(t)$.
By using the procedure previously described we have the central moment differential equations in the form

$$\dot{\mu}_r[Z^*] = r\, E\left[a(Z, t)\, \sigma_Z^{-1}(t)\, (Z^*)^{r-1}\right] - r\, \sigma_Z^{-1}(t)\left(\dot{\sigma}_Z(t)\, \mu_r[Z^*] + \dot{m}_Z(t)\, \mu_{r-1}[Z^*]\right)$$

$$+ \sum_{k=2}^{min\,(r,\, s)} \beta_{r,k}\, \mu_{r-k}[Z^*]\, \sigma_Z^{-k}(t)\, V^k(t)\, q_k(t) \;;\; r > 2 \tag{27}$$

Notice that to obtain the complete statistical solution it needs to associate to Eq. (27) the following differential equations

$$\dot{m}_1[Z] = E[a(Z, t)]$$

$$\dot{m}_2[Z] = 2E[a(Z, t)] + \frac{1}{2} V^2(t)\, q_2(t) \tag{28}$$

The probability density function of the standardized response Z^* can be expanded in the Gram-Charlier expansion, as follows:

$$p_{Z^*}(z^*;t) = \left(\sum_{j=0}^{\infty} h_j[Z^*] H_j(z^*) \right) p^0_{Z^*}(z^*;t) \tag{29}$$

where $H(z^*)$ are the Hermite polynomials and $p^0_{Z^*}(z^*;t)$ is the standard normal probability density function

$$p^0_{Z^*}(z^*;t) = \frac{1}{\sqrt{2\pi}} \exp\left(-(z^*)^2/2\right) \tag{30}$$

The Hermite moments of the standardized response $h_j[Z^*]$ are strictly related to the quasi-moments $b_j[Z^*]$ of the standardized response, that is

$$h_0[Z^*] = 1\,;\; h_1[Z^*] = h_2[Z^*] = 0\,;\quad h_j[Z^*] = \frac{1}{j!} b_j[Z^*] = \frac{1}{j!} E\left[H_j(Z^*)\right],\; j > 2 \tag{31}$$

By using Eq. (A.45) it can be easily show that the Hermite moments are related to the central moments by means of the following relationships

$$h_j[Z^*] = \frac{1}{j!}\left[\mu_j[Z^*] + (-1)^j \sum_{\substack{k=0,2\ (j=\text{even}) \\ k=3,5\ (j=\text{odd})}}^{j-1} B_{j,k}\, \mu_k[Z^*] \right] \;;\; j > 2 \tag{32}$$

where the coefficients $B_{j,k}$ are defined in Eq. (A.46).
In order to express the right-hand side of the Eq. (27) explicitly in terms of the central moments and to form a closed set of differential equation we have to evaluate

$$E\left[a(Z,t)\,\sigma_Z^{-1}(t)\,(Z^*)^{r-1}\right] = \sigma_Z^{-1}(t) \int_{-\infty}^{\infty} p_{Z^*}(z^*;t)\, a(\sigma_Z z^* + m_Z, t)\,(z^*)^{r-1}\, dz^* \tag{33}$$

which by using Eq. (29) becomes

$$E\left[a(Z,t)\,\sigma_Z^{-1}(t)\,(Z^*)^{r-1}\right] = \eta_{0,r-1}(m_Z, \sigma_Z, t) + \sum_{j=3}^{\infty} h_j[Z^*]\, \eta_{j,r-1}(m_Z, \sigma_Z, t) \tag{34}$$

where $\eta_{0,r-1}(\cdot)$ and $\eta_{j,r-1}(\cdot)$ are non-linear explicit functions of m_Z, σ_Z, given as

$$\eta_{0,r-1}(\sigma_Z, m_Z, t) = \sigma_Z^{-1}(t) \int_{-\infty}^{\infty} p^0_{Z^*}(z^*; t)\, a(z^* \sigma_Z + m_Z, t)\, (z^*)^{r-1}\, dz^*$$

$$\eta_{j,r-1}(\sigma_Z, m_Z, t) = \sigma_Z^{-1}(t) \int_{-\infty}^{\infty} p^0_{Z^*}(z^*; t)\, a(z^* \sigma_Z + m_Z, t)\, (z^*)^{r-1}\, H_j(z^*)\, dz^* \; ; \quad j > 2 \tag{35}$$

The integrals (35) can be seen as the stochastic average with normal standardized probability density function of non-linear functions and could be denoted as $E^0[a(Z^* \sigma_Z + m_Z, t)\, (Z^*)^{r-1}\, H_j(Z^*)]$.

Substituting Eq. (35) into Eq. (27), taking into account of Eq. (31) and operating a Hermite moments neglect closure of order N we have

$$\dot{\mu}_r[Z^*] = r\left[\eta_{0,r-1}(m_Z, \sigma_Z, t) + \sum_{j=3}^{N} \frac{(-1)^j}{j!} \left(\sum_{\substack{k=0,2 \; (j=\text{even}) \\ k=3,5 \; (j=\text{odd})}}^{j} B_{j,k}\, \mu_k[Z^*] \right) \eta_{j,r-1}(m_Z, \sigma_Z, t) \right]$$

$$- r\, \sigma_Z^{-1}(t) \left(\dot{\sigma}_Z(t)\, \mu_r[Z^*] + \dot{m}_Z(t)\, \mu_{r-1}[Z^*] \right)$$

$$+ \sum_{k=2}^{min\,(r,\,s)} \beta_{r,k}\, \mu_{r-k}[Z^*]\, \sigma_Z^{-k}(t)\, V^k(t)\, q_k(t) \; ; \quad r > 2 \tag{36}$$

In more suitable form Eq. (36) can be written as follows

$$\dot{\mu}_r[Z^*] = r\left[\eta_{0,r-1}(m_Z, \sigma_Z, t) + \sum_{j=0}^{N} \gamma_{j,r-1}(m_Z, \sigma_Z, t)\, \mu_j[Z^*] \right]$$

$$- r\, \sigma_Z^{-1}(t) \left(\dot{\sigma}_Z(t)\, \mu_r[Z^*] + \dot{m}_Z(t)\, \mu_{r-1}[Z^*] \right)$$

$$+ \sum_{k=2}^{min\,(r,\,s)} \beta_{r,k}\, \mu_{r-k}[Z^*]\, \sigma_Z^{-k}(t)\, V^k(t)\, q_k(t) \; ; \quad r > 2 \tag{37}$$

where $\mu_0[Z^*] = 1$, $\mu_1[Z^*] = 0$, $\mu_2[Z^*] = 1$ and

$$\gamma_{j,\,r-1}(m_Z, \sigma_Z, t) = (-1)^j \sum_{\substack{k=4,6\ (j=\text{even}) \\ k=3,5\ (j=\text{odd})}}^{N} \frac{1}{k!} B_{k,j}\, \eta_{k,\,r-1}(m_Z, \sigma_Z, t) \tag{38}$$

It follows that the Hermite neglect closure, in terms of standardized variable, gives a set of N-2 differential equations which is linear in $\mu_r[Z^*]$ and non-linear in $m_Z(t)$ and $\sigma_Z(t)$. Since the latter quantities are unknown, their evaluation depends on the solution of Eqs. (28) which depend on the probability density function $p_Z(z, t)$ strictly related to $p_{Z^*}(z^*, t)$ given in Eq. (29), that is

$$p_Z(z; t) = \frac{1}{\sigma_Z} p_{Z^*}(z^*; t) = \left[\sum_{j=0}^{\infty} h_j[Z^*]\, H_j\left(\frac{z - m_Z}{\sigma_Z}\right)\right] p_Z^0(z; t) \tag{39}$$

where $p_Z^0(z; t)$ is defined in Eq. (7).

The numerical solution of the problem can be obtained of a two step iterative procedure just described in the previous section that is:

(i) evaluate $m_1[Z]$ and $m_2[Z]$ by solving Eqs. (28) assuming a Gaussian neglect closure technique;

(ii) solve the set of linear differential Eqs. (36) or (37) assuming for $m_Z(t)$, $\dot{m}_Z(t)$, $\sigma_Z(t)$ and $\dot{\sigma}_Z(t)$ the functions evaluated by means of the Gaussian closure;

(iii) recalculate $m_1[Z]$ and $m_2[Z]$ by solving Eqs. (28) where $E[a(Z,t)]$ and $E[Z\,a(Z,t)]$ are given by the following relationships;

$$\begin{aligned} E[a(Z, t)] &= \int_{-\infty}^{\infty} p_Z(z; t)\, a(z, t)\, dz \\ E[Z\,a(Z, t)] &= \int_{-\infty}^{\infty} z\, p_Z(z; t)\, a(z, t)\, dz \end{aligned} \tag{40}$$

using for $p_Z(z; t)$ the following expression whose coefficients are the central moments of standardized variable

$$p_Z(z; t) = \frac{1}{\sigma_Z}\left[1 + \sum_{j=3}^{N}\left(\frac{(-1)^j}{j!} \sum_{\substack{k=0,2\ (j=\text{even}) \\ k=3,5\ (j=\text{odd})}}^{j} B_{j,k}\, \mu_k[Z^*]\right) H_j(z^*)\right] p_{Z^*}^0(z^*, t) \tag{41}$$

where $p^0_{Z^*}(z^*, t)$ is defined in Eq. (30). This equation is derived from Eq. (6) by using the relationship between Hermite moments and central moments given in Eq. (32).

(iv) solve the linear set of differential Eqs. (36) or (37);

(v) come back to (iii) and iterate until the accuracy required is obtained.

In the framework of the Hermite moment neglect closure methods one can also classify the method proposed by Liu and Davis [16-17]. In this method assuming a truncated Gram-Charlier series as approximate solution of the Fokker-Planck equation a set of Hermite moment differential equations which are non-linear in the mean value and in the variance and linear in the Hermite moments are obtained. It follows that the Hermite moment differential equations can be solved by means of the same two step approach described before.

Notice that for polynomials form non-linearities the Hermite moment neglect closure method leads to the same results of the quasi-moment neglect closure method.

It has to be emphasized that for stationary input all the statistical quantities are not time dependent. It follows that the differential equations become a set of algebraic equation which is a non-linear one by using the cumulant neglect closure. However by introducing the standardized variable and by adopting the quasi-moment or the Hermite moment neglect closure we have a set of non-linear algebraic equations into the two first moments and a set of linear equations in the central moments of the standardized variable. It follows that to obtain the solution it needs to solve the two sets of equations by using the two step procedure described for non-stationary input.

2.5 Stochastic linearization

The stochastic or statistical linearization can be considered as an extension of the equivalent linearization method proposed by Krylov and Bogoliubov [18] for the treatment of non-linear systems under deterministic excitations. Caughey [19] was one of the first researchers to extend this method to solve the problems of randomly excited non-linear systems. The basic idea of the stochastic linearization is to replace the original non-linear system by a linear one in such a way that the difference between the two system is minimized in some statistical sense. In this way, the parameters which appear in this linearized system involve unknown statistics of the response which are evaluated approximating the response to a Gaussian process, in accordance with the equivalent linearized system with unknown coefficients and subjected to a Gaussian excitations. It follows that the stochastic response is obtained after some iterations.

The stochastic linearization is perhaps one of the most common methods to solve non-linear systems under stochastic excitations and various versions of the method have been developed (see e.g. [5, 9, 20]). It is well known that for purely external excitations the stochastic linearization and the Gaussian closure lead to the same results, while for parametric excitations we have the same results, which give the best approximation, only if the coefficients of the Ito differential rule are linearized and not the differential equation of motion [10].

For the phylosophy of the method the stochastic linearization gives a good solution only if the real response is close to a Gaussian process. It follows that this method is very efficient for weak non linearity and for Gaussian excitations. Indeed, for delta-correlated excitations or generally for non-Gaussian excitations this method leads to different results with respect to those obtained by closure schemes which seem much more accurate comparing the results obtained with Monte-Carlo simulation [21].
For these reasons, in the following the stochastic linearization of non-linear systems for purely external Gaussian white noise excitation will be briefly illustrated. In this case the non-linear equation of motion is replaced by the following linearized one

$$dZ = [A(t)\, Z + v(t)]\, dt + v(t)\, dB(t) \tag{42}$$

The error made in linearizing the equation of motion is expressed by

$$e = A(t)\, Z + v(t) - a(Z, t) \tag{43}$$

The unknown coefficients A(t) and v(t) can be evaluated by minimizing the mean square error with respect to them, that is solving the following equations

$$\frac{\partial}{\partial A} E\left[e^2\right] = 0\,; \quad \frac{\partial}{\partial v} E\left[e^2\right] = 0 \tag{44}$$

obtaining

$$\begin{aligned} v(t) &= E\,[a(Z, t)] - A(t)\, m_1\,[Z] \\ A(t) &= \frac{E\,[a(Z, t)\, Z] - E\,[a(Z, t)]\, m_1\,[Z]}{m_2\,[Z] - m_1^2\,[Z]} \end{aligned} \tag{45}$$

where $m_1\,[Z]$ and $m_2\,[Z]$ can be obtained by applying the procedure previously described as the solution of the following differential equation:

$$\begin{aligned} \dot{m}_1\,[Z] &= A(t)\, m_1\,[Z] + v(t) \\ \dot{m}_2\,[Z] &= 2A(t)\, m_2\,[Z] + 2v(t)\, m_1\,[Z] + v^2(t)\, q_2(t) \end{aligned} \tag{46}$$

Notice that in order to evaluate $m_1\,[Z]$ and $m_2\,[Z]$ we have to know A(t) and v(t) which depends on $m_1\,[Z]$ and $m_2\,[Z]$. It follows that it needs an iterative procedure, assuming for the first iteration a given Gaussian probability density function $p_Z^{(1)}(z, t)$; to evaluate $E^{(1)}\,[a(Z, t)]$, $m_1^{(1)}\,[Z]$, $A^{(1)}(t)$, $v^{(1)}(t)$; to solve the differential equation (46); to evaluate the new probability density function and so on until a good accuracy is achieved. Generally the described procedure converge very fastly.
It is to be emphasized that for stationary input Eqs. (46) become a set of algebraic equations.

2.6 Applications

Two one-dimensional problems are solved to illustrate the described closure methods. The first one with cubic non-linearity the second one with quadratic non-linearity.

Example 1. Consider a one-dimensional diffusion process satisfying the following stochastic differential equation

$$dZ = \left(a_1 Z + a_3 Z^3\right) dt + V\, dB(t) \tag{E.1}$$

where a_1 and a_3 are negative constants, V a positive constant and $dB = W(t)\, dt$ is the increment of white noise process. It can be shown that all the odd moments of the process Z(t) are zero and the even moments up to sixth order can be evaluated as the solution of the following differential equations (with argument omitted)

$$\begin{aligned} \dot{m}_2 &= 2 a_1 m_2 + 2 a_3 m_4 + V^2 q_2 \\ \dot{m}_4 &= 4 a_1 m_4 + 4 a_3 m_6 + 6 V^2 q_2 m_2 \\ \dot{m}_6 &= 6 a_1 m_6 + 6 a_3 m_8 + 15 V^2 q_2 m_4 \end{aligned} \tag{E.2}$$

a) Stochastic linearization

The stochastic linearization gives the same results than the Gaussian closure. In the latter case we can write (see Eq. A.7)

$$m_j = (j - 1)\, m_2\, m_{j-2} \quad ; \quad j > 2 \tag{E.3}$$

and the first of Eq. (E.2) becomes

$$\dot{m}_2 = 2 a_1 m_2 + 6 a_3 m_2^2 + V^2 q_2 \tag{E.4}$$

which represents a non-linear differential equation.

b) Cumulant neglect closure

The second order cumulant neglect closure coincides with the stochastic linearization. Adopting a fourth order cumualnt neglect closure the cumulant of order greater than four are assumed to be zero. It follows that from Eq. (A.6) we can write

$$m_6 = 15\, m_2\, m_4 - 30\, m_2^3 \tag{E.5}$$

and the Eqs. (E.2) becomes

$$\begin{aligned} \dot{m}_2 &= 2\,a_1\,m_2 + 2\,a_3\,m_4 + V^2\,q_2 \\ \dot{m}_4 &= 4\,a_1\,m_4 + 4\,a_3\left(15\,m_2\,m_4 - 30\,m_2^3\right) + 6\,V^2\,q_2\,m_2 \end{aligned} \tag{E.6}$$

If a sixth order cumulant neglect closure is adopted by using Eqs. (A.6) we can write Eqs. (E.2) in the form

$$\begin{aligned} \dot{m}_2 &= 2\,a_1\,m_2 + 2\,a_3\,m_4 + V^2\,q_2 \\ \dot{m}_4 &= 4\,a_1\,m_4 + 4\,a_3\,m_6 + 6\,V^2\,q_2\,m_2 \\ \dot{m}_6 &= 6\,a_1\,m_6 + 4\,a_3\left(630\,m_2^4 - 420\,m_2^2\,m_4 + 35\,m_4^2 + 28\,m_2\,m_6\right) + 15\,V^2\,q_2\,m_4 \end{aligned} \tag{E.7}$$

c) Quasi-moment neglect closure

The second and the fourth order cumulant neglect closure coincides with the stochastic linearization and the fourth order quasi-moments neglect closure respectively since the cumulants and the quasi-moments until fifth order are the same.
In order to performe a sixth-order quasi-moments neglect closure we must use relationships (11) for j = 8 obtaining

$$m_8 = 315\,m_2^4 - 210\,m_2^2\,m_4 + 28\,m_6\,m_2 \tag{E.8}$$

It follows that a sixth-order quasi-moments neglect closure leads to the following differential equations

$$\begin{aligned} \dot{m}_2 &= 2\,a_1\,m_2 + 2\,a_3\,m_4 + V^2\,q_2 \\ \dot{m}_4 &= 4\,a_1\,m_4 + 4\,a_3\,m_6 + 6\,V^2\,q_2\,m_2 \\ \dot{m}_6 &= 6\,a_1\,m_6 + 6\,a_3\left(315\,m_2^4 - 210\,m_2^2\,m_4 + 28\,m_6\,m_2\right) + 15\,V^2\,q_2\,m_4 \end{aligned} \tag{E.9}$$

Notice that while the first two of these equations are the same of the cumulant neglect closure the latter is a different one.

d) Hermite moment neglect closure

Because of the non-linearity is in polynomial form the Hermite moment closure leads to the same differential equations than the quasi-moment closures if the standardized variable introduced. Since in this case $m_Z(t) = 0$, introducing the standardized variable $Z^* = Z/\sigma_Z$ with $\sigma_Z^2 = m_2[Z]$, we have

$$\dot{\mu}_r[Z^*] = r\,\sigma_Z^{-1}(t)\,E\left[a_1 Z (Z^*)^{r-1} + a_3 Z^3 (Z^*)^{r-1}\right] - r\,\sigma_Z^{-1}(t)\,\dot{\sigma}_Z(t)\,E\left[(Z^*)^r\right]$$

$$+\frac{1}{2} r(r-1)\,\mu_{r-2}[Z^*]\,\sigma_Z^{-2}(t)\,V^2(t)\,q_2(t) \quad ; \quad r > 2 \tag{E.10}$$

and Eq. (18) becomes

$$\dot{\mu}_r[Z^*] = r\left(a_1\,\mu_r[Z^*] + a_3\,\sigma_Z^2\,\mu_{r+2}[Z^*]\right) - r\,\sigma_Z^{-1}(t)\,\dot{\sigma}_Z(t)\,\mu_r[Z^*]$$

$$+\frac{1}{2} r(r-1)\,\mu_{r-2}[Z^*]\,\sigma_Z^{-2}\,V^2(t)\,q_2(t) \quad ; \quad r > 2 \tag{E.11}$$

Associating the first of Eqs. (E.9) to Eqs. (E.11) for r = 4 and 6, and observing that $\mu_2[Z^*] = 1$, we can write the following equations

$$\dot{m}_2[Z] = 2\,a_1\,m_2[Z] + 2\,a_3\,m_4[Z] + V^2\,q_2 \tag{E.12a}$$

$$\dot{\mu}_4[Z^*] = \left[4\,a_1 - \sigma_Z^{-1}(t)\,\dot{\sigma}_Z(t)\right]\mu_4[Z^*] + 4\,a_3\,\mu_6[Z^*]\,\sigma_Z^2 + 6\,\sigma_Z^{-2}\,V^2\,q_2 \tag{E.12b}$$

$$\dot{\mu}_6[Z^*] = \left[6\,a_1 - \sigma_Z^{-1}(t)\,\dot{\sigma}_Z(t)\right]\mu_6[Z^*] + 6\,a_3\,\sigma_Z^2\,\mu_8[Z^*] + 15\,\sigma_Z^{-2}\,\mu_4[Z^*]\,V^2\,q_2 \tag{E.12c}$$

Adopting the sixth order Hermite moment closure (wich coincides with the quasi-moment neglect closure on the standardized variable Z*) we have

$$\mu_8[Z^*] = 315 - 210\,\mu_4[Z^*] + 28\,\mu_6[Z^*] \tag{E.13}$$

Substituting this relationships into Eq. (E.12c) we have that this equation for a sixth order Hermite moment or quasi-moment neglect closure becomes

$$\dot{\mu}_6[Z^*] = \left[6\,a_1 - \sigma_Z^{-1}(t)\,\dot{\sigma}_Z(t)\right]\mu_6[Z^*] + 6\,a_3\,\sigma_Z^2(315 - 210\,\mu_4[Z^*] + 28\,\mu_6[Z^*])$$

$$+ 15\,\sigma_Z^{-2}\,\mu_4[Z^*]\,V^2\,q_2 \tag{E.14}$$

$$\dot{\mu}_6[Z^*] = \left[6\,a_1 - \sigma_Z^{-1}(t)\,\dot{\sigma}_Z(t)\right]\mu_6[Z^*] + 6\,a_3\,\sigma_Z^2\left(315 - 420\,\mu_4[Z^*] + 35\,\mu_4^2[Z^*]\right.$$

$$\left. + 28\,\mu_6[Z^*]\right) + 15\,\sigma_Z^{-2}\,\mu_4[Z^*]\,V^2\,q_2 \qquad \text{(E.15)}$$

Associating Eq. (E.14) to Eqs. (E.12b) we have a set of linear differential equations into the central moments where $\sigma_Z^2 = m_2[Z]$ must be obtained independently by solving the non-linear Eq. (E.12a). It follows that we have to adopt a two step iterative procedure. In the first iteration we can assume m_2 as the solution of Eq. (E.4) obtained by Eq. (E.12a) adopting a stochastic linearization and the successive iterations we can assume $m_4[Z] = \mu_4[Z^*]\,\sigma_Z^4$ until the whished convergence is obtained. Instead by using a cumulant neglect closure we have

$$\mu_8[Z^*] = 630 - 420\,\mu_4[Z^*] + 35\,\mu_4^2[Z^*] + 28\,\mu_6[Z^*] \qquad \text{(E.16)}$$

and Eq. (E.12c) for a sixth order comulant neglect closure becomes

$$\dot{\mu}_6[Z^*] = \left[6a_1 - \sigma_Z^{-1}(t)\,\dot{\sigma}_Z(t)\right]\mu_6[Z^*] + 6a_3\,\sigma_Z^2\left(315 - 420\,\mu_4[Z^*] + 35\,\mu_4^2[Z^*]\right.$$

$$\left. + 28\,\mu_6[Z^*]\right) + 15\,\sigma_Z^{-2}\,\mu_4[Z^*]\,V^2\,q_2(t) \qquad \text{(E.17)}$$

It follows that we have always a set of non-linear differential equations and the advantage in using the standardized variable is not so evident as in the quasi-moment closure.
Furthermore comparing the results of the two described neglect closures with the Monte-Carlo simulation we have, for this system, a higher accuracy using the Hermite closure rather than the cumulant closure of the same order.

Example 2. For quadratic non-linearities the stochastic differential equation can be written as follows

$$dZ = a_1\,Z + a_2\,Z^2 + V\,dB(t) \qquad \text{(E.18)}$$

where a_1 and a_2 are negative constant and V is a positive one. For this non-linearity the odd moments of Z are different from zero. In this case it seems more convenient, from a practical point of view, to adopt a Hermite moment closure writing the differential equations by using the standardized variable $Z^* = (Z - m_Z)/\sigma_Z$ where $m_Z = m_1[Z]$ and $\sigma_Z^2 = m_2 - m_Z^2$. In this case we can write Eqs. (14) in the form

$$\begin{aligned} \dot{m}_1[Z] &= \dot{m}_Z(t) = a_1\,m_1[Z] + a_2\,m_2[Z] \\ \dot{m}_2[Z] &= 2\,a_1\,m_2[Z] + a_2\,m_3[Z] + \frac{1}{2}V^2\,q_2 \end{aligned} \qquad \text{(E.19)}$$

and Eq. (18) becomes

$$\dot{\mu}_r[Z^*] = r\left(a_1 + 2\,a_2\,m_Z(t)\right)\mu_r[Z^*] + r\,a_2\,\sigma_Z(t)\,\mu_{r+1}[Z^*]$$

$$+ r\,\sigma_Z^{-1}(t)\left(a_1\,m_Z(t) + a_2\,m_Z^2(t)\right)\mu_{r-1}[Z^*] - r\,\sigma_Z^{-1}(t)\left(\dot{\sigma}_Z(t)\,\mu_r[Z^*] + \dot{m}_Z(t)\,\mu_{r-1}[Z^*]\right)$$

$$+ \frac{r(r-1)}{2}\,\mu_{r-2}[Z^*]\,\sigma_Z^2\,V^2\,q_2(t)\,; \qquad r > 2 \tag{E.20}$$

If a sixth order Hermite moment neglect closure is adopted we have to relate $\mu_7[Z^*]$ which appears in Eq. (E.20) for r = 6, with less order central moments by means of Eq. (32), for $h_7[Z^*] = 0$, that is

$$\mu_7[Z^*] = 21\,\mu_5[Z^*] - 105\,\mu_3[Z^*] \tag{E.21}$$

obtaining

$$\dot{\mu}_6[Z^*] = 6\left(a_1 + 2\,a_2\,m_Z(t)\right)\mu_7[Z^*] + 6\,a_2\,\sigma_Z(t)\left(21\,\mu_5[Z^*] - 105\,\mu_3[Z^*]\right)$$

$$+ 6\,\sigma_Z^{-1}(t)\left(a_1\,m_Z(t) + a_2\,m_Z^2(t)\right)\mu_5[Z^*] - 6\,\sigma_Z^{-1}(t)\left(\dot{\sigma}_Z(t)\,\mu_6[Z^*] + \dot{m}_Z(t)\,\mu_5[Z^*]\right)$$

$$+ 15\,\mu_4[Z^*]\,\sigma_Z^2\,V^2\,q_2(t) \tag{E.22}$$

which represents a linear equation into the central moments of standardized variable. Associating to this equation to Eqs. (E.20) for r = 3, 4, 5 we have a set of four linear differential equations, which together to non-linear Eqs. (E.19), gives the solution in terms of statistical moments of the response. Eqs. (E.19) and (E.20) can be solved by a two step iterative procedure assuming in the first step

$$m_3[Z] = 3\,m_Z\,m_2[Z] - 2\,m_Z^3 \tag{E.23}$$

as a consequence of the Gaussian closure and in the successive steps

$$m_3[Z] = \sigma_Z^3\,\mu_3[Z^*] + 3\,\sigma_Z^2\,m_Z + m_Z^3 \tag{E.24}$$

3 MULTIDIMENSIONAL LINEAR SYSTEMS.

3.1 Deterministic analysis

For sake of clarity, in this section some preliminary concepts of deterministic analysis, which are also useful for stochastic analysis, are summarized.
By using the finite element method, the equation of motion of an n-degree-of-freedom model of an elastic structural system with viscous damping subjected to a nodal forcing function vector $\mathbf{F}(t)$ is written as follows

$$\begin{aligned} &\mathbf{M}\,\ddot{\mathbf{X}} + \mathbf{C}\,\dot{\mathbf{X}} + \mathbf{K}\,\mathbf{X} = \mathbf{F}(t) \\ &\mathbf{X}(t_0) = \mathbf{X}_0\,; \quad \dot{\mathbf{X}}(t_0) = \dot{\mathbf{X}}_0 \end{aligned} \tag{48}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the inertia, damping and stiffness matrices respectively, $\ddot{\mathbf{X}}$, $\dot{\mathbf{X}}$ and $\mathbf{X}$ are the nodal acceleration, velocity and displacement vectors respectively, $\mathbf{X}_0$ and $\dot{\mathbf{X}}_0$ are the vectors of initial conditions; the upper dot means time differentiation. In order to reduce the number of equations, the dynamic response is evaluated in a reduced space by means of the following coordinate transformation

$$\mathbf{X} = \mathbf{\Phi}\,\mathbf{Y} \tag{49}$$

where $\mathbf{\Phi}$ is the modal matrix of the undamped system, containing the first few eigenvectors of the matrix $\mathbf{K}^{-1}\mathbf{M}$ normalized with respect to $\mathbf{M}$; in this way the matrix $\mathbf{\Phi}$ is of order n x m, m being the number of modes selected for the analysis ($m << n$). The matrix $\mathbf{\Phi}$ possesses the following properties

$$\mathbf{\Phi}^T\mathbf{M}\,\mathbf{\Phi} = \mathbf{I}_m\,; \quad \mathbf{\Phi}^T\mathbf{K}\,\mathbf{\Phi} = \mathbf{\Omega}^2 \tag{50}$$

where $\mathbf{I}_m$ is the identity matrix of order m and $\mathbf{\Omega}^2$ is the diagonal matrix listing the square of the natural radial frequencies ω_i^2 ($i = 1, 2, \ldots, m$). Letting $\mathbf{\Phi}^T\mathbf{C}\,\mathbf{\Phi} = \mathbf{\Xi}$, $\mathbf{\Xi}$ being an m x m symmetric matrix, Eq. (48) can be written in the reduced space of modal coordinates as follows

$$\begin{aligned} &\ddot{\mathbf{Y}} + \mathbf{\Xi}\,\dot{\mathbf{Y}} + \mathbf{\Omega}^2\,\mathbf{Y} = \mathbf{\Phi}^T\mathbf{F}(t) \\ &\mathbf{Y}_0 = \mathbf{\Phi}^T\mathbf{M}\,\mathbf{X}_0\,; \quad \dot{\mathbf{Y}}_0 = \mathbf{\Phi}^T\mathbf{M}\,\dot{\mathbf{X}}_0 \end{aligned} \tag{51}$$

If Ξ is a diagonal matrix, then the system possesses classical normal modes and Eq. (51) are given in decoupled form; in the most general case, when Ξ is a full matrix, a $2m$ dimension vector approach is commonly used. For this purpose Eq. (51) can be rewritten in the standard form

$$\dot{\mathbf{Z}} = \mathbf{A}\mathbf{Z} + \mathbf{V}\mathbf{F}(t) ; \quad \mathbf{Z}(t_0) = \mathbf{Z}_0 \tag{52}$$

where $\mathbf{Z}$ is the 2m-vector of modal state variables defined as

$$\mathbf{Z}^T = \left[\mathbf{Y}^T \ \dot{\mathbf{Y}}^T\right]; \quad \mathbf{Z}_0^T = \left[\mathbf{Y}_0^T \ \dot{\mathbf{Y}}_0^T\right] \tag{53}$$

and the matrices $\mathbf{A}$ and $\mathbf{V}$ are given by

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_m \\ -\boldsymbol{\Omega}^2 & -\boldsymbol{\Xi} \end{pmatrix}; \quad \mathbf{V} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\Phi}^T \end{pmatrix} \tag{54}$$

The vector solution $\mathbf{Z}(t)$ of Eq. (52) can be written as follows

$$\mathbf{Z}(t) = \boldsymbol{\Theta}(t - t_0)\, \mathbf{Z}(t_0) + \int_{t_0}^{t} \boldsymbol{\Theta}(t - \tau)\, \mathbf{V}\mathbf{F}(\tau)\, d\tau \tag{55}$$

where $\mathbf{Z}(t_0)$ is the vector of initial conditions and $\boldsymbol{\Theta}(t)$ is the so-called transition (or fundamental) matrix [22-23] of the system which, if all of the eigenvalues of $\mathbf{A}$ are distinct, can be evaluated as follows

$$\boldsymbol{\Theta}(t) = \boldsymbol{\psi}\, e^{t\,\boldsymbol{\Lambda}}\, \boldsymbol{\psi}^{-1} \tag{56}$$

where $\boldsymbol{\psi}$ and $\boldsymbol{\Lambda}$ are complex quantities that must be evaluated by solving the following eigenproblem

$$\mathbf{A}\,\boldsymbol{\psi} = \boldsymbol{\psi}\,\boldsymbol{\Lambda} \tag{57}$$

For classically damped systems the closed form of the matrix $\boldsymbol{\Theta}(t)$ can be easily obtained as

$$\boldsymbol{\Theta}(t) = \begin{pmatrix} -\mathbf{G}(t)\,\boldsymbol{\Omega}^2 & \mathbf{H}(t) \\ -\mathbf{H}(t)\,\boldsymbol{\Omega}^2 & \dot{\mathbf{H}}(t) \end{pmatrix} \tag{58}$$

where $\mathbf{H}(t)$ is a diagonal matrix, having the i-th diagonal element equal to the impulse response function of the i-th oscillator in modal coordinates given by

$$h_i(t) = \frac{1}{\overline{\omega}_i} \exp\left(-\xi_i \omega_i t\right) \sin\left(\overline{\omega}_i t\right); \quad t \geq 0$$
$$h_i(t) = 0; \quad t < 0 \tag{59}$$

and $\mathbf{G}(t)$ is a diagonal matrix such that $\dot{\mathbf{G}}(t) = \mathbf{H}(t)$. In Eq. (59) ξ_i and $\overline{\omega}_i = \omega_i \sqrt{1 - \xi_i^2}$ can be evaluated taking into account of the fact that $2\xi_i\omega_i$ is the i-th element of the diagonal matrix $\mathbf{\Xi}$.
In many cases of practical interest, the closed form solution of Eq. (52) cannot be computed and a numerical solution method must be applied. To this purpose, let the time space be divided into small intervals of equal length, Δt, and let $t_0 = 0, t_1, \ldots, t_{k-1}, t_k, \ldots, t_n$ be the division times. By imposing that the governing equation, Eq. (52), must be satisfied at discrete times, Eq. (55) can be written as follows

$$\mathbf{Z}(t_{k+1}) = \boldsymbol{\Theta}(\Delta t)\, \mathbf{Z}(t_k) + \int_{t_k}^{t_{k+1}} \boldsymbol{\Theta}(t_{k+1} - \tau)\, \mathbf{VF}(\tau)\, d\tau \tag{60}$$

Furthermore, by adopting a piecewise constant forcing function in each interval, after very simple algebra Eq. (60) becomes

$$\mathbf{Z}(t_{k+1}) = \boldsymbol{\Theta}(\Delta t)\, \mathbf{Z}(t_k) + \mathbf{L}(\Delta t)\, \mathbf{VF}(t_k) \tag{61}$$

where

$$\mathbf{L}(\Delta t) = \left[\boldsymbol{\Theta}(\Delta t) - \mathbf{I}_{2m}\right] \mathbf{A}^{-1} \tag{62}$$

is the so-called loading matrix [24].
Eq. (61) provides an unconditionally stable step-by-step procedure, where the only source of numerical errors is in the modelling of the forcing function $\mathbf{F}(t)$ as a stepwise function. A different choice is possible by modelling differently the forcing function in the interval [25].
The main computational drawbacks in evaluating the response of Eq. (52), using the proposed numerical procedure, is in the evaluation of the complex eigenproperties of matrix $\mathbf{A}$, required for the evaluation of the transition and loading matrices. Hence the solution of eigenproblem (57) represents the main computational aspect in the analysis of non-classically damped systems.
For non-classically damped systems, computation of complex eigeproperties of the matrix $\mathbf{A}$ can be avoided by directly evaluating the fundamental matrix by means of a Taylor expansion. Here a finite number N of terms in the expansion for $\boldsymbol{\Theta}(t)$, as well as for the load operator, are retained, that is

$$\hat{\Theta}(\Delta t) = \sum_{j=0}^{N} \frac{1}{j!}(\mathbf{A}\,\Delta t)^j; \quad \hat{\mathbf{L}}(\Delta t) = \Delta t\left[\sum_{j=0}^{N-1} \frac{1}{(j+1)!}(\mathbf{A}\,\Delta t)^j\right] \tag{63}$$

Notice that, by replacing in Eq. (61) the approximated operators given in Eq. (63), the unconditionally stable step-by-step procedure becomes conditionally stable [26, 27]. However, this is not a limitation on the effectiveness of the numerical procedure. Indeed the effects connected with the damping matrix Ξ only produce small changes in the frequencies of the damped structure with respect to the undamped one. Because the modal analysis of the undamped system has been performed and **A** has been written in a reduced space, the approximate value of the smallest period $T_m = 2\pi / \omega_m$ is already known, and this is essential for the choice of the time step length to ensure stability of the step-by-step integration method. In order to show this, let us introduce the spectral radius of $\hat{\Theta}(\Delta t)$ in the form

$$\rho\left[\hat{\Theta}(\Delta t)\right] = max_i\left[\left|\sum_{k=0}^{N} \frac{1}{k!}(\lambda_i\,\Delta t)^k\right|\right] = \left|\sum_{k=0}^{N} \frac{1}{k!}(\lambda_{max}\,\Delta t)^k\right| \tag{64}$$

where λ_{max} is the maximum among the eigenvalues λ_i of **A** (and is very close to the largest radial frequency ω_m of the undamped system). Eq. (64) shows that the spectral radii and therefore the stability of the integration method depend only on the step size selected and on the number of terms included in the Taylor expansion. In Figs. 1 the spectral radii $\rho(\hat{\Theta})$ of a single degree-of-freedom (SDOF) system with natural period T_0 and damping ratios $\xi = 0$, $\xi = 0.1$ versus the ratio $\Delta t/T_0$ with variations in the number of terms N are depicted. From these figures it is evident that the numerical procedure is stable using a time step $\Delta t < T_0/4$ if N = 7 and for any value of the damping ratios. The numerical results can be extended to non-classically damped systems using a natural period $T_0 = 2\pi / |\lambda_{max}| \cong 2\pi / \omega_m$.

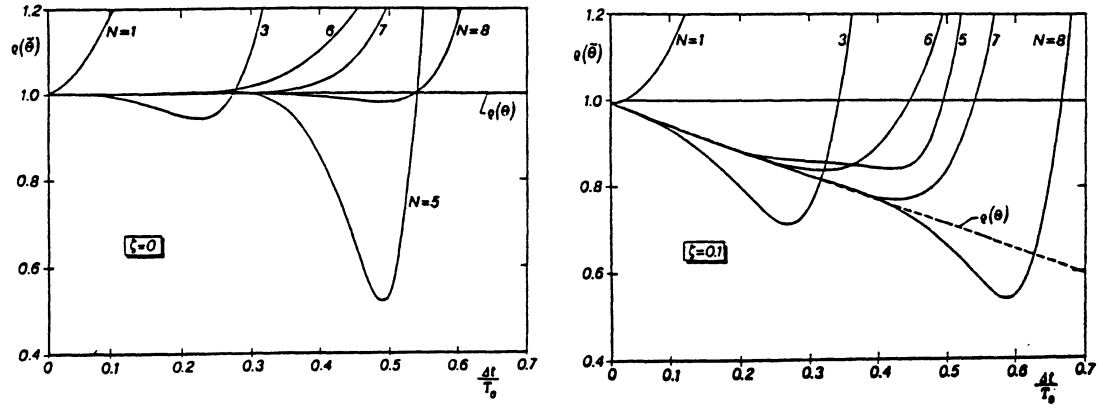

Fig. 1 - Solid line: spectral radii of approximate fundamental matrix; dashed line: spectral radii of exact fundamental matrix.

Other numerical integration methods are obtained in the Chapter four.

In order to compare the numerical integration method described here with the well-known other ones we evaluate the algorithmic counteparts $\bar{\xi}$ and $\bar{\omega}_0$ [28] of the damping ratio ξ and the radial frequency ω_0 of a SDOF as a function of the approximate transition matrix. These quantities are given as follows

$$\bar{\xi} = \frac{1}{2\,\bar{\omega}_0\,\Delta t}\, ln\, I_2\,; \quad \bar{\omega}_0 = \frac{1}{\Delta t\sqrt{1-\bar{\xi}^2}}\cos^{-1}\left[\frac{I_1}{\sqrt{I_2}}\right] \tag{65}$$

where $I_1 = (1/2)$ trace $[\Theta(\Delta t)]$ and $I_2 = \det[\Theta(\Delta t)]$.

It follows that as measures of the numerical dissipation and dispersion, we take the algorithmic damping ratio $\bar{\xi}$ and the relative period error $(\bar{T}_0 - T_0)/T_0$. These quantities are depicted in Figures 2(a) and 2(b), versus $\Delta t/T_0$, for N = 3-8, and are compared with the same quantities evaluated by well-known numerical integration methods. These figures show the great accuracy of the described method.

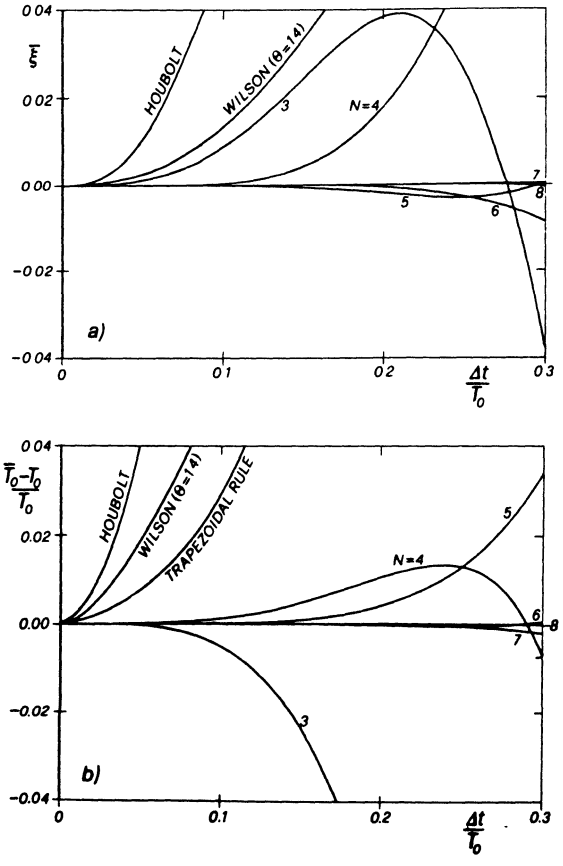

Fig. 2 - Accuracy tests: a) algorithmic damping ratio; b) relative period error.

3.2 White noise input

If $\mathbf{F}(t)$ is a stochastic vector process, then $\mathbf{X}(t)$ is a stochastic vector process too and it has to be characterized in probabilistic sense. If $\mathbf{F}(t)$ is a normal vector process, then also $\mathbf{X}(t)$, $\mathbf{Y}(t)$, $\mathbf{Z}(t)$ and $\dot{\mathbf{X}}(t)$, $\dot{\mathbf{Y}}(t)$, $\dot{\mathbf{Z}}(t)$ are normal vector processes and the complete probabilistic characterization of both input and output processes can be obtained by the knowledge of the first two correlation functions that is

$$\mathbf{m}_1[\mathbf{F}(t)] = E[\mathbf{F}(t)]; \quad \mathbf{m}_1[\mathbf{Z}(t)] = E[\mathbf{Z}(t)] \tag{66}$$

$$\begin{aligned} \mathbf{C}_2[\mathbf{F}(t_1), \mathbf{F}(t_2)] &= E\left[\mathbf{F}(t_1)\ \mathbf{F}(t_2)^T\right] - \mathbf{m}_1[\mathbf{F}(t_1)]\,\mathbf{m}_1^T[\mathbf{F}(t_2)] \\ \mathbf{C}_2[\mathbf{Z}(t_1), \mathbf{Z}(t_2)] &= E\left[\mathbf{Z}(t_1)\ \mathbf{Z}(t_2)^T\right] - \mathbf{m}_1[\mathbf{Z}(t_1)]\,\mathbf{m}_1^T[\mathbf{Z}(t_2)] \end{aligned} \tag{67}$$

where $E[\cdot]$ means stochastic average.
The goal of random vibration analysis of linear systems subjected to Gaussian input is finding $\mathbf{m}_1[\mathbf{Z}(t)]$ and $\mathbf{C}_2[\mathbf{Z}(t_1), \mathbf{Z}(t_2)]$, once that the analogous quantities of the input $\mathbf{F}(t)$ are known.
In this section the analysis is limited to the simpler case of a zero mean Gaussain white noise input $\mathbf{F}(t) = \mathbf{W}(t)$, characterized by having the first two correlation functions of the form

$$\mathbf{m}_1[\mathbf{W}(t)] = \mathbf{0}; \quad \mathbf{C}_2[\mathbf{W}(t_1), \mathbf{W}(t_2)] = \mathbf{Q}(t_1)\,\delta(t_1 - t_2) \tag{68}$$

where $\mathbf{Q}(t)$ is the matrix of the strength of the white noise and $\delta(\cdot)$ is the Dirac delta function.
Such kind of input radically simplifies the mathematical problems of analysis of the response which can be framed in the theory of diffusive Markov processes; this teory provides very effective tools, such as the Fokker-Planck-Kolmogorov equation, Ito's differential rule, etc.
Different approaches have been proposed to solve this problem, the simpler method for white noise excitation is the state space moment analysis. It consists in solving the differential equations for the covariances. By using this approach Bryson and Ho [29] gave the differential equation for the covariance function matrix of the response for zero mean white noise input in the Lyapunov form

$$\dot{\Sigma}_Z(t) = \mathbf{A}\,\Sigma_Z(t) + \Sigma_Z(t)\,\mathbf{A}^T + \mathbf{V}\,\mathbf{Q}(t)\,\mathbf{V}^T \tag{69}$$

where $\Sigma_Z(t)$ is the covariance function matrix defined as follows

$$\Sigma_Z(t) = \mathbf{C}_2[\mathbf{Z}(t), \mathbf{Z}(t)] \tag{70}$$

Further the correlation matrix $\mathbf{C}_2[\mathbf{Z}(t_1),\ \mathbf{Z}(t_2)]$ can be obtained in the form

$$\begin{aligned}\mathbf{C}_2[\mathbf{Z}(t_1),\ \mathbf{Z}(t_2)] &= \boldsymbol{\Theta}(t_1 - t_2)\,\boldsymbol{\Sigma}_{\mathbf{Z}}(t_1), \qquad t_1 \le t_2\,; \\ \mathbf{C}_2[\mathbf{Z}(t_1),\ \mathbf{Z}(t_2)] &= \boldsymbol{\Theta}(t_2 - t_1)\,\boldsymbol{\Sigma}_{\mathbf{Z}}(t_2), \qquad t_2 \le t_1\end{aligned} \tag{71}$$

Closed form solution of Eq. (52) for MDOF linear system having classical normal modes and for certain shape of the strength $\mathbf{Q}(t)$ [30,31] and a step-by-step method for evaluating the non-stationary response of a MODF system for a general shape of the intensity $\mathbf{Q}(t)$ [26] have been proposed.

An alternative approach in evaluating the second moments of the response can be obtained by using the Ito stochastic calculus. This approach has the advantage that can be directly extended to non-white input processes.

Since $\mathbf{W}(t)$ is a vector with rapidly fluctuating random term having infinite variance, it follows that [32] the element of process $\mathbf{W}(t)$ are not mean square Riemann integrable and, consequently Eq. (52) does not have mathematical meaning. However, we can write formally

$$d\mathbf{B}(t) = \mathbf{W}(t)\,dt \tag{72}$$

where $\mathbf{B}(t)$ is a 2n-vector of Wiener process such that the first two moments of $d\mathbf{B}(t)$ are

$$E[d\mathbf{B}] = \mathbf{0}\,; \quad E[d\mathbf{B} \otimes d\mathbf{B}] = \mathbf{q}_2(t)\,dt \tag{73}$$

while higher order moments are infinitesimal of order greater than dt. In equation (73) the symbol $\otimes$ means Kronecker product [33, 34] (see also Chapter two Appendix A), so that the vector $E[d\mathbf{B}\otimes d\mathbf{B}]$, of order $(2n)^2$, contains all possible moments of second order of the vector $d\mathbf{B}$ and

$$\mathbf{q}_2(t) = \mathrm{Vec}\,\{\mathbf{Q}(t)\} \tag{74}$$

where Vec $(\cdot)$ is a column vector obtained by putting each column of the matrix in parenthesis below each other. It follows, that Eq. (52) can be converted into an Ito type stochastic differential equation

$$d\mathbf{Z} = \mathbf{A}\,\mathbf{Z}\,dt + \mathbf{V}\,d\mathbf{B}(t) \tag{75}$$

Notice that $d\mathbf{B}$ is a vector listing infinitesimals of order $(dt)^{1/2}$ and consequently in Eq. (75) infinitesimal of different order appear. To evaluate the differential equation governing the evaluation

of the second order moments of $\mathbf{Z}(t)$ we observe that these quantities are listed in the following vector

$$E[\mathbf{Z}(t) \otimes \mathbf{Z}(t)] = \mathbf{m}_2[Z] = \text{Vec}\{\boldsymbol{\Sigma}_Z(t)\} = \text{Vec}\{E[\mathbf{Z}(t)\ \mathbf{Z}^T(t)]\} \tag{76}$$

Furthermore remembering that all the elements of $\mathbf{dB}$ are of order $(dt)^{1/2}$ we can evaluate the increment of vector

$$\mathbf{Z}^{[2]}(t) = \mathbf{Z}(t) \otimes \mathbf{Z}(t) \tag{77}$$

where the exponent in square brackets means Kronecker power, as follows

$$\Delta(\mathbf{Z}^{[2]}) = d(\mathbf{Z}^{[2]}) + \frac{1}{2} d^2(\mathbf{Z}^{[2]}) \tag{78}$$

Introducing the row differential vector

$$\nabla_Z^T = \left[\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \cdots \frac{\partial}{\partial z_{2m}}\right] \tag{79}$$

we can write [35]

$$\begin{aligned} d(\mathbf{Z}^{[2]}) &= (\nabla_Z^T \otimes \mathbf{Z}^{[2]})\, d\mathbf{Z} = \mathbf{Z} \otimes d\mathbf{Z} + d\mathbf{Z} \otimes \mathbf{Z} = \mathbf{Q}_{2,\,2m}(\mathbf{Z} \otimes \mathbf{I}_{2m})\, d\mathbf{Z} \\ d[d(\mathbf{Z}^{[2]})] &= (\nabla_Z^{T[2]} \otimes \mathbf{Z}^{[2]})(d\mathbf{Z})^{[2]} + [\nabla_Z^T \otimes \mathbf{Z}^{[2]}]\, d^2\mathbf{Z} = (d\mathbf{Z})^{[2]} \end{aligned} \tag{80}$$

where $\mathbf{Q}_{r,k}$ $(k = 2m)$ is the square matrix of order $(k)^r$ given in the form

$$\mathbf{Q}_{r,k} = \sum_{j=0}^{r-1} \mathbf{E}_{k^j,\,k^{r-j}} \tag{81}$$

$\mathbf{E}_{p,q}$ being permutant boolean matrices of order pq x pq consisting of q x p arrays of p x q dimensional elementary submatrices $\mathbf{E}^{is}$ [36] which have one in the (i, s)th position and zero in all other positions.

By using the two relationships (80), taking into account of Eq. (75) and neglecting infinitesimals of order greater than dt we can write Eq. (78) in the form

$$\Delta\left(\mathbf{Z}^{[2]}\right) = \mathbf{Q}_{r,\,2m}\left(\mathbf{Z}\otimes\mathbf{I}_{2m}\right)(\mathbf{A}\mathbf{Z}\,dt + \mathbf{V}\,d\mathbf{B}) + (\mathbf{A}\mathbf{Z}\,dt + \mathbf{V}\,d\mathbf{B})^{[2]}$$

$$= \mathbf{Q}_{r,\,2m}\left(\mathbf{Z}\otimes\mathbf{I}_{2m}\right)(\mathbf{A}\mathbf{Z}\,dt + \mathbf{V}\,d\mathbf{B}) + \mathbf{V}^{[2]}\,d\mathbf{B}^{[2]} \tag{82}$$

Since the following relationship holds

$$\mathbf{Q}_{r,\,2m}\left(\mathbf{I}_{2m}\otimes\mathbf{A}\right)\mathbf{Z}^{[2]} = \left(\mathbf{I}_{2m}\otimes\mathbf{A} + \mathbf{A}\otimes\mathbf{I}_{2m}\right)\mathbf{Z}^{[2]} = \mathbf{A}_2\,\mathbf{Z}^{[2]} \tag{83}$$

making the stochastic average of both sides of Eq. (82) and dividing by dt we obtain the differential equation governing the evolution of the second order moment which, for zero mean processes, fully characterized the response

$$\dot{\mathbf{m}}_2[\mathbf{Z}] = \mathbf{A}_2\,\mathbf{m}_2[\mathbf{Z}] + \mathbf{V}^{[2]}\,\mathbf{q}_2(t) \tag{84}$$

In obtaining this equation from Eq. (82) use has been made of the properties of process $d\mathbf{B}(t)$ given in Eq. (73) and of the fact that, according to Ito's stochastic caluclus, the following relationship holds

$$E\left[\mathbf{Z}^{[k]}\otimes(d\mathbf{B})^{[r]}\right] = E\left[\mathbf{Z}^{[k]}\right]\otimes E\left[(d\mathbf{B})^{[r]}\right] \tag{85}$$

It is worth noting that Eq. (84) is the vectorialized form of equation (69).

3.3 Filtered zero mean Gaussian input

3.3.1 Second order moments differential equations

The stochastic analysis of linear systems excited by stationary and non-stationary non-white Gaussian processes is usually performed by means of two different approaches. In the first one the input process is characterized by its autocorrelation function; in the second one the input process is the solution of a set of linear first order differential equations subjected to a white noise input processes. The second method is simpler from a numerical point of view than the first one. However, it requires the knowledge of the filter differential equations, which are not always known. By contrast, the first method requires only the knowledge of the autocorrelation function or alternatively of the power spectral density function of the input.
In this section in order to give a general overview of the approaches able to evaluate the response for filtered zero mean Gaussian input both methods are presented.
In the following these two approaches are indicated as Method 1 and Method 2 respectively.

a) Method 1

If $\mathbf{F}(t)$ is a non-white Gaussian vector process Eq. (52) has its own full mathematical sense and the differential of this equation can be written as follows

$$d\mathbf{Z} = \mathbf{A}\mathbf{Z}\,dt + \mathbf{V}\mathbf{F}(t)\,dt \tag{86}$$

and it is a summation of infinitesimal of the same order. It follows that

$$\Delta\left(\mathbf{Z}^{[2]}\right) = d\left(\mathbf{Z}^{[2]}\right) = \mathbf{Q}_{r,\,2m}\left(\mathbf{Z} \otimes \mathbf{I}_{2m}\right) d\mathbf{Z} = \mathbf{A}_2\,\mathbf{Z}^{[2]}\,dt + \mathbf{V}_2\left[\mathbf{Z} \otimes \mathbf{F}\right] dt \tag{87}$$

and consequently

$$\dot{\mathbf{m}}_2[\mathbf{Z}] = \mathbf{A}_2\,\mathbf{m}_2[\mathbf{Z}] + \mathbf{V}_2\,E\left[\mathbf{Z} \otimes \mathbf{F}\right] \tag{88}$$

where

$$\mathbf{V}_2 = \mathbf{Q}_{r,\,2m}\left(\mathbf{I}_{2m} \otimes \mathbf{V}\right) \tag{89}$$

In Eq. (87) $E\left[\mathbf{Z} \otimes \mathbf{F}\right]$ is a vector of order (2mn), listing all the cross-covariances between forcing function and response. The evaluation of these quantities will be discussed in the next section.

b) Method 2

The evaluation of the second order moments with Eq. (88) requires the evaluation of the cross-correlations between the input and the response. This computational problem can be avoided if the filtered input can be obtained as the solution of the following system of first order differential equation

$$\begin{aligned} \mathbf{F}(t) &= \mathbf{N}^T\,\mathbf{Z}_f(t) \\ \dot{\mathbf{Z}}_f &= \mathbf{A}_f\,\mathbf{Z}_f + \mathbf{V}_f\,\mathbf{W}(t) \end{aligned} \tag{90}$$

where $\mathbf{Z}_f$ and $\mathbf{V}_f$ are vectors of order m_f, $\mathbf{N}$ is a matrix of order m_f x n, $\mathbf{A}_f$ is a matrix of order m_f x m_f and $\mathbf{W}(t)$ is a zero mean Gaussian white noise vector process.
By associating Eqs. (90) to Eq. (52) we can write

$$\dot{\overline{\mathbf{Z}}} = \overline{\mathbf{A}}\,\overline{\mathbf{Z}} + \overline{\mathbf{V}}\,\mathbf{W}(t) \tag{91}$$

where

$$\overline{\mathbf{Z}} = \begin{bmatrix} \mathbf{Z} \\ \mathbf{Z}_f \end{bmatrix}; \quad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{V}\mathbf{N}^T \\ \mathbf{0} & \mathbf{A}_f \end{bmatrix}; \quad \mathbf{V} = \begin{bmatrix} \mathbf{0} \\ \mathbf{V}_f \end{bmatrix} \tag{92}$$

Eq. (91) is a first order differential equation with a modulated white noise input. As discussed in the previous section and by taking into account the main properties of the white noise, we can write the differential equation governing the evolution of the second order moments as follows

$$\dot{\mathbf{m}}_2[\overline{\mathbf{Z}}] = \overline{\mathbf{A}}_2 \mathbf{m}_2[\overline{\mathbf{Z}}] + \overline{\mathbf{V}}^{[2]} \mathbf{q}_2(t) \tag{93}$$

Eq. (93) is a set of p^2 ($p = 2m + m_f$) first order differential equations, in which

$$\mathbf{m}_2[\overline{\mathbf{Z}}] = E[\overline{\mathbf{Z}} \otimes \overline{\mathbf{Z}}]; \quad \overline{\mathbf{A}}_2 = \left[\mathbf{I}_p \otimes \overline{\mathbf{A}} + \overline{\mathbf{A}} \otimes \mathbf{I}_p\right] \tag{94}$$

By using the properties of Kronecker product we can write Eq. (93) in explicit form as follows [37]

$$\dot{\mathbf{m}}_2[\mathbf{Z}] = \mathbf{A}_2 \mathbf{m}_2[\mathbf{Z}] + \mathbf{V}_{2,f} E[\mathbf{Z} \otimes \mathbf{Z}_f] \tag{95a}$$

$$\dot{E}[\mathbf{Z} \otimes \mathbf{Z}_f] = \left(\mathbf{A} \otimes \mathbf{I}_{m_f} + \mathbf{I}_{2m} \otimes \mathbf{A}_f\right) E[\mathbf{Z} \otimes \mathbf{Z}_f] + \left[\mathbf{V}\mathbf{N}^T \otimes \mathbf{I}_{m_f}\right] \mathbf{m}_2[\mathbf{Z}_f] \tag{95b}$$

$$\dot{\mathbf{m}}_2[\mathbf{Z}_f] = \left(\mathbf{A}_f \otimes \mathbf{I}_{m_f} + \mathbf{I}_{m_f} \otimes \mathbf{A}_f\right) \mathbf{m}_2[\mathbf{Z}_f] + \mathbf{V}_f^{[2]} \mathbf{q}_2(t) \tag{95c}$$

where

$$\mathbf{V}_{2,f} = \mathbf{Q}_{r,2m}\left(\mathbf{I}_{2m} \otimes \mathbf{V}\mathbf{N}^T\right) = \mathbf{V}_2\left(\mathbf{I}_{2m} \otimes \mathbf{N}^T\right) \tag{96}$$

Eqs. (95) show that for filtered white noise input the response can be evaluated by solving a set of differential equations having for input the strenght of white noise, the second moments of the filter and the input-output cross-moments respectively. Although Eq. (93) is formally more compact than Eqs. (95), the solution of Eqs. (95) is more convenient from a numerical point of view. Indeed by using Eqs. (93) we have to solve a set of differential equations of order $(2m+m_f)^2$ while by using Eqs. (95) we have to solve three sets of differential equations of order $(2m)^2$, $(2m\ m_f)$ and $(m_f)^2$ respectively. Generally we have $(2m + m_f)^2 >> (2m)^2 + (2m\ m_f) + (m_f)^2$.
It has to be emphasized that Eqs. (95) can also be obtained by considering that: Eq. (95a) gives the second order moments for a non-white Gaussian input and consequently it is similar to Eq. (88); Eq. (95c) gives the second order moments for a white noise input and consequently formally coincides with Eq. (84). Eq. (95b) can be, then, obtained as follows

$$\dot{E}[\mathbf{Z} \otimes \mathbf{Z}_f] = E\left[\frac{d}{dt}(\mathbf{Z} \otimes \mathbf{Z}_f)\right] = E[d(\mathbf{Z} \otimes \mathbf{Z}_f)/dt] \tag{97}$$

where

$$d(\mathbf{Z} \otimes \mathbf{Z}_f) = d\mathbf{Z} \otimes \mathbf{Z}_f + \mathbf{Z} \otimes d\mathbf{Z}_f \tag{98}$$

where $d\mathbf{Z}$ is given from Eq. (86) in which we assume $\mathbf{F} = \mathbf{N}^T \mathbf{Z}_f$ and $d\mathbf{Z}_f$ is a summation of infinitesimal of different order given in a similar form to Eq. (75). It follows that

$$\begin{aligned} d(\mathbf{Z} \otimes \mathbf{Z}_f) &= (\mathbf{A}\mathbf{Z} + \mathbf{V}\mathbf{N}^T\mathbf{Z}_f)\,dt \otimes \mathbf{Z}_f + \mathbf{Z} \otimes (\mathbf{A}_f \mathbf{Z}_f\,dt + \mathbf{V}_f\,d\mathbf{B}) = \\ &= (\mathbf{A} \otimes \mathbf{I}_{m_f} + \mathbf{I}_{2m} \otimes \mathbf{A}_f)(\mathbf{Z} \otimes \mathbf{Z}_f)\,dt + \\ &+ (\mathbf{V}\mathbf{N}^T \otimes \mathbf{I}_{m_f})\mathbf{Z}_f^{[2]}\,dt + (\mathbf{I}_{2m} \otimes \mathbf{V}_f)(\mathbf{Z} \otimes d\mathbf{B}) \end{aligned} \tag{99}$$

Taking the stochastic average of both sides of Eq. (99), using the mode properties of process $d\mathbf{B}(t)$ and dividing by dt we obtain Eq. (95b).

3.3.2 Solution of second order moment differential equations

a) Method 1

As shown in the previous section, using Kronecker algebra the differential Eqs. (88) governing the evolution of the second order moments is a set of first order differential equations with deterministic forcing function; the cross-correlation vector between input and output. It follows that is formally identical to the differential equation of the deterministic case (given by Eq. (52)). Hence, in view of Eq. (55) the solution of Eq. (88) is

$$\mathbf{m}_2[\mathbf{Z}(t)] = \boldsymbol{\Theta}_2(t)\,\mathbf{m}_2[\mathbf{Z}(0)] + \int_0^t \boldsymbol{\Theta}_2(t-\tau)\,\mathbf{V}_2\,\mathbf{F}_2(\tau)\,d\tau \tag{100}$$

where $\mathbf{F}_2(t) = E[\mathbf{Z}(t) \otimes \mathbf{F}(t)]$ and $\boldsymbol{\Theta}_2(t)$ is the fundamental matrix of Eq. (88), which is related, in view of the particular form of the matrix $\mathbf{A}_2$, to the matrix $\boldsymbol{\Theta}(t)$ by means of the following relationship

$$\boldsymbol{\Theta}_2(t) = \boldsymbol{\Theta}^{[2]}(t) \tag{101}$$

Remembering that $\mathbf{Z}(t)$ is given by Eq. (55) and that $\mathbf{F}(t)$ is assumed to be a zero mean Gaussian process we can write

$$\mathbf{F}_2(\tau) = \mathrm{E}\left[\mathbf{Z}(\tau) \otimes \mathbf{F}(\tau)\right] = \mathrm{E}\left[\left(\boldsymbol{\Theta}(\tau)\,\mathbf{Z}(0) + \int_0^\tau \boldsymbol{\Theta}(\tau - \rho)\,\mathbf{V}\mathbf{F}(\rho)\,d\rho\right) \otimes \mathbf{F}(\tau)\right] =$$

$$= \int_0^\tau \boldsymbol{\Theta}(\tau - \rho)\left(\mathbf{V} \otimes \mathbf{I}_n\right)\mathbf{R}_2\left[\mathbf{F}(\rho), \mathbf{F}(\tau)\right] d\rho \tag{102}$$

where, for zero mean input processes,

$$\mathbf{R}_2\left[\mathbf{F}(\rho), \mathbf{F}(\tau)\right] = \mathrm{Vec}\left\{\mathbf{C}_2\left[\mathbf{F}(\rho), \mathbf{F}(\tau)\right]\right\} = \mathrm{E}\left[\mathbf{F}(\rho) \otimes \mathbf{F}(\tau)\right] \tag{103}$$

Notice that for stationary case we have

$$\mathbf{R}_2\left[\mathbf{F}(\rho), \mathbf{F}(\tau)\right] = \mathbf{R}_2(\tau - \rho) \tag{104}$$

it follows that Eq. (102) becomes

$$\mathbf{F}_2 = \int_{-\infty}^{\tau} \boldsymbol{\Theta}(\tau - \rho)\left(\mathbf{V} \otimes \mathbf{I}_n\right)\mathbf{R}_2(\tau - \rho)\,d\rho = \int_0^\infty \boldsymbol{\Theta}(\rho)\left(\mathbf{V} \otimes \mathbf{I}_n\right)\mathbf{R}_2(\rho)\,d\rho \tag{105}$$

That is the vector $\mathbf{F}_2$ can be evaluated by using a simple integral. Numerically $\mathbf{F}_2$ can be evaluated by using the trapezoidal rule

$$\mathbf{F}_2 = \frac{\Delta\rho}{2}\sum_{j=1}^{\infty} \boldsymbol{\Theta}^{j-1}(\Delta\rho)\left\{\left(\mathbf{V} \otimes \mathbf{I}_n\right)\mathbf{R}_2\left((j-1)\,\Delta\rho\right) + \boldsymbol{\Theta}(\Delta\rho)\left(\mathbf{V} \otimes \mathbf{I}_n\right)\mathbf{R}_2(j\,\Delta t)\right\} \tag{106}$$

Once the vector $\mathbf{F}_2$ is evaluated, Eq. (88) the stationary solution is simply obtained by setting $\dot{\mathbf{m}}_2[\mathbf{Z}] = \mathbf{0}$ in Eq. (88) and we have

$$\mathbf{m}_2[\mathbf{Z}] = -\mathbf{A}_2^{-1}\,\mathbf{V}_2\,\mathbf{F}_2 \tag{107}$$

For the non-stationary case it is not convenient to use the described procedure for the evaluation of the vector $\mathbf{F}_2(\tau)$. A more suitable approach consists in approximating the elements of the input correlation vector $\mathbf{R}_2[\cdot]$ by an approximate function expressed in terms of two-dimensional orthogonal Chebyshev polynomials [38]. Thus

$$\mathbf{R}_2[\mathbf{F}(\rho), \mathbf{F}(\tau)] \cong \tilde{\mathbf{R}}_2[\mathbf{F}(\rho), \mathbf{F}(\tau)] = \sum_{i=0}^{N_0} \sum_{j=0}^{N_1} \mathbf{c}_{ij}\, T_i(\rho')\, T_j(\tau') \tag{108}$$

where the normalized values τ' and ρ' of τ and ρ are respectively defined by

$$\tau' = \frac{2\tau}{\tau_{max}} - 1\,; \quad \rho' = \frac{2\rho}{\rho_{max}} - 1 \tag{109}$$

and $T_k(\cdot)$ $(k = i, j)$ are the Chebyshev polynomials which are given by following recurrence relationship

$$T_{n+1}(x) - 2x\, T_n(x) + T_{n-1}(x) = 0 \tag{110}$$

with $T_0(x) = 1$, $T_1(x) = x$.
In Eq. (109) $\tau_{max} = \rho_{max}$ denotes the time in which the input correlation function attains significant values.
The least-squares Chebyshev polynomial approximation $\tilde{\mathbf{R}}_2[\cdot\,]$ of $\mathbf{R}_2[\cdot]$ results in the coefficients vector $\mathbf{c}_{ij}$ of Eq. (108) which are given by

$$\mathbf{c}_{ij} = \alpha_{ij} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{\pi} \mathbf{R}_2\left(\cos^{-1}\xi_1, \cos^{-1}\xi_2\right) T_i(\theta)\, T_j(\varphi)\, d\varphi\, d\theta \tag{111}$$

with

$$\begin{aligned} &\xi_1 = \cos\theta\,, \quad \xi_2 = \cos\varphi \\ &\alpha_{ij} = (2/\pi)^2 \quad \text{for i and } j \neq 0\,; \quad \alpha_{ij} = (2/\pi^2) \quad \text{for i or } j = 0 \\ &\alpha_{ij} = (1/\pi^2) \quad \text{for } i = j = 0 \end{aligned} \tag{112}$$

Substituting Eq. (108) into Eq. (102) we can write

$$\mathbf{F}_2(\tau) = \sum_{i=0}^{N_0} \sum_{j=0}^{N_1} \mathbf{J}_i(\tau)\left(\mathbf{V} \otimes \mathbf{I}_n\right) \mathbf{C}_{ij}\, T_j(\tau) \tag{113}$$

where the matrix $\mathbf{J}_i(\tau)$ can be evaluated in closed form solution by solving the following matricial integral

$$J_i(\tau) = \int_{-1}^{\left(\frac{2\rho}{\rho_{max}} - 1\right)} \Theta\left(\tau - \frac{\rho_{max}}{2}(\xi + 1)\right) T_i(\xi)\, d\xi \tag{114}$$

Substituting Eq. (113) into Eq. (100) the second order moments of the response can be evaluated as simple convolution integrals.

b) Method 2

This method requires the solution of Eq. (93). Since it is a first order differential equation, its solution can be written as follows

$$\mathbf{m}_2[\overline{\mathbf{z}}(t)] = \overline{\boldsymbol{\Theta}}_2(t)\, \mathbf{m}_2[\overline{\mathbf{z}}(0)] + \int_0^t \overline{\boldsymbol{\Theta}}_2(t - \tau)\, \mathbf{V}^{[2]}\, \mathbf{q}_2(\tau)\, d\tau \tag{115}$$

where

$$\overline{\boldsymbol{\Theta}}_2(t) = \overline{\boldsymbol{\Theta}}^{[2]}(t) \tag{116}$$

$\overline{\boldsymbol{\Theta}}(t)$ being the fundamental matrix of differential Eq. (91) which can be evaluate by means of Eq. (56) once the eigenproperties $\overline{\boldsymbol{\Lambda}}$ and $\overline{\boldsymbol{\Psi}}$ of matrix $\overline{\mathbf{A}}$ are known. These eigenproperties can be evaluated in closed form as follows [37]

$$\overline{\boldsymbol{\Lambda}} = \begin{bmatrix} \boldsymbol{\Lambda} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_f \end{bmatrix}; \quad \overline{\boldsymbol{\Psi}} = \begin{bmatrix} \boldsymbol{\Psi} & \mathbf{N}_{sf}\boldsymbol{\Psi}_f \\ \mathbf{0} & \boldsymbol{\Psi}_f \end{bmatrix} \tag{117}$$

in which $\boldsymbol{\Lambda}_f$ and $\boldsymbol{\Psi}_f$ are the eigenproperties of matrix $\mathbf{A}_f$, and

$$\mathrm{Vec}\left(\mathbf{N}_{sf}^T\right) = -\left(\mathbf{A} \otimes \mathbf{I}_{m_f} - \mathbf{I}_{2m} \otimes \mathbf{A}_f^T\right)^{-1} \mathrm{Vec}\left(\mathbf{V}\,\mathbf{N}^T\right) \tag{118}$$

Hence the presence of the filter equations modifies only the eigenvectors while the eigenvalues of the system and filter do not change. This is due to the fact that the filter and the combined system can be considered as a cascaded composite system.

For the numerical solution of Eq. (93) it is possible to proceed in two ways (see e.g. [37]). The first is to solve numerically the three differential Eqs. (95), which are the explicit form of Eqs. (91); the second is connected to the very similar form of the mathematical structure governing the evaluation of the response of the state variables $\overline{\mathbf{Z}}(t)$ and the second order moments $\mathbf{m}_2[\overline{\mathbf{Z}}(t)]$. It follows that it is possible to perform the numerical solution of Eq. (91) as follows

$$\mathbf{m}_2[\bar{\mathbf{z}}(t_{k+1})] = \bar{\boldsymbol{\Theta}}_2(\Delta t)\,\mathbf{m}_2[\bar{\mathbf{z}}(t_k)] + \bar{\mathbf{L}}_2(\Delta t)\,\mathbf{V}^{[2]}\,\mathbf{q}_2(t_k) \tag{119}$$

where $\bar{\boldsymbol{\Theta}}_2(\Delta t)$ is the fundamental matrix given in Eq. (116) whose matrix $\bar{\boldsymbol{\Theta}}(\Delta t)$ is given as follows

$$\bar{\boldsymbol{\Theta}}(\Delta t) = \begin{bmatrix} \boldsymbol{\Theta}(t) & \boldsymbol{\Theta}_{sf}(\Delta t) \\ \mathbf{0} & \boldsymbol{\Theta}_f(\Delta t) \end{bmatrix} \tag{120}$$

where $\boldsymbol{\Theta}_f(\Delta t)$ is the fundamental matrix of differential Eq. (95c) and

$$\boldsymbol{\Theta}_{sf}(\Delta t) = \mathbf{N}_{sf}\,\boldsymbol{\Theta}_f(\Delta t) - \boldsymbol{\Theta}(\Delta t)\,\mathbf{N}_{sf} = \int_0^{\Delta t} \boldsymbol{\Theta}(\Delta t - \tau)\,\mathbf{V}\,\mathbf{N}^T\,\boldsymbol{\Theta}_f(\tau)\,d\tau \cong \mathbf{L}(\Delta t)\,\mathbf{V}\,\mathbf{N}^T \tag{121}$$

The approximation of $\boldsymbol{\Theta}_{sf}(\Delta t)$ is evaluated assuming that $\boldsymbol{\Theta}_f(\tau)$ is a constant matrix in the step Δt. In Eq. (119) $\bar{\mathbf{L}}_2(\Delta t)$ is a matrix of order $[(2m + m_f \times 2m + m_f)]^2$, given as

$$\bar{\mathbf{L}}_2(\Delta t) = \left[\bar{\boldsymbol{\Theta}}_2(\Delta t) - \mathbf{I}_r^{[2]}\right]\bar{\mathbf{A}}_2^{-1} \tag{122}$$

The numerical evaluation of the operators which appear in Eq. (119) can be alternatively performed by the approximate Taylor's expansion too, that is

$$\begin{aligned} \hat{\bar{\boldsymbol{\Theta}}}_2(\Delta t) &= \sum_{j=0}^{\infty} \frac{1}{j!}\left(\Delta t\,\bar{\mathbf{A}}_2\right)^j \cong \sum_{j=0}^{N_2} \frac{1}{j!}\left(\Delta t\,\bar{\mathbf{A}}_2\right)^j \\ \hat{\bar{\mathbf{L}}}_2(\Delta t) &= \Delta t\left[\sum_{j=0}^{\infty} \frac{1}{(j+1)!}\left(\Delta t\,\bar{\mathbf{A}}_2\right)^j\right] \cong \Delta t\left[\sum_{j=0}^{N_2-1} \frac{1}{(j+1)!}\left(\Delta t\,\bar{\mathbf{A}}_2\right)^j\right] \end{aligned} \tag{123}$$

where N_2 is an integer number that must be chosen according to stability and accuracy tests [27]. From Eq. (118) it is easy to see that the eigenvalues of $\bar{\mathbf{A}}$ are the eigenvalues of the system and of the filter separately taken. It follows that in order to evaluate the critical time step, it is necessary to know the eigenvalues of the composite system, which we expect to deviate slightly from those of the two subsystems separately taken, and of the filter.

3.4 Delta-correlated input

In many problems of structural dynamics the load consists of a series of impulses occurring at random time and with random strength. Examples are loads due to vehicular traffic flow on

highway bridges [39, 40], or excitation due to earthquake [41-44]. These stochastic input processes can be described by delta-correlated processes. As showed in the Chapter two the most common delta-correlated process is the Poisson white noise process.
Here the evaluation of the response moments of the following differential equation is treated

$$\dot{\mathbf{Z}} = \mathbf{A}\,\mathbf{Z} + \mathbf{V}\,\mathbf{W}(t) \tag{124}$$

where $\mathbf{W}(t)$ is a vector of delta-correlated processes here assumed up to s-th order, that is

$$\mathbf{C}_r\left[\mathbf{W}(t_1), \mathbf{W}(t_2), \ldots, \mathbf{W}(t_r)\right] = \mathbf{q}_r(t_1)\,\delta(t_1 - t_2)\,\delta(t_1 - t_3)\ldots\delta(t_1 - t_r)\ ,\ r \le s \tag{125}$$

In this equation $\mathbf{q}_r(t)$ is an $(2n)^r$ order vector containing the strenght of order r of the process $\mathbf{W}(t)$ and $\mathbf{C}_r[\cdot]$ is the vector of the correlations of order r of $\mathbf{W}(t)$ which for r = 1, 2, 3, ... can be evaluated as follows

$$\mathbf{C}_1\left[\mathbf{W}(t_1)\right] = \mathrm{E}\left[\mathbf{W}(t_1)\right] = \mathbf{q}_1(t_1) \tag{126a}$$

$$\mathbf{C}_2\left[\mathbf{W}(t_1), \mathbf{W}(t_2)\right] = \mathrm{E}\left[\mathbf{W}(t_1) \otimes \mathbf{W}(t_2)\right] - \mathrm{E}\left[\mathbf{W}(t_1)\right] \otimes \mathrm{E}\left[\mathbf{W}(t_2)\right] = \mathbf{q}_2(t_1)\,\delta(t_1 - t_2) \tag{126b}$$

$$\begin{aligned}\mathbf{C}_3\left[\mathbf{W}(t_1), \mathbf{W}(t_2), \mathbf{W}(t_3)\right] &= \mathrm{E}\left[\mathbf{W}(t_1) \otimes \mathbf{W}(t_2) \otimes \mathbf{W}(t_3)\right] - \mathbf{q}_1(t_1)\,\mathrm{E}\left[\mathbf{W}(t_2) \otimes \mathbf{W}(t_3)\right] \\ &- \mathrm{E}\left[\mathbf{W}(t_1) \otimes \mathbf{q}_1(t_2) \otimes \mathbf{W}(t_3)\right] - \mathrm{E}\left[\mathbf{W}(t_1) \otimes \mathbf{W}(t_2)\right] \otimes \mathbf{q}_1(t_3) - \mathbf{q}_1(t_1) \otimes \mathbf{q}_1(t_2) \otimes \mathbf{q}_1(t_3) = \\ &\mathbf{q}_3(t_1)\,\delta(t_1 - t_2)\,\delta(t_1 - t_3) \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\end{aligned} \tag{126c}$$

If $\mathbf{W}(t)$ is a zero mean delta-correlated process up to s-th order, Eq. (52) does not have mathematical meaning and it has to be substituted by the following equation:

$$d\mathbf{Z} = \mathbf{A}\,\mathbf{Z}dt + \mathbf{V}\,d\mathbf{L} \tag{127}$$

where $d\mathbf{L}(t)$ is the so-called Levy white noise vector process. The process, $d\mathbf{L}(t)$ is not mean square differentiable everywhere; however, we can write formally

$$d\mathbf{L}(t) = \mathbf{W}(t)\,dt \tag{128}$$

Extending the relationships of the one-dimensional case, it can be easily shown that the Levy vector process $d\mathbf{L}(t)$ has the following cumulants

$$\mathbf{k}_1[\mathrm{dL}(t)] = 0\,; \quad \mathbf{k}_r[\mathrm{dL}(t)] = \mathbf{q}_r(t)\,\mathrm{dt} \quad (r = 2, 3, \ldots, s)\,;$$
$$\mathbf{k}_{s+p}[\mathrm{dL}(t)] = 0 \quad (p = 1, 2, \ldots) \tag{129}$$

By taking into account of the relationships between moments and cumulants, it can be easily seen that, by neglecting infinitesimal of higher order than dt, the moments of dL coincide with the cumulants, that is

$$\mathbf{m}_r[\mathrm{dL}(t)] = \mathrm{E}[(\mathrm{dL}(t))^r] = \mathbf{k}_r[\mathrm{dL}(t)] = \mathbf{q}_r(t)\,\mathrm{dt}\,, \quad (r = 2, 3, \ldots) \tag{130}$$

Moreover, neglecting infinitesimals of higher order than dt, we can write

$$(\mathrm{dZ})^{[r]} = \mathbf{V}^{[r]}(\mathrm{dL})^{[r]}\,; \quad (r = 2, 3, \ldots, s)\,; \quad (\mathrm{dZ})^{[s+p]} = \mathbf{0}, \quad (p = 1, 2, \ldots)\,; \tag{131}$$

It follows that in the series expansion of the increment of $\Delta\left(\mathbf{Z}^{[r]}\right)$ we have to take into account of terms up to the s-th order, that is

$$\Delta\left(\mathbf{Z}^{[r]}\right) = \mathrm{d}\left(\mathbf{Z}^{[r]}\right) + \frac{1}{2!}\mathrm{d}^2\left(\mathbf{Z}^{[r]}\right) + \frac{1}{3!}\mathrm{d}^3\left(\mathbf{Z}^{[r]}\right) + \ldots + \frac{1}{s!}\mathrm{d}^s\left(\mathbf{Z}^{[r]}\right) \tag{132}$$

Since, by neglecting infinitesimal of order greater of dt, the following relationships hold

$$\mathrm{d}\mathbf{Z}^{[r]} = \left(\nabla_z^{\mathrm{T}} \otimes \mathbf{Z}^{[r]}\right)\mathrm{dZ} = \mathbf{Q}_{r,\,2m}\left(\mathbf{Z}^{[r-1]} \otimes \mathbf{I}_{2m}\right)\mathrm{dZ}$$
$$\mathrm{d}^k\mathbf{Z}^{[r]} = \mathbf{Q}_{r,\,2m}\left(\mathbf{Q}_{r-1,\,2m} \otimes \mathbf{I}_{2m}\right)\cdots\left(\mathbf{Q}_{r-k+1,\,2m} \otimes \mathbf{I}_{2m}^{[k-1]}\right)\left(\mathbf{Z}^{[r-k]} \otimes \mathbf{I}_{2m}^{[k]}\right)\mathbf{V}^{[k]}\left(\mathrm{dL}^{[k]}\right);$$
$$(1 < k \le r,\ k \le s)$$
$$\mathrm{d}^s\left(\mathbf{Z}^{[r]}\right) = 0\,; \quad (s > r) \tag{133}$$
$$\mathrm{d}^{s+p}\left(\mathbf{Z}^{[r]}\right) = 0\,; \quad (p = 1, 2, \ldots\,;\ (s + p) \le r)$$

Taking into account of Eq. (127) we can write Eq. (132) in the form

$$\Delta\left(\mathbf{Z}^{[r]}\right) = \mathbf{A}_r\,\mathbf{Z}^{[r]}\,\mathrm{dt} + \mathbf{V}_r\left(\mathbf{Z}^{[r-1]} \otimes \mathrm{dL}\right) + \sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk}\left(\mathbf{Z}^{[r-k]} \otimes \mathbf{I}_{2m}^{[k]}\right)(\mathrm{dL})^{[k]} \tag{134}$$

where

$$\mathbf{A}_r = \mathbf{A} \otimes \mathbf{I}_{2m}^{[r-1]} + \mathbf{I}_{2m} \otimes \mathbf{A}_{r-1} \; ; \quad \mathbf{V}_r = \mathbf{Q}_{r,\,2m} \left(\mathbf{I}_{2m}^{[r-1]} \otimes \mathbf{V} \right) \tag{135a}$$

$$\mathbf{V}_{rk} = \frac{1}{k!} \mathbf{Q}_{r,\,2m} \left(\mathbf{Q}_{r-1,\,2m} \otimes \mathbf{I}_{2m} \right) \cdots \left(\mathbf{Q}_{r-k+1,\,2m} \otimes \mathbf{I}_{2m}^{[k-1]} \right) \left(\mathbf{I}_{2m}^{[r-k]} \otimes \mathbf{V}^{[k]} \right) \tag{135b}$$

Making the stochastic average of both sides of Eq. (134), taking into account the properties of vector $\mathbf{dL}$ and that

$$E\left[\mathbf{Z}^k(t) \otimes \mathbf{dL}^{[r]} \right] = E\left[\mathbf{Z}^k \right] \otimes E\left[\mathbf{dL}^{[r]} \right] = \mathbf{m}_k\,[\mathbf{Z}] \otimes \mathbf{m}_r\,[\mathbf{dL}] \tag{136}$$

and dividing by dt we obtain the differential equations governing the evolution of the moments of every order in the form:

$$\dot{\mathbf{m}}_r[\mathbf{Z}] = \mathbf{A}_r\, \mathbf{m}_r[\mathbf{Z}] + \sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk} \left(\mathbf{m}_{r-k}\,[\mathbf{Z}] \otimes \mathbf{I}_{2m}^{[k]} \right) \mathbf{q}_k(t) \tag{137}$$

By using the multidimensional relationship between moments and cumulants given in the Appendix we can write the differential Eq. (137), in terms of cumulants as follows

$$\begin{aligned} \dot{\mathbf{k}}_1\,[\mathbf{Z}] &= \mathbf{A}_1\, \mathbf{k}_1\,[\mathbf{Z}] \\ \dot{\mathbf{k}}_r\,[\mathbf{Z}] &= \mathbf{A}_r\, \mathbf{k}_r\,[\mathbf{Z}] + \mathbf{V}^{[r]}\, \mathbf{q}_r(t) \; ; \quad 2 \le r \le s \\ \dot{\mathbf{k}}_r\,[\mathbf{Z}] &= \mathbf{A}_r\, \mathbf{k}_r\,[\mathbf{Z}] \; ; \quad r > s \end{aligned} \tag{138}$$

where $\mathbf{A}_r$ is given in equation (135). This equation shows that the r-th cumulant only depends on the r-th vector containing the strenght of order r of the process $\mathbf{W}(t)$. Furtherome, since we have that

$$\mathbf{k}_1\,[\mathbf{Z}(0)] = \mathbf{0} \; ; \quad \mathbf{k}_r\,[\mathbf{Z}(0)] = \mathbf{0} \, , \quad \forall r > s \tag{139}$$

only the first s-th cumulant are sufficient to characterize the response of a linear system subjected to a s-th delta-correlated input process. It follows that, since the r-th moment vector can be evaluated once the cumulant vectors until the r-th order are evaluated, in order to characterize the stochastic response of linear system subjected to s-th delta-correlated input, the first s-th differential equation (137) have to be solved. Indeed, the moment vectors of order greater than s can be evaluated by means of Eqs. (A.14) setting $\mathbf{k}_{s+p}\,[\mathbf{Z}] = \mathbf{0}$ with (p = 1, 2, ...).

In order to perform the numerical solution of Eqs. (137) and (138) the eigenproperties of $\mathbf{A}_r$ has to be evaluated. These quantities can be easily computed from the eigenvalues and eigenvectors of $\mathbf{A}$ as follows

$$\mathbf{A}_r \boldsymbol{\Psi}^{[r]} = \boldsymbol{\Psi}^{[r]} \boldsymbol{\Lambda}_r ; \quad \boldsymbol{\Lambda}_r = \boldsymbol{\Lambda} \otimes \mathbf{I}_{2m}^{[r-1]} + \mathbf{I}_{2m} \otimes \boldsymbol{\Lambda}_{r-1} \tag{140}$$

where $\boldsymbol{\Psi}$ is the matrix of eigenvectors of $\mathbf{A}$ and $\boldsymbol{\Lambda}$ is the diagonal matrix listing the eigenvalues of $\mathbf{A}$. It follows that if m modes are taken into account in the deterministic analysis, m^r modes are taken into account in the stochastic analysis for the evaluation of the moments of order r. These modes are the combination of the first m modes of the deterministic analysis. The fundamental matrices $\boldsymbol{\Theta}_r(\Delta t)$ of $\mathbf{A}_r$ are related to the fundamental matrix of $\mathbf{A}$ by means of the following relationship

$$\boldsymbol{\Theta}_r(\Delta t) = \boldsymbol{\Theta}^{[r]}(\Delta t) \tag{141}$$

Eqs. (140) and (141) show that, by means of the knowledge of the eigenproperties of $\mathbf{A}$, the corresponding eigenproperties of the matrix $\mathbf{A}_r$ can easily be evaluated without much effort. The numerical solution of Eq. (137) follows the step-by-step scheme given in Eq. (119) in the form

$$\mathbf{m}_r[\mathbf{Z}(t_{i+1})] = \boldsymbol{\Theta}^{[r]}(\Delta t)\, \mathbf{m}_r[\mathbf{Z}(t_i)] + \mathbf{L}_r(\Delta t) \sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk} \left(\mathbf{m}_{r-k}[\mathbf{Z}(t_i)] \otimes \mathbf{I}_{2m}^{[r]} \right) \mathbf{q}_r(t_i) \tag{142}$$

where

$$\mathbf{L}_r(t) = \left[\boldsymbol{\Theta}^{[r]}(\Delta t) - \mathbf{I}_{2m}^{[r]} \right] \mathbf{A}_r^{-1} \tag{143}$$

The inversion of $\mathbf{A}_r$ can be avoided by using the following relationship

$$\mathbf{A}^{-1} = \boldsymbol{\Psi}\, \boldsymbol{\Lambda}^{-1}\, \boldsymbol{\Psi}^{-1} ; \quad \mathbf{A}_r^{-1} = \boldsymbol{\Psi}^{[2]}\, \boldsymbol{\Lambda}_r^{-1}\, \boldsymbol{\Psi}^{[r]} \tag{144}$$

Notice that for classically damped systems $\boldsymbol{\Lambda}$ and $\boldsymbol{\Psi}$ are simply given as

$$\boldsymbol{\Lambda} = \begin{pmatrix} -\frac{1}{2}\boldsymbol{\Xi} + i\,\overline{\boldsymbol{\Omega}} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{2}\boldsymbol{\Xi} - i\,\overline{\boldsymbol{\Omega}} \end{pmatrix}$$

$$\boldsymbol{\Psi} = \begin{pmatrix} \mathbf{I}_m & \mathbf{I}_m \\ -\frac{1}{2}\boldsymbol{\Xi} + i\,\overline{\boldsymbol{\Omega}} & -\frac{1}{2}\boldsymbol{\Xi} - i\,\overline{\boldsymbol{\Omega}} \end{pmatrix} \tag{145}$$

where i is the imaginary unit and $\overline{\boldsymbol{\Omega}}$ is a diagonal matrix listing the damped radial frequencies.

For a non-classically damped system a further complex eigenproblem should be solved; but it is more convenient to avoid the complex eigensolution and to evaluate the fundamental matrix and the load operator by means of the described truncated Taylor expansion as follows

$$\widehat{\boldsymbol{\Theta}}_r(\Delta t) = \sum_{k=0}^{N_r} \frac{1}{k!}(\Delta t\, \mathbf{A}_r)^k \tag{146}$$

$$\widehat{\mathbf{L}}_r(\Delta t) = \Delta t \sum_{k=0}^{N_r - 1} \frac{1}{(k+1)!}(\Delta t\, \mathbf{A}_r)^k \tag{147}$$

where N_r is an integer number that must be chosen according to stability and accuracy tests [26].
We want to emphasize again that for both classically and non-classically damped systems, using Eq. (142), the step-by-step procedure for evaluating moments of every order of the stochastic vector process $\mathbf{Z}$ is unconditionally stable, if the matrix $\boldsymbol{\Theta}_r(\Delta t)$ is evaluated in an exact form. Instead, using the approximate fundamental matrix $\widehat{\boldsymbol{\Theta}}_r(\Delta t)$ the step-by-step procedure becomes a conditionally stable one, and after simple algebra the stability criterion based on the spectral radii of $\widehat{\boldsymbol{\Theta}}_r(\Delta t)$ can be written as

$$\rho\left[\widehat{\boldsymbol{\Theta}}_r(\Delta t)\right] = \left| \sum_{k=0}^{N} \left(\lambda_{max,\,r}\, \Delta t\right)^k \right| \leq 1 \tag{148}$$

where $\lambda_{max,\,r}$ is the maximum eigenvalue of $\mathbf{A}_r$ and, because of Eq. (140), is equal to r times the maximum eigenvalue of $\mathbf{A}$. It follows that, in order to ensure stability, the time step has to be selected as

$$\Delta t = \frac{2\pi}{4r\omega_m} \quad \text{for} \quad N_r = 7 \tag{149}$$

Hence, in the stochastic analysis of non-classically damped systems, if the fundamental and loading operators are evaluated in approximate form, the time step must be selected equal to the r-th part of the time step for the deterministic case. This is not a real limitation; indeed, the forcing vector in the stochastic analysis is $\mathbf{q}_r(t)$, that is a very smooth function varies much slowly than $\mathbf{F}(t)$, so that its frequency content is at very low frequency; on the other hand the natural frequencies of $\mathbf{A}_r$ are of order r-times the order of the natural frequencies of $\mathbf{A}$. It follows that, with the same order of accuracy, the number of modes selected for the stochastic analysis has to be less than the number of modes for the deterministic analysis.
Since the white noise processes can be considered as a delta-correlated process up to second order (or equivalently a normal delta-correlated process) we have that Eq. (137) is the natural extension of Eq. (84) to the case of non normal delta-correlated input processes.

Once the cumulants or the moments of order r have been evaluated, the correlations of $\mathbf{Z}$ can be obtained in the form [45]

$$\mathbf{C}_r[\mathbf{Z}(t_1), \mathbf{Z}(t_2), \ldots, \mathbf{Z}(t_r)] = [\mathbf{I}_{2m} \otimes \boldsymbol{\Theta}(t_2 - t_1) \otimes \ldots \otimes \boldsymbol{\Theta}(t_r - t_1)]\, \mathbf{k}_r[\mathbf{Z}(t_1)] \tag{150}$$

$$t_1 = \min(t_1, t_2, \ldots, t_r)$$

In this way the complete probabilistic characterization of the vector $\mathbf{Z}$ can be obtained.

3.5 Filtered delta-correlated input

The stochastic analysis of linear systems excited by non-Gaussian processes can be performed, similarly to Gaussian input, by means two different approaches. The first one requires all the correlation of the input, the second one the differential equation (excited by a delta-correlated process) governing the filtered input process. Similarly to the white noise input the two approaches are indicated as Method 1 and Method 2 in the following.

a) Method 1

If $\mathbf{F}(t)$ is a non-Gaussian process, Eq. (124) has its own full mathematical sense and the differential of $\mathbf{Z}$ is a summation of infinitesimals of the same order. It follow that

$$\Delta\left(\mathbf{Z}^{[r]}\right) = \mathrm{d}\left(\mathbf{Z}^{[r]}\right) = \mathbf{Q}_r\left(\mathbf{Z}^{[r-1]} \otimes \mathbf{I}_{2m}\right)\mathrm{d}\mathbf{Z} = \mathbf{A}_r\, \mathbf{Z}^{[r]}\, \mathrm{d}t + \mathbf{V}_r\, \mathbf{Z}^{[r-1]}\, \mathbf{F}(t)\, \mathrm{d}t \tag{151}$$

and consequently

$$\dot{\mathbf{m}}_r[Z] = \mathbf{A}_r\, \mathbf{m}_r[Z] + \mathbf{V}_r\, E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{F}(t)\right] \tag{152}$$

where $\mathbf{A}_r$ and $\mathbf{V}_r$ has been defined in Eqs. (135) and

$$E\left[\mathbf{Z}^{[r-1]}(t) \otimes \mathbf{F}(t)\right] = \int_0^t \ldots \int_0^t \boldsymbol{\Theta}(t - \tau_{r-1}) \ldots \boldsymbol{\Theta}(t - \tau_1)\, \mathbf{V}^{[r-1]}$$

$$E\left[\mathbf{F}(\tau_{r-1}) \ldots \mathbf{F}(\tau_1)\, \mathbf{F}(t)\right] \mathrm{d}\tau_{r-1} \ldots \mathrm{d}\tau_1 \tag{153}$$

It follows that in order to evaluate $E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{F}(t)\right]$ we have to evaluate the averages at multiple times of order r of the input process vector $\mathbf{F}(t)$ or equivalently the correlation of the same order of $\mathbf{F}(t)$. Since the expressions of the correlations of the input become more complicated by increasing their order, this approach is not useful from a numerical point of view. This approach has been recently developped in [46].

b) Method 2

According to this method we can write $\mathbf{F}(t)$ as the solution of a set of differential equation namely:

$$\mathbf{F}(t) = \mathbf{N}^T \mathbf{Z}_f(t) \tag{154a}$$

$$d\mathbf{Z}_f(t) = \mathbf{A}_f \mathbf{Z}_f dt + \mathbf{V}_f d\mathbf{L} \tag{154b}$$

It follows that in order to evaluate, the response moments we have solve the following set of differential equations derived from Eqs. (152) and (137) respectively

$$\dot{\mathbf{m}}_r[\mathbf{Z}] = \mathbf{A}_r \mathbf{m}_r[\mathbf{Z}] + \mathbf{V}_{rf} E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{Z}_f\right] \tag{155a}$$

$$\dot{\mathbf{m}}_r[\mathbf{Z}_f] = \mathbf{A}_{r,f} \mathbf{m}_r[\mathbf{Z}_f] + \sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk,f}\left(\mathbf{m}_{r-k}[\mathbf{Z}_f] \otimes \mathbf{I}_{2m}^{[k]}\right) \mathbf{q}_k(t) \tag{155b}$$

where

$$\begin{aligned} &\mathbf{V}_{rf} = \mathbf{V}_r\left(\mathbf{I}_{2m} \otimes \mathbf{N}^T\right), \quad \mathbf{A}_{r,f} = \mathbf{A}_f \otimes \mathbf{I}_{m_f}^{[r-1]} + \mathbf{I}_{m_f} \otimes \mathbf{A}_{r-1,f} \\ &\mathbf{V}_{rk,f} = \frac{1}{k!} \mathbf{Q}_{r,\,m_f}\left(\mathbf{Q}_{r-1,\,m_f} \otimes \mathbf{I}_{m_f}\right) \cdots \left(\mathbf{Q}_{r-k+1,\,m_f} \otimes \mathbf{I}_{m_f}^{[k-1]}\right)\left(\mathbf{I}_{m_f}^{[r-k]} \otimes \mathbf{V}_f^{[k]}\right) \end{aligned} \tag{156}$$

The vector $E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{Z}_f\right]$ can be evaluated as the solution of a differential equation obtained as follows

$$d\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{Z}_f\right] = \mathbf{Q}_{r-1,\,2m}\left(\mathbf{Z}^{[r-2]} \otimes \mathbf{I}_{2m}\right)\left[d\mathbf{Z} \otimes \mathbf{Z}_f\right] + \mathbf{Z}^{[r-1]} \otimes d\mathbf{Z}_f \tag{157}$$

By using for $d\mathbf{Z}$ and $d\mathbf{Z}_f$ their expression given in Eqs. (86) and (154b) and using the main properties of Levy process we have

$$d\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{Z}_f\right] = \left\{\mathbf{Q}_{r-1,2m}\left(\mathbf{Z}^{[r-2]} \otimes \mathbf{I}_{2m}\right)[\mathbf{A}\,\mathbf{Z}\,dt + \mathbf{V}\,\mathbf{F}\,dt]\right\} \otimes \mathbf{Z}_f$$

$$+ \mathbf{Z}^{[r-1]} \otimes \left(\mathbf{A}_f \mathbf{Z}_f\,dt + \mathbf{V}_f\,dL\right) = \mathbf{A}_{r-1,sf}\,\mathbf{Z}^{[r-1]}\,dt + \mathbf{V}_{r-1,sf}\left(\mathbf{Z}^{[r-2]} \otimes \mathbf{Z}_f^{[2]}\right)dt \qquad (158)$$

Taking the stochastic average and dividing by dt we obtain

$$\dot{E}\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{Z}_f\right] = \mathbf{A}_{r-1,sf}\,E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{Z}_f\right] + \mathbf{V}_{r-1,sf}\,E\left[\mathbf{Z}^{[r-2]} \otimes \mathbf{Z}_f^{[2]}\right] \qquad (159)$$

where

$$\mathbf{A}_{r-1,sf} = \mathbf{A}_{r-1} \otimes \mathbf{I}_{m_f} + \mathbf{I}_{2m}^{[r-1]} \otimes \mathbf{A}_f$$

$$\mathbf{V}_{r-1,sf} = \left[\mathbf{Q}_{r-1,2m}\left(\mathbf{I}_{2m}^{[r-2]} \otimes \mathbf{V}\,\mathbf{N}^T\right)\right] \otimes \mathbf{I}_{m_f} \qquad (160)$$

Equation (159) involves the evaluation of the stochastic average $E\,[\mathbf{Z}^{[r-2]} \otimes \mathbf{Z}_f^{[2]}]$ it follows that to solve the differential Eq. (159) we have to evaluate the stochastic average of the kind $E\,[\mathbf{Z}^{[j]} \otimes \mathbf{F}^{[l]}]$ which by using the procedure before described leads to

$$\dot{E}\left[\mathbf{Z}^{[j]} \otimes \mathbf{Z}_f^{[l]}\right] = \mathbf{A}_{jl,sf}\,E\left[\mathbf{Z}^{[j]} \otimes \mathbf{Z}_f^{[l]}\right] + \mathbf{V}_{jl,sf}\,E\left[\mathbf{Z}^{[j-1]} \otimes \mathbf{Z}_f^{[l+1]}\right] \qquad (161)$$

where

$$\mathbf{A}_{jl,sf} = \mathbf{A}_j \otimes \mathbf{I}_{mf}^{[l]} + \mathbf{I}_{2m}^{[j]} \otimes \mathbf{A}_{l,f}$$

$$\mathbf{V}_{jl,sf} = \left[\mathbf{Q}_{j,2m}\left(\mathbf{I}_{2m}^{[j-1]} \otimes \mathbf{V}\,\mathbf{N}^T\right)\right] \otimes \mathbf{I}_{mf}^{[l]} \qquad (162)$$

and

$$\mathbf{A}_{l,f} = \mathbf{I}_{mf} \otimes \mathbf{A}_{l-1,f} + \mathbf{A}_f \otimes \mathbf{I}_{mf}^{[l-1]}$$

$$\mathbf{A}_{j1,sf} = \mathbf{A}_{j,sf}\ ;\ \mathbf{V}_{j1,sf} = \mathbf{V}_{j,sf} \qquad (163)$$

Eqs. (161) represent a set of decoupled equations whose solution can be easily evaluated starting from j = 1. Indeed in this case we have

$$\dot{E}\left[\mathbf{Z} \otimes \mathbf{Z}_f^{[l]}\right] = \mathbf{A}_{1l,sf}\,E\left[\mathbf{Z} \otimes \mathbf{Z}_f^{[l]}\right] + \mathbf{V}_{1l,sf}\,\mathbf{m}_r[\mathbf{Z}_f] \qquad (164)$$

and for j = 2, j = 3, ... we have

$$\begin{aligned}
&\dot{E}\left[\mathbf{Z}^{[2]} \otimes \mathbf{Z}_f^{[l]}\right] = \mathbf{A}_{2l,\,sf}\, E\left[\mathbf{Z}^{[2]} \otimes \mathbf{Z}_f^{[l]}\right] + \mathbf{V}_{2l,\,sf}\, E\left[\mathbf{Z} \otimes \mathbf{Z}_f^{[l]}\right] \\
&\dot{E}\left[\mathbf{Z}^{[3]} \otimes \mathbf{Z}_f^{[l]}\right] = \mathbf{A}_{3l,\,sf}\, E\left[\mathbf{Z}^{[3]} \otimes \mathbf{Z}_f^{[l]}\right] + \mathbf{V}_{3l,\,sf}\, E\left[\mathbf{Z}^{[3]} \otimes \mathbf{Z}_f^{[l]}\right] \\
&\cdots\cdots\cdots
\end{aligned} \tag{165}$$

It follows that in order to solve Eq. (159) we have to solve the sequence of differential Eq. (161) starting from j = 1 and l = r - 1 until j = r - 1 and l = 1.
Equations (155a), (155b) and (161) are sets of first order differential equations whose numerical solution can be performed by means of the technique before described once the following transition and loading matrices can be evaluated as follows, respectively

$$\begin{aligned}
&\boldsymbol{\Theta}_r(\Delta t) = \boldsymbol{\Theta}^{[r]}(\Delta t);\quad \mathbf{L}_r(\Delta t) = \left[\boldsymbol{\Theta}^{[r]}(\Delta t) - \mathbf{I}_{2m}^{[r]}\right] \mathbf{A}_r^{-1} \\
&\boldsymbol{\Theta}_{r,f}(\Delta t) = \boldsymbol{\Theta}_f^{[r]}(\Delta t);\quad \mathbf{L}_{r,\,f}(\Delta t) = \left[\boldsymbol{\Theta}_f^{[r]}(\Delta t) - \mathbf{I}_{m_f}^{[r]}\right] \mathbf{A}_{r,\,f}^{-1} \\
&\boldsymbol{\Theta}_{jl,\,sf}(\Delta t) = \boldsymbol{\Theta}^{[j]}(\Delta t) \otimes \boldsymbol{\Theta}_f^{[l]}(\Delta t);\quad \mathbf{L}_{jl,\,sf}(\Delta t) = \left[\boldsymbol{\Theta}_{jl,\,sf}(\Delta t) - \mathbf{I}_{2m}^{[j]} \otimes \mathbf{I}_{m_f}^{[l]}\right] \mathbf{A}_{jl,\,sf}^{-1}
\end{aligned} \tag{166}$$

3.6 Stationary input

For stationary input the statistical moments are not time dependent quantities. It follows that the differential equations governing the evolution of these moments become a set of algebraic equation and the moments for both normal or non-normal delta-correlated input process from Eq. (137) can be obtained as follows

$$\mathbf{m}_r[\mathbf{Z}] = -\mathbf{A}_r^{-1}\left[\sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk}\left(\mathbf{m}_{r-k}[\mathbf{Z}] \otimes \mathbf{I}_{2m}^{[k]}\right)\mathbf{q}_k\right] \tag{167}$$

For filtered delta-correlated input process, according to the Method 1, we have from Eq. (152)

$$\mathbf{m}_r[\mathbf{Z}] = -\mathbf{A}_r^{-1}\,\mathbf{V}_r\,E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{F}\right] \tag{168}$$

where $E\,[\mathbf{Z}^{[r-1]} \otimes \mathbf{F}]$ must be calculate by solving a multiple integral; while, according to the Method 2, if the differential equations of the filter are known, we have, from Eqs. (155) and (159)

$$\mathbf{m}_r[\mathbf{Z}] = -\mathbf{A}_r^{-1}\,\mathbf{V}_{rf}\,E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{Z}_f\right]$$

$$E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{Z}_f\right] = -\mathbf{A}_{r-1,\,sf}^{-1}\,\mathbf{V}_{r-1,\,sf}\,E\left[\mathbf{Z}^{[r-2]} \otimes \mathbf{Z}_f^{[2]}\right] \tag{169}$$

$$\mathbf{m}_r[\mathbf{Z}] = -\mathbf{A}_{rf}^{-1}\left[\sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk,\,f}\left(\mathbf{m}_{r-k}[\mathbf{Z}_f] \otimes \mathbf{I}_{2m}^{[k]}\right)\mathbf{q}_k\right]$$

Since the input process is a zero mean process (that is $\mathbf{m}_1[\mathbf{Z}_f] = \mathbf{0}$) and consequently $\mathbf{m}_1[\mathbf{Z}] = \mathbf{0}$, the set of Eqs. (169) can be solved for r = 1, 2, ... , s starting from r = 2 . Indeed for r = 2 we have

$$\mathbf{m}_2[\mathbf{Z}] = -\mathbf{A}_2^{-1}\,\mathbf{V}_{2f}\,E\left[\mathbf{Z} \otimes \mathbf{Z}_f\right]$$

$$E\left[\mathbf{Z} \otimes \mathbf{Z}_f\right] = -\mathbf{A}_{1,\,sf}^{-1}\,\mathbf{V}_{1,\,sf}\,\mathbf{m}_2[\mathbf{Z}_f] \tag{170}$$

$$\mathbf{m}_2[\mathbf{Z}] = -\mathbf{A}_{2,\,f}^{-1}\,\mathbf{V}_{2r,\,f}\,\mathbf{q}_2$$

and for r = 3 we have

$$\mathbf{m}_3[\mathbf{Z}] = -\mathbf{A}_3^{-1}\,\mathbf{V}_{3f}\,E\left[\mathbf{Z}^{[2]} \otimes \mathbf{Z}_f\right]$$

$$E\left[\mathbf{Z}^{[2]} \otimes \mathbf{Z}_f\right] = -\mathbf{A}_{2,\,sf}^{-1}\,\mathbf{V}_{2,\,sf}\,E\left[\mathbf{Z} \otimes \mathbf{Z}_f^{[2]}\right] \tag{171}$$

$$\mathbf{m}_3[\mathbf{Z}_f] = -\mathbf{A}_{r,\,f}^{-1}\left[\sum_{k=2}^{3} \mathbf{V}_{3k,\,f}\left(\mathbf{m}_{r-k}[\mathbf{Z}_f] \otimes \mathbf{I}_{2m}^{[k]}\right)\mathbf{q}_k\right]$$

where $E\,[\mathbf{Z} \otimes \mathbf{Z}_f^{[2]}]$ can be obtained from Eq. (164)

$$E\left[\mathbf{Z} \otimes \mathbf{Z}_f^{[2]}\right] = -\mathbf{A}_{12,\,sf}^{-1}\,\mathbf{V}_{12,\,f}\,\mathbf{m}_3[\mathbf{Z}_f] \tag{172}$$

Following the same procedure and by using Eqs. (164) and (165) it is possible to evaluate the statistical moments until the s-th order. Notice that the inverse matrices which appear in these equations can be easily evaluated by using the eigenproperties of the structural system and of the filter [37].

3.7 Application

As an application in this Section the stochastic response of a simply supported beam excited by moving loads is considered. This model is correctly used for the evaluation of structural response of highway bridges subjected to traffic flow.
The force arrivals at the beam are assumed to constitute a Poisson process of events and the force amplitudes are random variables.
For clarity's sake, a simply supported beam of finite length L, that is loaded by a train of forces moving in the same direction, all with equal, constant velocity v is considered. The forces arrive at the beam at random times t_k, which constitute a stationary Poisson process, with constant parameter $\lambda > 0$.

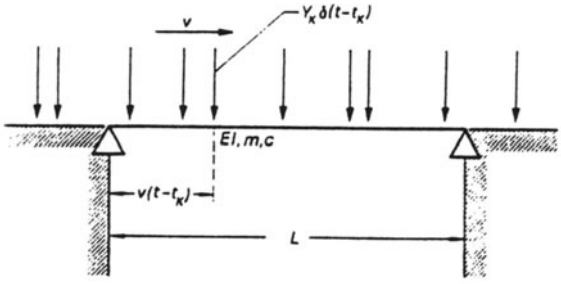

Fig. E.1 - Sketch of the bridge and its loadings.

The vibrations of the beam due to this stream of forces are described by the following equation [47]

$$E\,I\,w^{IV}(x,t) + c\,\dot{w}(x,t) + \rho\,A\,\ddot{w}(x,t) = \sum_{k=1}^{N(t)} Y_k\,\delta[x - (t - t_k)\,v]\,W(t - t_k, t_L) \qquad \text{(E.23)}$$

where w(x, t) denotes the transversal displacements of the beam, EI denotes the flexural rigidity of the beam, c denotes the damping coefficient, ρA denotes the mass density of unit length, $\delta(\cdot)$ is the Dirac delta function, roman numerals denote differentiation with respect to spatial coordinate x, while dots denote differentiation with respect to time t. The amplitudes Y_k are a family of identically distributed random variables, which are mutually independent and independent of the time instants t_k. Finally, N(t) is a counting Poisson process. To account for the finite length of the beam, on the right-hand-side of (E.23), the window function $W(\tau, t_L) = U(\tau)\,[1 - U(\tau - t_L)]$ appears, $U(\cdot)$ being the unit step function and $t_L = L/v$ the loading time. This loading formulation indicates that the k-th point load of magnitude Y_k at location $x = (t - t_k)v$ is effective to the beam only when the time satisfies $t_k \le t \le t_k + t_L$.
Using the normal mode approach, the deflection w(x, t) can be expanded in the following form (truncated at n-th term):

$$w(x, t) = \sum_{j=1}^{n} \Psi_j(x)\, q_j(t) \tag{E.24}$$

where $\Psi_j(x)$ are the normal modes of free vibration, depending on the boundary conditions, and $q_j(t)$ are the modal responses or generalized coordinates. For a simply supported beam $\Psi_j(x) = \sin(j \pi x / L)$.
Performing the usual coordinate transformation, we obtain the following differential equation governing the j-th modal response:

$$\ddot{q}_j(t) + 2\alpha_j \omega_j \dot{q}_j(t) + \omega_j^2 q_j(t) = \frac{2}{M} s_j(t) \tag{E.25}$$

where $\omega_j = (j \pi / L)^2 \sqrt{EI/\rho A}$ is the j-th radial natural frequency of the undamped structure, $\alpha_j = c/2\rho A\omega_j$ is the modal damping coefficient and $M = \rho AL$ is the mass of the beam.
The random forcing function $s_j(t)$ in (E.25) is given as:

$$s_j(t) = \sum_{k=1}^{N(t)} Y_k \phi_j(t - t_k), \qquad \phi_j(\tau) = \sin(\Omega_j \tau)\, W(\tau, t_L) \tag{E.26}$$

where $\Omega_j = j\pi v/L = j\pi / t_L$. The process $s_j(t)$ is a filtered Poisson process [48] and the deterministic function $\phi_j(t - t_k)$ describes the shape of k-th impulse which is a truncated sine function with period $T_j = 2t_L / j$ that starts at a random time t_k and with duration $t_L = j\, T_j / 2$.
The specific relationship between t_L and T_j suggests to express the loading function in a particular form [49]. Let us consider the following single undamped linear oscillator:

$$\ddot{\phi}_j(\tau) + \Omega_j^2 \phi_j(\tau) = \Omega_j \delta(\tau) + (-1)^{j+1} \Omega_j \delta(\tau - t_L) \tag{E.27}$$

The response due to the first forcing term on the right-hand-side of (E.27) is a sine function with frequency Ω_j which starts at time $\tau = 0$, that is: $\sin(\Omega_j \tau)\, U(\tau)$. The response due to the second term is a sine function which starts at time $\tau = t_L$, that is: $(-1)^{j+1} \sin[(\Omega_j (\tau - t_L)]\, U(\tau - t_L)$. A super-position of these two responses gives as solution the truncated sine function $\sin(\Omega_j \tau)\, W(\tau, t_L)$, as shown in Figg. E.2 for the first two modes.
It follows that the process $s_j(t)$ can be considered as the response of the following differential equation

$$\ddot{s}_j(t) + \Omega_j^2 s_j(t) = \Omega_j \left[\xi(t) + (-1)^{j+1} \xi(t - t_L)\right]; \quad \xi(t) = \sum_{k=1}^{N(t)} Y_k \delta(\tau - t_k) \tag{E.28}$$

$\xi(t)$ being a Poisson white noise process, that is a delta correlated process, characterized by the r-th correlation function as product of Dirac's delta functions, that is:

$$C_r[\xi; t_1, t_2, \ldots, t_r] = \lambda E[Y^r] \delta(t_2 - t_1) \delta(t_3 - t_1) \ldots \delta(t_r - t_1) \tag{E.29}$$

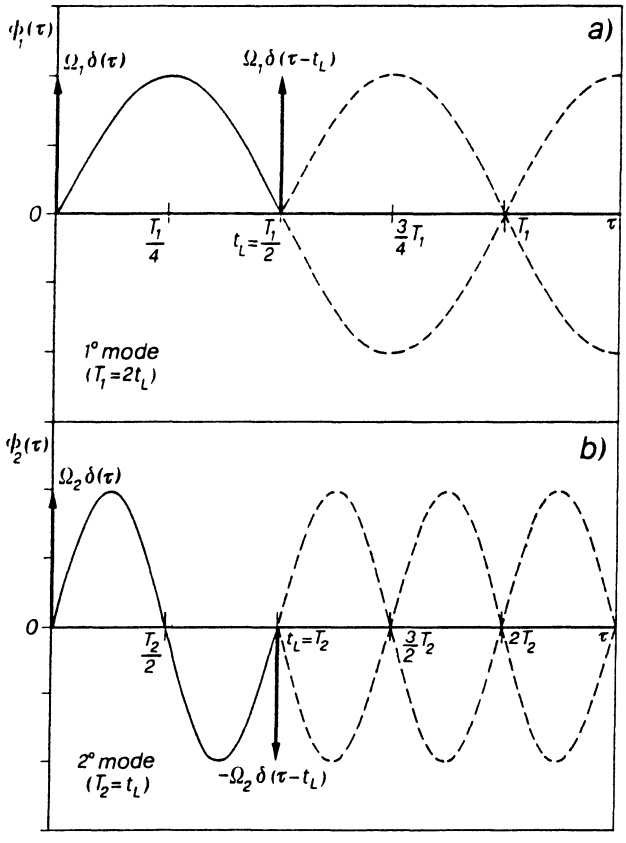

Fig. E.2 - Solution of differential equation (5) as truncated sine function: a) first mode; b) second mode

By using the state vector approach, Eqs. (E.25) and (E.28), written for j = 1 to n (where n is the number of modes included in the analysis), can be expressed in the following matrix form:

$$\dot{\mathbf{Z}} = \mathbf{A}\mathbf{Z} + \mathbf{v}\,\xi(t) + \overline{\mathbf{v}}(t)\,\xi(t - t_L) \tag{E.30}$$

where **Z** is the response vector in modal coordinates (of order m = 4n), given as

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_q \\ \mathbf{Z}_s \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \\ \mathbf{s} \\ \dot{\mathbf{s}} \end{bmatrix}; \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}; \quad \mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \tag{E.31}$$

and **A** is the dynamic matrix of order m x m, given as:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_q & \mathbf{A}_{qs} \\ \mathbf{0}_{2n} & \mathbf{A}_s \end{bmatrix}; \quad \mathbf{A}_q = \begin{bmatrix} \mathbf{0}_n & \mathbf{I}_n \\ -\boldsymbol{\omega}^2 & -\boldsymbol{\chi} \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_n & \mathbf{I}_n \\ -\boldsymbol{\Omega}^2 & \mathbf{0}_n \end{bmatrix}; \quad \mathbf{A}_{qs} = \begin{bmatrix} \mathbf{0}_n & \mathbf{0}_n \\ (2/M)\,\mathbf{I}_n & \mathbf{0}_n \end{bmatrix} \tag{E.32}$$

In Eq. (E.32), $\mathbf{0}_p$ and $\mathbf{I}_p$ are the zero matrix and the identity matrix of order p, respectively; $\boldsymbol{\omega}^2$, $\boldsymbol{\chi}$ and $\boldsymbol{\Omega}^2$ are diagonal matrices whose diagonal elements are ω_j^2, $2\alpha_j\omega_j$ and Ω_j^2, respectively. Moreover, the forcing vectors **v** and $\bar{\mathbf{v}}(t)$ are given as

$$\mathbf{v} = \begin{bmatrix} \bar{\mathbf{0}}_{3n} \\ \boldsymbol{\Omega}\, \boldsymbol{l} \end{bmatrix}; \quad \bar{\mathbf{v}}(t) = U(t - t_L) \begin{bmatrix} \bar{\mathbf{0}}_{3n} \\ \boldsymbol{\Omega}\, \bar{\boldsymbol{l}} \end{bmatrix} \tag{E.33}$$

where $\bar{\mathbf{0}}_{3n}$ is the zero column vector of order 3n, while $\boldsymbol{l}$ and $\bar{\boldsymbol{l}}$ are two influence vectors of order n given as: $\boldsymbol{l}^T = [1 \;\; 1 \;\; \ldots \;\; 1]$, $\bar{\boldsymbol{l}}^T = [1 \;\; -1 \;\; \ldots \;\; (-1)^{n+1}]$.
Eq. (E.30) can be considered to be formally equivalent to the following Ito stochastic differential equation

$$d\mathbf{Z} = \mathbf{A}\,\mathbf{Z}\,dt + \mathbf{v}\; d\gamma(t) + \bar{\mathbf{v}}(t)\; d\gamma(t - t_L) \tag{E.34}$$

where $\gamma(t)$ is a compound Poisson process and the formal derivative of $\gamma(t)$ is the process $\xi(t)$. Using the extension of Ito differential rule for non-normal input process [50], we can write the moment equations of the response. As an example, the differential equations of the first two moments are given as

$$\dot{\mathbf{m}}_1[\mathbf{z}(t)] = \mathbf{A}\,\mathbf{m}_1[\mathbf{Z}] + \lambda E[Y]\,\mathbf{V}(t)$$

$$\dot{\mathbf{m}}_2[\mathbf{Z}] = \mathbf{A}_2\,\mathbf{m}_2[\mathbf{z}(t)] + \lambda E[Y]\left\{\mathbf{m}_1[\mathbf{Z}] \otimes \mathbf{V}(t) + \mathbf{V}(t) \otimes \mathbf{m}_1[\mathbf{Z}]\right\} \tag{E.35}$$

$$+ \lambda E[Y^2]\left\{\mathbf{v}^{[2]} + \bar{\mathbf{v}}^{[2]}(t) + [\boldsymbol{\Theta}(t_L) \otimes \mathbf{I}_m]\,[\mathbf{v} \otimes \bar{\mathbf{v}}(t)] + [\mathbf{I}_m \otimes \boldsymbol{\Theta}(t_L)]\,[\bar{\mathbf{v}}(t) \otimes \mathbf{v}]\right\}$$

where $\mathbf{V}(t) = \mathbf{v} + \bar{\mathbf{v}}(t)$.

More compact form can be obtained in writing the differential equations in terms of cumulants, that is

$$\dot{\mathbf{k}}_r[Z] = \mathbf{A}_r \mathbf{k}_r[Z] + \mathbf{f}_r(t), \quad (r = 1, 2, \ldots) \tag{E.36}$$

where

$$\mathbf{A}_r = \mathbf{A}_{r-1} \otimes \mathbf{I}_m + \mathbf{I}_m^{[r-1]} \otimes \mathbf{A}; \quad \mathbf{A}_1 = \mathbf{A}$$

$$\mathbf{f}_r(t) = \lambda E[Y^r] \left\{ \left[\mathbf{I}_m^{[r]} - \boldsymbol{\Theta}^{[r]}(t_L) \right] \mathbf{v}^{[r]} + \left[\boldsymbol{\Theta}(t_L)\, \mathbf{v} + \overline{\mathbf{v}}(t) \right]^{[r]} \right\} \tag{E.37}$$

Eq. (E.36) provides the non-stationary stochastic response of a beam under moving loads in terms of cumulants of every order. Because of the particular form of the forcing vector $\mathbf{f}_r(t)$, Eq. (E.36) can be solved in closed form. Then, the r-th cumulant of the response for zero initial conditions is given as

$$\mathbf{k}_r[Z(t)] = \lambda E[Y^r] \mathbf{A}_r^{-1} \left\{ \left[\boldsymbol{\Theta}^{[r]}(t_L) - \mathbf{I}_m^{[r]} \right] \mathbf{v}^{[r]} + \left[\boldsymbol{\Theta}^{[r]}(t - t_L) - \mathbf{I}_m^{[r]} \right] \left[\boldsymbol{\Theta}(t_L)\, \mathbf{v} + \overline{\mathbf{v}}(t) \right]^{[r]} \right\} \tag{E.38}$$

By using this equation, the transient response of a bridge subjected to a random stream of moving forces arriving with expected rate λ (constant) has been determined. The geometry of the bridge studied and the structural properties are defined by the following data: L = 100m, EI = 4 x 10^{11} Nm^2, m = 12,000 kg/m. The first radial frequency is $\omega_1 = 5.698$ s^{-1} and the modal damping ratio assumed is 0.1. The weight of vehicles has been considered as a random variable uniformly distributed in the range 40 kN - 240 kN. The constant traffic speed is v = 25 m/s. The vehicle arrival rate, related to the average passage speed, has been calculated by the following expression [51]:

$$\lambda = 4 \lambda_{max} \left(\frac{v}{v_{max}} \right) \left(1 - \frac{v}{v_{max}} \right) \tag{E.39}$$

where $\lambda_{max} = 0.5$ s^{-1} is the maximum vehicle arrival rate (which corresponds to the capacity of about 1800 *vehicles/h*) and $v_{max} = 120$ km/h is the maximum speed. For v = 25 m/s, we have $\lambda = 0.375$ s^{-1}. The expected value $E[\overline{w}]$, the standard deviation $\sigma_{\overline{w}}$ the coefficient of asymmetry $\gamma_a[\overline{w}] = k_3[\overline{w}] / k_2[\overline{w}]^{3/2}$ and the coefficient of excess $\gamma_e[\overline{w}] = k_4[\overline{w}] / k_2[\overline{w}]^2$ of the midspan deflection $\overline{w}$ are plotted versus time in Figs. 3-6, for zero initial conditions and including six modes for the analysis.

The digital simulation method based on a Monte Carlo procedure (50.000 samples) has been applied to verify the results obtained by means of the proposed approach. The time instants t_k of impulse arrivals has been determined by means of Poisson number generator.

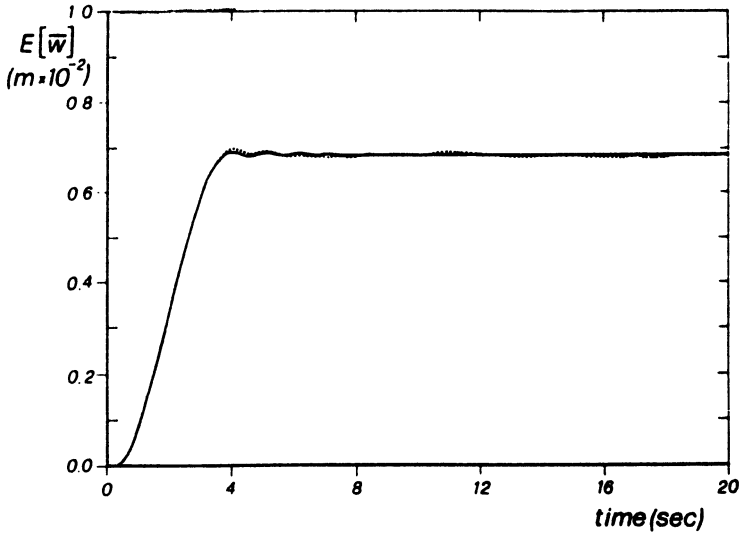

Fig. E.3 - Evolutionary expected value of midspan deflection E[w] by means of the proposed method (solid line) and the digital simulation method (dashed line).

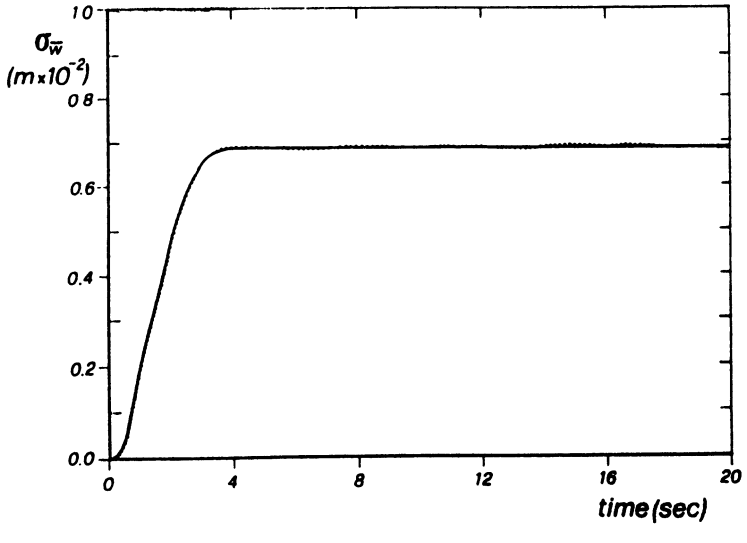

Fig. E.4 - Evolutionary standard deviation of midspan deflection $\sigma_{\dot{w}}$ by means of the proposed method (solid line) and the digital simulation method (dashed line).

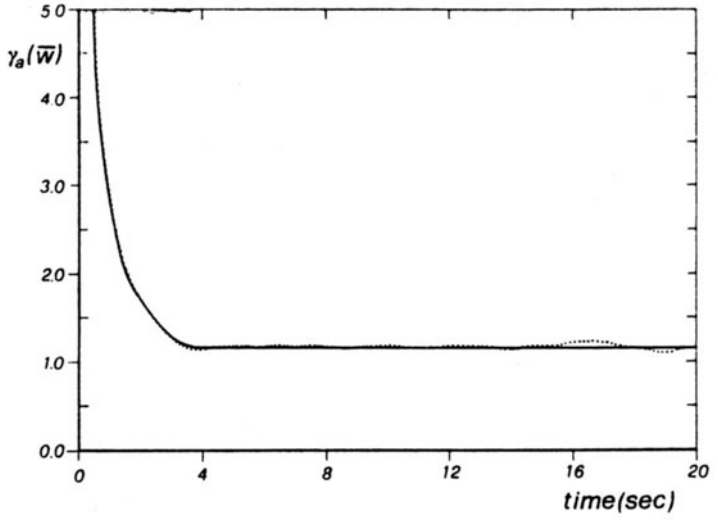

Fig. E.5 - Evolutionary coefficient of asymmetry of midspan deflection $\gamma_a(\overline{w})$ by means of the proposed method (solid line) and the digital simulation method (dashed line).

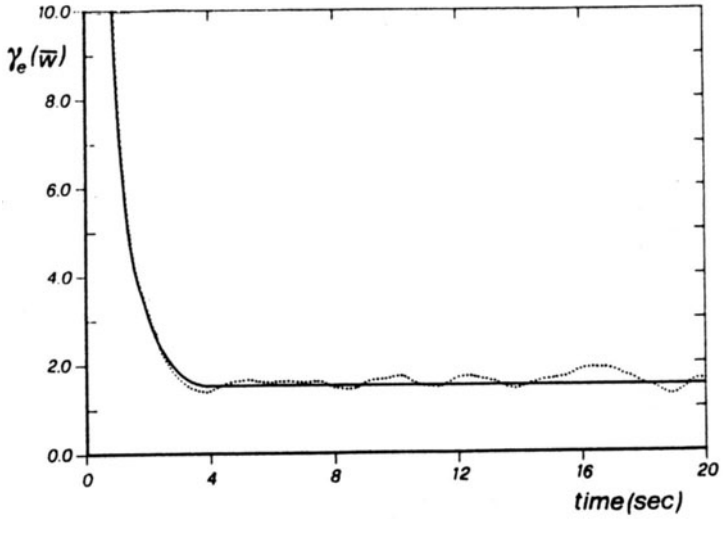

Fig. E.6 - Evolutionary coefficient of excess of midspan deflection $\gamma_e(\overline{w})$ by means of the proposed method (solid line) and the digital simulation method (dashed line).

4 MULTIDIMENSIONAL NON-LINEAR SYSTEMS

4.1 Numerical solution of non-linear differential equation for deterministic input

In this section a method for numerical treatment of deterministic non-linear equation of motion is presented in order to introduce a numerical procedure very useful in stochastic analysis.
Let us consider a non-linear system whose state differential equation can be written in the form

$$\dot{\mathbf{Z}} = \mathbf{a}(\mathbf{Z}(t), t) + \mathbf{V}(\mathbf{Z}(t), t)\,\mathbf{F}(t), \quad t \geq t_0 \tag{173}$$

where t_0 is the initial time of the motion, $\mathbf{Z}(t)$ is the 2n x 1 state vector and $\mathbf{F}(t)$ is the input vector of order f x 1. Numerical methods to solve non-linear differential equations has been proposed in literature (see e.g. [52]). In the follows the pseudo-force method [53-54] is described. According to last methods we replace the right hand side of Eq. (173) by power series representations. Adopting the Kronecker product notation, we write

$$\begin{aligned}\mathbf{a}(\mathbf{Z}, t) &= \mathbf{A}^{(1)}(t)\,\mathbf{Z}^{[1]} + \mathbf{A}^{(2)}(t)\,\mathbf{Z}^{[2]} + \ldots + \mathbf{A}^{(N)}(t)\,\mathbf{Z}^{[N]} + \ldots \\ &= \sum_{k=1}^{N_1} \mathbf{A}^{(k)}(t)\,\mathbf{Z}^{[k]}(t)\end{aligned} \tag{174a}$$

$$\begin{aligned}\mathbf{V}(\mathbf{Z}, t) &= \mathbf{V}^{(0)}(t) + \mathbf{V}^{(1)}(\mathbf{Z} \otimes \mathbf{I}_f) + \ldots + \mathbf{V}^{[N_2]}(t)\left(\mathbf{Z}^{[N_2]} \otimes \mathbf{I}_f\right) \\ &= \sum_{l=0}^{N_2} \mathbf{V}^{(l)}(t)\left(\mathbf{Z}^{[l]}(t) \otimes \mathbf{I}_f\right)\end{aligned} \tag{174b}$$

where we have explicity retained terms through degree N_1 in the expansion of $\mathbf{a}(\mathbf{Z}, t)$, and terms through degree N_2 in the expansion of $\mathbf{V}(\mathbf{Z}, t)$. That higher-degree terms in these expressions will not contribute to the response is taken in due course. According to Taylor series expansion we have

$$\begin{aligned}\mathbf{A}^{(k)}(t) &= \frac{1}{k!}\left[\nabla_Z^{T[k]} \otimes \mathbf{a}(\mathbf{Z}, t)\right]_{\mathbf{Z}=\mathbf{0}} \\ \mathbf{V}^{(k)}(t) &= \frac{1}{k!}\left[\nabla_Z^{T[k]} \otimes \mathbf{V}(\mathbf{Z}, t)\right]_{\mathbf{Z}=\mathbf{0}}\end{aligned} \tag{175}$$

By using Eqs. (174) we can write Eq. (173) in the form:

$$\dot{\mathbf{Z}} = \mathbf{A}(t)\,\mathbf{Z}(t) + \mathbf{V}(t)\,\mathbf{F}(t) + \mathbf{F}_{NL}(\mathbf{Z}, t) \tag{176}$$

where

$$\mathbf{A}(t) = \mathbf{A}^{(1)}(t); \quad \mathbf{V}(t) = \mathbf{V}^{(0)}(t);$$

$$\mathbf{F}_{NL}(\mathbf{Z}, t) = \sum_{k=2}^{N_1} \mathbf{A}^{(k)}(t)\, \mathbf{Z}^{[k]}(t) + \left[\sum_{l=1}^{N_2} \mathbf{V}^{(l)} \left(\mathbf{Z}^{[l]}(t) \otimes \mathbf{I}_f \right) \right] \mathbf{F}(t) \tag{177}$$

The solution of Eq. (176) can be written as follows [7]

$$\mathbf{Z}(t) = \mathbf{Z}_L(t) + \mathbf{Z}_{NL}(t) \tag{178}$$

where $\mathbf{Z}_L(t)$ and $\mathbf{Z}_{NL}(t)$ are the linear and non-linear contributions to the response respectively. These two terms can be obtained according in integral form to classical rules as follows

$$\begin{aligned} \mathbf{Z}_L(t) &= \boldsymbol{\Theta}(t - t_0)\, \mathbf{Z}_0 + \int_{t_0}^{t} \boldsymbol{\Theta}(t - \tau)\, \mathbf{V}(\tau)\, \mathbf{F}(\tau)\, d\tau \\ \mathbf{Z}_{NL}(t) &= \int_{t_0}^{t} \boldsymbol{\Theta}(t - \tau)\, \mathbf{F}_{NL}(\tau)\, d\tau \end{aligned} \tag{179}$$

where $\mathbf{Z}_0$ is the vector of the initial condition and $\boldsymbol{\Theta}(t)$ is the so-called transition fundamental matrix. In Eq. (179) $\mathbf{Z}_L(t)$ can be interpreted as the purely elastic response obtained by dropping out the non-linear term into Eq. (176), while $\mathbf{Z}_{NL}(t)$ is the response of the linear system subjected to the non-linear pseudo-forces $\mathbf{F}_{NL}(\mathbf{Z}, t)$ and having zero start conditions. For time-independent coefficients we can solve the convolution integrals of Eqs. (179) by using the numerical technique proposed in Sect. 3. In particular, discretizing the time axis into small intervals of equal length Δt, and assuming that the forcing vectors are constant in each step, we can write

$$\mathbf{Z}_L(t_j + \Delta t) = \boldsymbol{\Theta}(\Delta t)\, \mathbf{Z}_L(t_j) + \mathbf{L}(\Delta t)\, \mathbf{F}_L(t_j) \tag{180a}$$

$$\mathbf{Z}_{NL}(t_j + \Delta t) = \boldsymbol{\Theta}(\Delta t)\, \mathbf{Z}_{NL}(t_j) + \mathbf{L}(\Delta t)\, \mathbf{F}_{NL}\left(\mathbf{Z}(t_j), t_j\right) \tag{180b}$$

Notice that the elastic response system given in Eq. (180a) can be directly computed, while the non-linear part $\mathbf{Z}_{NL}(t)$, as the term $\mathbf{F}_{NL}$ appears, depends on the unknown state vector $\mathbf{Z}$ at each time step. Unless $\mathbf{Z}$ is computed $\mathbf{F}_{NL}$ cannot be determined. But the computation of $\mathbf{Z}$ requires that the value of $\mathbf{F}_{NL}$ be known. Therefore an iterative process is required to solve this problem. A possible strategy is the following: (i) compute the linear response $\mathbf{Z}_L(t)$ once the vector of initial coordinates $\mathbf{Z}_0$ is given; (ii) compute the pseudo-force vector assuming as first approximation

$$\mathbf{Z}^{(1)}(t_j + \Delta t) = \mathbf{Z}_L(t_j + \Delta t) + \mathbf{Z}_{NL}(t_j) \tag{181}$$

(iii) evaluate the vector $\mathbf{Z}_{NL}(t_j + \Delta t)$ by means of Eq. (180b) and consequently $\mathbf{Z}(t_j + \Delta t)$ by using Eq. (178); (iv) evaluate at i-th iteration the pseudo-force vector assuming

$$\mathbf{Z}^{(i)}(t_j + \Delta t) = \mathbf{Z}_L(t_j + \Delta t) + \mathbf{Z}_{NL}^{(i-1)}(t_j + \Delta t) \tag{182}$$

(v) iterate till the convergence, in terms of pseudo-force, is satisfied.

4.2 Normal and non-normal delta-correlated input process

In this section we study non-linear multidimensional systems to normal and non-normal delta-correlated up the s-th order purely external excitation governed by the following differential equation

$$\mathbf{M}\,\ddot{\mathbf{X}} + \mathbf{g}(\mathbf{X}, \dot{\mathbf{X}}) = \mathbf{F}(t) \tag{183}$$

where $\mathbf{g}(\mathbf{X}, \dot{\mathbf{X}})$ is the non-linear restoring force vector. However a general discussion of non-linear systems for both external and parametric excitations is described in the Chapter two.
For purely external delta-correlated up to s-th order excitation $\mathbf{F}(t)$, Eq. (183), by using the state variable vector approach, can be written as follows

$$d\mathbf{Z} = \Delta\mathbf{Z} = \mathbf{a}(\mathbf{Z}, t)\,dt + \mathbf{V}(t)\,d\mathbf{L} \tag{184}$$

where $\mathbf{Z}$ is a vector of order 2n and $d\mathbf{L}$ is the multidimensional s-th order zero mean Levy white noise vector process (defined in Sec. 3.4). It follows that, for purely external excitation, we can write

$$\begin{aligned}\Delta(\mathbf{Z}^{[r]}) &= d\mathbf{Z} + \sum_{k=2}^{min\,(r,\,s)} \frac{1}{k!}\,d^k(\mathbf{Z}^{[r]}) \\ &= \left(\nabla_{\mathbf{Z}}^{T} \otimes \mathbf{Z}^{[r]}\right) d\mathbf{Z} + \sum_{l=2}^{min\,(r,\,s)} \frac{1}{l!}\left(\nabla_{\mathbf{Z}}^{[l]\,T} \otimes \mathbf{Z}^{[r]}\right) d\left(\mathbf{Z}^{[k]}\right)\end{aligned} \tag{185}$$

This equation neglecting infinitesimals of greater order than dt can be written as follow

$$\begin{aligned}\Delta\left(\mathbf{Z}^{[r]}\right) &= \mathbf{Q}_{r,\,2n}\left(\mathbf{Z}^{[r-1]} \otimes \mathbf{a}(\mathbf{Z}, t)\right) dt + \mathbf{V}_r\left(\mathbf{Z}^{[r-1]} \otimes d\mathbf{L}\right) \\ &\quad + \sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk}\left(\mathbf{Z}^{[r-k]} \otimes \mathbf{I}_{2n}^{[k]}\right)(d\mathbf{L})^{[k]}\end{aligned} \tag{186}$$

where the matrices $\mathbf{V}_r$ and $\mathbf{V}_{rk}$ has been defined in Eqs. (135). Making the stochastic average of both sides of Eq. (186), taking into account of the properties of vector $d\mathbf{L}$ (given in Eqs. (129-131) and (136)) and dividing by dt we obtain the moment differential equation of non-linear system excited by purely external delta-correlated input up to s-th order

$$\dot{\mathbf{m}}_r[\mathbf{Z}] = \mathbf{Q}_{r,\,2n}\,E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{a}(\mathbf{Z}, t)\right] + \sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk}\left(\mathbf{m}_{r-k}[\mathbf{Z}] \otimes \mathbf{q}_k(t)\right) \tag{187}$$

Notice that for s = 2 we obtain the differential equation governing the evaluation of the r-th moment for normal delta correlated (the white noise) input.

4.3 Filtered normal and non-normal delta-correlated input process

Let us consider a structural non-linear system excited by a non-white Gaussian or non-Gaussian input. In this case Eq. (173) which describe the motion of the structure for both external and parametric excitation has its own mathematical meaning, that is

$$d\mathbf{Z} = \mathbf{a}(\mathbf{Z}(t), t)\,dt + \mathbf{V}(\mathbf{Z}(t), t)\,\mathbf{F}(t)\,dt \tag{188}$$

It follows that

$$\Delta\left(\mathbf{Z}^{[r]}\right) = d\left(\mathbf{Z}^{[r]}\right) = \left(\nabla_{\mathbf{Z}}^{T} \otimes \mathbf{Z}^{[r]}\right) d\mathbf{Z} = \mathbf{Q}_{r,\,2n}\left(\mathbf{Z}^{[r-1]} \otimes \mathbf{I}_{2n}\right) d\mathbf{Z} \tag{189}$$

Making the stochastic average of both sides of Eq. (189) and taking into account Eq. (188) and dividing the results by dt we can write

$$\dot{\mathbf{m}}_r[\mathbf{Z}] = \mathbf{Q}_{r,\,2n}\,E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{a}(\mathbf{Z}(t), t)\right] + \mathbf{Q}_{r,\,2n}\,E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{V}(\mathbf{Z}(t), t)\,\mathbf{F}\right] \tag{190}$$

which represents the differential equation governing the evaluation of the r-th moment.
The solution of Eq. (190) requires the stochastic average between the response and the input. These quantities are very difficult to evaluate for a general stochastic forcing function. It follows that the forcing function vector is assumed as the solution of a first order system of linear differential equations, which represent the filter differential equations, forced by a delta-correlated up to the s-th order input vector process. It follows that

$$d\mathbf{F} = \mathbf{A}_f\,\mathbf{F}\,dt + \mathbf{V}_f\,d\mathbf{L}(t) \tag{191}$$

Associating Eq. (191) to Eq. (188) we have a set of 2n+f non-linear differential equations with external forcing function. This property is more evident if it is possible to set

$$\mathbf{a}(\mathbf{Z}(t), t) = \mathbf{A}(\mathbf{Z}(t), t)\,\mathbf{Z} \tag{192}$$

and consequently we have

$$d\mathbf{Z}_c = \mathbf{A}_c(\mathbf{Z}_c(t), t)\,\mathbf{Z}_c\,dt + \mathbf{V}_c\,d\mathbf{L} \tag{193}$$

where

$$\mathbf{Z}_c = \begin{bmatrix} \mathbf{Z} \\ \mathbf{F} \end{bmatrix}; \quad \mathbf{A}_c(\mathbf{Z}_c(t), t) = \begin{bmatrix} \mathbf{A}(\mathbf{Z}(t), t) & \mathbf{V}(\mathbf{Z}(t), t) \\ \mathbf{0} & \mathbf{A}_f \end{bmatrix}; \quad \mathbf{V}_c = \begin{bmatrix} \mathbf{0} \\ \mathbf{V}_f \end{bmatrix} \tag{194}$$

It follows that operating in similar way to the case treated in the previous section we have the moment differential equations in terms of variable vector $\mathbf{Z}_c$ which are similar to Eq. (187), that is

$$\dot{\mathbf{m}}_r[\mathbf{Z}_c] = \mathbf{Q}_{r,\,2n+f}\left\{\mathrm{E}\left[\left(\mathbf{I}_{2n}^{[r-1]} \otimes \mathbf{A}_c(\mathbf{Z}(t), t)\right)\mathbf{Z}_c^{[r]}\right]\right\} + \sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{c,\,rk}\left(\mathbf{m}_{r-k}[\mathbf{Z}_c] \otimes \mathbf{q}_k(t)\right) \tag{195}$$

Notice that for both purely external and parametric excitation the moment differential equation of structural system excited by a filtered delta-correlated input have the same mathematical form.

4.4 Multidimensional quasi-moment and cumulant neglect closure techniques

In some structural problems the vector $\mathbf{a}(\mathbf{Z}, t)$ is expressed in a polynomial form, this happens, for example, in the case of geometrically non-linear system modelled by using the finite element method (see e.g. [7, 8]). In other structural problems the vector $\mathbf{a}(\mathbf{Z}, t)$ can be expressed, with good accuracy, by the Taylor's series expansion given in Eq. (174). It follows that, in these cases, substituting Eq. (174a) into Eq. (187) the differential equations governing the evaluation of the moments can be written as follows

$$\dot{\mathbf{m}}_r[\mathbf{Z}] = \mathbf{A}_r\,\mathbf{m}_r[\mathbf{Z}] + \sum_{l=2}^{N_1} \mathbf{A}_{r,\,l}\,\mathbf{m}_{r-1+l}[\mathbf{Z}] + \sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk}\left(\mathbf{m}_{r-k}[\mathbf{Z}] \otimes \mathbf{q}_k(t)\right) \tag{196}$$

where

$$\mathbf{A}_r = \mathbf{A}_{r-1} \otimes \mathbf{I}_{2n} + \mathbf{I}_{2n}^{[r-1]} \otimes \mathbf{A}^{(1)} \;; \quad \mathbf{A}_1 = \mathbf{A}^{(1)}$$
$$\mathbf{A}_{r,k} = \mathbf{Q}_{r,\,2n}\left(\mathbf{I}_{2n}^{[r-1]} \otimes \mathbf{A}^{(k)}\right). \tag{197}$$

Eq. (196) reveals a clear mathematical structure governing the moment equations of every order of the response of non-linear systems excited by external excitations. In particular this equation shows that the moment equation approach gives a linear infinite hierarchy of moment equations, in the sense that the equation of the r-th order moment involves moments of lesser and higher order. Notice that Eq. (196) becomes a very effective one, from a computational point of view, when the vector $\mathbf{a}(\mathbf{Z}, t)$ is expressed in a polynomial form.

Since for non-linear system the number of moments which we have to evaluate the exact probability density function is infinite and the numerical problems increase with the order r of the moment of the response, we have that, from a computational point of view, the solution of Eq. (196), which involves matrices of order $(2n)^r \times (2n)^r$ is very hard if r is big. In order to reduce the number of variables, neglect closure techniques are usually adopted.

The phylosophy of these techniques requires:

(i) express the probability density function (or alternatively the characteristic function) as a series expansion. If the Edgeworth series or the Gram-Charlier series is adopted we have respectively (see Appendix):

$$p_{\mathbf{Z}}(\mathbf{z}; t) = \left[1 + \sum_{j=3}^{\infty} \frac{1}{j!} \mathbf{b}_j^T [\mathbf{Z}] \left(\boldsymbol{\Sigma}_{\mathbf{Z}}^{-1/2}\right)^{[j]} \mathbf{H}_j\left(\boldsymbol{\Sigma}_{\mathbf{Z}}^{-1/2}(\mathbf{z} - \mathbf{m}_{\mathbf{Z}})\right)\right] p_{\mathbf{Z}}^0(\mathbf{z}; t) \tag{198}$$

$$p_{\mathbf{Z}}(\mathbf{z}; t) = \left[\sum_{j=0}^{\infty} \mathbf{h}_j^T [\mathbf{Z}] \; \mathbf{H}_j\left(\boldsymbol{\Sigma}_{\mathbf{Z}}^{-1/2}(\mathbf{z} - \mathbf{m}_{\mathbf{Z}})\right)\right] p_{\mathbf{Z}}^0(\mathbf{z}; t) \tag{199}$$

where $\boldsymbol{\Sigma}_{\mathbf{Z}}$ and $\mathbf{m}_{\mathbf{Z}}$ are respectively the covariance matrix and the mean value vector of the stochastic variable vector $\mathbf{Z}$; $\mathbf{H}_j(\cdot)$ are the multidimensional Hermite polynomials defined in Eq. (A.35) and $p_{\mathbf{Z}}^0(\mathbf{z}; t)$ is the normal multidimensional probability density function

$$p_{\mathbf{Z}}^0(\mathbf{z}; t) = \frac{1}{(2\pi)^{n/2} \, \mathrm{Det}\left(\boldsymbol{\Sigma}_{\mathbf{Z}}\right)^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{z} - \mathbf{m}_{\mathbf{Z}})^T \boldsymbol{\Sigma}_{\mathbf{Z}}^{-1} (\mathbf{z} - \mathbf{m}_{\mathbf{Z}})\right] \tag{200}$$

In Eq. (198) $\mathbf{b}_j[\mathbf{Z}]$ are the j-th quasi-moment vector, while in Eq. (199) $\mathbf{h}_j[\mathbf{Z}]$ are the multi-dimensional Hermite moments

$$\mathbf{h}_j[\mathbf{Z}] = \frac{1}{j!} \mathbf{C}_j = \frac{1}{j!} \mathrm{E}\left[\mathbf{H}_j\left(\boldsymbol{\Sigma}_{\mathbf{Z}}^{-1/2}(\mathbf{z} - \mathbf{m}_{\mathbf{Z}})\right)\right] \tag{201}$$

ii) truncate the expression series given in Eqs. (198) and (199) taking into account of the first N terms;

iii) relate the statistical moments $\mathbf{m}_r$, of order greater than N to statistical order of lesser order, by means of non-linear algebraic relationships between moments and quasi-moments and between moments and cumulants in the quasi-moments and cumulants neglect closure respectively;

iv) solve the non-linear set of differential moment equations by using the pseudo-force method described in Sec. 4.1.

Adopting the quasi-moment neglect closure of order N we relate the moments of order greater than N, with the lesser ones by means of the following relationship obtained from the (A.85)

$$\mathbf{m}_r = -(-1)^r \sum_{\substack{k=0,2\ (r=\text{even})\\ k=3,5\ (r=\text{odd})}}^{r-1} \widehat{\mathbf{B}}_{r,k}\left[\left(\mathbf{m}_2 - \mathbf{m}_1^{[r]}\right)^{[\alpha_{rk}]} \otimes \mathbf{m}_k\right]$$

$$-(-1)^r \sum_{\substack{k=0,2\ (r=\text{even})\\ k=3,5\ (r=\text{odd})}}^{r} \widehat{\mathbf{B}}_{r,k}\left\{\left[\left(\mathbf{m}_2 - \mathbf{m}_1^{[r]}\right)^{[\alpha_{rk}]}\right] \otimes \left[\frac{\mathbf{P}_{k,2n}}{k!} \sum_{j=1}^{k} (-1)^j \left(\frac{k!}{j!\,(k-j)!}\left(\mathbf{m}_{k-j} \otimes \mathbf{m}_1^{[j]}\right)\right)\right]\right\} \quad , \quad r > N \tag{202}$$

where $\alpha_{rk} = (r - k)/2$ and the matrices $\widehat{\mathbf{B}}_{r,k}$ are defined in Eq. (A.70).
In the cumulant neglect closure we have to set $\mathbf{k}_r = 0 \ \ \forall r > N$ obtaining from Eqs. (A.14) and (A.17)

$$\mathbf{m}_r = -\frac{\mathbf{P}_{r,2n}}{r!} \sum_{i=1}^{r-1} \frac{(r-1)!}{i!\,(r-1-i)!}\left(\mathbf{k}_{r-i} \otimes \widehat{\mathbf{k}}_i\right)\Big|_{\mathbf{k}_r = 0 \ \forall r > N} \quad ; \quad r > N \tag{203}$$

If the latter neglect closure is adopted we have to lay the quasi-moment and the cumulant by means of relationship (A.22) in order to obtain the Edgeworth series expression in terms of cumulants.
Once the neglect closure technique is chosen and the relationships between moments of order lesser and greater than N is established we can perform the numerical solution of Eq. (196) by using the procedure described in the Sect. 4.1 for deterministic input writing

$$\dot{\mathbf{m}}_r[Z] = \mathbf{A}_r \mathbf{m}_r[Z] + \mathbf{g}_{r,L} + \mathbf{g}_{r,NL} \tag{204}$$

where

$$\mathbf{g}_{r,L} = \sum_{k=2}^{min\,(r,\,s)} \mathbf{V}_{rk}(t)\left(\mathbf{m}_{r-k}[Z] \otimes \mathbf{q}_k(t)\right) + \sum_{k=2}^{N_1} \left(\mathbf{A}_{r,k}\,\mathbf{m}_{r-1+k}[Z]\right)\Big|_{r-1+k \,\le N}$$

$$\mathbf{g}_{r,NL} = \sum_{k=2}^{N_1} \left(\mathbf{A}_{r,k}\,\mathbf{m}_{r-1+k}[Z]\right)\Big|_{r-1+k>N} \tag{205}$$

The numerical solution of Eq. (204) can be easily evaluated because of between the fundamental matrix of Eq. (204) and the fundamental matrix of deterministic case by the following relationship holds

$$\boldsymbol{\Theta}_r(t) = \boldsymbol{\Theta}_1^{[r]}(t) \tag{206}$$

Eq. (204) reveals a perfect similarity to Eq. (176); it follows that the numerical solution of both equations can be performed in a similar way.

4.5 Hermite moment neglect closure technique by using the standardized variables

The quasi-moment and the cumulant neglect closure techniques are suitable for polynomials non-linearities or for non-linearities which can be expanded in Taylor's series. In the most general ca of non-linearities we have to evaluate the stochastic averages $E[\mathbf{Z}^{[r-1]} \otimes \mathbf{a}(\mathbf{Z}, t)]$ which appear in the moment differential equations (187) and (190). In order to do these averages by using the truncated Gram-Charlier expansion (199) we can write

$$E\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{a}(\mathbf{Z}, t)\right] = \boldsymbol{\psi}_{0,r-1}\left(\mathbf{m}_Z, \boldsymbol{\Sigma}_Z, t\right) + \sum_{j=3}^{N} \boldsymbol{\psi}_{j,r-1}^T\left(\mathbf{m}_Z, \boldsymbol{\Sigma}_Z, t\right) \mathbf{h}_j[\mathbf{Z}] \tag{2}$$

where

$$\boldsymbol{\psi}_{0,r-1}\left(\mathbf{m}_Z, \boldsymbol{\Sigma}_Z, t\right) = E^0\left[\mathbf{Z}^{[r-1]} \otimes \mathbf{a}(\mathbf{Z}, t)\right] =$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\mathbf{z}^{[r-1]} \otimes \mathbf{a}(\mathbf{z}, t)\right] p_Z^0(\mathbf{z}; t)\, dz_1 \ldots dz_{2n} \tag{208}$$

$$\boldsymbol{\psi}_{j,r-1}^T\left(\mathbf{m}_Z, \boldsymbol{\Sigma}_Z, t\right) = E^0\left[\left(\mathbf{Z}^{[r-1]} \otimes \mathbf{a}(\mathbf{Z}, t)\right) \mathbf{H}_j^T\left(\boldsymbol{\Sigma}_Z^{-1/2}\left(\mathbf{Z} - \mathbf{m}_Z\right)\right)\right] =$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\mathbf{z}^{[r-1]} \otimes \mathbf{a}(\mathbf{z}, t)\right] \mathbf{H}_j^T\left(\boldsymbol{\Sigma}_Z^{-1/2}\left(\mathbf{z} - \mathbf{m}_Z\right)\right) p_Z^0(\mathbf{z}; t)\, dz_1 \ldots dz_{2n}$$

where $E^0[\cdot]$ means stochastic average with normal probability density function.
By substituting Eq. (207) into Eq. (187) or (190) and by using the relationship between the statistical moments and the coefficients C_j, strictly related to the quasi-moments (see Eq. (A.68)) we have a set of 2n non-linear differential equations.

The main drawback of the quasi-moment, cumulant and Hermite moment neglect closure before decribed is due to the non-linearity of moment differential equations due to the neglect closure technique adopted. In Secs. 2.2 and 4 it has been shown that operating the quasi-moment or the Hermite moment neglect closure in terms of standardized variables we lead to a set of linear differential equations instead of a set of non-linear differential equations. Here we want to extend these techniques to multidimensional systems.

In order to show this we introduce the multidimensional modified standardized response process defined in Eq. (A.74), which is a vector of order 2n defined as follows (with argument omitted)

$$\hat{\mathbf{Z}}^* = \hat{\boldsymbol{\Sigma}}_Z^{-1/2} (\mathbf{Z} - \mathbf{m}_Z) \tag{209}$$

where $\mathbf{m}_Z$ is the mean value vector of actual response process $\mathbf{Z}(t)$ and $\hat{\boldsymbol{\Sigma}}_Z$ is a diagonal matrix whose elements are the diagonal elements of the covariance matrix $\boldsymbol{\Sigma}_Z$.

It follows that we can write the probability density function of the process $\mathbf{Z}(t)$ in the form

$$p_Z(\mathbf{z}; t) = J\left| \hat{\boldsymbol{\Sigma}}_Z^{-1/2} (\mathbf{Z} - \mathbf{m}_Z) \right| p_{\hat{\mathbf{Z}}^*}(\hat{\mathbf{z}}^*; t) = \left(\prod_{i=1}^{2n} \frac{1}{\sigma_{Z_i}} \right) p_{\hat{\mathbf{Z}}^*}(\hat{\mathbf{z}}^*; t) \tag{210}$$

where $J|\cdot|$ is the Jacobian of the translation and $\sigma_{Z_i}^2$ are the elements of the principal diagonal of the matrix $\boldsymbol{\Sigma}_Z$. Furthermore according to the modified Gram-Charlier expansion (see Appendix) we can write the non-Gaussian probability density function $p_{\hat{\mathbf{Z}}^*}(\hat{\mathbf{z}}^*; t)$ by means of the following expansion

$$p_{\hat{\mathbf{Z}}^*}(\hat{\mathbf{z}}^*; t) = \left(\sum_{j=0}^{\infty} \mathbf{H}_j^T(\hat{\mathbf{z}}^*)\, \mathbf{h}_j[\hat{\mathbf{Z}}^*] \right) p_{\hat{\mathbf{Z}}^*}^0(\hat{\mathbf{z}}^*; t) \tag{211}$$

where $\mathbf{H}(\cdot)$ is the multidimensional Hermite polynomials and $p_{\hat{\mathbf{Z}}^*}^0(\hat{\mathbf{z}}^*; t)$ is the multidimensional standardized normal probability density function of vector process $\hat{\mathbf{Z}}^*$ (of order 2n) given as

$$p_{\hat{\mathbf{Z}}^*}^0(\hat{\mathbf{z}}^*; t) = \frac{1}{(2\pi)^n} \exp\left[-\frac{1}{2} \hat{\mathbf{z}}^{*T} \hat{\mathbf{z}}^* \right] = \frac{1}{(2\pi)^n} \prod_{i=1}^{2n} \exp\left[-\frac{1}{2} (\hat{\mathbf{z}}^*)^2 \right] \tag{212}$$

In Eq. (211) the coefficients $\mathbf{h}_j[\mathbf{Z}^*]$ are the multidimensional Hermite moments which are strictly related to the coefficient $\hat{\mathbf{C}}_j$ of the modified standardized vector process by means of the following relationships

$$\mathbf{h}_j[\hat{\mathbf{Z}}^*] = \frac{1}{j!} \hat{\mathbf{C}}_j = \frac{1}{j!} E\left[\mathbf{H}_j(\hat{\mathbf{Z}}^*) \right] \tag{213}$$

By using these relationships we can write (see Appendix)

$$h_0\left[\hat{\mathbf{Z}}^*\right]=1\,;\;\mathbf{h}_1\left[\hat{\mathbf{Z}}^*\right]=0\,;\;\mathbf{h}_j\left[\hat{\mathbf{Z}}^*\right]=\frac{(-1)^j}{j!}\left\{\sum_{\substack{k=0,2\,(j=\text{even})\\k=3,5\,(j=\text{odd})}}^{j}\mathbf{B}_{j,k}\,\boldsymbol{\mu}_k\left[\hat{\mathbf{Z}}^*\right]\right\},\quad j>1 \tag{214}$$

where $\mathbf{B}_{j,k}$ $(j > k)$ are matrices of order $(2n)^j \times (2n)^k$ defined in Eq. (A.36) and $\boldsymbol{\mu}_k[\hat{\mathbf{Z}}^*]$ are the central moment vector of the modified standardized response process given in Eq. (210). Substituting Eqs. (211) and (214) into Eq. (210) we obtain the probability density function in terms of standardized variable vector as follows

$$p_{\mathbf{Z}}(\mathbf{z};t)=\left(\prod_{i=1}^{2n}\frac{1}{\sigma_{Z_i}}\right)p_{\mathbf{Z}^*}(\mathbf{z}^*,t)=$$

$$\left(\prod_{i=1}^{2n}\frac{1}{\sigma_{ij}}\right)\left\{1+\sum_{j=2}^{N}\left[\frac{(-1)^j}{j!}\mathbf{H}_j^{\mathrm{T}}(\hat{\mathbf{z}}^*)\left(\sum_{\substack{k=0,2\,(j=\text{even})\\k=3,5\,(j=\text{odd})}}^{j}\mathbf{B}_{j,k}\,\boldsymbol{\mu}_k[\hat{\mathbf{Z}}^*]\right)\right]\right\}p^{0}_{\mathbf{Z}^*}(\hat{\mathbf{z}}^*;t) \tag{215}$$

To evaluate $p_{\mathbf{Z}}(\mathbf{z};t)$ in the expansion given in Eq. (215) it needs to know $\mathbf{m}_{\mathbf{Z}}$, $\hat{\boldsymbol{\Sigma}}_{\mathbf{Z}}$ and the central moments $\boldsymbol{\mu}_k[\hat{\mathbf{Z}}^*]$.
Since the following relationships hold

$$\mathbf{m}_{\mathbf{Z}}(t)=\mathbf{m}_1[\mathbf{Z}]\,,\quad \mathrm{Vec}\left(\boldsymbol{\Sigma}_{\mathbf{Z}}(t)\right)=\boldsymbol{\mu}_2[\mathbf{Z}]=\mathbf{m}_2[\mathbf{Z}]-\mathbf{m}_1^{[2]}[\mathbf{Z}] \tag{216}$$

we can evaluate $\mathbf{m}_{\mathbf{Z}}(t)$ and $\hat{\boldsymbol{\Sigma}}_{\mathbf{Z}}(t)$ once the following differential equations are solved

$$\dot{\mathbf{m}}_1[\mathbf{Z}]=\mathrm{E}[\mathbf{a}(\mathbf{Z},t)] \tag{217a}$$

$$\dot{\mathbf{m}}_2[\mathbf{Z}]=\mathbf{Q}_{r,2n}\,\mathrm{E}[\mathbf{Z}\otimes\mathbf{a}(\mathbf{Z},t)]+\frac{1}{2}\mathbf{V}^{[2]}\,\mathbf{q}_2(t) \tag{217b}$$

For the standardized response process vector given in Eq. (209) the central moment vector are related to the moments as follows

$$\begin{aligned}
&\boldsymbol{\mu}_0\left[\hat{\mathbf{Z}}^*\right]=1\,,\quad \boldsymbol{\mu}_1\left[\hat{\mathbf{Z}}^*\right]=\mathbf{0}\,,\\
&\boldsymbol{\mu}_2\left[\hat{\mathbf{Z}}^*\right]=\left(\hat{\boldsymbol{\Sigma}}_{\mathbf{Z}}^{-1/2}\right)^{[2]}\left(\mathbf{m}_2[\mathbf{Z}]-\mathbf{m}_1^{[2]}[\mathbf{Z}]\right)\neq\mathrm{Vec}\left(\mathbf{I}_{2n}\right),\\
&\boldsymbol{\mu}_r\left[\hat{\mathbf{Z}}^*\right]=\mathbf{m}_r\left[\hat{\mathbf{Z}}^*\right]=\mathrm{E}\left\{\left[\hat{\boldsymbol{\Sigma}}_{\mathbf{Z}}^{-1/2}\left(\mathbf{Z}-\mathbf{m}_{\mathbf{Z}}\right)\right]^{[r]}\right\}\;;\quad r>2
\end{aligned} \tag{218}$$

It has to be emphasized that since $\hat{\Sigma}_Z$ is not the real covariance matrix we have $\mu_2[\hat{Z}^*] \neq \text{Vec}\,(I_{2n})$. It follows that , on the contrary of the one-dimensional case, $h_2[\hat{Z}^*] \neq 0$ and the lower limit of the summation of Eq. (215) starts from two.

In order to obtain the differential equations governing the evolution of the central moments we use Eq. (209) to write

$$\frac{d\mathbf{Z}}{dt} = \frac{d}{dt}\left(\hat{\Sigma}_Z^{1/2}\right)\hat{Z}^* + \hat{\Sigma}_Z^{1/2}\frac{d\hat{Z}^*}{dt} + \dot{m}_Z \tag{219}$$

It follows that taking into account Eq. (184) we have

$$d\hat{Z}^* = \hat{\Sigma}_Z^{-1/2}\left[a\left(\hat{\Sigma}_Z^{1/2}\hat{Z}^* + m_Z, t\right) - \frac{d}{dt}\left(\hat{\Sigma}_Z^{1/2}\right)\hat{Z}^* - \dot{m}_Z\right]dt + \hat{\Sigma}_Z^{-1/2}\,V(t)\,dL \tag{220}$$

and according to Eq. (185) we can write

$$\Delta\,(\hat{Z}^*)^{[r]} = \left[\nabla_{Z^*}^T \otimes (\hat{Z}^*)^{[r]}\right]d\hat{Z}^* + \sum_{k=2}^{min\,(r,\,s)}\frac{1}{k!}\left[\nabla_{Z^*}^T \otimes (\hat{Z}^*)^{[r]}\right]d(\hat{Z}^*)^{[k]} \tag{221}$$

By using the procedure described in Sec. 4.2 and neglecting infinitesimal of greater order than dt we obtain

$$\begin{aligned}\Delta\,(\hat{Z}^*)^{[r]} = {} & Q_{r,\,2n}\left[(\hat{Z}^*)^{[r-1]} \otimes \hat{\Sigma}_Z^{-1/2}\,a\left(\hat{\Sigma}_Z^{1/2}\hat{Z}^* + m_Z, t\right)\right]dt \\ & - Q_{r,\,2n}\left\{(\hat{Z}^*)^{[r-1]} \otimes \hat{\Sigma}_Z^{-1/2}\left[\frac{d}{dt}\left(\hat{\Sigma}_Z^{1/2}\right)\hat{Z}^* + \dot{m}_Z\right]\right\}dt \\ & + Q_{r,\,2n}\left[(\hat{Z}^*)^{[r-1]} \otimes \hat{\Sigma}_Z^{-1/2}\,V(t)\,dL\right] \\ & + \sum_{k=2}^{min\,(r,\,s)} S_{rk}(t)\left[(\hat{Z}^*)^{[r-k]} \otimes V^{[k]}(t)\right]dL^{[k]}\end{aligned} \tag{222}$$

Making the stochastic average of both sides of Eq. (222) and dividing by dt we have

$$\begin{aligned}\dot{\mu}_r\,[\hat{Z}^*] = {} & Q_{r,\,2n}\,E\left\{(\hat{Z}^*)^{[r-1]} \otimes \left[\hat{\Sigma}_Z^{-1/2}\,a\left(\hat{\Sigma}_Z^{1/2}\hat{Z}^* + m_Z, t\right)\right]\right\} - R_r'(t)\,\mu_r\,[\hat{Z}^*] \\ & - R_r''(t)\left(\mu_{r-1}\,[\hat{Z}^*] \otimes \dot{m}_Z\right) + \sum_{k=2}^{min\,(r,\,s)} S_{rk}(t)\left[\mu_{r-k}\,[\hat{Z}^*] \otimes \left(V^{[k]}\,q_k\right)\right];\quad r > 2\end{aligned} \tag{223}$$

where the matrices $S_{rk}(t)$, $R_r(t)$ and $R_r''(t)$ are defined as follows

$$\mathbf{S}_{rk}(t) = \frac{1}{k!}\mathbf{Q}_{r,\,2n}\left(\mathbf{Q}_{r-1,\,2n}\otimes\mathbf{I}_{2n}\right)\ldots\left(\mathbf{Q}_{r-k+1,\,2n}\otimes\mathbf{I}_{2n}^{[k-1]}\right)\left[\mathbf{I}_{2n}^{[r-k]}\otimes\left(\widehat{\boldsymbol{\Sigma}}_Z^{-1/2}(t)\right)^{[k]}\right]$$

$$\mathbf{R}_r'(t) = \mathbf{Q}_{r,\,2n}\left\{\mathbf{I}_{2n}^{[r-1]}\otimes\left[\widehat{\boldsymbol{\Sigma}}_Z^{-1/2}(t)\,\frac{d}{dt}\left(\widehat{\boldsymbol{\Sigma}}_Z^{1/2}(t)\right)\right]\right\} \tag{224}$$

$$\mathbf{R}_r''(t) = \mathbf{Q}_{r,\,2n}\left[\mathbf{I}_{2n}^{[r-1]}\otimes\widehat{\boldsymbol{\Sigma}}_Z^{-1/2}(t)\right]$$

Notice that to obtain the complete statistical solution it needs to associate Eqs. (217) to Eqs. (223). In order to express explicity the right-hand sides of Eq. (223) in terms of the central moments and to form a closed set of differential equation we have to evaluate

$$E\left\{(\widehat{\mathbf{Z}}^*)^{[r-1]}\otimes\left[\widehat{\boldsymbol{\Sigma}}_Z^{-1/2}(t)\,\mathbf{a}(\mathbf{Z},t)\right]\right\} = \left[\mathbf{I}_{2n}^{[r-1]}\otimes\widehat{\boldsymbol{\Sigma}}_Z^{-1/2}(t)\right]\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}p_{\widehat{\mathbf{Z}}^*}(\widehat{\mathbf{z}}^*;t)\left[(\widehat{\mathbf{z}}^*)^{[r-1]}\right.$$

$$\left.\otimes\,\mathbf{a}\left(\widehat{\boldsymbol{\Sigma}}_Z^{1/2}\,\widehat{\mathbf{z}}^*+\mathbf{m}_Z,t\right)\right]d\hat{z}_1^*\ldots d\hat{z}_{2n}^* \tag{225}$$

which by using Eq. (211) becomes

$$E\left\{(\widehat{\mathbf{Z}}^*)^{[r-1]}\otimes\left[\widehat{\boldsymbol{\Sigma}}_Z^{-1/2}(t)\,\mathbf{a}(\mathbf{Z},t)\right]\right\} = \boldsymbol{\eta}_{0,\,r-1}\left(\mathbf{m}_Z,\widehat{\boldsymbol{\Sigma}}_Z,t\right)$$

$$+\sum_{j=2}^{\infty}\boldsymbol{\eta}_{j,\,r-1}^T\left(\mathbf{m}_Z,\widehat{\boldsymbol{\Sigma}}_Z,t\right)\mathbf{h}_j\left[(\widehat{\mathbf{Z}}^*)\right] \tag{226}$$

where $\boldsymbol{\eta}_{0,\,r-1}(\cdot)$ and $\boldsymbol{\eta}_{j,\,r-1}(\cdot)$ are non-linear explicity function of $\widehat{\boldsymbol{\Sigma}}_Z$ and $\mathbf{m}_Z$, given as:

$$\boldsymbol{\eta}_{0,\,r-1}\left(\mathbf{m}_Z,\widehat{\boldsymbol{\Sigma}}_Z,t\right) = \left[\mathbf{I}_{2n}^{[r-1]}\otimes\widehat{\boldsymbol{\Sigma}}_Z^{-1/2}(t)\right]\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}p_{\widehat{\mathbf{Z}}^*}^{0}(\widehat{\mathbf{z}}^*;t)\left[(\widehat{\mathbf{z}}^*)^{[r-1]}\right.$$

$$\left.\otimes\,\mathbf{a}\left(\widehat{\boldsymbol{\Sigma}}_Z^{1/2}\,\widehat{\mathbf{z}}^*+\mathbf{m}_Z,t\right)\right]d\hat{z}_1^*\ldots d\hat{z}_{2n}^* \tag{227}$$

$$\boldsymbol{\eta}_{j,\,r-1}^T\left(\mathbf{m}_Z,\widehat{\boldsymbol{\Sigma}}_Z,t\right) = \left[\mathbf{I}_{2n}^{[r-1]}\otimes\widehat{\boldsymbol{\Sigma}}_Z^{-1/2}(t)\right]\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}p_{\widehat{\mathbf{Z}}^*}^{0}(\widehat{\mathbf{z}}^*;t)\left[(\widehat{\mathbf{z}}^*)^{[r-1]}\right.$$

$$\left.\otimes\,\mathbf{a}\left(\widehat{\boldsymbol{\Sigma}}_Z^{1/2}\,\widehat{\mathbf{z}}^*+\mathbf{m}_Z,t\right)\right]\mathbf{H}_j^T(\widehat{\mathbf{z}}^*)\,d\hat{z}_1^*\ldots d\hat{z}_{2n}^*$$

The integrals which appear into Eq. (227) can be seen as the stochastic average of non-linear functions with normal standardized probability density function. Substituting Eq. (226) into Eq. (223), taking into account of Eq. (214) and operating a Hermite moment neglect closure of order N, we have

$$\dot{\mu}_r[\hat{Z}^*] = Q_{r,\,2n}\left\{\eta_{0,\,r-1}\left(m_Z, \hat{\Sigma}_Z, t\right) + \sum_{j=2}^{N} \frac{(-1)^j}{j!} \eta^T_{j,\,r-1}\left(m_Z, \hat{\Sigma}_Z, t\right) \right.$$

$$\left. x\left[\sum_{\substack{k=0,\,2\ (j=\text{even})\\ k=3,\,5\ (j=\text{odd})}}^{j} B_{j,\,k}\,\mu_k[\hat{Z}^*]\right]\right\} - R'_r(t)\,\mu_r[\hat{Z}^*] \tag{228}$$

$$- R''_r(t)\left(\mu_{r-1}[\hat{Z}^*] \otimes \dot{m}_Z(t)\right) + \sum_{k=2}^{min\,(r,\,s)} S_{rk}(t)\left[\mu_{r-k}[\hat{Z}^*] \otimes \left(V^{[k]}\,q_k\right)\right] \;,\; 2 < r \le N$$

This equation after some algebra can be written in a more suitable form as follows

$$\dot{\mu}_r[\hat{Z}^*] = Q_{r,\,2n}\left[\eta_{0,\,r-1}\left(m_Z, \hat{\Sigma}_Z, t\right) + \sum_{j=0}^{N} \gamma^T_{j,\,r-1}\left(m_Z, \hat{\Sigma}_Z, t\right)\mu_j[\hat{Z}^*]\right]$$

$$- R'_r(t)\,\mu_r[\hat{Z}^*] - R''_r(t)\left(\mu_{r-1}[\hat{Z}^*] \otimes \dot{m}_Z(t)\right)$$

$$+ \sum_{k=2}^{min\,(r,\,s)} S_{rk}(t)\left[\mu_{r-k}[\hat{Z}^*] \otimes \left(V^{[k]}\,q_k\right)\right] \;,\; 2 < r \le N \tag{229}$$

where

$$\gamma^T_{j,\,r-1}\left(m_Z, \hat{\Sigma}_Z, t\right) = (-1)^j \sum_{\substack{k=2,\,4\ (j=\text{even})\\ k=3,\,5\ (j=\text{odd})}}^{N} \frac{1}{k!} \eta^T_{k,\,r-1}\left(m_Z, \hat{\Sigma}_Z, t\right) B_{k,j} \;;\; j > 2 \tag{230}$$

As stated in the one-dimensional case we have that the Hermite moment neglect closure here presented, gives a set of differential equations which is non-linear in $\mathbf{m}_Z(t)$ and $\mathbf{\Sigma}_Z(t)$ but linear in $\mu_r[Z^*]$. Since $\mathbf{m}_Z(t)$ and $\mathbf{\Sigma}_Z(t)$ depend on $\mathbf{m}_1[Z]$ and $\mathbf{m}_2[Z]$ which are unknown quantities whose evaluation depends on the probability density function $p_Z(z; t)$ strictly related, by means of the relationship (215), to $p_{\hat{Z}^*}(\hat{z}^*, t)$, we have the two step procedure solution described in Sec. 2.4, that is

(i) evaluate $\mathbf{m}_1[\mathbf{Z}]$ and $\mathbf{m}_2[\mathbf{Z}]$ by solving Eqs. (217) assuming a Gaussian neglect closure technique;

(ii) solve the set of linear differential Eq. (229) assuming for $\mathbf{m}_Z(t)$ and $\hat{\mathbf{\Sigma}}_Z(t)$ the functions evaluated by means of the Gaussian closure;

(iii) recalculate $\mathbf{m}_1[\mathbf{Z}]$ and $\mathbf{m}_2[\mathbf{Z}]$ by solving Eqs. (217) where

$$\mathrm{E}[\mathbf{a}(\mathbf{z},t)] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathbf{a}(\mathbf{z},t)\, p_Z(\mathbf{z};t)\, dz_1 \dots dz_{2n}\,;$$
$$\mathrm{E}[\mathbf{Z} \otimes \mathbf{a}(\mathbf{z};t)] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathbf{z} \otimes \mathbf{a}(\mathbf{z};t)\, p_Z(\mathbf{z};t)\, dz_1 \dots dz_{2n} \tag{231}$$

are evaluated by using for $p_Z(\mathbf{z};t)$ the expression (215);

(iv) solve the linear set of differential Eq. (229).

(v) come back to (iii) and iterate until the accuracy required is obtained.

It has to be emphasized that introducing the modified standardized variable we have to solve the set of $[2n + (2n)^2]$ non-linear differential Eqs. (217) and the set of $[\Sigma_{i=3}^{r} (2n)^i]$ linear differential Eqs. (229) instead of the set of $[\Sigma_{i=1}^{r} (2n)^i]$ non-linear differential equations. It is worth noting that because of it is very simple to evaluate the transition matrices of these differential equations it is possible to use the numerical procedure described in this Chapter. Alternatively, methods available in literature can be used (see e.g. Chapter four or Ref [52]).

Notice that for polynomials non-linearities or for non-linearities which can be expanded in Taylor's series the Hermite moment neglect closure leads to same results than the quasi-moments closure with standardized variables.

Furthermore for stationary input we have that the statistical moments are not time-dependent quantities, it follows that $\dot{\mathbf{E}}[\cdot] = \mathbf{0}$, $\dot{\mathbf{m}}_Z = \mathbf{0}$ and $d\left[\mathbf{\Sigma}_{Y/Z}(t)\right]/dt = 0$ and the differential Eqs. (217) and (229) become of the algebraic ones.

4.6 Stochastic linearization

Due to the same reasons arised in the Sec. 2.5, here this method is applied only in the case of purely external white noise excitations. So the non-linear systems considered are characterized by differential conditions of motion of the following type

$$d\mathbf{Z} = \mathbf{a}(\mathbf{Z},t)\,dt + \mathbf{v}(t)\,d\mathbf{B}(t) \tag{232}$$

Following the stochastic linearization method, Eq. (232) is replaced by the following linear one

$$d\mathbf{Z} = [\mathbf{A}(t)\,\mathbf{Z} + \mathbf{v}(t)]\,dt + \mathbf{V}(t)\,d\mathbf{B}(t) \tag{233}$$

where the components of the matrix $\mathbf{A}(t)$ and of the vector $\mathbf{v}(t)$ are choosen in such a way that the differences which arise passing form Eq. (232) to Eq. (233) are minimized in mean square value. These differences can be expressed by the following vector

$$\mathbf{e} = \mathbf{A}(t)\,\mathbf{Z} + \mathbf{v}(t) - \mathbf{a}\,(\mathbf{Z}, t) \tag{234}$$

and the minimization of their mean square values with respect to the components of $\mathbf{A}(t)$ and $\mathbf{v}(t)$ implies:

$$\nabla_A^T \otimes E\left[\mathbf{e}^T\,\mathbf{e}\right] = \mathbf{0} \tag{235}$$

$$\nabla_v^T \otimes E\left[\mathbf{e}^T\,\mathbf{e}\right] = \mathbf{0} \tag{236}$$

where ∇_A is a matricial differential operator whose (i, j)th element is the partial derivative with respect to the (i, j)th element of $\mathbf{A}(t)$ and ∇_v is a vectorial differential operator whose i-th element is the partial derivative with respect to the i-th element of $\mathbf{v}(t)$.
The solution of Eqs. (235) and (236) give the following expressions

$$\mathbf{A}(t) = \left\{E\left[\mathbf{a}\,(\mathbf{Z}, t)\,\mathbf{Z}^T\right] - E\left[\mathbf{a}\,(\mathbf{Z}, t)\right] E\,(\mathbf{Z}^T)\right\} \boldsymbol{\Sigma}_Z^{-1} \tag{237}$$

$$\mathbf{v}(t) = E\left[\mathbf{a}\,(\mathbf{Z}, t)\right] - \mathbf{A}(t)\,E\,[\mathbf{Z}] \tag{238}$$

As it can be seen from these relationships, the linearized terms depend on some statistical moments of the response that, now, can be considered as a Gaussian process. So, only the first two order moments need for the characterization of the response. It is easy to verify that these moments can be evaluated by means of an iterative method.

4.7 Application

As an application of non-linear system let us consider the following differential non-linear equation

$$\ddot{X} + 2\xi\omega\,\dot{X} + \omega^2\,X + \mu g\,\mathrm{sgn}\,(\dot{X}) = W(t) \tag{E.39}$$

which represents the equation of motion of a rigid structure with a Resilient-friction base isolator system (R-FBI) (Fig. E.7) [55, 56], subjected to a white noise excitation at its base. In Eq. (E.39) ω is the natural frequency of the rubber element of the base isolator, ξ is the damping ratio, g is the acceleration of gravity, μ is the dynamic friction coefficient and sgn $(\cdot)$ is the signum function.

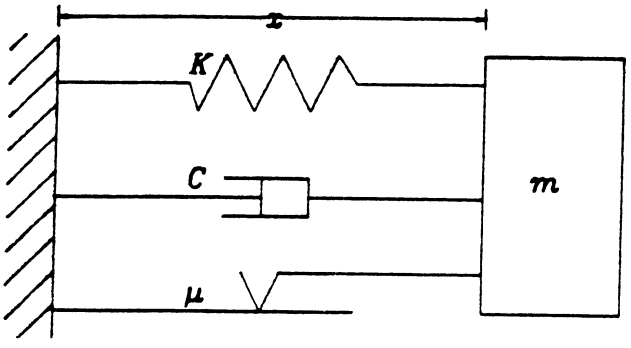

Fig. E.7 - Schematic diagram of R-FBI; $\omega^2 = k/m$, $2\xi\omega = c/m$.

According to the procedure before described, we write Eq. (E.39) as follows

$$\begin{aligned} \frac{dX}{dt} &= \dot{X} \\ \frac{d\dot{X}}{dt} &= -\omega^2 X - 2\xi\omega \dot{X} - \mu g \, \mathrm{sgn}(\dot{X}) + W(t) \end{aligned} \tag{E.40}$$

or in compact form

$$\dot{\mathbf{Z}} = \mathbf{A}_1 \mathbf{Z} + \mathbf{a}_2(\mathbf{Z}(t), t) + \mathbf{V} W(t) \tag{E.41}$$

where

$$\begin{aligned} \mathbf{Z} &= \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X \\ \dot{X} \end{bmatrix}; \quad \mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}; \\ \mathbf{a}_2(\mathbf{Z}(t), t) &= \begin{bmatrix} 0 \\ -\mu g \, \mathrm{sgn}(\mathbf{V}^T \mathbf{Z}) \end{bmatrix}; \quad \mathbf{V} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{E.42}$$

Because of the non-linearity is odd and the input process is a zero mean one, all odd moments of the response are zeros and the modified standardized variable vector can be defined as follows

$$\hat{\mathbf{Z}}^* = \begin{bmatrix} \hat{Z}_1^* \\ \hat{Z}_2^* \end{bmatrix} = \begin{bmatrix} X / \sigma_X \\ \dot{X} / \sigma_{\dot{X}} \end{bmatrix} = \hat{\boldsymbol{\Sigma}}_Z^{-1/2} \mathbf{Z}; \quad \hat{\boldsymbol{\Sigma}}_Z^{-1/2} = \begin{bmatrix} 1/\sigma_X & 0 \\ 0 & 1/\sigma_{\dot{X}} \end{bmatrix} \tag{E.43}$$

where $\sigma_X^2 = E[X^2]$ and $\sigma_{\dot{X}}^2 = E[\dot{X}^2]$. It follows that

$$\dot{Z}_1 = \frac{d}{dt}\left(\sigma_X \hat{Z}_1^*\right) = \dot{\sigma}_X \hat{Z}_1^* + \sigma_X \dot{\hat{Z}}_1^*$$

$$\dot{Z}_2 = \frac{d}{dt}\left(\sigma_{\dot{X}} \hat{Z}_2^*\right) = \dot{\sigma}_{\dot{X}} \hat{Z}_2^* + \sigma_{\dot{X}} \dot{\hat{Z}}_2^* \tag{E.44}$$

Substituting these relationships into Eqs. (E.40) where $X = Z_1$ and $\dot{X} = Z_2$ we obtain ofter very simple algebra

$$d\hat{Z}_1^* = \frac{\sigma_{\dot{X}}}{\sigma_X} \hat{Z}_2^* dt - \frac{\dot{\sigma}_X}{\sigma_X} \hat{Z}_1^* dt$$

$$d\hat{Z}_2^* = -\omega^2 \frac{\sigma_X}{\sigma_{\dot{X}}} \hat{Z}_1^* dt - 2\xi\omega \hat{Z}_2^* dt - \frac{\mu g}{\sigma_{\dot{X}}} \operatorname{sgn}\left(\hat{Z}_2^*\right) dt - \frac{\dot{\sigma}_{\dot{X}}}{\sigma_{\dot{X}}} \hat{Z}_2^* dt + \frac{1}{\sigma_{\dot{X}}} dL(t) \tag{E.45}$$

In order to evaluate the moments of the response we can particularize Eq. (223) for this case. However, since the Kronecker algebra is very efficient for a formal point of view and for systems having a lot of degree-of-freedom, in this case it is more convenient to evaluate the moment differential equations by using the traditional Ito rule. That is to expand the increment of a real valued scalar function of stochastic processes $\phi(\hat{Z}_1^*, \hat{Z}_2^*)$ into a Taylor series taking into account infinitesimal of order dt. It follow that we have

$$\Delta\phi\left(\hat{Z}_1^*, \hat{Z}_2^*\right) = d\phi\left(\hat{Z}_1^*, \hat{Z}_2^*\right) + \frac{1}{2!} d^2\phi\left(\hat{Z}_1^*, \hat{Z}_2^*\right) + \ldots$$

$$= \frac{\partial\phi}{\partial\hat{Z}_1^*} d\hat{Z}_1^* + \frac{\partial\phi}{\partial\hat{Z}_2^*} d\hat{Z}_2^* + \frac{1}{2}\left[\frac{\partial^2\phi}{\partial\hat{Z}_1^{*2}}\left(d\hat{Z}_1^*\right)^2 + 2\frac{\partial^2\phi}{\partial\hat{Z}_1^*\,\partial\hat{Z}_2^*} d\hat{Z}_1^*\, d\hat{Z}_2^* + \frac{\partial^2\phi}{\partial\hat{Z}_2^{*2}}\left(d\hat{Z}_2^*\right)^2\right] + \ldots \tag{E.46}$$

Substituting into this equation $d\hat{Z}_1^*$ and $d\hat{Z}_2^*$ given by Eq. (E.45), applying the stochastic average and remembering that dL is an infinitesimal of order $(dt)^{1/2}$ we can write the following differential equation where only infinitesimals of order dt appear

$$\dot{E}\left[\phi\left(\hat{Z}_1^*, \hat{Z}_2^*\right)\right] = E\left\{\frac{\partial\phi}{\partial\hat{Z}_1^*}\left[\frac{\sigma_{\dot{X}}}{\sigma_X}\hat{Z}_2^* - \frac{\dot{\sigma}_X}{\sigma_X}\hat{Z}_1^*\right]\right\}$$

$$+ E\left\{\frac{\partial\phi}{\partial\hat{Z}_2^*}\left[-\omega^2\frac{\sigma_X}{\sigma_{\dot{X}}}\hat{Z}_1^* - 2\xi\omega\hat{Z}_2^* - \frac{\mu g}{\sigma_{\dot{X}}}\operatorname{sgn}\left(\hat{Z}_2^*\right) - \frac{\dot{\sigma}_{\dot{X}}}{\sigma_{\dot{X}}}\hat{Z}_2^*\right]\right\} + E\left[\frac{\partial^2\phi}{\partial\hat{Z}_2^{*2}}\right]\frac{1}{\sigma_{\dot{X}}^2} q_2(t) \tag{E.47}$$

Generally we can write

$$\phi\left(\hat{Z}_1^*, \hat{Z}_2^*\right) = \frac{1}{\beta+1}\left(\hat{Z}_1^*\right)^{\alpha}\left(\hat{Z}_2^*\right)^{\beta+1} \tag{E.48}$$

It follows that in Eq. (E.47) the following averages appear

$$E\left[\hat{Z}_1^{*\alpha}\,\hat{Z}_2^{*\beta}\,\mathrm{sgn}\left(\hat{Z}_2^*\right)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{\hat{Z}_1^*\hat{Z}_2^*}\left(\hat{z}_1^*, \hat{z}_2^*; t\right)\hat{z}_1^{*\alpha}\,\hat{z}_2^{*\beta}\,\mathrm{sgn}\left(\hat{z}_2^*\right)d\hat{z}_1^*\,d\hat{z}_2^* \tag{E.49}$$

$p_{\hat{Z}_1^*\hat{Z}_2^*}\left(\hat{z}_1^*, \hat{z}_2^*; t\right)$ being the joint probability density function of the processes $\hat{Z}_1^*$ and $\hat{Z}_2^*$, that can be represented by the modified Gram-Charlier expression as follows (see Eq. (211))

$$p_{\hat{Z}_1^*\hat{Z}_2^*}\left(\hat{z}_1^*, \hat{z}_2^*; t\right) = p^0_{\hat{Z}_1^*}\left(\hat{z}_1^*; t\right)p^0_{\hat{Z}_2^*}\left(\hat{z}_2^*; t\right)\left[1 + \sum_{\substack{i,\,j\\ i+j=2}}^{\infty}\frac{1}{i!j!}\hat{C}_{ij}\,H_i\left[\hat{z}_1^*\right]H_j\left[\hat{z}_2^*\right]\right] \tag{E.50}$$

where $\hat{C}_{ij} = E\left\{H_i\left[\hat{Z}_1^*\right]H_j\left[\hat{Z}_2^*\right]\right\}$ and $H_k\left[\hat{z}^*\right]$ is the k-th Hermite polynomial (see Appendix). Substituting Eq. (E.50) into Eq. (E.49) we have

$$E\left[\hat{Z}_1^{*\alpha}\,\hat{Z}_2^{*\beta}\,\mathrm{sgn}\left(\hat{Z}_2^*\right)\right] = E^0\left[\hat{Z}_1^{*\alpha}\right]E^0\left[\hat{Z}_2^{*\beta}\,\mathrm{sgn}\left(\hat{Z}_2^*\right)\right]$$

$$+ \sum_{\substack{i,j\\ i+j=2}}^{\infty}\frac{1}{i!j!}\hat{C}_{ij}\,E^0\left\{\hat{Z}_1^{*\alpha}\,H_i\left[\hat{Z}_1^*\right]\right\}E^0\left\{\hat{Z}_2^{*\beta}\,H_j\left[\hat{Z}_2^*\right]\mathrm{sgn}\left(\hat{Z}_2^*\right)\right\} \tag{E.51}$$

where $E^0[\cdot]$ means stochastic average with normal one-dimensional probability density function $p^0_{Z^*}(\hat{z}^*; t)$. A method to evaluate these average has been recently proposed [57].
Notice that in order to apply the procedure described before we have to evaluate σ_X, $\sigma_{\dot{X}}$ and $E[X\ \dot{X}]$. In order to do this by applying the Ito rule to the function $\phi(Z_1, Z_2) = \phi(X, \dot{X})$ we can write the following differential equation

$$\dot{\sigma}_X^2 = 2\,E[X\ \dot{X}]$$

$$\dot{E}[X\ \dot{X}] = \sigma_{\dot{X}}^2 - \omega^2\sigma_X^2 - 2\,\xi\omega E[X\ \dot{X}] - \mu g\,E[X\,\mathrm{sgn}(\dot{X})] \tag{E.52}$$

$$\dot{\sigma}_{\dot{X}}^2 = -2\,\omega^2 E[X\ \dot{X}] - 4\,\xi\omega\sigma_{\dot{X}}^2 - 2\,\mu g\,E[\dot{X}\,\mathrm{sgn}(\dot{X})] + q_2(t)$$

According to the philosophy of the iterative procedure before obtained we can evaluate to the first iteration σ_X^2, $E[X\,\dot{X}]$ and $\sigma_{\dot{X}}^2$ by using a Gaussian closure. However to simplify the numerical procedure we can approximate the jointly Gaussian probability density function as follows

$$p^0_{X\dot{X}}(x, \dot{x}; t) \equiv p^0_X(x; t)\, p^0_{\dot{X}}(\dot{x}; t) \tag{E.53}$$

It follows that the quantities which appear in Eq. (E.52) become

$$\begin{aligned} &E[X \operatorname{sgn} \dot{X}] = 0; \\ &E[\dot{X} \operatorname{sgn}(\dot{X})] = \sigma_{\dot{X}} E^0[\dot{X} \operatorname{sgn}(\dot{X})] \\ &= \frac{1}{\sqrt{2}\,\pi\,\sigma_{\dot{X}}} \int_{-\infty}^{\infty} \dot{x} \operatorname{sgn}(\dot{X}) \exp\left[-\frac{\dot{x}^2}{2\sigma_{\dot{X}}^2}\right] d\dot{x} = \sigma_{\dot{X}} \sqrt{\frac{2}{\pi}} \end{aligned} \tag{E.54}$$

obtaining a set of non-linear differential equations. In the successive iteration we can evaluate $E[\dot{X} \operatorname{sgn}(\dot{X})]$ by using the modified Gram-Charlier expansion

$$p_{X\dot{X}}(x, \dot{x}; t) = \frac{1}{\sigma_X\,\sigma_{\dot{X}}} p^*_{\hat{Z}_1 \hat{Z}_2}\left(\hat{z}_1^*, \hat{z}_2^*; t\right) \tag{E.55}$$

where $p^*_{\hat{Z}_1 \hat{Z}_2}\left(\hat{z}_1^*, \hat{z}_2^*; t\right)$ is given in Eq. (E.50) obtaining

$$\begin{aligned} E[\dot{X} \operatorname{sgn}(\dot{X})] = \frac{1}{\sigma_X\,\sigma_{\dot{X}}} \Bigg\{ &\sigma_{\dot{X}} \sqrt{\frac{2}{\pi}} \\ &+ \sum_{\substack{i,j \\ i+j=2}}^{\infty} \frac{1}{i!j!} \hat{C}_{ij} E^0\left[H_i\left(\frac{X}{\sigma_X}\right)\right] E^0\left[\dot{X}\, H_j\left(\frac{\dot{X}}{\sigma_{\dot{X}}}\right) \operatorname{sgn}(\dot{X})\right] \Bigg\} \end{aligned} \tag{E.56}$$

As an example setting $\phi(\hat{Z}_1^*, \hat{Z}_2^*)$ equal to $\hat{Z}_1^{*4}$, $\hat{Z}_1^{*3}\hat{Z}_2^*$, $\hat{Z}_1^{*2}\hat{Z}_2^{*2}$, $\hat{Z}_1^*\hat{Z}_2^{*3}$, $\hat{Z}_2^{*4}$ we have all the fourth order moment differential equation.

If Hermite moment closure of the fourth order is adopted we have to assume $\hat{C}_{ij} = 0$ for $i+j > 4$. According to general theory presented in the previous Section we can relate $\hat{C}_{ij}$ for $i+j = 2, 4$ with the central moments of lower or equal order. It follows that Eq. (E.47) become a set of linear differential equation into the fourth order moments and non-linear σ_X and $\sigma_{\dot{X}}$, indeed by using the relationships (A.47) we can write

$$\hat{C}_{2\,0} = E\left[\hat{Z}_1^{*\,2}\right] - 1 = 0$$

$$\hat{C}_{1\,1} = E\left[\hat{Z}_1^{*}\,\hat{Z}_2^{*}\right]$$

$$\hat{C}_{0\,2} = E\left[\hat{Z}_1^{*\,2}\right] - 1 = 0$$

$$\hat{C}_{4\,0} = E\left[\hat{Z}_1^{*\,4}\right] - 6E\left[\hat{Z}_1^{*\,2}\right] + 3$$

$$\hat{C}_{3\,1} = E\left[\hat{Z}_1^{*\,3}\,\hat{Z}_2^{*}\right] - 3\left[\hat{Z}_1^{*}\,\hat{Z}_2^{*}\right] \tag{E.57}$$

$$\hat{C}_{2\,2} = E\left[\hat{Z}_1^{*\,2}\,\hat{Z}_2^{*\,2}\right] - E\left[\hat{Z}_1^{*\,2}\right] - E\left[\hat{Z}_2^{*\,2}\right] + 1$$

$$\hat{C}_{1\,3} = E\left[\hat{Z}_1^{*}\,\hat{Z}_2^{*\,3}\right] - 3\left[\hat{Z}_1^{*}\,\hat{Z}_2^{*}\right]$$

$$\hat{C}_{0\,4} = E\left[\hat{Z}_2^{*\,4}\right] - 6E\left[\hat{Z}_2^{*\,2}\right] + 3$$

where

$$E\left[\hat{Z}_1^{*\,2}\right] = 1\,;\quad E\left[\hat{Z}_1^{*}\,\hat{Z}_2^{*}\right] = \frac{1}{\sigma_X\,\sigma_{\dot{X}}}E\left[X\;\dot{X}\right]\,;\quad E\left[\hat{Z}_2^{*\,2}\right] = 1 \tag{E.58}$$

The some procedure can be applied if a sixth order Hermite moment neglect closure is adopted. In Fig. E.8, for a stationary white input process, the moments $E[X^2]$, $E[\dot{X}^2]$, $E[X^4]$, $E[X^2\,\dot{X}^2]$ and $E[\dot{X}^4]$ evaluated by means of a Gaussian closure, a fourth and sixth neglect are compared with the Monte Carlo simulation [57]. These Figures show that increasing the order of Hermite moment neglect closure increases the accuracy of the response.

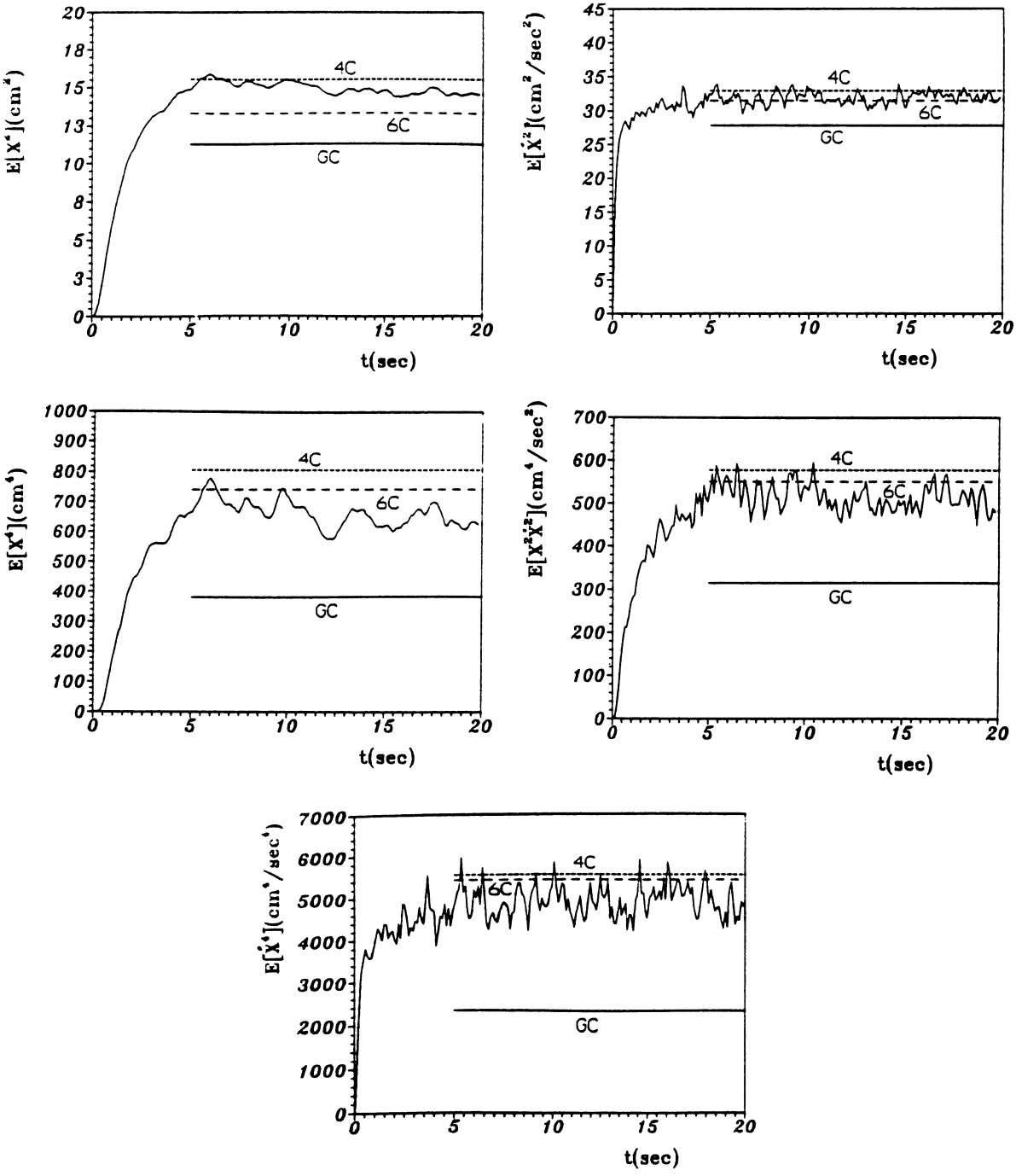

Fig. E.8 - Comparison of stationary moments of the response obtained by various order closure with Monte Carlo simulation (ξ = 0.1, S_0 = 55.44 cm²/sec³, μ = 0.04); Gaussian closure (GC, solid line), fourth order closure (4C, dottel line), sixth order closure (6C, dashed line).

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APPENDIX

A.1 Relationships between multidimensional moments and cumulants

Let Z(t) be a stochastic process; its probabilistic description at a fixed time t can be obtained by means of the knowledge of the *probability density function* $p_Z(z; t)$ or by its Fourier transform that is the so-called *characteristic function*, defined as

$$M_Z(\vartheta; t) = E[\exp(-i\,\vartheta\,Z)] = \int_{-\infty}^{\infty} p_Z(z; t)\, e^{-i\,\vartheta\,z}\, dz \tag{A.1}$$

where ϑ is a real parameter, i is the imaginary unit, and $E[\cdot]$ means stochastic average.
A Taylor expansion of the characteristic function gives

$$M_Z(\vartheta; t) = 1 + \sum_{j=1}^{\infty} \frac{(-i)^j}{j!}\, m_j[Z]\, \vartheta^j \tag{A.2}$$

where $m_j[Z]$ is the so-called *moment* (or statistical moment) of order j of the random variable Z(t), that is

$$m_j[Z] = E[Z^j] = \int_{-\infty}^{\infty} p_Z(z; t)\, z^j\, dz = \frac{1}{(-i)^j}\left[\frac{d^j M_Z(\vartheta; t)}{d\vartheta^j}\right]_{\vartheta=0} \tag{A.3}$$

A different representation of the characteristic function can be made by introducing the so-called *cumulants* or *semi-invariants* $k_j[Z]$ in the form

$$M_Z(\vartheta; t) = \exp\left(\sum_{j=1}^{\infty} \frac{(-i)^j}{j!}\, k_j[Z]\, \vartheta^j\right) \tag{A.4}$$

where $k_j[Z]$ is the j-th coefficient of the Taylor expansion of the *log-characteristic function*

$$k_j[Z] = \frac{1}{(-i)^j}\left[\frac{d^j\, ln\, M_Z(\vartheta; t)}{d\vartheta^j}\right]_{\vartheta=0} \tag{A.5}$$

moments and cumulants are related each other by means of non-linear algebraic recurrence relationship (with argument omitted),

$$
\begin{aligned}
m_1 &= k_1 \\
m_2 &= k_2 + k_1\, m_1 \\
m_3 &= k_3 + 2\, k_2\, m_1 + k_1\, m_2 \\
m_4 &= k_4 + 3\, k_3\, m_1 + 3\, k_2\, m_2 + k_1\, m_3 \\
m_5 &= k_5 + 4\, k_4\, m_1 + 6\, k_3\, m_2 + 4\, k_2\, m_3 + k_1\, m_4 \\
m_6 &= k_6 + 5\, k_5\, m_1 + 10\, k_4\, m_2 + 10\, k_3\, m_3 + 5\, k_2\, m_4 + k_1\, m_5 \\
&\vdots
\end{aligned}
\tag{A.6}
$$

where the coefficients in Eq. (A.6) are those of the Tartaglia triangle. From Eqs. (A.1), (A.2) and (A.4), it is evident that the complete probabilistic description of the stochastic process Z(t) at a fixed time t can be obtained indefferently by means of the knowledge of its probability density function or of its characteristic function or of its moments and/or cumulants of every order.
Many physical problems have small values of higher order cumulants and, for normal (Gaussian) processes they are exactly zero for order greater than two, thus in these cases the first cumulant $k_1[Z]$ (*mean value*) and the second cumulant $k_2[Z]$ (*variance*) fully define the stochastic process Z(t) from a probabilistic point of view at a fixed time t. The moments of order greater than two can be evaluated by means of the following relationships

$$
m_j = (j-1)\,(m_2 - m_1)^2\, m_{j-2} + m_1\, m_{j-1} \quad ; \quad j > 2 \tag{A.7}
$$

A stochastic vector process $\mathbf{Z}(t)$ can be characterized by means of the n-dimensional probability density function or by the n-dimensional characteristic function

$$
M_{\mathbf{Z}}(\boldsymbol{\vartheta}; t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-i\, \boldsymbol{\vartheta}^T \mathbf{z}\right) p_{\mathbf{Z}}(\mathbf{z}; t)\, dz_1\, dz_2 \cdots dz_s \tag{A.8}
$$

where $\boldsymbol{\vartheta}$ is an n-vector of real parameters (if n is the order of the vector $\mathbf{Z}$).
A Taylor expansion of the characteristic function gives

$$
M_{\mathbf{Z}}(\boldsymbol{\vartheta}; t) = 1 + \sum_{j=1}^{\infty} \frac{(-i)^j}{j!}\, \mathbf{m}_j^T[\mathbf{Z}]\, \boldsymbol{\vartheta}^{[j]} = 1 + \sum_{j=1}^{\infty} \frac{(-i)^j}{j!}\, \boldsymbol{\vartheta}^{[j]^T}\, \mathbf{m}_j[\mathbf{Z}] \tag{A.9}
$$

where $\mathbf{m}_j[\mathbf{Z}]$ is the vector of order n^j collecting all possible moments of order j of the random process Z_i (i = 1, 2, ... , n). In Eq. (A.9) the exponent into square brackets means Kronecker power and the symbol $\otimes$ means Kronecker product [33, 34] (see also Chapter two, Appendix A).
The vector of moments of order r of the vector $\mathbf{Z}$ can be obtained by the characteristic function as follows

$$\mathbf{m}_r[\mathbf{Z}] = \frac{1}{(-i)^r}\left[\nabla_{\vartheta}^{[r]} M_{\mathbf{Z}}(\vartheta; t)\right]_{\vartheta = \mathbf{0}} \tag{A.10}$$

where ∇_{ϑ} is the derivative vector defined as

$$\nabla_{\vartheta}^{T} = \left[\partial/\partial\vartheta_1 \;\; \partial/\partial\vartheta_2 \;\; \ldots \;\; \partial/\partial\vartheta_n\right] \tag{A.11}$$

A different representation of the characteristic function can be made by introducing the so-called cumulants or semi-invariants or order r, $\mathbf{k}_r[\mathbf{Z}]$. It follows that Eq. (A.9) can be rewritten in the form

$$M_{\mathbf{Z}}(\vartheta; t) = \exp\left(\sum_{j=1}^{\infty} \frac{(-i)^j}{j!} \mathbf{k}_j^T[\mathbf{Z}]\, \vartheta^{[j]}\right) = \exp\left(\sum_{j=1}^{\infty} \frac{(-i)^j}{j!} \vartheta^{[j]^T} \mathbf{k}_j[\mathbf{Z}]\right) \tag{A.12}$$

The entire set of cumulants of order r can be obtained in compact form as folows

$$\mathbf{k}_r[\mathbf{Z}] = \frac{1}{(-i)^r}\left[\nabla_{\vartheta}^{[r]} \ln M_{\mathbf{Z}}(\vartheta; t)\right]_{\vartheta = \mathbf{0}} \tag{A.13}$$

Inserting Eq. (A.12) into Eq. (A.10), after some very tedious algebra, we obtain the relationships between moments and cumulants in the following form [35]

$$\mathbf{m}_j[\mathbf{Z}] = \frac{1}{j!} \mathbf{P}_{j,n}\, \hat{\mathbf{k}}_j[\mathbf{Z}] \tag{A.14}$$

where $\mathbf{P}_{j,n}$ are singular (for $j > 1$) matrices of order $n^j \times n^j$ given as

$$\mathbf{P}_{1,n} = \mathbf{I}_n;\; \mathbf{P}_{2,n} = \mathbf{Q}_{2,n};\; \mathbf{P}_{j,n} = \mathbf{Q}_{j,n}\left(\mathbf{Q}_{j-1,n} \otimes \mathbf{I}_n\right) \ldots \left(\mathbf{Q}_2 \otimes \mathbf{I}_n^{[j-2]}\right),\; j > 3 \tag{A.15}$$

satisfying the following properties

$$\mathbf{P}_{j,n}^T = \mathbf{P}_{j,n};\; \mathbf{P}_{j,n}\, \mathbf{Z}^{[j]} = j!\, \mathbf{Z}^{[j]};\; \mathbf{P}_{j,n}\, \mathbf{k}_j = j!\, \mathbf{k}_j;\; \mathbf{P}_{j,n}\, \mathbf{D}_j = \mathbf{D}_j\, \mathbf{P}_{j,n} \tag{A.16}$$

In Eq. (A.14) $\hat{\mathbf{k}}_j[\mathbf{Z}]$ can be obtained by the following recursive relationships (with argument omitted)

$$\hat{\mathbf{k}}_j = \mathbf{k}_j + \sum_{r=1}^{j-1} \frac{(j-1)!}{r!\,(j-1-r)!}\left(\mathbf{k}_{j-r} \otimes \hat{\mathbf{k}}_r\right) \tag{A.17}$$

For example, the first seven terms of $\mathbf{k}_j$ can be written in explicit form as follows:

$$\begin{aligned}
\hat{\mathbf{k}}_1 &= \mathbf{k}_1 \; ; \; \hat{\mathbf{k}}_2 = \mathbf{k}_2 + \mathbf{k}_1 \otimes \hat{\mathbf{k}}_1 \\
\hat{\mathbf{k}}_3 &= \mathbf{k}_3 + 2\,\mathbf{k}_2 \otimes \hat{\mathbf{k}}_1 + \mathbf{k}_1 \otimes \hat{\mathbf{k}}_2 \\
\hat{\mathbf{k}}_4 &= \mathbf{k}_4 + 3\,\mathbf{k}_3 \otimes \hat{\mathbf{k}}_1 + 3\,\mathbf{k}_2 \otimes \hat{\mathbf{k}}_2 + \mathbf{k}_1 \otimes \hat{\mathbf{k}}_3 \\
\hat{\mathbf{k}}_5 &= \mathbf{k}_5 + 4\,\mathbf{k}_4 \otimes \hat{\mathbf{k}}_1 + 6\,\mathbf{k}_3 \otimes \hat{\mathbf{k}}_2 + 4\,\mathbf{k}_2 \otimes \hat{\mathbf{k}}_3 + \mathbf{k}_1 \otimes \hat{\mathbf{k}}_4 \\
\hat{\mathbf{k}}_6 &= \mathbf{k}_6 + 5\,\mathbf{k}_5 \otimes \hat{\mathbf{k}}_1 + 10\,\mathbf{k}_4 \otimes \hat{\mathbf{k}}_2 + 10\,\mathbf{k}_3 \otimes \hat{\mathbf{k}}_3 + 5\,\mathbf{k}_2 \otimes \hat{\mathbf{k}}_4 + \mathbf{k}_1 \otimes \hat{\mathbf{k}}_5 \\
\hat{\mathbf{k}}_7 &= \mathbf{k}_7 + 6\,\mathbf{k}_6 \otimes \hat{\mathbf{k}}_1 + 15\,\mathbf{k}_5 \otimes \hat{\mathbf{k}}_2 + 20\,\mathbf{k}_4 \otimes \hat{\mathbf{k}}_3 + 15\,\mathbf{k}_3 \otimes \hat{\mathbf{k}}_4 + 6\,\mathbf{k}_2 \otimes \hat{\mathbf{k}}_5 + \mathbf{k}_1 \otimes \hat{\mathbf{k}}_6
\end{aligned} \tag{A.18}$$

Notice that, since for the one-dimensional variable in which $\mathbf{P}_{j,\,n} = P_{j,\,1} = j!$ we have $m_j[Z] = \hat{k}_j[Z]$ and Eqs. (A.18) coincides with Eqs. (A.6).
For Gaussian input process we have $\mathbf{k}_j = 0 \;\; \forall j > 2$ and the vector $\hat{\mathbf{k}}_j$ can be evaluated in recursive form once $\mathbf{k}_1$ and $\mathbf{k}_2$ are known, that is

$$\hat{\mathbf{k}}_j = (j-1)\left(\mathbf{m}_2 - \mathbf{m}_1^{[2]}\right) \otimes \hat{\mathbf{k}}_{j-2} + \mathbf{m}_1 \otimes \hat{\mathbf{k}}_{j-1} \tag{A.19}$$

A.2 Relationships between multidimensional cumulants and quasi-moments

When the higher-order cumulants are different from zero, we have a non-Gaussian (or non-normal) random process. In this case we must take account of terms for which $j > 2$ in Eqs. (A.4) and (A.12). Although the characteristic function of a non-Gaussian process can immediately be written as an expansion form in terms of moments or cumulants, the calculation of the corresponding probability densities is diffucult since, in general, for $j > 2$ the Fourier transform of Eqs. (A.4) and (A.12) cannot be calculated directly.
In order to simplify the problem of finding the multidimensional distributions of a non-Gaussian process Kuznetsov et al. [58] introduced the quasi-moments functions. According to Stratonovich [59], these functions occupy an intermediate role between the cumulant and moment functions.
In the one dimensional and multidimensional cases respectively, the quasi-moments $\mathbf{b}_j[\mathbf{Z}]$ are related to the cumulants through the characteristic function

$$\exp\left(\sum_{j=3}^{\infty} \frac{(-i)^j}{j!}\, k_j[Z]\, \vartheta^j\right) = 1 + \sum_{j=3}^{\infty} \frac{(-i)^j}{j!}\, b_j[Z]\, \vartheta^j \tag{A.20}$$

and

$$\exp\left(\sum_{j=3}^{\infty}\frac{(-i)^j}{j!}\,\vartheta^{[j]\,T}\,\mathbf{k}_j\,[Z]\right)=1+\sum_{j=3}^{\infty}\frac{(-i)^j}{j!}\,\vartheta^{[j]\,T}\,\mathbf{b}_j\,[Z] \tag{A.21}$$

Formula (A.21) allows us to find explicit formulas relating the cumulants and the quasi-moments. Indeed, comparing this equation with Eq. (A.9) we have that these two equations are the same if we set $\mathbf{m}_1[Z]=\mathbf{0}$ and $\mathbf{m}_2[Z]=\mathbf{0}$. It follows that we can use Eq. (A.14) where $\mathbf{m}_1[Z]=\mathbf{k}_1[Z]=\widehat{\mathbf{k}}_1[Z]=\mathbf{0}$ and $\mathbf{m}_2[Z]=\widehat{\mathbf{k}}_2[Z]=\mathbf{k}_2[Z]=\mathbf{0}$ in order to establish the relationship between cumulants and quasi-moments in the form:

$$\mathbf{b}_j=\mathbf{k}_j+\frac{1}{j!}\,\mathbf{P}_{j,\,n}\left[\sum_{r=3}^{j-3}\frac{(j-1)!}{r!\,(j-r-1)!}\left(\mathbf{k}_{j-r}\otimes\widehat{\mathbf{k}}'_r\right)\right],\quad j>2 \tag{A.22}$$

where $\widehat{\mathbf{k}}'_r$ is evaluated by Eqs. (A.17) for $\widehat{\mathbf{k}}_1=\mathbf{0}$ and $\widehat{\mathbf{k}}_2=\mathbf{0}$.
Writing Eq. (A.22) in explicit form we observe that the difference between cumulants and quasi-moments begins with $\mathbf{b}_6$, that is

$$\begin{aligned}
&\mathbf{b}_3=\mathbf{k}_3;\quad \mathbf{b}_4=\mathbf{k}_4;\quad \mathbf{b}_5=\mathbf{k}_5\\
&\mathbf{b}_6=\mathbf{k}_6+\frac{10}{6!}\,\mathbf{P}_{6,\,n}\,\mathbf{k}_3^{[2]};\quad \mathbf{b}_7=\mathbf{k}_7+\frac{1}{7!}\,\mathbf{P}_{7,\,n}\left[20\,\mathbf{k}_4\otimes\mathbf{k}_3+15\,\mathbf{k}_3\otimes\mathbf{k}_4\right]\\
&\mathbf{b}_8=\mathbf{k}_8+\frac{1}{8!}\,\mathbf{P}_{8,\,n}\left[35\,\mathbf{k}_5\otimes\mathbf{k}_3+35\,\mathbf{k}_4^{[2]}+21\,\mathbf{k}_3\otimes\mathbf{k}_5\right]\\
&\mathbf{b}_9=\mathbf{k}_9+\frac{1}{9!}\,\mathbf{P}_{9,\,n}\left[56\,\mathbf{k}_6\otimes\mathbf{k}_3+70\,\mathbf{k}_5\otimes\mathbf{k}_4+56\,\mathbf{k}_4\otimes\mathbf{k}_5+28\,\mathbf{k}_3\otimes\left(\mathbf{k}_6+10\,\mathbf{k}_3^{[2]}\right)\right]\\
&\mathbf{b}_{10}=\mathbf{k}_{10}+\frac{1}{10!}\,\mathbf{P}_{10,\,n}\left[84\,\mathbf{k}_7\otimes\mathbf{k}_3+126\,\mathbf{k}_6\otimes\mathbf{k}_4+126\,\mathbf{k}_5^{[2]}+84\,\mathbf{k}_4\otimes\left(\mathbf{k}_6+10\,\mathbf{k}_3^{[2]}\right)\right.\\
&\qquad\left.+36\,\mathbf{k}_3\otimes\left(\mathbf{k}_7+20\,\mathbf{k}_4\otimes\mathbf{k}_3+15\,\mathbf{k}_3\otimes\mathbf{k}_4\right)\right]
\end{aligned} \tag{A.23}$$

Notice that these relationships are true for the one-dimensional variables in which $\mathbf{P}_{j,\,n}=\mathbf{P}_{j,\,1}=j!$ and the Kronecker product becomes the algebraic product.
By using Eq. (A.21) we have that the complete expression of the characteristic function in terms of the quasi-moments can be written as follows:

$$M_Z(\vartheta;t)=\left(1+\sum_{j=3}^{\infty}\frac{(-i)^j}{j!}\,\vartheta^{[j]\,T}\,\mathbf{b}_j[Z]\right)M_Z^0(\vartheta;t) \tag{A.24}$$

where $M_Z^0(\vartheta; t)$ is the characteristic function of a Gaussian process having $\mathbf{k}_1[Z]$ and $\mathbf{k}_2[Z]$ as the first two cumulants vectors, that is

$$M_Z^0(\vartheta; t) = \exp\left(-i\,\vartheta^T \mathbf{k}_1[Z] - \frac{1}{2}\vartheta^{[2]T} \mathbf{k}_2[Z]\right) \tag{A.25}$$

To calculate the n-dimensional probability density function we have to take the inverse Fourier transform of the characteristic function (A.25)

$$p_Z(z; t) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(i\,\vartheta^T Z)$$

$$x\left[1 + \sum_{j=3}^{\infty} \frac{(-i)^j}{j!}\vartheta^{[j]T}\mathbf{b}_j[Z]\right] M_Z^0(\vartheta; t)\, d\vartheta_1 \ldots d\vartheta_n \tag{A.26}$$

Observing that

$$\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(i\,\vartheta^T Z)\,(i)^j\,\vartheta^{[j]T}\mathbf{b}_j[Z]\,M_Z^0(\vartheta; t)\, d\vartheta_1 \ldots d\vartheta_n$$

$$= \mathbf{b}_j^T[Z]\left[\nabla_z^{[j]}\, p_Z^0(z; t)\right] \tag{A.27}$$

where ∇_Z is the differential operator vector and $p_Z^0(z, t)$ is the probability density of a Gaussian process with the same function $\mathbf{k}_1[Z]$ and $\mathbf{k}_2[Z]$ as the original non-Gaussian process, that is

$$p_Z^0(z; t) = \frac{1}{(2\pi)^{n/2}\ \mathrm{Det}\left(\Sigma_Z\right)^{1/2}} \exp\left[-\frac{1}{2}(z - \mathbf{m}_Z)^T \Sigma_Z^{-1}(z - \mathbf{m}_Z)\right] \tag{A.28}$$

where $\mathbf{m}_Z$ is the mean value vector and Σ_Z is the covariance function matrix such that

$$\mathbf{m}_Z = \mathbf{k}_1[Z]\,; \quad \mathrm{Vec}\left(\Sigma_Z\right) = \mathbf{k}_2[Z] \tag{A.29}$$

Interchanging the operations of summation and integration in the Eq. (A.26) and using Eq. (A.27) we obtain:

$$p_Z(z; t) = \left[1 + \sum_{j=3}^{\infty} \frac{(-1)^j}{j!}\mathbf{b}_j^T[Z]\,\nabla_z^{[j]}\right] p_Z^0(z; t) \tag{A.30}$$

A.3 Edgeworth asymptotic expression of the probability density function

In order to obtain a very suitable asymptotic expansion of the non-Gaussian probability density function we introduce the multidimensional Hermite polynomials.
The multidimensional Hermite polynomials can be obtained by using extensively the Kronecker differentiation rule starting from the following relationship:

$$\mathbf{H}_j(\mathbf{z}^*) = (-1)^j \exp\left(\frac{1}{2}\mathbf{z}^{*T}\mathbf{z}\right) \nabla_{\mathbf{z}^*}^{[j]} \exp\left(-\frac{1}{2}\mathbf{z}^{*T}\mathbf{z}^*\right) \tag{A.31}$$

where $\mathbf{z}^*$ is the standardized variable vector

$$\mathbf{z}^* = \boldsymbol{\Sigma}_{\mathbf{Z}}^{-1/2}\left(\mathbf{z} - \mathbf{m}_{\mathbf{Z}}\right) \tag{A.32}$$

By using Kronecker differentiation law we obtain the following recursive relationships

$$\mathbf{H}_j(\mathbf{z}^*) = \mathbf{z}^* \otimes \mathbf{H}_{j-1}(\mathbf{z}^*) - \nabla_{\mathbf{z}^*} \otimes \mathbf{H}_{j-1}(\mathbf{z}^*) \quad ; \quad j > 2 \tag{A.33}$$

where

$$\mathbf{H}_0(\mathbf{z}^*) = 1\,; \quad \mathbf{H}_1(\mathbf{z}^*) = \mathbf{z}^*; \quad \mathbf{H}_2(\mathbf{z}^*) = \mathbf{z}^{*[2]} - \mathrm{Vec}\left(\mathbf{I}_n\right) \tag{A.34}$$

By using extensively the Kronecker differentiation we can write the multidimensional Hermite polynomials given in Eq. (A.33) in an alternative form as follows [8]

$$\mathbf{H}_j(\mathbf{z}^*) = (-1)^j \sum_{\substack{k=0,2\ (j=\text{even}) \\ k=1,3\ (j=\text{odd})}}^{j} \mathbf{B}_{j,k}\, \mathbf{z}^{*[k]} \tag{A.35}$$

where $\mathbf{H}_j(\mathbf{z}^*)$ is a vector of order n^j and $\mathbf{B}_{j,k}$ are matrices of order $n^j \times n^k$ which can be evaluated in recursive form as follows

$$\mathbf{B}_{j,j} = (-1)^j \mathbf{I}_n^{[j]}\,; \quad \mathbf{B}_{j,k} = \left(\mathbf{I}_n \otimes \mathbf{B}_{j-1,k+1}\right)\mathbf{R}_{k+1} - \left(\mathbf{I}_n \otimes \mathbf{B}_{j-1,k-1}\right) \tag{A.36}$$

In the latter equation (true for $j > k$) the matrix $\mathbf{R}_j$ is given as

$$\mathbf{R}_k = \hat{\mathbf{R}}_k\left[\mathrm{Vec}\left(\mathbf{I}_n\right) \otimes \mathbf{I}_n^{[k-1]}\right] \tag{A.37}$$

$$\hat{\mathbf{R}}_k = \mathbf{I}_n \otimes \left(\mathbf{I}_n^{[k]} + \mathbf{E}_{n,n} \otimes \mathbf{I}_n^{[k-2]} + \mathbf{E}_{n,n^2} \otimes \mathbf{I}_n^{[k-3]} + \ldots + \mathbf{E}_{n,n^{k-1}} \right) \tag{A.38}$$

Once the multidimensional Hermite polynomials are introduced, observing that

$$\nabla_Z^{[j]} p_Z^0(z;t) = \frac{1}{\mathrm{Det}\left(\Sigma_Z^{-1/2}\right)} \left(\Sigma_Z^{-1/2}\right)^{[j]} \nabla_{z^*}^{[j]} p_{Z^*}^0(z^*;t) \tag{A.39}$$

where $p_{Z^*}^0(z^*, t)$ is the Gaussian probability density function of the standardized variable process $\mathbf{Z}^*$ given as

$$p_{Z^*}^0(z^*, t) = \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2} z^{*T} z^*\right] \tag{A.40}$$

and by using the relationship (A.31) between the Hermite polynomials and the probability density function of a Gaussian standardized variable, we can write

$$\nabla_Z^{[j]} p_Z^0(z;t) = \frac{1}{\mathrm{Det}\left(\Sigma_Z^{-1/2}\right)} (-1)^j \left(\Sigma_Z^{-1/2}\right)^{[j]} \mathbf{H}_j(z^*)\, p_{Z^*}^0(z^*;t) \tag{A.41}$$

Substituting this relationship into Eq. (A.30) we obtain the following multidimensional series expansion

$$p_Z(z;t) = \frac{1}{\mathrm{Det}\left(\Sigma_Z^{-1/2}\right)} \left[1 + \sum_{j=3}^{\infty} \frac{1}{j!} \mathbf{b}_j^T [Z] \left(\Sigma_Z^{-1/2}\right)^{[j]} \mathbf{H}_j(z^*)\right] p_{Z^*}^0(z^*;t)$$

$$= \left[1 + \sum_{j=3}^{\infty} \frac{1}{j!} \mathbf{b}_j^T [Z] \left(\Sigma_Z^{-1/2}\right)^{[j]} \mathbf{H}_j(z^*)\right] p_Z^0(z;t) \tag{A.42}$$

We want to emphasize that, as just seen, the quasi-moments function are the coefficients in an expansion of the non-Gaussian probability density in an infinite series of Hermite polynomials. Furthermore it can be shown that by using the series given in Eq. (A.42) we have

$$E[\mathbf{Z}] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathbf{z}\, p_{\mathbf{Z}}(\mathbf{z}; t)\, dz_1 \dots dz_n = \mathbf{k}_1[\mathbf{Z}]$$

$$E\left[\mathbf{Z}^{[2]}\right] = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mathbf{z}^{[2]}\, p_{\mathbf{Z}}(\mathbf{z}; t)\, dz_1 \dots dz_n = \mathbf{k}_2[\mathbf{Z}] + (\mathbf{k}_1[\mathbf{Z}])^{[2]} \tag{A.43}$$

The series which appears in Eq. (A.42) can be seen as the extension to multidimensional case of the one-dimensional series [59]. Indeed in the one-dimensional case the expressions (A.42) become

$$p_Z(z; t) = \frac{1}{\sigma_Z}\left[1 + \sum_{j=3}^{\infty} \frac{1}{j!} \frac{b_j[Z]}{\sigma_Z^j} H_j(z^*)\right] p_{Z^*}^0(z^*; t)$$

$$= \left[1 + \sum_{j=3}^{\infty} \frac{1}{j!} \frac{b_j[Z]}{\sigma_Z^j} H_j(z^*)\right] p_Z^0(z; t) \tag{A.44}$$

where σ_Z is the standard deviation and $H_j(z^*)$ is the well known Hermite polynomials which can be obtained by means of the following relationships derived from Eq. (A.35)

$$H_j(z^*) = (-1)^j \sum_{\substack{k = 0, 2\ (j = \text{even}) \\ k = 1, 3\ (j = \text{odd})}}^{j} B_{j,k}\, (z^*)^k \tag{A.45}$$

where

$$B_{j,j} = (-1)^j; \quad B_{j,k} = (k+1)\, B_{j-1,k+1} - B_{j-1,k-1} \tag{A.46}$$

Writing in explicit form these quantities[1] we have that the first five one-dimensional Hermite polynomials are given as follows

$$H_0(z^*) = 1; \quad H_1(z^*) = z^*; \quad H_2(z^*) = (z^*)^2 - 1$$

$$H_3(z^*) = (z^*)^3 - 3z^*; \quad H_4(z^*) = (z^*)^4 - 6(z^*)^2 + 3 \tag{A.47}$$

$$H_5(z^*) = (z^*)^5 - 10(z^*)^3 + 15$$

They satisfy the differentiation law

$$\frac{d}{dz^*} H_j(z^*) = j H_{j-1}(z^*) \tag{A.48}$$

and the recurrence relation

$$H_{j+1}(z^*) = z^* H_j(z^*) - j H_{j-1}(z^*) \tag{A.49}$$

Cramer [60] stated that rearrangering in a different way the Edgeworth expression we have that the accuracy increases quicly with the natural order of the terms and be proposed the following rearranged Edgeworth expansion

$$\begin{aligned} p_Z(z;t) &= \left[1 + \sum_{j=3}^{\infty} \frac{1}{j!} \frac{b_j}{\sigma_Z^j} H_j(z^*)\right] p_Z^0(z;t) \\ &= \left\{1 + \left[\frac{1}{3!}\frac{k_3}{\sigma_Z^3} H_3(z^*)\right] + \left[\frac{1}{4!}\frac{k_4}{\sigma_Z^4} H_4(z^*) + \frac{10}{6!}\frac{k_3^2}{\sigma_Z^6} H_6(z^*)\right]\right. \\ &\left. + \left[\frac{1}{5!}\frac{k_5}{\sigma_Z^5} H_5(z^*) + \frac{35}{7!}\frac{k_3 k_4}{\sigma_Z^7} H_7(z^*) + \frac{280}{9!}\frac{k_3^3}{\sigma_Z^9} H_9(z^*)\right] + \ldots\right\} p_Z^0(z;t) \end{aligned} \tag{A.50}$$

where k_i are the cumulants related to the quasi-moments by means of relationships (A.23) and the square brackets [·] denote terms of the some order of the expansion.

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(1)

$B_{0,0} = 1$; $B_{1,1} = -1$; $B_{2,2} = 1$; $B_{2,0} = -1$; $B_{3,3} = -1$; $B_{3,1} = 3$

$B_{4,4} = 1$; $B_{4,2} = -6$; $B_{4,0} = 3$; $B_{5,5} = -1$; $B_{5,3} = 10$; $B_{5,1} = -15$

$B_{6,6} = 1$; $B_{6,4} = -15$; $B_{6,2} = 45$; $B_{6,0} = -15$; $B_{7,7} = -1$; $B_{7,5} = 21$; $B_{7,3} = -105$; $B_{7,1} = 105$;

$B_{8,8} = 1$; $B_{8,6} = -28$; $B_{8,4} = 210$; $B_{8,2} = -420$; $B_{8,0} = 105$;

$B_{9,9} = -1$; $B_{9,7} = 36$; $B_{9,5} = -378$; $B_{9,3} = 1260$; $B_{9,1} = -945$;

$B_{10,10} = 1$; $B_{10,8} = -45$; $B_{10,6} = 630$; $B_{10,4} = -3150$; $B_{10,2} = 4275$; $B_{10,0} = -945$

A.4 Gram-Charlier asymptotic expansion of the probability density function

The one-dimensional non-Gaussian probability density function may be expresses by an orthogonal expansion, in terms of the derivatives of the normal density function, known as the Gram-Charlier expression [60].

$$p_Z(z;t) = \sum_{j=0}^{\infty} \frac{\sigma_Z^j}{j!} C_j \frac{d^j}{dz^j} p_Z^0(z;t) \tag{A.51}$$

where $p_Z^0(z;t)$ is a normal distribution, that is

$$p_Z^0(z;t) = \frac{1}{\sigma_Z \sqrt{2\pi}} \exp\left(-\frac{(z-m_Z)^2}{2\sigma_Z^2}\right) \tag{A.52}$$

By using the relationship between the Hermite polynomials and the normal density we can represent the non-normal distribution as a truncated Gram-Charlier expansion [61].

$$p_Z(z;t) = \left[1 + \sum_{j=3}^{N} \frac{C_j}{j!} H_j\left(\frac{z-m_Z}{\sigma_Z}\right)\right] p_Z^0(z;t) \tag{A.53}$$

where $H_j(z^*) = H_j\left((z-m_Z)/\sigma_Z\right)$ is the Hermite polynomial of order j introduced in the previous section. Eq. (A.53) can be interpreted as an expansion of a non-Gaussian probability density function as a series of a normal probability density. Both probability densities have unit area, mean m_Z and variance σ_Z^2 independently of the choice of the parameters C_j. This is the result of the following ortogonality relationship between Hermite polynomials and the standard normal distribution

$$\frac{1}{j!\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-(z^*)^2/2\right] H_j(Z^*) H_k(z^*)\, dz^* = \delta_{jk} \tag{A.54}$$

where δ_{jk} is the Kronecker delta and z^* is the standardized variable $z^* = (z - m_Z)/\sigma_Z$. Because of the ortogonality relation (A.54), the coefficients C_j can be evaluated as expectations of Hermite polynomials [61]

$$C_j = E\left[H_j(Z^*)\right] = E\left[H_j\left(\frac{Z-m_Z}{\sigma_Z}\right)\right] \tag{A.55}$$

By using for $H_j(Z^*)$ the expression given in Eq. (A.45) we obtain the relationship between the central moments $\mu_j[Z] = E[(Z - m_Z)^j]$ and the coefficients C_j

$$C_j = (-1)^j \sum_{\substack{k=0,2\ (j=\text{even}) \\ k=1,3\ (j=\text{odd})}}^{j} \sigma_Z^{-k} B_{j,k}\, \mu_k[Z] = (-1)^j \sum_{\substack{k=0,2\ (j=\text{even}) \\ k=1,3\ (j=\text{odd})}}^{j} B_{j,k}\, \mu_k[Z^*] \qquad (A.56)$$

where $\mu_k[Z^*] = E[((Z - m_Z)/\sigma_Z)^k]$.
Observing that $\mu_0[Z] = 1$; $\mu_1[Z] = 0$; $\mu_2[Z] = \sigma_Z^2$, we can write the first relationships (A.56) in explicit form as follows

$$C_0 = 1\ ;\ C_1 = C_2 = 0\ ;\ C_3 = \frac{\mu_3[Z]}{\sigma_Z^3}\ ;\ C_4 = \frac{\mu_4[Z]}{\sigma_Z^4} - 3\ ;\ C_5 = \frac{\mu_5[Z]}{\sigma_Z^5} - 10\frac{\mu_3[Z]}{\sigma_Z^3}$$

$$C_6 = \frac{\mu_6[Z]}{\sigma_Z^6} - 15\frac{\mu_4[Z]}{\sigma_Z^4} + 30\ ;\ C_7 = \frac{\mu_7[Z]}{\sigma_Z^7} - 21\frac{\mu_5[Z]}{\sigma_Z^5} + 105\frac{\mu_3[Z]}{\sigma_Z^3}$$

$$C_8 = \frac{\mu_8[Z]}{\sigma_Z^8} - 28\frac{\mu_6[Z]}{\sigma_Z^6} + 210\frac{\mu_4[Z]}{\sigma_Z^4} - 315 \qquad (A.57)$$

$$C_9 = \frac{\mu_9[Z]}{\sigma_Z^9} - 36\frac{\mu_7[Z]}{\sigma_Z^7} + 378\frac{\mu_5[Z]}{\sigma_Z^5} - 1260\frac{\mu_3[Z]}{\sigma_Z^3}$$

$$C_{10} = \frac{\mu_{10}[Z]}{\sigma_Z^{10}} - 15\frac{\mu_8[Z]}{\sigma_Z^8} + 630\frac{\mu_6[Z]}{\sigma_Z^6} - 3150\frac{\mu_4[Z]}{\sigma_Z^4} + 3330$$

Notice that the last two coefficients are different from the same ones obtained by Crandall [61].
Comparing Eq. (A.53) and Eq. (A.44) we can establish the following relationship between the coefficients C_j and the quasi-moments b_j

$$C_j = b_j / \sigma_Z^j\ ,\quad j > 2 \qquad (A.58)$$

It follows that by using Eq. (A.56), the quasi-moments are related to central moments by the following relationship

$$b_j[Z] = \mu_j[Z] + (-1)^j \sum_{\substack{k=0,2\ (j=\text{even}) \\ k=3,5\ (j=\text{odd})}}^{j-1} \sigma_Z^{j-k} B_{j,k} \mu_k[Z], \quad j > 2 \tag{A.59}$$

where the exponent (j - k) is always an even number. Since $\mu_0 = 1$ and $\mu_2 = \sigma_Z^2$ we can write the first relationships between quasi-moments and central moments as follows (with argument omitted)

$$\begin{aligned}
&b_3 = \mu_3 ; \; b_4 = \mu_4 - 3\mu_2^2 ; \; b_5 = \mu_5 - 10\mu_3\mu_2 \\
&b_6 = \mu_6 - 15\mu_4\mu_2 + 30\mu_2^3 ; \; b_7 = \mu_7 - 21\mu_5\mu_2 + 105\mu_3\mu_2^2 \\
&b_8 = \mu_8 - 28\mu_6\mu_2 + 210\mu_4\mu_2^2 - 315\mu_2^4 \\
&b_9 = \mu_9 - 36\mu_7\mu_2 + 375\mu_5\mu_2^2 - 1260\mu_3\mu_2^3 \\
&b_{10} = \mu_{10} - 45\mu_8\mu_2 + 630\mu_6\mu_2^2 - 3150\mu_4\mu_2^3 + 3330\mu_2^5
\end{aligned} \tag{A.60}$$

We want to emphasize that by means of Eqs. (A.59) the Edgeworth series (A.44) and the Gram-Charlier series (A.53) are practically coincident although they result from two completely different ideas. The first one is obtained as Fourier transform of the non-normal characteristic function and the second one as the orthogonal expansion of the probability density function. In principle any form of distribution can be represented by the two truncated series. Indeed, increasing the number N of the terms retained into the series we have a better approximation to the exact non-normal density. Obviously these series are more successful for density with little non-normality. When the first N terms are retained in the series we operate a so-called N-th order closure.
Notice that in principle any form of distribution with adjustable parameters could be employed to approximate the probability density function, indeed Charlier himself proposed the so-called Gram-Charlier series of type C (see e.g. [62])

$$p_Z(z,t) = \exp\left[\sum_{j=0}^{N} \gamma_j H_j(Z)\right] \tag{A.61}$$

Cramer [61] stated that the Gram-Charlier series (A.53) cannot be considered as a satisfactory solution of the expansion problem for $p_Z(z; t)$ and proposed the rearranged Edgeworth expansion (A.50). Assaf and Zirkle [12] stated that experiences with the rearranged Edgeworth expansions indicate that sufficiently accurate representations of the density functions it is possible if only the first four terms of the expansion are retained.

Indeed by using the rearranged Edgeworth expansion togheter with the fourth-order cumulant neglect closure, it appears three further terms with respect to fourth-order quasi-moments closure of the Gram-Charlier expansion

$$p_Z(z;t) = \left[1 + \frac{1}{3!}\frac{k_3}{\sigma_Z^3} H_3(z^*) + \frac{1}{4!}\frac{k_4}{\sigma_Z^4} H_4(z^*) + \frac{10}{6!}\frac{k_3^2}{\sigma_Z^6} H_6(z^*) + \frac{35}{7!}\frac{k_3 k_4}{\sigma_Z^7} H_7(z^*) + \frac{280}{9!}\frac{k_3^3}{\sigma_Z^9} H_9(z^*) + \ldots\right] p_Z^0(z;t) \qquad (A.62)$$

Notice that Eq. (A.62), on the contrary of Eq. (A.44), does not represent the exact Fourier transform, until the fourth cumulants, of the characteristic function. Indeed, further terms with cumulants of order three and four appears in the quasi-moments of order greater than eight.
For n-dimensional problems the Gram-Charlier expansion of the non-Gaussian multidi-mensional probability density function can be written as follows [8]

$$p_Z(z;t) = p_Z^0(z;t)\left[1 + \sum_{j=3}^{N} \frac{1}{j!} C_j^T H_j(z^*)\right] \qquad (A.63)$$

where z^* is the standardized variable defined in Eq. (A.32) and $p_Z^0(z,t)$ is the normal probability density given in Eq. (A.28). Because of the multidimensional polynomials are orthogonal with respect to the function $\exp(-1/2\, z^{*T} z)$, that is

$$\frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} z^{*T} z^*\right) H_j(z^*)\, H_k^T(z^*)\, dz^*_1 \ldots dz^*_n = \delta_{jk} P_{j,n} \qquad (A.64)$$

where $P_{j,n}$ are the matrices defined in Eqs. (A.15) and such that

$$P_{j,n} C_j = j!\, C_j \qquad (A.65)$$

Remembering that $\mu_0[Z] = 1$, $\mu_1[Z] = 0$; $\mu_2[Z] = \mathrm{Vec}[\Sigma_Z]$ we can evaluate the multidimensional coefficient C_j as follows [8]

$$C_j = E\left[H_j\left(\Sigma_Z^{-1/2}(Z - m_Z)\right)\right] = (-1)^j \sum_{\substack{k=0,2\ (j=\text{even}) \\ k=3,5\ (j=\text{odd})}}^{j} B_{j,k} \left(\Sigma_Z^{-1/2}\right)^{[k]} \mu_k[Z] \; ; \qquad (A.66a)$$

$$\mathbf{C}_j = E[\mathbf{H}_j(\mathbf{Z}^*)] = (-1)^j \sum_{\substack{k = 0, 2\ (j = \text{even}) \\ k = 3, 5\ (j = \text{odd})}}^{J} \mathbf{B}_{j,k}\, \boldsymbol{\mu}_k[\mathbf{Z}^*] \tag{A.66b}$$

These equations are obtained by using Eq. (A.35), which defines the multidimensional Hermite polynomials $\mathbf{H}_j(\mathbf{z}^*)$, and by introducing the followings multidimensional central moments $\boldsymbol{\mu}_j[\mathbf{Z}]$ and the standardized variable vector

$$\boldsymbol{\mu}_j[\mathbf{Z}] = E\left[(\mathbf{Z} - \mathbf{m}_Z[\mathbf{Z}])^{[j]}\right]; \quad \mathbf{Z}^* = \boldsymbol{\Sigma}_Z^{-1/2}(\mathbf{Z} - \mathbf{m}_Z[\mathbf{Z}]) \tag{A.67}$$

Comparing Eq. (A.63) with Eq. (A.42) we can establish the following relationship between the coefficient vectors $\mathbf{C}_j$ and the quasi-moments $\mathbf{b}_j[\mathbf{Z}]$

$$\mathbf{C}_j = \left(\boldsymbol{\Sigma}_Z^{-1/2}\right)^{[j]} \mathbf{b}_j[\mathbf{Z}]; \quad j > 2 \tag{A.68}$$

By combining Eqs. (A.66a) and (A.68) we obtain the multidimensional relationships between quasi-moments and central moments as follows

$$\mathbf{b}_j[\mathbf{Z}] = \boldsymbol{\mu}_j[\mathbf{Z}] + (-1)^j \sum_{\substack{k = 0, 2\ (j = \text{even}) \\ k = 3, 5\ (j = \text{odd})}}^{j-1} \widehat{\mathbf{B}}_{j,k}\left(\boldsymbol{\mu}_2^{[\alpha_{jk}]}[\mathbf{Z}] \otimes \boldsymbol{\mu}_k[\mathbf{Z}]\right) \quad ; \quad j > 2 \tag{A.69}$$

where $\alpha_{jk} = (j - k)/2$ is always an integer number and $\widehat{\mathbf{B}}_{j,k}$ $(j > k)$ are square matrices of order n^j

$$\begin{aligned} \widehat{\mathbf{B}}_{j,j} &= (-1)\, \mathbf{I}_n^{[j]} \\ \widehat{\mathbf{B}}_{j,k} &= \left(\mathbf{I}_n \otimes \widehat{\mathbf{B}}_{j-1,k+1}\right) \mathbf{E}_{n^{j-1},n} \left(\mathbf{I}_n^{[j-k-2]} \otimes \widehat{\mathbf{R}}_{k+1}\right) - \left(\mathbf{I}_n \otimes \widehat{\mathbf{B}}_{j-1,k-1}\right) \mathbf{E}_{n^{j-1},n} \end{aligned} \tag{A.70}$$

$\widehat{\mathbf{R}}_{k+1}$ being the matrix defined in Eq. (A.38). In the one-dimensional case Eq. (A.70) coincides with Eq. (A.58).

A.5 Modified Gram-Charlier asymptotic expansion

Significantly Eqs. (A.53) and (A.63) can be viewed as polynomial approximations to the ratio of probability density functions $p_Z(\mathbf{z}; t) / p_Z^0(\mathbf{z}; t)$. These approximations can be also interpreted as expansion of a non-Gaussian probability density function as a series of a normal distribution. Since this density, of unit area, has mean vector $\mathbf{m}_Z$ and covariance matrix $\boldsymbol{\Sigma}_Z$ independent of the choice of the vectors $\mathbf{C}_j$ some authors [12, 17, 63] to simplify numerical problems assume into

expansion (A.63) instead of $p_Z^0(z; t)$ (given in Eq. (A.28)) the following normal probability density function

$$\hat{p}_Z^0(z; t) = \frac{1}{(2\pi)^{n/2} \operatorname{Det}\left(\hat{\Sigma}_Z\right)^{1/2}} \exp\left[-\frac{1}{2}(z - m_Z)^T \hat{\Sigma}_Z^{-1} (z - m_Z)\right] \tag{A.71}$$

where $\hat{\Sigma}_Z$ is a diagonal matrix obtained form the matrix Σ_Z setting equal to zero the off-diagonal terms, it follows that we can write

$$\hat{p}_Z^0(z; t) = \prod_{i=1}^{n} p_{Z_i}^0(z_i; t) \tag{A.72}$$

where

$$p_{Z_i}^0(z_i; t) = \frac{1}{(2\pi)^{1/2} \sigma_{Z_i}} \exp\left[-\frac{(z_i - m_{Z_i})^2}{2\sigma_{Z_i}^2}\right] \tag{A.73}$$

where $\sigma_{Z_i}^2$ is the i-th diagonal element of the matrices $\hat{\Sigma}_Z$ and Σ_Z. It follows that in this case we have defined the standardized vector process in modified form as follows

$$\hat{Z}^* = \hat{\Sigma}_Z^{-1/2} (Z - m_Z) \tag{A.74}$$

and its elements can be written as follows

$$\hat{Z}_i^* = \frac{Z_i - m_{Z_i}}{\sigma_{Z_i}} \tag{A.75}$$

It follows that the non-Gaussian probability density function is now expanded in a new series here called the modified Gram-Charlier expansion, which can be written as follows

$$\begin{aligned} p_Z(z; t) &= \left[\sum_{j=0}^{\infty} h_j^T[\hat{Z}^*] \, H_j(\hat{z}^*)\right] \hat{p}_Z^0(z; t) \\ &= \frac{1}{\prod_{i=1}^{n} \sigma_{Z_i}} \left[\sum_{j=0}^{\infty} h_j^T[\hat{Z}^*] \, H_j(\hat{z}^*)\right] \hat{p}_{Z^*}^0(z^*; t) \end{aligned} \tag{A.76}$$

where $\hat{p}^0_{Z^*}(z^*; t)$ is given in Eq. (A.40) and $\mathbf{h}_j[\hat{\mathbf{Z}}^*]$ are the so-called Hermite moments [15] given as follows

$$\mathbf{h}_j[\hat{\mathbf{Z}}^*] = \frac{1}{j!}\hat{\mathbf{C}}_j = \frac{1}{j!}\mathrm{E}\left[\mathbf{H}_j(\hat{\mathbf{Z}}^*)\right] \tag{A.77}$$

It is easily to show that $\mathbf{h}_0[\hat{\mathbf{Z}}^*] = \hat{\mathbf{C}}_0 = 1$; $\mathbf{h}_1[\hat{\mathbf{Z}}^*] = \hat{\mathbf{C}}_1 = \mathbf{0}$; $\mathbf{h}_2[\hat{\mathbf{Z}}^*] = \frac{1}{2}\hat{\mathbf{C}}_1 \neq \mathbf{0}$.
By using the indical notation rather than the Kronecker one and Eq. (A.77), we can write Eqs. (A.76) in the form

$$p_Z(z; t) = \left\{1 + \sum_{\substack{i,j,k,\ \ldots \\ i+j+k+\ \ldots\ = 2}}^{\infty} \frac{\hat{C}_{ijk\ldots}}{i!j!k!\ldots}\left[H_i\left(\hat{z}_i^*\right) H_j\left(\hat{z}_j^*\right) H_k\left(\hat{z}_k^*\right)\ldots\right]\right\} \prod_{i=1}^{n} p^0_{Z_i}(z_i; t) \tag{A.78}$$

where

$$\hat{C}_{ijk\ldots} = \mathrm{E}\left[H_i\left(\hat{Z}_1^*\right) H_j\left(\hat{Z}_2^*\right) H_k\left(\hat{Z}_3^*\right)\ldots\right] = i!\, j!\, k!\, h_{ijk\ldots}[\hat{z}^*] \tag{A.79}$$

In writing Eq. (A.78) we have taking into account amount that $\hat{C}_{ijk\ldots} = 1$ for $i+j+k\ldots = 0$ and $\hat{C}_{ijk\ldots} = 0$ for $i+j+k\ldots = 1$.
For two variables Z_1 and Z_2 Eqs. (A.78) becomes

$$p_{Z_1 Z_2}(z_1, z_2; t) = \left\{1 + \sum_{\substack{i=0 \\ i+j=2}}^{\infty} \sum_{j=0}^{\infty} \frac{\hat{C}_{ij}}{i!j!}\left[H_i\left(\hat{Z}_1^*\right) H_j\left(\hat{Z}_2^*\right)\right]\right\} p^0_{Z_1}(z_1, t)\, p^0_{Z_2}(z_2, t) \tag{A.80}$$

The latter Eq. (A.80) can be written in explicit form as follows

$$\begin{aligned} p_{Z_1 Z_2}(z_1\, z_2; t) = \Bigg\{1 &+ \left[\frac{\hat{C}_{20}\, H_2(\hat{z}_1^*)}{2} + \hat{C}_{11}\, H_1(\hat{z}_1^*)\, H_1(\hat{z}_2^*) + \frac{\hat{C}_{02}\, H_2(\hat{z}_2^*)}{2}\right] \\ &+ \left[\frac{\hat{C}_{30}\, H_3\left(\hat{z}_1^*\right)}{3!} + \frac{\hat{C}_{21}\, H_2\left(\hat{z}_1^*\right) H_1\left(\hat{z}_2^*\right)}{2!} + \frac{\hat{C}_{12}\, H_1\left(\hat{z}_1^*\right) H_2\left(\hat{z}_2^*\right)}{2!}\right. \\ &\left. + \frac{\hat{C}_{03}\, H_3\left(\hat{z}_2^*\right)}{3!}\right] + \left[\frac{\hat{C}_{40}\, H_4\left(\hat{z}_1^*\right)}{4!} + \frac{\hat{C}_{31}\, H_3\left(\hat{z}_1^*\right) H_1\left(\hat{z}_2^*\right)}{3!} + \frac{\hat{C}_{22}\, H_2\left(\hat{z}_1^*\right) H_2\left(\hat{z}_2^*\right)}{4}\right. \\ &\left. + \frac{\hat{C}_{13}\, H_1\left(\hat{z}_1^*\right) H_3\left(\hat{z}_2^*\right)}{3!} + \frac{\hat{C}_{04}\, H_4\left(\hat{z}_2^*\right)}{4!}\right] + \ldots\Bigg\}\, p^0_{Z_1}(z_1, t)\, p^0_{Z_2}(z_2, t) \end{aligned} \tag{A.81}$$

This relationship can be furthely simplified observing that $\hat{C}_{20} = \hat{C}_{02} = 0$

A.6 Relationships between multidimensional central moments and moments and cumulants

In order to evaluate the relationship between the multidimensional central moment and moments we have to make the Kronecker power of $(\mathbf{Z} - \mathbf{m}_Z)^{[j]}$ and the stochastic average of the results, obtaining (with argument omitted)

$$\boldsymbol{\mu}_j = \mathbf{m}_j + \frac{1}{j!}\mathbf{P}_{j,n}\left[\sum_{r=1}^{j}(-1)^r \frac{j!}{r!\,(j-r)!}\left(\mathbf{m}_{j-r} \otimes \mathbf{m}_1^{[r]}\right)\right] \tag{A.82}$$

After some algebra by using the same procedure we can evaluate the inverse relationship to Eq. (A.82) in the form

$$\mathbf{m}_j = \boldsymbol{\mu}_j + \frac{1}{j!}\mathbf{P}_{j,n}\left[\sum_{r=1}^{j-2}\frac{j!}{r!\,(j-r)!}\left(\boldsymbol{\mu}_{j-r} \otimes \mathbf{m}_1^{[r]}\right)\right] + \mathbf{m}_1^{[j]} \tag{A.83}$$

Since the central moments can be interpreted as moments with zero mean relationship between central moments and cumulants can be obtained from Eq. (A.14) setting $\hat{\mathbf{k}}_1[\mathbf{Z}] = \mathbf{k}_1[\mathbf{Z}] = \mathbf{m}_1[\mathbf{Z}] = \mathbf{0}$ as follow

$$\boldsymbol{\mu}_j = \mathbf{k}_j + \frac{1}{j!}\mathbf{P}_{j,n}\left[\sum_{r=2}^{j-2}\frac{(j-1)!}{r!\,(j-r-1)!}\left(\mathbf{k}_{j-r} \otimes \hat{\mathbf{k}}''_r\right)\right], \quad j > 1 \tag{A.84}$$

where $\hat{\mathbf{k}}''_r$ is obtained from Eqs. (A.17) where $\hat{\mathbf{k}}_1 = \mathbf{0}$.
By means Eqs. (A.82) and (A.83) it is possible to obtain the corresponding relationship in the one-dimensional case setting $P_{j,n} = P_{j,1} = j!$
At the end substituting Eq. (A.82) into Eq. (A.69) we obtain the relationship between quasi-moments and statistical moments as follows

$$\begin{aligned}\mathbf{b}_j = \mathbf{m}_j &+ (-1)^j \sum_{\substack{k=0,2\,(j=\text{even})\\ k=3,5\,(k=\text{odd})}}^{j-1} \hat{\mathbf{B}}_{j,k}\left[\left(\mathbf{m}_2 - \mathbf{m}_1^{[2]}\right)^{[\alpha_{jk}]} \otimes \mathbf{m}_k\right] \\ &+ (-1)^j \sum_{\substack{k=0,2\,(j=\text{even})\\ k=3,5\,(j=\text{odd})}}^{j} \hat{\mathbf{B}}_{j,k}\left\{\left(\mathbf{m}_2 - \mathbf{m}_1^{[2]}\right)^{[\alpha_{jk}]} \right. \\ &\left. \otimes \left[\frac{\mathbf{P}_{k,n}}{k!}\sum_{r=1}^{k}(-1)^r\left(\frac{k!}{r!\,(k-r)!}\left(\mathbf{m}_{k-r} \otimes \mathbf{m}_1^{[r]}\right)\right)\right]\right\} \quad ; \quad j > 2\end{aligned} \tag{A.85}$$

Chapter 7

DYNAMIC ANALYSIS OF COMPLEX STRUCTURAL SYSTEMS

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1 INTRODUCTION

Attention is focused on the characterization of the stochastic response of nonlinear complex systems subjected to stochastic external excitations. Non-linearity arises from geometrical considerations and/or material properties. Since exact analytical solutions can be found only in the case of systems idealized by one or few degrees of freedom, approximate methods have been developed. For practical applications in structural engineering the techniques of the stochastic equivalent linearization and of the response surface appear as the most suitable. These methods are illustrated in this Chapter.

2 STOCHASTIC EQUIVALENT LINEARIZATION

The equivalent linearization technique has been very successful during the last decade in the stochastic dynamic analysis of nonlinear structural systems [1, 2]. Caughey [3, 4] was the first who applied equivalent linearization to the study of a nonlinear oscillator. An equivalent linear system is introduced, the coefficients of linearization having been found from a mean-square criterion. Atalik and Utku [5] considered a multi-degree-of-freedom nonlinear system described by the relation

$$\mathbf{L}(\ddot{\mathbf{u}}, \dot{\mathbf{u}}, \mathbf{u}) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{u}$ is the generalized displacement vector; $L_i()$ is the total internal force in the i-th degree-of-freedom and $\mathbf{f}(t)$ is a stationary Gaussian random excitation vector with zero

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mean.
The linearized system takes the form

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t) \tag{2}$$

where the mass, damping and stiffness matrices are determined by minimizing the mean-square error

$$E[\mathbf{e}^T\mathbf{e}] \rightarrow \min$$

with $\mathbf{e} = \mathbf{L} - \mathbf{M}\ddot{\mathbf{u}} - \mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u}$; $E[\ ]$ denotes the expected value.
The sufficient condition for minimization leads to

$$m_{ij} = E[\frac{\partial L_i}{\partial \ddot{u}_j}], \qquad c_{ij} = E[\frac{\partial L_i}{\partial \dot{u}_j}], \qquad k_{ij} = E[\frac{\partial L_i}{\partial u_j}]$$

The matrices of linearization depend on the response statistics. In case of higher order joint moments, Atalik and Utku proposed to express them by the following relation:

$$E[\mathbf{u}g(\mathbf{u})] = E[\nabla g(\mathbf{u})] \qquad \forall g$$

where ∇ is the gradient operator.
While the sufficient condition was given by Atalik and Utku , the necessary condition has been proved to require a jointly Gaussian random vector $\mathbf{u}$ [6].
Note that the linearization coefficients are functions of the response characteristics and the response is not "a priori" known. An iterative procedure is required: one starts assuming the system to be linear, uses response statistics to guess new coefficients and repeats the scheme either step by step during the time integration of the nonstationary excitations or until the solution converges in the stationary case.
Theoretical studies on equivalent linearization techniques are also in [7, 8]. But its success has been due to its coupling with the flexible, analytical expression of the hysteretic constitutive law proposed by Bouc and Wen [9, 10] i.e. by the introduction of appropriate 'endochronic models' of the force-displacement relationship. They are nonlinear hysteretic items that can be replaced by linear hysteretic items (see figure 1), thus preserving the hysteretic nature of the constitutive law after linearization.

The resulting technique is convenient for several applications: multistory frames in steel and reinforced concrete were analysed by considering the hysteretic constitutive law of each story described by a suitable endochronic model; masonry structures were analysed using a similar idealization; soil liquefaction problems were modelled [11]. Equivalent linearization was also connected with a plane finite-element discretization of the structural system [12, 13] and extensions to three-dimensional idealizations of the structure were pursued [14]. Both stationary and nonstationary cases were studied. The remaining sub-sections illustrate the method and its applications.

Criticism on the method is focused on the inability to describe the nonlinearity of the response. Nevertheless, for complex structural systems, stochastic equivalent linearization is the only method capable of providing results, even if approximate.

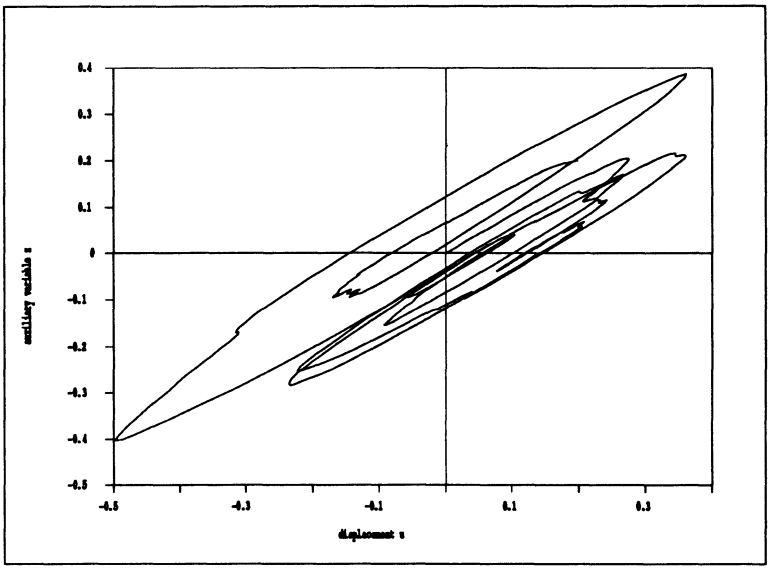

Figure 1: Behaviour of the linear hysteretic item by which the original nonlinear one is replaced after linearization

2.1 HYSTERETIC STRUCTURES

Let

$$\mathbf{M}\ddot{\mathbf{u}} + g(\dot{\mathbf{u}}, \mathbf{u}) = \mathbf{f}(t) \tag{3}$$

be the equations of motion of a general multi-degree-of-freedom nonlinear system.

a) The shear beam model

Consider a complex structural system with masses concentrated at girder levels and columns replaced by hysteretic springs with restoring forces given by

$$V_i = (1 - \alpha_{2_i})\alpha_{1_i} z_i + \alpha_{1_i}\alpha_{2_i} u_i \tag{4}$$

where V_i is the static quantity associated with the displacement interstorey u_i. The auxiliary variable z_i obeys the endochronic model

$$\dot{z}_i = \alpha_{3_i}\dot{u}_i - \alpha_{4_i} z_i |\dot{u}_i| |z_i|^{\alpha_{6_i}-1} - \alpha_{5_i}\dot{u}_i |z_i|^{\alpha_{6_i}} \tag{5}$$

where α_{1_i} to α_{6_i} are system parameters controlling the hysteresis shape of the system. Equation (5) can be linearized by the Atalik and Utku procedure, in the form

$$\dot{z}_i = C_i \dot{u}_i + H_i z_i \tag{6}$$

where the linearization coefficients C_i and H_i are given by

$$C_i = E[\frac{\partial \dot{z}_i}{\partial \dot{u}_i}]; \qquad H_i = E[\frac{\partial \dot{z}_i}{\partial z_i}] \tag{7}$$

if a Gaussian excitation is assumed.
Let the constitutive laws be written in absolute coordinate, say x_i (see figure 2a)). Moreover let

$$\mathbf{y} = \begin{Bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{Bmatrix} = \begin{Bmatrix} \mathbf{x} \\ \mathbf{z} \\ \dot{\mathbf{x}} \end{Bmatrix}$$

be the state vector. The dynamic of the system is described by

$$\frac{d}{dt}\mathbf{y} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{H} & -\mathbf{C} \\ \mathbf{M}^{-1}\mathbf{K_e} & \mathbf{M}^{-1}\mathbf{K_h} & \mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \mathbf{y} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\mathbf{r}\ddot{x}_g \end{Bmatrix} \tag{8}$$

where $\mathbf{M}$ is the diagonal mass matrix, $\mathbf{K_e}$ is the elastic stiffness matrix, $\mathbf{K_h}$ is the hysteretic stiffness matrix, $\mathbf{D}$ is the damping matrix, $\mathbf{C}$ and $\mathbf{H}$ are the linearization coefficients matrices and $\mathbf{r}$ is the vector of ground motion influence coefficients.

b) The hinged frame model

Consider a structural system discretized in perfectly elastic finite elements. Let the inelastic deformations $\mathbf{q}$ be concentrated at critical regions with hysteretic constitutive laws given by equation (4) (see figure 2b)). They can be assembled for the whole structure in the form [11]

$$\mathbf{V} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{q} \tag{9}$$

$\mathbf{z}$ being a vector of auxiliary variables related to $\mathbf{q}$ by means of a vector of equations which groups scalar equations of type (5). Let the linearized constitutive law be

$$\dot{\mathbf{z}} = \mathbf{C}\dot{\mathbf{q}} + \mathbf{H}\mathbf{z} \tag{10}$$

where $\mathbf{C}$ and $\mathbf{H}$ are matrices of linearization coefficients.
The generalized restoring force vector $\mathbf{g}$ in equation (1) can be linearly related to the internal force $\mathbf{V}$ in the form:

$$\mathbf{g} = \mathbf{E_3}\mathbf{V} + \mathbf{D}\dot{\mathbf{u}} \tag{11}$$

After the elastic and inelastic parts are separated into the global generalized displacement, the internal forces $\mathbf{V}$ can be expressed as:

$$\mathbf{V} = \mathbf{E_1}\mathbf{u} + \mathbf{E_2}\mathbf{q} \tag{12}$$

By introducing equation (12) in (11), $\mathbf{g}$ is given as:

$$\mathbf{g}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{E_3}\mathbf{E_1}\mathbf{u} + \mathbf{E_3}\mathbf{E_2}\mathbf{q} + \mathbf{D}\dot{\mathbf{u}} \tag{13}$$

where $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ are structural matrices. Equation (3) becomes:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{D}\dot{\mathbf{u}} + \mathbf{E_3}\mathbf{E_1}\mathbf{u} + \mathbf{E_3}\mathbf{E_2}\mathbf{q} = \mathbf{f}(t) \tag{14}$$

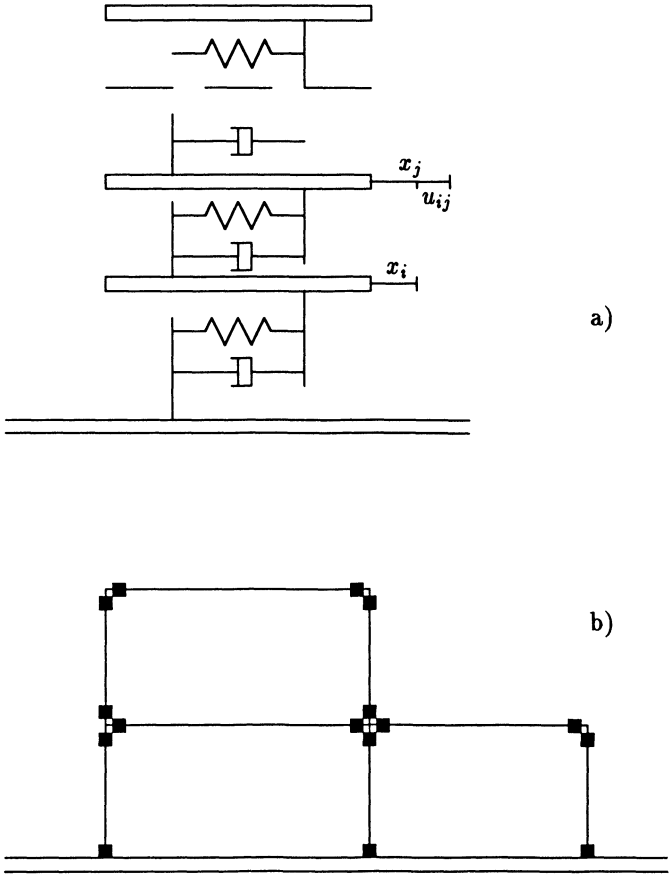

Figure 2: a) Shear-beam idealization of a multistorey building; b) Hinged frame model of an irregular building

The dynamics of the system may then be described as

$$\frac{d}{dt}\mathbf{y} + \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{E}_3\mathbf{E}_1 & \mathbf{M}^{-1}\mathbf{D} & \mathbf{M}^{-1}\mathbf{E}_3\mathbf{E}_2 \\ \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 \end{bmatrix} \mathbf{y} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t) \\ \mathbf{0} \end{Bmatrix} \tag{15}$$

Equation (8) requires the knowledge of the interstorey force-displacement relationships, while equation (15) makes use of the section constitutive laws which can be obtained from laboratory tests as well as from numerical modelling. Moreover, equation (8) is unable to analyse irregular buildings, as the one in figure 2b), while equation (15) does it.

c)Solution techniques

Equations (8) and (15) represent a linear system of equations of the type

$$\dot{\mathbf{y}} + \mathbf{G}\mathbf{y} = \mathbf{b} \tag{16}$$

which can be solved by classical methods of linear analysis. In (16) $\mathbf{G}$ is the resulting matrix of the coefficients and $\mathbf{b}$ is the corresponding r.h.s. vector. The matrix $\mathbf{G}$, however, contains the linearization coefficients C_i and H_i which are functions of response statistics. The covariance matrix $\mathbf{\Sigma}_{\dot{\mathbf{q}},\mathbf{z}}$ can be easily obtained from the covariance matrix $\mathbf{\Sigma}_\mathbf{y}$, as $\dot{\mathbf{q}}$ and $\mathbf{z}$ are linearly related to $\mathbf{y}$. Equation(15) can be written only when the covariance matrix $\mathbf{\Sigma}_\mathbf{y}$ has been calculated. From equation(16), classical results of random vibration theory lead to express the second moment description of $\mathbf{y}$ as the solution of the differential equation

$$\dot{\mathbf{\Sigma}}_\mathbf{y} + \mathbf{G}\mathbf{\Sigma}_\mathbf{y} + \mathbf{\Sigma}_\mathbf{y}\mathbf{G}^T = \mathbf{B} \tag{17}$$

where

$$\mathbf{B} = E[\mathbf{b}\mathbf{y}^T] + E[\mathbf{y}\mathbf{b}^T]$$

Matrix $\mathbf{B}$ is easly written in the case that the external excitation can be modelled as a Gaussian white-noise or shot-noise. By adding further differential equations, filtered white- and shot-noise can also be considered. For stationary excitations and non-deteriorating system, the response becomes stationary and can be obtained by solving the Liapunov matrix equation:

$$\mathbf{G}\mathbf{\Sigma}_\mathbf{y} + \mathbf{\Sigma}_\mathbf{y}\mathbf{G}^T = \mathbf{B} \tag{18}$$

The solution can be found by iteratively using the algorithm of reference [15]. An alternative frequency domain method can be applied in order to calculate the second order statistics. The possibibity of neglecting modes is one of the advantage to reduce the computational effort. A criterion to identify in the calculation process the meaningful terms, so that the others can be neglected is given in [16, 17].

3 A NEW STOCHASTIC LINEARIZATION TECHNIQUE

For the case of a non-linear stochastic dynamic system where the non-linearity appears in the term expressing the restoring force, the classical approach has been to use as objective function the mean square of the difference between the actual restoring-force and the restoring

force of the linearized system. Recently, new objective functions were proposed. For example, Elishakoff and Zhang [18] have suggested equalizing the mean square values of the potential energies and of the dissipation energies for the non-linear system and its linear equivalent. Casciati, Faravelli and Hasofer [19] proposed to choose the stochastic equivalent linearization coefficients by calculating the stationary upcrossing rates. This is an extension, to the time dependent situation, of what is known in static structural reliability as the "normal tail approximation" [20]. The proposed approach will be illustrated for the Duffing oscillator (for which the exact stationary distribution of the response and its derivative at time t is known). For a hysteretic oscillator the technique is extended by applying the stochastic averaging method and focusing attention on the stationary upcrossing to the energy envelope.

3.1 THE DUFFING OSCILLATOR

Consider the Duffing oscillator characterized by the following differential equation

$$\ddot{x} + 2\zeta\dot{x} + x(1 + \rho x^2) = f(t) \tag{19}$$

where ζ is the usual non-dimensional damping factor. The natural frequency ω_0 of the oscillation is taken equal to 1 and the restoring force is $x(1+\rho x^2)$. $f(t)$ is a Gaussian white noise with the delta-type correlation function

$$E[f(t)f(t+\tau)] = I\delta(\tau)$$

and I is the intensity function given by $I = 2\pi S_0$, being S_0 the constant spectral level of $f(t)$.
Let $Y_1 = x$ and $Y_2 = \dot{x}$. Equation (19) is equivalent to two first-order equations:

$$\dot{Y}_1 = Y_2$$
$$\dot{Y}_2 = -2\zeta Y_2 - Y_1 - \rho Y_1^3 + f(t)$$

Introduce the function $R(Y_1, Y_2) = 2\zeta Y_2 + r(Y_1)$ where $r(Y_1) = Y_1(1 + \rho Y_1^2)$. The stationary probability density function $p(Y_1, Y_2)$ of the system response is governed by the Fokker-Planck equation [21]:

$$Y_2 \frac{\partial p}{\partial Y_1} - \frac{\partial}{\partial Y_2}[pR(Y_1, Y_2)] - \pi S_0 \frac{\partial^2 p}{\partial Y_2^2} = 0 \tag{20}$$

Then it is well known that for a single degree system with nonlinear stiffness and linear damping subjected to Gaussian white-noise excitation the stationary displacement and velocity are independent and their joint probability density is given by the exact solution of equation (20):

$$p = k \exp\{-\frac{4\zeta}{2\pi S_0}[\frac{Y_2^2}{2} + \int_0^{Y_1} r(u)du]\}$$

For the Duffing oscillator, the stationary probability distribution of x and $\dot{x}$ is given by

$$w_D(\dot{x}, x) = K_D \exp\{-\frac{4\zeta}{I}[(\dot{x}^2/2) + (x^2/2) + \rho(x^4/4)]\} \tag{21}$$

where K_D is a normalization constant.
Let $\frac{I}{4\zeta} = \sigma^2$, K_D is a function of σ^2 and ρ. K_D cannot be calculated explicitly; in order to have some idea to its dependence we expand it in a power series in ρ:

$$K_D \simeq \frac{1}{2\pi\sigma^2}(1 - \frac{15}{2}\sigma^2\rho^2) + o(\rho^4)$$

To implement stochastic equivalent linearization we replace the term $x(1 + \rho x^2)$ by Cx and the damping coefficient ζ by ζ'. The linearized stationary probability distribution, $w_L(\dot{x}, x)$ now becomes

$$w_L(\dot{x}, x) = K_L \exp\{-\frac{4\zeta'}{I}[\frac{\dot{x}^2}{2} + C\frac{x^2}{2}]\} \tag{22}$$

Let $\frac{I}{4\zeta'} = \sigma_L^2$. The normalization constant is given by

$$K_L = \frac{\sqrt{C}}{2\pi\sigma_L^2}$$

Let now U_D be the upcrossing rate for the Duffing oscillator and U_L the upcrossing rate for the linearized system. Then

$$U_L = \int_0^\infty \dot{x} w_L(\dot{x}, u) d\dot{x} = \frac{\sqrt{C}}{2\pi} e^{-Cu^2/2(\sigma_L)^2} \tag{23}$$

$$U_D = \frac{1}{2\pi}(1 - \frac{15}{2}\sigma^2\rho^2) e^{-(1/\sigma^2)[(u^2/2 + \rho(u^4/4)]]} \tag{24}$$

There are two linearization parameters available namely C and ζ'; they can be evaluated equating the upcrossing rates at zero and at the critical level u:

$$\sqrt{C} = (1 - \frac{15}{2}\rho^2\sigma^2)$$
$$C \simeq 1 - 15\rho^2\sigma^2$$

and

$$\zeta' \simeq \zeta(\frac{1}{C} + \frac{\rho u^2}{2})$$

3.2 THE HYSTERETIC OSCILLATOR

The hysteretic oscillator can be defined by the following equation

$$\ddot{x} + 2\zeta\omega_o\dot{x} + \alpha\omega_o^2 x + (1 - \alpha)\omega_o^2 z = f(t) \tag{25}$$

where $f(t)$ is, as before, a white noise with covariance function $I\delta(\tau)$, and the auxiliary variable z solves the equation

$$\dot{z} = A\dot{x} - \beta|\dot{x}|z|z|^{r-1} - \delta\dot{x}|z|^r \tag{26}$$

Linearization is effected by putting

$$\dot{z} = C\dot{x} + Dz \tag{27}$$

where C and D are two arbitrary coefficients.
The basic steps of the approach are as follows [22, 23] .
The hysteretic variable z is split into two components: the "backbone" $g(x)$ and the purely hysteretic component $\epsilon(x)$. The equation of motion (25) is rewritten as

$$\ddot{x} + 2\zeta\omega_o\dot{x} + G(x) + \omega_o^2(1-\alpha)\epsilon = f(t) \tag{28}$$

where

$$G(x) = \omega_o^2[(1-\alpha)g(x) + \alpha x]$$

When the energy dissipation is small, the total energy E is given by

$$E = \frac{\dot{x}^2}{2} + V(x) \tag{29}$$

where $V(x)$ is the actual potential energy given approximately by

$$V(x) = \int_0^x G(\chi)d\chi$$

corresponding to the backbone potential energy. From equation (29)

$$\dot{x} = \sqrt{2[E - V(x)]}$$
$$dt = \frac{dx}{\sqrt{2[E - V(x)]}}$$

and the average period of vibration, $T(E)$, is given by

$$T(E) = 4\int_0^b \frac{dx}{\sqrt{2[E - V(x)}}$$

where $V(b) = E$. Mutiplying equation (28) by $\dot{x}$, one obtains

$$\dot{E} = -\dot{x}[2\zeta\omega_o\dot{x} + \omega_o(1-\alpha)\epsilon] + \dot{x}f(t) \tag{30}$$

Equation (30) is a power balance equation: it states that the rate of change of the total energy of the oscillator, with respect to time is equal to the power input due to the random excitation minus the power dissipation.
The stochastic averaging method is used to consider a one-dimensional Markov model for E and hence to write a FPK equation for the transition density function of the process. For this purpose averaging over one side the rate of energy dissipation, one has

$$H(E) = \frac{1}{T(E)}\oint \dot{x}[2\zeta\omega_o\dot{x} + \omega_o^2(1-\alpha)\epsilon]dt$$

The total average of energy dissipation is made up of two components $H_1(E)$ the energy dissipated in the damping and $H_2(E)$ the energy dissipated by the hysteretic loop:

$$H_1(E) = \frac{8\zeta\omega_0}{T(E)}\int_0^b \sqrt{(2[E - V(x)])}dx$$

and

$$H_2(E) = \frac{(1-\alpha)\omega_0^2}{T(E)} S(b)$$

where $S(b)$ is the area of the hysteretic loop when x varies between $-b$ and $+b$.
The mean and correlation functions of the energy input term are evaluated and then the average over one cycle $T(E)$ is performed to obtain an equivalent process of the form:

$$P_{in}(t) \simeq \frac{I}{2} + \sqrt{C(E)I}\phi(t)$$

where $P_{in}(t) = \dot{x}f(t)$, $\phi(t)$ being a white noise process of unit strength. The term $C(E)$ is given by

$$C(E) = [4/T(E)] \int_0^b \sqrt{(2[E - V(x)])} dx$$

Thus one can approximate the evolution of the energy envelope and its Itô equation is given by

$$dE = -[H(E) - \frac{I}{2}]dt + \sqrt{C(E)I}\, dW \tag{31}$$

where $dW = \phi(t)dt$, W being the standard Brownian motion.
Equation (31) shows that, when suitably averaged, the energy envelope is a one-dimensional Markov process. Then using standard methods, the FPK stationary equation is given by

$$\frac{\partial}{\partial E}[-[H(E) - \frac{I}{2}]p] + \frac{I}{2}\frac{\partial^2}{\partial E^2}[C(E)p] = 0$$

The final result for the probability density of E, $p(E)$ is

$$p(E) = kT(E)\exp\{-\frac{2}{I}\int_0^E \frac{H(\xi)}{C(\xi)} d\xi\}$$

where the constant k is obtained by using the normalizing conditions

$$\int_0^\infty p(E)dE = 1$$

The relationship between $p(E)$ and $w(x, \dot{x})$ is given by a variety of physical arguments. In particular, a physical argument for the approximation density of x given E states that it is inversely proportional to the velocity of $x(t)$ [24]. Normalizing one finds:

$$p(x|E) = \frac{2}{T(E)\dot{x}}$$

So now

$$p(x, E) = p(E)p(x|E)$$
$$2w(x, \dot{x})dxd\dot{x} = p(x, E)dxdE$$

Hence, the approximate joint density function $w(x, \dot{x})$ is given by

$$w(x, \dot{x}) = k\exp\{-\frac{2}{I}\int_0^E \frac{H(\xi)}{C(\xi)} d\xi\}$$

From the last expression the upcrossing rate at level u for the hysteretic oscillator, U_H, is given by

$$U_H = \int_0^\infty \dot{x} w(\dot{x}, x) d\dot{x} \tag{32}$$

Let

$$Q(E) = \int_0^E \frac{H(\xi)}{C(\xi)} d\xi \tag{33}$$

From equation (32) putting $\frac{\dot{x}^2}{2} = \kappa$ and $y = \kappa + V(u)$, $E = V(u)$

$$U_H = k \int_{V(u)}^\infty \exp[-\frac{2}{I} Q(y) dy] \tag{34}$$

From standard asymptotic results, for large u one has approximately [25]

$$U_H = k \int_{V(u)}^\infty \exp[-\frac{2}{I} Q(y) dy] \simeq \frac{I}{2Q'(V(u))} \exp[-\frac{2}{I} Q(V(u))] \tag{35}$$

where the apex denotes the function derivative. Hence

$$U_H = \frac{kIC[V(u)]}{2H[V(u)]} \exp[-(2/I) \int_0^{V(u)} H(\xi)/C(\xi) d\xi] \tag{36}$$

Given the parameters of the endochronic equation (2) it is comparatively easy to calculate numerically all the functions in equation (36).

Consider now the endochronic form in equation (27). In that case the backbone turns out to be linear and the shape of the hysteretic loop elliptic.
For a cycle of period $2\pi/\omega_1$, the slope of the backbone is given by

$$\beta = \frac{C\omega_1^2}{\omega_1^2 + D^2} \tag{37}$$

and the area of the loop $S(b)$ by $\pi\psi b^2$, where

$$\psi = \frac{-CD\omega_1}{\omega_1^2 + D^2} \tag{38}$$

Letting now

$$\begin{aligned} \omega_1 &= \gamma\omega_0 \\ \gamma^2 &= \alpha + (1-\alpha)\beta \end{aligned} \tag{39}$$

The joint density function $w(x, \dot{x})$ for the linearized system is

$$w(x, \dot{x}) = k \exp\{-[2\zeta + \frac{\psi}{\gamma}(1-\alpha)] \frac{2\omega_0}{I} [\frac{\dot{x}^2}{2} + \gamma^2 \omega_0^2 \frac{x^2}{2}]\}$$

Letting now

$$\zeta' = \zeta + \psi\frac{1-\alpha}{2\gamma} \tag{40}$$

The upcrossing rate at level u for the linearized system, U_L is given by

$$U_L = \frac{\gamma\omega_0}{2\pi}\exp\{-\frac{2\zeta'\gamma^2\omega_0^3}{I}u^2\} \tag{41}$$

In order to obtain the values of the two linearization parameters, C and D, one equates the envelope crossing rates at level u given by equations (36) and (41) and evaluates the parameter ζ' as:

$$\zeta' = -\frac{2I}{4\gamma^2\omega_0^3u^2}\ln[\frac{2\pi}{\gamma\omega_0}U_H] \tag{42}$$

Finally, from equations (37) and (38) the two linearization parameters are given by

$$D = -\psi\frac{\omega_1}{\beta} \tag{43}$$

$$C = \beta(1+\frac{\psi^2}{\beta^2}) \tag{44}$$

where $\omega_1 = \gamma\omega_0$, with γ given by equation (39), is 2π times the zero upcrossing rate of the linearized model. From equation (40) and for $\zeta' > \zeta$, ψ is

$$\psi = \frac{2\gamma(\zeta'-\zeta)}{1-\alpha}$$

with ζ' given by equation (42). For u larger than the yielding value of x, β is choosen to be either

$$\beta = \frac{z_{max}}{u}$$

or

$$\beta = \frac{z_{max}}{u+\alpha}$$

where z_{max} is the maximum value of the auxiliary variable z given by

$$z_{max} = [A/(\beta+\delta)]^{1/r}$$

4 STOCHASTIC FINITE-ELEMENTS

This section is mainly devoted to complex structural systems, i.e. to mechanical systems whose input-output relationship is governed by a numerical operator like a general-purpose finite-element computer code. Let the randomness of the system and of the excitation on it be appropriately defined. Attention is focused on the objectives the probabilistic structural analysis pursues. It can be a direct reliability assessment or just an uncertainty propagation. The result of the latter analysis provides, then, the input for a probabilistic model of lifetime prediction or damage accumulation. Both the studies are often denoted as "stochastic finite element analyses" [26].

The most direct approach to such problems is the Monte Carlo simulation that requires the

evaluation of the response for many sets of input values. These sets are obtained by random sampling from the probability distributions assumed for the input variables. The repeated response evaluation provides a valid description of the uncertainty in the output.

There are three main objections to a direct Monte Carlo approach: 1) its cost, since many thousands of computer runs are required; 2) the fact that it gives no clue as to which variables are the principal causes of uncertainty; 3) the circumstance that changes in the input distributions lead to new further Monte Carlo simulations independent of previous analyses. Two main different alternatives to Monte Carlo simulation can be envisaged: i) the numerical operator is disassembled and the effect of the uncertainty is incorporated at each single stage of the numerical solving process; ii) the numerical operator is regarded as a black box and the output uncertainty is estimated by means of planned (i.e. not random) repeated experiments.
One assumes here that the numerical operator is too complex to be opened. It follows that there are not alternatives to regarding it as a black-box. Repeated experiments have, therefore, to be carried out. The basic problem is to optimize the plan of these experiments in order to study the uncertainty propagation. The approach is based on the response surface methodology (r.s.m.) whose applications to the analysis of mechanical systems are reviewed in next section.

4.1 THE EXTENDED RESPONSE SURFACE METHOD

Let x_j $(j = 1, ..., J)$ be a random design variable of a mechanical system. They are assumed to be uncorrelated; if it is not so, a suitable transformation can be adopted in order to reach uncorrelation. Any response variable y is a random variable since it is a function of the input variables. Let this function be unknown from an analytical point of view.

The probability distribution of the response variable y can be approximately estimated by the statistical procedure proposed in [27], [28]. The same procedure was extended in [29] to the evaluation of the joint probability distribution of several response variables.

a) The r.s.m. scheme

Let some of the design variables x_j be functions of the spatial coordinates (random fields). When the continuous system is discretized, these functions become random vectors. Any vector χ_j, expressing the spatial variability of the design variable x_j, can be written in one of the two forms:

$$\begin{aligned} \chi_j &= x_{Aj}\chi'_j \\ \chi_j &= x_{Aj} + \chi'_j \end{aligned} \tag{45}$$

In equation (45), x_{Aj} is the central value of x_j and χ'_j denotes the deviations of χ_j from the central value x_{Aj}. The scalar quantity x_{Aj} has the nature of a random variable. The second factor on the r.h.s., χ'_j, is a random vector with a unit or zero central value.
Analogously, when x_j is a stochastic function of any parameter t in $(0, T)$, equation (45) still holds provided that the l.h.s. represents the vector resulting from the discretization of the

range $(0,T)$.
The extended response surface model is based on the relationship:

$$y = f'[\mathbf{x}_A, \boldsymbol{\theta}] + \epsilon \tag{46}$$

between y and the set of spatial averages $\mathbf{x}_A$, $\boldsymbol{\theta}$ being model parameters. The effect of the vectors χ'_j on y is accounted for in the random term ϵ, which collects the lack of fit and the randomness of neglected terms as in classical experiment theory. The same framework can be reached by selecting, instead of $\mathbf{x}_A$, a suitable subset of the x_j by appropriate sensitivity analyses.

Suitable transformations $Y = Y(y)$ and $\mathbf{X}_A = \mathbf{X}_A(\mathbf{x}_A)$ are introduced before equation (46) is written. This is useful in order:

i) to make linear the nonlinear relationships (on condition that the considered model is inherently linear in the regression coefficients). Let y be expressed by an exponential function with argument a second order polynomial form; then $\ln(y)$ is a second order polynomial form, i.e. a linear combination of the polynomial coefficients;

ii) to facilitate the selection of the experiments to be carried out.

In particular, for any Gaussian x_{Aj}, the standardized variable:

$$X_{Aj} = x^*_{A_j} = (x_{A_j} - E[x_{A_j}])/(Var[x_{A_j}])^{\frac{1}{2}} \tag{47}$$

is introduced. In equation (47) $E[\ ]$ denotes the mean value and $Var[\ ]$ the variance. The transformation $Y(y)$ will be discussed where the validation of the procedure is investigated.

Since the dependence of Y on $\mathbf{X}_A$ is conveniently described by a second order polynomial relationship, in matrix notation one writes:

$$\begin{aligned} Y = \theta_0 + \mathbf{X}_A^T\boldsymbol{\theta}_1 + \mathbf{X}_A^T\Theta\mathbf{X}_A + \epsilon_A = \\ = \mathbf{X}_{A1}^T\boldsymbol{\theta} + \epsilon_A = \tilde{\mathrm{Y}} + \epsilon_A \end{aligned} \tag{48}$$

where $\tilde{\mathrm{Y}}$ denotes the estimate of the response variable achieved by the response surface model. The coefficients θ_0, $\boldsymbol{\theta}_1$, Θ, which play the role of $\boldsymbol{\theta}$ in equation (46), can be computed by the regression analysis of the results obtained in numerical experiments appropriately planned [30] (see figure 3). In equation (48), the term ϵ_A takes into account the model error ϵ, the effects of the vectors χ'_j of the deviations from the spatial averages and their interactions.

b) Validation

Once the parameters of model (48) are evaluated, its validation can be pursued by the following scheme:

- several experiments at the same coordinates $\mathbf{X}_A$ are repeated with different simulated vectors χ'_j. For each experiment, different response values are found, but the estimate

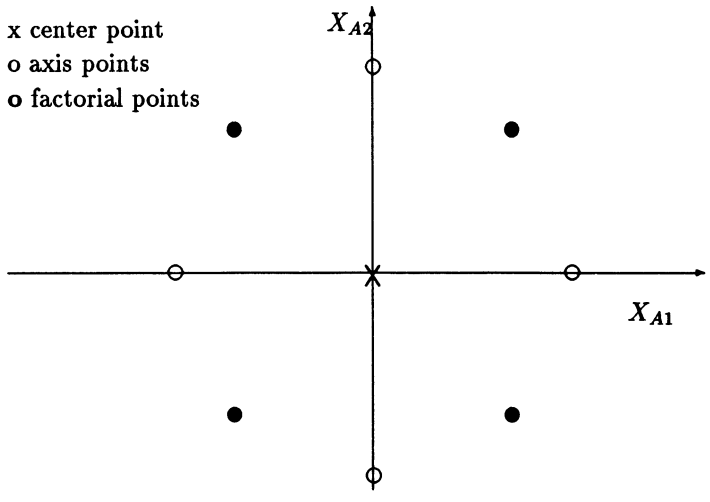

Figure 3: Central composite design for fitting a second order polynomial response surface

$\tilde{Y}$, obtained by equation (48), is always the same. This error of the model is called the "pure error". It is measured by the corresponding standard deviation s_ϵ.

- experiments are now conducted by changing both the coordinates $\mathbf{X}_A$ and the vectors χ'_j. Again discrepancies between Y and $\tilde{Y}$ are found. This error of the model is called the "lack of fit" error. It is measured by the standard deviation s_l which includes "the pure error" and the "lack of fit" of the second-order polynomial model.
- any transformation Y of y is better than another when the corresponding ratio:

$$\mathcal{T} = s_\ell / s_\epsilon \tag{49}$$

is closer to 1 [31]. This criterion states that the optimal response surface is characterized by a space-averaged error (s_ℓ) very close to the ensemble error (s_ϵ).

Note that the global pure error has to be almost constant over the region under consideration. Otherwise the analyser can conceive a piece-wise polynomial relationship $Y(\mathbf{X}_A)$ in different regions of the space of the variables x_{Aj}.

As a tool of further validation, simulation can also be employed for producing a cumulative frequency diagram to be compared graphically with the estimated cumulative distribution function of the response.

c) Probabilistic framework

An appropriate use of level-2 reliability methods (see next section) was shown to be convenient in view of the estimation of the cumulative distribution function (CDF) $P_Y(\psi)$ of the transformed response variable Y.

The basic idea introduced in [32] is to evaluate $P_Y(\psi)$ as $\Phi[-\beta(\mathbf{z})]$. Here Φ denotes the Gaussian CDF. Let $\mathbf{z}$ be the vector of all the random quantities on the right-hand side of equation (48), assumed to have normal distribution $N(\mu_z,\Sigma_z)$. Then β is given by:

$$\beta(\mathbf{z}) = \pm min[(\mathbf{z}-\mu_z)^T \Sigma_z^{-1}(\mathbf{z}-\mu_z)]^{\frac{1}{2}} \tag{50}$$

$$\mathbf{z} : Y(\mathbf{z}) = \psi \tag{51}$$

The procedure was tested by simulation. It can also be modified for non-normality [33].

d) **Covariance analysis**

In this way the marginal CDF's of the single response variables of interest can be estimated. However, one may also be interested in their joint distribution. For this purpose let Y_1 and Y_2 be the appropriate transformations of two response variables y_1 and y_2. Model (48) can be updated for Y_1 in the form:

$$Y_1 = \theta_0' + \mathbf{X}_A^T\theta_1' + \mathbf{X}_A^T\Theta'\mathbf{X}_A + C_1Y_2 + \epsilon_{A1} \tag{52}$$

where θ_0', θ_1', Θ' and C_1 are the coefficients of the new regression problem. The previous procedure, then, provides the conditional probability distribution of Y_1 given any assigned value for Y_2 .

5 RESPONSE SURFACE IN RELIABILITY ASSESSMENT

Nonlinear dynamics problems can generally be solved only in a numerical way. This prevents from a direct application of standard reliability methods. A technique which makes use of iterated response-surface analytical approximations of the system performance function in conjunction with asymptotic theory was therefore proposed in view of reliability assessment [34].
This response-surface iterative scheme can be used in the original space of the random variables, provided a maximum log-likelihood constrained optimization problem is solved. Moreover, asymptotic theory also provides a better estimate of the probability of failure of the dynamical system against any assigned limit state.

5.1 RELIABILITY ASSESSMENT

The problem of calculating failure probabilities involves in general the computation of multidimensional integrals with implicitly defined boundaries. Several approximation methods were developed to obtain more and more efficient solution methods for this problem.
The usual formulation of the problem is the following. The given items are a random vector $\mathbf{x}$ which describes the random influences on a structure and a limit state function $g(\mathbf{x})$ which

indicates the state of the structure, when the random vector $\mathbf{x}$ has the realization $x_1, ..., x_n$. If $g(\mathbf{x}) > 0$, the structure is safe, if $g(\mathbf{x}) \leq 0$, it is unsafe. If $p(\mathbf{x})$ is the joint probability density function (JPDF) of the random vector $\mathbf{x}$, the probability of failure $P(F)$, with the failure $F = \{\mathbf{x}; g(\mathbf{x}) \leq 0\}$, is then

$$P(F) = \int_{g(\mathbf{x})\leq 0} p(\mathbf{x})d\mathbf{x}$$

In general, the dimension n is large and the function $g(\mathbf{x})$ has no simple analytical form.
The first approximation methods, which were used to calculate this probability $P(F)$, are the so called FORM (First Order Reliability Methods) procedures. As shown in [33] every random vector $\mathbf{x}$ with independent components $x_1, ..., x_n$ and continuous PDF can be transformed into a standard Gaussian vector $\mathbf{u} = \{u_1, ..., u_n\}^T$ with independent components.
If a failure domain F is given in this standard normal space, an approximation for its probability content $P(F)$ was derived by simple geometric arguments. First, the point $\mathbf{x}^0 \in F$ with minimal distance to the origin was determined, this point must lie on the limit state surface $B = \{\mathbf{x}; g(\mathbf{x}) = 0\}$. Second, at this point a first order Taylor expansion of $g(\mathbf{x})$ was made and then $g(\mathbf{x})$ is replaced by the linear function $g_L(\mathbf{x})$ obtained by this expansion. Instead of $P(F)$ then the probability content of the domain $F_L = \{\mathbf{x}; g_L(\mathbf{x}) \leq 0\}$ is calculated. The probability content of such a domain is just $\Phi(-\beta)$ with β the distance of F_L to the origin, if F_L does not contain the origin in its interior. It can be shown that in this case $\beta = |\mathbf{x}^0|$.
Other approximation methods are the so-called SORM (Second Order Reliability Methods) procedures.These methods are analogous to the FORM's, but a second order Taylor expansion is made.
In the general case of dependent random variables the Rosenblatt transformation for transforming random vectors with dependent components into standard normal vectors is applicable only in special cases.
When an asymptotic procedure is used, it was proved in [35] that it is not necessary to transform random vectors into standard normal random vectors to be able to calculate asymptotic approximations. The basic idea is to understand the meaning of the minimization of the distance to the origin in FORM and SORM.
Consider the logarithm of the Gaussian PDF $p(\mathbf{u})$:

$$\ln(p(\mathbf{u})) = -n/2\ln(2\pi) - |\mathbf{u}|^2/2$$

The minimization of the distance $|\mathbf{u}|$ corresponds to the maximization of the logarithm of the PDF. In mathematical statistics this function is called the log-likelihood. Therefore the probabilistic meaning of this is the maximization of the log-likelihood.
Asymptotic approximations can be derived also for non-normal random vectors with a PDF $p(\mathbf{x})$ and log-likelihood $l(\mathbf{x}) = \ln(p(\mathbf{x}))$. The point $\mathbf{x}^*$ is calculated for which:

$$l(\mathbf{x}^*) = \max_{\mathbf{x}\in F} l(\mathbf{x})$$

Then a suitable expansion of the log-likelihood is made and as approximation [35]:

$$P(F) \sim (2\pi)^{(n-1)/2} \frac{p(\mathbf{x}^*)}{|\nabla l(\mathbf{x}^*)|\sqrt{|det\mathbf{H}^*|}} \tag{53}$$

Here the matrix $\mathbf{H}^*(\mathbf{x}^*)$ has the form

$$\mathbf{H}^* = \mathbf{P}\mathbf{H}\mathbf{P}^{\mathrm{T}} - \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}$$

with:

1) $\boldsymbol{\nu} = |\nabla l(\mathbf{x}^*)|^{-1}\nabla l(\mathbf{x}^*)$ the normal to the surface.

2) $\mathbf{P} = \mathbf{I_n} - \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}$ ($\mathbf{I}_n$ being the n-dimensional unity matrix).

3) $\mathbf{H} = (l_{,ij}(\mathbf{x}^*) - \frac{|\nabla l(\mathbf{x}^*|}{|\nabla g(\mathbf{x}^*)|} g_{,ij}(\mathbf{x}^*))$, $i,j = 1,...,n$ the Hessian of $l(\mathbf{x})$ in local coordinates.

An important advantage of the method of log-likelihood approximation is the fact that it can be used also for dependent random variables, if only the joint PDF $p(\mathbf{x})$ and its log-likelihood $l(\mathbf{x}) = \ln(p(\mathbf{x}))$ are known. In this case we calculate the point of maximum likelihood, then the first and second derivatives of the likelihood function at this point are computed and inserted into the approximation equation. Here no conditional distribution function, as needed in the Rosenblatt transformation, must be computed. Further we can calculate directly from the log-likelihood function in the original space importance and sensitivity factors [36].

5.2 RELIABILITY ASSESSMENT OF DYNAMICAL SYSTEMS

a) The method

The extended response surface model (46) is formulated as follows:

$$g(\mathbf{x}|\mathbf{x}_{ss}) = F_R(\mathbf{x}_v, \mathbf{x}_{sv}, \boldsymbol{\theta}) + \epsilon(\mathbf{x}_{ss}) \tag{54}$$

with $\mathbf{x}_v$ a vector of random variables, $\mathbf{x}_{sv}$ a vector of random central values and $\mathbf{x}_{ss}$ vectors of deviations from the corresponding central values; ϵ takes into account the error.
Consider equation (54) written in the central region of all the variables in order to identify the actual location of $G()$ in the standardized $\mathbf{z}$ space. Here the standardization is still introduced in order to preserve some useful features of the experiment planning. A linear polynomial form is preliminarly selected for the response surface. An appropriate fractional replicate of a factorial design centered at the origin of the space $\mathbf{z}$ defines the experiment plan [37] (see figure 4a)). This first response surface application (where the contribution of the quantities $\mathbf{x}_{ss}$ is neglected) leads to an estimation of the design point $\mathbf{z}^0_{(1)}$ and of the maximum likelihood point $\mathbf{z}^*_{(1)}$ for the linear model;

The sensitivity factors of each variable z_j can be estimated from the gradient of the log-likelihood function at the point $\mathbf{z}^*_{(1)}$ due to the fact that the probability of failure satisfies in first approximation the relation:

$$\partial P(F)/\partial z_i \sim (\partial l(\mathbf{z}^*)/\partial z_i) P(F)$$

No substantial error is generally introduced by considering the random variables with low sensitivity factors as fixed and equal to their means in the further calculations. Alternatively,

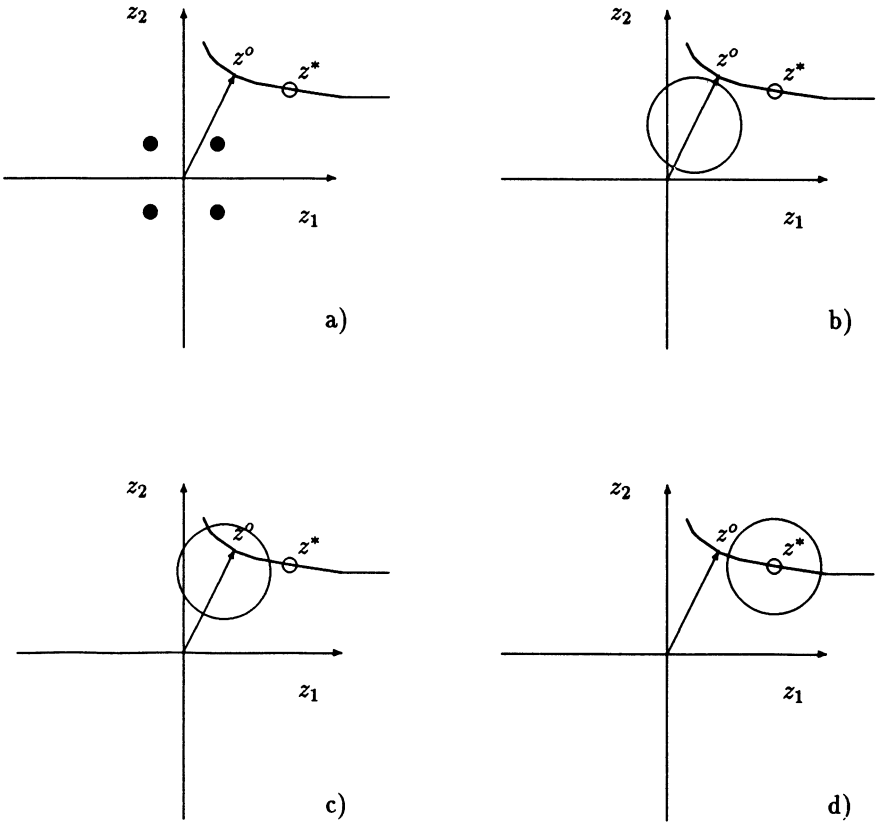

Figure 4: a) Factorial design for fitting a preliminary linear approximation of the response surface; b) and c) subsequent shifted composite central designs during the iterative procedure of reliability assessment; d) final composite central design in the estimated point of maximum likelihood.

one can account for their randomness in the error terms ϵ.

A better estimation is then pursued by a sequence of repeated response surface applications. Each of them consists of several deterministic structural analyses for different input data (see figure 4b) and c)). They form an experiment design conceived for both fitting second-order polynomial response surface and incorporating the terms $\mathbf{x}_{ss}$ (and the $\mathbf{x}_v$ and $\mathbf{x}_{vs}$ with low sensitivity factors).
At the i-th step, the resulting approximation for the response function is:

$$G_{\mathbf{v}}^{(i)}(\mathbf{v}_{(i)}) = F_R(\mathbf{v}_{(i)}, \Theta_{(i)}) + \epsilon_{(i)}(\mathbf{x}_{ss})$$

where

$$\mathbf{v}_{(i)} = \mathbf{z} - \mathbf{z}_{(i)}^{x} \tag{55}$$

By equation (55), the origin is shifted along the vector from the past origin to the relevant design point $\mathbf{z}_{(i-1)}^{0}$. The convergence to the actual value is pursued by selecting a point on the line between this point and the origin: this defines $\mathbf{z}_{(i)}^{x}$.
Practical reasons suggests that one should work, rather than in the space $\mathbf{v}$, in a particular space $\mathbf{u}$, obtained from it by rotation. The new first axis u_1 coincides with the normal to the function $G() = \text{const.}$ at the selected point. This iterative procedure stops when convergence on the distance of the design point from the origin is achieved. At each step one also computes the current estimate of the maximum-likelihood $\mathbf{z}_{(i)}^{*}$ by the algorithm of constrained nonlinear optimization.

b) A dynamics application

Consider the hysteretic oscillator governed by the equation [34]:

$$\begin{aligned} \dot{u} &= y \\ \dot{y} &= -\mu\omega_0^2 u - 2\zeta\omega_0 y - (1-\eta)\omega_0^2 z + f(t) \\ \dot{z} &= \dot{u} - c_1\dot{u}|z|^r - c_2 z|\dot{u}||z|^{(r-1)} \end{aligned}$$

where ω_0 is the circular frequency, ζ is the damping ratio, η is the ratio between the post and the pre-yielding stiffness and c_1 and c_2 are parameters of the hysteretic equation. r is a further parameter that is assumed equal to 1 in the following. The excitation $f(t)$ is a white noise of power spectral density G_0.
Let the parameters ζ, η, c_1 and c_2 be of a random nature. The oscillator becomes a system with uncertain parameters. The properties of such parameters are given in Table 1.

Let the performance function be assigned in the form

$$K - (1/2)[(Var[u]/.2)^{1/2} + Cov[z\dot{u}]/.5] \geq 0$$

$Cov[z\dot{u}]$ being the covariance of z and $\dot{u}$, i.e. a measure of the dissipated energy rate.

The correlated extreme distribution was selected to describe the joint probability density function (JPDF) of c_1 and c_2. The proposed procedure, in fact, just requires the analytical

or numerical knowledge of the first and second derivatives of the joint log-likelihood function. Therefore, there is not need for numerical transformations and/or for numerical calculations of conditional densities which may offer numerical difficulties. In a more general case the JPDF of non-normal dependent variables will be obtained by inversion of the characteristic function but the procedure will be still applicable.

The standardized variables are, as usual,

$$z_i = \frac{x_i - \mu_i}{\sigma_i}$$

Within the response surface scheme, experiments are planned according to the experiment design theory . For each set of variable values the nonlinear dynamic analysis is repeated and the response surface is built to fit the results in terms of the performance function $G(\mathbf{z})$. For the first linear interpolation a single white noise realization is considered. In the subsequent nonlinear approximations, some realizations are used according to the blocking theory. The governing equation becomes then a standard deterministic differential equation which is integrated numerically. The response variance $Var[u]$ is estimated over a period of 100 s after a time interval of $10/(2\zeta\omega)$ necessary to make negligible the effect of the initial conditions.

The results of the reliability analysis procedure proposed in this paper are summarized in Table II. The error term ϵ is due to the stochastic variability of the excitation.

Table I - Probabilisitc definition of the random variables of the example concerning the dynamic application. The variables ζ and η are described by their transformations $\zeta = 0.01\zeta' + 0.05$ and $\eta = 0.015\eta' + 0.05$, respectively (G_0 in sq.in./rad. s^3.

Symbol	Distribution	Mean Value	Standard dev.	Correl.
ζ'	Chi-square with 5 d.o.f.	0.	1.	-
η'	Chi-square with 5 d.o.f.	0.	1.	-
c_1	Joint extreme value distribution	1.25	0.25	0.75
c_2				
K	Gaussian	.1	.05	-
G_0	Gaussian	5.11	0.205	-

Table II - Maximum likelihood point estimates $\mathbf{z}^*_{(i)}$ and probability approximations for the dynamic example. The first step consider a linear response surface. The experiment design for the 4-th step is centered at $\mathbf{z}^0 = (-1.248, -.286, .476, -.090, -1.137, -1.860)$. The last step adopts an experiment design centered at $\mathbf{z}^*_{(4)}$.

Iteration	l_{ott}	z_1	z_2	z_3	z_4	z_5	z_6	ϵ	$P_F.10^4$
1 (linear)	- 9.230	- .997	-1.026	1.302	1.262	-1.226	1.534	-	-
2	-10.056	-1.169	- .745	.136	.075	-1.162	2.179	-1.394	6.254
3	- 9.793	-1.035	- .782	-.011	-.148	-1.181	6.845	-1.842	6.314
4	- 9.992	-1.031	- .779	-.001	-.167	-1.237	1.875	-1.880	5.473
final	-10.023	-1.070	- .837	-.204	-.323	-1.190	2.022	-1.808	5.171

6 CONCLUSIONS

This paper emphasizes the following aspects:

1. stochastic equivalent linearization is presently the tool of analysis more compatible with the special features of deterministic complex structural systems under stochastic excitation;

2. several criticisms on stochastic equivalent linearization can be overcome by an appropriate selection of the objective function which the selected linearized system must minimize;

3. systems with uncertain geometrical and mechanical properties can be conveniently studied by response surface schemes;

4. the reliability assessment for any expected performance of a dynamical system can be pursued by a maximum likelihood algorithm coupled with an iterative use of the response surface method.

All these approaches provide quantitative estimates satisfactory for the engineer decision making process. But the progress of higher accuracy techniques must be continuously followed and incorporated in the previous tools, toward a more and more robust standard of stochastic analysis.

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