Engineering Hydrology

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Oxford University Press

ISBN: 978-0-19-569461-1

Preface

Hydrology pertains to the scientific study of water, especially its movement in relation to land. In nature, water may exist in three states – solid, liquid, and vapour. Water is distributed unevenly; and it is in constant motion in the form of clouds, water vapour, as well as surface water bodies, such as rivers and groundwater. We are surrounded by water in all directions.

Various hydrologic processes, such as precipitation, evaporation, evapotranspiration, surface runoff, infiltration, and groundwater flow are important parameters for quantifying the movement and availability of water resources at a given site. For example, many hydraulic structures require information about the design flood which cannot be estimated without an in-depth knowledge of hydrology. Similarly, for food production, it is necessary to measure the crop water requirement for a proper planning of available water resources. Measuring evaporation from ground surface would be an uphill task unless one has a strong background in hydrology.

About the Book

Over the years, the groundwater level has declined significantly in many parts of India, and there is a growing concern over how to utilize the groundwater in an efficient way. The causes of declining groundwater level, its consequences and remedies, combined with the knowledge of groundwater balances, play a significant role in hydrology. These issues form the core subject matter of this book.

There are textbooks dealing with various topics related to hydrology. However, considering the present requirements regarding applied hydrology for emerging field problems, these books lack the much-needed rigour.

Engineering Hydrology is a step forward towards fulfilling the present-day needs of graduating engineers.

Content and Coverage

The book comprises eleven chapters, of which the first seven chapters deal with surface water hydrology, while the remaining deal with groundwater hydrology.

Chapter 1 deals with the fundamental topics of hydrology – water resources, hydrological cycle, water availability, and water balance.

Chapter 2 contains a detailed discussion about the elements of hydrologic cycle which comprises precipitation, infiltration, evaporation, and runoff. Stream flow measurements techniques are also discussed in this chapter.

Chapter 3 covers hyetographs and unit hydrograph derivation used for design flood computations as well as techniques to develop S-curves that are used to make hydrograph applications more flexible. It also covers a few popular conceptual models used for flood hydrograph derivations.

Chapter 4 deals with simulation and synthetic methods in hydrology. Advanced topics, such as Monte Carlo simulation and generation of random numbers using different distributions have been discussed in this chapter.

Chapter 5 throws light into the intricate details of rainfall-runoff relationships and different methods of flood routing.

Chapter 6 discusses statistical methods in hydrology. This chapter covers different types of distribution, error analysis, handling of data, and prediction of certain hydrologic processes with a certain degree of accuracy.

Chapter 7 contains an exhaustive discussion on flood frequency analysis. Different methods of parameter estimation including the probability of weighted moments (L-moments) are covered in this chapter.

Chapter 8 covers the fundamentals of groundwater flow. It also provides a basis for understanding the hydraulics of flow through porous media.

Chapter 9 deals with well hydraulics including abstraction of water not only from individual wells but also from a well field.

Chapter 10 includes the significance and applicability of groundwater budgeting. A keen effort has been made to enable the students to perform basic computations related to groundwater budgeting.

Chapter 11 introduces the use of numerical methods to deal with situations where analytical solutions are not feasible. The functionalities of Visual MODFLOW, the widely used software to calculate groundwater budgeting, are discussed in this chapter.

All the chapters contain a variety of examples, which will motivate the students towards practical applications. With such a wide coverage of concepts which form the part of this textbook, it is expected that the students will be better equipped to address various engineering problems related to hydrology.

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Contents

CHAPTER

1

Hydrology and Water Resources

1.1 INTRODUCTION

The objective of this chapter is to give an easily comprehensible introduction to hydrology and water balance calculations for engineering students and practitioners. The text has been compiled in order to give a holistic view of the water environment, i.e., hydrology seen as the water carrier in nature with human influence. The main hydrological components are treated with simple calculation methods to quantify water balances and mass transport.

Water is a chemical union between hydrogen and oxygen. Water is unique in the sense that it can exist in three phases at almost the same temperature: solid state (ice), liquid, and gas (water vapour). On Earth, about 2/3 of the surface is covered by water and about 1/3 by land. Water is a prerequisite for all known forms of life. A biological cell is usually made up of at least 70% water. Humans contain 55–60% water by weight (men about 60% and women about 55%).

1.1.1 Importance of Water

Water is a basic, natural resource for agriculture and industry. Water has always been intimately linked with human development, and its role is considered crucial for the transition of men from hunters to farmers. The earliest known human civilizations were developed at places having stable and regular access to water, e.g., the Nile valley, the Euphrates and Tigris, and the Indus valley. The river water provided water supply, food, fertile sediments, and easy transportation. Later, the water became a prerequisite for industrial development through the production of hydropower (mills, steam engines, etc.) and transportation routes. Consequently, for both agriculture as well as industry, water has become a key component as a natural resource, energy producer, solvent, and transporter. In many developed countries, people have constant access to water of good quality, and water is an indispensable component of clean outdoor environment. Water has, however, many other societal functions, and in many *arid* (dry) and semi-arid countries, lack of water is a limiting factor for economic and social development.

Water is a prerequisite for life, but it also creates many problems. Water-borne diseases (malaria, bilharzia, etc.), floods, as well as droughts cause innumerable casualties in excess of those due to all other disasters. About 1.2 billion people have to drink unhygienic water. This results in 5 million deaths every year. Proper management of water is, therefore, related not only to drinking water and a clean outdoor environment, but also to developments in public health, and social and economic development. Thus, the study of water is strongly interdisciplinary because all major societal areas and scientific disciplines are related to utilization of water resources.

1.1.2 Human Influence

Hydrology is associated with the circulation of water in nature and the human influence on this system. The water transport can be conceptualized as a combination of the natural circulation in an exterior system and an inner man-made system where humans tap water from the outer system and return it back after shorter or longer use, unfortunately quite polluted. In the natural water system, changes occur slowly (except in cases of natural disaster). The human influence, however, is now so large that even the exterior system is being affected significantly. Acidification of water resources and global warming are glaring examples of the damaging effect of human influence on nature. The entire water and energy system, with its exterior natural part and inner human-regulated part, is extremely complex and difficult to analyze. Figure 1.1 shows the two components and the transport of water from the natural system to and within a typical human community (after Anderberg, 1994). To supply the community's residents with water, a *water supply source* is connected to a *water treatment plant*. Consequently, water is taken from the natural system to the man-made water system. The water supply sources are open water reservoirs such as a large river or a lake.

1.1.3 Civic Use

However, the water supply can also come from *groundwater*, through a drilled or dug well. To secure a continuous supply, a *reservoir* is built that helps to even out seasonal differences in water availability, e.g., long dry periods during summer and snow melt and high flows during spring. A reservoir can be built by damming up a river in a suitable location so that an artificial lake is created.

From the reservoir, water is taken to the *treatment plant* where it is properly treated in order to make the water drinkable. After treatment, the water is sent through pipes to households and consumers. However, problems may arise because the use of water is extremely variable during the day. The quantity of

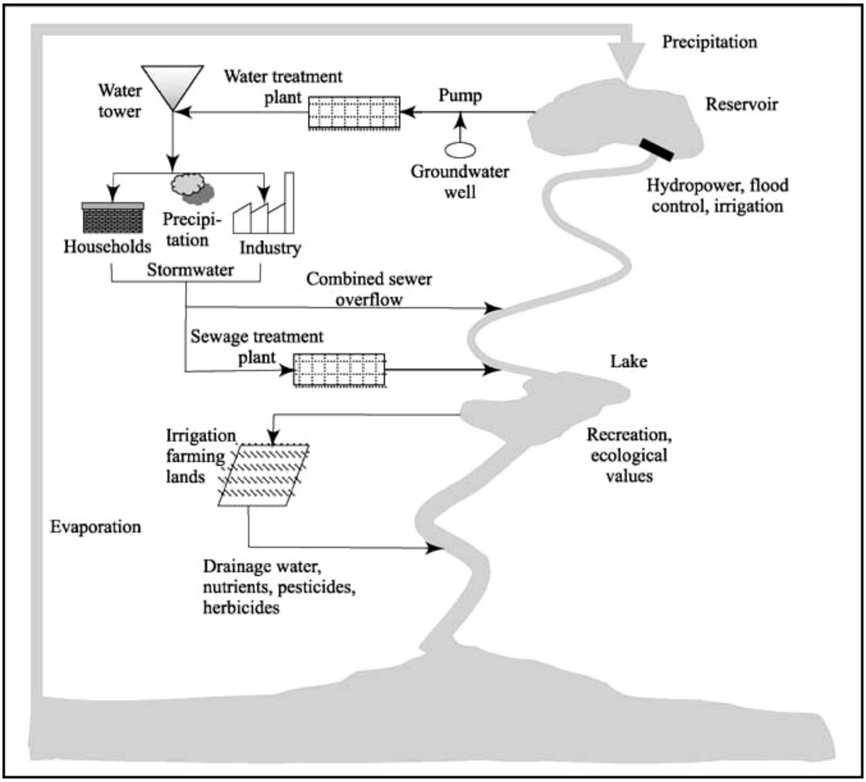


Fig. 1.1 *Circulation of water through an exterior natural and an inner man-modified system (after Anderberg, 1994)*

water used in households and industry during the night-time is relatively low; but water usage peaks, especially during morning and evening hours. To cope with such large daily variations, a short-term reservoir, usually a *water tower*, is built somewhere in between the consumer household and the treatment plant. The objective of the water tower is to keep as much water stored as is needed to cope with temporary consumption peaks without changing the production rate of the treatment plant. From the water tower, water is then distributed to consumers. Because the water is under constant pressure (from the water stored in the tower), continuous flow is provided to the water taps. The quantity of water consumed for drinking and cooking is considerably less in comparison to the quantity used for washing (rinsing soap, detergents, and washing powder), and flushing waste from the kitchen, toilet, and bathroom.

Water used in the kitchen, toilet, and bathroom is discharged from the buildings through common pipes. If the polluted water is discharged directly to the *recipient* (such as lakes and rivers) without treatment, it usually leads to serious deterioration of the ecological life in the water body. At the same time, the water becomes more or less unusable for downstream residents. Nowadays, the polluted water is usually treated in a *sewage treatment plant* before it is lead to the recipient. However, along with the sewage water from households and industry, *storm-water* (rain water from impermeable areas such parking lots, roofs, etc) also goes to the treatment plant in *combined sewer pipes* (which have sewage and storm-water in the same pipes). This is especially the case for the older central areas of cities built before the 1960s. Cities that are built after 1960s sometimes have *separated sewer pipes* (sewage water and storm-water in different pipes) instead, which direct the sewage water to the treatment plant and storm-water directly to the recipient without any treatment. Manholes and drains, which collect storm-water and lead it to recipients in order to avoid flooding problems after heavy rainfall, can be seen along the paved streets in some cities. However, storm-water, which has flushed off roofs, parking lots, and streets, can often carry quite a large pollution load from petrol, oil, heavy metals, sediments, etc. If a heavy rainfall occurs during a period of heavy water consumption in households and industry, the *combined sewer system* (sewage and storm-water in the same pipes) can be flooded and the polluted water may have to be discharged directly to the recipient without treatment (*combined sewer overflow*) in order to avoid serious flooding.

1.1.4 Environmental Fallout

Water is also used for other purposes. A major water consumer is agriculture. About 80% of the global water consumption is used in agriculture and *irrigation* of crops. To feed the human population, cereals and vegetables must be produced; and *fertilizers, pesticides, and herbicides* are to be used to keep the insects from destroying the crops. Heavy rainfall and irrigation water can then flush the soil and transport the fertilizers and pollutants such as *nitrogen* (N) and *phosphorous* (P) to nearby streams and lakes. Eventually, these will be transported to the sea, resulting in *eutrophication*, algae, and pollutant problems.

Evaporation occurs from water surface (and even from land surface). *Precipitation* closes the *hydrological cycle*. The evaporation mainly transports pure water, so substances in *runoff* accumulate in the sea, either as sediments in the sea bottom or as *dissolved matter* in the sea water. Precipitation forms runoff, and we are back to where we, began in the water supply and reservoir. The reservoir is a simple construction where the water is dammed by concrete obstruction in the water course itself (*dam building*). The water level upstream from the dam can then be regulated by releasing more or less water through gates or over a variable dam crest level. The difference in height between the upstream and the downstream water level can also be used to generate electricity in a *hydropower plant*. Reservoirs can be exclusively built for hydropower production, or they can combine this function with the control of water supply, irrigation, and/or flood protection.

As mentioned above, the total water system is very complex. If the objective is to improve the water environment, this cannot be achieved by improving the situation only in a part of the total system. The continuous circulation of water means that all parts are connected in one way or another. The water quality is affected by all the components related with the system. Problems in hydrologic engineering therefore cannot be solved without regarding the entire system.

The hydrological circulation and transportation of water as depicted in Fig. 1.1 also helps to envisage how a pollutant may be transported. Water itself is the most important transporter of chemical substances as well as biological organisms. Thus, knowledge about the hydrological circulation and transportation of water will empower us to stop potential pollutants from contaminating the system. With suitable assumptions, it is also possible to make predictions about the future of a pollutant. This may be a frequently done task for the hydrologic engineer.

1.2 HYDROLOGICAL CYCLE

The *hydrological cycle* is the most important carrier of water, energy, and matter (chemicals, biological material, sediments, etc), locally and globally (Fig. 1.2). The hydrological cycle acts like an enormous global pump that is driven mainly by two forces; *solar energy* and *gravitation pull*. Humans have ingeniously utilized this global and free pump to get irrigation water and to draw power from the enormous amount of energy that this cycle represents.

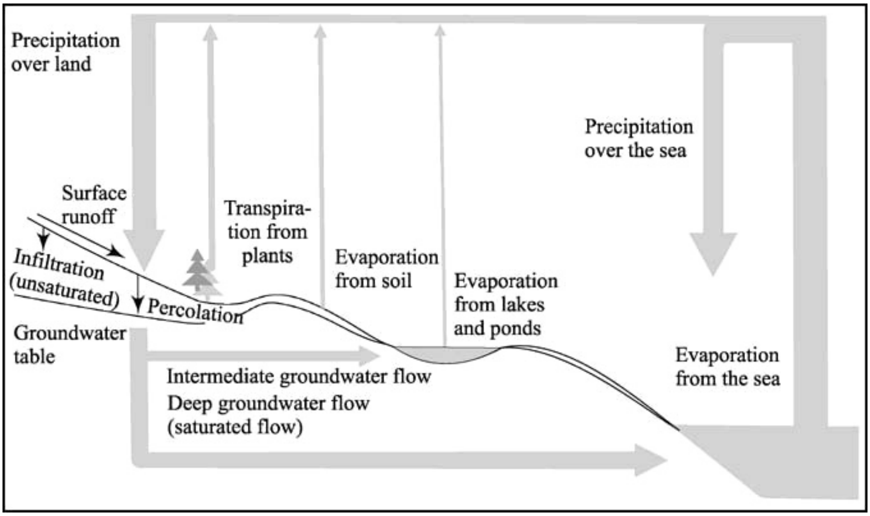


Fig. 1.2 *The hydrological cycle (after Bonnier World Map, 1975)*

The incoming solar energy forces water to evaporate from both land and sea. Much of this vapour condensates and falls directly over the sea surface again (globally about 7/8 of the rainwater falls over the oceans). The remainder of the rainwater falls over land (globally about 1/8), and it falls as *precipitation* (rainfall, snow, and/or hail). This forms *runoff* as creeks, rivers, and lakes on the soil surface. A major part, however, *infiltrates* through the soil surface and forms *soil water* (water in the upper soil layers above the *groundwater table*, also called the *unsaturated zone*) that may later *percolate* (deeper infiltration) down to the groundwater (groundwater zone also called the *saturated zone*) level.

In the ground, water can also be taken up by plant roots, and evaporate into the atmosphere through *transpiration* (evaporation through the plant leaves by plant respiration) or by direct *evaporation* from the soil. The total evaporation from both soil and plants is called *evapotranspiration*.

1.2.1 Carrier for Pollutants

The global cycle of water transports different types of chemical, biological, and sediment matter. Finally, these may be deposited in the sea because this is the lowest point in the system. If the release point for these constituents is known, it is often possible to predict the transport path by studying the local hydrology in the area. This is due to the fact that the pollutant often follows the same path as the water. However, chemical and/or biological transformation may also affect the pollutant.

Humans influence and change the general hydrological cycle to a great extent. Activities in the landscape directly affect the different components of the hydrological cycle. The chemical content of different hydrological parts is also increasingly affected by various activities such as industry, agriculture, and city life. Yet, the total amount of water on earth is constant. Water is neither created nor is disappearing from earth. However, the content of various biological and chemical elements can fluctuate, depending on the location of the hydrological cycle (Fig. 1.3).

1.2.2 Turnover Time

Only a fraction of the total water volume is fresh water (about 2.7%). And, a major part (2/3) of this fresh water is located around the poles as ice and glaciers. The total amount of fresh water resources is consequently limited, and desalination of seawater is still an expensive process. The water contained in different components (shown in Fig. 1.3) is continuously exchanged due to the constant movement of water.

The theoretical turnover time indicates the average time that it takes for the water volume to be exchanged once. For some components, e.g., water in rivers and atmosphere, the turnover time is very short, about one week (Table 1.1). This also indicates the theoretical transport time for pollutants released in various parts of the water cycle. A pollutant that accumulates on the glacial ice would theoretically surface again after about 8000 years (Table 1.1). A pollutant released in the atmosphere or a river would be flushed out after about a week. However, the average turnover is theoretical, and assumes that the pollutants are not adsorbed or transformed by biological and geological media through which they are transported. In any case, the turnover time can give a rough estimation in order to understand transport velocities in different parts of the hydrological cycle.

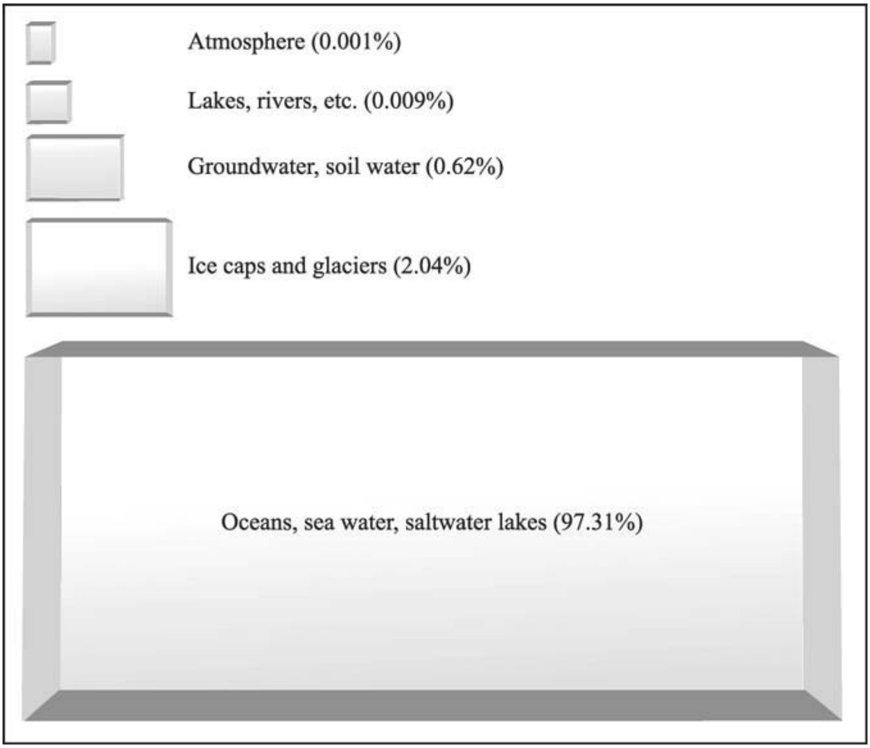


Fig. 1.3 *The global distribution of water in different hydrological parts (after Bonnier World Map, 1975)*

Table 1.1 Average turnover time for water in different hydrological parts

Hydrological part	*Volume (10^6 km^3)*	*%*	*Turnover time (year)*
Oceans	1370	94.2	3000
Groundwater	60	4.1	5000
Ice caps and glaciers	24	1.7	8000
Lakes	0.3	0.02	10
Soil water	0.1	0.006	1
Atmosphere	0.01	0.001	1 week
Surface water	0.001	0.0001	1 week

Consequently, the turnover time may give a rough but general idea of how quickly a water particle may travel through a water body. This can be compared to a more detailed picture as seen in Table 1.2. The table shows typical velocities by which a water molecule or a pollutant travelling with the same speed as water may travel. Note that these values are based on approximation. Large variations may be expected, depending on the hydrological situation, the hydraulic conductivity of the geologic medium, etc.

Table 1.2 Typical average velocity for a water molecule

Water body	*Soil type*	*Water molecule velocity*
Soil water (vertical)	—	1–3 m/year
Groundwater	Moraine close to soil surface	1–10 m/day
	1 m depth	0.1 m/day
	Deep	0.01 m/day
	Cracked bed rock	0–10 m/day
Creek	—	0.1 m/s ≈ 10 km/day
River	—	1 m/s ≈ 100 km/day

Humans interact with the natural hydrological system by diverting water for different activities. A major part of this water, about 80% on a global scale, is used for irrigation and production of agricultural products. The remaining 20% is used for industrial needs and domestic water supply. Figure 1.4 shows, in a schematic way, how water is used in households. It is seen that for a rich and developed country, a very small part (about 3%) is used for direct consumption (drinking water and cooking).The remaining part is used for sanitation purpose. For a poor country in the developing world, average per capita water consumption may be only 20 liters a day. The UN recommends that people need a minimum of 50 liters of water a day for drinking, washing, cooking, and sanitation.

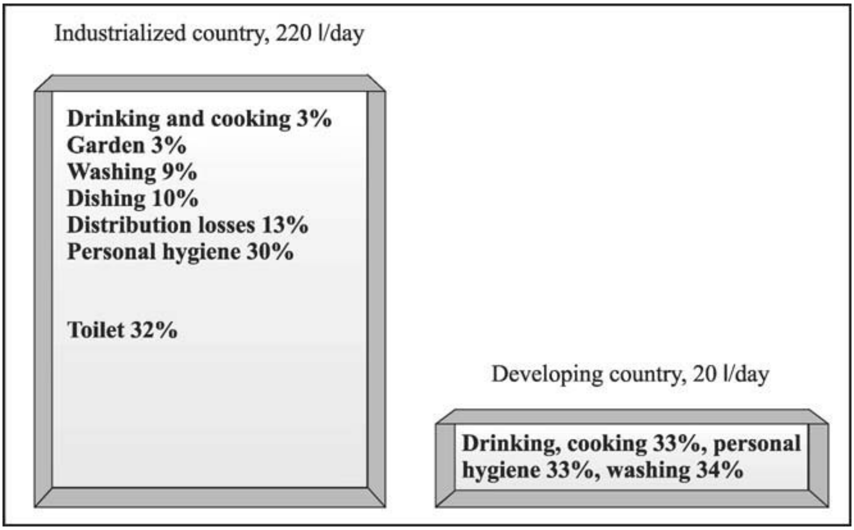


Fig. 1.4 *Domestic water consumption (after Bonnier World Map, 1975)*

1.3 CLIMATE AND WATER AVAILABILITY

As mentioned earlier, the global influx of solar energy is the main driving force of the hydrological cycle. The greater the influx of solar energy, the more water can evaporate over a specific area. And greater the evaporation, greater will be the water available for precipitation. The great effect of the *solar influx* is modified by the general atmospheric circulation. Winds from ocean bring moist air to land where precipitation occurs. Evaporation also occurs from land. The major influx of solar energy occurs along the equator (Fig. 1.5). The solar influx falls perpendicular to the soil surface at the equator. At higher latitudes, the solar influx falls at an oblique angle to the soil surface and over a larger area and thus brings less energy.

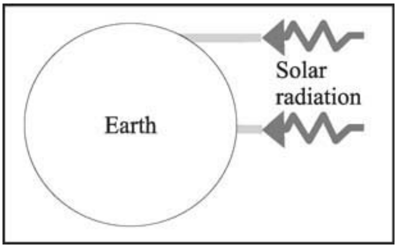


Fig. 1.5 *Solar influx (radiation)*

The moist air at the equator is warmed up by the solar influx, and due to density differences, a strong vertical uplift of the air occurs. This phenomenon is called *convection*. Convection is the result of the rising of expanding warm air of less density as compared to the surrounding cooler and denser air. The rising air parcels induce the flow of fresh air from the sides replacing the rising air. This gives rise to a general global atmospheric flow that, to a great extent, distributes air and precipitation over the entire globe.

Due to the rotation of earth, this general global atmospheric pattern is split up into six smaller atmospheric cell systems (Fig. 1.6). The whole system is, however, driven by the equatorial solar influx of energy. The friction between the cells (the cells are moving approximately as six inter-connected cogwheels) drives the circulation from the equator to the polar areas.

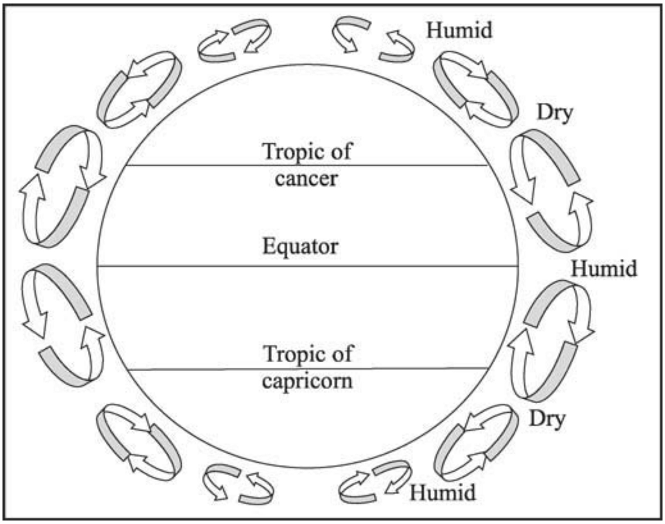


Fig. 1.6 *The global atmospheric circulation with six main cell systems (Arrows indicate cell rotation direction) (after Whipple, 1984)*

The atmospheric flow moving over ground surface can potentially take up moisture. When moisture in the air is driven up in to the upper part of the cells, the air is cooled off and precipitation may occur. Cooler air can hold less moisture as compared to warm air. The air in the upper part of the cell is then driven down again to the lower part of the cell. This air, however, does not contain any moisture. In turn, these down-turning cell areas with no moisture can explain much of the dry areas of the globe. The equator area as well as the tropics of Cancer and Capricorn have a large excess of precipitation. Areas where the dry cell air is descending to the ground contain the great deserts of the world: the *Sahara* and *Taklamakan* in the northern hemisphere and the deserts of Australia in the southern hemisphere. Further up towards the poles, the cells again bring rising moist air that precipitates over the *temperate areas* (e.g., northern Europe, North America, and southern South America). These areas have an excess of precipitation. However, this general precipitation pattern, is modified by local precipitation mechanisms.

The precipitation and therefore the availability of water also vary for different time scales. In general, precipitation and temperature may vary in the same manner. Higher the temperature, higher would be the evaporation (provided that there is water to evaporate). The precipitation at a single location is, however, difficult to predict because it depends on local conditions and local wind systems that modify the general large-scale atmospheric circulation. This also makes it very difficult to predict how precipitation will look after a hundred years from now. Figure 1.7 shows an example of how the monthly precipitation

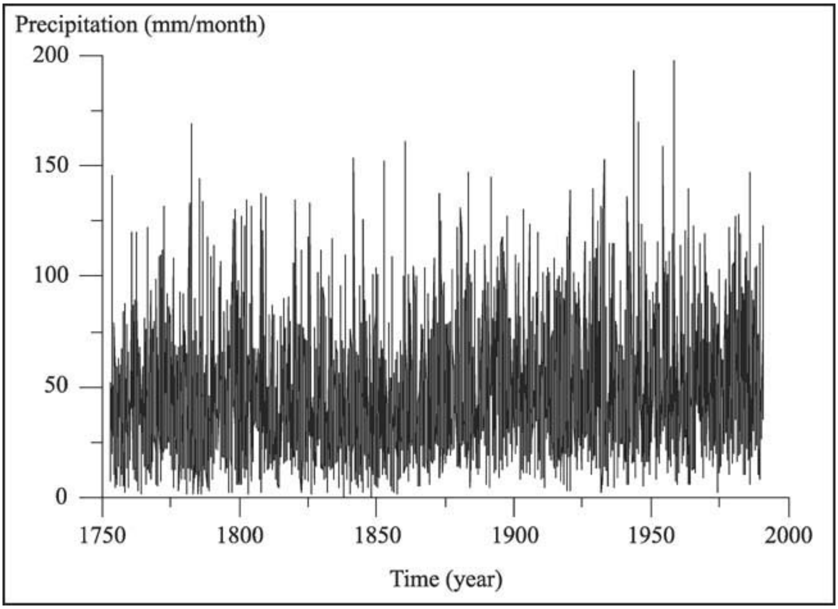


Fig. 1.7 *Monthly precipitation in Lund, Sweden, between 1750 and 2000*

has looked in Lund, Sweden, for the last 250 years. It is rather evident that the monthly precipitation can vary widely between 0 to 200 mm per month. At the same time, it is difficult to see if there is any long-term trend (general increase or decrease with time).

In general, the temperature and also the precipitation pattern are governed by both atmospheric and oceanic circulation. In short, the total *energy balance* over a specific area decides the general climate.

1.4 WATER BALANCES

The basis for availability and general transportation of water and pollutants for a specific area is called the *water balance* or *mass balance equation* or *continuity equation*. The water balance equation, in general, stipulates that all inflow minus all outflow to an area during a certain time period must be equal to the storage changes.

$$I - O = dS/dt \tag{1.1}$$

where, I is all inflow, O is all outflow, and dS is storage changes that occurred during time period dt. For a specific area and a specific time period, the water balance equation can be written as:

$$P - Q - E = \Delta S \tag{1.2}$$

where,

P = precipitation

Q = runoff

E = evaporation

ΔS = change in storage

Usually, the unit for water balance is mm/time (e.g., mm/month or mm/year). But, it can also be expressed in volume (for a specific period) or volume per time. The change in water storage (ΔS) will modify the total water storage (S) of the area. All the water that enters or leaves an area can be represented by these terms.

The precipitation (P) is the amount of water that falls as rain, snow, and/or hail. The runoff or discharge (Q) is the water that appears as surface water such as water in creeks, rivers, and/or lakes. The evaporation (E) can usually not be seen, but after condensation of the water vapour, we see the water as cloud drops. The total evaporation includes both evaporation from soil and water surfaces as well as transpiration from plants. The change in storage (ΔS) can be many things. It may be water that does not go directly as runoff. This may be snow and/or ice. It can also be infiltrating water that will appear as delayed runoff, and/or water that will percolate to the groundwater table, and/or water that will evaporate a little later.

1.4.1 Role of Water Balance Equation

The water balance equation is used for many basic analyses of water availability in an area, e.g., to find out how much water is available that can be used for drinking purpose or irrigation. This part is usually constituted by Q, i.e., runoff. The components of the water balance equation look different for different climatic conditions. It is logical that the total evaporation in warm areas is much larger as compared to cooler areas. Figure 1.8 shows, in a schematic way, an example of water balances for three different types of climatic conditions.

As seen in Fig. 1.8, the storage term (ΔS) is missing. This is due to the fact that for average values over longer time periods (e.g., several years) or over periods corresponding to multiples of a year, the storage is more or less likely to be constant. Thus, storage changes will be close to zero. We can, therefore, write the water balance equation as:

$$P - Q - E = 0 \tag{1.3}$$

This simplifies the calculations because there is one term less to determine when, for example, estimating available water (Q) for an area. The reason why the storage term can be assumed zero for longer periods is examplified in Fig. 1.9. When looking at the storage term of the water balance for a typical area over several years, it is usually noticed that it behaves in a periodic manner with a frequency equal to one year. That is, the typical period is one year and it is seen over many one-year periods that the change in storage is approximately

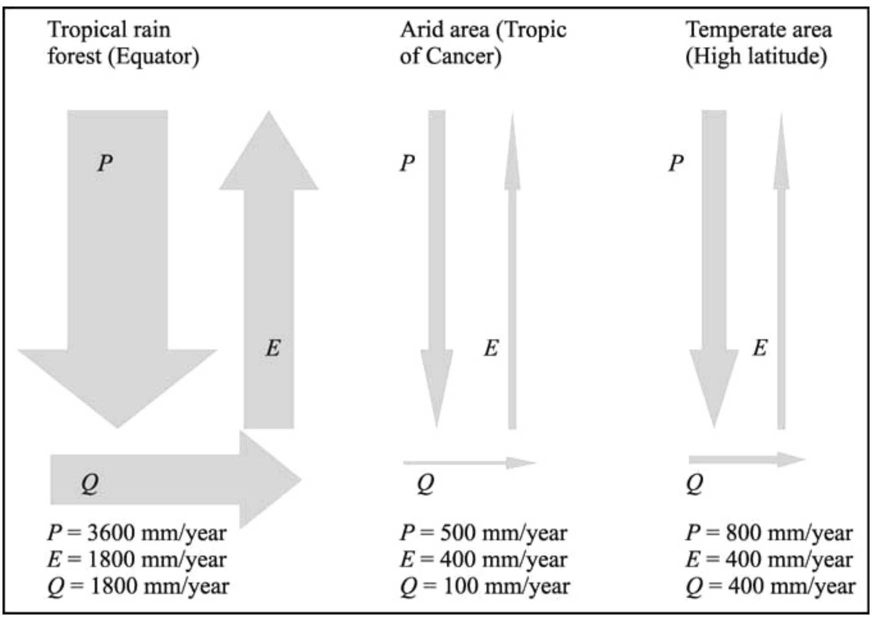


Fig. 1.8 *Example of water balances for three different types of climatic conditions*

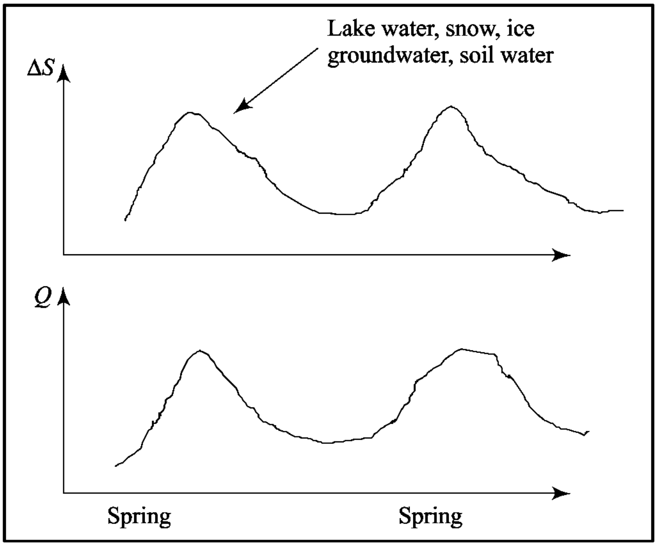


Fig. 1.9 *The storage term (ΔS) co-varies with precipitation and runoff in a cyclical manner over the year*

zero. This means that in water balance calculations involving a minimum of a one-year period, the storage term can be assumed to be zero ($\Delta S = 0$).

1.4.2 Catchment Area

Water balances are usually calculated for a specific area, which is called the *catchment area* (also called as *drainage basin*, *discharge area*, *precipitation area*, and *watershed*). A catchment is defined as the area upstream from a certain point in the water course that contributes to flow when precipitation falls. The size of the catchment depends on where this point is located in the stream and its topography or altitude situation.

In general, water flows from upper-lying areas to lower-lying areas in the catchment due to gravity. The area of the catchment is determined by the *water divide* (Fig. 1.10). The water divide is constituted by hills and peaks in the landscape over which the water cannot flow. Consequently, the catchment is completely surrounded by the water divide on all sides. All precipitation that falls within the catchment border (water divide) will tend to flow towards the lowest-lying point of the area which is the outflow point for the entire catchment. Some water, however, will infiltrate into the ground and become soil water or groundwater and be transported very slowly to the outflow point. Some water will never reach the outflow point because of evaporation. But, water reaching the outflow point must have come from the area within the catchment border. Consequently, other material and constituents transported by water, such as pollutants and nutrients that could come in contact with the water, will also reach the outflow point.

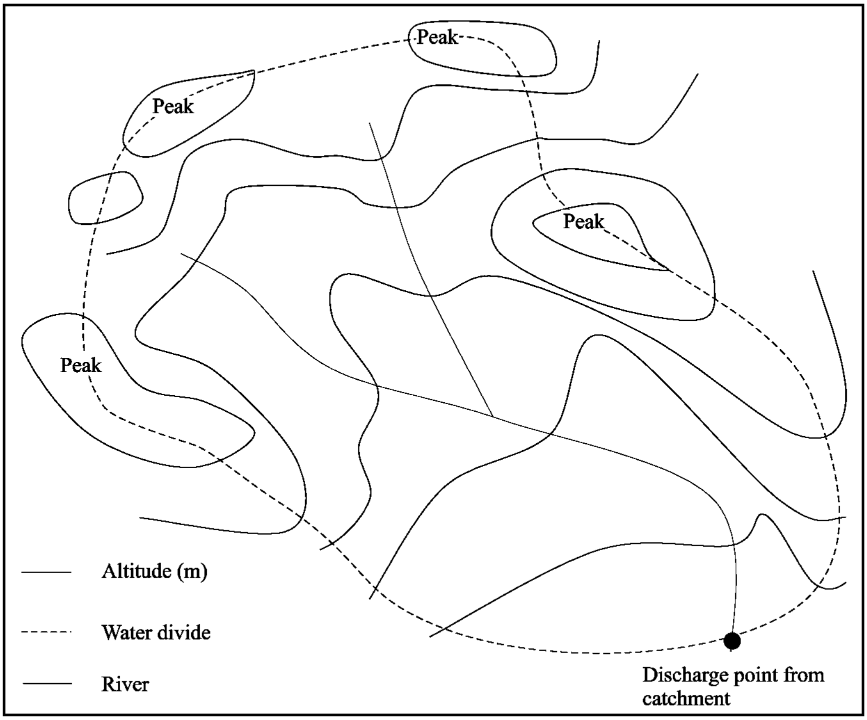


Fig. 1.10 *The water divide surrounds the catchment area and is defined by the topographical pattern*

From a topographical map (map with altitude levels) as in Fig. 1.10, the catchment area for a certain point in a stream can be determined. By following the altitude levels on the map, i.e., the highest points in the landscape (e.g., hills or mountain ridges) upstream from the point, one will eventually come back to the starting point, but on the other side of the water course. The water divide will always run perpendicular to the altitude levels on the map over the high peaks and ridges.

Usually, an outflow point is chosen so that it coincides with a location where water level or discharge measurement is performed. Once the catchment area is defined, it is possible to determine water balances for the area upstream from the discharge point. It is usually assumed that the water divide for the surface water coincides with the groundwater divide. This is usually the case; however, for some geological situations, (Fig. 1.11), the underground material's *hydraulic conductivity* (*rate of permeability*) can result in some discrepancies. For hydrologic engineering calculations it is often assumed in a relevant way that both surface water and groundwater follow the same water divide.

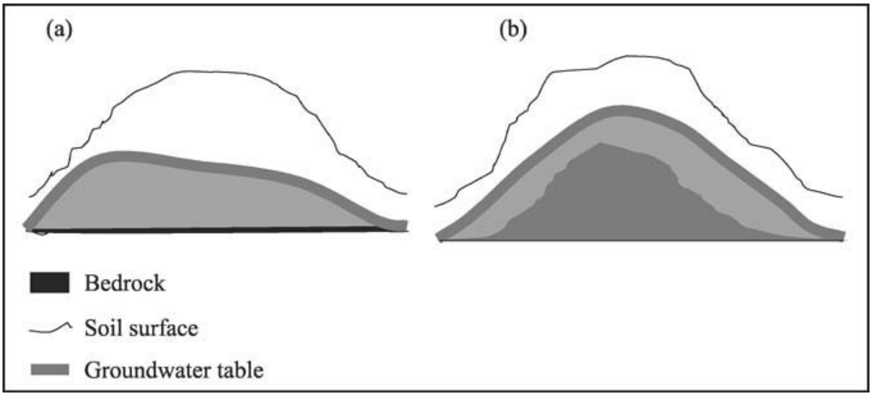


Fig. 1.11 *Example of a geological situation when the surface water divide does not coincide with the groundwater divide, (a) shows the case when the surface water and groundwater divide coincide, and (b) shows the case when the surface water and groundwater divide do not coincide (the underlying bedrock determines the groundwater divide instead) (Source: Grip and Rodhe 1991)*

The world's largest catchment is the Amazonas in South America (Table 1.3). Every 5th second, 1 Mm^3 of water is discharged into the Atlantic from this gigantic river. This represents 20% of all the world´s freshwater discharge from land areas. The largest river in India is the Brahmaputra with an average discharge of almost 20,000 m^3/s.

Table 1.3 The largest rivers in the world with catchment areas

River	*Catchment area (km^2)*	*Annual average discharge (m^3/s)*
Amazon	7,180,000	220,000
Kongo	4,014,500	39,600
Yangtze	1,942,500	22,000
Brahmaputra	935,000	19,800
Ganges	1,059,300	18,700
Jenisej	2,590,000	17,400
Mississippi	3,221,400	17,300
Orinoco	880,600	17,000
Lena	2,424,200	15,500
Parana	2,305,100	14,900
St. Lawrence	1,289,800	14,200
Irrawaddy	429,900	13,600
Ob	2,483,800	12,500
Mekong	802,900	11,000

1.4.3 Hydrological Data

To do a more detailed hydrological and environmental study, it is usually necessary to divide larger catchments into *sub-catchments* (smaller runoff catchments within the larger ones). This is done in the same way as above by defining smaller areas delimited by smaller water divides. In doing so, a more detailed picture can be obtained about water and environment problems of specific areas. Hydrological information can usually be found in national weather, hydrological, geological, and environmental surveys. Also, regional centers for water supply, hydropower, and/or irrigation usually keep hydrological data records. However, hydrological data can often be found at research centers, universities, river basin organizations, and various kinds of NGOs as well. Using this information, it is possible to find catchment discharge and use it, for example, to estimate the available water resources or pollutant transport.

Table 1.4 shows an example of a typical discharge statistics from a catchment. This type of information is typically necessary in order to estimate availability of water resources, the size of reservoirs, flood estimation, pollutant transport, etc. From the table, it is seen that the discharge is given both as dm^3/s and in $dm^3/(s\ km^2)$, i.e., runoff per unit area (km^2) within the catchment. It is also possible to see the *duration* of a specific flow (i.e., how many days per year), expressed in % of the total time that a certain runoff value is exceeded. This information is needed in order to decide the size of a reservoir to prevent flooding. Different minimum and maximum values (for different years and different months) for runoff are also given in the same table. This type of information is useful to determine if a certain organism with specific flow requirements would be able to survive in the water course (e.g., during dry summer months).

Table 1.4(a) Example of discharge statistics that can be used to solve water resources problems

Catchment Blue River Water course: Blue River; Drainage area 52 km², Lake percentage 11.5% Characteristic values of discharge											
	Maximum		*Mean*			*Minimum*		*Duration*			
	max	*mean*	*max*	*mean*	*min*	*mean*	*min*	*1%*	*50%*	*75%*	*95%*
dm^3/s	3208	1573	477	287	159	23	8.0	1612	119	46	21
$dm^3/s\ km^2$	62	30	9.2	5.5	3.1	0.45	0.15	31	2.3	0.88	0.40

Table 1.4(b) Mean duration of runoff 1965–1990

$dm^3/s\ km^2$	0.5	1	2	3	4	5	6	8	10	15	20	25	50	75	100	150	200
days/year	334	252	195	165	149	136	119	87	66	38	14	6	0	—	—	—	—

Table 1.4(c) Monthly and annual values of discharge and max and min values for the year in dm^3/s

	Jan	*Feb*	*Mar*	*Apr*	*May*	*Jun*	*Jul*	*Aug*	*Sep*	*Oct*	*Nov*	*Dec*	*Year*	*Max*	*Min*
1965	411	311	175	222	369	166	77	37	72	36	66	604	212	1182	8
1966	465	622	851	1181	883	370	125	106	139	118	130	745	477	2028	72
1967	850	903	790	689	38	217	92	54	62	98	106	454	390	1706	21
1968	1135	889	659	457	213	53	78	32	28	80	252	230	341	2256	21
1969	316	472	238	279	463	199	76	54	46	23	118	35	191	736	21
1970	34	26	599	2146	715	211	29	43	69	68	261	724	411	3208	20
1971	302	668	452	488	119	119	75	33	21	24	43	85	199	1518	21
1972	57	124	372	533	283	238	100	58	35	16	44	57	159	1034	8
1973	70	302	500	409	263	96	46	45	32	30	25	295	175	848	8
1974	713	743	354	95	45	34	35	22	37	106	508	1012	307	1512	21
1975	1076	750	453	632	286	110	50	47	48	54	55	52	299	1274	34
Mean	494	528	495	648	367	165	71	48	54	59	146	390	287	3208	8

Hydrologic information can also be published as maps of water balances for each month or average values as *isolines* (line joining points of equal values), e.g., *isohyets* (line joining points of equal precipitation on a map) for precipitation as exemplified in Fig. 1.12. The figure shows average precipitation in India. From maps like these, it is possible to get a quick overview of the water balance on an average for different areas in the country.

1.4.4 Role of Hydrological Engineer

Typical tasks of a hydrological engineer may be to estimate how much water or quantity of a specific pollutant is at hand at a specific location and time. This is an unusually concrete and specific question. More commonly, the contractor may provide the hydrological engineer with information on how to improve the flood situation or water environment in an area. To do this, the hydrological engineer has to create a logical map of what is governing the water environment of that particular area. Probably, first of all, the hydrological data and the land use information of the area have to be gathered. How much is the quantity of water and pollutants that enters and leaves the area? The hydrological engineer must find an answer to this question.

To solve this task, there is a set of commonly used calculation methods and techniques. These are mainly *empirical methods*, i.e., methods based on observations of water and material flows at various locations (usually not the specific location, one is interested in) during shorter or longer periods. These observations can be used to estimate the specific water and material balances for the area in question under given assumptions. The calculation methods are thus more or less based on physical quantities (conceptually imitating reality) and therefore be interpreted with great care. Most calculation techniques are

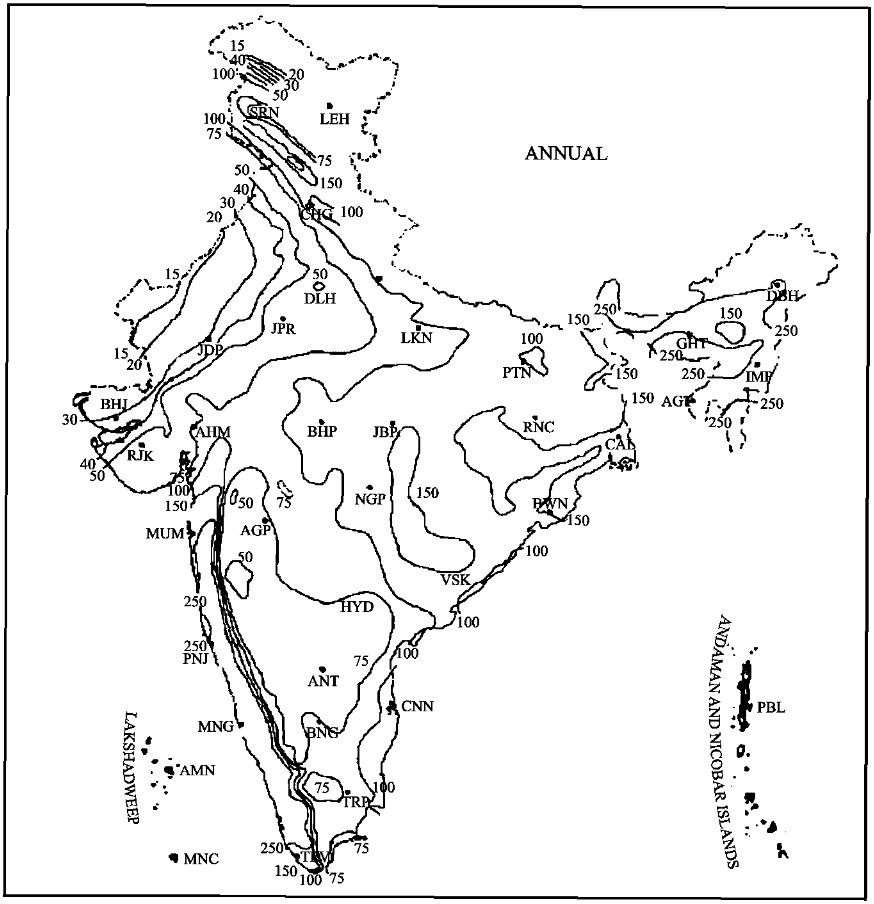


Fig. 1.12 *Precipitation isohyets in mm/year in India (Source: Indian Meteorological Department 2007)*

built upon simplifications of natural processes. It is, therefore, not evident that more or less general calculation techniques are valid for the specific area in question.

The most important task for the hydrological engineer is not to calculate some numbers for the above area but to *interpret* these numbers for the contractor. The interpretation is dependent upon what assumptions were made before the calculations, the degree of simplification, and the amount of background information acquired for the specific location.

Hydrological engineering can be said to be a typically *empirical* (based on observations in nature) science and not an exact science. This is obvious when numbers have to be estimated for water volumes entering an area under study. At the same time, typical questions from a contractor regarding hydrological problems are: (i) amount of pollutants, (ii) duration of flooding, (iii) frequency of flood, etc. Due to this, it is almost impossible to give any exact answers.

It is more important to try to give a reasonable interval for the answer to the questions. Primarily, *all hydrological calculations should be seen as a method to find a reasonable order of magnitude and/or interval and not a single number to falsely represent an absolute correct answer because this does not exist.*

The calculation methods in this text book are commonly used techniques that a hydrological engineer may use to solve practical problems. None of these methods give correct and definitive answer. Instead, every calculation must be complemented by assumptions made in the calculations.

Studying hydrology as an engineering subject means the emphasis is put on solving practical engineering problems involving different hydrological and environmental issues. The problem-solving also involves other aspects as listed below.

- Engineers often have to solve problems that are not well-defined by the contractor. Thus, more often than not, the engineer has to help himself or herself to formulate the correct questions. This involves finding the correct information or data which are required to solve the problem. For this reason, several assumptions and clarifications have to be made before the actual problem-solving can start. Each problem is therefore dependent on the starting assumptions that were made from the beginning, and these have to be clearly stated for every solution of a problem. The same problem may have several solutions depending on what assumptions were made.

- All engineering calculations and especially those involving hydrology, include uncertainties and errors. For this reason, all solutions to a problem should be given with an uncertainty interval. Especially, it is necessary to point out what factors can affect the final results. Calculations in hydrological and environmental problem-solving must not be given with unnecessary high (or low) numerical accuracy.

The components of the water balance are often given in different units depending on what part of the balance is in question. For example, precipitation is given in mm/day, mm/month, or mm/year; while runoff is usually given in l/s, dm^3/s, $dm^3/(s\ km^2)$, m^3/s, m^3/month, m^3/year, etc. When working with the water balance equation of a specific area, it is important to use the same unit in the equation. Some examples on how to use the water balance to solve some typical basic problems are discussed below.

SUMMARY

In this chapter, the basics of hydrology and the water balance and its components have been discussed. The water balance equation is the mass balance of water for a particular catchment. It states that all inflows of water to the catchment area minus all outflows from the catchment area during a certain time period must be equal to storage changes inside the catchment. For average values over longer periods, the storage change term may be set to zero.

The catchment is defined by the *water divide* and the area upstream from a certain point in the water course that contributes to flow when precipitation falls. The water divide may not always be the same for surface water and groundwater. In such cases, an error may be introduced in the water balance estimation. Hydrological information may be obtained from national and regional hydrological, meteorological, geological, and environmental surveys and agencies. Also, research institutes, universities, and NGOs may be able to provide useful hydrological information and data.

SOLVED EXAMPLES

Example 1.1 A sub-catchment within the Blue River Basin in Table 1.4 above has a catchment area of 200 ha. What is the average runoff?

Solution

From Table 1.4(a), mean value of average characteristic discharge

$= 5.5\ \text{dm}^3/(\text{s km}^2)$

$= (5.5/10^3)\ \text{m}^3/(\text{s km}^2)$

Sub-catchment area $= 200$ ha $= 200 \times 10^4\ \text{m}^2 = 200 \times 10^{-2}\ \text{km}^2$

Mean runoff = characteristics value of discharge × sub-cathchment area

$= (5.5/10^3)\ \text{m}^3/(\text{s km}^2) \times 200 \times 10^{-2}\ \text{km}^2$

$= 0.011\ \text{m}^3/\text{s}$

Example 1.2 The average runoff from the sub-catchment contains about 0.2 mg/l phosphorous. What is the annual transport of phosphorous from the sub-catchment?

Solution

From Example 1.1, we have:

Mean runoff $= 11 \times 10^{-3}\ \text{m}^3/\text{s}$

Transport of phorphorous with mean runoff $= 0.011\ \text{m}^3/\text{s} \times 0.2$ mg/l

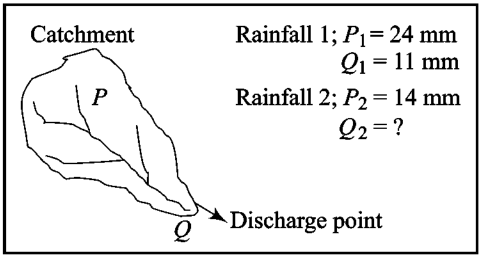
$= 0.011\ \text{m}^3/\text{s} \times (0.2/10^{-3})\ \text{mg/m}^3$

$= 2.2$ mg/sec

Annual transport $= 365 \times 24 \times 60 \times 60 \times 2.2$

$= 6.94 \times 10^7$ mg $= 69.4$ kg

Example 1.3 Two rainfall events, in close succession, fall over a catchment. After the first rainfall of 24 mm, a total runoff of 11 mm was observed. The second rainfall was measured at about 14 mm. How large a runoff can be expected after the second rainfall?



Solution

It has to be assumed that the two rainfalls were evenly distributed over the entire catchment. Under this assumption, the runoff from 24 mm becomes 11 mm from the first rainfall. This means that 13 mm "disappeared" on the way to the outflow point, i.e., this water either infiltrated and/or evaporated.

A simple way to solve this problem is to use the concept of *runoff coefficient.* The runoff coefficient, (c), is defined as runoff divided by precipitation for a specific precipitation event ($c = Q/P$), i.e., for the first rainfall, c becomes $11/24 = 0.46$.

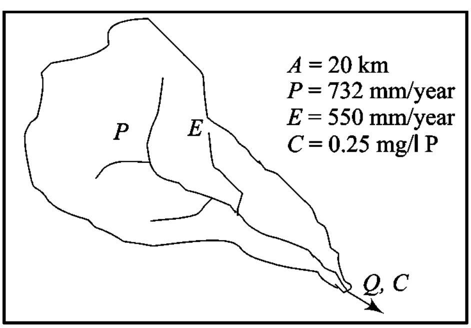
This means that from the first rainfall, just 46% was transformed into direct runoff. By assuming that the runoff coefficient is constant, i.e., no large changes will occur for losses such as infiltration and evaporation (the rain falls close in time and thus probably the physical conditions may be assumed to be the same for both rainfalls), we can put:

$$c = Q_1/P_1 = Q_2/P_2$$

$$\Rightarrow \quad Q_2 = (c \times P_2) = 0.46 \times 14 = 6.4 \text{ mm}$$

Answer Runoff from the second rainfall will be about 6.4 mm.

Example 1.4 For a catchment with an area 20 km^2, an average precipitation of 732 mm/year and a total evaporation of 550 mm/year have been observed during a 10-year period. Within the catchment, there is an old waste dump; and at the discharge point, high phosphorous (P) contents have been found, 0.25 mg/l of P. What is the annual mass transport of phosphorous in the water out from the catchment?



Solution

The water balance for the catchment is:

$$P - E - Q = \Delta S$$

In this case, the storage term S may be assumed to be zero because the calculations are made for average values (10 years).

Consequently, the water balance for the catchment is:

$$P = E + Q$$

Or, $\quad Q = P - E = 732 - 550 = 182$ mm/year

It has to be assumed that the concentration measurement of phosphorous is representative for the entire 10-year period, and this corresponds to 0.25 mg/L P. The runoff is transformed from mm/year to m^3/s by multiplying with the total catchment area.

Thus, the runoff becomes:

$$\begin{aligned} Q &= 182 \text{ mm/year} \times 20 \text{ km}^2 \\ &= (182 \times 10^{-3} \text{ m/year}) \times (20 \times 10^6) \text{ m}^2 \\ &= (3640 \times 10^3) \text{ m}^3\text{/year} = 3.64 \times 10^6 \text{ m}^3\text{/year} \end{aligned}$$

Let us transform this quantity to l/s.

$$\begin{aligned} Q &= 3.64 \times 10^6 \text{ m}^3\text{/ } (365 \times 24 \times 60 \times 60)\text{s} \\ &= 3.64 \times 10^6 \text{ m}^3\text{/}(31.54 \times 10^6)\text{s} \\ &= 0.115 \text{ m}^3\text{/s} = 115 \text{ l/s} \end{aligned}$$

This is consequently the average runoff every second for the 10-year period. Now, we can calculate the transport of phosphorous that discharges in the water from the catchment during one year.

$$Q \times C = 115 \text{ l/s} \times 0.25 \text{ mg/l} = 28.75 \text{ mg/s}$$

The phosphorous transport per year is

$$\begin{aligned} &= 28.75 \text{ mg/s} \times (60 \times 60 \times 24 \times 365) \text{ s/year} \\ &= 907 \text{ kg/year} \end{aligned}$$

This is probably not an exact figure, but we may assume the order of magnitude of phosphorous transport is about 900 kg/year.

Answer The phosphorous transport out from the catchment is about 900 kg/year.

Example 1.5 A lake has an area of 15 km^2. Observation of hydrological variables during a certain year has shown that:

$P = 700$ mm/year

Average inflow $Q_{in} = 1.4$ m^3/s

Average outflow $Q_{out} = 1.6$ m^3/s

Assume that there is no net water exchange between the lake and the groundwater. Determine the evaporation during this year.

Solution

(Σ inflows to the lake – Σ outflows from the lake) = change in volume

$$(P + Q_{in}) - (E + Q_{out}) = \Delta S$$

In this case, observations have been made covering a full year. Consequently, it may be assumed that the storage changes occurring over a year is following a regular sinus curve (see Fig. 1.9). The storage changes over one year can thus be assumed as zero. ($\Delta S = 0$)

$$(P + Q_{in}) - (E + Q_{out}) = 0$$

$$E = P + Q_{in} - Q_{out}$$

If we use mm as a common unit over the lake, then

$$P = 700 \text{ mm}$$

$$Q_{in} = 1.40 \times 3600 \times 24 \times 365/15.0 \times 10^6 = 2943.4 \text{ mm}$$

$$Q_{out} = 1.60 \times 3600 \times 24 \times 365/15.0 \times 10^6 = 3363.8 \text{ mm}$$

$$E = 700 + 2943.4 - 3363.8 = 279.6 \text{ mm}$$

Answer The evaporation from the lake during the year is about 280 mm.

Example 1.6 A pipe for discharging storm-water from a parking lot (area = 2500 m^2) in an urban area is to be constructed. The maximum rainfall that can occur is assumed to have an intensity of 60 mm/hr. Determine the corresponding discharge (m^3/s) and the dimension (diameter) of the pipe if the water velocity in the pipe is assumed to be 1 m/s.

Solution

Urban areas behave differently regarding infiltration and evaporation as compared to rural areas.

In urban areas, a large proportion of the land is often made impermeable (asphalt, roofs, etc). It is usually assumed that no infiltration can occur through an asphalted parking lot. Similarly, no evaporation can occur through this surface even though water on top of the surface may be evaporated.

Since no infiltration occurs through the surface, no losses through infiltration will occur. The runoff occurring on impermeable surfaces also occurs rather quickly, so that evaporation losses may be assumed zero.

If the discharge Q is assumed to run off instantly from the rainfall, the average discharge from the parking lot having an area of 2500 m^2 during an hour is:

$$Q = P \times A = (60 \times 10^{-3}) \times 2500/3600 = 0.042 \text{ m}^3/\text{s}$$

If the velocity $v = 1$ m/s, then the cross-sectional area A_c becomes:

$$A_c = Q/v = (0.042\ \text{m}^3/\text{s})/(1\ \text{m/s}) = 0.042\ \text{m}^2$$

$$A_c = \pi \times (D/2)^2, \text{ where } D \text{ is the diameter of the pipe}$$

$$\Rightarrow \quad D = 2 \times (0.042/\pi)^{0.5} = 0.23\ \text{m}$$

Answer The diameter of the pipe has to be at least 0.23 m.

Example 1.7 The evaporation from a lake is to be calculated by the water balance method. Inflow to the lake occurs through three small rivers A, B, and C. The outflow occurs through river D. Calculate the evaporation from the lake surface during summer (May–August) if the water level was at +571.04 m on May 1 and +571.10 m on August 31. The lake surface area is 100 km^2. The precipitation P during the period was 100 mm. Average inflows and outflow are given below.

River	*Catchment (km^2)*	*Q, average (m^3/s)*
A	150	15.0
B	120	20.0
C	130	17.0
D	—	45.0

Solution

Assuming that no other inflows or outflows (e.g., exchange with groundwater) are occurring than the ones stated above, we can define the water balance for the lake as:

$$(Q_{in} + P) - (Q_{out} + E) = S$$

In this example, it will be convenient to use the common unit, m/time = m/(May–Aug); May–August = 123 days

The total inflow from rivers to the lake during this period is:

$$Q_{in} = (15.0 + 20.0 + 17.0) \times (123 \times 24 \times 3600)/(100 \times 10^6) = 5.526\ \text{m}$$

$$P = 0.100\ \text{m}$$

$$Q_{out} = 45 \times (123 \times 24 \times 3600)/(100 \times 10^6) = 4.782\ \text{m}$$

$$\Delta S = 571.10 - 571.04 = 0.060\ \text{m}$$

Inserting the calculated data in the water balance equation gives:

$$(5.526 + 0.100) - (4.782 + E) = 0.060$$

$$\Rightarrow \quad E = 5.526 + 0.100 - 4.782 - 0.060 = 0.784\ \text{m}$$

Answer Evaporation during this period was 784 mm.

Example 1.8 A lake has a surface area of 7.0×10^5 m^2. During a given month, the mean inflow to the lake was 2.5 m^3/s. The increase in stored lake volume was observed to be 6.5×10^5 m^3. Precipitation during the same month was 250 mm and evaporation was 420 mm. Calculate the outflow from the lake for the same month.

Solution

We must assume that all the major inflows and outflows are known. Specifically, we need to assume that the exchange between groundwater and lake water is insignificant.

The water balance equation is:

$$(Q_{\text{in}} + P) - (Q_{\text{out}} + E) = \Delta S$$

In this equation, $Q_{\text{in}} = 2.5$ m^3/s, $P = 250$ mm, $E = 420$ mm, $\Delta S = 6.5 \times 10^5$ m^3, lake area $A = 7.0 \times 10^5$ m^2, $Q_{\text{out}} = ?$

Therefore, outflow from the lake becomes:

$$Q_{\text{out}} = Q_{\text{in}} + (P - E) - Q_{\text{out}} - \Delta S$$

$$Q_{\text{out}} = 2.5 + \frac{[(0.250 - 0.420) \times (7.0 \times 10^5)]}{(3600 \times 24 \times 30)} - \left[\frac{6.5 \times 10^5}{3600 \times 24 \times 30}\right]$$

$$Q_{\text{out}} = 2.5 - 0.046 - 0.251 = 2.20 \text{ m}^3/\text{s}$$

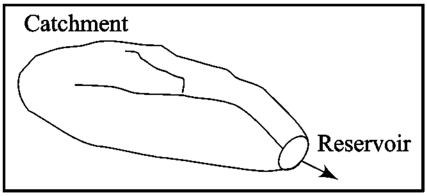
Answer The outflow during this month is 2.2 m^3/s.

Example 1.9 A retention reservoir is located so that all runoff from upstream catchment area passes through the reservoir according to the given schematic figure. The catchment area (reservoir not included) is 100 ha. The reservoir area is 1.0 ha. Infiltration through the bottom and sides of the reservoir is negligible. The discharge out from the reservoir is regulated at a constant $Q = 0.045$ m^3/s. Mean precipitation in July is 178 mm and mean inflow to the reservoir during the same month is 95000 m^3. Mean evaporation from the lake surface in the month of July is 3.1 mm/day.

(a) What is the average July runoff coefficient? ($c = Q/P$)

(b) In July 1991, precipitation was slightly higher than normal, $P = 220$ mm. What is the total inflow expected to the reservoir for this specific month?

(c) What is the expected change in water level of the reservoir during this month?



Solution

(a) The average runoff coefficient is:

$$c = Q/P = Q_{in}/P = 95000 / (0.178 \times 100 \times 10^4) = 0.53$$

(b) Here, we have to assume that the average runoff coefficient is also valid for the month of July, 1991.

Then, inflow to the reservoir becomes:

$$Q_{in} = c \times P = 0.53 \times 220 = 116.6 \text{ mm}$$

By multiplying this height with the area of the catchment, we convert the inflow to a volume.

$$0.1166 \times 100 \times 10^4 = 116600 \text{ m}^3$$

(c) By using a water balance for the reservoir, ΔS will be calculated.

$$(Q_{in} + P) - (Q_{out} + E) = \Delta S$$

Here, $Q_{in} = 95000$ m^3, $P = 220$ mm, $Q_{out} = 0.045$ m^3/s, and

$E = 3.1$ mm/day

This gives:

$$\Delta S = (95000 + 0.220 \times 10000) - [(0.045 \times 3600 \times 24 \times 31) + (0.0031 \times 31 \times 10000)]$$

$$\Delta S = 97200 - (120528 - 9610) = -13718 \text{ m}^3$$

The change in storage is negative indicating that the water level in the reservoir will decrease.

The decrease in water level is:

$$13718/10000 = 1.37 \text{ m}$$

Answer (a) The runoff coefficient is $c = 0.53$, (b) the inflow to the reservoir during July, 1991 is 116600 m^3, and (c) the reservoir level will decrease by 1.37 m.

Example 1.10 The average annual precipitation in a 3000 ha large catchment is 1200 mm. The corresponding value for actual evapotranspiration is 600 mm.

(a) What will be the average annual runoff from the catchment?

(b) A way to reduce nutrient and pollutant transport in runoff may be to create biological ponds where physical and biological processes reduce the nutrient content. The recommended turnover time for such a pond is about two days. Estimate the needed area for such a pond if half of the runoff from the above catchment is to be treated.

(c) Typical nutrient concentration in runoff water from the above catchment is 5–10 mg N/L and 0.1–0.3 mg P_{tot}/L. Calculate the nitrogen and phosphorous transport in kg/year from this catchment.

Solution

(a) As we are using average values over several years, we may estimate the average runoff according to:

$$P - E = Q$$

$$\Rightarrow \qquad 1200 - 600 = 600 \text{ mm/year}$$

This gives:

$$(0.6 \times 3000 \times 10^4)/(365 \times 24 \times 3600) = 0.57 \text{ m}^3/\text{s}$$

as an average discharge from the catchment during one year.

(b) Assume an average depth of the pond, e.g., 1 m.

The needed area is:

$(2 \times 0.5 \times 0.57) \times (365 \times 24 \times 3600) = 1798 \times 10^4 \text{ m}^2$

(c) The annual transport will be between ($5–10 \times 0.57 \times 10^3 \times 365 \times 24 \times 3600$) mg N and between ($0.1–0.3 \times 0.57 \times 10^3 \times 365 \times 24 \times 3600$) mg P per year.

This gives about $90–180 \times 10^3$ kg N and about 1800–5400 kg P_{tot} per year.

Answer (a) The discharge is 0.57 m^3/s, (b) the area necessary is about 1800 ha, and (c) about $90–180 \times 10^3$ kg N and about 1800–5400 kg P_{tot} per year.

Example 1.11 Derive a general relationship between the water level in a lake and the inflow/outflow from the lake depending on time.

Solution

To analyse the relationship, a general water balance for the lake can be used. The water balance during inflow/outflow will have the following appearance.

$$I - O = dS/dt$$

where, I and O are inflow and outflow, respectively; and dS is the temporary storage of water in the lake during the time, dt.

The storage (dS) depends mainly on the geometry of the lake, i.e., how deep and long/wide it is. In principle, the water stored is equal to:

$$dS = A(z)\, dz$$

where, $A(z)$ is the surface area of the lake, and dz is the change in depth of the lake. Due to the fact that the lake geometry is an unknown variable, we need to assume water-covered area $A(z)$ as a function of the water depth (z) in a reference point. This gives:

$$A(z)\, dz = (I - O)\, dt$$

$$\Rightarrow \qquad dz/dt = (I - O) / A(z)$$

$$\Rightarrow \qquad dz/dt = [Q_{in}(t) - Q_{out}(t)]/[A(z(t)]$$

Answer $dz/dt = [Q_{in}(t) - Q_{out}(t)]/A(z(t)$.

Example 1.12 Discuss how urbanization will affect the water balances and pollutant transport.

Discussion
Urbanization directly affects the runoff part of the water balance through decreasing infiltration, depending on the degree of impermeable surfaces of the area. Thereby, evaporation processes will be affected due to the fact that evaporation will occur only from wetted, impermeable surfaces. Thus, usually the runoff component increases greatly in terms of peak value and volume.

Urban areas also create heat islands and an increasing amount of dust particles, affecting precipitation amounts positively. Urban areas generate different types of pollutants as compared to rural areas. Typical pollutants from urban areas are heavy metals, suspended solids, oil, grease, etc.

Example 1.13 Evaporation losses from a reservoir can be estimated using the water balance method. Identify the necessary components to quantify this approach. How can the components be estimated and/or measured to calculate the actual evaporation?

Discussion
The general water balance equation reads $E = P - Q - \Delta S$. The rainfall is observed using a rain gauge. Losses of the groundwater can be quantified by observing groundwater table. Runoff in rivers is observed by water level and current meter, to and from the lake. Storage changes are determined by observing the lake water level.

Example 1.14 The water balance equation is often used to estimate average available runoff. Discuss the uncertainties and potential errors involved in such an approach.

Discussion
The available runoff is often determined using the water balance over a longer period of time. Thus, runoff is determined as the difference between average precipitation minus average evapotranspiration. This means that any error or uncertainty in the average precipitation and evapotranspiration carries over to the estimated runoff. For example, if the precipitation gauge is not representative for the area (may be located in a valley in a mountainous catchment), the gauge will probably underestimate the areal precipitation.

EXERCISES

Exercise 1.1 The evaporation is to be estimated from a lake, and therefore, a pan evaporimeter has been placed close to the lake. The water level is observed every morning, and then water is added to the pan if the water level comes close to 175 mm. Determine the daily potential evaporation for the first 5 days if the pan overestimates the real evaporation by $E_{\text{real}} = 0.70\ (\text{Pan})_{\text{value}}$.

Day	*Rainfall (mm)*	*Water level (mm)*	*Day*	*Rainfall (mm)*	*Water level (mm)*
1	0.0	203.2	8	0.3	192.4
2	5.8	205.9	9	0.0	187.6
3	14.2	217.5	10	0.0	182.8
4	1.3	214.3	11	0.0	178.3
5	0.3	208.3	12	0.3	200.9
6	0.0	201.7	13	0.0	195.6
7	0.5	196.7	14	0.5	190.1

Answer The corrected evaporation is: 2.2, 1.8, 3.2, 4.4, and 4.6 mm.

Exercise 1.2 The following series of precipitation and runoff has been observed for a large catchment in northern Europe (P: mm/month; Q: mm/month, area = 46,830 km^2 with 18.6% lake area) during January 1995–December 1996.

(a) How did the actual evaporation vary during the period? (*Tips: calculate the evaporation using the water balance for every month.)*

(b) Comment on the results.

Month	*P*	*Q*	*Month*	*P*	*Q*
1	32.5	26.6	13	25.5	35.4
2	30.0	30.7	14	19.5	36.5
3	42.0	31.1	15	52.0	35.2
4	55.5	34.7	16	48.5	39.2
5	37.5	33.7	17	17.0	38.5
6	67.0	34.0	18	63.5	38.7
7	35.5	31.6	19	127.5	38.5
8	148.5	28.8	20	79.5	35.5
9	84.5	30.4	21	61.5	34.1
10	28.5	32.5	22	56.0	32.4
11	61.5	35.6	23	31.0	33.2
12	57.0	37.7	24	32.0	32.3

Hint:

The water balance gives occasional negative values. This depends on the fact that the storage effect cannot be disregarded. Especially during winter when snow fall may occur, water is temporarily stored in the catchment.

Exercise 1.3 During a month, five rainfall events are recorded in a forest area. The potential evaporation for the forest is about 0.5 mm/hr. The forest also has a local storage (the canopy acts as a storage that can adsorb water up to 0.05 cm) of about 0.05 cm. The potential evaporation acts on this storage. Estimate the total amount of water that can pass through the canopy down to the soil surface during the observation period given below.

Rain	*amount (cm)*	*duration (hr)*
1	1	7
2	0.5	5
3	1	4
4	2	24
5	0.2	2

Answer Assuming an empty "canopy storage" from the beginning, Q becomes 0.6, 0.2, 0.75, 0.75, and 0.1 cm. Thus, approximately 50% of the rain can pass through the canopy down to the soil surface.

Exercise 1.4 A shallow circular pond of 2 m depth and radius of 50 m is having a water depth of 0.5 m. The concentration of phosphorous in pond is 2 mg/l. If the pond received an annual precipitation of 100 mm, what is the change in the total phosphorous concentration? Assume no phosphorous consumption in the pond and concentration of phosphorous in rainwater is zero. Neglect any losses due to seepage or evaporation of water.

Answer 1.66 mg/l

Exercise 1.5 Show the monthly variation of phosphorous concentration in the pond of Exercise 1.4 for the following monthly precipitation and evaporation data.

Month	*Jan*	*Feb*	*March*	*April*	*May*	*June*	*July*	*Aug*	*Sept*	*Oct*	*Nov*	*Dec*
Precipitation (mm)	4	8	0	0	5	50	85	110	45	10	4	4
Evaporation (mm/month)	1	1	15	20	25	35	15	12	10	7	2	2

OBJECTIVE QUESTIONS

1. What is the total amount of fresh water on earth in per cent of total water volume?
   (a) 0.001% (b) 0.1% (c) 2.7% (d) 10%
2. What is the approximate water velocity of water flow in a river?
   (a) 0.01 m/s (b) 0.1 m/s (c) 1 m/s (d) 5 m/s

3. What is the average domestic water consumption in a developed country?
   (a) 220 l/day (b) 20 l/day (c) 540 l/day (d) 1000 l/day
4. What is the average turnover time for water in ice caps and glaciers?
   (a) 50 years (b) 2000 years (c) 10,000 years (d) 8000 years
5. Convection is
   (a) Infiltration of water (b) Uplift of air
   (c) Radiation of energy (d) Plant respiration
6. The major climatic areas of the earth are mainly determined from
   (a) Location of south and north pole
   (b) Global atmosperic flow
   (c) Ocean currents
   (d) Planetary orbits
7. The water divide determines the
   (a) Catchment boundaries
   (b) Division between wastewater and drinking water
   (c) Areas collecting stormwater in a city
   (d) Division between water and snow in a snowpack
8. Which is the river basin with the largest discharge on earth?
   (a) The Amazon (b) The Ganga (c) Mississippi (d) Donau
9. Empirical means
   (a) Approximative (b) Theoretical
   (c) Obeying physics (d) Observation based
10. How many people die every year due to water-borne deseases and pollutants?
    (a) 5 million (b) 0.5 million (c) 25 million (d) 10 million

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CHAPTER 2

Elements of Hydrologic Cycle

2.1 INTRODUCTION

The objective of this chapter is to give an easily comprehensible overview of the hydrologic cycle with all the elements and their interconnected relationships. In general, precipitation is the driving component of the hydrologic cycle. After precipitation has fallen over the catchment, a number of losses or abstractions from the total precipitation volume occur through several processes. The more important ones are described below.

Interception is the process by which precipitation wets leaves and branches of trees and vegetation. Some of this water will evaporate back to the atmosphere and some water will follow the branches towards the soil surface. At the soil surface, water may *infiltrate*. Some of this infiltrated water will *percolate* deeper to the groundwater table. Infiltrated water may also *evaporate* from the upper soil sections, or water may be taken up by plant roots, and *transpiration* will occur through *photosynthesis*. The remaining water that is found at the catchment discharge point is usually termed *surface runoff* or *discharge*. This water in general contains both, water directly related to the precipitation event, and water drained from the groundwater table. For availability reasons, this water together with groundwater is generally regarded as potential water resource(s) that can readily be used as water supply or other types of human use.

Following is a more detailed description of how to estimate or calculate the various abstractions so as to be able to arrive at the surface runoff volume.

2.2 PRECIPITATION MECHANISMS

Precipitation is a summarizing term for atmospheric water that is deposited on ground, such as rainfall, drizzle, snow, and hail.

A common unit for precipitation is mm per time unit (e.g., mm/hour, mm/day, mm/month, mm/year). The unit indicates that the height of water is assumed to be evenly distributed over an area (usually a catchment). Then, 1 mm precipitation corresponds to 1 l/m^2 or 0.001 m^3/m^2 (1 mm distributed over 1 m^2).

2.2.1 Condensation

The global annual average precipitation has been estimated at about 900 mm over sea and 670 mm over land (National Encyclopedia, 1996). As mentioned in the previous chapter, there is an extreme variation of water occurrence in different parts of the Earth depending mainly on the general atmospheric circulation. The largest recorded annual precipitation was 26,000 mm in Cherrapunji in northern India during the period 1860–61. The highest precipitation observed during one minute (31.2 mm) was in Maryland, USA. For desert areas, the annual precipitation is usually about 50–200 mm. For precipitation to form, three simultaneous conditions are required, i.e., access to *water vapour*, *condensation nuclei,* and *cooling of the moist air mass*. Consequently, for precipitation to form, the water vapour in the air must be transformed to liquid drops. This is done by the presence of *condensation nuclei*.

A condensation nucleus is a microscopic dust particle that water vapour can precipitate on and create water droplets. These water droplets are in order of size (radius) less than 20×10^{-6} m (Fig. 2.1). For this to occur, however, cooling of the air mass is also required, i.e., cooler air can hold less water as compared to a warmer air mass.

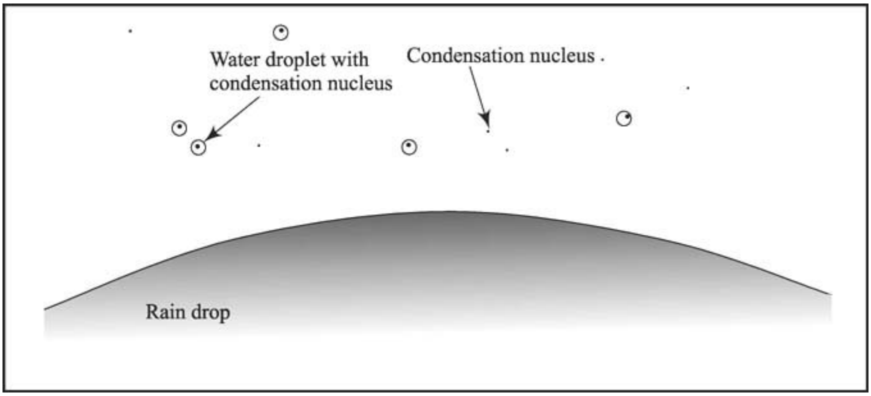


Fig. 2.1 *Comparison of size order of rain drops (>1 mm), water droplets (< 0.02 mm), and condensation nuclei (< 0.001 mm)*

2.2.2 Convection

Large cities generally create a local heat island around the urban area. This means that *local convection* is formed by hot air rising upwards and thus cools off (*adiabatic cooling*). Urban areas also generate air pollutant particles that may act as condensation nuclei and contribute to generate larger amounts of precipitation in these areas.

Local convection consequently means that small volumes of moist air heat up due to the solar influx or some other kind of local warming mechanism. Due to the heating, the air volume will expand and thus a decrease in density. The decreasing density means that the lighter air volumes will be lifted upwards in the atmosphere where it is cooler and then precipitation occurs.

Local convection is a way for an air mass to cool off and thereby form precipitation. Another way for moist air masses to cool off is *frontal lifting*. This mechanism is put in to work when a warm front meets a cold front in an atmospheric system. Due to the fact that the warm air in the warm front has a lower density than the cold air, the warm air is forced upwards on top of the cold front (Fig. 2.2). This also induces cooling. A third precipitation mechanism is when the wind forces the warm air up along and above a mountain ridge. Also, the air mass is forced upwards, into a cooler atmosphere. This is called *orographic lifting* (Fig. 2.2).

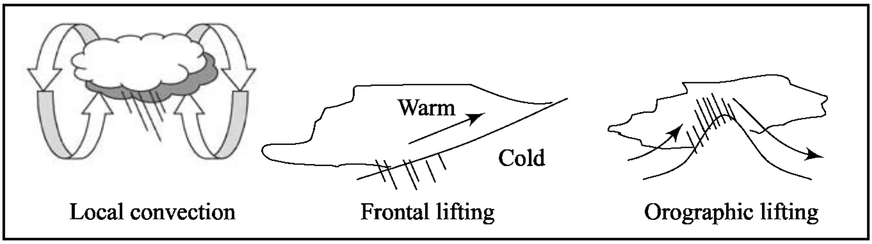


Fig. 2.2 *Three common causes of cooling of warm moist air and precipitation*

The orographic lifting suggests that there is often a topographic (altitude) dependence for precipitation pattern in the landscape. An example of this is shown in Fig. 2.3. Areas at higher altitudes, thus, receive larger precipitation amounts. Due to the effects of general wind direction, the leeward side of a mountain will receive much less precipitation as compared to the windward side for the same altitude.

2.3 MEASUREMENT OF PRECIPITATION

Precipitation is the driving component in the hydrological cycle. Without precipitation, there is no runoff, evaporation, or groundwater recharge. Due to this, it is important to make reliable precipitation observations, e.g., in water balance calculations. Precipitation is usually observed at many locations in important catchments. Precipitation is also fairly easy to observe.

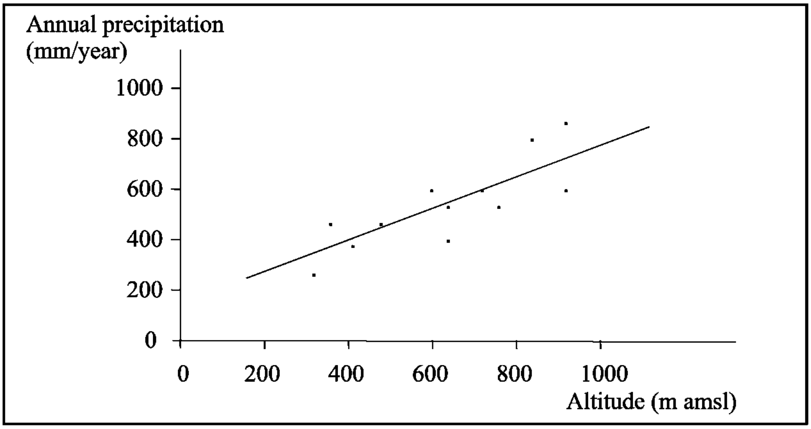


Fig. 2.3 *Example of topographic dependence of precipitation. The full line shows a theoretical linear equation fitted to the annual precipitation data vs. altitude*

Usually, a standard precipitation gauge consisting of a bucket-type collector is located at a specific height over the ground surface and read manually once a day. A common standard gauge has an opening of about 200 cm^2 and a surrounding windscreen (see Fig. 2.4). The windscreen is put around the gauge-opening to reduce errors due to wind. The stronger the wind, the stronger the deflection of rain, particularly snow. The gauge is emptied once a day (usually 8:30 am) and the content is measured with a specifically designed measuring glass. If there is ice or snow in the gauge, it has to be melted before measuring.

2.3.1 Short-term Measurement

Often, it is necessary to have a shorter-period observation than one day of precipitation. This is needed to estimate flooding risks in catchments. Also, in urban catchments, short intensive rainfall showers have a tendency to wash

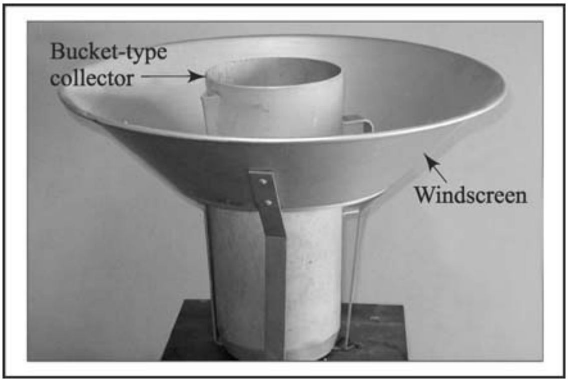


Fig. 2.4 *Standard precipitation gauge consisting of a bucket-type collector and a windscreen*

off accumulated pollutants from streets, parking lots, roofs, etc. Typical pollutants are oil, heavy metals, and other *toxic* substances.

To estimate the amount of pollutants transported by the surface water from precipitation, it is necessary to measure precipitation down to minute values. For such observations, non-recording gauges are not suitable. Instead, automatic or recording precipitation gauges are used. When short-term measurements of precipitation are done, usually some type of automatic gauge is used. A common type of gauge used for such measurements is the so-called tipping bucket gauge. This gauge contains a balancing bucket or a small container that tips as soon as half of the bucket is full (see Fig. 2.5). When the bucket tips, the water is emptied and the movement is registered automatically. The number of tips per unit time consequently corresponds to a known amount of water, which can be expressed in mm/minute.

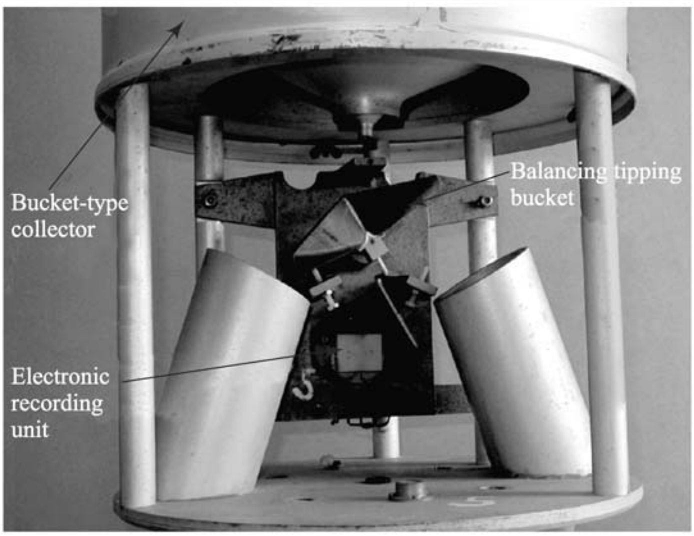


Fig. 2.5 *Tipping bucket rain gauge*

2.3.2 Error Analysis

All hydrological observations, including precipitation measurements, contain errors and uncertainties. The error in precipitation observations depends on the size and type of the gauge, the wind speed, whether the precipitation is constituted by rainfall or snow, the size and distance of shielding buildings and trees, reading mistakes, etc. The sum of all these errors will in turn carry over to a corresponding error in the water balance calculations.

In general, observational errors can be divided into random and systematic errors. *Random errors* may be less important and also more difficult to quantify. *Systematic errors* resulting from an unrecorded change in gauge location or shielding effect from trees or buildings from a certain date can be revealed with the help of a *double mass curve*. This technique will reveal if precipitation observations coming from a certain gauge are *statistically homogeneous* (if data belong to the same *statistical population*) as compared to other gauges in a certain area. The double mass analysis is done by comparing accumulated values (daily, monthly, or annual) of precipitation ($\sum P_X$) for a certain gauge X to the accumulated values of the mean for all other gauges in the same population ($\sum P_{\text{mean}}$). Figure 2.6 gives an example of such an analysis. According to the figure, the investigated rainfall gauge appeared to behave in a similar way in relation to other gauges in the area from 1991 to 1997. From 1998, however, the precipitation amounts suddenly decreased in a systematic way for gauge X, suggesting that the rainfall climate suddenly changed, perhaps due to shielding by an object near the gauge.

If a systematic error is confirmed for the gauge in the field, the change in slope of the curve after 1998, as shown in Fig. 2.6, can be used to correct the precipitation values. This test can be repeatedly performed for all gauges for a first quality check of the precipitation data.

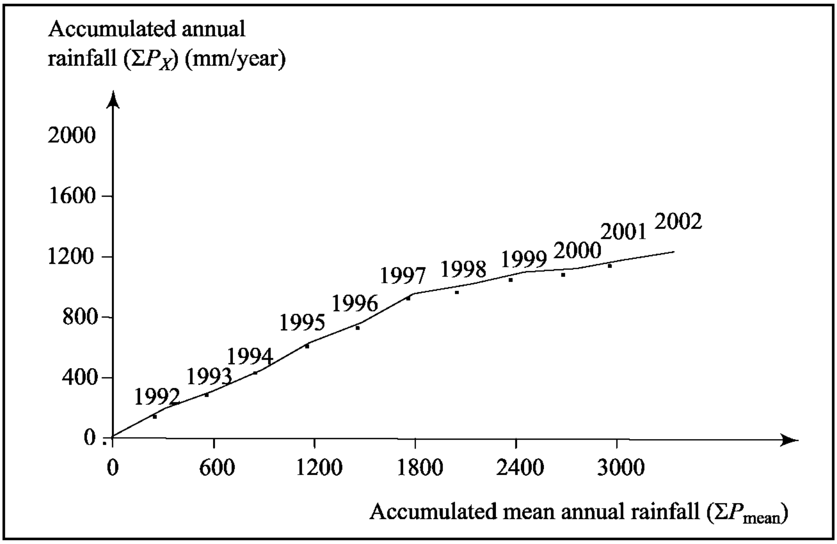


Fig. 2.6 *Example of a double mass curve analysis. The X-axis displays mean annual rainfall for a group of gauges. The Y-axis displays the annual rainfall at station X, one of the gauges in the group of gauges for a quantitaitve quantity check.*

2.3.3 Other Types of Measurement

Precipitation intensity can vary greatly during short periods of time and from one place to another. Consequently, it is difficult to get an areal estimate of precipitation if the number of gauges in an area is small. The problem according to the above is to calculate areal precipitation values based on the point measurements that the gauges represent. The collection area of the gauge is very small as compared to the area that the gauge is supposed to represent.

Therefore, some technique is needed to generalise the point measurements to areal estimates to be valid for. The areal estimates of precipitation are needed in order to be able to calculate water balances for the catchment. The simplest way to estimate areal values from point measurements is to calculate the *arithmetic mean of all gauges within the catchment* according to:

$$P_A = \Sigma P_i / n = (P_1 + P_2 + P_3 + ... + P_n)/n \tag{2.1}$$

where, P_A is the areal precipitation, P_i is the individual value for each precipitation gauge, and n is the number of precipitation gauges in the catchment. In this method, only those gauges which are inside the catchment border are used. The averaging method works rather well if the gauges are evenly distributed over the area. If the gauges have an uneven distribution, it would be better to use a method that weighs a representative area around each gauge. Such a method is called the *Thiessen method.*

$$P_A = \Sigma P_i a_i / A = (P_1 a_1 + P_2 a_2 + P_3 a_3 + ... + P_n a_n)/A \tag{2.2}$$

In this method, each gauge is given a representative area (a_i) of the total catchment area (A) that is assumed to reflect the average precipitation over the entire catchment. The method of dividing each partial area to respective gauge is exemplified in Fig. 2.7. This method also utilizes gauges outside the catchment border.

In case of snow storage in a catchment, it is useful to calculate the amount of water the snow represents. This calculation helps to estimate the spring flood. A common technique is to estimate the snow depth and snow density in the catchment using a *snow tube*. The snow tube is a simple device consisting of a long tube that can be used to sample snow different locations in a catchment. The sampled snow is weighed, and with diameter of the snow tube and height of the snow pack, the density of snow can be estimated. The *snow water equivalent* can be determined according to:

$$m_{\text{snow}} = m_{\text{water}} \tag{2.3}$$

where, m_{snow} and m_{water} are the weight of snow and water, respectively, in kg/m^2. Substituting m for density times volume (height) gives:

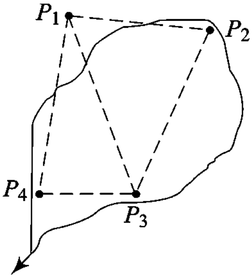
$$h_{\text{snow}} \times \rho_{\text{snow}} = h_{\text{water}} \times \rho_{\text{water}} \tag{2.4}$$

where, h_{snow} and h_{water} are depth of snow and the depth of the water equivalent, respectively, in metre; and ρ_{snow} and ρ_{water} are the density of snow and water, respectively (ρ_{water} = 1000 kg/m^3). Therefore,

$$h_{\text{water}} = h_{\text{snow}} \times \rho_{\text{snow}}/1000 \tag{2.5}$$

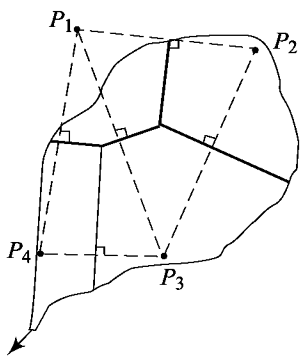
A simple way to calculate continuous melt from snow is to use the *degree-day method*. This is an *empirical equation* suggesting that the relationship is not physical, merely a statistical method and thereby the units in the left and right part of the equation do not coincide.

First, connect all gauges P_1–P_4 with help lines. These help lines should not cross. Also gauges outside the catchment border should be included.



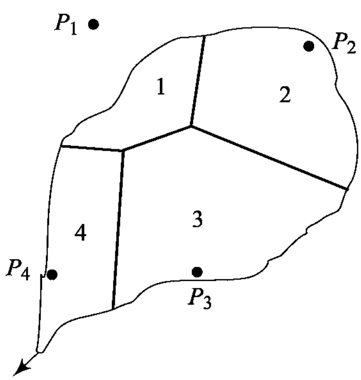
Help lines connecting all gauges

Mark the middle point of each help line. From this point, a perpendicular line is drawn until it connects to other lines. The lines now divide the catchment into areas that can be related to each gauge.



Dividing lines making an area associated with each gauge

The total catchment is now divided in to representative sub-areas 1–4 that can be associated to each gauge P_1–P_4. By measuring each sub-area and dividing it with the total catchment area, weight coefficients can be calculated to estimate an areally weighted value.



The catchment is now divided according to the Thiessen method

Fig. 2.7 *Division of catchment area according to Thiessen method*

The snow melt is calculated as:

$$S = (0.1 + 0.12 \times P + 0.8 \times a \times v)\ T + 2.0 \tag{2.6}$$

where, S = snow melt (mm/day)
P = rainfall (mm/day)
a = constant which is equal to about 0.3 for forest and 1.0 for open terrain
v = wind speed (m/s)
T = temperature (°C)

Due to the fact that the precipitation is much easier to observe as compared to runoff, the precipitation data are used to establish precipitation statistics for different time periods, which is consequently used to estimate runoff from various types of precipitation.

2.3.4 Precipitation Statistics

Precipitation statistics are usually used in three important parameters; (1) *rainfall average intensity*, (2) *rainfall duration*, and (3) *rainfall return period*. The meaning of average intensity and duration is depicted in Fig. 2.8. The figure shows the average rainfall intensity versus time. The rainfall intensity variation with time is also called a *rainfall hyetograph*. The rainfall hyetograph is usually shown as a bar chart.

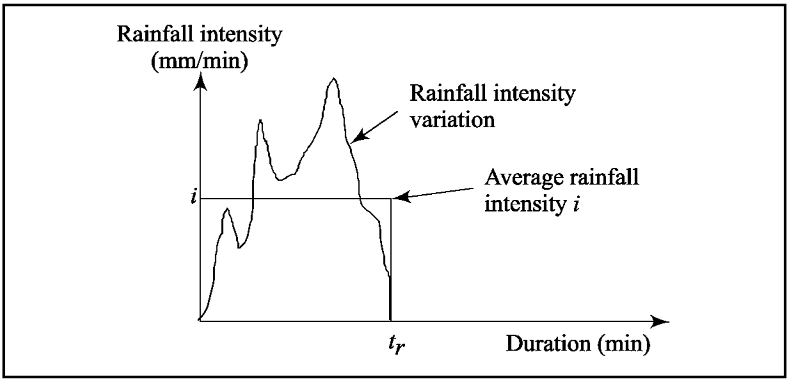


Fig. 2.8 *Definition of average rainfall intensity (i) and duration (t_r)*

The intensity of a rainfall (amount of rainfall per unit time) is highly dependent on the type of rainfall, e.g., thundershower or slow winter rain. But irrespective of the type and time of rainfall, a mean intensity and duration of every rain can be defined independently.

The duration is simply the time from the start until the end of the rainfall with an average intensity for this time. The return period is how often a certain type of rainfall intensity occurs. A rainfall with one-year return period occurs, on an average, just once a year. A rainfall with 10-year return period is so large that it occurs, on an average, just once in 10 years. It is also possible to express return period in terms of probability according to:

$$p = 1/R \tag{2.7}$$

where p is the probability and R is the return period. The probability of a certain rainfall, with a return period of 10 years, to occur during one specific year is 1/10 = 0.1. The probability for this rainfall to occur once in a 10-year period is 1.0.

The above statistical information for a certain area can be summarised with the help of an *intensity-duration-frequency diagram*. Figure 2.9 shows, in a schematic way, how such a diagram may look. Using this diagram, it is possible to determine the typical intensity of a rainfall of certain duration and return period (frequency). The diagram can be used to *design* (determine the size and appearance) various types of water structures, such as pipes and reservoirs. Usually, small water structures are designed according to a smaller return period rainfall, e.g., five-year rainfall; while larger structures may be designed with 100-year or even 10,000-year return period rainfall.

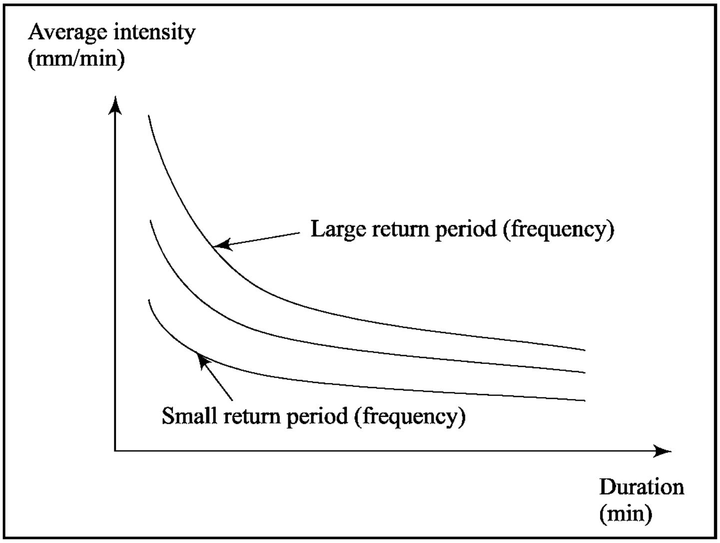


Fig. 2.9 *Schematic of intensity-duration-frequency (IDF) diagram for a specific region*

The number and type of rain gauges for a specific catchment depend upon the purpose and the use of precipitation data for that specific location. It is possible to calculate the number of rainfall gauges necessary in an area to keep a certain permissible error in mean precipitation estimation. The following equation can be used for this purpose.

$$N = [(t_{1-\alpha/2} \times s)/(x - \mu)]^2 \tag{2.8}$$

where, N = minimum number of rainfall gauges needed

$t_{1-\alpha/2}$ = tabulated Student's t-distribution at the chosen probability level (e.g., 0.05)

s = standard deviation of rainfall amounts

$x - \mu$ = the relative error margin (μ is the true mean to be estimated within the limits $+x$ and $-x$)

The expression $(x - \mu)^2$ can be replaced by $(0.1 \times m)$, where m is the mean value of the amount of rainfall and 0.1 denotes 10% error margin. The World Meteorological Organization (WMO) gives general recommendations regarding the minimum rain gauge density for various climates and topographical settings. It is recommended that for small mountainous islands, the minimum density of gauges should be 1 gauge per 25 km^2, about 250 km^2 per gauge for mountainous areas in general, and about 575 km^2 per gauge elsewhere in temperate, Mediterranean, and tropical climates.

2.4 INFILTRATION

Precipitation that *infiltrates* through the ground surface creates *soil water*; and if not evaporated, it may *percolate* further down to the groundwater table. Soil water is the water that is found between the soil surface and the groundwater table. This is called the *soil water zone* or the *unsaturated zone*. The unsaturated zone indicates that the pores in the soil are not saturated with water. Instead, the zone is a mix between soil particles, water, and gas-filled pores. Similarly, below the groundwater table, the soil is saturated with water, and hence it is called the *saturated zone*.

In general, vertical transport processes dominate in the unsaturated zone and horizontal transport processes in the saturated zone. However, large deviations occur from this general rule due to variation in topography and complex geological material. The soil and geological structure determine how quickly and what way the water will go once it has infiltrated through the soil surface. In the upper soil layers, a major part of the water may evaporate if there is energy available for this. Also, water may be taken up by plant roots and then transported to the atmosphere through *transpiration*. Water can also be transported upwards in the soil from the groundwater table through *capillary forces*. In general, it is the gravitation that determines the water movement in both the soil water and the groundwater zone. Thereby, the *topography* or the altitudinal characteristics of the area decide much of the water movement.

2.4.1 Transport Patterns

Depending on small topographical variations, groundwater can have different directions in the landscape. Shallow (young groundwater) and deep (old) groundwater at the same location can have different directions depending on small-scale and large-scale topography. Consequently, the age of groundwater and topography at different scales determine the general transport pattern. Shallow groundwater may also flow back to the surface water system, especially in low-lying areas. Typical areas where this phenomenon occurs are topographical depressions, such as marshy areas, bogs, etc. These areas are generally called *outflow areas*. Areas where surface water generally infiltrates are called *inflow areas*, e.g., topographical peaks, such as hills and slopes.

2.4.2 Measurement of Infiltration

The speed at which water infiltrates through the soil surface is vital because it has considerable effect on surface water occurrence and is thus strongly connected to the local environment and ecology. Measurement of the infiltration speed of ponded water (flooding conditions) can be done using a *ring infiltrometer* (Figs. 2.10 and 2.11).

Fig. 2.10 *Ring infiltrometer for measuring the infiltration capacity*

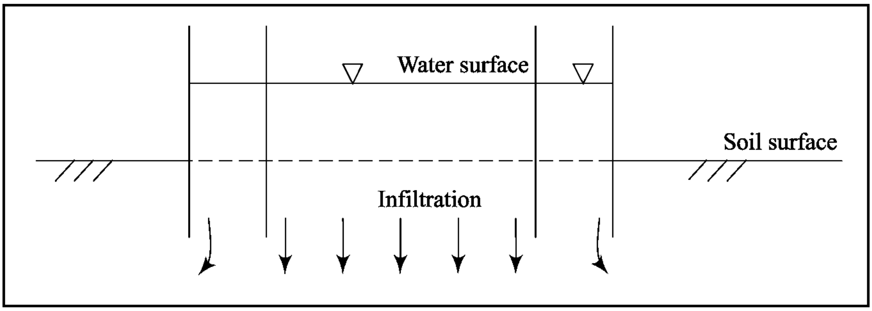


Fig. 2.11 *Infiltration through a double ring infiltrometer*

The ring infiltrometer usually consists of a 15 cm diameter steel cylinder that is open at both ends. The infiltrometer is pushed into the soil to about 10 cm depth. Thereafter, water is ponded inside the cylinder to about 5 cm water depth. This level is kept constant by adding fresh water continuously during the infiltration. The amount of added water is measured every minute and recorded as *infiltration capacity*. Usually, a second larger cylinder is pushed into the ground around the smaller one with the same water level in order to avoid errors due to horizontal flow. By definition, infiltration capacity concerns vertical flow only. Infiltration is measured only in the inner cylinder. Usually, the infiltration capacity is high at the beginning of experiments due to dry soil conditions. After some time, however, the infiltration curve levels out and becomes almost constant. The starting and ending value of measured infiltration is called *initial and final infiltration capacity* respectively. The *Horton infiltration equation* can be used to describe infiltration of ponded water.

$$f(t) = f_c + (f_o - f_c)e^{-kt} \tag{2.9}$$

where, f_o = infiltration capacity at $t = 0$ (dry soil conditions; mm/hr)

f_c = final infiltration capacity (mm/hr)

k = constant determining how quickly the infiltration capacity decreases (hr^{-1})

t = time (hr)

To calculate the totally infiltrated amount of water in mm., Eq. (2.9) is integrated from 0 to t.

$$\int_0^1 f(t)\,dt = F(t) = f_c\,t + \frac{f_o - f_c}{k}(1 - e^{-kt}) \tag{2.10}$$

Equation (2.10) gives the *accumulated infiltration* in mm after a specified time. Both Eq. (2.9) and (2.10) presume continuous *ponding of water*, i.e., a constant *head of water* on top of the soil. Equations (2.9) and (2.10) are exemplified in a schematic way in Fig. 2.12. The observations by ring infiltrometer can be used to calibrate the three parameters in Eqs. (2.9) and (2.10), to describe the infiltration process in time.

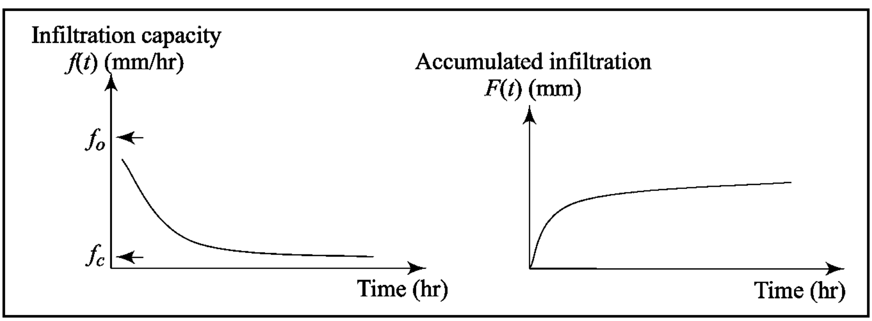


Fig. 2.12 *Schematic of the infiltration capacity and accumulated infiltration*

2.5 EVAPORATION AND TRANSPIRATION

Evaporation is the general term for describing water release from ground and water surfaces to the atmosphere. This includes evaporation from plants (*transpiration*) and the direct evaporation from soil and/or water surfaces (*evaporation*). A summarizing name for both evaporation and transpiration is *evapotranspiration*.

Evapotranspiration from an area is governed by the amount of energy that is available to transform water from liquid to gas. The amount of water that can theoretically evaporate (water is not a limiting factor, e.g., water can evaporate from a free water surface) is called *potential evaporation.* The evaporation that takes place when water is a limiting factor, e.g., when soil surface is only partially saturated, is called *actual evaporation.* Similar nomenclature can be applied to evapotranspiration. The actual evaporation is usually estimated by establishing a water balance for the area. The other components in the water balance must be known. Often, other components of the water balance are not known in which case evaporation has to be calculated or measured.

In general, two major factors govern the amount of water that can evaporate from a surface: *wind* and *temperature*. The temperature shows how much energy is available to transform the water in liquid form to gas form. At a specific temperature, the atmosphere over the ground or the water surface can hold a specific amount of water vapour, which is called *saturated vapour pressure*, e_s. The evaporation rate (E) will then be proportional to the difference in actual vapour pressure (e_a) and saturated vapour pressure (e_s).

$$E = c\,(e_s - e_a) \tag{2.11}$$

where, c is a constant. This relationship is called *Dalton's evaporation law.* The presence of wind transports the evaporated water away and replaces it by dry air. If $(e_s - e_a) > 0$, condensation will take place.

For shorter periods, it is necessary to use calculation methods to estimate evaporation. This can be done by the *Penman method.* This method was developed to determine the potential evapotranspiration (ETP) of a specific area, depending on its climatic and meteorological conditions. The Penman equation is a combination of energy balance and wind transfer, which reads:

$$ETP = (A\,H_n + E_a\gamma)/(A + \gamma) \tag{2.12}$$

where, ETP = daily potential evapotranspiration (mm/day)

A = slope of the saturation vapour pressure vs. temperature curve at mean air temperature (mm Hg/°C)

H_n = net radiation (mm/day) (Refer Table 2.2)

γ = the psychrometric constant (0.49 mm Hg/°C)

H_n is estimated from the following equation.

$$H_n = H_a\,(1 - r)(a + b\,n/N) - \sigma T_a^4\,(0.56 - 0.092\,\sqrt{e_a}\,)(0.10 + 0.90\,n/N) \tag{2.13}$$

where, H_a = incident solar radiation outside the atmosphere on a horizontal surface (mm/day; Table 2.3)

r = albedo of the surface giving the reflection at soil surface of incoming energy (Table 2.1)
a = constant equal to 0.29 cos ϕ, ϕ = latitude
b = constant approximately equal to 0.52
n = actual duration of bright sunshine (hours/day)
N = maximum possible duration of bright sunshine (hours/day; Table 2.4)
σ = Stefan-Boltzmann constant
T_a = mean air temperature in °K, i.e., (273 + °C)
e_a = actual mean vapour pressure in the air (mm Hg)

E_a is calculated according to

$$E_a = 0.35\,(1 + u_2/160)\,(e_s - e_a) \tag{2.14}$$

where, u_2 = mean wind speed at 2 m above the ground (km/day)
e_s = saturated vapour pressure in the air (mm Hg) (Refer Table 2.2)

The albedo (r) can be estimated according to the type of surface (Table 2.1).

Table 2.1 Typical albedo values (reflection of incoming energy) for different surface types

Type of surface	*The albedo (r) may vary between*
Plant covered ground surface	0.15–0.25
Bare ground surface	0.05–0.45
Water surface	0.05
Snow	0.45–0.95

Table 2.2 gives saturated vapour pressure values of water depending on the temperature.

Table 2.2 Saturated vapour pressure of water (e_s), and slope of saturation vapour pressure vs. temperature A

Temperature (°C)	*Saturated vapour pressure e_s (mm Hg)*	*Slope of saturation vapour pressure vs. temperature A (mm/°C)*
0	4.58	0.30
5.0	6.54	0.45
7.5	7.78	0.54
10.0	9.21	0.60
12.5	10.87	0.71
15.0	12.79	0.80
17.5	15.00	0.95
20.0	17.54	1.05

(*Contd.*)

(*Contd.*)

22.5	20.44	1.24
25.0	23.76	1.40
27.5	27.54	1.61
30.0	31.82	1.85
32.5	36.68	2.07
35.0	42.81	2.35
37.5	48.36	2.62
40.0	55.32	2.95

Table 2.3 gives values of H_a (mean solar radiation outside the atmosphere on a horizontal surface; mm/day) and Table 2.4 gives the mean monthly values of possible sunshine hours, N.

Table 2.3 Mean solar radiation outside the atmosphere on a horizontal surface, H_a (mm/day)

North Latitude	*Jan*	*Feb*	*Mar*	*Apr*	*May*	*Jun*	*Jul*	*Aug*	*Sep*	*Oct*	*Nov*	*Dec*
0°	14.5	15.0	15.2	14.7	13.9	13.4	13.5	14.2	14.9	15.0	14.6	14.3
10°	12.8	13.9	14.8	15.2	15.0	14.8	14.8	15.0	14.9	14.1	13.1	12.4
20°	10.8	12.3	13.9	15.2	15.7	15.8	15.7	15.3	14.4	12.9	11.2	10.3
30°	8.5	10.5	12.7	14.8	16.0	16.5	16.2	15.3	13.5	11.3	9.1	7.9
40°	6.0	8.3	11.0	13.9	15.9	16.7	16.3	14.8	12.2	9.3	6.7	5.4
50°	3.6	5.9	9.1	12.7	15.4	16.7	16.1	13.9	10.5	7.1	4.3	3.0

Table 2.4 Mean monthly possible sunshine hours, N (hrs/day)

North Latitude	*Jan*	*Feb*	*Mar*	*Apr*	*May*	*Jun*	*Jul*	*Aug*	*Sep*	*Oct*	*Nov*	*Dec*
0°	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1	12.1
10°	11.6	11.8	12.1	12.4	12.6	12.7	12.6	12.4	12.2	11.9	11.7	11.5
20°	11.1	11.5	12.0	12.6	13.1	13.3	13.2	12.8	12.3	11.7	11.2	10.9
30°	10.4	11.1	12.0	12.9	13.7	14.1	13.9	13.2	12.4	11.5	10.6	10.2
40°	9.6	10.7	11.9	13.2	14.4	15.0	14.7	13.8	12.5	11.2	10.0	9.4
50°	8.6	10.1	11.8	13.8	15.4	16.4	16.0	14.5	12.7	10.8	9.1	8.1

In order to calculate *ETP* according to Penman, the values of n, e_a, u_2, mean air temperature, and r are needed as local values. These can usually be obtained from observations at the nearest climatic station. All other variables and constants can be estimated from the tables given. According to Penman, *ETP* may represent potential evaporation as well as potential evapotranspiration, in practical applications.

In most cases, the soil surface is not water saturated; in other words, water is a limiting factor for the evaporation process. In such cases, actual evaporation will be less than its potential, and a further calculation step is necessary to arrive at the actual evaporation. In general, the actual evaporation from the soil surface will depend on capacity of the soil to hold water. The actual evaporation will be equal to the potential evaporation as long as water is not a limiting factor for the evaporation process. However, when the water surface is not saturated, the capacity of the soil to transport water to the surface will be the determining factor for the evaporation process. Figure 2.13 shows this general behaviour for two different types of soil.

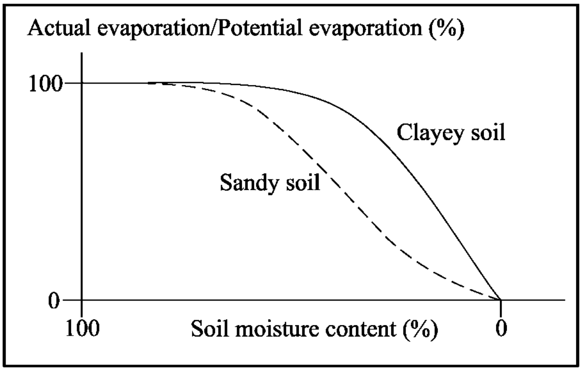


Fig. 2.13 *Variation of actual evaporation with soil moisture content*

In general, sandy soils have a small capillary rise; and thus as soon as the soil surface becomes unsaturated, evaporation will quickly decrease with soil moisture content. On the other hand, clayey soils, due to a finer *soil texture* (soil particle size), have a large capillary rise and may transport water from large depths to the soil surface. Hence, with decreasing saturation, the actual evaporation drops slowly as compared to the sandy soil. Another contributing factor is that the clayey soils have larger *porosity* (void volume) as compared to sandy soils, and they may contain larger amounts of water.

It is also possible to measure the potential evaporation from a free water surface by a *pan-evaporimeter* (Fig. 2.14). However, the pan-evaporimeters often overestimate the potential evaporation. This is because the much smaller water volume in the pan easily heats up to a warmer temperature as compared to the surrounding soil and therefore evaporates more than a larger water body. To overcome this problem, pan-evaporimeters have to be corrected by multiplying with a pan correction coefficient.

The actual evaporation from soil during short periods of time can be measured by a *lysimeter*. A lysimeter consists of an earth filled container that

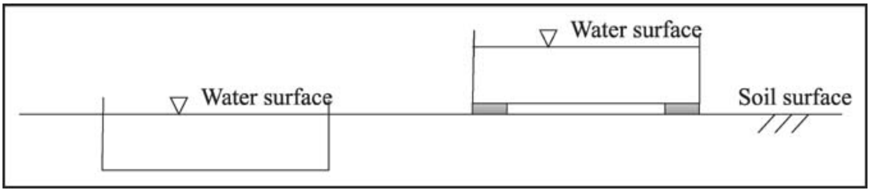


Fig. 2.14 *Schematic of two types of pan-evaporimeter; (i) buried and (ii) above soil surface. The pan-evaporimeter above soil surface has the largest pan correction coefficient.*

can be weighed, and the percolating water can be collected at the bottom. Usually, lysimeter observations are combined with rainfall gauge observations. Thereby, it is possible to establish a water balance for the soil in the lysimeter. The advantage of lysimeters is that it is also possible to include the transpiration term in the observations by keeping representative plants in the lysimeter soil. Information on evaporation and climatic data are usually available at the national meteorological agency.

2.6 SURFACE RUNOFF

Surface runoff, discharge, or stream flow is usually defined as visible water on the ground, i.e., the *surface water*. This may include ponds, brooks, creeks, rivers, lakes, reservoirs, etc. Surface water is valuable for water supply and environmental concerns. For example, a lake is a natural water reservoir that can be used for water supply as well as irrigation.

All surface runoff water (and groundwater) is in constant movement towards the sea (usually the lowest point). However, the flow of water gets obstructed due to infiltration; also a major volume of water, depending on the season, gets lost into the atmosphere through evaporation. Groundwater and soil water are often in a dynamic relationship with the surface water. Groundwater may be drained back to the surface water depending on water levels around the lake and/or river. This is partly a seasonal phenomenon as seen in Fig. 2.15. When the groundwater level is lower than the lake or river water level, a significant amount of water may be transported from the lake or river to the surrounding groundwater. This usually occurs during summer and winter, due to long dry periods. During spring and autumn, with heavy rainfall and slow evaporation, the groundwater level becomes higher than the river water level, and the groundwater is drained into the river.

The intensive drainage of natural land in urban areas has considerably decreased the *residence time* of water; and thereby pollutants are more or less flushed through the system without the possibility of sedimentation or being reduced by plant uptake or other kinds of biological transformation. In hydrological sense, drainage means that the flow has changed from a slow

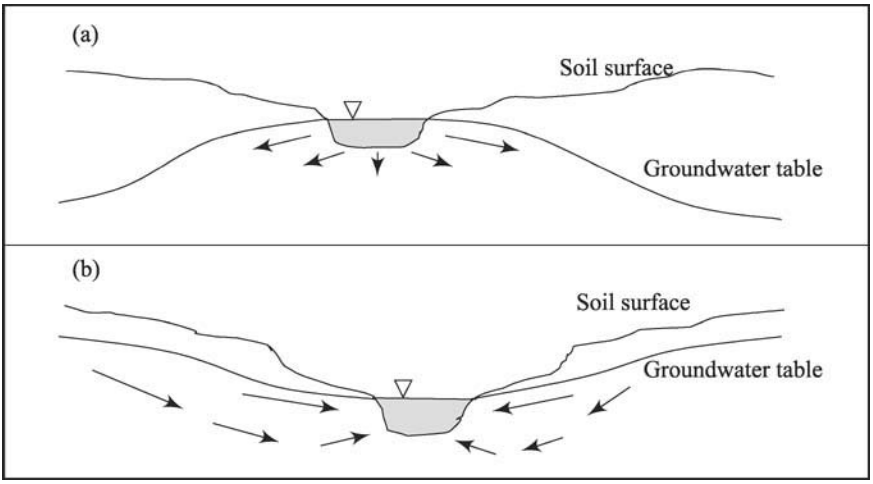


Fig. 2.15 *(a) Infiltration to the groundwater from a river or lake and (b) outflow of groundwater to a river or lake (Knutsson and Morfeldt, 1973; Geol. Survey of Canada, 1967). Situation (a) shows a typical dry period when water can infiltrate into the groundwater. Situation (b) shows a typical wet period when water can be drained into the river or lake.*

process with low peak flows to a quick runoff process with high peaks. Urban areas are usually efficiently drained through underground stormwater system pipes. Runoff occurs quickly from asphalted and other impermeable areas. Almost no water can infiltrate, and the amount of water evaporated is less due to the rapid runoff. The water volume transported increases due to less evaporation and short runoff time. The result is a runoff hydrograph with a short time to peak after rainfall, a large peak runoff, and a large runoff volume. This is shown schematically in Fig. 2.16.

In comparison to urban areas, the amount of water infiltrated and evaporated is more in drained agricultural areas. Therefore, the time to peak is longer, and the peak flow is smaller as is the total runoff volume. In contrast to urban areas, the runoff is a slow process through wetlands and the naturally meandering water course. The slow runoff also contributes to a rich biodiversity and good pollution reduction through sedimentation and biological processes. Slow runoff also means more wetland and ponds and lakes. Water passing through a wetland or lake will be naturally treated to some extent through biological processes and sedimentation. A theoretical *residence time* (T_{resid}) for the water passing through a lake or any type of water body can be calculated by comparing the total inflow of water (ΣQ_{inflow}) with the total lake volume (V_{lake}). Assuming that as much water flows out as flows in, i.e., storage effects are ignored) the residence time in the lake for the water becomes

$$T_{resid} = V_{lake} / \Sigma Q_{inflow} \tag{2.15}$$

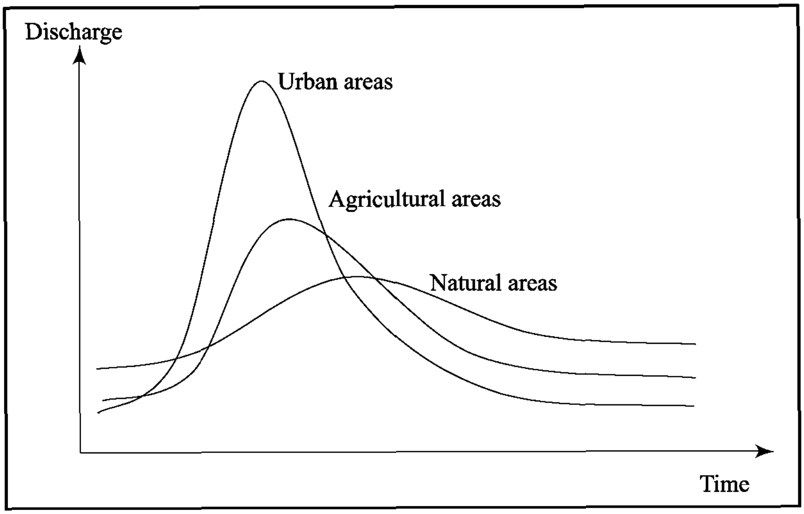


Fig. 2.16 *Schematic of how drainage of developed lands changes the runoff hydrograph after a rainfall event*

Correct evaluation of the amount of discharge is important because it has its effects on water supply, irrigation, generation of hydropower, and environment. An indirect way to measure discharge is to observe the water level in the river or creek. The discharge (Q) is a function of the water level (h). According to Fig. 2.17, $Q_1 = f(h_1)$, $Q_2 = f(h_2)$, $Q_3 = f(h_3)$, ... where $Q_1 > Q_2 > Q_3$.

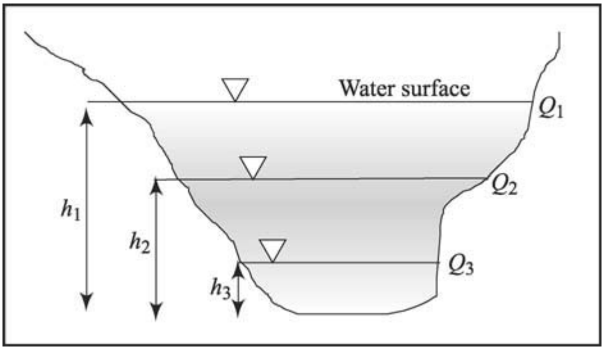


Fig. 2.17 *Relationship between discharge in a water course and interconnected water level. The figure shows a water-filled transect of the water course at different water levels with corresponding discharges*

The function $f(h)$ is usually exponential which can be easily observed by plotting observed Q against h (Fig. 2.18). This relationship is called *stage-runoff* or *stage-discharge curve*. Then, a theoretical equation for $f(h)$ can be adapted to the points, usually of the type:

$$Q(h) = f(h) = a \times h^b \tag{2.16}$$

where, a and b are constants, h is the water level (m), and Q is the discharge (m^3/s). The theoretical curve is adapted to the observations by changing the values of a and b, a *calibration* of the theoretical model given by Eq. (2.16).

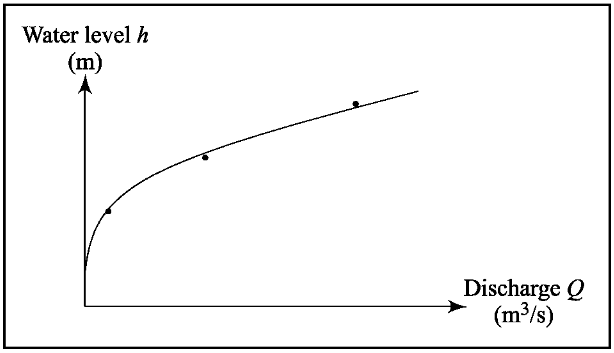


Fig. 2.18 *Example of the relationship Q = f(h), discharge (or runoff) against water level. The dots represent observed disharge at different water levels.*

The measured values in Fig. 2.18 come from observations of discharge at different water levels. Often, this is done by the *current meter method*. A current meter is a device which is used to measure water velocity by putting it into the water at different depths (see Fig. 2.19). In principle, the current meter consists of a propeller that rotates in relation to the water velocity. The velocity from the current meter is determined along different representative sections of the wet water course transect. The water velocity will vary depending on different locations in this transect. The highest velocity is obtained in the middle of the transect, close to the surface. Due to *friction*, the velocity decreases towards the bottom and the edges. The different representative sections a_i (m^2) and corresponding velocities v_i (m/s) can be weighted to give a total representative average velocity for the whole wet transect area.

The total discharge (m^3/s) is calculated by multiplying the representative sections a_i with their corresponding velocities v_i, and dividing the total by the whole wet transect area A.

$$Q(h)_{\text{estimated}} = \sum a_i v_i / A = (a_1 v_1 + a_2 v_2 + a_3 v_3 + \ldots + a_n v_n)/A \tag{2.17}$$

The resulting $Q(h)_{\text{estimated}}$ corresponds to a point in Fig. 2.18 that can be used to calibrate the theoretical curve.

Fig. 2.19 *Rodheld portable current meter manufactured by Hydro-Bios Intern. Ltd. Co.*

For smaller water courses, such as brooks and creeks, it is difficult to use water current meter; instead, a measuring *weir*, e.g., the V-shaped weir is used (Fig. 2.20). Just upstream from the weir, a water level gauge registers the water level above the lowest point in the weir. It is possible to use a similar equation as in Eq. (2.16); but as the geometry of the V-shaped weir is known, a better discharge estimation can be achieved.

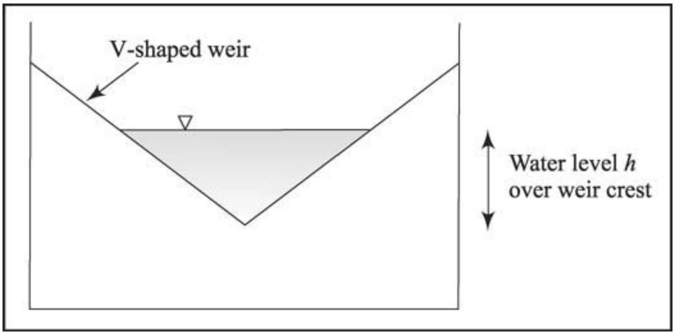


Fig. 2.20 *Schematic of a V-shaped sharp-crested weir*

In many cases, it is necessary to calculate the runoff from individual rainfall events. A *hydrograph analysis* is then applied to determine the relationship between the time sequence of rainfall and the runoff. The variation of runoff or discharge along the time axis is called the *runoff hydrograph*. The corresponding rainfall variation is called *rainfall hyetograph*. Together with these variables, the chemical contents [e.g., lead (Pb) or biological oxygen demand (BOD_7)] in the runoff can also be analyzed, as a *pollutograph*. These variables are displayed in Fig. 2.21. Note that different variables have different units, but their time scale remains unchanged.

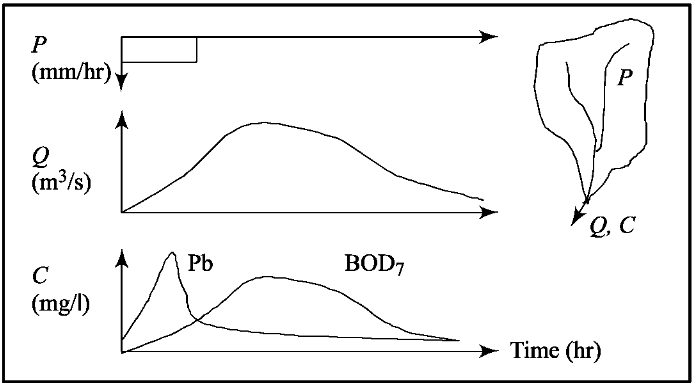


Fig. 2.21 *Definition of important analysis tools in catchment hydrology, hyetograph P(t), hydrograph Q(t), and pollutograph C(t)*

Apart from the *direct runoff* which is directly related to a rainfall event, there is another factor that contributes to the total discharge from a catchment area. Usually, there is a diffuse flow of water infiltrating from the surrounding area even if there is no rainfall. This type of flow is called *base flow*. The base flow varies according to season and size of the catchment. The source of base flow is the water from prior rainfalls that have infiltrated into the soil. The infiltrated groundwater slowly gets drained into the river. We can thus define the *total runoff* from a catchment as:

$$Q_{\text{total}} = Q_{\text{direct}} + Q_{\text{base flow}} \tag{2.18}$$

where direct runoff (Q_{direct}) corresponds to that part of the rainfall that is not lost through any catchment losses, such as interception, infiltration, and/or evapotranspiration. We can call this rainfall, the *effective rainfall* $P_{\text{effective}}$. The effective rainfall must be equal to the direct runoff.

$$Q_{\text{direct}} = P_{\text{effective}} \tag{2.19}$$

Corresponding to Eq. (2.18), we can define the losses for the total rainfall that do not contribute to direct runoff as ϕ_{index}.

$$P_{\text{total}} = P_{\text{effective}} + \phi_{\text{index}} \tag{2.20}$$

The ϕ_{index} denotes an average loss from the total rainfall (P_{total}) during the rainfall event, in mm per time. Note that the above equations indicate volumes of water for a specific time period. While using these equations, it is important to use the same units. Figure 2.22 shows the relation among runoff, catchment losses, and total rainfall.

In order to calculate direct runoff, the base flow is subtracted from the total runoff. There are different ways in which the base flow can be subtracted. The simplest way is to assume a constant base flow (Fig. 2.22). However, it is also possible to separate the base flow by a straight line from where discharge begins to increase to a selected point on the recession limb of the hydrograph (Fig. 2.23). This point may be chosen as the point of greatest curvature near the lower end of the recession part of the hydrograph.

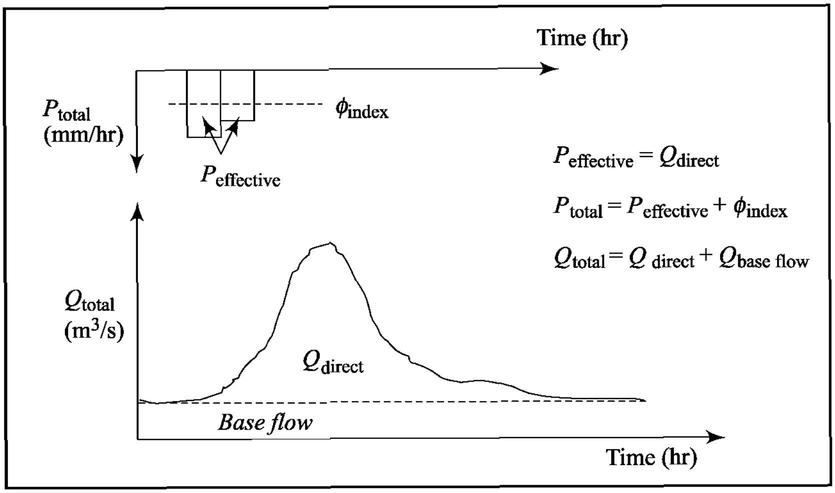


Fig. 2.22 *Example of observed rainfall and runoff and relationship among P_{total}, $P_{effective}$, ϕ_{index}, Q_{total}, $Q_{base\ flow}$, and Q_{direct} (note that for the equations units of P and Q need to be the same).*

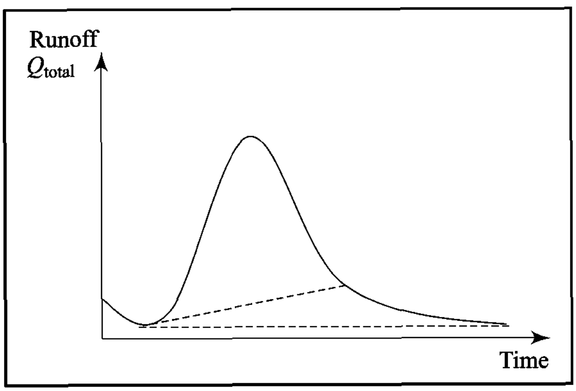


Fig. 2.23 *Two ways of separating base flow from total discharge*

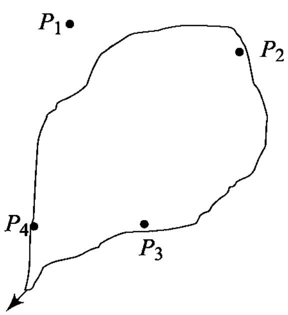
SUMMARY

Precipitation is the driving force behind the hydrological cycle. The great variation in space and time for the precipitation means that this variation will carry over to other hydrological cycle components. Similarly, there are difficulties to observe various hydrological components with great accuracy. Consequently, all hydrological and water balance calculations have to be based on the fact that the hydrological observations contain a varying degree of uncertainties and errors. It is, for example, not uncommon that areal precipitation estimates contain errors of 1–5%. Similarly, error and uncertainty levels for discharge observations may be 5–20%. Thus, all hydrological calculations will also be affected by these uncertainties and errors.

Indicating a high degree of numerical accuracy in the calculations that are based on uncertain observations is a serious flaw. It is important that all engineering calculations and solutions openly present these uncertainties. Therefore, it is often much better to give hydrological calculation results as an interval instead of a single numerical value.

SOLVED EXAMPLES

Example 2.1 A catchment with an area $A = 100$ km^2, has four rainfall gauges ($P_1 - P_4$) according to the figure. Use the rainfall data in the table to determine the areal rainfall over the catchment.

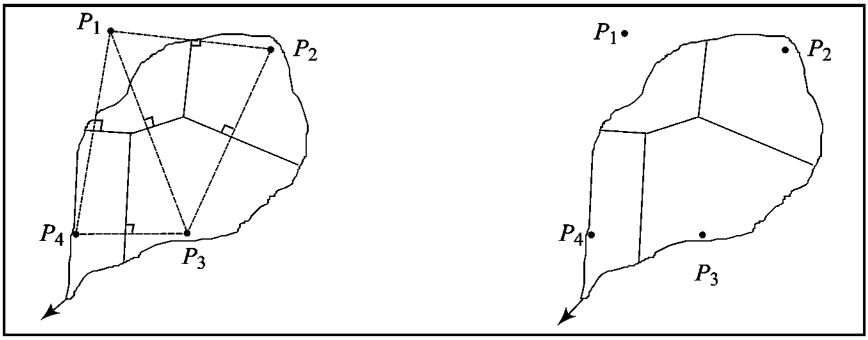


Rainfall gauge	Rainfall (mm)
P_1	25.0
P_2	10.0
P_3	5.0
P_4	15.0

(a) Use the arithmetic averaging method to calculate areal rainfall.
(b) Use the Thiessen method to calculate areal rainfall.

Solution

(a) The averaging method gives (10.0 + 5.0 + 15.0)/3 = 10.0 mm.
The P_1 gauge is not included since it is outside the catchment border.

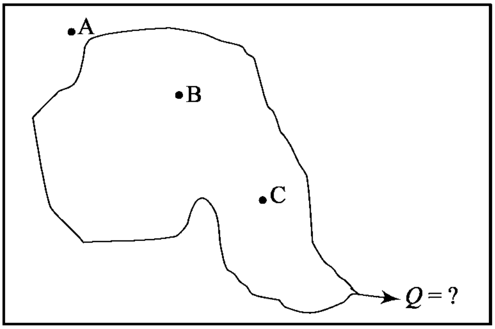


(b) The Thiessen method gives representative areas according to the figure below.
Estimation of sub-areas for every gauge gives approximately P_1: 15%, P_2: 30%, P_3: 35%, and P_4: 20% of the total area.
This gives $(25.0 \times 0.15) + (10.0 \times 0.30) + (5.0 \times 0.35) + (15.0 \times 0.20) =$ 11.5 mm.

Example 2.2 A catchment has a total area of 5 ha. Snow sampling with snow tube has been made at three points according to the below figure and the following snow depths and densities were found.

Point A: $h_A = 30$ cm, $\rho_A = 150$ kg/m^3
Point B: $h_B = 20$ cm, $\rho_B = 175$ kg/m^3
Point C: $h_C = 15$ cm, $\rho_C = 225$ kg/m^3

Estimate the runoff from the catchment when all the snow melts.



Solution

First, the water equivalent has to be calculated for the snow. The sampling points appear to be rather evenly distributed over the catchment area and thus we may take an average of all the measurements directly (an alternative would have been to apply the thiessen method to the measuring points). The average becomes:

Average depth of snow: $(30 + 20 + 15)/3 = 21.7$ cm
Average density of snow: $(150 + 175 + 225)/3 = 183.3$ kg/m^3

Then, we use Eq. (2.5) to calculate the water equivalent:

$$h_{\text{water}} = h_{\text{snow}} \times \rho_{\text{snow}} /1000 = (0.217 \text{ m} \times 183.3 \text{ kg/m}^3)/(1000 \text{ kg/m}^3) \approx 40 \text{ mm}$$

Now, we know how much water the snow pack corresponds to. This water may be assumed to be evenly distributed over the 5 ha catchment area. We don't know how much water will infiltrate and/or evaporate during the snowmelt. We therefore have to make some assumptions. We also do not know how long the snow will take to melt. We may, however, assume that 21 cm of snow could perhaps melt off in about 3 days.

The soil is probably frozen, so most likely, very little water could infiltrate. Because it is winter, maybe the evaporation will not be so large. Consequently, we may perhaps assume no losses. This assumption may be said to correspond to the security principle, e.g., if the calculations concern flood risks, we assume that in the worst case all water will runoff.

The runoff becomes:

$$Q = (0.04 \text{ m} \times 50000 \text{ m}^2)/(3 \times 24 \times 60 \times 60 \text{ s}) = 0.0077 \text{ m}^3/\text{s}$$

Under the given assumptions, the runoff will be, on an average, about 0.0077 m^3/s over a period of 3 days.

Example 2.3 A parking lot with an area of 2000 m^2 in a city is drained to a small water course without subsequent treatment. It is suspected that heavy metal pollutants may affect the biological life in the water. Therefore, some sampling of the water is done and high lead concentrations are found, about 0.10 mg/l on an average. The authorities want to determine how much lead could be transported to the water course instantaneously during a 10-year period, e.g., in connection with a heavy summer rain when the flow in the river is slow. The average annual precipitation for the area is 700 mm and IDF-relationships for a nearby area shows that the 10-year rainfall is 55 l/(s·ha).

Solution

The parking lot has an asphalted surface. Because of this, it can be assumed that no infiltration occurs. Also, since runoff occurs quickly after rainfall, it may be assumed that little evaporation takes place. Consequently, we may put $P \approx Q$.

We need to know the amount of rainfall to calculate the runoff. When we have the runoff, we can multiply it with the average concentration of lead to get the total lead transport.

This has to be done under the assumption that the concentration is constant and not dependent on the type of rainfall. We know the maximum rainfall amount during the 10-year period. We need to assume that the nearby area's IDF-relationship is also valid for our area. We also need to assume a suitable duration for a 10-year rainfall. Let us assume one hour duration for a 10-year rainfall. This appears to be a suitable duration for a summer rain.

This gives the following runoff:

$$Q = 55 \text{ l/(s·ha)} \times 0.2 \text{ ha} = 11 \text{ l/s}$$

We assumed that the rain occurs for one hour and since the parking lot is small, we can assume that the runoff also occurs for one hour. The total water volume then becomes:

$$Q_{vol} = 11 \text{ l/s} \times 60 \times 60 \text{ s} = 39600 \text{ l}$$

The amount of lead in this water volume is:

$$m_{lead} = 0.1 \text{ mg/l} \times 39600 \text{ l} = 3960 \text{ mg} = 3.96 \text{ g} \approx 4 \text{ g}$$

Consequently, a maximum of about 4 g can be transported out of the parking lot during one rainfall event. This, however, does not say anything about the continuous transport of lead from all other rainfalls during the 10-year period. We know that the annual average precipitation is about 700 mm/year. (Also, evaporation and infiltration are assumed to be negligible.) Therefore, the total runoff from the parking lot during the 10-year period is given by:

$$Q_{10\text{-year}} = 10 \times 0.70 \text{ m} \times 2000 \text{ m}^2 = 14000 \text{ m}^3 = 14000000 \text{ l}$$

This gives a total lead content of

$$m_{lead\ (10\text{-year})} = (0.1 \text{ mg/l}) \times 14000000 \text{ l} = 1400000 \text{ mg} = 1400 \text{ g}$$

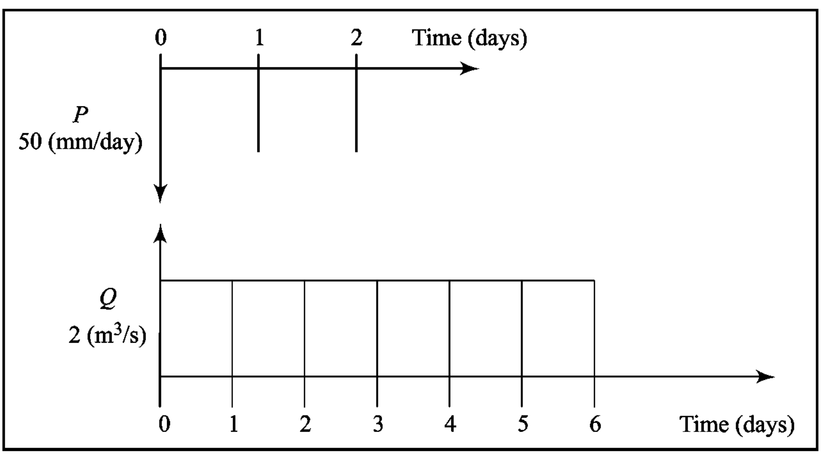
That is, the long-term transport of lead from the parking lot is about 350 times larger than the instantaneous runoff from the largest rainfall during the period.

Example 2.4 Over a period of two days, (50 + 50) mm rain fell over a catchment (A = 50 km^2). The runoff in the water course draining the area was $Q = 0$ at the start of rainfall; and over six days, $Q_{mean} = 2$ m^3/s; and thereafter $Q = 0$ again. Answer the below questions:

(a) How large were the losses from the total precipitation (in mm)?
(b) How large was the effective rainfall (in mm)?
(c) What is the value of ϕ_{index} (in mm/day)?

Solution

To best understand the problem, let us plot the prerequisites!



(a) The effective rainfall is equal to the direct runoff. From the prerequisites, it is seen that the runoff was zero just before the rainfall. Therefore, the base flow can be assumed to be zero.
The direct runoff becomes: $2\ m^3/s \times 6 \times 24 \times 60 \times 60\ s = 1.0368 \times 10^6\ m^3$
Therefore, the effective rainfall per unit area is given by:
$1.0368 \times 10^6\ m^3 / (50 \times 10^6\ m^2) = 0.0207\ m = 20.7\ mm$

(b) The total losses were $P_{total} - P_{effective} = (100 - 20.7)\ mm = 79.3\ mm$

(c) ϕ_{index} is (79.3 / 2) mm/day, i.e., 39.7 mm/day

Example 2.5 Runoff (l/s) according to the below table was observed from a 500 ha large catchment after a total rainfall of 10 mm for a period of 30 min. What was the effective rainfall in mm? What were the losses?

Timc (min)	0	30	60	90	120	150	180	210	240	270	300	330	360
Q (l/s)	5.0	43.3	158.3	464.8	694.4	1154.5	1001.2	847.5	694.7	311.5	158.3	81.7	5.0

Solution

Q according to the above table corresponds to the total runoff, i.e., $Q_{total} = Q_{direct} + Q_{base\ flow}$.

The total rainfall was 10 mm that corresponds to $P_{total} = P_{effective} + \phi_{index}$. Once we have Q_{direct}, we can calculate $P_{effective}$.

Therefore, we first have to subtract the base flow from the total flow according to the table below. We assume a constant base flow of 5 l/s. Having done this, we know that we have the direct runoff resulting from the effective rainfall. To calculate the effective precipitation, we have to add all direct runoff and then multiply it with the total runoff time to get the direct runoff in m^3. This volume can then be divided by the total catchment area to get the direct runoff equal to the effective rainfall in mm.

According to the table below, the total direct runoff volume becomes: $5555.5\ l/s \times 30\ min = 5555.5\ l/s \times 30 \times 60\ s = 9999.9\ m^3$ (i.e., about $10000\ m^3$). The total direct runoff divided by the total catchment area gives the effective rainfall, i.e., $10000\ m^3/(500 \times 10^4\ m^2) = 0.002\ m = 2\ mm$.

That is, the effective rainfall was 2 mm. Consequently, the total losses were 8 mm.

Time (min)	Q_{total} *(l/s)*	$Q_{total} - Q_{base}$ *(l/s)*	Q_{mean} *(l/s)*
0	5.0	0.0	—
30	43.3	38.3	19.2
60	158.3	153.3	95.8
90	464.8	459.8	306.6
120	694.4	689.4	574.6
150	1154.5	1149.5	919.5

(*Contd.*)

(*Contd.*)

180	1001.2	996.2	1072.9
210	847.5	842.5	919.4
240	694.7	689.7	766.1
270	311.5	306.5	498.1
300	158.3	153.3	229.9
330	81.7	76.7	115.0
360	5.0	0.0	38.4
		Total = 5555.5 l/s	

Example 2.6 To determine the infiltration capacity of a soil, the following experiments were made with a ring infiltrometer having a diameter D = 20 cm. Water was applied in order to maintain a constant water level inside the ring.

(a) Calculate the infiltration capacity, accumulated infiltration, and determine the parameters in the Horton equation with the help of the observations given in the following table:

(b) Use the results in (a) to calculate effective rainfall from 45 mm rain during a period of 30 min.

Time (min)	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90
Applied water (ml)	105	69	50	39	33	30	28	27	27

Solution

(a) The area of the ring is $0.10 \times 0.10 \times 3.14 = 0.0314\ m^2$; $1\ ml = 1 \times 10^{-6}\ m^3$. The amount of applied water = $105 \times 10^{-6}\ m^3/0.0314\ m^2 = 3.3$ mm/min, as an average for the first 10 min.

By plotting these infiltration capacities in the middle of each time interval in a diagram extrapolation gives $f_o = 0.46$ mm/min (for $t = 0$), $f_c = 0.086$ mm/min (for $t > 80$ min).

Time (min)	*Applied water (ml)*	*Infiltration capacity f(t) (mm/min)*	*Accumulated infiltration F(t) (mm)*
0–10	105	0.334	3.34
10–20	69	0.220	5.54
20–30	50	0.159	7.13
30–40	39	0.124	8.37
40–50	33	0.105	9.42
50–60	30	0.095	10.37
60–70	28	0.089	11.26
70–80	27	0.086	12.12
80–90	27	0.086	12.98

By putting these parameters in the Horton equation:

$$f(t) = f_c + (f_o - f_c)\, e^{-kt}$$
$$f(t) = 0.086 + (0.46 - 0.086)\, e^{-kt}$$
$$f(t) = 0.086 + 0.374\, e^{-kt}$$
$$f(t) = 0.159 \text{ and } t = 25$$
$$0.159 = 0.086 + 0.374\, e^{-k \times 25}$$
$$0.073 = 0.374\, e^{-k \times 25}$$
$$0.1952 = e^{-k \times 25}$$
$$\ln(0.1952) = -k \times 25$$
$$-1.63373 = -k \times 25$$
$$k = 1.63373/25$$
$$k \approx 0.065 \text{ min}^{-1}.$$

Testing a few other points on the curve gives an average, $k \approx 0.062$ min^{-1}

(b) The rainfall supplies, on an average, 45/30 = 1.5 mm/min of water to the soil surface. When compared to the infiltration capacity in the above table, it is seen that the average rainfall is always larger than the infiltration capacity for the first 30 min. Consequently, the Horton assumption of continuous flooding is fulfilled.

Accumulated infiltration for the first 30 min is 7.13 mm.

Consequently, (45 – 7.13) mm gives 37.87 mm as the effective rainfall.

Example 2.7 A 15 mm rainfall is recorded over a small catchment according to the table below. Infiltration tests in the area have given the following Horton equation parameters:

$f_o = 27$ mm/hr; $f_c = 6$ mm/hr; $k = 3.5$ hr.

Time (hr)	*Rainfall intensity (mm/hr)*
0.00–0.25	40
0.25–0.50	3
0.50–0.75	12
0.75–1.00	16

(a) Calculate the infiltration capacity as a function of time. It is assumed that the soil surface is always ponded with water.

(b) Can the Horton equation be applied to calculate infiltration for this rainfall event?

Solution

The infiltration capacity is given by: $f(t) = 6 + (27 - 6)e^{-3.5t}$ mm/hr

The respective infiltration capacities for the given time durations are calculated in the table below. (For the first time-step 0–0.25 hr; average infiltration is (27 + f(0.25 hr))/2 = (27 + 12.8)/2 = 19.9 mm

Time (hr)	*Infiltration capacity (mm/hr)*	*Rainfall intensity (mm/hr)*
0.00–0.25	19.9	30
0.25–0.50	11.6	2
0.50–0.75	6.7	10
0.75–1.00	8.2	15

From the above table, it can be observed that the average rainfall intensity always exceeds the average infiltration capacity, except for the time duration 0.25–0.50 hr. During this period, infiltration capacity is larger than rainfall intensity and thus the water surface is not ponded, and hence the prerequisite of the Horton equation is not fulfilled.

Example 2.8 Calculate the potential evapotranspiration using Penman's method, for a location with the following available local hydrometeorological data:
Latitude: 28° 4′ N
Altitude: 230 m amsl
Average monthly temperature: 20°C (November)
Average relative humidity: 75%
Average sunshine hours per day: 9 hrs
Average wind velocity at 2 m above ground: 85 km/day
Surface cover: Plant-covered ground

Solution
For an average temperature of 20°C, A = 1.05 mm/°C and e_w = 17.54 mm Hg. (Refer Table 2.2)

For a latitude of 28° 4′ N and November, H_a = 9.506 mm water per day. (Refer Table 2.3)

For a latitude of 28° 4′ N and November, N = 10.7 hrs per day;
and n/N = 9/10.7 = 0.84. (Table 2.4)
Further:

$$e_a = e_w \times 75\% = 17.54 \times 0.75 = 13.16 \text{ mm Hg}$$
$$a = 0.29 \cos \phi = 0.29 \times \cos(28°\,4') = 0.2559$$
$$b = 0.52$$
$$\sigma = 2.01 \times 10^{-9} \text{ mm/day}$$
$$T_a = 273 + 20°\text{C} = 293 \text{ K}$$

$\sigma \cdot T_a^4 = 2.01 \times 10^{-9} \times 293^4 = 14.814$

$r = 0.25$

Applying Eq. (2.13) gives:

$$H_n = H_a (1 - r)(a + b\, n/N) - \sigma T_a^4\ (0.56 - 0.092 \sqrt{e_a})\ (0.10 + 0.90\, n/N)$$

$$H_n = 9.506(1 - 0.25) \times (0.2559 + 0.52 \times 0.84) - 14.814 \times (0.56 - 0.092 \sqrt{13.16}) \times (0.10 + 0.9 \times 0.84)$$

$$= 4.939 - 2.869 = 2.07 \text{ mm } H_2O \text{ per day}$$

Applying Eq. (2.14) gives:

$$E_a = 0.35\ (1 + u_2/160)\ (e_s - e_a)$$

$$E_a = 0.35\ (1 + 85/160)\ (16.50 - 13.16) = 1.78 \text{ mm } H_2O \text{ per day}$$

Applying Eq. (2.12) gives:

$$ETP = (A\ H_n + E_a\ \gamma) / (A + \gamma)$$

$$ETP = (1.05 \times 2.07) + (1.78 \times 0.49)/(1.05 + 0.49) = 2.59 \text{ mm } H_2O \text{ per day.}$$

Example 2.9 A catchment has an area equal to 1 km^2 and a runoff coefficient of 0.8. A 10 mm rainfall is recorded, evenly over the catchment area.

(a) How much runoff volume can be expected out of the catchment from this rainfall?

(b) Samples are taken from the runoff and these are found to contain an average of 10 mg/l nitrogen. What is the expected amount of nitrogen to be transported out of the catchment in case of a rainfall of 20 mm?

Solution

(a) If the catchment has a runoff coefficient of 0.8, it means that 80% of the rainfall runs off directly Consequently, from a 10 mm rainfall, 8 mm will form the direct runoff.

8 mm corresponds to: $0.008 \times 1.0 \times 10^6 = 8000$ m^3

The expected runoff is 8 mm or 8000 m^3.

(b) Assuming that the same runoff coefficient can be used for the 20 mm rainfall, runoff will be equal to $20 \times 0.8 = 16$ mm.

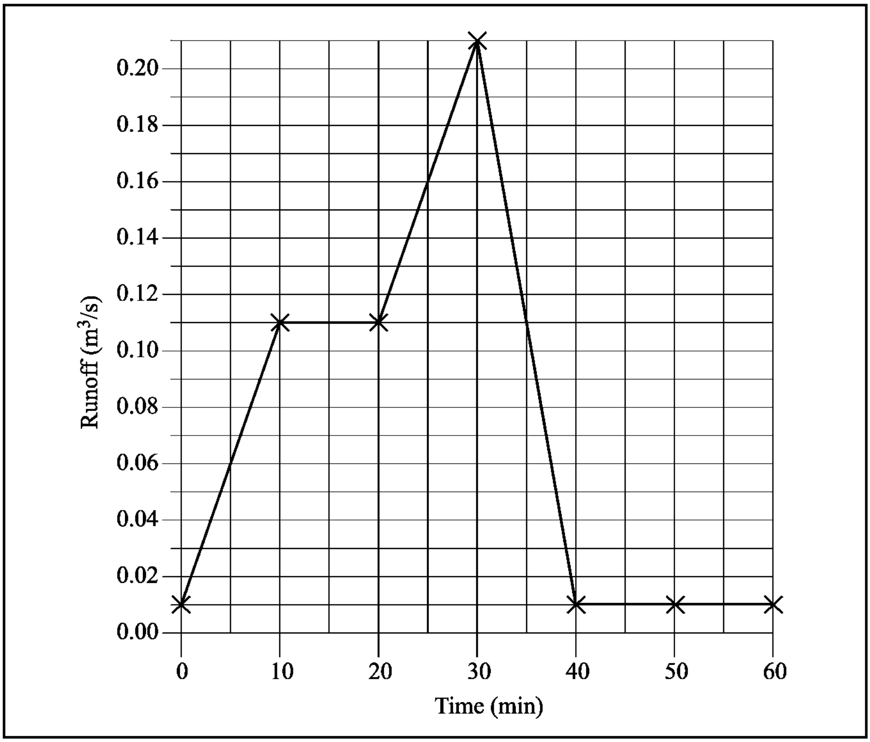
This corresponds to: $0.016 \times 1.0 \times 10^6 = 16000$ m^3

Assuming that the sampling from the 10 mm rainfall is also representative for the 20 mm rainfall, the nitrogen transport in direct runoff from the 20 mm rainfall becomes:

10×10^{-6} kg / (10^{-3} m^3) $\times$ 16000 m^3 $= 0.010 \times 16000$ kg $= 160$ kg N.

Example 2.10 A community has hired you to make a hydro-ecological investigation of a 4.8 ha large catchment area to improve the ecology by designing artificial wetlands. The following hydrograph (total runoff in m^3/s every 10 min) depicts a 15 mm uniform rainfall over the catchment that was recorded over a period of 10 min. Determine the following:

(a) How large was the base flow?
(b) How much was the direct runoff in m^3?
(c) What was the effective rainfall in mm?
(d) How large were the losses in mm?
(e) How large was the runoff coefficient?
(f) What was the water balance for this rainfall event?



From water sampling, it is found that there is nitrogen leakage from the area. Analyses indicate that water leaving the area, on an average, has a content of about 20 mg N per litre runoff.

(a) How much nitrogen can be expected out from the area in runoff for a design rainfall of 100 mm?
(b) By designing a series of ponds and a wetland, the community hopes that the area's hydrology and ecology will improve. After these changes, how do you think the runoff hydrograph will change?

Solution

(a) By assuming a constant base flow, this can be estimated to 0.01 m^3/s.
(b) See the table below:

Time (min)	Q_{total} (m^3/s)	Q_{total} – *base flow* (m^3/s)	Q_{mean} (m^3/s)
0	0.010	0	0
10	0.110	0.10	0.05
20	0.110	0.10	0.10
30	0.210	0.20	0.15
40	0.010	0	0.10
50	0.010	0	0
Sum	—	—	0.40

According to the above table, the sum of Q_{mean} for the first 40 min is $0.40\ m^3/s$. Thus, the volume of direct runoff in m^3 is:

$$Q_{\text{direct}} = 0.40\ m^3/s \times 10 \times 60\ s = 240\ m^3$$

(c) The effective rainfall is equal to the direct runoff.
Hence, $P_{\text{effective}} = 240\ m^3/(4.8 \times 10^4\ m^2) = 0.005\ m = 5.0\ mm$

(d) The total losses were: $15 - 5 = 10$ mm

(e) The runoff coefficient is $Q/P = 5/15 = 0.33$

(f) The water balance $P = Q + E$ (losses); $P = 15$ mm, $Q = 5$ mm, and E (losses) = 10 mm.

(g) Assuming the same runoff coefficient of 0.33, the direct runoff from a 100 mm rainfall would be 33.3 mm. Then the volume of direct runoff will be: $0.0333\ m \times 4.8 \times 10^4\ m^2 \approx 1600\ m^3$
This water is expected to contain 20 mg N per litre. The total nitrogen transport from 100 mm rainfall would then become:

$$20 \times 10^{-6}\ kg/(10^{-3}\ m^3) \times 1600\ m^3 = 0.020 \times 1600\ kg = 32\ kg\ N.$$

(h) By designing an artificial wetland with ponds, *the runoff will be delayed and the peak runoff will be lower*. Possibly, *more evaporation will occur, thereby decreasing the total runoff volume*. Exactly how the shape of the hydrograph will look after the introduction of ponds will depend on the detailed flow conditions.

Example 2.11 A heavy rainfall event in a catchment lasted 12 hours and a total of 55 mm of rain was recorded during this period. The direct runoff corresponding to this rainfall was 30 mm.

(a) Calculate ϕ_{index}.

(b) During another rainfall event in the same catchment, 20 mm of rainfall was recorded during the first 6 hours and 10 mm during the following 6 hours. Calculate the losses according to the ϕ_{index} method [use the calculated ϕ_{index} from (a)].

Solution

(a) In this case, the ϕ_{index} can be calculated as follows:

$$\phi_{\text{index}} = (P - Q)/t = (55 - 30)/12 = 2.1 \text{ mm/hr}$$

(b) The loss during each period is the lowest value of either (i) the rainfall or (ii) $\phi \times t$, where t is the duration of the time period.

During the first 6 hours, the rainfall was equal to 20 mm, and the losses were $\phi \times t = 2.1 \times 6 = 12.6$ mm. During the following 6 hours, the rainfall was equal to 10 mm, and the losses were $\phi \times t = 2.1 \times 6 = 12.6$ mm.

Consequently, the rainfall was less than the potential losses and only 10 mm were lost. Thus, the loss was 12.6 mm during the first 6 hours and 10 mm during the following 6 hours. The total loss was 22.6 mm.

Example 2.12 The Horton infiltration equation $f(t)$ describes how the infiltration capacity (mm/hr) for the soil is changing with time as the soil becomes wetter. The accumulated infiltrated water (in mm) can be calculated using the integral of $f(t)$, i.e., $F(t)$.

$$f(t) = f_c + (f_o - f_c)e^{-kt}$$

$$F(t) = f_c t + \frac{f_o - f_c}{k}(1 - e^{-kt})$$

where, f_o = initial infiltration capacity of dry soil (mm/hr)

f_c = final infiltration capacity when the soil has become saturated soil (mm/hr)

k = time constant (hr^{-1})

Note that the Horton equation is only valid for ponding conditions.

In a specific soil, the following parameters are known:

$f_o = 35$ (mm/hr)

$f_c = 6$ (mm/hr)

$k = 2$ (hr^{-1})

(a) What is the infiltration capacity after 15, 30, and 60 min?

(b) How much water has infiltrated after 15, 30, and 60 min?

Solution

(a) By inserting the above parameter values into the Horton equation, the infiltration capacity becomes:

$$f(t) = f(15 \text{ min}) = f(0.25 \text{ hr}) = 6 + (35 - 6)\, e^{-2\times 0.25}$$

$$f(0.25 \text{ hr}) = 6 + 29\, e^{-0.5} = 6 + 29/1.649 = 23.6 \text{ mm/hr.}$$

Corresponding calculation gives $f(30 \text{ min}) = 16.7$ mm/hr and $f(60 \text{ min}) = 9.9$ mm/hr.

(b) By inserting the above parameter values in to the Horton equation, the accumulated infiltration becomes:

$F(t) = F(15 \text{ min}) = F(0.25 \text{ hr}) = 6 \times 0.25 + [(35 - 6)/2] \times (1 - e^{-2 \times 0.25})$

$F(0.25 \text{ hr}) = 6 \times 0.25 + [(35 - 6)/2] \times (1 - e^{-2 \times 0.25}) = 1.5 + 14.5 \times (1 - 0.606) = 7.2 \text{ mm}.$

Corresponding calculation gives $f(30 \text{ min}) = 12.2$ mm and $f(60 \text{ min}) = 18.5$ mm.

Example 2.13 A catchment has an area, $A = 2.26 \text{ km}^2$. Find the ϕ_{index} for a given rainfall event according to the table given below. Given, the direct runoff volume is $5.6 \times 10^4 \text{ m}^3$.

Time (hr)	*Rainfall intensity (mm/hr)*
0–2	7.1
2–5	11.7
5–7	5.6
7–10	3.6
10–12	1.5

Solution

For the entire 12 hr period, the direct runoff was:

$$Q_{\text{direct}} = (5.6 \times 10^4 \text{ m}^3)/2.26 \text{ km}^2 = 0.02478 \text{ m} = 24.8 \text{ mm}$$

The total rainfall was:

$$P_{\text{total}} = (2 \times 7.1) + (3 \times 11.7) + (2 \times 5.6) + (3 \times 3.6) + (2 \times 1.5) = 74.3 \text{ mm}$$

The losses were:

$$P_{\text{total}} - P_{\text{effective}} = 74.3 - 24.8 = 49.5 \text{ mm}$$

This means that the average loss rate was 49.5/12 = 4.13 mm/hr.

However, during the time period 7–12 hr, the rainfall intensity was lower than 4.13 mm/hr, which implies that the actual loss rate during this period was less than the ϕ_{index}.

In practical terms, there was no runoff during this period. Hence, the total runoff must have occurred between 0–7 hr with a total direct runoff equal to 24.8 mm.

The total rainfall during this period (0–7 hr) was:

$$P_{\text{total}} = (2 \times 7.1) + (3 \times 11.7) + (2 \times 5.6) = 60.5 \text{ mm}$$

The total losses during this period were:

$$P_{\text{total}} - \text{losses} = 60.5 - 24.8 = 35.7 \text{ mm}$$

This gives an average loss rate = 35.7/7 = 5.10 mm/hr. The rainfall intensity is always larger; therefore, $\phi_{\text{index}} = 5.10$ mm/hr.

Example 2.14 From a hydrologic viewpoint, a catchment can be described as a system that transforms input (precipitation) to output (runoff). Mention some (maximum five) important factors that influence this transformation. Describe *how* and *why* these factors influence the output.

Discussion

Catchment geometry influences the time to peak and the total length of the runoff hydrograph. The longer the total length of water courses, the longer the runoff time and the hydrograph will be.

Antecedent conditions in the catchment affect the total losses that occur in the catchment. If a long and dry period precedes a rainfall-runoff event, then more losses will occur as compared to the situation wherein the antecedent period was wet. These conditions also affect the base flow from the catchment.

The catchment geomorphology has a considerable effect on runoff conditions and total water balance. If soil and bedrock consist of highly pervious material, large volumes of water will be infiltrated and thus much water will be stored underground. On the other hand, if surface cover is impervious to a great extent; then the runoff will be quick with less loss, greater runoff volume, and shorter time to peak.

Surface cover and use of catchment: Plant cover and human use greatly influence the runoff conditions. The runoff conditions depend on whether the catchment falls under natural, agricultural, or urban area. The most important aspect that alters the runoff conditions is how well the catchment is drained.

Lake percentage of catchment area: The number of lakes and the total area occupied by the lakes in relation to the total catchment area greatly influence the runoff. A lake is a temporary storage that tends to delay runoff. The greater the lake percentage, the greater will be the storage effects.

Example 2.15

(a) What is *potential evapotranspiration?* How does it affect the water balance?
(b) What is *areal precipitation*? Why is its importance?
(c) What is *unsaturated zone*? How does it affect the hydrologic circulation?

Discussion

(a) Potential evapotranspiration denotes the maximum possible sum of evaporation and transpiration from an area, when water is not a delimiting factor. Actual evapotranspiration will equal potential evapotranspiration as long as the soil surface is saturated with water. When water is a delimiting factor for the evapotranspiration process, i.e., when the soil surface is no longer saturated with water, properties of the soil and ground will determine the actual evapotranspiration. Properties like capillary rise, texture and structure of the soil, and type of plants will then determine how much water will evaporate and transpire through plants. In general, more the potential evapotranspiration, greater is the water loss and less is the runoff volume.

(b) The areal precipitation has to be determined essentially from the point measurements that the precipitation gauges represent. Areal precipitation estimations are used in all kinds of hydrological calculations, such as water balances, runoff estimations, water availability, flood risk estimations, etc.

(c) The unsaturated zone is the soil section between the soil surface and the groundwater table. The term 'unsaturated' indicates that the pores and void volume in the soil are not saturated with water. Instead, the zone contains water, gas, and soil material. The unsaturated zone has a large effect on runoff and evaporation processes in the catchment. Surface water may infiltrate through the soil surface to the unsaturated zone and further percolate down to the groundwater table.

Example 2.16 The water circulation in a lake may be studied by the water balance. Establish a water balance for a lake (note: no numeric values or calculations).

(a) How can you estimate or measure the different components of the water balance for the lake?

(b) If you want to calculate the phosphorous mass balance for the lake, what other information do you need besides the water balance?

Discussion

(a) The average water balance for a lake may be given as: $P = Q + E$, where P is the average precipitation, Q is the average sum of all inflows minus all outflows, and E is the average evaporation. P can be measured by precipitation gauges; all inflows and outflows in water courses need to be measured by continuous water level readings together with stage-discharge relationships; and E can be calculated from the water balance.

It may be assumed that no significant inflow or outflow via groundwater is occurring. Usually, this is a seasonal phenomenon that can be ignored for average long-term values. Outflow and/or inflow via groundwater may, however, be observed through groundwater level observations around the lake.

(b) To establish phosphorous mass balances through transport in water, concentrations of different parts of the water balance are needed. All parts of the water balance have to be used, except for evaporation which does not contribute significantly to phosphorous transport. Also, transport via groundwater may be assumed negligible.

Example 2.17 Explain the following briefly:

(a) Stormwater

(b) Sewage water

(c) Soil water

(d) Storage term of the water balance

Discussion

(a) Stormwater is the term for runoff and surface water in urban areas. Essentially, this water is rainwater but the stormwater is often polluted by typical urban pollutants, such as heavy metals, oil, etc. The stormwater is usually led to the recipient without prior treatment.
(b) Sewage water is the collective name for the water coming from urban uses, such as kitchen, bath, toilets, washing clothes and dishes. It may be led to water treatment plants in separated (separated from stormwater) or combined (combined with stormwater) sewers.
(c) Soil water is water in the unsaturated zone between soil surface and groundwater table.
(d) The storage term denotes temporary storages in snow, ice, lakes, soil, groundwater, etc., that need to be considered in the water balance for time periods shorter than one year.

Example 2.18 A lake is a usually a natural water reservoir that is continuously replenished by water from the inflow, such as rivers and other water courses. Depending on the residence time that this water spends in the lake, pollutants in the water may either undergo natural biological treatment and/or sedimentation.

(a) If the residence time increases, how will the pollutants be affected in the lake water?
(b) If you know the total inflow of water to the lake and the total lake volume, what can you say about the theoretical residence time?
(c) If you want to measure the inflow to the lake with the help of a water level gauge, what kind of information and/or measurements do you need?

Discussion

(a) If the residence time increases, it means that the water is spending a longer time in the lake. This increases the chances that natural biological treatment processes have enough time to decompose pollutants in the water and also that pollutants may sediment.
(b) The theoretical residence time will be the total lake volume divided by the total inflow of water to the lake.
(c) You need a stage-discharge curve that relates the water level in the river to the discharge in m^3/s. The stage-discharge curve can be established by plotting the observed water level versus the discharge. Consequently, the discharge has to be estimated, e.g., with a current meter for representative sections of the water course. By plotting the observed water levels with the estimated discharges, a theoretical relationship can be established.

Example 2.19 The rapid drainage of agricultural areas and urban areas during the last few hundred years have dramatically changed the hydrological conditions of large areas. Try to give examples of how these changes have modified the following:

(a) hydrology
(b) transport of nutrients and pollutants
(c) biodiversity

Discussion

(a) The total water balances have been changed by increasing the speed at which the water is discharged. The drainage has meant that water is more rapidly transported to the sea. Thereby, evaporation and the residence time of water have decreased. The runoff hydrograph has become shorter in time, with larger peaks, and larger runoff volume. This has also increased the flood risks to a major extent.
(b) Since the residence time has decreased, natural biological processes have less time to treat the water naturally. Also, pollutants have less time to sediment if the discharge is occurring more quickly. Pollutants are flushed out from surfaces in agricultural and urban areas, thereby increasing the pollution load of the sea.
(c) The biodiversity has changed in relation to fewer occurrences of natural wetland and ponds/lakes. Also, less variation in surface use has meant less variation in species.

Example 2.20 A newly graduated hydrologic engineer is to investigate why the community's groundwater supply varies so much during the year. It is extremely cold and there is much snow on ground when the engineer goes out to have a look at the groundwater wells in the area. The engineer finds that the water levels are all very low. In a well, not far from a small river, it is noticed that the water level is much lower than in the river.

(a) How can it be explained that the groundwater level is lower than that of the river?
(b) If the engineer had been out during other seasons (spring, summer, autumn), what could the relationship between water levels in the river and groundwater have looked like? Try to draw figures.
(c) The engineer suspects that the storage term in the water balance plays a key role in the explanation. What is the storage term and how does it vary for different seasons?

Discussion

(a) See Fig. 2.15. During long, dry summers and/or snow-rich cold winters, there is no water that can replenish the groundwater. Consequently, due to evapotranspiration, the groundwater level may be lowered. Compared to stream water levels (the stream may be fed by water from the upper parts of the catchment with more precipitation and/or glacial melt), the groundwater level may be lower, and thus water is transported out of the stream to the groundwater.

(b) During wet periods with less evaporation, the situation may be reversed, e.g., autumn and spring periods.

(c) Probably much water is stored as groundwater during autumn and spring. This storage decreases during summer and winter.

EXERCISES

Exercise 2.1 State whether the following statements are correct or not.

(a) The probability for a 100-year return rainfall to occur during an individual year is 0.01.

(b) 1 l/(s · ha) is the same as 0.36 mm/hr.

(c) Direct runoff plus base flow gives the total runoff.

(d) In combined sewage pipes, stormwater and sewage water are discharged together.

(e) For longer time periods (years), the storage term of the water balance can be ignored.

(f) Thiessen method is a way to calculate the areal values from point values.

(g) The reason why the water velocity decreases towards the bottom and the sides of the wet transect of a water course is friction.

(h) The $_{\text{index}}$ describes how much water percolates to the groundwater during runoff.

(i) Frontal lifting often results in a precipitation increase with altitude.

Answer

(a) Correct (b) Correct

(c) Correct (d) Correct

(e) Correct (f) Correct

(g) Incorrect (h) Incorrect

Exercise 2.2 The water balance for a lake can be used to explain lake water level variations. For a specific lake, it rained 40 mm in a month. At the same time, the evaporation from the lake was 20 mm. The only inflow to the lake was equal to 3000 m^3, and the only outflow from the lake was through another river which was equal to 4000 m^3. During this month, the lake water level increased by 10 mm. Calculate the area of the lake. What assumptions are necessary?

Answer

The lake area is 30 ha. Other inflows and outflows such as groundwater flow must be ignored.

Exercise 2.3 A quadratic catchment area has its four corners in the following (x, y) coordinates; (0, 0), (1000, 0), (0, 1000), and (1000, 1000) with both x and y in metres. Two rain gauges are located in the area. Gauge A has the coordinates (500, 500), and gauge B has the coordinates (1100, 500). During a rainfall, 100 mm were recorded in gauge A and 0 mm in gauge B.

(a) Calculate the areal rainfall over the catchment using arithmetic average method.
(b) What would be the areal rainfall if Thiessen method is used?
(c) During a previous rainfall, a runoff coefficient of 0.8 was recorded. What is the runoff expected for the rainfall in (a)?
(d) What is the expected rainfall in (b)?
(e) Observed mass transport of phosphorous from the catchment was 100 kg. What was the concentration of phosphorous in the water?

Answer

(a) 100 mm
(b) 80 mm
(c) 80 mm
(d) 64 mm
(e) Between 1.25–1.56 g/m^3

Exercise 2.4 (after Larsson 2000) The discharge from a catchment with an area A = 1000 ha, was measured during a rainfall event. The discharge values (m^3/s) are given below for every 30 min, starting from the time 10:00.

(a) Estimate the times for start and finish of the direct runoff.
(b) What is the base flow?
(c) What is the volume corresponding to the direct runoff?
(d) What is the effective rainfall?

Q = 4.9, 4.7, 4.5, 7.3, 11.8, 17.2, 21.1, 25.5, 29.7, 32.4, 32.2, 30.9, 28.7, 26.2, 23.3, 20.5, 18.0, 15.7, 13.8, 12.3, 11.1, 10.0, 9.1, 8.4, 7.7, 7.2, 6.7, 6.3, 5.9, 5.5, 5.2, 4.9, 4.6, 4.4, 4.1, 3.9, 3.7, 3.5, 3.4, 3.2, 3.0

Answer

(a) 11:00–02:30
(b) about 4.5 m^3/s if a constant base flow is assumed
(c) 511560 m^3
(d) $P_{\text{effective}} = 51$ mm

Exercise 2.5 Using Eq. (2.14), obtain a and b for the following observed pairs of stage (m) and discharge (m^3/s).

Stage (m)	*Discharge (m^3/s)*
1	5
2	7.1
3	8.7
4	10
6.25	12.5

Answer $a = 5$, $b = 0.5$

OBJECTIVE QUESTIONS

1. Orographic lifting results in
   (a) Topographical dependence of precipitation
   (b) Lifting of soil surface after freezing
   (c) Higher ocean water level
   (d) A warmer climate

2. A precipitation gauge without windscreen as compared to a gauge with windscreen records
   (a) Less precipitation amount
   (b) More snow precipitation
   (c) Same amount of rainfall
   (d) Areal precipitation

3. Snow water equivalent shows
   (a) The depth of snow
   (b) Ratio snow/rainfall in precipitation
   (c) Climatic area in cold areas
   (d) The amount of water in a snow pack

4. The Thiessen method is used to
   (a) Estimate convective part of rainfall
   (b) Estimate areal precipitation
   (c) Calculate wind correction for rainfall
   (d) Divide catchment into sub-catchments

5. What is the probability that a 100-year flood occurs in a single year?
   (a) 0 (b) 0.001 (c) 0.01 (d) 1.0

6. 1 mm water on an area represents
   (a) 1 g/m^2 (b) 10 kg (c) 1 mm/m^2 (d) 1 dm^3/m^2

7. Horton equation is used to estimate
   (a) Amount of evaporation (b) Infiltration
   (c) Soil water content (d) Runoff

8. A higher albedo means
   (a) Higher reflection (b) Less reflection
   (c) Saturated surface (d) Less snow

9. Capillary rise will be largest in
   (a) Warm soil (b) Sandy soil (c) Clayey soil (d) Saturated soil

10. A lysimeter measures
    (a) Actual evaporation (b) Potential evaporation
    (c) Transpiration (d) Penman evaporation

11. A pollutograph shows the
    (a) Pollutant content in runoff (b) Pollutant reduction in soil
    (c) Pollution concentration in cities (d) Graph with unit kg/s

12. Which of the following statements is wrong?
    (a) $P_{\text{effective}} = Q_{\text{direct}}$ (b) $Q_{\text{direct}} = Q_{\text{total}} -$ base flow
    (c) Base flow $= \Delta S$ (d) $P_{\text{total}} = P_{\text{effective}} + F_{\text{index}}$

13. Estimate the rate of evaporation in mm/day if saturated vapour pressure of water at 30°C is 31.82 mm of mercury and relative humidity is 35%. Use the value of constant 0.5 in Dalton's law.
    (a) 5.67 (b) 8.70 (c) 10.34 (d) 12.45

14. While measuring the velocity at a point in the flow cross-section of shallow streams having depth upto 3.0 m, the average velocity is taken as the velocity measured at:
    (a) Water surface
    (b) Bottom
    (c) 0.6 times the depth of flow below the water surface
    (d) Anywhere between the water surface to bottom of streams

15. The relation between the stages G and discharge Q in a river is expressed as:
    (a) $Q = C_r(G + a)^{\beta}$ (b) $Q = C_r(G - a)^{\beta}$
    (c) $Q = \dfrac{C_r}{(G - a)^{\beta}}$ (d) $Q = C_r(Ga)^{\beta}$

    [Note: C_r and β are rating constants and a is a constant which represent the guage reading corresponding to zero diacharge]

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CHAPTER
3

Hyetograph and Hydrograph Analysis

3.1 INTRODUCTION

Not all the rain that falls over a catchment contribute to the runoff volume at the catchment outlet. A part of it is lost and the remainder of the rainfall, termed *excess rainfall*, is the driving force causing runoff (or flow or discharge) at the outlet of the catchment (or basin). A good understanding of these factors that preclude rainfall from running off is necessary in order to fully use the capabilities of observed hydrographs. Much of the rain falling during the first part of a storm is stored on the vegetal cover as interception and in surface depressions (puddles) as depression storage. As the rain continues, the soil surface gets saturated and the water starts to run off towards the channel. Thus, the part of storm rainfall which does not appear either as infiltration or as surface runoff is known as *surface retention*. *Interception* due to vegetal cover is relatively unimportant in moderate-to-severe floods, though it may be considerable over a period of time.

Almost immediately after the beginning of rainfall, the small depressions become full and overland flow begins. Some of the flows move towards larger depressions, while the major overland flows contribute to the streams. *Infiltration* is the phenomenon of water penetration from the ground surface to the sub-surface layers. By far, infiltration is the major contributing factor responsible for the water losses from rainfall which does not appear as runoff in the stream. Rainwater infiltrates through the soil surface once it exceeds the infiltration capacity of the soil. Thus, only excess rainfall will lead to runoff. Concepts related to precipitation, effective precipitation, relevant losses, runoff and base-flow were briefly introduced in the previous chapter. Here, a detailed coverage of hyetograph and hydrograph is considered with the main objective of estimating runoff, giving equal emphasis to single-storm as well as multiple-storm events.

3.2 HYETOGRAPHS

The graph that plots the rainfall over the basin along a time scale is called the *hyetograph*. Although a number of techniques are available for separating the losses from a rainfall hyetograph, the *infiltration indices method* is the simplest and the most popular technique used for this purpose. In hydrological calculations involving floods, it is convenient to use a constant value of infiltration rate (including all losses from the total rainfall hyetograph) for the duration of the storm.

The average infiltration rate is called *infiltration index*, and two types of indices are commonly used, they are: (i) ϕ-index and (ii) W-index. These indices account for infiltration and other losses. It basically involves a trial and error procedure, which does not differentiate between types of losses, e.g., interception, depression, and infiltration. Certain background on ϕ-index along with its computations has already been provided in the preceding chapter. However, for clarifying distinction between ϕ-index and W-index, both of these are explained below.

3.2.1 ϕ-Index

The ϕ-index or constant loss rate is defined as the rate of rainfall above which the rainfall volume equals the runoff volume (Fig. 3.1). Mathematically, the ϕ-index can be expressed as the average rainfall above which the rainfall volume is equal to the direct runoff volume. The ϕ-index is derived from the rainfall

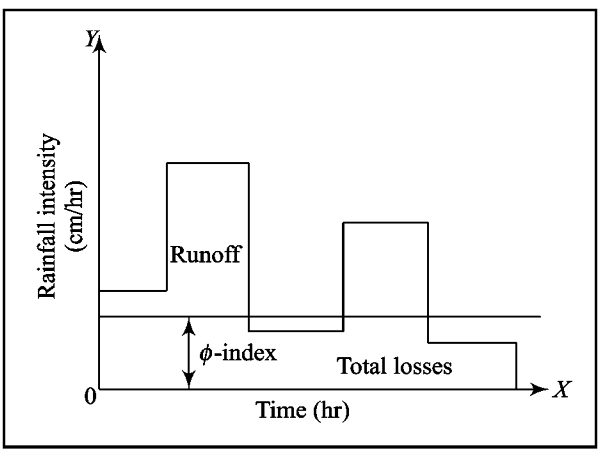


Fig. 3.1 *A generalized rainfall hyetograph showing ϕ-index in relation to total rainfall*

hyetograph which also helps to calculate the direct runoff volume. The ϕ-value is found by treating it as a constant infiltration capacity. If the rainfall intensity is less than ϕ, then the infiltration rate is equal to the rainfall intensity. However, if the rainfall intensity is greater than ϕ, then the difference between rainfall and infiltration in an interval of time represents the direct runoff volume. The amount of rainfall in excess of the index is called *excess rainfall.* The ϕ-index thus accounts for the abstraction of total losses and enables runoff magnitudes to be estimated for a given rainfall hyetograph. The example given below illustrates the calculation of ϕ-index.

Mathematically, the ϕ-index can be expressed as:

$$\phi = \frac{P - R}{t_e} \tag{3.1a}$$

where, P = total storm precipitation (mm or cm)

R = total direct surface runoff (mm or cm)

t_e = duration of the excess rainfall, i.e., the total time in which the total intensity is greater than ϕ (in hours), and

ϕ = uniform rate of infiltration (mm/hr or cm/hr)

If antecedent precipitation conditions in the catchment is such that other losses due to interception, depression, and evaporation, etc. are considerable in the beginning of the storm, then these losses may have to be subtracted from the rainfall, before applying the uniform loss rate method (ϕ-index) for computing the excess rainfall. Since infiltration accounts for majority of the water losses in heavy storms, the ϕ-index method must be interpreted for all practical purposes as a constant infiltration capacity method. It is a well-known fact that the infiltration capacity decreases with time. Therefore, the ϕ-index method is not that realistic due to varying infiltration capacity with time. It underestimates the infiltration for early rainfall and overestimates it for later rainfall. In addition, one does not know how to extrapolate the ϕ-value determined for one storm to another storm. In spite of these drawbacks, this simple method is often used for computing the excess rainfall hyetograph. The derivation of ϕ-index is simple, and the approach is described below with the help of two examples.

Example 3.1 A storm with 10 cm precipitation produced a direct surface runoff of 5.8 cm in the equivalent depth unit. The time distribution of the storm is given in Table 3.1(a). Estimate the ϕ-index of the storm and the excess rainfall hyetograph.

Table 3.1(a)

Time from start (hr)	1	2	3	4	5	6	7	8
Incremental rainfall in each hour (cm)	0.4	0.9	1.5	2.3	1.8	1.6	1.0	0.5

Solution

(i) Compute total infiltration, i.e.,
Total infiltration = Total rainfall – Direct surface runoff
= 10 – 5.8 = 4.2 cm

(ii) Assume duration of the excess rainfall (t_e) is equal to the total duration of the storm for the first trial, i.e., $t_e = 8$ hr

(iii) Compute the trial value of ϕ as:

$$\phi = \frac{P - R}{t_e} = \frac{4.2}{8} = 0.525 \text{ cm/hr}$$

(iv) Compare the above ϕ value with individual blocks of rainfall. The ϕ value makes the rainfall of the first hour and the eighth hour ineffective as their magnitudes are less than 0.525 cm/hr. Therefore, value of t_e is modified.

(v) Assume $t_e = 6$ hour in the second trial and modified value of total rainfall for 6 hour duration will be 10 – (0.4 + 0.5) = 9.1 cm

(vi) Compute a modified value of infiltration,
i.e., infiltration = 9.1 – 5.8 = 3.3 cm

(vii) Compute second trial value of ϕ, i.e., $\phi = 3.3/6 = 0.55$ cm/hr.
This value of ϕ is satisfactory as it gives $t_e = 6$ hr

(viii) Calculate the excess rainfall, subtracting the uniform loss from each block. The excess rainfall hyetograph is given below:

Table 3.1(b) Calculation of excess rainfall

Time from start (hr)	1	2	3	4	5	6	7	8
Excess rainfall (cm)	0	0.35	0.95	1.75	1.25	1.05	0.45	0

Note that the total excess rainfall = 5.8 cm = direct surface runoff (cm)

Example 3.2 A storm with 12.0 cm precipitation produced a direct runoff of 6.8 cm. The time distribution of the storm is given in Table 3.2(a). Estimate ϕ-index.

Table 3.2(a)

Time from start (hr)	1	2	3	4	5	6	7	8
Incremental rainfall in each hour (cm)	0.56	0.95	1.9	2.8	2.0	1.8	1.2	0.61

Solution

Total infiltration = 12.0 – 6.8 = 5.2 cm

Assume t_e = duration of excess rainfall = 8 hr for the first trial.

Then, $\phi = 5.2/8 = 0.65$ cm/hr

But, this value of ϕ makes the rainfalls of the first hour and the eighth hour ineffective as their magnitude is less than 0.65 cm/hr. Hence, the value of t_e is required to be modified.

Assume $t_e = 6$ hr for the second trial.
In this period:

$$\text{Infiltration} = 12.0 - (0.56 + 061 + 6.8) = 4.03$$

$$\phi = 4.03/6 = 0.67 \text{ cm/hr}$$

This value of ϕ is satisfactory as it gives $t_e = 6$ hr.

Table 3.2(b) Calculation of excess rainfall

Time from start (hr)	1	2	3	4	5	6	7	8
Excess rainfall (cm)	0	0.28	1.23	2.13	1.33	1.13	0.53	0

Total excess rainfall = 6.8 cm = total runoff.

3.2.2 W-Index

In an attempt to refine the ϕ-index, the initial losses are separated from the total abstractions and an average value of infiltration rate (called the W-index) is calculated as given below:

$$W = (P - R - I_a)/t_e \tag{3.1b}$$

where P = total storm precipitation (cm)
R = total storm runoff (cm)
I_a = initial losses (cm)
t_e = duration of the excess rainfall (in hours), i.e., the total time in which the rainfall intensity is greater than *infiltration capacity*
and W = average rate of infiltration (cm/hr)

Example 3.3 Determine ϕ-index and W-index for a watershed SW-12 with an area of 100 km^2. A rainfall with the specifications given in the following table occurred in the watershed.

Time (min)	*Intensity (cm/hr)*
0–30	0.300
60–120	1.205
120–150	1.500

The average flow measured at the outlet of the watershed was 100 m^3/s. Assume the retention to be 10% of rainfall.

Solution

Total rainfall $P = 0.150 + 1.205 + 0.75 = 2.105$ cm

Total runoff $V_Q = (100 \text{ m}^3/\text{s}) \times (150 \times 60 \text{ s})/(100 \times 10^6 \text{ m}^2) = 0.009 \text{ m} = 0.9$ cm

Total loss = 2.105 – 0.9 = 1.205 cm

Total time $T = 150/60 = 2.5$ hr

The estimated value of ϕ is = 1.205/2.5 = 0.482

Consider ϕ_{initial} = 0.7 cm /hr

Net rainfall is obtained as follows:

Table 3.3 Calculation of excess rainfall

Intensity (cm/hr)	$\phi_{initial}$ *(cm/hr)*	*Net rainfall in cm [(1) – (2)] × (duration of rainfall in hr)*
0.3	0.7	0
1.205	0.7	0.505
1.500	0.7	0.4
		Sum = 0.905 cm runoff

This shows that ϕ = 0.7 cm/hr is acceptable.

As $\quad W = (P - R - I_a)/t_e$

$$W_{\text{index}} = \frac{2.105 - 0.9 - (0.2 \times 2.105)}{1.5} = 0.52 \text{ cm/hr}$$

In the calculation, 20% of rainfall is taken as initial losses.

3.3 HYDROGRAPHS

Design of structures, e.g., determining the height of a dam, spillway design, design of bridge culverts, sizing the capacity of outlet works, etc., require information about extreme magnitudes of floods. It requires quantitative information on both peak flows and time distribution of runoff or hydrographs. Thus, a hydrograph gives the temporal variation of flow at the outlet of the catchment or basin.

The flow may be expressed in different units, e.g., in terms of discharge [L^3/T], or depth of flow per unit time [L/T]. The hydrograph yields time distribution of runoff from the excess rainfall occurring over a catchment for any given event. The runoff is generally measured at a gauging site located at the outlet of a catchment. In India, systematic gauging records are maintained by Central Water Commission and different state agencies for most of the major and medium basins.

The unit hydrograph theory proposed by Sherman (1932) is primarily based on *principle of linearity* and *time and space invariance*. For the gauged catchments, unit hydrographs can be derived from the analysis of available rainfall-runoff records. The procedures used to derive a unit hydrograph depend on the type of storm, whether it is a single-period storm or a multi-period storm. Thunderstorms, generally being intense and of short duration, are treated as single-period storms. On the other hand, frontal storms of usually longer duration are treated as multi-period storms.

A unit hydrograph converts the rainfall (excess rainfall) to runoff (direct surface runoff). For its derivation, the duration of the unit hydrograph is generally taken as the duration of excess rainfall block. However, *S*-curve and superimposition methods can be utilized for deriving a unit hydrograph of desired duration. The former approach is more general than the latter, because the former approach changes the duration of available unit hydrograph to any duration length.

This chapter includes description of unit hydrograph theory, its assumptions, and limitations; derivation of unit hydrograph and factors influencing its derivation, instantaneous unit hydrograph (IUH), *S*-curve, change of unit hydrograph duration, and derivation of average unit hydrograph.

The unit hydrograph derived from an event usually differs from another unit hydrograph derived from a different event. There are two possible approaches available for the derivation of the representative unit hydrograph for a watershed. The first approach averages the unit hydrographs derived for various events using conventional arithmetic averaging. The second approach, however, accumulates various events as a collective unit and then derives a representative single unit hydrograph for the watershed.

3.3.1 Definition of Unit Hydrograph

A unit hydrograph (UH) is a hydrograph of direct surface runoff from a unit excess rainfall occurring in unit time uniformly over the catchment area. The excess rainfall (ER) excludes losses, i.e., hydrological abstractions, infiltration losses from total rainfall; and unit excess rainfall volume equalling 1 mm may be considered as a standard value since rainfall is usually measured in mm. However, in certain cases, UH is derived for 1 cm ER. The selection of unit time depends on the duration of storm and size of the catchment area. For example, for small catchments, periods of 1 or 2 hours can be assumed; and for larger catchments, 3, 4, 6, or even 12 hours can be adopted.

A unit hydrograph can be interpreted as a multiplier that converts excess rainfall to direct surface runoff. The direct surface runoff (DSRO) is the streamflow hydrograph excluding base-flow contribution. Since, a unit hydrograph depicts the time distribution of flows, its multiplying effect varies with time. In real-world application, the unit hydrograph is applied to each block of excess rainfall; and the resulting hydrographs from each block are added for computing direct surface runoff hydrographs, to which base-flows are further added to obtain total flood hydrographs.

3.3.2 Unit Hydrograph Theory and Assumptions

The unit hydrograph theory can be described using the following underlying assumptions:

Constant Base Length Assumption

The base of direct surface runoff (t_B) corresponding to a rainfall amount of a given duration (T) is constant for a catchment and does not depend on the total runoff volume. For example, if T is the duration of excess rainfall, the base length of the DSRO for all rainfall events of this duration is the same for all events and is equal to t_B, as shown in Fig. 3.1.

Proportional Ordinate Assumption

For storms of same duration, the ordinate of DSRO at a time, i.e., $Q(t)$ is proportional to the total volume of excess rainfall. Thus, $Q_1(t) = k\,P_{E1}$ and $Q_2(t) = k\,P_{E2}$, where $Q_i(t)$ is the ordinate of DSRO at any given time t for the i^{th} storm having an excess rainfall volume of P_{Ei}, and k is proportionality constant. Similarly, the excess rainfall of another event, P_{E3}, over a time period T is equal to $P_{E3}\,T$. Therefore, at a given time t, the following proportionality holds.

$$\frac{Q_1(t)}{Q_3(t)} = \frac{P_{E1} \times T}{P_{E3} \times T} = \frac{P_{E1}}{P_{E3}} \tag{3.2}$$

Figure 3.2 shows the surface runoff hydrographs for three different excess rainfall volumes of 10, 20, and 30 mm for the same unit period. The ratio of q10:q20:q30 = 10:20:30 = 1:2:3. Thus, the ordinates of the DSRO produced by 60 mm excess rainfall will be 0.6 times the available 100 mm unit hydrograph.

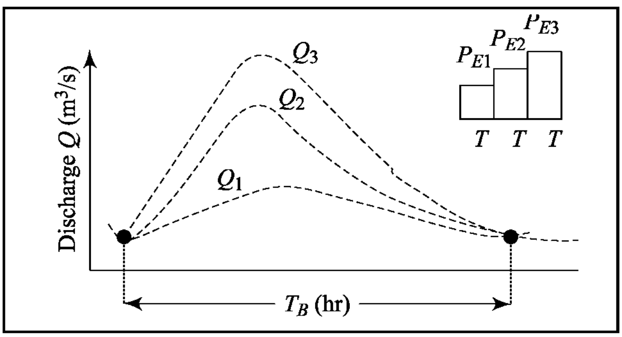


Fig. 3.2(a) *Three different unit hydrographs with the same base but varying peaks*

Concurrent Flow Assumption

The hydrograph of surface runoff that results from a particular portion of storm rainfall is not affected by the concurrent runoff resulting from other portions of the storm. In other words, the total hydrograph of surface runoff is the sum of the surface runoff produced by the individual portions of the excess rainfall. Figure 3.3 shows two hydrographs: one resulting from excess rainfall P_{E1} and another from the excess rainfall P_{E2}. The DSRO (Q_1) from the first excess rainfall (P_{E1}) is not affected by the DSRO (Q_2) from the second excess rainfall (P_{E2}). The total DSRO is, therefore, the sum of Q_1 and Q_2.

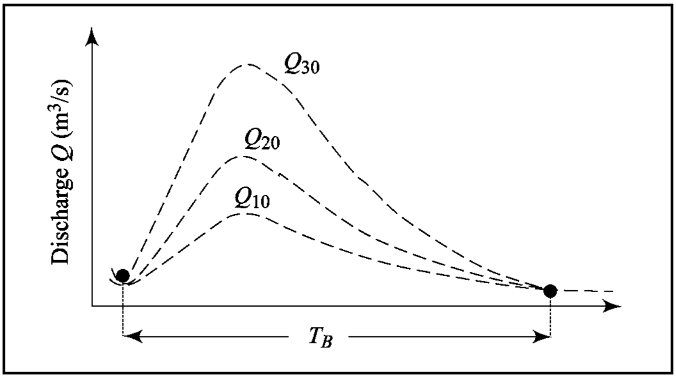


Fig. 3.2(b) *Surface runoff hydrographs for three different excess rainfall volumes of same unit period*

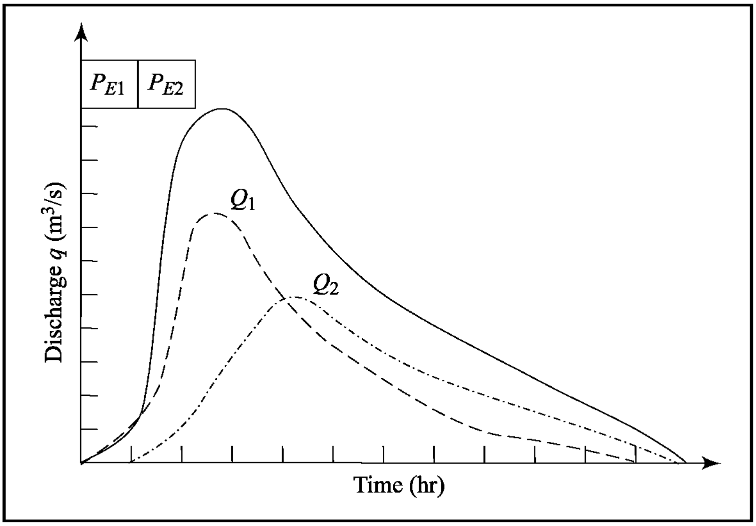


Fig. 3.3 *The total hydrograph resulting from consecutive storms*

Uniform Excess Rainfall in Time Assumption

It is assumed that the excess rainfall is the result of constant intensity of excess rainfall, which is equal to the excess rainfall divided by time duration *T*. The assumption requires that: (i) the selected storm must be intense and of short duration, and (ii) the resulting hydrograph be of short time base and single peaked.

Uniform Excess Rainfall in Space Assumption

It is assumed that the intensity of excess rainfall is uniform over the whole catchment. For the validity of this assumption, the catchment considered should be small. Therefore, the unit hydrograph theory is not applicable to large watersheds (more than 5000 sq. km). It can, however, be applied in components by subdividing the large watersheds.

Time Invariance Assumption

It is assumed that the unit hydrograph derived for a watershed does not vary with time. This principle holds when: (i) the physical characteristics of the watershed do not change with time and (ii) storm pattern and its movement do not change with time.

3.3.3 Factors Affecting Unit Hydrograph Shape

The factors affecting the shape of unit hydrograph are the rainfall distribution over the catchment and physiographic elements of the catchment, viz., shape, slope, vegetation, soil type, etc.

Rainfall Distribution

A variation in the areal pattern of rainfall, rainfall duration, and time intensity pattern greatly affects the shape of the hydrograph. For example, a hydrograph resulting from a rainfall concentrated in the lower part of a basin, i.e., nearer to the outlet of the basin (gauging site) will have a rapid rise since most of the rainfall volume will reach the outlet in a short time. This yields a sharp peak and a rapid recession. On the other hand, rainfall concentrated in the upper part of the same basin will yield a slow rising and receding hydrograph having a broad peak. Therefore, it is natural that unit hydrographs developed from rainfall of different areal distributions will exhibit differing shapes. This is one reason why the unit hydrograph technique is restricted to small basins. Small basins generally have a uniform structure throughout, which reduces the difference between the shapes of two unit hydrographs of equal excess rainfalls having same duration but different areal extensions.

For a given amount of runoff, the time base of the unit hydrograph increases and the peak lowers as the duration of rainfall increases. This means that two rainfall events having different intensities yield two different shapes of hydrographs. On large basins, changes in storm intensity should last for several hours to produce distinguishable effects on the hydrograph. The storage capacity of large basins tends to eliminate the effects of short time intensities and only major variations in the time intensity pattern is reflected in the hydrograph. On the other hand, on small basins, short bursts of excess rainfall lasting only for a few minutes may yield clearly defined peaks in the hydrographs. Thus, the effects of variation in the time intensity pattern can be lessened by selecting a short computational interval so that the variation in patterns from one computational interval to the next is not significant.

Physiography of the Catchment

Physical characteristics of a watershed change with time due to man-made interfaces, change in vegetative cover, and degradation of soil surface due to changing physiographical effects. Due to this, the shape of the derived unit hydrograph also changes. Due to high velocity of flow, steep catchment slopes produce runoff peaks earlier than flatter slopes; consequently, giving rise to earlier hydrograph peaks than that of flatter slopes. Seasonal and long-term changes in vegetation or other causes, such as fire, also change the physical characteristics of the watershed. It resorts to developing a regional relationship between unit hydrograph parameters and existing basin characteristics, for deriving the unit hydrograph in the changed environment.

3.3.4 Base-flow Separation

The main objective while deriving a unit hydrograph (hereafter, we shall refer it as UH) is to establish a relationship between surface flow hydrographs and the effective rainfall. In other words, the response of the basin to any input such as rainfall (in this case) is established using the available rainfall-runoff (discharge) data. The surface flow hydrograph is obtained from the total storm hydrograph by separating the quick-response flow from the slow-response runoff. There are three methods available to separate the base flow. Some of these methods have already been introduced in the previous chapter.

Method I

This is the simplest method and is generally practiced by field engineers. In this method, the base-flow is separated by drawing a straight line from the beginning of the surface runoff to a point on the recession limb representing the end of direct runoff. In Fig. 3.4, point A represents the end of direct runoff, identified by the sharp end of direct runoff at that point; and point B marks the end of direct runoff. An empirical equation for the time interval of N days from the peak to the point B is given by:

$$N = 0.83\ A^{0.2} \tag{3.3}$$

where, A = drainage area in km^2, and N in days

The value of N is approximate, and position of B should be decided by considering the number of hydrographs.

Method II

In this method, base-flow existing prior to the commencement of the surface runoff is extended until it intersects with the ordinate drawn at the peak point C. This point is joined to point B by a straight line. The segments AC and CB demarcate the base-flow and surface runoff. This method is most widely used in engineering practises.

Method III

In this method, the base-flow recession curve, after depletion of flood water, is extended backwards till it intersects with the ordinate at the point of inflection

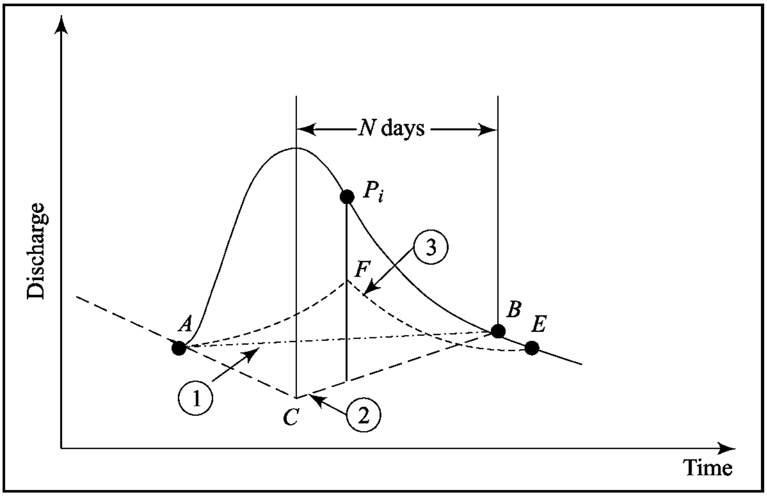


Fig. 3.4(a) *Schematic diagram showing various salient points used for base-flow separation*

(line *EF* in Fig. 3.4(a)). This method is realistic in situations where groundwater contributions are significant and reach the stream quickly.

All the above methods of base-flow separation are arbitrary, and the selection of any of these methods is dependant on local practise and successful predictions achieved in the past.

Example 3.4 The ordinates of discharge for a typical storm of a catchment having an area of 450 km^2 are given below:

Table 3.4(a) Ordinates of discharge

Time (hrs)	*Discharge (m^3/s)*	*Time (hrs)*	*Discharge (m^3/s)*	*Time (hrs)*	*Discharge (m^3/s)*
0	12	36	90	72	27
6	15	42	75	78	22
12	25	48	63	84	15
18	75	54	45	90	13
24	120	60	35	96	13
30	110	66	30	102	13

Find ordinates of the direct surface runoff hydrograph using straight-line technique for base-flow separation.

Solution

The computational steps involved are:

(i) Plot the recession limb of the discharge hydrograph on a semilog graph paper, keeping time on arithmetic scale and discharge values on log scale.

(ii) Locate the recession point in the above plot.

(iii) Draw a straight line from the rising point of the hydrograph to the recession limb of the hydrograph for base-flow separation.

(iv) Subtract the base-flow ordinates from the corresponding discharge hydrograph ordinates to estimate direct surface runoff hydrograph ordinates.

Table 3.4(b) shows the above computations. Here the base-flow ordinates, given in column (3) are subtracted from the discharge hydrograph ordinates, given in column (2). The resulting hydrograph ordinates known as direct surface runoff hydrograph ordinates are given in column (4).

It can be seen from the above plotted hydrograph (Fig. 3.4(b)) that the storm hydrograph has a base-flow component. Using the straight line technique for base-flow separation, we can find the time interval (N days) from the peak to the point B.

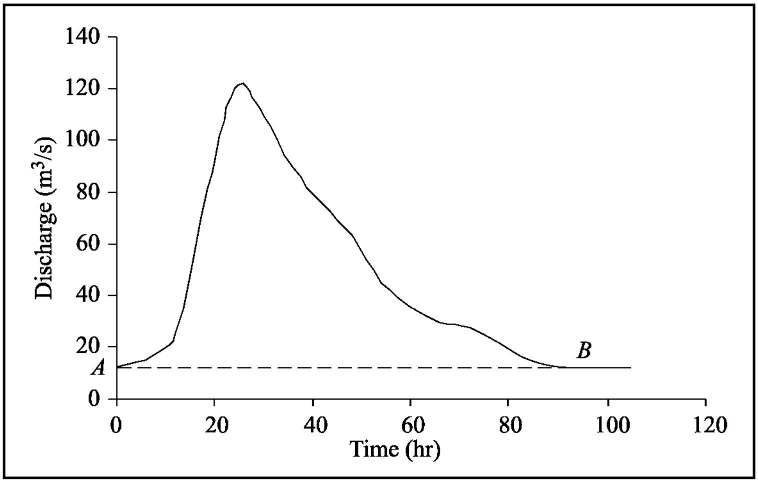


Fig. 3.4(b) *Base flow separation*

Using Eq. (3.3):

$$N = 0.83\ A^{0.2}$$
$$= 2.8165 \text{ days} = 67.6 \text{ hrs}$$

N can be calculated by judgement.

Beginning of DRH	$t = 0$
End of DRH	$t = 90$ hr
Peak	$t = 24$ hr

Hence, $N = 90 - 24 = 66$ hr

For convenience, N is taken as 66 hrs. A straight line is drawn between the points A and B which demarcates the base-flow and surface runoff.

Table 3.4(b) Estimation of direct surface runoff

Time (hrs) (1)	*Discharge Hydrograph (m^3/s) (2)*	*Base-flow (m^3/s) (3)*	*Direct Surface Runoff (m^3/s) (4) = (2) – (3)*
0	12	12	0
6	15	12	3
12	25	12	13
18	75	12	63
24	120	12	108
30	110	12.5	97.5
36	90	12.5	77.5
42	75	12.5	62.5
48	63	12.5	50.5
54	45	12.5	32.5
60	35	12.5	22.5
66	30	12.5	17.5
72	27	13	14
78	22	13	9
84	15.0	13	2
90	13	13	0
96	13	13	0
102	13	13	0

Example 3.5 The ordinates for discharge of a typical storm from a catchment having an area of 50 km^2 are given below. Estimate the direct surface runoff and the volume of excess rainfall for the given event.

Table 3.5(a) Estimation of direct surface runoff

t (hrs.) (1)	*Q (m^3/s) (2)*	*dQ/dt (3)*	*t (hrs.) (1)*	*Q (m^3/s) (2)*	*dQ/dt (3)*	*t (hrs.) (1)*	*t (hrs.) (2)*	*dQ/dt (3)*
0	61.67		24	378	–20	49	138.88	–13.34
1	63.67	2	25	356	–22	50	134.04	–12.92
2	64.5	0.83	26	341.75	–14.25	51	120.6	–4.84
3	88.74	24.24	27	344.42	2.67	52	122.1	–13.44
4	103.04	14.3	28	339.1	–5.32	53	99.01	1.5
5	146.04	43	29	296.8	–42.3	54	98.11	–23.09
6	189.53	43.49	30	289.8	–7	55	96.55	–0.9
7	224.56	35.03	31	280.03	–9.77	56	95.63	–1.56
8	292.8	68.24	32	280.03	0	57	94.4	–0.92
9	389.3	96.5	33	284.87	4.84	58	94.4	–1.23

(*Contd.*)

(*Contd.*)

10	419.9	30.6	34	243.9	–40.97	59	88.75	0
11	422.8	2.9	35	216.03	–27.87	60	88.7	–5.65
12	480.4	57.6	36	214	–2.03	61	89.91	–0.05
13	617.6	137.2	37	211.95	–2.05	62	89.91	1.21
14	**641.87**	24.27	38	210	–1.95	63	87.56	0
15	569	–72.87	39	198.17	–11.83	64	86.75	–2.35
16	514	–55	40	196.51	–1.66	65	85.0	–3.81
17	487	–27	41	194.6	–1.91	**66**	**83.8**	0.05
18	462.85	–24.15	42	190.3	–4.3	67	82.7	0
19	450	–12.85	43	187.1	–3.2	68	82.7	–1.1
20	**425.1**	–24.9	44	177.11	–9.99	69	82.7	0
21	422.2	–2.9	45	171.74	–5.37	70	81.61	0
22	412	–10.2	46	165.4	–6.34	71	73.83	–1.09
23	398	–14	47	165.14	–0.26	72	71.83	–7.78
			48	152.22	–12.92			

Solution

The flood hydrograph for the above data is shown in Fig. 3.5. The peak of the hydrograph is at $t = 14$ hours and the peak discharge (Q_p) is equal to 641.87 m^3/s.

Column (3) shows the variation of slope of the hydrograph at different time ordinates. This is used to mark the point of inflection (P_i), which marks the change of hydrograph slope. dQ/dt changes from –24.9 to –2.9 at $t = 20$ hours.

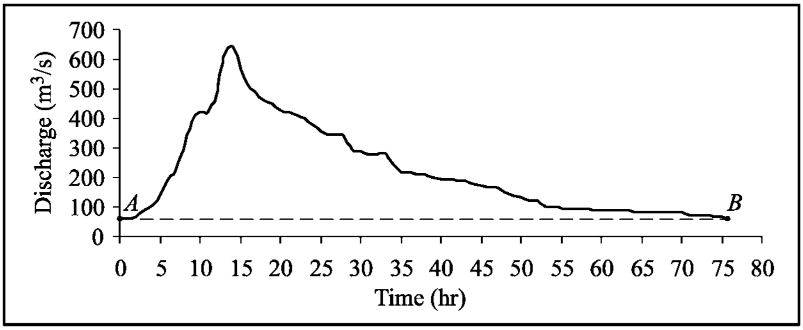


Fig. 3.5 *Flood hydrograph*

This is taken as the point of inflection.

Using Eq. (3.3), $N = 0.82 \times 50^{0.2} = 1.81$ days

Therefore, point B (refer Fig. 3.5) is at $t = 66$ hours. The corresponding discharge at point B is 83.5 m^3/s (marked bold in table). From the data, it can be seen that point A falls nearly at 3 hours. The ordinate of direct surface runoff hydrograph using the straight-line technique is given in Table 3.5(b).

Table 3.5(b)

t (hrs.)	*Q* (m^3/s)	*DSRO* (m^3/s)	*t* (hrs.)	*Q* (m^3/s)	*DSRO* (m^3/s)	*t* (hrs.)	*Q* (m^3/s)	*DSRO* (m^3/s)
0	61.67	—	24	378	294.2	49	138.88	68
1	63.67	—	25	356	272.2	50	134.04	55.08
2	64.5	—	26	341.75	258	51	120.6	50.24
3	88.74	4.94	27	344.42	260.6	52	122.1	36.8
4	103.04	19.24	28	339.1	255.3	53	99.01	38.3
5	146.04	62.24	29	296.8	213	54	98.11	15.21
6	189.53	105.7	30	289.8	206	55	96.55	14.31
7	224.56	140.8	31	280.03	196.2	56	95.63	12.75
8	292.8	209	32	280.03	196.2	57	94.4	11.83
9	389.3	305.5	33	284.87	201.1	58	94.4	10.6
10	419.9	336.1	34	243.9	160.1	59	88.75	10.6
11	422.8	339	35	216.03	132.2	60	88.7	4.95
12	480.4	396.6	36	214	130.2	61	89.91	4.9
13	617.6	533.8	37	211.95	128.2	62	89.91	6.11
14	**641.87**	558.1	38	210	126.2	63	87.56	6.11
15	569	485.2	39	198.17	114.4	64	86.75	2.95
16	514	430.2	40	196.51	112.7	65	85.0	1.2
17	487	403.2	41	194.6	110.8	**66**	**83.8**	0
18	462.85	379.1	42	190.3	106.5	67	82.7	
19	450	366.2	43	187.1	103.3	68	82.7	
20	**425.1**	341.3	44	177.11	93.31	69	82.7	
21	422.2	338.4	45	171.74	87.94	70	81.61	
22	412	328.2	46	165.4	81.6	71	73.83	
23	398	314.2	47	165.14	81.34	72	71.83	
			48	152.22	74.88			

The summation of the DSRO is 10669 m^3/s or 10669×3600 m^3/hr.

In terms of depth, this is equal to $(10669 \times 3600)/(50 \times 10^6) = 0.000213$ m = 0.213 mm. This is the excess rainfall depth.

3.3.5 Features of UH and Distribution Graph

The distribution graph is a non-dimensional form of the unit hydrograph, generally available in percent form. The sum of all percentage values should be equal to 100. The main features of a unit hydrograph or a distribution graph are:

- It shows the time distribution of the discharge hydrograph of a watershed produced by a uniform excess rainfall of given depth.

- It transforms the excess rainfall to time-varying discharge at the outlet assuming a linear process.
- It represents the watershed characteristics by exhibiting the integrated effect of the surface features on the routing of the excess rainfall through the catchment.

It is worth mentioning that the derivation of a unit hydrograph usually excludes records of moderate floods; and the principles of unit hydrograph can be applied for estimating the design flood, supplementing missing flood records, and short-term flood forecasting that is based on recorded rainfall.

3.4 DERIVATION OF UH

Unit hydrograph is extensively used in computation of surface runoffs resulting due to storm events. Similarly, for known direct runoffs and storm characteristics including magnitude and duration, unit hydrograph can be derived. In this section, we discuss the procedures to derive the unit hydrograph.

3.4.1 Conventional Method using Single-Peak Storm

The unit hydrograph for a gauging station is derived from a single-period storm as follows:

(i) Extract hydrographs for major floods from stream-flow records using observed gauge-discharge or rating curves.

(ii) Separate base-flow from total hydrograph to produce direct surface runoff hydrograph. Compute the surface runoff volume Q in mm. In practice, floods of runoff volume greater than 20 mm are analysed. If records of higher floods are not available, lower flow values can also be used, but with caution.

(iii) Examine the rainfall records available for the rain gauge stations in the catchment and compute average rainfall for the storm under investigation.

(iv) Derive the hydrograph and the total average rainfall for the time period which corresponds to the desired unit period of the unit hydrograph.

(v) Determine the loss rate subtracting the direct surface runoff from the total rainfall to estimate the excess rainfall in each of the unit periods.

(vi) Since the unit hydrograph is a multiplying factor to convert excess rainfall to DSRO, the procedure for its derivation is as follows:
   - Plot the surface runoff hydrograph.
   - Determine the volume of excess rainfall, P_E, in the single unit period. (P_E also equals the volume of the direct surface runoff hydrograph Q_s).
   - Relate the value of P_E to the unit volume of the unit hydrograph, Q_{UH}, as demonstrated in the following example.

It is important to understand precisely the term "unit volume of the unit hydrograph". Normally, many textbooks use the word "unit hydrograph" to imply that it is due to 1 cm effective rainfall. This may not always hold good. Unit hydrograph may also imply that it is due to 1 mm effective rainfall, or for that matter, any other unit, say 1 feet, 1 inch, etc. In this book, we use unit hydrograph in the sense that it is due to 1 mm effective rainfall.

If 1 mm effective rainfall occurs uniformly over a catchment for a unit time, the volume of UH will be equal to (1 mm × area of catchment). If we divide this volume by the catchment area A, this will be just equal to 1 mm (the unit used for effective or excess rainfall). Thus, the term "unit volume of unit hydrograph" implies that the volume under the UH is equal to one unit used for effective rainfall of one unit.

Example 3.6 For a given P_E of 22.5 mm, obtain proportionality factor (F) to derive a unit hydrograph for 10 mm unit volume.

Solution

Give unit volume of unit hydrograph Q_{UH} = 10 mm, the proportionality factor F can be computed as:

$$F = \frac{P_E}{Q_{UH}} = \frac{22.5}{10} = 2.25$$

Ordinates of the UH can be obtained by dividing the direct surface runoff hydrographs by F. Please note that unit volume of such UH will be equal 10 mm.

Example 3.7 A thunderstorm of 6-hr duration with the excess rainfall of 154 mm produces the following surface runoff hydrograph:

Table 3.6(a) Data for Example 3.7

Date	*June 15*				*June 16*				*June 17*			
Time (Hrs)	0600	1200	1800	M.N.	0600	1200	1800	M.N.	0600	1200	1800	M.N.
DSRO (m^3/s)	10	500	1600	3500	5200	3100	1500	650	250	0	0	0

Find the ordinates of a 6-hr UH of unit volume equal to 100 mm.

Solution

Since P_E = 154 mm and Q_{UH} = 100 mm, the conversion factor F = 154/100 = 1.54.

The ordinates of unit hydrograph are computed dividing the direct surface runoff hydrograph ordinates by the conversion factor. The calculation of 6-hr unit hydrograph for the above storm is shown in Table 3.6(b).

Table 3.6(b) Computation of 6-hr unit hydrograph

Date	*Time (hr)*	*Direct surface runoff (m^3/s)*	*6-hr 100 mm unit hydrograph (m^3/s)*
		Q	*Q/F*
June 15	0600	10	7
	1200	500	325
	1800	1600	1039
	2400	3500	2272
June 16	0600	5200	3377
	1200	3100	2013
	1800	1500	974
	2400	650	422
June 17	0600	250	162
	1200	0	0
	1800	0	0

3.4.2 UH Derivation from a Multi-period Storm

When a storm is of short duration and of fairly uniform intensity, the unit hydrograph can be derived by simply applying the single period storm technique. However, for a storm of long duration or a shorter one with variable intensity, the storm should be treated as consisting of a series of storms and the derivation of the unit hydrograph gets slightly more complicated. There are a number of methods to derive unit hydrographs from multi-period storms, but all are based on unit hydrograph theory as follows.

Let P_{ei} = intensity of excess rainfall (mm/hr), P_{Ei} = volume of excess rainfall (mm), U_i = ordinates of unit hydrograph (m^3/sec), Q_i = ordinates of direct surface runoff hydrograph (m^3/sec), and i = an integer value for an ordinate. The underlying procedure of calculating the direct surface runoff hydrograph from a multi-period storm is described below.

If P_{ei} = rainfall rate (mm/hr) in period T, then P_{Ei} = volume of rainfall $= P_{ei} \times T$. For example, if $P_{e_i} = 2$ mm/hr and $T = 2$ hr, then $P_{E_i} = 2 \times 2 = 4$ mm. To determine the direct surface runoff hydrograph from 4 mm of rainfall that occurred in 2 hrs, the 10 mm unit hydrograph of 2-hr duration having 7 unit periods (time base = 7 × 2 = 14 hrs) is used. Thus,

(i) the base length of the direct surface runoff hydrograph for rainfalls in excess of 2-hr duration is 14 hr

(ii) for a 10 mm unit hydrograph, the average runoff in the various periods for the first rainfall in excess of 4 mm is computed using the proportionality principle, as given below:

$$Q_1 = 0.4 \times U_1 \quad Q_3 = 0.4 \times U_3 \quad Q_5 = 0.4 \times U_5 \quad Q_7 = 0.4 \times U_7$$
$$Q_2 = 0.4 \times U_2 \quad Q_4 = 0.4 \times U_4 \quad Q_6 = 0.4 \times U_6$$

(iii) If P_{E2} = 4 mm/hr (2nd period of rainfall) or $P_{E2} = 4 \times 2 = 8$ mm, the ordinates for the second period of rainfall will be as follows:

$$Q_1 = 0.8 \times U_1, \quad Q_2 = 0.8 \times U_2, \quad \text{and so on.}$$

It is to be noted that the excess rainfall in the second period is lagged by T = 2 hr. Thus, Q_1 of the second period of rainfall is added to the second period of runoff for first hydrograph ordinate. This principle continues for the third and subsequent rainfall values. The equations for the combined surface runoff can, therefore, be generalised as follows:

$$Q_1 = P_{E1}\,U_1$$
$$Q_2 = P_{E1}\,U_2 + P_{E2}\,U_1$$
$$Q_3 = P_{E1}\,U_3 + P_{E2}\,U_2 + P_{E3}\,U_1$$
$$Q_4 = P_{E1}\,U_4 + P_{E2}\,U_3 + P_{E3}\,U_2$$
$$Q_5 = P_{E1}\,U_5 + P_{E2}\,U_4 + P_{E3}\,U_3$$
$$Q_6 = P_{E1}\,U_6 + P_{E2}\,U_5 + P_{E3}\,U_4$$
$$Q_7 = P_{E1}\,U_7 + P_{E2}\,U_6 + P_{E3}\,U_5$$
$$Q_8 = 0 + P_{E2}\,U_7 + P_{E3}\,U_6$$
$$Q_9 = 0 + 0 + P_{E3}\,U_7$$

All the following procedures used for deriving unit hydrographs from multi-period storms use the principles of proportionality and superposition.

Method of Single Division

This is basically a trial and error approach. The following procedure involves solving each of the above equations relating to the unit hydrograph ordinates, as shown in Fig. 3.6. For the first rainfall, both Q_1 and P_{E1} are known; Q_1 from the surface runoff hydrograph and P_{E1} from the hyetograph excess rainfall.

Thus, $\quad Q_1 = P_{E1} \times U_1 \quad \text{or} \quad U_1 = \dfrac{Q_1}{P_{E1}}$

For the second rainfall, Q_2 is known from the hydrograph; and P_{E1}, U_1 and P_{E2} are already known.

Since $\quad Q_2 = P_{E2}\,U_2 + P_{E2}\,U_1, \; U_2 = \dfrac{Q_2 - P_{E2}U_1}{P_{E1}}, \; U_3 = \dfrac{Q_3 - P_{E3}U_2}{P_{E2}}$, and so on.

Thus, the unit hydrograph ordinates (U_i) are computed.

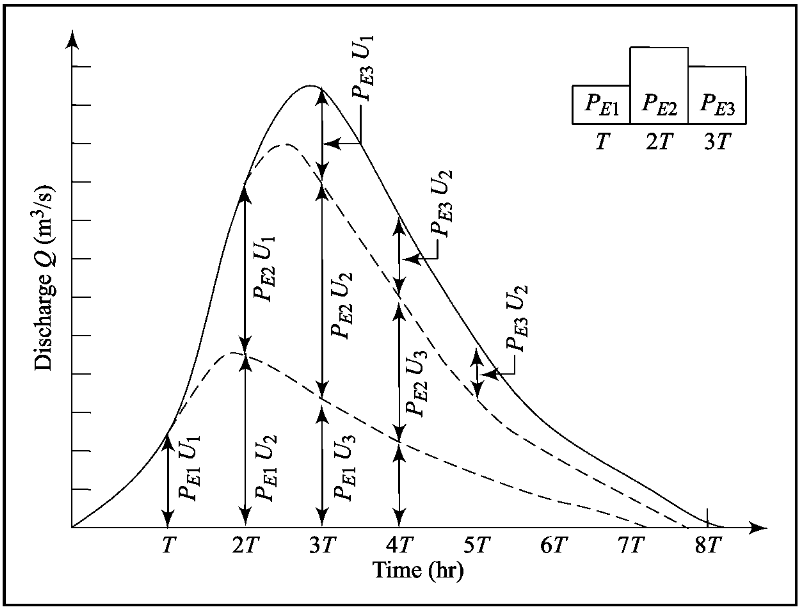


Fig. 3.6 *Unit hydrograph derivation from a multi-period storm*

Since there are more equations than unknown values, the equations relating Q_8 and Q_9 to U_6 and U_7, respectively, may not be in balance. Therefore, it is necessary to apply an 'averaging' procedure. It involves calculation of U_1, U_2,..., U_7 from Q_1, Q_2,...,Q_7 (forward run); and the calculation of U_7, U_6,...,U_1 from Q_9, Q_8,...,Q_1 (backward run) as illustrated in Fig. 3.6 through a typical example. The unit hydrograph ordinates are then derived by 'averaging' the values of U_1, U_2,...,U_7 computed from the two runs. The derived unit hydrograph is then checked for producing a hydrograph that compares well with the recorded hydrograph. This test is recommended in all unit hydrograph derivations.

The direct surface runoff hydrograph computed using the above average unit hydrograph does not match satisfactorily with the observed direct surface runoff hydrograph. This deviation may be attributed to small errors in U_1, U_2, U_6, and U_7, which are the leading and trailing ordinates, respectively; and these values propagate in calculations of other ordinates. Computations rely more on U_1 and U_2 in forward run and U_6 and U_7 in backward run. Furthermore, it is possible that (i) the records of rainfall and runoff may not be representative for the actual situation, (ii) the unit hydrograph theory may not be applicable to the storm, and (iii) storms may not be uniformly distributed over the catchment. These circumstances create difficulties in derivation of unit hydrograph.

Least Squares Method

As is evident in single division method, there are more equations than unknown values and the solution adopted depends on the selected set of equations. The least squares method is a statistical curve fitting procedure which yields a 'line of best fit'.

The basic principle of the least squares method lies in the fact that the line of best fit produces the best reproduction of the hydrograph, computed from the application of the unit hydrograph to the excess rainfall, as shown in Fig. 3.7. The developed unit hydrograph yields minimum deviation of the ordinates of the reproduced hydrograph from the ordinates of the actual hydrograph.

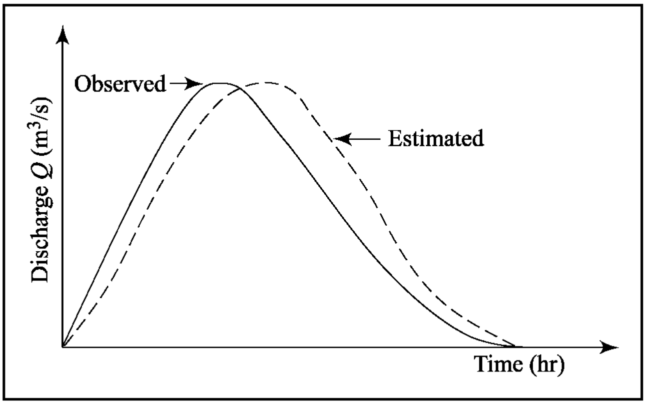


Fig. 3.7 *Goodness-of-fit of observed and computed hydrographs*

As this method involves a number of tedious computations to minimize the errors, it is not a practical proposition to carry out computations manually. The readily available computer packages can be employed. This method is suitable subject to the following criteria:

- the hydrograph represents the basin and storm characteristics,
- the rainfall and hydrograph ordinates are accurate, and
- the unit hydrograph theory is applicable to the catchment.

If these criteria are not met, the 'best fit' unit hydrograph can assume any shape and may yield even negative ordinates. Therefore, it is necessary to stress that the derived unit hydrograph is more mathematical rather than just a representative of the characteristics of runoff peculiar to a particular catchment. Kuchment (1967) suggested a procedure for deriving a smooth unit hydrograph using a regularization factor determined by trial and error.

Collins' Method

This is a trial and error approach. Assume that the unit hydrograph has four ordinates, viz., U_1, U_2, U_3, and U_4 corresponding to t values equal to T, $2T$, $3T$ and $4T$; and there are three periods of time T of excess rainfall, viz., P_{E1}, P_{E2}, and P_{E3} with P_{E2} representing the maximum rainfall volume. The surface runoff ordinates are determined as described earlier in Section 3.2. It is to note that in the previous example, the unit hydrograph had 7 ordinates; whereas in this example, there are only 4 ordinates. It is not necessary that the unit hydrograph ordinates are at the same time interval as the excess rainfall. The following equations hold good for computations of Q_1 to Q_4:

$$Q_1 = P_{E1}\, U_1$$
$$Q_2 = P_{E1}\, U_2 + P_{E2}\, U_1$$
$$Q_3 = P_{E1}\, U_3 + P_{E2}\, U_2 + P_{E3}\, U_1$$
$$Q_4 = P_{E1}\, U_4 + P_{E2}\, U_3 + P_{E3}\, U_2$$

Computations for other Q values can be done in a similar manner. For a gauged catchment, a unit hydrograph can be derived using Collins' method, according to the following steps:

(i) Derive the flood hydrographs for major flood events from stream flow records using rating curves.

(ii) Estimate the direct surface runoff by separating the base-flow, using a suitable approach, from the total hydrograph.

(iii) Estimate the average rainfall hyetograph.

(iv) Compute the effective rainfall hydrograph by separating the losses from the total rainfall hyetograph.

(v) Ignore the terms in the equation that contain the maximum rainfall [P_{E2}, termed $P_{E\,(\mathrm{MAX})}$].

(vi) Assume a first trial unit hydrograph, i.e., a set of U-values, which appears reasonable. To this end, use either constant value for unit hydrograph ordinates or single division procedure for the first estimate of unit hydrograph. The number of unit hydrograph ordinates can be determined using the following relation:

Periods in unit hydrograph base = (Periods in direct surface runoff base – number of periods of excess rainfall blocks) + 1

Arrange this set of values in Col. 3 of Table 3.7.

(vii) Apply the initial trial unit hydrograph to all periods of excess rainfall except $P_{E\,(\mathrm{MAX})}$ and determine the resulting hydrograph, as shown in Columns 2 through 4 of Table 3.7.

(viii) Deduct the sum of $P_{Ei}U_i$ from the actual direct surface runoff hydrograph ordinates, and estimate the runoff from the maximum rainfall $P_{E\,(\mathrm{MAX})}$, as shown in Column 6 of Table 3.7. Retain only those values having the term $P_{E\,(\mathrm{MAX})}$, as the resultant DSRO is the response of $P_{E\,(\mathrm{MAX})}$.

Table 3.7 Collins' method

Time Period	*Excess rainfall (mm)*	*Trial unit hydrograph (m^3/s)*				$\Sigma P_{Ei}U_i$ *(excludes* $P_{E(MAX)}$*)* *(m^3/s)*	*Actual DSRO (m^3/s)*	$Q_i - \Sigma P_{Ei}U_i$ *(m^3/s)*	$\left[\dfrac{Q_i - \Sigma P_{Ei}U_i}{P_{E\,(MAX)}}\right]$ *(m^3/s)*	$\bar{U}$ *(m^3/s)*	*Adjusted* $\bar{U}$
		U_1	U_2	U_3	U_4						
(1)	*(2)*	*(3)*				*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9) = 8/F*
0							Q_0	—			
1	P_{E1}	$P_{E1}U_1$				$P_{E1}U_1$	Q_1	$Q_1 - P_{E1}U_1$			
2	$P_{E(MAX)}$	—	$P_{E1}U_2$			$P_{E1}U_2$	Q_2	$Q_2 - P_{E1}U_2$	U_{1c}		
3	P_{E2}	$P_{E3}U_1$	—	$P_{E1}U_3$		$P_{E3}U_1 + P_{E1}U_3$	Q_3	$Q_3 - (P_{E3}U_1 + P_{E1}U_3)$	U_{2c}		
4			$P_{E3}U_2$	—	$P_{E1}U_4$	$P_{E3}U_2 + P_{E1}U_4$	Q_4	$Q_4 - (P_{E3}U_2 + P_{E1}U_4)$	U_{3c}		
5				$P_{E3}U_3$	—	$P_{E3}U_3$	Q_5	$Q_5 - P_{E3}U_3$	U_{4c}		
6					$P_{E3}U_4$	$P_{E3}U_4$	Q_6	$Q_6 - P_{E3}U_4$			

(ix) Divide the ordinates computed at step (viii) by $P_{E\,(\text{MAX})}$ and obtain another estimate of the unit hydrograph ordinates, as shown in Column 7.

(x) Compare the derived unit hydrograph ordinates (U_{ic}) with the original trial ordinates (U_i).

(xi) If the comparison is not satisfactory, take an average of the unit hydrograph ordinates of the first trial for computing unit hydrograph values for the second trial, as given below:

$$\overline{U} = \frac{MU_i + NU_{ic}}{M + N} \tag{3.4}$$

where, M = total excess rainfall (= $P_{E1} + P_{E2} + \ldots$) except $P_{E\,(\text{MAX})}$, and $N = P_{E\,(\text{MAX})}$. Equation (3.4) ensures that the trial ordinates of the unit hydrograph correspond to the unit runoff volume. Note that the unit runoff volume can be any multiple of 1 mm. The conversion factor (F) is applied as shown in Column 9 of Table 3.7. It is computed as follows:

$$F = \frac{\Sigma U_i}{\Sigma U_{ic}} \tag{3.5}$$

where, ΣU_i is the sum of trial unit hydrograph ordinates, and ΣU_{ic} is the sum computed unit hydrograph ordinates falling between the limits of $P_{E\,(\text{MAX})}$ influence.

(xii) Repeat step (vii) to (xi) until the calculated unit hydrograph agrees with the trial unit hydrograph. The application is illustrated with the help of an example.

Example 3.8 The excess rainfall and surface runoff ordinates for a storm of a typical catchment of 1,70,000 ha are given below. Derive a 6-hr 100 mm unit hydrograph using Collins' method.

Table 3.8 Data for Example 3.8

Time Period (6 hr)	P_{Ei} *(mm)*	q_i *(m³/s/6 hr)*
1	40	250
2	100	1050
3	60	2050
4		4350
5		4150
6		2300
7		1070
8		450
9		120

Solution

Tables 3.9 and 3.10 present computations for two trials of unit hydrograph derivation using Collins' method.

In Table 3.9, number of direct surface runoff periods = 9 (6-hr periods) and rainfall periods = 3 (6-hr periods).

Therefore, the total period of UH = (9 – 3) + 1 = 7 (6-hr periods).

The first trial assumes a uniform unit hydrograph, determined as follows:

Calculate total surface runoff (from volume under hydrograph) or by planimetering area. It is equal to 340×10^6 m^3.

Since the catchment area = 170,000 ha = 1700×10^6 m^2, the unit runoff of 1 mm corresponds to = $1700 \times 10^6 \times 10^{-3}$ (= 1.7×10^6) m^3.

Therefore, unit runoff volume of 100 mm unit hydrograph will be equal to 170×10^6 m^3, and 340×10^6 m^3 of direct surface runoff (DSRO) volume will correspond to 200 mm depth, which is equal to the sum of excess rainfall depths (40 mm + 60 mm + 100 mm). Thus, the DSRO volume and the volume of excess rainfall are perfectly matched.

Since a unit hydrograph of 100 mm is to be developed, it is necessary that each value of excess rainfall be divided by 100. Thus, for P_{E1} = 40 mm, the result is 0.4; for P_{E2} = 100 mm, it is equal to 1.0; and for P_{E3} = 60 mm, it is 0.6. It is to note that if the objective had been to develop a unit hydrograph of 1 mm, all values of P_E would have been unchanged.

For the first trial, assume a uniform rectangular-shaped unit hydrograph. The magnitude of ordinates can be computed as $U_i = 170 \times 10^6/(42 \times 3600)$ = 1125 m^3/sec. Having computed the trial values of the 100 mm unit hydrograph ordinates, steps (vii) through (xii) can be followed.

The computations are shown in Table 3.9. In this table, Column 1 presents the time steps and Column 2 presents the excess rainfall values which are divided by 100 and the resulting values are shown in Column 3. Column 4 presents the trial unit hydrograph ordinates and computations of direct surface runoff hydrograph from P_{E1} and P_{E2} values of Column 3. The respective sum of the ordinates which is the response of P_{E1} and P_{E2} is shown in Column 5. Column 6 presents the actual observed direct surface runoff and the difference of Column 6 and Column 5. Column 7 lists the values of Column 6 divided by $P_{E\,(MAX)}$, which yields the ordinates of the 100 mm unit hydrograph derived from only $P_{E\,(MAX)}$.

It is to note that the first negative value (Row 1 of Col. 7) is excluded because the effect of $P_{E\,(MAX)}$ starts from second time-step onwards only. Since these ordinates deviate far from the trial values, rainfall weighted ordinate values are computed and presented in Column 8.

Since $M = P_{E1} + P_{E2} = 0.4 + 0.6 = 1.0$ and N = 1.00, the division of values of Column 7 by 2 gives the values shown in Column 8. Here, it is necessary to check the sum of trial unit hydrograph ordinates, to verify whether or not it

Table 3.9 Example of UH derivation using Collins' method (Trial **1**)

Time Period	*Excess rainfall (mm)*	*Excess rainfall for 100 mm UH*	*Trial 1 unit hydrograph (m^3/s)*							$\Sigma P_{Ei}U_i$ *(excludes $P_{E\,(MAX)}$) (m^3/s)*	*Actual DSRO (m^3/s)*	$Q_i - \Sigma P_{Ei}U_i$ *(m^3/s)*	$\bar{U}$ *(m^3/s)*	*Adjusted* $\bar{U}$
			U_1 *1125*	U_2 *1125*	U_3 *1125*	U_4 *1125*	U_5 *1125*	U_6 *1125*	U_7 *1125*					
(1)	*(2)*	*(3)*	*(4)*							*(5)*	*(6)*	*(7)*	*(8)*	*(9) = 8/F*
1	40	0.4	450							450	250	—	—	—
2	100	1.00	—	450						450	1050	600	863	821
3	60	0.60	675	—	450					1125	2050	925	1025	975
4				675	—	450				1125	4350	3225	2175	2069
5					675	—	450			1125	4150	3025	2075	1974
6						675	—	450		1125	2300	1175	1150	1094
7							675	—	450	1125	1070	–55	535	509
8								675	—	675	450	–225	450	425
9									675	675	120	—	—	—
Total	200												8273	7870
													7870	
													$F = 1.05$	

Table 3.10 Example of unit hydrograph derivation using Collins' method (Trial 2)

Time Period	*Excess rainfall (mm)*	*Excess rainfall for 100 mm UH*	*Trial 2 unit hydrograph (m^3/s)*							$\Sigma P_{Ei} U_i$ – $P_{E\,(MAX)}$ (m^3/s)	*Actual DSRO (m^3/s)*	$Q_i - \Sigma P_{Ei} U_i$ (m^3/s)	$\bar{U}$ (m^3/s)	*Adjusted* $\bar{U}$
			U_1 821	U_2 975	U_3 2069	U_4 1974	U_5 1094	U_6 509	U_7 428					
1	*2*	*3*	*4*							*5*	*6*	*7*	*8*	*9 = 8/F*
1	40	0.4	328.4							328.4	250	—	—	—
2	100	1.00	—	390						390.0	1050	660	741	729
3	60	0.60	429.6	—	827.6					1320.2	2050	729.8	852	838
4				585	—	789.6				1374.6	4350	2975.4	2522	2480
5					1241.4	—	437.6			1679.0	4150	2471.0	2223	2186
6						1184.4	—	203.6		1388.0	2300	912.0	1003	986
7							656.4	—	171.2	827.6	1070	242.4	376	509
8								305.4	—	305.4	450	144.6	286	425
9									256.8	256.8	120	—	—	—
Total	200												8003	7870
													7870	
													$F = 1.02$	

equals that of Column 8 values. The sum of trial ordinates is 7875 whereas the sum of values of Column 8 is 8273. Therefore, the conversion factor F is computed as 1.05, and the ordinates of Column 8 are divided by 1.05, and the resulting values are shown in Column 9. These values are the trial values for the second trial, as shown in Table 3.10. Figure 3.8 represents the unit hydrograph, derived using Collins' method.

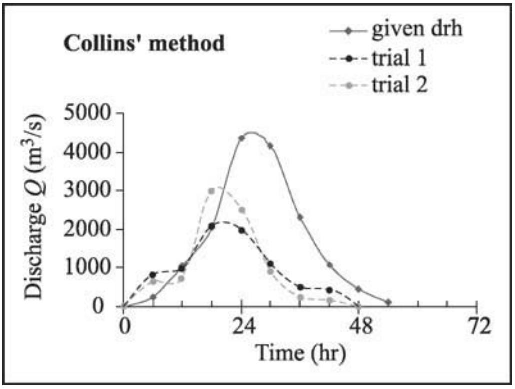


Fig. 3.8 *UH derived using Collins' method*

3.4.3 Selection of Unit Time Period of UH

The time period (T) should be determined using the following criteria:

(i) It should be short enough to define the hyetograph and the hydrograph reasonably.
(ii) It should be a simple fraction of 24 hour. Use of time periods that are not a whole fraction of 24 hours may not be used, for example, 5-hr, 7-hr, etc.
(iii) Conventionally, a unit period of approximately 1/3 to 1/4 of the period of rise of the hydrograph is used.

3.4.4 Instantaneous UH

Given the excess rainfall amount, if the period of a unit hydrograph is reduced, peak flow magnitude of direct surface runoff hydrograph increases and the base length decreases. If the period of excess rainfall reduces to zero and the excess rainfall depth remains the same, the intensity of the excess rainfall reaches infinity. This corresponds to the instantaneous application of a sheet of water over the entire catchment area. This water drains off the basin through gravity, as shown in the instantaneous unit hydrograph (IUH) in Fig. 3.9. Thus, an IUH is purely a theoretical concept and represents the unit hydrograph derived from the unit excess rainfall volume that occurred instantaneously. An IUH represents a characteristic of a catchment and is not affected by time duration. It is a useful tool in regional unit hydrograph studies.

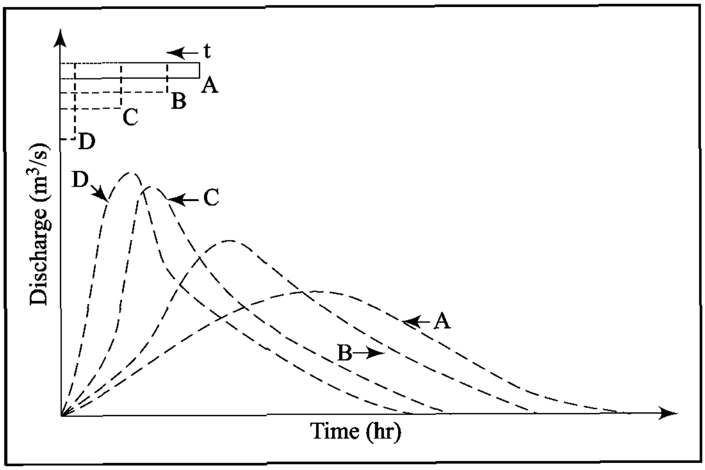


Fig. 3.9 *Derivation of Instantaneous unit hydrograph*

IUH is a single-peaked hydrograph with a finite base width and its properties are as follows.

1. $0 \le u(t) \le a$ positive value, for $t > 0$
2. $u(t) = 0$, for $t \le 0$
3. $u(t) \to 0$, as $t \to \infty$
4. $\int_0^\infty u(t)\,dt$ = unit depth over catchment

3.5 S-CURVE

An S-curve of T-hr unit hydrograph is the graph that results from $1/T$ intensity of excess rainfall which occurs for an infinite period, as shown in Fig. 3.10. The S-curve may be derived as follows:

(i) Plot the unit hydrograph of unit volume at time $t = 0$.
(ii) Plot the unit hydrograph successively lagged by T-hr.
(iii) Add the ordinates of each unit hydrograph, taking the time lag into account.
(iv) At time t_b (base length of the first unit hydrograph), the above sum of ordinates reaches a constant value.
(v) Plot the sum of the ordinates as ordinate and time as abscissa. It is an S-curve. Thus, the S-curve for the unit period T-hr is determined. Sometimes, the S-curve fluctuates in the region approaching a constant value. It is a problematic feature of the S-curve derivation. To avoid it, a curve of best fit through the points is drawn, for using the values derived from the fitted curve.
(vi) The maximum ordinate of the S-curve corresponds to the equilibrium discharge, and it is defined as:

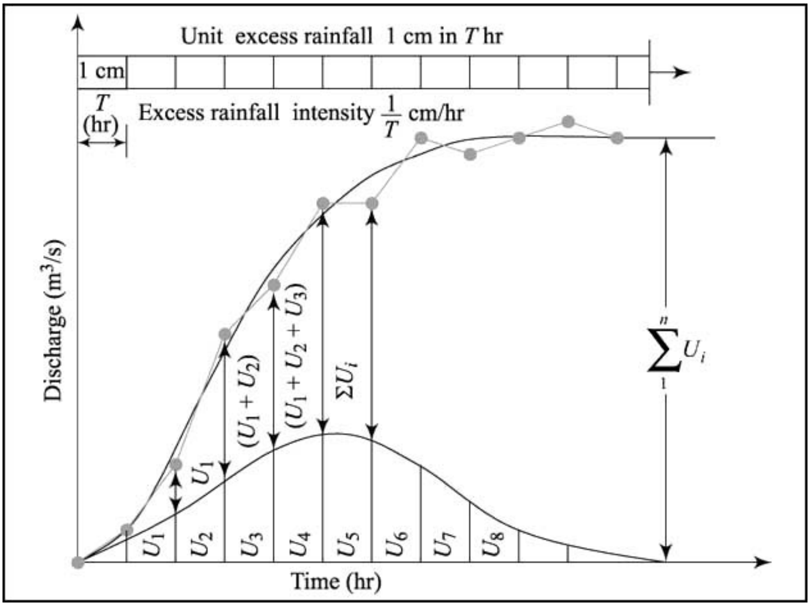


Fig. 3.10 *S-curve*

$$Q_{max} = \frac{0.2778(A)\,(Q_{UH})}{T} \tag{3.6}$$

where, A is the catchment area (sq. km), Q_{UH} is the unit volume (mm), and T is the time duration of the unit hydrograph (hr). The derivation of S-curve from a given unit hydrograph is explained using the following example.

Example 3.9 For a typical catchment, the ordinates of 6-hr UH of 10 mm volume are given below. If the equilibrium discharge is 16250 m^3/s, compute the S-curve ordinates.

Table 3.11 Data for Example 3.9

Date	*Time (hr)*	*Time interval (hrs)*	*6 hr, 10 mm UH (m³/s)*
June 15	0600	0	0
	0900	3	200
	1200	6	500
	1500	9	1000
	1800	12	1600
	2100	15	2400
	2400	18	3500

(*Contd.*)

(*Contd.*)

June 16	0300	21	4200
	0600	24	5200
	0900	27	4400
	1200	30	3100
	1500	33	2300
	1800	36	1500
	2100	39	1000
	2400	42	650
June 17	0300	45	400
	0600	48	250
	0900	51	150
	1200	54	0

Solution

The steps followed with reference to Table 3.7 are:

(i) Enter the ordinates of 6-hr UH in Column 4.

(ii) Lag the 6-hr UH by 6-hr, 12-hr, etc. till 54 hours, which is the time base of the UH; and enter these values in Columns 5 through 13.

(iii) Derive *S*-curve ordinates summing the values of Columns 5 through 13 at each time interval, as shown in Column 14.

(iv) Plot the *S*-curve obtained at step (iii).

(v) If the plotted *S*-curve shows hunting effect at the upper end, plot a best fit curve through these points. Figure 3.11 shows the smoothened *S*-curve.

(vi) Read the discretized values of the *S*-curve, obtained at step (v), and enter in the table, as in Column 15. This is the *S*-curve of intensity 10/6 mm/hr.

(vii) Multiply the ordinates of Column 15 by 6 to get the *S*-curve of intensity 10 mm/hr, shown in Column 16.

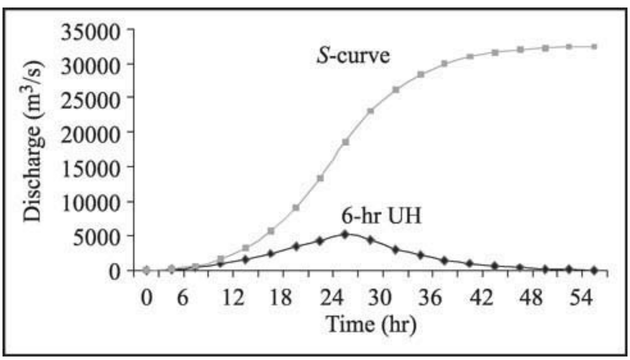


Fig. 3.11 *Smoothed* S-*curve*

3.5.1 Relation between S-Curve and UH

There exists a simple relationship between the *S*-curve of a watershed and its unit hydrograph, as shown in Fig. 3.12, where S_t is the time-varying *S*-curve for a continuous rain of *i* cm/hr and S_{t-T} is the same curve shifted by *T* hr to the right. The difference in the ordinates of the two *S*-curves (*S* and S_{t-T}) represent the ordinates of a hydrograph resulting due to excess rainfall depth of *iT* cm. Therefore, *T*-hr and *d*-mm unit hydrograph ordinates (U_t) can be derived as:

$$U_t = \frac{d}{iT}(S_t - S_{t-T}) \tag{3.7}$$

If unit depth is equal to 1 mm and intensity (*i*) of excess rainfall is 1 mm/hr, then

$$U_t = \frac{S_t - S_{t-T}}{T} \tag{3.8}$$

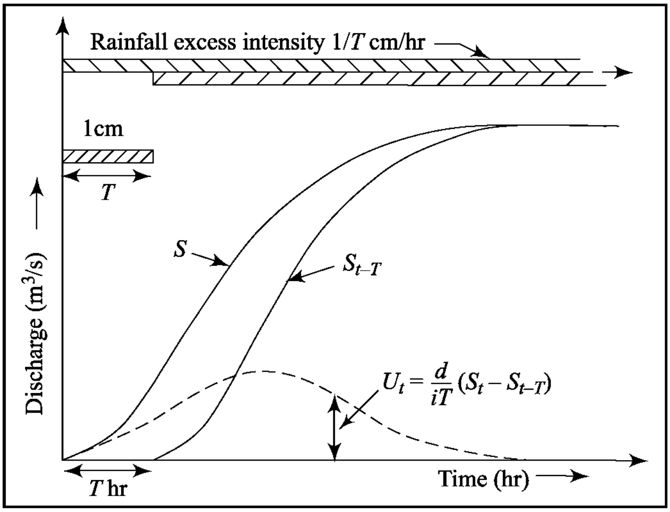


Fig. 3.12 *S-curve and UH*

3.5.2 Relation between S-curve and IUH

Since $U_d(T, t) = [d(S_t - S_{t-T})/(iT)]$; as *T* approaches zero, $U_1(T, t)$ approaches $U_1(0, t)$, and $U_d(T, t)$ approaches (dS_t/dt) (d/i). Here, $U_1(0, t)$ and $U_d(T, t)$ represent 1-mm and *d*-mm *T*-hr unit hydrographs, respectively. Thus,

$$U_d(0, t) = u(t) = \frac{dS_t}{dt}\frac{d}{i} \tag{3.9}$$

Or, alternatively,

$$S_t = \frac{i}{d}\int_0^t u(t)\, dt \tag{3.10}$$

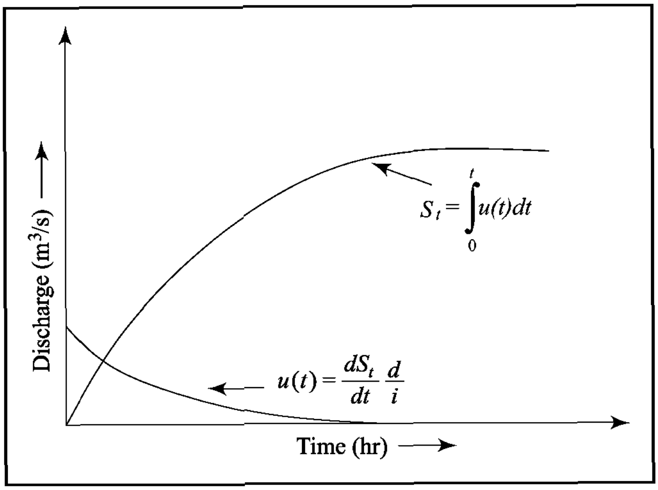


Fig. 3.13 *S-curve and IUH*

If $i = 1$ and $d = 1$, the above equation reduces to

$$S_t = \int_0^t u(t)\,dt \tag{3.11}$$

3.5.3 Relation between UH and IUH

The relationship between IUH [$u(t)$] and T-hr UH [$U(T, t)$], both of the same unit depth, can be derived using the two S-curves, as shown in Fig. 3.12. *Note*: here, the subscript (i) of U is dropped for convenience.

Since, $$S_t = \int_0^t u(t)\,dt \tag{3.12}$$

$$T \times U(T, t) = S_t - S_{t-T}$$

$$= \int_0^t u(t)\,dt - \int_0^{t-T} u(t)\,dt$$

$$= \int_0^{t-T} u(t)\,dt \tag{3.13}$$

which equals the area between the two S-curves. The T-hr unit hydrograph with a unit depth is given by:

$$U(T, t) = \frac{\int_{t-T}^{t} u(t)\,dt}{T} \tag{3.14}$$

Thus, a T-hr UH can be derived from an IUH. If the IUH is available, the ordinate of the T-hr unit hydrograph at the end of the time unit T is equal to the average ordinate of the instantaneous unit hydrograph over the T-hr period.

3.6 CHANGE OF UNIT PERIOD OF UH

Having derived a unit hydrograph for a particular unit period (say, 6 hrs), one may want to change the time period and derive a new hydrograph. The following two methods are used for this purpose:

1. Superimposition method, and
2. *S*-curve method.

3.6.1 Superimposition Method

This method is suitable only when the new duration of the UH is an integer multiple of the given unit duration. For example, the given duration is 6 hr, and the hydrograph is to be converted to a 12-hr duration. In general, a unit hydrograph of $2T$-duration can be derived from a UH of T-duration as follows:

(i) Add the ordinates of the T-hr UH to the ordinates of an identical UH lagged by T-hr.
(ii) Divide the ordinates of the resulting hydrograph of step (i) by 2 to obtain a UH for unit duration of $2T$. Figure 3.14 shows the conversion of a UH of T-duration to $2T$-duration.

Similarly, the UH of nT-duration, where n is an integer, can be derived by successive lagging of the T-duration unit hydrograph n times and then dividing the resulting hydrograph by n, as illustrated in the following example.

Example 3.10 The ordinates of 6-hr UH are given below. Derive a 12-hr UH of the same unit volume as of the 6-hr UH.

Table 3.12 Data for Example 3.10

Time (hr)	*6-hr UH (m^3/s)*	*Time (hr)*	*6-hr UH (m^3/s)*
0	0	30	3100
3	200	33	2300
6	500	36	1500
9	1000	39	1000
12	1600	42	650
15	2400	45	400
18	3500	48	250
21	4200	51	150
24	5200	54	0
27	4400		

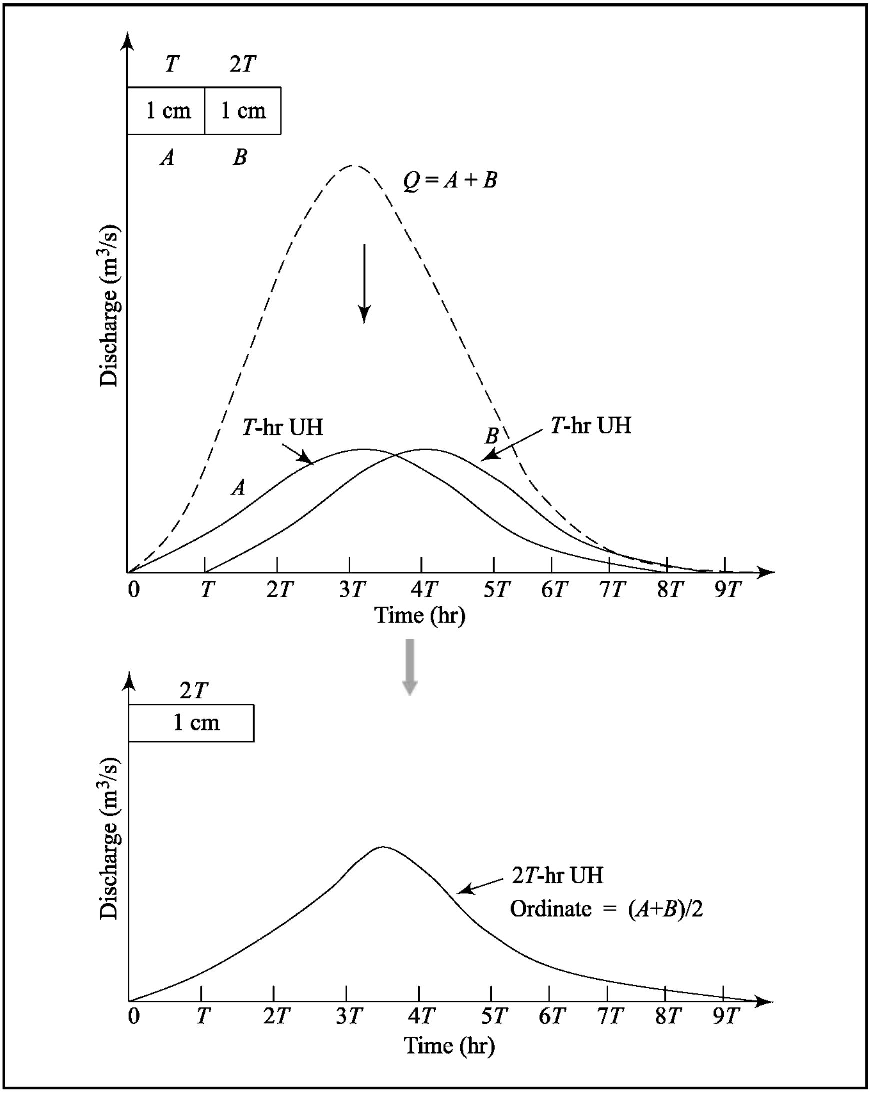


Fig. 3.14 *Change of unit period of UH*

Solution

With reference to Table 3.13, the computational steps are:

(i) Enter the ordinates of 6-hr UH in Column 2.

(ii) Lag the 6-hr UH by 6-hr and enter these values in Column 3.

(iii) Add the respective ordinates of Columns 2 and 3. The resulting hydrograph ordinates are shown in Column 4.

(iv) Divide the ordinates of the hydrograph derived from step (iii) by 2 to compute UH of 12-hr duration. The ordinates of this UH are given in Column 5.

Table 3.13 Derivation of 12-hr UH from 6-hr UH

Time (hrs) (m^3/s)	*6-hr UH ordinates (m^3/s)*	*UH ordinates lagged by 6-hr (m^3/s)*	*Superimposed hydrograph*	*12-hr UH (m^3/s)*
0	0	—	0	0
3	200	—	200	100
6	500	0	500	250
9	1000	200	1200	600
12	1600	500	2100	1050
15	2400	1000	3400	1700
18	3500	1600	5100	2550
21	4200	2400	6600	3300
24	5200	3500	8700	4350
27	4400	4200	8600	4300
30	3100	5200	8300	4150
33	2300	4400	6700	3350
36	1500	3100	4600	2300
39	1000	2300	3300	1650
42	650	1500	2150	1075
45	400	1000	1400	700
48	250	650	900	450
51	150	400	550	275
54	0	250	250	125
57	—	150	150	75
60	—	0	0	0

3.6.2 S-Curve method

This is a more general method than the method of superimposition. After having derived the *S*-curve of unit (or some other) intensity (for example, 1 mm/hr) from *T*-duration unit hydrograph, the unit hydrograph of *T*-duration can be obtained as follows:

(i) Shift the *S*-curve by *T* hrs to obtain another *S*-curve, as shown in Fig. 3.15.
(ii) Subtract another *S*-curve ordinate from the original *S*-curve.
(iii) The difference between the two *S*-curves represent the unit hydrograph for time *T* with a unit volume equal to $iT = T$ mm (as $i = 1$ mm/hr).

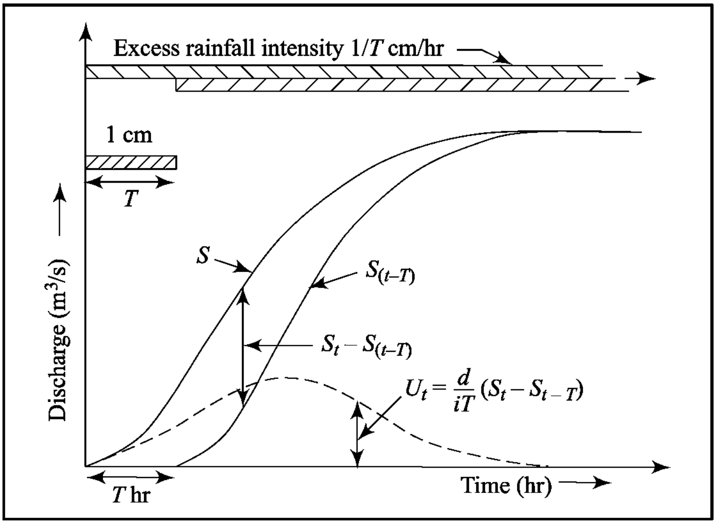


Fig. 3.15 *Changing UH duration using S-curve method*

(iv) To derive a 1 mm UH of duration T divide the difference of step (iii) by T.

Example 3.11 The *S*-curve ordinates of intensity 1 cm/hr derived from 6-hr 1 cm UH are given below. Derive a UH of 3-hr duration and 1 cm depth.

Table 3.14 Data for Example 3.11

Time (hr)	S-*curve hydrograph ordinates* (m^3/s)	*Time (hr)*	S-*curve hydrograph ordinates* (m^3/s)
0	0	30	82800
3	1200	33	88200
6	3000	36	91800
9	7200	39	94200
12	12600	42	95400
15	21600	45	96600
18	33600	48	97200
21	47400	51	97380
24	62400	54	97500
27	74400	57	97500

Solution

With reference to Table 3.15, the computational steps are as follows:

(i) Enter the *S*-curve ordinates in Column 2. This *S*-curve is derived from 6-hr UH used in Example 3.4 of *S*-curve derivation.

(ii) Shift the *S*-curve by 3-hr and enter these values in Column 3.

(iii) Subtract the two *S*-curves [Column (2) minus Column (3)] and enter these values in Column (4). This represents the UH of 3-hr duration with a unit volume equal to 3 cm.

(iv) Compute the 3-hr UH of unit volume equal to 1 cm by dividing the hydrograph obtained from step (iii) by 3. The ordinates of the resulting UH are given in Column 5.

Table 3.15 Derivation of 3-hr UH from 6-hr UH (*S*-curve method)

Time (hrs)	*S-curve ordinates of excess rainfall intensity 1 cm/hr* (m^3/s)	*S-curve lag-ged by 3 hrs* (m^3/s)	*3-hr UH with 3-cm depth* (m^3/s)	*3-hr UH of 3-cm depth* (m^3/s)
(1)	*(2)*	*(3)*	*(4) = (2) – (3)*	*(5) = (4)/3*
0	0	—	0	0
3	1200	0	1200	400
6	3000	1200	1800	600
9	7200	3000	4200	1400
12	12600	7200	5400	1800
15	21600	12600	9000	3000
18	33600	21600	12000	4000
21	47400	33600	13800	4600
24	62400	47400	15000	5000
27	74400	62400	12000	4000
30	82800	74400	8400	2800
33	88200	82800	5400	1800
36	91800	88200	3600	1200
39	94200	91800	2400	800
42	95400	94200	1200	400
45	96600	95400	1200	400
48	97200	96600	600	200
51	97380	97200	180	60
54	97500	97300	120	40
57	97500	97500	0	0

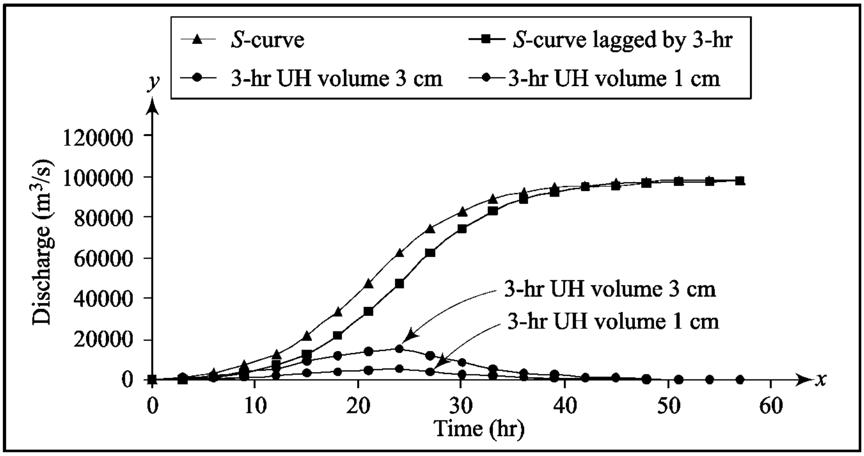


Fig. 3.16 *UH derivation*

3.7 DERIVATION OF AN AVERAGE UH

In practice, unit hydrographs derived from different rainfall-runoff events of a particular catchment differ significantly from one another. These unit hydrographs are averaged for computing a basin-representative average unit hydrograph.

3.7.1 Graphical Method

The graphical method (Wilson, 1974) of deriving an average UH can be applied by the following steps:

(i) Plot the unit hydrographs derived from a number of individual events on a single plotting paper.
(ii) Mark a point such that it corresponds to the average peak discharge ordinates and average time to peak, as shown in Fig. 3.17.
(iii) Draw an average unit hydrograph such that it passes through the average peak point, has unit volume, and generally conforms to the characteristic shapes of the individual unit hydrographs. Some trials for adjusting the ordinates of the derived unit hydrograph may be required for preserving the volume.

3.7.2 Statistical Method

The statistical approach characterizes unit hydrographs by their shapes, and the shapes are described by volume, mean, coefficient of variation, skewness, and peakedness. These five parameters are termed as *shape factors*. It is to note that the volume and mean, however, represent the scale and location characteristics of a distribution (or unit hydrograph), respectively. The steps for using shape factors for determining an average unit hydrograph are described below.

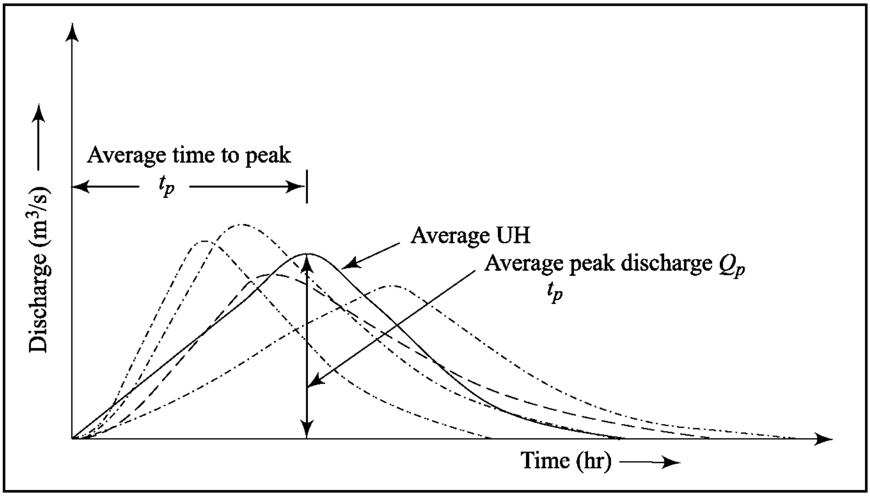


Fig. 3.17 *Averaging UH by the graphical method*

(i) Calculate the values of shape factors for each derived unit hydrograph and average them. Averaging based on median values minimizes the effect of outliers.

(ii) The values of shape factors of each unit hydrograph are investigated and that unit hydrograph is selected the parameters of which closely match with the average value of the parameters.

(iii) The unit hydrograph derived from step (ii) represents the average unit hydrograph for the catchment.

Further details on statistical methods are given in Chapter 4.

3.8 CONCEPTUAL MODELS

The conceptual models occupy an intermediate position between the fully physical approach and empirical black-box analysis. Such models are formulated on the basis of a relatively small number of components, each of which is a simplified representation of one process element in the system being modeled. The purpose of this breakdown is primarily to enable the runoff from catchments to be estimated using standard parameters together with actual data. A detailed discussion of two popular conceptual models, i.e., Nash model and Clark's model are discussed below.

3.8.1 Nash Model

The derivation of the Nash IUH is described using the unit impulse function theory. Let I, Q, and S represent the rainfall, runoff (discharge), and storage capacity of a basin at any arbitrary time t; then the following water balance equation holds:

$$I - Q = dS/dt$$

Since for any time-invariant system $S = KQ$ (refer Chapter 5),

$$dS/dt = K(dQ/dt)$$

Or $$I = Q + K(dQ/dt)$$

Or $$dQ/dt + Q/K = I/K$$

By multiplying the factor $e^{(t/K)}$ to both sides of the above equation, we have:

$$\frac{d}{dt}(Qe^{t/K}) = (I/K)e^{t/K}$$

This means:

$$Qe^{t/K} = (I/K)\int e^{t/K}\, dt$$

Or $$Qe^{t/K} = Ie^{t/K} + A$$

where A is the integration constant which can be derived as follows:

At $t = 0$, $Q = 0$, therefore $A = -I(e)^{(-0/K)} = I$

Or $$Q = I(1 - e^{-t/K})$$

For $I = 1$, $Q = 1 - e^{-t/K}$

This equation gives the unit impulse, i.e., the output for given unit input ($I = 1$). The Q in the equation represents the ordinates of the S-hydrograph when there is a continuous input of $I = 1$.

Therefore, the IUH can be computed by deriving the equation as:

$$q(t) = dQ(t)/dt = -\frac{-e^{-t/K}}{K}$$

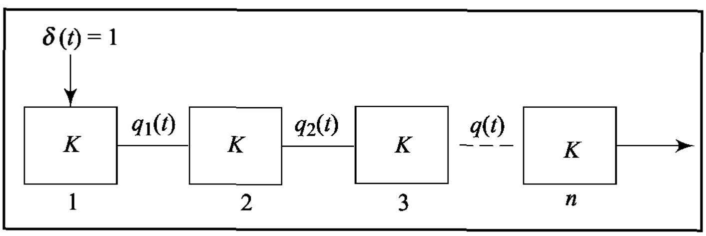


Fig. 3.18 *Concept of Nash Model*

If there are n reservoirs in series, and there is an input of unit rainfall occurring within a small time, i.e., $\Delta t \to 0$, the output forms the ordinates $q(t)$ of an IUH.

For the first reservoir, the output $q_1(t) = -\frac{-e^{-t/K}}{K}$. This forms the input to the second reservoir, therefore, the output from second reservoir is:

$$q_2(t) = (-e^{-t/K}/K)(1 - e^{-t/K})$$

In a similar way, the output from the n^{th} reservoir is derived in form of a gamma distribution function as follows:

$$q_n(t) = \frac{1}{K\Gamma(n)}\left(\frac{t}{K}\right)^{n-1} e^{\frac{-t}{K}}$$

This is the Nash model. Table of $\Gamma(n)$ is given in Appendix-I. Explicit estimation of Nash parameters is also explained in Chapter 4.

Computational steps for deriving UH by Nash Model

The following steps are involved for deriving the unit hydrograph by the Nash Model using the rainfall–runoff data of a particular storm:

(i) Obtain mean rainfall values at each computational interval taking the weighted mean of the observed values at different stations.
(ii) Estimate direct surface runoff separating the base-flow from the discharge hydrograph using one of the base-flow separation techniques.
(iii) Estimate the excess rainfall hyetograph separating the loss from total rainfall hyetograph.
(iv) Estimate the first and second moment of effective rainfall hyetograph about the origin.
(v) Estimate the first and second moment of direct surface runoff hydrograph about the origin.
(vi) Find out the parameters n and K using the values of moments obtained from step (iv) and (v).
(vii) Estimate the unit hydrograph of duration T hours using Nash model.

Example 3.12 The ordinate of excess rainfall hyetograph and direct surface runoff hydrograph for a storm of typical catchment of the size 1700 sq. km. are given below. Find the Nash model parameters.

Table 3.16 Data for Example 3.12

Time (hrs)	*Excess rainfall (mm)*	*Direct surface runoff (m^3/sec)*
0	0	0
6	40.209	250
12	100.209	1050
18	60.209	2050
24	0	4350
30	0	4150
36	0	2300
42	0	1070
48	0	450
54	0	120

Solution

(i) First and second moment of effective rainfall:

$${}_1M'x = \frac{\sum_{i=1}^{M} X_i t_i}{\sum_{i=1}^{M} X_i}$$

$$= \frac{(40.209 \times 3) + (100.209 \times 9) + (60.209 \times 15)}{40.209 + 100.209 + 60.209}$$

$$= \frac{1925.643}{200.627} = 9.598 \text{ hr}$$

$${}_2M'x = \frac{\sum_{i=1}^{M} X_i t_i^2}{\sum_{i=1}^{M} X_i}$$

$$= \frac{[40.209 \times (3)^2] + [100.209 \times (9)^2] + [60.209 \times (15)^2]}{40.209 + 100.209 + 60.239}$$

$$= \frac{22.25.135}{200.627} = 109.765 \text{ hr}$$

(ii) First and second moment of direct surface runoff:

$${}_1M'y = \frac{\sum_{i=1}^{N} \frac{Y_i + Y_{i+1}}{2} t_{ir}}{\sum_{i=1}^{N} \frac{Y_i + Y_{i+1}}{2}} = \frac{\sum_{i=1}^{N} (Y_i + Y_{i+1}) t_{ir}}{\sum_{i=1}^{N} (Y_i + Y_{i+1})}$$

$$= \frac{250}{2} \times 3 + \frac{(250 + 1050)}{2} \times 9 + \frac{(1050 + 2050)}{2} \times 15$$

$$+ \frac{(2050 + 4350)}{2} \times 21 + \frac{(4350 + 4150)}{2} \times 27$$

$$+ \frac{(4150 + 2300)}{2} \times 33 + \frac{(2300 + 1070)}{2} \times 39$$

$$+ \frac{(1070 + 450)}{2} \times 45 + \frac{(450 + 120)}{2} \times 51 + \frac{120}{2} \times 57$$

$${}_1M'y = \frac{250}{2} + \frac{(250 + 1050)}{2} + \frac{(1050 + 2050)}{2} + \frac{(2050 + 4350)}{2}$$

$$+ \frac{(4350 + 4150)}{2} + \frac{(4150 + 2300)}{2} + \frac{(2300 + 1070)}{2}$$

$$+\frac{(1070+450)}{2}+\frac{(450+120)}{2}+\frac{(120)}{2}$$

$$=\frac{435720}{15790}=27.595 \text{ hr}$$

To compute second moment, t_i is to be replaced by square of t_i in the above expression, i.e.,

$$\frac{250}{2}\times 3^2+\frac{(250+1050)}{2}\times 9^2+\frac{(1050+2050)}{2}\times 15^2$$

$$+\frac{(2050+4350)}{2}\times 21^2+\frac{(4350+4150)}{2}\times 27^2+\frac{(4150+2300)}{2}\times 33^2$$

$$+\frac{(2300+1070)}{2}\times 39^2+\frac{(1070+450)}{2}\times 45^2+\frac{(450+120)}{2}\times 51^2$$

$$+\frac{120}{2}\times 57^2$$

$${}_2M'y=\frac{13462110}{15790}=852.572 \text{ hr}^2$$

(iii) Solve the following equations to get the parameters n and K.

$$nK={}_1M'y-{}_2M'x$$

$$n(n+1)K^2+2nK{}_1M'x={}_2M'y-{}_2M'x$$

where ${}_1M'x = 9.598$ hour and ${}_2M'x = 109.785$ hour^2

${}_1M'y = 27.395$ hour, and ${}_2M'y = 852.572$ hour^2

$\therefore \quad nK = 27.595 - 9.598 = 17.997$

and $\quad n\,(n+1)\,K^2 + 2\,n\,K \times 9.598 = 852.572 - 109.785$

$n\,(n+1)\,K^2 + 19.196\,n\,K = 742.787$

$(n\,K)^2 + n\,K^2 + 19.196\,n\,K = 742.787$

$(17.997)^2 + 17.997\,K + (19.196 \times 17.997) = 742.787$

$323.892 + 17.997\,K + 345.470 = 742.787$

$17.997\,K = 73.425$

$K = 4.08$

$$\therefore \quad n=\frac{17.997}{4.08}=4.411$$

(iv) Compute the IUH using n and K. The procedure to convert IUH into S-curve or unit hydrograph is already explained. The procedures used to compute moments in case of rainfall and runoff are illustrative only and can be used interchangeably. Please check if estimates of n and K will be sensitive to the use of a particular procedure of computing moment and if so, up to what extent?

3.8.2 Clark's Method

Similar to Nash model discussed in previous section, Clark's method routs the time-area histogram to derive an IUH. The time area diagram is developed for an instantaneous excess rainfall over a catchment and represents the relationship between the areas that contributes to the runoff at the outlet versus the time of travel. It is assumed that the excess rainfall first undergoes pure translation by a travel time-area histogram, and the attenuation is attained by routing the results of the above through a linear reservoir at the outlet. The details are discussed below.

Time-Area Curve

The time of concentration is the time taken for a droplet of water to travel from the upper boundary of a catchment to the outlet point. This is generally computed using empirical formulae which relate the time of concentration to geomorphological parameters of the catchment. In gauged areas, the time interval between the end of the excess rainfall and the point of inflection of the resulting surface runoff provides a good way of estimating t_c from known rainfall-runoff data. In simple terms, the total catchment area drains into the outlet in t_c hours.

First, the catchment is drawn using a toposheet map, and then the points on the map having equal time of travel, (say, t_{1c} hr where $t_{1c} < t_c$), are considered and located on a map of the catchment. The t_{1c} is calculated using the length of stream, length of centroid of the area from the outlet, average slope, and the area (from empirical equations). Subsequently, a line is joined to all such points having same t_{1c}, to get an *isochrones* (or runoff isochrones). These are shown in an example below.

To make this procedure simple, one can start with selection of the longest water course. Then, its profile plotted as elevation versus distance from the outlet; the distance is then divided into N parts and the elevations of the subparts measured on the profile transferred to the contour map of the catchment.

The areas between two isochrones denoted as A_1, A_2, ..., A_N are used to construct a travel time-area histogram. If an excess rainfall of 1 cm occurs instantaneously and uniformly over the catchment area, this time-area histogram represents the sequence in which the volume of rainfall will be moved out of the catchment and arrive at the outlet. For example, a sub-area of A_{ic} km^2 represents a volume of A_{ic} km$^2 \times 1$ cm $= A_{ic} \times 10^4$ (m^3) moving out in time $\Delta t_{1c} = \frac{t_c}{N}$ hours. Clark routed this time-area diagram through a linear reservoir which he assumed to be hypothetically available at the outlet to provide the requisite attenuation.

Routing

The linear reservoir at the outlet is assumed to be described by $S = KQ$, where K is the storage time constant. The continuity equation can be written as:

$$I - Q = \frac{ds}{dt}$$

Or $$-Q = \frac{ds}{dt} = K\frac{dQ}{dt}$$

Or $$K = -Q_i/(dQ/dt)I \tag{3.15}$$

where, suffix '*i*' refers to the point of inflection, and K can be estimated from a known surface runoff hydrograph of the catchment. The K can also be estimated from the data on the recession limb of the hydrograph. Knowing K of the linear reservoir, the inflows at various times are routed by the *Muskingum method*. The Muskingum equation can be written as:

$$S = K[xI + (1 - x)Q] \quad \text{(for details, refer Chapter 5)} \tag{3.16}$$

where, x is known as the weighting factor. For a linear reservoir, value of $x = 0$.

The inflow rate between an inter-isochrones area of A_{1c} km^2 with a time interval Δt_{1c} hr is:

$$I = \frac{A_r \times 10^4}{3600(\Delta t_c)} = 2.78\frac{A_{1c}}{\Delta t_{1c}} \text{ (m}^3\text{/s)}$$

The Muskingum routing equation can be written as:

$$Q_2 = C_0 I_2 + C_1 I_1 + C_2 Q_1 \tag{3.17}$$

where $C_0 = (0.5\,\Delta t_c)/(K + 0.5\,\Delta t_c)$

$C_1 = (0.5\,\Delta t_c)/(K + 0.5\,\Delta t_c)$

$C_2 = (K - 0.5\,\Delta t_c)/(K + 0.5\,\Delta t_c)$

i.e., $C_0 = C_1$. Also, since the inflows are derived from the histogram, $I_1 = I_2$ for each interval. Thus, Eq. (3.17) becomes:

$$Q_2 = 2C_1 I_1 + C_2 Q_1 \tag{3.18}$$

Routing of the time-area histogram by Eq. (3.18) gives the ordinates of IUH for the catchment. Using this IUH, any other D-hr UH can be derived.

The computational steps involved in Clark's model are as follows:

(i) Make first estimate of Clark's model parameters, t_c and R, from the excess rainfall hyetograph and direct surface runoff hydrograph.

(ii) Construct the time-area curve, taking the t_c value obtained from step (i), using the procedure described.

(iii) Measure the area between each pair of isochrones using planimeter.

(iv) Plot the curve of time versus cumulative area. Note that the abscissa is expressed in percent of t_c. Tabulate increments between points that are at computational interval Δt apart.

(v) Route the inflow using Eqs. (3.17) and (3.18) to get IUH ordinates.

(vi) Compute the unit hydrograph of the excess rainfall duration using the given equation:

$UH_i = 1/n\ [0.5U_{i-n} + U_{i-n+1} + \ldots\ldots.. U_{i-1} + 0.5\ U_i]$

where, $n = D/\Delta t$, and D is duration of the UH.

Example 3.13 Clark's model parameters, t_c and R, derived from a short-duration storm excess rainfall hyetograph and direct surface runoff hydrograph, are 8 and 7.5 hours respectively for a typical catchment of area 250 km^2. The ordinates of time-area diagram are:

Time (hr)	1	2	3	4	5	6	7	8
Area (km^2)	10	23	39	43	42	40	35	18

Derive a 2-hr UH using Clark's model.

Solution

Computational steps are: (Ref. Table 3.17)

(i) Convert the units of inflow using the equation: $I_i = Ka_i / \Delta t$

where a_i = the ordinates of time-area curve in m^2

I_i = the ordinates of the same time-area curve in m^3/s

K = conversion factor

Δt = computational interval (hours; 1 hour in this example)

If the volume under the time-area curve is taken equivalent to 1 cm excess rainfall, then the conversion factor K would be:

$$K = \frac{1 \times 10^6 \times 10^{-2}}{3.6 \times 10^3} = \frac{10}{3.6} = 2.778 \text{ m}^3\text{/s/km}^2$$

Therefore, the above equation may be written as:

$I_i = 2.778\ a_i/1 = 2.778\ a_i$

(ii) Compute the routing coefficent (C) using the following equation:

$$C = \frac{\Delta t}{R + 0.5\Delta t} = \frac{1}{7.5 + 0.5} = 0.125$$

$\therefore\ 1 - C = 1 - 0.125 = 0.875$

(iii) Route the time-area curve obtained from step (i); (see Columns 3, 4 and 5)

$U_i = CI_i + (1 - C)\ U_i - 1$

$U_i = 0.125\ I_i + 0.875\ U_i - 1$

where U_i = the i^{th} ordinate of IUH in m^3/s.

(iv) Compute 2-hr unit hydrograph (see Column 6).

$$H_i = 1/n\ [0.5U_{i-n} + U_{i-n+1} + \ldots\ldots U_{i-1} + 0.5\ U_i]$$

For this example:

$D = 2$ hours

$t = 1$ hour

$$\therefore\ n = \frac{D}{\Delta t} = 2$$

$$\therefore\ UH_i = \frac{1}{2}(0.5\ U_{i-2} + U_{i-1} + 0.5U_i]$$

See Table 3.17 for the detailed computation.

Table 3.17 Clark's model computation

Time (hrs) (1)	*Time area diagram (km^2)* (2)	*$0.125\ I_1 = 2.78 \times 0.125 \times$ Col (2) (m^3/s)* (3)	*$0.875 \times$ Col(5) (m^3/s)* (4)	*Col (3) + (4) IUH ordinates (m^3/s)* (5)	*2-hr UH ordinates (m^3/s)* (6)
0	0	0	0	0	0
1	10	3.5	0	3.5	0.875
2	23	8.0	3.1	11.1	4.525
3	39	13.5	9.7	23.2	12.225
4	43	14.9	20.3	35.2	23.175
5	42	14.6	30.8	45.4	34.750
6	40	13.9	39.6	53.5	44.875
7	35	12.1	46.8	58.9	52.825
8	18	6.2	51.4	57.6	57.225
9	0	0	50.5	50.5	56.150
10	0	0	44.1	44.1	50.675
11	0	0	38.6	38.6	44.325
12	0	0	33.8	33.8	38.775
13	0	0	29.6	29.6	33.950
14	0	0	25.9	25.9	29.725
15	0	0	22.7	22.7	26.025

SUMMARY

In this chapter, an attempt has been made to introduce the concept of hydrograph analyses. Understanding of different terms like Unit Hydrograph, IUH and *S*-curve is very relevant in this context. While analysis of single storm runoff is relatively simpler than that of complex storm runoff, one is often confronted with analysis of complex storms. From the analysis of single runoff storm hydrograph, based on application of Z-transform, it is also possible to identify hyetograph and unit hydrograph. However, such an approach has not been included here. A catchment normally has different orders of streams, and for a given channel network, geomorphological instantaneous unit hydrograph (GIUH) can be derived using known catchment characteristics. One can also use synthetic approaches to develop unit hydrograph.

SOLVED EXAMPLES

Example 3.14 In a catchment, the following 1-hr UH due to 1 mm effective rainfall is known:

Time (hr)	0	1	2	3	4
QUH (m^3/s)	0	3	2	1	0

(a) Estimate the direct runoff if the effective rainfall during 1-hr is 5 mm.
(b) Estimate the direct runoff if the effective rainfall is 5 mm during the first hour and 10 mm during the second.

Solution

(a) UH ordinates need to be multiplied by 5 because effective rainfall is 5 mm.

Time (m^3/s)	*UH (m^3/s)*	Q_{direct}
0	0	0
1	3	15
2	2	10
3	1	15
4	0	0

(b) We need to obtain two DRHs corresponding to 5 mm and 10 mm effecvtive rainfall. Columns 3 and 4 show these at 1-hr lag.

Time (m^3/s)	*UH (m^3/s)*	$Q_{direct1}$ *(m^3/s)*	$Q_{direct2}$ *(m^3/s)*	$Q_{direct\ total}$
0	0	0	0	
1	3	15	0	15
2	2	10	30	40
3	1	5	20	25
4	0	0	10	10
5	0		0	0

Example 3.15 During and after a rainfall event, the following discharge was measured in a river. Plot the hydrograph both in linear/linear and linear/log scale.

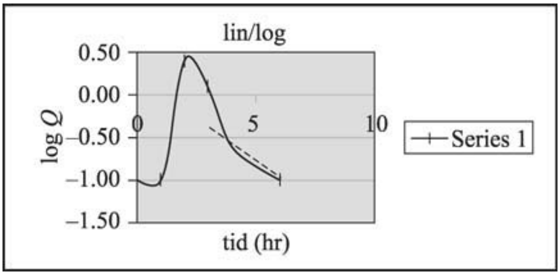
(a) What was the base flow if it can be assumed to be constant?
(b) What was the base flow if it can be assumed to follow the recession curve?

Tid (hr)	0	1	2	3	4	5
Q_{total} (m^3/s)	0.1	0.1	2.50	1.30	0.25	0.1

Solution

(a) Base flow constant = 0.1 m^3/s

(b)



The direct runoff ends when the flow follows the recession curve. This point can be found when the flow follows a straight line in a lin/log plot. Thus, the base flow increases from ($t = 2$, $Q = 0.1$) to ($t = 4$, $Q = 0.25$), after the point $Q = Q_{base}$.

Example 3.16 A catchment in Egypt experiences a long period without rain. The discharge in the river which drains the catchment is 100 m^3/s 10 days after the long period without rain, and 50 m^3/s after another 30 days. What flow can be expected to occur on day 120 if there is no rainfall during this period?

Solution

The recession can normally be assumed to follow an exponential function given by:

$$Q = Q_0\, e^{-kt}$$

Data given:

$t = 10$ days, $\quad Q = 100\ m^3/s$

$t = 40$ days, $\quad Q = 50\ m^3/s$

Applying the above function, we have:

$$100 = Q_0\, e^{-k10}$$

$$50 = Q_0\, e^{-k40}$$

Therefore, $100/50 = e^{-10k + 40k}$

$$2 = e^{30k}$$

$$k = 0.0231$$

For $t = 120$ days:

$$Q_{40}/Q_{120} = e^{-40k + 120k} = e^{80k} = 6.347$$

$$Q_{120} = 50/6.347 = 7.9 \text{ m}^3/\text{s}$$

Answer Flow after 120 days would be 7.9 m^3/s.

Example 3.17 Given below are the discharge values for a 30-min UH.

(a) What is the size of the catchment area?

(b) Calculate and plot the hydrograph for a 60 minute rain with an effective intensity of 10 mm/hr.

Time (min)	0	15	30	45	60	75	90	105	120	135	150
Discharge (m^3/s)	0	4.5	10	12.5	11	9	6.5	4	2.5	1	0

Solution

Time (min)	*Discharge (m^3/s)*	*Volume (m^3)*
0	0	0
15	4.5	4050
30	10	9000
45	12.5	11250
60	11	9900
75	9	8100
90	6.5	5850
105	4	3600
120	2.5	2250
135	1	900
150	0	0
		Sum = 54900

(a) Total volume of runoff is given above.
Totally, 1 mm effective rain has fallen over the catchment.
Thus, the area can be calculated by $54900/0.001 = 5.49 \times 10^7 \text{ m}^2 = 54.9 \text{ km}^2$

(b) The given hydrograph has a duration of 30 min and is valid for an intensity of 2 mm/hr.
Add two hydrographs multiplied by 5 with a 30 min lag between the hydrographs.

Time (min)	*UH 2 mm/hr (m^3/s)*	*UH × 5 (m^3/s)*	*UH × 5 (m^3/s)*	*Sum (m^3/s)*
0	0	0		
15	4.5	22.5		22.5
30	10	50	0	50
45	12.5	62.5	22.5	85
60	11	55	50	105
75	9	45	62.5	107.5
90	6.5	32.5	55	87.5
105	4	20	45	65
120	2.5	12.5	32.5	45
135	1	5	20	25
150	0	0	12.5	12.5
165			5	5
180			0	0

Example 3.18 A catchment received a continuous rainfall for 3 hours. The rain is distributed in time with 17 mm during the first hour, 22 mm during the second hour, and 14 mm during the final hour.

ϕ-index is 12 mm/hr and the measured hydrograph is given below.

(a) Find the 1-hr UH. (Assume the base flow to be constant.)

(b) What is the time base (duration) and the lag-time between the maximum rainfall and the maximum flow of the hydrograph?

Time (hr)	0	1	2	3	4	5	6	7	8	9	10	11
Discharge (m^3/s)	0	1	5.5	12.4	15.4	13	9.6	6.2	3.05	0.9	0.1	0

Solution

The measured runoff can be described by the sum of three unit hydrographs scaled to the rainfall intensity ϕ.

Subtract the ϕ-index from the rainfall intensities to get the effective rainfall.

(a) For the first hour,

effective rainfall = 17 mm – 12 mm = 5 mm

For the second hour,

effective rainfall = 22 mm – 12 mm = 10 mm

For the third hour,

effecitve rainfall = 14 mm – 12 mm = 2 mm

(b) Columns (2), (3) and (4) are the multiplied values of UH ordinates with the effective rainfalls of 5 mm, 10 mm and 20 mm respectively. The values are placed at 1-hr lag.

Column (5) will be the sum of the values in columns (2), (3), and (4). This implies

5 UH0 = 0

5UH1 + 10 UH0 = 1

5 UH2 + 10 UH1 = 2 UH0 = 5.5

and so on.

The solution of these equations is given in the following table.

Time (hr)	*5 × UH (m^3/s)*	*10 × UH (m^3/s)*	*2 × UH (m^3/s)*	*Sum (m^3/s)*
0	5 × UH0			0
1	5 × UH1	10 × UH0		1
2	5 × UH2	10 × UH1	2 × UH0	5.5
3	5 × UH3	10 × UH2	2 × UH1	12.4
4	5 × UH4	10 × UH3	2 × UH2	15.4
5	5 × UH5	10 × UH4	2 × UH3	13
6	5 × UH6	10 × UH5	2 × UH4	9.6
7	5 × UH7	10 × UH6	2 × UH5	6.2
8	5 × UH8	10 × UH7	2 × UH6	3.05
9	5 × UH9	10 × UH8	2 × UH7	0.9
10	5 × UH10	10 × UH9	2 × UH8	0.1
11	5 × UH11	10 × UH10	2 × UH9	0

UH	*Solving for UH_n*	*(m^3/s)*
UH0	0/5	0
UH1	1/5	0.2
UH2	(5.5 – 10 × 0.2 – 0)/5	0.7
UH3	(12.4 – 10 × 0.7 – 2 × 0.2)/5	1
UH4	(15.4 – 10 × 1 – 2 × 0.7)/5	0.8
UH5	(13 – 10 × 0.8 – 2 × 1)/5	0.6
UH6	(9.6 – 10 × 0.6 – 2 × 0.8)/5	0.4
UH7	(6.2 – 10 × 0.4 – 2 × 0.6)/5	0.2
UH8	(3.05 – 10 × 0.2 – 2 × 0.4)/5	0.05
UH9	(0.9 – 10 × 0.05 – 2 × 0.2)/5	0
UH10	(0.1 – 10 × 0.2 × 0.05)/5	0

EXERCISES

3.1 Explain UH theory with its basic assumptions and applications.

3.2 Describe the governing factors affecting the shape of the unit hydrograph. How is the UH different from the distribution graph?

3.3 Describe the methods of UH derivation from rainfall-runoff data of an event.

3.4 In Question 3, how is the conventional method advantageous over other methods? If the available rainfall-runoff data correspond to multi-period storms, which methods are suitable for deriving a UH of the desired duration?

3.5 Describe Collins' method. State the reasons for the occurrence of oscillating ordinates in the recession limb of the derived UH. What are the procedures followed to do away with the oscillating ordinates?

3.6 Describe the methods of single division and least squares for deriving a UH and discuss their advantages and disadvantages.

3.7 What are the unit period and volume of the UH? State the theory for changing the unit volume of the UH of a specific duration.

3.8 Describe IUH and *S*-curve hydrograph and their relationship. How are these related with the UH?

3.9 What is superimposition method and how is it advantageous over the *S*-curve method?

3.10 Discuss various methods of averaging unit hydrographs.

3.11 A thunderstorm of 1-hr duration with 20 mm excess rainfall volume resulted in the following direct surface runoff (DSRO) hydrograph at the outlet of the catchment. Find out the UH of 1 mm excess rainfall volume and 1-hr duration.

Time (hr)	*DSRO (m^3/s)*	*Time (hr)*	*DSRO (m^3/s)*
0	0	12	91.6
1	61.0	13	61.4
2	314.6	14	40.6
3	561.0	15	26.4
4	673.4	16	17.0
5	645.0	17	11.0
6	584.2	18	7.0

(*Contd.*)

(*Contd.*)

7	475.0	19	4.4
8	365.4	20	2.8
9	269.8	21	1.6
10	192.8	22	1.0
11	134.4		

3.12 The excess rainfall hyetograph and the direct surface runoff hydrograph ordinates for an event of Godavary basin Sub-zone 3f (bridge no. 807/1) are given below. The catchment area is 824 sq. km. Determine 1-hr UH of 1 mm volume using Collins' method.

Time (hr)	*Excess Rainfall (mm)*	*DSRO (m^3/s)*	*Time (hr)*	*DSRO (m^3/s)*
0	3.58	0	12	91.6
1	4.07	61.0	13	61.4
2	2.54	314.6	14	40.6
3	—	561.0	15	26.4
4	—	673.4	16	17.0
5	—	645.0	17	11.0
6	—	584.2	18	7.0
7	—	475.0	19	4.4
8	—	365.4	20	2.8
9	—	269.8	21	1.6
10	—	192.8	22	1.0
11	—	134.4		

3.13 The ordinates of a 2-hr UH for a catchment (area = 824 sq. km) are given below. Determine 1-hr UH using *S*-curve method.

Time (hr)	*UH coordinates (m^3/s)*	*Time (hr)*	*UH coordinates (m^3/s)*
1	1.53	13	3.83
2	9.39	14	2.55
3	21.89	15	1.68
4	30.86	16	1.09
5	33.46	17	0.70
6	31.23	18	0.45
7	26.48	19	0.29
8	21.01	20	0.18
9	15.88	21	0.11
10	11.57	22	0.07
11	8.18	23	0.03
12	5.65		

3.14 The ordinates of 1-hr UH for a catchment (area = 824 sq. km) are given below.

Time (hr)	*UH coordinates (m^3/s)*	*Time hr*	*UH coordinates (m^3/s)*
1	3.05	12	4.58
2	15.73	13	3.07
3	28.05	14	2.03
4	33.67	15	1.32
5	32.25	16	0.85
6	29.21	17	0.55
7	23.75	18	0.35
8	18.27	19	0.22
9	13.49	20	0.14
10	9.64	21	0.08
11	6.72	22	0.05

(a) Determine the *S*-curve hydrograph for the excess rainfall of unit intensity (1 mm/hr).
(b) Find out the ordinates of 2-hr UH (1 mm) using *S*-curve and superimposition methods.
(c) Discuss the advantages of using superimposition method over the *S*-curve method for the given problem. State the circumstances under which the superimposition method fails.

3.15 A catchment (area = 300 km^2) receives an uniform rainfall of 30 mm in one day. During thenext few days, the discharge is observed in the river that drains the catchment. The readings are given in the following table.

T (days)	0	1	2	3	4	5
Q (m^3/s)	2	5	10	8	4	2

The base-flow is assumed constant $Q_{\text{base}} = 2$ m^3/s during the discharge period.

(a) How much was the direct runoff during the period (answer in m^3/s)?
(b) How much was the total direct runoff during the period (answer in mm)?
(c) How much was the total losses for the rainfall (answer in mm)?
(d) What was the maximum discharge in the water course during the period?

Answer

(a) 3, 8, 6 and 2 m^3/s or an average of 4.75 m^3/s
(b) $4.75 \cong 4 \cong 24 \cong 3600/300 \cong 10^6$ m = 5.5 mm
(c) 24.5 mm (d) 10 m^3/s

OBJECTIVE QUESTIONS

1. The inflection point on the recession side of the hydrograph indicates the end of
   (a) Base flow (b) Direct runoff
   (c) Overland flow (d) Rainfall
2. The concept of unit hydrograph was first introduced by
   (a) Dalton (b) Sherman
   (c) Horton (d) Thiessen
3. UH is the graphical relation between the time distributions of
   (a) Total rainfall and total runoff
   (b) Total rainfall and direct runoff
   (c) Effective rainfall and total runoff
   (d) Effective rainfall and direct runoff
4. The word "unit" in unit hydrograph refers to the
   (a) Unit depth of runoff
   (b) Unit duration of the storm
   (c) Unit base period of the hydrograph
   (d) Unit area of the basin
5. The range of the area of the basin where the unit hydrograph is applicable is
   (a) 200 hectar–5000 km^2 (b) 100 hectar–10000 km^2
   (c) 100 km^2–5000 km^2 (d) 250 km^2–7500 km^2
6. The basic principles of unit hydrograph theory are
   (a) Principle of superposition (b) Principle of time invariance
   (c) Both (a) and (b) (d) None of these
7. The *S*-curve hydrograph is the summation of the
   (a) Unit hydrograph (b) Total runoff hydrograph
   (c) Effective rainfall hyetograph (d) Base-flow recession curve
8. The *S*-curve hydrograph is used to
   (a) Estimate the peak flood flow of a basin resulting from a given storm
   (b) Develop synthetic unit hydrograph
   (c) Convert the UH of any given duration into a UH of any other desired duration
   (d) Derive the UH from complex storms
9. The lag-time of the basin is the time interval between the
   (a) Centroid of the rainfall diagram and the peak of the hydrograph
   (b) Beginning and end of direct runoff

(c) Beginning and end of effective rainfall
(d) Inflection points on the rising and recession limbs of the hydrograph

10. The direct runoff hydrograph of a basin can be approximated as a triangle with a base period of 90 hr and a peak flow of 230 m^3/sec during the 18^{th} hour. If the area of the basin is 1863 km^2, what is the depth of runoff indicated by the hydrograph?
(a) 1 cm (b) 2 cm (c) 5 cm (d) 10 cm

11. The 6-hr UH of a basin can be approximated as a triangle with a base period of 40 hr and peak ordinate of 150 m^3/s. Then the area of the basin is
(a) 2160 km^2 (b) 1080 km^2 (c) 540 km^2 (d) 1280 km^2

12. The peak discharge in 2-hr and 4-hr UHs of a basin occurs at t_1 and t_2. What can you infer from this information?
(a) $t_1 = t_2$ (b) $t_1 > t_2$ (c) $t_1 < t_2$ (d) $t_1 <= t_2$

13. The 4-hr and 8-hr UH of a basin having the same base period is plotted. Which of the following statement is true?
(a) The peak discharge at 4^{th} hr is greater than 8^{th} hr
(b) The peak discharge at 8^{th} hr is greater than 4^{th} hr
(c) Both the peak occurs at the same time
(d) None of the above

14. A 6-hr storm produced rainfall intensities of 7, 16, 20, 14, 8, and 5 mm/hr in successive one hour intervals over a basin of 1000 sq km. The resulting runoff is observed to be 2600 ha. m. Determine ϕ-index for the basin.
(a) 8 mm/hr (b) 9 mm/hr (c) 10 mm/hr (d) 11 mm/hr

15. In an 8-hr storm, the total depth of precipitation is 80 mm. Total storm runoff is 55 mm. Losses due to depression storage and interception is 5mm. Find out the W-index.
(a) 2.0 mm/hr (b) 2.5 mm/hr (c) 3.0 mm/hr (d) 4.0 mm/hr

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CHAPTER

4

Simulation and Synthetic Methods in Hydrology

4.1 INTRODUCTION

Why should we use simulation in hydrology? What is wrong with conventional techniques which have been in use for many years? Doesn't simulation require a lot more data? Doesn't it cost more and take more time? Is it really worthwhile to use computer simulation when we have so little data? The answer to the first question is specific, not only to hydrology but also to all other fields which necessitates experimental data to validate a system or to authenticate the outcome of certain experiments. Many a time, we find ourselves in situations when we cannot build a model that depicts the original object of our interest in the laboratory, e.g., a river channel, a watershed, or a reservoir system. In such situations, we try to fit a model in our laboratory by compressing or sizing the original scales of the object. Such a model is called a *prototype* which is compressed to a size that fits in our laboratory. This is called *scaling* or *model simulation*. This is about the simulating behaviour of physical models, but in hydrology, we more often resort to mathematical models as has been discussed in Chapter 1. In such situations, we try to build models which replicate the behaviour of the system which assists the analyst to record the response of the system to different conditions that may be imposed on it or that may occur in the future.

There might be a scenario when we do not have sufficient data to proceed with certain analysis, or we may face some situation wherein measurement of actual data is difficult. For example, the depth of water in a river channel during high-flood events or the height of water rise in an ocean during an event like the *tsunami* or a high-flood event coupled with cyclone is difficult to measure. Similarly, routine measurement of gauge data in a remote hilly watershed is difficult. It is inaccessible and the cost of experiments and maintenance are high. In such situations, data generated using simulations is the sole alternative to augment the data shortage.

Hence, in simple terms, *simulation* is the process of duplicating the behaviour of an existing or proposed system. It consists of designing a model of the system and conducting experiments with this model, either for better understanding of the functioning of the system or for evaluating various strategies for its management. *It must be noted that simulation is a statistical experiment, and hence its output must be tested with suitable statistical tests.*

4.2 TRADITIONAL ANALYSIS VS. HYDROLOGIC SIMULATION

Prior to the development of simulation, a number of techniques were used in hydrology to arrive at the estimates of flow when no observed values existed. These procedures utilized approximations because the work had to be done with pencil and paper, and it was not possible to deal in detail with the problem and expect a solution within a reasonable time. Classical methods resorted to simplification of procedures. For example, in the unit hydrograph technique for constructing a graph of streamflow, it was assumed that flow rate was in linear proportion to runoff volume. This assumption conveniently permits superposition of runoff from several storm increments. This is not absolutely correct; however, in watersheds where large flood plains exist, the errors may be very large. The computer permits a much more detailed calculation using kinematic routing which relies on the actual stream cross-sections and the theory of flow in channels. Similarly, in estimating the runoff produced due to a rainfall event, relatively simple relationships were employed. The most complex of these relationships used parameters, such as time of year, an antecedent precipitation index, and the duration of rain in a statistically derived relationship. By contrast, simulation allows the continuous calculation of soil moisture, infiltration, and the movement of moisture in overland flow to the stream. A detailed computation represents the actual physical processes realistically, and consequently provides a more reliable basin for extrapolation in time or space.

Preliminary research at Stanford University around 1960 had indicated that it was feasible to write a computer program which would simulate hydrologic processes and compute streamflow from rainfall data. This instigated the development of simulation techniques in hydrology. The concept of infiltration which is the heart of continuous simulation was first proposed by Horton in 1933. Similarly, the routing to construct the runoff hydrograph was first introduced by Zoch in 1934. The idea of continuous soil moisture accounting was suggested by Linsley and Ackermann in 1942. Slowly, the basic concepts of computer simulation were recognized as preferable approaches to hydrologic analysis, long before the computational ability was available.

4.3 ISSUES OF DATA AND COST

The comments above offer many reasons why simulation is preferred over conventional methods. However, simulation techniques are expensive and difficult to implement. Simulation requires input of a considerably larger quantity of data than is normally used in most conventional hydrologic studies. Even in case of conventional studies, an equal volume of data is usually reviewed, but due to the constraints of manual computation, only selected data (much less than 10% of the available data) items are actually used. It is a relatively simple matter to get the required streamflow and rainfall data from responsible federal agencies on magnetic tape.

It is often implied that one should use approximate methods because of the restrictions posed by limited data. However, there is nothing about approximate methods that makes better use of the limited data, and most approximate methods have been demonstrated to be highly unreliable. There are many techniques which can be used to adapt the limited data to simulation. By careful use of a simulation program, data of poor quality can be checked, missing records can be completed, and a considerable extension of the record can be made. The most critical data for simulation are the rainfall data. Without rainfall data, it is impossible to carry out a simulation study.

Simulation techniques cannot compete with the use of empirical formulae for computation of design flows on the basis of time and cost, but they can easily compete with conventional methods, such as rainfall-runoff relations and unit hydrographs. Usually, it takes less time to develop necessary data for simulation than to develop the desired estimates. Meanwhile, one simulation run will provide an abundance of data which can answer many hydrologic problems. Although the data from the run may not be required in all its detail for a study of flood flows, it may subsequently have considerable value in dealing with streamflow volumes or minimum flows. If one wishes to explore the effect of changing vegetal cover on the watershed, or increasing the amount of urban land use, or other possible land use changes, this is easily done with simulation. Using conventional methods, it would be difficult, if not impossible, to estimate the effect of such changes.

4.4 MONTE CARLO SIMULATION

A popular simulation technique that is frequently used nowadays is *Monte Carlo simulation*. The technique was initially developed for managing troops and ammunitions during World War II. The name "Monte Carlo" is inspired from the city Monte Carlo, USA, famous for gambling where a wheel was used to generate random numbers. The scheme is aimed at estimating stochastic or deterministic parameters based on random sampling. The advantage of this technique lies in the fact that the time element is not a pertinent factor, e.g., evaluation of multiple integration, solving for the parameters of a nonlinear complex equation, matrix inversions, etc. Design of real-world systems is generally based on observed historical data. For example, the observed streamflow data are used in checking the robustness of flood frequency analysis methods in return period flood quantile estimation, parameter estimation methods for conceptual models, historical traffic data is used in design of highways, observed data are used in design of customer services, etc. However, the historical records are not enough in most cases, and the observed pattern of data is not likely to repeat exactly. The performance of a system critically depends on the extreme values of input variables, and the historical data may not contain the entire range of input variables. There are many instances when a flood with peak value has crossed a gauging site without the notice of the observer or a peak exceeding the historical record enters a reservoir without being taken into account by the observer. In such cases, the design for extreme events becomes faulty.

The conclusion that can be drawn from the above examples is that one does not get a complete picture of the system performance and risks involved when actual historical data are absent in evaluation. Thus, for instance, the planner cannot determine the risks of a designing system because this requires a very large sample of data which are not commonly available. For many systems, some or all inputs are random, system parameters are random, initial conditions as well as boundary condition(s) may also be random in nature. The probabilistic properties of these parameters are known. For such systems, simulation experiments are conducted using a set of inputs which are synthetically (artificially) generated. While generating inputs, it is ensured that the statistical properties of the random variables are preserved. Each simulation experiment with a particular set of inputs gives an answer. When many such experiments are conducted with different sets of inputs, a set of answers is obtained. These answers are statistically analyzed to understand or forecast the behaviour of the system. This approach is known as Monte Carlo simulation.

The main steps of Monte Carlo simulation are:

1. Generating the inputs
2. Developing a prototype of the system
3. Running the model using the inputs generated at (1)
4. Assembling different set of input-output samples
5. Conducting goodness-of-fit test or other statistical tests to verify the results obtained at (4)

4.4.1 Random Numbers having Uniform Distribution

In simulation, many events appear to occur at random or to involve attributes whose values must be assigned randomly. This is because most of the simulations are based on knowledge expressed as general or historical relationships. For example, in many situations, the duration of event is known to lie in a fixed range or the respective formula involves a term that varies from 0 to 1. These variables may be assigned by a probability density function (PDF) or some *discrete functions*. For these reasons, one of the requirements of any computer system is to have some facility for generating random numbers.

Nearly all modern compilers have built-in routines to generate uniformly distributed random numbers between 0 and 1. The most popular random number generation method is the *congruence method*. To start the process, a number known as *seed* is input to the equation which gives a random number. This number is again input to the equation to generate another number, and so on. When this process is repeated n times, n random numbers are obtained. The recursive equation that is commonly used to generate random numbers in the *linear congruential generator* (LCG) is given as:

$$R_i = (aR_{i-1} + b) \text{ (modulo } d) \tag{4.1}$$

where, R_i is the integer variable; and a, b, and d are positive integer constants which depend on the properties of the computer. The word 'modulo' denotes that the variable to the left of this word is divided by the variable to the right (in this case d), and the remainder is assigned the value R_i. The desired uniformly distributed random number is obtained as R_i/d. The initial value of the variable (R_0) in Eq. (4.1) is called the *seed*. It is to note that the properties of the generated numbers may depend on the values of constants a, b, and d. The value of constant a needs to be sufficiently high; low values may not give good results. Constants b and d should not have any common factors. The positive integers R_0, a, b, and c are chosen such that $d > 0$, $a < d$, and $b < d$.

Example 4.1 Generate a sequence of four random numbers using Eq. (4.1). Assume seed as zero, and assume suitable values of a, b, and d as per the criteria given above.

Solution

Let $a = 2$, $b = 3$, $d = 10$ and $R_0 = 0$ (Please note that $d > 0$, $a < d$, and $b < d$)

$$R_0 = 0$$

$$R_1 = [(2 \times 0) + 3)] \text{ mod } 10 = 3$$

$$R_2 = [(2 \times 3) + 3)] \text{ mod } 10 = 9$$

$$R_3 = [(2 \times 9) + 3)] \text{ mod } 10 = 1$$

$$R_4 = [(2 \times 1) + 3)] \text{ mod } 10 = 5$$

Similarly, there is *additive congruential generator* (ACG) and *quadratic congruential generator* (QCG).

Algorithm for ACG is:

$$R_j = [R_{j-1} + R_{j-n}] \bmod (m) \tag{4.2}$$

To extend a series with 1, 2, 4, 8, and 6 with the same initial values of above example,

$$R_1 = 1$$
$$R_2 = 2$$
$$R_3 = 4$$
$$R_4 = 6$$
$$R_5 = 8$$
$$R_6 = [R_5 + R_1] \bmod (10) = 7$$

Algorithm for QCG is:

$$R_{n+1} = [R_n (R_n + 1)] \bmod (m) \tag{4.3}$$

Pseudo random number (PRN) is useful for generating numbers in the range of 0–1. Since these algorithms of random number generation are deterministic, the generated numbers can be duplicated again. Therefore, these numbers are not random in a strict sense and are called *pseudo random numbers.*

A good random number generator should produce uniformly distributed numbers that don't have any correlation with each other, and it should be able to exactly reproduce a given stream of random numbers. Besides, a good random number generator is fast and consumes less memory space. It is also required that the generator should be capable of generating several separate streams of random numbers.

Let $X_{n+1} = [a];\ a = 10^p\, c\, X_n$ (4.4)

where, $[a] = [10^p\ c\ X_n]$ denotes the fractional part of a, p is the number of digits of *pseudo random number*, c is the constant multiplier such that $(0 < c < 1)$.

It has been established that $c = 10^{-p}\,[200\,A \pm B]$, where A is a non-negative integer and B is any number from {3, 11, 13, 19, 21, 27, 29, 37, 53, 59, 61, 67, 69, 77, 83, 91}. The *seed* for PRN is denoted by X_0 and is equal to $10^{-p}\,k$, where k is any integer *not divisible* by 2 or 5 such that $0 < k < 10^p$.

Example 4.2 Using Eq. (4.2) and a seed value of 0.33, generate a sequence of four random numbers containing fractions up to two digits. Consider $A = 0$ and $B = 11$.

Solution

Let $x_0 = 0.33$, $A = 0$, and $B = 11$. For generating fractions up to two digit, $p = 2$, which means $c = 10^{-2}\,[11.0] = 0.11$.

Therefore, $x_0 = 0.33$

$$x_1 = [100\,(0.11)\,(0.33)] = 0.63$$

Please note that the product of 100, 0.11, and 0.33 is 3.63. However, we have to consider only the fractional part for obtaining x_2

Thus, $x_2 = [100\ (0.11)\ (0.63)] = 0.93$

$x_3 = [100\ (0.11)\ (0.93)] = 0.23$

$x_4 = [100\ (0.11)\ (0.23)] = 0.53$

Thus, the generated sequence of random numbers is 0.63, 0.93, 0.23, and 0.53.

The main disadvantage of this method is that it is slow. Hence, for general purpose simulating systems, random number generators are used which may incorporate a large number of PRN generators, each producing one sequence. A variety of statistical tests, such as Frequency tests, Serial test, Kolmogrov-Smirnov test, and Run test are conducted to check the validity of these random number generators. Some of these tests are included in later chapters.

4.4.2 Transformation of Random Numbers

All the generators used thus far are designed to generate uniformly distributed random numbers following a *uniform distribution*. Any form of input data used in a system shall have certain statistical properties and a certain probability distribution. Therefore, our task is to convert these uniformly distributed random numbers such that they follow the desired probability distribution. The variables involved may either be continuous or discrete random variables. Methods to generate random variate that follow a given distribution are described next. If the inverse form of a distribution can be easily expressed analytically, then inverse transformation is the simplest method. If an analytical expression for the inverse of the concerned distribution is not known, special algorithms are employed to efficiently generate numbers with such a distribution. The meanings of continuous and discrete distributions are explained in detail in Section 4.5.

4.4.3 Inverse Transformation Method

The inverse transformation technique is used to transform a standard uniform variate into any other distribution. It is particularly useful when the distribution is empirical. Suppose we wish to generate a PRN from a distribution given by $F(x)$, where F satisfies all the properties of a cumulative distribution function (CDF). To achieve this, first a uniformly distributed random number x_i in the range [0, 1] is generated. Thereafter, X can be generated applying the following formula:

$$X = F^{-1}\ (x) \tag{4.5}$$

where, F^{-1} is the inverse of the CDF of the random variable x. This method is also known as *the inverse transformation method*. It is useful when the inverse of the CDF of the random variable can be expressed analytically. This is a simple and computationally efficient method. However, it can be used for only those distributions whose inverse form can be easily expressed. The following examples will clarify the application of direct and inverse transformation methods in simulations or data generation.

Example 4.3 The standard uniform random variables are given as: 0.1021, 0.2162, and 0.7621. Transform these into a distribution given by the following relation:

$$F(x) = \begin{cases} 0 & \text{for} \quad (x < 0) \\ x & \text{for} \quad (0 < x < 1/4) \\ \dfrac{3x+1}{7} & \text{for} \quad (1/4 < x < 2) \\ 1 & \text{for} \quad (x > 2) \end{cases}$$

Solution

$$F^{-1}(a) = \begin{cases} a & \text{for} \quad (0 < a < 1/4) \\ (7a-1)/3 & \text{for} \quad (1/4 < a < 1) \end{cases}$$

Or,

$$F^{-1}(0.1021) = 0.1021$$
$$F^{-1}(0.2162) = 0.2162$$
$$F^{-1}(0.7621) = 1.4449$$

4.5 GENERATION OF RANDOM NUMBERS

Random numbers can be generated using discrete as well as continuous distributions. For some distributions, random variables can be expressed as functions of other random variables that can be easily generated. This property can be exploited for generation of random variables. For instance, the gamma distribution is nothing but the sum of exponential distributions. The generation of random numbers using discrete as well as continuous distribution is as follows;

4.5.1 Using Discrete Distributions

Poisson distribution, binomial distribution, negative binomial distribution, Bernoulli distribution, geometric distribution and hypergeometric distribution pertain to class of discrete distributions. Generations of random numbers which follow these distributions are often of interest to hydrologists. Generation of Poisson Distributed Random Numbers is explained here. Generation of random numbers using all other types of distribution can be done similarly.

To generate Poisson distributed random numbers, cumulative distribution function of the Poisson-distribution is used along with uniformly distributed random numbers. Values of uniformly distributed random numbers are used in place of CDF to get corresponding variate. Such variates will be random numbers having Poisson-distribution.

Example 4.4 Corresponding to a uniformly distributed random number 0.14, obtain a Poisson distributed random number having Poisson-distribution with $\lambda = 2.5$.

Solution

The CDF of the Poisson-distribution is given by:

$$F(x) = \sum_{x=1}^{N} \frac{e^{-\lambda}\lambda^{x}}{x!} \tag{4.6}$$

For $x = 0$, $F(0) = e^{-2.5}\,(2.5)^0/(0!) = 0.082085$

For $x = 1$, $F(1) = [e^{-2.5}\,(2.5)^0/(0!)] + [e^{-2.5}\,(2.5)^1/(1!)]$

$= 0.082085 + 0.2052 = 0.2873$

Similarly, $F(2)$, $F(3)$ and $F(4)$ are computed as shown in Table 4.1

Table 4.1 Generation of Poisson distributed variates

x	0	1	2	3	4
F	0.0821	0.2873	0.5438	0.7576	0.8912

For a standard uniform random number $u = 0.14$, the Poisson-distributed number $x = 1$. It is to note from Table 4.1 that for F greater than 0.0821 and up to 0.2873, $x = 1$.

4.5.2 Using Continuous Distributions

In this section, random numbers generation using continuous distribution such as normal distribution, exponential distribution, gamma distribution, generalized extreme value distribution, and generalized pareto distribution are explained.

4.5.2.1 Normally Distributed Random Numbers

Normally, distributed random numbers are obtained from uniformly distributed random numbers after applying transformation. Two methods, namely (i) *Box-Muller method*, and (ii) method based on *central limit theorem*, are used for this purpose.

Box-Muller Method Box-Muller method requires the prior generation of two series (X_1 and X_2) rectangular distributed (0, 1) random numbers. These are then transformed to normally distributed series (Y_1 and Y_2) by the following equation:

$$Y_1 = [-2\,\mathrm{Log}_e X_1]^{1/2} \cos(2\pi X_2); \text{ and } Y_2 = [-2\,\mathrm{Log}_e X_1]^{1/2} \sin(2\pi X_2) \tag{4.7}$$

The values Y_1 and Y_2 are normally and independently distributed with zero mean and unit variance.

Method Based on Central Limit Theorem The central limit theorem states that under certain very broad conditions, the sum of a sequence of independent random variables approximates a normal distribution, irrespective of the distribution of the random variables in the sequence. The larger the number of random variables making up the sum, the better is the normal approximation. In particular, the central limit theorem holds good when the independent random variables summed are sampled from a rectangular distribution. The values x_1, $x_2, \ldots x_n$ are therefore generated from a rectangular distribution over the interval (0, 1), and the quantity y is then calculated by:

$$y = x_1 + x_2 \ldots + x_n - 0.5n \tag{4.8}$$

If $n = 12$, then the distribution of the values of y closely approximates a normal distribution with zero mean and unit variance. These random numbers are also represented by $N(0, 1)$. The normally distributed random numbers with zero mean and unit variance, $N(0, 1)$, can be transformed to normally distributed random numbers with any mean μ and variance σ^2, i.e., $N(\mu, \sigma^2)$ as follows:

$$z_i = \mu + \sigma y_i$$

where, z_i is $N(\mu, \sigma^2)$ and y_i is $N(0, 1)$.

Example 4.5 Consider the standard uniform random numbers as: {0.1602, 0.1124, 0.7642, 0.4314, 0.6241, 0.9443, 0.8121, 0.2419, 0.3124, 0.5412, 0.6212, and 0.0021}. Using these numbers, obtain a normal variate having a mean of 25 and variance 9.

Solution

Adding the 12 standard uniform random numbers gives $y = 5.5675$

This number is from a normal distribution with mean = 6 and variance = 1

The corresponding standard normal number is $z = y - 6 = -0.4325$

Now, transformation of this to $N(25, 9)$ is done by:

$$x = \mu + \sigma z = 25 + 3(-0.4325) = 23.7025$$

4.5.2.2 Exponential Distributed Random Numbers

Exponential function is invertible and this facilitates computations of random numbers having exponential distribution. For this purpose, CDF of exponential function is equated to uniformly distributed random number and the variate x is obtained.

Example 4.6 Generate exponential distributed variates for a given specified parameter $\alpha = 1.5$ for the following exponential distribution:

$$F(x) = 1 - \exp\left[-\frac{x}{\alpha}\right] \tag{4.9}$$

Solution

$F(x)$ is the CDF corresponding to the PDF of exponential distribution function. $F(x)$ takes a value in the range of 0 to 1.

Equation (4.9) can be transformed to the following form:

$$x = (-\alpha)\ \ln[1-F(x)] \tag{4.10}$$

The values of exponential distributed variates (x) are calculated substituting the generated F in Eq. (4.10) as shown in the following table.

Table 4.2 Generation of exponential distributed random variates **x**

F	0.310	0.790	0.874	0.539	0.367	0.161	0.883	0.032	0.522	0.340	0.075	0.486	0.831
x	0.557	2.343	3.113	1.161	0.686	0.264	3.225	0.049	1.108	0.623	0.116	0.997	2.667

It is to note that F also represents uniformly distributed random numbers.

4.5.2.3 Gamma-Distributed Random Numbers

As Gamma distribution is the sum of exponentially distributed functions, the following expression can be written for a variable x following a gamma distribution with parameters (k, λ)

$$x = \sum_{i=1}^{k} x_i \tag{4.11}$$

where, $x_1, x_2, \ldots, x_k$ are independent exponentially distributed random variables with parameter λ; and x_i is given by:

$$x_i = -\ln(u_i)/\lambda \tag{4.12}$$

The procedure to generate gamma-distributed random variate following this method consists of generating random variate that follows exponential distributions having parameter λ. Now, k such variates are added to obtain one gamma distributed variate. The inverse transform method can be used to generate exponentially distributed random variates. We first generate a uniformly distributed random number u in the range [0, 1]. Using it, a random variate that follows the exponential distribution with parameter λ can be obtained.

Example 4.7 Generate gamma-distributed random variates with parameters (4, 0.8). Assume four uniformly distributed random numbers, u_1, u_2, u_3, u_4 as 0.5371, 0.1814, 0.6193, and 0.1319.

Solution

Corresponding to four uniformly distributed random numbers, the corresponding exponentially distributed random variates with parameter $\lambda = 0.8$ will be 0.777, 2.134, 0.599, and 2.532.

Therefore, the gamma-distributed random number is:

$$x = \frac{1}{\lambda}\sum_{i=1}^{4}[-\ln(u_i)] = 0.777 + 2.134 + 0.599 + 2.532 = 6.042 \qquad (4.13)$$

4.5.2.4 Generalized Extreme Value (GEV) Distributed Random Numbers

These distributions find application in describing series of maximum or minimum values from a set of data. For example, if maximum annual flood is recorded at a site for 20 years, one has a set of 20 such data. Such a series will be suited to analysis using GEV distribution which has the following CDF:

$$F(x) = \exp\left[-\left(1-\kappa\frac{x-\xi}{\alpha}\right)^{1/\kappa}\right] \qquad (4.14)$$

where, x represents the extremes to be analysed, α is the scale parameter, κ is the shape parameter, and ξ is the location parameter. The case $\kappa = 0$ corresponds to the Gumbel distribution and is also known as Type I EV distribution, Type II and III EV distributions correspond to κ less than or greater than zero.

The above equation of GEV distribution is transformed to a form which gives x as:

$$x = \xi + \frac{\alpha}{\kappa}(1 - [-\ln F]^{\kappa}] \qquad (4.15)$$

The procedure of generating random numbers having GEV distribution is straightforward. Substituting for F as uniformly distributed random number will give random variate x, conforming to GEV distribution.

Example 4.8 Compute the GEV variates using the parameters: $\kappa = -0.1$, $\alpha = 100$, and $\xi = 25$. The sequence of uniformly distributed random numbers is given in Table 4.3.

Solution

In Table 4.3, F values represent uniformly distributed random numbers.

Substituting the values of F in Eq. (4.15) and using the given parameters, the values of x are calculated, which are shown in the following Table.

Table 4.3 Generation of GEV random variates

F	0.592	0.250	0.890	0.810	0.508	0.409	0.564	0.920	0.010	0.162	0.881	0.031	0.704
x	91.709	–7.135	264.8	193.501	64.737	36.263	82.321	307.014	–116.6	–33.134	254.48	–92.082	135.381

4.5.2.5 GP Distributed Random Numbers

Similar to GEV distributions, random numbers conforming to generalized Pareto (GP) distribution can be easily computed as CDF is invertible. GP distribution was developed to describe the distribution of wealth among individuals. Needless to say that certain individuals are very rich while some are poor, and the difference between the wealth of a rich and a poor man is too big. The same situation may also arise in hydrology. For example, one may encounter very large variation in flows. Here, the objective is not to describe this variation in detail. Rather, emphasis is given to generate random variate. A typical GP distribution [Lama et al. (2006)] for variate x can be represented as follows.

$$y = \begin{cases} -\dfrac{1}{\kappa}\ln\left\{1-\kappa\dfrac{(x-\xi)}{\alpha}\right\}, & \kappa \neq 0 \\ \dfrac{(x-\xi)}{\alpha}, & \kappa = 0 \end{cases} \tag{4.16a}$$

$$F(y) = 1 - \exp(-y) \tag{4.16b}$$

In Eq. (4.16b), $F(y)$ is the CDF for the GP distribution. The values of GP-distributed variates, x, are calculated by transforming the above equation, and taking x to the left hand side as:

$$x = \xi + \frac{\alpha}{\kappa}\ \{1 - (1 - F)^{\kappa}\} \tag{4.17}$$

GP-distributed random numbers can be generated using uniformly distributed random numbers in place of F in Eq. (4.17).

Example 4.9 Compute the GP-distributed variates for the following parameters: $\alpha = 2$, $\xi = 0$, and $\kappa = 0.2$. Uniformly distributed random numbers are given in the first row of Table 4.4.

Solution

Corresponding to $F = 0.303$, $\alpha = 2$, and $\kappa = 0.2$, Eq. (4.17) gives

$$x = (2/0.2)\ \{1 - (1 - 0.303)^{0.2}\}$$

On substituting the values of F in Eq. (4.17), the values of x are calculated as shown in Table 4.4.

Table 4.4 Generation of GP random variates

F	0.303	0.479	0.842	0.483	0.959	0.110	0.724	0.823	0.895	0.355	0.335	0.210	0.591
x	0.696	1.222	3.085	1.236	4.72	0.23	2.269	2.927	3.628	0.839	0.783	0.46	1.637

4.6 SIMULATION OF SYSTEMS WITH RANDOM INPUTS

Often, hydrologists confront situations where data regarding hydrologic variables, such as precipitation, runoff, and stream flows are limited. For a better planning of water resources, it is necessary to extend these data. Here, an insight into generation of additional data is provided.

4.6.1 Generation of Annual Flows

The annual flows generally exhibit persistence, i.e., the tendency of high flows to follow high flows, and low flows to follow low flows. The generation of annual flow sequences depends whether the persistence is present or absent in historic record. For judging the presence of persistence, *turning point test* or *Anderson's correlogram test* may be used.

Persistence Absent Case When persistence is absent in historic flow record, i.e., the annual flows are random, the alternate annual flow sequence may be generated using the probability distribution of annual flows. Generally, annual flows follow normal distribution. The procedure for generating normally distributed random number has been explained in the earlier section. The following steps summarize the procedure for generating synthetic sequences of annual flows.

(i) Plot the historic sequence of annual flows as a histogram, and decide the theoretical distribution to be used to represent them.
(ii) Estimate the parameters of the distribution.
(iii) Sample from this distribution to obtain the generated sequence.

Persistence Present Case When the annual flows exhibit dependence, the flows are generated using either auto-regressive (AR) models or auto-regressive integrated moving average (ARIMA) models. These models along with *turning point test* or *Anderson's correlogram test* can be found in any textbook on Stochastic Hydrology.

4.6.2 Generation of Monthly Stream Flows

Monthly stream flows are used for deciding capacity of the reservoir and planning water resources. Over the past three to four decades, a number of models have been developed to generate monthly flows. These models can be broadly classified into two categories, namely (i) univariate, and (ii) multivariate.

Univariate models are used when synthetic sequence of flows is required at one particular site, and the cross-correlation properties with flows are not required to be preserved at other sites. In this category, there are basically two models, namely (i) Univariate Thomas-Fiering Model, and (ii) Disaggregation process-based models.

Multivariate models are used when it is necessary to generate simultaneous sequences of several hydrological variables or simultaneous sequences of a hydrological variable at many sites. Multivariate models are important as water resources planning are concerned with determining the size and operation of not one but several reservoirs. In such cases, the use of univariate models for generating synthetic flows at each site of interest is limited, because the flows of various sites tend to be interdependent. The following are some of the models in this category: (i) Bi-variate Thomas-Fiering model, (ii) Multi-site Disaggregation models, (iii) Multivariate Matalas model, and (iv) HEC- 4 model.

The present section covers Thomas-Fiering model and its modified forms to deal with negative flows and zero flows in the generated data. The model requires generation of normally distributed random numbers.

4.6.2.1 Thomas–Fiering Model

The method of Thomas and Fiering implicitly allows for the periodicity observed in the monthly discharge data. The method is based on AR(1) model and is used when month-to-month correlation structure is *non-stationary*. In its simplest form, this method uses twelve linear regression equations. If twenty years of monthly flows are available, then all the January and December flows are abstracted, and January flows are regressed upon December flows. Similarly, February flows are regressed upon January flows, and so on for each month of the year. Using the Thomas-Fiering notation, the model may be written as follows:

$$Q_{i+1} - \bar{Q}_{j+1} = b_j[Q_i - \bar{Q}_j] + Z_i S_{j+1}(1 - r_j^2)^{1/2} \tag{4.18}$$

In Eq. (4.18), Q_i and Q_{i+1} are the discharges during the i^{th} and $(i + 1)^{th}$ months, respectively; $\bar{Q}_{j+1}$, $\bar{Q}_j$ are the mean monthly discharges during the $(j+1)^{th}$ and j^{th} months, respectively, within a repetitive annual cycle of 12 months; Z_i is a random normal variate with zero mean and unit variance; S_{j+1} and S_j are the standard deviations of discharges in the $(j+1)^{th}$ and j^{th} months, respectively; r_j is the correlation coefficient between flows in the j^{th} and $(j+1)^{th}$ months; and b_j is the regression coefficient for estimating discharge in the $(j+1)^{th}$ month from the j^{th} month and is given by:

$$b_j = r_j \times (S_j + 1)/S_j \tag{4.19}$$

The method should be used with caution if fewer than twelve years of data are available. When the Thomas-Fiering model is fitted to monthly stream flows, it may be found that values in the generated sequence are sometimes negative. The negative flows in the generated data may be avoided if the model is fitted either to log-transformed or square root transformed data. Once the flows are generated, antilog or square of flows is taken. R. T. Clarke, in his book *Mathematical Models in Hydrology*, has given a computer program for fitting the Thomas-Fiering model.

Generation of Monthly Flows having Zero Flow It is sometimes necessary to generate synthetic discharge sequences for streams having no discharge during the dry season. The procedure to be adopted in such cases is described below:

Suppose, we have N years of data record for each month j (j =1, 2,...,12) of the year. If n_j is the number of years (out of N) for which the flow is not zero, then compute P_j as $P_j = n_j/N$.

1. Calculate the mean monthly flow and the variance of flows for each month j.
2. Fit a Thomas-Fiering model to those successive pairs of months for which flow is not zero.
3. Generate synthetic sequences of monthly flows as follows:
   (a) For month j, choose a pseudo-random number, having a rectangular distribution over (0, 1). If this number is less than P_j (but greater than zero), then flow is to occur in month j; otherwise zero flow.
   (b) If no flow is to occur in month j, repeat the steps for month (j + 1).
   (c) If flow is to occur in month j, and it is the first month of the year for which flow is to occur, then select a pseudo-random normal variate for a distribution with mean and variance equal to the mean monthly flow and variance of flows for month j.
   (d) If flow is to occur in j^{th} month and flow has also occurred in $(j-1)^{th}$ month, then use the regression equation of the Thomas-Fiering model to obtain the flow for j^{th} month.

Example 4.10 For a tributary, the mean monthly flow, standard deviation of the $(j+1)^{th}$ month, and correlation of the j and $(j+1)^{th}$ month are given in Table 4.5. Develop equations for monthly flow generation.

Table 4.5 Mean monthly flow, standard deviation of $(j + 1)^{th}$ month, and correlation of j^{th} and $(j+1)^{th}$ month

Month (j)	q_{mean} $[m^3/s]$	$S_{(j+1)}$	r_j	$[1-r_j^2]^{1/2} S_{(j+1)}$	$b_j = r_j \times (S_j+1)/S_j$
1	12	8.6	0.413	7.83	0.355
2	7.9	7.4	0.98	1.47	0.967
3	7.3	7.3	0.991	0.98	0.991
4	9.1	7.3	0.762	4.73	1.221
5	19	11.7	0.555	9.73	3.567
6	84	75.2	0.498	65.21	0.587
7	215	88.7	–0.073	88.46	0.09
8	187	112.1	0.537	94.57	0.178
9	75	37.2	0.602	29.70	0.550
10	70	34	0.741	22.83	1.853
11	71.4	85	–0.072	84.78	–0.009
12	20.3	10.3	0.266	9.93	0.03

Solution

The general form of Thomas-Fiering model is:

$$Q_{i+1} - \bar{Q}_{j+1} = b_j[Q_i - \bar{Q}_j] + Z_i S_{j+1}[1 - r_j)^2]^{1/2}$$

where, $b_j = r_j \times (S_j+1)/S_j$

Regression equations for 4 months to generate the monthly flows are:

$$Q_{\text{jan}} - 12 = 0.355 \quad [Q_{\text{dec}} - 20.3] + 7.83Z_i$$
$$Q_{\text{feb}} - 7.9 = 0.83 \quad [Q_{\text{jan}} - 12] + 1.472Z_i$$
$$Q_{\text{mar}} - 7.3 = 0.98 \quad [Q_{\text{feb}} - 7.9] + 0.977Z_i$$
$$Q_{\text{apr}} - 9.1 = 0.76 \quad [Q_{\text{mar}} - 7.3] + 4.727Z_i$$

If the sequences of random normal variates are: 2.289, –0.445, –1.2, 0.83, and Q_{dec} = 20.3 m^3/s, then the flow values for the month of January, February, March, and April are: 29.9, 22.1, 20, and 22.7 m^3/s, respectively. Similar regression equations for other months can be generated.

4.7 DEVELOPING SYNTHETIC UNIT HYDROGRAPHS

Most of the available methods for derivation of synthetic unit hydrograph (SUH) involve manual, subjective fitting of a hydrograph through the data points. Because cumbersome trials are involved, the generated unit hydrograph is often left unadjusted for unit runoff volume. In last few decades, the use of probability distribution functions in developing SUH has received much attention because of its similarity with unit hydrograph properties. In this section, the potentials of two-parameter gamma distribution to derive SUH shall be discussed. Simple formulae will be derived using analytical and numerical schemes to compute the distribution parameters, and their validity will be examined using an example. Before moving onto the application of PDFs in deriving synthetic unit hydrographs, the existing popular techniques used for the same purpose are discussed.

4.7.1 Snyder's Method

Snyder (1938) used three parameters to describe the hydrograph, using data from the catchments of Appalachian Highlands. These are lag-to-peak, t_{Lp}; peak discharge, Q_p; and base time, t_B; as shown in Fig. 4.1.

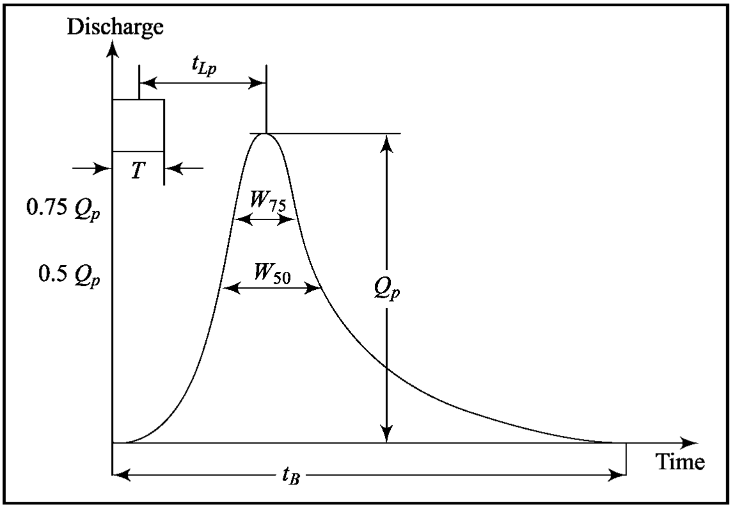

Fig. 4.1 *Elements of a synthetic unit hydrograph*

Empirical relationships were suggested by Snyder to estimate these parameters using following equations:

$$t_{Lp} = C_T (L\, L_{CA})^{0.3} \quad (4.20)$$

$$Q_p = 640\, C_p\, A / t_{Lp} \text{ (FPS unit)} \quad (4.21a)$$

$$t_B = 72 + 3\, t_{Lp} \quad (4.22)$$

where, L is the length of the main stream from the outlet to the catchment boundary in miles, L_{CA} is the distance from the outlet to a point on the stream nearest to the centroid of the catchment in miles, C_T is a coefficient that depends upon units and basin characteristics (it varies from 1.8 to 2.2 for FPS unit and from 1.32 to 1.65 for SI units), C_p is also a coefficient that depends upon units and basin characteristics (it varies from 0.56 to 0.69), A is the area of the catchment in square miles, t_{Lp} is in hrs, Q_p is in ft^3/s, and t_B is in hrs.

It is necessary to exercise caution with the use of units of involved parameters. If SI unit is followed, coefficients in the empirical equations need to be adjusted. For example, if SI unit is used, Eq. (4.21a) becomes:

$$Q_p = 2.778\, C_p\, A / t_{Lp} \text{ (SI unit)} \quad (4.21b)$$

Here, A is in km^2 and Q_p is in m^3/s. Equations (4.20) to (4.22) are valid for excess rainfall of standard duration T_{SD} (hr) equal to t_{Lp} /5.5. In case, excess rainfall duration T is not equal to $T_{SD} = t_p$ /5.5, the time to peak needs to be adjusted using Eq. (4.23), as shown below:

$$t_{pR} = t_{Lp} + (T - T_{SD})/4 \tag{4.23}$$

where, t_{pR} is the revised lag-time in hours and T is the actual rainfall duration. It is to note that the revised time to peak will correspond to Q_p. These relations provide a complete shape to the SUH. Since the base length of the hydrograph obtained is always greater than 3 days, Snyder's method is applicable to fairly large catchments only. Similar to Eq. (4.22), empirical expressions are given by Langbein et al. (1947), Taylor and Schwarz (1952), and Gray (1961) among others.

Linsley et al. (1958) revised the equation of time-to-lag (or peak) given by Eq. (4.20) as follows:

$$t_{Lp} = C_T \left(\frac{L\, L_{CA}}{\sqrt{S}} \right)^n \tag{4.24}$$

where n is a coefficient for a region. This equation can be transformed to the following form as:

$$\log\,(t_{Lp}) = \log\,(C_T) + n \log\,(L\, L_{CA}/\sqrt{S}) \tag{4.25}$$

Available observed hydrographs in a region are used to calculate the coefficient C_T.

Empirical expressions for the time base can be generalized as:

$$t_B = T + a\, A^b \tag{4.26}$$

where, a and b are the coefficient and the exponent, respectively; and A is the area of the catchment in km^2.

According to Linsley et al. (1975), $a = 0.8$ and $b = 0.2$. The US Army Corps of Engineers used this approach with added empirical expressions for W_{50} and W_{75} (in hr) as functions of $q_p = Q_p/A$ in m^3/s/km^2, which represent the width of UH at $0.5Q_p$ and $0.75Q_p$, respectively as:

$$W_{50} = 5.6/(q_p)^{1.08} \tag{4.27}$$

$$W_{75} = 3.21/(q_p)^{1.08} \tag{4.28}$$

In literature, the coefficient values in Eqs. (4.27) and (4.28) are also reported differently. In place of 5.6, 5.87 is also used. However, the ratio between W_{75} and W_{50} remains 1.75. To take care of variation in coefficient values, it has to be ensured that the area under the UH is 1. It is to note further that coefficients C_T and C_p may also vary over a range of 10 times and Snyder's method may give reasonable flood estimates by deriving the relations for these coefficients in the region of interest.

Synthetic unit hydrograph for an ungauged watershed can be derived by determining the Snyder coefficients for the adjoining gauged watershed by comparing its derived unit hydrograph from observed data with that computed by Snyder's method. The values of Snyder's coefficients so obtained can then be applied to ungauged watershed to determine the unit hydrograph.

Example 4.11 The Snyder's parameters for a catchment are given as: $C_p = 0.65$, $C_T = 1.5$, $L = 25$ km, L_{CA} (distance from the outlet to a point on the stream, nearest to the centroid of the catchment in miles) = 12 km, and catchment area A = 1295 km^2. Using Snyder's method, develop a 2-hr UH for the catchment.

Solution

Time-to-peak of the UH, computed using Eq. (4.20) is:

$$t_{Lp} = 1.5 \times (25 \times 15)^{0.3} = 8.877 \text{ hr}$$

Therefore, time of standard excess-rainfall duration,

$$T_{SD} = \frac{8.877}{5.5} = 1.614 \text{ hr}$$

Since, $T_{SD} \neq T$ (= 2 hr.), Eq. (4.23) is used to calculate the adjusted time-lag, i.e.,

$$t_{pR} = 8.877 + \left(\frac{2.0 - 1.614}{4}\right) = 8.973 \text{ hr}$$

$$Q_P = \frac{2.778 \times 0.65 \times 1295}{8.973}$$

$$= 260.6 \text{ m}^3\text{/sec}$$

$$q_p = \frac{260.6}{1295} = 0.201 \text{ m}^3\text{/s/km}^2$$

Using Eqs (4.27) and (4.28),

$$W_{50} = 31.67 \text{ hrs}; \; W_{75} = 18.157 \text{ hrs}$$

t_B is calculated using Eq. (4.22) as equal to 72 + (3 × 8.877) = 98.63 hr. A smooth curve is drawn through these points to get a UH. This is shown in Fig. 4.2.

Example 4.12 In a hydrological homogeneous region, i.e., where the food producing mechanisms are similar for all the catchments, observed catchment characteristics are as follows:

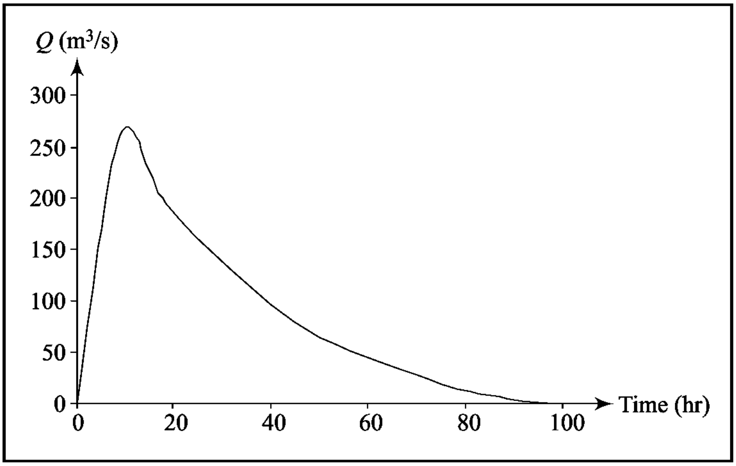

Fig. 4.2 *UH using Snyder's method*

Table 4.6

Catchment No. (hrs)	t_{Lp} *(hrs)*	*L* *(miles)*	L_{CA} *(miles)*	*S* *(%)*
1	28	231	116	0.6
2	34	229	96.3	0.46
3	24	161	70	0.83
4	22	148	57	1.03
5	18	123	60	0.43

Compute the value of C_T for this region.

Solution

Prepare Table 4.7 using Table 4.6 and plot a graph between Col (3) and Col (4)

Table 4.7 Computations for C_T

Catchment No.	t_{Lp} *in hrs.*	*log* (t_{Lp})	*log* $[L_{CA}/\sqrt{S}]$
1	28	1.447	4.649
2	34	1.531	4.680
3	24	1.380	4.132
4	22	1.342	3.913
5	18	1.255	4.234

Figure 4.3 shows $\log(L\,L_{CA}/\sqrt{S})$ and log (t_{Lp}) on X and Y axis, respectively. Fitting a straight line to the scattered points, the coefficients of the best-fit line equation gives the value of n and C_T as follows:

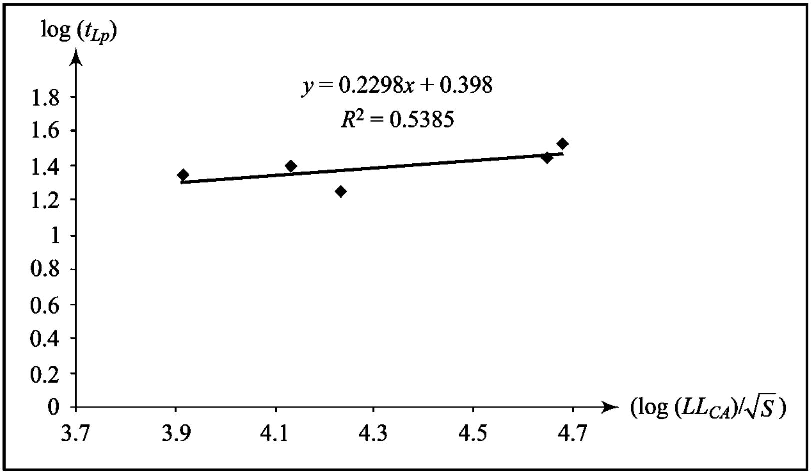

Fig. 4.3 *Best-fit-line between log-transformed ($LL_{CA}/\sqrt{S}$) and t_{Lp} values*

$$C_T = (10)^{0.398} = 2.5 \quad \text{and} \quad n = 0.2298$$

These are the computed regional parameters of Eq. (4.32). The coefficients can be used to calculate lag time-to-peak of a unit hydrograph where no data is available, e.g., if the catchment has $(L\,L_{CA}/\sqrt{S})$ value of 3.1, then $t_{Lp} = 2.5 \times (3.1)^{0.2298} = 3.242$ hours.

Example 4.13 In a hydrological homogeneous region, i.e., where the food producing mechanisms are similar for all the catchments, observed catchment characteristics are as follows:

Table 4.8 Catchment characteristics

Catchment No.	*A in Sq. miles*	*t_{Lp} in hrs.*	*Q_p in ft^3/s*
1	986	28	1101
2	700	34	496
3	430	24	417
4	354	22	401
5	336	18	301

Compute the value of C_p.

Solution

Taking log-transform of Eq (4.25), we have:

$$\log (Q_p) = \log (C_p) + \log (640A/t_{Lp})$$

Using the given data and following the same procedure as above, the best-fit-line between log (640 A/t_p) and log (Q_p) is shown in Fig. 4.4.

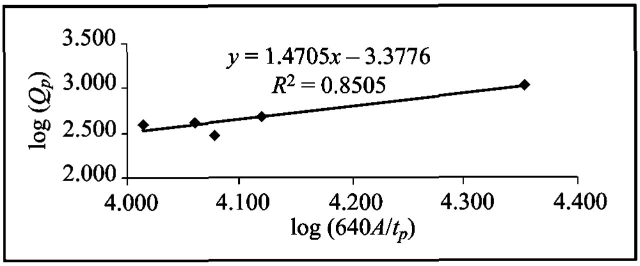

Fig. 4.4 *Best-fit-line between log-transformed ($640A/t_{Lp}$) and Q_p values*

Table 4.9 Computation procedure for C_p

Catchment No.	t_{Lp} *in hrs.*	*log* (Q_p)	*log* ($640A/t_{Lp}$)
1	28	3.042	4.353
2	34	2.695	4.120
3	24	2.620	4.059
4	22	2.603	4.013
5	18	2.479	4.077

From the best-fit-line equation $C_p = 10^{-3.3} = 0.0005$. It is to be noted that this value is catchment specific.

4.7.2 SCS Method

SCS method is also known as Soil Conservation Service method. The SCS method is widely used for estimating floods on small to medium sized ungauged drainage basins. In this method, the shape of SUH is determined from an average dimensionless Q/Q_p versus t/t_p hydrograph, as given in Table 4.10.

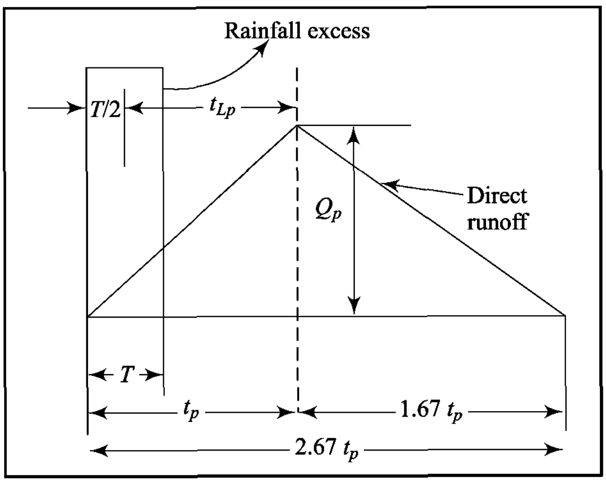

Fig. 4.5 *SCS triangular hydrograph*

Table 4.10 SCS dimensionless UH values of U.S. Dept. of Agriculture (1957)

t/t_p	Q/Q_p	t/t_p	Q/Q_p
0	0	1.5	0.66
0.1	0.015	1.6	0.56
0.2	0.075	1.8	0.42
0.3	0.16	2	0.32
0.4	0.28	2.2	0.24
0.5	0.43	2.4	0.18
0.6	0.6	2.6	0.13
0.7	0.77	2.8	0.098
0.8	0.89	3	0.075
0.9	0.97	3.5	0.036
1	1	4	0.018
1.1	0.98	4.5	0.009
1.2	0.92	5	0.004
1.3	0.84	5.5	0.002
1.4	0.75	6	0

The time-to-peak (t_p) and peak discharge (Q_p) are computed, respectively, as:

$$t_p = T/2 + t_{Lp} \tag{4.29}$$

and

$$Q_p = 484\, AV/t_p \tag{4.30}$$

where, t_{Lp} is the lag-time (hr) from centroid of rainfall to peak discharge Q_p, T is the duration of rainfall (hour), Q_p is in ft^3/s, A is the area in square miles, t_p is in hour, and V is in inches. V is the volume estimated from the rainfall which can be described as the net excess rainfall, i.e., $\frac{1}{2} Q_p t_B$, t_B is the base length of the hydrograph and is assumed equal to $2.67t_p$.

The lagtime (t_{Lp}) from the centroid of rainfall excess to the peak of the unit hydrograph is assumed to be 0.6 t_c, where t_c is the time of concentration. In SI units, with Q_p in m^3/s, basin size in km^2, and V in mm; the coefficient 484 in Eq. (4.30) is replaced by 0.208. The lag-time (t_{Lp}) in Eq. (4.29) is given by:

$$t_{Lp} = \frac{l^{0.8}(S_0 + 1)^{0.7}}{1900\ S^{0.5}} \tag{4.31}$$

where, l is the length to divide in feet, S is the average slope in percent, and S_0 is the potential maximum retention derived from curve number (CN) in inches as:

$$S_0 = (1000/\text{CN}) - 10 \tag{4.32}$$

The value of CN can be derived from National Engineering Handbook (NEH-4) tables (SCS, 1957), utilizing land use, soil type, hydrologic condition, and antecedent moisture condition of the watershed. The Soil Conservation Service Curve Number (SCS-CN) method uses the concept of Antecedent Moisture Condition (AMC) into three types: AMC I, a dry condition; AMC II, an average condition; and AMC III, a wet condition. Depending on the total 5-day antecedent rainfall (cm), Table 4.11 provides a basis for classification of AMC.

Table 4.11 Antecedent soil moisture condition (AMC)

AMC	*Total 5-day antecedent rainfall (cm)*	
	Dormant season	Growing season
I	Less than 1.3	Less than 3.6
II	1.3 to 2.8	3.6 to 5.3
III	More than 2.8	More than 5.3

Similarly, depending on the minimum infiltration rate, hydrologic soil group are of four types, as shown in Table 4.12.

Table 4.12 Description of hydrologic groups

Hydrologic Soil group	*Minimum infiltration rate (inch/hr)*
A	0.30–0.45
B	0.15–0.30
C	0.05–0.15
D	0–0.05

CN also depends on hydrologic conditions which are based on vegetation. Depending on the extent and characteristics of vegetal cover, Table 4.13 lists the classification used by SCS in computation of CN.

Table 4.13(a) Classification of hydrologic condition for woods

Vegetation condition	*Hydrologic condition*
Heavily grazed or regularly burned. Litter, small trees, and brush are destroyed	Poor
Grazed but not burned. Some litter exists, but these woods not protected.	Fair
Protected from grazing and litter and shrubs cover the soil.	Good

Table 4.13(b) Classification of hydrologic condition for native pasture

Vegetation condition	*Hydrologic condition*
Heavily grazed and no mulch or plant cover on less than ½ of the area.	Poor
Not heavily grazed and plant cover on less than ½ to ¾ of the area	Fair
Lightly grazed and plant cover on more than ¾ of the area	Good

Table 4.13(c) Classification of hydrologic condition for Open spaces, lawns, parks, golf courses, etc.

Vegetation condition	*Hydrologic condition*
Grass cover on 75% or more of the area	Good
Grass cover on 50% to 75% of the area	Fair

For known hydrologic conditions, land use and hydrologic soil group, Table 4.14 lists CN values for Antecedent Moisture Condition II. It is to note that CN values also differ with AMC type.

Table 4.14 Runoff curve numbers for hydrologic cover complexes

Land use description	*Hydrologic condition*	*Hydrologic soil groups*			
		A	*B*	*C*	*D*
Pasture	Poor	68	79	86	89
	Fair	49	69	79	84
	Good	39	61	74	80
Woods or Forest land	Poor	45	66	77	83
	Fair	36	60	73	79
	Good	25	55	70	77
Open spaces, lawns, parks, golf courses	Good	39	61	74	80
	Fair	49	69	79	84

For a given catchment, an average value of CN can be obtained. For example, if in a catchment of area A, a part of it, say A_1, has CN number as CN_1 and $(A - A_1)$ has CN number as CN_2, the average CN_{avg} is:

$$CN_{avg} = CN_1(A_1/A) + CN_2[(A - A_1)/A)] \tag{4.33}$$

With known Q_p, t_p, and the specified dimensionless UH, an SUH can be derived. A word of caution is necessary here regarding the use of SCS method. One has to check the method against observed flood data in each region where the method is applied.

Example 4.14 Compute the area-weighted curve number for a hypothetical watershed of two type of soils covering 500 sq. miles and 300 sq. miles and exhibiting curve number 60 and 45, respectively. Slope of watershed is 0.6% and hydraulic length of watershed is 10,000 ft. Net rainfall is 2 inches during the rainfall of 6 hr. Compute the parameters of the SCS triangular hydrograph.

Solution

The area-weighted CN is:

$$CN = \frac{500 \times 60 + 300 \times 45}{500 + 300} = 54.3 \approx 54$$

Potential maximum retention (S_0) is calculated from Eq. (4.32) as:

$$S_0 = \frac{1000}{54} - 10 = 8.518$$

The lag-time can be computed from Eq. (4.31) as:

$$t_{Lp} = \frac{10000^{0.8}(8.518 + 1)^{0.7}}{1900(0.6)^{0.5}} = 5.213 \text{ hr}$$

Time to peak is can be computed using Eq.(4.29) as:

$$t_p = \frac{6}{2} + 5.213 = 8.213 \text{ hr}$$

Base length of the hydrograph can be computed as:

$$t_B = 2.67\ t_p = 2.67 \times 8.213 = 21.92 \text{ hr}$$

The ratio of Peak discharge to volume of runoff can be computed using Eq. (4.30).

$$Q_p = \frac{484 \times 800}{8.213} \times 2$$

$$= 94289.54 \text{ ft}^3/\text{s}$$

The hydrograph is shown in Fig. 4.6.

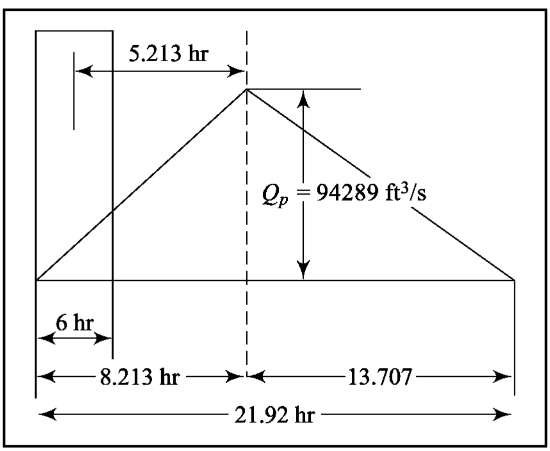

Fig. 4.6 *SCS triangular hydrograph*

In reality, the shape of hydrograph cannot be a triangle. So, one can get a more precise hydrograph using Table 4.10 by multiplying the value of Q_p with Q/Q_p and t/t_p with t_p to plot the SCS hydrograph. The values in Table 4.10 are also sensitive to different t/t_p values, as indicated by SCS. For detail, one can refer to comprehensive tables provided by SCS. Table 4.10 represents only the average variation between Q and t.

4.7.3 Geomorphological Instantaneous Unit Hydrograph (GIUH) Model

Figure 4.7 shows a typical catchment along with a network of streams of different orders. Order 1 stream is the starting stream. When two 1st order streams meet, the resultant stream is said to be of order 2. The order of a stream will not change unless a stream of same order joins. For example, if a 1st order stream meets a 2nd order stream, the resultant stream will remain as a 2nd order stream only. When two 2nd order streams join, the resultant stream is of order 3.

In Fig. 4.7, as the highest order stream is of 3rd order, such a catchment is also known as a order 3 catchment. It is to note that each stream irrespective of its order has a corresponding area from which the flow is contributed to it. Similarly, each stream order has a certain length associated with it. Of course, in a catchment, the number of streams of different orders will vary.

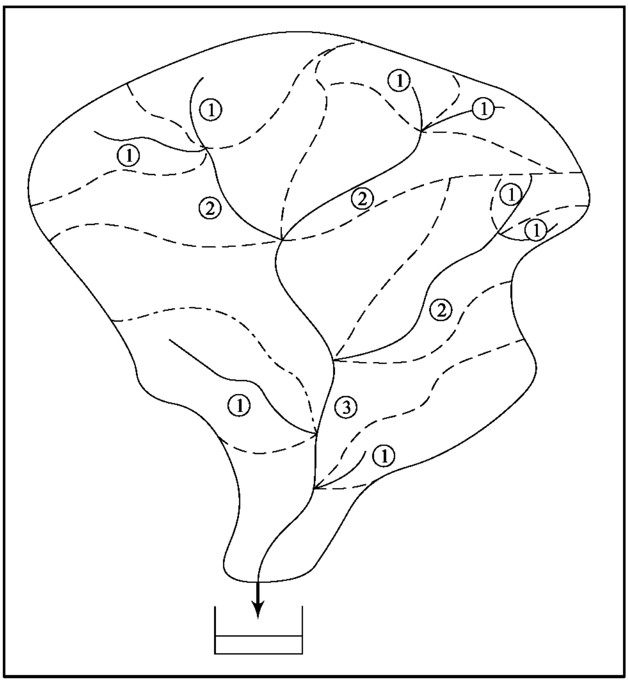

Fig. 4.7 *Third-order basin*

To obtain IUH from catchments with a given pattern of stream network, normally three geomorphic parameters R_A, R_B, and R_L representing the area ratio, bifurcation ratio, and the length ratio are obtained. These are represented as:

$$R_A = \frac{\lambda_i}{\lambda_{i-1}} \tag{4.34}$$

$$R_B = \frac{N_i}{N_{i+1}} \tag{4.35}$$

$$R_L = \frac{L_i}{L_{i-1}} \tag{4.36}$$

where, N_i is the number of streams of order i, L_i is the mean length of streams of order i, and λ_i is the mean area of the basin of order i.

The GIUH model, proposed by Rodriguez-Iturbe and Valdes (1979), can be interpreted as the PDF of the water travel times within the catchment, i.e.,

$$q(t) = \sum_{\text{All paths}} f_{Ts}(t) \times P(s) \tag{4.37}$$

where, $q(t)$ is the runoff ordinate of the IUH at any given time $t, f_{Ts}(t)$ is PDF of the total path travel times within the catchment, and $P(s)$ is the probability that a water droplet will follow the specific paths. Rodriguez-Iturbe and Valdes assumed the form of the PDF of total path travel times to be an exponential function.

$$f_{Ts}(t) = K_i\, e^{(-K_i t)} \tag{4.38}$$

where, K_i is the mean travel time in channels of order i, and can be approximated by:

$$K_i = v/L_i \tag{4.39}$$

where, v is the 'characteristic velocity', assumed constant throughout the network; and L_i is the average length of streams of order i.

A simplified procedure based upon two assumptions: (i) the shape of the IUH is taken to be triangular, and therefore fully specified by its peak, q_p, t_p, and t_B [T]; and (ii) the rate of excess rainfall is essentially constant throughout its duration. Under these conditions, relationships between the peak runoff and the time-to-peak of the GUH versus R_A, R_B, R_L (which are area, bifurcation, and length ratios, respectively), and vL^{-1} have been derived through regression analysis. These are given as follows:

$$q_P = \frac{1.31v}{L_\Omega R_L^{0.43}} \tag{4.40}$$

$$t_p = 0.44 L_\Omega\, R_B^{0.55}\, R_A^{-0.55}\, R_L^{-0.38}\, v^{-1} \tag{4.41}$$

where, Ω is the order of the basin, θ represents the slope of the line q_p (hr^{-1}) versus v (m/s), L_Ω is the scale variable in km, t_p is in hr, q_p is expressed in terms of depth of flow volume for unit time per unit excess rainfall [L T^{-1} L^{-1}].

$$t_B = 2/q_p \tag{4.42}$$

t_p and t_B are expressed in hours, v in km/hr, and L is in km.

The product of Eqs. (4.40) and (4.41) yields a dimensionless term $\beta = q_p\, t_p$, given as:

$$q_p\, t_p = 0.58\, (R_B/R_A)^{0.55}\, (R_L)^{0.05} \tag{4.43}$$

It is clear from Eq. (4.43) that β depends only on the catchment characteristics.

Example 4.15 For a third-order catchment, obtain the GIUH from the following data: $L_3 = 33.6$ km, $R_A = 4.844$, $R_B = 4.215$, and $R_L = 2.775$. Assume $v = 2$ m/s.

Solution

Using the data in Eqs (4.41) and (4.42), we have:

$$q_p = 0.05(1/\text{hr}), \quad t_p = 4.65 \text{ hr}, \quad \text{and} \quad q_p\, t_p = 0.232$$

With the known values of q_p ,t_p and t_b, the ordinates of the triangular GIUH can be obtained.

4.7.4 Two-Parameter Gamma Distribution for IUH

In general, a triangular shape of GIUH may not hold good. Based on the concept of n-linear reservoirs of equal storage coefficient (K), Nash and Dooge (1959) derived the instantaneous unit hydrograph in terms of a gamma function as given below:

$$q(t) = \frac{1}{K\Gamma(n)}\left(\frac{t}{K}\right)^{n-1} e^{\frac{-t}{K}} \tag{4.44}$$

Here, n and K define the shape of IUH and q is the depth of runoff per unit time per unit effective rainfall expressed in hr^{-1}. The mean and variance are described as:

$$\mu = n\,K; \quad \sigma^2 = n\,K^2 \tag{4.45}$$

Equation (4.44) is used for deriving SUH from known n and K. Chow (1964) relates n and K as:

$$K = t_p/(n-1) \tag{4.46}$$

Defining a non-dimensional parameter β as a product of q_p and t_p, Eqs. (4.44) and (4.46) are combined into the following simpler form:

$$\beta = \frac{(n-1)^{(n-1)}\, e^{-(n-1)}}{\Gamma(n-1)} \tag{4.47}$$

Equation (4.47) can be solved using Sterling's formula (Abramowitz and Stegun, 1964) expanding the gamma function as:

$$\Gamma(x) = (e^{-x})(x^{x-1/2})(2\pi)^{1/2}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} \cdots\right) \tag{4.48}$$

where, $x \rightarrow$ in $|\arg(x)| < \pi$

Substituting (n–1) by x in Eq. (4.47), and considering the first two terms of Eq. (4.48) for deriving $\Gamma(x)$, Eq. (4.47) leads to the following form:

$$\beta = \frac{x^x\, e^{-x}\sqrt{x}}{e^{-x}\; x^x\; \sqrt{2\pi}\,\sqrt{(1+1/12x)}} \cong \frac{\sqrt{x}}{\sqrt{2\pi(1+1/12x)}} \tag{4.49}$$

Assuming $\left(1+\dfrac{1}{12x}\right) \approx \left(1+\dfrac{1}{6x}\right)^{1/2}$, the above equation can be simplified as follows:

$$\beta \cong \frac{\sqrt{6x(x)}}{\sqrt{2\pi(1+6x)}} \tag{4.50}$$

which, on further simplification, can be written in the form of roots of a quadratic function as:

$$n \cong 1 + \pi\beta^2 + \beta\sqrt{1+(\pi\beta)^2} \tag{4.51}$$

A simple numerical procedure was used by Bhunya et al. (2003) to get an approximate solution of Eq. (4.47) given as:

$$\begin{aligned} n &= 5.53\,\beta^{1.75} + 1.04 \qquad &&\text{for } (0.01 < \beta < 0.35)\\ n &= 6.29\,\beta^{1.998} + 1.157 \qquad &&\text{for } (\beta \geq 0.35) \end{aligned} \tag{4.52}$$

These equations were derived using numerical simulation and optimization. For known values of q_p and t_p, n and K can be obtained. With known n and K, Eq. (4.44) can be used to generate a more realistic IUH shape in preference to a triangular one. The procedure to obtain SUH from an IUH is described in the preceding chapter. Bhunya et al. (2003) assumed that an IUH can be approximated as UH of unit duration. It is to note that for short durations, IUH can be approximated as UH. However, it may be advisable to check the validity of such assumptions under specific situations.

SUMMARY

In hydrological analysis, one is frequently confronted with a situation where synthetic data has to be generated. In this context, use of random numbers is a prerequisite. With this in view, a variety of random number generators have been used.

The importance and application areas of simulation are discussed in this chapter. Use of simulation has been increasing in many hydrological investigations, Through a detailed analysis of rainfall-runoff in a catchment, ways to derive unit hydrographs of varying durations were the focus in the preceding chapter. However, one faces the situations where due to paucity of data on rainfall-runoff, other methods of hydrograph derivation are needed. Such approaches which can cope well with limited amount of data are also described herein. We have also covered the specific approaches adopted in different countries and regions for synthetic unit hydrograph generation.

EXERCISES

4.1 What are the methods available for random number generation? Write appropriate algorithms for generating the same.

4.2 Explain the procedure of deriving synthetic hydrograph using Snyder's method.

4.3 What is the necessity of Monte Carlo simulation in hydrology?

4.4 What is GIUH? Consider a test catchment and apply GIUH.

4.5 Assuming $x_0 = 0.14$, $A = 0$, $B = 11$, generate five real number fractions up to two digit. (Hint: $p = 2$, which means $c = 10^{-2}$ [11.0] = 0.11)

[*Ans.* 0.54, 0.94, 0.34, 0.74, 0.14]

4.6 Suppose the standard uniform random variables are given as: 0.231, 0.132, and 0.611. Transform them into a distribution given by:

$$F(x) = \begin{cases} 0 & (x < 0) \\ (x+1)/3 & (0 < x < 2) \\ 1 & (x > 2) \end{cases}$$

[Hint: $F^{-1}(a) = 3/(a+1)$ for $(0 < a < 2)$; 1 for $(a > 2)$]

[*Ans.* 2.437, 2.65, 1.862]

4.7 It is required to generate a random number applying Poisson distribution with $\lambda = 2$. Suppose we have already generated standard uniform random numbers, such as: 0.516, 0.918, 0.40, and 0.13. Using the procedure given in Example 4.4, generate four Poisson variates.

[*Ans.* 2, 4, 1, 0]

4.8 Generate five numbers using the exponential distribution function whose CDF is given by:

$$F(x) = 1 - \exp\left[-\frac{x}{1.3}\right]$$

(Hint: generate random number and substitute for F to estimate x.)

4.9 Derive a synthetic unit hydrograph using gamma distribution. The salient points of the UH like q_p and t_p may be obtained from GIUH model. Compare the UH with SCS method. The following geomorphological data may be used: L = 32 km, R_A = 3.844, R_B = 3.415, R_L = 2.225. Assume v = 2 m/s.

4.10 Derive a synthetic unit hydrograph using GIUH method using the data given in the previous question.

4.11 What do you feel is the best method for deriving a UH under no data availability situations? Give appropriate reasons to justify your answer. What are the advantages of GIUH over two-parameter gamma distribution methods?

OBJECTIVE QUESTIONS

1. The word *seed* in generating the random numbers in the *linear congruential generator* (LCG) is
   (a) Initial number (b) Final number
   (c) Any number (d) None of these

2. A good random number generator
   (a) Should produce uniformly distributed numbers that don't have any correlation with each other
   (b) Should be fast
   (c) Should not require large memory
   (d) All of the above

3. Let $a = 3$, $b = 4$, $d = 20$, and $R_0 = 0$. The random numbers generated are
   (a) 1, 2, 3, 4 (b) 4, 8, 2, 6 (c) 2, 4, 6, 8 (d) 1, 3, 5, 7

4. Among the following which one is used as a random number generators
   (a) Linear congruential generator
   (b) Additive congruential generator
   (c) Quadratic congruential generator
   (d) All of the above

5. Snyder's synthetic unit hydrograph is applicable to
   (a) Small catchments (b) Large catchments
   (c) Moderate size catchments (d) Catchments of any size

6. Basin lag is the time dfference between the
   (a) Rainfall and runoff of two basins
   (b) Excess rainfall and surface runoff
   (c) Centroid of excess rainfall and surface runoff
   (d) None of the above

7. Which is/are the parameter(s) that Snyder used to describe the hydrograph using data from the catchments of Appalachian Highlands?
   (a) Lag-to-peak (b) Peak discharge
   (c) Base time (d) All of these

8. Using Snyder's synthetic hydrograph, calculate the basin lag in hours if the length of the main stream from the outlet to the catchment boundary is 5 miles, and the distance from the outlet to a point on the stream nearest to the centroid of the catchment is 3 miles. Assume coefficient is 2.
   (a) 4.0 (b) 4.5 (c) 5.0 (d) 6.0

9. Which is not a synthetic unit hydrograph?
   (a) Snyder's method (b) SCS method
   (c) Gray method (d) IUH method

10. SCS method assumes the hydrograph of
    (a) Triangular shape (b) Circular shape
    (c) Parabolic shape (d) Any shape

11. Gray method is based on
    (a) Incomplete beta distribution
    (b) Incomplete gamma distribution
    (c) Incomplete alpha distribution
    (d) None of the above

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CHAPTER

5

Rainfall-Runoff Relationships and Flood Routing

5.1 INTRODUCTION

Many processes that are of interest from a hydrologic point of view are often difficult to observe routinely and unambiguously. Streamflow measurement is one such variable that can only be measured at a gauging site of a basin with some confidence. However, from a broader perspective, major river basins or catchments, especially in developing countries like India, have been gauged for determination of hydrological variables, while medium and small size catchments are mostly ungauged. Several major catchments in different parts of the country still remain ungauged; and in some catchments, the existing gauging networks are being discontinued due to economic constraints, lack of regular manpower, and inaccessible reasons. Thus, there is a need for methods that can be utilized for realistic estimation of such hydrological variables in ungauged catchments.

One of the popular methods is to make use of the available rainfall-runoff data to develop a relationship and use the same for extrapolating the rainfall series to generate runoff. These relationships, at times, are used for homogeneous regions where flood-producing patterns are similar. In this chapter, we will discuss how and why these relationships are developed. While such relationships are useful to estimate mean flood, a more rigorous approach is often needed which involves routing of runoffs at different levels of a catchment to produce a flood hydrograph at the outlet of a catchment. In such an approach, runoff from different sub-catchments needs to be routed through a network of channels. With this in view, concepts of routing are also covered here.

5.2 DEVELOPMENT OF RAINFALL-RUNOFF RELATIONSHIPS

One may often need to develop such relationships in case the existing approaches may not hold good. In this section, we will review the existing approaches and develop procedures to establish new relationships. In many real field problems, stream flow records are short or rarely available, and in such cases, they need to be extended.

In general, the following data situations or scenarios are encountered:

(i) long-term precipitation record along with a streamflow data for a few years at the site are available;

(ii) long-term precipitation record is available at the site along with precipitation and streamflow data for a few years at a neighboring site;

(iii) only precipitation record at the site is available; and

(iv) no record of any kind is available.

All these cases require a mathematical method that can be used for the available data and extend the streamflow series. A common method is to use simple models based on linear regression, which will be discussed in the following pages. To emphasize the importance of developing rainfall-runoff relationships, we begin with the existing approaches.

5.2.1 Existing Approaches

These fall into two categories. In the first, the objective is to compute total runoff using total rainfall in a given time. In the second type, the emphasis is to compute only peak value of runoff.

5.2.1.1 Total runoff in relation to total rainfall

Variety of methods to relate total runoff and rainfall are available in India alone. Many of these are empirical in nature and these tend to be linear or non-linear relationships. Here, we discuss only two methods, viz., SCS method and Khosla's method because of their relevance and adaptability at a larger scale.

SCS Method SCS method relates the direct runoff Q with the rainfall P as follows:

$$Q = \frac{(P - I_a)^2}{P - I_a + S_0} \quad \text{when} \quad P \geq I_a; \quad S_0 \geq (I_a + F) \tag{5.1}$$

$$F = P - I_a - Q \tag{5.2}$$

where, I_a is initial abstractions, S_0 is storage potential of soil, and F is the infiltration. The empirical relation $I_a = 0.2S_0$ is the best approximation and thus, Eq. (5.1) converts to:

$$Q = \frac{(P - 0.2S_0)^2}{P + 0.8S_0} \tag{5.3}$$

Khosla's Formula Khosla's formula relates the runoff and rainfall through an empirical relationship which is applicable for catchments of both India and USA. Generation of synthetic runoff data from historical rainfall and temperature data is also possible from the Khosla's formula. The relationship for monthly runoff is:

$$R_m = P_m - L_m \tag{5.4a}$$

$$L_m = 0.48T_m \quad \text{for} \quad (T_m > 4.5°\text{C}) \tag{5.4b}$$

where, R_m is monthly runoff in cm and $R_m \geq 0$, P_m is monthly rainfall in cm, L_m is monthly losses in cm, and T_m is mean monthly temperature of the catchment in °C. For $T_m \leq 4.5$°C, the loss L_m with mean monthly temperature is shown in Table 5.1.

Table 5.1 Variation of monthly loss with mean monthly temperature of the catchment

T(°C)	4.5	–1	–6.5
L_m	2.17	1.78	1.52

5.2.1.2 Relationships for Peak Runoff Computations

Any precipitation event over a catchment may lead to occurrence of a runoff which varies with time. In addition to knowing the total runoff, hydrologists are also interested to know the peak runoff, as it is the peak runoff which may act as a precursor to a flood. Here, we cover rational approach and empirical approaches for peak runoff computations

Rational Approach The rational method is developed in mid-nineteenth century. The rational approach considers the following

1. The peak rate of runoff is a function of the average rainfall rate during the time of concentration, which is the time taken for the water to reach at the outlet from the farthest point of the catchment.
2. Rainfall intensity is constant during the rainfall.

Peak runoff Q_p from a catchment of area (A) for a given rainfall intensity (i) can be computed using:

$$Q_p = CAi \quad \text{for} \quad (t \geq t_c) \tag{5.5}$$

In Eq. (5.5), C is a runoff coefficient representing surface characteristics and ranges between 0 and 1. It is to note that for an impervious surface, C equals 1. Sometimes, drainage areas consist of more than one area of different drainage characteristics. Under such conditions,

$$Q_p = i \sum_{j=1}^{m} C_j A_j \tag{5.6}$$

In Eq. (5.6), m is the number of sub-catchments, Q_p is peak discharge in m^3/s, C is coefficient of runoff, $i_{t_c, p}$ is the mean intensity of precipitation in mm/hr for a duration equal to time of concentration t_c and an exceedence probability p, A is drainage area in km^2. To estimate t_c, some empirical formulae are listed in Table 5.2.

Table 5.2 List of formulae to estimate time of concentration t_c (*Source:* Subramanya, 2003)

Name of formula	*Time of concentration*	*Comments*
As per Linsley et al. (1958)	$t_c = C_{tL}\left(\frac{L\,L_{ca}}{\sqrt{S}}\right)$	t_c is time of concentration in hours, C_{tL} and n are basin constants, L = basin length in km, L_{ca} is the distance along the main water course from gauging station to a point opposite the watershed centroid in km, S is the basin slope
Kirpich Equation (1940)	$t_c = 0.01947\ L^{0.77}\ S^{-0.385}$	t_c is time of concentration in minutes, L is maximum length of travel of water (m), s is the slope of catchment

It is to note that for estimation of peak runoff, duration of rainfall must be equal to or greater than the time of concentration. In case of non-availability of rainfall data, rainfall-frequency duration relationship for the given catchment area can be used. As per this relationship, the rainfall intensity corresponding to a duration t_c and return period T can be expressed as:

$$i = \frac{aT^c}{(t_c + b)^m} \tag{5.7}$$

where, a, b, c, and m are constants. It is to note that a, b, c, and m are catchment specific. The peak runoff corresponding to the occurrence of precipitation for a time equal to or greater than time of concentration is also termed as *peak flood*. Further details on rational method including variability of runoff coefficient (C) can be had from Chow et al. (1988). The functional form given by Eq. (5.7) is not unique.

Empirical Relationships The empirical relationships are generally applicable for a particular region where the relationships are developed by taking into account the effect of data of that region. In all other areas, it can give best approximate values. Table 5.3 shows some of the empirical relations developed for Indian conditions.

Table 5.3 List of formulae to estimate peak discharge for Indian conditions (*Source*: Subramanya, 2003)

Name of formula	*Peak discharge*	*Comments*
Dickens' Formula (1865) (For North-Indian plain, Central India, coastal Andhra, and Orissa)	$Q_p = C_D A^{3/4}$	Dickens' constant C_D varies from 6–30, Q_p = maximum flood discharge (m^3/s), A is the catchment area (km^2); $C_D = 6$ for North-Indian plain, 11–14 for North-Indian hilly regions, 14–28 for Central India, 22–28 for Coastal Andhra and Orissa
Ryves Formula (1884) (For Tamil Nadu and parts of Karnataka and Andhra Pradesh)	$Q_p = C_R A^{2/3}$	Ryves coefficient $C_R = 6.8$ for areas within 80 km from east coast, 8.5 for areas which are 80–160 km from east coast, and 10.2 for limited areas near hills
Inglis Formula (1930) (For Western Ghat in Maharastra)	$Q_p = \frac{124A}{\sqrt{A+10.4}}$	Q_p = maximum flood discharge (m^3/s), A is the catchment area (km^2)

Example 5.1 A 20 cm storm occurred for 6 hrs in a catchment having a CN of 50. Estimate the net rainfall in cm using SCS method.

Solution

Potential maximum retention for the catchment is:

$$S_0 = \frac{1000}{50} - 10 = 10$$

Considering $I_a = 0.2S_0$, the net rainfall is:

$$Q = \frac{(20 - 0.2 \times 10)^2}{20 + 0.8 \times 10} = 11.57 \text{ cm}$$

Example 5.2 For a catchment in India, the mean monthly rainfall and temperature is given in Table 5.4. Calculate the annual runoff and the annual runoff coefficient by Khosla's formula.

Solution

Table 5.4 Mean monthly rainfall and temperature for Example 5.2

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Temp (°C)	8	10	15	24	30	35	32	30	30	22	15	10
Rainfall (cm)	2	1	0	0	1	8	25	30	10	4	1	1
Monthly losses	3.84	4.8	7.2	11.52	14.4	16.8	15.36	14.4	14.4	10.56	7.2	4.8
Runoff (cm)	0	0	0	0	0	0	9.64	15.6	0	0	0	0

Annual runoff = 9.64 + 15.6 = 25.24 cm
Annual rainfall = 83cm

$$\text{Annual runoff coefficient} = \frac{\text{Annual runoff}}{\text{Annual rainfall}} = \frac{25.24}{83} = 0.304$$

Example 5.3 Estimate the maximum flood flow for a catchment having an area of 50 km^2 by using empirical formula for the following regions.

(i) Catchment located in Western Ghat, Maharastra
(ii) Catchment located in coastal Andhra and Orissa
(iii) Catchment located in Tamil Nadu region

Solution

(i) For Western ghat, Maharastra, Inglis formula is used to calculate peak discharge.

$$Q_p = \frac{124 \times 50}{\sqrt{50 + 10.4}} = 797.76 \text{ m}^3\text{/sec}$$

(ii) If the catchment is located in coastal Andhra and Orissa, Dickens' formula is used. Assume Dickens' constant as 25.

$$Q_p = 25 \times 50^{3/4} = 470.07 \text{ m}^3\text{/sec}$$

(iii) If the catchment is located in Tamil Nadu region, Ryve's formula is used. Using Ryve's constant as 8.5:

$$Q_p = 8.5 \times 50^{2/3} = 115.51 \text{ m}^3\text{/sec}$$

It is to note that there is lot of variability in estimation of Q_p for the same size of catchment area and each region has a different relationship. If such relationships are not available at a given site, development of new rainfall—runoff relationships become very important.

5.2.2 Linear and Nonlinear Rainfall-Runoff Relationship

In case of linear regression, a statistical correlation between observed monthly rainfall and monthly runoff is established. The relationship is then plotted in a log-log graph for each month. In the next step, a straight line is fitted using the linear relationship as follows:

$$Q = a\,P - C \tag{5.8}$$

where, Q = runoff in mm, cm, or feet; P = rainfall in mm, cm, or feet; C is a coefficient to be determined that accounts for losses; and a is a reduction factor which accounts for losses such as interception, depression storage, etc. The following example shows the application of this method.

Example 5.4 Data of monthly rainfall and runoff are available, as shown in Table 5.5. Derive a linear *P-Q* relationship using this data.

Table 5.5 Data for Example 5.4

Month	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
P (cm)	4	40	30	25	20	15	10	5	50
Q (cm)	0.2	9	5	4.5	2.5	2	1	0.5	14

Solution

The plot between P and Q is shown in Fig. 5.1. Also, the best-fit line is shown along with its equation.

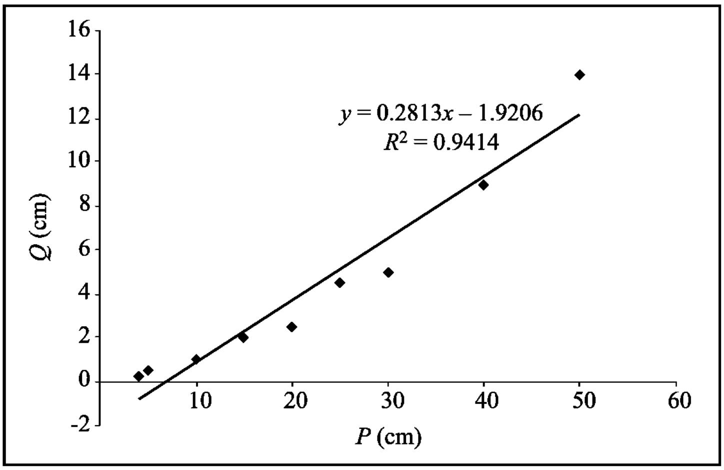

Fig. 5.1 *Line fitted to the data of Example 5.4*

The fitted relationship is obtained using EXCEL spreadsheet which reads:

$$Q = 0.2813\ P - 1.9206$$

When the rainfall and runoff data do not follow a linear fit, then the following relationship is used.

$$Q_t = K\,(P_t - P_0)^n \tag{5.9}$$

where, K and n are constants, and P_0 is the initial loss. The following example shows the application of this method.

Example 5.5 Develop a nonlinear *P–Q* relationship using the data given in Table 5.6. Assume P_0 as 1.2 cm.

Table 5.6 Data for Example 5.5

Month	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
$P - P_0$	2.8	38.8	28.8	23.8	18.8	13.8	8.8	3.8	48.8
Q (cm)	0.2	9	5	4.5	2.5	2	1	0.5	14

Solution

An EXCEL spreadsheet is used to develop the equation for the given data, as follows:

$$\log(Q) = 1.368 \log (P - P_0) - 1.2449$$

This can be transformed to:

$$Q = 0.0568 (P - 1.2)^{1.368}$$

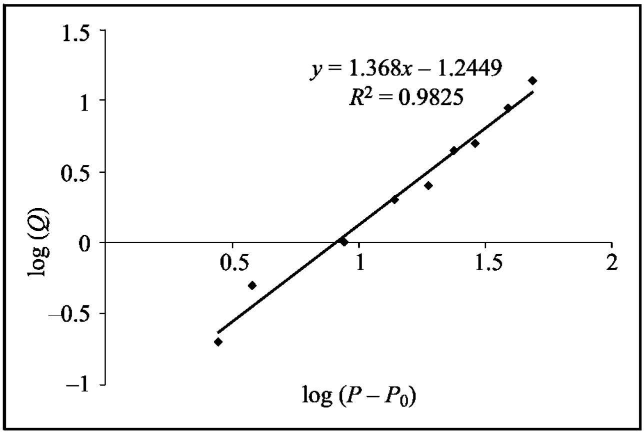

Fig. 5.2 *Line fitted to the transformed logarithmic data of Example 5.5*

It can be observed that goodness-of-fit, i.e., R^2 has increased from 0.94 in linear model to 0.98 in case of the nonlinear model.

A better rainfall-runoff relationship is obtained by accounting for the effect of preceding month's rainfall using the following relationship:

$$Q_t = a + b\,P_t + c\,P_{t-1} \tag{5.10}$$

where,

Q_t = Volume of flow in the t^{th} month,

P_t = Rainfall in the t^{th} month,

P_{t-1} = Rainfall in $(t-1)^{\text{th}}$ month,

a = Constant (always a negative quantity to account for the initial losses), and b and c are constants which are non-negative and less than one.

Example 5.6 The monthly mean rainfall of a basin for the month of June and July are given in Table 5.6 along with the corresponding runoff for July. Use the available data for the period 1980–89 to develop a rainfall-runoff relationship and estimate the runoff for the years 1990 and 1991.

Table 5.7 Calculations of Example 5.6

Year	*Monthly rainfall in mm*		*Monthly runoff in $m^3 \times 10^3$*	*Estimated using*	
	June	*July*	*July*	*Eq. (5.10 a)*	*Eq. (5.10 b)*
1980	296	385	1304	766.461	1101.615
1981	136	493	1030	1127.894	1022.487
1982	217	427	770	907.0182	1027.053
1983	180	445	652	967.257	986.895
1984	182	446	1172	970.6036	995.249
1985	104	273	227	391.6418	249.487
1986	118	393	805	793.2338	662.107
1987	104	467	991	1040.882	857.483
1988	112	272	222	388.2952	267.233
1989	230	328	756	575.7048	750.717
1990	211	234	—	261.1244	406.531
1991	323	553	—	1328.69	1698.597

Solution

As the objective is to compute runoff for the years 1990 and 1991, the same can be achieved by calibrating the runoff and rainfall data given in Table 5.3. To estimate runoff, two types of relationships have been calibrated. The first type of relationship is given by Eq. (5.10 a).

$$Q_{\text{July}} = -521.983 + 3.346\, P_{\text{July}}, \tag{5.10a}$$

Correlation coefficient (R^2) = 0.5317

The second type of relationship is a multiple linear relationship involving input variables as the rainfall data for the months of June and July. This relationship is given by Eq. (5.10 b).

$$Q_{\text{July}} = -877.535 + 2.613\, P_{\text{June}} + 3.134\, P_{\text{July}} \tag{5.10b}$$

Correlation coefficient (R^2) = 0.75

The estimated mean flow for the years 1990 and 1991 are shown in Table 5.7.

5.2.3 Extension of Stream Flow Record

At many sites in our country, the long-term data required for the design and planning of water resources projects are not available. The only options available for the hydrologist are to:

1. extend the short-term data somehow, or
2. generate the data with the properties of observed historical data.

Some popular methods used under such circumstances are:

1. Double-mass curve method
2. Correlation with catchment areas
3. Regression analysis between the flows at base and index stations

Double-mass Curve Method The steps followed in this method are as follows:

1. Plot the cumulative stream flow of the base station under consideration and the index station on a graph.
2. The slope of the double-mass curve gives the relation of the stream flows at two stations.

The following example illustrates this method.

Example 5.7 Stream flow data for two stations are given in Table 5.8. Determine the annual flow for 1987 and 1988 at Station-2 using the double-mass curve method.

Table 5.8 Annual streamflow data at two stations

Year	*Annual flow at Station-1 (Mm^3)*	*Annual flow at Station-2 (Mm^3)*	*Cumulative annual flow at Station-1 (Mm^3)*	*Cumulative annual flow at Station-2 (Mm^3)*
1980	94	54	94	54
1981	82	43	176	97
1982	45	23	221	120
1983	20	10	241	130
1984	26	16	267	146
1985	43	24	310	170
1986	90	50	400	220
1987	110			
1988	86			

Solution

Table 5.8 shows the calculations for cumulative values of annual flow values at stations 1 and 2.

For example, in 1981, cumulative value for Station-1 is 94 + 82 = 176; in 1982, it is 176 + 45 = 221; and so on.

The cumulative values of annual flows at stations 1 and 2 are used to plot the graph in Fig. 5.3. The equation of the straight line fit is:

$$Q(\text{Station-2}) = 0.5435\ Q(\text{Station-1}) + 1.1556\ ; \quad R^2 = 0.9993$$

This derived equation is used to calculate annual flow values in 1987 and 1988 as:

$$Q(\text{Station-2 in 1987}) = 0.5435 \times (110) + 1.1556 = 60.94$$

$$Q(\text{Station-2 in 1988}) = 0.5435 \times (86) + 1.1556 = 47.89$$

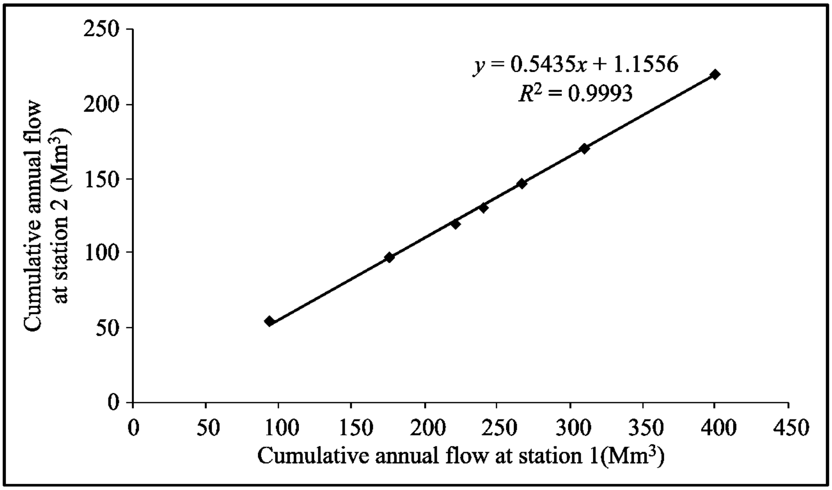

Fig. 5.3 *Cumulative annual flow values*

Correlation with Catchment Areas

$$Q_s = Q_i\,(A_s/A_i)^a \tag{5.11}$$

where, Q is the flow values and A is the catchment area, and a is a coefficient which is generally taken as 0.75. It may be noted here that the area unit does not have any effect on the final outcome of Q.

For example, if monthly flow at station-1 is Q_1 = 100 Mm^3, area of catchment-1 is A_1 = 500 km^2, and area of catchment-2 is A_2 = 50 km^2; then monthly flow at Station-2 is:

$$Q_2 = (Q_1)\,(A_2/A_1)^a = 59.46\text{ Mm}^3$$

Regression Analysis between Flows at Base and Index Stations

If Q_s is dependent variable and Q_i is independent variable, then:

$$Q_s = a + bQ_i \tag{5.12}$$

where, a and b are regression coefficients and can be obtained by using linear regression analysis.

5.3 FLOW DURATION CURVES

Flow duration curves are useful for estimating available water resources for different uses, such as hydropower generation, study of the flood control, computing the sediment load and dissolved solid load of a stream, design of the drainage system, comparing the adjacent catchments with a view to extend streamflow data, planning of water resources engineering projects, etc. They also reflect the nature of the streams. Whenever adequate lengths of records are available, the flow duration may be developed, analysing the available flow records at the site. The length of data required for the development of flow duration curves depends upon the type of scheme, type of development, and variability of inputs. General guidelines regarding the minimum length of data required for some of the projects are given below in Table 5.9.

Table 5.9 Guidelines for the minimum length of data required for flow duration curves

Type of Project	*Minimum length of data*
Diversion project	10 yrs.
Within the year storage projects	25 yrs.
Over the year storage projects	40 yrs.
Complex systems involving combination of above	Depending upon the predominant element

The above guidelines are only illustrative and not exhaustive. Sometimes the rainfall records are available for a long period; however, the runoff records are available for a short period. In such a situation, the rainfall-runoff relationships may be developed based on the available rainfall-runoff records for the concurrent period. Then, these relationships are used to generate the long-term runoff records corresponding to the available long-term rainfall records. Often, all such data are not generally available, and it becomes necessary to use data of nearby site(s) also. Considering these aspects, the data requirements for major water resources projects may be summarized as:

(a) Streamflow data of the desired specific duration (daily, weekly, monthly, etc.) at the proposed site for at least 40 to 50 years, or
(b) Rainfall data of specific duration for at least 40 to 50 years for rain gauge stations influencing the catchment of the proposed site as well as streamflow data of specific duration at the proposed site for the last 5 to 10 years, or
(c) Rainfall data of specific duration for the catchment of the proposed site for the last 40 to 50 years and flow data of the specific duration and concurrent rainfall data of the existing work on a nearby river or located at upstream or downstream of the proposed site for the last 5 to 10 years or more; provided orographic conditions of the catchment at the works are similar to that of the proposed site.

In case the flow data are disturbed because of construction of water resources projects upstream of the gauging site, the operational data such as reservoir regulation which include the outflows from spillway and releases for various uses is required. If the flow data time series consists of the records for the period prior as well as after the construction of the structure, the flow series is considered to be non-homogeneous. Necessary modifications have to be made to the records in order to make them homogeneous.

For the development of flow duration curve and computation of dependable flows for ungauged catchments, some important catchment and climatic characteristics are required. The catchment characteristics are derived from the toposheet covering the drainage area of the catchment. The basic statistics such as mean and standard deviation of the rainfall data represent the climatological characteristics.

5.3.1 Development of Flow Duration Curves

Case 1

For gauged catchments, if the available data correspond to situation (a), as discussed, the flow duration curves from daily flow data may be developed in the following steps:

(i) Choose a constant width class interval (c_i) such that about 25 to 30 classes are formed.
(ii) Assign each day's discharge to its appropriate class interval.
(iii) Count the total number of days in each class interval.
(iv) Cumulate the number of days in each class interval to get the number of days above the lower limit of each class interval.
(v) Compute the probabilities of exceedence dividing the quantities obtained from step (iv) by the total number of days in the record (for example, 365 if one year record is considered for the construction of flow duration curve).
(vi) Multiply the probabilities of exceedence obtained from step (v) by 100 to get percentage exceedence.
(vii) Plot the probabilities of exceedence in percentage against the corresponding lower bound of class interval on linear graph paper. Sometimes the flow duration curve better approximates to a straight line if lognormal probability paper is used in place of linear graph paper.

Case 2

In case the data items are not sufficient enough to define the class intervals, the flow duration curves (from monthly flow data or any other duration larger than 'daily') may be developed in the following steps:

(i) Arrange the flow data in descending order
(ii) Assign the probability of exceedences to each data item obtained from step (i) using the *Weibull plotting position formula*:

$$P = \frac{m}{N+1} \times 100 \tag{5.13}$$

Here, $m = 1$ for the highest flow values, and N is the number of data items (or variate).

Note: If the flow duration curve is required to be linearized on normal probability paper or lognormal probability paper, the probability of exceedences may be assigned using *Blom's plotting position formula*:

$$P = \frac{m - 0.375}{N + 0.250} \times 100 \tag{5.14}$$

(iii) Plot the ranked flow values against the probabilities of exceedence (computed using Eq. 5.13) on linear graph paper to get the flow duration curve.

Note: Use normal probability paper if the required dependable flow (or probability of exceedence) is to be extrapolated. Try to fit either normal distribution or lognormal distribution in order to linearize the flow duration curve. Here the probabilities of exceedence may be computed using Eq. (5.14) for the purpose of plotting. Fitting of other theoretical frequency distribution may also be tried.

Case 3

If limited runoff data and long series of rainfall data are available for site, the steps are:

(i) Develop the rainfall-runoff relationship for the specific duration utilizing the available data for the concurrent period.
(ii) Compute the long-term flow data of the specific duration using the developed relationship at step (i) and long-term available rainfall data.
(iii) Develop the flow duration curve using the procedure stated in Case 1 or Case 2.

Case 4

If no data is available, then the following steps may be used for the development of the flow duration curves.

(i) Develop the rainfall-runoff relationship for the existing site for the specific duration, analyzing the available rainfall-runoff records of concurrent periods.
(ii) Develop the flow duration curve using the procedure described either in Case 1 or Case 2.
(iii) Divide the flow values of flow duration curve by the catchment area of the existing project site.
(iv) Multiply the flow values obtained from step (iii) by the catchment area of the proposed site for which the flow duration curve is required to be developed.

Example 5.8 Develop a flow duration curve for a streamflow data given in first two columns of Table 5.10.The daily flow data are tabulated in eight class intervals of 50 TCM (thousand cubic metres).

Table 5.10 Daily flow at the proposed site for 36 years

Class interval (TCM)	*No. of days flow in each class interval*	*Cumulative no. of days*	*Probability of flow in the class interval equaled or exceeded* $P_p = \left(\frac{m}{N+1}\right) \times 100\%$
> 350	2	2	5.405
300–350	3	5	13.513
250–300	2	7	18.918
200–250	3	10	27.027
150–200	4	14	37.837
100–150	5	19	51.351
50–100	7	26	70.270
0–50	10	36	97.297

Solution

Total number of days, $N = 36$

Considering the first data in Col. (3), $m = 2$

First entry in Col. 4 = 100[2/(36+1)] = 5.405. This way, all the entries are completed in Column 4.

A plot of data pairs (350, 5.405), (300, 13.513) ··· (50, 70.27) is shown in Fig. 5.4. As marking of Q with zero is not possible, a very small value of Q as 1 m^3/s is taken corresponding to a P_p value of 97.297 and the same is shown in Fig. 5.4.

As the data in the first column is in descending order, Columns 1 and 4 are used to develop Fig. 5.4.

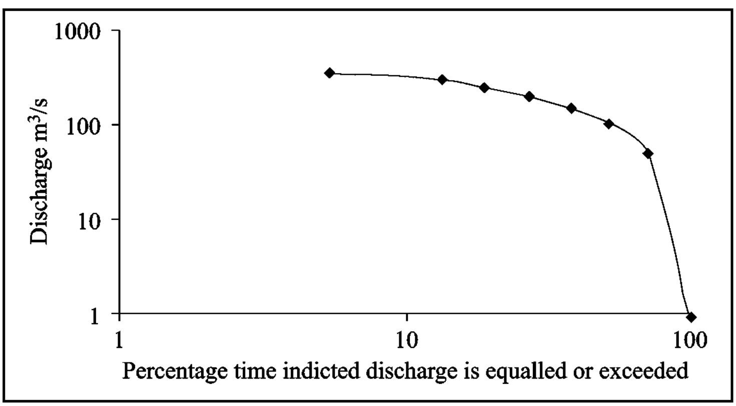

Fig. 5.4 *Flow duration curve*

Example 5.9 Mean monthly flows along with catchment area and length of stream for 5 catchments in a homogeneous region are given in Table 5.11. Obtain dependable flows with different probabilities in an ungauged catchment of the region having a catchment area of 1000 km^2 and stream length equal to 60 km.

Table 5.11(a) Mean monthly flows in m^3/s

Catchment-1	*Catchment-2*	*Catchment-3*	*Catchment-4*	*Catchment-5*
24	17	9	21	11
14	22	17	29	22
18	23	23	33	32
35	37	54	65	41
37	23	68	132	33
46	56	143	156	76
67	69	243	123	78
100	112	116	98	198
57	58	98	78	76
32	44	56	45	44
22	13	23	33	34
12	10	17	11	10

Table 5.11(b) Catchment characteristics

	Catchment-1	*Catchment-2*	*Catchment-3*	*Catchment-4*	*Catchment-5*
A (km^2)	450	520	1034	546	475
L (km)	50	53	65	57	43
Q_{mcan}(m^3/s)	38.66	40.3	72.35	68.7	54.8

Solution

We can use the following steps to obtain dependable flows:

1. Mean flow of each catchment is calculated and given in Table 5.11(b).
2. Q/Q_{mean} is calculated for each catchment.
3. Arrange all the Q/Q_{mean} value in descending order in an excel spreadsheet. Determine the probability of the Q/Q_{mean} value being equaled or exceeded.
4. Determine the best fitting curve of the graph between probability of Q/Q_{mean} value being equaled or exceeded vs Q/Q_{mean} and determine the value of R^2.
5. Determine the value of Q/Q_{mean} for different probability which is shown in Table 5.11(c).
6. Multiply the value of Q_{mean} for ungauged catchment for different probability to get the discharge for ungauged catchment which is shown in Table 5.11(d).

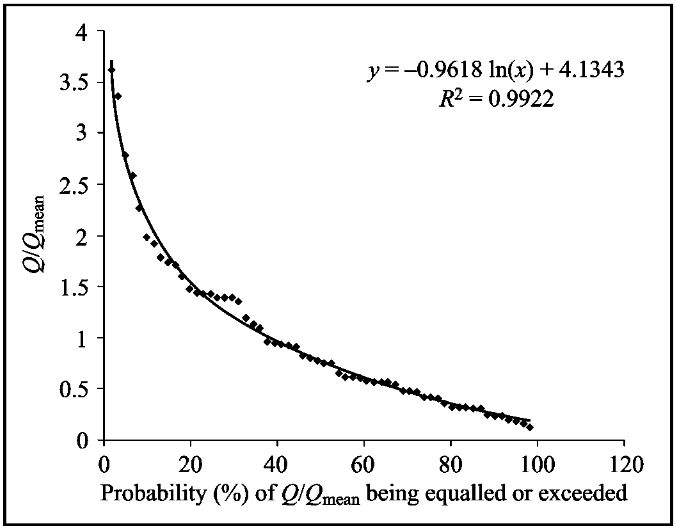

Fig. 5.5 *Flow in relation to probability of non-exceedence*

Table 5.11 (c) Value of Q/Q_{mean} for different probability

P (%)	Q/Q_{mean}
25	1.36
50	0.7629
75	0.4134
90	0.2563

To get the flow duration curve for the ungauged catchment, the values of Table 5.11(b) are used to develop the following relationship

$$Q_{mean} = 1.72\, A^{0.5}\, L^{0.07}, \quad R^2 = 0.6$$

It is to note that R^2 is not very high and we consider this only for illustrative purposes.

Q_{mean} for the ungauged catchment = $1.72(1000)^{0.5}(60)^{0.07} = 72.44$ m^3/s. Multiplying Q_{mean} obtained above with the values of Q/Q_{mean} in Table 5.11(c) gives the dependable flows for various probabilities as shown in Table 5.11(d).

Table 5.11(d) Value of discharge for different probability for ungauged catchment

P (%)	*Q (m^3/s)*
25	98.6
50	55.31
75	29.97
90	18.58

Example 5.10 Derive a flow duration curve for a catchment having 350 km^2 area having no runoff data from the regional analysis. Use the Regional formula: $X_T = [35.46 - 12.41\ (1/T)^{0.323}]\ A^{0.54}$, where A is the area of catchment, T is the return period in year for occurrence of a flood of magnitude X_T.

Solution

Use regional formula to prepare Table 15.2.

Table 5.12 Computational results

Return period (T) (years)	Probability of equaled or exceeded (%)	Discharge (X_T) (m^3/s)
300	0.33	792.062
100	1.0	772.2542
5	20.0	664.0603
3	30.33	632.7564
2	50.0	603.9583
1.5	66.67	581.1137

Using Columns (2) and (3) of Table 5.12, the flow duration curve can be plotted, as shown in Fig. 5.6.

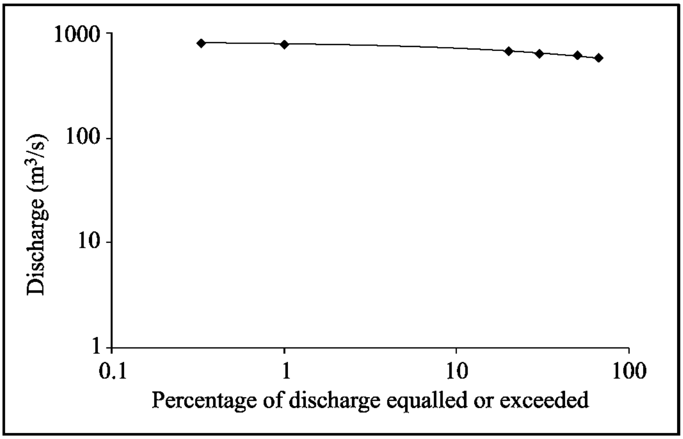

Fig. 5.6 *Flow duration curve for Example 5.10*

5.4 FLOOD ROUTING

Flood routing is a technique of determining the flood hydrograph at a section of a river by utilizing the data of flood flow at one or more upstream sections. The two broad categories of routing are *reservoir routing* and *channel routing*. The difference between these two is that in the reservoir routing, the water surface slope is zero; while in channel routing, it is a non-zero value. In this section, we will be concentrating only on channel routing.

5.4.1 Channel Routing

In *channel routing*, the flood hydrograph at various sections of the reach is predicted by considering a channel reach and an input hydrograph at the upstream end. Information on the flood-peak attenuation and the duration of high-water levels obtained by channel routing is of utmost importance in flood-forecasting operations and flood-protection works. As the flood wave moves down the river, the shape of the wave gets modified due to various factors, such as channel storage, resistance, lateral addition or withdrawal of flows, etc. Flood waves passing down a river have their peaks attenuated due to friction if there is no lateral inflow. The addition of lateral inflows can cause a reduction of attenuation or even amplification of a flood wave.

A variety of routing methods are available, and they can be broadly classified into two categories as: (i) hydrologic routing, and (ii) hydraulic routing. Hydrologic routing methods essentially employ the equation of continuity. Hydraulic methods, on the other hand, employ the continuity equation together with the equation of motion of unsteady flow.

5.4.1.1 Basic Equations

In open channel hydraulics, the passage of a flood hydrograph through a channel is classified as gradually varied unsteady flow. In all hydrologic routing, the equation of continuity is used. The equation of continuity states that the rate of change of storage is the difference between the inflow and outflow rate.

Let I = inflow rate, Q = outflow rate, and S = storage capacity of the reservoir. The continuity equation in the differential form for the reservoir is given by:

$$I - Q = \frac{dS}{dt} \tag{5.15}$$

Alternatively, in a small time interval (Δt), the difference between the total inflow and total outflow in a reach is equal to the change in storage in that reach.

Let $\bar{I}$ = average inflow in time Δt, $\bar{Q}$ = average outflow in time Δt, and ΔS = change in storage.

$$\bar{I}\,\Delta t - \bar{Q}\,\Delta t = \Delta S \tag{5.16}$$

where, $\bar{I} = (I_1 + I_2)/2$, $\bar{Q} = (Q_1 + Q_2)/2$, and $\Delta S = S_2 - S_1$ with subscripts 1 and 2 to denote the beginning and end of time interval Δt.

Therefore, Eq. (5.16) can be written as:

$$\left(\frac{I_1 + I_2}{2}\right)\Delta t - \left(\frac{Q_1 + Q_2}{2}\right)\Delta t = S_2 - S_1 \tag{5.17}$$

The time interval (Δt) should be sufficiently short so that the inflow and outflow hydrographs can be assumed to be straight lines in that time interval. Further, Δt must be shorter than the time of transit of the flood wave through the reach. In the differential form, the equation of continuity for unsteady flow in a reach with no lateral flow is given by:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0 \tag{5.18}$$

where, u = velocity of flow, and h = depth of flow

The equation of motion for a flood wave is derived from the application of the momentum equation as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} - (S_0 - S_f)g = 0 \tag{5.19}$$

where, S_0 = channel bed slope, and S_f = slope of the energy line. The continuity equation [Eq. (5.16)] and the equation of motion [Eq. (5.19)] are used in hydraulic channel routing.

5.4.1.2 Storages in the Channel

Prism Storage It is the volume that would exist if uniform flow occurred at the downstream depth, i.e., the volume formed by an imaginary plane parallel to the channel bed drawn at the outflow section of water surface (see Fig. 5.7).

Wedge Storage It is the wedge-like volume formed between the actual water surface profile and the top surface of the prism storage (see Fig. 5.7).

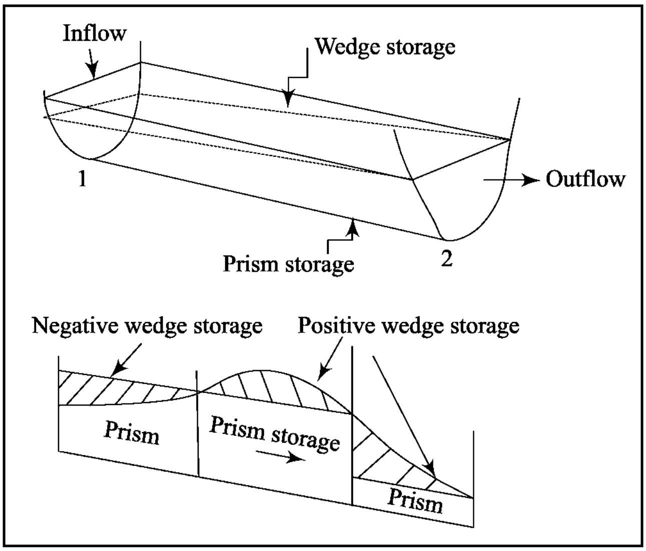

Fig. 5.7 *Storage in a channel reach*

At a fixed depth at a downstream section of a river reach, the prism storage is constant while the wedge storage changes from a positive value at an advancing flood to a negative value during a receding flood. The prism storage (S_p) is similar to a reservoir and can be expressed as a function of the outflow discharge, $S_p = f(Q)$. The wedge storage can be accounted for by expressing it as $S_w = f(I)$. The total storage in the channel reach can then be expressed as:

$$S = K\,[x\,I^m + (1 - x)\,Q^m] \tag{5.20}$$

where, K and x are coefficients, and m is a constant exponent. It has been found that the value of m varies from 0.6 for rectangular channels to about 1.0 for natural channels.

5.4.1.3 Muskingum Equation

Using $m = 1.0$, Eq. (5.20) reduces to a linear relationship for S in terms of I and Q as:

$$S = K[xI + (1 - x)Q] \tag{5.21}$$

and this relationship is known as the *Muskingum equation*. In this, the parameter x is known as *weighting factor* and it takes a value between 0 and 0.5. It accounts for the storage portion of the routing. When $x = 0$, the storage is only the function of discharge and Eq. (5.21) reduces to:

$$S = KQ \tag{5.22}$$

Such storage is known as *linear storage* or *linear reservoir*. When $x = 0.5$, both the inflow and the outflow become equally important in determining the storage.

The coefficient K is known as *storage-time constant* and has the dimensions of time. It is a function of the flow and channel characteristics. It is approximately equal to the time of travel of a flood wave through the channel reach.

Estimation of K and x Figure 5.8 shows a typical inflow and outflow hydrograph through a channel reach. Note that the outflow peak does not occur at the point of intersection of the inflow and outflow hydrographs. Using the continuity equation [Eq. (5.23)], the increment in storage at any time t and time element Δt can be calculated.

$$(I_1 + I_2)\frac{\Delta t}{2} - (Q_1 + Q_2)\frac{\Delta t}{2} = \Delta S \tag{5.23}$$

Summation of various incremental storage values enables to find the channel storage S vs. time relationship as shown in Fig. 5.8.

If an inflow and outflow hydrograph set is available for a given reach, values of S at various time intervals can be determined by the above equation. By choosing a trial value of x, values of S at any time t are plotted against corresponding $[xI + (1 - x)Q]$ values. If the value of x is chosen correctly, a straight-line relationship as given by Eq. (5.14) will result. However, if an incorrect value of x is used, the plotted points will trace a looping curve.

By trial and error, a value of x is chosen so that the data describe a straight line (Fig. 5.8). The inverse slope of this straight line will give the value of K. Normally, for natural channels, the value of x lies between 0 and 0.3. For a

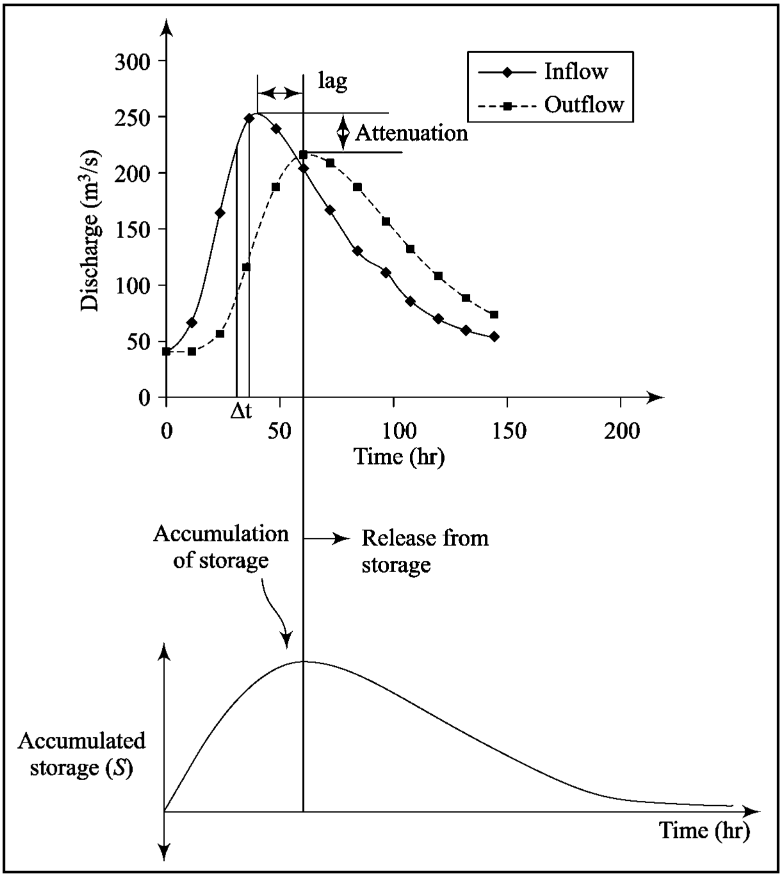

Fig. 5.8 *Hydrographs and storage in channel routing*

given reach, the values of x and K are assumed to be constant. A calibration can be performed with several flood events to arrive at constant values of x and K for a particular reach.

Example 5.11 The following inflow and outflow hydrographs were observed in a river reach. Estimate the values of K and x applicable to this reach.

Table 5.13 Data of Inflow and outflow hydrographs for Example 5.11

Time (hr)	0	6	12	18	24	30	36	42	48	54	60	66
Inflow (m^3/s)	20	80	210	240	215	170	130	90	60	40	28	16
Outflow (m^3/s)	20	20	50	150	200	210	185	155	120	85	55	23

Solution

Using a time increment Δt = 6 hr, the calculations are performed in a tabular manner as shown in Table 5.14. The incremental storage ΔS and S are calculated in columns 6 and 7, respectively. It is advantageous to use the units [(m^3/s) × hr] for storage terms.

As a first trial, $x = 0.35$ is selected and the value of $xI + (1 - x)Q]$ is evaluated (column 8) and plotted against S in Fig. 5.9. Since a looped curve is obtained, further trials are performed with x = 0.30. It is seen from Fig. 5.9 that for x = 0.3, the plot describes a straight line. Hence, x = 0.3 is taken as the appropriate value for the reach. From Fig. 5.9, K = 9.314 hr.

Table 5.14 Determination of K and x

Δt = 6 hr							*Storage in (m^3/s) × hr*	
Time (hr)	*I (m^3/s)*	*Q (m^3/s)*	*(I–Q)*	*Average (I–Q)*	*ΔS = Col.5 × Δt (m^3/s × hr)*	*S = ΣΔS (m^3/s × hr)*	*[xI + (1–x)Q]m^3/s*	
							x = 0.35	*x = 0.30*
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
0	20	20	0			0	20.0	20.0
				30.0	180.0			
6	80	20	60			180.0	41.0	38.0
				110.0	660.0			
12	210	50	160			840.0	106.0	98.0
				125.0	750.0			
18	240	150	90			1590.0	181.5	177.0
				52.5	315.0			
24	215	200	15			1905.0	205.25	204.5
				–12.5	–75.0			
30	170	20	–40			1830.0	196.0	198.0
				–47.5	–285.0			
36	130	185	–55			1545.0	165.75	168.5
				–60.0	–360.0			
42	90	55	–65			1185.0	109.5	135.5
				–62.5	–375.0			
48	60	120	–60			810.0	99.0	102.0
				–52.5	–315.0			
54	40	85	–45			495.0	69.25	71.5
				–36.0	–216.0			
60	28	55	–27			279.0	45.55	46.9
				–17.0	–102.0			
66	16	23	–7			177.0	20.55	20.9

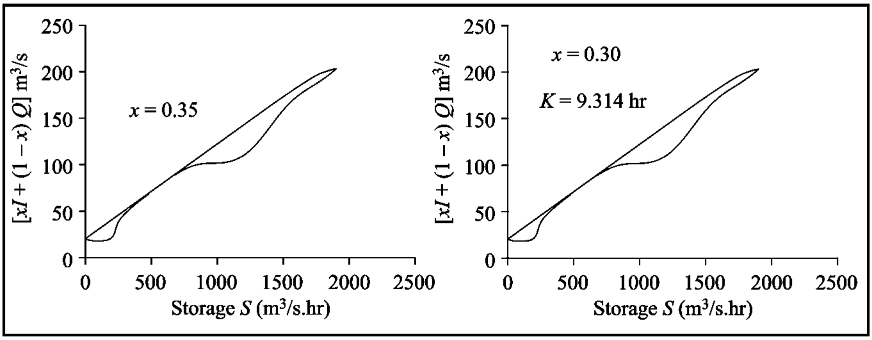

Fig. 5.9 *Determination of K and x for a channel reach*

5.4.1.4 Muskingum Method of Routing

The Muskingum method of flood routing was introduced by McCarthy and his colleagues (U.S. Army Corps of Engineers, 1960) in connection with the flood control studies of the Muskingum River Basin in Ohio, U.S.A. Since its development, this method has been widely used in river engineering practice.

For a given channel reach, the change in storage can be calculated by selecting a routing interval Δt and using the Muskingum equation, as given below:

$$S_2 - S_1 = K[x(I_2 - I_1) + (1 - x)(Q_2 - Q_1)] \tag{5.24}$$

where, subscripts 1 and 2 refer to the conditions before and after the time interval Δt. The continuity equation for the reach is:

$$S_2 - S_1 = \left(\frac{I_1 + I_2}{2}\right)\Delta t - \left(\frac{Q_1 + Q_2}{2}\right)\Delta t \tag{5.25}$$

From Eqs. (5.24) and (5.25), Q_2 is evaluated as:

$$Q_2 = C_0 I_2 + C_1 I_1 + C_2 Q_1 \tag{5.26}$$

where, $$C_0 = \frac{-Kx + 0.5\Delta t}{K - Kx + 0.5\Delta t} \tag{5.27a}$$

$$C_1 = \frac{Kx + 0.5\Delta t}{K - Kx + 0.5\Delta t} \tag{5.27b}$$

$$C_1 = \frac{K - Kx - 0.5\Delta t}{K - Kx + 0.5\Delta t} \tag{5.27c}$$

Note that $C_0 + C_1 + C_2 = 1.0$. Therefore, Eq. (5.26) can be written in a general form (for the n^{th} time step) as:

$$Q_n = C_0 I_n + C_1 I_{n-1} + C_2 Q_{n-1} \tag{5.28}$$

Equation (5.28) is known as *Muskingum Routing Equation*, and it provides a simple linear equation for channel routing. For best possible results, the routing interval Δt should be chosen such that $K > \Delta t > 2Kx$.

If $\Delta t < 2\,Kx$, the coefficient C_0 will be negative. Generally, negative values of coefficients are avoided by choosing appropriate values of Δt.

To use the Muskingum equation to route a given inflow hydrograph through a reach, the values of K and x for the reach and the value of the outflow Q_1 from the reach at the start are needed. The procedure is as follows:

(a) Knowing K and x, select an appropriate value of Δt.
(b) Calculate C_0, C_1, and C_2.
(c) Calculate Q_2, using the initial conditions I_1, Q_1, and I_2 at the end of the first time step Δt.
(d) The outflow calculated in step (c) becomes the known initial outflow for the next time step. Repeat the calculations for the entire inflow hydrograph.

Example 5.12 Route the following hydrograph through a river reach for which $K = 22.0$ hr and $x = 0.25$. At the start of the inflow flood, the outflow discharge is 40 m^3/s.

Table 5.15 Data of a inflow hydrograph through a river

Time (hr)	0	12	24	36	48	60	72	84	96	108	120	132	144
Inflow (m^3/s)	40	65	165	250	240	205	170	130	115	85	70	60	54

Solution

Since $K = 22$ hr and $2Kx = 2 \times 22 \times 0.25 = 11$ hr, Δt should be such that 22 hr $> \Delta t >$ 11 hr.

In the present case, $\Delta t = 12$ hr is selected to suit the given inflow hydrograph ordinate interval.

Using Eq. (5.20), the coefficients C_0, C_1, and C_2 are calculated as:

$$C_0 = \frac{-22 \times 0.25 + 0.5 \times 12}{22 - 22 \times 0.25 + 0.5 \times 12} = \frac{0.5}{22.5} = 0.0222$$

$$C_1 = \frac{22 \times 0.25 + 0.5 \times 12}{22 - 22 \times 0.25 + 0.5 \times 12} = \frac{11.5}{22.5} = 0.511$$

$$C_2 = \frac{22 - 22 \times 0.25 - 0.5 \times 12}{22 - 22 \times 0.25 + 0.5 \times 12} = \frac{10.5}{22.5} = 0.466$$

For the first time interval (0 to 12 hr),

$$I_1 = 40.0 \qquad C_1\,I_1 = 20.44$$

$$I_2 = 65.0 \qquad C_0\,I_2 = 1.443$$

$$Q_1 = 40.0 \qquad C_2\,Q_1 = 18.64$$

From Eq. (5.19), $Q_2 = C_0 I_2 + C_1 I_1 + C_2 Q_1 = 40.52$ m^3/s

The procedure is repeated for the entire duration of the inflow hydrograph. The computations are done in a tabular form as shown in Table 5.16. By plotting the inflow and outflow hydrographs, the attenuation and peak lag are found to be 35 m^3/s and 21 hr, respectively.

Table 5.16 Muskingum method of routing (Δt = 12 hr)

Time (hr) *(1)*	*I (m^3/s)* *(2)*	*0.0222 I_2* *(3)*	*0.511 I_1* *(4)*	*0.466 Q_1* *(5)*	*Q (m^3/s)* *(6) (sum of col. 3,4 and 5)*
0	40				40.00
		1.433	20.44	18.64	
12	65				40.52
		3.663	33.215	18.882	
24	165				55.76
		5.55	84.315	25.984	
36	250				115.849
		5.328	127.75	53.985	
48	240				187.063
		4.551	122.64	87.171	
60	205				214.362
		3.774	104.755	99.892	
72	170				208.421
		2.886	86.87	97.124	
84	130				186.88
		2.553	66.43	87.086	
96	115				156.069
		1.887	58.765	72.728	
108	85				133.38
		1.554	43.435	62.155	
120	70				107.144
		1.332	35.77	49.929	
132	60				87.031
		1.198	30.66	40.556	
144	54				72.414

5.4.2 Reservoir Routing

The passage of flood hydrograph through a reservoir is an unsteady flow phenomenon. The equation of continuity is used in all hydrologic routing methods as a primary equation. According to this equation, the difference between the inflow and outflow is equal to the rate of change of storage, i.e.,

$$I - Q = dS/dt \tag{5.29}$$

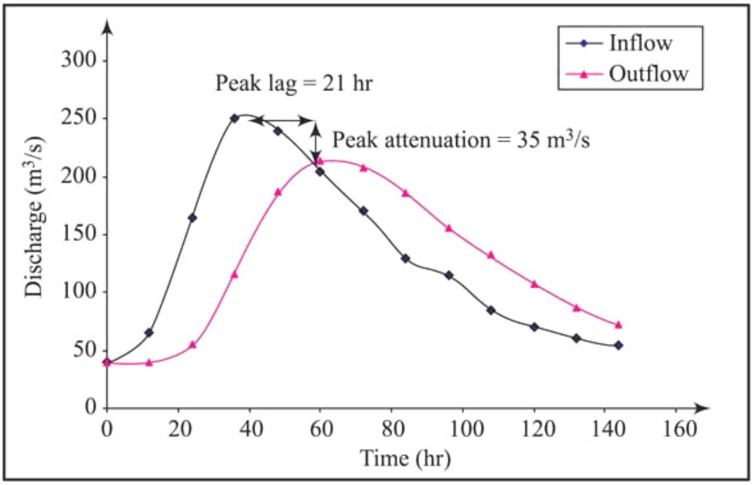

Fig. 5.10 *Variation of inflow and outflow discharge*

where, I = inflow to the reservoir, Q = outflow from the reservoir, S = storage in the reservoir, and t = time.

For the sake of clarity, it is necessary to introduce the frequently used terms, e.g., translation and attenuation characteristics of the flood wave propagation. The translation is taken as the time difference between the occurrence of inflow and outflow peak discharge, and the attenuation as the difference between the inflow and outflow peak discharge. Using the peak discharge for the outflow hydrograph, the attenuation and the translation are computed as:

Attenuation = (*Inflow peak discharge*) – (*Outflow peak discharge*)

Translation = (*Time-to-peak of inflow*) – (*Time-to-peak of outflow*)

Over a small time interval Δt, the difference between total inflow and total outflow in the reach is equal to the change in storage in that reach. Hence, Eq. (5.29) can be written as:

$$I_m \Delta t - Q_m \Delta t = \Delta S \quad (5.30)$$

where, I_m, Q_m, and ΔS denote average inflow, average outflow, and change in storage during time period Δt, respectively.

5.4.2.1 Routing Techniques

The reservoirs can be either controlled or uncontrolled. The controlled reservoirs have spillways with gates to release water at the time of need. The uncontrolled reservoirs are those whose spillways are not controlled by gate operation. Reservoir routing requires the relationship among the reservoir elevation, storage, and discharge to be known. This relationship is a function of the topography of the reservoir site and the characteristics of the outlet facility. It is important to establish the relationship among elevation, storage, and discharge as a single function; because topography of the reservoir is susceptible to change which will consequently affect the elevation-discharge characteristics. McCuen (1989) has discussed this aspect in detail.

Using the basic Eq. (5.29), several methods have been developed for routing a flood wave through a reservoir, namely: (i) Mass Curve Method, (ii) Pulse Method, (iii) Modified Pulse Method, (iv) Wisler-Brater Method, (v) Goodrich Method, (vi) Steinberg Method, and (vii) Coefficient Method. However, we will keep our discussion limited to Mass Curve Method and Modified Pulse Method.

A brief description of each of these methods follows. The schematic representation of reservoir routing is given in Fig. 5.11.

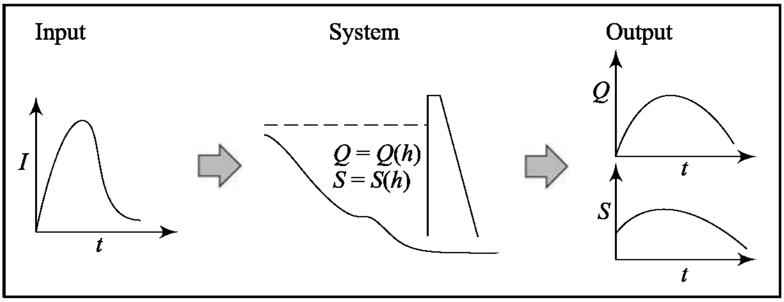

Fig. 5.11 *Schematic representation of reservoir routing*

Mass Curve Method This is one of the most versatile methods of reservoir routing, various versions of which include: (i) direct, (ii) trial and error, and (iii) graphical technique. Here, the trial-and-error version is described in detail. For solution by trial-and-error method, Eq. (5.30) can be rewritten as:

$$M_2 - (V_1 + Q_m \Delta t) = S_2 \tag{5.31}$$

where, M is the accumulated mass inflow, and V is the accumulated mass outflow.

A storage-discharge relationship and the mass curve of inflow should be plotted before obtaining trial-and-error solution. Necessary adjustments are made to show zero storage at the beginning elevation and, correspondingly, spillway discharge is obtained. The following steps are involved in trial-and-error solution:

(a) A time is chosen and Δt is computed. Mass inflow is also computed.
(b) Mass outflow is assumed. As a guideline, it is a function of accumulated mass inflow.
(c) Reservoir storage is computed by deducting mass outflow from mass inflow.
(d) The instantaneous and average spillway discharges are calculated.

(e) Outflow for the time period Δt is computed by multiplying Δt with average discharge. Then, the mass outflow is computed.
(f) Finally, computed mass outflow is compared with assumed mass outflow. If the two values agree within an acceptable degree of accuracy, then the routing is complete. If this agreement is not acceptable, then another mass outflow is assumed and the above procedure is repeated.

Modified Pulse Method The basic law used in modified pulse method states: *The inflow minus outflow is equal to the rate of change in storage*. This is also referred to as the *Storage-Indication* method.
Assuming $I_m = (I_1 + I_2)/2$, $Q_m = (Q_1 + Q_2)/2$, and $\Delta S = S_2 - S_1$, Eq. (5.30) is written as:

$$(I_1 + I_2)\Delta t/2 - (Q_1 + Q_2)\Delta t/2 = S_2 - S_1 \tag{5.32}$$

where, subscripts 1 and 2 denote the beginning and end of time interval Δt, and Q may incorporate controlled discharge as well as uncontrolled discharge. Here, the time interval Δt must be sufficiently small so that the inflow and outflow hydrographs can be assumed to be linear in that time interval. Further, Δt must be shorter than the time of transit of flood wave through the reservoir. Separating the known quantities from the unknown ones and rearranging:

$$(I_1 + I_2) + (2S_1/\Delta t - Q_1) = (2S_2/\Delta t + Q_2) \tag{5.33a}$$

Or

$$(I_1 + I_2)\,\Delta t/2 + (S_1 - Q_1\,\Delta t/2) = (S_2 + Q_2\,\Delta t/2) \tag{5.33b}$$

Here, the known quantities are I_1 (inflow at time 1), I_2 (inflow at time 2), Q_1 (outflow at time 1), and S_1 (storage in the reservoir at time 1); and the unknown quantity are S_2 and Q_2. Since one equation with two unknown parameters cannot be solved, one must have another relation that relates storage S, and outflow Q.

As the outflow from the reservoir takes place through the spillway, the discharge passing through the spillway can be conveniently related with the reservoir elevation which, in turn, can be related to the reservoir storage. Such a relationship is invariably available for any reservoir. Also, it can be computed from the following relation:

$$Q = C_d\,L\,H^{1.5} \tag{5.34}$$

where, Q is the outflow discharge (m^3/s), C_d is the coefficient of discharge (1.70 in metric unit), L is the length of spillway (m), and H is the depth of flow above the spillway crest (m).

Thus, the left side of Eq. (5.33) contains the known terms and the right side terms are unknown. The inflow hydrograph is known. The discharge Q, which may pass through the turbines, outlet works, or over the spillway, is also known. The uncontrolled discharge goes freely over the spillway. It depends upon the depth of flow over the spillway and the spillway geometry. Further, the depth of flow over the spillway depends upon the level of water in the reservoir.

Therefore, $S = S\,(Y)$

$$Q = Q\,(Y) \tag{5.35}$$

where, Y represents the water surface elevation. The right side of Eq. (5.33a) can be written as:

$$2S/\Delta t + Q = f(Y) \tag{5.36}$$

Adding the crest elevation with the depth of flow, the elevation for which storage in the reservoir is known can be computed. Therefore, one can develop a relation between storage and outflow. This storage outflow relation is used to develop the storage indication $[(2S/\Delta t) + Q]$ vs. outflow relation. To develop this relation, it is necessary to select a time interval such that the resulting linearization of the inflow hydrograph remains a close approximation of the actual nonlinear (continuous time varying) shape of the hydrograph. For smoothly rising hydrographs, a minimum value of $t_p/\Delta t = 5$ is recommended, in which t_p is the time to peak of the inflow hydrograph. In practice, a computer aided calculation would normally use a much greater ratio $(t_p/\Delta t)$, say 10 to 20.

In order to utilize Eq. (5.33), the elevation storage and elevation-discharge relationship must be known. Before routing, the curves of $(2S/\Delta t \pm Q)$ vs. Q are constructed. The routing is now very simple and can be performed using the above equation.

The computations are performed as follows. At the starting of flood routing, the initial storage and outflow discharge are known. In Eq. (5.32), all the terms in the left hand side are known at the beginning of time step Δt. Hence, the value of $(S_2 + Q_2\, \Delta t/2)$ at the end of the time step is calculated by Eq. (5.33). Since the relation $S = S(h)$ and $Q = Q(h)$ are known, $(S_2 + Q_2\, \Delta t/2)$ will enable one to determine the reservoir elevation and hence the discharge at the end of the time step. This procedure is repeated to cover the full inflow hydrograph.

Example 5.13 A reservoir has the following elevation, discharge, and storage relationships:

Table 5.17 Reservoir data for Example 5.13

Elevation (m)	*Storage (10^6 m^3)*	*Outflow discharge (m^3/s)*
101.00	4.550	0
101.50	4.575	15
102.10	4.880	31
102.50	5.284	52
103.00	5.927	86
103.50	6.370	120
103.75	6.527	127
104.00	6.856	140

When the reservoir level was at 101.50 m, the following flood hydrograph entered the reservoir.

Table 5.18 Flood hydrograph for Example 5.13

Time (hrs.)	0	6	12	18	24	30	36	42	48	54	60	66	72
Discharge (m^3/s)	12	22	57	85	70	52	41	33	21	15	14	12	11

Route the flood, and obtain the outflow hydrograph and the reservoir elevation-time curve for the flood passage.

Solution

The time step used for the hydrograph is taken for the analysis, i.e., $\Delta t = 6$ hr
The elevation-discharge ($S + Q\,\Delta t/2$) is prepared as follows:

$$\Delta t = 6 \times 3600 = 216000 = 0.0216 \times 10^6 \text{ sec}$$

Table 5.19 Calculations of elevation versus ($S + Q\,\Delta t/2$)

Elevation (m)	*Storage (S) (10^6 m^3)*	*Outflow discharge (Q) in (m^3/s)*	*$S + Q\,\Delta t/2$ (Mm^3)*
101.00	4.550	0	4.55
101.50	4.575	15	4.737
102.10	4.880	31	5.2148
102.50	5.284	52	5.8456
103.00	5.927	86	6.855
103.50	6.370	120	7.666
103.75	6.527	127	7.898
104.00	6.856	140	8.368

The plot showing elevation vs. ($S + Q\,\Delta t/2$) is shown in Fig. 5.12. Figure 5.13 shows the variation of elevation with outflow.

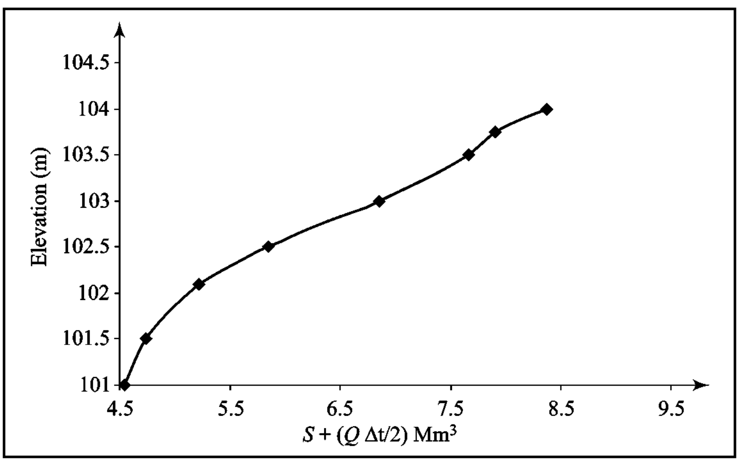

Fig. 5.12 *Plot showing Elevation vs. ($S + Q\,\Delta t/2$)*

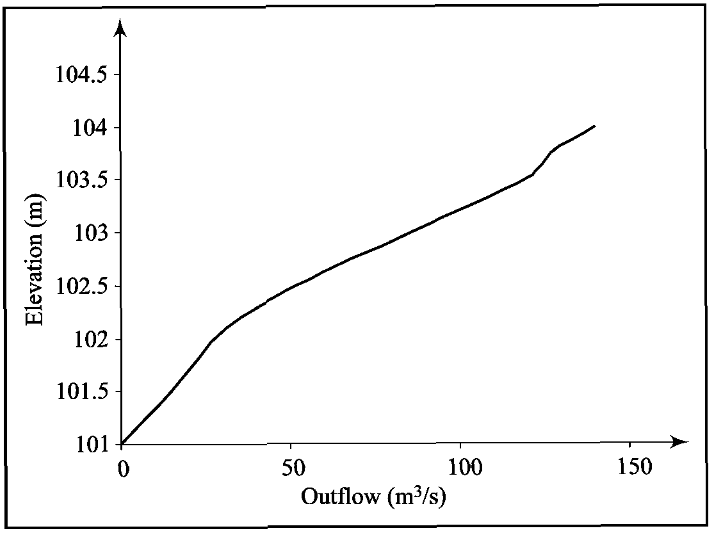

Fig. 5.13 *Plot showing elevation vs. outflow*

At the start of the elevation when the flood wave enters the reservoir, the outflow discharge is 15 m^3/s and $S = 4.575$ Mm^3.
Therefore, $(S - Q\,\Delta t/2)_1 = 4.575 - (15 \times 0.0216)/2 = 4.413$ Mm^3.

Using this initial value of $(S - Q\,\Delta t/2)$, Eq. (5.33b) is used to get $(S + Q\,\Delta t/2)$.

$$\left(S + \frac{Q\Delta t}{2}\right)_2 = (I_1 + I_2)\frac{\Delta t}{2} + \left(S - \frac{Q\Delta t}{2}\right)_1$$
$$= (12 + 22) \times (0.0216/2) + 4.413$$
$$= 4.7802$$

From Fig. 5.12, the water surface elevation corresponding to $(S + Q\,\Delta t/2)_2 = 4.7802$ is 101.55 m and from Fig. 5.13, the outflow discharge Q_2 corresponding to elevation 101.55 m is 16.33 m^3/s. For the next time step,

$$(S - Q\,\Delta t/2)_2 = (S + Q\,\Delta t/2)_2 \text{ of the previous step} - Q_2\,\Delta t$$
$$= 4.7802 - (16.33 \times 0.0216)$$
$$= 4.4274 \text{ Mm}^3$$

$$\left(S + \frac{Q\Delta t}{2}\right)_3 = (I_2 + I_3)\frac{\Delta t}{2} + \left(S - \frac{Q\Delta t}{2}\right)_2$$
$$= (22 + 57) \times 0.0216/2 + 4.4274$$
$$= 0.8532 + 4.4274$$
$$= 5.2806$$

From Fig. 5.12, the water surface elevation corresponding to $(S + Q\,\Delta t/2)_3 = 5.2806$ is 102.14 m and from Fig. 5.13, the outflow discharge Q_3 corresponding to elevation 102.14 m is 33.1 m^3/s. Use of similar sequence of computations will enable outflow discharge computations at the remaining time steps.

Table 5.20 Calculations related to Example 5.13

Time (*hr*)	*Inflow L* (m^3/s)	$\bar{I}$ (m^3/s)	$\bar{I}\,\Delta t$ (Mm^3)	$S - \frac{Q\Delta t}{2}$ (Mm^3)	$S + \frac{Q\Delta t}{2}$ (Mm^3)	*Elevation* (*m*)	*Q* (m^3/s)
1	*2*	*3*	*4*	*5*	*6*	*7*	*8*
0	12					101.5	**15**
		17	0.3672	4.413	4.7802		
6	22					101.55	16.33
		39.5	0.8532	4.4274	5.2806		
12	57					102.14	33.1
		71	1.5336	4.565	6.0986		
18	85					102.62	60.16
		77.5	1.674	4.80	6.474		
24	70					102.81	73.08
		61	1.3176	4.895	6.2126		
30	52					102.68	62.24
		46.5	1.0044	4.825	5.83		
36	41					102.49	51.5
		37	0.7992	4.7176	5.5168		
42	33					102.31	42.0
		27	0.5832	4.61	5.19		
48	21					102.07	30.2
		18	0.3888	4.537	4.925		
54	15					101.73	22.36
		14.5	0.3132	4.442	4.755		
60	14					101.52	15.64
		13	0.2808	4.417	4.697		
66	12					101.39	11.7
		11.5	0.2484	4.444	4.692		
72	11					101.37	11.1

Figures 5.14 and 5.15 show the variation of inflow, outflow, and reservoir elevation (level of water surface) with time.. From Fig. 5.14, it can be seen that peak lag and peak attenuation are 7 hr and 12 m^3/s, respectively.

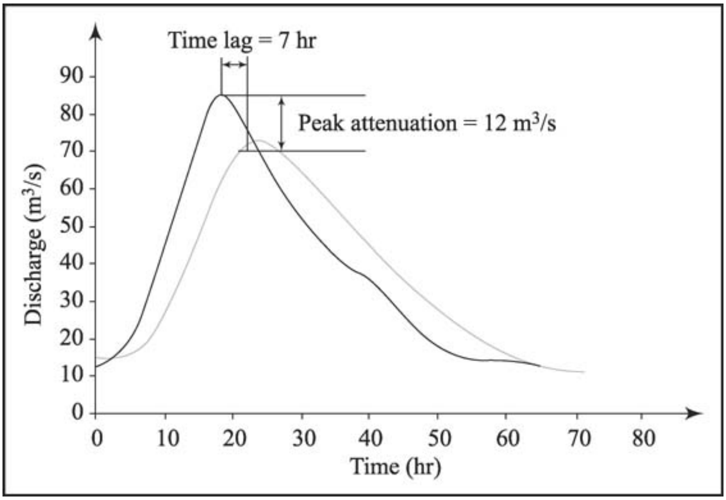

Fig. 5.14 *Variation of inflow and outflow discharge*

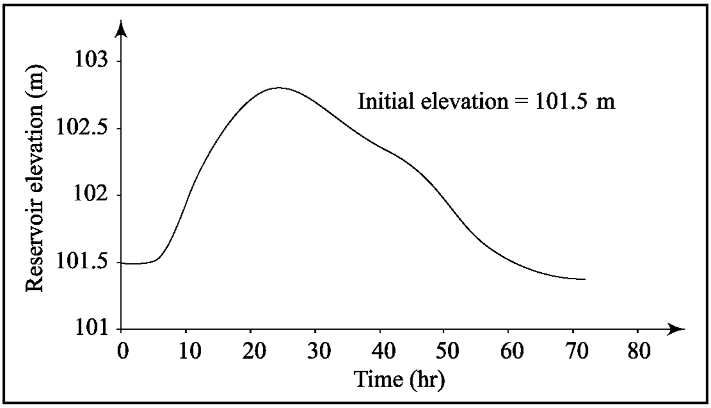

Fig. 5.15 *Variation of reservoir elevation with time*

SUMMARY

Rainfall-runoff relationship is generally nonlinear in nature and may vary from place to place. It may be useful to compile such relationships and study their merits and demerits. In recent years, a lot of work on representation of such nonlinearity has also been reported.

Runoff leads to flood, and it is important to understand the propagation of flood. Flood routing is a useful means of studying this propagation. Flood routing in open channels is considered using the Muskingum approach. It is possible to extend this approach to a channel network which is typically encountered at a catchment scale. To route flow in such situations, flow at each junction has to be computed. This may include adding routed outflow from a channel or a series of secondary channels joining at the junction.

EXERCISES

5.1 Why is double-mass curve method used? Explain the detailed procedure of this method.

5.2 What is a flow duration curve? Explain the detailed procedure to plot the flow duration curve.

5.3 Define routing. What are the methods of routings? Explain these methods in detail.

5.4 What do you understand by prism storage and wedge storage?

5.5 Explain the procedure of estimating coefficients K and x in Muskingum method of flood routing.

5.6 Write an algorithm for the Muskingum method of flood routing.

5.7 For an assumed inflow hydrograph and channel dimensions, write a program using Muskingum method to generate outflow hydrograph at a prescribed section.

5.8 What is the utility of mass curve method?

5.9 What is the difference between reservoir routing and channel routing?

5.10 Route the following flood hydrograph through a stream using Muskingum method. The initial outflow discharge is 10 (m^3/s). The Muskingum coefficient are K= 6 hr and x = 0.3.

Time (hr)	0	2	4	6	8	10	12	14	16
Inflow (m^3/s)	15	25	40	75	125	185	235	215	160

Time (hr)	18	20	22	24	26	28	30
Inflow (m^3/s)	105	75	60	45	30	20	10

OBJECTIVE QUESTIONS

1. In reservoir routing, the storage is a function of
   (a) Inflow discharge (b) Outflow discharge
   (c) Both (a) and (b) (d) None of these

2. In channel routing, the storage is a function of
   (a) Inflow discharge (b) Outflow discharge
   (c) Both (a) and (b) (d) None of these

3. The flow of a river during a flood is a
   (a) Gradually varied steady flow (b) Gradually varied unsteady flow
   (c) Rapidly varied unsteady flow (d) Uniform flow

4. If the outflow from a storage reservoir is uncontrolled, the peak of the outflow hydrograph will occur
   (a) After the point of intersection of the inflow and outflow curves
   (b) Before the point of intersection of the inflow and outflow curves
   (c) At the point of intersection of the inflow and outflow curves
   (d) At any point

5. The prism storage is a function of
   (a) Inflow discharge (b) Outflow discharge
   (c) Both (a) and (b) (d) None of these

6. The wedge storage is a function of
   (a) Inflow discharge (b) Outflow discharge
   (c) Both (a) and (b) (d) None of these

7. The total storage in the channel reach can be expressed as
   (a) $S = m\,[\,x\,I^K + (1-x)\,Q^K]$ (b) $S = K\,[\,x\,I^m + (1-x)\,Q^m]$
   (c) $S = K\,[\,x\,Q^m + (1-x)\,I^m]$ (d) $S = m\,[\,x\,Q^K + (1-x)\,I^K]$

8. In a linear reservoir
   (a) Storage is directly proportional to the discharge
   (b) Storage is inversely proportional to the discharge
   (c) There is no relation between the storage and discharge
   (d) None of the above

9. The Muskingum method of flood routing is a
   (a) Hydrologic storage routing
   (b) Hydrologic channel routing
   (c) Hydraulic method of flood routing
   (d) Both (a) and (b)

10. The hydrologic routing method uses
    (a) Continuity equation (b) Momentum equation
    (c) Energy equation (d) All of these

11. In Muskingum method of routing, the sum of three coefficients $C_0 + C_1 + C_2$ is equal to
    (a) 0.5 (b) 1 (c) 2 (d) 3

12. In Muskingum method of routing, the routing interval Δt should be chosen such that
    (a) $K > 2K_x > \Delta t$ (b) $2K_x > K > \Delta t$
    (c) $K > \Delta t > 2K_x$ (d) $\Delta t < 2K_x$

13. For natural stream channels, the value of the Muskingum parameter *x* will generally be
    (a) Equal to 0 (b) Equal to 1
    (c) Equal to 0.5 (d) Between 0 and 0.3

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CHAPTER 6

Hydrologic Statistics

6.1 INTRODUCTION

Statistical analyses in hydrology can be classified into two categories—univariate and multivariate. Problems dealing with the data of a single random variable fall under the category of univariate analysis. The univariate analysis helps identify the statistical population, from which the sample measurements are derived, and future occurrences of the random variable are predicted based on the assumed population. In analyses of multivariable data with more than one variable, the functional form of the inherent relationship is derived. The linear regression analysis is generally used to develop the suitable linear form of multiple variable models. Here, a dependent variable takes the values caused by variations in one or more independent or predictor variables. Such models are found to be of great use in hydrological prediction. Development of rainfall runoff relationship has been considered in the preceding chapter. However, one needs to know various statistical terms relevant to any regression relationship.

The objective of this chapter is to:

(a) define some of the important terms of statistical and probabilistic hydrology;
(b) describe measures of location, dispersion, and symmetry of both grouped and ungrouped data;
(c) describe standard errors of mean, standard deviation, and coefficient of skewness;
(d) describe performance evaluation measures including normal reduced variate, chi-square, student's t-distribution, and F-distribution, and
(e) describe the detailed procedures for deriving linear and multiple regression relationships.

6.2 SAMPLE STATISTICS

The following terms are generally used in statistical hydrology.

Sample data: A sample consists of the data derived from the observation of an event.

Population: A population can be defined as a combination of infinite number of discrete samples.

Random events: Those events whose occurrences are not affected by their earlier occurrences are called random events.

Probability density function (PDF): It is the probability of occurrence of an event.

Cumulative density function (CDF): It is the probability of non- of the event.

Probability paper: It is a special graph paper on which the ordinate usually represents the magnitude of the variate, and the abscissa usually represents the probability P or the return period T. Scales of the ordinate and the abscissa are taken such that the plotted distribution is close to a straight line.

Plotting position: It refers to the probability assigned to a data point.

In any analysis of statistical data in general and hydrologic data in particular, it is important to determine the basic properties of data. The sample mean and variance are the most important statistical characteristics of a dataset. The sample statistics provides, in general, the basic information about the variability of a given data set about its mean.

The commonly used sample characteristics include:

(i) the central tendency or the value around which all other values are clustered,
(ii) the spread of sample values about the mean,
(iii) the asymmetry or skewness of the frequency distribution, and
(iv) the flatness of the frequency distribution.

6.3 MEASURES OF STATISTICAL PROPERTIES

Some of the important statistical measures of location include the following:

(i) Mid-range It is the average of the minimum and maximum values of the sample (or population).

$$\text{Mid-range} = (\text{minimum value} + \text{maximum value})/2$$

(ii) Dispersion or Variation The important measures of dispersion or variation include:

Range: It is the difference between the maximum and the minimum values.

$$\text{Range} = \text{maximum value} - \text{minimum value}$$

Interquartile Range: It is defined as $I_3 - I_1$, where I_1 is the value separating the lowest quarter of the ranked data from the second quarter. In other words, interquartile range is between 25% and 75% of cumulative frequency values and contains 50% of the values.

Mean Deviation: Dispersion about the arithmetic mean is the mean deviation. Thus,

$$\text{Mean deviation} = \frac{\sum_{i=0}^{N} (x_i - \bar{x})}{N} \tag{6.1}$$

where, x_i is the i^{th} data point, $\bar{x}$ is the mean of all data points, and N is the total number of data points in a sample.

Variance (S^2): It represents the dispersion about the mean. Expressed mathematically as,

$$S^2 = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N} \tag{6.2}$$

Standard deviation (S): An unbiased estimate of standard deviation (S) of the population can be derived from the sample data and it is given as the square root of the variance.

$$S = [1/(N-1) \sum_{i=1}^{N} (x_i - \bar{x})^2]^{1/2} \tag{6.3}$$

Coefficient of variation (C_v): The coefficient of variation is a dimensionless dispersion parameter and is equal to the ratio of the standard deviation to the mean.

$$C_v = S/\bar{x} \tag{6.4}$$

This coefficient is extensively used in hydrology, particularly as a regionalization parameter.

Example 6.1 The monthly rainfall (in mm) records at a station for a year are given in table below. Find the mean rainfall, variance, standard deviation and coefficient of variation.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Rainfall in mm	12	10	5	1	2	8	15	25	22	15	12	12

Solution

$$\text{Mean rainfall } \bar{x} = \frac{12+10+5+1+2+8+15+25+22+15+12+12}{12}$$

$$= 11.58 \text{ mm}$$

$$\text{Variance } S^2 = \frac{\begin{bmatrix} (12-11.58)^2+(10-11.58)^2+(5-11.58)^2+(1-11.58)^2 \\ +(2-11.58)^2+(8-11.58)^2+(15-11.58)^2+(25-11.58)^2 \\ (22-11.58)^2+(15-11.58)^2+(12-11.58)^2+(12-11.58)^2 \end{bmatrix}}{12-1}$$

$$= \frac{\begin{bmatrix} 0.1764+2.4964+43.2964+111.9364 \\ +91.7764+12.8164+11.6964+180.0964 \\ +108.5764+11.6964+0.1764+0.1764 \end{bmatrix}}{11}$$

$$= 52.26$$

Standard deviation $S = 7.229$
Coefficient of variation = 7.229/11.58 = 0.624

Discussion: Range and mean deviation are of the same dimension as the original data. Variance is of squared dimension and, therefore, cannot be directly compared with the data; instead, standard deviation is used. In many samples of hydrological data, especially in flood hydrology, the largest sample value may be much larger than the second largest value. In such situations, range may not better represent the scatter in whole data. Though mean deviation is a good measure of spread, its handling is not easy in mathematical statistics due to its absolute sign, and so is the case with interquartile range though it describes the spread well. Since variance can be used conveniently, it has a prominent place in statistical hydrology.

(iii) Symmetry If data are symmetrically displaced about the mean, the measure of symmetry is equal to zero. On the other hand, if these are displaced more to the right of the mean than those on the left, then, by convention, the asymmetry is positive; and it is negative for otherwise asymmetry.

The commonly used measures of asymmetry are:

Interquartile measure of asymmetry (I_{as}): It is defined as:

$$I_{as} = |I_3 - I_2| - |I_1 - I_2| \tag{6.5}$$

where I_1, I_2, and I_3 are the lower, median, and upper quartiles, respectively.

Third central moment (M_3): The third moment of sample data about the mean is given by:

$$M_3 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^3 \tag{6.6}$$

If data are symmetrical, $M_3 = 0$; it is positive or negative otherwise.

Skewness coefficient (C_s): It is a non-dimensional measure of asymmetry of the frequency distribution fitted to the data. Its unbiased estimate is given by:

$$C_s = \frac{N\sum_{i=1}^{N}(x_i - \bar{x})^3}{(N-1)(N-2)S^3} \tag{6.7}$$

Skewness coefficient is an important indicator of the symmetry of distribution fitted to data points. Symmetrical frequency distributions exhibit a very low value of sample skewness coefficient C_s, whereas asymmetrical frequency distributions show a positive or negative value. Smaller value of C_s implies more closeness of the distribution to the normal distribution, because $C_s = 0$ for normal distribution. The third central moment having the dimension of the cube of the variable is not of direct use, as described above. On the other hand, C_s does not suffer from this limitation and is, therefore, generally preferred.

(iv) Peakedness or Flatness The peakedness or the flatness of the frequency distribution near its center is measured by kurtosis coefficient (C_k), which is expressed as:

$$C_k = \frac{N^2 \sum_{i=1}^{N} (x_i - \bar{x})^4}{(N-1)(N-2)(N-3)\, S^4} \tag{6.8}$$

The excess coefficient (E) defined as: $E = C_k - 3$, is a variant of C_k. Positive E-values indicate a frequency distribution to be more peaked around its center than the normal distribution. Such a frequency distribution is known as *leptokurtic*. On the other hand, negative values of E indicate that a given frequency distribution is more flat around its center than the normal. These are called *platykurtic* distributions. Normal distribution is knows as *mesokurtic*. It is worth emphasizing that the use of C_k is not much popular in statistical hydrology.

(v) Maxima or Minima Maxima or minima refer to data extremes or extreme values of data, whether high or low. These data are commonly used in frequency analyses, for they are taken to be independent of each other or random data. It is also for the reason that the data observed for an event sometimes exhibit dependence. Examples of sample data extremes include annual maximum peak flood series, annual low-flow series for a specific duration, and so on.

Example 6.2 Determine the skewness coefficient for the data given in Example 6.1

Solution

$$C_s = 12 \times \frac{\begin{bmatrix} (12-11.58)^3 + (10-11.58)^3 + (5-11.58)^3 \\ + (1-11.58)^3 + (2-11.58)^3 + (8-11.58)^3 \\ + (15-11.58)^3 + (25-11.58)^3 + (22-11.58)^3 \\ + (15-11.58)^3 + (12-11.58)^3 + (12-11.58)^3 \end{bmatrix}}{(12-1)(12-2)(7.229)^3}$$

$$= 12 \times \frac{\begin{bmatrix} 0.074 - 3.9443 - 284.8903 - 1184.2871 \\ - 879.2179 - 45.8827 + 40.00 + 2416.8936 \\ + 1131.366 + 40.00 + 0.074 + 0.074 \end{bmatrix}}{11 \times 10 \times 377.77}$$

$$= 0.355$$

Exercise: Compute kurtosis coefficient for the data given in Example 6.1.

6.4 STANDARD ERRORS

Statistical characteristics, such as mean and standard deviation, derived from a short period of sample record represent only an estimate of the population statistic. The reliability of such estimates can be evaluated from their standard errors (SE). It is of common experience that there exists a probability of about 68% that the population statistic lies within one standard error of the value estimated from the available (or sample) data. The standard errors of commonly used mean, SE($\bar{x}$), standard deviation, SE(S); as given below.

$$\text{SE}(\bar{x}) = S/\sqrt{N} \tag{6.9}$$

$$\text{SE}(S) = S/\sqrt{(2N)} \tag{6.10}$$

Example 6.3 Compute standard error of mean and standard deviation of the data given in Example 6.1.

Solution

Given, $S = 7.229$ and $n = 12$

Using Eq (6.9), SE of mean $= \dfrac{7.229}{\sqrt{12}} = 2.086$

Using Eq. (6.10), SE of standard deviation $= \dfrac{7.229}{\sqrt{2 \times 12}} = 1.475$

6.5 GRAPHICAL PRESENTATION OF GROUPED DATA

Grouped data are graphically presented as histograms or cumulative histograms of frequency (or relative frequency or probability). A frequency table is prepared first. The range of the data variable is divided into a number of intervals of convenient size, and the frequency (f) of the variable is derived for each interval. It is noted that if the class interval is very large, the table will become compact, but it will lose detail. On the other hand, if the interval is too small, the table may become too bulky to handle. Therefore, an optimal choice of the class interval is important in statistical analyses.

The following two criteria are generally used in practice. According to Brooks and Carruthers (1953), the number of classes are taken greater than five times the logarithm of the total number of data points: i.e.,

number of classes > 5 log(number of values)

According to Charlier and Bortkiewicz, (1947), the size of the class interval (w) can be taken as one-twentieth of the difference between the maximum and the minimum values: i.e.,

w = (maximum value – minimum value)/20

where, w is the size of class interval. The number of classes generally vary from 15–25.

In steps, the frequency table can be prepared as follows:

(i) Order the values of variable (x) in increasing or decreasing order of magnitude.

(ii) Determine the size (w) and number (NC) of the class interval using the above guidelines.

(iii) Divide the ordered observations x into NC intervals (or groups).

(iv) Determine the frequency (n) of each class interval by counting the observations that fall within the respective intervals.

(v) Determine the corresponding relative frequencies (n/N) for all NC intervals.

(vi) Compute the cumulative relative frequency F for each class interval. These approximate the probabilities as:

$F = F(x < X)$ if order is increasing

Or, $F = F(x > X)$ if order is decreasing

where, x is a random variable, and X is its value at any point. It is to note that many textbooks use X as a random variable and x as its value. In this chapter, [X] is also used as a matrix of x-values.

(vii) Prepare plots of relative frequencies or cumulative relative frequencies on simple graph papers, taking the group interval as abscissa, and the relative frequencies or cumulative relative frequencies as ordinate.

It is to note that in the preceding chapter, the section on flow-duration curve involves use of the steps mentioned here.

6.6 STATISTICS OF GROUPED DATA

The sample statistics mean (m_1), variance (m_2), coefficient of skewness (m_3), and coefficients of kurtosis (m_4) are derived from the grouped data, respectively, as:

$$m_1 = \frac{\sum_{i=1}^{NC} f_i\, x_i}{\sum_{i=1}^{NC} f_i} \tag{6.11}$$

$$m_2 = \frac{1}{(N-1)} \sum_{i=1}^{NC} (x_i - m_1)^2 f_i \tag{6.12}$$

$$m_3 = \frac{\sum_{i=1}^{NC} f_i \, (x_i - m_1)^3}{\sum_{i=1}^{NC} f_i} \tag{6.13}$$

$$m_4 = \frac{\sum_{i=1}^{NC} f_i \, (x_i - m_1)^4}{\sum_{i=1}^{NC} f_i} \tag{6.14}$$

where, x_i is the mid-value of the i^{th} class interval, f_i is the number of values (n) in the i^{th} class, and N is the total number of values. Mean represents the first moment about origin; variance, the second moment of the grouped data about the mean; skewness, the third moment about the mean; and kurtosis, the fourth moment about the mean.

It is to note that standard deviation is the square root of the variance, and coefficient of variation is the ratio of standard deviation and mean of the grouped data.

6.7 PROBABILITY DISTRIBUTIONS

A distribution is an attribute of a statistical population. If each element of a population has a value of X, then the distribution describes the constitution of the population through its X-values. It reveals the following:

(a) whether X-values are too large or too small, and their location on the axis;
(b) whether the data are bunched together or spread out, and whether or not they are symmetrically disposed on the X-axis (these information can also be derived by mean, standard deviation, and skewness);
(c) whether the relative frequency or proportion of various X-values in the population similar to a histogram; and
(d) whether the relative frequency or probability, $P_r(x < X)$, that the an element x drawn randomly from the population will be less than a particular value of X.

When population is sufficiently large, the histogram of its X-values prepared using a very small class interval can be replaced by a smooth curve. The area enclosed between any two vertical ordinates yields the relative frequency or probability of X-values between those ordinates. Such an interpretation leads to the description of relative frequency distribution as probability distribution, and the curve describing the distribution is called a *probability density function* (PDF) whose cumulative function is known as the *distribution function*. In flood frequency analysis, the sample data is used to fit probability distribution, which in turn is used to extrapolate recorded events and design events either graphically or analytically by estimating the parameters of the distribution.

6.7.1 Continuous Probability Distribution

A number of frequency distributions are available in literature. Some of the distributions commonly used in hydrology and water resources are: normal, lognormal (two- and three-parameters), extreme value type-I (Gumbel or EV1), Pearson type-III, log-Pearson type-III, general extreme value (GEV), gamma, and exponential distributions. The probability density functions, cumulative density functions, and other properties of some of the commonly used theoretical frequency distributions are described below.

(i) Normal Distribution

The normal distribution is one of the most important distributions in statistical hydrology. This is a bell-shaped symmetrical distribution having coefficient of skewness equal to zero. The normal distribution enjoys unique position in the field of statistics due to central limit theorem. This theorem states that under certain broad conditions, the distribution of the sum of random variables tends to a normal distribution irrespective of the distribution of random variables, as the number of terms in the sum increases. The PDF and CDF of the distribution are given as:

$$\text{P.D.F.: } f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{6.15}$$

$$\text{C.D.F.: } F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx \tag{6.16}$$

In terms of reduced variate: $z = \dfrac{(x-\mu)}{\sigma}$

Using mean of reduce variate equal to zero and standard deviation of reduce variate equal to 1, Eqs. (6.15) and (6.16) become:

$$\text{P.D.F.: } f(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) \tag{6.17}$$

$$\text{C.D.F.: } F(z) = -\int_{\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz \tag{6.18}$$

Example 6.4 From a mean monthly rainfall data of 10 years, the mean is 65mm and standard deviation is 7.0. Find the percentage of magnitude of rainfall which is:

(a) between 70 mm and 80 mm

(b) at least 80 mm

Assume that the rainfall data are normally distributed. Use Appendix II to find the value of variate z.

Solution

(a) $z_1 = (70 - 65)/7 = -0.71$

$z_2 = (80 - 65)/7 = 2.14$

Here $(0 < z_1 < z_2)$; therefore,

$$P(70 \le x \le 80) = P(0.71 \le z \le 2.14)$$
$$= F(2.14) - F(0.71) = 0.9838 - 0.7612 = 0.2226$$

That is, approximately 22.26% of rainfall occurred having a magnitude range of 70 mm to 80 mm.

(b) $z_1 = (90 - 65)/7 = 3.57$

Here $0 < z_1$; therefore,

$$P(x \ge 90) = P(z \ge 3.57) = 1 - F(3.57) = 1 - 0.9998 = 0.0002$$

That is, approximately 0.02% of rainfall occurred having a magnitude of at least 90 mm.

Example 6.5 Compute $F(z)$ for z values of 0.25, 0.5, 1.0, 1.2, 1.5, and 2.5. Use the following approximations of $F(z)$:

(a) Approximation proposed by Swamee and Rathie, 2007:

$$F(z) = [\exp(-1.7255\, z\, |z|^{0.12}) + 1]^{-1}$$

(b) Approximation proposed by Abramowitz and Stegun, 1965:

$$B = \frac{1}{2}[1 + 0.196854|z| + 0.115194|z|^2 + 0.000344|z|^3 + 0.019527|z|^4]^{-4}$$

$$F(z) = \begin{cases} B & \text{for } (z < 0) \\ 1 - B & \text{for } (z \ge 0) \end{cases}$$

Solution

z	*Swamee and Rathie*	*Abramowitz and Stegun*
0.25	0.5903	0.5986
0.5	0.6886	0.6916
1.0	0.8488	0.8411
1.2	0.8924	0.8847
1.5	0.9380	0.9332
2.5	0.9919	0.9936

Exercise: Obtain $F(z)$ using the values given in Appendix II and compute the percentage error.

(ii) Lognormal Distribution (Two-parameter)

The causative factors for many hydrologic variables act multiplicatively rather than additively; and so, the logarithms of these variables which are the product of these causative factors follow the normal distribution.

If $y = \ln(x)$ follows normal distribution, then x is said to follow lognormal distribution. The PDF and CDF of the distribution are given as:

$$\text{P.D.F.:} \quad f(x) = \frac{1}{x\sigma_L\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu_L}{\sigma_L}\right)^2\right] \tag{6.19}$$

$$\text{C.D.F.:} \quad F(x) = \frac{1}{\sigma_L\sqrt{2\pi}}\int_0^x \frac{1}{x}\exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu_L}{\sigma_L}\right)^2\right]dx \tag{6.20}$$

where, μ_L = *mean of logarithms of* x, and σ_L = standard deviation of $\ln(x)$. Using mean and standard deviation of the reduced variate, 0 and 1 respectively; Eqs. (6.19) and (6.20) become:

$$\text{P.D.F.:} \quad f(z) = \frac{1}{\sqrt{2\pi}}\exp(-z^2/2) \tag{6.21}$$

$$\text{C.D.F.:} \quad F(z) = \int_0^z \frac{1}{\sqrt{2\pi}}\exp(-z^2/2)dz \tag{6.22}$$

Swamee and Rathie (2007) approximated the lognormal distribution as given below.

$$F(x) = \left\{\exp\left[-\frac{1.7255}{(\ln\sigma_g)^{1.12}}(\ln x - \ln\mu_g)|\ln x - \ln\mu_g|^{0.12}\right]+1\right\}^{-1} \tag{6.23}$$

where, μ_g is the geometric mean.

Example 6.6 If $x_1, x_2, x_3, \ldots x_n$ are log distributed, show that $\mu_L = \ln(\mu_g)$

Solution

$$\begin{aligned}\mu_L &= \frac{1}{N}\sum_{i=1}^{n}\ln x_i = \frac{1}{N}(\ln x_1 + \ln x_2 + \cdots + \ln x_n)\\ &= \frac{1}{N}\ln\prod_{i=1}^{n}(x_i)\\ &= \ln\left[\prod_{i=1}^{n}x_i\right]^{1/N}\\ &= \ln\mu_g\end{aligned}$$

Exercise: Compute $F(z)$ for 10 random numbers using Eq. (6.22) and compare it with the approximations proposed by Swamee and Rathie (2007).

(iii) Pearson type-III Distribution

Pearson type-III (PT3) distribution is a three-parameter distribution. This is also known as *gamma distribution* with three parameters. The PDF and CDF of the distribution are given as:

$$\text{P.D.F.:} \quad f(x) = \frac{(x-u)^{\gamma-1} e^{-(x-u)/\beta}}{\beta^{\gamma}\,\Gamma(\gamma)} \tag{6.24}$$

$$\text{C.D.F.:} \; F(x) = \int_u^x \frac{(x-u)^{\gamma-1} e^{-(x-u)/\beta}}{\beta^{\gamma}\,\Gamma(\gamma)} dx \tag{6.25}$$

In this distribution, u, β and γ are also known as location, scale and shape parameters, respectively.

In terms of reduced variate: $z = \dfrac{x-u}{\beta}$

$$\text{P.D.F.:} \quad f(z) = \frac{1}{|\beta|\,\Gamma(\gamma)} (z)^{\gamma-1} e^{-z} \tag{6.26}$$

$$\text{C.D.F.:} \; F(z) = \int_0^z \frac{1}{|\beta|\,\Gamma(\gamma)} (z)^{\gamma-1} e^{-z} \tag{6.27}$$

(iv) Exponential Distribution

Exponential distribution is a special case of Pearson type-III distribution wherein the shape parameter $\gamma = 1$. The PDF and CDF are given as:

$$\text{P.D.F.:} \quad f(x) = \frac{1}{\beta} \exp\left[\frac{-(x-u)}{\beta}\right] \tag{6.28}$$

$$\text{C.D.F.:} \; F(x) = 1 - \exp\left[\frac{-(x-u)}{\beta}\right] \tag{6.29}$$

In Eqs. (6.28) and (6.29), u = location parameter, and β = scale parameter.

In terms of reduced variate: $z = \dfrac{x-u}{\beta}$

$$\text{P.D.F.:} \quad f(z) = e^{-z} \tag{6.30}$$

$$\text{C.D.F.:} \; F(z) = 1 - e^{-z} \tag{6.31}$$

(v) General Extreme Value Distribution

The PDF and CDF of the distribution are given below:

$$\text{P.D.F.:} \quad f(x) = \frac{1}{\beta}\left[1-k\left(\frac{x-u}{\beta}\right)\right]^{\frac{1}{k}-1} \exp\left(-\left[1-k\left(\frac{x-u}{\beta}\right)\right]\right)^{-\frac{1}{k}} \tag{6.32}$$

$$\text{C.D.F.: } F(x) = \left[-\left(1 - k\left(\frac{x-u}{\beta}\right)\right)^{\frac{1}{k}} \right] \tag{6.33}$$

In Eqs (6.32) and (6.33), u = location parameter, β = scale parameter, and k = shape parameter.

If $k = 0$, it leads to EV-I distribution,

$k < 0$, it leads to EV-II distribution,

$k > 0$, it leads to EV-III distribution.

(vi) Log-Pearson Type-III (LP3) Distribution

If $y = \ln(x)$ follows Pearson type-III distribution, then x is said to follow log-Pearson type-III distribution. In 1967, the U.S. Water Resources Council recommended that LP3 distribution should be adopted as the standard flood frequency distribution by all U.S. federal government agencies. The PDF and CDF of the distribution are given as:

$$\text{P.D.F.: } f(x) = \frac{(\ln x - u)^{\gamma-1} \exp[-(\ln x - u)/\beta]}{|\beta|\,\Gamma(\gamma)} \tag{6.34}$$

$$\text{C.D.F.: } F(x) = \int_u^x f(x)\,dx \tag{6.35}$$

In Eqs (6.34) and (6.35), u = location parameter, β = scale parameter, and γ = shape parameter.

In terms of reduced variate: $z = \dfrac{\ln x - u}{\beta}$

$$\text{P.D.F.: } f(z) = \frac{1}{|\beta|\,\gamma}(z)^{\gamma-1}e^{-z} \tag{6.36}$$

$$\text{C.D.F.: } F(z) = \int_0^z f(z)\,dz \tag{6.37}$$

(vii) Gamma Distribution

It is a special case of PT3 distribution. The PDF and CDF are expressed as:

$$\text{P.D.F.: } f(x) = \frac{(x)^{\gamma-1}e^{-x/\beta}}{\beta^{\gamma}\Gamma(\gamma)} \tag{6.38}$$

$$\text{C.D.F.: } F(x) = \int_0^x \frac{(x)^{\gamma-1}e^{-x/\beta}}{\beta^{\gamma}\Gamma(\gamma)}\,dx \tag{6.39}$$

In Eqs. (6.38) and (6.39), β = scale parameter, and γ = shape parameter.

In terms of Gamma reduced variate: $z = \dfrac{x}{\beta}$

$$\text{P.D.F.:}\quad f(z) = \frac{1}{|\beta|\Gamma(\gamma)}(z)^{\gamma-1}e^{-z} \tag{6.40}$$

$$\text{C.D.F.:}\quad F(z) = \int_0^z \frac{1}{|\beta|\Gamma(\gamma)}(z)^{\gamma-1}e^{-z} \tag{6.41}$$

(viii) Gumbel Extreme Value Type-I (EV1) Distribution

One of the most commonly used distributions in flood frequency analysis is the double exponential distribution (also known as Gumbel distribution or extreme value type-I or Gumbel EV1 distribution). The PDF and CDF of the distribution are given below.

$$\text{P.D.F.:}\quad f(x) = \frac{1}{\beta}\exp\left[-\left(\frac{x-u}{\beta}\right) - \exp\left\{-\left(\frac{x-u}{\beta}\right)\right\}\right] \tag{6.42}$$

$$\text{C.D.F.:}\quad F(x) = e^{-\exp\left[-\left(\frac{x-u}{\beta}\right)\right]} \tag{6.43}$$

In terms of reduced variate: $z = \dfrac{x-u}{\beta}$

$$\text{P.D.F.:}\quad f(z) = e^{-z-(e^{-z})} \tag{6.44}$$

$$\text{C.D.F.:}\quad F(z) = e^{-e^{-z}} \tag{6.45}$$

Example 6.7 Show the effect of location parameter in the Gumbel extreme value distribution.

Solution

Using the different values of x, find the value of $f(x)$ using Eq. (6.42). Make the value of u twice by keeping α as constant. Plot the values of x vs. $f(x)$. From the Fig. 6.1, it can be concluded that by changing the location parameter u from 0.18 to 0.36 and keeping shape parameter $\beta = 0.26$ constant, the peak value is shifted to right side.

Example 6.8 Find out the value of the reduced variates for EV1 distribution corresponding to probability of non-exceedence of 0.30 from Appendix II.

Solution

For EV1 distribution, the reduced variate corresponding to probability of non-exceedence of 0.3 can be obtained as:

$$0.30 = e^{-(e^{-z})}$$

$$z = -\ln[-\ln(0.30)] = -0.186$$

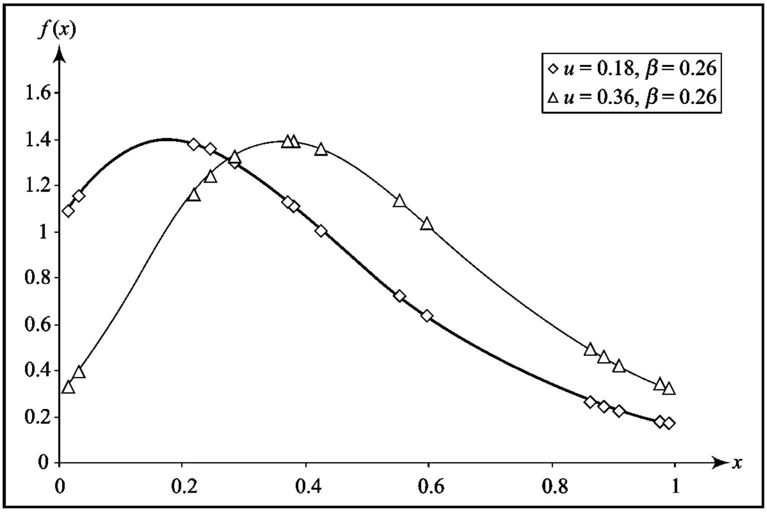

Fig. 6.1 *Effect of location parameter in the Gumbel extreme value distribution*

Exercise: Examine the effect of shape parameter β in GEV distribution.

6.7.2 Discrete Probability Distributions

The use of discrete probability distributions is restricted generally to those random events for which the outcome can be described as a success or a failure. Discrete probability distribution is useful for those experiments wherein only two mutually exclusive events exist. Furthermore, successive trials are independent and the probability of success remains constant from one trial to another. Binomial and Poisson distributions are commonly used in discrete probability distribution analyses.

(i) Binomial distribution

The binomial distribution is based on the binomial theorem which states—probability of exactly r successes in n trials is:

$$P_{r,n} = {}^{n}C_r (P^r)(q^{n-r}) = \frac{n!}{r!\,(n-r)!}\, p^r q^{n-r} \tag{6.46}$$

where, $P_{r,n}$ = probability of a random hydrologic event of a given magnitude and exceedence occurring r times in n successive years, p = probability of exceedence of a single event, q = probability of non-exceedence or failure which is equal to $(1 - p)$, r = number of exceedence/successes, and n = total number of events.

The assumptions for binomial distribution are the same as for Bernoulli trials, which, for example, are based on tossing of a coin or drawing a card from a pack. Bernoulli trials operate under three conditions:

(i) the outcome of any trial can be two mutually exclusive events only, i.e., any trial can have either success or failure, true or false;
(ii) successive trials are independent; and
(iii) probabilities are stable.

The application of Eq. (6.46) can also be explained as follows:

(i) The probability of an event of exceedence occurring z times in n successive years will be:

$$P_{z,n} = \frac{n!}{z!(n-z)!} p^z q^{n-z} \tag{6.47}$$

(ii) The probability of the event not occurring at all in n successive years will be:

$$P_{0,n} = q^n = (1-p)^n \tag{6.48}$$

(iii) The probability of the event occurring at least once in n successive years will be:

$$p_1 = 1 - q^n = 1 - (1-p)^n \tag{6.49}$$

Example 6.9 A maximum 1-day rainfall depth of 300 mm at a station shows a return period of 50 years. Determine the probability of 1-day rainfall depth equal to or greater than 300 mm occurring (a) once in 14 successive years, (b) twice in 10 successive years, and (c) at least once in 25 successive years.

Solution

Here, $P = 1/50 = 0.02$

Probabilities for the given three cases can be determined as follows:

Case (a): $n = 14$ and $r = 1$

Therefore, $P_{1,14} = \dfrac{14!}{1!(14-1)!}(0.02)^1\,(0.98)^{13}$

$$= 14 \times 0.02 \times 0.7093 = 0.2153$$

Case (b): $n = 10$ and $r = 2$

Therefore, $P_{2,10} = \dfrac{10!}{2!(10-2)!}(0.02)^2\,(0.98)^8$

$$= \frac{10 \times 9}{2} \times (0.02)^2 \times (0.98)^8 = 0.0153$$

Case (c): From Eq. (6.49), $P_1 = 1 - (1 - 0.02)^{25} = 0.3965$

(ii) Poisson Distribution

It is sometimes difficult to compute the binomial distribution for large values of n. If n is large (> 30) and p is small (< 0.1), then the binomial distribution tends to behave as Poisson distribution which is defined as:

$$P_{r,n} = \frac{\lambda^r e^{-\lambda}}{r!} \tag{6.50}$$

where, $\lambda = n\,p$. Conditions for this approximation are:

(i) the number of events is a discrete integer,
(ii) two events cannot coincide,
(iii) the mean number of events in unit time is constant, and
(iv) events are independent.

Binomial random variable is a count of the number of successes in certain number of trials whereas the negative binomial distributed variable is literally opposite of that; wherein the number of successes is predetermined, and the number of trials is random.

(iii) Negative Binomial Distribution

The negative binomial distribution is the probability of having to wait $X (= r + k - 1)$ trials to obtain $(r - 1)$ successes, and a success in $(r + k)$ trials. If x has a NB distribution with parameters p and r, the probability mass function is given by:

$$P(x) = \binom{r+k-1}{r-1} p^r (1-p)^k; \quad (r > 0) \text{ and } (0 \le p \le 1) \tag{6.51}$$

In Eq. (6.51), p is the constant probability of a success in any independent trial. $P(x = r)$ gives the probability that the variable x is equal to r. Because at least r trials are required to get r successes, the range of Eq. (6.51) is from r to ∞. In a special case where r is equal to 1, the negative binomial random variable becomes a geometric random variable. The probability that there is no flood in n years of record is obtained by making $r = 1$ in Eq. (6.51) as:

$$P[(r-1) = 0] = \binom{k}{0} (1-p)^k p = (1-p)^k\, p \tag{6.52}$$

Equation (6.52) gives the probability of no success in $(r + k - 1)$ trials.

6.8 PROBABILITY FUNCTIONS FOR HYPOTHESIS TESTING

Probability functions frequently used for hypothesis testing in hydrologic analyses include the following distributions: normal, student's t-distribution, chi-square, and F-distributions. The last three are primarily used to statistically test a hypothesis of concern. The normal distribution is, however, used for both prediction of the variable and testing. This testing is based on the derivation of critical values, derivable from generally available tables.

6.8.1 Normal Distribution

For normal distribution, the PDF and CDF are derived in Section 6.7.1. It is to be noted that μ and σ represent the population parameters of the distribution. The best estimates of μ and σ for the sample data are taken as the sample $\bar{x}$ and the standard deviation (S), respectively. Thus, for computing the probability based on sample statistics, $\bar{x}$ and S are substituted for μ and σ, respectively. It is possible to compute probabilities by integrating the PDF over a range of x-values; however, since there exist an infinite number of values of $\bar{x}$ and S for the infinite number of samples in the population, it is rather impossible to integrate numerically. It can, however, be circumvented by transformation of the random variable x, as follows.

If the random variable x has a normal distribution, a new random variable z can be defined as:

$$z = \frac{x-\mu}{\sigma} \tag{6.53}$$

where, z is the standardized variate or normal reduced variate with mean and standard deviation of 0 and 1, respectively. This transformation enables PDF to be expressed as a function of z, which is called the *standard normal distribution*.

Probabilities can be computed using Appendix II. In this table, z-values down the left margin are incremented by 0.1 and by 0.01 across the top. The corresponding $F(z)$ value is also listed in the table and can be used to compute probability of z lying within any interval of z. While adopting these values, one should make use of symmetric nature of normal distribution about $z = 0$.

6.8.2 Chi-square Distribution

Let $z_1, z_2, z_3 \ldots, z_k$ be normally and independently distributed random variables, with mean $\mu = 0$ and variance $\sigma^2 = 1$. Then, the random variable χ^2 $(= z_1^2 + z_2^2 + z_3^2 + \cdots + z_k^2)$ has the following PDF:

$$f(u) = \begin{cases} \dfrac{1}{2^{k/2}\Gamma\left(\dfrac{k}{2}\right)} u^{\frac{k}{2}-1} e^{-\frac{u}{2}}, & u > 0 \\ 0 & , \text{ otherwise} \end{cases} \tag{6.54}$$

which is called as chi-square distribution with k degrees of freedom and can be abbreviated as χ_k^2. The mean and variance of the χ_k^2 distribution are $\mu = k$ and $\sigma^2 = 2k$. It is to note that the chi-square random variable is non-negative, and the probability distribution is skewed to the right. As k increases, the distribution becomes more symmetric, and as $k \to \infty$, the limiting form of the chi-square distribution behaves like normal distribution.

The chi-square (χ^2) distribution is similar to t-distribution in that it is a function of a single parameter n, which denotes the degrees of freedom. At the same time, it differs from t-distribution in that the distribution is not symmetric. The table of chi-square values (Appendix IV) is identical in structure to the table for t-distribution (Appendix III), with n down the left margin, the probability across the top, and χ^2-values in the table. For example, for 11 degrees of freedom and 5% of the area at the left tail, χ^2 varies from 0 to 4.575. For 17 degrees of freedom, there is a probability of 0.025 that χ^2 will lie between 30.191 and ∞.

6.8.3 Student's t-Distribution

Let $z \sim N(0,1)$ and v be a chi-square random variable with k degrees of freedom. If z and v are independent, then the random variable t is given by:

$$t = \frac{z}{\sqrt{v/k}}$$

The PDF of the random variable is:

$$f(t) = \frac{\Gamma[(k+1)/2]}{\sqrt{\pi k}\,\Gamma(k/2)} \frac{1}{[(t^2/k)+1]^{(k+1)/2}}, \quad (-\infty < t < \infty) \tag{6.55}$$

This is called as t-distribution with k degrees of freedom. It is abbreviated as t_k. The student's t-distribution is also a symmetric distribution, similar to the normal distribution. However, it has only one parameter, v, which represents degrees of freedom. This parameter controls the spread of t-distribution. Thus, the structure of the table of t-values (Appendix III) slightly differs from the normal table. In Appendix III, the value of v is given at the left margin, and the probability at the right tail of the distribution is given across the top.

For example, for $v = 7$ and a probability of 0.05, the t-value is 1.895. Since t-distribution is symmetric, 5% of the area at the left tail is to the left of a t-value of -1.895 for $v = 7$. As a result, for 5% of the area (2.5% at each tail), critical t-values are -2.365 and 2.365 for $v = 7$. It is noted that for $v > 30$, the normal distribution is generally used in place of t-distribution.

6.8.4 F-Distribution

Let w and y be independent chi-square random variables with u and v degrees of freedom, respectively. Then the ratio $f = \dfrac{w/u}{y/v}$ has the following probability density function:

$$h(f) = \frac{\Gamma\left(\frac{u+v}{2}\right)\left(\frac{u}{v}\right)^{u/2} f^{(u/2)-1}}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)\left[\frac{u}{v}f+1\right]^{(u+v)/2}}, \quad (0 < f < \infty) \tag{6.56}$$

This is called as F-distribution with u degrees of freedom in the numerator and v degrees of freedom in the denominator. It can be abbreviated as $F_{u,v}$.

The F-distribution is a function of two parameters, m and n, and these are tabulated in Appendix V. The value of F corresponding to the appropriate probability in the right tail of the distribution can be derived from this table. For example, if $m = 10$ and $n = 20$, there is 5% chance that F will be greater than 2.35. It is noted that the table is not symmetric. For $m = 20$ and $n = 10$, the critical value for 5% area in the right tail is 2.77. Thus, the values of m and n must not be switched.

Example 6.10 Estimate the value of location parameter (u) and scale parameter (β) for the density function given below.

$$f(x) = \begin{cases} \dfrac{1}{\beta - u} & (u \le x \le \beta) \\ 0 & \text{otherwise} \end{cases}$$

Solution

The first moment or mean of the distribution is:

$$\begin{aligned} \mu_1 &= \int_{-u}^{\infty} x f(x)\, dx \\ &= \int_u^{\beta} x\left(\frac{1}{\beta - u}\right) dx \\ &= \left(\frac{1}{\beta - u}\right)\int_u^{\beta} x\, dx \\ &= \left(\frac{1}{\beta - u}\right)\left[\frac{x^2}{2}\right]_u^{\beta} \\ &= \frac{\beta^2 - u^2}{2(\beta - u)} = \frac{\beta + u}{2} \end{aligned}$$

The second moment about the mean is variance, and it can be determined as follows:

$\sigma^2 = \mu_2 - (\mu_1)^2$, where μ_2 is the second moment about the origin.

$$\begin{aligned} \sigma^2 &= \int_u^{\beta} x^2 \frac{1}{\beta - u}\, dx - \left(\frac{\beta + u}{2}\right)^2 \\ &= \frac{1}{\beta - u}\left[\frac{x^3}{3}\right]_{\alpha}^{\beta} - \left(\frac{\beta + u}{2}\right)^2 \\ &= \frac{(\beta - u)^2}{12} \end{aligned}$$

where, α and β are the two unknowns. By solving the two equations, we can find out the value of α and β. After solving, we get:

$$u = \mu - \sqrt{3}\sigma$$

$$\beta = \mu + \sqrt{3}\sigma$$

6.9 STATISTICAL ANALYSIS FOR LINEAR REGRESSION

The simple linear form of the equation is:

$$Y = a + bX \tag{6.57}$$

where, a and b are regression coefficients, computed respectively, as:

$$b = \frac{\sum_{i=1}^{n} (X_i - \bar{x})(Y_i - \bar{y})}{\sum_{i=1}^{n} (X_i - \bar{x})^2} \tag{6.58}$$

and $$a = \bar{y} - b\bar{x} \tag{6.59}$$

where, $$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \text{ and } \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{6.60}$$

6.9.1 Best-fit Evaluation Criteria

(a) *Coefficient of determination* (r^2): It describes the extent of best-fit and is expressed as:

$$r^2 = \frac{[\Sigma(x_i - \bar{x})\ (y_i - \bar{y})]^2}{[\Sigma(x_i - \bar{x})^2 \Sigma(y_i - \bar{y})^2]} \tag{6.61}$$

(b) *Coefficient of correlation* (r): It is the square root of the coefficient of determination (r^2):

$$r = \sqrt{r^2} = \frac{[\Sigma(x_i - \bar{x})(y_i - \bar{y})]}{[\Sigma(x_i - \bar{x})^2 (y_i - \bar{y})^2]^{1/2}} \tag{6.62}$$

(c) *Efficiency* (EFF): The efficiency of the best-fit is given as:

$$\text{EFF} = 1 - \frac{S^2}{S_y^2} \tag{6.63}$$

where, $$S^2 = \frac{\Sigma(y_i - \hat{y}_i)^2}{(n-2)} \tag{6.64}$$

$$S_y^2 = \frac{\Sigma(y_i - \bar{y}_i)^2}{(n-1)} \tag{6.65}$$

where, $\hat{y}_i$ is the computed value of y_i, and n is the total number of data points (x_i, y_i).

(d) *Standard error* (SE): Standard errors of the estimated regression coefficients a and b are computed, respectively as:

$$SE_a = S\left(\frac{1}{n} + \frac{(\bar{x})^2}{\Sigma(x_i - \bar{x})^2}\right)^{1/2} \tag{6.66}$$

$$SE_b = \frac{S}{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]^{1/2}} \tag{6.67}$$

where, SE_a and SE_b are standard errors of coefficients a and b, respectively. Square of standard errors in Eqs. (6.66) and (6.67) will give variance of the parameters a and b.

(e) *Confidence interval*: The confidence interval for a is given as:

$$l_a = a - t_{\left(1-\frac{\alpha}{2}\right),\,(n-2)} SE_a \tag{6.68}$$

$$u_a = a + t_{\left(1-\frac{\alpha}{2}\right),\,(n-2)} SE_a \tag{6.69}$$

and for b, it is given as:

$$l_b = b - t_{\left(1-\frac{\alpha}{2}\right),\,(n-2)} SE_b \tag{6.70}$$

$$u_b = b + t_{\left(1-\frac{\alpha}{2}\right),\,(n-2)} SE_b \tag{6.71}$$

where, l_a and l_b denote lower confidence limits of a and b, respectively; u_a and u_b denote upper confidence limits of a and b, respectively; α is the confidence level; and $t_{\left(1-\frac{\alpha}{2}\right),\,(n-2)}$ represent t-values corresponding to $(1 - \alpha/2)$ confidence limits and $(n - 2)$ degrees of freedom.

The requisite t-values can be obtained from the table given in Appendix III by referring to the column corresponding to $\frac{\alpha}{2}$ and $(n - 2)$. In many textbooks, e.g., Hines et al. (2003), expressions are available for $100\,(1 - \alpha)$ % confidence interval. Thus, the t-value will correspond to $\frac{\alpha}{2}$ and $(n - 2)$ also, as the t-distribution is symmetric about zero.

For describing the confidence interval, $t_{\frac{\alpha}{2},\,(n-2)}$ can also be substituted in place of $t_{\left(1-\frac{\alpha}{2}\right),\,(n-2)}$ in view of the symmetric distribution (Hines et al., 2003). Thus, the upper 5% of the t-distribution with 10 degrees of freedom is $t_{0.05,\,10} = 1.812$. Similarly, the lower-tail point $t_{0.95,\,10} = -t_{0.05,\,10} = 1.812$.

6.9.2 Test of Hypothesis

To test whether regression equation is acceptable, regression coefficients a and b are utilized to compute t-statistics and these are subjected to test of hypothesis. Hypothesis tests are of following types: H_o, and H. In test H_o, the concern is to test whether a parameter is equal to true value. In case, H_o is rejected, H holds good. We use notations H_a and H_b to represent hypothesis H for regression parameters a and b.

Hypothesis H_o: $a = a_0$ versus H_a: $a \neq a_0$ is tested by computing:

$$t = (a - a_0)/SE_a \tag{6.72}$$

where, H_o is the null hypothesis, and represents a statement of equality (=). The alternate hypothesis (H_a) is a statement of inequality which can be either one sided (< or >) or two sided (≠).

H_o is rejected if $|t| > t_{\left(1-\frac{\alpha}{2}\right),(n-2)}$

The t-statistic for b, is given by $t_b = (b - b_0)/SE_b$

Hypothesis H_o: $b = b_0$ is tested by computing:

$$t = (b - b_0)/SE_b \tag{6.73}$$

H_o is rejected if $|t| > t_{\left(1-\frac{\alpha}{2}\right),(n-2)}$

Normally, a_0 and b_0 are assumed as zero in hypothesis testing related to linear regression. This helps in computing t. It is to note that a_0 and b_0 will not always be zero.

Hypothesis H_o: $b = 0$ can be tested by computing:

$$t = (b - 0)/SE_b \tag{6.74}$$

H_o is rejected if:

(a) $|t| > t_{\left(1-\frac{\alpha}{2}\right),(n-2)}$, and

(b) the regression equation explaining a significant amount of variation in Y.

Example 6.11 Using the data of mean monthly rainfall given in Example 6.1, can we conclude that mean rainfall is 12 mm. Assume the level of significance is 0.05.

Solution

Even though the sample mean is less than the hypothesized mean of 12 mm, the monthly rainfall exceeded 4 times 12 mm. Thus, one sided alternative hypothesis is considered.

H_o: Mean = 12 mm

H_A: Mean > 12 mm

$\alpha = 0.05$

From Example 6.1, we know that sample mean = 11.58, $S = 7.229$, and $n = 12$.

$$t = \frac{11.58 - 12.0}{7.229/\sqrt{12}} = -0.2012$$

Degree of freedom $v = n - 1 = 12 - 1 = 11$

For $v = 11$ and $\alpha = 0.05$, $t_a = 1.796$ (See Appendix III)

Since $t_a > t$, H_o is accepted. Acceptance of null hypothesis indicates rejection of alternate hypothesis. The test indicates that the true mean is likely to be 12 mm rather than some value significantly greater than 12 mm.

6.9.3 Confidence Intervals for Line of Regression

The lower and upper confidence intervals are expressed, respectively, as:

$$L = \hat{y}_k - SE_{\hat{y}_k} t_{\left(1-\frac{\alpha}{2}\right), (n-2)} \tag{6.75}$$

$$U = \hat{y}_k + SE_{\hat{y}_k} t_{\left(1-\frac{\alpha}{2}\right), (n-2)} \tag{6.76}$$

where, L and U represent lower and upper confidence limits, respectively and

$$\hat{y}_k = a + bx_k \tag{6.77}$$

and
$$SE_{\hat{y}_k} = S\left[\frac{1}{n} + \frac{(x_k - \bar{x})^2}{\sum(x_i - \bar{x})^2}\right]^{1/2} \tag{6.78}$$

which is the standard error of $\hat{y}_k$.

6.9.4 Confidence Intervals of Individual Predicted Values

Confidence intervals of individual predicted values of y are given as:

$$L' = \hat{y}_k - SE_{\hat{y}_k} t_{\left(1-\frac{\alpha}{2}\right), (n-2)} \tag{6.79}$$

$$U' = \hat{y}_k + SE_{\hat{y}_k} t_{\left(1-\frac{\alpha}{2}\right), (n-2)} \tag{6.80}$$

$$SE_{\hat{y}_k} = S\left[1 + \frac{1}{n} + \frac{(x_k - \bar{x})^2}{\sum(x_i - \bar{x})^2}\right]^{1/2} \tag{6.81}$$

The above statistical analysis for the linear regression is illustrated below.

Example 6.12 For a typical catchment, the following precipitation and corresponding runoff data were observed in the month of July:

Table 6.1 Year-wise precipitation and runoff values

Year	1953	1954	1955	1956	1957	1958	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968
Precipitation (mm)	42.39	33.48	47.67	50.24	43.28	52.60	31.06	50.02	47.08	47.08	40.89	37.31	37.15	40.38	45.39	41.03
Runoff (mm)	13.26	3.31	15.17	15.50	14.22	21.20	7.70	17.64	22.91	18.89	12.82	11.58	15.17	10.40	18.02	16.25

(a) Develop a linear rainfall-runoff relationship using the above data.
(b) How much variation in runoff is accounted for by the developed relationship?
(c) Compute 95% confidence interval for regression coefficients.
(d) Test the hypotheses that (i) the intercept is equal to zero, and (ii) the tangent is equal to 0.500.
(e) Calculate 95% confidence limits for the fitted line.
(f) Calculate 95% confidence interval for an individual predicted value of the dependent variable.
(g) Comment on the extrapolation of the developed linear relationship.

Solution

(a) Assume the form of linear equation is: $Y = a + bX$, where Y is the runoff, X is the rainfall, and a and b are regression coefficients. The coefficients can be determined using Eqs (6.58) and (6.59).

$$b = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{369.432}{570.0559} = 0.648$$

and $a = \bar{y} - b\bar{x} = 14.63 - (0.648 \times 42.94) = -13.1951$

Thus, the regression equation is: $Y = -13.1951 + 0.648\,X$.

(b) The variation in Y accounted by the regression is determined using the coefficient of determination (r^2) as:

$$r^2 = \frac{[\sum(x_i - \bar{x})(y_i - \bar{y})]^2}{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}$$

$$= \frac{[\sum(x_i - \bar{x})(y_i - \bar{y})]}{\sum(x_i - \bar{x})^2} \times \frac{[\sum(x_i - \bar{x})(y_i - \bar{y})]}{\sum(y_i - \bar{y})^2}$$

$$= b \times \frac{[\sum(x_i - \bar{x})(y_i - \bar{y})]}{\sum(y_i - \bar{y})^2} = 0.648 \times \frac{369.432}{363.0714} = 0.66$$

$r^2 = 0.66$ also implies that the regression equation explains 66% of variation of Y.

The coefficient of correlation (r) can be given by the square root of (r^2) and can be computed as: $r = \sqrt{0.66} = 0.81$. It is noted that r^2 can vary between 0 and 1, whereas r can vary from –1 to +1.

(c) The 95% confidence interval for coefficients a and b is determined as follows:

(i) Compute the standard error of the regression equation as:

$$S^2 = \Sigma(y_i - \hat{y}_i)2/(n-2) = (123.7)/(16-2) = 8.83$$

$$\Rightarrow \quad S = 2.97$$

(ii) Compute standard error of a using Eq. (6.66):

$$SE_a = S\left(\frac{1}{n} + \frac{\bar{x}^2}{\sum(x_i - \bar{x})^2}\right)^{1/2}$$

$$= 2.97\left(\frac{1}{16} + \frac{42.94 \times 42.94}{570.0559}\right)^{1/2}$$

$$= 5.39$$

(iii) Compute standard error of b (SE_b) using Eq. (6.67):

$$SE_b = \frac{S}{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]^{1/2}} = \frac{2.97}{(570.0559)^{1/2}}$$

$$= 0.124$$

(iv) Compute $t_{\left(1-\frac{\alpha}{2}\right),(n-2)}$ from t-table (Appendix III) for $\alpha = 0.05$ and $n = 16$. It is equal to 2.145.

(v) Compute 95% confidence interval for a using Eqs (6.68) and (6.69).

$$l_a = a - t_{\left(1-\frac{\alpha}{2}\right),(n-2)} SE_a$$

$$= -13.1951 - (2.145 \times 5.39) = 24.76$$

$$u_a = a + t_{\left(1-\frac{\alpha}{2}\right),(n-2)} SE_a$$

$$= -13.1951 + (2.145 \times 5.39) = -1.63$$

(vi) Similarly, compute 95% confidence interval for b using Eqs. (6.70) and (6.71).

$$l_b = b - t_{\left(1-\frac{\alpha}{2}\right),(n-2)} SE_b = 0.648 - (2.145 \times 0.124) = 0.38$$

$$u_b = b + t_{\left(1-\frac{\alpha}{2}\right),(n-2)} SE_b = 0.648 + (2.145 \times 0.124) = 0.91$$

(d) In the assumed linear relation: $Y = a + bX$, where a is the intercept, and b is the tangent the first hypothesis is tested for case (i) that $a = 0$ versus $a \neq 0$ (or H_o: $a = 0$ versus $a \neq 0$). To this end, the t-statistic is computed:

$$t = (a - a_0)/SE_a = (-13.1951 - 0.0)/5.39 = -2.44$$

Note that $a_0 = 0$. Then, determine the critical t-value for $\alpha = 95\%$ and $n = 16$, which is equal to 2.145, as computed in step c (iv).

Since $|t| > t_{\left(1-\frac{\alpha}{2}\right),(n-2)}$ $(= t_{0.975,\,14})$, reject H_o: $a = 0$. It implies that a cannot be equal to zero.

Similarly, the hypothesis H_o: $b = 0.5$ versus $a \neq 0.5$ can be tested as (Eq. 6.73):

$$t = (b - b_0)/SE_b = (0.648 - 0.5)/0.124 = 1.193$$

which is less than $t_{0.975,\,14}$ (= 2.145). H_o: $b = 0.5$ cannot be rejected.

The above tests imply that the intercept is significantly different from zero whereas b is not significantly different from 0.5. The significance of the overall regression can be evaluated by testing H_o: $b = 0$ by computing t for $b_0 = 0$, which comes out to be 5.184. Since this value of 5.184 is greater than the above critical t-value of 2.145, the hypothesis H_o: $b = 0$ is rejected.

(e) The 95% limits for the regression line are computed in the following steps:

Compute the standard error $\hat{Y}_k$ as (Eq. 6.78):

$$SE_{\hat{y}_k} = S\left[\frac{1}{n} + \frac{(x_k - \bar{x})^2}{\sum(x_i - \bar{x})^2}\right]^{1/2}$$

$$= 2.97 \times \left[\frac{1}{16} + \frac{(x_k - 42.94)^2}{570.0559}\right]^{1/2}$$

Thus, the lower and upper limits can be given by Eqs. (6.75) and (6.76), respectively, as:

$$L = \hat{y}_k - SE_{\hat{y}_k}\, t_{\left(1-\frac{\alpha}{2}\right),(n-2)}$$

$$U = \hat{y}_{y_k} + SE_{\hat{y}_k}\, t_{\left(1-\frac{\alpha}{2}\right),(n-2)}$$

The numerical values of these limits for different values of x are given in Table 6.2.

Table 6.2 Computation of lower and upper limits for the fitted line

x	y	$\hat{y}_k$	L	U
42.39	13.26	14.27	12.67	15.87
33.48	3.31	8.50	5.52	11.48
47.67	15.17	17.69	15.66	19.72
50.24	15.50	19.36	16.85	21.87
43.28	14.22	14.85	13.25	16.44
52.60	21.20	20.89	17.86	23.91
31.06	7.70	6.93	3.39	10.47
50.02	17.64	19.22	16.75	21.68
47.08	22.91	17.31	15.38	19.25
47.08	18.89	17.31	15.38	19.25
40.89	12.82	13.30	11.62	14.98
37.31	11.58	10.98	8.79	13.16
37.15	15.17	10.87	8.66	13.09
40.38	10.40	12.97	11.24	14.70
45.39	18.02	16.21	14.50	17.93
41.03	16.20	13.39	11.72	15.06

(f) The computation of 95% confidence limits for individual predicted Y-values can be performed as follows:

(i) Compute the standard error of individual predicted values as (Eq. 6.81):

$$SE_{\hat{y}_k} = S\left[1 + \frac{1}{n} + \frac{(x_k - \bar{x})^2}{\sum (x_i - \bar{x})^2}\right]^{1/2}$$

$$= 2.97 \times \left[1 + \frac{1}{16} + \frac{(x_k - 42.94)^2}{570.0559}\right]^{1/2}$$

(ii) Compute 95% confidence limits for individual predicted Y-values as (Eqs. 6.79 and 6.80):

$$L' = \hat{y}_k - SE_{\hat{y}_k} t_{\left(1-\frac{\alpha}{2}\right),(n-2)}$$

$$= (-13.1951 + 0.648\,x_k)$$

$$-\,2.97 \times \left[1.0625 + \frac{(x_k - 42.94)^2}{570.0559}\right]^{1/2} \times 2.145$$

$$U' = \hat{y}_k - SE_{\hat{y}_k} t_{\left(1-\frac{\alpha}{2}\right),(n-2)}$$

$$= (-13.1951 + 0.648\,x_k)$$

$$+\,2.97 \times \left[1.0625 + \frac{(x_k - 42.94)^2}{570.0559}\right]^{1/2} \times 2.145$$

The computed numerical values for the lower and upper limits are shown in Table 6.3.

Table 6.3 Computation of lower and upper limits for the fitted line

x	y	$\hat{y}_k$	L'	U'
42.39	13.26	14.27	7.71	20.83
33.48	3.31	8.50	1.47	15.52
47.67	15.17	17.69	11.02	24.37
50.24	15.50	19.36	12.52	26.20
43.28	14.22	14.85	8.29	21.40
52.60	21.20	20.89	13.84	27.93
31.06	7.70	6.93	-0.35	14.21
50.02	17.64	19.22	12.39	26.04
47.08	22.91	17.31	10.66	23.96
47.08	18.89	17.31	10.66	23.96
40.89	12.82	13.30	6.72	19.88
37.31	11.58	10.98	4.25	17.70
37.15	15.17	10.87	4.14	17.61
40.38	10.40	12.97	6.38	19.56
45.39	18.02	16.21	9.63	22.80
41.03	16.20	13.39	6.81	19.97

(g) The extrapolation of a regression equation beyond the range of x used in estimating a and b are generally discouraged for two reasons. First, the confidence interval of a fitted line increases as x-values increase beyond their mean; secondly the relation between Y and X may be nonlinear, whereas a linear relation is fitted for the prescribed range of x-values.

6.10 MULTIPLE LINEAR REGRESSION

When a hydrological variable depends on more than one independent variable, the prediction model can be developed using the multiple regression approach.

Forms of Equation

The linear form of equations is given as:

$$
\begin{aligned}
y_1 &= \beta_1 x_{1,1} + \beta_2 x_{1,2} + \ldots + \beta_p x_{1,p} \\
y_2 &= \beta_1 x_{2,1} + \beta_2 x_{2,2} + \ldots + \beta_p x_{2,p} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
y_n &= \beta_1 x_{n,1} + \beta_2 x_{n,2} + \ldots + \beta_p x_{n,p}
\end{aligned}
\tag{6.82}
$$

where, y_i is the i^{th} observation on the dependent variable.

$$y_i = \sum_{j=1}^{p} \beta_j \, x_{i,j} \tag{6.83}$$

in which p is the number of independent variables. In matrix notation,

$$(Y)_{n \times 1} = (X)_{n \times p} (\beta)_{p \times 1} \tag{6.84}$$

Regression Coefficients

The vector β which represents the set of regression coefficient values is computed using the matrix operation in the following relationship:

$$(\beta)_{p \times 1} = [(X^T)_{p \times n} (X)_{n \times p}]^{-1}_{p \times p} (X^T)_{p \times n} (Y)_{n \times 1} \tag{6.85}$$

where, X^T is transpose of the matrix X of size $(n \times p)$.

Correlation Matrix of the Independent Variables

The correlation among the independent variables is necessary in order to have a better prediction equation for dependent variable. This correlation matrix is computed as:

$$\text{Let } Z_{i,j} = (X_{i,j} - \overline{x}_j)/S_j \tag{6.86}$$

where, $\overline{x}_j$ and S_j are mean and standard deviation of the j^{th} independent variable, respectively.

$$Z = [Z_{i,j}] \tag{6.87}$$

So, the correlation matrix is:

$$R = \frac{Z^T Z}{n-1} = [R_{i,j}] \tag{6.88}$$

where, $R_{i,j}$ is the correlation between the i^{th} and j^{th} independent variables. R is a symmetric matrix since $R_{i,j} = R_{j,i}$

Coefficient of Determination

The statistic "coefficient of determination" is defined as the proportion of the total variability in Y, as explained by the regression model. This statistic, denoted by R^2, is defined by:

$$R^2 = (\beta^T X^T Y - n\overline{Y}^2)/(Y^T Y - n\overline{Y}^2) \tag{6.89}$$

Here, β^T is the transpose of vector β, and Y^T is the transpose of vector Y. In case of multiple regression, the word "coefficient of determination" is also referred as "coefficient of multiple determination".

Coefficient of Correlation

The coefficient of correlation is defined as the square root of the coefficient of determination. It is denoted by R.

Efficiency (EFF)

The efficiency of the fitted regression model is defined as:

$$\text{EFF} = 1 - \frac{S^2}{S_y^2} \tag{6.90}$$

where, $S^2 = \Sigma(y_i - \hat{y}_i)^2/(n-p)$ (6.91)

$S_y^2 = \Sigma(y_i - \bar{y})^2/(n-1)$ (6.92)

Inferences on Regression Coefficients

(i) Standard errors S_{β_i} of β_i:

Let $C = X^T X$, then

$C^{-1} = (X^T X)^{-1}$ and $\mathrm{Var}(\beta_i) = C_i^{-1} S^2$ (6.93)

where, C_i^{-1} is the i^{th} diagonal element of $(X^T X)^{-1}$.

Therefore, $S_{\beta_i}^2 = C_i^{-1} S^2$ (6.94)

(ii) Confidence intervals on β_i:

$$L_{\beta_i} = \beta_i - t_{\left(1-\frac{\alpha}{2}\right),\,(n-2)} S_{\beta_i} \tag{6.95}$$

$$U_{\beta_i} = \beta_i + t_{\left(1-\frac{\alpha}{2}\right),\,(n-2)} S_{\beta_i} \tag{6.96}$$

(iii) Test of hypothesis concerning β_i:

Hypothesis H_o: $\beta_i = \beta_0$ versus H: $\beta_i \neq \beta_o$ is tested by computing:

$$t = \frac{\beta_i - \beta_0}{S_{\beta_i}} \tag{6.97}$$

H_o is rejected if $|t| > t_{\left(1-\frac{\alpha}{2}\right),\,(n-p)}$

(iv) Test of hypothesis that the i^{th} independent variable is not contributing significantly in explaining the variation in the dependent variable.

Hypothesis H_o: $b_i = 0$ versus H: $b \neq 0$ is tested by computing:

$$t = \frac{\beta_i}{S_{\beta_i}} \tag{6.98}$$

H is accepted if $|t| > t_{\left(1-\frac{\alpha}{2}\right),\,(n-p)}$

In such a situation, it is advisable to delete the i^{th} independent variable from the model.

It can be noted that once the multiple regression equation is developed for a given input and output matrix, the steps involved in defining the confidence interval for regression coefficients as well as regression line are very much similar to the approach adopted in case of simple regression. Therefore, Example 6.12 also provides a basis for carrying out necessary computations in this case.

Example 6.13 The following table shows the value of peak discharge Q_p for a catchment area A and rainfall intensity I respectively. Develop a multiple regression model for Q_p as a linear function of catchment area A and rainfall intensity I.

Q_p	23	45	44	64	68	62
I	3.2	4.6	5.1	3.8	6.1	7.4
A	12	21	18	32	24	16

Solution

Let the linear relationship be represented as

$$Q_p = a + bI + cA$$

Consider:

$$X = \begin{pmatrix} 1 & 3.2 & 12 \\ 1 & 4.6 & 21 \\ 1 & 5.1 & 18 \\ 1 & 3.8 & 32 \\ 1 & 6.1 & 24 \\ 1 & 7.4 & 16 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 23 \\ 45 \\ 44 \\ 64 \\ 68 \\ 62 \end{pmatrix}$$

Using the equation $(\beta)_{3\times1} = [(X^T)_{3\times6}(X)_{6\times3}]^{-1}_{3\times3}\ (X^T)_{3\times6}\ (Y)_{6\times1}$ and performing the matrix operations, we get the value of coefficients a, b, c as:

$$\beta = \begin{pmatrix} -27.082 \\ 7.847 \\ 1.882 \end{pmatrix}$$

Thus, the regression equation is $y = -27.082 + 7.847\,I + 1.882\,A$

Using the equation $R^2 = (\beta^T X^T Y - n\bar{Y}^2)/(Y^T Y - n\bar{Y}^2)$ where $n = 6$ and $\bar{Y} = 51$, we get the value of R^2 as 0.9775.

SUMMARY

The hydrological processes such as rainfall, snowfall, floods, or droughts, are usually investigated by analysing their records of observation. Hydrologists use statistical methods to analyse the records. The reliability of hydrological estimates is further evaluated by statistical measures. In hydrologic analysis, sometimes it is important to determine the statistical properties of the hydrologic data. Therefore, knowledge of statistical analysis of inputs and outputs (hydrologic data) is necessary to appreciate both, the limitations and accuracy of hydrologic analysis.

In many cases, theoretical analyses are used to formulate a methodology (model), but it is made operational only after hydrological analysis under statistical framework. Thus, statistical methods are essential for collecting, organizing, summarizing, and analysing the quantitative hydrological information.

This chapter covered the presentation and analysis of both, grouped data and continuous data, using probability distribution to represent the probability of occurrence of a random variable. We discussed a variety of discrete and continuous probability distribution functions including the parameter estimation procedures. In the last section, the statistical hypothesis testing tool for making decisions about descriptive statistics of a random variable (mean and variance) is discussed, which includes the use of t-distribution and chi-square distributions for making statistical inferences. The chapter concludes with a discussion on regression analysis, which helps to represent a dataset with an equation, and tests their reliability using different statistical criteria.

EXERCISES

6.1 **(a)** What do you understand by 'skewness'? What is the basic difference between positive and negative skewness?

(b) If a distribution has a long tail on the left side, then is it positive or negative skewed?

6.2 For a rain gauge station, the annual precipitation data for 12 years were recorded.

Annual precipitation data (mm)

Year	1	2	3	4	5	6
Precipitation (mm)	142.39	133.48	147.67	125.24	143.28	152.6
Year	7	8	9	10	11	12
Precipitation (mm)	130.15	150.02	147.08	129.08	140.89	137.31

Find the probability that:

(a) the annual precipitation in any year will be less than 130 mm;

Ans. 0.167

(b) the annual precipitation in any year will be greater than 140 mm;

Ans. 0.583

(c) the annual precipitation in any year will be greater than 140 mm and less than 130 mm; **Ans.** 0.250

(d) there will be three successive years of precipitation less than 130 mm. (Assume that the annual precipitations are independent.) **Ans.** 0.005

6.3 Calculate the mean and standard deviation of the precipitation data given in Problem 6.2. **Ans. (i)** 139.93 mm **(ii)** 8.89 mm

6.4 For the precipitation data given in Problem 6.2, find:

(a) standard error of mean, $SE(\bar{x})$

(b) standard error of deviation, $SE(S)$ **Ans. (i)** 2.566 **(ii)** 1.815

6.5 Determine the probability of rainfall depth equal to or greater than 1000 mm occurring

(a) once in 5 successive years,

(b) twice in 10 successive years, and

(c) at least once in 5 successive years,

(d) not at all in 5 years.

Assume that the rainfall depth of 1000 mm has a 100-years return period.

Ans. (a) 0.048 **(b)** 0.0042 **(c)** 0.049 **(d)** 0.951

6.6 The following table shows the annual maximum peak flow rate (m^3/s) records at a gauging site of a stream for 30 years.

Year	1	2	3	4	5	6	7	8	9	10
Peak flow rate	100	121	125	130	150	132	165	185	124	160
Year	11	12	13	14	15	16	17	18	19	20
Peak flow rate	356	374	385	198	251	354	175	190	174	142
Year	21	22	23	24	25	26	27	28	29	30
Peak flow rate	121	125	214	322	151	204	302	311	288	304

Arrange the peak flow rates as grouped data, and find:

(a) mean (m_1)

(b) variance (m_2)

(c) coefficient of skewness (m_3)

(d) coefficient of kurtosis (m_4)

6.7 For the data given in Problem 6.6, find:

(a) standard error of mean $SE(\bar{x})$

(b) standard deviation, $SE(S)$

6.8 Draw the histogram and the cumulative histogram for the data given in Problem 6.6.

6.9 Use Poisson distribution to find the probability of fewer than 3 occurrences of a 20-year storm in a 150-year period. **Ans.** 0.0203

6.10 For two catchments, the following runoff data were obtained at the outlet:

X_A(mm)	13.26	3.31	15.17	15.5	14.22	21.2	7.7	17.64	22.91	18.89
X_B(mm)	16.97	11.20	20.39	22.06	17.55	23.58	10.63	22.91	21.01	21.01
X_A(mm)	12.82	11.58	15.17	10.4	18.02	16.25	9.93	11.94	14.58	16.85
X_B(mm)	17.0	14.68	14.57	16.67	16.91	14.09	10.63	12.64	15.28	17.55
X_A(mm)	20.59	17.7	12.68	11.2	13.92	8.05	7.44	16.83	6.17	22.51
X_B(mm)	21.29	20.4	15.32	13.9	16.62	10.75	10.14	19.53	8.87	25.21

(i) Develop a simple linear rainfall-runoff model.

Ans. $X_B = 5.341 + 0.7985\ X_A$

(ii) Estimate goodness-of-fit criteria for the model developed in terms of:

(a) Coefficient of determination (r^2)
(b) Coefficient of correlation (r)
(c) Efficiency (EFF)
(d) Standard error (S)

Ans. (a) 0.7668 **(b)** 0.8757 **(c)** 50.86 **(d)** 2.17

OBJECTIVE QUESTIONS

1. Probability density function is
   (a) Mean of the event
   (b) Standard deviation of the event
   (c) Probability of occurrence of an event
   (d) None of the above

2. Coefficient of variation is the
   (a) Ratio of the mean to the standard deviation
   (b) Ratio of the standard deviation to the mean
   (c) Ratio of the square root of the standard deviation to the mean
   (d) Product of the standard deviation and the mean

3. Variance represents the
   (a) Dispersion about mean (b) Dispersion about medium
   (c) Dispersion about mode (d) None of these

4. Skewness coefficient is
   (a) An indicator of symmetry of the distribution fitted to data points
   (b) Measure of asymmetry of the frequency distribution fitted to the data
   (c) Both (a) and (b)
   (d) None of the above

5. "Skewness coefficient is equal to zero" indicates
   (a) Normal distribution
   (b) Binomial distribution
   (c) Gaussian distribution
   (d) Gumbel distribution

6. Kurtosis coefficient measures the
   (a) Peakedness or the flatness of the frequency distribution at the beginning
   (b) Peakedness or the flatness of the frequency distribution near its centre
   (c) Peakedness or the flatness of the frequency distribution at the end
   (d) All of the above

7. Excess coefficient is defined as
   (a) Kurtosis coefficient – 1
   (b) Kurtosis coefficient – 2
   (c) Kurtosis coefficient – 3
   (d) Kurtosis coefficient – 4

8. Positive excess coefficient indicates
   (a) LEPTOKURTIC distribution
   (b) MESOKURTIC distribution
   (c) PLATYKURTIC distribution
   (d) None of these

9. Negative excess coefficient indicates
   (a) LEPTOKURTIC distribution
   (b) MESOKURTIC distribution
   (c) PLATYKURTIC distribution
   (d) None of these

10. Normal distribution is known as
    (a) LEPTOKURTIC
    (b) MESOKURTIC
    (c) PLATYKURTIC
    (d) None of these

11. Standard deviation is the
    (a) Square of the variance
    (b) Square root of the variance
    (c) Square of the coefficient of variance
    (d) Square root of the coefficient of variance

12. Pearson type-III distribution
    (a) Is a three parameter distribution
    (b) Is also known as gamma distribution with three parameters
    (c) Both (a) and (b)
    (d) None of the above

13. Exponential distribution is a special case of Pearson type-III distribution when the shape parameter,
    (a) $\gamma = 0$
    (b) $\gamma = 1$
    (c) $\gamma = 2$
    (d) $\gamma = \infty$

14. Probability functions used for hypothesis testing in hydrologic analyses include the
    (a) F-distributions
    (b) student t-distributions
    (c) Chi-square distributions
    (d) All of these

15. Cumulative density function is the
    (a) Mean of non-exceedence of the event
    (b) Median of non-exceedence of the event
    (c) Probability of non-exceedence of the event.
    (d) None of the above

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CHAPTER 7

Flood Frequency Analysis

7.1 INTRODUCTION

There are numerous problems in hydrology wherein the data mostly contains measurements on a single random variable. Hence, univariate analysis and univariate estimation are important keys to solve hydrological problems. The objective of univariate analysis is to analyse measurements on the random variable, which is called sample information; and identify the statistical population from which we can reasonably expect the sample measurements to have come from. After the underlying population has been identified, one can make probabilistic statements about the future occurrences of the random variable; this represents univariate estimation. It is important to remember that univariate estimation is based on the assumed population and not the sample; the sample is used only to identify the population.

Hydrologic processes, such as rainfall, snowfall, floods, droughts, etc. are usually investigated by analysing their records of observations. Many characteristics of these processes may not represent definite relationship. For example, if annual instantaneous peak discharges of a river are plotted, then a rather erratic graph would be obtained. The variation of peak discharge from one year to another cannot be explained by fitting a definite relationship, which we call as *deterministic relationship*. In hydrologic analysis, the annual peak discharge is considered to be a random variable. Probability and statistical methods are employed for analysis of random variables. In this chapter, some elementary probability distributions are presented, which are used for frequency analysis in hydrology.

In order to have meaningful estimates from flood frequency analysis, the following assumptions are implicit.

- The data to be analysed describe random events.
- The data is homogeneous.
- The population parameters can be estimated from the sample data.
- It is of good quality.

If the data available for analysis do not satisfy any of the listed assumptions, then the estimates are not considered reliable. Moreover, the data should be (i) relevant, (ii) adequate, and (iii) accurate. For flood frequency analysis, either annual flood series or partial duration series may be used.

In general, an array of annual peak flood series may be considered as a sample of random and independent events. The non-randomness of the peak series will increase the degree of uncertainty in the derived frequency relationship. Various tests are available to check the randomness of the peak flow data. The annual maximum flood series can generally be regarded as consisting of random events as the mean interval of each observed flood peak is 1 year. However, in case data is used for partial duration series analysis, then independence among the data is doubtful. The peaks are selected in such a way that they constitute a random sample.

The term 'relevant' means that the data must deal with the problem. For example, if the problem is of the duration of flooding, then the data series should represent the duration of flows in excess of some critical value. If the problem is of interior drainage of an area, then the data series must consist of the volume of water above a particular threshold.

The term 'adequate' primarily refers to the length of data. The length of data primarily depends upon variability of data; and hence, there is no guideline for the length of data to be used for frequency analysis.

The term 'accurate' primarily refers to the homogeneity of data and accuracy of the discharge figures. The data used for analysis should not have any effect of man-made changes. Changes in the stage discharge relationship may render stage records non-homogeneous and unsuitable for frequency analysis. It is therefore preferable to work with discharges; and if stage frequencies are required, then most recent rating curve is used.

Watershed history and flood records should be carefully examined to ensure that no major watershed changes have occurred during the period of record. Only those records, which represent relatively constant watershed conditions, should be used for frequency analysis. The fundamental terms related to statistics are explained in Chapter 6. In the following section, some of the popular methods used in flood frequency analysis are explained with suitable examples.

7.2 METHODS OF PARAMETER ESTIMATION

Parameter estimation is important in flood frequency analysis. Any set of flood occurrences can be described using some statistical distribution. Statistical distribution may be mathematically represented through functional relationships which may contain a set of parameters. Some of these distributions have been described in preceding chapters. There are four well-known parameter estimation techniques, viz.:

- Graphical method,
- Frequency factor method,
- Method of moments, and
- Method of probability weighted moments and L-moments.

7.2.1 Graphical Method

In graphical method of parameter estimation, the variate under consideration is regarded as a function of a reduced variate of a known distribution. The steps involved in the graphical method are as follows:

(i) Arrange the variates of annual maximum flood series in ascending order and assign different ranks to individual variates.

(ii) Assign plotting positions to each of the variates. The plotting position formula may be used depending upon the type of distribution being fitted. The recommended plotting position formulae for normal, lognormal, Gumbel EV-I, Pearson type-III, and log-Pearson type-III distributions are given in Table. 7.1

(iii) Estimate the reduced variates for the selected distribution corresponding to different plotting positions, which represent the probability of non-exceedence.

Table 7.1 Commonly used unbiased plotting position formulae

Distributions	*Recommended plotting position formula*	*Form of the plotting position formulae* $F(X \le x) = 1 - \frac{1}{T}$
Normal and Lognormal	Blom	$(i - 0.375)/(N + 0.25)$
Gumbel EV-I	Gringorton	$(i - 0.4)/(N + 0.12)$
Any distribution	Weibull	$i/(N + 1)$
PT3 and LP3	Cunnane	$(i - 0.4)/(N + 0.2)$

['i' indicates the rank number. $i = 1$ for the smallest observation, and $i = N$ for the largest observation, when flood series is arranged in ascending order.]

The reduced variates for normal and lognormal distributions are computed with the help of the table given in Appendix II. For Gumbel (EV-I) distribution, the reduced variates are computed using the relationships given in Chapter 6. In case of Pearson type-III and log-Pearson type-III which are three-parameter distributions, different sets of reduced variates are obtained for different coefficients of skewness. Graphical method is not widely used nowadays. However, to illustrate its application in parameter estimation, an example using Gringrorton plotting position formula is given here. Let the relation among x_T (discharge corresponding to a return period T), location parameter u, and scale parameter β be of the type:

$$x_T = u + \beta y_T \tag{7.1}$$

For Gumbel distribution,

$$y_T = -\ln\left[\ln\left(\frac{T}{T-1}\right)\right] \tag{7.2}$$

For two discharges x_{T1} and x_{T2} corresponding to return period T_1 and T_2, let y_T be y_{T1} and y_{T2}. Thus,

$$x_{T1} = \mu + y_{T1}\beta \tag{7.3}$$

$$x_{T2} = \mu + y_{T2}\beta \tag{7.4}$$

From Eqs. (7.3) and (7.4),

$$\beta = \frac{x_{T1} - x_{T2}}{(y_{T1} - y_{T2})} = \text{slope of straight line} \tag{7.5a}$$

$$u = x_T \text{ corresponding to } y_T = 0 \tag{7.5b}$$

Example 7.1 The annual maximum series of a certain gauge and discharge site are given below. Table 7.2(a) shows the ranks of the AMS in descending order. Estimate the flood and the parameters for a return period of 1000 years using Gringrorton plotting position formula.

Table 7.2(a)

m	*Q*	*m*	*Q*	*m*	*Q*
1	2294	11	1095	21	675
2	1880	12	915	22	467
3	1850	13	889	23	467
4	1700	14	869	24	392
5	1500	15	867	25	371
6	1466	16	860	26	325
7	1395	17	792	27	288
8	1366	18	720	28	212
9	1175	19	693	29	170
10	1100	20	675	30	120

Solution

Table 7.2(b) Calculation of recurrence interval T using Gringrorton plotting position formula

Rank	*Q* (m^3/s)	$\frac{(i-0.44)}{(N+0.12)}$	*T*	*Rank*	*Q* (m^3/s)	$\frac{(i-0.44)}{(N+0.12)}$	*T*
1	120	0.01859	1.018945	16	867	0.5166	2.068681
2	170	0.05179	1.054622	17	869	0.5498	2.221239
3	212	0.08499	1.092888	18	889	0.583	2.398089
4	288	0.11819	1.134036	19	915	0.6162	2.605536
5	325	0.15139	1.178404	20	1095	0.6494	2.852273
6	371	0.18459	1.226384	21	1100	0.6826	3.150628
7	392	0.21779	1.278438	22	1175	0.7158	3.518692
8	467	0.25099	1.335106	23	1366	0.749	3.984127
9	467	0.28419	—	24	1395	0.7822	4.591463
10	675	0.31739	1.464981	25	1466	0.8154	5.417266
11	675	0.35059	—	26	1500	0.8486	6.605263
12	693	0.38379	1.622845	27	1700	0.8818	8.460674
13	720	0.41699	1.715262	28	1850	0.915	11.76563
14	792	0.45019	1.818841	29	1880	0.9482	19.30769
15	860	0.48339	1.935733	30	2294	0.9814	53.78571

For $i = 1$, $(1 - 1/T) = (1 - 0.44) / (30 + 0.12) = 0.01859$. Therefore, $T = 1.018945$.

Similarly, return periods for all the discharge can be determined. Since the discharge values are in ascending order, the lower rank of a discharge is taken into account where repetition of discharge has taken place.

The graph is plotted on a Gumbel probability paper between Q and T. In Gumbel probability paper, the graph will always be a straight line because there exists a linear relationship between return period T and discharge Q. Reduced variate values are shown below the scale of return period.

Using Eq. (7.2),

For $T = 53.78$, $\quad y_T = 3.9756$

For $T = 1.018945$, $\quad y_T = -1.3825$

Using Eq. (7.5a),

$$\text{slope} = \beta = \frac{(2294 - 120)}{[3.9756 - (-1.3825)]} = 405.74$$

Corresponding to $y_T = 0$, intercept $u = 688.0\ m^3/s$

From Fig. 7.1, it can be seen that after extending the straight line, the 1000 year return flood comes to be 3800 m^3/s.

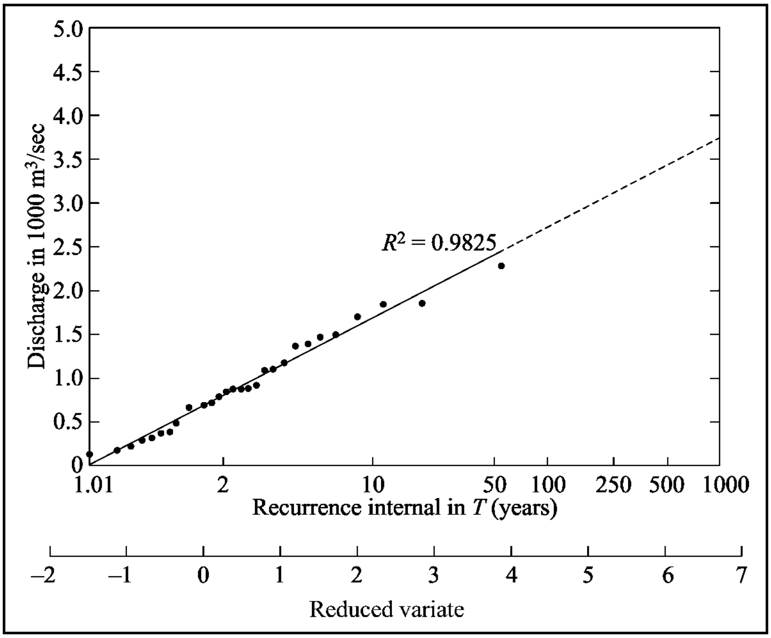

Fig. 7.1 *Gumbel probability paper using Gringrorton plotting position formula*

Example 7.2 Estimate u and β in the previous example by considering the following plotting position formula $T = (N + 1)/i$, where N is the sample size and i is the order of flood arranged in descending manner.

Solution

Table 7.3 contains the necessary computations and Fig. 7.2 shows the plot of x_T versus reduced variate.

Table 7.3

Rank	*Q (m³/s)*	$\frac{(N+1)}{i}$	*Rank*	*Q (m³/s)*	$\frac{(N+1)}{i}$	*Rank*	*Q (m³/s)*	$\frac{(N+1)}{i}$
1	2294	31	11	1095	2.818	21	675	1.4762
2	1880	15.5	12	915	2.583	22	467	—
3	1850	10.33	13	889	2.384	23	467	1.3478
4	1700	7.75	14	869	2.214	24	392	1.2916
5	1500	6.2	15	867	2.067	25	371	1.24
6	1466	5.166	16	860	1.9375	26	325	1.1923
7	1395	4.428	17	792	1.823	27	288	1.1481
8	1366	3.875	18	720	1.722	28	212	1.1071
9	1175	3.444	19	693	1.6315	29	170	1.0689
10	1100	3.1	20	675	—	30	120	1.0333

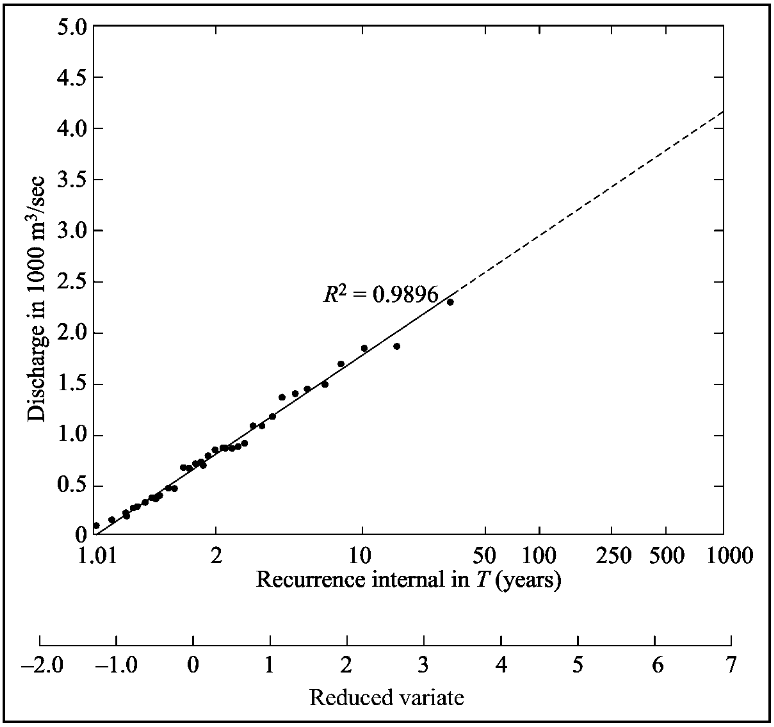

Fig. 7.2 *Gumbel probability paper using Weibull plotting position formula*

Using Eq. (7.2),

For $T = 31$, $y_T = 3.4176$ and

For $T = 1.033$, $y_T = -1.234$

Using Eq. (7.5a),

$$\text{slope} = \beta = \frac{(2294 - 120)}{[3.4176 - (-1.234)]} = 467.36$$

Corresponding to $y_T = 0$, intercept $u = 682.0\ \text{m}^3/\text{s}$

From Fig. 7.2, it can be seen that after extending the straight line, the 1000 year return flood comes to be 4200 m^3/s.

It is to note that plotting position formula influences estimates of model parameters. Thus, one has to be careful in selecting a plotting formula for a particular distribution.

7.2.2 Frequency Factor

Similar to Eq. (7.1), a general frequency equation proposed by Chow (1964), and applicable for different distributions, is in the following form:

$$x_T = m + K_T s \tag{7.6}$$

where, x_T = magnitude of the different peak flood series, or magnitude of the flood at the required return period T; K_T = frequency factor corresponding to x_T; m and s = mean and standard deviation of the population, respectively, which would be replaced by the sample statistics. The similarity between Eqs. (7.1) and (7.6) is evident. Table 7.4 provides expressions to determine K_T for certain probability distributions.

Table 7.4 Frequency factor for different distributions

Normal distribution	$w = \left[\ln\left(\frac{1}{p^2}\right)\right]^{1/2}$ $z = w - \frac{2.515517 + 0.802853w + 0.010328w^2}{1 + 1.432788w + 0.189269w^2 + 0.001308w^3}$ $K_T = z$
EV-I distribution	$K_T = -\frac{\sqrt{6}}{\pi}\left\{0.5772 + \ln\left[\ln\left(\frac{T}{T-1}\right)\right]\right\}$
Log-Pearson type-III distribution	$K_T = z + (z^2 - 1)k + \frac{1}{3}(z^3 - 6z)k^2 - (z^2 - 1)k^3 + zk^4 + \frac{1}{3}K^5$ where $k = g/6$ and g is skewness

Example 7.3 Calculate the frequency factor for an event with a return period of 100 years using normal distribution and EV-I distribution.

Solution

Normal distribution

$$T = 100 \text{ years}, \quad p = 1/100 = 0.01$$

$$W = \left[\ln\left(\frac{1}{p^2}\right)\right]^{1/2}$$

$$= \left[\ln\left[\frac{1}{(0.01)^2}\right]\right]^{1/2}$$

$$= 3.034$$

$$z = w - \frac{2.515517 + 0.802853w + 0.010328w^2}{1 + 1.432788w + 0.189269w^2 + 0.001308w^3}$$

$$= 3.034 - \frac{2.515517 + (0.802853 \times 3.034) + [0.010328 \times (3.034)^2]}{1 + (1.432788 \times 3.034) + [0.189269 \times (3.034)^2] + [0.001308 \times (3.034)^3]}$$

$$= 2.325$$

EV-I distribution

$$K_T = -\frac{\sqrt{6}}{\pi}\left\{0.5772 + \ln\left[\ln\left(\frac{T}{T-1}\right)\right]\right\}$$

$$K_T = -\frac{\sqrt{6}}{\pi}\left\{0.5772 + \ln\left[\ln\left(\frac{100}{100-1}\right)\right]\right\}$$

$$= 3.136$$

7.2.3 Method of Moments (MOM)

The method of moments makes use of the fact that if all the moments of a distribution are known then everything about the distribution is known. For all the distributions in common usage, four moments or fewer are sufficient to specify all the moments. For instance, two moments, the first together with any moment of even order are sufficient to specify all the moments of the normal distribution, and therefore the entire distribution. Similarly, in Gumbel EV-I distribution, the first two moments are sufficient to specify all the moments and hence the distribution. For Pearson type-III distribution three moments are sufficient, always taken as the first three, to specify all the moments. In these cases, the number of moments needed to specify all the moments and hence the distribution equals the number of parameters.

This method is dependent on the assumption that the distribution of variate values in the sample is representative of the population distribution. Therefore, a representation of the former provides an estimate of the later. Given that the form of the distribution is known or assumed, the distribution that the sample follows is specified by its first two or three moments calculated from the data. Table 7.5 provides relationships between model parameters and moments for a variety of distributions.

Table 7.5 Probability distribution parameters in relation to sample moments

Distribution	*Probability Density Function*	*Range*	*Equation of parameters in terms of the sample moments*
Exponential	$f(x) = \frac{1}{\beta}\exp\left[\frac{-(x)}{\beta}\right]$	$x \geq 0$	$\beta = \bar{x}$
Extreme value type-I	$f(x) = \frac{1}{\beta}\exp\left[-\frac{x-u}{\beta} - \exp\left(-\frac{x-u}{\beta}\right)\right]$	$(-\infty <$ $x < \infty)$	$u = \bar{x} - 0.5772\beta$ $\beta = \frac{\sqrt{6}\,s_x}{\pi}$
Gamma	$f(x) = \frac{(x)^{\gamma-1}e^{-x/\beta}}{\beta^{\gamma}\,\Gamma(\gamma)}$	$x \geq 0$	$\beta = \frac{(s_x)^2}{\bar{x}}$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	$(-\infty \leq$ $x \leq \infty)$	$\mu = \bar{x}$, $\sigma = s_x$
Lognormal	$f(x) = \frac{1}{x\sigma_L\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu_L}{\sigma_L}\right)^2\right]$	$x > 0$	$\mu_y = \bar{y}$, $\sigma_y = s_y$
Pearson type-III	$f(x) = \frac{(x-u)^{\gamma-1}\exp\left[\frac{-(x-u)}{\beta}\right]}{\beta^{\gamma}\Gamma(\gamma)}$	$x \geq u$	$u = \bar{x} - s_x\sqrt{\gamma}$, $\beta = \frac{\sqrt{\gamma}}{s_x}$, $\gamma = \left(\frac{2}{g}\right)^2$
Log-Pearson type-III	$f(x) = \frac{(\ln x - u)^{\gamma-1}\exp[-(\ln x - u)/\beta]}{\lvert\beta\rvert\,\Gamma(\gamma)}$	$\ln x \geq u$	$u = \bar{y} - s_y\sqrt{\gamma}$ $\beta = \frac{\sqrt{\gamma}}{s_y}$, $\gamma = \left(\frac{2}{g_y}\right)^2$

Table 7.5 does not include other GEV distributions. Stedinger et al. (1993) have given MOM estimators for three types of GEV distributions.

Parameter Estimates using MOM for GEV Distribution

$$F(q) = \exp\left[-\left(1-\gamma\frac{q-u}{\beta}\right)^{\frac{1}{\gamma}}\right] \quad \text{for } \gamma \neq 0 \tag{7.7}$$

$$F(q) = \exp\left[-\exp\left(-\frac{q-u}{\beta}\right)\right] \quad \text{for } \gamma = 0 \tag{7.8}$$

where, q is the flood exceedence value, u is a location parameter, β is a scale parameter, and γ is the shape parameter. The range of the variable, q, depends on the sign of γ.

When γ is negative ($g > 1.1369$), then the variable q can take values in the range of $[(u + \beta/\gamma) < q < \infty]$ which make it suitable for flood frequency analysis.

When γ is positive, then the variable q becomes upper bounded; and for $\gamma = 0$, the distribution reduces to two-parameter extreme value type-I or Gumbel's distribution. The MOM estimators of the GEV parameters are given by Stedinger et al., (1993) as:

$$u = \mu + \frac{\beta[\Gamma(1+\gamma)-1]}{\gamma} \tag{7.9}$$

$$\beta = \frac{\sigma\gamma}{\{[\Gamma(1+2\gamma)]-[\Gamma(1+\gamma)]^2\}^{1/2}} \tag{7.10}$$

$$g = \text{sign}\,(\gamma)\frac{[\{-\Gamma(1+3\gamma)+3\Gamma(1+\gamma)\Gamma(1+2\gamma)-2(\Gamma(1+\gamma))^3\}]}{[\Gamma(1+2\gamma)-(\Gamma(1+\gamma))^2]^{3/2}} \tag{7.11}$$

where, sign is '+' or '–' depending upon the value of γ-estimate, and g is the skewness.

Parameter Estimates using MOM for Generalized Pareto (GP) Distribution

The GP distribution introduced by Pickands (1975), has the following CDF:

$$F(q) = 1 - \left[1-\gamma\frac{q-u}{\beta}\right]^{1/\gamma}; \quad \gamma \neq 0 \tag{7.12}$$

where, β is the scale parameter, u is the location parameter, and γ is the shape parameter. For $\gamma = 0$, exponential distribution is obtained as a special case. For $\gamma \le 0$, the range of u is $(u \le q \le \infty)$ and for $\gamma \ge 0$ the upper bound exists: $u \le q \le (u + \beta/\gamma)$. The parameter estimates of the GP distribution using MOM estimation method are given in the following relations:

$$g = \frac{2(1-\gamma)(1+2\gamma)^{1/2}}{(1+3\gamma)} \tag{7.13}$$

Given the skewness of the AMS data, γ can be computed numerically. The other parameters are then obtained as follows:

$$\beta = \sigma[(1+\gamma)^2(1+2\gamma)]^{1/2} \tag{7.14}$$

$$u = \mu - \frac{\beta}{(1+\gamma)} \tag{7.15}$$

where, μ and σ are the mean and standard deviation, respectively.

Example 7.4 Estimate the parameters using method of moments for the given data in Example 7.1 considering GEV distribution and GP distribution.

Solution

Parameter Estimates for GEV Distribution

Using the given data, the following statistics are calculated:

Mean (μ) = 919.6 m^3/s

Standard deviation (σ) = 561.88 m^3/s

Skewness coefficient (g) = 0.6348

Using skewness coefficient g and Eq. (7.11), one can estimate γ to decide the type of GEV distribution.

Skewness coefficient (g) for AMS is 0.6348.

Using the equation given by Stedinger et al., (1993), the value of γ is computed as 0.10083 using the method of iteration.

Equation (7.10) yields $\beta/\sigma = 0.8743$.

The value of σ which is the standard deviation of AMS data is 561.88 m^3/s, therefore $\beta = 491.31$ m^3/s.

Hence, using Eq. (7.9), we have:

$$u = \mu + \frac{\beta[\Gamma(1+\gamma) - 1]}{\gamma}$$

$$u = 919.6 + \frac{491.31 \times [\Gamma(1.10083) - 1]}{0.10083}$$

$$= 680.91 \text{ m}^3/\text{s}$$

Thus, the parameters estimated using method of moments u, β, and γ are 680.91, 491.31, and 0.10083, respectively.

Parameter Estimates for GP Distribution

Skewness coefficient (g) for AMS is 0.6348.

Using Eq. (7.13), the value of γ is computed as 0.456 using the method of iteration.

Using $\beta = \sigma[(1 + \gamma)^2(1 + 2\gamma)]^{1/2} = 561.88[(1.456)^2\ (1 + 2 \times 0.456)]^{1/2}$

$$= 1131.22 \text{ m}^3/\text{s}$$

$$u = \mu - \beta/(1 + \gamma) = 919.6 - [1131.22/1.456]$$

$$= 142.66 \text{ m}^3/\text{s}$$

7.2.4 Probability Weighted Moment (PWM) and Linear (L) Moments

Greenwood et al. (1979) introduced the method of probability weighted moment (PWM) and showed its usefulness in deriving explicit expressions for parameters of distributions whose inverse forms $x = x(F)$ can be explicitly defined. If PDF be denoted by $F = F(x) = P[X \le x]$, then PWMs are the moment of the function $x(F)$; and they are expressed as:

$$M_{i j, k} \equiv E[x^i F^j (1-F)^k] = \int_0^1 [x(F)^i F^j (1-F)^k \, dF \tag{7.12}$$

It can be shown that,

$$M_{i,0,k} = \sum_{j=0}^{k} \binom{k}{j} (-1)^j M_{i,j,0} \tag{7.13}$$

where, $M_{i,j,k}$ is the PWM of the order of i, j, and k; and E is the expectation operator. $M_{i,0,0}$ represents the conventional moment about the origin of order i. These can than be related to the distribution parameters and the resultant relation may be a simpler structure than the conventional moments and the parameters. For example, the parameters of Gumbel EV-I distribution using PWMs are as follows:

$$x(F) = u - \beta \ln[-\ln(F)] \tag{7.14}$$

$$M_{i,j,0} = \int_0^1 [u - \beta \ln(-\ln F)]^i F^j (1-F)^0 \, dF \tag{7.15}$$

$$= \int_0^1 [u F^j] dF - \int_0^1 \beta \ln(-\ln F)]^i F^j \, dF \tag{7.16}$$

$$= \frac{u}{1+j} + \beta \frac{\ln(1+j) + \varepsilon}{1+j} \tag{7.17}$$

where, ε = Euler's constant = 0.5772

$$M_{1,0,0} = u + \beta\varepsilon \tag{7.18}$$

$$M_{1,1,0} = \frac{u}{2} + \beta\left[\frac{\ln(2) + \varepsilon}{2}\right] \tag{7.19}$$

Solving Eqs. (7.18) and (7.19):

$$\beta = \frac{2M_{1,1,0} - M_{1,0,0}}{\ln(2)} \tag{7.20}$$

and $$u = M_{1,0,0} - 0.5772\,\beta \tag{7.21}$$

L-moments are linear combination of order statistics which are robust to outliers and are unbiased for small samples, making them suitable for flood frequency analysis, including identification of distribution and parameter estimation (Hosking, 1990, Hosking and Wallis, 1997). L-moments are identified as linear combination of probability weighted moment (PWM)

$$M_{1,r,0} = \beta_r = E\{X\,[F(x)]^r\} \tag{7.22}$$

where, $F(x)$ is the cumulative distribution function of x. When $r = 0$, β_0 is the mean. The first four L-moments expressed as linear combination of PWMs are:

$$\lambda_1 = \beta_0 \tag{7.23a}$$

$$\lambda_2 = 2\,\beta_1 - \beta_0 \tag{7.23b}$$

$$\lambda_3 = 6\,\beta_2 - 6\,\beta_1 + \beta_0 \tag{7.23c}$$

$$\lambda_4 = 20\,\beta_3 - 30\,\beta_2 + 12\,\beta_1 - \beta_0 \tag{7.23d}$$

Landwehr et al. (1979) recommends the use of biased estimates of PWMs and L-moments, since such estimates often produce quintile estimates with lower root mean square error than unbiased alternatives. Nevertheless, unbiased estimators are preferred in moment diagrams for goodness-of-fit for less bias and are invariant when the data is multiplied by a constant. Unbiased estimates of PWMs for any distribution can be computed as follows:

$$b_0 = \frac{1}{n}\sum_{j=1}^{n} x_j \tag{7.24a}$$

$$b_1 = \sum_{j=1}^{n-1}\left[\frac{(n-j)}{(n)(n-1)}\right]x_j \tag{7.24b}$$

$$b_2 = \sum_{j=1}^{n-2}\left[\frac{(n-j)(n-j-1)}{n(n-1)(n-2)}\right]x_j \tag{7.24c}$$

$$b_3 = \sum_{j=1}^{n-3}\left[\frac{(n-j)(n-j-1)(n-j-2)}{n(n-1)(n-2)(n-3)}\right]x_j \tag{7.24d}$$

where, x_j represents the ordered streamflow with x_1 as the largest streamflow and x_n as the smallest. The L-moment ratios which are used for expressing the parameter estimates are expressed as:

L-coefficient of variation $(\tau) = \lambda_1$

L-skewness $(\tau_3) = \lambda_3/\lambda_2$

and L-kurtosis $(\tau_4) = \lambda_4/\lambda_3$

The sample estimates of L-moments are calculated by replacing b_0, b_1, b_2, and b_3 for β_0, β_1, β_2, and β_2 in Eq. (7.23). So, the parameters of Gumbel EV-I distribution in the above example can be written in terms of L-moment as follows:

$$\beta = \lambda_2/\log 2$$

$$u = \lambda_0 - 0.5772\beta$$

The relationship equations between sample moment parameters and population parameter estimation for different distributions using L-moment are given in Hosking and Wallis (1997). Although the theory and application of L-moment parallels those of conventional moment, there are certain advantages of

L-moment. Since sample estimators of L-moments are always a combination of ranked observations, they are subjected to less bias than ordinary product moments. This is because ordinary product moments require squaring and cubing, giving greater weight to observations far from the mean resulting in bias.

Parameter Estimates using PWM Method for GEV Distribution

The PWM estimators for GEV distribution are given by Hosking et al., (1985) as:

$$u = \lambda_1 - \frac{\beta[1-\Gamma(1+\gamma)]}{\gamma} \tag{7.25}$$

$$\beta = \frac{\lambda_2}{\Gamma(1+\gamma)(1-2^{-\gamma})}\gamma \tag{7.26}$$

$$\gamma = 7.859c + 2.9554c^2; \quad c = \frac{2}{(\tau_3+3) - \ln(2)/\ln(3)} \tag{7.27}$$

Parameter Estimates using PWM Method for GP Distribution

Similarly, the parameter estimates by the PWM method is given by Hosking and Wallis (1997) as follows:

$$\gamma = (1 - 3\tau_3)/(1+\tau_3); \quad \beta = \lambda_2(1+\gamma)(2+\gamma); \quad u = \lambda_1 - \lambda_2(2+\gamma) \tag{7.28}$$

where, τ_3 is the L-skewness, λ_1 and λ_2 are the first and second L-moments for the sample data. Hosking and Wallis (1997) have given the PWM estimator equations as follows:

$$\gamma = (\lambda_1 - q_0)/\lambda_2 - 2 \quad \beta = (1+\gamma)(\lambda_1 - q_0) \tag{7.29}$$

where, λ_i are the L-moments, which are defined for $\gamma > -1$.

Example 7.5 Estimate the parameters using probability weighted moments for the given data in Example 7.1 considering GEV distribution and GP distribution.

Solution

Parameter estimates using GEV distribution

Table 7.6 Calculation of probability weighted moments

Rank	*Q* (m^3/s)	b_1	b_2	b_3	*Rank*	*Q* (m^3/s)	b_1	b_2	b_3
1	2294	76.467	76.467	76.47	16	860	13.839	6.425	2.86
2	1880	60.506	58.345	56.18	17	792	11.834	5.072	2.07
3	1850	57.414	53.313	49.36	18	720	9.931	3.901	1.44
4	1700	50.805	45.361	40.32	19	693	8.762	3.129	1.04

(*Contd.*)

(*Contd.*)

5	1500	43.103	36.946	31.47	20	675	7.759	2.494	0.74
6	1466	40.441	33.220	27.07	21	675	6.983	1.995	0.52
7	1395	36.879	28.977	22.54	22	467	4.294	1.074	0.24
8	1366	34.543	25.907	19.19	23	467	3.757	0.805	0.15
9	1175	28.362	20.259	14.26	24	392	2.703	0.483	0.07
10	1100	25.287	17.159	11.44	25	371	2.132	0.305	0.03
11	1095	23.914	15.373	9.68	26	325	1.494	0.160	0.01
12	915	18.931	11.494	6.81	27	288	0.993	0.071	0.00
13	889	17.371	9.926	5.51	28	212	0.487	0.017	0.00
14	869	15.982	8.562	4.44	29	170	0.195	0.000	0.00
15	867	14.948	7.474	3.60	30	120	0.000	0.000	0.00
							620.118	**474.713**	**387.516**

First of all, the probability weighted moments are calculated using Eq. (7.24) and the calculations are shown in Table 7.6. From the table, the summation of the values in columns (3), (4), and (5) gives the values of b_1, b_2, and b_3, respectively; and b_0 is the mean of AMS. The L-moments are calculated as under.

$$\lambda_1 = b_0 = 919.6$$

$$\lambda_2 = 2\, b_1 - b_0 = 320.63$$

$$\lambda_3 = 6\, b_2 - 6\, b_1 + b_0 = 47.17$$

$$\lambda_4 = 20\, b_3 - 30\, b_2 + 12\, b_1 - b_0 = 30.75$$

$$\tau_3 = \lambda_3/\lambda_2 = 0.147$$

Therefore, $c = [2/(\tau_3 + 3)] - [\ln(2)/\ln(3)] = [2/(3.147)] - 0.6309 = 0.00459$

$$\gamma = 7.859c + 2.9554c^2 = 0.03617$$

$$\beta = \frac{320.63 \times 0.03617}{\Gamma(1 + 0.03617)(1 - 2^{-0.03617})} = 476.20 \text{ m}^3/\text{s}$$

And $$u = \lambda_1 - \frac{\beta[1 - \Gamma(1+\gamma)]}{\gamma} = 705.0 \text{ m}^3/\text{s}$$

Thus, the parameters estimated using probability weighted moments u, β, and γ are 705.0, 476.20, and 0.03617, respectively.

Parameter estimates using GP distribution

$$\gamma = \frac{(1 - 3\tau_3)}{(1 + \tau_3)} = [1 - (3 \times 0.147)]/(1.147) = 0.487$$

$$\beta = \lambda_2(1 + \gamma)(2 + \gamma) = 320.63\ (1.487)\ (2.487) = 1185.74 \text{ m}^3/\text{s}$$

$$u = \lambda_1 - \lambda_2(2 + \gamma) = 919.6 - 320.63\ (2.487) = 122.19 \text{ m}^3/\text{s}$$

7. 3 RETURN PERIOD FLOOD ESTIMATION

A flood of certain magnitude is expected to occur after a certain recurrence interval. Expected or average value of this recurrence interval is also known as return period. Given the flood records at any site, it is of interest to compute the magnitude of floods which correspond to different return periods. One possible and most widely used approach is to identify the particular statistical distributions, and use these to predict a flood of desired return period. In this section, we discuss the use of different statistical or probabilistic distributions in return period flood estimation.

7.3.1 Normal Distribution

The parameters of the normal distribution, μ and σ, which describe the characteristics of the given set, are computed as:

$$\mu \equiv \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{7.30}$$

$$\sigma \equiv s = \frac{\sqrt{\sum (x_i - \bar{x})^2}}{(N-1)} \qquad 1(7.31)$$

After computing the parameters, μ and σ, the T-year flood estimates can be obtained using the normal distribution. The steps are:

(i) Compute $\bar{x}$ and s from the sample data. $\bar{x}$ and s provide estimates for μ and σ.

(ii) Compute the probability of non-exceedence using the relation:

$$F(x < X) = 1 - 1/T \tag{7.32}$$

(iii) Compute the normal reduced variate (z_T) from the statistical table, corresponding to the probability of non-exceedence computed in step (2).

(iv) Estimate the flood for T-year recurrence interval using the following equation:

$$x_T = \mu + \sigma z_T \tag{7.33}$$

Example 7.6 Using the data given in Example 7.1, estimate the flood for a return period of 1000 years using the formula given by Swamee and Rathie (2007) for normal distribution.

If the data follows normal distribution, then according to Swamee and Rathie (2007):

$$k = \begin{cases} -0.6144[-\ln(T-1)]^{1/1.12} & \text{for } (1 \le T \le 2) \\ -0.6144[\ln(T-1)]^{1/1.12} & \text{for } (T \ge 2) \end{cases}$$

$$k = \frac{x - \mu}{\sigma}$$

Solution

Since $T = 1000$ years, using the formula given above $k = 3.449$

Using the formula $k = \dfrac{x - \mu}{\sigma}$, where $\mu = 919.6$ and $s = 561.8827$, the value of flood discharge is equal to $x = 3.449 \times 561.8827 + 919.6 = 2857.53$ m^3/s.

7.3.2 Lognormal Distribution (Two parameter)

In lognormal two-parameter distribution, the variates are transformed to the log domain by taking logarithm of each variate, and then the mean and standard deviation of the transformed series are computed. The flood at required recurrence interval can be computed in the following steps after fitting the lognormal distribution with the sample data.

(i) Transform the original series of peak flood data into log domain by taking logarithm of each variate to base e.

(ii) Compute the mean ($\bar{y}$) and the standard deviation (s_y) from the log-transformed series. $\bar{y}$ and s_y provide estimates for μ_y and σ_y, respectively.

(iii) Compute the probability of non-exceedence for the given recurrence interval (T).

(iv) Compute the normal reduced variate (z_T) corresponding to the probability of non-exceedence computed in step (3).

(v) Estimate the flood for T-year recurrence interval in log domain using the following equation:

$$y_T = \mu_y + \mu_y\, z_T \tag{7.34}$$

(vi) Transform the estimated T-year flood in original domain by computing its exponent, i.e.,

$$x_T = \exp(y_T) \tag{7.35}$$

Example 7.7 Estimate 1000-years floods assuming that the peak discharge data follow lognormal distribution for the data given in Example 7.1.

Solution

Since $\mu_y = 6.60$ and $\sigma_y = 0.743$

Probability of non-exceedence, $F(z) = 1 - 1/1000 = 0.999$

$$z_T = 3.10 \text{ for } T = 1000 \text{ years}$$

$$y_T = \mu_y + \sigma_y\, z_T$$

$$y_{1000} = 6.60 + (0.743 \times 3.10) = 8.9033$$

$$x_{1000} = (e)^{y_{1000}} = (e)^{8.9033} = 7356.2 \text{ m}^3/\text{s}$$

7.3.3 Gumbel's Extreme Value Type-I Distribution

The relationship between the parameters and the statistical moments of the data are given by the following equations:

$$\beta = \frac{\sqrt{6}\sigma}{\pi} \tag{7.36}$$

$$u = \mu - 0.45\sigma \tag{7.37}$$

The population statistics, μ and σ would be replaced by the sample statistics, $\bar{x}$ and s, respectively. The step-by-step procedure for computing the T-year flood using EV-I distribution is given below:

(i) Compute mean ($\bar{x}$) and standard deviation (s) from the sample. $\bar{x}$ and s provide estimates for μ and σ, respectively.
(ii) Compute the parameters, u and β, using Eqs. (7.36) and (7.37).
(iii) Compute the probability of non-exceedence corresponding to T-year recurrence interval.
(iv) Compute the EV-I reduced variate (y_T) using the relationship:

$$y_T = -\ln\left[\ln\frac{T}{T-1}\right] \tag{7.38}$$

(v) Estimate T-year recurrence interval flood using the EV-I distribution as follows:

$$x_T = u + \beta y_T \tag{7.39}$$

Sample Size Correction

To estimate frequency factor, Chow (1964) has given the frequency factor which is given in Table 7.4. Literature does indicate that frequency factor depends upon the sample size (Subramanya, 1984, Raghunath, 2007).

$$K_T = \frac{y_T - y_n}{s_n}$$

$$x_T = \mu + \sigma K_T$$

where, μ and σ are mean and standard deviation respectively, y_n = reduced mean and s_n = reduced standard deviation which is a function of sample size. As $N \rightarrow \infty$, $y_n = 0.577$ and $s_n = 1.2825$. Tables 7.7 and 7.8 give the values of reduced mean and reduced standard deviation as per the sample size, respectively.

Table 7.7 Reduced mean (y_n) in Gumbel EV-I distribution

N	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
10	0.4952	0.4996	0.5035	0.507	0.51	0.5128	0.5157	0.5181	0.5202	0.522
20	0.5236	0.5252	0.5268	0.5283	0.5296	0.5309	0.532	0.5332	0.5343	0.5353

(*Contd.*)

(*Contd.*)

30	0.5362	0.5371	0.5380	0.5388	0.5396	0.5402	0.5410	0.5418	0.5424	0.5430
40	0.5436	0.5442	0.5448	0.5453	0.5458	0.5463	0.5468	0.5473	0.5477	0.5481
50	0.5485	0.5489	0.5493	0.5497	0.5501	0.5504	0.5508	0.5511	0.5515	0.5518
60	0.5521	0.5524	0.5527	0.5530	0.5533	0.5535	0.5538	0.5540	0.5543	0.5545
70	0.5548	0.5550	0.5552	0.5555	0.5557	0.5559	0.5561	0.5563	0.5565	0.5567
80	0.5569	0.5570	0.5572	0.5574	0.5576	0.5578	0.5580	0.5581	0.5583	0.5585
90	0.5586	0.5587	0.5589	0.5591	0.5592	0.5593	0.5595	0.5596	0.5598	0.5599
100	0.56									

Table 7.8 Reduced standard deviation (s_n) in Gumbel EV1 distribution

N	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
10	0.9496	0.9676	0.9833	0.9971	1.0095	1.0206	1.0316	1.0411	1.0493	1.0565
20	1.0628	1.0696	1.0754	1.0811	1.0864	1.0915	1.0961	1.1004	1.1047	1.1086
30	1.1124	1.1159	1.1193	1.1226	1.1255	1.1285	1.1313	1.1339	1.1363	1.1388
40	1.1413	1.1436	1.1458	1.1480	1.1499	1.1519	1.1538	1.1557	1.1574	1.1590
50	1.1607	1.1623	1.1638	1.1658	1.1667	1.1681	1.1696	1.1708	1.1721	1.1734
60	1.1747	1.1759	1.1770	1.1782	1.1793	1.1803	1.1814	1.1824	1.1834	1.1844
70	1.1854	1.1863	1.1873	1.1881	1.1890	1.1898	1.1906	1.1915	1.1923	1.1930
80	1.1938	1.1945	1.1953	1.1959	1.1967	1.1973	1.1980	1.1987	1.1994	1.2001
90	1.2007	1.2013	1.2020	1.2026	1.2032	1.2038	1.2044	1.2049	1.2055	1.2060
100	1.2065									

Example 7.8 Using the data given in Example 7.1, estimate 1000-years floods assuming that the peak discharge data follow Gumbel's EV-I distribution.

Solution

Since $\mu = 919.6$

$\sigma = 561.8827$, and

$\alpha = 0.7797 \times \sigma = 0.7797 \times 561.8827 = 438.09$

$u = \mu - 0.45\sigma$

$u = 919.6 - (0.45 \times 561.8827) = 666.75$

$$y_T = -\ln\left[\ln\frac{T}{T-1}\right]$$

$y_{1000} = -\ln\,[\ln\,(1000/999)] = 6.907255$

$x_{1000} = u + \beta y_{1000}$

$x_{1000} = 666.75 + (438.09 \times 6.907255) = 3692.75\ \text{m}^3/\text{s}$

Consideration of sample size

$\mu = 919.6$, $\sigma = 561.8827$,

$y_{1000} = -\ln[\ln(1000/999)] = 6.907255$

For $N = 30$, $y_n = 0.5362$ and $s_n = 1.1124$

$K_T = (6.907255 - 0.5362)/1.1124 = 5.7273$

$x_{1000} = 919.6 + (561.8827 \times 5.7273) = 4137.67 \text{ m}^3/\text{s}$

It is to note that estimation of return flood is influenced by the sample size.

7.3.4 Pearson Type-III (PT-III) Distribution

PT-III distribution is a three-parameter distribution. Therefore, three moments are needed for computing the parameters. Mean, standard deviation, and skewness computed from the sample data describe the measures for the first three moments of the sample data. The following steps are usually involved in computing the T-year flood using the PT-III distribution.

(i) Compute the mean, μ and standard deviation σ. Compute the coefficient of skewness, g, from the sample using the following equation:

$$g = \frac{N}{(N-1)(N-2)} \sum_{i=1}^{N} \frac{(x_i - \mu)^3}{\sigma} \tag{7.40}$$

(ii) Compute the probability of exceedence for the given recurrence interval, T, which equals to $1/T$.

(iii) Estimate the frequency factor (K_T) from the table corresponding to the computed coefficient of skewness (g) and the probability of exceedence.

(iv) Estimate T-year flood using the equation:

$$x_T = \mu + \sigma K_T \tag{7.41}$$

Example 7.9 Using the data given in Example 7.1, estimate 1000-years floods assuming that the peak discharge data follow PT-III distribution.

Solution

Since $\mu = 919.6$

$\mu = 561.8827$, and

$g = 0.634$

Probability of exceedence = 1/1000 = 0.001

Frequency factor, $K_T = 4.00$ (using table given in Appendix VI)

$$x_T = \mu + \sigma K_T$$

$$= 919.6 + (561.8827 \times 4.00)$$

$$= 3172.35 \text{ m}^3/\text{s}$$

7.3.5 Generalized Extreme Value (GEV) Distribution

The L-moment estimators λ_1, λ_2, λ_3, and τ_3 are obtained by unbiased estimators of the first three PWMs. The maximum likelihood estimators are determined numerically using a modified Newton-Raphson algorithm (Hosking et al., 1985). The return period flood is given as:

$$q(T) = u + \frac{\beta[1-(-\log F)^{\gamma}]}{\gamma} \quad ; \quad F = 1 - 1/T \tag{7.42}$$

Example 7.10 Using the data of Example 7.1 estimate the flood for a return period of 1000 years using (i) method of moments, and (ii) PWM considering both GEV and GP distributions.

Solution

Using MOM

From Example 7.4, the parameters estimated using method of moments u, β, and γ are 680.91, 491.31, and 0.10083, respectively. Using these parameter values in Eq (7.42):

$$q(T) = u + \frac{\beta[1-(-\log F)^{\gamma}]}{\gamma}$$

$$q(T) = 680.91 + \frac{491.31[1-(-\log 0.999)^{0.10083}]}{0.10083}$$

$$= 3321.14 \text{ m}^3/\text{s}$$

Using PWM

From Example 7.5, the parameters estimated using probability weighted moments u, β, and γ are 705.0, 476.20, and 0.03617, respectively. Using these values of the parameters in Eq (7.42), we get:

$$q(T) = 705.0 + \frac{476.2[1-(-\log 0.999)^{0.03617}]}{0.03617}$$

$$= 3920.27 \text{ m}^3/\text{s}$$

7.3.6 Generalized Pareto (GP) Distribution

The return period flood is given as follows:

$$q(T) = u + \frac{\beta[1-(1-F)^{\gamma}]}{\gamma} \quad ; \quad F = 1 - 1/T \tag{7.43}$$

Example 7.11 Estimate the flood for a return period of 1000 years using (i) method of moments and (ii) PWM using GP distributions using the data given in Example 7.1.

Solution

Using MOM

From Example 7.4, the parameters estimated using method of moments u, β, and γ are 142.66, 1131.22, and 0.456, respectively. Using these values of the parameters in Eq. (7.43), we get:

$$q(T) = u + \frac{\beta[1-(1-F)^{\gamma}]}{\gamma}$$

$$F = 1 - (1/1000) = 0.999$$

$$q(T) = 142.66 + \frac{1131.22[1-(1-0.999)^{0.456}]}{0.456}$$

$$= 251.7 \text{ m}^3/\text{s}$$

Using PWM

From Example 7.5, the parameters estimated using probability weighted moments u, β, and γ are 122.19, 1185.74, and 0.487, respectively. Using these values of the parameters in Eq (7.43), we get:

$$q(T) = 122.19 + \frac{1185.74[1-(1-0.999)^{0.487}]}{0.487}$$

$$= 2472.74 \text{ m}^3/\text{s}$$

7.4 ESTIMATION OF FLOOD DISCHARGE FOR A CONFIDENCE INTERVAL

It is often of interest to estimate confidence interval for a predicted return flood. The procedure to estimate confidence interval involves the following steps.

1. Compute standard error (S_E) for a given distribution as given below.

For normal distribution

$$S_E = \sigma\left(\frac{2+z^2}{n}\right)^{1/2} \tag{7.44}$$

For extreme value type-I

$$S_E = \sigma\left[\frac{1}{n}[1+1.1396K_T+1.1(K_T)^2]\right]^{1/2} \tag{7.45}$$

where, K_T is the frequency factor.

2. Compute flood discharge for a particular confidence limits as the formula given below. Table 7.9 provides the value of $f(c)$ for a particular confidence interval.

Confidence Limits

$$x_{\text{conf}} = x_T \pm f(c)\, S_E \tag{7.46}$$

where, $f(c)$ is the function of confidence probability.

Table 7.9 Determination of $f(c)$ for a particular confidence interval

c (%)	50	68	80	90	95	99
$f(c)$	0.674	1.0	1.282	1.645	1.96	2.58

The following example illustrates the estimation of flood discharge for a given confidence interval.

Example7.12 Data covering a 50 years for a river at a station yielded: mean = 5000 m^3/s and standard deviation = 1200 m^3/s. Using Gumbel's method, estimate the flood discharge for T = 250 years. What are the 95% and 80% confidence limits for this estimate?

Solution

$F(z) = 1 - 1/250 = 0.996$

Using the formula $K_T = -\dfrac{\sqrt{6}}{\pi}\left\{0.5772 + \ln\left[\ln\left(\dfrac{T}{T-1}\right)\right]\right\}$, we have:

$$K_T = -\frac{\sqrt{6}}{\pi}\left\{0.5772 + \ln\left[\ln\left(\frac{250}{250-1}\right)\right]\right\}$$

$$= 3.853$$

$x_T = \mu + K_T\sigma = 5000 + 3.853 \times 1200 = 9624.15$ m^3/s

Calculation of standard error

$$S_E = \sigma\left[\frac{1}{n}(1 + 1.1396K_T + 1.1K_T)^2\right]^{1/2}$$

$$S_E = 1200\left[\frac{1}{50}(1 + 1.1396 \times 3.853 + 1.1 \times 3.853^2)\right]^{1/2} = 790.2$$

For c = 95 %, $f(c)$ = 1.96

$x_{95\%} = 9624.15 \pm (1.96 \times 790.92) = 11174.35$ m^3/s and 8073.94 m^3/s

$x_{80\%} = 9624.15 \pm (1.282 \times 790.92) = 10638.1$ m^3/s and 8610.19 m^3/s

7.5 REGIONAL FLOOD FREQUENCY ANALYSIS

The purpose of frequency analysis is to estimate the design flood for desired recurrence interval, assuming the sample data follow a theoretical frequency distribution. It is assumed that the sample data is a true representative of the population. It is generally seen that a minimum of 30 to 40 years of records are needed in order to carry out flood frequency analysis to the at-site data for estimating the floods in extrapolation range, somewhat within the desired accuracy.

In case the length of records is too short, it represents inadequate data situation; then at-site flood frequency analysis fails to provide the reliable and consistent flood estimates. Under inadequate data situation, regional flood frequency curves together with at-site mean provides more reliable and consistent estimates of floods. For ungauged catchments, the regional flood frequency analysis approach is the only way to estimate the flood for a desired recurrence interval, for which a regional relationship between mean annual peak flood and catchment characteristics is developed along with the regional frequency curves.

7.5.1 Index-Flood Method

Basically, the index-flood method (Dalrymple, 1960) extrapolates statistical information of runoff events for flood frequency analysis from gauged catchments to ungauged catchment in the vicinity having similar catchments and hydrologic characteristics. Given below are the sequential steps which are to be followed to estimate return period flood using index-flood method.

(i) Select the gauged catchments within the region having similar characteristics to the ungauged catchment.
(ii) Determine the base period to be used for the study.
(iii) Establish the flood frequency curve prepared with the ranked annual series and estimate the corresponding plotting position using *Gringortan plotting* formula.
(iv) $Q_{T=2.33\text{ yr}}$ (annual mean flood) is estimated using EV-I parameters.
(v) Establish the relationship between the mean annual flood and the basin area.
(vi) Rank the ratios of floods of selected return periods to the mean annual flood at each site and for a given return period, obtain the ratio of flood (having return period T) to mean annual flood for each site.
(vii) For N sites, there will be N values of such ratios for a given return period.
(viii) Use these ratios to compute median flood ratio for each of the selected return period and multiply each median flood ratio by the estimated mean annual flood of the ungauged catchment, and plot these values versus recurrence interval on Gumbel probability paper.

The curves obtained in the last step are the flood frequency curves for ungauged catchments.

Example 7.13 Flood frequency analysis for five homogeneous catchments provides the following estimates for mean annual flood and 50-year return flood.

Table 7.10 Data for Example 7.13

Catchment	*Mean annual flood*	*50-year return flood*
A	20	100
B	35	150
C	25	120
D	50	170
E	10	90

An ungauged catchment which is having similar characteristics to catchments A to E is having a mean annual flood of 40 m^3/s. Estimate the 50-year return flood for the ungauged catchment.

Solution

Table 7.11 Solution table for Example 7.13

Catchment	*Ratio of 50-year return flood to mean annual flood*
A	5
B	4.285
C	4.8
D	3.4
E	9.0

Thus, median value of ratio of 50-year return flood to mean annual flood = 4.8
As the mean annual flood for ungauged catchment = 40 m^3/s
The 50-year return flood for ungauged catchment = $40 \times 4.8 = 192$ m^3/s

7.5.2 Method of L-Moments

The following sequential steps are followed:

(i) Test regional homogeneity for the selected gauged catchments using the procedure described by Dalrymple (1960), and discard the catchments which are not homogeneous.
(ii) Arrange the flood series, and compute L-moments.
(iii) Let the L-moments are $\lambda_{1,i}$, $\lambda_{2,i}$, $\lambda_{3,i}$, $\lambda_{4,i}$. where i is the index of sites. At each site i, the no. of records of flood data are n_i.
(iv) The total data of all the site is N where N is equal to $\sum_{i=1}^{m} n_i$ where m is the number of sites.
(v) Scale each of the linear moments by multiplying with the ratio n_i/N.
(vi) Sum all such scale $\lambda_{1,i}$ and this gives the mean annual flood.

(vii) Normalize the mean annual flood so that it becomes unity. In other words assume the mean flood is unity. This implies aggregated value of λ_1 is 1.

(viii) Corresponding to $\lambda_1 = 1$, obtain λ_2 and λ_3. Use these λ values to estimate PDF parameters.

(ix) Use PDF or CDF with the known parameters to compute a flood of return period T such that the mean of such computed return period floods is unity.

(x) Multiply the flood in step (ix) with the mean annual flood of ungauged catchment. The product is the magnitude of the flood of ungauged catchment having return period T.

To explain steps (i) to (x), an example is provided here.

Example 7.14 For North Cascades drainage basin, the lengths and L-moment ratios for 19 sites are given in Table 7.12. Estimate regional parameters assuming that PT-III distribution fits well to the given data.

Table 7.12 Summary statistics of North Cascades precipitation data (*Source*: Hoskings and Wallis, 1997)

n	λ_1	τ	τ_3	*Col (2)× Col (1)/1378*	*Col (3) × Col (1)/1378*	*Col (4) × Col (1)/1378*
98	19.69	0.1209	0.0488	1.400305	0.008598	0.003471
59	62.58	0.0915	0.0105	2.679405	0.003918	0.00045
90	40.85	0.1124	0.0614	2.667997	0.007341	0.00401
61	46.05	0.1032	0.0417	2.038498	0.004568	0.001846
65	45.02	0.0967	–0.0134	2.123585	0.004561	–0.00063
86	31.04	0.1328	–0.0176	1.937184	0.008288	–0.0011
78	80.14	0.1008	0.0943	4.536226	0.005706	0.005338
72	41.31	0.1143	0.0555	2.158433	0.005972	0.0029
67	30.59	0.1107	0.0478	1.487322	0.005382	0.002324
99	32.93	0.1179	0.0492	2.365798	0.00847	0.003535
49	17.56	0.1308	0.094	0.624412	0.004651	0.003343
61	69.52	0.1119	–0.0429	3.077446	0.004953	–0.0019
69	47.65	0.1018	0.0435	2.385958	0.005097	0.002178
73	102.5	0.1025	0.0182	5.429971	0.00543	0.000964
70	52.41	0.1054	–0.0224	2.662337	0.005354	–0.00114
66	79.7	0.1174	0.0124	3.817271	0.005623	0.000594
59	44.64	0.1115	–0.0346	1.911292	0.004774	–0.00148
74	58.66	0.1003	0.0446	3.150102	0.005386	0.002395
82	39.02	0.1046	0.0128	2.321945	0.006224	0.000762
Sum =1378	941.86			Avg. $\lambda_1 = 48.775$	Avg. $\tau = 0.11029$	Avg. $\tau_3 = 0.027859$

The regional normalized average mean, L-CV, L-skewness are: $\lambda_1 = 1$, $\tau = 0.11029$, $\tau_3 = 0.027859$

Solution

For the data of Table 7.12, it is reported that PT-III distribution gives close fit to the regional average L-moments. Thus, it is of interest to estimate parameters of PT-III distribution.

To estimate these parameters, we use the following set of equations given by Hoskings and Wallis (1997):

$$\beta \approx \begin{cases} \dfrac{1+0.2906z}{z+0.1882z^2+0.0442z^3} & \left(0<|\tau_3|<\dfrac{1}{3}\right) \\ \dfrac{0.36067z-0.59567z^2+0.25361z^3}{1-2.78861z+2.56096z^2-0.77045z^3} & \left(\dfrac{1}{3}\le|\tau_3|<1\right) \end{cases} \tag{7.47}$$

where, $z = 3\pi(\tau_3)^2$

and, $$g = 2\,\beta^{-1/2}\,\text{sign}(\tau_3), \quad \sigma = \frac{\lambda_2 \pi^{1/2}\beta^{1/2}\Gamma(\beta)}{\Gamma\left(\beta+\dfrac{1}{2}\right)}, \quad \mu = \lambda_1 \tag{7.48}$$

We find,

$$z = 3\pi\,(0.027859)^2 = 0.00731479$$

Using the value of z in Eq (7.47), the value of $\beta = 136.811$

Using the value of β in Eq (7.48), the parameters are obtained as $\mu = 1.0$, $\sigma = 0.1196$, and $g = 0.1709$

[Hint: Using skewness value and recurrence interval, determine K_T and thus $x_T = 1 + K_T\,\sigma$. For 50-year return flood, $K_T = 2.143$.

Thus, $x_T = 1.2563$. If the mean annual flood of ungauged catchment is Q m^3/s, then the 50-year return flood for the ungauged catchment will be $1.2563Q$ m^3/s.]

7.6 RISK AND RELIABILITY CONCEPTS

Basic questions to be answered in design are as follows:

1. Why should we do risk analysis?
2. What should be the return period for which the structure should be designed?
3. What is the risk involved when we design a structure having a design life of n years for a T-year return period flood?
4. How much risk is permissible?

7.6.1 Risk Analysis Parameters

If this design event has a return period of T years and a corresponding annual probability of exceedence of p, then:

$$p = \frac{1}{T} \tag{7.49}$$

The probability of non-occurrence in any one year is:

$$q = 1 - \frac{1}{T} \tag{7.50}$$

The probability of non-occurrence in n years is:

$$q = \left(1 - \frac{1}{T}\right)^n \tag{7.51}$$

Hence, the probability that X will occur at least once in n years, i.e., the risk of failure R is:

$$R = 1 - \left(1 - \frac{1}{T}\right)^n \tag{7.52}$$

where, n is the design life of the structure.

Example 7.15 What will be the risk involved for a hydraulic structure having a design life of 100 years if it is designed for: (i) 50-year return period flood, and (ii) 1000-year return period flood?

Solution

(i) $n = 100$ years, $T = 50$ years, and the risk (R) involved may be computed by Eq. (7.52) as:

$$R = 1 - \left(1 - \frac{1}{50}\right)^{100}$$

$$= 0.867 = 86.7\%$$

(ii) $n = 100$ years, $T = 1000$ years, and the risk involved is:

$$R = 1 - \left(1 - \frac{1}{1000}\right)^{100}$$

$$= 0.095 = 9.5\%$$

Table 7.13 Return periods associated with various degrees of risk and expected design life

Risk %	*Expected Design Life*							
	2	*5*	*10*	*15*	*20*	*25*	*50*	*100*
75	2.00	4.02	6.69	11.0	14.9	18.0	35.6	72.7
50	3.43	7.74	11.9	22.1	29.4	36.6	72.6	144
40	4.44	10.3	20.1	29.9	39.7	49.5	98.4	196
30	6.12	14.5	28.5	42.6	56.5	70.6	140	281

(*Contd.*)

(*Contd.*)

25	7.46	17.9	35.3	52.6	70.0	87.4	174	348
20	9.47	22.9	45.3	67.7	90.1	112	224	449
15	12.8	31.3	62.0	90.8	123	154	308	616
10	19.5	48.1	95.4	142	190	238	475	950
5	39.5	98.0	195	292	390	488	976	194
2	99.5	248	496	743	990	1238	2475	4950
1	198	498	996	1492	1992	2488	4975	9953

Based on the risk acceptable, the return period for which the structure should be designed can be ascertained.

Example 7.16 What is the return period that a highway engineer must use in his design of a critical underpass drain if he is willing to accept only:

(i) 10% risk that the flooding will occur in the next 5 years?

(ii) 20% risk that the flooding will occur in the next 2 years?

Solution

(i) The risk involved (R) is 0.10 and $n = 2$ years. Substituting these values in the following equation:

$$R = 1 - \left(1 - \frac{1}{T}\right)^n$$

$$0.10 = 1 - \left(1 - \frac{1}{T}\right)^5$$

Or, $T = 48.1$ years

This means that there is 10% chance that a 48.1-year flood will occur once or more in the next 5 years.

(ii) The risk involved (R) is 0.20 and $n = 2$ years.

$$0.20 = 1 - \left(1 - \frac{1}{T}\right)^2$$

Or, $T = 9.47$ years

It shows that there is a chance of 20% that a flood corresponding to a return period of 9.47 years will occur in the next 2 years. These values may also be read from the Table 7.13.

7.7 BINOMIAL DISTRIBUTION

The Binomial distribution is a discrete distribution and is based on the Binomial theorem which states that probability of exactly x successes in n trials is:

$$P(x) = \binom{n}{x} p^x q^{n-x} \tag{7.53}$$

where, $$\binom{n}{x} = \frac{n!}{x!(n-x)!},\ q = 1 - p \tag{7.54}$$

p = probability of exceedence/success

q = probability of non-exceedence/failure

x = number of exceedence/successes

n = total number of events

The assumptions for binomial distribution are same as those for Bernoulli trials. Tossing a coin and drawing a card from a pack are the common examples of Bernoulli trials which operate under the following three conditions:

(i) Any trial can have success or failure (not both), true or false, rain or no rain.
(ii) Successive trials are independent.
(iii) Probabilities are stable.

Binomial distribution is valid under the above three conditions.

Example 7.17 If a dam is having a project life of 50 years, then what is the probability that a flood with a return period of 100 years will occur (i) once, and (ii) twice during the life of the dam?

Solution

(i) Given, $n = 50$ years, $T = 100$ years, and p = probability of exceedence

Here, $p = \dfrac{1}{T} = \dfrac{1}{100} = 0.01$

$q = (1 - p) = (1 - 0.01) = 0.99$

$x = 1$

$n - x = 50 - 1 = 49$

$$P(1) = \binom{50}{1} (0.01)^1\ (0.99)^{49}$$

$$= \frac{50!}{1!(49)!} (0.01)^1\ (0.99)^{49}$$

$$= 0.306 = 30.6\%$$

(ii) Given, $n = 50$ years, $T = 100$ years, and $x = 2$

$$n - x = 50 - 2 = 48$$

$$P(2) = \binom{50}{2}(0.01)^2(0.99)^{48}$$

$$= \frac{50!}{2!(48)!}(0.01)^2(0.99)^{48}$$

$$= 0.0756 = 7.56\%$$

This means that there is a 30.6% chance that a 100-year return period flood will occur once during the project life, and the chances of its twice occurrence is 7.55%.

7.8 POISSON DISTRIBUTION

The terms of binomial expansion are a little inconvenient to compute for large numbers. If n is large (> 30) and p is small (< 0.1), then binomial distribution tends to Poisson distribution.

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!} \tag{7.55}$$

where, $\lambda = n\,p$

The conditions for this approximation are:

(i) The number of events is discrete.
(ii) Two events cannot coincide.
(iii) The mean number of events in unit time is constant.
(iv) Events are independent.

Example 7.18 Solve Example 7.16 using Poisson distribution.

Solution

$$n = 50 \quad \text{and} \quad p = 1/100$$

(i) $$P(1) = \frac{\left(50 \times \frac{1}{100}\right)^1 e^{-(50/100)}}{1!} = 0.303 = 30.3\%$$

(ii) $$P(2) = \frac{\left(50 \times \frac{1}{100}\right)^2 e^{-(50/100)}}{2!} = 0.0758 = 7.58\%$$

The results obtained by Binomial distribution and Poisson distribution are almost same.

7.9 DESIGN FREQUENCIES

The designer is generally concerned with the return period for which the structure should be designed. The design frequencies shown in Table 7.4 are typical of levels generally encountered in minor structure design (Viessman et al., 1977).

Probable Maximum Flood (PMP) Probable maximum flood is the flood caused by probable maximum precipitation. Probable maximum flood is generally obtained using unit hydrograph and rainfall estimates of the PMP.

Standard Project Floods (SPF) Standard project flood is the flood caused by standard project storm which is generally obtained from a survey of severe storms in the general vicinity of the drainage basin or severe storms experienced in meteorologically similar areas.

Table 7.14 Minor structure design frequencies

Type of Minor Structures	*Return Period (in T years)*	*Frequency* $\left(P = \frac{1}{T}\right)$
Highway crossroad drainage	10	0.1
0–400 ADT*	10–25	0.1–0.04
400–1700 ADT	25	0.04
1700–5000 ADT	50	0.02
5000 ADT	5	0.2
Airfields	2–10	0.5–0.1
Storm drainage	2–5	05–0.02
Levees	5–50	0.20–0.02

*Average Daily Traffic

Table 7.15 Spillway design flood for dams

<table>
<tr><th rowspan="3">Category</th><th colspan="2">Impoundment danger potential</th><th colspan="3">Failure damage</th></tr>
<tr><th rowspan="2">Storage (Acre-feet)</th><th rowspan="2">Height (ft.)</th><th colspan="2">Potential</th><th rowspan="2">Spillway design flood</th></tr>
<tr><th>Loss of life</th><th>Damage</th></tr>
<tr><td>Major</td><td>50,000</td><td>60</td><td>Considerable</td><td>Excessive or as a matter of policy</td><td>Probable maximum flood (a)</td></tr>
<tr><td>Intermediate</td><td>1000–50,000</td><td>40–100</td><td>Possible but small</td><td>—</td><td>Standard project flood (b)</td></tr>
<tr><td>Minor</td><td>1000</td><td>50</td><td>None</td><td>Of same magnitude as cost of dam</td><td>50–100 years occurrence interval</td></tr>
</table>

Table 7.16 Frequency of return period advocated by IBS code

Structure	*Return Period of design flood (years)*
Major dams with storage more than 6000 ha.m. (50,000 ac. ft.)	1000
Minor dams with storage less than 6000 ha.m.	100*
Barrages and pick up weirs	
(a) Free board	500+
(b) Items other than free board	50–100
River Training Works (Calculation of Scour)	50–100@
Water way of bridges	50

* IS 5477 – Method of fixing capacities of reservoirs–Part IV
+ IS 6966 – Criteria for hydraulic design of barrages and weirs
@ IS 3408 – Criteria for river training works for barrages and weirs in alluvium

In case of spillways for major and medium reservoirs with storage capacities of more than 6000 ha.m., designing is generally done for the probable maximum flood which is the physical upper limit of the flood in the catchment.

Table 7.17 CWC, India recommends the following design frequencies

Structure	*Return period*
Major dam with storage more than 50,000 acre feet	Probable maximum flood or frequency not less than once in 100 years
Permanent barrages and minor dams with storage less than 50,000 acre feet	Standard project flood or 100 year flood whichever is higher
Pick up weirs	50–100 years
Canal aqueducts:	
(a) Waterways	50–100 years
(b) Foundations and freeboard	50–500 years

7.10 PEAK OVER THRESHOLD (POT) MODELS

Two approaches commonly used for probabilistic analysis of extreme flood magnitudes are based on annual maximum series (AMS) and partial duration series (PDS). The AMS model considers the annual maximum flood of each year in a series that has as many elements as there are years in the data record. In contrast, the PDS model considers all flood peaks above a certain threshold level. Clearly, in PDS model, more than one flood per year may be included; hence, it is also referred as peak over threshold (POT) method. For this reason, PDS models are used for flood frequency analysis (FFA) where the flood record is short.

In the limit, when the threshold is increased, recurrence interval of large events computed using AMS and PDS models tend to converge. The methods followed for formulating these models are quite different; e.g., an AMS model uses a cumulative distribution function (CDF) to model the exceedences, whereas the PDS model uses two probabilistic models: (a) one for the probability of occurrence of peaks above a threshold, and (b) the CDF, for modelling the flood exceedences. Thus, for a fixed size of observation years, the sample size in PDS method is a random variable. We shall consider the approach that is employed in formulation of such a model (Madsen et al. 1997).

7.10.1 Models for PDS Analysis

In PDS model, all peak events above a threshold level are considered, and the number of these occurrences (x) in a given year is assumed to follow Poisson distribution having the probability mass function:

$$P(x = r) = \frac{e^{-\mu}\mu^r}{r!}; \quad r = 0, 1, 2, \ldots \tag{7.56}$$

where, $P(x = r)$ is the probability of r events in a year, μ is the average number of threshold exceedences. For n number of observed exceedences in N years, μ is equal to n/N, however, Eq. (7.56) is true as long as the flood peaks are independent. For Poisson distribution, PWM estimations and the population estimate of μ is given by:

$$\mu = \frac{n}{N} \tag{7.57}$$

The mean and variance of r are defined as follows:

$$E(r) = \mu = n/N = \text{Var}(r) = \sigma^2; \quad (n \geq 5) \tag{7.58}$$

For $\mu > 5$, the distribution of r is symmetrical, and it asymptotically approaches a normal distribution (Stark and Woods, 1986). For no flood exceedences to occur in a given year, substituting $r = 0$ in Eq. (7.56), the following is obtained.

$$P(r = 0) = \frac{e^{-\mu}(\mu)^0}{0!} = e^{-\mu} \tag{7.59}$$

which is the PDF of an exponential distribution. The cumulative mass function of Eq. (7.56) for N years is given by:

$$F = 1 - \mu e^{-\mu N} \quad ; \quad (N > 0) \tag{7.60}$$

Assume that a threshold level q_0 is chosen, corresponding to a mean annual number of exceedences of Poisson distribution μ; then at any higher threshold $(q > q_0)$, the number of exceedences r' is Poisson distributed with the following parameter (Flood Studies Report, 1975):

$$\mu' = \mu[1 - F(q)] \tag{7.61}$$

Since $F(q) \leq 1$, it follows that $(\mu' \leq \mu)$, i.e., the truncated mean of exceedences above the new threshold q decreases, which is quite obvious.

The AMS model used for determination of T-year return period flood $[q(T)]$ requires identification of a statistical distribution so as to obtain the $[q(T) - T]$ relationship in explicit form.

If n floods exceed q_0 in N years of record in PDS, the average rate of occurrence is $\mu = n/N$. Consequently, the flood which has a return period T' sampling units in this PDS has a return period of approximately $T = T'/\mu$ years. Thus, the PDS method uses two probabilistic models:

(a) one for the probability of occurrence (P_λ) for μ exceedences in each year, and
(b) the statistical distribution $F_{\mu,y}$ of the maximum of n exceedences, where $y = q - q_0$.

The probability of annual maxima is obtained using the CDF, computed using the total probability theorem (Shane and Lynn, 1964) as:

$$F(q) = \sum_{r=0}^{\infty} P_\mu(r)\,(F_{\mu,y})^r \tag{7.62}$$

If P_μ follows Poisson distribution, and $F_{\mu,y}$ follows an exponential distribution $= 1 - \exp[-(q - q_0)/\beta]$ with parameter β, then

$$F(q) = \sum (F_{\mu,y})^r \frac{1}{r!}(\mu^r e^{-\mu}) = \exp\left[-\exp\left(-\frac{1}{\beta}(q - q_0)\right) + \ln(\mu n)\right] \tag{7.63}$$

$F(q)$ given by Eq. (7.63) is the Gumbel or EV-I distribution that is widely used in FFA. From Eq. (7.63), the following PDS model can be obtained.

$$q(T) = q_0 + \beta[\ln(\mu)] + \beta y_T \tag{7.64}$$

where, $y_T = -\ln[-\ln(1 - 1/T)]$ is the Gumbel reduced variate.

Since the T-year return period is defined as the average interval of time within which the magnitude of q of the flood event will be equaled or exceeded once, the POT model should be preferred over the AMS model for estimating the q-T relationship. Thus, a comparison of the AMS and PDS model is of practical interest.

Example 7.19 The daily flow records of a catchment were available for 18 years. From the data record, all the flow values crossing a threshold of 40 m^3/s are given in Table 7.18 These thresholds are to be used to develop a PDS model using Poisson distribution (PD) for counting the peaks exceeding the threshold and exponential distribution (ED) is to be used for modelling the exceedences. Use the derived PDS/PD-ED model to find the 50-year return period flood. Assume mean exceedences (μ) = 3.

Table 7.18 Flow records and steps to find 50-year return period flood

Rank (i)	*Q (m^3/s)*	*F*	y_i	*Rank (i)*	*Q (m^3/s)*	*F*	y_i
1	276.92	0.991	4.659	31	63.36	0.483	0.660
2	237.5	0.974	3.635	32	63.36	0.466	0.628
3	136.35	0.957	3.140	33	62.95	0.449	0.596
4	131.35	0.940	2.810	34	62.95	0.432	0.566
5	124.04	0.923	2.562	35	62.54	0.415	0.537
6	118.12	0.906	2.364	36	61.31	0.399	0.508
7	116.38	0.889	2.199	37	61	0.382	0.481
8	114.07	0.872	2.057	38	61	0.365	0.454
9	109.54	0.855	1.932	39	61	0.348	0.427
10	97.61	0.838	1.822	40	59.3	0.331	0.402
11	96.04	0.821	1.722	41	59.3	0.314	0.377
12	93.58	0.804	1.632	42	59	0.297	0.352
13	91.91	0.788	1.549	43	57.32	0.280	0.329
14	90.91	0.771	1.472	44	56.15	0.263	0.305
15	90.91	0.754	1.401	45	55	0.246	0.283
16	86.92	0.737	1.335	46	55	0.229	0.261
17	85.94	0.720	1.273	47	54.23	0.212	0.239
18	84.96	0.703	1.214	48	54.23	0.196	0.218
19	82.07	0.686	1.159	49	54.23	0.179	0.197
20	81.12	0.669	1.106	50	52.35	0.162	0.176
21	80.17	0.652	1.056	51	51.61	0.145	0.156
22	80.17	0.635	1.009	52	51.61	0.128	0.137
23	78.3	0.618	0.963	53	51.24	0.111	0.118
24	71.94	0.601	0.920	**54**	**50.87**	**0.094**	**0.099**
25	70.61	0.585	0.878	55	50.14	0.077	0.080
26	67.57	0.568	0.839	56	49.78	0.060	0.062
27	67.14	0.551	0.800	57	49.06	0.043	0.044
28	67.14	0.534	0.763	58	49.06	0.026	0.027
29	65.45	0.517	0.728	59	48	0.009	0.010
30	64.61	0.500	0.693				

Solution

Mean of the partial duration series ($\bar{q}$) = Mean of the data in Column (2) = 82.23 m^3/s.

N = (No. of years of AMS data) × μ = 3 × 18 = 54

That means the threshold (q_0) from the table is equal to 50.87 m^3/s (marked as bold in the Table).

By MOM estimation: $\beta = N(\bar{q} - q_0)/(N - 1) = 31.95$ m^3/s

Hence, q (50 y) = 50.87 + 31.95 [ln (3)] + 31.95 y_{50} = 210.3 m^3/s.
[$y_{50} = -\ln\{-\ln(1 - 1/50)\} = 3.9$]

SUMMARY

In this chapter, different methods of flood parameter estimation have been discussed. Flood estimation using normal, lognormal, Pearson type, log-Pearson, and Gumbel distribution have been explained in detail. For designing any hydraulic structure, flood frequency analysis and risk factor and reliability of the structure in frequency analysis are generally considered. All these areas have been covered in this chapter.

The designer is generally concerned with the return period for which the structure should be designed. Depending on the available data at any site, it is possible to fit a suitable probability distribution function. Subsequently, this function can be used to estimate flood of an assumed return period. Use of PDS models is also highlighted. Using such models, all flood peaks above a certain threshold level can be analysed. This approach can be more useful when the data on annual maximum flood is limited.

EXERCISES

7.1 The following table shows the estimated flood peaks using Gumbel method and corresponding return periods.

Peak Flood (m^3/s)	*Return Period (Years)*
350	50
275	25

Find out the amount of peak flood corresponding to 100-years return period. **Ans.** 384.76 m^3/s

7.2 A bridge is to be constructed over a river for a design flood of 500 m^3/s. The mean and the standard deviation for annual flood series are 250 m^3/s and 100 m^3/s, respectively. Calculate the return period corres-ponding to this design flood using Gumbel method.

Ans. 44.47 years

7.3 Let the probability of rainfall occurrence on any day in the month of May is 0.3. If you are wishing a no-rainfall 7-day period as you are planning for a 7-day study tour, (i) find the probability that the rainfall will not occur in these 7 days; (ii) find the probability of 1 rainy day during these 7 days. **Ans. (i)** 0.0824 **(ii)** 0.2471

7.4 What would be the return period of a design storm to be used for the design of a hydraulic structure? Let there be a probability of 25% that the storm will occur in the next 25 years. **Ans.** 87.40 years

7.5 The return period (T) and risk of failure (R) are related as:

$$R = 1 - \left(1 - \frac{1}{T}\right)^n$$

where, n is the number of years. Plot a graph between R and T for $n = 10, 25, 50, 100$, and 150.

(i) Observe the nature of the graphs.
(ii) Find the risk of failure that a 50-year flood will occur at least once in the next 10 years. **Ans.** 0.183 or 18.3%

7.6 A stream has the flood peaks of 1200 m^3/s and 1060 m^3/s for the return periods of 100 and 50 years, respectively, using Gumbel method for a dataset spanning 30 years. Find (i) mean and standard deviation; and (ii) flood peak corresponding to a return period of 500 years.
Ans. 425.85 m^3/s, 257.24 m^3/s, 1523.7 m^3/s

7.7 The mean, standard deviation, and coefficient of skewness of log-transformed annual maximum peak flood series of a typical gauging site are given below:

Mean (m^3/s)	700.25
Standard deviation (m^3/s)	255.6
Coefficient of skewness	2.31

Estimate 100-year floods assuming that the peak discharge data follow EV-I distribution. Assume the peak discharge follows other distributions and compare their results. **Ans.** 1501.97 m^3/s

7.8 Using Poisson distribution, find: (i) the probability that a 6-year flood will occur 7 times in a period of 20 years, (ii) the probability that it will not occur at all, and (iii) how many floods of this magnitude will occur on an average during 20 years? Compare the results with those obtained from binomial distribution. **Ans. (i)** 0.0322 **(ii)** 0.0358 **(iii)** 0 3.33

7.9 Suppose you are going to design a check dam for a flood having a return period of 100 years. What are the risks that you are going to accept for flooding to occur in the next 5 years? **Ans.** 0.049

7.10 A temporary dam is to be built to protect a 5-year construction activity for a hydroelectric power plant. If the dam is designed to withstand 25-year flood, determine the risk that the structure will be overtopped: (i) at least once in the 5-year construction period, and (ii) not at all during the 5-year period. **Ans. (i)** 0.185 **(ii)** 0.815

OBJECTIVE QUESTIONS

1. Which is not a parameter estimation technique?
   (a) Graphical method (b) Least squares method
   (c) Method of moments (d) None of these
2. For normal and lognormal distribution, the recommended plotting position formula is
   (a) Blom (b) Gringorton
   (c) Cunnane (d) None of these
3. Coefficient of correlation is
   (a) Equal to the coefficient of determination
   (b) Square root of the coefficient of determination
   (c) Cubic root of the coefficient of determination
   (d) None of the above
4. The relationship between the return period (T) and the probability of exceedence (p) is
   (a) $p = 1 - T$ (b) $p = \frac{1}{T}$
   (c) $T = 1 - p$ (d) None of these
5. A dam is having a project life of 50 years. What is the probability of flood that does not exceed having a return period of 100 years?
   (a) 0.01 (b) 0.99
   (c) 0.9 (d) None of these
6. What is the return period that a designer must use in a dam design if he/she is willing to accept: (i) only 5% risk that the flooding will occur in the next 25 years.
   (a) 100 (b) 250 (c) 487 (d) 500
7. For an annual flood series arranged in decreasing order of magnitude, the return period for a magnitude listed at position m in a total of N entries is
   (a) m/N (b) $m/(N + 1)$ (c) $(N + 1)/m$ (d) $N/(m + 1)$
8. Probable maximum flood is
   (a) An extremely large flood
   (b) The flood caused by probable maximum precipitation
   (c) Standard project flood
   (d) All of the above

9. As per the CWC recommendation, in the design of permanent barrages and minor dams with storage capacity of less than 50,000 acre feet, the return period is considered as
   (a) Probable maximum flood
   (b) Standard project flood
   (c) Standard project flood or 100-year flood, whichever is higher
   (d) None of the above
10. The regional flood frequency analysis approach is used to estimate the flood
   (a) For gauged catchment (b) For ungauged catchment
   (c) Both (a) and (b) (d) None of these

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CHAPTER

8

Principles of Groundwater Flow

8.1 INTRODUCTION

Groundwater has been a popular resource of water in many tropical countries. Groundwater is easy to extract, and it remains well protected from the hazards of pollution that the surface water has to put up with. However, situations wherein we have encountered overexploitation of groundwater resources are not uncommon. Lack of detailed knowledge about the basics of groundwater is the primary reason why we have not been able to use groundwater resources to their full extent. Thus, there is a growing emphasis on groundwater management.

In this chapter, we intend to cover the basics of groundwater which includes a description of the potential, the flow from higher potential to lower potential by Darcy's law along with its application. Some of the preliminary terms related to groundwater are also defined. The basic equations of groundwater flow for steady and unsteady saturated flow are derived. An insight into solutions of the groundwater flow equations is also provided for a variety of flow conditions.

8.2 MECHANICAL ENERGY AND FLUID POTENTIAL

The flow always occurs from higher potential quantity to lower potential quantity, the nature of the physical phenomena notwithstanding. These potential quantities differ from process to process. For example, heat flows from higher temperature to lower temperature, and current flows from higher voltage to lower voltage; temperature and voltage being the potential quantities. The difference of the potential quantity per unit distance between two places is known as the *potential gradient*. As the potential gradient increases, the rate of flow increases.

Hubbert (1940) defines potential as "a physical quantity, capable of measurement at every point in a flow system, whose properties are such that flow always occurs from regions in which the quantity has higher values to those in which it has lower values, regardless in the direction of space". According to classical definition, potential is work done during the flow process.

The groundwater also flows from higher to lower fluid potential. Groundwater flow is a mechanical process; forces driving the fluid must overcome the frictional forces between porous media and the fluid. This indicates that some energy is lost during the flow process. That energy is converted into thermal energy. "Work" is defined as the mechanical energy per unit mass, required to move a fluid from point $z = 0$ to point z.

Since the physical quantity (fluid potential) satisfies both Hubbert's definition and classical definition,

Fluid potential is mechanical energy per unit mass = Work done to move unit mass

Figure 8.1 shows an arbitrary standard state; at elevation $z = 0$, pressure $p = p_0$, where p_0 is atmospheric pressure.

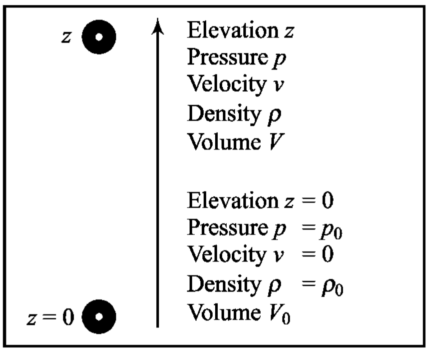

Fig. 8.1 *Calculation of mechanical energy of unit mass of fluid*

Our aim is to calculate the work required to lift the unit mass of the fluid of density ρ_0 and volume V_0 from point $z = 0$ to point z. Fluid potential is the mechanical energy per unit mass. Fluid potential at z = (fluid potential at datum) + (work done from $z = 0$ to point z)

The work done to move a unit mass of water has three components.

1. Work done to lift the mass from elevation $z = 0$ to elevation z.

$$w_1 = mgz \tag{8.1}$$

2. Work done to accelerate the fluid from velocity $v = 0$ to velocity v.

$$w_2 = \frac{mv^2}{2} \tag{8.2}$$

3. Work done to raise the fluid pressure from p_0 to p.

$$w_3 = \int_{p_0}^{p} V\,dp = m\int_{p_0}^{p} \frac{V}{m}\,dp = m\int_{p_0}^{p} \frac{dp}{\rho} \tag{8.3}$$

The fluid potential (the mechanical energy per unit mass, $m = 1$) is sum of the above three works, i.e., loss in potential energy, loss in kinetic energy, and loss in pressure energy.

So, we have:

$$\phi = gz + \frac{v^2}{2} + \frac{dp}{\rho} \tag{8.4}$$

The velocity term is not important for groundwater flow because the velocities are extremely low. So, the second term can be neglected. Density is constant for incompressible fluid. So, the above equation can be simplified to give:

$$\phi = gz + \frac{p - p_0}{p} \tag{8.5}$$

8.3 FLUID POTENTIAL AND HYDRAULIC HEAD

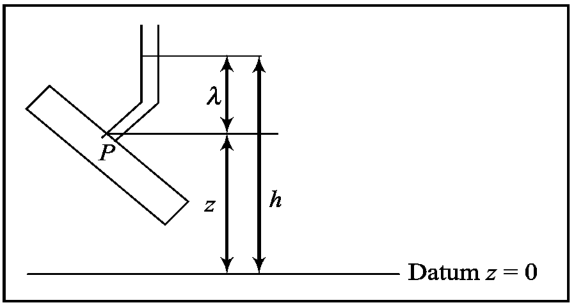

Fig. 8.2 *Manometer showing hydraulic head*

Figure 8.2 shows a manometer. Let p_0 be the atmospheric pressure. The fluid pressure p at point P is given by:

$$p = \rho g\lambda + p_0 \tag{8.6}$$

where λ is the height of the liquid column above point P.

From Fig. 8.2,

$$\lambda = h - z \tag{8.7}$$

Substituting the value of λ in Eq. (8.6), we have:

$$p = \rho g(h - z) + p_0 \tag{8.8}$$

Substituting Eq. (8.8) in Eq. (8.5):

$$\phi = gz + \frac{[\rho g(h - z) + p_0] - p_0}{\rho}$$

Or $\quad \phi = gh \quad$ (8.9)

Equation (8.9) shows the relationship of the fluid potential and hydraulic head. Since acceleration due to gravity (g) is constant, the hydraulic head and the fluid potential (ϕ) at any point P in a porous medium are perfectly correlated. Therefore, the hydraulic head (h) is suited as potential function and is also used in groundwater flow.

Normally, the atmospheric pressure is assumed as zero. The idea is to work in *gage pressures* (i.e., pressures with respect to atmospheric pressure). Thus, one can also write:

$$\phi = gz + \frac{p}{\rho} = gh \tag{8.10}$$

or
$$h = z + \frac{p}{\rho g} \tag{8.11}$$

Using Eq. (8.6), Eq. (8.11) reduces to:

$$h = z + \lambda \tag{8.12}$$

Equation (8.12) shows that the hydraulic head is sum of the elevation head and the pressure head.

8.4 DARCY'S LAW

In the year 1856, Henry Darcy, a French hydraulic engineer, conducted the experiments in vertical homogeneous sand filter. He was investigating the flow of water through sand. In his experimental setup, Darcy had used a vertical cylinder. But, the Darcy's law can be understood by using an inclined cylinder.

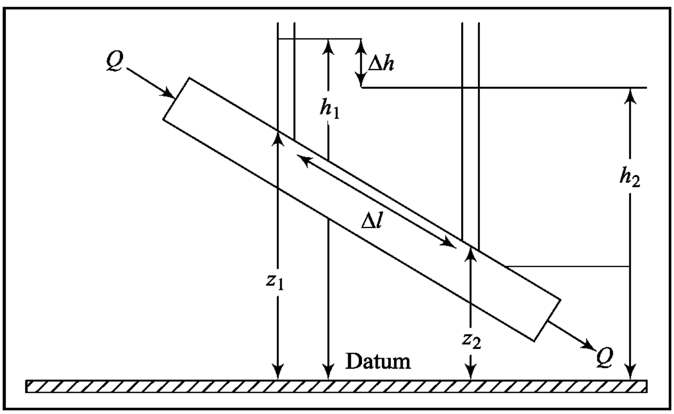

Fig. 8.3 *Experimental apparatus to demonstrate Darcy's law*

In Fig. 8.3, we have shown an inclined circular cylinder of cross-section A. The cylinder is filled with sand. It is fitted with a pair of manometers, separated by a distance Δl. Water is entered into the cylinder at a known rate Q [L^3/T]. It is allowed to flow through the cylinder for some time such that all the pores are filled with water, and the rate of flow at the outlet is equal to rate of flow at the inlet (Q). An arbitrary datum is set at an elevation $z = 0$.

z_1, z_2 = Elevation of intake of manometers from the datum

h_1, h_2 = Elevation of water levels of manometers from the datum

From this experiment, Darcy has concluded that the rate of flow (Q) is:

(i) directly proportional to the cross-sectional area, $Q \propto A$
(ii) directly proportional to the difference of heads between the two points, $Q \propto (h_1 - h_2)$
(iii) inversely proportional to the length between the two points, $Q \propto \left(\frac{1}{\Delta l}\right)$

Therefore, Darcy's law can be written as:

$$Q = AK\left(\frac{h_1 - h_2}{\Delta l}\right) \tag{8.13}$$

Or $$v = -K\left(\frac{\Delta h}{\Delta l}\right) \tag{8.14}$$

where, $\Delta h = h_2 - h_1$ and $v = \frac{Q}{A}$

Or, in differential form:

$$v = -K\frac{dh}{dl} \tag{8.15}$$

Equation (8.15) can be further simplified by putting $\frac{dh}{dl} = i$

$$v = Ki \tag{8.16}$$

where, K is a constant of proportionality known as the *hydraulic conductivity* [L/T], h is known as the *hydraulic head* [L], $\frac{dh}{dl}$ or i is known as the hydraulic gradient, and v is known as the specific discharge or Darcy's velocity [L/T].

The part of the cross-sectional area of the soil column, A, is occupied by solid matrix, and the remaining part is void. Actual flow of water takes place through the void. Since the specific discharge, v, considers the total cross-sectional area A, the Darcy's velocity is not equal to the actual velocity of flow through the pores. Due to the irregular pore geometry, the actual velocity of flow varies from point to point. So, the term "pore velocity" is defined as the actual velocity of water in the porous medium, and it is expressed as:

$$v_{\text{act}} = \frac{v}{\eta} \tag{8.17}$$

where, v_{act} = actual velocity of the flow, and η = volumetric porosity

The parameter hydraulic conductivity, K, in Darcy's law is a function of both fluid property and property of porous medium. By conducting the Darcy's experiment for different types of fluids of density and dynamic viscosity for a constant hydraulic gradient, it can also be observed that:

(i) $v \propto d^2$ (ii) $v \propto \rho g$ (iii) $v \propto \frac{1}{\mu}$

where, d = mean grain diameter.

Incorporating these three relationships, Darcy's law can be written as:

$$v = -\frac{Cd^2 \rho g}{\mu}\frac{dh}{dl} \tag{8.18}$$

where, C is a constant of proportionality.

Comparing Eq. (8.18) with the original Darcy's equation (Eq. 8.15), it can be inferred that:

$$K = \frac{Cd^2 \rho g}{\mu} \tag{8.19}$$

In Eq. (8.19), and are function of the fluid, and Cd^2 is a function of the medium.

If we define $k = Cd^2$, Eq. (8.19) can be written as:

$$K = \frac{k \rho g}{\mu} \tag{8.20}$$

The parameter, k, is known as the *intrinsic permeability*. The unit of k is m^2 or cm^2. The unit of intrinsic permeability can also be written as "darcy". 1 darcy = 9.87×10^{-9} cm^2.

Example 8.1 In a Darcy's experiment, water of viscosity 0.879×10^{-3} Ns/m^2 flows through a soil having hydraulic conductivity 0.01 cm/sec. Calculate intrinsic permeability of the soil.

Solution

Specific weight of water = ρg = 9810 N/m^3 = 9.81×10^{-3} N/cm^3

Hydraulic conductivity = 0.01 cm/sec

Viscosity of water = 0.879×10^{-3} Ns/m^2 = 8.79×10^{-8} Ns/cm^2

We know that:

$$k = \frac{K\mu}{\rho g}$$

$$k = \frac{0.01 \times 8.79 \times 10^{-8}}{9.81 \times 10^{-3}} = 8.96 \times 10^{-8} \text{ cm}^2 = 9.078 \text{ darcy}$$

Example 8.2 Groundwater flows through an aquifer with a cross-sectional area of 1.0×10^4 m^2 and a length of 1500 m. Hydraulic heads are 300 m and 250 m at the groundwater entry and exit points in the aquifer, respectively. Groundwater discharges into a stream at the rate of 1500 m^3/day. What is the hydraulic conductivity of the aquifer? If the porosity of the material is 0.3, what is the pore velocity of water?

Solution

$$\text{Darcy's velocity: } v = \frac{Q}{A} = \frac{1500}{10^4} = 0.15 \text{ m/day}$$

$$\text{Hydraulic gradient: } -\frac{dh}{dl} = \frac{300-250}{1500} = 0.0333$$

$$\text{Hydraulic conductivity: } K = -\frac{v}{dh/dl} = \frac{0.15}{0.0333} = 4.5 \text{ m/day}$$

$$\text{Pore velocity of water: } v_{\text{act}} = \frac{v}{\eta} = \frac{0.15}{0.3} = 0.5 \text{ m/day}$$

8.4.1 Heterogeneity and Anisotropy

Hydraulic characteristics usually vary either spatially or in different directions within a geologic formation. The hydraulic characteristics are hydraulic conductivity, permeability, etc. If these hydraulic characteristics vary spatially, then the geologic formation is known as heterogeneous; otherwise it is known as homogeneous.

For example:

$K(x, y, z) = C$	Formation is homogeneous.
$K(x, y, z) \neq C$	Formation is heterogeneous.

Here, K is the hydraulic conductivity; x, y, and z are the hydraulic characteristics; and C is a constant.

If these hydraulic characteristics vary with the direction of measurement at a point, then the geologic formation is known as *anisotropic* at that point; otherwise it is known as *isotropic* at that point.

For example:

$K_x = K_y = K_z$	Formation is isotropic.
$K_x \neq K_y \neq K_z$	Formation is anisotropic.

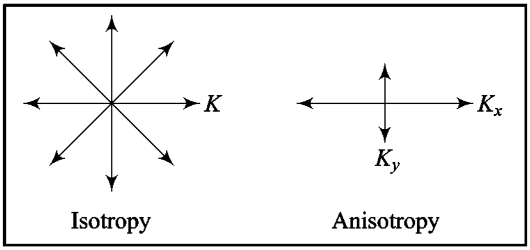

Fig. 8.4 *Property of isotropic and anisotropic aquifer*

8.4.2 Generalization of Darcy's Law

It is necessary to generalize the one-dimensional form of Darcy's law presented in Eq. (8.15). Generally, in a layered structure of soil, the resistance to flow in vertical and horizontal directions is not equal. So, in that case, the soil medium is said to be anisotropic. The Darcy's law can be represented for anisotropic aquifer as follows:

$$v = -K\, grad\,(h) = -K\,\nabla h \tag{8.21}$$

Here, v is Darcy's velocity vector, and K is the hydraulic conductivity matrix tensor.

$$v = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{8.22}$$

$$-\nabla h = = \frac{\partial h}{\partial x}\hat{i} + \frac{\partial h}{\partial y}\hat{j} + \frac{\partial h}{\partial z}\hat{k} \tag{8.23}$$

$$K = \begin{pmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{pmatrix} \tag{8.24}$$

In three dimensions, the Darcy's velocity vector can be represented as:

$$\begin{Bmatrix} v_x = -K_{xx}\dfrac{\partial h}{\partial x} - K_{xy}\dfrac{\partial h}{\partial y} - K_{xz}\dfrac{\partial h}{\partial z} \\ v_y = -K_{yx}\dfrac{\partial h}{\partial x} - K_{yy}\dfrac{\partial h}{\partial y} - K_{yz}\dfrac{\partial h}{\partial z} \\ v_z = -K_{zx}\dfrac{\partial h}{\partial x} - K_{zy}\dfrac{\partial h}{\partial y} - K_{zz}\dfrac{\partial h}{\partial z} \end{Bmatrix} \tag{8.25}$$

If the principal direction of anisotropy coincides with x, y, z axes, then:

$$K_{xy} = K_{yx} = K_{xz} = K_{zx} = K_{yz} = K_{zy} = 0$$

With this, Darcy's velocities in three dimensions for anisotropic media can be represented as:

$$\begin{Bmatrix} v_x = K_x\dfrac{\partial h}{\partial x} \\ v_y = K_y\dfrac{\partial h}{\partial y} \\ v_z = K_z\dfrac{\partial h}{\partial z} \end{Bmatrix} \tag{8.26}$$

8.4.3 Range of Validity of Darcy's Law

Darcy's law is a linear law; however, this law may not hold good in certain situations. To identify such situations, the Reynolds number is used as the criterion for determining the range of validity of Darcy's law. Reynolds number is a dimensionless number which is defined as the ratio of inertia to the viscous force. The Reynolds number for flow through porous media is defined as:

$$R_e = \frac{vd}{\upsilon} \tag{8.27}$$

where, v = average velocity (not pore velocity), υ = kinematic viscosity of the fluid, and d = representative length of grains (mean grain diameter). Sometimes, d_{10}, the diameter such that 10 percent by weight of the grains are smaller than that diameter, is taken as representative grain diameter.

Darcy's law is not valid when the Reynolds number exceeds the range of 1 to 10. One has to be careful about the value of d being used in defining Reynolds number. Limits of applicability of Darcy's law will differ with the use of d_{10} or d_{50}. Usually, with the use of an average diameter-based Reynolds number, limit of applicability of Darcy's law is considered to be less than or equal to unity. It will be a useful exercise to perform an experiment on verification of Darcy's law and identify the conditions for its validity.

The relation between hydraulic gradient (i) and velocity (v) for fully turbulent flow can be represented as:

$$-i = av + bv^2 \tag{8.28}$$

where, a and b are positive constants.
Solving v from Eq. (8.28), we get:

$$v = -\frac{a}{2b} + \frac{\sqrt{a^2 - 4bi}}{2b} = -\frac{a}{2b} + \frac{a}{2b}\left[1 - \frac{4bi}{a^2}\right]^{1/2} \cong -\frac{i}{a} - \frac{bi^2}{a^3} \tag{8.29}$$

For a constant hydraulic gradient, the velocity as well as specific capacity of a well, resulting from turbulent flow condition, is less than the velocity resulting from laminar flow condition.

Example 8.3 A fully penetrating well with radius r_w in a confined aquifer is located at the centre of a circular groundwater basin having constant head boundary condition at the outer periphery. The well is recharged maintaining a constant head at the well face. Find the recharge rate per unit rise at the well face considering: (i) flow is laminar, (ii) flow is turbulent.

Solution

Under steady state flow condition, at any radial distance (r) from the well, the radial flow is given by:

$$Q = 2\pi r/Dv_r$$

where, D = thickness of aquifer, and v_r = radial Darcy's velocity.

(i) Consider the flow is laminar. So, the relation between hydraulic gradient (i) and velocity (v) for laminar flow can be represented as:

$$-i = av$$

So, $$v = \frac{-i}{a}$$

Substituting the value of v in $Q = 2\pi r/Dv_r$ and using $i = \frac{dh}{dr}$, we get:

$$\frac{Qa}{2\pi D}\left(\frac{dr}{r}\right) = -dh$$

Integrating and applying the boundary conditions, $h(r_w) = h_w$ and $h(R) = h_R$

$$h_w - h_R = \frac{Qa}{2\pi D}\ln\left(\frac{R}{r_w}\right)$$

(ii) Considering the flow is turbulent, the relation between hydraulic gradient (i) and velocity (v) for fully turbulent flow can be represented as:

$$-i = av + bv^2$$

From Eq. 8.29:

$$v_r = \frac{-a}{2b} + \frac{\sqrt{a^2 - 4b\frac{dh}{dr}}}{2b}$$

Incorporating v_r in the above equation and simplifying it, we have:

$$\left(\frac{bQ}{\pi D}\right)^2 \frac{1}{r^2} - \left(\frac{2abQ}{\pi D}\right)\frac{1}{r} = -4b\frac{dh}{dr}$$

Integrating and applying the boundary conditions, $h(r_w) = h_w$ and $h(R) = h_R$

$$h_w - h_R = b\left(\frac{Q}{2\pi D}\right)^2\left[\frac{1}{r_w} - \frac{1}{R}\right] + \frac{aQ}{\pi D}\log_e\left(\frac{R}{r_w}\right)$$

The first part of head loss is due to turbulence, and second part is due to viscous resistance.

8.5 GRADIENT OF HYDRAULIC HEAD

We have seen earlier that the fluid flows from the regions of higher hydraulic head to the regions of lower hydraulic head, i.e., there must be difference in hydraulic head so that flow will occur. The difference in hydraulic head per unit distance is known as the *hydraulic gradient.*

$$i = \frac{h_1 - h_2}{\Delta l} = \frac{dh}{dl} \tag{8.30}$$

where, h_1 and h_2 are the hydraulic heads at points 1 and 2, respectively; and Δl = distance between the points.

8.5.1 Relationship of Groundwater Flow Direction to Hydraulic Gradient

Figure 8.5 shows three wells, A, B, and C. Our aim is to find out the direction of groundwater flow.

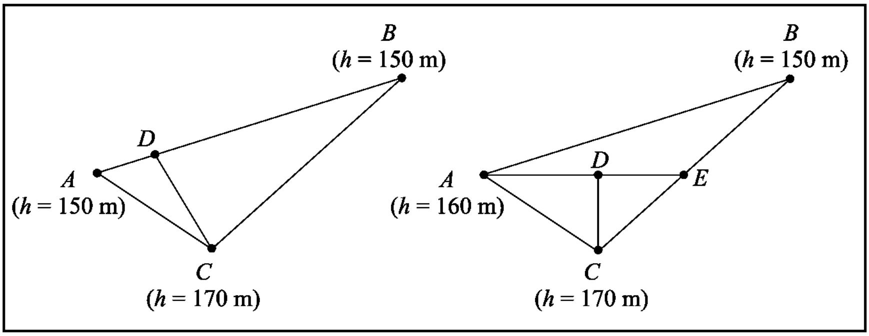

Fig. 8.5 *Determination of flow direction*

To find out the direction of groundwater flow, first of all, we should know the hydraulic heads of the three wells.

1. With the help of piezometer, measure the hydraulic head in three wells. The hydraulic heads of three wells should not be equal.

2. Consider hydraulic heads of two wells are equal. In Fig. 8.5(a), we have shown that the hydraulic head of A and B are equal. In such a case, draw a line perpendicular to AB through C. Let it be D. If the hydraulic head of C is greater than the hydraulic head of A and B, then the flow will occur from C to D.

3. Consider the hydraulic heads are different at the three wells. In Fig.. 8.5(b), we have shown that:

   $$h_C > h_A > h_B$$

   where h_C, h_A, and h_B are hydraulic heads of wells C, A, and B, respectively. Locate another point E in between B (lowest head) and C (Highest head), such that the hydraulic head at E will be equal to hydraulic head at A (intermediate head). Draw a line between A and E and another line through C, intercepting the line AE. The flow occurs in the direction from C to D.

8.6 AQUIFER

An aquifer is the saturated, permeable, geological formation or group of formations which contains, transmits, and yields significant quantity of water. The flow in the aquifer is mainly horizontal, and the source of water is mainly through infiltration.

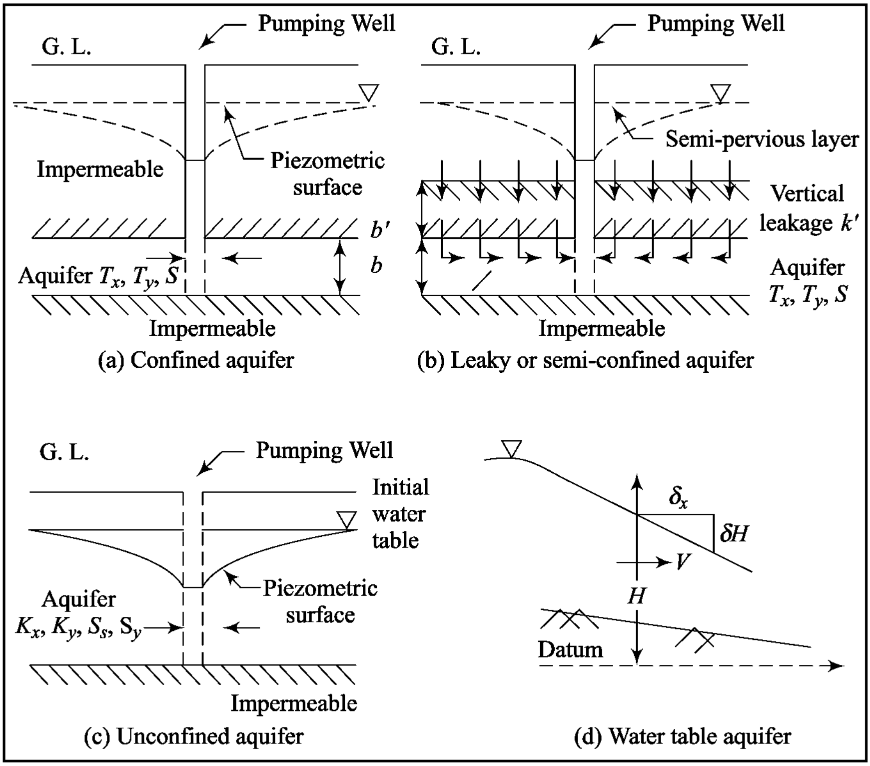

Fig. 8.6 *Different types of aquifer*

Aquifers can be classified as: (i) confined aquifer, (ii) semi-confined aquifer, and (iii) unconfined aquifer. These aquifers are schematically shown below in Fig. 8.6.

A confined aquifer or *artesian aquifer* is underlain and overlain by impermeable layer, so that water is under pressure like conduit flow. The piezometric surface is above the confining layer (Fig. 8.6 a). In a leaky or a semi-confined aquifer, there is a leakage from overlain or underlain aquitard into the aquifer, which is due to the difference of piezometric heads in the aquifer and the aquitard. In other words, the aquitard is not fully impervious, i.e., it is semi-impervious (Fig. 8.6 b). Unconfined or water table aquifer is one in which the groundwater has a free surface open to the atmosphere. The upper surface zone is called *groundwater table* or *water table* (Fig. 8.6 c and d).

8.6.1 Stratification

If the aquifer consists of different strata with different hydraulic conductivities, and flow is occurring parallel to the strata; then equivalent hydraulic conductivity of aquifer is given by:

$$K_e = \frac{\sum_1^n K_i B_i}{\sum_1^n B_i} \tag{8.31}$$

where, B_i = width of i^{th} stratum, K_i = hydraulic conductivity of i^{th} stratum, K_e = equivalent hydraulic conductivity, and n = number of strata.

If the flow is occurring normal to the strata, then equivalent hydraulic conductivity of aquifer is:

$$K_e = \frac{\sum_1^n L_i}{\sum_1^n (L_i/K_i)} \tag{8.32}$$

where, L = length of each stratum.

Example 8.4 A confined aquifer has 3 layers. The bottom layer consists of coarse sand of thickness 4.0 m, having hydraulic conductivity of 0.01 cm/sec. The middle layer consists of fine sand of thickness 3.0 m, having hydraulic conductivity of 0.002 cm/sec. The top layer consists of gravel of 4.0 m having hydraulic conductivity of 2.0 cm/sec. Calculate the equivalent hydraulic conductivity of the confined aquifer if (i) the flow is along the stratification, and (ii) the flow is normal to the stratification.

Solution

(i) If the flow is along the stratification, then the equivalent hydraulic conductivity of the confined aquifer is:

$$K_e = \frac{\sum_1^n K_i B_i}{\sum_1^n B_i} = \frac{(4\times100)\times0.01+(3\times100)\times0.002+(4\times100)\times2}{400+300+400}$$

$$= \frac{4+0.6+800}{1100} = 0.7315 \text{ cm/sec}$$

(ii) If the flow is normal to the stratification, then the equivalent hydraulic conductivity of the confined aquifer is:

$$K_e = \frac{\sum_1^n L_i}{\sum_1^n (L_i/K_i)} = \frac{(4\times100)+(3\times100)+(4\times100)}{\left(\frac{4\times100}{0.01}\right)+\left(\frac{3\times100}{0.002}\right)+\left(\frac{4\times100}{2.0}\right)}$$

$$= \frac{1100}{40,000+1,50,000+200} = 5.783\times10^{-3} \text{ cm/sec}$$

8.6.2 Porosity

The porosity of the aquifer is the ratio of the volume of voids to the total volume of the aquifer material. The soil mainly contains silt, sand, rock pebbles, or clay. The porosity depends on the shape, size, and the packing of the grain size. Porosity is a measure of water-bearing capacity of the formation, and all this water cannot be released by gravity or pumping from wells.

$$\eta = \frac{V_v}{V_T} \tag{8.33}$$

Porosity is also defined as the sum of specific retention and specific yield.

$$\eta = S_r + S_y \tag{8.34}$$

where, η is the porosity, V_v is volume of void, V_T is the total volume of aquifer formation, S_r is specific retention, and S_y is the specific yield.

Porosity and specific yield of selected formations

Formation	*Porosity (%)*	*Specific yield (%)*
Clay	45–55	1–10
Sand	35–40	10–30
Gravel	30–40	15–30
Sand stone	10–20	5–15
Shale	1–10	0.5–5
Lime stone	1–10	0.5–5

Example 8.5 In a field test, a tracer took 8 hours to travel between two observation wells which are 56 m apart. The difference in water table elevations in these well were 0.7 m. The volume of the void of the aquifer is 30% of the total volume of the aquifer. Calculate the hydraulic conductivity and intrinsic permeability of the aquifer. Viscosity of water is 0.995×10^{-3} Ns/m^2.

Solution

Since it is a tracer test, it records the actual velocity of water.

$$\text{Actual velocity of water} = v_{\text{act}} = \frac{56 \times 100}{8 \times 3600} = 0.1945 \text{ cm/sec}$$

As volume of the void of the aquifer is 30% of the total volume of the aquifer, porosity of the aquifer is 30%.

$$\text{Darcy's velocity} = \eta v_{\text{act}} = 0.3 \times 0.1945 = 0.0583 \text{ cm/sec}$$

$$\text{Hydraulic gradient} = \frac{dh}{dl} = \frac{0.7}{56} = 0.0125$$

$$\text{Hydraulic conductivity} = K = \frac{v}{dh/dl} = \frac{0.0583}{0.0125} = 4.664 \text{ cm/sec}$$

Specific weight of water = ρg = 9810 N/m^3 = 9.81×10^{-3} N/cm^3

Viscosity of water = 0.995 × 10^{-3} Ns/m^2 = 9.95 × 10^{-8} Ns/cm^2

Intrinsic permeability = $k = \dfrac{K\mu}{\rho g} = \dfrac{4.664 \times 9.95 \times 10^{-8}}{9.81 \times 10^{-3}}$ = 4.73 × 10^{-5} cm^2

8.6.3 Compressibility of aquifers

Compressibility is the inverse of modulus of elasticity.

$$\text{Compressibility} = \frac{\text{Change in strain}}{\text{Change in stress}} = \frac{d\varepsilon}{d\sigma} \tag{8.35}$$

Range of compressibility of the pore for some formation materials

Material	*Bulk modulus of elasticity E_s (N/cm^2)*
Loose clay	100–500
Stiff clay	1000–10,000
Loose sand	1000–2000
Dense sand	5000–8000
Dense sandy gravel	10^4 to 2×10^4
Fissured and jointed rock	1.5×10^4 to 3×10^5

Compressibility of Water

Let the change in pressure be dp, change in volume of water be dv_w, original volume of water be v_w, density of the fluid be ρ, and compressibility of the water be β.

Compressibility of water is defined as:

$$\beta = -\frac{dv_w/v_w}{dp} \tag{8.36}$$

By conservation of mass:

$$\rho\, v_w = \text{constant} \tag{8.37}$$

Differentiating Eq. (8.37), we get:

$$\rho\, dv_w + v_w\, d\rho = 0 \tag{8.38}$$

Substituting Eq. (8.38), Eq. (8.36) reduces to:

$$\beta = \frac{d\rho/\rho}{dp} \tag{8.39}$$

$$\beta\, dp = \frac{d\rho}{\rho} \tag{8.40}$$

Let p_0 be the atmospheric pressure and ρ_0 be the fluid density at pressure p_0. Integration of Eq. (8.40) between limits p_0 to p for dp and ρ_0 to ρ for $d\rho$ gives:

$$\beta \int_{p_0}^{p} dp = \int_{\rho_0}^{\rho} \frac{d\rho}{\rho} \tag{8.41}$$

$$\rho = \rho_0 e^{[\beta(p-p_0)]} \tag{8.42}$$

For an incompressible fluid, $\beta = 0$. So, $\rho = \rho_0 =$ constant.

Compressibility of Porous Medium

Let the change in total volume of porous medium be dV_t, the total volume of porous medium be $V_t =$ (volume of solids + volume of voids), change in effective stress be $d\sigma_e$, and compressibility of porous medium be α.

Compressibility of porous medium is defined as:

$$\alpha = -\frac{dV_t/V_t}{d\sigma_e} \tag{8.43}$$

The compression of the porous medium is governed by three mechanisms:

1. compression of water
2. compression of grains
3. rearrangement of the soil grain

Assume that soil grains are incompressible. So, when a stress σ is applied to a saturated geological formation, part of the stress is borne by the fluid and part of the stress is borne by the grains of the porous medium. The effective stress is the stress borne by the grains of the porous medium.

$$\sigma = \sigma_e + p \tag{8.44}$$

Let $d\sigma$ be the increase in stress. Therefore,

$$d\sigma = d\sigma_e + dp \tag{8.45}$$

Many of the subsurface flow problems generally do not involve the changes in stress. So, $d = 0$. Therefore,

$$d\sigma_e = -dp$$

This indicates that when fluid pressure increases, the effective stress decreases with the same amount of effective stress. So, rearrangement of sand grain and volumetric deformation are caused due to the change in effective stress and are controlled by the fluid pressure.

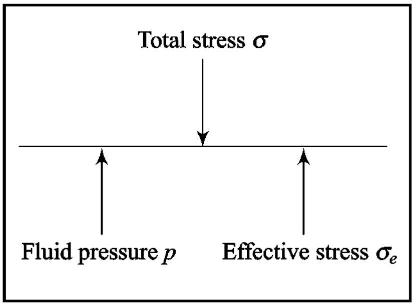

Fig. 8.7 *Diagram showing the total stress, effective stress, and fluid pressure*

Substituting $p = \rho g \varphi$, the effective stress will be:

$$d\sigma_e = -\rho g d\varphi = -\rho g dh \quad (8.46)$$

[Since $\varphi = h - z$ and z being constant]

8.7 AQUIFER PROPERTIES

To understand the flow through aquifers, it is relevant to know aquifer properties. Speicific storage, storativity, transmissivity, and hydraulic diffusivity are extensively used terms related to aquifer properties. Certain properties, such as porosity and compressibility have been addressed in preceding section.

8.7.1 Specific Storage

The volume of water released from unit volume of a saturated aquifer under unit change in head is known as *specific storage*.

In Fig. 8.8, we have shown a confined aquifer. Consider that a unit decline of hydraulic head is occurred due to the pumping of well. Due to decrease in head, the fluid pressure (p) decreases and the effective stress (σ_e) increases. Therefore, the compaction of aquifer and the expansion of water are due to the decrease in head. The total volume of water released due to decrease in head comprises the value of water released due to:

1. compaction of aquifer
2. expansion of water

Amount of water produced due to the compaction of aquifer =

$$dV_w = -dV_t = \alpha\, V_t\, d\sigma_e \quad (8.47)$$

Since $d\sigma_e = -\rho g dh$

$$dV_w = \alpha\, V_t\, \rho\, g \quad (8.48)$$

Amount of water produced due to the expansion of water =

$$dV_w = -\beta\, V_w\, dp \quad (8.49)$$

But $dp = \rho\, g\, d\varphi = \rho\, g\, dh$ [Since $\varphi = hz$, and z being constant]

Since $dh = -1$ and $V_w = \eta V_t$ [where η = porosity]

$$dV_w = \beta \eta \rho g\, V_t \quad (8.50)$$

Specific storage = Amount of water produced due to (compaction of aquifer + expansion of water)

$$S_s = \rho g V_t\,(\alpha + \eta\beta) \quad (8.51)$$

With $V_t = 1$,

$$S_s = \rho g\,(\alpha + \eta\beta) \quad (8.52)$$

The dimension of specific storage is [L^{-1}].

8.7.2 Storativity

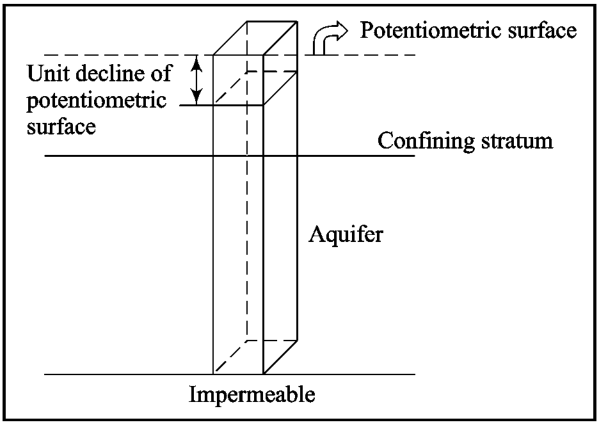

Fig. 8.8 *Representation of storativity in a confined aquifer*

The volume of water released from storage per unit surface area of aquifer per unit decline in the component of hydraulic head normal to that surface is known as *storativity*.

$$S = \frac{\text{Volume of water}}{\text{Unit area} \times \text{Unit head change}}$$

The storativity, also known as storage coefficient, and specific storage are related as:

$$S = S_s b \tag{8.53}$$

where, S is the storativity, S_s is the specific storage, and b is the thickness of confined aquifer.

Using Eqs. (8.52) and (8.53),

$$S = \rho g b(\alpha + \eta \beta) \tag{8.54}$$

Storativity is dimensionless. In most confined aquifers, the values fall in the range between 0.00005 and 0.005. For unconfined aquifers, the storativity or storage coefficient is the same as specific yield and may range between 0.05 and 0.30.

Example 8.6 A confined aquifer has a thickness of 25 m and a porosity of 30%. If the bulk modulus of elasticity of water and the formation material are 2.12×10^5 N/cm^2 and 5000 N/cm^2, respectively, calculate the storage coefficient.

Solution

Bulk modulus of elasticity of water = $E_w = 2.12 \times 10^5$ N/cm^2

Compressibility of water = $\beta = 1/E_w = 4.717 \times 10^{-6}$ cm^2/N

Bulk modulus of formation = E_s = 5000 N/cm^2

Compressibility of water = $\alpha = 1/E_s = 2.0 \times 10^{-4}$ cm^2/N

Specific weight of water = $\rho g = 9.81 \times 10^{-3}$ N/cm^3

Specific storage = $S_s = \rho\, g(\alpha + \eta\beta)$

$= 9.81 \times 10^{-3}\ [(2.0 \times 10^{-4}) + (4.717 \times 10^{-6} \times 0.3)]$

$= 1.976 \times 10^{-6}$ /cm

Storage coefficient = $S_s\, b = 1.976 \times 10^{-6} \times 2500 = 4.94 \times 10^{-3}$

8.7.3 Transmissivity

The ability of movement of water through an aquifer is described by the term *transmissivity*. It is measured across the vertical thickness of an aquifer.

$$T = bK \tag{8.55}$$

The dimension of transmissivity is [L^2/T]. SI unit of transmissivity is m^2/day.

8.7.4 Hydraulic Diffusivity

The hydraulic diffusivity is defined as the ratio of transmissivity and storativity.

$$D = \frac{T}{S} = \frac{K}{S_s} \tag{8.56}$$

Example 8.7 Calculate the transmissivity of the confined aquifer if the flow is along the stratification in Example 8.4.

Solution

Transmissivity = $(4 \times 100) \times 0.01 + (3 \times 100) \times 0.002 + (4 \times 100) \times 2.0$

$= 4 + 0.6 + 800$

$= 804.6$ cm^2/sec

8.8 EQUATION OF GROUNDWATER FLOW

There are two types of equations of groundwater flow: transient saturated flow and transient unsaturated flow. For transient saturated flow, Darcy's law combined with the equation of continuity, results into equations for groundwater flow. Transient unsaturated flow is considered in Chapter 10.

8.8.1 Mass Conservation Equation

In Fig. 8.9, we have shown a control volume of dimensions δx, δy, δz, centered at point $P(x, y, z)$. Our objective is to derive a conservation of mass equation which implies:

Mass inflow rate – mass outflow rate = change of mass storage with time

Rate of flow of mass entering the face ABCD is:

$$= \rho\, v_x\, dy\, dz - \frac{\partial(\rho\, v_x)}{\partial x}\frac{dx}{2}\ dy \times dz \tag{8.57}$$

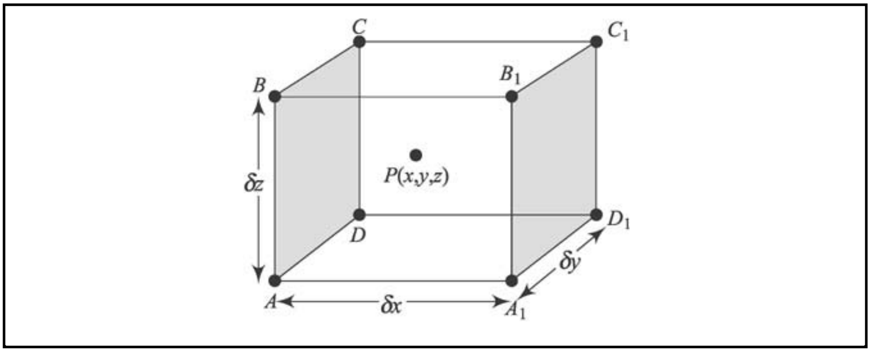

Fig. 8.9 *Control volume for conservation of mass*

Rate of flow of mass leaving the face $A_1B_1C_1D_1$ is

$$= \rho\, v_x\, dy\, dz + \frac{\partial(\rho v_x)}{\partial x}\frac{dx}{2}\, dy\, dz \tag{8.58}$$

The net mass rate of flow being retained in the control volume along x-axis is

= Mass inflow rate – mass outflow rate

$$= \left[\rho v_x dydz - \frac{\partial(\rho v_x)}{\partial x}\frac{dx}{2}dydz\right] - \left[\rho v_x dydz + \frac{\partial(\rho v_x)}{\partial x}\frac{dx}{2}dydz\right] \tag{8.59}$$

$$= -\frac{\partial(\rho v_x)}{\partial x}\, dx\, dy\, dz \tag{8.60}$$

Similarly, the net gain of mass per unit time along y-axis will be

$$= -\frac{\partial(\rho v_y)}{\partial y}\, dx\, dy\, dz \tag{8.61}$$

And, the net gain of mass per unit time along z-axis will be

$$= -\frac{\partial(\rho v_z)}{\partial z}\, dx\, dy\, dz \tag{8.62}$$

Therefore, the total net gain of mass per unit time is

$$= -\left\{\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right\} dx\, dy\, dz \tag{8.63}$$

The change in mass of the control volume per unit time will be

$$= \frac{\partial(\rho\, \eta)}{\partial t}\, dx\, dy\, dz \tag{8.64}$$

Hence, $$\frac{\partial(\rho\eta)}{\partial t}\,dx\,dy\,dz = \left\{\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right\} dx\,dy\,dz \tag{8.65}$$

Dividing both sides of Eq. (8.65) by the volume ($dx\,dy\,dz$) of the element, and taking the limit as dx, dy, dz approaches zero, the continuity equation at a point (centre of the control volume) becomes:

$$-\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = \frac{\partial(\rho\eta)}{\partial t} \tag{8.66}$$

Equation (8.66) is known as the *mass conservation equation* for transient flow in a saturated porous medium.

8.8.2 Flow Equation for Transient State Saturated Flow

The mass conservation equation for transient flow in a saturated porous medium is given in Eq. (8.66). The terms on the right-hand side of Eq. (8.66) can be written as:

$$\frac{\partial(\rho n)}{\partial t} = \rho\left(\frac{\partial n}{\partial t}\right) + \eta\left(\frac{\partial \rho}{\partial t}\right) \tag{8.67}$$

$$\frac{\partial(\rho n)}{\partial t} = \left(\rho\frac{\partial n}{\partial p} + \eta\frac{\partial \rho}{\partial p}\right)\frac{\partial p}{\partial t} \tag{8.68}$$

$$\frac{\partial \eta}{\partial p} = \alpha = \text{Aquifer compressibility} \tag{8.69}$$

$$\frac{\partial \rho}{\partial p} = \rho\beta \qquad \text{(where } \beta \text{ is fluid compressibility)} \tag{8.70}$$

Substituting Eqs. (8.69) and (8.70), Eq. (8.68) becomes:

$$\frac{\partial(\rho\eta)}{\partial t} = \rho(\alpha + \eta\beta)\frac{\partial p}{\partial t} \tag{8.71}$$

But, $\rho g\,(\alpha + \eta\beta) = S_s$ = specific storage and $p = \rho g h$, therefore Eq. (8.66) becomes:

$$-\frac{\partial(\rho v_x)}{\partial x} - \frac{\partial(\rho v_y)}{\partial y} - \frac{\partial(\rho v_z)}{\partial z} = \rho S_s\frac{\partial h}{\partial t} \tag{8.72}$$

As $\frac{\partial(\rho v_x)}{\partial x} = \rho\left(\frac{\partial v_x}{\partial x}\right) + v_x\left(\frac{\partial \rho}{\partial x}\right)$, and since the term $\rho\left(\frac{\partial v_x}{\partial x}\right)$ is much greater than $v_x\left(\frac{\partial \rho}{\partial x}\right)$, one can neglect $v_x\left(\frac{\partial \rho}{\partial x}\right)$. Taking this into consideration using Darcy's law in Eq. (8.72) leads to:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = S_s \frac{\partial h}{\partial t} \tag{8.73}$$

Equation (8.73) is the flow equation for transient state flow through saturated anisotropic porous medium.

Particular Cases

1. For transient flow in homogeneous and isotropic medium, Eq. (8.73) reduces to:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K}\frac{\partial h}{\partial t} \tag{8.74}$$

2. For steady flow of incompressible fluids (ρ = constant), Eq. (8.73) reduces to:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = 0 \tag{8.75}$$

3. Equation (8.75) is the flow equation for steady, anisotropic, saturated, homogeneous, porous medium. For an isotropic homogeneous medium,

$$K_x = K_y = K_z = K$$

With this, Eq. (8.75) reduces to flow equation for steady state flow through an isotropic saturated homogeneous porous medium.

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \tag{8.76}$$

Equation (8.76) is also known as *Laplace equation.*

8.9 RADIAL FLOW OF CONFINED AQUIFER

In Fig. (8.10), we have shown a well in a confined aquifer. Let Q be the rate of discharge of the well, and b be the thickness of the aquifer. Consider the cylindrical section of the well having inner and outer radii r and $(r + dr)$, respectively. Let the hydraulic gradient across the cylindrical section of infinitesimal thickness dr be $\frac{\partial h}{\partial r}$.

The discharge through a cylindrical section having thickness dr is given by Darcy's law as:

$$Q = -2\pi r\, b\, K \frac{\partial h}{\partial r} \tag{8.77}$$

Differentiating Eq. (8.77) with respect to ∂r, we get:

$$\frac{\partial Q}{\partial r} = -\,2\pi b\, K\left(\frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}\right) \tag{8.78}$$

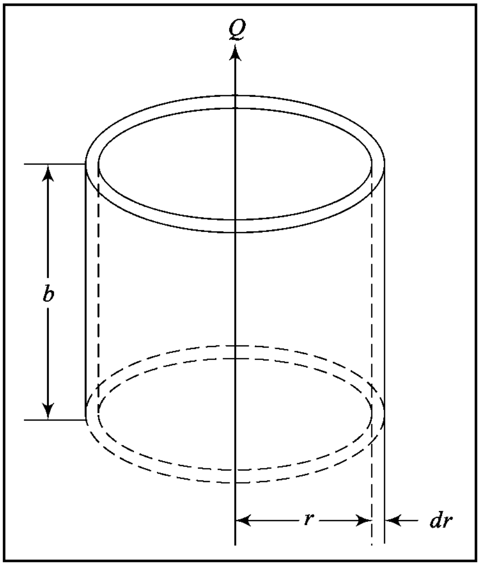

Fig. 8.10 *Flow through the cylindrical section of a confined aquifer*

The change in storage within the cylindrical section is due to the volumetric rate of release of water in storage corresponding to rate of decrease of head.

$$\frac{\partial V}{\partial t} = 2\pi\, r\, dr\, S \frac{\partial h}{\partial t} \tag{8.79}$$

Discharge through the cylindrical section of thickness dr $\left(= \frac{\partial Q}{\partial r} dr\right)$ must be equal to change in storage.

$$-2\pi b\, K\left(\frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}\right) dr = 2\pi r\, dr\, S\left(\frac{\partial h}{\partial t}\right) \tag{8.80}$$

Dividing both sides by $2\pi rbK$, we get:

$$\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = \frac{S}{bK}\frac{\partial h}{\partial t} \tag{8.81}$$

Putting $T = bK$, where T = transmissivity, Eq. (8.81) reduces to:

$$\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = \frac{S}{T}\left(\frac{\partial h}{\partial t}\right) \tag{8.82}$$

Equation (8.82) is the flow equation in a confined aquifer in radial coordinate. For steady radial flow, Eq. (8.82) reduces to:

$$\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = 0 \tag{8.83}$$

8.10 DUPUIT ASSUMPTIONS

In Fig. 8.11, we have shown an unconfined aquifer. In an unconfined aquifer, water table is the upper boundary which acts as a flow line. Since water table is a flow line, the tangent to this flow line at every point gives the specific discharge, which is given by Darcy's law.

$$v = -K\left(\frac{dh}{dl}\right) \quad (8.84)$$

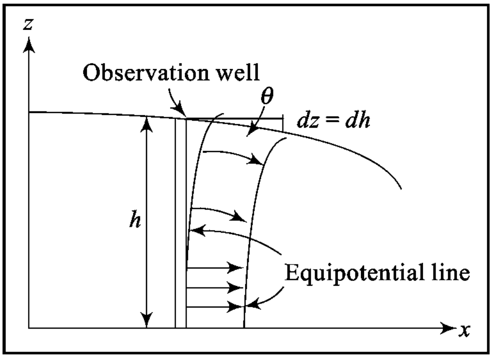

Fig. 8.11 *Phreatic surface in an unconfined aquifer*

where, K = hydraulic conductivity.

Along the water table, surface pressure head = 0. So, hydraulic head, $h = z$.

$$v = -K\left(\frac{dz}{dl}\right) = -K \sin\theta \quad (8.85)$$

Since θ is very small, Dupuit used $\tan\theta = \frac{dh}{dl}$ instead of $\sin\theta = \frac{dz}{dl}$. The assumption of θ being small also means that equipotential surfaces are vertical. This indicates that flow becomes horizontal. But in actual case, equipotential surfaces are never vertical.

Thus, Dupuit assumed the following:

1. Flow is horizontal.
2. Hydraulic gradient line does not vary with depth. It is equal to slope of the free surface.

Based on this assumption, the specific discharge can be expressed by:

$$v_x = -K\left(\frac{dh}{dx}\right) \text{ and } v_y = -K\left(\frac{dh}{dy}\right) \quad (8.86)$$

8.11 UNCONFINED FLOW WITH RECHARGE

In Fig. 8.12, we have shown an unconfined aquifer having recharge of water at a rate R m^3/sec/m^2 of area due to the infiltration. Our aim is to derive the flow equation for the recharge condition in an unconfined aquifer.

Rate of flow of mass entering face $ABCD = \rho v_x h dy$ (8.87)

Rate of flow of mass leaving the face $A_1B_1C_1D_1$

$$= \rho v_x h dy + \frac{\partial(\rho v_x h dy)}{\partial x} dx \quad (8.88)$$

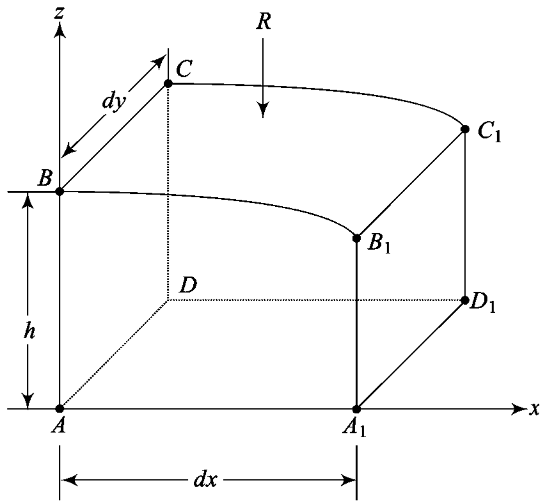

Fig. 8.12 *Unconfined flows with recharge*

The net mass rate of flow being retained in the element:

$$= \text{mass inflow rate} - \text{mass outflow rate}$$

$$= \rho v_x h\, dy - \left[\frac{\partial(\rho v_x\, h\, dy)}{\partial x} dx + \rho v_x\, h\, dy \right]$$

$$= -\frac{\partial(\rho v_x\, hdy)}{\partial x} dx \tag{8.89}$$

Similarly, the net gain of mass per unit time in the element along y-axis will be:

$$= -\frac{\partial(\rho v_y\, hdx)}{\partial y} dy \tag{8.90}$$

Since there is recharge in z-direction, the net inflow into the element in z-direction will be

$$= \rho\, R\, dx\, dy \tag{8.91}$$

Therefore, the total net gain of mass per unit time is:

$$= -\left[\frac{\partial(\rho v_x\, hdy)}{\partial x} dx + \frac{\partial(\rho v_y\, hdx)}{\partial y} dy + \rho R\, dx\, dy \right] \tag{8.92}$$

For steady, incompressible flow, the total net gain of mass per unit time is zero.

Therefore,

$$-\left[\frac{\partial(\rho v_x h dy)}{\partial x}dx + \frac{\partial(\rho v_y\, h dx)}{\partial y}dy\right] + \rho\, R\, dx\, dy = 0 \tag{8.93}$$

$$\Rightarrow -\left[\frac{\partial(v_x h)}{\partial x}dx + \frac{\partial(v_y h)}{\partial y}dy\right] + R\, dx\, dy = 0 \tag{8.94}$$

Making use of Dupuit assumption and invoking Eq. (8.86), Eq. (8.94) reduces to:

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = -\frac{2R}{K} \tag{8.95}$$

Equation (8.95) is the flow equation for unconfined aquifer with recharge.

Particular Case

If there is no recharge, $R = 0$ and Eq. (8.95) reduces to:

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = 0 \tag{8.96}$$

Equation (8.96) is the steady, incompressible flow equation for unconfined aquifer.

8.12 SOLUTION TO FLOW EQUATIONS

The methods to solve groundwater flow problems comprise the following:

- Understanding the physical problem
- Replacement of the physical problem by an equivalent mathematical equation
- Solution to the mathematical equation applying the technique of mathematics
- Interpretation of the results in relation to physical problem

The groundwater flow equations presented above have infinite number of solutions as they do not provide any information unless solved. To get a particular solution from these many solutions, some additional information is needed which may include:

- Geometry of the domain
- Values of all relevant parameters
- Initial condition
- Boundary condition

First of all, initial conditions and boundary conditions are determined, either from available information in the field or from past experience.

Let us consider a flow domain D. In three-dimensional coordinates system, it is bounded by a surface S, as shown in Fig. 8.13.

In two dimensions, it is bounded by a curve C. Sometimes, unbounded domain is also considered, i.e., flow domain is imagined to extend to infinity.

Consider h (hydraulic head) = dependent variable = $h(x, y, z, t)$. We have to specify our initial and boundary conditions in terms of h. Initial condition is the assigned value of h at all points within the domain at initial time $t = 0$.

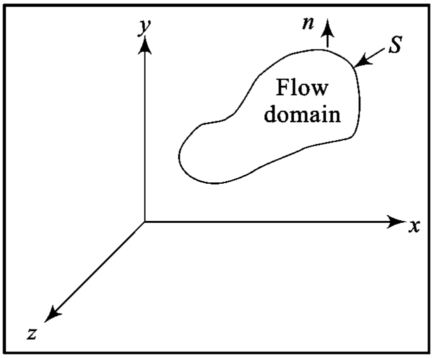

Fig. 8.13 *A 3-D flow domain enclosed by a surface (S) with an outward normal (n)*

$h = f(x, y, z, 0)$ for all points x, y, z inside D.

There are three types of boundary conditions encountered in flow through porous media.

- Dirichlet boundary condition
- Neumann boundary condition
- Cauchy boundary condition

In *Dirichlet boundary condition*, the hydraulic head h is assigned for all points of the boundary. This can be written as:

$h = f(x, y, z, t)$ on surface S, where f is a known function.

In Neumann boundary condition, the flux normal to boundary surface is assigned for all points. This can be written as:

$q_n = f(x, y, z, t)$ on boundary surface S

where, n is an outward normal to the boundary; and f is a known function.

Cauchy boundary condition is a mixed boundary condition. If there are two domains separated by a relatively thin semi-pervious layer, then Cauchy boundary condition is applied.

$$q_n = \frac{K_m}{M} [h(x, y, z, t) - h_m(x, y, z, t)]$$

where, q_n is the component of the specific discharge q, normal to the surface S,

K_m is the hydraulic conductivity of the boundary,

M is the thickness of the boundary,

h is the hydraulic head in the considered domain, and

h_m is the hydraulic head in the external domain.

An example of the application of Cauchy boundary condition is the *stream aquifer interaction*.

To calculate the value of the hydraulic head h, we have to solve the governing flow equation subject to initial and boundary conditions. There are three methods to solve such type of problems.

1. Analytical methods
2. Graphical methods
3. Numerical methods

8.12.1 Solution by analytical methods

Analytical method is superior to other methods. The advantage of this method is that the solution can be applied to different values of inputs and parameters. But, the disadvantage is that the parameters cannot be presented in the form of analytical solution in case of heterogeneous soil or irregular aquifer boundaries. This method can be applied to simple one- or two-dimensional cases of well hydraulics, steady flow to confined and unconfined aquifer, etc. Some of the cases are illustrated here.

Steady Flow in a Confined Aquifer

In Fig. 8.14, we have shown a confined aquifer of uniform thickness.

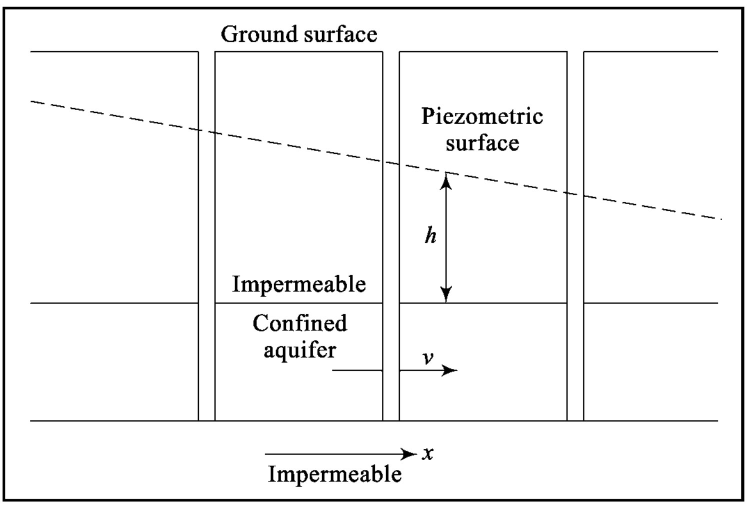

Fig. 8.14 *Steady flow in a confined aquifer*

Let v be the velocity of groundwater in the x-direction, and h be the hydraulic head. For steady and one-dimensional flow, Eq. (8.76) of groundwater flow reduces to:

$$\frac{\partial^2 h}{\partial x^2} = 0 \tag{8.97}$$

The given boundary conditions are:

$$h = h_0 \text{ at } x = 0, \text{ and} \tag{8.98a}$$

$$h = h_f \text{ at } x = 1 \tag{8.98b}$$

The solution to the above equation is:

$$h = C_1 x + C_2 \tag{8.99}$$

where, C_1 and C_2 are constants of integration.

Substituting the boundary condition of Eq. (8.98a) in Eq. (8.99), we will get:

$$C_2 = h_0 \tag{8.100}$$

Substituting the boundary conditions of Eq. (8.98b) and Eq. (8.100) in Eq. (8.99), we will get:

$$C_1 = \frac{h_f - h_0}{l} \tag{8.101}$$

Substituting Eqs. (8.100) and (8.101) in Eq. (8.99):

$$h = h_0 + (h_f - h_0)\frac{x}{l} \tag{8.102}$$

Equation (8.102) indicates that the variation of head h is linear for the confined aquifer. The variation of piezometric surface is linear, as shown in Fig. 8.13, which also contains piezometers. It is noted that there is no abstraction of water taking place from piezometers.

Steady Flow in an Unconfined Aquifer

In Fig. 8.15, we have shown an unconfined aquifer. Since water table is a flow line, direct analytic solution is impossible in an unconfined aquifer.

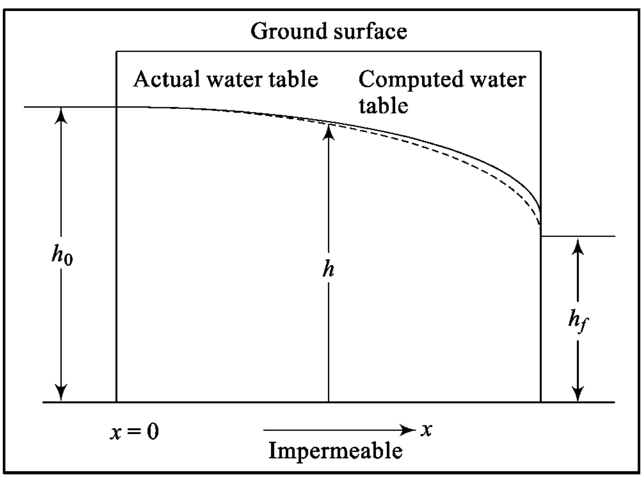

Fig. 8.15 *Steady flow in an unconfined aquifer*

Based on Dupuit assumption, the specific discharge can be expressed as:

$$v_s = -K\frac{dh}{dx} \tag{8.103}$$

The discharge per unit width (q) can be represented as:

$$q = Kh\frac{dh}{dx} \tag{8.104}$$

The given boundary conditions are:

$$h = h_0 \text{ at } x = 0, \text{ and} \tag{8.105}$$

$$h = h_f \text{ at } x = l \tag{8.106}$$

Integrating Eq. (8.104), we get:

$$qx = -K\left(\frac{h^2}{2}\right) + C \tag{8.107}$$

Putting the boundary condition of Eq. (8.105) in Eq. (8.107), we get:

$$C = K\left(\frac{h_0^2}{2}\right) \tag{8.108}$$

Putting the boundary condition of Eq. (8.106) and Eq. (8.108), Eq. (8.107) reduces to:

$$q = \frac{K}{2l}(h_0^2 - h_f^2) \tag{8.109}$$

Equation (8.109) indicates that the water table is parabolic.

Example 8.8 In an area underlain by an unconfined aquifer having hydraulic conductivity of 30 m/day, the distance between observation wells located on the banks of a canal and drain is 3.0 km and heads recorded in them are 60m and 50m above impermeable base. Calculate the quantity of seepage into the drain.

Solution

$$h_0 = 60 \text{ m}, h_f = 50 \text{ m}, l = 1500 \text{ m}, K = 30 \text{ m/day}$$

The quantity of seepage into the drain is given by Eq. (8.88).

$$q = \frac{K}{2l}(h_0^2 - h_f^2) = \frac{30}{2 \times 3000}(60^2 - 50^2) = 5.5 \text{ m}^3\text{/day per metre}$$

8.12.2 Solution by Graphical methods

Graphical methods have been in use much before the advent of computers. Field engineers still prefer to use graphical methods. However, it must be realized that these methods are only approximate.

Flow Lines and Flow Net

A flow line is an imaginary line drawn in a flow field such that a tangent drawn at any point on this line represents the direction of the specific discharge vector, v. From this definition, it is clear that there can be no flow across a flow line. The flow lines indicate the path followed by a particle of water.

The equation of a flow line is

$$v \times ds = 0 \tag{8.110}$$

where '×' denotes cross product.

$$ds = \hat{i}\, dx + \hat{j}\, dy = \text{element of length along the stream line}$$

In Cartesian coordinates, the equation of flow line can be written as:

$$v_y dx - v_x dy = 0 \tag{8.111}$$

Flow net in Homogeneous and Isotropic Media

The Laplace equation for two dimensional flows through a homogeneous isotropic medium is:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{8.112}$$

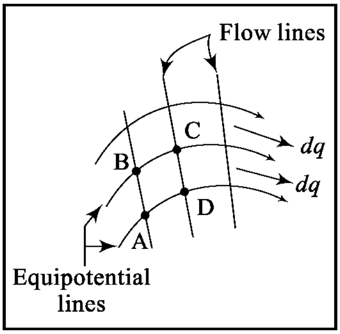

Fig. 8.16 *Diagram showing the flow net*

A flow net is a graphical solution of Laplace equation for two dimensions, given in Eq. (8.112). The plot of flow lines and equipotential lines for specified boundary conditions is known as *flow net*. The equipotential lines are the contour of equal head.

In Fig. 8.16 we have shown a flow net. The area between two flow lines is known as *stream tube*. Our aim is to calculate the discharge in a flow system.

Discharge in a flow system = (Discharge in a single stream tube) × (no. of stream tubes)

Let us consider flow through the region ABCD.

Let the loss of hydraulic head between AB and CD be dh, the distance between two flow lines be ds, the distance between two equipotential lines be dl, total no of flow lines in the whole system be n_f, total no of head drop in the whole system be n_d.

Total head loss in the whole system be $H = n_d dh$ (8.113)

The discharge across this region (considering unit depth) is:

$$dq = K\frac{dh}{dl}\,(ds \times 1) \tag{8.114}$$

The discharge across any plane within the stream tube is also equal to dq under steady state condition.

If $ds = dl$ (square flow net), then Eq. (8.114) becomes:

$$dq = K\,dh \tag{8.115}$$

The total discharge in the whole system is equal to discharge in a single stream tube multiplied by number of stream tubes.

$$Q = K\,dh\,n_f \tag{8.116}$$

But, from Eq. (8.113):

$$dh = \frac{H}{n_d} \tag{8.117}$$

Using Eq. (8.117), Eq. (8.116) reduces to:

$$Q = KH\left(\frac{n_f}{n_d}\right) \tag{8.118}$$

Example 8.9 In a flow net analysis, the number of flow line is 18 and the number of hydraulic head drop is 6. Flow is occurring in a medium having hydraulic conductivity 0.05 cm/sec and head loss is 30 m. Calculate the average discharge.

Solution

The average discharge is:

$$Q = KH\left(\frac{n_f}{n_d}\right)$$

$$= 0.05 \times (30 \times 100) \times \frac{18}{6} = 450 \text{ cm}^3\text{/sec}$$

Properties of Flow Net

For homogeneous and isotropic media, the properties of the flow net are:

1. Flow lines and equipotential lines must intersect at right angles.
2. Equipotential lines must meet impermeable boundaries at right angles.

It should be further noted that in case of earth dam, surface of seepage develops at the downstream. It is neither a stream line nor an equipotential line. Similarly, in case of vertical recharge taking place and approaching towards a water table, the water table is neither a streamline nor an equipotential line despite the presence of atmospheric pressure everywhere on the water table.

Refraction of Flow Lines

In Fig. 8.17, we have shown a heterogeneous system in which a stream tube is passing from a formation of lower hydraulic conductivity K_1 to a formation of higher hydraulic conductivity K_2. In such condition, the flow lines refract when they cross from one medium to another medium. In groundwater, refraction obeys tangent law.

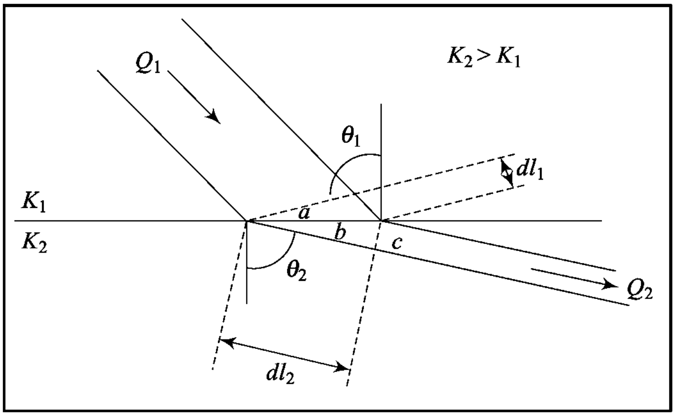

Fig. 8.17 *Refraction of flow lines at a geologic boundary*

Let θ_1 = angle made by the flow lines to vertical, θ_2 = refracted angle of flow lines to vertical after passing through a region of change of hydraulic conductivity, the distance between two equipotential lines in the region where hydraulic conductivity is K_1 be dl_1, the distance between two equipotential lines in the region where hydraulic conductivity is K_2 be dl_2, the distance between two flow lines in the region where hydraulic conductivity is K_1 be a, the distance between two flow lines in the region where hydraulic conductivity is K_2 be c, the horizontal distance between two flow lines be b, head drop across the distance dl_1 be dh_1, and head drop across the distance dl_2 be dh_2. For steady flow, the inflow Q_1 must be equal to Q_2.

From Darcy's law:

$$K_1 a\left(\frac{dh_1}{dl_1}\right) = K_2 c\left(\frac{dh_2}{dl_2}\right) \tag{8.119}$$

From Fig. 8.17, it can be shown that:

$$a = b\cos\theta_1 \tag{8.120}$$

$$c = b\cos\theta_2 \tag{8.121}$$

$$\sin\theta_1 = \frac{dl_1}{b} \tag{8.122}$$

$$\sin\theta_2 = \frac{dl_2}{b} \tag{8.123}$$

Using Eqs. (8.120) to (8.123), Eq. (8.119) becomes:

$$\frac{K_1}{K_2} = \frac{\tan\theta_1}{\tan\theta_2} \tag{8.124}$$

Equation (8.124) is the tangent law for the refraction of groundwater flow lines in heterogeneous media. We can find out the refracted angle θ_2 if we know the hydraulic conductivity of two regions and angle of incidence of flow lines.

8.12.3 Solution by Numerical Methods

Nowadays, numerical methods are practically major tool for solving large scale groundwater forecasting problems. In this method, the governing equation is solved by numerically using various methodologies and techniques. The most used methodologies are:

1. Finite difference method
2. Finite volume method
3. Finite element method

Out of these, the finite difference method is widely used to solve a variety of groundwater flow problems. A detailed coverage to use of finite difference method is given in Chapter 11.

SUMMARY

In this chapter, the preliminary terms related to groundwater have been defined. Darcy's law and its range of validity have been explained. The governing equations of groundwater flow for different cases have been derived. Flow towards wells is also briefly discussed to give an idea about the radial flow equations. More on radial flow is included in next chapter.

An attempt has been made to solve the groundwater flow equation analytically in different types of aquifer, and information has been given regarding different methods of numerical solution. Also, the graphical method of solution has been explained. It is expected that this chapter will provide necessary background for a better understanding of the subsequent chapters.

EXERCISES

8.1 Differentiate and explain (a) homogeneous and non-homogeneous (b) isotropy and anisotropy.

8.2 Explain the different types of aquifers.

8.3 Derive the groundwater flow equation for saturated (a) steady flow (b) unsteady flow.

8.4 Explain the procedure to find out the direction of groundwater flow with a suitable example.

8.5 Define and explain following terms.
(a) Transmissibility
(b) Storativity
(c) Compressibility
(d) Specific storage
(e) Hydraulic diffusivity

8.6 Derive the equation for steady flow of groundwater in a (a) confined aquifer (b) unconfined aquifer.

8.7 Derive the relation between the hydraulic conductivity of soil and angle made by flow lines when flow occurs through a heterogeneous formation.

8.8 Discuss briefly about flow net and its importance.

8.9 A tracer took 15 hr to travel from well A to B which are 120 m apart. The difference in their water table elevation was 0.9m. The porosity of the aquifer is 30%. Calculate (a) hydraulic conductivity, and (b) intrinsic permeability.

8.10 A confined stratified aquifer has a total thickness of 15 m and is made up of 3 layers. The bottom layer of thickness 4.5 m has a hydraulic conductivity of 25 m/day. The middle layer of thickness 4.5 m has a hydraulic conductivity of 35 m/day. The top layer has a hydraulic conductivity of 40 m/day.
Calculate the equivalent hydraulic conductivity of the confined aquifer, (i) if the flow is along the stratification, (ii) if the flow is normal to the stratification. Also, calculate the transmissivity of the confined aquifer, if the flow is along the stratification.

8.11 A confined aquifer is 30 m thick and 1 km wide. The heads of two observations well located 500 m apart are 50 m and 28 m. The hydraulic conductivity of the aquifer is 25 m/day. Calculate the total daily flow through the aquifer.

8.12 Three wells A, B, and C tap the same horizontal aquifer. The distances AB =1100 m and BC = 800 m. The well B is exactly south of well A and well C lives to the west of well B. The following are the ground surface elevation and depth of water below the ground surface in the three wells.

Well	*Surface elevation (meters above datum)*	*Depth of water table (m)*
A	184	12
B	180	6
C	189	18

Determine the direction of groundwater flow in aquifer in the area ABC of the wells.

8.13 Two parallel rivers, *A* and *B*, are separated by a heterogeneous aquifer at a separation distance of 2000 m. The bottom one is having a width of 15 m and hydraulic conductivity of 20 m/day and the top one having the hydraulic conductivity of 35 m/day. Heads recorded in them are 40 m and 25 m above impermeable base. Estimate the seepage discharge per unit length from river *A* to river *B*.

8.14 Solve the previous question 8.13 when a rainfall of 10 mm/hr occurs in the area between the rivers. Assume a suitable time for rainfall occurrence.

8.15 In a flow net analysis of heterogeneous system, it has been found that flow lines are crossing from a medium having hydraulic conductivity of 40 m/day to another medium having hydraulic conductivity of 25 m/day. Angle made by the flow line to vertical before refracting was 30°. Calculate the refraction angle of flow lines after passing the medium.

OBJECTIVE QUESTIONS

1. The unit of transmissivity is
   (a) m/day (b) m^2/sec (c) m.day (d) Imensionless

2. The unit of storativity is
   (a) m/day (b) m^2/sec (c) m.day (d) Dimensionless

3. The hydraulic conductivity of a confined aquifer having thickness of 5 m is 10 m/day. Determine the value of transmissivity in m^2/day.
   (a) 2 (b) 50 (c) 10 (d) 0

4. Hydraulic diffusivity is defined as (S = storativity, T= transmissivity, S_s = specific storage)
   (a) S/T (b) S/K (c) K/S (d) K/S_s

5. In confined aquifers, the range of storativity is
   (a) 0–1 (b) 0.05–0.5
   (c) 0.005–0.05 (d) 0.005–0.00005

6. Compressibility is defined as
   (a) Ratio of strain to stress (b) Inverse of modulus of elasticity
   (c) Equal to effective stress (d) Both (a) and (b)

7. The relation between void ratio and porosity is
   (a) $e = \dfrac{\eta}{1-\eta}$ (b) $\eta = \dfrac{e}{1+e}$
   (c) $\eta = e(1+e)$ (d) both (a) and (b)

8. The range of the void ratio of soil is
   (a) 0–1 (b) 0–2 (c) 0–3 (d) 0–4
9. A geologic formation is said to be heterogeneous when
   (a) The hydraulic conductivity is independent of direction of measurement at a point in a geologic formation
   (b) The hydraulic conductivity is independent of position within a geologic formation
   (c) The hydraulic conductivity varies with the direction of measurement at a point in a geologic formation
   (d) The hydraulic conductivity is dependent of position within a geologic formation
10. A geologic formation is said to be anisotropy when
    (a) The hydraulic conductivity is independent of direction of measurement at a point in a geologic formation
    (b) The hydraulic conductivity is independent of position within a geologic formation
    (c) The hydraulic conductivity varies with the direction of measurement at a point in a geologic formation
    (d) The hydraulic conductivity is dependent on position within a geologic formation
11. The hydraulic head is defined as the summation of
    (a) Elevation head and velocity head
    (b) Pressure head and velocity head
    (c) Elevation head and pressure head
    (d) Elevation head, pressure head and velocity head
12. For an aquifer, a zone of 800 sq km is bounded by confined aquifer of 20 m thick. The average maximum and minimum piezometric level variation range is between 6–15 m. Taking storage coefficient as 0.001, calculate the annual rechargeable groundwater storage from the area.
    (a) 6.0 Mm^3 (b) 7.0 Mm^3 (c) 7.2 Mm^3 (d) 7.5 Mm^3
13. Calculate the discharge per unit width of a barrage if the difference of head between upstream and downstream water level is 12 m and coefficient of permeability is 0.9 cm/sec. There are 25 flow lines and 40 equipotential drops.
    (a) 6.75 m^2/sec (b) 0.675 m^2/sec
    (c) 0.0675 m^2/sec (d) 0.00675 m^2/sec
14. If at the foundation of the reservoir hydraulic conductivity in horizontal and vertical direction are 50 m/day and 5 m/day. Determine the equivalent hydraulic conductivity.
    (a) 15.81 m/day (b) 3.162 m/day (c) 10.0 m/day (d) 45 m/day

15. The actual velocity of water flowing in a medium is observed as 0.25 cm/sec. The porosity of the medium is 0.30. Determine the Darcy's velocity.
    (a) 1.2 cm/sec (b) 0.83 cm/sec (c) 0.075 cm/sec (d) 0.05 cm/sec

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CHAPTER 9

Well Hydraulics

9.1 INTRODUCTION

Wells are used for abstraction of water from aquifers. In regions having significant decline in groundwater level, wells are also used to recharge the aquifers. Depending on the diameter of the well, these can be called as small-diameter well (diameter up to 1 m) or large-diameter well (usually having diameter above 1 m). Wells of very small diameter (say up to 0.3 m) are also known as borewell. In rural India as well as in regions having hard rocks, large-diameter wells are common.

If the well is being used for abstraction of water, it is also called as *pumping well* or *abstraction well*. However, if the well is being used for recharge, it is known as *recharge well*. If the well is being used only for measurements, it is also known as *monitoring* or *observation well*.

Todd (2004) presents several classifications of wells which also include the classification based on the method of digging a well. Abstraction of water from a well may lead to lowering of groundwater level (in case of unconfined aquifer) or piezometric levels (in case of confined aquifers) if the rate of replenishment from the aquifer is less than the rate of abstraction. Usually, the recharge from the aquifer is not instantaneous to replenish the decline of water level in the well and often, the *drawdown* or decline of the groundwater level continues in the well and its vicinity.

The drawdown is maximum in the well and reduces as one moves away from the well, unless interference effects get pronounced due to presence of other wells. The decline of water level in the well is a matter of concern. Excessive abstraction of groundwater has resulted in lowering of water table by more than 150 m in certain parts of India. Lowering of water table means higher investment on pumping costs. A good knowledge of well hydraulics is indispensable for optimum abstraction from wells.

9.2 DRAWDOWN DUE TO ABSTRACTION

To compute drawdown in and around the abstraction well, knowledge of the following is essential:

(i) Type of well—borewell or large-diameter well
(ii) Type of aquifer—confined, unconfined, or semi-confined
(iii) Extent of penetration of the well into the aquifer—fully penetrating or partially penetrating
(iv) Number of aquifers contributing water to the well—single or multiple
(v) Rate of abstraction—constant or variable with time
(vi) Presence of recharge (river, canal, etc.) or barrier boundaries
(vii) Presence of other abstraction wells
(viii) Presence of laterals in abstraction wells
(ix) Variation in aquifer properties in and around the well—consideration of homogeneity as well as heterogeneity of aquifer properties
(x) Variation in the thickness of the aquifer in the region of influence

This list provides only an idea about the type of situations which can be encountered. To provide an insight into the computations of drawdown, only a few specific conditions corresponding to steady as well as transient abstraction are considered.

9.2.1 Steady Abstraction

Abstraction from wells may not necessarily be steady. In fact, maintaining steady abstractions requires a certain amount of control over the regulating valves located above the delivery pipe. In this section, three typical cases of steady abstraction are considered. They are:

(i) Steady abstraction from a fully penetrating borewell located in a confined aquifer, and leading to an unsteady state of drawdown in and around the well,
(ii) Steady abstraction from a fully penetrating borewell located in a confined aquifer, and leading to a steady state of drawdown in and around the well, and
(iii) Steady abstraction from a fully penetrating borewell located in an unconfined aquifer, and leading to a steady state of drawdown in and around the well.

Case 1

Figure 9.1 represents this case. The basic assumptions applicable to this case are given in the following list:

(i) The aquifer medium is homogeneous and isotropic.
(ii) The aquifer thickness is constant.

(iii) Darcy's equation is valid, i.e., flow is within Darcy's range.
(iv) The fluid and the medium are incompressible.
(v) There is no vertical component of flow.
(vi) The well is fully penetrating.
(vii) The pumping rate is constant.

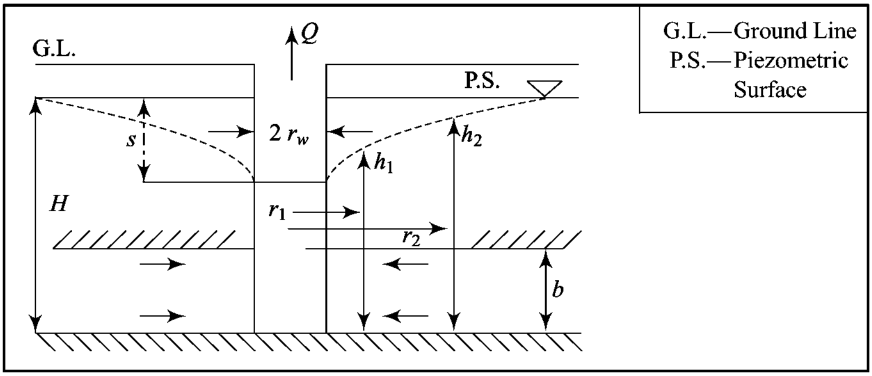

Fig. 9.1 *Abstraction from a confined aquifer using a borewell*

Let H be the height of initial piezometric level measured w.r.t. the confined aquifer bottom. For a given drawdown or drop in piezometric level by s, the height of the piezometric level will reduce by s. Let the new height be $h = H - s$. If S and T are storage coefficient and transmissivity of the confined aquifer, Boussinesq's equation for unsteady state in (x, y, z) coordinates can be written as:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S}{T}\frac{\partial h}{\partial t} \tag{9.1}$$

As flow towards the well is radial, in (r, θ, z) coordinates, Eq. (9.1) can be rewritten as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S}{T}\frac{\partial h}{\partial t} \tag{9.2}$$

Invoking the assumption of flow being horizontal,

$$\frac{\partial h}{\partial z} = 0 \tag{9.3}$$

Since the flow is radially symmetric,

$$\frac{\partial h}{\partial \theta} = 0 \tag{9.4}$$

Thus, Eq. (9.2) further simplifies to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = \frac{S}{T}\frac{\partial h}{\partial t} \tag{9.5}$$

Equation (9.5) was solved by Jacob by treating the well as a sink of constant strength and infinitesimal diameter, and subject to the following boundary conditions:

$$H(r, 0) = H \qquad \text{for } (t = 0) \tag{9.6a}$$

$$H = H \text{ as } (r \to \infty) \quad \text{for } (t \geq 0) \tag{9.6b}$$

$$\lim_{r \to 0}\left(r\frac{\partial h}{\partial r}\right) = \frac{Q}{2\pi T} \tag{9.6c}$$

Solution of Eq. (9.5) subject to Eq. (9.6) leads to the following distribution of drawdown s with r and t, i.e.,

$$s(r, t) = H - h = \frac{Q}{4\pi T}\int_u^{\infty}\frac{e^{-u}}{u}du = \frac{Q}{4\pi T}W(u) \tag{9.7}$$

where, $u = \dfrac{r^2 S}{4Tt}$ (9.7a)

Using a series expansion of exponential term and integrating, the well function $W(u)$ can be represented as:

$$W(u) = \left[-0.577216 - \ln u + u - \frac{u^2}{2 \times 2!} + \frac{u^3}{3 \times 3!} - \frac{u^4}{4 \times 4!} + \ldots\right] \tag{9.8}$$

For ($u < 0.01$), Jacob approximated Eq. (9.7) as:

$$S = \frac{Q}{4\pi T}[-0.577216 - \ln u] \tag{9.9}$$

An algebraic approximation of $W(u)$ was given by Swamee and Ojha (1990a) as:

$$W(u) = \left(\left\{\ln\left[\left(\frac{c_1}{u} + 0.65\right)(1 + u)\right]\right\}^{-7.7} + u^4 e^{7.7u}(2 + u)^{3.7}\right)^{-0.13} \tag{9.10}$$

For different values of u, this well function will generate nearly the same value of $W(u)$ as given in reported tabular solution (Walton, 1970). The maximum error in approximation is limited to 1%.

Example 9.1 Compute $W(u)$ for the given values of u, (a) $u = 0.004$, and (b) $u = 0.089$. Use Jacob's approximation as well as Swamee and Ojha approximation.

Solution

(a) As $u < 0.01$, using Jacob's approximation,

$W(u) = [-0.577216 - \ln(u)] = 4.944$

Using the algebraic approximation of $W(u)$ given in Eq. (9.10),

$$W(u) = \left(\left\{\ln\left[\left(\frac{c_1}{u} + 0.65\right)(1+u)\right]\right\}^{-7.7} + u^4 e^{7.7u}(2+u)^{3.7}\right)^{-0.13}$$

(for $c_1 = 0.561459$)

$= 4.96028$

(b) As $u > 0.01$, using Jacob's approximation,

$$W(u) = \left[-0.577216 - \ln u + u - \frac{u^2}{2\times 2!} + \frac{u^3}{3\times 3!} - \frac{u^4}{4\times 4!} + \ldots\right]$$

$= 1.9289$

Using the algebraic approximation of $W(u)$ given in Eq. (9.10),

$$W(u) = \left(\left\{\ln\left[\left(\frac{c_1}{u} + 0.65\right)(1+u)\right]\right\}^{-7.7} + u^4 e^{7.7u}(2+u)^{3.7}\right)^{-0.13}$$

(for $c_1 = 0.561459$)

$= 1.93364$

Example 9.2 In case of radial unsteady flow towards a pumping well located in a confined aquifer having transmissivity of 360 m^2/day and storage coefficient of 2.1×10^{-4}, two monitoring wells are placed 50 m and 100 m apart from the pumping well with a constant discharge of 900 m^3/day. Find the drawdowns at two different monitoring wells after half an hour of pumping?

Solution

Given, $T = 360$ m^2/day $= 0.25$ m^2/min

$S = 2.1 \times 10^{-4}$

$Q = 900$ m^3/day $= 0.625$ m^3/min

$r_1 = 50$ m

$r_2 = 100$ m

$t = 30$ min

In case of unsteady state flow towards wells in a confined aquifer,

$$u_1 = \frac{(r_1)^2 S}{4Tt} = \frac{(50)^2 \times (2.1\times 10^{-4})}{4\times(0.25)\times 30} = 0.0175$$

$$u_2 = \frac{(r_2)^2 S}{4Tt} = \frac{(100)^2 \times (2.1\times 10^{-4})}{4\times(0.25)\times 30} = 0.07$$

Let the drawdowns at monitoring wells be s_1 and s_2.

From Eq. (9.8),

$$s = \frac{Q}{4\pi T}\left[-0.577216 - \ln(u) + u - \frac{u^2}{2 \times 2!} + \frac{u^3}{3 \times 3!} - \frac{u^4}{4 \times 4!} + \ldots\right]$$

Substituting for u_1 and u_2 in Eq. (9.8):

$$s_1 = 0.6935 \text{ m}$$

And $\quad s_2 = 0.4278$ m

Case 2

The assumptions applicable to Case 1 are equally valid in this case too. As the drawdown in and around the abstraction well attains a steady state:

$$\frac{\partial h}{\partial t} = 0 \tag{9.11}$$

With the attainment of steady state, Eq. (9.5) converts into:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = 0 \tag{9.12}$$

As ($r \neq 0$) in the domain of interest, one can also write Eq. (9.12) as:

$$\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = 0 \tag{9.13}$$

Integration of Eq. (9.13) w.r.t. r leads to:

$$r\frac{\partial h}{\partial r} = c_1 \tag{9.14}$$

In Eq. (9.14), c_1 is a constant. From Eq. (9.6c), $c_1 = Q/(2\pi T)$. Further integration of Eq. (9.14) w.r.t. r leads to:

$$h = \frac{Q}{2\pi T}\ln r + c_2 \tag{9.15}$$

At ($r = r_w$), $h = h_w$

Thus, $\quad c_2 = h_w - \dfrac{Q}{2\pi T}\ln r_w \qquad (9.16)$

Substituting Eq. (9.16) in Eq. (9.15):

$$h - h_w = \frac{Q}{2\pi T}\ln\left(\frac{r}{r_w}\right) \tag{9.17}$$

Equation (9.17) provides the distribution of drawdown around the abstraction well when a steady state of the cone of depression has been obtained.

Example 9.3 A fully penetrating artesian well of 0.3 m diameter was pumped with a constant rate of 1800 m^3/day under a steady state condition. If the transmissivity of the aquifer is 0.145 m^2/min and the drawdown observed in the well is 8.92 m, find the radius of influence of the well.

Solution

Given:

$$Q = 1800 \text{ m}^3/\text{day} = \frac{1800}{24 \times 60} \text{ m}^3/\text{min}$$

$$T = 0.145 \text{ m}^2/\text{min}, \quad h - h_w = 8.92 \text{ m}, \quad \text{and} \quad r_w = 0.3 \text{ m}$$

Using $h - h_w = \dfrac{Q}{2\pi T} \ln\left(\dfrac{r}{r_w}\right)$

$$8.92 = \frac{\left(\dfrac{1800}{24 \times 60}\right)}{2 \times \pi \times 0.145} \ln\left(\frac{r}{0.3}\right)$$

Or $r = 199.8$ m

Case 3

Figure 9.2 represents Case 3. The basic assumptions remain same as for Cases 1 and 2. It should be noted that in case of unconfined flow, the term S/T is replaced by S_s/K with S_s as the specific yield.

For a steady state of drawdown, Eqs. (9.11) to (9.14) still hold good. One can proceed in a similar way, starting from Eq. (9.11). However, c_1 needs to be expressed as:

$$c_1 = \frac{Q}{2\pi Kh} \tag{9.18}$$

Replacement of T as Kh can be noted here. Here, K is the hydraulic conductivity. In case of confined aquifer, transmissivity (T) can be expressed as Kb where b is the thickness of the confined aquifer. However, in case of unconfined aquifer, T must be expressed as Kh, as it is the width (h) of the unconfined aquifer through which the flow is taking place.

Coupling of Eqs. (9.14) and (9.18) leads to:

$$h\partial h = \frac{Q}{2\pi K} \frac{\partial r}{r} \tag{9.19}$$

Integrating both sides of Eq. (9.19) between the limits $h = h_w$ at $r = r_w$ and h = head at any distance r,

$$h^2 - h_w^2 = \frac{Q}{\pi K} \ln\left(\frac{r}{r_w}\right) \tag{9.20}$$

Equation (9.20) can be used to compute drawdown around the well.

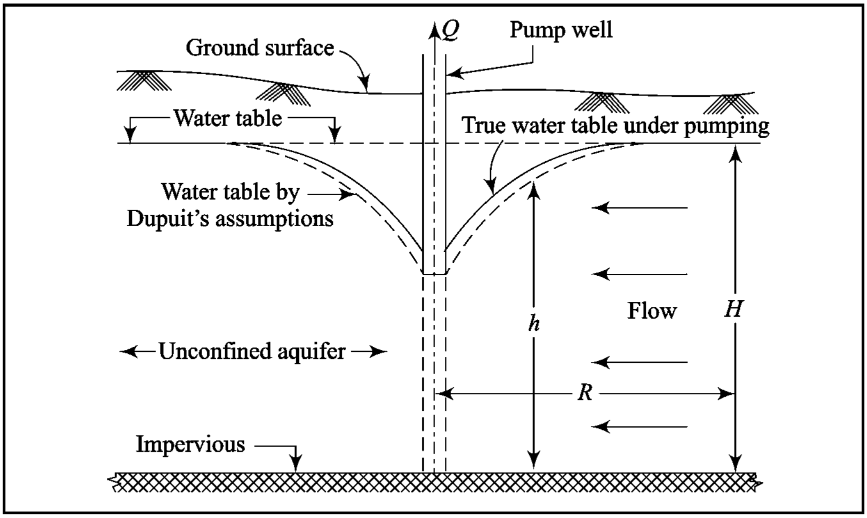

Fig. 9.2 *Abstraction from an unconfined aquifer using a borewell*

Example 9.4 A fully penetrating well of diameter 0.3 m is located in an unconfined aquifer of saturated depth 45 m. If the drawdown in the well is 15 m for the discharge of 1200 m^3/day and the radius of influence is 300 m, compute hydraulic conductivity?

Solution

Given:

diameter = 0.3 m

$r_w = 0.15$ m

$h = 45$ m

$h_w = 30$ m

$Q = 1200$ m^3/day

$r = 300$ m

From Eq. (9.20):

$$h^2 - h_w^2 = \frac{Q}{\pi K} \ln\left(\frac{r}{r_w}\right)$$

Or $$K = \frac{Q}{\pi(h^2 - h_w^2)} \ln\left(\frac{r}{r_w}\right)$$

$$K = \frac{(1200 \text{ m}^2/\text{day})}{\pi[(45 \text{ m})^2 - (30 \text{ m})^2]} \ln\left(\frac{300 \text{ m}}{0.15 \text{ m}}\right)$$

Or $K = 2.5807$ m/day

9.2.2 Transient Abstraction

As discussed earlier, drawdown in and round the well can also be computed in case of transient abstraction. A perusal of Cases 1 to 3 as discussed above also implies that steady state of drawdown is not relevant because under transient abstraction, the drawdown will generally be under an unsteady state.

Case 4

(Transient abstraction from a fully penetrating borewell located in a confined aquifer and leading to an unsteady state of drawdown in and around the well)

To explain the drawdown development around a well under transient abstraction, Case 1 is considered as the basis. Application of the concepts presented here is possible in different scenarios, as drawdown is proportional to abstraction.

For computation of drawdown under transient abstraction, the following steps are to be followed:

(a) Obtain drawdown due to a pulse (a constant abstraction of unit magnitude for a unit time step)
(b) Represent a given sequence of abstractions as a linear combinations of such pulses
(c) Superimpose drawdowns of each of the pulses of different magnitudes and durations

The stated abstraction pattern can also be represented as shown in Fig. 9.3.

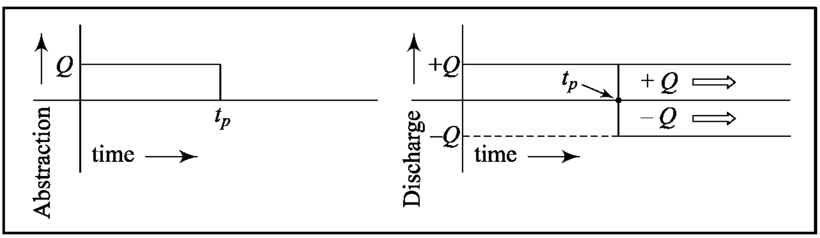

Fig. 9.3 *Representation of abstraction pattern*

Step 1: To obtain drawdown (s_{dt}) due to a unit abstraction for a unit time dt, starting at $t = 0$, Eq. (9.21) can be used.

$$s_{dt} = \frac{1}{4\pi T}\left[W\left(\frac{r^2}{4Tt}\right) - W\left(\frac{r^2}{4T(t-dt)}\right)\right] \tag{9.21}$$

Equation (9.21) consists of two terms. First term represents the drawdown due to a unit abstraction up to time t. The second term represents the recovery of drawdown due to a unit recharge beginning at time $(t - dt)$. The resultant of such an abstraction and recharge combination is a unit abstraction for a time step dt. Equation (9.21) can be used as the basis to obtain drawdowns of other pulses beginning at other times.

Step 2: Any unsteady abstraction $Q(t)$ can be assumed as a combination of pulses of varying magnitude and unit duration. Any convenient unit of time can be considered as a time step. For better accuracy, smaller time steps, such as 5, 10, 15, 30, and 60 minutes are preferable. One should ascertain that the results are not sensitive to the selection of step size. A few trials are enough to decide the appropriate step size.

Step 3: The drawdown due to a unit pulse beginning at $t = 0$ to dt can be obtained from Eq. (9.21). Similarly, drawdown due to pulses beginning at ($n\ dt$) and lasting up to $(n + 1)dt$ can be obtained. Here, n may vary from 1 to the total number of time steps during the abstraction period. The resultant drawdown is the sum of drawdowns due to all the pulses constituting $Q(t)$.

Example 9.5 An abstraction borewell located in a confined aquifer is subjected to the following abstraction pattern: (a) abstraction at rate Q for a time t_p and (b) zero abstraction for ($t > t_p$). Obtain the drawdown variation with time when the abstraction is stopped.

Solution

Using Eq. (9.7), drawdown s_1 due to abstraction beginning at $t = 0$ is given as:

$$s_1 = \frac{Q}{4\pi T} W\left(\frac{r^2}{4Tt}\right)$$

Similarly, the drawdown s_2 due to negative abstraction beginning at $t = t_p$ is given as:

$$s_2 = \frac{-Q}{4\pi T} W\left(\frac{r^2}{4T(t - t_p)}\right)$$

Thus, the resultant drawdown (s) is:

$$s = s_1 \qquad \text{for } (0 \le t \le t_p)$$

And $$s = s_1 + s_2 \qquad \text{for } (t > t_p)$$

The situation depicted in this example is very common as the abstraction continues only for a limited time. After the abstraction ceases, there is a gradual restoration of the original water level due to replenishment from the aquifer.

9.3 DRAWDOWN DUE TO ABSTRACTION FROM MULTIPLE WELLS

One of the underlying assumptions in analysis of abstraction from a single well is that, it is located in an infinite aquifer and thereby its performance is not influenced by any other source or sink. In nature, such an ideal situation is very rare. Wells are located very close to rivers or water bodies so that they can get replenished from these sources. Whenever, a well is located near a recharge boundary or barrier boundary, the drawdown expression for single well will not hold good.

Another common scenario is that, there can be several wells located in a particular region. In certain wells, such a situation is very common where a series of borewells can be observed. Each land owner or farmer will normally have his own borewell or tubewell. These wells are dug many times without much planning as a result of which optimum utilization of groundwater is not achieved.

To achieve optimum utilization of groundwater, drawdown computations for multiple wells are important. To provide an insight into drawdown computations in the presence of a hydrogeological boundary, following cases are considered.

9.3.1 Abstraction Well near a Recharge Boundary

Figure 9.4(a) shows an abstraction well near a recharge boundary. The pattern of stream or flow lines and equipotential lines is also shown in the figure.

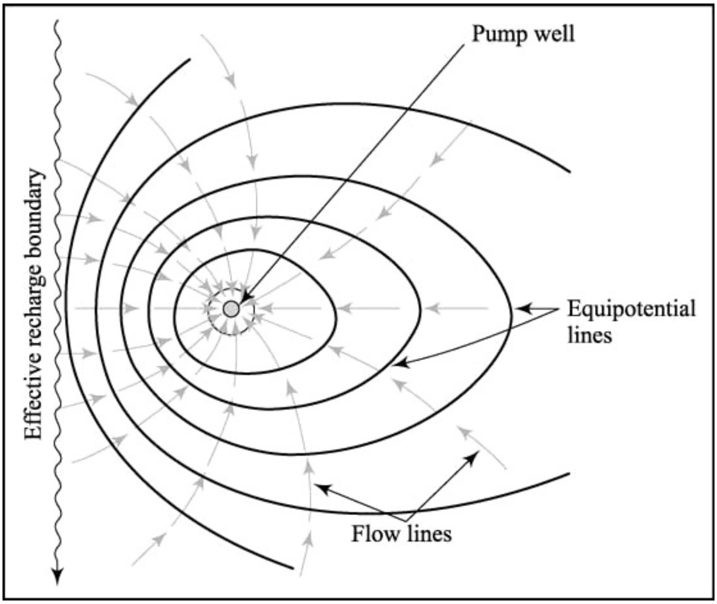

Fig. 9.4 (a) *Abstraction well near a recharge boundary (Source: Sharma and Chawla, 1977)*

As abstraction from the well begins, the cone of depression starts developing; and it may reach the recharge boundary after sometime. Irrespective of the progression of cone, the drop in the level at the boundary will always be zero.

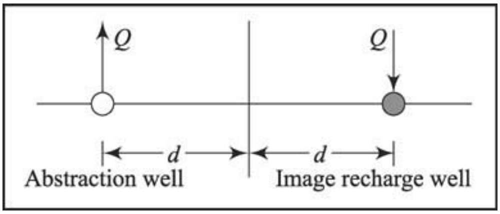

Fig. 9.4(b) *Representation of image recharge well*

Consideration of observation taken only from a single well will indicate that, there is some drawdown at the recharge boundary. However, in reality, the drawdown is zero. Thus, to get zero drawdown, a certain amount of replenishment of water level has to be brought in. This is intelligently done by bringing in another well which is recharging in nature and is located on the other side of the boundary. It is noted that the *recharge well* has to be on the other side of the boundary. The recharge well is also known as *image well* despite its reverse nature.

Thus, Fig. 9.4(a) is replaced by Fig. 9.4(b) for analysis. The drawdown at any point in the flow domain is simply the algebraic sum of the drawdowns due to abstraction as well as its image well, which is a recharge well.

For a confined aquifer with no recharge, applying Jacob's equation, the drawdown (s) at any given time (t) and at any distance (r), can be expressed as:

$$s = \frac{2.3\,Q}{4\pi T}\log\left(\frac{2.25\,T\,t}{r^2 S}\right)$$

Or $$s = \frac{2.3\,Q}{4\pi T}\log\left(\frac{C}{r^2}\right) \quad \text{where,} \quad C = \frac{2.25\,T\,t}{S}$$

The drawdown (s_w) at the well face due to the pumping well and the image well can be expressed as:

$$s_w = \frac{2.3\,Q}{4\pi T}\left[\log\left(\frac{C}{r_w^{\,2}}\right) - \log\left(\frac{C}{(2a)^2}\right)\right]$$

Further simplification of this expression leads to:

$$s_w = \frac{2.3\,Q}{2\pi T}\log\left(\frac{2a}{r_w}\right)$$

where, a is the distance between the pumping well and the recharge boundary.

Example 9.6 A well of 30 cm diameter is pumped at the rate of 1900 lpm. The well is located at a distance of 110 m from a recharging canal. Find out the drawdown in the well if the transmissivity of the aquifer is 0.015 m^2/sec.

Solution

Given:

$$Q = 1900 \text{ lpm},\ r_w = 15 \text{ cm},\ a = 110 \text{ m, and } T = 0.015 \text{ m}^2\text{/sec}$$

$$s_w = \frac{2.3\,Q}{2\pi T}\log\left(\frac{2a}{r_w}\right)$$

$$s_w = 2.476 \text{ m}$$

9.3.2 Abstraction Well near a Barrier Boundary

Figure 9.5 shows an abstraction well near a barrier boundary. Unlike recharge boundary, the drawdown at the barrier boundary may exist. However, as the flow cannot take place at the barrier boundary, such a situation is possible only when the drawdown on both sides of the boundary is equal. To achieve this, an equivalent drawdown has to be introduced on the other side of the boundary. This is made possible by placing another abstraction well at the same distance on the other side of the boundary.

Thus, Fig. 9.5(a) is replaced by Fig. 9.5(b).

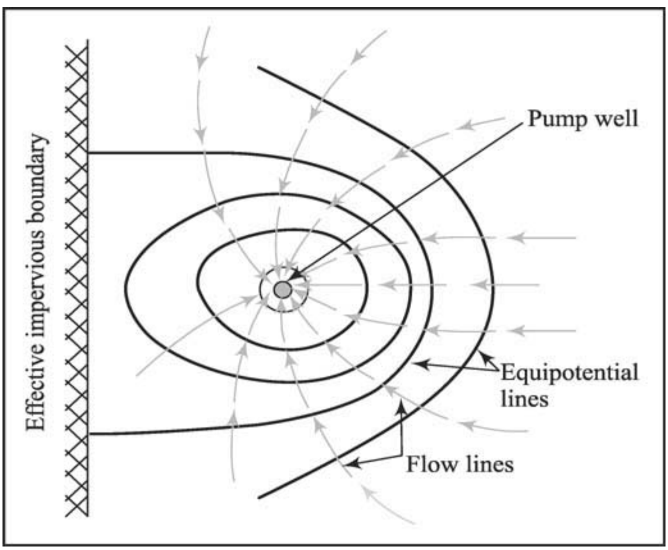

Fig. 9.5(a) *Abstraction well near a barrier boundary (Source: Sharma and Chawla, 1977)*

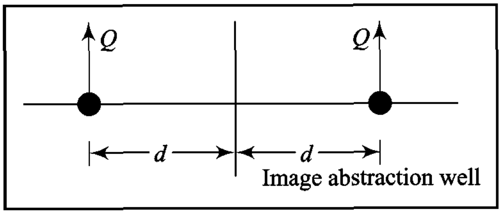

Fig. 9.5(b) *Representation of image abstraction well*

Figure 9.5 can be used to write the expressions of drawdown at any observation well. The drawdown at the barrier boundary which is located at a distance of d from pumping well and image abstraction well, as shown in Fig. 9.5(b), will record a drawdown which is two times of the drawdown caused by the pumping well had there not been a barrier boundary.

9.3.3 Well Interference

Let us consider a system of three wells A, B, and C. Let us assume that A is located at the origin, and B and C are located at (x_b, y_b) and (x_c, y_c), respectively. Let the wells A, B, and C operate for a time t with abstraction rates Q_A, Q_B, and Q_C; and let the drawdowns of the three wells be s_a, s_b, and s_c, respectively. Then, the drawdown s_0 at a point O with coordinates (x_0, y_0) can be expressed as:

$$s_0 = \frac{Q_A}{4\pi T} W\left[\frac{\{(x_0 - x_a)^2 + (y_0 - y_a)^2\}}{4Tt} S\right] + \frac{Q_B}{4\pi T} W\left[\frac{\{(x_0 - x_b)^2 + (y_0 - y_b)^2\}}{4Tt} S\right] + \frac{Q_C}{4\pi T} W\left[\frac{\{(x_0 - x_c)^2 + (y_0 - y_c)^2\}}{4Tt} S\right] \tag{9.22}$$

From Eq. (9.22), it can be seen that the drawdown at point O is the sum of the drawdowns due to the wells A, B, and C.

Equation (9.22) provides a basis to compute the drawdown at either of the wells. The drawdown at well A will not only be due to the drawdown caused by abstraction of discharge Q_A from well A, but it will also be influenced by the drawdown at well A due to abstractions exerted by wells B and C. Thus, drawdown at well A of radius r_w can be written as:

$$s_a = \frac{Q_A}{4\pi T} W\left[\frac{r_w^2}{4Tt} S\right] + \frac{Q_B}{4\pi T} W\left[\frac{(x_b^2 + y_b^2)}{4Tt} S\right] + \frac{Q_C}{4\pi T} W\left[\frac{(x_c^2 + y_c^2)}{4Tt} S\right] \tag{9.23}$$

In the absence of wells, B and C, the second and third terms of Eq. (9.23) will be zero. Due to the presence of the wells, B and C, the magnitude of S_a is increased. This effect of change in the magnitude of S_a because of the presence of other wells in the cone of depression is also called as *interference effect*.

The process of drawdown computations can be adapted for any well field. It is not necessary that all the wells in the region will only be borewells. Analysis of well fields consisting of borewells and large-diameter wells is also possible using the *superposition principle* (Sahu, 1989).

9.4 PUMPING TESTS

Pumping tests are essential to know the aquifer parameters. The main objective of a pumping test is to test the response of aquifers when they are subjected to certain excitation, either in the form of abstraction or recharge. Prior to execution of pumping tests, the following factors must be considered.

(i) Selection of site
(ii) Design of pumping and observation well
(iii) Well spacing
(iv) Duration of pumping test
(v) A set of measurements

Selection of site

Before selecting a site, it is important to know the geological and hydrological conditions of the test site. The subsurface lithology and aquifer geometry are also helpful in interpretation of pump test data. A preliminary estimate of transmissivity can also be done from surface lithology and aquifer thickness.

The presence of river, lake, or any other hydrological boundary plays an important role because of stream aquifer interaction. Pumping tests are avoided near highway or railway track because of water level fluctuations.

Design considerations of pumping and observation well

(i) The pumping test includes a pumping well and at least one monitoring well.
(ii) The pumping well diameter should be at least 150 mm, so that an electric pump can be easily installed.
(iii) The pumping well should be bored to the complete thickness of the aquifer, so that partial penetration can be avoided.
(iv) The monitoring well should also penetrate the same depth as that of the pumping well.
(v) The length and position of a screen can affect the amount of drawdown.
(vi) The pumped water should be conveyed and discharged at a distance of 200 m away from the pumping well.

Well spacing

In case of confined aquifer, the distance of monitoring well from the pumping well can be up to 200 to 250 m. In case of unconfined aquifer, the cone of depression is comparatively small and monitoring well should be within a distance of 100m. In case of high transmissivity, monitoring well can be put further away and vice versa in case of low transmissivity.

Duration of pumping test

The duration of pumping test depends on type of the aquifer. The duration of constant rate pumping test in confined, leaky confined, and unconfined aquifers could be about 24 hr, 20 hr, and 72 hr, respectively (Kruesman and de Ridder, 1990).

Set of measurements

Measurements for groundwater levels and measurement of well discharge are very important to be observed during pumping test. Initial drawdown is very fast so the water level observations should be made at an interval of 0.5–1 min for the first 10 minutes of pumping. The time intervals can be increased by 2, 5, and 10 minutes and later the time interval can be in hours. After the pumping is stopped, the initial rate of recovery is very fast so the initial water level measurement can be taken at shorter time intervals and later, the time intervals can be increased. Well discharge can be measured using different types of weirs, circular orifice weir being the most common (Discroll, 1986). Nowadays, well discharge can also be measured using electronic discharge meters.

9.5 ESTIMATION OF AQUIFER PARAMETERS

There are various approaches to estimate aquifer parameters. The parameters can be estimated using time-drawdown data of: (i) abstraction phase, and (ii) recovery phase. Some of the popular approaches and the steps involved in determining the parameters are discussed in the following pages.
The approaches that use time-drawdown data of abstraction phase are:

(i) Graphical approach
(ii) Analytical approach
(iii) Combination of graphical and semi-analytical approach
(iv) Optimization-based approach

9.5.1 Graphical Approach

Use of graphical approaches has been very common for estimation of aquifer parameters. Algebraic approximations of many well functions are still not available. Thus, this approach is the oldest one in terms of its application. The application of this approach is explained for Case 1 which deals with unsteady state drawdown situation of a well, located in a confined aquifer.

Using Eqs. (9.7) and (9.7a):

$$\log s = \log W(u) + \log\left[\frac{Q}{4\pi T}\right] \tag{9.24}$$

$$\log\left(\frac{r^2}{t}\right) = \log u + \log\left(\frac{4T}{S}\right) \tag{9.25}$$

With this background, the steps involved in this approach, also known as *type-curve matching method*, were suggested by Theis. The steps are as follows:

(i) A type curve is drawn plotting log [$W(u)$] versus log (u).
(ii) A data curve is drawn between log (s) versus log (r^2/t). Same scale has to be maintained in drawing the curves in steps 1 and 2.
(iii) The data curve is superimposed on the type curve and the sheet is moved so as to keep the axes parallel to each other, and the best-fit position is obtained.
(iv) A point is chosen on the best-matching segment of the two curves. The values of s, (r^2/t), u, and $W(u)$ corresponding to the selected match point are obtained from the graph.
(v) Transmissivity is computed as:

$$T = \frac{Q}{4\pi s} W(u) \tag{9.26}$$

(vi) Storage coefficient is computed as:

$$S = \frac{4T}{\left(\frac{r^2}{t}\right)} u \tag{9.27}$$

Table 9.1 $W(u)$ values in relation to u

u	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
× 1	0.219	0.049	0.013	0.0038	0.0011	0.00036	0.00012	0.000038	0.000012
$\times 10^{-1}$	1.82	1.22	0.91	0.70	0.56	0.45	0.37	0.31	0.26
$\times 10^{-2}$	4.04	3.35	2.96	2.68	2.47	2.30	2.15	2.03	1.92
$\times 10^{-3}$	6.33	5.64	5.22	4.95	4.73	4.54	4.39	4.26	4.14
$\times 10^{-4}$	8.63	7.94	7.53	7.25	7.02	6.84	6.69	6.55	6.44
$\times 10^{-5}$	10.94	10.24	9.84	9.55	9.33	9.14	8.99	8.86	8.74
$\times 10^{-6}$	13.24	12.55	12.14	11.85	11.63	11.45	11.29	11.16	11.04
$\times 10^{-7}$	15.54	14.85	14.44	14.15	13.93	13.75	13.60	13.46	13.34
$\times 10^{-8}$	17.84	17.15	16.74	16.46	16.23	16.05	15.90	15.76	15.65
$\times 10^{-9}$	20.15	19.45	19.05	18.76	18.54	18.35	18.20	18.07	17.95
$\times 10^{-10}$	22.45	21.76	21.35	21.06	20.84	20.66	20.50	20.37	20.25

(*Contd.*)

(*Contd.*)

$\times 10^{-11}$	24.75	24.06	23.65	23.36	23.14	22.96	22.81	22.67	22.55
$\times 10^{-12}$	27.05	26.36	25.96	25.67	25.44	25.26	25.11	24.97	24.86
$\times 10^{-13}$	29.36	28.66	28.26	27.97	27.75	27.56	27.41	27.28	27.16
$\times 10^{-14}$	31.66	30.97	30.56	30.27	30.05	29.87	29.71	29.58	29.46
$\times 10^{-15}$	33.96	33.27	32.86	32.58	32.35	32.17	32.02	31.88	31.76

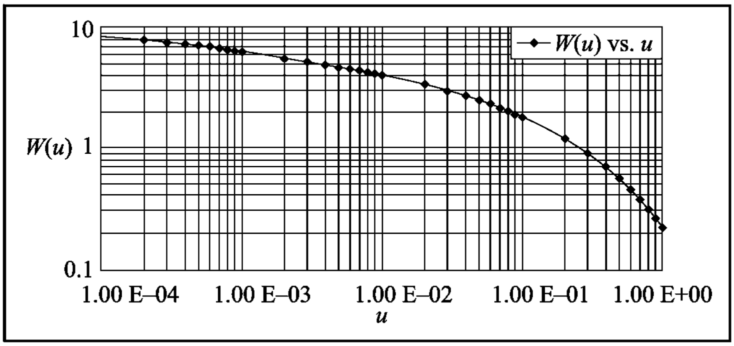

Fig. 9.6 *Type curve showing variation of W(u) vs. u*

Example 9.7 A pump test was conducted with a constant discharge of 2.90 m^3/min. The observation well was located at a distance of 19.20 m. Estimate T and S using the time-drawdown data given in Table 9.2.

Table 9.2 Time-drawdown data

Time since pumping began (min)	*Drawdown (m)*	*Time since pumping began (min)*	*Drawdown (m)*
0	0.0000	25	0.8809
0.5	0.3596	30	0.9083
1	0.4298	35	0.9296
1.5	0.4785	40	0.9449
2	0.5090	45	0.9693
2.5	0.5394	50	0.9906
3	0.5608	55	1.0089
3.5	0.5791	60	1.0150
4	0.6096	70	1.0211
4.5	0.6309	80	1.0302
5	0.6401	90	1.0363
6	0.6706	100	1.0363
7	0.6888	110	1.0424

(*Contd.*)

(*Contd.*)

8	0.7102	120	1.0516
9	0.7254	135	1.0607
10	0.7407	150	1.0668
12	0.7742	165	1.0820
16	0.8108	180	1.0942
18	0.8209	210	1.1003
20	0.8504	217	Pump stopped

Solution

The steps are as follows:

(i) A type curve is plotted between u versus $W(u)$ on log-log graph.

(ii) A data curve is plotted between (r^2/t) versus s on log-log graph.

(iii) The data curve is matched with the type curve, keeping x and y axes parallel; and a matching zone is determined.

(iv) From the matching zone a point is selected and corresponding values of s, (r^2/t), u, and $W(u)$ are determined from the matching point.

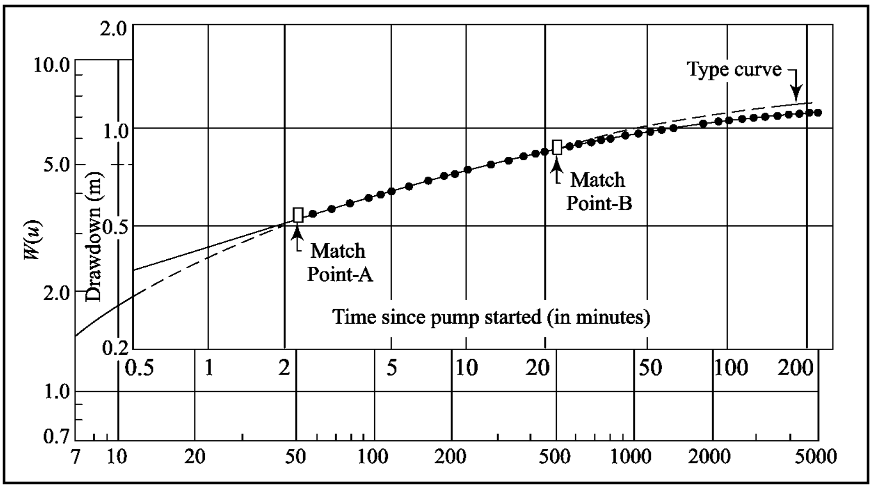

Fig. 9.7 *Type curve and data curve showing the match point*

For the match point at A,

$$W(u) = 3.4, \quad u = 0.02, \quad t = 2.2 \text{ min}, \quad s = 0.5182 \text{ m}$$

Given, $Q = 2.90 \text{ m}^3/\text{min}$

$r = 19.20$ m

Therefore, $$T = \frac{Q}{4\pi s} W(u) = 1.515 \text{ m}^2/\text{min}$$

And $$S = \frac{4Ttu}{r^2} = 7.238 \times 10^{-4}$$

9.5.2 Analytical Approach

This approach is applicable to cases 2 and 3. With known pairs of (r, h), and (r_w, h_w), T can be evaluated using Eq. (9.7) for a confined aquifer; whereas K can be evaluated using Eq. (9.20) in case of an unconfined aquifer. Examples 9.3 and 9.4 illustrate the use of analytical approach.

9.5.3 Combination of Graphical and Semi-analytical Approach

Plotting a type curve is time consuming and is subject to error due to human judgement. To avoid this, Jacob used approximation of well function, as given in Eq. (9.9) for the situation depicted in Case 1. His analytical approach is based on Eqs. (9.7) and (9.9) which leads to:

$$s = \frac{Q}{4\pi T}\left[-0.577216 - \ln\left(\frac{r^2 S}{4Tt}\right)\right] \tag{9.28}$$

Using time drawdown pairs (s_1, t_1) and (s_2, t_2):

$$\Delta s = (s_2 - s_1) = \frac{2.303Q}{4\pi T}\log\left(\frac{t_2}{t_1}\right) \tag{9.29}$$

It is noted that a change in drawdown $\Delta s = s_2 - s_1$ takes place in a time Δ (log t) = log (t_2/t_1). [Note: ln(t) = 2.303 log t] Here, ln(t) is logarithm on exponential base, whereas log refers to logarithm on base 10.

Equation (9.29) is also used to find an estimate of t at which drawdown s is still zero. We know that at $t = 0$, s is zero. However, Eq. (9.29) also provides a value of t, say at $t = t_0$ at which s is zero. Using Eq. (9.28) and solving for $s = 0$ leads to:

$$S = \frac{2.257\ T\ t_0}{r^2} \tag{9.30}$$

With this background, the steps involved in Jacob's method are as follows:

(i) A graph is plotted between drawdown versus time on a semi-log graph. Time is represented on a logarithmic scale.

(ii) From the semi-log graph, let the drawdown be s_1 for $t_1 = 10$ min, and s_2 for $t_2 = 100$ min.
Hence, $\Delta s = s_2 - s_1$ for $\Delta(\log t) = 1$

(iii) Using this value of Δs, T is computed using Eq. (9.27).

(iv) From the graph in Step 1, the value of t_0 is obtained at $s = 0$. For this purpose, the linear plot is extended to intersect the time axis. The intersected value of time is t_0. S is calculated using the value of t_0 in Eq. (9.27).

In Jacob's method, the transmissivity T and S are determined by the following formulae,

$$T = \frac{2.3Q}{4\pi\Delta s}$$

$$S = \frac{2.25\,T\,t_0}{r^2}$$

where, t_0 is time intercept on zero drawdown.

Example 9.8 Using Jacob's method, obtain S and T for a given aquifer at Patmangalam, Tamil Nadu, India. The time-drawdown data is given in Table 9.3 for which the observation well is 12.3 m from the production well (Source: Raghunath, H.M.).

Table 9.3 Time and drawdown data

Time t (min)	*Drawdown s (m)*	*Time t (min)*	*Drawdown s (m)*
1	0.24	22	0.53
2	0.24	28	0.54
3	0.28	35	0.57
4	0.32	45	0.64
6	0.37	55	0.65
8	0.41	65	0.68
10	0.45	80	0.69
14	0.49	100	0.74
18	0.51	120	0.76

Solution

The steps are as follows:

(i) A graph is plotted between drawdown versus time on a semi-log graph. Time is represented on a logarithmic scale.

(ii) From the semi-log graph, let the drawdown be s_1 for $t_1 = 10$ min, and s_2 for $t_2 = 100$ min.

$$\Delta s = s_2 - s_1 = 0.73 - 0.43 = 0.3 \text{ m}$$

(iii) Using Jacob's method, T and S are determined by the following formulae,

$$T = \frac{2.3Q}{4\pi\Delta s} = 1008 \text{ m}^2\text{/day}$$

$$S = \frac{2.25Tt_0}{r^2} = 0.0037$$

9.5.4 Optimization-based Approach

Jacob's method of parameter estimation is simple. However, it suffers from the limitation that the well argument u has to be less than 0.1. Many a time, this is not so, and the computed values of S and T do violate this assumption. Further, Jacob's method involves graphical plotting, which is a cumbersome and time-consuming process.

To overcome the drawbacks of Jacob's method, Swamee and Ojha (1990a) developed a full range of approximation for the well function; and suggested a procedure in which an error function, based on observed and computed drawdowns for any assumed pair of S and T, has to be minimized by variations of S and T in a certain feasible range.

Using Eqs. (9.7) and (9.10):

$$s = \frac{Q}{4\pi T}\left(\left\{\ln\left[\left(\frac{c_1}{u} + 0.65\right)(1+u)\right]\right\}^{-7.7} + u^4 e^{7.7u}(2+u)^{3.7}\right)^{-0.13} \tag{9.31}$$

For a set of S and T values, the error in i^{th} observation is:

$$E_i = s_i - \frac{Q}{4\pi T}\left(\left\{\ln\left[\left(\frac{c_1}{u_i} + 0.65\right)(1+u_i)\right]\right\}^{-7.7} + u_i^4 e^{7.7u_i}(2+u_i)^{3.7}\right)^{-0.13} \tag{9.32}$$

where, $u_i = r^2\left(\dfrac{S}{4Tt_i}\right)$ (and s_i is observed at t_i) (9.33)

The sum of square of errors overemphasizes large errors that are associated with the later part of the pump test. S and T can be obtained by varying them so that the sum of square of errors for the entire dataset is minimized. Thus, the data will not be properly used for the initial part of the pump test. To use the entire dataset to its full effect, the proportionate error (e_i) is considered instead of the errors themselves.

$$e_i = \frac{E_i}{s_i} \tag{9.34}$$

To determine aquifer parameters, summation of a criteria function $f(e_i)$ has to be minimized.

$$F = \sum_{i=1}^{n} f(e_i) \tag{9.35}$$

Several criteria functions have been proposed from time to time. The most common among them is a square function.

$$f(e_i) = (e_i)^2 \tag{9.36}$$

The criteria function given in Eq. (9.36) is popularly known as the *least square method* which was introduced by Gauss in 1768. Another criteria function for $f(e_i)$ is an absolute function.

$$f(e_i) = |e_i| \tag{9.37}$$

By varying S and T in a feasible range, several values of F can be obtained. The value of S and T at which F is minimum will be the correct estimate of S and T.

Example 9.9 In Table 9.2, the observed drawdown at $t = 2$ min is 0.5090 m. Use the following data to compute drawdown, and estimate the percentage error in the computed drawdown.

$$S = 7.238 \times 10^{-4}$$

$$T = 1.515 \text{ m}^2/\text{min}$$

$$r = 19.2 \text{ m}$$

$$Q = 2.90 \text{ m}^3/\text{min}$$

Also, estimate $f(e_i)$ as per Eq. (9.36).

Solution

$$S = \frac{4Ttu}{r^2}$$

Or $$7.238 \times 10^{-4} = \frac{4 \times 1.515 \times 2 \times u}{(19.2)^2}$$

$\Rightarrow$ $u = 0.0220$

$\Rightarrow$ $W(u) = 3.346$

From the equation, $T = \dfrac{Q}{4\pi s} W(u)$, we have:

$$s = \frac{Q}{4\pi T} W(u)$$

$$s = 0.50968 \text{ m}$$

Hence, percentage error

$$= \frac{\text{Observed drawdown} - \text{Computed drawdown}}{\text{Observed drawdown}} \times 100$$

$$= 0.13359$$

Unsolved: Obtain F, as per Eq. (9.35) using the given in Table 9.2. Study the effect of varying T and S values. Comment whether the use of S and T, as considered in Example 9.9, is the best choice.

There are some other approaches which are used to estimate aquifer parameters. These include computations of time derivatives of drawdown and estimating the time at which time derivative of drawdown becomes zero. Aquifer parameters are obtained using this time at which time derivative of drawdown is zero [Singh (2000)]. Another approach which estimates aquifer transmissivity using specific capacity is described here. The idea of presenting this is also to introduce widely used term of specific capacity.

9.5.5 Estimating Aquifer Transmissivity using Specific Capacity

Specific capacity is a widely used term in well hydraulics. It is defined as the ratio of abstraction discharge Q to drawdown s_w in the well. For a steady abstraction from a fully penetrating borewell located in a confined aquifer and leading to a steady state of drawdown in and around the well, specific capacity C_{sp} can be computed as follows.

Using Eq. (9.17), for $r = R$ with R as the radius of influence where h approaches H (the original depth of piezometric surface)

$$s_w = H - h_w = \frac{Q}{2\pi T} \ln\left(\frac{R}{r_w}\right) \tag{9.38}$$

Thus,

$$C_{sp} = \frac{Q}{s_w} = \frac{2\pi}{\ln\left(\dfrac{R}{r_w}\right)} T \tag{9.39}$$

From Eq. (9.39), invariance of specific capacity with time is apparent for steady state. It also indicates that the aquifer transmissivity is related with specific capacity. This has provided motivation for relating T with specific capacity. For example, Huntley et al. (Singhal and Gupta, 1999) proposed:

$$T = A\left(\frac{Q}{s_w}\right)^{1.18} \tag{9.40}$$

where, the value of A is 0.12 if both T and (Q/s_w) are expressed in $m^2 day^{-1}$. In this case, it is assumed that the well loss is negligible, which will underestimate the transmissivity value in alluvial aquifers. Normally, T values estimated from specific data are greater than those obtained from aquifer tests. To appreciate this, one has to realize that the drawdown computed using aquifer response at $(r = r_w)$ is not necessarily equal to the drawdown in the well. As the water enters into the well, it may encounter additional flow resistance as it is flowing through the well screen. In addition, turbulent flows may occur near the well. Therefore, the piezometric surface drops further.

The drop in piezometric surface occurs due to (i) headloss resulting from laminar flow in the formation, also known as formation loss, (ii) headloss due to turbulent flow in the zone close to the well face, (iii) headloss through the well screen, and (iv) headloss in the well casing. The drawdown s_w in the well is also expressed using the following relationship.

$$s_w = B\,Q + CQ^n \tag{9.41}$$

In Eq. (9.41), B is also known as coefficient of formation loss, and the term BQ accounts for headloss resulting from laminar flow in the formation. The second term (CQ^n) collectively represents the headlosses due to factors (ii) to (iv), as described above. C represents a well loss coefficient and n may vary from 1 to 2, although higher values of n are also reported.

Using Eq. (9.41), one can express C_v as:

$$C_v = \frac{1}{B + CQ^{n-1}} \tag{9.42}$$

From Eq. (9.42), one can see that a plot of s_w/Q versus Q will be a straight line and having slope as C for $n = 2$. Similarly, the intercept at $Q = 0$ will give B. Thus, with B and C being known, it is also possible to compute specific capacity with a priori assumption of $n = 2$.

If the s_w/Q versus Q plot is not linear for $n = 2$, either the plots have to be attempted for different n values or B, C, and n have to be obtained so that the error between computed s_w using Eq. (9.41) and observed drawdown is minimized.

It is also important to realize the difference between two expressions of specific capacity, as given by Eqs. (9.39) and (9.42). Equation (9.39) only considers the formation loss. Here, no well loss is considered. If we neglect well loss component in Eq. (9.42), C_v will be equal to $1/B$.

The specific capacity which is computed by ignoring the well loss is also known as *theoretical specific capacity*. The ratio of specific capacity for the observed drawdown to theoretical specific capacity is also known as well efficiency (η_{well}).

$$\eta_{\text{well}} = \frac{Q/s_w}{Q/(BQ)} = \frac{BQ}{s_w} \times 100\% \tag{9.43}$$

It is to note that in Eq. (9.43), s_w given by Eq. (9.41) needs to be used. A high specific capacity indicates an efficient good yielding well.

The difference of the piezometric surface at the outer periphery of the well (as one approaches towards well from aquifer side) and the actual level in the well is also called *seepage face* in large-diameter wells. It is to note that seepage face and well loss are not same.

In case of large-diameter wells and in the absence of any well casing, seepage face may be same as well loss under a steady state situation. The purpose of introducing seepage face is to indicate how it leads to similar expressions of drawdown in the case of a large-diameter well.

Ojha and Gopal (1999) observed that the seepage face (f) is related with the entrance velocity of flow into the well (v_f) as:

$$f = a(v_f)^n \tag{9.44}$$

The values of exponent n were found to range from 1 to 2. Thus, if the *seepage face* is considered, Eq. (9.38) will modify to:

$$s_w = \frac{Q}{2\pi T} \ln\left(\frac{R}{r_w}\right) + a(v_f)^n \tag{9.45}$$

As v_f is expressible in terms of Q, Eq. (9.45) can also be expressed as:

$$s_w = C_1 Q + C_2 Q^n \tag{9.46}$$

In Eq. (9.46), $C_1\,Q$ is known as formation loss and $C_2\,Q^n$ is known as well loss. Normally, n as 2 is adopted in the literature which may not always hold good. Thus, if the well loss is ignored, a smaller value of s_w is being used in Eq. (9.40); and thus, T is overestimated.

Example 9.10 A 0.3 m diameter well completely penetrating an infinite artesian aquifer is pumped at the rate 60 m^3/hour. The transmissivity and the storativity of the aquifer are 600 m^2/day and 0.0004, respectively. Assuming a well loss coefficient C_2 of 0.5 min^2/m^5 and index $n = 2$, find the specific capacity of the pumping well after 1 day of pumping. [Hint: Take care of units]

Solution

As $\quad s_w = C_1\,Q + C_2\,Q^n$

$$s_w = \frac{2.3\,Q}{4\pi T} \log\left[\frac{2.25Tt}{(r_w)^2\,S}\right] + C_2\,Q^n$$

Or
$$\frac{Q}{s_w} = \frac{1}{\dfrac{2.3}{4\pi T} \log\left(\dfrac{2.25Tt}{(r_w)^2\,S}\right) + C_2\,Q^{n-1}}$$

Given:

$r = 0.15$ m

$Q = 60$ m^3/hr $= 1$ m^3/min $= 1/60$ m^3/s

$T = 600$ m^2/day $= 25$ m^3/hr $= 0.006944$ m^3/s

$C_2 = 0.5$ min^2/m^5 $= 1800$ s^2/m^5

$C_2\,Q^{n-1} = 1800$ s^2/m^5 $\times$ 1/60 m^3/s

$= 30$ s/m^2

Thus, $\dfrac{Q}{s_w} = \dfrac{1}{\left[\dfrac{2.3}{4\pi(0.006944)}\log\left(\dfrac{2.25\times 600\times 1}{(0.15)^2(0.0004)}\right)+30\right]} = 0.0040757 \text{ m}^2\text{/s}$

Unsolved: Study the effect of varying t on specific capacity computation.

9.5.6 Coupling of Steady and Unsteady State Relationships

Beginning with an appropriate value of radius of influence (between 50 to 200 m), steady state equation [Eq. (9.17)] is used to estimate T using first-time drawdown pair of unsteady state case. This value of T is substituted in second-time drawdown pair to compute radius of influence. A successive application of this process leads to different estimates of transmissivity. As time increases, the transmissivity values progressively tend towards a constant value. With this value of T, one can use drawdown at later stages for which T has been stabilized to compute storage coefficient S.
The advantages of using this approach are:

(i) No graphical construction is involved,
(ii) The values of S and T obtained can be used further for tuning aquifer parameters in optimization approach, and
(iii) The search range of S and T is considerably reduced, which significantly enhances the efficiency of parameter estimation using optimization procedures.

Aquifer parameters are also estimated using time-drawdown data of recovery phase. When the pumping is discontinued, flow from aquifer towards the well still continues. As a result, water level in the abstraction well keeps on rising and drawdown in the well at the time of closure of pumping starts decreasing. If the time-drawdown observations are recorded during recovery period, these can also be utilized to estimate aquifer parameters.

If the abstraction continues only for a time t_p, for $(t > t_p)$, the process of recovery of drawdown will take place. Under the assumption that aquifer properties do not change after the abstraction ceases, one can write the following expression to calculate drawdown for $(t > t_p)$.

$$s = \frac{Q}{4\pi T}W\left(\frac{r^2S}{4Tt}\right) - \frac{Q}{4\pi T}W\left[\frac{r^2S}{4T(t-t_p)}\right] \tag{9.47}$$

Using Jacob's approximation of well function, Eq. (9.47) simplifies to:

$$s = \frac{2.3Q}{4\pi T}\log\left(\frac{t}{t-t_p}\right) \tag{9.48}$$

From a semi-logarithmic plot between s and $t/(t - t_p)$, an intercept of s for one log cycle of $t/(t - t_p)$ is noted, and this is used in Eq. (9.48) to estimate T.

Example 9.11 A production well, fully penetrating an artesian well, is pumped at a constant rate of 2100 m^3/day. If the transmissivity and the storage coefficient be 0.145 m^2/min and 4×10^{-4}, obtain the drawdown at a distance of 3 m after 1 hour of pumping.

Solution

Given:

$$r = 3 \text{ m and } t = 1 \text{ hour}$$

$$u = \frac{r^2 S}{4Tt} = \frac{3^2 \times 4 \times 10^{-4}}{4 \times 0.145 \times 60 \times 1} = 0.000103$$

$$\Rightarrow \quad W(u) = 8.62$$

$$\Rightarrow \quad s = \frac{Q}{4\pi T} W(u) = 6.9 \text{ m}$$

Example 9.12 Three tubewells of 25 cm diameter each are located at the three vertices of an equilateral triangle of side 100 m. Each tubewell penetrates fully in a confined aquifer of thickness 25 m. Assume the radius of influence for these wells and the coefficient of permeability of aquifer as 300 m and 40 m/day, respectively.

(i) Calculate the discharge when only one well is discharging under a depression head of 3 m.

(ii) What will be the percentage change in the discharge of this well, if all three wells were to pump such that the depression head is 3 m in all the wells?

Solution

(i) When only one well is pumping at steady state

Given:

$$s = 3 \text{ m}, r_w = 0.125 \text{ m}, R = 300 \text{ m}, K = 40 \text{ m/day, and}$$

$$b = 25 \text{ m}$$

$$T = K\,b = 1000 \text{ m}^2\text{/day}$$

$$s = \frac{Q}{2\pi T} \ln\left(\frac{R}{r_w}\right)$$

$$Q = 2421.82 \text{ m}^3\text{/day}$$

(ii) When all the three wells are pumping at steady state

$$s_{\text{total}} = \sum_{i=1}^{3} s_i = \sum \frac{Q_j}{2\pi T} F(r_{ij})$$

$$Q_1 = Q_2 = Q_3$$

$$s_t = \frac{3Q}{2\pi T} \ln\left(\frac{R}{r^*}\right) \quad \text{where,} \quad r^* = (r_{11} \times r_{12} \times r_{13})^{1/3}$$

$$r_{11} = 0.125 \text{ m}, \quad r_{12} = r_{13} = 100 \text{ m}, \quad \text{and} \quad s_t = 3 \text{ m}$$

$$Q = 1875.25 \text{ m}^3/\text{day}$$

$$\Rightarrow \quad \text{Percentage reduction in discharge} = \frac{2421.82 - 1875.25}{2421.82} \times 100 = 22.57$$

Example 9.13 A pumping well of diameter 30 cm is being pumped at the rate of 2240 lpm. The incremental drawdown observed in an observation well located at 15 m from the pumping well corresponding to one log cycle is 0.35 m. Obtain transmissivity of the aquifer using Jacob's method.

Solution

$$s = \frac{2.3\,Q}{4\pi T} \log\left(\frac{2.25Tt}{r^2 S}\right)$$

$$\Delta s = \frac{2.3\,Q}{4\pi T} \Delta(\log t)$$

Given:

$$\Delta(\log t) = 1, \quad \Delta s = 0.35 \text{ m}, \quad \text{and} \quad Q = 2.240 \text{ m}^3/\text{min}$$

$$\Rightarrow \quad T = 1686.24 \text{ m}^2/\text{day}$$

Example 9.14 Show that for a given aquifer, the time of occurrence of equal drawdown in different observation wells vary directly as the squares of the distances of the observation wells from the pumping wells.

Solution

Let there be two observation wells at distances r_1 and r_2 from the pumping well. Using Jacob's approximation of the well function, one can write the following expressions for drawdown s_1 and s_2 in these observation wells at time t_1 and t_2.

$$s_1 = \frac{2.303\,Q}{4\pi T} \log_{10}\left[\frac{2.25\,T\,t_1}{(r_1)^2 S}\right]$$

Similarly,

$$S_2 = \frac{2.303\,Q}{4\pi T} \log_{10}\left[\frac{2.25\,T\,t_1}{(r_2)^2 S}\right]$$

From $s_1 = s_2$, we get $\dfrac{t_1}{(r_1)^2} = \dfrac{t_2}{(r_2)^2}$, which is the desired result.

SUMMARY

Considering the importance of well hydraulics, an exposure to some of the commonly used situations of abstraction has been provided in this chapter. Principle of superposition can be used to obtain resultant drawdown in a variety of situations including the presence of multiple wells, and the abstraction well being located near the recharge or barrier boundary.

To project the effect of abstraction in and around the abstraction well, various approaches which are used to estimate aquifer parameters are described. Keeping pace with the modern developments, much emphasis is placed to estimate aquifer parameters using error minimization approaches.

Knowledge of aquifer parameters may be helpful in computing the drawdown in a well or well field and in estimating the maximum safe yield of a well or a well field which also equals the capacity of aquifer to supply water without continuous lowering of the water table or piezometric surface. Therefore, if the abstraction from an aquifer is replenished by some recharge, either from a rainfall or other means, excessive lowering of groundwater levels can be certainly avoided.

EXERCISES

9.1 An abstraction well and a recharge well are situated at a distance r apart. Assume that these are fully penetrating a confined aquifer having formation constants T and S. Identify the conditions leading to interference among these wells.

9.2 Define image well. Draw the pattern of drawdown curve resulting from an abstraction well, placed at an equal distance from a barrier boundary and a recharge boundary.

9.3 Define and explain the well loss. How is well loss related to specific capacity of the well?

9.4 Derive an expression for drawdown due to steady abstraction from a fully penetrating borewell located in a confined aquifer and leading to an unsteady state of drawdown in and around the well.

9.5 Prepare a summary of new approaches developed in literature to estimate aquifer parameters for a situation described in Problem 9.4.

9.6 Study the effect of using minimization of square of errors, absolute of average percentage error, and cutoff error concept of Swamee and Ojha (1990) in estimation of aquifer parameters.

9.7 An observation well of 40 cm diameter located at 40 m from a river is pumped for 15 hrs at a rate of 2580 lpm. The drawdown in the observation well is given in the table below. Determine the aquifer constant.

Time (t) in min, since pumping began	*Drawdown in the observation well, in (min)*	*Time (t) in min, since pumping began*	*Drawdown in the observation well, in (min)*
15	0.716	180	0.838
30	0.732	210	0.844
45	0.753	240	0.860
60	0.771	300	0.866
75	0.783	390	0.875
90	0.792	480	0.881
110	0.802	600	0.884
130	0.814	720	0.890
150	0.826	900	0.893

9.8 What will be the drawdown after (i) 1 day, and (ii) 5 days, if the pumping has to be continued for 10 days in Problem 9.7?

9.9 If the average error in the drawdowns in the observation well of Problem 9.7 is 1%, determine the aquifer constant and compare it with the result of Problem 9.7.

9.10 There are four wells, viz. *A*, *B*, *C*, *D* of 20 cm diameter each on four sides of a square having sides of 20 m. Pumping has been started at a rate of 2500 lpm from another well of 20 cm which is located at the centre of the square. Storage coefficient of the aquifer is 0.005. Determine the drawdowns of the four wells, *A*, *B*, *C*, and *D*. Transmissibility of the aquifer is 1800 lpm/m.

9.11 A production well has been discharging at a constant rate for 8 years. A recharge boundary is located at a distance of 12 km from the well. The transmissibility and storage coefficient of the aquifers are 1500 lpm/m and 0.0005, respectively. Compute the percentage of the well yield from the source of recharge.

OBJECTIVE QUESTIONS

1. The maximum safe yield of a well is the capacity of the aquifer to supply water
   (a) Without causing a continuous lowering of the water table
   (b) Without causing a continuous lowering of the piezometric surface

(c) Is limited to the rate at which groundwater is replenished by rainfall
(d) All of these

2. Dimension of transmissivity is
   (a) $[M^0L^0T^{-1}]$ (b) $[M^0L^2T^{-1}]$ (c) $[M^0L^1T^{-1}]$ (d) $[M^0L^0T^0]$

3. Dimension of storage coefficient is
   (a) $[M^0L^0T^{-1}]$ (b) $[M^{-1}L^{-1}T^{-1}]$ (c) $[M^0L^{-1}T^{-1}]$ (d) $[M^0L^0T^0]$

4. If T is transmissivity and b is the width of aquifer, the hydraulic conductivity K is
   (a) $K = T \times b$ (c) $K = T/b$ (c) $K = T + b$ (c) $K = T - b$

5. An artesian aquifer is one in which
   (a) Underground water is at atmospheric pressure
   (b) Water table serves as upper surface of zone of saturation
   (c) Water is under pressure between two impervious data
   (d) None of the above

6. The basic assumptions applicable to pumping tests are
   (a) The aquifer medium is homogeneous and isotropic; and thickness is constant
   (b) Darcy's equation is valid, and the fluid and medium are incompressible
   (c) No vertical component of flow; the well is fully penetrating; and the pumping rate is constant
   (d) All of the above

7. When abstraction from a well begins, the cone of depression starts developing; and in case of the recharge boundary, the drawdown at the recharge boundary is
   (a) Less than the abstraction well (b) Greater than the abstraction well
   (c) Zero (d) None of these

8. What will be the drawdown at mid-point in steady state condition of a pumping well and a recharging well, given that pumping and recharging rates are same? Assume the drawdown in pumping well as s.
   (a) $2s$ (b) $0.5s$ (c) s (d) zero

9. In Example 8.15, obtain S if the drawdown $s = 0.45$ m at the observation well after 5 minutes.
   (a) 3.04×10^{-3} (b) 3.04×10^{-5} (c) 5.69×10^{-3} (d) 5.69×10^{-5}

10. For a pumping well, tapping an unconfined aquifer flow, Dupuit's assumption holds good when
    (a) Stream lines are converging, and two-dimensional flow condition prevails

(b) Stream lines are diverging, and radial flow condition prevails
(c) Stream lines are parallel, and radial flow condition prevails
(d) None of the above

11. An aquifer of aerial extent 50 km^2 is overlain by two strata with the following details:

Strata	*Thickness*	*Hydraulic conductivity*
1 (Top)	10 m	0.40 m/day
2 (Bottom)	20 m	0.25 m/day

The transmissivity of aquifer is
(a) 10.57 m^2/day (b) 9.57 m^2/day
(c) 8.57 m^2/day (d) None of these

12. In case of well interference between two tubewells
(a) The spacing between two tubewells should be considered
(b) The radius of influence with safe margin should be considered
(c) The pumping rate of two tubewells should be considered
(d) All the above are considered

13. Specific capacity indicates
(a) Productivity of both aquifer and well
(b) Productivity of aquifer
(c) Productivity of well
(d) None of the above

14. A plot of s_w versus Q yields a straight line. The intercept and slope of this line indicate
(a) Specific capacity and well loss
(b) Specific capacity and formation loss
(c) Formation loss coefficient and well loss coefficient
(d) None of the above

15. The specific capacity is not constant but decreases
(a) With increasing Q (b) With decreasing Q
(c) Does not depend on Q (d) None of these

16. In case of specific capacity computations, the drawdown in the pumping well is
(a) Equal to formation loss
(b) Equal to well loss
(c) Sum of formation loss and well loss
(d) None of the above

17. Fluctuations in groundwater level are caused by
    (a) Pumping well in the vicinity
    (b) Earthquake, change in atmospheric pressure
    (b) Both (a) and (b)
    (b) None of the above

18. Assumptions in Theis equation are
    (a) Homogeneous, isotropic aquifer of uniform saturated thickness
    (b) Discharge of well is constant, radius of well is infinitesimal, and radial flow with no entrance loss
    (c) Both (a) and (b)
    (d) None of the above

19. For unsteady flow, where $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, and h is the head causing flow;
    (a) $\nabla^2 h = 0$ (b) $\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t}$
    (c) Both (a) and (b) (d) None of these

20. In case of two parallel boundaries, two image planes are considered, which leads to
    (a) Two sets of images (b) Four sets of images
    (c) Nine sets of images (d) Infinite sets of images

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CHAPTER

10

Subsurface Water

10.1 INTRODUCTION

In Chapters 8 and 9, the focus has been primarily on the basics of groundwater and well hydraulics. This chapter covers subsurface water in greater detail. Not all subsurface water is groundwater. If a hole is dug, moist or even saturated soil may be encountered. As long as this water does not seep freely into the hole, it is not groundwater. True groundwater is reached only when the water begins to flow into the hole. Since air in the hole is at atmospheric pressure, the groundwater pressure must be at least equal to or more than the atmospheric pressure if it has to flow into the hole. Similarly, the pressure of subsurface water that does not flow into the hole must be less than the atmospheric pressure. Thus, what distinguishes groundwater from rest of the subsurface water is: its pressure is greater than atmospheric pressure. Since groundwater moves freely under the force of gravity into the wells, it is also called *free water* or *gravitational water*.

The zone between the ground surface and the top of the groundwater is called the *vadose zone* or *unsaturated zone*. Water in this zone (very little in dry climates) is held by soil particles or other underground material which can absorb water due to capillary action. While this water is still able to move within the vadose zone, it cannot move out of the zone into wells or other places that are exposed to atmospheric pressure. Aquifer and vadose zones are basically porous media that store and transmit water; but the major factor that differentiates vadose zone from aquifer is its inability to yield water. Clays and other unconsolidated materials may compress or expand, depending on whether groundwater is withdrawn or added. In order to quantify these processes, physical properties of vadose zone have often been expressed in terms of a number of parameters.

Having recognized these aspects, groundwater is classified as either *saturated* or *unsaturated*. Surface tension and gravity are the driving forces for unsaturated groundwater flow. Knowledge of unsaturated groundwater flow is very important in various aspects of groundwater hydrology, viz:

(a) soil water movement in the upper unsaturated zone above the groundwater table,
(b) evaporation at the land surface and rainfall, and
(c) infiltration within soil structures, such as dikes and levees.

As groundwater hydrology is defined as the science of occurrence, movement, and distribution of water below the ground surface; knowledge of vadose zone is very much needed. The present chapter deals with vadose zone and its related topics, such as soil moisture and the governing equations of flow through unsaturated media along with recharge estimation. Also, when the groundwater rises, it occupies a part of previously existing unsaturated zone; and when it falls, it extends the existing unsaturated zone. For this reason, the subject of groundwater balance is included here. (Groundwater budget)

10.2 POROSITY AND WATER CONTENT

The subsurface water is soaked in by the openings in rock or soil. That portion of the soil which is not occupied by solids is known as "voids, interstices, pores, or pore space". Effective porosity may be defined as the porosity due to interconnected pores that can be filled with and emptied of water. Henceforth, whenever porosity is used, effective porosity will be assumed. The pore space is of fundamental importance to the study of subsurface water, since it serves as conduits for water. The pores are usually characterized by their shape, size, irregularity, and distribution.

The total unit volume of a soil (V_T) can be thought as a sum of the volume of solids (V_s) and the volume of voids (V_v) as shown in Fig. 10.1. Further, the volume of voids V_v can be divided into two portions; volume of voids occupied by water (V_w) and the volume of voids occupied by air (V_a). Then, the porosity (η) of a soil is defined as the ratio of volume of voids and the total volume, i.e.,

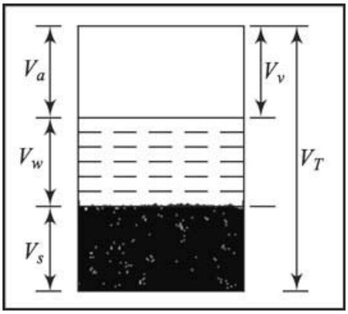

Fig. 10.1 *Three-phase soil system*

$$\eta = \frac{V_v}{V_T} = \frac{V_w + V_a}{V_T} \tag{10.1}$$

The water content (θ) is defined as the ratio of volume of water V_w present in a soil and the total volume.

$$\theta = \frac{V_w}{V_T} \tag{10.2}$$

If all the voids are occupied by water, then the soil is said to be fully saturated. In saturated condition, V_w becomes V_v and θ becomes η.

The ratio of volume of water to volume of voids is defined as the degree of saturation (S), i.e.,

$$S = \frac{V_w}{V_v} = \frac{V_w}{V_w + V_a} \tag{10.3}$$

Using Eqs. (10.1), (10.2), and (10.3), it can be shown that:

$$\theta = S\eta \tag{10.4}$$

Whenever field soil samples are collected, the soil moisture is reported as the ratio of weight of water in the sample and the dry weight of the sample; and this ratio is called the *dry weight moisture fraction* (W).

$$W = \frac{W_w}{W_d} = \frac{\text{(weight of wet sample)} - \text{(weight of dry sample)}}{\text{weight of dry sample}} \tag{10.5}$$

where, W_w is the weight of water in the sample, and W_d is the dry weight of the sample.

Dry weight soil moisture fraction (W) and the water content (θ) are related as:

$$\theta = \gamma_b W \tag{10.6}$$

where, γ_b is the bulk specific weight of the dry soil. Also, γ_b is related to the specific weight of the soil particles (γ_s) by the following relation:

$$\gamma_b = \gamma_s(1 - \eta) \tag{10.7}$$

Example 10.1 A wet soil sample has a volume of 470 cm^3 and a weight of 798 gm. The dry weight is 740 gm, and the specific weight of the soil particles is 2.66. Determine the porosity, soil moisture fraction, water content, and degree of saturation.

Solution

$$\gamma_b = \frac{\text{weight of dry sample}}{\text{volume of the sample} \times \text{weight of sample water}}$$

$$= \frac{740 \text{ gm}}{470 \text{ cm}^3 \left(1 \text{ gm/cm}^3\right)} = 1.57$$

Solving for porosity (η) from Eq. (10.7), we get:

$$\eta = 1 - \frac{\gamma_b}{\gamma_s} = 1 - \frac{1.57}{2.66} = 1 - 0.59 = 0.41$$

From Eq. (10.5), soil moisture fraction $W = \dfrac{798 - 740}{740} = 0.078$

From Eq. (10.6), water content $\theta = \gamma_b\, W = 1.57 \times 0.078 = 0.122 = 12.2\%$

From Eq. (10.4), degree of saturation $S = \dfrac{\theta}{\eta} = \dfrac{0.122}{0.41} = 0.298$

Example 10.2 The following data is collected from the soil moisture zone of a field site. Calculate the volume of water available in the entire soil profile.

Soil depth (mm)	γ_b	*W*
0–250	1.35	0.15
250–500	1.55	0.20
500–850	1.45	0.16
850–1100	1.50	0.13

Solution

The water content (θ) in each layer is computed using Eq. (10.6). The volume of water in each layer is computed by multiplying θ with the layer thickness.

Soil depth (mm)	γ_b	*W*	θ	*Volume of water (mm)*
250	1.35	0.15	0.2025	50.625
250	1.55	0.2	0.31	77.5
350	1.45	0.16	0.232	81.2
250	1.50	0.13	0.195	48.75
				Water available in the entire soil profile = 268.075

Example 10.3 The soil core was drawn with a core sampler having an inside dimension of 5 cm diameter and 15 cm length from a field, 2 days after the irrigation event when the soil water was near field capacity. The weight of the core sampler with fresh soil sample was 1.95 kg and the weight of the same on oven drying was 1.84 kg. The empty core sampler weighed 1.4 kg. Calculate the (a) bulk density of the soil, (b) water holding capacity of the soil in % on volume basis, and (c) depth of water held per metre depth of soil.

Solution

Weight of the moist soil core = 1.95 – 1.4 = 0.55 kg
Weight of oven-dried soil core = 1.84 – 1.4 = 0.44 kg
Therefore, soil water content = (0.55 – 0.44)/0.44 = (0.11/0.44) × 100 = 25%

(a) Volume of soil core = $\pi r^2 h = 294.64$ cm^3
Therefore, the bulk density = (0.44 × 1000)/294.64 = 1.51 gm/cm^3

(b) Water holding capacity of the soil
= (soil water content on weight basis) × (bulk density)
= 25 % × 1.51
= 37.75 %

(c) Water holding capacity of the soil per meter depth of soil = 37.75 cm

10.3 MEASUREMENT OF SOIL WATER

Numerous techniques have been developed for measuring soil moisture content (θ) in the soil profile. Soil water content can be determined by the following two methods:

(1) Direct method, and
(2) Indirect method.

10.3.1 Direct Method

Direct method is based on measurements of soil moisture by experimental means. Here, the output of experiment is soil moisture and is obtained directly.

Gravimetric Method

The gravimetric method of soil water determination is the most standard method to which all other methods are compared. This method requires a simple equipment, but it is time-consuming and involves destructive procedure. This method consists of the following steps:

(i) Soil samples are collected, weighing in between 100 and 200 gm.
(ii) The soil samples are put in a container of known weight and sealed airtight.
(iii) Then, the samples are taken to the lab and opened, put in an oven and dried at 105 °C until all the water has evaporated.
(iv) After drying, the samples are weighed again.
(v) The moisture content on a weight basis is the difference between wet and dry weights divided by dry weight.

10.3.2 Indirect Methods

In case of indirect methods, experiments produce outputs which need to be converted into soil moisture using some additional relationship or equations, as explained below.

Time-Domain Reflectometry

This method is very useful in the field to make continuous measurements on the same area. There has been a rapidly growing interest in non-radioactive methods to determine soil water content using geophysical properties of the soil. One of these methods, time domain reflectometry (TDR), is based on measuring the dielectric constant of soil. Topp et al. (1980), who first proposed the method for soil water investigations, showed that the dielectric constant of soil is dependent primarily upon the water content through a nearly universal calibration equation that is very insensitive to the soil type. Most soil minerals have a dielectric constant of less than 5, whereas water has an exceptionally high dielectric constant of about 78. Thus, the water content variations should be easily detectable from measured variations in dielectric constants.

To measure the dielectric constants in situ using TDR requires determining the propagation velocity of an electromagnetic wave through the soil. The velocity of travel along the transmission lines is described by the following equation:

$$v = c/\sqrt{K_{ad}}$$

where, c is the propagation velocity of an electromagnetic wave in free space (3×10^8 m/s), and K_{ad} is the apparent dielectric constant. The velocity is determined by dividing twice the length of the transmission line by the duration of travel. Since v and c are known, K_{ad} can be easily derived from the above equation. Then, one can use the generic calibration equation given below, to compute the water content:

$$\theta = -0.053 + 0.029 K_{ad} - [(5.5 \times 10^{-4})\,(K_{ad})^2 + [(4.3 \times 10^{-6})(K_{ad})^3]$$

It is to note that water content is not measured directly. The velocity of electromagnetic energy through soil is measured from the above experiment. Using the relations given above, the soil moisture content can be obtained

Neutron Probe

When a source of fast neutrons is placed in a medium of moist soil, the emitted neutrons collide with the nuclei of the medium, thereby losing energy. Neutrons that have been slowed down are called *thermal* or *slow neutrons*. The ability of the nuclei in the moist soil medium to reduce the speed of neutrons varies considerably, but the ability of the soil nuclei is less compared to hydrogen. Therefore, the presence of more hydrogen nuclei in the soil will slow down more emitted neutrons (emitted from the source), and a greater number of slow neutrons will be concentrated near the neutron source. If the soil medium surrounding the neutron source is low in soil moisture (which means that the soil medium is also low in hydrogen nuclei), then the cloud of slow neutrons surrounding the neutron sources will be less dense compared to soil medium of higher moisture content. The manufacturers of neutron probe equipment furnish a calibrating curve that relates neutron count rate to volumetric soil moisture content (θ).

Frequency Domain Reflectrometry

Like time domain reflectometry, frequency domain reflectrometry (FDR) is a non-nuclear method to determine in situ soil water content from the dielectric properties. FDR is a borehole logging technique in which the sensor, comprised of two conductive electrodes separated by an insulating spacer, is lowered in a boring lined with non-conductive plastic casing (thin-walled polyvinyl chloride). The sensor establishes a resonating electromagnetic field, the frequency of which depends upon the dielectric constant of the soil. The water content is determined through calibration with the apparent dielectric constant.

10.4 VERTICAL DISTRIBUTION OF SUBSURFACE WATER

The occurrence of subsurface water below the ground surface may broadly be divided into two zones:

(1) Zone of aeration, and
(2) Zone of saturation.

In zone of aeration, soil pores are partly filled with water and partly with air. In zone of saturation, all the soil pores are filled water under hydrostatic pressure.

10.4.1 Zone of Saturation

The saturated zone extends from the upper surface of saturation down to underlying impermeable rock; as shown in Fig. 10.2. In the absence of overlying impermeable layer, water table or phreatic surface becomes the upper surface of zone of saturation. The water table or phreatic surface is defined as the surface where the water pressure equals atmospheric pressure. The water table appears as the level at which the water stands in a penetrating well. The water appearing in the zone of saturation is termed as groundwater. The pressure in the zone of saturation is always more than or equal to atmosphere pressure.

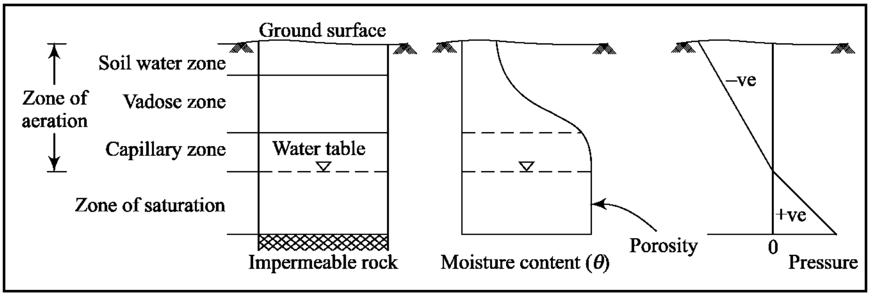

Fig. 10.2 *Vertical distribution of subsurface water*

10.4.2 Zone of Aeration

In the zone of aeration, *vadose water* (gravitational water or free water) occurs. This zone is further classified into three zones: (i) soil water zone, (ii) vadose zone, and (iii) capillary zone, as shown in Fig. 10.2.

Soil Water Zone

Soil water zone extends from the ground surface down through the major root zone. It is the zone which encompasses the roots and its thickness depends on the type of vegetation. Water in the soil water zone exists at less than saturation except temporarily when excess water reaches the ground surface as rainfall or irrigation. Since the soil water zone contains the plant roots, agriculturists and soil scientists have studied soil moisture distribution and movement extensively in this zone.

Vadose Zone or Unsaturated Zone

The vadose zone extends from the bottom of soil water zone to the upper surface of the capillary zone (Fig. 10.2). The thickness of vadose zone may vary from few centimeters to more than 100 m depending upon the depth of water table. This zone acts as a connection between the soil water zone near the ground surface and the zone of saturation. The pressure in the unsaturated zone is less than atmospheric (negative). This reflects the fact that water in the unsaturated zone is held in the soil pores under surface tension forces. The negative pressures in the unsaturated zone are usually determined with tensiometers. In principle, a tensiometer consists of a porous cup attached to an airtight, water filled tube. The porous cup, when inserted into the soil at the desired position, comes in contact with the soil water and reaches hydraulic equilibrium. The process involves the passage of water through the porous cup from the tube into the soil. The vacuum created at the top of the airtight tube is a measure of the pressure in the soil. The pressure is usually measured by a vacuum gage attached to the tube.

Capillary Zone

The capillary zone, also called capillary fringe, extends from the water table up to the height of capillary rise of water. The water is held in the capillary zone due to surface tension forces. In capillary fringe, the pores are saturated, but the pressure is less than atmospheric. If a soil pore could be considered as a capillary tube, the capillary rise h_c (Fig.10.3) can be derived from the equilibrium between surface tension of water and the weight of water raised.

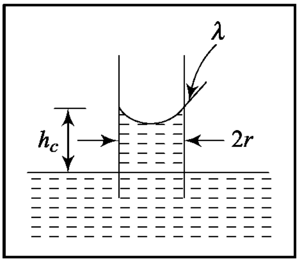

Fig. 10.3 *Capillary rise in a capillary tube*

$$\left.\begin{array}{l}\text{Component of surface tension}\\ \text{force in vertical direction}\end{array}\right\} = 2\pi r \tau \cos\lambda \tag{10.8}$$

Here, τ is surface tension, r is radius of the capillary tube, and λ is angle of contact between the meniscus and the wall of the tube.

$$\left.\begin{array}{l}\text{Weight of water raised}\\ \text{in the capillary tube}\end{array}\right\} = \pi r^2 h_c \gamma_w \tag{10.9}$$

where, γ_w is the specific weight of water.

Equating Eqs. (10.8) and (10.9) for equilibrium, we have:

$$h_c = 2\pi r\,\tau \cos\lambda = \pi r^2\, h_c\, \gamma_w$$

$$h_c = \frac{2\tau\cos\lambda}{r\gamma_w} = \frac{2\tau}{f\gamma_w}\cos\lambda \tag{10.10}$$

It follows from Eq. (10.10) that the capillary rise will vary inversely with the radius of capillary tube or the pore size. Table 10.1 presents the measurements of capillary rise in unconsolidated materials.

Table 10.1 Capillary rise in samples of unconsolidated materials (Lohman, 1972)

Material	*Grain Size (mm)*	*Capillary Rise (cm)*
Fine Gravel	5–2	2.5
Very Coarse Sand	2–1	6.5
Coarse Sand	1–0.5	13.5
Medium Sand	0.5–0.2	24.6
Fine Sand	0.2–0.1	42.8
Silt	0.1–0.05	105.5
Silt	0.05–0.02	200

10.5 EQUATION OF UNSATURATED FLOW

The equation governing the unsaturated flow through a soil can be obtained by combining Darcy's law with the continuity equation. This section describes the development of equations for (i) unsteady state unsaturated flow, and (ii) steady state unsaturated flow.

Consider an elemental control volume of length *dx*, *dy*, and *dz* in co-ordinate directions as shown in Fig. 10.4.

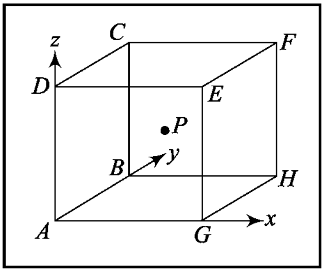

Fig. 10.4 *Elemental control volume*

Let *P* be the centre of the elemental volume, and let h and θ be the hydraulic head and the moisture content in the elemental volume, respectively. Also, let u, v, and ω be the Darcy velocities at *P* in x, y, and z directions, respectively.

$$\left.\begin{array}{l}\text{Volume of water entering}\\ \text{the face ABCD to the frome}\\ \text{lemental volume in time } dt\end{array}\right\} = \left(u - \frac{\partial u}{\partial x}\cdot\frac{dx}{2}\right)(dy\ dz)\ dt \tag{10.11}$$

$$\left.\begin{array}{l}\text{Volume of water leaving from}\\ \text{the face EFGH out of the}\\ \text{elemental volume in time } dt\end{array}\right\} = \left(u + \frac{\partial u}{\partial x}\cdot\frac{dx}{2}\right)(dy\ dz)\ dt \tag{10.12}$$

$$\left.\begin{array}{l}\text{Net volume of water entering}\\ \text{the elemental volume in}\\ x\text{–direction in time } dt\end{array}\right\} = \frac{-\partial u}{\partial x}\,(dx\; dy\; dz)\; dt \tag{10.13}$$

$$\left.\begin{array}{l}\text{Net volume of water entering}\\ \text{the elemental volume in}\\ y\text{–direction in time } dt\end{array}\right\} = \frac{-\partial v}{\partial y}\,(dx\; dy\; dz)\; dt \tag{10.14}$$

$$\left.\begin{array}{l}\text{Net volume of water entering}\\ \text{the element volume in}\\ z\text{–direction in time } dt\end{array}\right\} = \frac{-\partial \omega}{\partial z}\,(dx\; dy\; dz)\; dt \tag{10.15}$$

$$\left.\begin{array}{l}\text{Net volume of water}\\ \text{entering the elemental}\\ \text{volume in all directions}\\ \text{in time } dt\end{array}\right\} = \left(-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial \omega}{\partial z}\right)(dx\; dy\; dz)\; dt \tag{10.16}$$

If $\partial\theta$ is the change in the moisture content in the control volume in time *dt* due to the imbalance in the inflows and outflows, then

$$\left.\begin{array}{l}\text{Change in storage in the}\\ \text{elemental volume in time } dt\end{array}\right\} = (\partial\theta)\,(dx\; dy\; dz) \tag{10.17}$$

From the equation of continuity, the change in storage in time *dt* must be equal to the net inflows in time *dt*. Thus, equating Eqs. (10.16) and (10.17) and simplifying, we have:

$$-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial \omega}{\partial z} = \frac{\partial \theta}{\partial t} \tag{10.18}$$

The Darcy velocities, u, v, and ω, are related to the hydraulic head (h) as follows:

$$u = (-k_x)\frac{\partial h}{\partial x}, \quad v = (-k_y)\frac{\partial h}{\partial y}, \quad \text{and} \quad \omega = (-k_z)\frac{\partial h}{\partial z} \tag{10.19}$$

where, k_x, k_y, and k_z are the unsaturated hydraulic conductivities in x, y, and z directions, respectively.

Substituting the values of u, v, and ω from Eq. (10.19) into Eq. (10.18), we have:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial h}{\partial z}\right) = \frac{\partial \theta}{\partial t} \tag{10.20}$$

The total head (h) is the sum of pressure head (ψ) and the elevation head (z). Since the velocity is very small, velocity head is neglected.

$$h = \psi + z \tag{10.21}$$

Substituting for h in Eq. (10.20), we have:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial \psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y \frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z \frac{\partial \psi}{\partial z}+1\right)=\frac{\partial \theta}{\partial t} \tag{10.22}$$

Equation (10.22) is also known as *Richards' equation*. Equation (10.22) is the governing equation for three-dimensional unsaturated flow in a soil. Equation (10.22) is nonlinear in nature as the unsaturated hydraulic conductivities K (k_x, k_y, and k_z) and the water content (θ) are functions of the pressure head (ψ). Further, it has been observed experimentally that ($\theta - \psi$) relationship is hysteretic; it has a different shape when soils are wetting than when they are draining (Poulovassilis 1962). The variation of moisture content θ and hydraulic conductivity K with pressure head ψ was studied by Brooks and Corey (1964) and Van Genuchten (1980).

10.5.1 Brooks and Corey Relationships

Brooks and Corey (1964), after laboratory tests of many soils, found that the pressure head can be expressed as a logarithmic function of effective saturation S_e and proposed the following relation.

$$S_e = \begin{cases} \left(\dfrac{\psi_b}{\psi}\right)^{\lambda} & \text{for } (\psi \le \psi_b) \\ 1 & \text{for } (\psi > \psi_b) \end{cases} \tag{10.23}$$

where, ψ_b is the bubbling pressure, λ is the pore-size index, and S_e is the effective saturation defined as:

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} \tag{10.24}$$

where, θ_s is saturated water content, and θ_r is the residual water content. θ_s is generally taken as the porosity (η) of the soil. θ_r is the water content of the soil after it has been thoroughly drained.

They also proposed the following relationship between K and ψ.

$$K = \begin{cases} K_{\text{sat}}\left(\dfrac{\psi_b}{\psi}\right)^{3+\lambda} & \text{for } (\psi \le \psi_b) \\ K_{\text{sat}} & \text{for } (\psi > \psi_b) \end{cases} \tag{10.25}$$

where, K_{sat} is the saturated hydraulic conductivity of the soil.

10.5.2 Van Genuchten Relationships

The main disadvantage of the Brooks and Corey relationship is the discontinuity in the relationship at ψ_b. Van Genuchten (1980) proposed an exponential relationship which overcomes the limitation of Brooks and Corey relationship.

The relationship is as follows:

$$S_e = \begin{cases} \left[\dfrac{1}{1+|\alpha\psi|^n}\right]^m & \text{for } (\psi \le 0) \\ 1 & \text{for } (\psi > 0) \end{cases} \tag{10.26}$$

where, α and n are unsaturated soil parameters with $m = \left(1 - \dfrac{1}{n}\right)$, and S_e is the effective saturation as defined earlier.

He also presented the following relationship between K and S_e as:

$$K = K_{\text{sat}} (S_e)^{1/2}\left[1 - \{1 - (S_e)^{1/m}\}^m\right]^2 \tag{10.27}$$

10.5.3 Rawls and Brakensiek approach

Let c be % clay ($5 < \% < 60$), s as % sand ($5 < \% < 70$), and ϕ = porosity (volume fraction). One can use regression equations developed by Rawls and Brakensiek (1982) to compute a variety of parameters.

Brooks-Corey Bubbling Pressure

$$\begin{aligned} h_b = \exp\,[&5.3396738 + 0.1845038(c) - 2.48394546(\phi) \\ &- 0.00213853(c)^2 - 0.04356349(s)(\phi) \\ &- 0.61745089(c)(\phi) + 0.00143598(s^2)(\phi^2) \\ &- 0.00855375(c^2)(\phi^2) - 0.00001282(s^2)(c) \\ &+ 0.00895359(c^2)(\phi) - 0.00072472(s^2)(\phi) \\ &+ 0.0000054(c^2)(s) + 0.50028060(\phi^2)(s) \\ &+ 0.50028060\,(\phi^2)(c)] \end{aligned}$$

Brooks-Corey Pore-size Distribution Index

$$\begin{aligned} \lambda = \exp\,[&-0.7842831 + 0.0177544(s) - 1.062498(\phi) \\ &- 0.00005304(s^2) - 0.00273493(c^2) + 1.11134946(\phi^2) \\ &- 0.03088295(s)(\phi) + 0.00026587(s^2)(\phi^2) \\ &- 0.00610522(c^2)(\phi^2) - 0.00000235(s^2)(c) \\ &+ 0.00798746(c^2)(\phi) - 0.00674491(\phi^2)(c)] \end{aligned}$$

Brooks-Corey Residual Water Content (volume fraction)

$$\begin{aligned} \theta_r = [&-0.0182482 + 0.00087269(s) + 0.00513488(c) \\ &+ 0.02939286(\phi) - 0.00015395(c^2) - 0.0010827(s)(\phi) \\ &- 0.00018233(c^2)(\phi^2) + 0.00030703(c^2)(\phi) \\ &- 0.0023584(\phi^2)(c)] \end{aligned}$$

Brooks-Corey Effective Saturation

$$\begin{aligned}\theta = [&0.01162 - 0.001473(s) - 0.002236(c) + 0.98402(\phi)\\ &+ 0.0000987(c^2) + 0.03616(s)(\phi) - 0.010859(c)(\phi)\\ &-0.00096(c^2)(\phi) - 0.0024372(\phi)(s)\\ &+ 0.0115395(\phi^2)(c)]\end{aligned}$$

Saturated Hydraulic Conductivity

$$\begin{aligned}K_s = \exp [&19.52348(\phi) - 8.96847 - 0.028212(c) + 0.00018107(s)^2\\ &- 0.0094125(c) - 8.395215(\phi^2) + 0.077718(s)(\phi)\\ &- 0.00298(s^2)(\phi^2) - 0.019492(c^2)(\phi^2)\\ &+ 0.0000173(s^2)(c) + 0.02733(c^2)(\phi) + 0.001434(s^2)(\phi)\\ &- 0.0000035(c^2)(s)]\end{aligned}$$

Example 10.4 The relevant unsaturated soil parameters for clay are given in the table below. Plot the variation of water content (θ) and the unsaturated hydraulic conductivity (K) as a function of pressure head (ψ) for ψ values ranging from –1000 cm to 0 cm. Use Van Genuchten relationships.

Soil	α (m^{-1})	η	K_{sat} (m/sec)	θ_s	θ_r
Clay	0.8	1.09	5.56×10^{-7}	0.46	0.068

Solution

Water content θ and unsaturated hydraulic conductivity K are computed using Eqs. (10.26) and (10.27). They are plotted in Figs. 10.5 and 10.6, respectively.

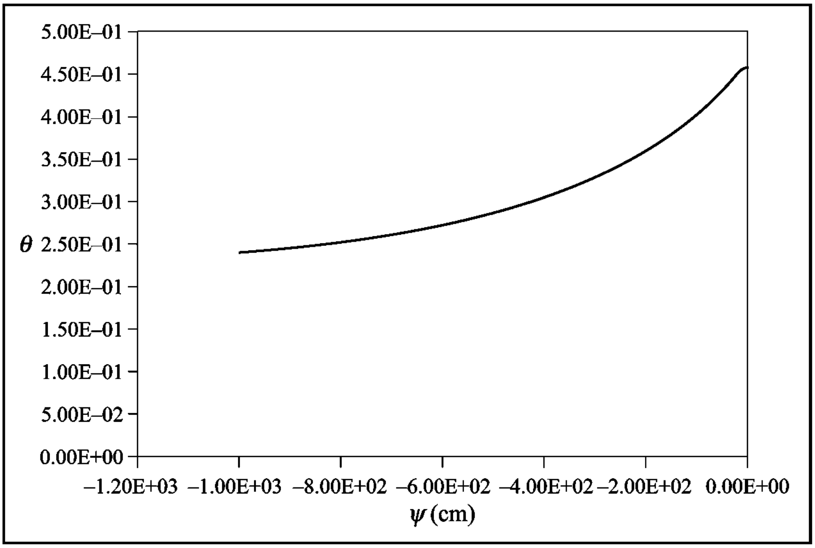

Fig. 10.5 *ψ vs. θ for clay*

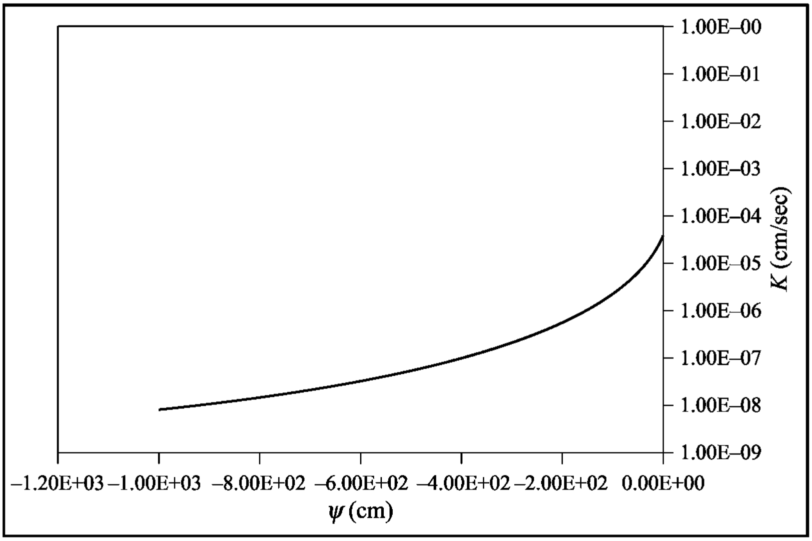

Fig. 10.6 *ψ vs. K for clay*

10.6 GROUNDWATER RECHARGE ESTIMATION

Groundwater recharge is the amount of water which reaches the water table, either by direct contact in the *riparian zone* (transitional area between the water and the surrounding lands) or by downward percolation through the overlying zone of aeration. Groundwater recharge estimation can be made using direct and indirect methods. In the direct method, tracers are used to obtain the estimate of recharge. This method is localized and its applicability to regional studies offer difficulties. The indirect methods used to estimate rates can be divided mainly into three groups [Sophocleous, 1991].

1. Hydrometerologic and soil crop data processing to determine the soil moisture balance. In this method, the recharge is usually estimated as the remainder when the losses identified in the form of runoff and evaporation have been deduced from the precipitation.
2. Hydrological data interpretation including analysis of water table fluctuations. In this method, the changes in seasonal water table level are utilized to estimate groundwater recharge.
3. Soil physics measurements involving the estimation of water flux beneath the root zone using unsaturated hydraulic conductivity and hydraulic potential gradients.

The third indirect method of estimating groundwater recharge is of relevance to this chapter.

Computation of Soil Water Flux

The water flux (q) at any depth can be computed using Darcy's law, as:

$$q = -K\left(\frac{\partial h}{\partial z}\right) \tag{10.28}$$

where, K is the hydraulic conductivity, h is the hydraulic head, and z is the vertical coordinate. Neglecting the velocity head, the hydraulic head (h) can be treated as the sum of pressure head (ψ) and elevation head (z).

$$h = \psi + z \tag{10.29}$$

Substituting the value of h from Eq. (10.29) in Eq. (10.28), we obtain:

$$q = -K\left[\frac{\partial}{\partial z}(\psi + z)\right] \tag{10.30}$$

Soil water flux can be determined using either Eq. (10.28) or (10.30), given the measurements of pressure head (ψ) at different depths (z) in the soil, and the relationship between hydraulic conductivity (K) and the pressure head (ψ) are known.

Example 10.5 The data of hydraulic heads h and pressure heads ψ at two different depths are given at weekly intervals in the following table. For this soil, the relationship between hydraulic conductivity and the pressure head is $K = 240(-\psi)^{-2.1}$ where K is in cm/day and ψ is in cm. Compute the daily soil moisture flux and the total groundwater recharge per unit area.

Week	*Hydraulic head h at 0.8 m (cm)*	*Hydraulic head h at 1.8 m (cm)*	*Pressure head ψ at 0.8 m (cm)*	*Pressure head ψ at 1.8 m (cm)*
1	–140	–235	–60	–55
2	–165	–240	–85	–50
3	–135	–240	–65	–60
4	–140	–240	–70	–65
5	–125	–230	–45	–70
6	–100	–225	–25	–55
7	–120	–215	–30	–60
8	–110	–230	–30	–70

Solution

The average soil water flux between two measurement points 1 and 2 is written using Eq. (10.28) as:

$$q_{12} = -k\left(\frac{h_1 - h_2}{z_1 - z_2}\right)$$

Considering point 1 is at 0.8 m and point 2 at 1.8 m, we have:

$$z_1 = -80 \text{ cm} \quad \text{and} \quad z_2 = -180 \text{ cm}$$

Hence, $z_1 - z_2 = -80 - (-180) = 100$ cm

The hydraulic heads (h_1 and h_2) and the pressure heads (ψ_1 and ψ_2) at these depths are given.

The hydraulic conductivity K varies with ψ, so the value corresponding to the average value of ψ values at 0.8 m and 1.8 m is used.

The computations are as shown below:

Week	*Total head h_1 at 0.8 m*	*Total head h_2 at 1.8 m (cm)*	*Pressure head ψ_1 at 0.8 m (cm)*	*Pressure head ψ_2 at 1.8 m (cm)*	*Average ψ (cm)*	*Average K (cm/day)*	*Head difference $(h_1 - h_2)$ (cm)*	*Water flux (cm/day)*
1	–140	–235	–60	–55	–57.5	0.04841	95	–0.04599
2	–165	–240	–85	–50	–67.5	0.03457	75	–0.02593
3	–135	–240	–65	–60	–62.5	0.04063	105	–0.04266
4	–140	–240	–70	–65	–67.5	0.03457	100	–0.03457
5	–125	–230	–45	–70	–57.5	0.04841	105	–0.05083
6	–100	–225	–25	–55	–40	0.10373	125	–0.12966
7	–120	–215	–30	–60	–45	0.081	95	–0.07695
8	–110	–230	–30	–70	–50	0.06492	120	–0.07790
								Total flux = –0.4845

The flux is negative because the water is flowing downward.

Groundwater recharge per unit area in eight weeks = $q \times t$

$$= (-0.4845) \times 8 \times 7$$
$$= -27.13 \text{ cm}$$

Soil water flux can be determined using either Eq. (10.28) or (10.30), given the required parameters are known.

Exercise: Compute the groundwater recharge by computing K values corresponding to ψ_1 and ψ_2 and taking average K. It is to note that average K is computed based on average ψ.

The properties of zone of saturation and zone of aeration are summarized in the following table.

Zone of Saturation	*Zone of Aeration*
• It occurs below the water table.	• It occurs above the water table and above the capillary fringe.
• The soils pores are completely filled with water and the moisture content θ equals the porosity η.	• The soil pores are partly filled with water and partly with air. The moisture content θ is less than the porosity η.
• The water pressure is more than atmospheric and the pressure head ψ is greater than zero.	• The water pressure is less than atmospheric and the pressure head ψ is less than zero.
• Piezometers are used to measure the hydraulic head.	• Tensiometers are used to measure the hydraulic head.
• The hydraulic conductivity is constant, and it does not depend on pressure head ψ.	• The hydraulic conductivity k and the moisture content θ are both functions of the pressure head ψ.

10.7 GROUNDWATER BUDGETING

The groundwater potential of an area depends mainly on geological features of that area. The existence of highly pervious or permeable rock indicate significant yield of water. The presence of fissures, joints, bedding planes, faults, shear zones and cleavages, solution cavities, etc. contribute to the perviousness of the rock. Weathered basalts and sand stones form a good source of water. The most important water yielding formations are unconsolidated gravels, sands, alluvium, lake sediment, glacial deposits, etc. So, if there is available information about the geology of the area, one can be aware of the possible potential or strength of the groundwater reservoirs.

Due to limitations of water resources available and because of population explosion or poor rainfall, hydrologists all over the world are trying to utilize these groundwater reservoirs to their full extent. In certain locations, the groundwater levels have been depleting very fast. A groundwater level, let it be the water table of an unconfined aquifer or the piezometric surface of a confined aquifer, indicates the elevation up to which the water table will rise in a well.

Essentially, a groundwater balance study is a mathematical representation of various types of inflows and outflows, which may help in deciding spatial and temporal variation of groundwater levels. Thus, models of groundwater balance may be numerous, depending on their use in quantifying different types of inflows and outflows. However, in this chapter, we restrict the class of models to the estimation of groundwater inflows and outflows. It is needless to emphasize that groundwater balance models are equally useful to the overall water balance of a watershed.

10.7.1 Computation of Groundwater Flows

The groundwater outflow for different seasons along the boundaries of the study area can be calculated using measured water levels and transmissivity estimates from pumping tests. To explain the computation of groundwater flow, a study area (Ojha, 1984) is considered as shown in Fig. 10.7. There are several borewells (BW) and dug wells or large-diameter wells (DW) in the study area. It also consists of a marshy land, which is formed due to standing water from waste water drain and lies in between DW1 and DW2. Based on pumping test analysis of several wells, it had been possible to estimate transmissivity.

Referring to Fig. 10.7 and based on Darcy's law, the net flow across the boundary is given by:

$$\text{Net outflow} = \int_{t_0}^{t_1} \int_{c} T \frac{\partial h}{\partial n} \, dl dt \tag{10.31}$$

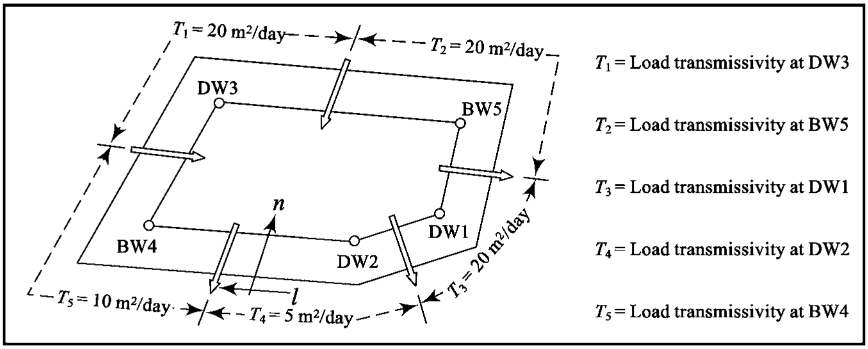

Fig. 10.7 *Definition sketch for net outflow calculation*

where, the net outflow is calculated for the duration from t_0 to t_1 days, l and n are directions along the boundary and normal to the boundary (n is the inward normal), h is the piezometric head, and T is the local transmissivity. Thus, to determine the groundwater level, contours have to be drawn for different periods. To achieve this, available groundwater level variations in different wells of the study area can be used.

Exercise: Identify the inflow and outflow boundaries in Fig. 10.7. Ascertain whether there will be depletion in groundwater level. (Hint: Compute inflows and outflows for a feasible distribution of piezometric heads in and around the study area. Depletion will occur if total outflow is more than total inflow. Please remember that no other source of recharge is considered here.)

10.7.2 Groundwater Balance Equation

If a groundwater balance equation is written for a closed region for any specified period, then

$$\text{Total recharge} = (\text{Area} \times h_r \times S_y) + (\text{Area} \times V_p \times f) + \text{Net outflow} \tag{10.32}$$

where, h_r is the average rise in water level in the region over the area in the specified period, S_y is the specific yield, V_p is the annual draft in the region for unit area, f is the proportion of the annual draft withdrawn in the specified period, and net outflow refers to groundwater outflow from the region.

Equation (10.32) ignores evapotranspiration. However, if significant, it can be added to the right-hand side of Eq. (10.32). Under ideal conditions, Eq. (10.32) can be used to calculate specific yield, S_y, for periods in which no rainfall takes place. If recharge can be taken as zero during dry periods; the two components, i.e., pumping and net outflow can be calculated. The specific yield, S_y, can be calculated by equating the sum of these two components to the decrease in storage. These computations are done most conveniently in a tabular format, as shown in Table 10.2.

Table 10.2 Determination of recharge from rainfall

Period (from) – (to)	Δt
Total pumping (TP) over the period (m^3)	TP
Average change in groundwater level over the period (m)	h_f
Average change in groundwater storage (ACGWS)	ACGWS = $(\Delta H)(S_y)$ Area
Net groundwater outflow (NGWO) for the period (m^3)	NGWO
Waste water recharge (WWR) in m^3	WWR
Recharge from rainfall (RR) over the study area (m^3)	RR = (TP + ACGWS + NGWO) – WRR

Example 10.6 Determine the net groundwater outflow if the total pumping is 1.5 m^3 and the waste water recharge is 1.5 m^3 over an area of 500 m^2 during a certain period. The rainfall recorded during that period is 100 mm. Average rise in groundwater level over that period is 0.5 m. The specific yield of that aquifer is 0.05.

Solution

Average change in groundwater storage (ACGWS)

$$= (\Delta H)\ (S_y)\ \text{Area}$$
$$= 0.5 \times 0.05 \times 500$$
$$= 12.5\ \text{m}^3$$

Since there is 100 mm rainfall during that period, recharge from rainfall (RR) = $0.1 \times 500 = 50.0$ m^3

Waste water recharge (WWR) = 1.5 m^3

$$\text{Net groundwater outflow (NGWO)} = [\text{RR} - (\text{TP} + \text{ACGWS})] + \text{WWR}$$
$$= [50.0 - (1.5 + 12.5)] + 1.5$$
$$= 37.5\ \text{m}^3$$

Exercise: Estimate the net groundwater outflow when average fall in groundwater level is 0.5 m. Other data remain same except the fact that there is a drop in groundwater level. (Hint: Consider ΔH as negative)

Example 10.7 During a specific period, the average rise in water level in a region of 300 m^2 is 0.7 m. Net groundwater outflow during that period is 10 m^3. There is 5cm of rainfall during that period. Total pumping during that period is estimated at 0.85 m^3. Determine the specific yield of the aquifer.

Solution

Average change in groundwater storage (ACGWS)

$$= \text{RR} + \text{WRR} - \text{TP} - \text{NGWO}$$
$$= 300(0.05) + 0.0 - 0.85 - 10.0$$
$$= 4.15\ \text{m}^3$$

$$\begin{aligned}\text{Specific yield of the aquifer} &= \text{ACGWS}/(h_r \times \text{area})\\ &= 4.15/(0.7 \times 300)\\ &= 0.0197\end{aligned}$$

Example 10.8 During a specific period, the average rise in water level in a region of 400 m^2 is 0.5 m. Specific yield of that aquifer is 0.08. Annual draft in that region is 0.25 m. Proportion of annual draft withdrawn in the specified period is 0.3, and net outflow from that region is 2.0 m^3. Determine the total recharge.

Solution

$$\begin{aligned}\text{Total recharge} &= (\text{Area} \times h_r \times S_y) + (\text{Area} \times V_p \times f) + \text{Net outflow}\\ &= (400 \times 0.5 \times 0.08) + (400 \times 0.25 \times 0.3) + 2.0\\ &= 48.0 \text{ m}^3\end{aligned}$$

SUMMARY

Declining groundwater levels have become a matter of great concern; however, hydrologists all over the world are making a constant headway towards finding better methods to utilize underground water resources. This chapter gives a broad overview of the factors responsible for receding underground water levels.

In recent years, use of rainwater harvesting and other means of groundwater recharging, such as waste water flooding have become relevant.

Regression relations pertaining to soil moisture must be used with caution as these are not extensively tested. The flow may not be idealized as one-dimensional in case of field problems. Use of numerical approaches may be adhered while dealing with real-field problems.

Recharge from basins may also lead to development of groundwater mounds which are not covered in this chapter. However, a list of references provided herein is expected to motivate further reading.

EXERCISES

10.1 Define the following terms.

(a) Porosity
(b) Bulk unit weight
(c) Water content
(d) Degree of saturation

10.2 Describe soil as a three-phase system.

10.3 How will you measure the moisture content of soil in a field?

10.4 Derive Richards' equation for the flow of water through unsaturated soil.

10.5 Explain the relationships given by: (a) Van Genuchten, and (b) Brooks and Corey.

10.6 Collect a soil sample from your area and develop moisture versus head relationship.

10.7 Derive the relationship among bulk unit weight, dry unit weight, and water content of the soil.

10.8 A wet soil sample has a volume of 600 cm^3 and a weight of 1450 gm. The dry weight is 1275 gm and the specific weight of the soil particles is 2.68. Determine the porosity, soil moisture fraction, water content, and degree of saturation.

10.9 The following data is collected from the soil moisture zone of a field site. Calculate the volume of water available in the entire soil profile.

Soil depth (mm)	*Bulk unit weight*	*W*
0–200	1.70	0.18
200–450	1.85	0.25
450–800	1.80	0.23
800–1200	1.75	0.20

10.10 Water in a circular tube filled with uniform soil takes x minutes to rise 10% of the maximum height of the capillary rise. How long will it take to reach the same height in the same tube if the soil particles are twice their present diameter? Assume the contact angle is zero and K varies directly with the square of the diameter of the soil particles.

10.11 Determine the net groundwater outflow if the total pumping is 2.0 m^3 and waste water recharge is 1.5 m^3 over an area of 600 m^2 during a certain period. The rainfall recorded during that period is 200 mm. Average fall in groundwater level over that period is 0.3 m. Specific yield of the aquifer is 0.08.

10.12 During a specific period, the average rise in water level in a region of 600 m^2 is 0.25 m. Specific yield of that aquifer is 0.05. Annual draft in that region is 0.25 m. Proportion of annual draft withdrawn in the specified period is 0.25 and the net outflow from that region is 4.0 m^3. Determine the total recharge.

10.13 The relevant unsaturated soil parameters are given below.

$\lambda = 2.89$

Bubbling pressure = –50 cm

Porosity = 0.374

Effective porosity = 0.354

Saturated hydraulic conductivity = 11.78 cm/hr

Plot the following graphs using Brooks-Corey relationships.

(a) ψ values ranging from –750 cm to 0 cm

(b) Soil suction head (ψ) vs. effective saturation (S_e)

(c) Variation of water content (θ) vs. pressure head (ψ)

(d) Variation of water content (θ) vs. the unsaturated hydraulic conductivity (K)

10.14 The relevant unsaturated soil parameters for sand are given below. Plot the variation of water content θ versus the unsaturated hydraulic conductivity K as function of pressure head ψ, for ψ values ranging from –1000 cm to 0 cm. Use van Genuchten relationships. Observe the trend and compare it with the soil parameters of clay plotted in Example 10.4.

Soil	α (m^{-1})	η	K_{sat} (m/sec)	θ_s	θ_r
Sand	14.5	2.68	8.25×10^{-5}	0.43	0.045

10.15 The data of hydraulic heads h and pressure head ψ at two different depths are given at weekly intervals in the Table shown below. For this soil, the relationship between hydraulic conductivity and the pressure head is $K = 250(-\psi)^{-2.11}$ where K is in cm/day and ψ is in cm. Compute the dialing soil moisture flux and the total groundwater recharge per unit area.

Week	*Hydraulic head h at 0.6 m (cm)*	*Hydraulic head h at 1.5 m (cm)*	*Pressure head ψ at 0.6 m (cm)*	*Pressure head ψ at 1.5 m (cm)*
1	–150	–240	–65	–50
2	–160	–250	–75	–55
3	–180	–255	–75	–60
4	–165	–255	–70	–60
5	–140	–235	–55	–35
6	–165	–245	–45	–55
7	–170	–245	–30	–50
8	–180	–250	–60	–55
9	–200	–260	–80	–75
10	–210	–265	–100	–90

OBJECTIVE QUESTIONS

1. If water content and porosity of a soil are 0.345 and 0.4, respectively, what will be its degree of saturation?
(a) 0.754 (b) 0.862 (c) 0.913 (d) 0.897

2. The water content of a fully saturated soil is 0.4 and the total volume of soil is 470 cm^3. Determine its porosity.
(a) 0.188 (b) 0.085 (c) 0.4 (d) 0.3

3. Determine the bulk specific weight of dry soil if specific weight of the soil particles is 2.70 and porosity of that soil is 0.38.
(a) 1.553 (b) 1.674 (c) 1.695 (d) 1.638

4. Determine the value of effective saturation of the soil if pressure head at a point is 1.5 m and van Genuchten soil parameters are: $\alpha = 14.5$ m^{-1} and $n = 2.68$.
(a) 1.0 (b) 0.0056 (c) 0.001 (d) 1.5

5. Determine the value of effective saturation of the soil at a point if saturated moisture content of the soil is 0.45 and residual moisture content of the soil is 0.037. Moisture content at that point at a particular time is 0.15.
(a) 0.5 (b) 0.405 (c) 0.3015 (d) 0.2736

6. The total head at a point, which is at 0.5 m, is 1.5 m. The total head at another point, which is at 1.3 m, is 2.9 m. The hydraulic conductivity of that soil is 7.9 m/day. Calculate the average soil water flux between the two points.
(a) 13.825 (b) 12.543 (c) 15.87 (d) 11.897

7. Determine the value of hydraulic conductivity if the saturated hydraulic conductivity is 7.23 m/day, effective saturation is 0.05, and the soil property $n = 2.0$.
(a) 0.215 (b) 0.235 (c) 0.255 (d) 0.275

8. Calculate the specific yield of an aquifer if the average rise in water level in a region of 200 m^2 is 0.55m and the average change in groundwater storage is 12.5 m^3.
(a) 0.135 (b) 0.113 (c) 0.146 (d) 0.158

9. The total pumping over a specific period of time is 2.5 m^3; waste water recharge is 0.5 m^3; average change in groundwater storage is 5 m^3; and recharge from rainfall is 25 m^3. Calculate the net groundwater outflow over the specified period of time.
(a) 15.0 (b) 18.0 (c) 21.0 (d) 24.0

10. Phreatic surface is the surface where
    (a) Water pressure is equal to the atmospheric pressure
    (b) Water pressure is greater than the atmospheric pressure
    (c) Water pressure is less than the atmospheric pressure
    (d) None of the above

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CHAPTER

11 Numerical Modelling of Groundwater Flow

11.1 INTRODUCTION

We have discussed a number of analytical solutions that are being used for solving various groundwater problems. Even though these analytical solutions are exact solutions, they cannot be applied for problems involving complex boundaries and time-varying boundary conditions.

However, in the real world, the aquifer geometry is irregular and the boundary conditions vary with time. In such situations, analytical solutions will be of little help and one has to resort to numerical techniques. Several types of numerical methods, such as: finite difference, finite element, finite volume, and boundary element methods have been widely used to solve groundwater flow problems.

All numerical methods convert the governing differential equation into a set of linear or nonlinear algebraic equations. This system of equations can be solved using matrix algebra. The nature of conversion from differential form to algebraic form varies from one numerical method to another. For example, finite difference methods make use of Taylor Series approximation for conversion, while the finite element methods use basis functions for the same purpose. In this chapter, *finite difference method* will be discussed in detail to obtain solutions for different types of groundwater flow problems.

11.2 FINITE DIFFERENCE METHOD

Richardson (1910) introduced the finite difference method to calculate an approximate solution to partial differential equations. The purpose of this chapter is to introduce the terminology involved in finite difference methods and apply these methods to solve a few simple problems.

The basic philosophy of the finite difference method is to replace the derivatives at a point by a ratio of the changes in appropriate variables over a small but finite interval. These approximations are then used in place of derivatives in the governing equation which converts the differential equation into a system of algebraic equations.

At first, we discuss the simplest finite difference scheme and progress to those more useful in solving groundwater flow problems. It is to be noted that finite difference methods make use of approximations. However, the resulting inaccuracies in the solution can be made negligibly small through proper use of the methods.

11.3 FINITE DIFFERENCE APPROXIMATIONS

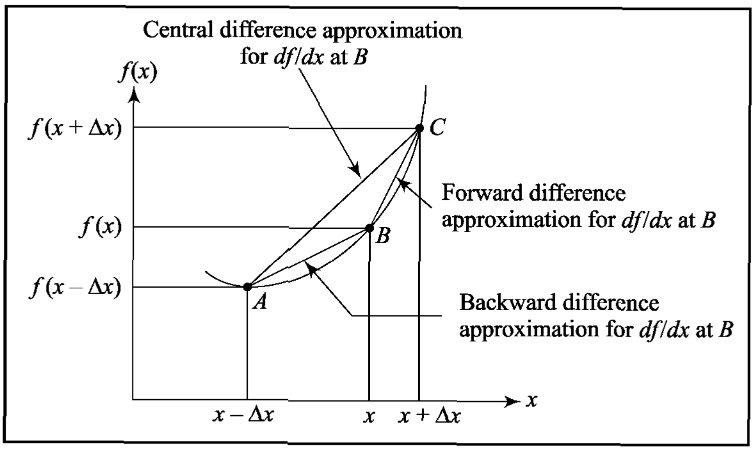


Fig. 11.1 *Geometric interpretation of a finite difference approximation*

Consider a function $f(x)$, sufficiently smooth as shown in Fig. 11.1. Then $f(x)$ may be expanded in a Taylor series about x in the positive direction as:

$$f(x + \Delta x) = f(x) + \frac{df}{dx}\Delta x + \frac{1}{2!}\frac{d^2 f}{dx^2}(\Delta x)^2 + \frac{1}{3!}\frac{d^3 f}{dx^3}(\Delta x)^3 + \cdots \quad (11.1)$$

Equation (11.1) can be solved for $\frac{df}{dx}$ as:

$$\frac{df}{dx} = \frac{f(x + \Delta x) - f(x)}{\Delta x} + 0(\Delta x) \quad (11.2)$$

The term $0(\Delta x)$ denotes the remaining terms of the series. The term A is said to be of the order $(\Delta x)^n$, denoted as $0(\Delta x)^n$; if a positive constant K, independent of Δx, exists such that $|A| < K|(\Delta x)^n|$. As a result, as $|(\Delta x)^n| \to 0$, $A \to 0$.

The forward difference approximation of $\frac{df}{dx}$ is obtained by dropping the $0(\Delta x)$ term in Eq. (11.2), i.e.,

$$\frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{11.3}$$

Similarly, the backward difference approximation of $\frac{df}{dx}$ can be obtained by writing the Taylor series about x in the negative direction.

$$f(x-\Delta x) = f(x) - \Delta x \frac{df}{dx} + \frac{1}{2!}\frac{d^2 f}{dx^2}(\Delta x)^2 - \frac{1}{3!}\frac{d^3 f}{dx^3}(\Delta x)^3 + \cdots \tag{11.4}$$

Solving for $\frac{df}{dx}$ and dropping $0(\Delta x)$, the backward difference approximation for $\frac{df}{dx}$ is written as:

$$\frac{df}{dx} \approx \frac{f(x) - f(x-\Delta x)}{\Delta x} \tag{11.5}$$

The central difference approximation of $\frac{df}{dx}$ is obtained by subtracting Eq. (11.4) from Eq. (11.1) and dropping $0(\Delta x)^2$ term as:

$$\frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2(\Delta x)} \tag{11.6}$$

The central difference approximation is more accurate than forward and backward difference approximations, since it has a truncation error $0(\Delta x)^2$.

The finite difference approximation for the second derivative $\frac{d^2 f}{dx^2}$ is obtained by adding Eqs. (11.1) and (11.4), solving for $\frac{d^2 f}{dx^2}$ and dropping $0(\Delta x)^2$ as:

$$\frac{d^2 f}{dx^2} \approx \frac{f(x-\Delta x) - 2f(x) + f(x+\Delta x)}{(\Delta x)^2} \tag{11.7}$$

Approximations to higher-order derivatives can be obtained in a similar way. Since differential equations encountered in surface and subsurface flows are of second order, approximating to first and second-order derivatives are considered here.

11.4 GEOMETRIC INTERPRETATION

The geometric interpretation clarifies the ideas involved in the finite difference approximation. Figure 11.1 shows the slope at point B and its approximations by forward, backward, and central difference approximations.

It is clear from Fig. 11.1 that the central difference scheme approximates $\frac{df}{dx}$ at B more accurately than forward and backward difference approximations. It can also be seen from Fig. 11.1 that the differences among approximations decrease with Δx.

11.5 FINITE DIFFERENCE SOLUTION OF 1-D GROUNDWATER FLOW EQUATION

The finite difference approximations for the first and second derivatives were explained in the previous section. In the present section, these approximations are used to solve one-dimensional groundwater flow equation.

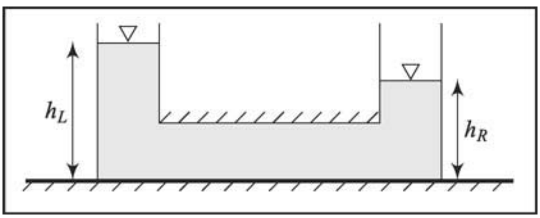


Fig. 11.2 *Confined aquifer connecting two reservoirs*

Figure 11.2 shows an isotropic and homogeneous confined aquifer with transmissivity T and storage coefficient S connecting two reservoirs. The governing equation for one-dimensional groundwater flow in the aquifer is:

$$T\frac{\partial^2 h}{\partial x^2} = S\frac{\partial h}{\partial t} \tag{11.8}$$

The initial and boundary condiions are:

$$h = h_o, \quad t = 0, \quad (0 \le x \le L) \tag{11.9}$$

$$h = h_L, \quad t \ge 0, \quad x = 0 \tag{11.10a}$$

$$h = h_R, \quad t \ge 0, \quad x = L \tag{11.10b}$$

where, h is the piezometric head in the aquifer, h_o is the initial piezometric head, and h_L and h_R are the water levels in left and right reservoirs, respectively.

The first step in obtaining the finite difference solution is to superimpose a finite difference grid in the x-t plane as shown in Fig. 11.3. Each intersection is termed a *node* or a *mesh point*.

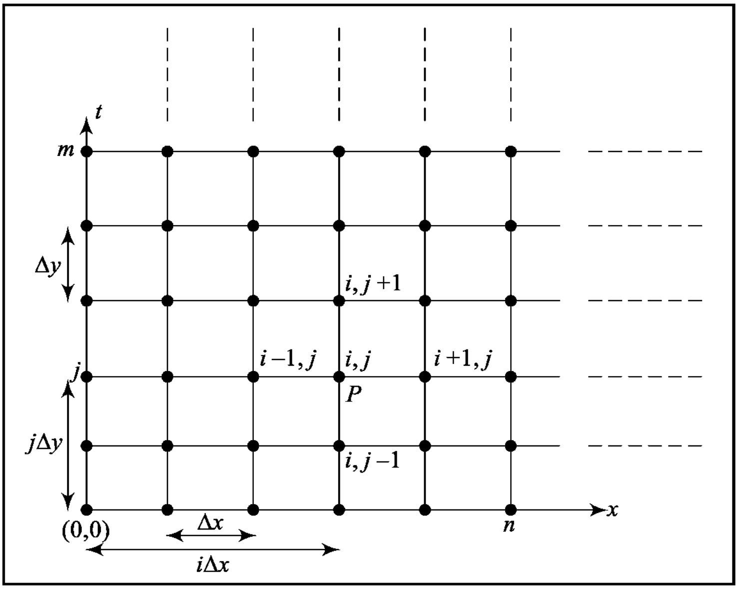


Fig. 11.3 *Grid pattern explaining the finite difference solution*

Let i denote the index in x direction and j denote the index in t direction. So, the co-ordinates of a mesh point is given by (x_i, t_j), where:

$$x_i = i\,\Delta x, \quad i = 0, 1, \ldots, n$$

and $$t_j = j\,\Delta t, j = 0, 1, \ldots, m$$

Δx and Δt represent space and time increments, respectively. The piezometric head h at mesh point (x_i, t_j) is denoted $h(x_i, t_j)$ or simply $h_{i,j}$.

11.6 EXPLICIT APPROXIMATION

Consider the node $P_{i,j}$ as shown in Fig.11.3. The finite difference approximation of space and time derivatives in Eq. (11.8), at node P can be written as:

$$T_i\left(\frac{h_{i-1,\,j} - 2h_{i,\,j} + h_{i+1,\,j}}{(\Delta x)^2}\right) = S_i\left(\frac{h_{i,\,j+1} - h_{j,\,1}}{\Delta t}\right) \tag{11.11}$$

Equation (11.11) is a linear algebraic equation which can be solved for $h_{i,j+1}$ as:

$$h_{i,j+1} = \left(\frac{T_i}{S_i}\right)\frac{\Delta t}{(\Delta x)^2}(h_{i-1,\,j} - 2h_{i,\,j} + h_{i+1,\,j}) + h_{i,\,j} \tag{11.12}$$

At the left and right boundaries, we have $h_{0,j} = h_L$ and $h_{w,j} = h_R$. Starting from $t = 0$ in Fig. 11.3, Eq. (11.12) is used to solve for h after the first time increment (i.e., $j = 1$) since $h_{i-1,0}$, $h_{i,0}$, $h_{i+1,0}$ are all known from the initial condition.

Once the values of $h_{i,1}$ are computed, the values of $h_{i,2}$ (i.e., h values after two time increments) are computed using Eq. (11.12), since the values of $h_{i,1}$ are already obtained from the previous solution.

In this manner, the solution is marched forward in time. The scheme is called as "explicit" since the finite difference Eq. (11.12) provides an explicit solution for the unknown $h_{i,j+1}$ in terms of the known values $h_{i-1,j}$, $h_{i,j}$, and $h_{i+1,j}$.

Since explicit scheme provides an explicit solution for the unknown, it is very easy to program such a scheme. However, the scheme has serious limitations and is not suitable in many applications. The main limitation of an explicit scheme is that the space and time increment Δx and Δt cannot be chosen independently in such a scheme. It has been shown analytically that the explicit scheme provides stable and meaningful results only when the following inequality is satisfied.

$$\Delta t \le \frac{1}{2}\frac{S}{T}(\Delta x)^2 \tag{11.13}$$

The mathematical treatment needed to derive Eq. (11.13) is beyond the scope of this book. Equation (11.13) suggests that the time increment Δt cannot be chosen independently of Δx, which makes the scheme unsuitable for many applications.

For the explicit scheme to be reasonably accurate, Δx must be kept small. Because Δt must be kept less than $\frac{1}{2}\frac{S}{T}(\Delta x)^2$ the number of computations required to reach a time level m becomes extremely large.

11.7 IMPLICIT APPROXIMATION FOR 1-D FLOW DOMAIN

In contrast to the explicit approximation, in implicit approximation, the finite difference approximation of the space derivative at node P is written at the time level $(j + 1)$ where the solution is sought.

Accordingly, the finite difference approximation of Eq. (11.8) is written as:

$$T_i\left(\frac{h_{i-1,\,j+1} - 2h_{i,\,j+1} + h_{i+1,\,j+1}}{(\Delta x)^2}\right) = S_i\left(\frac{h_{i,\,j+1} - h_{i,\,j}}{\Delta t}\right) \tag{11.14}$$

From Eq. (11.14), it can be seen that to solve for the unknown $h_{i,j+1}$, one needs the values $h_{i-1,j+1}$ and $h_{i+1,j+1}$ which themselves are not known. As such, Eq. (11.14) is implicit in the sense that the solution $h_{i,j+1}$ at node i depends on the solutions $h_{i-1,j+1}$ and $h_{i+1,j+1}$ at the adjacent nodes i–1 and i+1.

The solution at the unknown time level j+1 can be obtained by writing Eq. (11.14) at every node of the solution domain and assembling these equations in a matrix. Equation (11.14) is written in matrix form as:

$$\begin{bmatrix} \dfrac{T_i}{(\Delta x)^2} & -\left(\dfrac{2T_i}{(\Delta x)^2}+\dfrac{S_i}{\Delta t}\right) & \dfrac{T_i}{(\Delta x)^2} \end{bmatrix} \begin{bmatrix} h_{i-1,\,j+1} \\ h_{i,\,j+1} \\ h_{i+1,\,j+1} \end{bmatrix} = \begin{bmatrix} -\dfrac{S_i}{\Delta t} h_{i,\,j} \end{bmatrix} \quad (11.15)$$

In Eq. (11.15), the row vector on the left hand side is the vector containing the coefficients, the second vector is the column vector of unknown piezometric heads at an unknown time level j + 1, and right hand side vector contains the known term at the time level j.

Matrix equations similar to Eq. (11.15) are written for every node, and these equations are assembled to form a system of algebraic equations which is written in matrix form as:

$$A\,H = R \quad (11.16)$$

where, A is an ($n \times n$) matrix containing the coefficients, H is the vector of unknown piezometric heads at the time level j + 1, and R is the vector containing the known terms at the time level j.

The algebraic system of Eq. (11.16) can be solved using matrix algebra to obtain H vector at the time level j+1. Having solved for piezometric heads at time level j+1, these values can be used to obtain the piezometric heads at time level j+2. Similarly, the solution can be obtained up to the desired time level m.

The implicit approximation looks more complicated than the explicit approximation. However, the main advantage of the implicit approximation is that the approximation is unconditionally stable, and as such the time increment Δt can be chosen independent of the space increment Δx. This leads to a considerable reduction in the number of computations required to reach time level m, thus saving the computation time.

11.8 INCORPORATION OF BOUNDARY CONDITIONS

The finite difference approximation in Eq. (11.15) is valid for all the interior nodes. For the boundary nodes, one has to write the equations representing the boundary condition. Usually, we come across two types of boundary conditions, namely

(i) Specified head boundary condition (also called Dirichlet type boundary condition) where one specifies the head value at the boundary as in Eqs. (11.10a) and (11.10b), and

(ii) Specified flux boundary condition (also called Neumann type boundary condition) where one specifies the flux at the boundary.

The nodal equations for the boundary nodes have to be modified accordingly before solving the system of algebraic Eq. (11.16).

Specified Head Boundary Condition

To incorporate such a condition described by Eq. (11.10a), the coefficient matrix A and the right hand side vector R are modified. The diagonal term in the coefficient matrix A corresponding to the row representing the boundary node (in this case, the first row) is set to unity, while the off-diagonal terms are set to zero.

In the right hand side vector R , the element corresponding to the Dirichlet node (in this case, the first element) is set to the specified head given by the boundary condition, i.e., h_L. Similarly, the boundary condition as Eq. (11.10b) can also be incorporated by suitably changing the nodal equation of the last node.

Specified Flux Boundary Condition

Suppose that Eq. (11.10a) is changed from specified head boundary condition to a specified flux boundary condition as:

$$-T\frac{\partial h}{\partial x} = q_L, \quad t \geq 0, \quad x = 0 \tag{11.17}$$

where, q_L is the specified flux at the left boundary. To incorporate Eq. (11.17) at the left boundary, an imaginary node is introduced slightly outside the domain at a distance Δx from the left boundary as shown in Fig. 11.4 with the spatial index of the node being 0. Let the piezometric head at this node be denoted as $h_{0,j+1}$.

The finite difference approximation of Eq. (11.17) is written as:

$$-T_i\left(\frac{h_{2,j+1} - h_{0,j+1}}{2\Delta x}\right) = q_L \tag{11.18}$$

Equation (11.18) contains an unknown piezometric head $h_{0,j+1}$ at the imaginary node. To eliminate the peizometric head at the imaginary node, one needs an additional equation which is obtained by writing the finite difference equation at the left boundary treating the left boundary as an interior node as:

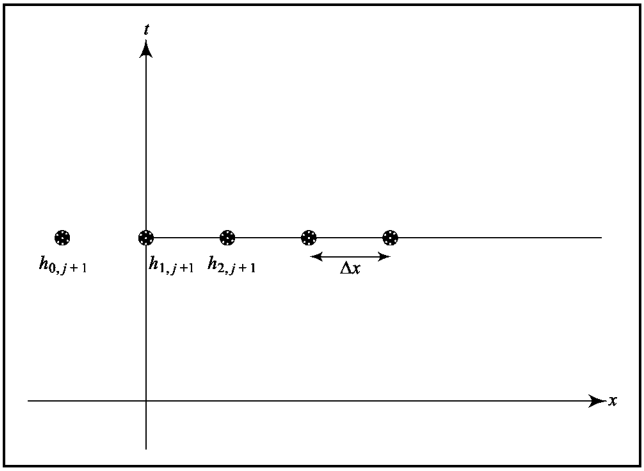


Fig. 11.4 *Diagram showing imaginary node for specified flux boundary conditions*

$$T_1\left(\frac{h_{0,\,j+1}-2h_{1,\,j+1}+h_{2,\,j+1}}{(\Delta x)^2}\right)=S_1\left(\frac{h_{1,\,j+1}-h_{1,\,j}}{\Delta t}\right) \tag{11.19}$$

The nodal equation for the boundary node is obtained by eliminating $h_0, j+1$ from Eqs. (11.18) and (11.19) and bringing all the unknown terms to the left-hand side and known terms to the right-hand side.

11.9 CRANK NICHOLSON SCHEME

The Crank Nicholson scheme makes use of both explicit and implicit approximations and is unconditionally stable. The advantage of this scheme over the implicit approximation is that the truncation errors are minimized leading to a more accurate solution.

In this scheme, the space derivate at any node is approximated as the average of explicit and implicit finite difference approximations. For a typical interior node i, the finite difference approximation of Eq. (11.8) is written as:

$$T_i\left(\frac{\dfrac{h_{i-1,\,j}-2h_{i,\,j}+h_{i+1,\,j}}{(\Delta x)^2}+\dfrac{h_{i-1,\,j+1}-2h_{i,\,j+1}+h_{i+1,\,j+1}}{(\Delta x)^2}}{2}\right)$$

$$=S_i\left(\frac{h_{i,\,j+1}-h_{i,\,j}}{\Delta t}\right) \tag{11.20}$$

Rearranging Eq. (11.20) to bring all the unknown terms on the left-hand side and all the known terms on the right-hand side, we get:

$$T_i\left(\frac{h_{i-1,\,j+1}-2h_{i,\,j+1}+h_{i+1,\,j+1}}{2(\Delta x)^2}\right)-S_i\frac{h_{i,\,j+1}}{\Delta t}$$

$$=-T_i\left(\frac{h_{i-1,\,j}-2h_{i,\,j}+h_{i+1,\,j}}{2(\Delta x)^2}\right)-S_i\frac{h_{i,\,j}}{\Delta t} \qquad (11.21)$$

The matrix form of Eq. (11.21) can be written as:

$$\begin{bmatrix}\dfrac{T_i}{2(\Delta x)^2} & -\left(\dfrac{2T_i}{2(\Delta x)^2}+\dfrac{S_i}{\Delta t}\right) & \dfrac{T_i}{2(\Delta x)^2}\end{bmatrix}\begin{bmatrix}h_{i-1,\,j+1}\\ h_{i,\,j+1}\\ h_{i+1,\,j+1}\end{bmatrix}$$

$$=\left[-T_i\left(\frac{h_{i-1,\,j}-2h_{i,\,j}+h_{i+1,\,j}}{2(\Delta x)^2}\right)-S_i\frac{h_{i,\,j}}{\Delta t}\right] \qquad (11.22)$$

Matrix equations similar to Eq. (11.22) are written for every node and these equations are assembled to form a system of algebraic equations of the form as Eq. (11.16), which can be solved for the unknown piezometric heads at the time level $j + 1$.

11.10 SOLUTION OF MATRIX EQUATIONS

As discussed earlier, implicit and Crank Nicholson schemes result in a system of algebraic equations which have to be solved to get the solution. A variety of methods such as Gaussian Elimination method, Gauss Siedel method, LU decomposition, and QR decomposition (Press et al., 2001) can be used to solve these algebraic equations.

In this section, a simple method to solve the system of equations resulting from the finite difference approximation of one-dimensional groundwater flow problems is discussed. It can be seen that the coefficient matrix A resulting from the finite difference approximation is tri-diagonal in nature with elements being non-zero on the main diagonal, the upper, and the lower off-diagonals. The other elements are equal to zero. This is due to the fact that the finite difference approximation at a node has contributions only from the immediately adjacent two nodes.

Due to the tri-diagonal nature of the coefficient matrix, a simple algorithm can be developed to solve the system equation by using the diagonal and off-diagonal elements in the solution.

11.11 SOLUTION OF TRI-DIAGONAL SYSTEM OF EQUATIONS

Consider an $(n \times n)$ tri-diagonal system of equations as shown in Eq. (11.23):

$$
\begin{array}{ccc} A & H & R \\
\begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & \cdots & \cdots & a_n & a_n \end{bmatrix} & \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ \vdots \\ \vdots \\ h_{n-1} \\ h_n \end{bmatrix} = & \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ \vdots \\ \vdots \\ r_{n-1} \\ r_n \end{bmatrix} \end{array} \tag{11.23}
$$

where, A is the tri-diagonal coefficient matrix, H is the column vector containing the unknown values, and R is the known right hand side matrix. The solution of Eq. (11.23) is as follows.

First, the coefficient matrix A is written as a product of two $(n \times n)$ matrices L and U. Here, L is a lower triangular matrix, and U is an upper triangular matrix given as:

$$
L = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ a_2 & \alpha_2 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a_3 & \alpha_3 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{n-1} & \alpha_{n-1} & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & a_n & \alpha_n \end{bmatrix} \tag{11.24}
$$

And
$$
U = \begin{bmatrix} 1 & \beta_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \beta_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \beta_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 & \beta_{n-1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{11.25}
$$

Since $LU = A$, we have the following relations:

$$\alpha_1 = b_1$$

$$\beta_i = \frac{c_i}{\alpha_i} \qquad (i = 1, 2, \ldots, n-1) \tag{11.26}$$

$$\alpha_i = b_i - a_i\,\beta_{i-1} \quad (i = 2, 3, \ldots, n)$$

Since α and β are known, the solution of Eq. (11.23) is obtained in two steps. Equation (11.23) is written as:

$$AH = R \quad \text{or} \quad LUH = R \quad \text{or} \quad LY = R \tag{11.27}$$

where, $Y = UH$. Since L is a lower triangular matrix, the system of equations $LY = R$ is solved for Y as:

$$y_1 = \frac{r_1}{\alpha_1}$$

$$y_i = \frac{r_i - a_i y_{i-1}}{\alpha_i} \quad (i = 2, 3, \ldots, n) \tag{11.28}$$

This is called the *forward sweep*. Since $Y = UH$ and U is upper triangular, H is obtained with a *backward sweep* as:

$$h_n = y_n$$

$$h_i = y_i - \beta_i\, y_{i+1} \quad (i = n-1, n-2, \ldots, 1) \tag{11.29}$$

The algorithm is simple to use and needs the storing of only the non-zero elements of the coefficient matrix.

11.12 FINITE DIFFERENCE SOLUTION OF 2-D GROUNDWATER FLOW EQUATION

Figure 11.5 shows an anisotropic and homogeneous confined aquifer *ABCD* with transmissivities T_x and T_y in principal directions and storage coefficient *S*.

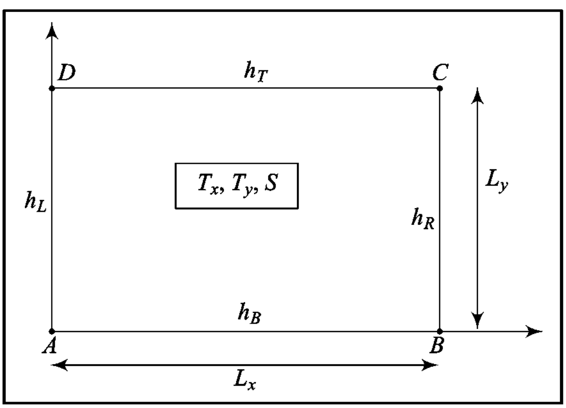


Fig. 11.5 *Two-dimensional groundwater flow in a rectangular confined aquifer*

The aquifer is subject to different piezometric heads along its boundaries. The governing equation, initial, and boundary conditions are:

$$T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} = S \frac{\partial h}{\partial t} \tag{11.30}$$

Such that $h = h_o, \quad t = 0, \quad 0 \le x \le L_x, \quad 0 \le y \le L_y$ (11.31)

$$h = h_L, \quad t \ge 0, \quad x = 0, \quad 0 \le y \le L_y \tag{11.32a}$$

$$h = h_R, \quad t \ge 0, \quad x = L_x, \quad 0 \le y \le L_y \tag{11.32b}$$

$$h = h_B, \quad t \ge 0, \quad y = 0, \quad 0 \le y \le L_x \tag{11.32c}$$

$$h = h_T, \quad t \ge 0, \quad y = L_y, \quad 0 \le y \le L_x \tag{11.32d}$$

11.13 IMPLICIT APPROXIMATION FOR 2-D FLOW DOMAIN

A finite difference grid is superimposed on the solution domain to obtain the finite difference approximations of the space and time derivatives in Eq. (11.30) as shown in Fig. 11.6.

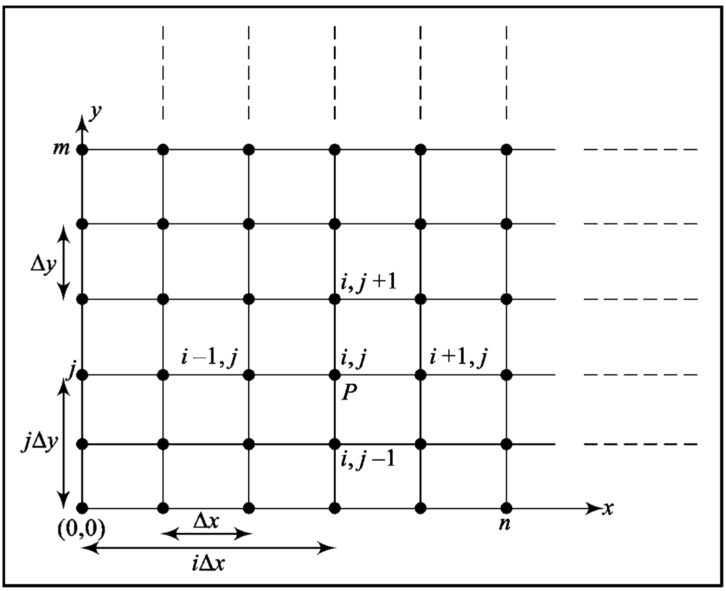


Fig. 11.6 *Spatial finite difference discretization for two-dimensional groundwater flow*

Let i denote the index in x-direction, j the index in y-direction, and k the index in t-direction, so that the coordinate of a mesh point is given by (x_i, y_j, t_k), where:

$$x_i = i\Delta x, \quad (i = 0, 1, \ldots, n)$$

$$y_j = j\Delta y, \quad (j = 0, 1, \ldots, m)$$

and $$t_k = k\Delta t, \quad (k = 0, 1, \ldots, p)$$

Here, Δx and Δy are the space increments in x and y directions, respectively, and Δt is the time increment.

The piezometric head h at mesh point (x_i, y_j, t_k) is denoted $h(x_i, y_j, t_k)$ or simply $h_{i,j,k}$. For an interior node $P_{i,j,k}$ in Fig. 11.6, the finite implicit difference approximation of Eq. (11.30) is written as:

$$T_{x,i,j}\left(\frac{h_{i-1,j,k+1} - 2h_{i,j,k+1} + h_{i+1,j,k+1}}{(\Delta x)^2}\right) + T_{y,i,j}\left(\frac{h_{i,j-1,k+1} - 2h_{i,j,k+1} + h_{i,j+1,k+1}}{(\Delta y)^2}\right) = S_{i,j}\left(\frac{h_{i,j,k+1} - h_{i,j,k}}{\Delta t}\right) \quad (11.33)$$

The solution at the unknown time level $k + 1$ can be obtained by writing Eq. (11.33) at every node of the solution domain and assembling these equations in a matrix. Equation (11.33) is written in matrix form as:

$$\begin{bmatrix} \frac{T_{x,i,j}}{(\Delta x)^2} & \frac{T_{x,i,j}}{(\Delta x)^2} & -\left(\frac{2T_{x,i,j}}{(\Delta x)^2} + \frac{2T_{y,i,j}}{(\Delta y)^2} + \frac{S_{i,j}}{\Delta t}\right) & \frac{T_{y,i,j}}{(\Delta y)^2} & \frac{T_{y,i,j}}{(\Delta y)^2} \end{bmatrix} \begin{bmatrix} h_{i-1,j,k+1} \\ h_{i+1,j,k+1} \\ h_{i,j,k+1} \\ h_{i,j-1,k+1} \\ h_{i,j+1,k+1} \end{bmatrix} = \begin{bmatrix} -\frac{S_{i,j}}{\Delta t} h_{i,j,k} \end{bmatrix} \quad (11.34)$$

In Eq. (11.34), the row vector on the left hand side is the vector containing the coefficients, the second vector is the column vector of unknown piezometric heads at time level $k + 1$ and right hand side vector contains the known term at time level k.

Matrix equations similar to Eq. (11.34) are written for every node, and these equations are assembled to form a system of algebraic equations which is written in matrix form as:

$$A\,H = R \tag{11.35}$$

where, A is an $(mn \times mn)$ matrix containing the coefficients, H is the vector of unknown piezometric heads at the time level $k+1$ and R is the vector containing the known terms at time level k.

The algebraic system of Eq. (11.35) can be solved using matrix algebra to obtain H vector at time step $k+1$. Having solved for piezometric heads at time step $k+1$, these values can be used to obtain the piezometric heads at time step $k+2$. Similarly, the solution can be obtained up to the desired time step p.

11.14 ALTERNATING DIRECTION IMPLICIT METHOD

Unlike the one-dimensional problem discussed earlier, the coefficient matrix A in Eq. (11.35) is not tri-diagonal in nature, and the algorithm for solving a tri-diagonal system of equations cannot be used as such for the solution of Eq. (11.35). With the use of Alternating Direction Implicit (ADI) Method, the coefficient matrix A in Eq. (11.35) can be made tri-diagonal in nature as discussed below.

In ADI, the space derivative in one direction (say x-direction) is written at the time step k, where the solution is known; and the space derivative in the other direction (in y-direction) is written at the time step $k+1$, where the solution is sought. In such a formulation, the space derivative becomes explicit in x-direction and implicit in y-direction.

11.15 EXPLICIT IN x AND IMPLICIT IN y-DIRECTION

For such an approximation, the implicit finite difference formulation of Eq. (11.30) is written as:

$$T_{xi,j}\left(\frac{h_{i-1,j,k} - 2h_{i,j,k} + h_{i+1,j,k}}{(\Delta x)^2}\right) + T_{y,i,j}\left(\frac{h_{i,j-1,k+1} - 2h_{i,j,k+1} + h_{i,j+1,k+1}}{(\Delta y)^2}\right) = S_{i,j}\left(\frac{h_{i,j,k+1} - h_{i,j,k}}{\Delta t}\right) \tag{11.36}$$

Equation (11.36) contains three unknowns, namely $h_{i,j-1,k+1}$, $h_{i,j,k+1}$, $h_{i,j+1,k+1}$ and can be written in matrix form as:

$$\begin{bmatrix} \dfrac{T_{y,i,j}}{(\Delta y)^2} & -\left(\dfrac{2T_{y,i,j}}{(\Delta y)^2} + \dfrac{S_{i,j}}{\Delta t}\right) & \dfrac{T_{y,i,j}}{(\Delta y)^2} \end{bmatrix} \begin{bmatrix} h_{i,j-1,k+1} \\ h_{i,j,k+1} \\ h_{i,j+1,k+1} \end{bmatrix}$$

$$= \left[-\frac{S_{i,j}}{\Delta t} h_{i,j,k} - T_{x,i,j} \left(\frac{h_{i-1,j,k} - 2h_{i,j,k} + h_{i+1,j,k}}{(\Delta x)^2} \right) \right] \quad (11.37)$$

Finite difference equations similar to Eq. (11.37) can be written for all nodes of the i^{th} column. Assembling all the nodal equations of the i^{th} column, a system of algebraic equations is formed according to:

$$A_i H_i = R_i \quad (11.38)$$

where, A_i is the coefficient matrix for the i^{th} column, H_i is the vector containing unknown piezometric heads of all the nodes in the i^{th} column, and R_i is the right hand side matrix for the i^{th} column.

A_i in Eq. (11.38) is tri-diagonal in nature and can be solved easily using the algorithm discussed in Section 11.11. Matrix equations similar to Eq. (11.38) can be written for all the columns starting from $i = 1$ to $i = m$ and can be solved for unknown nodal piezometric heads in all the columns, thereby obtaining the piezometric heads at time level $k + 1$.

To obtain the unknown piezometric heads at time $k + 2$ from the known values at time $k + 1$, the direction in which the explicit and implicit approximations are written is interchanged, i.e., the space derivative is written implicitly at time level $k + 2$ in x-direction and explicitly at time level $k + 1$ in y-direction.

11.16 EXPLICIT IN y AND IMPLICIT IN x-DIRECTION

For such an approximation, the implicit finite difference formulation of Eq. (11.30) is written as:

$$T_{xi,j} \left(\frac{h_{i-1,j,k+2} - 2h_{i,j,k+2} + h_{i+1,j,k+2}}{(\Delta x)^2} \right) + T_{y,i,j} \left(\frac{h_{i,j-1,k+1} - 2h_{i,j,k+1} + h_{i,j+1,k+1}}{(\Delta y)^2} \right) = S_{i,j} \left(\frac{h_{i,j,k+2} - h_{i,j,k+1}}{\Delta t} \right) \quad (11.39)$$

Equation (11.39) contains three unknowns, namely $h_{i-1,j,k+2}$, $h_{i,j,k+2}$, $h_{i+1,j,k+2}$ and can be written in matrix form as:

$$\begin{bmatrix} \dfrac{T_{x,i,j}}{(\Delta x)^2} & -\left(\dfrac{2T_{x,i,j}}{(\Delta x)^2} + \dfrac{S_{i,j}}{\Delta t}\right) & \dfrac{T_{x,i,j}}{(\Delta x)^2} \end{bmatrix} \begin{bmatrix} h_{i-1,j,k+2} \\ h_{i,k,k+2} \\ h_{i+1,j,k+2} \end{bmatrix}$$

$$= \left[-\frac{S_{i,j}}{\Delta t} h_{i,j,k+1} - T_{y,i,j} \left(\frac{h_{i,j-1,k+1} - 2h_{i,j,k+1} + h_{i,j+1,k+1}}{(\Delta y)^2} \right) \right] \quad (11.40)$$

Finite difference equations similar to Eq. (11.40) can be written for all the nodes of the j^{th} row. Assembling all the nodal equations of the j^{th} row results in a system of algebraic equations according to:

$$A_j H_j = R_j \quad (11.41)$$

where, A_j is the coefficient matrix for the j^{th} row, H_j is the vector containing unknown piezometric heads of all nodes in the j^{th} row, and R_j is the right hand side matrix for the j^{th} row.

The A_j in Eq. (11.41) is tri-diagonal in nature and can be easily solved using the algorithm outlined in Section 11.11. Matrix equations similar to Eq. (11.41) can be written for all rows starting from $j = 1$ to $j = n$ and can be solved for unknown nodal piezometric heads in all columns, thereby obtaining the piezometric heads at time k+2. In this manner, the solution can be obtained for the desired time p by writing implicit finite difference approximation in alternating directions during successive time steps.

Example 11.1 A confined aquifer of length 100 m has a transmissivity of 0.02 m^2/min and a storage coefficient of 0.002. The initial piezometric head in the aquifer is 16 m. If the piezometric head at the right boundary drops to 11 m. Compute the piezometric head in the aquifer as a function of space and time using explicit finite difference method.

Solution

The initial condition is $h|_{x,0} = 16$ m and the boundary conditions are $h|_{0,t} = 16$ m and $h|_{100,t} = 11$ m

The discretized groundwater flow equation for the explicit finite difference method (shown in Eq. 11.12) can be written as:

$$h_{i,j+1} = \left(\frac{T_i}{S_i}\right) \frac{\Delta t}{(\Delta x)^2} (h_{i-1,j} - 2h_{i,j} + h_{i+1,j}) + h_{i,j}$$

$$\frac{T}{S} = \frac{0.02}{0.002} = 10$$

Let us divide the 100 m length aquifer into 6 grid nodes.

$$\text{Therefore, } \Delta x = \frac{100}{6-1} = 20 \text{ m}$$

$$\text{Since} \qquad \Delta t \leq \frac{1}{2}\frac{S}{T}(\Delta x)^2$$

$$\text{Therefore, } \Delta t \leq \frac{1}{2} \times \frac{1}{10} \times (20)^2 = 20 \text{ min}$$

In Eq. (11.12), i is the grid number and j is the time.

Take Δt =20 min

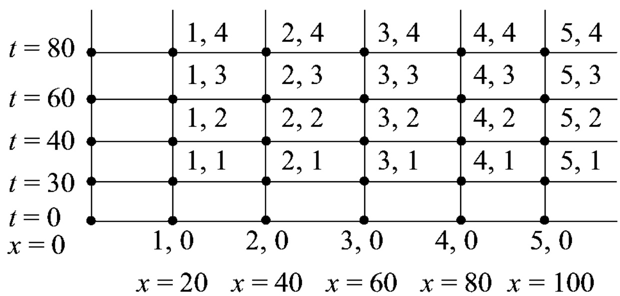
$i_1 = 20$ $i_2 = 40$ m $i_3 = 60$ m $i_4 = 80$ m, and $i_5 = 100$ m

$j_1 = 20$ min, $j_2 = 40$ min, $j_3 = 60$ min, $j_4 = 80$ min, and $j_5 = 100$ min

At $t = 0$, $h_1 = h_2 = h_3 = h_4 = h_5 = 16$ m (initial condition)

$h|_{100,t} = 11$ m (boundary condition)

Now using Eq. (11.12), we can find out the head at different distance of the aquifer.



For the first time step ($j = 0$), only grid 5 ($i = 4$) is affected.

Take $i = 4, j = 0$

$$h_{4,1} = 10 \times \frac{20}{20^2}(h_{3,0} - 2h_{4,0} + h_{5,0}) + h_{4,0}$$

$$= 10 \times \frac{20}{400}(16 - 32 + 11) + 16$$

$$= 13.5 \text{ m}$$

Similarly, $h_{1,1} = h_{2,1} = h_{3,1}$ =16 m, and $h_{5,1}$ = 11m (boundary condition)

For the second time step ($j = 1$), $t + \Delta t = 40$ min

Take $i = 4, j = 1$

$$h_{4,2} = 10 \times \frac{20}{20^2}(h_{3,1} - 2h_{4,1} + h_{5,1}) + h_{4,1}$$

$$= 10 \times \frac{20}{400}(16 - 27 + 11) + 13.5$$

$$= 13.5 \text{ m}$$

Take $i = 3, j = 1$

$$h_{3,2} = 10 \times \frac{20}{20^2}(h_{2,1} - 2h_{3,1} + h_{4,1}) + h_{3,1}$$

$$= 10 \times \frac{20}{400}(16 - 32 + 13.5) + 16$$

$$= 14.75 \text{ m}$$

Similarly, $h_{1,2} = h_{2,2} = 16$ m, and $h_{5,2} = 11$ m (boundary condition).

For the third time step ($j = 2$), $t + 2\Delta t = 60$ min

Take $i = 2, j = 2$

$$h_{2,3} = 10 \times \frac{20}{20^2}(h_{1,2} - 2h_{2,2} + h_{3,2}) + h_{2,2}$$

$$= 10 \times \frac{20}{400}(16 - 32 + 14.75) + 16$$

$$= 15.375 \text{ m}$$

Take $i = 3, j = 2$

$$h_{3,3} = 10 \times \frac{20}{20^2}(h_{2,2} - 2h_{3,2} + h_{4,2}) + h_{3,2}$$

$$= 10 \times \frac{20}{400}(16 - 29.5 + 13.5) + 14.75$$

$$= 14.75 \text{ m}$$

Similarly $h_{1,3} = 16$ m, $h_{4,3} = 13.5$ m and $h_{5,3} = 11$ m (boundary condition)

For the fourth time step ($j = 3$), $t + 3\,\Delta t = 80$ min

Take $i = 1, j = 3$

$$h_{1,4} = 10 \times \frac{20}{20^2}(h_{0,3} - 2h_{1,3} + h_{2,3}) + h_{1,3}$$

$$= 10 \times \frac{20}{400}(16 - 32 + 15.375) + 16$$

$$= 15.6875 \text{ m}$$

Similarly, $h_{2,4}$ =15.375 m, $h_{3,4}$ =14.75 m, $h_{4,4}$ =13.5 m, and $h_{5,4}$ =11 m (boundary condition). We can proceed further increasing the time by Δt.

Example 11.2 Solve Example 11.1 using implicit finite difference method.

Solution

The discretized groundwater flow equation for implicit finite difference method can be written as:

$$\left[\frac{T_i}{2(\Delta x)^2} - \left(\frac{2T_i}{2(\Delta x)^2} + \frac{S_i}{\Delta t}\right) + \frac{T_i}{2(\Delta x)^2}\right]\begin{bmatrix} h_{i-1,j+1} \\ h_{i,j+1} \\ h_{i+1,j+1}\end{bmatrix}$$

$$= \left[-T_i\left(\frac{h_{i-1,j} - 2h_{i,j} + h_{i+1,j}}{2(\Delta x)^2}\right) - S_i\frac{h_{i,j}}{\Delta t}\right]$$

$$\frac{T}{2(\Delta x)^2} = \frac{0.02}{2 \times 20^2} = 0.000025$$

$$\frac{2T}{2(\Delta x)^2} + \frac{S}{\Delta t} = \frac{2 \times 0.02}{2 \times 20^2} + \frac{0.002}{20} = 0.00015$$

Take $i = 1, j = 0$

$$0.000025\ h_{0,1} - 0.00015\ h_{1,1} + 0.000025\ h_{2,1}$$
$$= [0.000025(h_{0,0} - 2h_{1,0} + h_{2,0}) - 0.0001\ h_{1,0}]$$

$$0.000025 \times 16 - 0.00015\ h_{1,1} + 0.000025\ h_{2,1}$$
$$= [0.000025(16 - 32 + 16) - 0.0001 \times 16]$$

$$-0.00015\ h_{1,1} + 0.000025\ h_{2,1} = -0.002 \tag{11.42}$$

Take $i = 2, j = 0$

$$0.000025\ h_{1,1} - 0.00015\ h_{2,1} + 0.000025\ h_{3,1}$$
$$= [0.000025(h_{1,0} - 2h_{2,0} + h_{3,0}) - 0.0001\ h_{2,0}]$$

$$0.000025\ h_{1,1} - 0.00015\ h_{2,1} + 0.000025\ h_{3,1}$$
$$= [0.000025(16 - 32 + 16) - 0.0001 \times 16]$$

$$0.000025\ h_{1,1} - 0.00015\ h_{2,1} + 0.000025\ h_{3,1} = -0.0016 \tag{11.43}$$

Take $i = 3, j = 0$

$$0.000025\ h_{2,1} - 0.00015\ h_{3,1} + 0.000025\ h_{4,1}$$
$$= [0.000025(h_{2,0} - 2h_{3,0} + h_{4,0}) - 0.0001\ h_{3,0}]$$

$$0.000025\ h_{2,1} - 0.00015\ h_{3,1} + 0.000025\ h_{4,1}$$
$$= [0.000025(16 - 32 + 16) - 0.0001 \times 16]$$

$$0.000025\ h_{2,1} - 0.00015\ h_{3,1} + 0.000025\ h_{4,1} = -0.0016 \tag{11.44}$$

Take $i = 4, j = 0$

$$0.000025\, h_{3,1} - 0.00015\, h_{4,1} + 0.000025\, h_{5,1} = [0.000025(h_{3,0} - 2h_{4,0} + h_{5,0}) - 0.0001\, h_{4,0}]$$

$$0.000025\, h_{3,1} - 0.00015\, h_{4,1} + 0.000025 \times 11 = [0.000025(16 - 32 + 162) - 0.0001 \times 16]$$

$$0.000025\, h_{3,1} - 0.00015\, h_{4,1} = - 0.001875 \quad (11.45)$$

There are four equations and four unknowns. This forms a tri-diagonal matrix that can be solved by Thomas algorithm.

After solving the equations, we get:

$h_{1,1} = 16.0$ m (*boundary condition*)

$h_{2,1} = 15.974$ m $h_{3,1} = 15.852$ m $h_{4,1} = 15.142$ m

$h_{5,1} = 11$ m (*boundary condition*)

Similarly, we can proceed with increasing the time step further.

Example 11.3 Solve the following set of equations.

$$2x_1 + 3x_2 = 8.0$$
$$x_1 + x_2 + 2x_3 = 9.0$$
$$2x_2 + x_3 + 4x_4 = 23.0$$
$$2x_3 + 2x_4 + x_5 = 19.0$$
$$x_4 + x_5 = 9.0$$

Solution

Here,

$a_2 = 1.0$	$a_3 = 2.0$	$a_4 = 2.0$	$a_5 = 1.0$	
$b_1 = 2.0$	$b_2 = 1.0$	$b_3 = 1.0$	$b_4 = 2.0$	$b_5 = 1.0$
$c_1 = 3.0$	$c_2 = 2.0$	$c_3 = 4.0$	$c_4 = 1.0$	
$r_1 = 8.0$	$r_2 = 9.0$	$r_3 = 23.0$	$r_4 = 19.0$	$r_5 = 9.0$

After solving the above system of equations using the algorithm as explained in Section 11.11, we get:

$\alpha_1 = 2.0$	$\alpha_2 = -0.5$	$\alpha_3 = 9.0$	$\alpha_4 = 1.11$	$\alpha_5 = 0.1$
$\beta_1 = 1.5$	$\beta_2 = -4.0$	$\beta_3 = 0.444$	$\beta_4 = 0.9$	$\beta_5 = 0.0$
$y_1 = 4.0$	$y_2 = -10.0$	$y_3 = 4.778$	$y_4 = 8.5$	$y_5 = 5.0$

Now, after back substitution, we will get:

$x_1 = 1.0$ $x_2 = 2.0$ $x_3 = 3.0$ $x_4 = 4.0$ $x_5 = 5.0$

11.17 VISUAL MODFLOW AND PMWIN

Visual MODFLOW is a three-dimensional finite-difference groundwater model that was first published in 1984 by USGS. It has a modular structure that allows it to be easily modified to adapt the code for a particular application.

The modular structure of Visual MODFLOW consists of a *Main Program* and a series of highly-independent subroutines called *modules*. The modules are grouped in sub-modules. Each sub-module deals with a specific feature of the hydrologic system which is to be simulated such as flow from rivers or flow into drains or with a specific method of solving linear equations which describe the flow system. The division of Visual MODFLOW into modules permits the user to examine specific hydrologic features of the model independently.

Visual MODFLOW simulates steady and unsteady flow in an irregular flow system in which aquifer layers can be confined, unconfined, or a combination of both. Flow to wells, areal recharge, evapotranspiration, flow to drains, and flow through river beds, can be easily simulated. Aquifer conditions having different combinations of homogeneous, heterogeneous, isotropic and anisotropic soils can be simulated. Different types of boundary conditions can be specified to solve various types of groundwater flow problems.

The groundwater flow equation is solved using the finite-difference approximation. The flow region is subdivided into blocks in which the medium properties are assumed to be uniform. In plan view, the blocks are made from a grid of mutually perpendicular lines that may be variably spaced. Model layers can have varying thickness. A flow equation is written for each block, called a *cell.*

Several solvers are provided for solving the resulting matrix problem; the user can choose the best solver for the particular problem. Flow-rate and cumulative-volume balances from each type of inflow and outflow are computed for each time step. Initial conditions, hydraulic properties, and stresses must be specified for every model cell in the finite-difference grid. In addition to simulating groundwater flow, Visual MODFLOW can be used to solve solute transport equation and groundwater management problems.

The PMWIN software is similar to Visual MODFLOW and is also freely available on Internet and contains many solved problems of practical significance. Stefan et al. (2006) had explained step-by-step application of the PMWIN software in solving a variety of groundwater flow problems.

SUMMARY

In recent years, simulation and mathematical models have often been used to analyse groundwater systems. This chapter is useful to solve real field problems in groundwater flow using numerical methods. Since field experiments are difficult to perform, mathematical models play an important role to analyse the complex geometry of the aquifers. Consequently, an attempt has been made to analyse these complex mathematical models using different types of available numerical methods. Also, a brief discussion on available groundwater flow software has been included in this chapter. These software can be of great use to solve a variety of complex problems.

EXERCISES

11.1 A confined aquifer has a length of 1000 m, transmissivity of 500 m^2/day, and storage coefficient of 0.0002. The head is initially uniform at 20 m. The head increases to 50 m at the inlet boundary and to 40 m at the outlet boundary. Solve the one-dimensional transient groundwater flow equation using:

(a) Explicit finite difference method
(b) Implicit finite difference method
(c) Crank Nicholson method

How will your results be affected if $\Delta t > \frac{1}{2}\frac{S}{T}(\Delta x)^2$?

11.2 For two-dimensional flow in an aquifer having a length of 1000 m and width of 800 m, find out the head at different points. The properties of the aquifer, initial conditions, and boundary conditions are given below.

Transmissivity in x-direction = 500 m^2/day

Transmissivity in y-direction = 700 m^2/day

Storativity in x-direction = 0.0002

Storativity in y-direction = 0.0002

Initial condition: $h = 60$ m, $t = 0$, $0 \le x \le 1000$ m, $0 \le y \le 800$ m

Boundary conditions:

$$h = 100 \text{ m}, \quad t \ge 0, \quad 0 \le x \le 1000 \text{ m}, \quad y = 0$$
$$h = 80 \text{ m}, \quad t \ge 0, \quad 0 \le x \le 1000 \text{ m}, \quad y = 800$$
$$h = 70 \text{ m}, \quad t \ge 0, \quad 0 \le y \le 800 \text{ m}, \quad x = 0$$
$$h = 90 \text{ m}, \quad t \ge 0, \quad 0 \le y \le 800 \text{ m}, \quad x = 100$$

11.3 Solve the following system of tri-diagonal equations.

$$x_1 + x_2 = 6.0$$
$$2x_1 + x_2 + 3x_3 = 26.0$$
$$x_2 + x_3 + 3x_4 = 13.0$$
$$x_3 + 4x_4 + 2x_5 = 20.0$$
$$2x_4 + x_5 = 7.0$$

11.4 Solve the one-dimensional heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Subject to initial conditions

$$u(x, 0) = \begin{cases} x^2 & \text{for } \left(0 \le x \le \frac{1}{2}\right) \\ 2x & \text{for } \left(\frac{1}{2} \le x \le 1\right) \end{cases}$$

and the conditions $u(0, t) = u(1, t) = 0$.

11.5 Solve the two-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Subject to initial conditions $u(x, y, 0) = \sin\frac{\pi x}{2}\sin\frac{\pi y}{2}$, $(0 \le x \le 1)$ and the conditions $u(x, y, t) = 0$, $t > 0$ on the boundaries. Calculate results for one time step.

11.6 Using PMWIN or MODFLOW

A town is planning to increase its water supply by constructing a borehole in an unconfined aquifer consisting of sand and gravel (Fig.11.7). The borehole is designed to pump constantly at a rate of 20,000 m^3 day^{-1}. However, the Society for the Preservation of the Green Swamp has presented an objection. The Society claims the pumping would "significantly reduce" the groundwater seepage to the swamp and threaten the swamp biohabitat. The town claims that the river and the groundwater divide, located somewhere near the centre of the valley would prevent any change in groundwater flow to the Green Swamp.

Investigate the above problem by experimenting with different types of aquifer boundaries, under a condition of steady-state, to calculate groundwater flow and groundwater heads at the river and swamp boundaries.

The hydraulic conductivity and specific yield are:

$K_x = 50$ m/day $\quad K_y = 50$ m/day $\quad K_z = 5$ m/day $\quad S_y = 0.1$

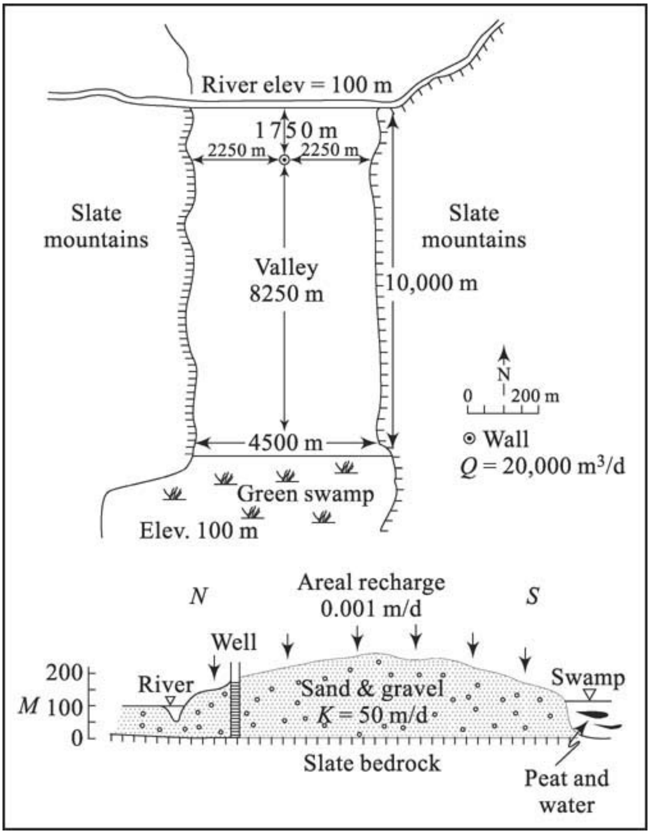


Fig. 11.7 *Bore hole in an unconfined aquifer*

OBJECTIVE QUESTIONS

1. The criteria for the time increment for the successive iteration in an explicit finite difference scheme is

(a) $\Delta t \geq \frac{1}{2}\frac{S}{T}(\Delta x)^2$ (b) $\Delta t \leq \frac{1}{2}\frac{S}{T}(\Delta x)^2$

(c) $\Delta t \leq \frac{1}{2}\frac{T}{S}(\Delta x)^2$ (d) $\Delta t \geq \frac{1}{2}\frac{T}{S}(\Delta x)^2$

2. A confined aquifer has a length of 200 m, transmissivity of 0.015 m^2/min, and a storage coefficient of 0.002. The aquifer is divided into 101 nodes. What must be the time increment between two successive iterations to solve the one-dimensional transient groundwater flow equation using the explicit finite difference method?

(a) $\Delta t \leq 0.26$ min (b) $\Delta t \leq 0.26$ hr

(c) $\Delta t \leq 15$ min (d) Any value can be taken

3. Determine the value of x in $f(x) = e^{-x}$ corresponding to the first four terms of the Taylor series about $x_0 = 0$ and rounding error to be less than 10^{-6}.
   (a) 0.01 (b) 0.06 (c) 0.03 (d) 0.1

4. In the previous question, determine the number of terms to be used in the Taylor series approximation to restrict the rounding off error to 10^{-10}.
   (a) ≤ 10 (b) ≤ 7 (c) ≤ 14 (d) $= 10$

5. To convert the governing equation from differential form to algebraic form, finite element method uses
   (a) Taylor series approximation (b) Euler method
   (c) Basis function (d) All of these.

6. Crank Nicholson scheme uses
   (a) Explicit approximation (b) Implicit approximation
   (c) Both (a) and (b) (d) None of these.

7. In Crank Nicholson scheme
   (a) Roundoff errors are minimized (b) Truncation errors are minimized
   (c) Both (a) and (b) (d) None of these

8. In implicit approximation
   (a) The approximation is unconditionally stable and Δt can not be chosen independent of the space increment Δx
   (b) The approximation is unconditionally stable and Δt can be chosen independent of the space increment Δx
   (c) The approximation is unstable and Δt can not be chosen independent of the space increment Δx
   (d) The approximation is unstable and Δt can be chosen independent of the space increment Δt

9. If the value of head is given in the boundary, the boundary condition is called as
   (a) Dirichlet type (b) Neumann type
   (c) Cauchy type (d) All of these

10. In the Neumann type of boundary condition, at the boundary node
   (a) The value of head is known (b) Flux is known
   (c) Both (a) and (b) (d) None of these.

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APPENDIX

I

Gamma Function

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
1.00	1.000000	1.34	0.892216	1.68	0.905001
1.02	0.988844	1.36	0.890185	1.70	0.908639
1.04	0.978438	1.38	0.888537	1.72	0.912581
1.06	0.968744	1.40	0.887264	1.74	0.916826
1.08	0.959725	1.42	0.886356	1.76	0.921375
1.10	0.951351	1.44	0.885805	1.78	0.926227
1.12	0.943590	1.46	0.885604	1.80	0.931384
1.14	0.936416	1.48	0.885747	1.82	0.936845
1.16	0.929803	1.50	0.886227	1.84	0.942612
1.18	0.923728	1.52	0.887039	1.86	0.948687
1.20	0.918169	1.54	0.888178	1.88	0.955071
1.22	0.913106	1.56	0.889639	1.90	0.961766
1.24	0.908521	1.58	0.891420	1.92	0.968774
1.26	0.904397	1.60	0.893515	1.94	0.976099
1.28	0.900718	1.62	0.895924	1.96	0.983743
1.30	0.897471	1.64	0.898642	1.98	0.991708
1.32	0.894640	1.66	0.901668	2.00	1.000000

Note: $\Gamma(n+1) = n\,\Gamma(n)$

$$\begin{aligned}\Gamma(6.4) &= \Gamma(5.4+1) = 5.4\,\Gamma(5.4)\\ &= 5.4 \times 4.4 \times 3.4 \times 2.4 \times 1.4 \times \Gamma(1.4)\\ &= 240.833\end{aligned}$$

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