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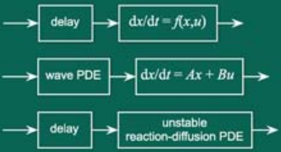
Systems & Control: Foundations and Applications

Miroslav Krstic

Delay Compensation for Nonlinear, Adaptive, and PDE Systems



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Delay Compensation for Nonlinear, Adaptive, and PDE Systems

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Preface

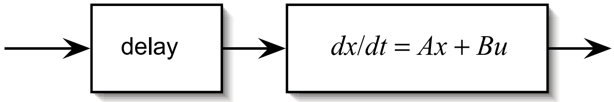
This year is the 50th anniversary of Otto J. Smith's 1959 publication of a control design idea commonly referred as the *Smith predictor* for the compensation of actuator delays. Actuator and sensor delays are among the most common dynamic phenomena that arise in engineering practice but fall outside the scope of the standard finite-dimensional systems.

Predictor-based feedbacks and other controllers for systems with input and output delays have been (and continue to be) an active area of research during the last five decades. Several books exist that focus on the mathematical and engineering problems in this area. The goals of this book are not to duplicate the material in those books nor to present a comprehensive account about control of systems with input and output delays. Instead, the book's goal is to shed light on new opportunities for predictor feedback, through extensions to nonlinear systems, delay-adaptive control, and actuator dynamics modeled by PDEs more complex than transport (pure delay) PDEs.

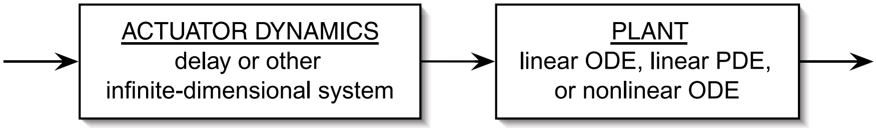
*   *   *

What Does the Book Cover? This book is a research monograph that introduces the treatment of systems with input delays as PDE-ODE cascade systems with *boundary control*. The PDE-based approach yields Lyapunov–Krasovskii functionals that make the control design constructive and enables stability analysis with quantitative estimates, which leads to the resolution of several long-standing problems in predictor feedback for linear time-invariant (LTI) systems. More importantly, the PDE-based approach enables the extension of predictor feedback design to nonlinear systems and to adaptive control for systems with unknown delays.

However, the book's treatment of input and output delays as transport PDEs allows it to aim even further, in expanding the predictor feedback ideas to systems with other types of infinite-dimensional actuator dynamics and sensor dynamics. We develop methods for compensating heat PDE and wave PDE dynamics at the input of an arbitrary, possibly unstable, LTI-ODE plant. Similarly, we develop observers for LTI-ODE systems with similar types of sensor dynamics. Finally, we introduce problems for PDE-PDE cascades, such as, for example, the notoriously



The standard problem of linear ODE plant with input delay, leading to conventional predictor feedback control.



The problems considered in this book.

difficult problem of a wave PDE with input delay where, if the delay is left uncompensated, an arbitrarily short delay destroys the closed-loop stability (as shown by Datko in 1988).

* * *

Who Is the Book For? The book should be of interest to all researchers working on control of delay systems—engineers, graduate students, and delay systems specialists in academia. The latter group will especially benefit from this book, as it opens several new paradigms for delay research. Many opportunities present themselves to extend the present results to systems that contain state delays (discrete and/or distributed) in addition to input delays.

Mathematicians with interest in the broad area of control of distributed parameter systems, and PDEs in particular, will find the book stimulating because it tackles nonlinear ODEs simultaneously with linear PDEs, as well as PDEs from different classes. These problems present many stimulating challenges for further research on the stabilization of ever-expanding classes of unstable infinite-dimensional systems.

Chemical engineers and process dynamics researchers, who have traditionally been users of the Smith predictor and related approaches, should find the various extensions of this methodology that the book presents (adaptive, nonlinear, other PDEs) to be useful and exciting. Engineers from other areas—electrical and computer engineering (telecommunication systems and networks), mechanical and aerospace engineering (combustion systems and machining), and civil/structural engineering—have no doubt faced problems with actuator delays and other distributed parameter input dynamics and will appreciate the advances introduced by this book.

This book is not meant to be a standalone textbook for any individual graduate course. However, its parts can be used as supplemental material in lectures or projects in many graduate courses:

- general distributed parameter systems (Chapters 2, 3, 6, 14–20),
- linear delay systems (Chapters 2, 3, 6, 18, and 19),
- partial differential equations (Chapters 14–20),
- nonlinear control (Chapters 10–13),

- state estimators/observers (Chapters 3 and 17),
- adaptive control (Chapters 7–9), and
- robust control (Chapters 4 and 5),
- linear time-varying (LTV) systems (Chapter 6).

The background required to read this book includes little beyond the basics of function spaces and Lyapunov theory for ODEs. However, the basics of the Poincaré and Agmon inequalities, Lyapunov and input-to-state stability, parameter projection for adaptive control, and Bessell functions are summarized in appendices for the reader's convenience.

I hope that the reader will not view the book as a collection of problems that have been solved, but will focus on it as a collection of tools and techniques that are applicable in open problems, many more of which exist than have been solved in this book, particularly in the areas of interconnected systems of ODEs and PDEs, systems with simultaneous input and state delays, nonlinear delay systems, and systems with unknown delays.

In no book are all chapters equal in value for the reader. My personal recommendations to a reader on a time budget are Chapters 7, 10, 16, and 18 if the reader is interested mainly in feedback design problems and tools. A reader primarily interested in analysis and robustness problems for delay systems might also enjoy Chapter 5.

Acknowledgments. I would like to thank Delphine Bresch-Pietri, Andrey Smyshlyaev, and Rafael Vazquez for their contributions in Chapters 8, 9, 11, 14, and elsewhere.

I am also grateful to Mrdjan Jankovic for exchanges of ideas and his guidance through the area of control of delay systems. If it were not for Mrdjan's superb and innovative papers on control of nonlinear delay systems, I would never have been enticed to start to work on these problems. I am also pleased to express special gratitude to Iasson Karafyllys for some helpful and inspiring discussions.

Many thanks to Manfred Morari, Silviu Niculescu, Galip Ulsoy, and Qing-Chang Zhong for discussions on delay systems and on the Smith predictor. I would also like to thank Anu Annaswammy for getting me intrigued with her papers on adaptive control of delay systems.

Finally, Petar Kokotovic's encouragement and interest in new research results are priceless—often a key difference between deciding whether or not to spend time on writing a new book.

Over the course of writing this book, I had the pleasure to meet Otto J. M. Smith on the occasion of my visit to the University of California at Berkeley in October 2008. "Predictably," I chose the results on predictor feedback as the topic of my Nokia Distinguished Lecture. Otto Smith was a professor at Berkeley from 1947 until his retirement in 1988. I have never met a 91-year-old person with as sharp a mind as Otto Smith's. Truth be told, I have met few 30-year-olds who would be worth a comparison. Even at this age, Otto Smith was every bit the inventor and creative engineer as his list of patents indicates. I had the pleasure of hearing about his favorite designs, from the HP function generator to his current interest in solar

energy turbine power plants with controlled focusing via heliostats. We never got to discuss the "Smith predictor"—there was so much else worth hearing about from Otto Smith's bank of engineering knowledge. I am grateful to Alex Bayen and Dean Shankar Sastry for arranging for Otto Smith to come to campus that day. I also thank Masayoshi Tomizuka for sharing many thoughts on Otto Smith during my Springer Professor sabbatical stay at Berkeley in the fall of 2007.

Sadly, Otto Smith passed away on May 10, 2009 as a result of a fall at his home. In the five decades since the publication of his influential paper on compensation of dead time, he had seen his idea become one of the most commonly used tools in control practice.

I am grateful to Cymer (Bob Akins, Danny Brown, and other friends) and General Atomics (Mike Reed, Sam Gurol, Bogdan Borowy, Dick Thome, Linden Blue, and other friends) for their support through the *Cymer Center for Control Systems and Dynamics* at UC San Diego. I also very much appreciate the support by Bosch (Nalin Chaturvedi and Aleksandar Kojic) and the National Science Foundation (Kishan Baheti and Suhada Jayasuriya).

Finally, for all the hundreds of evening and weekend hours that were spent on this book and not with my family, for all the mathematics homework that I was excused from helping with, my gratitude and love go to Alexandra, Victoria, and Angela.

La Jolla, California *Miroslav Krstic*
May 2009

Contents

Chapter 1
Introduction

1.1 Delay Systems

Time delays are ubiquitous in physical systems and engineering applications. A limited list of control applications in which delays arise includes

- chemical process control,
- combustion engines,
- rolling mills,
- control over communication networks/Internet and MPEG video transmission,
- telesurgery,
- machine tool "chatter,"
- road traffic systems.

For this reason, it is no surprise that delay systems have been an active area of research in control engineering for more than 60 years. The first work of methodological significance may have been Tsypkin's paper [225] in 1946.

Thousands of papers and dozens of books have appeared since. The most recent significant books and surveys are those by Niculescu [170], Michiels and Niculescu [153], Zhong [250], Richard [193], and Gu and Niculescu [60].

Several major bursts of activity on control of delay systems occurred throughout the 1970s, in the context of systems over rings, finite spectrum assignment, and the early efforts in control of distributed parameter systems. While this is becoming a fairly forgotten era in the development of control theory, many notable researchers (Artstein, Kalman, Khargonekar, Mitter, Morse, Sontag, Tannenbaum, etc.) contributed to those delay system-related developments in the 1970s and the early 1980s. Another burst of research activity occurred in the 1990s after the introduction of linear matrix inequalities (LMIs). Still, many basic problems in control of delay systems remain unsolved. Control of delay systems remains a very active area of research.

For a reader trained mainly in finite-dimensional systems, several basic points should be noted about delay systems:

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- delay systems are infinite-dimensional;
- the state is not a vector but a function (or a vector of functions);
- the characteristic equation is not a polynomial; it involves exponentials;
- stability analysis requires Krasovskii *functionals* rather than Lyapunov functions.

At least one (or all) of these issues is reflected in any work on control of delay systems.

To understand the multitude of possible problems in control of delay systems, it is useful to consider the following three scalar "toy problems":

- System with state delay only:

$$\dot{X}(t) = X(t-D) + U(t)\,.$$

  The plant is open-loop unstable for large D, but the stabilization problem for this system is easy and can be solved either by cancellation or by high gain. However, systems with state delays only (and no input delays) can be challenging in higher dimensions, especially if multiple delays occur. Still, these problems are the easiest in our list as they can often be solved using finite-dimensional feedback laws. The LMI-based methods have been quite successful in solving some classes of problems with state delays.
- System with input delay only:

$$\dot{X}(t) = X(t) + U(t-D)\,.$$

  This is a nontrivial problem and requires infinite-dimensional feedback for large D. The prototypical design for this problem employs predictor-based feedback. The actuator's distributed (delayed) state is employed in the feedback law. We focus on this class of problems in this book.
- System with both input delay and state delay:

$$\dot{X}(t) = X(t-D_1) + U(t-D_2)\,, \quad D_2 > D_1\,.$$

  Despite the (seemingly) scalar character of this problem, this problem is rather challenging. It requires the use of predictor feedback, but, unlike the predictor feedback for a problem with input delay only where the "semigroup" employed in the distributed delay part of the feedback is finite-dimensional, in this problem the associated semigroup is infinite-dimensional and cannot be written explicitly. The area of control design for systems with simultaneous input and state delay is underdeveloped. We do not study such systems in this book.

1.2 How Does the Difficulty of Delay Systems Compare with PDEs?

Both *delay differential equations* (DDEs) and *partial differential equations* (PDEs) belong to the broad class of infinite-dimensional or distributed parameter systems.

A delay can be modeled as a transport PDE, which is the simplest PDE in existence and belongs to the class of first-order hyperbolic PDEs. Hence, a DDE is in fact a system of interconnected ODEs and PDEs (of the simple transport type).

Because it is a system of finite-dimensional and infinite-dimensional differential equations, a DDE system can be a challenging problem, possibly (in some cases) more challenging than a single PDE in one dimension. However, in general, the PDE world presents a much richer set of challenges. It would be fair to say that going from control of PDEs to control of DDEs is much easier than the other way around.

When it comes to systems with input delays only, it is worth noting that this is actually a boundary control problem if one represents it as a cascade of the transport PDE with an ODE. We will exploit this observation throughout this book to recover some classical results on predictor feedback design and to extend this method to broader classes of systems.

The boundary control approach that we will employ is the so-called backstepping method [116]. It was originally developed for parabolic and second-order hyperbolic PDEs, as well as for several challenging physical problems such as turbulent flows and magnetohydrodynamics [228]. However, when applied to ODEs with input delay, it recovers classical results on predictor feedback, providing an explicit Lyapunov–Krasovskii functional as a bonus, which we will exploit in this book.

1.3 A Short History of Backstepping

The following is a chronological account of the development of the backstepping method with a focus on the development of tools employed in this book:

- The backstepping approach was originally developed in the 1990s for adaptive and robust (deterministic and stochastic) control of nonlinear ODEs [112, 109].
- In the last 10 years, we have been developing a backstepping approach for PDEs. In [202] we first developed a continuum backstepping approach for stabilization of parabolic linear PDEs.
- In [227] we introduced a backstepping design for linearized (but linearly unstable) Navier–Stokes PDEs.
- In [110, 111] we extended the backstepping approach to second-order hyperbolic PDEs (wave equations and beams).
- In [117, 205, 206] we developed the first adaptive designs for boundary control of linear parabolic PDEs with unknown parameters.
- In [118] we developed the backstepping designs for first-order hyperbolic PDEs and presented a design for LTI-ODEs with actuator delays, which recovers the classical predictor designs (finite spectrum assignment, modified Smith predictor, Artstein's reduction approach).
- In [229, 230] we introduced the first boundary control designs for nonlinear PDEs, focusing on a class of parabolic PDEs with nonlinear functions and Volterra series nonlinear operators on the right-hand side.

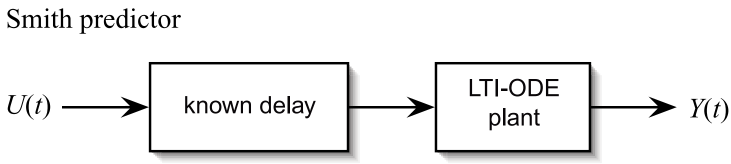

Fig. 1.1 The problem addressed by the Smith predictor and its variants (modified Smith predictor, finite spectrum assignment, Artstein's "reduction" approach).

1.4 From Predictor Feedbacks for LTI-ODE Systems to the Results in This Book

The main objective of this book is to expand the applicability of predictor-like design ideas, engendered in the Smith predictor, to other problems that are modeled as cascades of two systems in which at least one is infinite-dimensional and the other is potentially open-loop unstable.

The problem addressed with the Smith predictor and other related design frameworks (modified Smith predictor, finite spectrum assignment, Artstein's "reduction" approach) is given in Fig. 1.1.

We expand the design scope to problems shown in Fig. 1.2: adaptive control for systems with unknown delays, nonlinear plants, heat PDE and wave PDE dynamics at the input of an arbitrary, possibly unstable, LTI-ODE plant, and predictor feedback design for PDE plants.

In addition, we provide the studies of robustness to delay mismatch and external disturbances and also develop observers for LTI-ODE systems with delay or PDE sensor dynamics (Fig. 1.3).

1.5 Organization of the Book

The book is structured as follows.

1. *Part I.* Chapters 2 through 6 deal with the predictor-based controller and observer design for LTI systems, their robustness, inverse optimality, and extension to time-varying delays.
2. *Part II.* Chapters 7, 8, and 9 introduce adaptive predictor controllers for systems with unknown input delay and ODE parameters.
3. *Part III.* Chapters 10 through 13 introduce extensions of the predictor feedback design to nonlinear plants, which yield global results for nonlinear plants that are forward-complete or in the strict-feedforward form, and a regional result for plants that may exhibit finite escape in open loop.
4. *Part IV.* Chapter 14 extends the predictor design from a pure transport PDE (pure input delay) problem to the problem where the input dynamics are still of first-order hyperbolic type but are more complex, involving nonconvective effects.

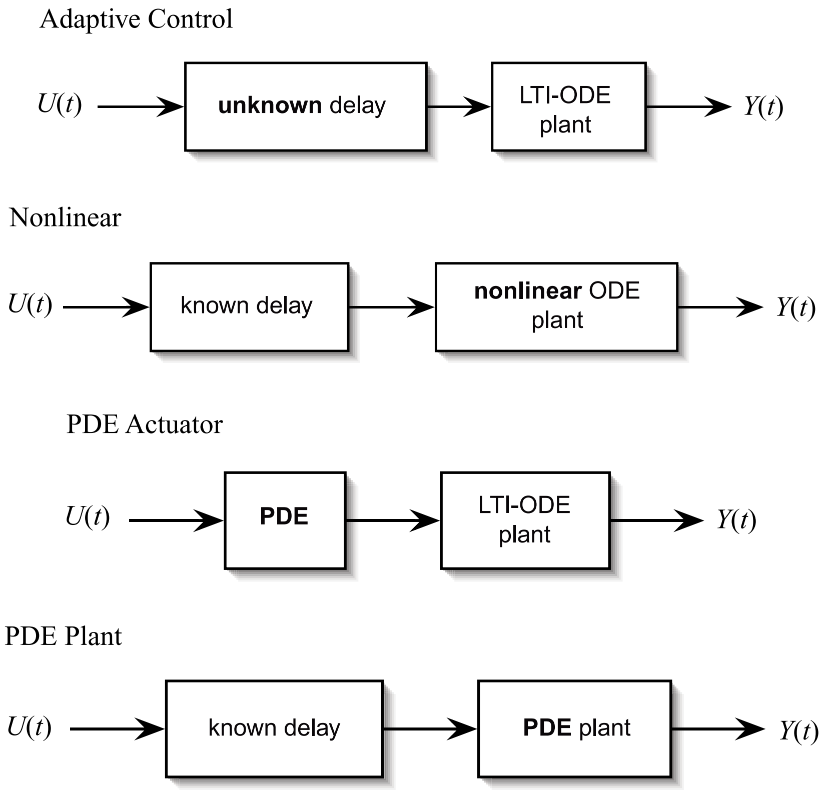

Fig. 1.2 The major problems we consider in this book.

Chapters 15 and 16 develop predictor-like feedback for the compensation of input dynamics modeled by PDEs that are yet more complex than the transport equations—the heat PDE and the wave PDE. Chapter 17 deals with observer designs for ODEs that contain sensor dynamics modeled by heat or wave PDEs.

5. *Part V*. Chapters 18 and 19 deal, respectively, with heat and wave equations with a pure delay at the input. Chapter 20 tackles PDE-PDE cascades, notably wave-heat and heat-wave cascades where the last subsystem in the cascade is unstable.

1.6 Use of Examples

Throughout the book we provide illustrations through the following three unstable or marginally stable examples:

$$G_1(s) = \frac{e^{-Ds}}{s-1}, \tag{1.1}$$

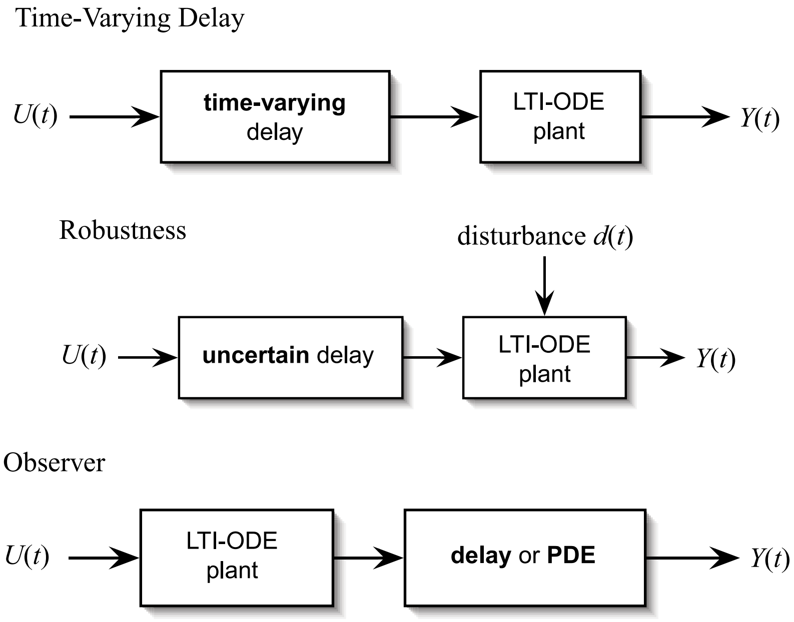

Fig. 1.3 Additional problems we consider in this book.

$$G_2(s) = \frac{e^{-Ds}}{s^2+1}, \tag{1.2}$$

$$G_3(s) = \frac{e^{-Ds}}{s(s-1)}. \tag{1.3}$$

These examples are chosen for possessing relevant properties such as open-loop instability and/or the order and relative degree higher than one in the ODE part, while being sufficiently simple so that the controller and observer designs being illustrated can be worked out by hand and in closed form.

In the chapters of the book dealing with heat and wave PDEs, we provide illustrations using, respectively, the examples

$$G_4(s) = \frac{1}{(s-1)\cosh(D\sqrt{s})}, \tag{1.4}$$

$$G_5(s) = \frac{1}{(s^2+1)\sinh(Ds)}, \tag{1.5}$$

where the hyperbolic sine and cosine functions come from the PDE actuator dynamics.

We also extensively use examples in our chapters for nonlinear systems. Our set of examples includes the first-order example (as an example of a nonlinear system that is not forward-complete)

$$\dot{Z}(t) = Z^2(t) + U(t-D), \tag{1.6}$$

the second-order example (as an example of a nonlinear system that is forward-complete but not in the strict-feedforward form)

$$\dot{Z}_1(t) = Z_2(t), \tag{1.7}$$
$$\dot{Z}_2(t) = \sin(Z_1(t)) + U(t-D), \tag{1.8}$$

the second-order example (as an example of a linearizable system in the strict-feedforward form)

$$\dot{Z}_1(t) = Z_2(t) - Z_2^2(t)U(t-D), \tag{1.9}$$
$$\dot{Z}_2(t) = U(t-D), \tag{1.10}$$

and the third-order example (as an example of a strict-feedforward system that is not linearizable)

$$\dot{Z}_1(t) = Z_2(t) + Z_3^2(t), \tag{1.11}$$
$$\dot{Z}_2(t) = Z_3(t) + Z_3(t)U(t-D), \tag{1.12}$$
$$\dot{Z}_3(t) = U(t-D). \tag{1.13}$$

1.7 Krasovskii Theorem or Direct Stability Estimates?

Krasovskii's extension of Lyapunov's theorem to systems (linear or nonlinear) that include (discrete or distributed) delay(s) on the state is broadly used in the area of control and analysis of delay systems. This theorem was first published in Krasovskii's book [100] and is also well covered in several other references [59, 60, 92, 95, 193]. Some important extensions are available in [81, 187, 188].

The Krasovskii theorem is applicable to systems with state delays, but it does not apply, at least not in an obvious manner, or in a manner that offers some advantages, to systems with input delays. Feedback systems with input delays, which are the focus of this book, contain a control law where the input signal $U(t)$ is given by an integral equation over the time interval $[t-D,t]$. The infinite-dimensional state of the system is not associated with a delay on the ODE state, $X(t)$, but with a delay on the input signal $U(t)$. This, along with the implicit character of the control law for $U(t)$, results in the state of the system not being defined in a manner that is consistent with the classical Krasovskii theorem.

A natural way to study stability of systems with input delay comes from the set of techniques commonly used in boundary control of PDEs. In our developments the state of the overall system typically consists of the plant state (which is finite-dimensional when the plant is an ODE) and the actuator state (which is always infinite-dimensional when we have actuator delay). Similarly, when we have sensor delay, an infinite-dimensional sensor state will be included. As is common in the area of stabilization of PDEs, we don't use a ready-made Lyapunov-like theorem, but rather build our Lyapunov functionals and compute their stability estimates (exponential or otherwise) in an ad hoc manner. There is no disadvantage to this approach since the bulk of the effort in applying any Lyapunov–Krasovskii-like theorem for infinite-dimensional systems is in verifying the conditions of such a theorem, not in proving the theorem. In other words, the step of *inferring a decay estimate* (via an off-the-shelf Lyapunov–Krasovskii theorem) from

- a differential inequality on the Lyapunov functional, and from
- upper and lower bounds on the Lyapunov functional in terms of the system norm

is a tiny fraction of the overall endeavor of deriving a stability estimate, so we do not add any significant burden to our work by conducting the stability studies in an ad hoc manner.

The advantage of not relying on a ready-made Lyapunov–Krasovskii-like theorem is that all the steps of our analysis are presented explicitly; thus, we obtain explicit estimates, both for the decay rates and for the overshoot coefficients (in linear problems).

An additional advantage in not having to rely on the state-of-the-art in existing stability theorems for systems involving delays is that we consider some relatively unconventional problems, including PDE-ODE cascades, delay-PDE cascades, predictor feedback with a perturbation on the length of the delay state, problems with a time-varying delay, and delay-adaptive control where asymptotic stability is not achieved, but only stability with regulation of the plant state and the actuator state (the parameter estimation error does not necessarily converge).

Lyapunov analysis and computation of stability estimates are not the only key elements of our presentation. Equally important, if not the most important, is the construction of our "backstepping" transformations. Our transformed systems (which we refer to as "target systems" in the backstepping approach) will have an extremely simple structure of a cascade of an unforced transport PDE, feeding into an asymptotically stable system. The Lyapunov functional for such a transformed system will be extremely simple, consisting of a Lyapunov function(al) for the driven system and of the square of a weighted L_2 norm for the transport PDE. This simplicity is achieved by design. However, if one looks at the dependence of the Lyapunov–Krasovskii functional on the original (distributed) state of the actuator, this functional is quite complex and contains integrals of weighted quadratic forms, as well as nested weighted integrals, on the actuator state. It also contains cross terms between the plant state and the weighted integral of the actuator state.

A reader well versed in the Lyapunov–Krasovskii techniques for systems with state delays only will find our Lyapunov functionals involving the actuator state to be familiar and not surprising in retrospect given the structure of such functionals for systems with state delays only. While we have emphasized that we do not rely on Krasovskii's *theorem* [100], we will refer throughout the text to the Lyapunov-like functionals that we construct as "Lyapunov–Krasovskii functionals." This usage of the term refers to the structure of these functionals rather than to their relation with any off-the-shelf theorems on stability of systems with delays.

1.8 DDE or Transport PDE Representation of the Actuator/Sensor State?

When dealing with the case of a known constant delay at the input of an LTI system, it is a matter of choice whether one will use a delay differential equation (DDE) model or treat the input delay as a transport PDE. The entire analysis can be conducted in either of the two notational settings and the Lyapunov–Krasovskii functional can be represented in either of the two formats.

However, the advantage of using the PDE representation starts becoming clearer in problems where the delay dynamics start being replaced by more complex PDE dynamics, or where the problem starts calling for the use of norms on the infinite-dimensional state other than the standard L_2 norm. We pursue the transport PDE representation in problems with input delays so that we can smoothly transition to problems with nondelay infinite-dimensional dynamics. This provides the book with notational uniformity and a consistency in the use of similar technical tools in different parts of the book. The reader accustomed to the conventional treatment of delay systems, particularly to standard representations of Krasovskii functionals, will need to apply a little effort to get used to our PDE-inspired approach but will benefit from this in latter parts of the book. For clarity and interpretation, whenever appropriate, we give stability results involving the transport PDE state as well as the more conventional representation using the input variable $U(\theta)$ over the time window $\theta \in [t-D,t]$.

1.9 Notation, Spaces, Norms, and Solutions

Since we model actuator and sensor dynamics via transport PDEs (in case of delays) or via heat or wave PDEs, the actuator/sensor dynamics are modeled by a PDE whose state is $u(x,t)$, where t is time and x is a spatial variable that takes values in the interval $[0,D]$. When the plant being controlled (or whose state is estimated) is an ODE, then the overall state of the system is the state of the ODE, $X(t)$, along with the state of the PDE, $u(x,t), x \in [0,D]$. Typically, we study stability in the sense of a 2-norm on $X(t)$ and an $L_2[0,D]$ norm on $u(x,t)$, but this is not always the case.

The 2-norm of a finite-dimensional vector $X(t)$ is denoted by single bars, $|X(t)|$. In contrast, norms of functions (of x) are denoted by double bars. By default, $\|\cdot\|$ denotes the $L_2[0,D]$ norm, i.e., $\|\cdot\| = \|\cdot\|_{L_2[0,D]}$. For other norms, we emphasize which norm we are referring to by using a subscript, for example, $\|\cdot\|_{L_p[0,D]}$ or $\|\cdot\|_{L_\infty[0,D]}$.

Since the PDE state variable $u(x,t)$ is a function of two arguments, x and t, we should emphasize that taking a norm in one of the variables, for example, in x, makes the norm a function of the other variable. For example,

$$\|u(t)\| = \left(\int_0^D u^2(x,t)dx \right)^{1/2} \tag{1.14}$$

or

$$\|u(t)\|_{L_\infty[0,D]} = \sup_{x\in[0,D]} |u(x,t)|. \tag{1.15}$$

In some situations we will also make use of weighted norms, such as, for example,

$$\|u(t)\|_{c,\infty} = \sup_{x\in[0,D]} \mathrm{e}^{cx}|u(x,t)|, \tag{1.16}$$

for some $c > 0$.

In addition to norms in x, we will occasionally be using norms in time, for example,

$$\|u(x)\|_{L_2[0,\infty)} = \left(\int_0^\infty u^2(x,t)dt\right)^{1/2} \tag{1.17}$$

or

$$\|u\|_{L_2[0,\infty)} = \left(\int_0^\infty \int_0^D u^2(x,t)dxdt\right)^{1/2}. \tag{1.18}$$

In some situations the stability statements will not be pursued in the sense of norms $L_2[0,D]$ or $L_\infty[0,D]$, which are very common in delay systems, but instead in Sobolev norms such as $H_1[0,D]$ or even $H_2[0,D]$. For a function $u(x,t)$, these norms are defined as

$$\|u(t)\|_{H_1[0,D]} = \left(\int_0^D \left[u^2(x,t) + u_x^2(x,t)\right] dx\right)^{1/2} \tag{1.19}$$

and

$$\|u(t)\|_{H_2[0,D]} = \left(\int_0^D \left[u^2(x,t) + u_x^2(x,t) + u_{xx}^2(x,t)\right] dx\right)^{1/2}, \tag{1.20}$$

respectively.

While the transport PDE has only one boundary condition, second-order (in space) PDEs like the heat equation and the wave equation have two boundary conditions, one at $x = 0$ and the other at $x = D$. When neither of the two conditions is of the homogeneous Dirichlet type (it may be the Neumann type, or the mixed/Robin type, or the damping/antidamping type), the norm on the actuation/sensor dynamics has to be defined slightly differently. The norm in that case has to include a boundary value of the state at $x = 0$ or $x = D$, such as, for example,

$$\left(u^2(0,t) + \int_0^D \left[u_x^2(x,t) + u_t^2(x,t)\right] dx\right)^{1/2} \tag{1.21}$$

in the case of a wave equation with a Neumann boundary condition on the "free end" and with Neumann actuation on the controlled end. The reader should note that since the wave equation is not only second-order in x but also second-order in t, the state of the PDE includes both the "shear" variable $u_x(x,t)$ and the velocity variable $u_t(x,t)$. Likewise, the norm of the system includes both the potential and the kinetic energy.

A comment is in order on the issue of the existence and uniqueness of solutions. The closed-loop system in our work has the so-called classical solution, namely, a solution that has continuous derivatives in all of the independent variables of the model (time and space) with respect to which derivatives appear in the model, provided the initial condition and the initial control value are *consistent* with the feedback law. Otherwise, namely, if the initial control and plant state do not satisfy the consistency condition, the closed-loop system has a solution that is only continuous in time, with states taking values in the function space in which the stability estimate is stated. We do not belabor this issue in our presentation, as it tends to distract the reader from the main purpose of our presentation—stability and the design of stabilizing feedback laws—and it tends to give a less seasoned reader the impression that the mathematical barrier for understanding the material is higher that it really is.

1.10 Beyond This Book

Before we proceed with the material in this book, we briefly discuss its key limitations and future opportunities for extensions of the results of this book. The problems we discuss next are not merely "rhetorical" open problem. For most of them, we do not have any preliminary results or highly promising leads. A reader inspired by the tools presented in the book is more than welcome to pursue the solution of these problems.

Systems with Both Input and State Delays

The single most important open area in control of delay systems is control of systems with simultaneous input and state delays. This opinion was expressed in [193], and we agree with it. The literature on control of delay systems prior to this book exhaustively covers systems with state delays, both of the discrete kind and distributed delays. Even nonlinear systems with state delays represent a fairly advanced area at the moment, not to mention linear systems with state delays. With this book being focused exclusively on systems with input delays, the major research opportunities exist in marrying the techniques from two subproblems and solving stabilization problems for systems with simultaneous state and input delays. This is by no means a simple extension; in fact, the challenges grow by an order of magnitude relative to state or input delays alone.

Systems with State-Dependent Input Delays

The reaction time of a driver is often modeled as a pure delay. This time is larger when the driver is tired or intoxicated. However, in practice the reaction time is not constant. It depends on the intensity of the disturbance, or the size of the tracking error, to which the driver is reacting. It is plausible to assume that a very large disturbance or tracking error (resulting, for example, from an unexpected patch of ice on the road) can have a nearly "paralyzing" effect on the driver, extending his or her reaction time considerably relative to a nominal value. It is of interest, both practically and mathematically, to study such problems. There may seem to be some relation between this problem and the problem of time-varying delay studied in Chapter 6; however, it is not clear that the results for known time-varying delays can be directly applied to the case of state-dependent delays since even if the delay is known as a function of the state, it is not clear how the *prediction time*, denoted in Chapter 6 as $\phi^{-1}(t) - t$, would be determined.

Delay-Adaptive Control for Nonlinear Systems

Given the rich literature on adaptive control of nonlinear systems that emerged in the 1990s, it would be exciting to consider problems of nonlinear control in the presence of highly uncertain input delays. The approach we introduced in Chapter 7 for linear systems does not extend in an obvious way to nonlinear plants since we use the linear boundedness of the plant model in our stability proof. Other approaches should be sought and devised for delay-adaptive control for select classes of nonlinear systems, particularly for strict-feedforward systems with unknown input delays.

Nonlinear Systems with Input Dynamics Governed by Heat or Wave PDEs

While we solve the stabilization problem for several classes of nonlinear systems with input delays in Part III of the book, similar problems can be pursued for nonlinear systems with other kinds of distributed-parameter input dynamics, particularly for input dynamics modeled by heat and wave PDEs. The promise for solving such problems comes from the recent results for feedback linearization of nonlinear parabolic PDEs in [229, 230], where nonlinear Volterra series were employed, with integration in x, to construct the stabilizing feedback laws and the backstepping transformations. It would seem promising to try a similar approach for nonlinear systems with input dynamics modeled by PDEs. Things are likely to get much more complex than when the input dynamics are of the delay type, where the predictor idea provides an intuitive design tool. While the form of the feedback law will be the same irrespective of whether the input dynamics are of, say, the transport PDE type or the heat PDE type, the process of arriving at the feedback law will be much more complex in the latter case.

Multi-Input Systems

Many of the results in the book seem to easily (or even trivially) extend to systems with multiple inputs when the delay is the same in all the input channels. It is worth considering problems with several inputs with different delays, different diffusion coefficients in individual channels, different wave propagation speeds in individual channels, or distributed coupling between the input channels. The most elementary in the last set of problems is the case of an ODE with two inputs and the actuator dynamics modeled by the Schrodinger PDE.

PDE-PDE Cascades

While in Chapter 20 we only hint at the opportunities that exist in extending the results from Part IV of the book to PDE-PDE cascades, much still remains to be done even on the topic of studying the implementability of the control laws in Chapter 20, and certainly much more remains to be done on the topic of control of cascades of PDEs from different families that are interconnected through boundary conditions.

Part I
Linear Delay-ODE Cascades

Chapter 2
Basic Predictor Feedback

In this chapter we introduce the basic idea of a PDE backstepping design for systems with input delay. We treat the input delay as a transport PDE, an elementary first-order hyperbolic PDE. Our design yields a classical control formula obtained through various other approaches—modified Smith predictor (mSP), finite spectrum assignment (FSA), and the Artstein–Kwon–Pierson "reduction" approach. The backstepping approach is distinct because it provides a construction of an infinite-dimensional transformation of the actuator state, which yields a cascade system of transformed stable actuator dynamics and stabilized plant dynamics. Our design results in the construction of an explicit Lyapunov–Krasovskii functional and an explicit exponential stability estimate.

The basic ideas introduced in this chapter are the core for all the developments in the rest of the book. They are made parameter-adaptive, when the delay is unknown, in Part II, extended to nonlinear plants in Part III, and extended to PDE plants in Part IV. They are also converted to solve dual problems, such as observer design in the presence of sensor delay in Chapter 3.

We do not deal with the original "Smith predictor" (SP) [201] in detail in this chapter, as it is a rather different tool than the mSP/FSA/reduction approach. While these approaches are inspired mainly by full-state feedback considerations (though they do extend to output feedback problems), the original Smith predictor is a frequency-domain idea, inspired by different considerations than the ones we pursue here. However, in Sections 3.4 and 3.5 we present a side discussion that connects an observer-based predictor feedback design for systems with input delay with the classical Smith predictor.

We start with a basic idea of predictor feedback in Section 2.1. We then introduce a backstepping-based predictor design in Section 2.2 and explain in Section 2.3 that it results in the same controller as the mSP/FSA/reduction approaches, but with an additional benefit of providing a Lyapunov function. The predictor feedback design is illustrated with examples in Section 2.5. The heart of this chapter is stability analysis, which is presented in Section 2.4, with the aid of a backstepping-based Lyapunov function, and in Section 2.6 without using a Lyapunov function.

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Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_2,

2.1 Basic Idea of Predictor Feedback Design for ODE Systems with Actuator Delay

We consider the linear infinite-dimensional system

$$\dot{X}(t) = AX(t) + BU(t-D), \tag{2.1}$$

where $X \in \mathbb{R}^n$, (A,B) is a controllable pair, and the input signal $U(t)$ is delayed by D units of time.

Given a stabilizing gain vector K for the undelayed system, namely, given a vector K such that the matrix $A+BK$ is Hurwitz, our wish is to have a control that achieves

$$U(t-D) = KX(t). \tag{2.2}$$

This control can be alternatively written as

$$U(t) = KX(t+D), \tag{2.3}$$

and it appears to be nonimplementable since it requires future values of the state. However, with the variation-of-constants formula, treating the current state $X(t)$ as the initial condition, we have

$$X(t+D) = \mathrm{e}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta, \quad \forall t \geq 0. \tag{2.4}$$

This yields a feedback law

$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right], \quad \forall t \geq 0, \tag{2.5}$$

which is implementable, but it is infinite-dimensional, since it contains the distributed delay term involving past controls, $\int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta$. The closed-loop system is delay-compensated,

$$\dot{X}(t) = (A+BK)X(t), \quad t \geq D, \tag{2.6}$$

but this is true only after the control "kicks in" at $t = D$. During the interval $t \in [0,D]$, the system state is governed by

$$X(t) = \mathrm{e}^{At}X(0) + \int_{0}^{t} \mathrm{e}^{A(t-\tau)}BU(\tau-D)d\tau, \quad \forall t \in [0,D]. \tag{2.7}$$

The feedback law (2.5) was introduced within the framework of "finite spectrum assignment" [121, 135] and the "reduction approach" [8]. In the next section we derive the same control law, but in a considerably more complicated way, which will pay dividends later on by providing us with an explicit Lyapunov–Krasovskii function and the ability to conduct stability analysis in the time domain.

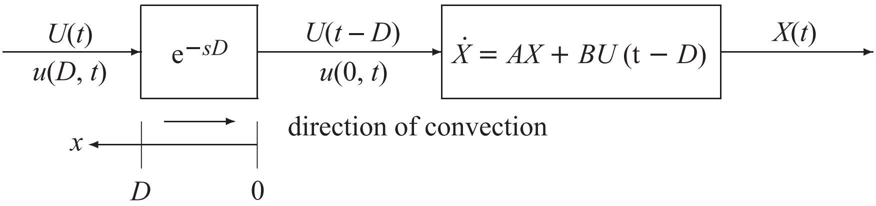

Fig. 2.1 Linear system $\dot{X}(t) = AX(t) + BU(t-D)$ with actuator delay D.

2.2 Backstepping Design Via the Transport PDE

The delay in the system (2.1) can be modeled by the following first-order hyperbolic PDE, also referred to as the "transport PDE":

$$u_t(x,t) = u_x(x,t), \tag{2.8}$$

$$u(D,t) = U(t). \tag{2.9}$$

The solution to this equation is

$$u(x,t) = U(t+x-D), \tag{2.10}$$

and therefore the output

$$u(0,t) = U(t-D) \tag{2.11}$$

gives the delayed input. The system (2.1) can now be written as

$$\dot{X}(t) = AX(t) + Bu(0,t). \tag{2.12}$$

Equations (2.8)–(2.12) form an ODE–PDE cascade that is driven by the input U from the boundary of the PDE (Fig. 2.1).

Suppose a static state feedback control has been designed for a system with no delay (i.e., with $D = 0$) such that

$$U(t) = KX(t) \tag{2.13}$$

is a stabilizing controller; i.e., the matrix $(A+BK)$ is Hurwitz. Consider the backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy - \gamma(x)^T X(t) \tag{2.14}$$

with which we want to map the system (2.8)–(2.12) into the target system

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \tag{2.15}$$

$$w_t(x,t) = w_x(x,t), \tag{2.16}$$
$$w(D,t) = 0\,. \tag{2.17}$$

The reason for selecting the transformation (2.14) is the following. The ODE-PDE system

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{2.18}$$
$$u_t(x,t) = u_x(x,t), \tag{2.19}$$
$$u(D,t) = U(t) \tag{2.20}$$

has a block-lower-triangular structure, where the key "off-diagonal" component is the potentially unstable plant dynamics $AX(t)$. The transformation (2.14) is selected also to have a lower-triangular part. This transformation is to be understood as a part of the complete 2×2 transformation

$$(X,u) \mapsto (X,w), \tag{2.21}$$

which has a lower-triangular form

$$\begin{bmatrix} X \\ w \end{bmatrix} = \begin{bmatrix} I_{n\times n} & 0_{n\times[0,D]} \\ \Gamma & \mathscr{Q} + I_{[0,D]\times[0,D]} \end{bmatrix} \begin{bmatrix} X \\ u \end{bmatrix}, \tag{2.22}$$

where $I_{n\times n}$ denotes the identity matrix, $I_{[0,D]\times[0,D]}$ denotes the identity operator on the functions $u(x,t)$ of the argument $x \in [0,D]$, the symbol Γ denotes the operator

$$\Gamma:\ X(t) \mapsto \gamma(x)^T X(t), \tag{2.23}$$

and the symbol $\mathscr{Q}$ denotes the Volterra operator

$$\mathscr{Q}:\ u(x,t) \mapsto \int_0^x q(x,y)u(y,t)dy. \tag{2.24}$$

So, due to the lower-triangularity of $\mathscr{Q}$, the overall transformation $(X,u) \mapsto (X,w)$ is lower-triangular. Furthermore, the diagonal of this transformation is the identity operator,

$$\mathrm{Id} = \mathrm{diag}\left\{I_{n\times n},\ I_{[0,D]\times[0,D]}\right\}. \tag{2.25}$$

Due to the triangular structure, the transformation (2.22) is not only suitable for converting the system (2.18)–(2.20) into the target form (2.15)–(2.17)—with the help of an appropriate boundary feedback law—but is also invertible, as we shall see soon.

Let us now calculate the time and spatial derivatives of the transformation (2.14):

$$\begin{aligned} w_x(x,t) = u_x(x,t) - q(x,x)u(x,t) - \int_0^x q_x(x,y)u(y,t)dy \\ - \gamma'(x)^T X(t), \end{aligned} \tag{2.26}$$

$$w_t(x,t) = u_t(x,t) - \int_0^x q(x,y)u_t(y,t)dy$$
$$- \gamma(x)^T [AX + Bu(0)] \tag{2.27}$$
$$= u_x(x,t) - q(x,x)u(x,t) + q(x,0)u(0,t)$$
$$+ \int_0^x q_y(x,y)u(y,t)dy - \gamma(x)^T [AX + Bu(0,t)]. \tag{2.28}$$

Subtracting (2.26) from (2.28), we get

$$\int_0^x (q_x(x,y) + q_y(x,y))u(y,t)dy$$
$$+ \left[q(x,0) - \gamma(x)^T B\right] u(0,t)$$
$$+ \left[\gamma'(x)^T - \gamma(x)^T A\right] X(t) = 0. \tag{2.29}$$

This equation should be valid for all u and X, so we have three conditions:

$$q_x(x,y) + q_y(x,y) = 0, \tag{2.30}$$
$$q(x,0) = \gamma(x)^T B, \tag{2.31}$$
$$\gamma'(x) = A^T \gamma(x). \tag{2.32}$$

The first two conditions form a first-order hyperbolic PDE and the third one is a simple ODE. To find the initial condition for this ODE, let us set $x = 0$ in (2.14), which gives

$$w(0,t) = u(0,t) - \gamma(0)^T X(t). \tag{2.33}$$

Substituting this expression into (2.15), we get

$$\dot{X}(t) = AX(t) + Bu(0,t) + B\left(K - \gamma(0)^T\right) X(t). \tag{2.34}$$

Comparing this equation with (2.12), we have

$$\gamma(0) = K^T. \tag{2.35}$$

Therefore, the solution to the ODE (2.32) is $\gamma(x) = \mathrm{e}^{A^T x} K^T$, which gives

$$\gamma(x)^T = K\mathrm{e}^{Ax}. \tag{2.36}$$

A general solution to (2.30) is

$$q(x,y) = \phi(x-y), \tag{2.37}$$

where the function ϕ is determined from (2.31). We get

$$q(x,y) = K\mathrm{e}^{A(x-y)}B. \tag{2.38}$$

We can now plug the gains $\gamma(x)$ and $q(x,y)$ into the transformation (2.14) and set $x = D$ to get the control law:

$$u(D,t) = \int_0^D K e^{A(D-y)} B u(y,t) dy + K e^{AD} X(t) . \tag{2.39}$$

2.3 On the Relation Among the Backstepping Design, the FSA/Reduction Design, and the Original Smith Controller

The controller (2.39) is given in terms of the transport delay state $u(y,t)$. Using (2.10), one can also derive the representation in terms of the input signal $U(t)$:

$$U(t) = K \left[e^{AD} X(t) + \int_{t-D}^{t} e^{A(t-\theta)} B U(\theta) d\theta \right] , \tag{2.40}$$

which is identical to the controller (2.5) in Section 2.1.

The controller (2.40) was first derived in the years 1978–1982 in the framework of “finite spectrum assignment” [121, 135] and the “reduction approach” [8]. The idea of the reduction approach is to introduce the “predictor state”

$$P(t) = X(t + D) , \tag{2.41}$$

which is alternatively defined as

$$P(t) = e^{AD} X(t) + \int_{t-D}^{t} e^{A(t-\theta)} B U(\theta) d\theta , \tag{2.42}$$

and study the control of the reduced finite-dimensional system

$$\dot{P}(t) = AP(t) + BU(t) , \quad t \geq 0 . \tag{2.43}$$

The resulting control law is simply

$$U(t) = KP(t) , \quad t \geq 0 , \tag{2.44}$$

and it is also written as (2.40). The predictor transformation (2.42) and the simple, intuitive design based on the reduction approach do not equip the designer with a tool for Lyapunov–Krasovskii stability analysis. The reason for this is that the transformation $P(t)$ is only a transformation of the ODE state $X(t)$, rather than also providing a suitable change of variable for the infinite-dimensional actuator state $u(x,t)$. As a result, the analysis in [8, 121, 135] does not capture the entire system consisting of the ODE plant and the infinite-dimensional subsystem of the input delay. As we shall see in Section 2.4, the backstepping construction permits a stability analysis of the complete feedback system with the cascade PDE-ODE plant

(2.15)–(2.17) and the infinite-dimenasional control law (2.40), resulting in an exponential stability estimate in the appropriate norm for this system.

It is important also to appreciate the difference between the original Smith predictor [201] and the modified Smith predictors such as "finite spectrum assignment" [176, 121, 135], the "reduction" approach [8], and the latest incarnation of the same design, in which it appears in this book—"backstepping." The original Smith predictor [201] was a frequency-domain design, so it is not trivial to establish a parallel between it and the subsequent modifications. However, the main idea of the Smith predictor, if one were to develop it for a state-space problem, leads to feedback of the form

$$U(t) = K\left[X(t) + \int_0^t e^{A(t-\theta)}BU(\theta)d\theta - \int_{-D}^{t-D} e^{A(t-D-\theta)}BU(\theta)d\theta\right]. \tag{2.45}$$

This feedback law is different than the (modified) predictor feedback pursued in this book, which is given in the form (2.40). The Smith predictor is suitable for compensating the effect of input delay on set-point regulation problems for stable plants. However, it is well known that the Smith predictor offers no stability guarantee for unstable plants.

We do not pursue the explanation for the potential feedback instability under the original Smith predictor feedback for unstable plants. This argument gets complex in the same manner as the positive stability argument for the feedback (2.40), which is given in the next section.

2.4 Stability of Predictor Feedback

Now we study closed-loop stability, both in the transformed variables where exponential stability is nearly obvious, and in the original variables where it is less easily evident.

From this point on, and throughout the book, we use the following notion of exponential stability.

Definition 2.1. Consider the (evolution equation) system

$$\dot{z}(t) = \mathscr{A}z(t), \tag{2.46}$$

where $z(t)$ belongs to a (possibly infinite-dimensional) Banach space $\mathscr{B}$, and $\mathscr{A}$ is the system's infinitesimal generator. Let $\|\cdot\|_{\mathscr{B}}$ denote a norm associated with $\mathscr{B}$. The equilibrium $z = 0$ of system (2.46) is said to be *exponentially stable* if there exist positive constants ρ and α such that

$$\|z(t)\|_{\mathscr{B}} \leq \rho e^{-\alpha t}\|z_0\|_{\mathscr{B}}, \quad \forall t \geq 0, \tag{2.47}$$

where z_0 denotes the initial condition $z(0)$.

Now we state and prove an exponential stability result for the closed-loop system with the predictor feedback.

Theorem 2.1. *The closed-loop system consisting of the plant* (2.8), (2.9), (2.12) *with the controller* (2.39) *is exponentially stable at the origin in the sense of the norm*

$$\left(|X(t)|^2+\int_0^D u(x,t)^2dx\right)^{1/2}. \tag{2.48}$$

Proof. First we prove that the origin of the target system (2.15)–(2.17) is exponentially stable. Consider a Lyapunov–Krasovskii functional

$$V(t)=X(t)^TPX(t)+\frac{a}{2}\int_0^D(1+x)w(x,t)^2\,dx, \tag{2.49}$$

where $P=P^T>0$ is the solution to the Lyapunov equation

$$P(A+BK)+(A+BK)^TP=-Q \tag{2.50}$$

for some $Q=Q^T>0$, and the parameter $a>0$ is to be chosen later. We have

$$\begin{aligned}\dot V(t)&=X(t)^T((A+BK)^TP+P(A+BK))X(t)\\&\quad+2X(t)^TPBw(0,t)-\frac{a}{2}w(0,t)^2-\frac{a}{2}\int_0^D w(x,t)^2\,dx\\&\le -X(t)^TQX(t)+\frac{2}{a}|X(t)^TPB|^2-\frac{a}{2}\int_0^D w(x,t)^2\,dx.\end{aligned} \tag{2.51}$$

Let us choose

$$a=\frac{4\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q)}, \tag{2.52}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of the corresponding matrices. Then

$$\begin{aligned}\dot V(t)&\le -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2-\frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q)}\int_0^D w(x,t)^2\,dx\\&\le -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2-\frac{2\lambda_{\max}(PBB^TP)}{(1+D)\lambda_{\min}(Q)}\int_0^D(1+x)w(x,t)^2\,dx\\&= -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2-\frac{a}{2(1+D)}\int_0^D(1+x)w(x,t)^2\,dx\\&\le -\min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)},\frac{1}{1+D}\right\}V(t).\end{aligned} \tag{2.53}$$

So we obtain

$$\dot V(t)\le -\mu V(t), \tag{2.54}$$

where

$$\mu = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{1}{1+D}\right\}. \tag{2.55}$$

Thus, the closed-loop system is exponentially stable in the sense of the full state norm

$$\left(|X(t)|^2 + \int_0^D w(x,t)^2 dx\right)^{1/2}, \tag{2.56}$$

i.e., in the transformed variable (X,w). To show exponential stability in the sense of the norm $(|X(t)|^2 + \int_0^D u(x,t)^2 dx)^{1/2}$, we need the inverse of the transformation (2.14). One can show with calculations similar to (2.28)–(2.38) that such a transformation is

$$u(x,t) = w(x,t) + \int_0^x K\mathrm{e}^{(A+BK)(x-y)} Bw(y,t)\,dy + K\mathrm{e}^{(A+BK)x} X(t). \tag{2.57}$$

Let us now denote the backstepping transformation and its inverse in compact form as

$$w(x,t) = u(x,t) - \int_0^x m(x-y)u(y,t)dy - KM(x)X(t), \tag{2.58}$$

$$u(x,t) = w(x,t) + \int_0^x n(x-y)w(y,t)dy + KN(x)X(t), \tag{2.59}$$

where

$$m(s) = KM(s)B, \tag{2.60}$$

$$n(s) = KN(s)B, \tag{2.61}$$

$$M(x) = \mathrm{e}^{Ax}, \tag{2.62}$$

$$N(x) = \mathrm{e}^{(A+BK)x}, \tag{2.63}$$

or even more compactly as

$$w(x,t) = u(x,t) - m(x) \star u(x,t) - KM(x)X(t), \tag{2.64}$$

$$u(x,t) = w(x,t) + n(x) \star w(x,t) + KN(x)X(t), \tag{2.65}$$

where $\star$ denotes the convolution operation in x. To derive a stability bound, we need to relate the norm

$$\left(|X(t)|^2 + \textstyle\int_0^D u(x,t)^2 dx\right)^{1/2}$$

to the norm

$$\left(|X(t)|^2 + \textstyle\int_0^D w(x,t)^2 dx\right)^{1/2},$$

and then the norm

$$\left(|X(t)|^2 + \int_0^D w(x,t)^2 dx\right)^{1/2}$$

to

$$\sqrt{V(t)} = \left(X(t)^T P X(t) + \frac{a}{2}\int_0^D (1+x) w(x,t)^2\, dx\right)^{1/2}.$$

We start from the latter, as it is easier, and obtain

$$\psi_1\left(|X(t)|^2 + \int_0^D w(x,t)^2 dx\right) \le V(t) \le \psi_2\left(|X(t)|^2 + \int_0^D w(x,t)^2 dx\right), \quad (2.66)$$

where

$$\psi_1 = \min\left\{\lambda_{\min}(P), \frac{a}{2}\right\}, \quad (2.67)$$

$$\psi_2 = \max\left\{\lambda_{\max}(P), \frac{a(1+D)}{2}\right\}. \quad (2.68)$$

It is easy to show, using (2.64) and (2.65), that

$$\int_0^D w(x,t)^2 dx \le \alpha_1 \int_0^D u(x,t)^2 dx + \alpha_2 |X(t)|^2, \quad (2.69)$$

$$\int_0^D u(x,t)^2 dx \le \beta_1 \int_0^D w(x,t)^2 dx + \beta_2 |X(t)|^2, \quad (2.70)$$

where

$$\alpha_1 = 3\left(1 + D\|m\|^2\right), \quad (2.71)$$

$$\alpha_2 = 3\|KM\|^2, \quad (2.72)$$

$$\beta_1 = 3\left(1 + D\|n\|^2\right), \quad (2.73)$$

$$\beta_2 = 3\|KN\|^2, \quad (2.74)$$

and $\|\cdot\|$ denotes the $L_2[0,D]$ norm. Hence, we obtain

$$\phi_1\left(|X(t)|^2 + \int_0^D u(x,t)^2 dx\right) \le |X(t)|^2 + \int_0^D w(x,t)^2 dx, \quad (2.75)$$

$$|X(t)|^2 + \int_0^D w(x,t)^2 dx \le \phi_2\left(|X(t)|^2 + \int_0^D u(x,t)^2 dx\right), \quad (2.76)$$

where

$$\phi_1 = \frac{1}{\max\{\beta_1, \beta_2 + 1\}}, \quad (2.77)$$

$$\phi_2 = \max\{\alpha_1, \alpha_2 + 1\}. \quad (2.78)$$

Combining the above inequalities, we get

$$\phi_1\psi_1\left(|X(t)|^2+\int_0^D u(x,t)^2dx\right)\le V(t)\le \phi_2\psi_2\left(|X(t)|^2+\int_0^D u(x,t)^2dx\right). \tag{2.79}$$

Hence, with (2.54), we get

$$|X(t)|^2+\int_0^D u(x,t)^2dx\le \frac{\phi_2\psi_2}{\phi_1\psi_1}\mathrm{e}^{-\mu t}\left(|X(0)|^2+\int_0^D u(x,0)^2dx\right), \tag{2.80}$$

which completes the proof of exponential stability. □

Remark 2.1. It is clear that the exponential stability estimate (2.80) is conservative. The decay rate $\mu/2$, where μ is defined in (2.55), seems like it could be much lower than $\min_i\{\mathrm{Re}\{-\lambda_i(A+BK)\}\}$. The overshoot coefficient $\phi_2\psi_2/\phi_1\psi_1$ looks equally conservative, though it is clear that its value must be large since the plant runs in an open loop until the control kicks in at $t=D$. Despite the conservatism in the Lyapunov analysis, it does quantitatively capture the dependence on time and on the initial conditions in the chosen norm (note that this choice is not unique). This cannot be said for [121, 135, 8], where stability is not even claimed in precise terms, but, instead, only a statement on eigenvalues is made.

2.5 Examples of Predictor Feedback Design

Example 2.1. Consider the second-order plant

$$Y(s)=\frac{\mathrm{e}^{-Ds}}{s^2+1}U(s), \tag{2.81}$$

i.e., the system

$$\ddot\eta(t)+\eta(t)=U(t-D). \tag{2.82}$$

This is a neutrally stable system with eigenvalues on the imaginary axis. Its state-space form is

$$\dot\xi_1(t)=\xi_2(t), \tag{2.83}$$

$$\dot\xi_2(t)=-\xi_1(t)+U(t-D), \tag{2.84}$$

where

$$\xi_1=\eta, \tag{2.85}$$

$$\xi_2=\dot\eta, \tag{2.86}$$

and

$$X=\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}. \tag{2.87}$$

The objective with this system would be to add some damping, so the nominal feedback would be just a simple derivative control,

$$U(t) = -h\dot{\eta}(t), \qquad h > 0; \tag{2.88}$$

i.e., the nominal feedback gain vector is

$$K = [0 \quad -h]. \tag{2.89}$$

The predictor-based version of this feedback employs a matrix exponential

$$e^{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(t-\theta)} = \begin{bmatrix} \cos(t-\theta) & \sin(t-\theta) \\ -\sin(t-\theta) & \cos(t-\theta) \end{bmatrix} \tag{2.90}$$

and can be obtained as

$$U(t) = h\eta(t)\sin(D) - h\dot{\eta}(t)\cos D - h\int_{t-D}^{t} \cos(t-\theta)U(\theta)d\theta. \tag{2.91}$$

Note that for $D = 0$, this feedback reduces to $U = -h\dot{\eta}$. The time response of the closed-loop system (2.83)–(2.91) is given by

$$\eta(t) = \cos(t)\eta(0) + \sin(t)\dot{\eta}(0) + \int_{t-D}^{t} \sin(t-\theta)U(\theta)d\theta, \tag{2.92}$$

$$\dot{\eta}(t) = -\sin(t)\eta(0) + \cos(t)\dot{\eta}(0) + \int_{t-D}^{t} \cos(t-\theta)U(\theta)d\theta \tag{2.93}$$

until $t = D$, and then an exponentially damped oscillatory response

$$\begin{bmatrix} \eta(t) \\ \dot{\eta}(t) \end{bmatrix} = e^{\begin{bmatrix} 0 & 1 \\ -1 & -h \end{bmatrix}(t-D)} \begin{bmatrix} \eta(D) \\ \dot{\eta}(D) \end{bmatrix} \tag{2.94}$$

for $t \geq D$.

Example 2.2. Figure 2.2 presents the simulation results for system (2.1) for

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & -2 \\ 0 & 1 & -1 \end{bmatrix}, \tag{2.95}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.96}$$

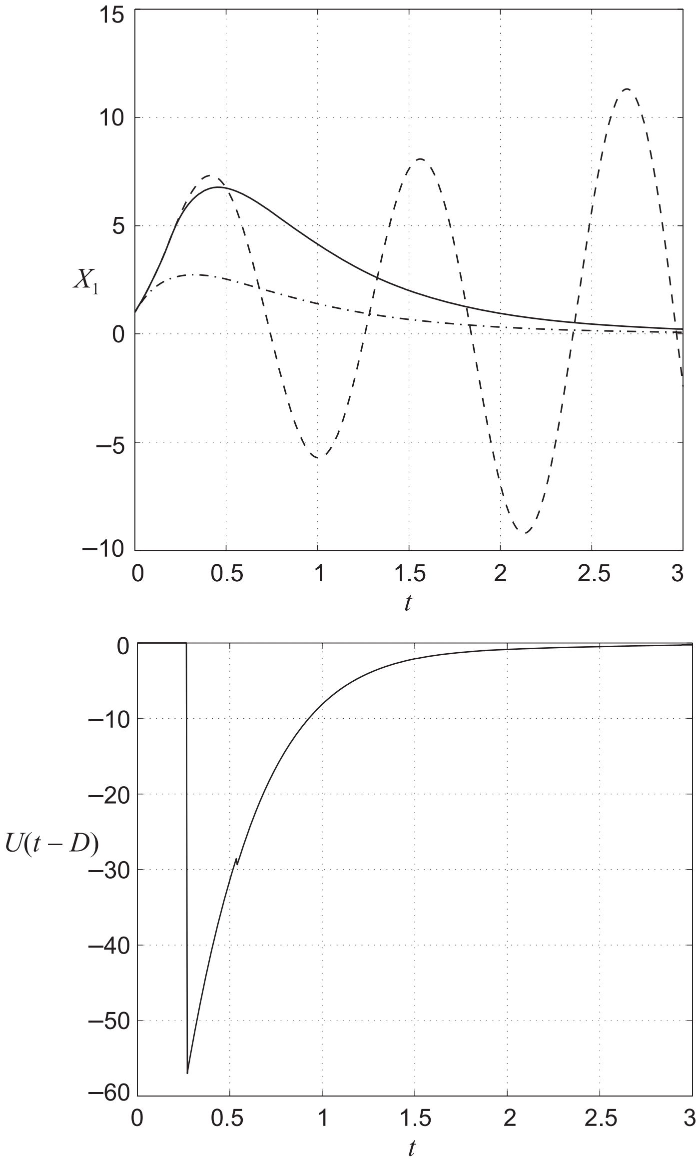

Fig. 2.2 The closed-loop response of the finite-dimensional system with actuator delay. Top: state evolution with nominal LQR controller in the absence of the delay (dash-dotted); with nominal LQR controller in the presence of the delay (dashed); with the backstepping controller in the presence of the delay (solid). Bottom: delayed control input.

This system is unstable at the origin with eigenvalues given by

$$\sigma_1 = 2, \tag{2.97}$$

$$\sigma_{2,3} = -1.5 \pm 1.4j. \tag{2.98}$$

One can see that the nominal LQR controller, with

$$Q = I_{3\times 3}, \tag{2.99}$$
$$R = 1, \tag{2.100}$$

does not stabilize the system when a small delay

$$D = 0.3 \tag{2.101}$$

is present. The predictor controller (2.39), which compensates the input delay, stabilizes the system. The larger transient is due to the fact that in the beginning the input to the system is zero because of the delay.

2.6 Stability Proof Without a Lyapunov Function

In this section we consider the problem of stability analysis without relying on the backstepping transformation and on a Lyapunov–Krasovskii function. This is a somewhat more compact proof of exponential stability; however, it comes with two caveats:

- This form of stability analysis does not extend from delay systems to systems involving more complex PDEs.
- A stability proof that avoids a construction of a Lyapunov function deprives the designer from the usual benefits of having a Lyapunov function—a study of robustness to modeling uncertainties, quantification of disturbance attenuation gains, inverse optimal redesign, and adaptive control design.

Nevertheless, we present this stability analysis so that the reader is aware of multiple options and alternatives for obtaining time-domain estimates for exponential stability.

We consider the closed-loop system

$$\dot{X}(t) = AX(t) + BU(t-D), \tag{2.102}$$
$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right] \tag{2.103}$$

and prove the following result.

Theorem 2.2. *Let $A+BK$ be a Hurwitz matrix such that*

$$\left|\mathrm{e}^{(A+BK)(t-t_0)}\right| \le G\mathrm{e}^{-g(t-t_0)}, \quad \forall t \ge t_0, \tag{2.104}$$

where $g>0$, $G\ge 1$, and t_0 has any finite value. The solutions to system (2.102), (2.103) satisfy

$$\Xi(t) \le \Gamma \mathrm{e}^{-gt}\Xi(0), \quad \forall t \ge 0, \tag{2.105}$$

where $\Xi(t)$ denotes

$$\Xi(t) = |X(t)| + \sup_{\tau \in [t-D,t]} |U(\tau)| \tag{2.106}$$

and the overshoot coefficient Γ is given by

$$\Gamma = (1 + |B|D)\left(1 + (1 + |K|)G\mathrm{e}^{|A|D}\right)\mathrm{e}^{gD}. \tag{2.107}$$

Proof. With the variation-of-constants formula, we write the solution to (2.102) as

$$X(t) = \mathrm{e}^{At}X_0 + \int_0^t \mathrm{e}^{A(t-\tau)}BU(\tau - D)d\tau. \tag{2.108}$$

Taking a 2-norm of both sides, we obtain

$$\begin{aligned} |X(t)| &\le \mathrm{e}^{|A|t}\left(|X_0| + \int_0^t |B||U(\tau - D)|d\tau\right) \\ &\le \mathrm{e}^{|A|t}\left(|X_0| + |B|t \sup_{\theta \in [-D,t-D]} |U(\theta)|\right), \quad \forall t \ge 0. \end{aligned} \tag{2.109}$$

This yields

$$\begin{aligned} |X(t)| &\le \mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta \in [-D,0]} |U(\theta)|\right) \\ &\le \mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta \in [-D,0]} |U(\theta)|\right)\mathrm{e}^{g(D-t)}, \quad \forall t \in [0,D], \end{aligned} \tag{2.110}$$

and, in particular,

$$|X(D)| \le \mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta \in [-D,0]} |U(\theta)|\right). \tag{2.111}$$

With the predictor feedback (2.103), we have

$$\dot{X}(t) = (A + BK)X(t), \quad \forall t \ge D. \tag{2.112}$$

Hence, with (2.104), we have

$$|X(t)| \le G|X(D)|\mathrm{e}^{-g(t-D)}, \quad \forall t \ge D. \tag{2.113}$$

Substituting (2.111), we get

$$|X(t)| \le G\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta \in [-D,0]} |U(\theta)|\right)\mathrm{e}^{g(D-t)}, \quad \forall t \ge D. \tag{2.114}$$

Since $G \geq 0$, in view of (2.110), the same inequality holds for $t \in [0,D]$, so we obtain

$$|X(t)| \leq G\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]}|U(\theta)|\right)\mathrm{e}^{g(D-t)}, \quad \forall t \geq 0. \tag{2.115}$$

We have arrived at an inequality from which one immediately gets an exponentially decaying (in time) bound on $|X(t)|$ in terms of the norm $\Xi(0)$. However, we need a bound on $\Xi(t)$ that incorporates the entire state of the closed-loop system. Toward this end, we observe that the control input (2.103) actually represents

$$U(t) = KX(t+D), \quad \forall t \geq 0. \tag{2.116}$$

With (2.115), we get

$$|U(t)| \leq G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]}|U(\theta)|\right)\mathrm{e}^{-gt}, \quad \forall t \geq 0. \tag{2.117}$$

However, we need an estimate in terms of $\sup_{\theta\in[t-D,t]}|U(\theta)|$. For $t \geq D$, such an estimate immediately follows from (2.117), namely,

$$\sup_{\theta\in[t-D,t]}|U(\theta)| \leq G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]}|U(\theta)|\right)\mathrm{e}^{g(D-t)}, \quad \forall t \geq D. \tag{2.118}$$

Now we turn our attention to estimating $\sup_{\theta\in[t-D,t]}|U(\theta)|$ over $t \in [0,D]$. We split the interval $[t-D,t]$ in the following manner:

$$\begin{aligned}
\sup_{\theta\in[t-D,t]}|U(\theta)| &\leq \sup_{\theta\in[t-D,0]}|U(\theta)| + \sup_{\theta\in[0,t]}|U(\theta)| \\
&\leq \sup_{\theta\in[-D,0]}|U(\theta)| + \sup_{\theta\in[0,t]}|U(\theta)| \\
&\leq \sup_{\theta\in[-D,0]}|U(\theta)| + G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|D \sup_{\theta\in[-D,0]}|U(\theta)|\right) \\
&\leq G|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + G|K||B|D\mathrm{e}^{|A|D}\right)\sup_{\theta\in[-D,0]}|U(\theta)|, \\
&\qquad \forall t \in [0,D].
\end{aligned} \tag{2.119}$$

We upper-bound this expression to achieve uniformity with (2.118):

$$\sup_{\theta\in[t-D,t]}|U(\theta)| \leq \left[G|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + G|K||B|D\mathrm{e}^{|A|D}\right)\sup_{\theta\in[-D,0]}|U(\theta)|\right]\mathrm{e}^{g(D-t)}, \quad \forall t \in [0,D]. \tag{2.120}$$

In fact, the same bound holds for both $t \in [0,D]$ (2.120) and $t \geq D$ (2.118):

$$\sup_{\theta\in[t-D,t]} |U(\theta)| \leq \left[G|K|\mathrm{e}^{|A|D}|X_0| + \left(1+G|K||B|D\mathrm{e}^{|A|D}\right) \sup_{\theta\in[-D,0]} |U(\theta)| \right] \mathrm{e}^{g(D-t)}, \quad \forall t \geq 0. \tag{2.121}$$

Now adding the bound (2.115), we get

$$\begin{aligned} &|X(t)| + \sup_{\theta\in[t-D,t]} |U(\theta)| \\ &\quad \leq \mathrm{e}^{g(D-t)} \Big[(1+|K|)G\mathrm{e}^{|A|D}|X_0| \\ &\qquad + \left(1+(1+|K|)G\mathrm{e}^{|A|D}|B|D\right) \sup_{\theta\in[-D,0]} |U(\theta)| \Big], \qquad \forall t \geq 0. \end{aligned} \tag{2.122}$$

By majorizing this expression to extract a factor of $|X_0| + \sup_{\theta\in[-D,0]} |U(\theta)|$ on the right-hand side, we obtain

$$\begin{aligned} &|X(t)| + \sup_{\theta\in[t-D,t]} |U(\theta)| \\ &\quad \leq (1+|B|D)\left(1+(1+|K|)G\mathrm{e}^{|A|D}\right)\mathrm{e}^{g(D-t)} \left(|X_0| + \sup_{\theta\in[-D,0]} |U(\theta)| \right), \quad \forall t \geq 0, \end{aligned} \tag{2.123}$$

which completes the proof of the theorem. □

While Theorem 2.2 establishes exponential stability in terms of the $l_2 \times L_\infty[t-D,t]$ norm $|X(t)| + \sup_{\tau\in[t-D,t]} |U(\tau)|$, one might also be interested in how stability would be proved in the sense of the $l_2 \times L_2[t-D,t]$ norm $|X(t)| + \left(\int_{t-D}^{t} U^2(\theta)d\theta\right)^{1/2}$. This result is established next.

Theorem 2.3. *Let (2.104) hold. Then the solutions to system (2.102), (2.103) satisfy*

$$\Upsilon(t) \leq \Gamma \mathrm{e}^{-gt}\Upsilon(0), \quad \forall t \geq 0, \tag{2.124}$$

where $\Upsilon(t)$ denotes

$$\Upsilon(t) = |X(t)| + \left(\int_{t-D}^{t} U^2(\theta)d\theta \right)^{1/2} \tag{2.125}$$

and the overshoot coefficient Γ is given by

$$\Gamma = \left(1+|B|\sqrt{D}\right)\left(1+\left(1+\frac{|K|}{\sqrt{2g}}\right)G\mathrm{e}^{|A|D}\right)\mathrm{e}^{gD}. \tag{2.126}$$

Proof. With the variation-of-constants formula and the Cauchy–Schwartz inequality, we obtain

$$
\begin{aligned}
|X(t)| &\le \mathrm{e}^{|A|t}\left(|X_0| + \int_0^t |B||U(\tau - D)|d\tau\right) \\
&\le \mathrm{e}^{|A|t}\left(|X_0| + |B|\sqrt{t}\left(\int_{-D}^{t-D} U^2(\theta)d\theta\right)^{1/2}\right), \quad \forall t \ge 0. \qquad (2.127)
\end{aligned}
$$

This yields

$$
\begin{aligned}
|X(t)| &\le \mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right) \\
&\le \mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\mathrm{e}^{g(D-t)}, \quad \forall t \in [0,D], \qquad (2.128)
\end{aligned}
$$

and, in particular,

$$
|X(D)| \le \mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right). \qquad (2.129)
$$

With the predictor feedback (2.103), we have for $t \ge D$

$$
\begin{aligned}
|X(t)| &\le G|X(D)|\mathrm{e}^{-g(t-D)} \\
&\le G\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\mathrm{e}^{g(D-t)}, \quad \forall t \ge D. \qquad (2.130)
\end{aligned}
$$

Since $G \ge 0$, in view of (2.128), the same inequality holds for $t \in [0,D]$, so we obtain

$$
|X(t)| \le G\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\mathrm{e}^{g(D-t)}, \quad \forall t \ge 0. \qquad (2.131)
$$

From (2.116) and with (2.131), we get

$$
|U(t)| \le G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\mathrm{e}^{-gt}, \quad \forall t \ge 0. \qquad (2.132)
$$

However, we ultimately need an estimate of $\|U\|_{L_2[t-D,t]}$. For $t \ge D$, such an estimate immediately follows from (2.132), namely,

$$
\begin{aligned}
\|U\|_{L_2[t-D,t]} &\le G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\left(\int_{t-D}^{t} \mathrm{e}^{-2g\tau}d\tau\right)^{1/2} \\
&= G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\left(\frac{1}{2g}\left(\mathrm{e}^{2g(D-t)} - \mathrm{e}^{-2gt}\right)\right)^{1/2} \\
&\le G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\frac{1}{\sqrt{2g}}\mathrm{e}^{g(D-t)}, \quad \forall t \ge D.
\end{aligned}
$$

(2.133)

Now we turn our attention to estimating $\|U\|_{L_2[t-D,t]}$ over $t \in [0,D]$. We split the interval $[t-D,t]$ in the following manner:

$$\begin{aligned}
\|U\|_{L_2[t-D,t]} &\leq \|U\|_{L_2[t-D,0]} + \|U\|_{L_2[0,t]} \\
&\leq \|U\|_{L_2[-D,0]} + \|U\|_{L_2[0,t]} \\
&\leq \|U\|_{L_2[-D,0]} + G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\left(\int_0^t \mathrm{e}^{-2g\tau}d\tau\right)^{1/2} \\
&\leq \|U\|_{L_2[-D,0]} + G|K|\mathrm{e}^{|A|D}\left(|X_0| + |B|\sqrt{D}\|U\|_{L_2[-D,0]}\right)\frac{1}{\sqrt{2g}} \\
&\leq \frac{G}{\sqrt{2g}}|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + \frac{G}{\sqrt{2g}}|K||B|\sqrt{D}\mathrm{e}^{|A|D}\right)\|U\|_{L_2[-D,0]}\,, \\
&\qquad \forall t \in [0,D]\,.
\end{aligned} \tag{2.134}$$

We upper-bound this expression to achieve uniformity with (2.133):

$$\begin{aligned}
&\|U\|_{L_2[t-D,t]} \\
&\leq \left[\frac{G}{\sqrt{2g}}|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + \frac{G}{\sqrt{2g}}|K||B|\sqrt{D}\mathrm{e}^{|A|D}\right)\|U\|_{L_2[-D,0]}\right]\mathrm{e}^{g(D-t)}\,, \\
&\qquad \forall t \in [0,D]\,.
\end{aligned} \tag{2.135}$$

In fact, the same bound holds for both $t \in [0,D]$ (2.135) and $t \geq D$ (2.133):

$$\begin{aligned}
&\|U\|_{L_2[t-D,t]} \\
&\leq \left[\frac{G}{\sqrt{2g}}|K|\mathrm{e}^{|A|D}|X_0| + \left(1 + \frac{G}{\sqrt{2g}}|K||B|\sqrt{D}\mathrm{e}^{|A|D}\right)\|U\|_{L_2[-D,0]}\right]\mathrm{e}^{g(D-t)}\,, \\
&\qquad \forall t \geq 0\,.
\end{aligned} \tag{2.136}$$

Now adding the bound (2.131), we get

$$\begin{aligned}
&|X(t)| + \|U\|_{L_2[t-D,t]} \\
&\leq \mathrm{e}^{g(D-t)}\left[\left(1 + \frac{|K|}{\sqrt{2g}}\right)G\mathrm{e}^{|A|D}|X_0|\right. \\
&\quad \left. + \left(1 + \left(1 + \frac{|K|}{\sqrt{2g}}\right)G|B|\sqrt{D}\mathrm{e}^{|A|D}\right)\|U\|_{L_2[-D,0]}\right]\,, \qquad \forall t \geq 0\,.
\end{aligned} \tag{2.137}$$

By majorizing this expression to extract a factor of $|X_0| + \|U\|_{L_2[-D,0]}$ on the right-hand side, we obtain

$$|X(t)| + \|U\|_{L_2[t-D,t]} \leq \left(1+|B|\sqrt{D}\right)\left(1+\left(1+\frac{|K|}{\sqrt{2g}}\right)Ge^{|A|D}\right)e^{g(D-t)}\left(|X_0| + \|U\|_{L_2[-D,0]}\right), \quad \forall t \geq 0, \tag{2.138}$$

which completes the proof of the theorem. □

2.7 Backstepping Transformation in the Standard Delay Notation

In Section 2.2 we derived the backstepping transformation and its inverse, respectively, as

$$w(x,t) = u(x,t) - \int_0^x Ke^{A(x-y)}Bu(y,t)dy - Ke^{Ax}X(t), \tag{2.139}$$

$$u(x,t) = w(x,t) + \int_0^x Ke^{(A+BK)(x-y)}Bw(y,t)\,dy + Ke^{(A+BK)x}X(t). \tag{2.140}$$

This derivation was performed using the transport PDE representation (2.8), (2.9) of the delay dynamics, which provides both a physical intuition and a mathematically clear setting for formulating the backstepping transformation and the subsequent stability analysis in Section 2.4.

However, as most readers in the field of control of delay systems are not accustomed to the PDE notation, we present here an alternative view of the backstepping transformation, based purely on standard delay notation. For the reader's convenience, we repeat here the closed-loop system consisting of the plant

$$\dot{X}(t) = AX(t) + BU(t-D) \tag{2.141}$$

and of the predictor feedback law

$$U(t) = K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]. \tag{2.142}$$

We express the future state $X(t+D)$ using the current state $X(t)$ as the initial condition, and using the controls $U(\theta)$ from the past time window $[t-D,t]$, based on the variation of constants formula applied to (9.1) as

$$\begin{aligned} X(t+D) &= e^{AD}X(t) + \int_t^{t+D} e^{A(t+D-\eta)}BU(\eta-D)d\eta \\ &= e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta. \end{aligned} \tag{2.143}$$

Hence, the feedback law (2.142) is simply

$$U(t) = KX(t+D), \qquad \text{for all } t \geq 0. \tag{2.144}$$

We start by noting that, while $U(t)$ is the value of the control signal at time t, the function $U(\theta), \theta \in [t-D, t]$, namely, the function $U(\cdot)$ on the entire sliding window $[t-D, t]$, is the state of the actuator. The state $U(\theta), \theta \in [t-D, t]$, is infinite-dimensional, since it is a function, rather than a vector. To introduce the backstepping transformation in the standard delay notation, we consider the function

$$W(\theta) = U(\theta) - KX(\theta + D) \tag{2.145}$$

at time $\theta \in [t-D, t]$. From (2.144), with $\theta = t$, it follows that

$$W(t) = 0, \qquad \text{for all } t \geq 0. \tag{2.146}$$

However, in general we have

$$W(\theta) \neq 0, \qquad \text{for } \theta \in [-D, 0]. \tag{2.147}$$

We develop next an alternative representation of $W(\theta)$, different than (2.145), which will serve as a definition of the actuator state for all $\theta \in [t-D, t]$ and all $t \geq 0$. Towards that end, similar to (2.143), using the variation-of-constants formula with initial time $t - D$, initial state $X(t)$, and current time θ, from (2.141) we obtain

$$X(\theta + D) = \mathrm{e}^{A(\theta+D-t)}X(t) + \int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma, \tag{2.148}$$

for all $\theta \in [t-D, t]$ and all $t \geq 0$.

We are now ready to introduce the backstepping transformation $U \mapsto W$ of the actuator state. By substituting (2.148) into (2.145), we arrive at

$$W(\theta) = U(\theta) - K\left[\int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma + \mathrm{e}^{A(\theta+D-t)}X(t)\right], \tag{2.149}$$

where $\theta \in [t-D, t]$ and $t \geq 0$. In the transformation (2.149), it is not helpful to view $W(\theta)$ as a value of a function but as a transformation of a function $W(\theta), [t-D, t]$, into another function $U(\theta), \theta \in [t-D, t]$.

Next we introduce a representation of the closed-loop system where U is replaced by W, which is an alternative representation of the target system (2.18)–(2.20). Setting $\theta = t - D$ in (2.149), solving the resulting equation as $U(t-D) = KX(t) + W(t-D)$, and substituting this expression into (2.141), we arrive at the target system

$$\dot{X}(t) = (A + BK)X(t) + BW(t-D), \tag{2.150}$$

$$W(t) = 0, \tag{2.151}$$

which is satisfied for all $t \geq 0$ and where (2.146) is repeated for the sake of clarity in further exposition. Thanks to the property (2.151), from (2.150) we obtain that

$$\dot{X}(t) = (A+BK)X(t), \qquad \text{for all } t \geq D, \tag{2.152}$$

which means that the delay is perfectly compensated in D seconds, namely, the system evolves as if the delay were absent after D seconds.

Since $W(\cdot)$ has a finite support $[-D,0]$ and the X-system (2.150) is exponentially stable, the target system (2.150), (2.151) is exponentially stable. We proved this fact in Section 2.4. In order for exponential stability to also hold for the original system, namely, for the system whose state is $X(t), U(\theta), \theta \in [t-D,t]$, it is necessary that the transformation (2.149) be invertible. The inverse of (2.149) is given explicitly as

$$U(\theta) = W(\theta) + K\left[\int_{t-D}^{\theta} \mathrm{e}^{(A+BK)(\theta-\sigma)} BW(\sigma)d\sigma + \mathrm{e}^{(A+BK)(\theta+D-t)}X(t)\right], \tag{2.153}$$

where $\theta \in [t-D,t]$ and $t \geq 0$. Hence, since the target system (2.150), (2.151) is exponentially stable, the actuator state (2.153) also exponentially converges to zero.

To rigorously prove exponential stability, in the standard delay system notation, a Lyapunov–Krasovskii functional of the target system (2.150), (2.151) is given as

$$V(t) = X(t)^T PX(t) + \frac{a}{2}\int_{t-D}^{t} (1+\theta+D-t)W(\theta)^2 d\theta, \tag{2.154}$$

where

$$a = \frac{4\lambda_{\max}(PBB^T P)}{\lambda_{\min}(Q)}. \tag{2.155}$$

The Lyapunov functional (2.154) depends on the state variables (X,W) in a simple diagonal-like manner with no cross-terms involving X and W and with a dependence on W, which is only a temporally scaled norm of this quantity. However, V is a functional of the state of the original system $X(t), U(\theta), \theta \in [t-D,t]$, and its expanded form is far from simple. The dependence of (2.154) on the variables (X,U), through the transformation (2.149), is

$$\begin{aligned} V(t) = {} & X(t)^T PX(t) + \frac{a}{2}\int_{t-D}^{t} (1+\theta+D-t)U(\theta)^2 d\theta \\ & + \frac{a}{2}X^T(t)\left(\int_{t-D}^{t} (1+\theta+D-t)\mathrm{e}^{A^T(\theta+D-t)}K^T K \mathrm{e}^{A(\theta+D-t)} d\theta\right)X(t) \\ & + \frac{a}{2}\int_{t-D}^{t} (1+\theta+D-t)\left(K\int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma\right)^2 d\theta \\ & - 2\frac{a}{2}\int_{t-D}^{t} (1+\theta+D-t)U(\theta)K\int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma d\theta \end{aligned}$$

$$
\begin{aligned}
&-2\frac{a}{2}\left(\int_{t-D}^{t}(1+\theta+D-t)U(\theta)Ke^{A(\theta+D-t)}d\theta\right)X(t)\\
&+2\frac{a}{2}\left(\int_{t-D}^{t}(1+\theta+D-t)K\int_{t-D}^{\theta}e^{A(\theta-\sigma)}BU(\sigma)d\sigma Ke^{A(\theta+D-t)}d\theta\right)X(t),
\end{aligned}
\tag{2.156}
$$

containing cross terms involving X and several nested integrals of U. A Lyapunov functional of a form such as (2.156) would not be possible to obtain without first constructing the backstepping transformation (2.149).

2.8 Notes and References

This chapter is an expanded version of the backstepping desing results for LTI-ODE systems with input and output delays in [118]. We recover the classical predictor-based feedback law studied in numerous papers [176, 121, 135, 8, 165, 45, 234, 89] (see also [60, 193] for recent surveys). The backstepping construction enables a quantification of exponential stability in a norm suitable for the closed-loop infinite-dimensional system.

We want to explain our preference for the term “predictor feedback” over the rather common term “finite spectrum assignment,” which we consider to be somewhat misleading. It neglects the fact that the system

$$w_t(x,t) = w_x(x,t), \tag{2.157}$$

$$w(D,t) = 0 \tag{2.158}$$

has its own spectrum, with complex poles whose real parts are at negative infinity. Even if one accepted that having a spectrum at negative infinity is somehow equivalent to not having a spectrum at all, stability characterization based on spectrum alone is imprecise as it neglects the effect of eigenvectors and eigenfunctions.

Related to the issue of spectrum and stability, one should note that it can be proved that

$$\|w(t)\|_{L_p[0,D]} \le e^{b(D-t)}\|w_0\|_{L_p[0,D]} \tag{2.159}$$

for any $b > 0$ and any $p \in [1,\infty]$ (Section 11.4). This is not a well-known fact, and its importance is that it reflects the trade-off between the decay rate and the overshoot coefficient. The decay rate b can be viewed as arbitrarily fast, at the expense of having a very large overshoot coefficient e^{bD}.

The bound (2.159) is tight for

$$t = D, \tag{2.160}$$

$$p = \infty, \tag{2.161}$$

$$w_0(x) \equiv \text{const}. \tag{2.162}$$

The bound is very conservative for both $t \gg D$ and $t \ll D$. The conservativeness for $t \gg D$ comes from the fact that $w(x,t)$ is identically zero for $t > D$. The conservativeness for $t \ll D$ comes from the fact that

$$||w(t)||_{L_p[0,D]} \leq ||w_0||_{L_p[0,D]} \tag{2.163}$$

actually holds for all $t \leq D$.

The stability of the entire infinite-dimensional state of a predictor-based feedback system, quantified in Section 2.4 in a $2 \times L_2$ norm, can be characterized in the Lyapunov sense using any of the $L_p[0,D]$ norms for the actuator state. For instance, in Section 2.6 we provided a stability characterization in a $2{\times}L_\infty$ norm, though not with the aid of a Lyapunov function. A Lyapunov function can be constructed in that and other norms using the tools we introduce in Section 11.4.

Sections 2.4 and 2.6 contrast two options we have in performing a time-domain exponential stability analysis of the feedback system with predictor feedback. The approach pursued in Section 2.6, which avoids the use of a Lyapunov function with the help of the facts that

$$\begin{aligned} \dot{X}(t) &= (A+BK)X(t)\,, && t \geq D\,, \\ U(t) &= KX(t+D)\,, && t \geq 0\,, \end{aligned} \tag{2.164}$$

does not extend (in an obvious way) to the case where one encounters some modeling uncertainties (either parametric or additive disturbances) and does not endow the designer with a tool for inverse optimal redesign or for adaptive control design. Furthermore, the approach to proving stability without a Lyapunov function, given in Section 2.6, does not extend to the cases where the delay input dynamics are replaced by input dynamics modeled by a more complex PDE such as a heat equation (Chapter 15) or a wave equation (Chapter 16). In those cases we do not have a finite-time effect of the input dynamics, and therefore the stability analysis cannot be performed by calculating estimates over two distinct time intervals, $[0,D]$ and $[D,\infty)$. In those cases we employ Lyapunov functions constructed using the backstepping approach.

Chapter 3
Predictor Observers

In this chapter we develop a result that is an exact dual of the predictor feedback design in Chapter 2. We develop an observer that compensates for a sensor delay, namely, a delay at the plant's output. Our design is based on observer designs for PDEs with boundary sensors [203].

The main subject of this chapter is an observer design for ODEs with sensor delay. This result is the focus of Sections 3.1, 3.2, and 3.3.

In Sections 3.4, 3.5, and 3.6 we present a side discussion that connects an observer-based predictor feedback design for systems with input delay with the classical Smith controller [201]. In this discussion we focus on structural similarities as well as the specific differences between the two designs, and present a stability analysis showing that the separation principle holds for the observer-based predictor feedback (Section 3.6).

3.1 Observers for ODE Systems with Sensor Delay

We consider the system

$$\dot{X}(t) = AX(t)\,, \tag{3.1}$$

$$Y(t) = CX(t-D)\,, \tag{3.2}$$

where (A,C) is an observable pair. We omit the input term $BU(t)$ for simplicity. In (3.54)–(3.56) we show how to add $BU(t)$ to the observer.

The output equation (3.2), i.e., the sensor delay equation, can be represented through the first-order hyperbolic PDE as

$$u_t(x,t) = u_x(x,t)\,, \tag{3.3}$$

$$u(D,t) = CX(t)\,, \tag{3.4}$$

$$Y(t) = u(0,t)\,. \tag{3.5}$$

We have the following result for (3.1), (3.3)–(3.5).

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_3,

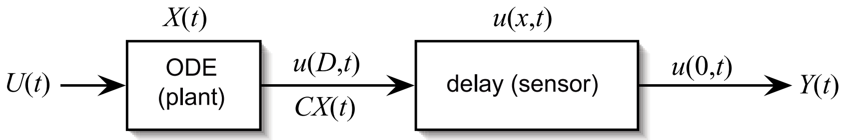

Fig. 3.1 Linear system $\dot{X} = AX + BU$ with sensor delay D. [For notational simplicity, we first consider the problem with $U = 0$ and then provide an extension for systems with a nonzero input in (3.54)–(3.56)].

Theorem 3.1. *The observer*

$$\dot{\hat{X}}(t) = A\hat{X}(t) + e^{AD}L(Y(t) - \hat{u}(0,t)), \tag{3.6}$$

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) + Ce^{Ax}L(Y(t) - \hat{u}(0,t)), \tag{3.7}$$

$$\hat{u}(D,t) = C\hat{X}(t), \tag{3.8}$$

where L is chosen such that $A - LC$ is Hurwitz, guarantees that $\hat{X}$, $\hat{u}$ exponentially converge to X, u; i.e., more specifically, that the observer error system is exponentially stable in the sense of the norm

$$\left(|X(t) - \hat{X}(t)|^2 + \int_0^D (u(x,t) - \hat{u}(x,t))^2 dx\right)^{1/2}.$$

Proof. Introducing the error variables

$$\tilde{X}(t) = X(t) - \hat{X}(t), \tag{3.9}$$

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t), \tag{3.10}$$

we obtain

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - e^{AD}L\tilde{u}(0,t), \tag{3.11}$$

$$\tilde{u}_t(x,t) = \tilde{u}_x(x,t) - Ce^{Ax}L\tilde{u}(0,t), \tag{3.12}$$

$$\tilde{u}(D,t) = C\tilde{X}(t). \tag{3.13}$$

Consider the transformation

$$\tilde{w}(x,t) = \tilde{u}(x,t) - Ce^{A(x-D)}\tilde{X}(t). \tag{3.14}$$

We have

$$\begin{aligned} \tilde{w}_t(x,t) - \tilde{w}_x(x,t) &= \tilde{u}_x(x,t) - Ce^{Ax}L\tilde{u}(0,t) \\ &\quad - Ce^{A(x-D)}(A\tilde{X}(t) - e^{AD}L\tilde{u}(0,t)) \\ &\quad - \tilde{u}_x(x,t) + Ce^{A(x-D)}A\tilde{X}(t) \\ &= 0 \end{aligned} \tag{3.15}$$

and

$$\tilde{w}(D,t) = \tilde{u}(D,t) - C\tilde{X}(t) = 0, \tag{3.16}$$

which means that it converges to zero in finite time. Equation (3.11) can be written as

$$\begin{aligned}
\dot{\tilde{X}}(t) &= A\tilde{X}(t) - \mathrm{e}^{AD}L\tilde{u}(0,t) \\
&= A\tilde{X}(t) - \mathrm{e}^{AD}L(\tilde{w}(0,t) + C\mathrm{e}^{-AD}\tilde{X}(t)) \\
&= (A - \mathrm{e}^{AD}LC\mathrm{e}^{-AD})\tilde{X}(t) - \mathrm{e}^{AD}L\tilde{w}(0,t)\,.
\end{aligned} \tag{3.17}$$

So, the observer error system is given by

$$\dot{\tilde{X}}(t) = (A - \mathrm{e}^{AD}LC\mathrm{e}^{-AD})\tilde{X}(t) - \mathrm{e}^{AD}L\tilde{w}(0,t)\,, \tag{3.18}$$

$$\tilde{w}_t(x,t) = \tilde{w}_x(x,t)\,, \tag{3.19}$$

$$\tilde{w}(D,t) = 0\,. \tag{3.20}$$

The matrix $A - \mathrm{e}^{AD}LC\mathrm{e}^{-AD}$ is Hurwitz, which can easily be seen by using a similarity transformation e^{AD}, which commutes with A. With a Lyapunov function

$$V(t) = \tilde{X}(t)^T\mathrm{e}^{-A^TD}P\mathrm{e}^{-AD}\tilde{X}(t) + \frac{a}{2}\int_0^D (1+x)\tilde{w}(x,t)^2\,dx, \tag{3.21}$$

where $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A - LC) + (A - LC)^T P = -Q \tag{3.22}$$

for some $Q = Q^T > 0$ and a is sufficiently large, one can show that

$$\dot{V}(t) \leq -\mu V(t) \tag{3.23}$$

for some $\mu > 0$; i.e., the $(\tilde{X},\tilde{w})$-system is exponentially stable at the origin. From (3.14) we get exponential stability in the sense of $(|\tilde{X}(t)|^2 + \int_0^D \tilde{u}(x,t)^2dx)^{1/2}$. The derivation of an estimate in this particular norm, from an estimate that directly follows from (3.23), is obtained in the same way as in the proof of Theorem 2.1. □

The next result provides a realization of the observer that involves a distributed-delay integral feedback term in the output injection, akin to the predictor feedback in Chapter 2.

Corollary 3.1. *The equivalent representation of the observer* (3.6)–(3.8) *in terms of the output* Y *is*

$$\dot{\hat{X}}(t) = A\hat{X}(t) + \mathrm{e}^{AD}L(Y(t) - \hat{Y}(t))\,, \tag{3.24}$$

$$\hat{Y}(t) = C\hat{X}(t-D) + C\int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}L(Y(\theta) - \hat{Y}(\theta))\,d\theta\,. \tag{3.25}$$

Proof. Let us take the Laplace transform of the $\hat{u}$-system in time, for a zero initial condition on the sensor delay state, $\hat{u}(x,0) = 0$,

$$s\hat{u}(x,s) = \hat{u}'(x,s) + Ce^{Ax}L(Y(s) - \hat{Y}(s)), \tag{3.26}$$

$$\hat{u}(0,s) = \hat{Y}(s). \tag{3.27}$$

The solution to this ODE is

$$\hat{u}(x,s) = \hat{Y}(s)e^{sx} - \int_0^x e^{s(x-\xi)}Ce^{A\xi}L(Y(s) - \hat{Y}(s))\,d\xi. \tag{3.28}$$

Since $\hat{u}(D,s) = C\hat{X}(s)$, we get

$$\hat{Y}(s) = C\hat{X}(s)e^{-sD} + \int_0^D e^{-s\xi}Ce^{A\xi}L(Y(s) - \hat{Y}(s))\,d\xi. \tag{3.29}$$

Taking the inverse Laplace transform, we obtain

$$\hat{Y}(t) = C\hat{X}(t-D) + \int_0^D Ce^{A\xi}L(Y(t-\xi) - \hat{Y}(t-\xi))\,d\xi. \tag{3.30}$$

Finally, after a change of variables $\theta = t - \xi$, we have

$$\hat{Y}(t) = C\hat{X}(t-D) + C\int_{t-D}^t e^{A(t-\theta)}L(Y(\theta) - \hat{Y}(\theta))\,d\theta. \tag{3.31}$$

□

An appealing quality of the observer form (3.24), (3.25) is the resemblance with the PDE-free form of the predictor feedback, (2.40). However, the disadvantage of the form (3.24), (3.25) is that it does not yield an estimate of the sensor state.

It is possible that the predictor observer in the PDE-free form (3.24), (3.25) may be implicitly *obtainable* within the framework of the general infinite-dimensional observer form in [233, Theorem 4.1]; however, it is not clear that [233] actually contains a constructive procedure that leads to the design (3.24), (3.25).

3.2 Example: Predictor Observer

We return to Example 2.1, but with a delay on the sensor rather than on the actuator:

$$\dot{\xi}_1(t) = \xi_2(t), \tag{3.32}$$

$$\dot{\xi}_2(t) = -\xi_1(t) + U(t), \tag{3.33}$$

$$Y(t) = \xi_1(t-D). \tag{3.34}$$

To summarize, we have

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.35}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.36}$$

$$C = [1 \;\; 0]. \tag{3.37}$$

Let us take the nominal observer gain vector as

$$L = \begin{bmatrix} g \\ 0 \end{bmatrix}, \qquad g > 0, \tag{3.38}$$

so that the nominal (undelayed) observer error system is governed by the system matrix

$$A - LC = \begin{bmatrix} -g & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.39}$$

which is Hurwitz. We now develop an observer in the form (3.24), (3.25) with an input added, i.e.,

$$\dot{\hat{X}} = A\hat{X} + BU + e^{AD}L(Y - \hat{Y}), \tag{3.40}$$

$$\hat{Y}(t) = C\hat{X}(t-D) + C\int_{t-D}^{t} e^{A(t-\theta)}L(Y(\theta) - \hat{Y}(\theta))\,d\theta. \tag{3.41}$$

For our choice of L, we have

$$e^{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}D} L = g\begin{bmatrix} \cos D \\ -\sin D \end{bmatrix} \tag{3.42}$$

and

$$Ce^{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(t-\theta)} L = g\cos(t-\theta). \tag{3.43}$$

Hence, our observer is given by

$$\dot{\hat{\xi}}_1(t) = \hat{\xi}_2(t) + g\cos D\left(Y(t) - \hat{Y}(t)\right), \tag{3.44}$$

$$\dot{\hat{\xi}}_2(t) = -\hat{\xi}_1(t) + U(t) - g\sin D\left(Y(t) - \hat{Y}(t)\right), \tag{3.45}$$

$$\hat{Y}(t) = \hat{\xi}_1(t-D) + g\int_{t-D}^{t} \cos(t-\theta)(Y(\theta) - \hat{Y}(\theta))\,d\theta. \tag{3.46}$$

This observer's state is guaranteed to exponentially converge to the state of the actual system (3.32)–(3.34), despite the output delay, and for any initial condition $\xi_1(0), \xi_2(0), Y(\theta), \theta \in [-D, 0]$. This convergence result is inferred from the observer error system

$$\dot{\tilde{\xi}}_1(t) = \tilde{\xi}_2(t) - g\sin D\tilde{Y}(t), \tag{3.47}$$

$$\dot{\tilde{\xi}}_2(t) = -\tilde{\xi}_1(t) + g\cos D\tilde{Y}(t), \tag{3.48}$$

$$\tilde{Y}(t) = \tilde{\xi}_1(t-D) - g\int_{t-D}^{t} \sin(t-\theta)\tilde{Y}(\theta)\,d\theta \tag{3.49}$$

using the backstepping construction of a Lyapunov–Krasovskii functional as in the proof of Theorem 3.1.

3.3 On Observers That Do Not Estimate the Sensor State

In contrast to the observer that we presented in Section 3.1, the classical delay-compensating observer results in [89, 234] take a different approach. Those designs arrive at an observer of the form

$$\dot{\hat{X}}(t) = A\hat{X}(t) + \mathrm{e}^{AD}L\left(Y(t) - C\mathrm{e}^{-AD}\hat{X}(t)\right), \tag{3.50}$$

where the gain vector L is selected to make the matrix $A - LC$ Hurwitz (which is equivalent to making $A - \mathrm{e}^{AD}LC\mathrm{e}^{-AD}$ Hurwitz).

The observer (3.50) differs from our observer (3.6)–(3.8) in the way that the estimate of $Y(t)$ is introduced in the estimation error. While (3.50) uses $C\mathrm{e}^{-AD}\hat{X}(t)$ in lieu of an estimate of $Y(t)$, we use a distributed estimator $\hat{u}(x,t)$ of $Y(t+x)$, $x \in [0,D]$, given by (3.7), (3.8), with output injection, which can also be viewed as the estimator of the actual plant output $CX(\theta)$ over the window $\theta \in [t-D,t]$. In other words, our observer generates not only a convergent estimate $\hat{X}(t)$ of $X(t)$, but also a (quantifiably) convergent estimate $\hat{Y}(t+x) = \hat{u}(x,t)$ of $Y(t+x) = CX(t+x-D)$ for $x \in [0,D]$.

Since our observer is infinite-dimensional, whereas the observer (3.50) is finite-dimensional, it is valid to ask whether the additional dimensionality is of any value. One should first note that (3.50) is a classical *reduced-order observer* for the plant (3.1), (3.2), which treats the infinite-dimensional "sensor state" $Y(t+x)$, $x \in [0,D]$, as known (in the future), and does not "waste" dynamic order to estimate it.

In contrast, our observer is a full-order observer, which estimates both the plant state X and the sensor state. One benefit of employing a full-order observer over a reduced-order observer is that reduced-order observers are well known to be overly sensitive to measurement noise.

An additional comment in favor of our full-order observer approach is that the idea that allows the reduced-order observer (3.50) does not extend to more general sensor dynamics (whether finite- or infinite-dimensional). It works only with delays because of the special form of their dynamics (pure "time-shift") and also because the transport delay dynamics are exponentially stable; hence, output injection is not necessary to stabilize their observer error system.

The dimensionality advantage of the reduced-order observer (3.50) disappears the moment one adds an input into the plant (3.1), (3.2), i.e.,

$$\dot{X}(t) = AX(t) + BU(t), \tag{3.51}$$

$$Y(t) = CX(t-D). \tag{3.52}$$

Then the observer (3.50) assumes the form

$$\begin{aligned}\dot{\hat{X}}(t) &= A\hat{X}(t) + BU(t)\\&\quad + \mathrm{e}^{AD}L\left(Y(t) - C\mathrm{e}^{-AD}\hat{X}(t) + C\int_{t-D}^{t}\mathrm{e}^{A(t-D-\theta)}BU(\theta)d\theta\right).\end{aligned} \tag{3.53}$$

Note that, even though infinite-dimensional, this is still a reduced-order observer because it does not attempt to estimate the sensor state.

Our observer (3.6)–(3.8) needs only a slight modification when the term $BU(t)$ is added to the plant and its order doesn't increase:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \mathrm{e}^{AD}L(Y(t) - \hat{u}(0,t)), \tag{3.54}$$

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) + C\mathrm{e}^{Ax}L(Y(t) - \hat{u}(0,t)), \tag{3.55}$$

$$\hat{u}(D,t) = C\hat{X}(t). \tag{3.56}$$

An alternative implementation of the reduced-order observer (3.53) is

$$\dot{\Xi}(t) = A\Xi(t) + BU(t-D) + L(Y(t) - C\Xi(t)), \tag{3.57}$$

$$\hat{X}(t) = \mathrm{e}^{AD}\Xi(t) + \int_{t-D}^{t}\mathrm{e}^{A(t-\theta)}BU(\theta)d\theta. \tag{3.58}$$

So the classical [89, 234] observer essentially estimates the past state from D seconds back and then advances it in an open-loop manner D seconds into the future.

Even though the observers (3.53) and (3.57), (3.58), which are equivalent, are designed only to estimate the ODE state $X(t)$, via the estimate $\hat{X}(t)$, they can be augmented to estimate the distributed state of the sensor. The estimate of the sensor state can be defined either as

$$\hat{u}(x,t) = C\left[\mathrm{e}^{AD(x-1)}\hat{X}(t) - \int_{t+D(x-1)}^{t}\mathrm{e}^{A(t+D(x-1)-\theta)}BU(\theta)d\theta\right], \tag{3.59}$$

which can be used along with the observer (3.53), or as

$$\hat{u}(x,t) = C\left[\mathrm{e}^{ADx}\Xi(t) + \int_{t-D}^{t+D(x-1)}\mathrm{e}^{A(t+D(x-1)-\theta)}BU(\theta)d\theta\right], \tag{3.60}$$

which is an equivalent realization and can be used along with the observer (3.57), (3.58).

By substituting either the observer (3.53), (3.59) or the observer (3.57), (3.58), (3.60) into the system (3.54)–(3.56), we see that both observers satisfy the system (3.54)–(3.56). Thus, the three observers,

1. (3.54)–(3.56),
2. (3.53), (3.59),
3. (3.57), (3.58), (3.60),

are equivalent, up to an exponentially decaying term due to the effect of the initial condition $\hat{u}(x,0)$.

3.4 Observer-Based Predictor Feedback for Systems with Input Delay

In this section we turn our attention to the question of developing an observer-based version of the predictor feedback from Chapter 2. This question is different than the question of observer design for systems with sensor delay, which is studied in the rest of this chapter. In this section we deal with systems with input delay only.

Consider the plant

$$\dot{X}(t) = AX(t) + BU(t-D)\,, \tag{3.61}$$

$$Y(t) = CX(t)\,, \tag{3.62}$$

where only the output $Y(t)$ is available for measurement. Let us associate with this system a standard finite-dimensional observer

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t-D) + L\left(Y(t) - C\hat{X}(t)\right)\,, \tag{3.63}$$

where the vector L is chosen so that the matrix $A - LC$ is Hurwitz. With this observer, the observer error system

$$\dot{\tilde{X}}(t) = (A - LC)\tilde{X}(t), \tag{3.64}$$

whose state is defined as

$$\tilde{X}(t) = X(t) - \hat{X}(t), \tag{3.65}$$

is exponentially stable.

We now consider an observer-based predictor feedback given by

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t-D) + L\left(Y(t) - C\hat{X}(t)\right)\,, \tag{3.66}$$

$$U(t) = K\left[\mathrm{e}^{AD}\hat{X}(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)} BU(\theta)d\theta\right]\,. \tag{3.67}$$

The transfer function of this compensator can be derived in the form

$$\begin{aligned} U(s) = {} & K\mathrm{e}^{AD}\left(sI - A - \mathrm{e}^{-AD}BK\mathrm{e}^{AD} + LC\right)^{-1} L \\ & \times \left\{Y(s) + C(sI - A)^{-1}\left(\mathrm{e}^{-AD}B - B\mathrm{e}^{-sD}\right)U(s)\right\}\,. \end{aligned} \tag{3.68}$$

In the next section we discuss the relation of this compensator with the original Smith controller.

3.5 The Relation with the Original Smith Controller

We recall that the original Smith predictor (also referred to in the literature as the Smith controller, perhaps to accentuate the fact that the feedback law that employs perfect *prediction* of the state in the control law is actually the "modified Smith predictor," or the "finite spectrum assignment controller") is given by [201]

$$U(s) = \mathscr{C}(s)\left\{Y(s) + P(s)\left(1 - \mathrm{e}^{-sD}\right)U(s)\right\}, \tag{3.69}$$

where

$$P(s) = C(sI - A)^{-1}B \tag{3.70}$$

and where the compensator can be chosen based on an observer-based controller, namely, as

$$\mathscr{C}(s) = K(sI - A - BK + LC)^{-1}L \tag{3.71}$$

(we point out that at the time of the development of the Smith controller, the state observer theory did not yet exist).

To explain the relation between the Smith controller and the observer-based predictor feedback (3.66), (3.67), we introduce the following quantities:

$$B_{-D} = \mathrm{e}^{-AD}B, \tag{3.72}$$

$$K_D = K\mathrm{e}^{AD}. \tag{3.73}$$

Then the compensator (3.66)–(3.67) is written as

$$U(s) = \mathscr{C}_D(s)\left\{Y(s) + \left(P_{-D}(s) - P(s)\mathrm{e}^{-sD}\right)U(s)\right\}, \tag{3.74}$$

where the transfer function $P_{-D}(s)$ is given by

$$P_{-D}(s) = C(sI - A)^{-1}B_{-D} \tag{3.75}$$

and where the delay-adjusted compensator $\mathscr{C}_D(s)$ is given by

$$\mathscr{C}_D(s) = K_D(sI - A - B_{-D}K_D + LC)^{-1}L. \tag{3.76}$$

Comparing the original Smith controller (3.69)–(3.71) with the observer-based predictor feedback (3.74)–(3.76), we observe subtle but significant differences in how input delay is being compensated, though the structure is the same, particularly in how the only infinite-dimensional element, the delay e^{-sD}, appears in the control laws.

3.6 Separation Principle: Stability Under Observer-Based Predictor Feedback

The closed-loop stability, even when the plant is unstable, can be proved for the observer-based predictor feedback (3.74)–(3.76). Such a result does not hold for the original Smith controller (3.69)–(3.71).

The stability proof is very similar to the stability proof in Chapter 2. First, we denote

$$u(x,t) = U(t+x-D) \tag{3.77}$$

and introduce a backstepping transformation involving the observer state $\hat{X}$ (rather than the plant state X):

$$\hat{w}(x,t) = u(x,t) - \int_0^x \mathrm{e}^{A(x-y)} u(y,t)dy - K\mathrm{e}^{Ax}\hat{X}(t). \tag{3.78}$$

This transformation yields a closed-loop system in the form

$$\dot{\hat{X}}(t) = (A+BK)\hat{X}(t) + B\hat{w}(0,t) + LC\tilde{X}(t)\,, \tag{3.79}$$

$$\dot{\tilde{X}}(t) = (A-LC)\tilde{X}(t)\,, \tag{3.80}$$

$$\hat{w}_t(x,t) = \hat{w}_x(x,t)\,, \tag{3.81}$$

$$\hat{w}(D,t) = 0\,. \tag{3.82}$$

The closed-loop system $(\hat{X}, \tilde{X}, \hat{w})$ has a cascade structure, where the autonomous exponentially stable systems $\tilde{X}$ and $\hat{w}$ drive the exponentially stable system $\hat{X}$. We use a similar Lyapunov–Krasovskii functional as in Chapter 2,

$$V(t) = \hat{X}(t)^T P\hat{X}(t) + \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q_1)} \int_0^D (1+x)\hat{w}(x,t)^2\,dx$$
$$+\, b\tilde{X}(t)^T \Pi \tilde{X}(t)\,, \tag{3.83}$$

where $P = P^T > 0$ and $\Pi = \Pi^T > 0$ are, respectively, the solutions to the Lyapunov equations

$$P(A+BK) + (A+BK)^T P = -Q_1\,, \tag{3.84}$$

$$\Pi(A-LC) + (A-LC)^T \Pi = -Q_2 \tag{3.85}$$

for some $Q_1 = Q_1^T > 0$ and $Q_2 = Q_2^T > 0$, and a positive parameter b to be chosen in the analysis. We have

$$\dot{V} \leq -\frac{\lambda_{\min}(Q_1)}{2}|\hat{X}|^2 - \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q_1)} \int_0^D \hat{w}(x)^2\,dx$$
$$+\, 2\hat{X}^T \Pi LC\tilde{X} - b\tilde{X}^T Q_2 \tilde{X}$$
$$\leq -\frac{\lambda_{\min}(Q_1)}{2}|\hat{X}|^2 - \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q_1)} \int_0^D \hat{w}(x)^2\,dx$$
$$+\, \frac{\lambda_{\min}(Q_1)}{4}|\hat{X}|^2 2 + \frac{4\|\Pi LC\|^2}{\lambda_{\min}(Q_1)}|\tilde{X}|^2 - b\lambda_{\min}(Q_2)|\tilde{X}|^2$$

$$\leq -\frac{\lambda_{\min}(Q_1)}{4}|\hat{X}|^2 - \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q_1)}\int_0^D \hat{w}(x)^2\,dx + \left(\frac{4\|\Pi LC\|^2}{\lambda_{\min}(Q_1)} - b\lambda_{\min}(Q_2)\right)|\tilde{X}|^2. \tag{3.86}$$

Picking

$$b = \frac{8\|\Pi LC\|^2}{\lambda_{\min}(Q_1)\lambda_{\min}(Q_2)}, \tag{3.87}$$

we obtain

$$\dot{V} \leq -\frac{\lambda_{\min}(Q_1)}{4}|\hat{X}|^2 - \frac{2\lambda_{\max}(PBB^TP)}{\lambda_{\min}(Q_1)}\int_0^D \hat{w}(x)^2\,dx - \frac{4\|\Pi LC\|^2}{\lambda_{\min}(Q_1)}|\tilde{X}|^2. \tag{3.88}$$

Next, we observe that

$$\dot{V} \leq -\frac{\lambda_{\min}(Q_1)}{4\lambda_{\min}(P)}\hat{X}^TP\hat{X} - \frac{2\lambda_{\max}(PBB^TP)}{(1+D)\lambda_{\min}(Q_1)}\int_0^D (1+x)\hat{w}(x)^2\,dx - \frac{\lambda_{\min}(Q_2)}{2\lambda_{\min}(\Pi)}\frac{8\|\Pi LC\|^2}{\lambda_{\min}(Q_1)\lambda_{\min}(Q_2)}\tilde{X}^T\Pi\tilde{X}. \tag{3.89}$$

So we obtain

$$\dot{V} \leq -\mu V, \tag{3.90}$$

where

$$\mu = \min\left\{\frac{\lambda_{\min}(Q_1)}{4\lambda_{\max}(P)}, \frac{\lambda_{\min}(Q_2)}{2\lambda_{\min}(\Pi)}, \frac{1}{1+D}\right\}. \tag{3.91}$$

This establishes the exponential stability of the system $(\hat{X}, \tilde{X}, \hat{w})$. To establish the exponential stability of the system $(\hat{X}, X, u)$, we use the inverse backstepping transformation, as in Chapter 2, and also the fact that $X = \tilde{X} + \hat{X}$.

The above analysis results in the following stability theorem.

Theorem 3.2. *The closed-loop system*

$$\dot{X}(t) = AX(t) + BU(t-D), \tag{3.92}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t-D) + L\left(Y(t) - C\hat{X}(t)\right), \tag{3.93}$$

$$U(t) = K\left[e^{AD}\hat{X}(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right], \tag{3.94}$$

$$Y(t) = CX(t) \tag{3.95}$$

is exponentially stable at the origin in the sense of the norm

$$\left(|X(t)|^2 + |\hat{X}(t)|^2 + \int_{t-D}^{t} U^2(\theta)d\theta\right)^{1/2}. \tag{3.96}$$

3.7 Notes and References

The basic existing references on observer design for systems with output delay are [89, 234]. Our design in this chapter is based on observer designs for PDEs with boundary sensors [203] and, as we shall see in Chapter 17, extends to problems where the sensor dynamics are governed by more complex PDEs. Our observer introduced in this chapter is equivalent to the classical observers [89, 234] for systems with a sensor delay, but it allows estimation of the sensor state in addition to estimation of the plant state.

Chapter 4
Inverse Optimal Redesign

In Chapter 2 we studied the system

$$\dot{X}(t) = AX(t) + BU(t-D) \tag{4.1}$$

with the controller

$$U(t) = K\left[{}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right], \tag{4.2}$$

and, using the backstepping method for PDEs, we constructed a Lyapunov–Krasovskii functional for the closed-loop system (4.1), (4.2). This Lyapunov–Krasovskii functional is given by

$$V(t) = X(t)^T PX(t) + \frac{a}{2}\int_{t-D}^{t}(1+\theta+D-t)W(\theta)^2 d\theta, \tag{4.3}$$

where P is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^{\mathrm{T}}P = -Q, \tag{4.4}$$

P and Q are positive definite and symmetric, the constant $a > 0$ is sufficiently large, and $W(\theta)$ is defined as

$$W(\theta) = U(\theta) - K\left[\int_{t-D}^{\theta} \mathrm{e}^{A(\theta-\sigma)}BU(\sigma)d\sigma + \mathrm{e}^{A(\theta+D-t)}X(t)\right], \tag{4.5}$$

with $-D \leq t-D \leq \theta \leq t$.

The main purpose of a Lyapunov function is the establishment of Lyapunov stability. But what else might a Lyapunov function be useful for? We explore this question in the present chapter and in Chapter 5.

As we shall see, the utility of a Lyapunov function is in quantitative studies of robustness, to additive disturbance and to parameters, as well as in achieving inverse optimality in addition to stabilization.

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_4,

In this chapter we highlight inverse optimality and disturbance attenuation, which are achieved with the help of the transformation (4.5) and the Lyapunov function (4.3).

We first derive an inverse-optimal controller, which incorporates a penalty not only on the ODE state $X(t)$ and the input $U(t)$ but also on the delay state. The inverse optimal feedback that we design is of the form (where, for brevity and conceptual clarity, we mix the frequency and time domains, i.e., the lag transfer function on the right should be understood as an operator)

$$U(t) = \frac{c}{s+c}\left\{K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right]\right\}, \tag{4.6}$$

where $c > 0$ is sufficiently large; i.e., the inverse-optimal feedback is of the form of a low-pass filtered version of (4.2).

In Section 4.1 we establish the inverse optimality of the feedback law (4.6) and its stabilization property for sufficiently large c. In Section 4.3 we consider the plant (4.1) in the presence of an additive disturbance and establish the inverse optimality of the feedback (4.6) in the sense of solving a meaningful differential game problem, and we quantify its L_∞ disturbance attenuation property.

4.1 Inverse Optimal Redesign

In the formulation of the inverse optimality problem we will consider $\dot{U}(t)$ as the input to the system, whereas $U(t)$ is still the actuated variable. Hence, our inverse optimal design will be implementable after integration in time, i.e., as dynamic feedback. Treating $\dot{U}(t)$ as an input is the same as adding an integrator, which has been observed as being beneficial in the control design for delay systems in [69].

Theorem 4.1. *There exists c^* such that the feedback system (4.1), (4.6) is exponentially stable in the sense of the norm*

$$N(t) = \left(|X(t)|^2 + \int_{t-D}^{t} U(\theta)^2 d\theta + U(t)^2\right)^{1/2} \tag{4.7}$$

for all $c > c^*$. Furthermore, there exists $c^{**} > c^*$ such that for any $c \geq c^{**}$, the feedback (4.6) minimizes the cost functional

$$J = \int_0^\infty \left(\mathscr{L}(t) + \dot{U}(t)^2\right) dt\,, \tag{4.8}$$

where $\mathscr{L}$ is a functional of $(X(t), U(\theta)), \theta \in [t-D, t]$, and such that

$$\mathscr{L}(t) \geq \mu N(t)^2 \tag{4.9}$$

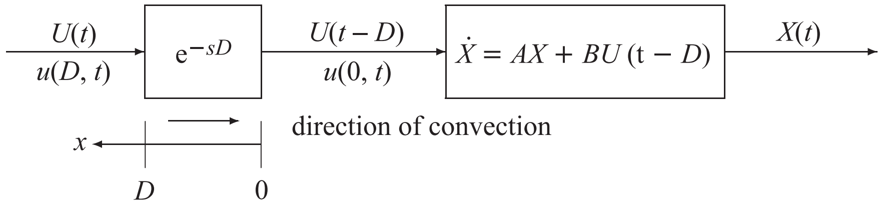

Fig. 4.1 Linear system $\dot{X}(t) = AX(t) + BU(t-D)$ with actuator delay D.

for some $\mu(c) > 0$ *with a property that*

$$\mu(c) \to \infty \;\; as \;\; c \to \infty. \tag{4.10}$$

Proof. We start by writing (4.1) as the ODE-PDE system

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{4.11}$$

$$u_t(x,t) = u_x(x,t), \tag{4.12}$$

$$u(D,t) = U(t), \tag{4.13}$$

where

$$u(x,t) = U(t+x-D), \tag{4.14}$$

and therefore the output

$$u(0,t) = U(t-D) \tag{4.15}$$

gives the delayed input (see Fig. 4.1).

Consider the infinite-dimensional backstepping transformation of the delay state (Chapter 2)

$$w(x,t) = u(x,t) - \left[\int_0^x K e^{A(x-y)} Bu(y,t)dy + Ke^{Ax}X(t)\right], \tag{4.16}$$

which satisfies

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \tag{4.17}$$

$$w_t(x,t) = w_x(x,t). \tag{4.18}$$

Let us now consider $w(D,t)$. It is easily seen that

$$w_t(D,t) = u_t(D,t) - K\left[Bu(D,t) + \int_0^D e^{A(D-y)} Bu(y,t)dy + Ae^{AD}X(t)\right]. \tag{4.19}$$

Note that

$$u_t(D,t) = \dot{U}(t), \tag{4.20}$$

which is designated as the control input penalized in (4.8). The inverse of (4.16) is given by[1]

$$u(x,t) = w(x,t) + \int_0^x K\mathrm{e}^{(A+BK)(x-y)} Bw(y,t)\, dy + K\mathrm{e}^{(A+BK)x} X(t)\,. \tag{4.21}$$

Plugging (4.21) into (4.19), after a lengthy calculation that involves a change of the order of integration in a double integral, we get

$$w_t(D,t) = u_t(D,t) - KBw(D,t) - K(A+BK)\left[\int_0^D M(y)Bw(y,t)dy + M(0)X(t)\right], \tag{4.22}$$

where

$$M(y) = \int_y^D \mathrm{e}^{A(D-\sigma)} BK\mathrm{e}^{(A+BK)(\sigma-y)} d\sigma + \mathrm{e}^{A(D-y)} = \mathrm{e}^{(A+BK)(D-y)} \tag{4.23}$$

is a matrix-valued function defined for $y \in [0,D]$. Note that $N : [0,D] \to \mathbb{R}^{n\times n}$ is in both $L_\infty[0,D]$ and $L_2[0,D]$.

Now consider a Lyapunov function

$$V(t) = X(t)^T PX(t) + \frac{a}{2}\int_0^D (1+x)w(x,t)^2\, dx + \frac{1}{2}w(D,t)^2\,, \tag{4.24}$$

where $P > 0$ is defined in (4.4) and the parameter $a > 0$ is to be chosen later. We have

$$\dot V(t) = X^T(t)((A+BK)^T P + P(A+BK))X(t) + 2X^T(t)PBw(0,t) + \frac{a}{2}\int_0^D (1+x)w(x,t)w_x(x,t)dx + w(D,t)w_t(D,t)\,. \tag{4.25}$$

After the substitution of the Lyapunov equation, we obtain

$$\dot V(t) = -X^T(t)QX(t) + 2X^T(t)PBw(0,t) + \frac{a}{2}(1+D)w(D,t)^2 - \frac{a}{2}w(0,t)^2 - \frac{a}{2}\int_0^D w(x,t)^2\, dx + w(D,t)w_t(D,t)\,, \tag{4.26}$$

[1] The fact that (4.21) is the inverse of (4.16) can be seen in various ways, including a direct substitution and manipulation of integrals, as well as by using a Laplace transform in x and employing the identity $(\sigma I - A - BK)^{-1}\left(I - BK(\sigma I - A)^{-1}\right) = (\sigma I - A)^{-1}$, where σ is the argument of the Laplace transform in x.

which gives

$$\dot{V}(t) \leq -X^T(t)QX(t) + \frac{2}{a}|X^T PB|^2 - \frac{a}{2}\int_0^D w(x,t)^2\,dx + w(D,t)\left(w_t(D,t) + \frac{a(1+D)}{2}w(D,t)\right), \tag{4.27}$$

and finally,

$$\dot{V}(t) \leq -\frac{1}{2}X^T(t)QX(t) - \frac{a}{2}\int_0^D w(x,t)^2\,dx + w(D,t)\left(w_t(D,t) + \frac{a(1+D)}{2}w(D,t)\right), \tag{4.28}$$

where we have chosen

$$a = 4\frac{\lambda_{\max}(PBB^T P)}{\lambda_{\min}(Q)}, \tag{4.29}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of the corresponding matrices. Now we consider (4.28) along with (4.22). With a completion of squares, we obtain

$$\begin{aligned}\dot{V}(t) \leq& -\frac{1}{4}X^T(t)QX(t) - \frac{a}{4}\int_0^D w(x,t)^2\,dx \\ &+ \frac{|K(A+BK)M(0)|^2}{\lambda_{\min}(Q)}w(D,t)^2 \\ &+ \frac{\|K(A+BK)MB\|^2}{a}w(D,t)^2 \\ &+ \left(\frac{a(1+D)}{2} - KB\right)w(D,t)^2 \\ &+ w(D,t)u_t(D,t).\end{aligned} \tag{4.30}$$

[We suppress the details of this step in the calculation but provide the details on the part that may be the hardest to see:

$$\begin{aligned}&-w(D,t)\langle K(A+BK)MB, w(t)\rangle \\ &\leq |w(D,t)|\|K(A+BK)MB\|\|w(t)\| \\ &\leq \frac{a}{4}\|w(t)\|^2 + \frac{\|K(A+BK)MB\|^2}{a}w(D,t)^2,\end{aligned} \tag{4.31}$$

where the first inequality is the Cauchy–Schwartz and the second is Young's, the notation $\langle\cdot,\cdot\rangle$ denotes the inner product in the spatial variable y, on which both $M(y)$ and $w(y,t)$ depend, and $\|\cdot\|$ denotes the L_2 norm in y.]

Then, choosing

$$u_t(D,t) = -cw(D,t)\,, \tag{4.32}$$

we arrive at

$$\dot{V}(t) \le -\frac{1}{4}X(t)^T QX(t) - \frac{a}{4}\int_0^D w(x,t)^2\,dx - (c-c^*)w(D,t)^2\,, \tag{4.33}$$

where

$$c^* = \frac{a(1+D)}{2} - KB + \frac{|K(A+BK)M(0)|^2}{\lambda_{\min}(Q)} + \frac{\|K(A+BK)MB\|^2}{a}\,. \tag{4.34}$$

Using (4.16) for $x = D$ and the fact that $u(D,t) = U(t)$, from (4.32) we get (4.6). Hence, from (4.33), the first statement of the theorem is proved if we can show that there exist positive numbers α_1 and α_2 such that

$$\alpha_1 N^2(t) \le V(t) \le \alpha_2 N^2(t)\,, \tag{4.35}$$

where

$$N(t)^2 = |X(t)|^2 + \int_0^D u(x,t)^2\,dx + u(D,t)^2\,. \tag{4.36}$$

This is straightforward to establish by using (4.16), (4.21), and (4.24), and employing the Cauchy–Schwartz inequality and other calculations, following a pattern of a similar computation in [202]. Thus, the first part of the theorem is proved.

The second part of the theorem is established in a manner very similar to the lengthy proof of Theorem 6 in [202], which is based on the idea of the proof of Theorem 2.8 in [109]. We choose

$$c^{**} = 4c^* \tag{4.37}$$

and

$$\begin{aligned}\mathscr{L}(t) =& -2c\dot{V}(t)\big|_{\text{(4.26) with (4.22), (4.32), and } c=2c^*} \\ &+ c(c-4c^*)w(D,t)^2 \\ \ge&\, c\left(\frac{1}{2}X^T(t)QX(t) + \frac{a}{2}\int_0^D w(x,t)^2\,dx + (c-2c^*)w(D,t)^2\right)\,. \end{aligned} \tag{4.38}$$

We have

$$\mathscr{L}(t) \ge \mu N(t)^2 \tag{4.39}$$

for the same reason that (4.35) holds. This completes the proof of inverse optimality. □

Remark 4.1. We have established the stability robustness to varying the parameter c from some large value c^* to ∞, recovering in the limit the basic, unfiltered predictor-based feedback (4.2). This robustness property might be intuitively expected from a singular perturbation idea, though an off-the-shelf theorem for establishing this property would be highly unlikely to be found in the literature due to the infinite dimensionality and the special hybrid (ODE-PDE-ODE) structure of the system at hand.

Remark 4.2. The feedback (4.2) is not inverse optimal, but the feedback (4.6) is, for any $c \in [c^{**}, \infty)$. Its optimality holds for a relevant cost functional, which is underbounded by the temporal $L_2[0,\infty)$ norm of the ODE state $X(t)$, the norm of the control $U(t)$, as well as the norm of its derivative $\dot{U}(t)$ [in addition to $\int_{-D}^{0} U(\theta)^2 d\theta$, which is fixed because feedback has no influence on it]. The controller (4.6) is stabilizing for $c = \infty$, namely, in its nominal form (4.2); however, since $\mu(\infty) = \infty$, it is not optimal with respect to a cost functional that includes a penalty on $\dot{U}(t)$.

Remark 4.3. Having obtained inverse optimality, one would be tempted to conclude that the controller (4.6) has an infinite gain margin and a phase margin of 60°. This is unfortunately not true, at least not in the sense of multiplicative (frequency-domain) perturbations of the feedback law. These properties can be claimed only for the feedback law (4.32), i.e., $\dot{U}(t) = -cW(t)$. The meaning of the phase margin is that the feedback

$$\dot{U}(t) = -c(1 + P(s))\{W(t)\} \tag{4.40}$$

is also stabilizing for any $P(s)$ that is strictly positive real. For example, the feedback of the form (4.6) but with $c/(s+c)$ replaced by

$$\frac{c(s+\nu+\omega)}{s^2+(c+\omega)s+c(\nu+\omega)}, \tag{4.41}$$

which may be a lightly damped transfer function for some ν, ω, is stabilizing for all ν and ω and for $c > c^*$. This result is not obvious but can be obtained by mimicking the proof of Theorem 2.17 from [109].

4.2 Is Direct Optimality Possible Without Solving Operator Riccati Equations?

In general, for infinite-dimensional systems, direct optimal control formulations lead to operator Riccati equations, which are infinite-dimensional nonlinear algebraic problems that can only be approached numerically; i.e., they cannot be simplified to finite-dimensional problems. The class of delay systems is an exception to this rule.

It is useful to mention here an elegant result on *direct* optimal control in the presence of actuator delay by Tadmor [210] (see also [250, Chapter 7]).

For the class of systems (4.1), it was shown in [210] that the predictor-based "nominally optimal" feedback law

$$U(t) = -B^T \Pi \left[e^{AD} X(t) + \int_{t-D}^{t} e^{A(t-\theta)} BU(\theta) d\theta \right], \tag{4.42}$$

where Π is a positive-definite and symmetric $n \times n$ solution to the *matrix* Riccati equation

$$\Pi A + A^T \Pi - \Pi B B^T \Pi + Q = 0 \tag{4.43}$$

(for a positive-definite and symmetric matrix Q), is actually the minimizer of the cost functional

$$J = \int_0^\infty \left(X(t)^T Q X(t) + U(t)^2 \right) dt \,. \tag{4.44}$$

This is a striking and subtle result, as the control $U(t)$ is penalized in (4.44) as both the control input and the infinite-dimensional state of the actuator. Our inverse optimality result, whose cost functional (4.8) is such that

$$J \geq \int_0^\infty \left(\mu |X(t)|^2 + \mu U(t)^2 + \dot{U}(t)^2 \right) dt \,, \tag{4.45}$$

is far less general and its only advantage is that the optimal value function (4.24) is actually a legitimate Lyapunov function that can be used for proving exponential stability. In contrast, the optimal value function in [210] is given by

$$V\left(X(0), U([-D,0])\right) = X(D)^T \Pi X(D) + \int_0^D X(t)^T Q X(t) dt, \tag{4.46}$$

where

$$X(t) = \mathrm{e}^{At} X(0) + \int_{-D}^{t-D} \mathrm{e}^{A(t-D-\theta)} B U(\theta) d\theta \,. \tag{4.47}$$

It is clear that (4.46) is positive *semi*definite, but in general it is not clear (nor claimed in [210]) that it is *positive definite* in $(X(0), U([-D,0]))$, i.e., that it is lower-bounded in terms of $|X(0)|^2 + \int_{-D}^0 U(\theta)^2 d\theta$; hence, it may not be a valid Lyapunov function. So, for the controller (4.42), the Lyapunov function defined by (4.3), (4.5) with

$$K = -B^T \Pi \,, \tag{4.48}$$

is the first Lyapunov function made available for proving exponential stability. Note that in [210] exponential stability in the strict Lyapunov sense, namely a characterization that involves a dependence on the norm of the infinite-dimensional state for $t \geq 0$, is neither stated nor quantified. Only "exponential decay to zero" (in time) is claimed and argued qualitatively.

4.3 Disturbance Attenuation

Consider the following system:

$$\dot{X}(t) = AX(t) + BU(t-D) + Gd(t) \,, \tag{4.49}$$

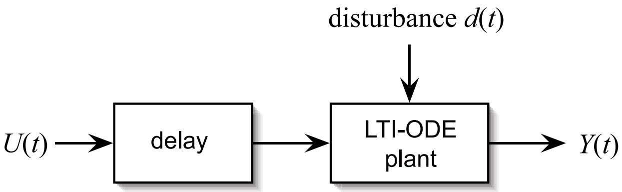

Fig. 4.2 Disturbance attenuation in the presence of input delay.

where $d(t)$ is an unmeasurable disturbance signal and G is a vector (see Fig. 4.2). In this section the availability of the Lyapunov function (4.24) lets us establish the disturbance attenuation properties of the controller (4.6), which we pursue in a differential game setting.

Theorem 4.2. *There exists c^* such that for all $c > c^*$, the feedback system (4.49), (4.6) is L_∞-stable; i.e., there exist positive constants $\beta_1, \beta_2, \gamma_1$ such that*

$$N(t) \le \beta_1 \mathrm{e}^{-\beta_2 t} N(0) + \gamma_1 \sup_{\tau\in[0,t]} |d(\tau)| \,. \tag{4.50}$$

Furthermore, there exists $c^{**} > c^*$ such that for any $c \ge c^{**}$, the feedback (4.6) minimizes the cost functional

$$J = \sup_{d\in\mathscr{D}} \lim_{t\to\infty} \left[2cV(t) + \int_0^t \left(\mathscr{L}(\tau) + \dot{U}(\tau)^2 - c\gamma_2 d(\tau)^2 \right) d\tau \right] \tag{4.51}$$

for any

$$\gamma_2 \ge \gamma_2^{**} = 8\frac{\lambda_{\max}(PBB^T P)}{\lambda_{\min}(Q)} \,, \tag{4.52}$$

where $\mathscr{L}$ is a functional of $(X(t), U(\theta)), \theta \in [t-D, t]$, and such that (4.9) holds for some $\mu(c,\gamma_2) > 0$ with a property that

$$\mu(c,\gamma_2) \to \infty \ \ as \ \ c \to \infty \,, \tag{4.53}$$

and $\mathscr{D}$ is the set of linear scalar-valued functions of X.

Proof. First, with a slight modification of the calculations leading to (4.33), we get

$$\begin{aligned} \dot{V}(t) \le & -\frac{1}{8} X(t)^T Q X(t) - \frac{a}{4}\int_0^D w(x,t)^2 \, dx \\ & - (c - c^*) w(D,t)^2 + \gamma_2^{**} d(t)^2 \,. \end{aligned} \tag{4.54}$$

From here, a straightforward, though lengthy, calculation gives the L_∞ stability result.

The proof of inverse optimality is obtained by specializing the proof of Theorem 2.8 in [109] to the present case. The function $\mathscr{L}(t)$ is defined as

$$\mathscr{L}(t) = -2c\Omega(t) + 8c|PG|^2 \frac{\gamma_2 - \gamma_2^{**}}{\gamma_2 \gamma_2^{**}} |X(t)|^2 + c(c - 4c^*)w(D,t)^2 , \tag{4.55}$$

where $\Omega(t)$ is defined as

$$\begin{aligned}\Omega(t) = & -X(t)^T QX(t) + 2X(t)^T PBw(0,t) + \frac{1}{\gamma_2^{**}} X(t)^T PGG^T PX(t) \\ & + \frac{a}{2}(1+D)w(D,t)^2 - \frac{a}{2}w(0,t)^2 \\ & - \frac{a}{2}\int_0^D w(x,t)^2\,dx - (2c^* + KB)w(D,t)^2 \\ & - K(A+BK)\left[\int_0^D M(y)Bw(y,t)dy + M(0)X(t)\right] w(D,t) . \end{aligned} \tag{4.56}$$

It is easy to see that

$$\Omega(t) \le -\frac{1}{8} X^T QX - \frac{a}{4}\int_0^D w(x,t)^2\,dx - 2c^* w(D,t)^2 . \tag{4.57}$$

Therefore,

$$\mathscr{L}(t) \ge c\left(\frac{\gamma_2 - \gamma_2^{**}/2}{\gamma_2}\lambda_{\min}(Q)|X(t)|^2 + \frac{a}{2}\int_0^D w(x,t)^2\,dx + (c - 2c^*)w(D,t)^2\right) , \tag{4.58}$$

which is lower-bounded by $\mu N(t)^2$ as in the proof of Theorem 4.1.

To complete the proof of inverse optimality, one can then show, by direct verification, that the cost of the two-player $(\dot{U}, d)$ differential game (4.51), along the solutions of the system, is

$$\begin{aligned} J = & \, 2cV(0) + \int_0^\infty (u_t(D,t) - u_t^*(D,t))^2\,dt \\ & + c\gamma_2 \sup_{d\in\mathscr{D}} \left\{ -\int_0^\infty (d(t) - d^*(t))^2\,dt \right\} , \end{aligned} \tag{4.59}$$

where

$$u_t^*(D,t) = -cw(D,t) \tag{4.60}$$

represents the optimal control as in (4.32), and $d^*(t)$ represents the "worst-case disturbance"

$$d^*(t) = \frac{2}{\gamma_2} G^T PX(t) . \tag{4.61}$$

The choice

$$d(t) = d^*(t) \tag{4.62}$$

achieves the supremum in the last term in (4.59), whereas the choice

$$u_t(D,t) = u_t^*(D,t)\,, \tag{4.63}$$

i.e., the choice given by (4.6), minimizes J. This completes the proof. □

Remark 4.4. Similar to the last point in Remark 4.2, the nominal predictor feedback (4.2), though not inverse optimal, is L_∞-stabilizing. This is seen with a different Lyapunov function,

$$V(t) = X(t)^T PX(t) + \frac{a}{2}\int_0^D (1+x)w(x,t)^2\,dx\,, \tag{4.64}$$

which yields

$$\dot{V}(t) \le -\frac{1}{4}X(t)^T QX(t) - \frac{a}{2}\int_0^D w(x,t)^2\,dx + \frac{\gamma_2^{**}}{2}d(t)^2\,. \tag{4.65}$$

4.4 Notes and References

Inverse optimality, as an objective in designing controllers for delay systems, was pursued by Jankovic [69, 70].

The low-pass filter modification, proposed here for inverse optimality, has already been proposed in [165] as a tool for helping robustness to the discretization of the integral term in (4.2). This low-pass filtering is not required for robustness to discretization, as shown in [250, Chapter 11] and [249, 251], but it is helpful.

It is worth noting that, due to the constructive character of the proofs of Theorems 4.1 and 4.2, all of the constants in their statements $(c^*, c^{**}, \beta_1, \beta_2, \gamma_1, \gamma_2^{**})$ can be given as explicit (albeit conservative) estimates.

Chapter 5
Robustness to Delay Mismatch

As we have seen in Chapter 2, the backstepping method helps construct an explicit Lyapunov–Krasovskii functional for the predictor feedback. One benefit of this construction is the possibility of inverse-optimal and disturbance attenuation designs, both presented in Chapter 4.

The second major benefit of the Lyapunov construction is that one can prove robustness of exponential stability of the predictor feedback to a *small* mismatch in the actuator delay, in both the positive and negative directions.

At first, this may seem a rather intuitive result. However, as explained in Section 5.4, a negative result (zero robustness) would be just as intuitive based on existing technical results for hyperbolic PDE systems.

Since predictor feedback employs an integral (distributed-delay) operator, with an integration interval based on the assumed delay length, an underestimation of the delay length does not change the dimension of the system, while an overestimation of the delay length adds to the dimension of the system. Clearly, this kind of a perturbation, occurring through an infinite-dimensional feedback law (of "mistuned dimension") is unlike other common perturbations (singular perturbation, occurrence of input delay in a finite-dimensional feedback design, etc.).

Since we are dealing with stability robustness, it is important that we clearly define the system norm through which we characterize stability. There is more than one choice in this respect. A more standard L_2 norm yields a rather different analysis than a stronger H_1 norm (which bounds the supremum of the delay state). First, in Section 5.1 we establish stability robustness in the sense of the L_2 norm on the actuator delay state. Then in Section 5.3 we establish stability robustness in the H_1 norm.

5.1 Robustness in the L_2 Norm

We consider the feedback system

$$\dot{X}(t) = AX(t) + BU(t - D_0 - \Delta D), \tag{5.1}$$

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_5,

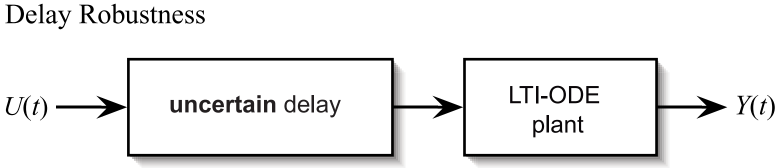

Fig. 5.1 Robustness study for small errors in the input delay.

$$U(t) = K\left[e^{AD_0}X(t) + \int_{t-D_0}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]. \tag{5.2}$$

The reader should note that the actual actuator delay has a mismatch of ΔD (see Fig. 5.1), which can be either positive or negative, relative to the assumed plant delay $D_0 > 0$, with the obvious necessary condition that $D_0 + \Delta D \geq 0$.

Being in the possession of a Lyapunov function, we are able to prove the following result in Theorem 5.1. In this theorem we consider the case where $D_0 + \Delta D > 0$, namely, the case where there is an input delay. The case where $D_0 + \Delta D = 0$, namely, where the input delay is actually zero, though the designer assumes that some positive delay exists and applies predictor feedback, is considered in Section 5.2.

Theorem 5.1. *There exists* $\delta > 0$ *such that for all*

$$\Delta D \in (-\delta, \delta), \tag{5.3}$$

the system (5.1), (5.2) is exponentially stable in the sense of the state norm

$$N_2(t) = \left(|X(t)|^2 + \int_{t-\bar{D}}^{t} U(\theta)^2 d\theta\right)^{1/2}, \tag{5.4}$$

where

$$\bar{D} = D_0 + \max\{0, \Delta D\}. \tag{5.5}$$

Proof. We use the same transport PDE formalism as in Chapters 2 and 4; i.e., we represent the system by

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{5.6}$$

$$u_t(x,t) = u_x(x,t), \tag{5.7}$$

$$u(D_0 + \Delta D, t) = U(t), \tag{5.8}$$

where the spatial domain of the PDE is defined as

$$x \in (\min\{0, \Delta D\}, D_0 + \Delta D], \tag{5.9}$$

and where

$$u(x,t) = U(t + x - D_0 - \Delta D), \tag{5.10}$$

from which it follows that the output is

$$u(0,t) = U(t - D_0 - \Delta D). \tag{5.11}$$

We use the same transformations as in Chapter 2, i.e.,

$$w(x,t) = u(x,t) - \left[\int_0^x K\mathrm{e}^{A(x-y)}Bu(y,t)dy + K\mathrm{e}^{Ax}X(t)\right], \tag{5.12}$$

$$u(x,t) = w(x,t) + \int_0^x K\mathrm{e}^{(A+BK)(x-y)}Bw(y,t)\,dy + K\mathrm{e}^{(A+BK)x}X(t). \tag{5.13}$$

The target system is given by

$$\dot{X}(t) = (A+K)X(t) + Bw(0,t), \tag{5.14}$$

$$w_t(x,t) = w_x(x,t), \tag{5.15}$$

and with the boundary condition for $w(D_0 + \Delta D, t)$ to be defined next. First, we note that the feedback (5.2) is written as

$$u(D_0 + \Delta D, t) = K\left[\mathrm{e}^{AD_0}X(t) + \int_{\Delta D}^{D_0+\Delta D} \mathrm{e}^{A(D_0+\Delta D-y)}Bu(y,t)dy\right], \tag{5.16}$$

which, using (5.12) for $x = D_0 + \Delta D$, gives us

$$w(D_0 + \Delta D, t) = K\mathrm{e}^{AD_0}\left[\left(I - \mathrm{e}^{A\Delta D}\right)X(t) - \int_0^{\Delta D} \mathrm{e}^{A(\Delta D-y)}Bu(y,t)dy\right]. \tag{5.17}$$

Then, employing (4.21) under the integral and performing certain calculations, we obtain

$$\begin{aligned} w(D_0 + \Delta D, t) = K\mathrm{e}^{AD_0}\Bigg[&\left(I - \mathrm{e}^{(A+BK)\Delta D}\right)X(t) \\ &- \int_0^{\Delta D} \mathrm{e}^{(A+BK)(\Delta D-y)}Bw(y,t)dy\Bigg]. \end{aligned} \tag{5.18}$$

One then shows that

$$w(D_0 + \Delta D, t)^2 \le 2q_1|X|^2 + 2q_2\int_{\min\{0,\Delta D\}}^{\max\{0,\Delta D\}} w(x,t)^2dx, \tag{5.19}$$

where the functions $q_1(\Delta D)$ and $q_2(\Delta D)$ are

$$q_1(\Delta D) = \left|K\mathrm{e}^{AD_0}\left(I - \mathrm{e}^{(A+BK)\Delta D}\right)\right|^2, \tag{5.20}$$

$$q_2(\Delta D) = \int_{\min\{0,\Delta D\}}^{\max\{0,\Delta D\}} \left(K\mathrm{e}^{AD_0}\mathrm{e}^{(A+BK)(\Delta D-y)}B\right)^2 dy. \tag{5.21}$$

Note that

$$q_1(0) = q_2(0) = 0 \tag{5.22}$$

and that q_1 and q_2 are both continuous functions of ΔD (note that the two integral terms in q_2 are both zero at zero, and continuous in ΔD).

The cases $\Delta D > 0$ and $\Delta D < 0$ have to be considered separately. The case $\Delta D > 0$ is easier and the state of the system is $X(t), u(x,t), x \in [0, D_0 + \Delta D]$, i.e., $X(t), U(\theta), \theta \in [t - D_0 - \Delta D, t]$. The case $\Delta D < 0$ is more intricate, as the state of the system is $X(t), u(x,t), x \in [\Delta D, D_0 + \Delta D]$, i.e., $X(t), U(\theta), \theta \in [t - D_0, t]$.

Case $\Delta D > 0$. We take the Lyapunov function

$$V(t) = X(t)^T P X(t) + \frac{a}{2} \int_0^{D_0 + \Delta D} (1 + x) w(x,t)^2 \, dx. \tag{5.23}$$

A calculation similar to that at the beginning of the proof of Theorem 4.1 gives

$$\begin{aligned} \dot{V} = &-X^T Q X + 2X^T P B w(0,t) + \frac{a}{2}(1 + D) w(D_0 + \Delta D, t)^2 \\ &- \frac{a}{2} w(0,t)^2 - \frac{a}{2} \int_0^{D_0 + \Delta D} w(x,t)^2 \, dx \end{aligned} \tag{5.24}$$

$$\begin{aligned} \leq &-\left(\frac{\lambda_{\min}(Q)}{2} - a(1 + D) q_1(\Delta D) \right) |X|^2 \\ &- \left(\frac{a}{2} - \frac{2|PB|^2}{\lambda_{\min}(Q)} \right) w(0,t)^2 \\ &- a \left(\frac{1}{2} - (1 + D) q_2(\Delta D) \right) \int_0^{D_0 + \Delta D} w(x,t)^2 \, dx, \end{aligned} \tag{5.25}$$

where we have denoted

$$D = D_0 + \Delta D \tag{5.26}$$

for brevity. This proves the exponential stability of the origin of the $(X(t), w(x,t), x \in [0, D_0 + \Delta D])$ system, for sufficiently small ΔD, by choosing

$$a > \frac{4|PB|^2}{\lambda_{\min}(Q)}, \tag{5.27}$$

and then choosing the sufficiently small $\delta > 0$ as the largest value of $|\Delta D|$ so that

$$\frac{\lambda_{\min}(Q)}{2} > a(1 + D) q_1(\Delta D) \tag{5.28}$$

and

$$\frac{1}{2} > (1 + D) q_2(\Delta D). \tag{5.29}$$

Exponential stability in the norm $N_2(t)$ is obtained using the same technique as in the proof of Theorem 2.1 for over- and under-bounding $V(t)$ by a linear function of

$N_2^2(t)$, where, for $\Delta D > 0$,

$$N_2(t) = \left(|X(t)|^2 + \int_{t-D_0-\Delta D}^{t} U(\theta)^2 d\theta \right)^{1/2}. \tag{5.30}$$

Case $\Delta D < 0$. In this case we use a different Lyapunov function,

$$\begin{aligned} V(t) = X(t)^T P X(t) + \frac{a}{2} \int_0^{D_0+\Delta D} (1+x) w(x,t)^2 \, dx \\ + \frac{1}{2} \int_{\Delta D}^{0} (D_0 + x) w(x,t)^2 dx, \end{aligned} \tag{5.31}$$

and obtain

$$\begin{aligned} \dot{V} \le & -\left(\frac{\lambda_{\min}(Q)}{2} - a(1+D) q_1(\Delta D) \right) |X|^2 \\ & - \left(\frac{a}{2} - \frac{D_0}{2} - \frac{2|PB|^2}{\lambda_{\min}(Q)} \right) w(0,t)^2 \\ & - \left(\frac{1}{2} - a(1+D) q_2(\Delta D) \right) \int_{\Delta D}^{0} w(x,t)^2 dx \\ & - \frac{D}{2} w(\Delta D, t)^2 - \frac{\max\{a,1\}}{4} \int_{\Delta D}^{D_0+\Delta D} w(x,t)^2 dx. \end{aligned} \tag{5.32}$$

This quantity is made negative definite by first choosing

$$a > D_0 + \frac{4|PB|^2}{\lambda_{\min}(Q)} \tag{5.33}$$

and then choosing the sufficiently small $\delta > 0$ as the largest value of $|\Delta D|$ so that

$$\frac{\lambda_{\min}(Q)}{2} > a(1+D) q_1(\Delta D) \tag{5.34}$$

and

$$\frac{1}{2} > a(1+D) q_2(\Delta D). \tag{5.35}$$

One thus gets

$$\begin{aligned} \dot{V} \le & -\left(\frac{\lambda_{\min}(Q)}{2} - a(1+D) q_1(\Delta D) \right) |X|^2 \\ & - \left(\frac{1}{2} - a(1+D) q_2(\Delta D) \right) \int_{\Delta D}^{0} w(x,t)^2 dx \\ & - \frac{\max\{a,1\}}{4} \int_{\Delta D}^{D_0+\Delta D} w(x,t)^2 dx \end{aligned}$$

$$\begin{aligned}
&\leq -\left(\frac{\lambda_{\min}(Q)}{2} - a(1+D)q_1(\Delta D)\right)\frac{1}{\lambda_{\max}(P)}X^T PX \\
&\quad -\left(\frac{1}{2} - a(1+D)q_2(\Delta D)\right)\frac{2}{D_0}\frac{1}{2}\int_{\Delta D}^{0}(D_0+x)w(x,t)^2 dx \\
&\quad -\frac{\max\{a,1\}}{4}\frac{2}{a(1+D)}\frac{a}{2}\int_0^{D_0+\Delta D}(1+x)w(x,t)^2 dx, \qquad (5.36)
\end{aligned}$$

which yields

$$\dot{V} \leq -\mu V, \tag{5.37}$$

where

$$\begin{aligned}
\mu = \min\Bigg\{ &\left(\frac{\lambda_{\min}(Q)}{2} - a(1+D)q_1(\Delta D)\right)\frac{1}{\lambda_{\max}(P)}, \\
&\left(\frac{1}{2} - a(1+D)q_2(\Delta D)\right)\frac{2}{D_0}, \\
&\frac{\max\{a,1\}}{2a(1+D)}\Bigg\}. \qquad (5.38)
\end{aligned}$$

Hence, we get an exponential stability estimate in terms of $|X(t)|^2 + \int_{\Delta D}^{D_0+\Delta D} w(x,t)^2 dx$. With some further work, we also get an estimate in terms of $|X(t)|^2 + \int_{\Delta D}^{D_0+\Delta D} u(x,t)^2 dx$, i.e., in terms of $|X(t)|^2 + \int_{t-D_0}^{t} U(\theta)^2 d\theta$. We start from

$$\psi_1\left(|X(t)|^2 + \int_{\Delta D}^{D} w(x,t)^2 dx\right) \leq V(t) \leq \psi_2\left(|X(t)|^2 + \int_{\Delta D}^{D} w(x,t)^2 dx\right), \tag{5.39}$$

where

$$\psi_1 = \min\left\{\lambda_{\min}(P), \frac{a}{2}, \frac{D}{2}\right\}, \tag{5.40}$$

$$\psi_2 = \max\left\{\lambda_{\max}(P), \frac{a(1+D)}{2}, \frac{D_0}{2}\right\}. \tag{5.41}$$

Let us now consider

$$w(x,t) = u(x,t) - m(x) \star u(x,t) - KM(x)X(t), \tag{5.42}$$

$$u(x,t) = w(x,t) + n(x) \star w(x,t) + KN(x)X(t), \tag{5.43}$$

where $\star$ denotes the convolution operation in x and

$$m(s) = KM(s)B, \tag{5.44}$$

$$n(s) = KN(s)B, \tag{5.45}$$

$$M(x) = \mathrm{e}^{Ax}, \tag{5.46}$$

$$N(x) = \mathrm{e}^{(A+BK)x}. \tag{5.47}$$

It is easy to show, using (5.42) and (5.43), that

$$\int_{\Delta D}^{D} w(x,t)^2 dx \le \alpha_1 \int_{\Delta D}^{D} u(x,t)^2 dx + \alpha_2 |X(t)|^2, \tag{5.48}$$

$$\int_{\Delta D}^{D} u(x,t)^2 dx \le \beta_1 \int_{\Delta D}^{D} w(x,t)^2 dx + \beta_2 |X(t)|^2, \tag{5.49}$$

where

$$\alpha_1 = 3\left(1 + D_0 \|m\|^2\right), \tag{5.50}$$

$$\alpha_2 = 3\|KM\|^2, \tag{5.51}$$

$$\beta_1 = 3\left(1 + D_0 \|n\|^2\right), \tag{5.52}$$

$$\beta_2 = 3\|KN\|^2 \tag{5.53}$$

and $\|\cdot\|$ denotes the $L_2[\Delta D, D]$ norm. Hence, we obtain

$$\phi_1 \left(|X(t)|^2 + \int_{\Delta D}^{D} u(x,t)^2 dx \right) \le |X(t)|^2 + \int_{\Delta D}^{D} w(x,t)^2 dx, \tag{5.54}$$

$$|X(t)|^2 + \int_{\Delta D}^{D} w(x,t)^2 dx \le \phi_2 \left(|X(t)|^2 + \int_{\Delta D}^{D} u(x,t)^2 dx \right), \tag{5.55}$$

where

$$\phi_1 = \frac{1}{\max\{\beta_1, \beta_2 + 1\}}, \tag{5.56}$$

$$\phi_2 = \max\{\alpha_1, \alpha_2 + 1\}. \tag{5.57}$$

Combining the above inequalities, we get

$$\phi_1 \psi_1 \left(|X(t)|^2 + \int_{\Delta D}^{D} u(x,t)^2 dx \right) \le V(t) \le \phi_2 \psi_2 \left(|X(t)|^2 + \int_{\Delta D}^{D} u(x,t)^2 dx \right), \tag{5.58}$$

i.e.,

$$\phi_1 \psi_1 N_2^2(t) \le V(t) \le \phi_2 \psi_2 N_2^2(t), \tag{5.59}$$

where, for $\Delta D < 0$,

$$N_2(t) = \left(|X(t)|^2 + \int_{t-D_0}^{t} U(\theta)^2 d\theta \right)^{1/2}. \tag{5.60}$$

Hence, with (5.37), we get

$$N_2^2(t) \le \frac{\phi_2 \psi_2}{\phi_1 \psi_1} N_2^2(0) \mathrm{e}^{-\mu t}, \tag{5.61}$$

which completes the proof of exponential stability. □

Remark 5.1. The result of Theorem 5.1 is fairly subtle. The case when $\Delta D > 0$ is clear; the robustness to a "surplus" of actuator delay is a result that already holds for ODEs [221]. The case $\Delta D < 0$ is trickier. The controller, which overestimates the delay to be $D_0 > D_0 + \Delta D$, introduces the delayed inputs from the time interval $[t - D_0, t - D_0 - \Delta D]$ into the overall dynamic system, making its state consist of control inputs $U(\theta)$ from the entire interval $\theta \in [t - D_0, t]$, even though the actual actuator delay $D_0 + \Delta D$ is shorter. This peculiarity results in more complicated analysis for $\Delta D < 0$, with different weights on the Krasovskii functionals for the different parts of the delay interval (with lesser weight on the subinterval that represents the delay "mismatch"). The greater difficulty in proving the result for $\Delta D < 0$ leads us to conjecture that the predictor-based controllers *may* exhibit greater sensitivity[1] to delay mismatch in the cases where the delay is "overestimated" (and thus "overcompensated") rather than when it is "underestimated." This means that while there is no question that predictor-based delay compensation is indispensable for dealing with long actuator delay and, thus, that "some amount" of delay compensation is better than none, when faced with a delay of uncertain length—if our conjecture is true—"less" may be better than "more"; i.e., it may be better to err on the side of caution and design for the lower end of the delay range expected.

5.2 Aside: Robustness to Predictor for Systems That Do Not Need It

The brief result in this section, for

$$\Delta D = -D_0 > 0\,, \tag{5.62}$$

shows that even if the system has no actuator delay, it is robust to a small amount of predictor feedback.

Corollary 5.1. *There exists* $\delta > 0$ *such that for all*

$$D_0 \in [0, \delta), \tag{5.63}$$

the system

$$\dot{X}(t) = AX(t) + BU(t), \tag{5.64}$$

$$U(t) = K\left[\mathrm{e}^{AD_0}X(t) + \int_{t-D_0}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right] \tag{5.65}$$

is exponentially stable in the sense of the state norm $(|X(t)|^2 + \int_{t-D_0}^{t} U(\theta)^2 d\theta)^{1/2}$.

---

[1] This is to be ascertained by a separate study, which may be hard to conduct analytically and may have to be mainly numerical, for select examples.

Proof. The closed-loop system is

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \tag{5.66}$$

with

$$w_t = w_x \tag{5.67}$$

evolving over $x \in [-D_0, 0]$ and $w(0,t)$ satisfying the relations (5.18)–(5.21) for

$$D_0 + \Delta D = 0. \tag{5.68}$$

The Lyapunov function

$$V(t) = X(t)^T PX(t) + \frac{1}{2}\int_{-D_0}^{0} (D_0 + b + x) w(x,t)^2 dx, \tag{5.69}$$

where $b > 0$, satisfies

$$\dot{V} \leq -\left(\frac{\lambda_{\min}(Q)}{2} - \Omega q_1\right)|X|^2 - \left(\frac{1}{2} - \Omega q_2\right)\int_{-D_0}^{0} w(x)^2 dx - \frac{b}{2} w(-D_0)^2, \tag{5.70}$$

where

$$\Omega = \frac{4|PB|}{\lambda_{\min}(Q)} + D_0 + b. \tag{5.71}$$

Then D_0 can be chosen sufficiently small to make q_1 and q_2 arbitrarily small and achieve exponential stability. □

5.3 Robustness in the H_1 Norm

In this section we formulate the problem differently than in Section 5.1. We write system (5.1) as a transport PDE

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{5.72}$$

$$Du_t(x,t) = u_x(x,t), \tag{5.73}$$

$$u(1,t) = U(t), \tag{5.74}$$

where the uncertain delay D denotes the same quantity as $D_0 + \Delta D$, i.e.,

$$D = D_0 + \Delta D, \tag{5.75}$$

the nominal delay D_0 will be alternatively denoted as $\hat{D} = D_0$ (the hat symbolizes an estimate of the delay), and the delay mismatch ΔD will be alternatively denoted as

$$\tilde{D} = \Delta D = D - \hat{D}. \tag{5.76}$$

We adopt this different notation in the present section, relative to Section 5.1, because the results that we develop here will also be used in the adaptive developments in Chapter 8.

We note that in the representation (5.72)–(5.74) the actuator state $u(x,t)$ has a different meaning than in Section 5.1 because the domain length has been normalized to unity, i.e., $x \in [0,1]$. Note that in the representation in this section the quantity $1/D$ represents the propagation speed and is unknown. The actuator state is related to the input through the following equation:

$$u(x,t) = U(t + D(x-1)), \tag{5.77}$$

which, in particular, gives

$$u(1,t) = U(t), \tag{5.78}$$

$$u(0,t) = U(t-D). \tag{5.79}$$

The control law around which we build a delay-adaptation mechanism is the predictor-based feedback law written as

$$U(t) = K\left[e^{AD}X(t) + D\int_0^1 e^{AD(1-y)}Bu(y,t)dy\right]. \tag{5.80}$$

In addition to proving robustness to a small delay mismatch, in this section we also remove the requirement for the measurement of the full state of the transport PDE (which was present in Section 5.1). Instead of the true transport PDE state variable (5.77), we use its estimate

$$\hat{u}(x,t) = U(t + \hat{D}(x-1)), \tag{5.81}$$

where $\hat{D}$ is the estimate of D. The variable $\hat{u}(x,t)$ is governed by the transport equation

$$\hat{D}\hat{u}_t(x,t) = \hat{u}_x(x,t), \tag{5.82}$$

$$\hat{u}(1,t) = U(t). \tag{5.83}$$

Remark 5.2. The reader may want to view (5.82)–(5.83) as an *open-loop certainty equivalence* observer of the transport equation (5.73)–(5.74). By "certainty equivalence," we are referring to the fact that the parameter D is replaced by the estimate $\hat{D}$, whereas by "open-loop," we are referring to the fact that no output injection is used in the observer since the transport equation is open-loop exponentially stable.

The control $U(t)$ is given in terms of $\hat{D}$ and $\hat{u}(x,t)$ as

$$U(t) = K\left(e^{A\hat{D}}X(t) + \hat{D}\int_0^1 e^{A\hat{D}(1-y)}B\hat{u}(y,t)dy\right). \tag{5.84}$$

Next, we establish the following delay mismatch robustness result for the controller (5.84).

Theorem 5.2. *Consider the closed-loop system consisting of the plant (5.72)–(5.74), observer (5.82)–(5.83), and control law (5.84). There exists $\delta^* > 0$ such that for any $|\tilde{D}| < \delta^*$, i.e., for any $\hat{D} \in (D - \delta^*, D + \delta^*)$, the zero solution of the system $(X, u, \hat{u})$ is exponentially stable, namely, there exist positive constants R and ρ such that for all initial conditions satisfying $(X_0, u_0, \hat{u}_0) \in \mathbb{R}^n \times L_2(0,1) \times H_1(0,1)$, the following holds:*

$$\Gamma(t) \leq R\Gamma(0)\mathrm{e}^{-\rho t}, \tag{5.85}$$

where

$$\Gamma(t) = |X(t)|^2 + \int_0^1 u(x,t)^2 dx + \int_0^1 \hat{u}(x,t)^2 dx + \int_0^1 \hat{u}_x(x,t)^2 dx. \tag{5.86}$$

This result can also be rephrased in terms of the more standard delay system notation (namely, not involving transport PDEs). Toward this end, we denote

$$\omega(\theta) = U(\theta), \qquad \theta \in [-\max\left\{D, \hat{D}\right\}, 0]. \tag{5.87}$$

Corollary 5.2. *Consider the dynamic system consisting of the plant*

$$\dot{X}(t) = AX(t) + BU(t - D) \tag{5.88}$$

and the controller

$$U(t) = K\left(\mathrm{e}^{A\hat{D}}X(t) + \int_{t-\hat{D}}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right). \tag{5.89}$$

There exists $\delta^* > 0$ such that for any

$$|\tilde{D}| = |D - \hat{D}| < \delta^*, \tag{5.90}$$

there exists a positive constant R' such that for all $X_0 \in R^n$, $\omega \in L_2[-\max\left\{D, \hat{D}\right\}, 0] \cap H_1[-\hat{D}, 0]$, the following holds:

$$\Pi(t) \leq R'\Pi(0)\mathrm{e}^{-\rho t}, \tag{5.91}$$

where

$$\Pi(t) = |X(t)|^2 + \int_{t-\max\left\{D,\hat{D}\right\}}^{t} U(\theta)^2 d\theta + \int_{t-\hat{D}}^{t} \dot{U}(\theta)^2 d\theta. \tag{5.92}$$

We start the proof of these results with the following lemma.

Lemma 5.1. *Consider the transformations*

$$\hat{w}(x,t) = \hat{u}(x,t) - \hat{D}\int_0^x K\mathrm{e}^{A\hat{D}(x-y)}B\hat{u}(y,t)dy - K\mathrm{e}^{A\hat{D}x}X(t), \tag{5.93}$$

$$\hat{u}(x,t) = \hat{w}(x,t) + \hat{D}\int_0^x K\mathrm{e}^{(A+BK)\hat{D}(x-y)}B\hat{w}(y,t)dy + K\mathrm{e}^{(A+BK)\hat{D}x}X(t) \tag{5.94}$$

and the observer error state

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)\,. \tag{5.95}$$

The closed-loop system (5.72)–(5.74), (5.82)–(5.83), (5.84) is equivalent to the system in which the X-subsystem is represented as

$$\dot{X}(t) = (A+BK)X(t) + B\hat{w}(0,t) + B\tilde{u}(0,t), \tag{5.96}$$

the $\tilde{u}$-subsystem is represented as

$$D\tilde{u}_t(x,t) = \tilde{u}_x(x,t), -\tilde{D}r(x,t), \tag{5.97}$$

$$\tilde{u}(1,t) = 0, \tag{5.98}$$

with

$$r(x,t) = \frac{\hat{w}_x(x,t)}{\hat{D}} + KB\hat{w}(x,t) + \hat{D}\int_0^x K(A+BK)\mathrm{e}^{(A+BK)\hat{D}(x-y)}B\hat{w}(y,t)dy$$
$$+ K\mathrm{e}^{(A+BK)\hat{D}x}(A+BK)X(t)\,, \tag{5.99}$$

and the $\hat{w}$-subsystem is represented as

$$\hat{D}\hat{w}_t(x,t) = \hat{w}_x(x,t) - \hat{D}K\mathrm{e}^{A\hat{D}x}B\tilde{u}(0,t), \tag{5.100}$$

$$\hat{w}(1,t) = 0\,. \tag{5.101}$$

Furthermore,

$$\hat{D}\hat{w}_{xt}(x,t) = \hat{w}_{xx}(x,t) - \hat{D}^2K\mathrm{e}^{A\hat{D}x}AB\tilde{u}(0,t), \tag{5.102}$$

$$\hat{w}_x(1,t) = \hat{D}K\mathrm{e}^{A\hat{D}}\tilde{u}(0,t)\,. \tag{5.103}$$

This lemma is proved by a lengthy but straightforward verification.

It is of crucial importance for the subsequent analysis to observe the interconnection structure of the overall $(X,\tilde{u},\hat{w})$-system, which is displayed in Fig. 5.2. The $\tilde{D}$-connections are "weak" when the delay estimation error is small. They disappear when $\hat{D} = D$. The connections that are "strong" are $\tilde{u} \to X$ and $\tilde{u} \to \hat{w} \to X$. These interconnections are present even when $\hat{D} = D$, in which case we have two parallel cascades of exponentially stable subsystems $\tilde{u}$, $\hat{w}$, and X. The analysis will capture the fact that the potentially destabilizing feedback connections through $\tilde{D}$ can be suppressed by making $\tilde{D}$ small. One additional serious difficulty is that an "unbounded" connection from $\hat{w}_x$ to $\tilde{u}$ exists. We will deal with it by including an H_1 norm in the stability analysis.

Due to the need to deal with the $\hat{w}_x$-system (5.102), the following lemma, which is obtained by differentiating (5.93) and (5.94) and using integration by parts, is important in our future analysis.

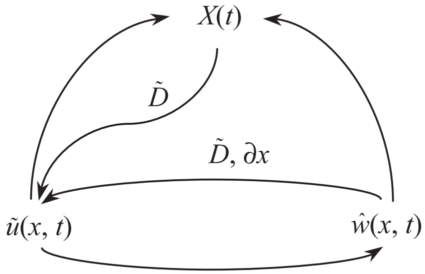

Fig. 5.2 Interconnections among the different variables.

Lemma 5.2. *The following holds for the transformations (5.93) and (5.94):*

$$\hat{u}_x(x,t) = \hat{w}_x(x,t) + \hat{D}KB\hat{w}(x,t) + \hat{D}\int_0^x K(A+BK)\hat{D}\mathrm{e}^{(A+BK)\hat{D}(x-y)}B\hat{w}(y,t)dy$$
$$+ K(A+BK)\hat{D}\mathrm{e}^{(A+BK)\hat{D}x}X(t), \tag{5.104}$$
$$\hat{w}_x(x,t) = \hat{u}_x(x,t) - \hat{D}KB\hat{u}(x,t) - \hat{D}\int_0^x KA\hat{D}\mathrm{e}^{A\hat{D}(x-y)}B\hat{u}(y,t)dy$$
$$- KA\hat{D}\mathrm{e}^{A\hat{D}x}X(t). \tag{5.105}$$

In our Lyapunov analysis we will need the relations between the norms of $\hat{u}$ and $\hat{w}$ as well as between the norms of their partial derivatives with respect to x. Using the Cauchy–Schwartz and Young inequalities, after a lengthy calculation, we obtain the following lemma.

Lemma 5.3. *The following holds for the transformations (5.93), (5.94), (5.104), and (5.105):*

$$\|\hat{u}(t)\|^2 \leq p_1\|\hat{w}(t)\|^2 + p_2|X(t)|^2, \tag{5.106}$$
$$\|\hat{u}_x(t)\|^2 \leq 4\|\hat{w}_x(t)\|^2 + p_3\|\hat{w}(t)\|^2 + p_4|X(t)|^2, \tag{5.107}$$
$$\|\hat{w}(t)\|^2 \leq q_1\|\hat{u}(t)\|^2 + q_2|X(t)|^2, \tag{5.108}$$
$$\|\hat{w}_x(t)\|^2 \leq 4\|\hat{u}_x(t)\|^2 + q_3\|\hat{u}(t)\|^2 + q_4|X(t)|^2, \tag{5.109}$$

where

$$p_1\left(\hat{D}\right) = 3\left(1 + \hat{D}^2|K|^2\mathrm{e}^{2|A+BK|\hat{D}}|B|^2\right), \tag{5.110}$$
$$p_2\left(\hat{D}\right) = 3|K|^2\mathrm{e}^{2|A+BK|\hat{D}}, \tag{5.111}$$
$$p_3\left(\hat{D}\right) = 4\hat{D}^2|K|^2|B|^2\left(1 + \hat{D}^2|A+BK|^2\mathrm{e}^{2\hat{D}|A+BK|}\right), \tag{5.112}$$
$$p_4\left(\hat{D}\right) = 4|K|^2\hat{D}^2|A+BK|^2\mathrm{e}^{2\hat{D}|A+BK|}, \tag{5.113}$$

$$q_1\left(\hat{D}\right) = 3\left(1+\hat{D}^2|K|^2\mathrm{e}^{2|A|\hat{D}}|B|^2\right), \tag{5.114}$$

$$q_2\left(\hat{D}\right) = 3|K|^2\mathrm{e}^{2|A|\hat{D}}, \tag{5.115}$$

$$q_3\left(\hat{D}\right) = 4\hat{D}^2|K|^2|B|^2\left(1+\hat{D}^2|A|^2\mathrm{e}^{2\hat{D}|A|}\right), \tag{5.116}$$

$$q_4\left(\hat{D}\right) = 4|K|^2\hat{D}^2|A|^2\mathrm{e}^{2\hat{D}|A|}. \tag{5.117}$$

The central part of our proof is the following Lyapunov result.

Lemma 5.4. *Consider the Lyapunov function*

$$\begin{aligned} V(t) &= X^T(t)PX(t) + b_1 D\int_0^1 (1+x)\tilde{u}(x,t)^2 dx \\ &+ b_2\hat{D}\left(\int_0^1 (1+x)\hat{w}(x,t)^2 dx + \int_0^1 (1+x)\hat{w}_x(x,t)^2 dx\right). \end{aligned} \tag{5.118}$$

There exist positive constants b_1, b_2, ρ, and δ^* such that for any $|\tilde{D}| < \delta^*$, the following holds:

$$\dot{V} \le -\rho V. \tag{5.119}$$

Proof. Differentiating (5.118) along the solutions to (5.96), (5.97), (5.100), and (5.102), and using integration by parts in x, we obtain

$$\begin{aligned} \dot{V} = &-X^T(t)QX(t) + 2X^T(t)PB(\hat{w}(0,t)+\tilde{u}(0,t)) \\ &+ 2b_1 D\int_0^1 (1+x)\tilde{u}(x,t)\tilde{u}_t(x,t)dx \\ &+ 2b_2\hat{D}\left(\int_0^1 (1+x)\hat{w}(x,t)\hat{w}_t(x,t)dx + \int_0^1 (1+x)\hat{w}_x(x,t)\hat{w}_{xt}(x,t)dx\right) \\ = &-X^TQX + 2X^TPB(\hat{w}(0,t)+\tilde{u}(0,t)) \\ &- 2b_1\left(\frac{\tilde{u}(0,t)^2}{2} + \frac{\|\tilde{u}(t)\|^2}{2} + \tilde{D}\int_0^1 (1+x)\tilde{u}(x,t)r(x,t)dx\right) \\ &- 2b_2\left(\frac{\hat{w}(0,t)^2}{2} + \frac{\|\hat{w}(t)\|^2}{2} + \hat{D}K\int_0^1 (1+x)\hat{w}(x,t)\mathrm{e}^{A\hat{D}x}\tilde{u}(0,t)dx\right) \\ &- 2b_2\left(\frac{\hat{w}_x(0,t)^2}{2} - \hat{w}_x(1,t)^2 + \frac{\|\hat{w}_x(t)\|^2}{2}\right. \\ &\left. + \hat{D}^2KA\int_0^1 (1+x)\hat{w}_x(x,t)\mathrm{e}^{A\hat{D}x}B\tilde{u}(0,t)dx\right). \end{aligned} \tag{5.120}$$

Let us define the following constants:

$$c_1 = |KB|, \tag{5.121}$$

$$c_2 = \hat{D}|K(A+BK)|\mathrm{e}^{|A+BK|\hat{D}}|B|, \tag{5.122}$$

$$c_3 = c_2/(\hat{D}|B|), \tag{5.123}$$

$$c_4 = 2\hat{D}|K|\mathrm{e}^{|A|\hat{D}}, \tag{5.124}$$

$$c_5 = c_4/2, \tag{5.125}$$

$$c_6 = 2\hat{D}^2|KA||B|\mathrm{e}^{2|A|\hat{D}} \tag{5.126}$$

and choose

$$b_2 \geq 8|PB|/\lambda_{\min}(Q)\,. \tag{5.127}$$

Using the Cauchy–Schwartz and Young inequalities, we have

$$\begin{aligned}
\dot{V}(t) \leq & -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 + \frac{b_2}{2}(\hat{w}(0,t)^2 + \tilde{u}(0,t)^2) \\
& - 2b_1\left(\frac{\tilde{u}(0,t)^2}{2} + \frac{\|\tilde{u}(t)\|^2}{2} - \frac{|\tilde{D}|}{\hat{D}}\|\tilde{u}(t)\|^2\right. \\
& - \frac{|\tilde{D}|}{\hat{D}}\|\hat{w}_x(t)\|^2 - |\tilde{D}|c_1^2\|\tilde{u}(t)\|^2 - |\tilde{D}|\|\hat{w}(t)\|^2 \\
& \left. - |\tilde{D}|(c_2^2\|\tilde{u}(t)\|^2 + \|\hat{w}(t)\|)^2 - |\tilde{D}|(|X(t)|^2 + c_3^2\|\tilde{u}(t)\|^2)\right) \\
& - 2b_2\left(\frac{\hat{w}(0,t)^2}{2} + \frac{\|\hat{w}(t)\|^2}{2} - c_4^2\tilde{u}(0,t)^2 - \frac{\|\hat{w}(t)\|^2}{4}\right) \\
& - 2b_2\left(\frac{\hat{w}_x(0,t)^2}{2} + \frac{\|\hat{w}_x(t)\|^2}{2} - c_6^2\tilde{u}(0,t)^2 - \frac{\|\hat{w}_x(t)\|^2}{4} - c_5^2\tilde{u}(0,t)^2\right).
\end{aligned} \tag{5.128}$$

Grouping like terms, we obtain

$$\begin{aligned}
\dot{V}(t) \leq & -\left(\frac{\lambda_{\min}(Q)}{2} - 2|\tilde{D}|b_1\right)|X(t)|^2 \\
& - b_1\left(1 - 2|\tilde{D}|\left(\frac{1}{\hat{D}} + c_1^2 + c_2^2 + c_3^2\right)\right)\|\tilde{u}(t)\|^2 \\
& - \left(b_1 - 2b_2\left(\frac{1}{4} + c_4^2 + c_5^2 + c_6^2\right)\right)\tilde{u}(0,t)^2 \\
& - \left(b_2 - \frac{2|\tilde{D}|}{\hat{D}}b_1\right)\|\hat{w}_x(t)\|^2 - \left(\frac{b_2}{2} - 2b_1|\tilde{D}|\right)\|\hat{w}(t)\|^2 \\
& - \frac{b_2}{2}\hat{w}(0,t)^2 - b_2\hat{w}_x(0,t)^2,
\end{aligned} \tag{5.129}$$

and, with some further majorizations (for $|\tilde{D}| < D$), we get

$$
\begin{aligned}
\dot{V}(t) \leq &-\left(\frac{\lambda_{\min}(Q)}{2} - 2|\tilde{D}|b_1\right)|X(t)|^2 \\
&- b_1\left(1 - 2\frac{|\tilde{D}|(1 + D(c_1^2 + c_2^2 + c_3^2))}{D - |\tilde{D}|}\right)\|\tilde{u}(t)\|^2 \\
&- \left(b_1 - 2b_2\left(\frac{1}{4} + c_4^2 + c_5^2 + c_6^2\right)\right)\tilde{u}(0,t)^2 \\
&- \left(b_2 - \frac{2|\tilde{D}|}{D - |\tilde{D}|}b_1\right)\|\hat{w}_x(t)\|^2 - \left(\frac{b_2}{2} - 2b_1|\tilde{D}|\right)\|\hat{w}(t)\|^2 \\
&- \frac{b_2}{2}\hat{w}(0,t)^2 - b_2\hat{w}_x(0,t)^2 .
\end{aligned} \tag{5.130}
$$

Assuming that

$$
|\tilde{D}| < \delta^* = \min\left\{\frac{D}{3 + 2(c_1^2 + c_2^2 + c_3^2)}, \frac{Db_2}{4b_1 + b_2}, \frac{b_2}{4b_1}, \frac{\lambda_{\min}(Q)D}{4b_1}\right\} \tag{5.131}
$$

and

$$
b_1 > \frac{2}{\frac{1}{4} + c_4^2 + c_5^2 + c_6^2} b_2, \tag{5.132}
$$

from (5.129) we obtain a bound for $\dot{V}$:

$$
\begin{aligned}
\dot{V} \leq &-\eta\left(|X(t)|^2 + \|\tilde{u}(t)\|^2 + \tilde{u}(0,t)^2 + \|\hat{w}_x(t)\|^2\right) \\
&+ \|\hat{w}(t)\|^2 + \hat{w}(0,t)^2 + \hat{w}_x(0,t)^2) \qquad (5.133) \\
\leq &-\eta\Gamma_0, \qquad (5.134)
\end{aligned}
$$

where

$$
\Gamma_0(t) = |X(t)|^2 + \|\tilde{u}(t)\|^2 + \|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2 \tag{5.135}
$$

and

$$
\begin{aligned}
\eta = \min\Bigg\{ &\frac{\lambda_{\min}(Q)}{2} - 2|\tilde{D}|b_1, \\
&b_1\left(1 - 2\tilde{D}\left(\frac{1}{\hat{D}} + c_1^2 + c_2^2 + c_3^2\right)\right), \\
&b_1 - 2b_2\left(\frac{1}{4} + c_4^2 + c_5^2 + c_6^2\right), \\
&b_2 - \frac{2|\tilde{D}|}{\hat{D}}b_1, \frac{b_2}{2} - 2b_1|\tilde{D}|, \frac{b_2}{2}\Bigg\}
\end{aligned} \tag{5.136}
$$

is a constant. The constant η is strictly positive whenever the conditions (5.131) and (5.132) are satisfied. Having obtained (5.134), to complete the proof of (5.119), we first obtain the following inequalities from (5.118):

$$\begin{aligned} V(t) &\geq \lambda_{\min}(P)|X(t)|^2 + b_1 D\|\tilde{u}(t)\|^2 \\ &\quad + b_2\hat{D}(\|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2) && (5.137) \\ &\geq \min\left\{\lambda_{\min}(P), b_1 D, b_2\hat{D}\right\} \Gamma_0(t)\,. && (5.138) \end{aligned}$$

From (5.134) and (5.138), we complete the proof of (5.119) with

$$\rho = \frac{\eta}{\min\left\{\lambda_{\min}(P), b_1 D, b_2\hat{D}\right\}}\,. \tag{5.139}$$

□

Lemma 5.5. *There exist positive constants d_1 and d_2 such that the following holds for the functions (5.86) and (5.118):*

$$d_1\Gamma(t) \leq V(t) \leq d_2\Gamma(t)\,. \tag{5.140}$$

Proof. From (5.106)–(5.109), we get

$$\begin{aligned} \Gamma(t) &\leq |X(t)|^2 + 2(\|\tilde{u}(t)\|^2 + \|\tilde{u}_x(t)\|^2) \\ &\quad + \|\hat{u}(t)\|^2 + \|\hat{u}_x(t)\|^2 && (5.141) \\ &\leq |X(t)|^2 + 2\|\tilde{u}(t)\|^2 + 3(p_1\|\hat{w}(t)\|^2 + p_2|X(t)|^2) \\ &\quad + 4\|\hat{w}_x(t)\|^2 + p_3\|\hat{w}(t)\|^2 + p_4|X(t)|^2 && (5.142) \\ &\leq \max\{1 + 3p_2 + p_4, 2, 3p_1 + p_3, 4\}\,\Gamma_0(t) && (5.143) \\ &\leq \frac{1}{d_1}V(t) && (5.144) \end{aligned}$$

and

$$\begin{aligned} V(t) &\leq \lambda_{\max}(P)|X(t)|^2 + 2(\|\tilde{u}(t)\|^2 + \|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2) && (5.145) \\ &\leq \lambda_{\max}(P)|X(t)|^2 + 4(\|u(t)\|^2 + \|\hat{u}\|^2) \\ &\quad + 2(q_1\|\hat{u}(t)\|^2 + q_2|X(t)|^2) \\ &\quad + 2(4\|\hat{u}_x(t)\|^2 + q_3\|\hat{u}(t)\|^2 + q_4|X(t)|^2) && (5.146) \\ &\leq d_2\Gamma(t), && (5.147) \end{aligned}$$

with

$$d_1 = \frac{\max\{1 + 3p_2 + p_4, 3p_1 + p_3\}}{\min\left\{\lambda_{\min}(P), b_1 D, b_2\hat{D}\right\}}, \tag{5.148}$$

$$d_2 = \max\left\{\lambda_{\max}(P) + 2q_2 + 2q_4, 4 + 2q_1 + 2q_3\right\}. \tag{5.149}$$

□

We are now ready to complete the proof of Theorem 5.2. From Lemma 5.4, it follows that

$$V(t) \le V(0)\mathrm{e}^{-\rho t}. \tag{5.150}$$

Then, from Lemma 5.5, we get

$$\Gamma(t) \le \frac{d_2}{d_1}\Gamma(0)\mathrm{e}^{-\rho t}, \tag{5.151}$$

so $R = d_2/d_1$, which completes the proof of the theorem. □

To give the result of Theorem 5.2 in terms of the more standard delay system notation, namely, to prove Corollary 5.2, we introduce the following lemma.

Lemma 5.6. *There exist positive constants d_3 and d_4 such that the following holds for the functions (5.86) and (5.92):*

$$d_3\Gamma(t) \le \Pi(t) \le d_4\Gamma(t). \tag{5.152}$$

Proof. By substituting (5.77) in (5.86) and using an appropriate change of the time variable, we express $\Gamma(t)$ as follows:

$$\Gamma(t) = |X(t)|^2 + \frac{1}{D}\int_{t-D}^{t} U(\theta)^2 d\theta + \frac{1}{\hat{D}}\int_{t-\hat{D}}^{t} U(\theta)^2 d\theta + \hat{D}\int_{t-\hat{D}}^{t} \dot{U}(\theta)^2 d\theta. \tag{5.153}$$

Thus, we obtain (5.152) with

$$d_3 = \frac{1}{2\max\left\{1, \frac{1}{D}, \frac{1}{\hat{D}}, \hat{D}\right\}}, \tag{5.154}$$

$$d_4 = \max\left\{1, D, \hat{D}, \frac{1}{\hat{D}}\right\}, \tag{5.155}$$

which completes the proof of Lemma 5.6. □

By combining Theorem 5.2 with Lemma 5.6, we complete the proof of Corollary 5.2 with

$$R' = \frac{d_4}{d_3}R. \tag{5.156}$$

Remark 5.3. Corollary 5.2 establishes a similar robustness result to delay mismatch as in Theorem 5.1. The significance of the proof in this section is that it uses the same Lyapunov function for both of the distinct cases $\hat{D} > D$ and $\hat{D} < D$, i.e., for $\Delta > 0$ and $\Delta < 0$. Another difference is that the result in Corollary 5.2 requires $D > 0$, whereas the result in Theorem 5.1 holds even for $D = 0$.

5.4 Notes and References

Predictor-based feedbacks are known to be sensitive to errors in the knowledge of the value of actuator delay. This problem is discussed in [60, 152, 170, 165] and other references. Despite the sensitivity, the predictor feedbacks are an "irreplaceable and widely used tool" [193].

The existing studies of robustness to delay mismatch are frequency-domain studies. We are not aware of robustness analyses performed using Lyapunov techniques. The result in [221] answers the delay robustness question for ODE plants with finite-dimensional controllers, but it does not apply to the present case where the nominal case (without delay mismatch) is infinite-dimensional and the feedback law is also infinite-dimensional.

It is worth noting that due to the constructive character of the proofs of Theorem 5.1 and Corollary 5.1, δ can be given through an explicit (albeit conservative) estimate.

The robustness results for delay mismatch are best appreciated if one is aware of negative results on the robustness of infinite-dimensional systems with actuator delay. In [32, 33, 34, 35, 36, 37] Datko and coworkers revealed that exponentially stabilizing results for hyperbolic PDE systems (such as wave and beam equations) have zero robustness to delay in the feedback loop—an arbitrarily small $D > 0$ produces eigenvalues in the right half-plane, no matter how "deeply" in the left half-plane the closed-loop eigenvalues are, for $D = 0$ (note that the addition of the delay $D > 0$ introduces more eigenvalues, i.e., this result contains no discontinuity in the dependence of the eigenvalues on D). Due to this result for hyperbolic PDEs, and given that the actuator delay in our problem is also a hyperbolic (though first-order) PDE system, at the start of the research effort leading to these results, we did not know, even on an intuitive level, if the predictor feedback would actually have a positive robustness margin to delay uncertainty.

Chapter 6
Time-Varying Delay

Before we complete Part I of the book, on predictor feedback for linear systems, and before we move on to delay-adaptive control in Part II, we examine a somewhat related problem of linear (time-invariant) systems with time-varying input or output delays.

Predictor-based control for time-varying delays has a rather intuitive extension from the case of constant delay. The key is to calculate the state prediction over a nonconstant window, starting with the current state as an initial condition, and using past controls over a window of time of nonconstant length.

The real complexity arises in the study of stability, where one has to construct a Lyapunov–Krasovskii functional using a backstepping transformation with time-varying kernels, and transforming the actuator state into a transport PDE with a convection speed coefficient that varies with both space and time. An additional challenge is how to define the state of the transport PDE modeling the actuator state using the past input signal.

We start in Section 6.1 with an intuitive introduction of the predictor feedback. Then in Section 6.2 we present a stability study. In Section 6.3 we present an observer for the case of a plant with time-varying sensor delay. Finally, in Section 6.4 we present several examples, including a numerical example with a scalar unstable plant and with an oscillating time-varying input delay.

6.1 Predictor Feedback Design with Time-Varying Actuator Delay

We consider the system

$$\dot{X}(t) = AX(t) + BU(\phi(t)), \tag{6.1}$$

where $X \in \mathbb{R}^n$ is the state, U is the control input, and $\phi(t)$ is a continuously differentiable function that incorporates the actuator delay. This function will have to

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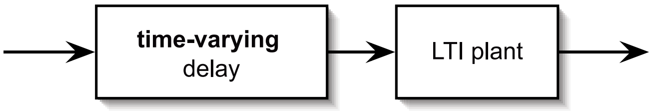

Fig. 6.1 Linear system $\dot{X}(t) = AX(t) + BU(\phi(t))$ with time-varying actuator delay $\delta(t) = t - \phi(t)$.

satisfy certain conditions that we shall impose in our development, in particular that

$$\phi(t) \leq t, \quad \forall t \geq 0. \tag{6.2}$$

One can alternatively view the function $\phi(t)$ in the more standard form

$$\phi(t) = t - D(t), \tag{6.3}$$

where $D(t) \geq 0$ is a time-varying delay (see Fig. 6.1). However, the formalism involving the function $\phi(t)$ turns out to be more convenient, particularly because the predictor problem requires the inverse function of $\phi(t)$, i.e., $\phi^{-1}(t)$, so we will proceed with the model (6.1). The invertibility of $\phi(\cdot)$ will be ensured by imposing the following assumption.

Assumption 6.1. $\phi : \mathbb{R}_+ \to \mathbb{R}$ *is a continuously differentiable function that satisfies*

$$\phi'(t) > 0, \quad \forall t \geq 0, \tag{6.4}$$

and such that

$$\pi_1^* \frac{1}{\sup_{\vartheta \geq \phi^{-1}(0)} \phi'(\vartheta)} > 0. \tag{6.5}$$

The meaning of the assumption is that the function $\phi(t)$ is strictly increasing, which, as we shall see, we need in several elements of our analysis.

The main premise of the predictor-based design is that one generates the control input

$$U(\phi(t)) = KX(t), \quad \forall \phi(t) \geq 0, \tag{6.6}$$

so that the closed-loop system is

$$\dot{X}(t) = (A + BK)X(t), \quad \forall \phi(t) \geq 0, \tag{6.7}$$

or, alternatively, using the inverse of $\phi(\cdot)$,

$$\dot{X}(t) = (A + BK)X(t), \quad \forall t \geq \phi^{-1}(0). \tag{6.8}$$

The gain vector K is selected so that the system matrix $A + BK$ is Hurwitz.

We now rewrite (6.6) as

$$U(t) = KX\left(\phi^{-1}(t)\right), \quad \forall t \geq 0. \tag{6.9}$$

With the help of the model (6.1) and the variation-of-constants formula, the quantity $X\left(\phi^{-1}(t)\right)$ is written as

$$X\left(\phi^{-1}(t)\right) = \mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}X(t) + \int_t^{\phi^{-1}(t)} \mathrm{e}^{A\left(\phi^{-1}(t)-\tau\right)}BU(\phi(\tau))d\tau . \tag{6.10}$$

To express the integral in terms of the signal $U(\cdot)$ rather than the signal $U(\phi(\cdot))$, we introduce the change of the integration variable,

$$\theta = \phi(\tau) , \tag{6.11}$$

$$\tau = \phi^{-1}(\theta) . \tag{6.12}$$

Recalling the basic differentiation rule for the inverse of a function,

$$\frac{d}{d\theta}\phi^{-1}(\theta) = \frac{1}{\phi'\left(\phi^{-1}(\theta)\right)} , \tag{6.13}$$

where $\phi'(\cdot)$ denotes the derivative of the function $\phi(\cdot)$, we get

$$\begin{aligned} X\left(\phi^{-1}(t)\right) = {} & \mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}X(t) \\ & + \int_{\phi(t)}^{t} \mathrm{e}^{A\left(\phi^{-1}(t)-\phi^{-1}(\theta)\right)}B\frac{U(\theta)}{\phi'\left(\phi^{-1}(\theta)\right)}d\theta . \end{aligned} \tag{6.14}$$

Substituting this expression into the control law (6.9), we obtain the predictor feedback

$$U(t) = K\left[\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}X(t) + \int_{\phi(t)}^{t} \mathrm{e}^{A\left(\phi^{-1}(t)-\phi^{-1}(\theta)\right)}B\frac{U(\theta)}{\phi'\left(\phi^{-1}(\theta)\right)}d\theta\right] , \quad \forall t \geq 0 . \tag{6.15}$$

The division by $\phi'\left(\phi^{-1}(\theta)\right)$ in this compensator is safe thanks to the assumption (6.4).

We refer to the quantity

$$t - \phi(t) \tag{6.16}$$

as the *delay time* and to the quantity

$$\phi^{-1}(t) - t \tag{6.17}$$

as the *prediction time*.

Remark 6.1. To make sure the above discussion is completely clear, we point out that when the system has a constant delay,

$$\phi(t) = t - D , \tag{6.18}$$

we have

$$\phi^{-1}(t) = t + D \tag{6.19}$$

and

$$\phi'\left(\phi^{-1}(\theta)\right) = 1\,. \tag{6.20}$$

Hence, controller (6.15) reduces to (2.40).

6.2 Stability Analysis

In our stability analysis we will use the transport equation representation of the delay and a Lyapunov construction.

First, we introduce the following fairly nonobvious choice for the state of the transport equation:

$$u(x,t) = U(\phi(t + x(\phi^{-1}(t) - t)))\,. \tag{6.21}$$

This choice yields boundary values

$$u(0,t) = U(\phi(t))\,, \tag{6.22}$$
$$u(1,t) = U(t)\,. \tag{6.23}$$

System (6.1) can now be represented as

$$\dot{X}(t) = AX(t) + Bu(0,t)\,, \tag{6.24}$$
$$u_t(x,t) = \pi(x,t)u_x(x,t)\,, \tag{6.25}$$
$$u(1,t) = U(t)\,, \tag{6.26}$$

where the speed of propagation of the transport equation is given by

$$\pi(x,t) = \frac{1 + x\left(\dfrac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\phi^{-1}(t) - t}\,. \tag{6.27}$$

To obtain a meaningful stability result, we need the propagation speed function $\pi(x,t)$ to be strictly positive and uniformly bounded from below and from above by finite constants. Guided by the concern for boundedness from above, we examine the denominator $\phi^{-1}(t) - t$. Since we assumed that $\phi(t)$ is strictly increasing (and continuous), so is $\phi^{-1}(t)$. We also recall assumption (6.2). We need to make this inequality strict, since if $\phi(t) = t$, i.e., $\phi^{-1}(t) = t$, for any t, the propagation speed is infinite at that time instant and the transport PDE representation does not make sense for the study of the stability problem. Hence, we make the following assumption.

Assumption 6.2. $\phi : \mathbb{R}_+ \to \mathbb{R}$ *is a continuously differentiable function that satisfies*

$$\phi(t) < t, \quad t \geq 0\,, \tag{6.28}$$

and such that

$$\pi_0^* = \frac{1}{\sup_{\vartheta \geq \phi^{-1}(0)} (\vartheta - \phi(\vartheta))} > 0. \tag{6.29}$$

Assumption 6.2 can be alternatively stated as

$$\phi^{-1}(t) - t > 0. \tag{6.30}$$

The implication of the assumption on the delay-time and prediction-time functions is that they are both positive and uniformly bounded.

Now we return to system (6.24)–(6.26), the definition of the transport PDE state (6.21), and the control law (6.15). The control law (6.15) is written in terms of $u(x,t)$ as

$$\begin{aligned} u(1,t) = K \Bigg[& \mathrm{e}^{A\left(\phi^{-1}(t)-t\right)} X(t) \\ & + \int_0^1 \mathrm{e}^{A(1-y)\left(\phi^{-1}(t)-t\right)} Bu(y,t) \left(\phi^{-1}(t) - t\right) dy \Bigg]. \end{aligned} \tag{6.31}$$

In order to study the exponential stability of the system $(X(t), u(x,t), x \in [0,1])$, we introduce the initial condition

$$u_0(x) = u(x,0) = U\left(\phi\left(\phi^{-1}(0)x\right)\right), \quad x \in [0,1], \tag{6.32}$$

and $X_0 = X(0)$.

Now we establish the following stability result.

Theorem 6.1. *Consider the closed-loop system consisting of the plant (6.24)–(6.26) and the controller (6.31) and let Assumptions 6.1 and 6.2 hold. There exist a positive constant G, and a positive constant g independent of the function $\phi(\cdot)$, such that*

$$|X(t)|^2 + \|u(t)\|^2 \leq G\mathrm{e}^{-gt} \left(|X_0|^2 + \|u_0\|^2\right), \quad \forall t \geq 0. \tag{6.33}$$

Proof. Consider the transformation of the transport PDE state given by

$$\begin{aligned} w(x,t) = {} & u(x,t) - K\mathrm{e}^{Ax\left(\phi^{-1}(t)-t\right)} X(t) \\ & - K \int_0^x \mathrm{e}^{A(x-y)\left(\phi^{-1}(t)-t\right)} Bu(y,t) \left(\phi^{-1}(t) - t\right) dy. \end{aligned} \tag{6.34}$$

Taking the derivatives of $w(x,t)$ with respect to t and x, we get

$$\begin{aligned} w_t(x,t) = {} & u_t(x,t) - KAx \left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1 \right) \mathrm{e}^{Ax\left(\phi^{-1}(t)-t\right)} X(t) \\ & - K\mathrm{e}^{Ax\left(\phi^{-1}(t)-t\right)} \left(X(t) + Bu(0,t)\right) \end{aligned}$$

$$
\begin{aligned}
&-K\int_0^x e^{A(x-y)\left(\phi^{-1}(t)-t\right)}\left(A(x-y)\left(\phi^{-1}(t)-t\right)+I\right)\\
&\times\left(\frac{d\left(\phi^{-1}(t)\right)}{dt}-1\right)Bu(y,t)dy\\
&-K\int_0^x e^{A(x-y)\left(\phi^{-1}(t)-t\right)}Bu_t(y,t)\left(\phi^{-1}(t)-t\right)dy\\
=\ & u_t(x,t)-\left(1+x\left(\frac{d\left(\phi^{-1}(t)\right)}{dt}-1\right)\right)K\Big[Ae^{Ax\left(\phi^{-1}(t)-t\right)}X(t)\\
&+A\int_0^x e^{A(x-y)\left(\phi^{-1}(t)-t\right)}Bu(y,t)\left(\phi^{-1}(t)-t\right)dy\\
&+Bu(x,t)\Big],
\end{aligned}
\tag{6.35}
$$

where we have used integration by parts, and

$$
\begin{aligned}
w_x(x,t)=u_x(x,t)-\left(\phi^{-1}(t)-t\right)K\Big[&Ae^{Ax\left(\phi^{-1}(t)-t\right)}X(t)\\
&+A\int_0^x e^{A(x-y)\left(\phi^{-1}(t)-t\right)}Bu(y,t)\left(\phi^{-1}(t)-t\right)dy\\
&+Bu(x,t)\Big].
\end{aligned}
\tag{6.36}
$$

With the help of (6.31), we also obtain $w(1,t)=0$; hence, we arrive at the "target system"

$$\dot{X}(t)=(A+BK)X(t)+Bw(0,t)\,, \tag{6.37}$$

$$w_t(x,t)=\pi(x,t)w_x(x,t)\,, \tag{6.38}$$

$$w(1,t)=0\,. \tag{6.39}$$

This is a standard cascade configuration

$$w\rightarrow X \tag{6.40}$$

that we have encountered many times before. We focus first on the Lyapunov analysis of the w-subsystem. We take a Lyapunov function

$$L(t)=\frac{1}{2}\int_0^1 e^{bx}w^2(x,t)dx\,, \tag{6.41}$$

where b is any positive constant. The time derivative of $L(t)$ is

$$\dot{L}(t)=\int_0^1 e^{bx}w(x,t)w_t(x,t)dx$$

$$
\begin{aligned}
&= \int_0^1 e^{bx} w(x,t)\pi(x,t)w_x(x,t)dx \\
&= \frac{1}{2}\int_0^1 e^{bx}\pi(x,t)dw^2(x,t) \\
&= \frac{e^{bx}}{2}\pi(x,t)w^2(x,t)\Big|_0^1 \\
&\quad - \frac{1}{2}\int_0^1 \left(b\pi(x,t)+\pi_x(x,t)\right)e^{bx}w^2(x,t)dx \\
&= -\frac{\pi(0,t)}{2}w^2(0,t) \\
&\quad - \frac{1}{2}\int_0^1 \left(b\pi(x,t)+\pi_x(x,t)\right)e^{bx}w^2(x,t)dx\,.
\end{aligned} \tag{6.42}
$$

Noting that

$$
\pi(0,t) = \frac{1}{\phi^{-1}(t)-t} \geq \pi_0^*\,, \tag{6.43}
$$

we get

$$
\dot{L}(t) \leq -\frac{\pi_0^*}{2}w^2(0,t) - \frac{1}{2}\int_0^1 \left(b\pi(x,t)+\pi_x(x,t)\right)e^{bx}w^2(x,t)dx\,. \tag{6.44}
$$

Next, we observe that

$$
\pi_x(x,t) = \frac{\dfrac{d\left(\phi^{-1}(t)\right)}{dt}-1}{\phi^{-1}(t)-t} \tag{6.45}
$$

is a function of t only. Hence,

$$
\begin{aligned}
b\pi(x,t)+\pi_x(x,t) &= \frac{b\left[1+x\left(\dfrac{d\left(\phi^{-1}(t)\right)}{dt}-1\right)\right]+\dfrac{d\left(\phi^{-1}(t)\right)}{dt}-1}{\phi^{-1}(t)-t} \\
&= \frac{b-1+\dfrac{d\left(\phi^{-1}(t)\right)}{dt}+bx\left(\dfrac{d\left(\phi^{-1}(t)\right)}{dt}-1\right)}{\phi^{-1}(t)-t}\,.
\end{aligned} \tag{6.46}
$$

Since this is a linear function of x, it follows that it has a minimum at either $x = 0$ or $x = 1$, so we get

$$
b\pi(x,t)+\pi_x(x,t) \geq \frac{\min\left\{b-1+\dfrac{d\left(\phi^{-1}(t)\right)}{dt},\ (b+1)\dfrac{d\left(\phi^{-1}(t)\right)}{dt}-1\right\}}{\phi^{-1}(t)-t}\,. \tag{6.47}
$$

Next, we note that

$$\frac{d\left(\phi^{-1}(t)\right)}{dt} = \frac{1}{\phi'\left(\phi^{-1}(t)\right)} \geq \frac{1}{\sup_{\vartheta \geq \phi^{-1}(0)} \phi'(\vartheta)} = \pi_1^*, \tag{6.48}$$

which yields

$$b\pi(x,t) + \pi_x(x,t) \geq \frac{\min\{b-1+\pi_1^*,\ (b+1)\pi_1^* - 1\}}{\phi^{-1}(t) - t}. \tag{6.49}$$

Choosing

$$b \geq (1-\pi_1^*)\max\left\{1,\ \frac{1}{\pi_1^*}\right\}, \tag{6.50}$$

we get

$$b\pi(x,t) + \pi_x(x,t) \geq \pi_0^*\beta^*, \tag{6.51}$$

where

$$\beta^* = \min\{b-1+\pi_1^*,\ (b+1)\pi_1^* - 1\} > 0. \tag{6.52}$$

So, returning to $\dot{L}(t)$, we have

$$\dot{L}(t) \leq -\frac{\pi_0^*}{2} w^2(0,t) - \pi_0^*\beta^* L(t). \tag{6.53}$$

Let us now turn our attention to the X-subsystem. We have

$$\frac{d}{dt}\left(X(t)^T P X(t)\right) = -X^T(t) Q X(t) + 2X^T(t) P B w(0,t), \tag{6.54}$$

where P satisfies a Lyapunov equation

$$P(A+BK) + (A+BK)^T P = -Q. \tag{6.55}$$

With a usual completion of squares, we get

$$\frac{d}{dt}\left(X(t)^T P X(t)\right) \leq -\lambda_{\min}(Q)|X(t)|^2 + \frac{2|PB|^2}{\lambda_{\min}(Q)} w^2(0,t). \tag{6.56}$$

Now we take the Lyapunov–Krasovskii functional

$$V(t) = X(t)^T P X(t) + \frac{4|PB|^2}{\pi_0^* \lambda_{\min}(Q)} L(t). \tag{6.57}$$

Its derivative is

$$\dot{V}(t) \le -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \pi_0^*\beta^*\frac{4|PB|^2}{\pi_0^*\lambda_{\min}(Q)}L(t)\,. \tag{6.58}$$

Finally, with the definition of $V(t)$, we get

$$\dot{V}(t) \le -\mu V(t)\,, \tag{6.59}$$

where

$$\mu = \min\left\{\pi_0^*\beta^*,\ \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}\right\}. \tag{6.60}$$

Thus, we obtain

$$V(t) \le \mathrm{e}^{-\mu t}V(0), \quad \forall t \ge 0\,. \tag{6.61}$$

Let us now denote

$$\Omega(t) = |X(t)|^2 + \int_0^1 w^2(x,t)dx\,. \tag{6.62}$$

The following relation holds between $V(t)$ and $\Omega(t)$:

$$\psi_1\Omega(t) \le V(t) \le \psi_2\Omega(t)\,, \tag{6.63}$$

where

$$\psi_1 = \min\left\{\lambda_{\min}(P), \frac{2|PB|^2}{\pi_0^*\lambda_{\min}(Q)}\right\}, \tag{6.64}$$

$$\psi_2 = \max\left\{\lambda_{\max}(P), \frac{2|PB|^2}{\pi_0^*\lambda_{\min}(Q)}\mathrm{e}^{b}\right\}. \tag{6.65}$$

It then follows that

$$\Omega(t) \le \frac{\psi_2}{\psi_1}\mathrm{e}^{-\mu t}\Omega(0), \quad \forall t \ge 0\,. \tag{6.66}$$

Now we consider the norm

$$\Xi(t) = |X(t)|^2 + \int_0^1 u^2(x,t)dx\,. \tag{6.67}$$

We recall the backstepping transformation (6.34) and introduce its inverse,

$$\begin{aligned} u(x,t) &= w(x,t) + K\mathrm{e}^{(A+BK)x\left(\phi^{-1}(t)-t\right)}X(t) \\ &\quad + K\int_0^x \mathrm{e}^{(A+BK)(x-y)\left(\phi^{-1}(t)-t\right)}Bu(y,t)\left(\phi^{-1}(t)-t\right)dy\,. \end{aligned} \tag{6.68}$$

It can be shown that

$$\|w(t)\|^2 \le \alpha_1(t)\|u(t)\|^2 + \alpha_2|X(t)|^2\,, \tag{6.69}$$

$$\|u(t)\|^2 \le \beta_1(t)\|w(t)\|^2 + \beta_2|X(t)|^2\,, \tag{6.70}$$

where

$$\alpha_1(t) = 3\left(1 + \int_0^1 \left(KM\left(x\left(\phi^{-1}(t)-t\right)\right)B\left(\phi^{-1}(t)-t\right)\right)^2 dx\right), \tag{6.71}$$

$$\alpha_2(t) = 3\int_0^1 \left|KM\left(x\left(\phi^{-1}(t)-t\right)\right)\right|^2 dx, \tag{6.72}$$

$$\beta_1(t) = 3\left(1 + \int_0^1 \left(KN\left(x\left(\phi^{-1}(t)-t\right)\right)B\left(\phi^{-1}(t)-t\right)\right)^2 dx\right), \tag{6.73}$$

$$\beta_2(t) = 3\int_0^1 \left|KN\left(x\left(\phi^{-1}(t),-t\right)\right)\right|^2 dx, \tag{6.74}$$

and where

$$M(s) = \mathrm{e}^{As}, \tag{6.75}$$

$$N(s) = \mathrm{e}^{(A+BK)s}. \tag{6.76}$$

Furthermore, we can show that

$$\alpha_1(t) \le \bar{\alpha}_1 = 3\left(1 + |K|^2|B|^2 \frac{\mathrm{e}^{\frac{2|A|}{\pi_0^*}} - 1}{2\pi_0^*|A|}\right), \tag{6.77}$$

$$\alpha_2(t) \le \bar{\alpha}_2 = 3|K|^2\pi_0^* \frac{\mathrm{e}^{\frac{2|A|}{\pi_0^*}} - 1}{2|A|}, \tag{6.78}$$

$$\beta_1(t) \le \bar{\beta}_1 = 3\left(1 + |K|^2|B|^2 \frac{\mathrm{e}^{\frac{2|A+BK|}{\pi_0^*}} - 1}{2\pi_0^*|A+BK|}\right), \tag{6.79}$$

$$\beta_2(t) \le \bar{\beta}_2 = 3|K|^2\pi_0^* \frac{\mathrm{e}^{\frac{2|A+BK|}{\pi_0^*}} - 1}{2|A+BK|}. \tag{6.80}$$

With a few substitutions, we obtain

$$\phi_1 \Xi(t) \le \Omega(t) \le \phi_2 \Xi(t), \tag{6.81}$$

where

$$\phi_1 = \frac{1}{\max\left\{\bar{\beta}_1, \bar{\beta}_2 + 1\right\}}, \tag{6.82}$$

$$\phi_2 = \max\left\{\bar{\alpha}_1, \bar{\alpha}_2 + 1\right\}. \tag{6.83}$$

Finally, we get

$$\Xi(t) \le \frac{\phi_2\psi_2}{\phi_1\psi_1}\mathrm{e}^{-\mu t}\Xi(0), \quad \forall t \ge 0, \tag{6.84}$$

which completes the proof of the theorem with

$$G = \frac{\phi_2 \psi_2}{\phi_1 \psi_1} \tag{6.85}$$

and

$$g = \mu \,. \tag{6.86}$$

By choosing

$$\beta^* \geq \frac{\lambda_{\min}(Q)}{2\pi_0^* \lambda_{\max}(P)}\,, \tag{6.87}$$

i.e., by picking b positive and such that

$$b \geq \left(1 - \pi_1^* + \frac{\lambda_{\min}(Q)}{2\pi_0^* \lambda_{\max}(P)}\right) \max\left\{1,\ \frac{1}{\pi_1^*}\right\}, \tag{6.88}$$

we get

$$g = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}\,, \tag{6.89}$$

which means that g is independent of the function $\phi(\cdot)$. □

While Theorem 6.1 provides a stability result in terms of the system norm

$$|X(t)|^2 + \int_0^1 u^2(x,t)dx\,, \tag{6.90}$$

we would like also to get a stability result in terms of the norm

$$|X(t)|^2 + \int_{\phi(t)}^t U^2(\theta)d\theta\,. \tag{6.91}$$

Toward that end, we first observe that

$$\int_{\phi(t)}^t U^2(\theta)d\theta = \left(\phi^{-1}(t) - t\right)\int_0^1 \phi'\left(t + x\left(\phi^{-1}(t) - t\right)\right) u^2(x,t)dx, \tag{6.92}$$

$$\int_0^2 u_0^2(x)dx = \frac{1}{\phi^{-1}(0)} \int_{\phi(0)}^0 \frac{1}{\phi'\left(\phi^{-1}(\theta)\right)} U^2(\theta)d\theta. \tag{6.93}$$

With these identities, we obtain the following theorem from Theorem 6.1.

Theorem 6.2. *Consider the closed-loop system consisting of the plant (6.24)–(6.26) and the controller (6.31) and let Assumptions 6.1 and 6.2 hold. There exist positive constants G and g (the latter one being independent of ϕ) such that*

$$|X(t)|^2 + \int_{\phi(t)}^t U^2(\theta)d\theta \leq hG\mathrm{e}^{-gt}\left(|X_0|^2 + \int_{\phi(0)}^0 U^2(\theta)d\theta\right), \quad \forall t \geq 0\,, \tag{6.94}$$

where G is the same as in the proof of Theorem 6.1 and

$$h = \frac{\sup_{\tau \geq 0} \phi'(\tau)}{\pi_0^* \phi^{-1}(0) \inf_{\tau \in [0, \phi^{-1}(0)]} \phi'(\tau)}. \tag{6.95}$$

6.3 Observer Design with Time-Varying Sensor Delay

We give a brief presentation of an observer design for an LTI system with a time-varying sensor delay:

$$\dot{X}(t) = AX(t) + BU(t), \tag{6.96}$$

$$Y(t) = CX(\phi(t)). \tag{6.97}$$

We approach the observer design problem in a two-step manner:

- Design an observer for the delay state $X(\phi(t))$ since the output $Y(t) = CX(\phi(t))$ is delayed.
- Use a model-based predictor to advance the estimate of $X(\phi(t))$ by the delay time $t - \phi(t)$.

We start by writing (6.96) as

$$\frac{dX(\phi(t))}{d\phi(t)} = AX(\phi(t)) + BU(\phi(t)). \tag{6.98}$$

Then we introduce a state estimator $\Sigma(\phi(t))$ for $X(\phi(t))$ as

$$\frac{d\Sigma(\phi(t))}{d\phi(t)} = A\Sigma(\phi(t)) + BU(\phi(t)) + L(Y(t) - C\Sigma(\phi(t))), \tag{6.99}$$

where L is selected so that the matrix $A - LC$ is Hurwitz, i.e., so that the system

$$\frac{d\zeta(\phi(t))}{d\phi(t)} = (A - LC)\zeta(\phi(t)), \tag{6.100}$$

where

$$\zeta(\phi(t)) = X(\phi(t)) - \Sigma(\phi(t)), \tag{6.101}$$

is exponentially stable in the time variable $\phi(t)$.

Let us now denote

$$\xi(t) = \Sigma(\phi(t)). \tag{6.102}$$

This variable is governed by the differential equation

$$\dot{\xi}(t) = \frac{d\Sigma(\phi(t))}{d\phi(t)} \phi'(t)$$

$$= \phi'(t)\left[A\Sigma(\phi(t)) + BU(\phi(t)) + L(Y(t) - C\Sigma(\phi(t)))\right]$$
$$= \phi'(t)\left[A\xi(t) + BU(\phi(t)) + L(Y(t) - C\xi(t))\right]. \tag{6.103}$$

Now we take $\xi(t) = \Sigma(\phi(t))$, which is an estimate of the past state $X(\phi(t))$, and advance it by the delay time $t - \phi(t)$, obtaining the estimate of the current state $X(t)$, which we denote by $\hat{X}(t)$:

$$\hat{X}(t) = \mathrm{e}^{A(t-\phi(t))}\xi(t) + \int_{\phi(t)}^{t} \mathrm{e}^{A(t-\tau)}BU(\tau)d\tau. \tag{6.104}$$

To summarize, the observer is given by the equations

$$\dot{\xi}(t) = \phi'(t)\left[A\xi(t) + BU(\phi(t)) + L(Y(t) - C\xi(t))\right], \tag{6.105}$$

$$\hat{X}(t) = \mathrm{e}^{A(t-\phi(t))}\xi(t) + \int_{\phi(t)}^{t} \mathrm{e}^{A(t-\tau)}BU(\tau)d\tau. \tag{6.106}$$

This observer has a structure that displays duality with respect to the predictor-based controller (6.15) in two interesting ways:

- While controller (6.15) employs prediction over the future period $\left[t, \phi^{-1}(t)\right]$, the observer (6.105)–(6.106) employs prediction over the past period $[\phi(t), t]$.
- While controller (6.15) involves a time derivative of $\phi^{-1}(t)$, the observer (6.105)–(6.106) involves a time derivative of $\phi(t)$.

In the case of a constant sensor delay, namely, $\phi(t) = t - D$, the observer (6.105)–(6.106) reduces to (3.57)–(3.58).

6.4 Examples

We illustrate the control law (6.15),

$$U(t) = K\left[\mathrm{e}^{A\left(\phi^{-1}(t)-t\right)}X(t) + \int_{\phi(t)}^{t} \frac{d\left(\phi^{-1}(\theta)\right)}{d\theta}\mathrm{e}^{A\left(\phi^{-1}(t)-\phi^{-1}(\theta)\right)}BU(\theta)d\theta\right], \tag{6.107}$$

through several examples. The first two examples will actually violate some of the basic assumptions of the theory but will be valuable in illustrating the design principle. The other two examples will fit the assumptions.

Example 6.1 (Linearly growing delay). We consider

$$\phi_1(t) = \frac{t}{2}, \tag{6.108}$$

which means that the delay time

$$\delta(t) = t - \phi_1(t) = \frac{t}{2} \tag{6.109}$$

is linearly growing and unbounded. (In addition, the assumption that the delay is strictly positive for all time is violated, but this assumption is less essential.) The inverse of $\phi_1(t)$ is

$$\phi_1^{-1}(t) = 2t\,, \tag{6.110}$$

so the prediction time is

$$\phi_1^{-1}(t) - t = t\,. \tag{6.111}$$

The derivative of $\phi_1(t)$ is

$$\frac{d\left(\phi_1^{-1}(t)\right)}{dt} = 2\,. \tag{6.112}$$

So the predictor feedback (6.107) assumes the form

$$U(t) = K\left[\mathrm{e}^{At}X(t) + \int_{t/2}^{t} 2\mathrm{e}^{2A(t-\theta)}BU(\theta)d\theta\right]. \tag{6.113}$$

It is interesting to observe that this system has zero dead time, since the initial delay is zero and the controller continues to compensate the delay for $t \geq 0$. The control signal is $U(t) = KX(2t)$ and, since $X(t) = \mathrm{e}^{(A+BK)t}X_0$, the control signal remains a bounded function $U(t) = K\mathrm{e}^{(A+BK)2t}X_0$ in spite of the delay growing unbounded. This is potentially confusing, as some difficulty should arise in a system where the delay is growing unbounded. The difficulty manifests itself when the system is subject to a disturbance or modeling error. In that case the control signal will not be given by $U(t) = K\mathrm{e}^{(A+BK)2t}X_0$ but will be governed by the feedback law (6.113). In this feedback law the gains grow exponentially with time. In the presence of a persistent disturbance, which prevents $X(t)$ from settling, the control signal will grow unbounded as its gains grow unbounded.

Example 6.2 (Prediction time grows exponentially). We consider

$$\phi_2(t) = \ln(1+t)\,. \tag{6.114}$$

In this case we get

$$\phi_2^{-1}(t) = \mathrm{e}^t - 1\,, \tag{6.115}$$

$$\phi_2^{-1}(t) - t = \mathrm{e}^t - 1 - t\,, \tag{6.116}$$

$$\frac{d\left(\phi_2^{-1}(t)\right)}{dt} = \mathrm{e}^t\,, \tag{6.117}$$

$$\phi_2'(t) = \frac{1}{1+t}\,. \tag{6.118}$$

In this case

$$U(t)=K\left[\mathrm{e}^{A(\mathrm{e}^{t}-1-t)}X(t)+\int_{\ln(1+t)}^{t}\mathrm{e}^{-\theta}\mathrm{e}^{A\left(\mathrm{e}^{t}-\mathrm{e}^{\theta}\right)}BU(\theta)d\theta\right]. \tag{6.119}$$

The gain growth with time is even more pronounced here than in Example 6.2—the gains grow as an exponential of an exponential.

Example 6.3 (Bounded delay with a constant limit). We consider

$$\phi_3(t)=t-\frac{1+t}{1+2t}. \tag{6.120}$$

In this case the delay time $\delta(t)=t-\phi_3(t)$ is such that

$$\delta(0)=1, \tag{6.121}$$

$$\delta(\infty)=\frac{1}{2}. \tag{6.122}$$

In addition, we get

$$\phi_3^{-1}(t)=\frac{t+\sqrt{(t+1)^2+1}}{2}, \tag{6.123}$$

$$\phi_3^{-1}(t)-t=\frac{-t+\sqrt{(t+1)^2+1}}{2}, \tag{6.124}$$

$$\frac{d\left(\phi_3^{-1}(t)\right)}{dt}=\frac{1}{2}\left(1+\frac{1}{\sqrt{1+\frac{1}{(1+t)^2}}}\right). \tag{6.125}$$

Let us examine the prediction time (6.124):

$$\phi_3^{-1}(t)-t=\frac{-t+\sqrt{(t+1)^2+1}}{2}=\frac{t+1}{\sqrt{(t+1)^2+1}+t}. \tag{6.126}$$

The initial value of the prediction time is $\sqrt{2}/2$, the final value is $1/2$, and the uniform bound on the prediction time is $\sqrt{2}/2$. Furthermore, the uniform bound on the quantity (6.125) is 1. Hence, the feedback law

$$U(t)=K\Bigg[\mathrm{e}^{A\left(\phi_3^{-1}(t)-t\right)}X(t)+\int_{\phi_3(t)}^{t}\frac{d\left(\phi_3^{-1}(\theta)\right)}{d\theta}\mathrm{e}^{A\left(\phi_3^{-1}(t)-\phi_3^{-1}(\theta)\right)}BU(\theta)d\theta\Bigg] \tag{6.127}$$

employs bounded gains and achieves exponential stability (it also achieves a finite disturbance-to-state gain).

Example 6.4 (Bounded delay function without a limit). In the past three examples the function $\phi'(t)$ was monotonic and had a limit [in two of the three examples the function $\phi(t)$ itself also had a limit]. Now we consider an example where $\phi(t)$ is oscillatory. Let

$$\rho(t) = t + 1 + \frac{1}{2}\cos t \tag{6.128}$$

and denote

$$\phi_4(t) = \rho^{-1}(t). \tag{6.129}$$

So

$$\phi_4^{-1}(t) = \rho(t) = t + 1 + \frac{1}{2}\cos t. \tag{6.130}$$

Furthermore, we have

$$\phi_4^{-1}(t) - t = 1 + \frac{1}{2}\cos t, \tag{6.131}$$

$$\frac{d\left(\phi_4^{-1}(t)\right)}{dt} = 1 - \frac{1}{2}\sin t. \tag{6.132}$$

Thus, both of the quantities involved in the gains of the predictor feedback

$$\begin{aligned} U(t) = K\Bigg[& e^{A\left(1+\frac{1}{2}\cos t\right)}X(t) \\ & + \int_{\rho^{-1}(t)}^{t}\left(1-\frac{1}{2}\sin\theta\right)e^{A\left(t+\frac{1}{2}\cos t-\theta-\frac{1}{2}\cos\theta\right)}BU(\theta)d\theta\Bigg] \end{aligned} \tag{6.133}$$

are uniformly bounded. Now we consider a specific first-order example

$$\dot{X}(t) = X(t) + U\left(\rho^{-1}(t)\right), \tag{6.134}$$

namely, an example with

$$A = B = 1. \tag{6.135}$$

In a closed loop with the control law

$$\begin{aligned} U(t) = -(1+c)\Bigg[& e^{1+\frac{1}{2}\cos t}X(t) \\ & + \int_{\rho^{-1}(t)}^{t}\left(1-\frac{1}{2}\sin\theta\right)e^{t+\frac{1}{2}\cos t-\theta-\frac{1}{2}\cos\theta}U(\theta)d\theta\Bigg], \end{aligned} \tag{6.136}$$

where $c > 0$, the plant (6.134) has an explicit solution

$$X(t) = X_0\begin{cases} e^{t}, & t \in [0, \rho(0)), \\ e^{-c(t-\rho(0))+\rho(0)}, & t \geq \rho(0)|. \end{cases} \tag{6.137}$$

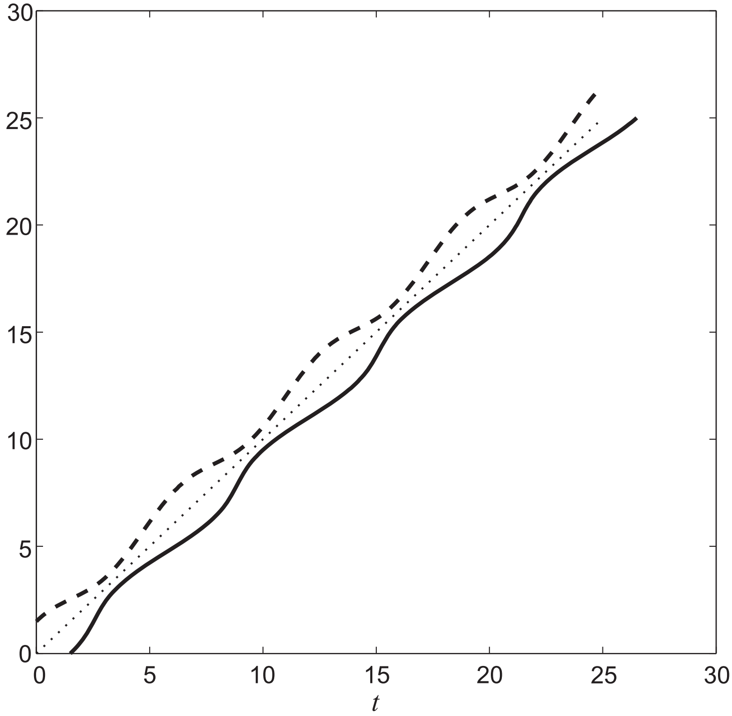

Fig. 6.2 Oscillating delay function in Example 6.4. Solid: $\phi_4(t)$; dashed: $\rho(t)$.

The explicit form of the control signal is

$$U(t) = -(1+c)\mathrm{e}^{\rho(0)-c\rho(t)}X_0, \quad \forall t \geq 0\,. \tag{6.138}$$

The explicit formulas for both $X(t)$ and $U(t)$ require $\rho(0)$, which is given by

$$\rho(0) = \frac{3}{2}\,. \tag{6.139}$$

Figures 6.2, 6.3, and 6.4 show the graphs of the delay, state, and control functions in this example. The gain is chosen as $c = 0.13$ to achieve visual clarity about the LTV character of the overall system, particularly about the response of $U(t)$, which has a "wavy" character to compensate for the oscillating delay function.

6.5 Notes and References

We have presented predictor feedback for LTI systems with a time-varying delay and proved exponential stability under several nonrestictive conditions on the delay function

$$\delta(t) = t - \phi(t)\,. \tag{6.140}$$

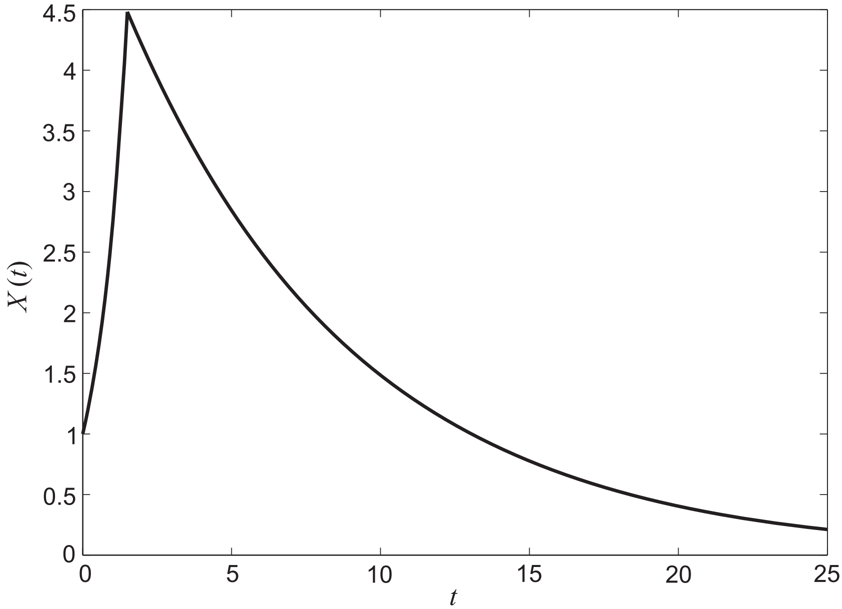

Fig. 6.3 State evolution for Example 6.4. The control "kicks in" at $\rho(0) = 3/2$ s.

These conditions are as follows:

- the delay function $\delta(t)$ is strictly positive (technical condition ensuring that the state space of the input dynamics can be defined);
- the delay function $\delta(t)$ is uniformly bounded from above;
- the delay-rate function $\delta'(t)$ is strictly smaller than 1; i.e., the delay may *increase* at a rate smaller than 1;
- the delay-rate function $\delta'(t)$ is uniformly bounded from below (by a possibly negative finite constant); i.e., the delay may *decrease* at a uniformly bounded rate.

All these conditions need to be satisfied simultaneously. This has two implications. First, the delay can grow at a rate strictly smaller than 1 but not indefinitely, because the delay must remain uniformly bounded. Second, the delay may decrease at any uniformly bounded rate but not indefinitely, because it must remain positive.

Perhaps the most interesting elements of our construction are the choice of the transport PDE state (6.21) and the backstepping transformation (6.34). They are quite nontrivial to guess because one has to simultaneously guess the relationship among $u(x,t)$ and $U(\cdot)$, the convection speed function $\pi(x,t)$ of the transport PDE in the plant and the target system, and the kernels of the backstepping transformation. Even if one has already made a "lucky" guess about the definition of the transport PDE state (6.21), the backstepping transformation is difficult to find because it has the form

$$w(x,t) = u(x,t) - \gamma(x,t)X(t) - \int_0^x m(x-y,t)u(y,t)dy, \tag{6.141}$$

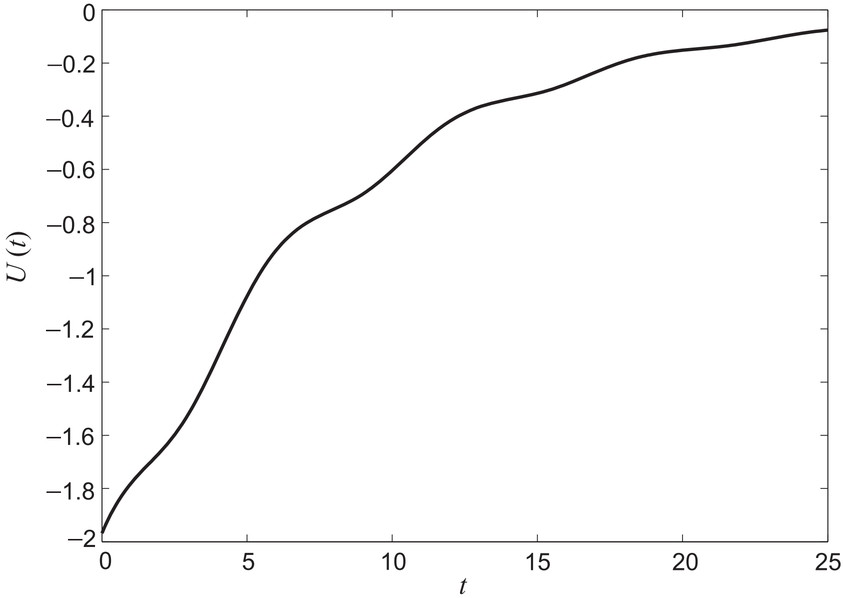

Fig. 6.4 Control signal for Example 6.4. The "waviness" serves to compensate for the time-varying (oscillating) delay.

where the time-varying kernel functions $\gamma(x,t)$ and $m(x-y,t)$ have to satisfy certain PDEs in space and time, with coefficients that depend on space and time through the convection speed function $\pi(x,t)$. We constructed the transport PDE state (6.21) and the backstepping transformation (6.34) through several "educated guesses," including, for example, that x and y in the backstepping transformation (4.16) should be prepaced by $x\left(\phi^{-1}(t)-t\right)$ and $y\left(\phi^{-1}(t)-t\right)$.

The control design that we studied in this chapter was introduced by Nihtila [174]. A more general framework for LTV systems with variable delay was presented in [8], but without a control design or stability study.

Part II
Adaptive Control

Chapter 7
Delay-Adaptive Full-State Predictor Feedback

Adaptive control in the presence of actuator delays is a challenging problem. Over the last 20 years, several control designs have been developed that address this problem. However, the existing results deal only with the problem where the plant has unknown parameters but the delay value is known. The remaining theoretical frontier, and a problem of great practical relevance, is the case where the actuator delay value is unknown and highly uncertain (of completely unknown value). This problem is open in general even in the case where no parametric uncertainty exists in the ODE plant.

In this chapter we present a Lyapunov-based adaptive control design that achieves global stability, without a requirement that the delay estimate be near the true delay value. We solve the problem by employing a framework where the actuator delay is represented as a transport PDE, by estimating the delay value as the reciprocal of the convection speed in the transport PDE, and by using full-state predictor-based feedback.

We focus on the system

$$\dot{X}(t) = AX(t) + BU(t-D)\,, \tag{7.1}$$

where the full state—both the ODE plant state $X \in \mathbb{R}^n$ and the infinite-dimensional actuator state $U(\eta), \eta \in [t-D, t]$—is available for measurement, and where the ODE plant parameters are known, but where the delay length D is unknown (though constant) and can have an arbitrarily large value. This problem can be formulated around an actuator delay model given by a *transport equation* (convective/first-order hyperbolic PDE), namely,

$$\dot{X}(t) = AX(t) + Bu(0,t)\,, \tag{7.2}$$

$$Du_t(x,t) = u_x(x,t)\,, \tag{7.3}$$

$$u(1,t) = U(t)\,, \tag{7.4}$$

where $u(x,t)$ is the state of the actuator, the domain length is known (unity), but the propagation speed $1/D$ is unknown. The actuator state is related to the input through

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_7,

the following equation:

$$u(x,t) = U(t + D(x-1)), \tag{7.5}$$

which, in particular, gives $u(1,t) = U(t)$ and $u(0,t) = U(t-D)$. The control law around which we build a delay-adaptation mechanism is a predictor-based feedback law

$$U(t) = K\left[e^{AD}X(t) + D\int_0^1 e^{AD(1-y)}Bu(y,t)dy \right], \tag{7.6}$$

which achieves exponential stability at $u \equiv 0, X = 0$ by performing perfect compensation of the actuator delay.

Within this framework, in this chapter we obtain a global adaptive stabilization design for an arbitrarily large and unknown actuator delay value.

Without a question, an even more relevant and challenging problem is the one where the full state is not available for measurement, more specifically, when the state of the transport PDE $u(x,t)$, i.e., the actuator state, is not measured. A 'yet more challenging problem is when, in addition, only an output of the ODE system

$$Y(t) = CX(t) \tag{7.7}$$

is measured, rather than the full state $X(t)$, and, finally, the most challenging in this string of problems is when the ODE plant has parametric uncertainty, i.e., $A(\theta), B(\theta), C(\theta)$, where θ is unknown. (For an exhaustive categorization of adaptive control problems with actuator delay, see Section 7.1.) However, as restrictive as the requirement for the measurement of $u(x,t)$ may seem, we do not believe that any delay-adaptive problem without the measurement of $u(x,t)$ is solvable globally because it cannot be formulated as linearly parametrized in the unknown delay D. As a consequence, when the controller uses an estimate of $u(x,t)$, not only do the initial values of the ODE state and the actuator state have to be small, but the initial value of the delay estimation error also has to be small (the delay value is allowed to be large, but the initial value of its estimate has to be close to the true value of the delay).

Such a *local* adaptive result is fundamentally weaker in terms of uncertainty management, so we consider our full-state design in this chapter, which gives a global result, to be the key result. However, we also present the local adaptive result for contrast (Chapter 8) as well as document the difference between the "desirable" and "possible" in the research literature on this important topic, for the benefit of future researchers attempting this problem.

In our global full-state feedback design we require only one bit of a priori knowledge regarding the length of the delay:

Assumption 7.1. *An upper bound $\bar{D}$ on the unknown $D > 0$ is known.*

This upper bound is used in two ways. An adaptation algorithm employing projection keeps the delay estimate below the a priori bound. In addition, based on the upper bound for the delay length, the adaptation gain is selected to be sufficiently *small*, and a normalization parameter is selected to be sufficiently large, to ensure that adaptation is sufficiently slow to guarantee closed-loop stability. The approach for update law design (Section 7.2) and for the corresponding stability analysis (Section 7.3) is based on the ideas that we introduced in [117] for Lyapunov-based adaptive control of parabolic PDEs. The adaptation and normalization gain choices are conservative. The relevant part of the design is the structure of the adaptation law, not the exact gain values employed in the analysis.

In this chapter the only parametric uncertainty considered is the unknown delay. This is done for clarity of presentation, as the presence of unknown parameters in the plant would obscure the presentation of new tools for handling the unknown delay. In Chapter 9 we present an extension with unknown plant parameters and where the control objective is not regulation to zero but trajectory tracking.

We start the chapter with Section 7.1, in which we categorize all the combinations of delay-adaptive, ODE parameter-adaptive, full-state, and output-feedback problems arising in the area of adaptive control in the presence of delay.

7.1 Categorization of Adaptive Control Problems with Actuator Delay

A finite-dimensional system with actuator delay may come with the following four types of basic uncertainties:

- unknown delay (D),
- unmeasured actuator state (u),
- unknown parameters in the finite-dimensional part of the plant (A),
- unmeasured state of the finite-dimensional part of the plant (X).

Each of these situations introduces a design difficulty that needs to be dealt with by using an estimator (a parameter estimator or a state estimator). We point out that a state estimator of the actuator state is trivial when the delay is known (one gets the full state by waiting one delay period); however, this estimation problem is far from trivial when the delay is also unknown.

The symbols D,u,A, and X will be helpful as we try to categorize all the problems in which one, two, three, or all four of these design difficulties may arise. For example, (D,u,X) denotes the case where only the ODE plant parameters are known, whereas the delay is unknown and the state of the actuator and the ODE are unmeasurable.

A total of 14 combinations arises from the four basic problems, (D), (u), (A), and (X). We focus exclusively on problems where the delay is present and is of significant length to require the use of predictor feedback (rather than being treated as a small perturbation through some form of small gain argument). The following list categorizes the 15 control problems and gives the status of each of them:

1. (X), (u), (u,X)—nonadaptive problems solvable using observer-based predictor feedback (Chapter 2);
2. (A,X), (A)—solved in [182, 171, 44] but with relative degree limitations;
3. (u,A), (u,A,X)—tractable using the techniques from [182, 171, 44];

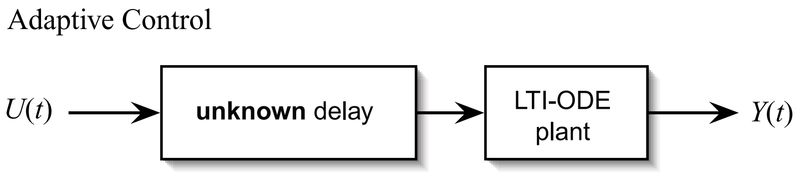

Fig. 7.1 An adaptive control problem for a long, unknown input delay.

4. (D)—present chapter (see Fig. 7.1);
5. (D,X)—tractable as in point 4 (by adding a standard ODE observer) but not highly relevant;
6. (D,A)—the subject of Chapter 9;
7. (D,A,X)—tractable using the techniques in point 6 combined with adaptive backstepping and Kreisslemeier observers;
8. (D,u), (D,u,A), $(\square,u,X)$, (D,u,A,X)—not tractable globally because of the lack of linear parametrization in any situation involving (D) and (u) simultaneously; the case (D,u) is studied in Chapter 8.

If this combinatorial complexity hasn't already overwhelmed the reader, we should point out that in each of the cases involving unknown parameters, namely, (D) and (A), multiple choices exist in terms of design methodology (Lyapunov-based, estimation/swapping-based, passivity/observer-based, direct, indirect, pole placement, etc.). In addition, in output feedback adaptive problems, namely, problems involving (A) and (X), the relative degree plays a major role in determining the difficulty of a problem. Finally, trajectory tracking requires additional tools compared to problems of regulation to zero.

So this book addresses only a subset among important problems in adaptive control with actuator delay, but in our opinion it addresses the most relevant among the tractable problems.

7.2 Delay-Adaptive Predictor Feedback with Full-State Measurement

We consider system (7.2)–(7.4) where the pair (A,B) is completely controllable. Before we proceed, for a reader familiar with our prior work, we point out that the representation (7.3), (7.4) is different from the representation

$$\breve{u}_t(\breve{x},t) = \breve{u}_x(\breve{x},t)\,, \tag{7.8}$$

$$\breve{u}(D,t) = U(t)\,, \tag{7.9}$$

$$\breve{u}(0,t) = U(t-D)\,, \tag{7.10}$$

which yields

$$u(\breve{x},t) = U(t+x-D)\,, \tag{7.11}$$

and which we used in Chapter 2 and would be less convenient for adaptive control as it is not linearly parametrized in D.

When D is unknown, we replace (7.6) by the adaptive controller

$$U(t) = K\left[\mathrm{e}^{A\hat{D}(t)}X(t) + \hat{D}(t)\int_0^1 \mathrm{e}^{A\hat{D}(t)(1-y)}Bu(y,t)dy\right] \tag{7.12}$$

with an estimate $\hat{D}$ governed by the update law

$$\dot{\hat{D}}(t) = \gamma \mathrm{Proj}_{[0,\bar{D}]}\{\tau(t)\}, \tag{7.13}$$

where

$$\tau(t) = -\frac{\int_0^1 (1+x)w(x,t)K\mathrm{e}^{A\hat{D}(t)x}dx\,(AX(t)+Bu(0,t))}{1+X(t)^TPX(t)+b\int_0^1(1+x)w(x,t)^2dx}, \tag{7.14}$$

the standard projection operator is given by

$$\mathrm{Proj}_{[0,\bar{D}]}\{\tau\} = \tau\begin{cases} 0, & \hat{D}=0 \text{ and } \tau<0, \\ 0, & \hat{D}=\bar{D} \text{ and } \tau>0, \\ 1, & \text{else}, \end{cases} \tag{7.15}$$

the matrix P is the positive-definite and symmetric solution of the Lyapunov equation

$$P(A+BK)+(A+BK)^TP = -Q \tag{7.16}$$

for any positive-definite and symmetric matrix Q, the constant b is chosen to satisfy the inequality

$$b \geq \frac{4|PB|^2\bar{D}}{\lambda_{\min}(Q)}, \tag{7.17}$$

the transformed state of the actuator is given by

$$\begin{aligned} w(x,t) = u(x,t) - \hat{D}(t)\int_0^x K\mathrm{e}^{A\hat{D}(t)(x-y)}Bu(y,t)dy \\ - K\mathrm{e}^{A\hat{D}(t)x}X(t), \end{aligned} \tag{7.18}$$

and the positive adaptation gain γ is chosen "sufficiently small."

For this adaptive controller, which is shown in Figure 7.2, the following result holds.

Theorem 7.1. *Consider the closed-loop system consisting of the plant (7.2)–(7.4), the control law (7.12), and the parameter update law defined through (7.13)–(7.18). Let Assumption 7.1 hold. There exists $\gamma^* > 0$ such that for any $\gamma \in (0,\gamma^*)$, the zero solution of the system $(X,u,\hat{D}-D)$ is stable in the sense that there exist positive constants R and ρ (independent of the initial conditions) such that for all initial conditions satisfying $(X_0,u_0,\hat{D}_0) \in \mathbb{R}^n \times L_2(0,1) \times [0,\bar{D}]$, the following holds:*

$$\Upsilon(t) \leq R\left(\mathrm{e}^{\rho\Upsilon(0)} - 1\right), \quad \forall t \geq 0, \tag{7.19}$$

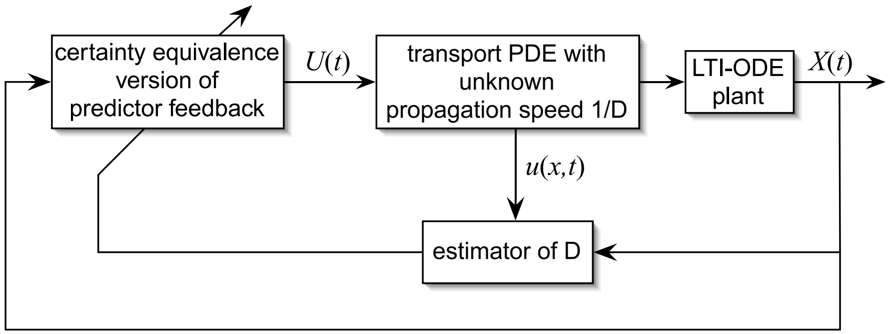

Fig. 7.2 Adaptive control diagram for an ODE with a transport PDE with unknown propagation speed $1/D$ at its input.

where

$$\Upsilon(t) = |X(t)|^2 + \int_0^1 u(x,t)^2 dx + \tilde{D}(t)^2 \,. \tag{7.20}$$

Furthermore,

$$\lim_{t\to\infty} X(t) = 0\,, \quad \lim_{t\to\infty} U(t) = 0\,. \tag{7.21}$$

7.3 Proof of Stability for Full-State Feedback

In this section we prove Theorem 7.1. We start by considering the transformation (7.18) along with its inverse

$$\begin{aligned} u(x,t) = w(x,t) + \hat{D}(t) \int_0^x K\mathrm{e}^{(A+BK)\hat{D}(t)(x-y)} Bw(y,t)dy \\ + K\mathrm{e}^{(A+BK)\hat{D}(t)x} X(t)\,. \end{aligned} \tag{7.22}$$

After a careful calculation, the transformed system can be written as

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t)\,, \tag{7.23}$$

$$Dw_t(x,t) = w_x(x,t) - \tilde{D}(t)p(x,t) - D\dot{\hat{D}}(t)q(x,t)\,, \tag{7.24}$$

$$w(1,t) = 0\,, \tag{7.25}$$

where

$$\tilde{D}(t) = D - \hat{D}(t) \tag{7.26}$$

is the parameter estimation error, and

$$\begin{aligned} p(x,t) &= K\mathrm{e}^{A\hat{D}(t)x} \left(AX(t) + Bu(0,t)\right) \\ &= K\mathrm{e}^{A\hat{D}(t)x} \left((A+BK)X(t) + Bw(0,t)\right), \end{aligned} \tag{7.27}$$

$$\begin{aligned}q(x,t) &= \int_0^x K(I+A\hat{D}(t)(x-y))\mathrm{e}^{A\hat{D}(t)(x-y)}Bu(y,t)dy \\ &\quad + KAx\mathrm{e}^{A\hat{D}(t)x}X(t) \\ &= \int_0^x w(y,t)\Big[K\left(I+A\hat{D}(t)(x-y)\right)\mathrm{e}^{A\hat{D}(t)(x-y)}B \\ &\quad + \hat{D}(t)\int_y^x K\left(I+A\hat{D}(t)(x-\xi)\right)\mathrm{e}^{A\hat{D}(t)(x-\xi)}BK\mathrm{e}^{(A+BK)\hat{D}(t)(\xi-y)}Bd\xi\Big]dy \\ &\quad + \Big[KAx\mathrm{e}^{A\hat{D}(t)x} + \int_0^x K\left(I+A\hat{D}(t)(x-y)\right) \\ &\quad \times \mathrm{e}^{A\hat{D}(t)(x-y)}BK\mathrm{e}^{(A+BK)\hat{D}(t)y}dy\Big]X(t)\,. \end{aligned} \tag{7.28}$$

Now we consider a Lyapunov–Krasovskii-type (nonquadratic) functional

$$V(t) = D\log N(t) + \frac{b}{\gamma}\tilde{D}(t)^2\,, \tag{7.29}$$

where

$$N(t) = 1 + X(t)^T PX(t) + b\int_0^1 (1+x)w(x,t)^2 dx\,. \tag{7.30}$$

Taking a time derivative of $V(t)$, we obtain

$$\begin{aligned}\dot{V}(t) &= -\frac{2b}{\gamma}\tilde{D}(t)\left(\dot{\hat{D}}(t)-\gamma\tau(t)\right) \\ &\quad + \frac{D}{N(t)}\Bigg(-X(t)^T QX(t) + 2X(t)^T PBw(0,t) \\ &\quad - \frac{b}{D}w(0,t)^2 - \frac{b}{D}\|w(t)\|^2 \\ &\quad - 2b\dot{\hat{D}}(t)\int_0^1 (1+x)w(x,t)q(x,t)dx\Bigg)\,, \end{aligned} \tag{7.31}$$

where we have used integration by parts and $\|w(t)\|^2$ denotes $\int_0^1 w(x,t)^2 dx$. Using the assumption that $\hat{D}(0) \in [0,\bar{D}]$ and the update law (7.13)–(7.15) with the help of Lemma E.1, we get

$$\begin{aligned}\dot{V}(t) &\le \frac{D}{N(t)}\Bigg(-X(t)^T QX(t) + 2X(t)^T PBw(0,t) - \frac{b}{D}w(0,t)^2 - \frac{b}{D}\|w(t)\|^2 \\ &\quad - 2b\dot{\hat{D}}(t)\int_0^1 (1+x)w(x,t)q(x,t)dx\Bigg) \end{aligned} \tag{7.32}$$

as well as

$$\hat{D}(t) \in [0,\bar{D}]\,, \quad \forall t \ge 0\,, \tag{7.33}$$

and

$$\dot{\hat{D}}^2 \le \gamma^2\tau^2\,. \tag{7.34}$$

Then, applying Young's inequality and employing (7.17), we obtain

$$\dot{V}(t) \leq -\frac{D}{2N(t)}\left(\lambda_{\min}(Q)|X(t)|^2 + \frac{b}{D}w(0,t)^2 + 2\frac{b}{D}\|w(t)\|^2 \right. \\ \left. + 4b\dot{\hat{D}}(t)\int_0^1 (1+x)w(x,t)q(x,t)dx\right), \tag{7.35}$$

and, finally, substituting (7.13), we arrive at

$$\dot{V}(t) \leq -\frac{D}{2N(t)}\left(\lambda_{\min}(Q)|X(t)|^2 + \frac{b}{D}w(0,t)^2 + 2\frac{b}{D}\|w(t)\|^2\right) \\ + 2Db\gamma\frac{\int_0^1 (1+x)|w(x,t)||p(x,t)|dx}{N(t)}\frac{\int_0^1 (1+x)|w(x,t)||q(x,t)|dx}{N(t)}. \tag{7.36}$$

Then a lengthy but straightforward calculation, employing the Cauchy–Schwartz and Young inequalities, along with (7.27) and (7.28), yields

$$\int_0^1 (1+x)|w(x,t)||p(x,t)|dx \leq M\mathrm{e}^{m\hat{D}(t)}\left(|X(t)|^2 + \|w(t)\|^2 + w(0,t)^2\right) \tag{7.37}$$

and

$$\int_0^1 (1+x)|w(x,t)||q(x,t)|dx \leq M\mathrm{e}^{m\hat{D}(t)}\left(|X(t)|^2 + \|w(t)\|^2\right), \tag{7.38}$$

where M and m are sufficiently large positive constants given by

$$M = \max\left\{2|K|^2|A+BK|^2, 2|K|^2|B|^2, \right. \\ 1 + 2|K|(1+|A|\bar{D})|B|(1+\bar{D}|BK|), \\ \left. |K|^2(|A| + |(1+|A|\bar{D})|BK|)^2\right\}, \tag{7.39}$$

$$m = |A| + |A+BK|. \tag{7.40}$$

Introducing these two bounds into (7.36), we get

$$\dot{V}(t) \leq -\frac{D}{2N(t)}\left(\lambda_{\min}(Q)|X(t)^2 + \frac{b}{D}w(0,t)^2 + 2\frac{b}{D}\|w(t)\|^2 \right. \\ \left. -\gamma\frac{4bM^2\mathrm{e}^{2m\bar{D}}}{\min\{\lambda_{\min}(P), b\}}\left(|X(t)|^2 + \|w(t)\|^2 + w(0,t)^2\right)\right), \tag{7.41}$$

and, finally,

$$\dot{V}(t) \leq -\frac{D}{2}\left(\min\left\{\lambda_{\min}(Q), \frac{b}{\bar{D}}\right\} - \gamma\frac{4bM^2\mathrm{e}^{2m\bar{D}}}{\min\{\lambda_{\min}(P), b\}}\right) \\ \times\frac{|X(t)|^2 + \|w(t)\|^2 + w(0,t)^2}{N(t)}. \tag{7.42}$$

By choosing

$$\gamma^* = \frac{\min\left\{\lambda_{\min}(Q), \frac{b}{\bar{D}}\right\} \min\left\{\lambda_{\min}(P), b\right\}}{4bM^2 \mathrm{e}^{2m\bar{D}}} \tag{7.43}$$

and $\gamma \in (0, \gamma^*)$, we make $\dot{V}$ negative semidefinite, and hence

$$V(t) \le V(0), \quad \forall t \ge 0. \tag{7.44}$$

From this result, we now derive a stability estimate.

From (7.18) and (7.22), we show that

$$\|u(t)\|^2 \le r_1 \|w(t)\|^2 + r_2 |X(t)|^2, \tag{7.45}$$

$$\|w(t)\|^2 \le s_1 \|u(t)\|^2 + s_2 |X(t)|^2, \tag{7.46}$$

where r_1, r_2, s_1, and s_2 are sufficiently large positive constants given by

$$r_1 = 3\left(1 + \bar{D}^2 |K|^2 \mathrm{e}^{2|A+BK|\bar{D}} |B|^2\right), \tag{7.47}$$

$$r_2 = 3|K|^2 \mathrm{e}^{2|A+BK|\bar{D}}, \tag{7.48}$$

$$s_1 = 3\left(1 + \bar{D}^2 |K|^2 \mathrm{e}^{2|A|\bar{D}} |B|^2\right), \tag{7.49}$$

$$s_2 = 3|K|^2 \mathrm{e}^{2|A|\bar{D}}. \tag{7.50}$$

The following two inequalities readily follow from (7.29), (7.30):

$$\tilde{D}^2 \le \frac{\gamma}{b} V, \tag{7.51}$$

$$|X|^2 \le \frac{1}{\lambda_{\min}(P)} \left(\mathrm{e}^{V/D} - 1\right). \tag{7.52}$$

Furthermore, from (7.29), (7.30), and (7.45), it follows that

$$\|u\|^2 \le \frac{r_1}{b}\left(\mathrm{e}^{V/D} - 1\right) + r_2 |X|^2. \tag{7.53}$$

Combining (7.51)–(7.53), we get

$$\Upsilon(t) \le \left(\frac{1 + r_2}{\lambda_{\min}(P)} + \frac{r_1}{b} + \frac{\gamma}{Db}\right)\left(\mathrm{e}^{V(t)/D} - 1\right). \tag{7.54}$$

So, we have bounded $\Upsilon(t)$ in terms of $V(t)$ and thus, using (7.44), in terms of $V(0)$. Now we have to bound $V(0)$ in terms of $\Upsilon(0)$. First, from (7.29), (7.30), it follows that

$$V \le D\left(\lambda_{\max}(P)|X|^2 + 2b\|w\|^2\right) + \frac{b}{\gamma}\tilde{D}^2. \tag{7.55}$$

Using (7.46), we get

$$V \le \left(D\lambda_{\max}(P) + 2bDs_2\right)|X|^2 + 2bDs_1\|u\|^2 + \frac{b}{\gamma}\tilde{D}^2 \tag{7.56}$$

and hence,

$$V(0) \leq \left(D\lambda_{\max}(P) + 2bDs_2 + 2bDs_1 + \frac{b}{\gamma} \right) \Upsilon(0). \tag{7.57}$$

Denoting

$$R = \frac{1+r_2}{\lambda_{\min}(P)} + \frac{r_1}{b} + \frac{\gamma D}{b}, \tag{7.58}$$

$$\rho = \lambda_{\max}(P) + 2bs_2 + 2bs_1 + \frac{b}{\gamma D}, \tag{7.59}$$

we complete the proof of the stability estimate (7.19).

Finally, to prove the regulation result, we will use (7.42) and Barbalat's lemma. However, we first discuss the boundedness of the relevant signals. By integrating (7.44) from $t = 0$ to $t = \infty$, and by noting that $N(t)$ is uniformly bounded, it follows that $X(t)$, $\|w(t)\|$, and $\hat{D}(t)$ are uniformly bounded in time. Using (7.45), we also get the uniform boundedness of $\|u(t)\|$ in time. With the Cauchy–Schwartz inequality, from (7.12) we get the uniform boundedness of $U(t)$ for $t \geq 0$. From (7.5), we get the uniform boundedness of $u(0,t)$ for $t \geq D$. Using (7.2), we get the uniform boundedness of $d|X(t)|^2/dt$ for $t \geq D$. From (7.42), it follows that $X(t)$ is square integrable in time. From this fact, along with the uniform boundedness of $d|X(t)|^2/dt$ for $t \geq D$, by Barbalat's lemma we get that $X(t) \to 0$ as $t \to \infty$.

What remains is to prove the regulation of $U(t)$. From (7.42), it follows that $\|w(t)\|$ is square integrable in time. Using (7.45), we get that $\|u(t)\|$ is also square integrable in time. With the Cauchy–Schwartz inequality, from (7.12) we get that $U(t)$ is also square integrable. To complete the proof of regulation of $U(t)$ by Barbalat's lemma, all that remains to show is that $dU(t)^2/dt$ is uniformly bounded. Toward this end, we calculate

$$\frac{d}{dt}U(t)^2 = 2U(t)K\left[e^{A\hat{D}(t)}\dot{X}(t) + \dot{\hat{D}}(t)G_1(t) + \frac{\hat{D}(t)}{D}G_2(t) \right], \tag{7.60}$$

where

$$G_1(t) = Ae^{A\hat{D}(t)}X(t) + \int_0^1 \left(I + A\hat{D}(t)(1-y) \right) g(y,t)dy, \tag{7.61}$$

$$G_2(t) = BU(t) - Be^{A\hat{D}(t)}u(0,t) + \int_0^1 A\hat{D}(t)g(y,t)dy, \tag{7.62}$$

and

$$g(y,t) = e^{A\hat{D}(t)(1-y)}Bu(y,t). \tag{7.63}$$

The signal $\dot{\hat{D}}(t)$ is uniformly bounded over $t \geq 0$ according to (7.13)–(7.15). By also using the uniform boundedness of $X(t), \dot{X}(t), \|u(t)\|, U(t)$ over $t \geq 0$, and of $u(0,t)$ over $t \geq D$, we get the uniform boundedness of $dU(t)^2/dt$ over $t \geq D$. Then, by Barbalat's lemma, it follows that $U(t) \to 0$ as $t \to \infty$.

7.4 Simulations

We present the simulation results for the state feedback scheme in Section 7.2, namely, for the closed-loop system consisting of the plant (7.2)–(7.4), the control law (7.12), and the parameter update law defined through (7.13)–(7.18).

We focus on highlighting the most important aspect of our scheme—the ability to handle long delays, in the presence of a large uncertainty on the delay. For this reason, we focus on the case of a scalar but unstable ODE (7.2), with $A = 0.75$ and $B = 1$. We take the delay as $D = 1$, which is larger than A. So, the system's transfer function is

$$\frac{X(s)}{U(s)} = \frac{\mathrm{e}^{-s}}{s - 0.75}\,. \tag{7.64}$$

We assume that the known upper bound on the delay is $\bar{D} = 2$. We take the nominal control gain as $K = -A - 1 = -1.75$ (which means that $P = 1, Q = 2$). We take the adaptation gain as $\gamma = 23$ and the normalization coefficient as $b = 4|PB|^2\bar{D}/\lambda_{\min}(Q) = 2\bar{D} = 4$. We take the actuator initial condition as $u_0(x) \equiv 0$, i.e., as $U(\theta) \equiv 0, \forall\theta \in [-D, 0]$, and the plant initial condition as $X(0) = 0.5$.

Hence, the closed-loop system responds to $X(0)$ and to $\hat{D}(0)$. We perform our tests for two distinctly different values of $\hat{D}(0)$—at one extreme, we take $\hat{D}(0) = 0$, and at the other extreme, we take $\hat{D}(0) = \bar{D}$.

The responses are shown in Fig. 7.3.

- First, they show that for both initial estimates, the adaptive controller achieves regulation of the state and input to zero.
- Second, they show that in both cases the estimate $\hat{D}(t)$ converges toward the true D and settles in its vicinity. The perfect convergence is not achieved in either of the two cases, since the regulation problem does not provide persistence of excitation for parameter convergence.
- Third, the dashed plot for $\hat{D}(t)$ shows that the projection operator is active during the first 0.7 seconds.
- Fourth, we can observe that by about 3 second, the evolution of the estimate $\hat{D}(t)$ has been completed.
- Fifth, the plots for $X(t)$ and $U(t)$ are very informative in showing four distinct intervals of behavior of the controller and of the closed-loop system.

  – During the first second, the delay precludes any influence of the control on the plant, so $X(t)$ shows an exponential open-loop growth.
  – At 1 second, the plant starts responding to the control and its evolution changes qualitatively, resulting also in a qualitative change of the control signal. When the estimation of $\hat{D}(t)$ ends at about 3 seconds, the controller structure becomes linear.
  – However, due to the delay, the plant state $X(t)$ continues to evolve based on the inputs from 1 second earlier, so, a nonmonotonic transient continues until about 4 seconds.

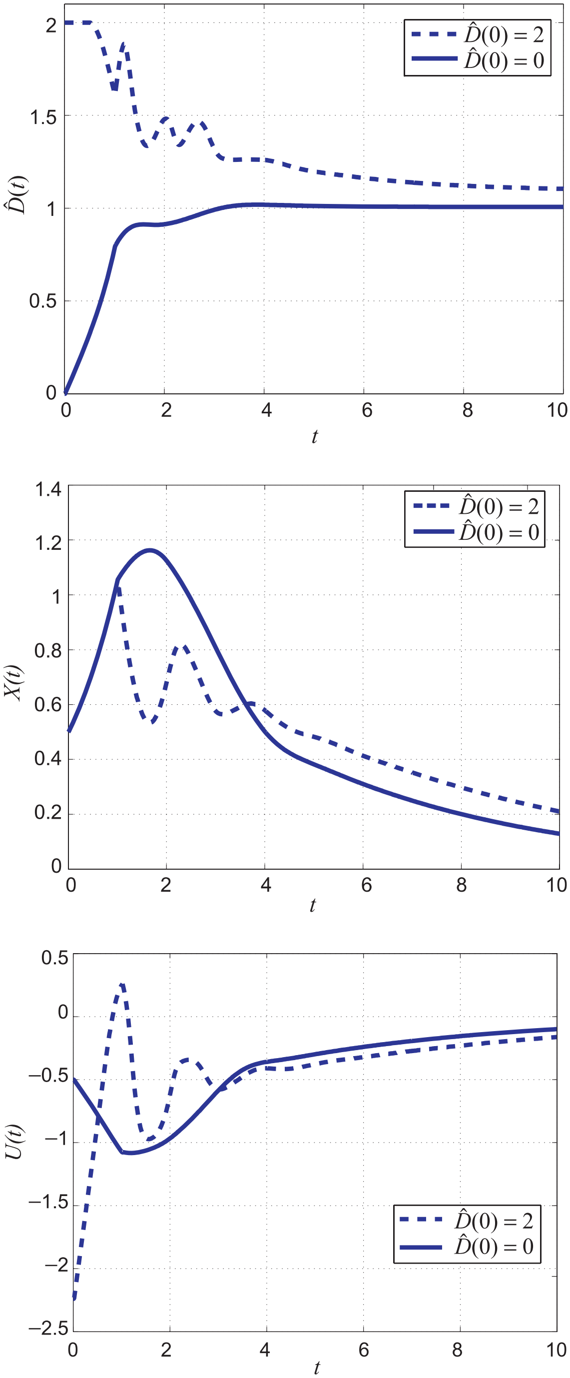

Fig. 7.3 The system response of the system (7.2)–(7.4), (7.12)–(7.18) for $D = 1$ and for two dramatically different values of the initial estimate, $\hat{D}(0) = 0$ and $\hat{D}(0) = \bar{D} = 2D = 2$.

– From about 4 seconds onwards, the (X,U)-system is linear and the delay is sufficiently well compensated, so the response of $X(t)$ and $U(t)$ shows a monotonically decaying exponential trend of a first-order system.

We want to stress that the plots presented here do not show the best performance achievable with the scheme. Quite on the contrary, the plots have been selected to illustrate the less-than-perfect behaviors, with nonmonotonic evolution of all the states in the closed-loop system, that one would obtain when γ and b are not highly tuned.

The simulations do show the effectiveness of the Lyapunov-based adaptive controller. Whether the initial estimate of the delay is zero or 100% above the true value, the estimator drives the estimate toward the true value, which in turn results in the stabilization of the closed-loop system by the predictor-based adaptive controller.

7.5 Notes and References

To our knowledge, the only existing results on adaptive control in the presence of actuator delays are the 1988 result by Ortega and Lozano [182] and the 2003 results by Niculescu and Annaswamy [171] and Evesque et al. [44]. These results deal with the problem where the plant has unknown parameters but the delay value is known.

The importance of problems with unknown delays was highlighted in [41], where a simple scheme for delay estimation and controller gain adjustment to preserve closed-loop stability was also presented. An attempt at adaptive design for unknown delay was also made in [108] by applying the Padé approximation; however, while the design was (predictably) successful for the approximate problem, it was not successful for a model with an actual delay of significant length.

As we explained in Section 7.1, the problem of full-state stabilization with known ODE plant parameters but with unknown delay is the central problem in adaptive control of systems with actuator delays. The other problems in the lengthy catalog of problems are extensions of this central problem. Some of them are solvable globally (this chapter and Chapter 9) and some of them are solvable only locally (Chapter 8).

Chapter 8
Delay-Adaptive Predictor with Estimation of Actuator State

In Chapter 7 we solved the problem of adaptive stabilization in the presence of a long and unknown actuator delay, under the assumption that the actuator delay state is available for measurement. In this chapter we dispose of this assumption.

The result that we obtain in this chapter is not global, as the problem where the actuator state is not measurable and the delay value is unknown at the same time is not solvable globally, since the problem is not linearly parametrized.

We want to state up front that, in a practical sense, the stability result we prove in this chapter is not a highly satisfactory result since it is local both in the initial state and in the initial parameter error. This means that the initial delay estimate needs to be sufficiently close to the true delay. (The delay can be long, but it needs to be known quite closely.) Under such an assumption, we would argue, one might as well use a linear controller and rely on robustness of the feedback law to small errors in the assumed delay value.

Nevertheless, we present the local result here as it highlights quite clearly why a global result is not obtainable when both the delay value and the delay state are unavailable. Hence, this result amplifies the significance of the global result in Chapter 7 and highlights the importance of employing the measurement of the delay state, when available, such as when the delay is the result of a physical transport process.

8.1 Adaptive Control with Estimation of the Transport PDE State

In this section we return to the adaptive problem from Section 7.2 and prove that the adaptive controller designed there assuming the full-state measurement of the transport PDE state,

$$u(x,t) = U(t + D(x-1)), \tag{8.1}$$

remains *locally* stabilizing when it is replaced by the estimate

$$\hat{u}(x,t) = U(t + \hat{D}(t)(x-1)), \tag{8.2}$$

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_8,

where $\hat{D}(t)$ is the estimate of the unknown delay. The global result is not obtainable because the solution $u(x,t) = U(t + D(x-1))$ is not linearly parametrizable (statically or dynamically) in D. Linear parametrization is a requirement for the design of a globally valid adaptive observer.

In this section we slightly strengthen the assumption on a priori knowledge regarding the delay D.

Assumption 8.1. *A lower bound $\underline{D}$ and an upper bound $\bar{D}$ on the unknown delay D are known.*

This assumption is still unrestrictive and allows a large uncertainty ($\bar{D} - \underline{D}$) on the delay.

The estimate in (8.2) is governed by the following *transport equation* representation:

$$\hat{D}(t)\hat{u}_t(x,t) = \hat{u}_x(x,t) + \dot{\hat{D}}(t)(x-1)\hat{u}_x(x,t)\,, \tag{8.3}$$

$$\hat{u}(1,t) = U(t)\,. \tag{8.4}$$

This is an adaptive state estimator of the transport PDE with an unknown propagation speed. Let

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t) \tag{8.5}$$

denote the transport PDE state estimation error. Then the ODE portion of the plant is represented as

$$\dot{X}(t) = AX(t) + B\hat{u}(0,t) + B\tilde{u}(0,t)\,. \tag{8.6}$$

The adaptive controller is now written as a certainty-equivalence version [with respect to the unmeasured $u(x,t)$] of the adaptive controller (7.12):

$$U(t) = K\left(\mathrm{e}^{A\hat{D}(t)}X(t) + \hat{D}(t)\int_0^1 \mathrm{e}^{A\hat{D}(t)(1-y)}B\hat{u}(y,t)dy\right)\,. \tag{8.7}$$

The update law for $\hat{D}$ is chosen as

$$\dot{\hat{D}}(t) = \gamma \mathrm{Proj}_{[\underline{D};\bar{D}]}\left\{\tau(t)\right\}\,, \tag{8.8}$$

where $\gamma > 0$ and

$$\tau(t) = -\int_0^1 (1+x)\hat{w}(x,t)K\mathrm{e}^{A\hat{D}(t)x}dx(AX(t) + B\hat{u}(0,t))\,. \tag{8.9}$$

The backstepping-transformed state of the actuator, used in the implementation of the update law (8.9) but also in subsequent analysis, is given by

$$\hat{w}(x,t) = \hat{u}(x,t) - \hat{D}(t)\int_0^x K\mathrm{e}^{A\hat{D}(t)(x-y)}B\hat{u}(y,t)dy - K\mathrm{e}^{A\hat{D}(t)x}X(t)\,. \tag{8.10}$$

8.2 Local Stability and Regulation

Now we state and prove a result on stability and regulation for initial conditions that are sufficiently small in an appropriate norm.

Theorem 8.1. *Consider the closed-loop system consisting of the plant (9.1)–(9.26), (8.3), (8.4), the control law (8.7), and the update law defined by (8.8)–(8.10). Let Assumption 8.1 hold and let*

$$\begin{aligned}\Upsilon(t) = {} & |X(t)|^2 \\ & + \int_0^1 u(x,t)^2 dx \\ & + \int_0^1 \hat{u}(x,t)^2 dx + \int_0^1 \hat{u}_x(x,t)^2 dx \\ & + \tilde{D}(t)^2 \end{aligned} \tag{8.11}$$

denote the norm of the overall state of the closed-loop system. There exist positive constants ρ and R such that if the initial state $(X_0, u_0, \hat{u}_0, \hat{D}_0)$ is such that $\Upsilon(0) < \rho$, then

$$\Upsilon(t) \le R\Upsilon(0) \tag{8.12}$$

and

$$\lim_{t\to\infty} X(t) = 0\,, \tag{8.13}$$

$$\lim_{t\to\infty} U(t) = 0\,. \tag{8.14}$$

In the proof of this result we concentrate on the details that are different from the proofs of the results in Chapters 5 and 7.

Proof. Mimicking the calculations used to prove Lemma 5.1 (omitted due to their length), we obtain the equations of the overall closed-loop system in the $(X, \tilde{u}, \hat{w})$ variables as

$$\dot{X}(t) = (A+BK)X(t) + B\hat{w}(0,t) + B\tilde{u}(0,t)\,, \tag{8.15}$$

$$D\tilde{u}_t(x,t) = \tilde{u}_x(x,t) - \tilde{D}(t)r(x,t) - D\dot{\hat{D}}(t)(x-1)r(x,t)\,, \tag{8.16}$$

$$\tilde{u}(1,t) = 0\,, \tag{8.17}$$

$$\hat{D}(t)\hat{w}_t(x,t) = \hat{w}_x(x,t) - \hat{D}(t)\dot{\hat{D}}(t)s(x,t)\,, -\hat{D}(t)K\mathrm{e}^{A\hat{D}(t)x}B\tilde{u}(0,t)\,, \tag{8.18}$$

$$\hat{w}(1,t) = 0\,, \tag{8.19}$$

where

$$\begin{aligned} r(x,t) = {} & \frac{\hat{w}_x(x,t)}{\hat{D}(t)} + KB\hat{w}(x,t) \\ & + \int_0^x K(A+BK)\hat{D}(t)\mathrm{e}^{(A+BK)\hat{D}(t)(x-y)}B\hat{w}(y,t)dy \\ & + K(A+BK)\mathrm{e}^{(A+BK)\hat{D}(t)x}X(t) \end{aligned} \tag{8.20}$$

and

$$\begin{aligned} s(x,t) &= (1-x)\left(\frac{\hat{w}_x(x,t)}{\hat{D}(t)} + K\mathrm{e}^{A\hat{D}(t)x}B(KX(t)+\hat{w}(0,t))\right) \\ &\quad + KA\mathrm{e}^{A\hat{D}(t)x}X(t) \\ &\quad + \int_0^x K(I+A\hat{D}(t)(x-y))\mathrm{e}^{A\hat{D}(t)(x-y)}B\hat{u}(y,t)dy \qquad (8.21) \\ &= (1-x)\left(\frac{\hat{w}_x(x,t)}{\hat{D}(t)} + K\mathrm{e}^{A\hat{D}(t)x}B(KX(t)+\hat{w}(0,t))\right) \\ &\quad + \int_0^x \hat{w}(y,t)\Big[K(I+A\hat{D}(t)(x-y))\mathrm{e}^{A\hat{D}(t)(x-y)}B \\ &\quad + \hat{D}(t)\int_y^x K(I+A\hat{D}(t)(x-\xi))\mathrm{e}^{A\hat{D}(t)(x-\xi)} \\ &\quad \times BK\mathrm{e}^{(A+BK)\hat{D}(t)(\xi-y)}Bd\xi\Big]dy \\ &\quad + \Bigg(KA\mathrm{e}^{A\hat{D}(t)x} + \int_0^x K(I+A\hat{D}(t)(x-y)) \\ &\quad \times \mathrm{e}^{A\hat{D}(t)(x-y)}BK\mathrm{e}^{(A+BK)\hat{D}(t)y}dy\Bigg)X(t)\,. \qquad (8.22) \end{aligned}$$

Since our Lyapunov analysis will also involve an H_1 norm of $\hat{w}$ (in addition to the L_2 norms of $\tilde{u}$ and $\hat{w}$), we also need the governing equations of the $\hat{w}_x$-system:

$$\hat{D}(t)\hat{w}_{xt}(x,t) = \hat{w}_{xx}(x,t) - \hat{D}\dot{\hat{D}}(t)s_x(x,t) - KA\hat{D}^2\mathrm{e}^{A\hat{D}(t)x}B\tilde{u}(0,t)\,, \qquad (8.23)$$

$$\begin{aligned} \hat{w}_x(1,t) &= \hat{D}(t)\dot{\hat{D}}(t)s(1,t) + \hat{D}(t)K\mathrm{e}^{A\hat{D}(t)}B\tilde{u}(0,t) \qquad (8.24) \\ &= \hat{D}(t)\dot{\hat{D}}(t)\Bigg(\Bigg[KA\mathrm{e}^{A\hat{D}(t)} + \int_0^1 K(I+A\hat{D}(t)(1-y)\mathrm{e}^{A\hat{D}(t)(1-y)}BK \\ &\quad \times \mathrm{e}^{(A+BK)\hat{D}(t)y}dy\Bigg]X(t) \\ &\quad + \int_0^1 \hat{w}(y,t)\Big[K(I+A\hat{D}(t)(1-y))\mathrm{e}^{A\hat{D}(t)(1-y)} \\ &\quad + \int_y^1 K(I+A\hat{D}(t)(1-\xi))\mathrm{e}^{A\hat{D}(t)(1-\xi)} \\ &\quad \times BK\mathrm{e}^{(A+BK)\hat{D}(t)(\xi-y)}d\xi\Big]Bdy\Bigg) \\ &\quad + \hat{D}(t)K\mathrm{e}^{A\hat{D}(t)}B\tilde{u}(0,t)\,, \qquad (8.25) \end{aligned}$$

where

$$
\begin{aligned}
s_x(x,t) = & \frac{1}{\hat{D}(t)}\left[(1-x)\hat{w}_{xx}(x,t) - \hat{w}_x(x,t)\right] \\
& + K\mathrm{e}^{A\hat{D}(t)x}\left(A\hat{D}(t)(1-x) - I\right)B(KX(t) + \hat{w}(0,t)) \\
& + \hat{D}(t)\int_0^x \hat{w}(y,t)\left[K(BK + A(2I + A\hat{D}(t)(x-y)))\mathrm{e}^{A\hat{D}(t)(x-y)}B\right. \\
& + \hat{D}(t)\int_y^x K\left(BK + A(2I + A\hat{D}(t)(x-\xi))\right) \\
& \left.\times\, \mathrm{e}^{A\hat{D}(t)(x-\xi)}BK\mathrm{e}^{(A+BK)\hat{D}(t)(\xi-y)}Bd\xi\right] dy \\
& + KB\hat{w}(x,t) + \left[K(A^2\hat{D}(t) + BK)\mathrm{e}^{A\hat{D}(t)x}\right. \\
& + \hat{D}(t)\int_0^x K\left(BK + A(2I + A\hat{D}(t)(x-y))\right) \\
& \left.\times\, \mathrm{e}^{A\hat{D}(t)(x-y)}BK\mathrm{e}^{(A+BK)\hat{D}(t)y}dy\right] X(t)\,.
\end{aligned}
\tag{8.26}
$$

The inverse backstepping transformation

$$
\begin{aligned}
\hat{u}(x,t) = \hat{w}(x,t) + \hat{D}(t)\int_0^x K\mathrm{e}^{(A+BK)\hat{D}(t)(x-y)}B\hat{w}(y,t)dy \\
+ K\mathrm{e}^{(A+BK)\hat{D}(t)x}X(t)
\end{aligned}
\tag{8.27}
$$

is crucial in replacing all the occurrences of $\hat{u}$ by $\hat{w}$ in the derivation of the above equations. It should be pointed out that the derivation of (8.15)–(8.26) is even lengthier than the proof of Lemma 10.1 due to the time-varying character of $\hat{D}(t)$. We now start our Lyapunov analysis by introducing

$$
\begin{aligned}
V(t) = & X^T(t)PX(t) \\
& + b_1 D\int_0^1 (1+x)\tilde{u}(x,t)^2 dx \\
& + b_2\hat{D}(t)\left(\int_0^1 (1+x)\hat{w}(x,t)^2 dx + \int_0^1 (1+x)\hat{w}_x(x,t)^2 dx\right) \\
& + b_3\tilde{D}(t)^2\,,
\end{aligned}
\tag{8.28}
$$

where P is the positive-definite and symmetric solution of the Lyapunov equation:

$$
P(A+BK) + (A+BK)^T P = -Q \tag{8.29}
$$

for any positive-definite and symmetric matrix Q. Differentiating $V(t)$ along the solutions of (8.15)–(8.26), we get

$$\begin{aligned}
\dot{V}(t) = & -X^T(t)QX(t) + 2X^T(t)PB(\hat{w}(0,t) + \tilde{u}(0,t)) \\
& + b_1\Bigg(- \tilde{u}(0,t)^2 - \|\tilde{u}(t)\|^2 - 2\tilde{D}(t)\int_0^1 (1+x)\tilde{u}(x,t)r(x,t)dx \\
& - 2D\dot{\hat{D}}(t)\int_0^1 (x^2-1)\tilde{u}(x,t)r(x,t)dx\Bigg) \\
& + b_2\Bigg(-\hat{w}(0,t)^2 - \|\hat{w}(t)\|^2 - 2\hat{D}(t)\dot{\hat{D}}(t)\int_0^1 (1+x)\hat{w}(x,t)s(x,t)dx \\
& - 2\hat{D}(t)\int_0^1 (1+x)K\mathrm{e}^{A\hat{D}(t)x}B\tilde{u}(0,t)\hat{w}(x,t)dx\Bigg) \\
& + b_2\Bigg(2\hat{w}_x(1,t)^2 - \hat{w}_x(0,t)^2 - \|\hat{w}_x(t)\|^2 \\
& - 2\hat{D}(t)\dot{\hat{D}}(t)\int_0^1 (1+x)\hat{w}_x(x,t)s_x(x,t)dx \\
& - 2\hat{D}(t)^2\int_0^1 (1+x)AK\mathrm{e}^{A\hat{D}(t)x}B\tilde{u}(0,t)\hat{w}_x(x,t)dx\Bigg) \\
& + \dot{\hat{D}}(t)b_2\Bigg(\int_0^1 (1+x)\hat{w}(x,t)^2dx \\
& + \int_0^1 (1+x)\hat{w}_x(x,t)^2dx\Bigg) - 2b_3\tilde{D}(t)\dot{\hat{D}}(t)\,. \qquad (8.30)
\end{aligned}$$

Using (8.8), (8.20), (8.21), (8.24), (8.26), the standard properties of the projection operator, and Agmon's inequality

$$\hat{w}(0,t)^2 \leq 4\|\hat{w}_x(t)\|^2 \qquad (8.31)$$

[with the help of the fact that $\hat{w}(1,t) = 0$], after lengthy calculations we show that there exist constants $M_1, M_2, \ldots, M_8$ (independent of initial conditions) such that the following inequalities hold:

$$\left|\int_0^1 (1+x)\tilde{u}(x,t)r(x,t)dx\right| \leq M_1(\|\tilde{u}(t)\|^2 + \|\hat{w}_x(t)\|^2 + \|\hat{w}(t)\|^2 + |X(t)|^2)\,, \qquad (8.32)$$

$$\left|\int_0^1 (x^2-1)\tilde{u}(x,t)r(x,t)dx\right| \leq M_1(\|\tilde{u}(t)\|^2 + \|\hat{w}_x(t)\|^2 + \|\hat{w}(t)\|^2 + |X(t)|^2)\,, \qquad (8.33)$$

and

$$\begin{aligned}
\left|2\hat{D}(t)\int_0^1 (1+x)\hat{w}(x,t)s(x,t)dx\right| \leq & \; 4M_2(\|\hat{w}(t)\|^2 \\
& + \|\hat{w}_x(t)\|^2 + |X(t)|^2)\,, \qquad (8.34)
\end{aligned}$$

$$\left|2\hat{D}\int_0^1 (1+x)K\mathrm{e}^{A\hat{D}(t)x}B\tilde{u}(0,t)\hat{w}(x,t)dx\right| \leq M_3\tilde{u}(0,t)^2 + \frac{\|\hat{w(t)}\|^2}{4}\,, \qquad (8.35)$$

$$\left|2\hat{D}(t)\int_0^1 (1+x)\hat{w}_x(x,t)s_x(x,t)dx\right| \leq 4M_4(\|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2 + |X(t)|^2)\,, \quad (8.36)$$

$$\left|2\hat{D}(t)^2\int_0^1 (1+x)AK\mathrm{e}^{A\hat{D}(t)x}B\tilde{u}(0,t)\hat{w}_x(x,t)dx\right| \leq M_5\tilde{u}(0,t)^2 + \frac{\|\hat{w}_x(t)\|^2}{4}\,, \quad (8.37)$$

and

$$|\dot{\hat{D}}(t)| \leq 4M_6(|X(t)|^2 + \|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2)\,, \quad (8.38)$$

$$2\hat{w}_x(1,t)^2 \leq \dot{\hat{D}}(t)^2 M_7(|X(t)|^2 + \|\hat{w}(t)\|^2) + M_8\tilde{u}(0,t)^2\,. \quad (8.39)$$

Then

$$\begin{aligned}\dot{V}(t) \leq &-\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \left(b_1 - b_2\left(\frac{1}{2}+M_3+M_5+M_8\right)\right)\tilde{u}(0,t)^2 \\ &- b_1\|\tilde{u}(t)\|^2 - \frac{b_2}{2}\hat{w}(0,t)^2 - b_2\hat{w}_x(0,t)^2 \\ &- \left(b_2 - \frac{b_2}{4}\right)\|\hat{w}(t)\|^2 - \left(b_2 - \frac{b_2}{4}\right)\|\hat{w}_x(t)\|^2 \\ &+ b_2M_7\dot{\hat{D}}(t)^2\left(|X(t)|^2 + \|\hat{w}(t)\|^2\right) + 2b_3|\tilde{D}(t)||\dot{\hat{D}}(t)| \\ &+ b_1M_1\bar{D}|\dot{\hat{D}}(t)|\left(\|\tilde{u}(t)\|^2 + \|\hat{w}_x(t)\|^2 + \|\hat{w}(t)\|^2 + |X(t)|^2\right) \\ &+ b_1M_1|\tilde{D}(t)|\left(\|\tilde{u}(t)\|^2 + \|\hat{w}_x(t)\|^2 + \|\hat{w}(t)\|^2 + |X(t)|^2\right) \\ &+ |\dot{\hat{D}}(t)|\Big(4b_2M_2(\|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2 + |X(t)|^2) \\ &+ 4b_2M_4(\|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2 + |X(t)|^2) \\ &+ 2b_2(\|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2)\Big)\,, \end{aligned} \quad (8.40)$$

where, as in the nonadaptive analysis, we have chosen

$$b_2 \geq 8|PB|/\lambda_{\min}(Q)\,. \quad (8.41)$$

By choosing b_1 such that

$$b_1 > b_2\left(\frac{1}{2}+M_3+M_5+M_8\right) \quad (8.42)$$

and defining

$$\eta = \min\left\{\frac{\lambda_{\min}(Q)}{2}, \frac{b_2}{2}, b_1 - b_2\left(\frac{1}{2}+M_3+M_5+M_8\right)\right\} > 0 \quad (8.43)$$

and

$$V_0(t) = |X(t)|^2 + \|\tilde{u}(t)\|^2 + \|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2, \tag{8.44}$$

we obtain

$$\begin{aligned}
\dot{V}(t) \leq & -\eta(|X(t)|^2 + \|\tilde{u}(t)\|^2 + \|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2 \\
& + \tilde{u}(0,t)^2 + \hat{w}(0,t)^2 + \hat{w}_x(0,t)) \\
& + 16b_2M_7M_6^2V_0(t)^3 + 4b_1M_1M_6\bar{D}V_0(t)^2 \\
& + (8b_3M_6 + b_1M_1)|\tilde{D}(t)|V_0(t) \\
& + 4M_6(4b_2M_2 + 4b_2M_4 + 2b_2)V_0(t)^2 \qquad (8.45) \\
\leq & -\eta V_0(t) + 4M_6(b_1M_1\bar{D} + 2b_2(2M_2 + 2M_4 + 1)V_0(t)^2 \\
& + (8b_3M_6 + b_1M_1)|\tilde{D}(t)|V_0(t) + 16b_2M_7M_6^2V_0(t)^3. \qquad (8.46)
\end{aligned}$$

This is a nonlinear differential inequality with which we shall deal in a moment. Before we do, to eliminate the parameter error term, we employ the bound

$$\begin{aligned}
|\tilde{D}(t)| \leq & \frac{\varepsilon}{2} + \frac{\tilde{D}(t)^2}{2\varepsilon} \qquad (8.47) \\
\leq & \frac{\varepsilon}{2} + \frac{1}{2b_3\varepsilon}\left(V(t) - \lambda_{\min}(P)|X(t)|^2 - b_1\underline{D}\|\tilde{u}(t)\|^2\right. \\
& \left. - b_2\underline{D}\|\hat{w}(t)\|^2 - b_2\underline{D}\|\hat{w}_x(t)\|^2\right) \qquad (8.48) \\
\leq & \frac{\varepsilon}{2} + \frac{1}{2b_3\varepsilon}\left(V(t) - \min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D},\}V_0(t)\right), \qquad (8.49)
\end{aligned}$$

which yields

$$\begin{aligned}
\dot{V}(t) \leq & -\left(\eta - (8b_3M_6 + b_1M_1)\left(\frac{\varepsilon}{2} + \frac{1}{2b_3\varepsilon}V(t)\right)\right)V_0(t) \\
& - \left(\frac{(8b_3M_6 + b_1M_1)\min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D}\}}{2b_3\varepsilon}\right. \\
& - 4M_6(b_1M_1\bar{D} + 2b_2(2M_2 + 2M_4 + 1)) \\
& \left. - 16b_2M_7M_6^2V_0(t)\right)V_0(t)^2. \qquad (8.50)
\end{aligned}$$

If we choose the analysis parameter ε as

$$\begin{aligned}
\varepsilon \leq \min\Bigg\{ & \frac{2\eta}{8b_3M_6 + b_1M_1}, \\
& \frac{(8b_3M_6 + b_1M_1)\min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D}\}}{8b_3M_6(b_1M_1\bar{D} + 2b_2(2M_2 + 2M_4 + 1))}\Bigg\} \qquad (8.51)
\end{aligned}$$

and restrict the initial conditions so that

$$
\begin{aligned}
V(0) \le \min \Bigg\{ & 2b_3\varepsilon \left(\frac{\eta}{8b_3M_6 + b_1M_1} - \frac{\varepsilon}{2} \right), \\
& \frac{\min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D}\}}{16b_2M_7M_6^2} \\
& \times \Bigg(\frac{(8b_3M_6 + b_1M_1)\min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D}\}}{2b_3\varepsilon} \\
& - 4M_6(b_1M_1\bar{D} + 2b_2(2M_2 + 2M_4 + 1)) \Bigg) \Bigg\}, \qquad (8.52)
\end{aligned}
$$

we obtain

$$
\dot{V}(t) \le -\mu_1(t)V_0(t) - \mu_2(t)V_0(t)^2, \qquad (8.53)
$$

where

$$
\mu_1(t) = \eta - (8b_3M_6 + b_1M_1)\left(\frac{\varepsilon}{2} + \frac{1}{2b_3\varepsilon}V(t) \right), \qquad (8.54)
$$

$$
\begin{aligned}
\mu_2(t) = & \frac{(8b_3M_6 + b_1M_1)\min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D}\}}{2b_3\varepsilon} \\
& - 4M_6(b_1M_1\bar{D} + 2b_2(2M_2 + 2M_4 + 1)) \\
& - \frac{16b_2M_7M_6^2}{\min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D}\}}V(t) \qquad (8.55)
\end{aligned}
$$

are nonnegative functions if the initial conditions are constrained as in (128). Hence,

$$
V(t) \le V(0), \qquad \forall t \ge 0. \qquad (8.56)
$$

From this result for $V(t)$, we now obtain the result concerning $\Upsilon(t)$. Using Lemma 10.3 [which holds both when $\hat{D}$ is constant and with a time-varying $\hat{D}(t)$], we obtain

$$
\begin{aligned}
\Upsilon(t) = & |X(t)|^2 + \|u(t)\|^2 + \|\hat{u}(t)\|^2 + \|\hat{u}_x(t)\|^2 + \tilde{D}(t)^2 \qquad (8.57) \\
\le & |X(t)|^2 + 2(\|\tilde{u}(t)\|^2 + \|\hat{u}(t)\|^2) \\
& + \|\hat{u}(t)\|^2 + \|\hat{u}_x(t)\|^2 + \tilde{D}(t)^2 \\
\le & (1 + 3p_2 + p_4)|X(t)|^2 + 2\|\tilde{u}(t)\|^2 \\
& + (3p_1 + p_3)\|\hat{w}(t)\|^2 + 4\|\hat{w}_x(t)\|^2 + \tilde{D}(t)^2 \\
\le & \max\{1 + 3p_2 + p_4, 3p_1 + p_3, 4\} \\
& \times (|X(t)|^2 + \|\tilde{u}(t)\|^2 + \|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2 + \tilde{D}(t)^2)
\end{aligned}
$$

$$\leq \frac{\max\{1+3p_2+p_4, 3p_1+p_3\}}{\min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D}, b_3\}} V(t)$$
$$\leq \frac{\max\{1+3p_2+p_4, 3p_1+p_3\}}{\min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D}, b_3\}} V(0).$$

Similarly, using Lemma 10.3, we show that

$$V(0) \leq \max\{\lambda_{\max}(P), 2b_1\bar{D}, 2b_2\bar{D}, b_3\} \times (|X(t)|^2 + \|\tilde{u}(t)\|^2 + \|\hat{w}(t)\|^2 + \|\hat{w}_x(t)\|^2) \tag{8.58}$$
$$\leq \max\{\lambda_{\max}(P), 2b_1\bar{D}, 2b_2\bar{D}, b_3\} \times \max\{1+q_2+q_4, 2, 2+q_1+q_3, 4\}\Upsilon(0). \tag{8.59}$$

Then, using (8.57) and (8.58), we complete the local stability part of the proof by defining

$$R = \frac{\max\{1+3p_2+p_4, 3p_1+p_3\}}{\min\{\lambda_{\min}(P), b_1\underline{D}, b_2\underline{D}, b_3\}} \times \max\{\lambda_{\max}(P), 2b_1\bar{D}, 2b_2\bar{D}, b_3\} \times \max\{1+q_2+q_4, 2+q_1+q_3\}. \tag{8.60}$$

Now we proceed to prove local regulation, under the initial condition restriction (8.52). From (8.56), it follows that $X(t), \|\tilde{u}\|, \|\hat{w}\|, \|\hat{w}_x\|$, and $\hat{D}(t)$ are uniformly bounded in time. Then, from (8.27), using the Cauchy–Schwartz inequality, we obtain the uniform boundedness of $\|\hat{u}(t)\|$ and consequently also the uniform boundedness of $U(t)$ for $t \geq 0$ from (8.7). Thus, $u(0,t) = U(t-D)$ is uniformly bounded for $t \geq D$. Using (9.1), we get that $d|X(t)|^2/dt$ is uniformly bounded for $t \geq D$. From (8.50), it follows that $|X(t)|$ is square integrable in time. Finally, by Barbalat's lemma, we get that $X(t) \to 0$ when $t \to \infty$. To also prove the regulation of $U(t)$, we start by deducing from (8.50) the square integrability of $\|\hat{w}(t)\|$. Then, from Lemma 10.3, we have the square integrability of $\|\hat{u}\|$ and, from (8.7), using the Cauchy–Schwartz inequality, the square integrability of $U(t)$. To establish the boundedness of $dU(t)^2/dt$, we compute it as

$$\frac{d}{dt}U(t)^2 = 2U(t)K\left(\mathrm{e}^{A\hat{D}(t)}\dot{X}(t) + \dot{\hat{D}}(t)G_1(t) + \hat{D}(t)G_2(t)\right) \tag{8.61}$$

with

$$G_1(t) = A\mathrm{e}^{A\hat{D}(t)}X(t) + \int_0^1 (I + A\hat{D}(t)(1-y))\mathrm{e}^{A\hat{D}(t)(1-y)}B\hat{u}(y,t)dy + (x-1)\int_0^1 \mathrm{e}^{A\hat{D}(t)(1-y)}B\hat{u}_x(y,t)dy, \tag{8.62}$$

$$G_2(t) = \int_0^1 \mathrm{e}^{A\hat{D}(t)(1-y)}B\hat{u}_x(y,t)dy. \tag{8.63}$$

The signal $\dot{\hat{D}}(t)$ is uniformly bounded for $t \geq D$ according to (8.8). By using the boundedness of $\dot{X}(t), X(t), \|\hat{u}(t)\|$, and $\|\hat{u}_x(t)\|$ over $t \geq D$, we get the boundedness of $dU(t)^2/dt$ for $t \geq D$. Then, by Barbalat's lemma, $U(t) \to 0$ when $t \to \infty$. This completes the proof of Theorem 8.1. □

8.3 Simulations

We return to the example

$$X(s) = \frac{e^{-Ds}}{s-a} U(s) \tag{8.64}$$

for $D = 1$ and $a = 0.75$, namely, where the plant is unstable and the delay is on the order of the plant's unstable pole (and, in fact, larger).

We have conducted simulations of the adaptive predictor-based feedback where $u(x,t)$ is not measured and is replaced by the estimate $\hat{u}(x,t)$, as in Section 8.1.

As expected, the result is local and doesn't allow a large $X(0)$ or a $\hat{D}(0)$ that is too far from the true D. Figure 8.1 shows one simulation example with $D = 1$, $A = 0.5, B = 1, K = -1.5, P = 1, Q = 2$, and $\gamma = 5$.

Figure 8.2 shows the actuator state estimation error signal $\tilde{u}(0,t) = u(0,t) - \hat{u}(0,t) = U(t-D) - U(t-\hat{D}(t))$, which represents the crucial difference between the design where the actuator state is measured and the design used in Fig. 8.1 where the actuator state is estimated by (8.2), i.e., by (8.3)–(8.4).

8.4 Notes and References

In this chapter we provided a proof of local stability for the case of adaptive predictor-based feedback where the delay state is not available for measurement, but it is estimated, in a certainty-equivalence manner. We provide the proof of this local result mainly to illustrate the sacrifice one is making by not using the actuator state measurement when it is available.

The reader might have noted in (8.9) that we do not employ normalization in the update law in this chapter. This is because the purpose of the update law normalization in Chapter 7 was to achieve a global result. When both the delay state and the delay value are unmeasurable, a global result is not achievable independent of whether or not we employ update law normalization. So, for simplicity, in this chapter we have forgone normalization.

The local adaptive result that we achieve in this chapter should not be too surprising (though it is quite nontrivial to prove) given the robustness result in Corollary 5.2. A consequence of that result for adaptive control is that any estimator of $\hat{D}(t)$ employing projection to $[D-\delta^*, D+\delta^*]$, for sufficiently small δ^*, and with a sufficiently small adaptation gain, should be stabilizing. One such update law, based on the approximation

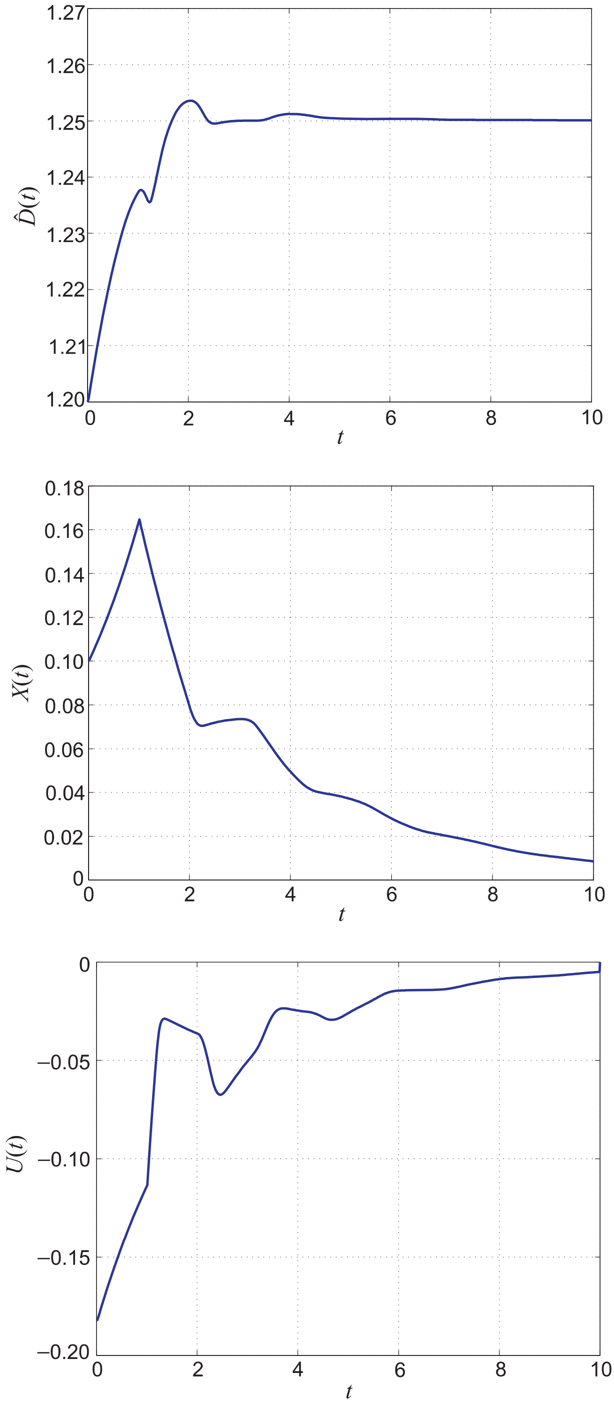

Fig. 8.1 The system response of the system (7.2)–(7.4), (8.3), (8.4), and (8.7)–(8.10) for $D = 1$.

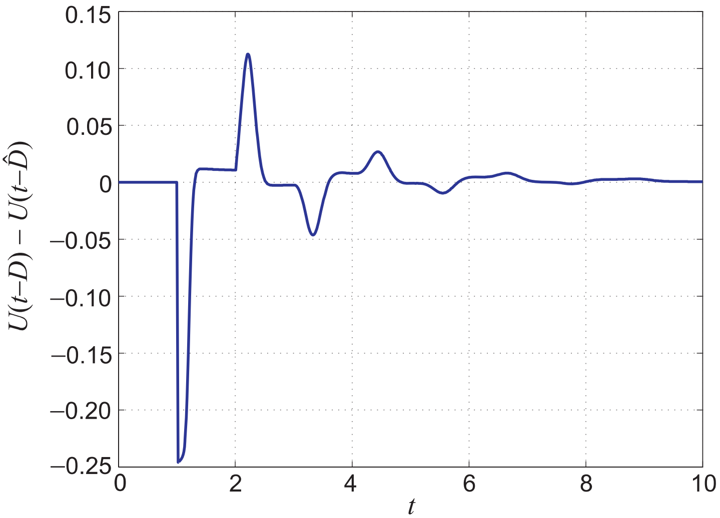

Fig. 8.2 The estimation error of the actuator state, $\tilde{u}(0,t) = u(0,t) - \hat{u}(0,t) = U(t-D) - U(t-\hat{D}(t))$, for the system (7.2)–(7.4), (8.3), (8.4), and (8.7)–(8.10).

$$\mathrm{e}^{-(D-\hat{D})s} \approx 1 - (D-\hat{D})s\,, \tag{8.65}$$

is

$$\dot{\hat{D}}(t) = \gamma \frac{\mathrm{Proj}_{[D-\delta^*, D+\delta^*]}\{e(t)\omega(t)\}}{1+\omega(t)^2}\,, \tag{8.66}$$

$$e(t) = x(t) - (\lambda I + A)\frac{1}{s+\lambda}[x(t)] - B\frac{1}{s+\lambda}\left[u\left(t-\hat{D}(t)\right)\right]\,, \tag{8.67}$$

$$\omega(t) = \lambda\frac{1}{s+\lambda}\left[u\left(t-\hat{D}(t)\right)\right] - u\left(t-\hat{D}(t)\right)\,. \tag{8.68}$$

Unfortunately, closed-loop stability with this adaptive scheme is only a conjecture, which looks difficult to prove due to the presence of the time-varying, state-variable-dependent delay [note that $\hat{D}(t)$ is a state variable of the closed-loop system]. As we noted earlier, the update law normalization in (8.66) is not essential, because global stability is not achievable, whether or not we employ the normalization.

Chapter 9
Trajectory Tracking Under Unknown Delay and ODE Parameters

In Chapter 7 we presented an adaptive control design for an ODE system with a possibly large actuator delay of unknown length. We achieved global stability under full-state feedback.

In this chapter we generalize the design from Chapter 7 in two major ways: We extend it to ODEs with unknown parameters and extend it from equilibrium regulation to trajectory tracking. These issues were not pursued immediately in Chapter 7 for pedagogical reasons, to prevent the tool from achieving global adaptivity in the infinite-dimensional (delay) context from being buried under standard but nevertheless complicated details of ODE adaptive control.

A significant number of new technical issues arise in this chapter. The estimation error of the ODE parameters appears in the error models of both the ODE and the infinite-dimensional (delay) subsystem, which is also reflected in the update law. The update law also has to deal appropriately with ensuring stabilizability with the parameter estimates, for which projection is employed. Finally, our approach for dealing with delay adaptation involves normalized Lyapunov-based tuning, a rather nonstandard approach compared to finite-dimensional adaptive control. In this framework we need to bound numerous terms involving parameter adaptation rates (both for the delay and for the ODE parameters) in the Lyapunov analysis. Some of these terms are vanishing (when the tracking error is zero), while the others (which are due to the reference trajectory) are nonvanishing. These terms receive different treatment though both are bounded by normalization and their size is controlled with the adaptation gain.

We begin in Section 9.1 by defining the problem and present the adaptive control design and the main stability theorem in Section 9.2. Simulation results are shown in Section 9.3, followed by the stability proof in Section 9.4.

9.1 Problem Formulation

We consider the following system:

$$\dot{X}(t) = A(\theta)X(t) + B(\theta)U(t-D), \tag{9.1}$$

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Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_9,

$$Y(t) = CX(t)\,, \tag{9.2}$$

where $X \in \mathbb{R}^n$ is the ODE state, U is the scalar input to the entire system, $D > 0$ is an unknown constant delay, the system matrix $A(\theta)$ and the input vector $B(\theta)$ in (9.1) are linearly parametrized, i.e.,

$$A(\theta) = A_0 + \sum_{i=1}^{p} \theta_i A_i, \tag{9.3}$$

$$B(\theta) = B_0 + \sum_{i=1}^{p} \theta_i B_i\,, \tag{9.4}$$

and θ is an unknown but constant-parameter vector that belongs to the following convex set:

$$\Pi = \{\theta \in \mathbb{R}^p | \mathscr{P}(\theta) \le 0\}\,, \tag{9.5}$$

where, by assuming that the convex function $\mathscr{P} : \mathbb{R}^p \to \mathbb{R}$ is smooth, we ensure that the boundary $\partial\Pi$ of Π is smooth.

Assumption 9.1. *The set Π is bounded and known. A constant $\bar{D}$ is known such that $D \in]0;\bar{D}]$.*

Assumption 9.2. *The pair, $(A(\theta),B(\theta))$ is completely controllable for each θ and there exists a triple of vector-/matrix-valued functions $(K(\theta),P(\theta),Q(\theta))$ such that $(K,P) \in C^1(\Pi)^2$, $Q \in C^0(\Pi)$, the matrices $P(\theta)$ and $Q(\theta)$ are positive definite and symmetric, and the following Lyapunov equation is satisfied for all $\theta \in \Pi$:*

$$P(\theta)(A+BK)(\theta) + (A+BK)(\theta)^T P(\theta) = -Q(\theta)\,. \tag{9.6}$$

Example 9.1. Consider the potentially unstable plant

$$\dot{X}_1(t) = \theta X_1(t) + X_2(t), \tag{9.7}$$

$$\dot{X}_2(t) = U(t-D), \tag{9.8}$$

$$Y(t) = X_1(t)\,, \tag{9.9}$$

where we assume $\Pi = [-\underline{\theta};\bar{\theta}]$, and define

$$A(\theta) = A_0 + \theta A_1, \tag{9.10}$$

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{9.11}$$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{9.12}$$

$$B = B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9.13}$$

Using the backstepping method, we construct (K,P,Q) as

$$K(\theta) = -\left(\, 1 + (\theta+1)^2 \;\; \theta+2 \,\right), \tag{9.14}$$

$$P(\theta) = \frac{1}{2}Q(\theta) = \begin{pmatrix} 1+(1+\theta)^2 & 1+\theta \\ 1+\theta & 1 \end{pmatrix}, \tag{9.15}$$

which satisfies the Lyapunov equation (9.6).

Assumption 9.3. *The quantities*

$$\underline{\lambda} = \inf_{\theta \in \Pi} \min \{\lambda_{\min}(P(\theta)), \lambda_{\min}(Q(\theta))\}, \tag{9.16}$$

$$\bar{\lambda} = \inf_{\theta \in \Pi} \lambda_{\max}(P(\theta)) \tag{9.17}$$

exist and are known.

Example 9.2 (Example 9.1 continued). One can show that

$$\lambda_{\max}(P(\theta)) = \frac{2+(\theta+1)^2 + |\theta+1|\sqrt{(\theta+1)^2+1}}{2}, \tag{9.18}$$

$$\lambda_{\min}(P(\theta)) = \frac{1}{\lambda_{\max} P(\theta)}, \tag{9.19}$$

from which $\underline{\lambda}$ and $\bar{\lambda}$ are readily obtained over the set $\Pi = [-\underline{\theta}; \bar{\theta}]$.

Assumption 9.4. *For a given smooth function $Y^r(t)$, there exist known functions $X^r(t,\theta)$ and $U^r(t,\theta)$, which are bounded in t and continuously differentiable in the unknown argument θ on Π, and which satisfy*

$$\dot{X}^r(t,\theta) = A(\theta)X^r(t,\theta) + B(\theta)U^r(t,\theta), \tag{9.20}$$

$$Y^r(t) = CX^r(t). \tag{9.21}$$

Example 9.3 (Example 9.2 continued). Take $Y^r(t) = \sin(t)$. Then the reference trajectory pair for the state and input is

$$X^r(t,\theta) = \begin{pmatrix} \sin(t) \\ \cos(t) - \theta \sin(t) \end{pmatrix}, \tag{9.22}$$

$$U^r(t,\theta) = -\sin(t+D) - \theta\cos(t+D), \tag{9.23}$$

bounded in t and continuously differentiable in θ.

9.2 Control Design

We first represent the plant as

$$\dot{X}(t) = A(\theta)X(t) + B(\theta)u(0,t), \tag{9.24}$$

$$Y(t) = CX(t), \tag{9.25}$$

$$Du_t(x,t) = u_x(x,t), \tag{9.26}$$

$$u(1,t) = U(t), \tag{9.27}$$

where the delay is represented as a transport PDE and

$$u(x,t) = U(t + D(x-1)). \tag{9.28}$$

We consider reference trajectories $X^r(t)$ and $U^r(t)$, such as described in Assumption 9.4. Let us introduce the following tracking error variables:

$$\tilde{X}(t) = X(t) - X^r(t,\hat{\theta}), \tag{9.29}$$

$$\tilde{U}(t) = U(t) - U^r(t,\hat{\theta}), \tag{9.30}$$

$$e(x,t) = u(x,t) - u^r(x,t,\hat{\theta}), \tag{9.31}$$

with an estimate $\hat{\theta}$ of the unknown θ. When D and θ are known, one can show that the control law

$$\begin{aligned} U(t) = {} & U^r(t) - KX^r(t+D) \\ & + K\left[\mathrm{e}^{AD}X(t) + D\int_0^1 \mathrm{e}^{AD(1-y)}Bu(y,t)dy\right] \end{aligned} \tag{9.32}$$

achieves exponential stability of the equilibrium $(\tilde{X},e) = 0$, compensating the effects of the delay D.

When D and θ are unknown, we employ the control law

$$\begin{aligned} U(t) = {} & U^r(t,\hat{\theta}) - K(\hat{\theta})X^r(t+\hat{D},\hat{\theta}) \\ & + K(\hat{\theta})\left[\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}X(t) + \hat{D}(t)\int_0^1 \mathrm{e}^{A(\hat{\theta})\hat{D}(t)(1-y)}B(\hat{\theta})u(y,t)dy\right], \end{aligned} \tag{9.33}$$

based on the certainty-equivalence principle. The update laws for the estimates $\hat{D}$ and $\hat{\theta}$ are chosen based on the Lyapunov analysis (presented in Section 9.4) as

$$\dot{\hat{D}}(t) = \gamma_1 \mathrm{Proj}_{[0,\bar{D}]}\{\tau_D(t)\}, \tag{9.34}$$

$$\dot{\hat{\theta}}(t) = \gamma_2 \mathrm{Proj}_{\Pi}\{\tau_\theta(t)\}, \tag{9.35}$$

with adaptation gains γ_1 and γ_2 chosen as positive, the update law $\tau_D(t)$ chosen as

$$\tau_D(t) = -\frac{1}{N(t)}\left(\int_0^1 (1+x)w(x,t)K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)x}dx\right)\varepsilon_D(t), \tag{9.36}$$

the components of the update law $\tau_\theta(t)$ chosen as

$$\tau_{\theta i}(t) = \frac{1}{N(t)}\left(\frac{2\tilde{X}(t)^T P(\hat{\theta})}{b} - \int_0^1 (1+x)w(x,t)K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)x}dx\right)\varepsilon_i(t) \quad (9.37)$$

for $i = 1,2,\ldots,p$, the normalization signal is

$$N(t) = 1 + \tilde{X}(t)^T P(\hat{\theta})\tilde{X}(t) + b\int_0^1 (1+x)w(x,t)^2 dx, \quad (9.38)$$

and the error signals driving the update laws are

$$\varepsilon_D(t) = (A+BK)(\hat{\theta})\tilde{X}(t) + B(\hat{\theta})w(0,t), \quad (9.39)$$

$$\varepsilon_i(t) = A_i X(t) + B_i u(0,t), \qquad 1 \le i \le p. \quad (9.40)$$

The matrix P is defined in Assumption 9.2. The standard projector operators are given by

$$\mathrm{Proj}_{[0,\bar{D}]}\{\tau_D\} = \tau_D \begin{cases} 0, & \hat{D} = 0 \text{ and } \tau_D < 0, \\ 0, & \hat{D} = \bar{D} \text{ and } \tau_D > 0, \\ 1, & \text{otherwise} \end{cases} \quad (9.41)$$

for the scalar (delay case) and by

$$\mathrm{Proj}_{\Pi}\{\tau_\theta\} = \tau_\theta \begin{cases} I, & \hat{\theta} \in \mathring{\Pi} \text{ or } \nabla_{\hat{\theta}}\mathscr{P}^T\tau \le 0, \\ I - \frac{\nabla_{\hat{\theta}}\mathscr{P}\nabla_\theta\mathscr{P}^T}{\nabla_{\hat{\theta}}\mathscr{P}^T\nabla_\theta\mathscr{P}}, & \hat{\theta} \in \partial\Pi \text{ and } \nabla_{\hat{\theta}}\mathscr{P}^T\tau > 0, \end{cases} \quad (9.42)$$

for the vector (plant parameter) case.

The transformed state of the actuator is

$$w(x,t) = e(x,t) - \hat{D}(t)\int_0^x K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)(x-y)}B(\hat{\theta})e(y,t)dy$$
$$- K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)x}\tilde{X}(t), \quad (9.43)$$

and the constant b is chosen such that

$$b \ge 4\sup_{\hat{\theta}\in\Pi}|PB|^2(\hat{\theta})\frac{\bar{D}}{\underline{\lambda}}. \quad (9.44)$$

Theorem 9.1. *Let Assumptions 9.1–9.4 hold and consider the closed-loop system consisting of (9.24)–(9.27), the control law (9.33), and the update laws defined by (9.34)–(9.44). There exists $\gamma^* > 0$ such that for any $\gamma \in [0,\gamma^*[$, there exist positive constants R and ρ (independent of the initial conditions) such that for all initial conditions satisfying $(X^0, u^0, \hat{D}^0, \theta^0) \in \mathbb{R}^n \times L_2(0,1) \times]0,\bar{D}] \times \Pi$, the following holds:*

$$\Upsilon(t) \le R\left(\mathrm{e}^{\rho\Upsilon(0)} - 1\right), \quad \forall t \ge 0, \quad (9.45)$$

where

$$\Upsilon(t) = |\tilde{X}(t)|^2 + \int_0^1 e(x,t)^2 dx + \tilde{D}(t)^2 + \tilde{\theta}(t)^T \tilde{\theta}(t). \tag{9.46}$$

Furthermore, asymptotic tracking is achieved, i.e.,

$$\lim_{t\to\infty} \tilde{X}(t) = 0, \tag{9.47}$$

$$\lim_{t\to\infty} \tilde{U}(t) = 0. \tag{9.48}$$

9.3 Simulations

We return to the system from Examples 9.1–9.3, namely, to the unstable example system with a transfer function

$$Y(t) = \frac{\mathrm{e}^{-Ds}}{s(s-\theta)}[U(t)]. \tag{9.49}$$

We focus on the issues arising from the large uncertainties in D and θ and from the tracking problem with the reference trajectory (9.22)–(9.23).

We take $D = 1$, $\theta = 0.5$, $\bar{D} = 2D = 2$, $\underline{\theta} = 0$, and $\bar{\theta} = 2\theta = 1$. We pick the adaptation gains as $\gamma_1 = 10$, $\gamma_2 = 2.3$ and the normalization coefficient as $b = 4\overline{|PB|}^2\bar{D}/\underline{\lambda} = 3200$. We show simulation results for $X_1(0) = X_2(0) = 0.5$, $\hat{\theta}(0) = 0$, and two different values of $\hat{D}(0)$.

In Figs. 9.1 and 9.2, the tracking of $X^r(t)$ is achieved for both simulations, as Theorem 9.1 predicts. In Fig. 9.2 we observe that $\hat{\theta}(t)$ converges to the true θ, whereas this is not the case with $\hat{D}(t)$. This is consistent with the theory. By examining the error systems (9.51), (9.52), with the help of the persistence of excitation, we could infer the convergence of $\hat{\theta}(t)$ but not of $\hat{D}(t)$.

9.4 Proof of Global Stability and Tracking

In this section we prove Theorem 9.1. We start by considering the transformation (9.43) along with its inverse

$$e(x,t) = w(x,t) + \hat{D}(t)\int_0^x K(\hat{\theta})\mathrm{e}^{(A+BK)(\hat{\theta})\hat{D}(t)(x-y)}B(\hat{\theta})w(y,t)dy + K(\hat{\theta})\mathrm{e}^{(A+BK)(\hat{\theta})\hat{D}(t)x}\tilde{X}(t). \tag{9.50}$$

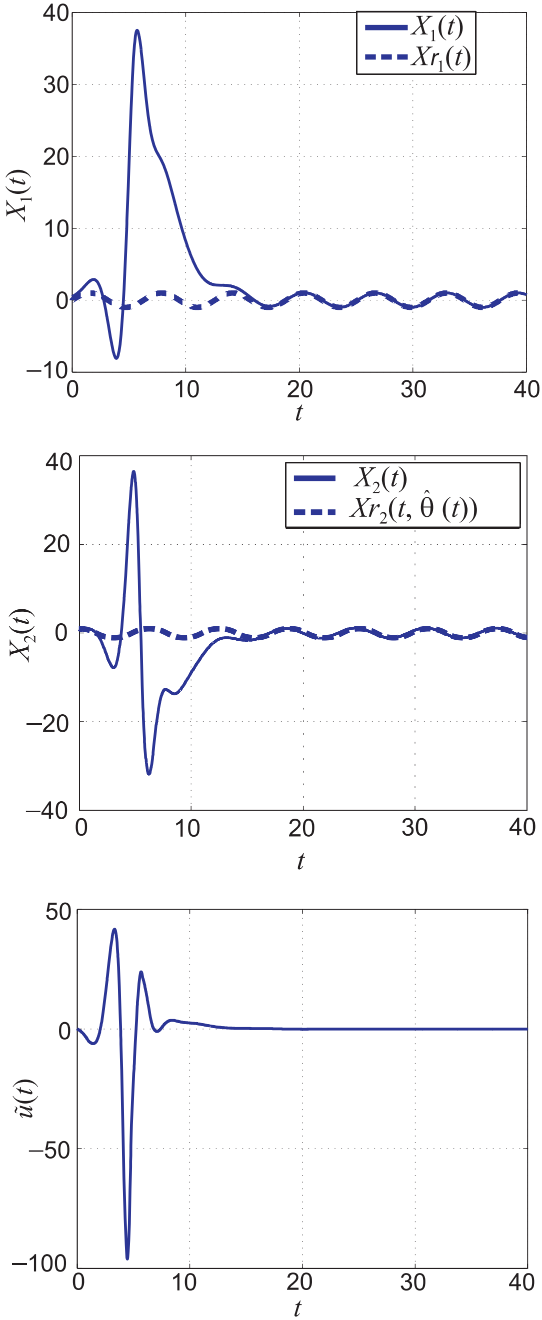

Fig. 9.1 The system response of system (9.1)–(9.2) with the reference trajectory (9.22)–(9.23) for $D = 1$, $\theta = 0.5$, $\hat{\theta}(0) = 0$, and $\hat{D}(0) = 0$.

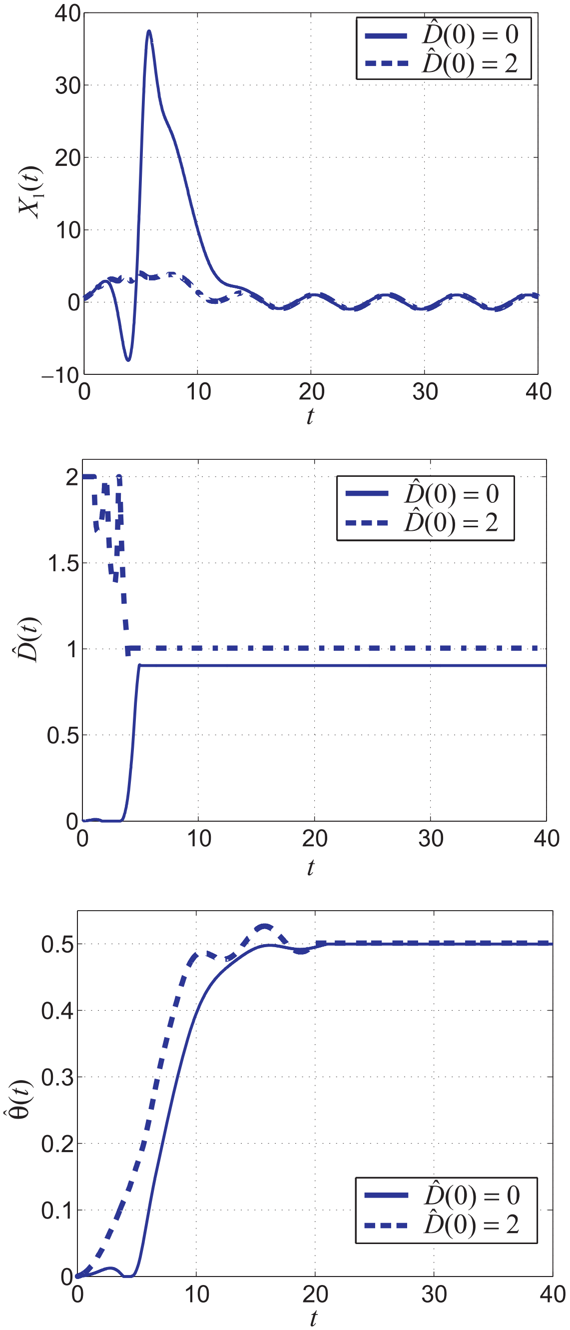

Fig. 9.2 The system response of system (9.1)–(9.2) with the reference trajectory (9.22)–(9.23) for $D = 1$, $\theta = 0.5$, $\hat{\theta}(0) = 0$, and two dramatically different initial conditions for $\hat{D}$: $\hat{D}(0) = 0$ and $\hat{D}(0) = \bar{D} = 2D = 2$. Note that the solid plots in this figure correspond to the same simulation in Fig. 9.1.

Using these transformations and the models (9.1) and (9.20), the transformed system is written as

$$\begin{aligned}\dot{\tilde{X}}(t) &= (A+BK)(\hat{\theta})\tilde{X}(t) + B(\hat{\theta})w(0,t) \\ &\quad + A(\tilde{\theta})X(t) + B(\tilde{\theta})u(0,t) \\ &\quad - \frac{\partial X^r}{\partial \hat{\theta}}(t,\hat{\theta})\dot{\hat{\theta}}(t), \qquad (9.51)\\ Dw_t(x,t) &= w_x(x,t) - \tilde{D}(t)p_0(x,t) - D\dot{\hat{D}}(t)q_0(x,t) \\ &\quad - D\tilde{\theta}(t)^T p(x,t) - D\dot{\hat{\theta}}(t)^T q(x,t), \qquad (9.52)\\ w(1,t) &= 0, \qquad (9.53)\end{aligned}$$

where

$$\tilde{D}(t) = D - \hat{D}(t) \qquad (9.54)$$

is the estimation error of the delay, the quantities

$$A(\tilde{\theta}) = \sum_{i=1}^{p} \tilde{\theta}_i A_i = \sum_{i=1}^{p} (\theta_i - \hat{\theta}_i(t))A_i, \qquad (9.55)$$

$$B(\tilde{\theta}) = \sum_{i=1}^{p} \tilde{\theta}_i B_i \qquad (9.56)$$

are linear in the parameter error

$$\tilde{\theta}(t) = \theta - \hat{\theta}(t), \qquad (9.57)$$

the quantities

$$\begin{aligned}p_0(x,t) &= K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)x}((A+BK)(\hat{\theta})\tilde{X}(t) \\ &\quad + B(\hat{\theta})w(0,t)), \qquad (9.58)\\ q_0(x,t) &= \int_0^x K(\hat{\theta})\left(I + A(\hat{\theta})\hat{D}(t)(x-y)\right)\mathrm{e}^{A(\hat{\theta})\hat{D}(t)(x-y)}B(\hat{\theta})\mathrm{e}(y,t)dy \\ &\quad + KA(\hat{\theta})x\mathrm{e}^{A(\hat{\theta})\hat{D}(t)x}\tilde{X}(t) \qquad (9.59)\end{aligned}$$

and the vector-valued functions $q(x,t)$ and $p(x,t)$ are defined through their coefficients as follows for $1 \le i \le p$:

$$\begin{aligned}p_i(x,t) &= K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)x}(A_iX(t) + B_iu(0,t)) \qquad (9.60)\\ &= K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)x}((A_i + B_iK(\hat{\theta}))\tilde{X}(t) + B_iw(0,t) \\ &\quad + A_iX^r(t,\hat{\theta}) + B_iu^r(0,t,\hat{\theta})), \qquad (9.61)\end{aligned}$$

$$
\begin{aligned}
q_i(x,t) = \hat{D}(t)\int_0^x \Bigg(& \left[\frac{\partial K}{\partial \hat{\theta}_i}(\hat{\theta}) + K(\hat{\theta})A_i\hat{D}(t)(x-y)\right] \mathrm{e}^{A(\hat{\theta})\hat{D}(t)(x-y)}B(\hat{\theta}) \\
& + K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)(x-y)}B_i \Bigg) e(y,t)dy \\
& + \left(\frac{\partial K}{\partial \hat{\theta}_i} + K(\hat{\theta})A_i\hat{D}(t)x\right)\mathrm{e}^{A(\hat{\theta})\hat{D}(t)x}\tilde{X}(t) \\
& - K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)x}\frac{\partial X^r}{\partial \hat{\theta}_i}(t,\hat{\theta}) + \frac{\partial u^r}{\partial \hat{\theta}_i}(x,t,\hat{\theta}) \\
& - \hat{D}(t)\int_0^x K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)(x-y)}B(\hat{\theta})\frac{\partial u^r}{\partial \hat{\theta}_i}(y,t,\hat{\theta})dy. \qquad (9.62)
\end{aligned}
$$

Now we define the following Lyapunov function:

$$
V(t) = D\log(N(t)) + \frac{b}{\gamma_1}\tilde{D}(t)^2 + \frac{bD}{\gamma_2}\tilde{\theta}(t)^T\tilde{\theta}(t). \qquad (9.63)
$$

Taking a time derivative of $V(t)$, we obtain

$$
\begin{aligned}
\dot{V}(t) = & -\frac{2b}{\gamma_1}\tilde{D}(t)(\dot{\hat{D}}(t) - \gamma_1\tau_D(t)) \\
& - \frac{2bD}{\gamma_2}\tilde{\theta}(t)^T(\dot{\hat{\theta}}(t) - \gamma_2\tau_\theta(t)) \\
& + \frac{D}{N(t)}\Bigg(\sum_{i=1}^{p}\dot{\hat{\theta}}_i(t)\left(\tilde{X}(t)^T\frac{\partial P}{\partial \hat{\theta}_i}(\hat{\theta})\tilde{X}(t) - \tilde{X}(t)^TP(\hat{\theta})\frac{\partial X^r}{\partial \hat{\theta}_i}(t,\hat{\theta})\right) \\
& - \tilde{X}(t)^TQ(\hat{\theta})\tilde{X}(t) + 2\tilde{X}(t)^TPB(\hat{\theta})w(0,t) \\
& - \frac{b}{D}\|w\|^2 - \frac{b}{D}w(0,t)^2 \\
& - 2b\dot{\hat{D}}(t)\int_0^1(1+x)w(x,t)q_0(x,t)dx \\
& - 2b\dot{\hat{\theta}}(t)^T\int_0^1(1+x)w(x,t)q(x,t)dx\Bigg), \qquad (9.64)
\end{aligned}
$$

where we have used integration by parts. Using the assumptions that $\hat{D}(0) \in]0;\bar{D}]$ and $\hat{\theta}(0) \in \Pi$, the update laws (9.34)–(9.35) with the properties of the projection operator, while substituting the expressions of (9.34)–(9.35) and using (9.44) with the Young inequality, we obtain

$$
\begin{aligned}
\dot{V}(t) \leq & -\frac{D}{2N(t)}\left(\lambda_{\min}(Q)|\tilde{X}|^2 + \frac{b}{D}\|w\|^2 + 2\frac{b}{D}w(0,t)^2\right) \\
& + 2Db\gamma_1\frac{\int_0^1(1+x)|w(x,t)||p_0(x,t)|dx}{N(t)}\frac{\int_0^1(1+x)|w(x)||q_0(x,t)|dx}{N(t)}
\end{aligned}
$$

$$+ D\gamma_2 \sum_{i=1}^{p} \left(\frac{\int_0^1 (1+x)|w(x,t)||p_i(x,t)|dx}{N(t)} \right.$$
$$\left. + \frac{2|\tilde{X}(t)^T P(\hat{\theta})/b||A_i X(t) + B_i u(0,t)|}{N(t)} \right)$$
$$\times \frac{1}{N(t)} \left(\left| \tilde{X}(t)^T \frac{\partial P}{\partial \hat{\theta}_i}(\hat{\theta}) \tilde{X}(t) \right| + \left| \tilde{X}(t)^T P(\hat{\theta}) \frac{\partial X^r}{\partial \hat{\theta}_i}(t,\hat{\theta}) \right| \right.$$
$$\left. + 2b \int_0^1 (1+x)|w(x,t)||q_i(x,t)|dx \right) . \tag{9.65}$$

Furthermore, each signal depending on $\hat{\theta}$, namely, A, B, K, P, $\partial P/\partial \hat{\theta}_i, \partial X^r/\partial \hat{\theta}_i$, and $\partial u^r/\partial \hat{\theta}_i$, is given as continuous in $\hat{\theta}$. Since $\hat{\theta}$ remains in Π, a closed and bounded subset of $\mathbb{R}^p$, each signal is bounded in terms of $\hat{\theta}$ and admits a finite upper bound. We denote

$$M_A = \sup_{\hat{\theta} \in \Pi} |A(\hat{\theta})| \tag{9.66}$$

and define M_P, M_B, M_K, M_{A+BK}, and $M_{\partial K/\partial \hat{\theta}}$ similarly. Therefore, substituting the expression of $e(x,t)$ in (9.59) and (9.62) with the inverse transformation (9.50), we arrive, with the help of the Cauchy–Schwartz and Young inequalities, along with (9.58)–(9.59) first and (9.60)–(9.62) next, at the inequalities

$$\int_0^1 (1+x)|w(x,t)||p_0(x,t)|dx \leq M_0(|\tilde{X}(t)|^2 + \|w(t)\|^2 + w(0,t)^2), \tag{9.67}$$
$$\int_0^1 (1+x)|w(x,t)||q_0(x,t)|dx \leq M_0(|\tilde{X}(t)|^2 + \|w(t)\|^2), \tag{9.68}$$

and

$$\left(\int_0^1 (1+x)|w(x,t)||p_i(x,t)|dx + 2|\tilde{X}(t)^T P(\hat{\theta})/b||A_i X(t) + B_i u(0,t)| \right)$$
$$\leq M_i(|\tilde{X}(t)|^2 + \|w(t)\|^2 + w(0,t)^2 + \|w(t)\|) \tag{9.69}$$
$$\int_0^1 (1+x)|w(x,t)||q_i(x,t)|dx \leq M_i(|\tilde{X}(t)|^2 + \|w(t)\|^2 + \|w(t)\|), \tag{9.70}$$

where M_0 and M_i $(1 \leq i \leq p)$ are sufficiently large constants given by

$$M_0 = M_K \max\{M_{A+BK} + M_A,$$
$$2M_K((1+M_A\bar{D})(M_B + M_B M_K(1+\bar{D})) + M_A)\}\, \mathrm{e}^{(M_A+M_{A+BK})\bar{D}}, \tag{9.71}$$
$$M_i = \max\left\{ |A_i| + |B_i|M_K + |B_i| + 2M_P/b, \right.$$
$$2 \sup_{(t,\hat{\theta}) \in \mathbb{R} \times \Pi} (|A_i||X^r(t,\hat{\theta})| + |B_i||u^r(0,t,\hat{\theta})|),$$

$$2M_K \sup_{(t,\hat{\theta})\in\mathbb{R}\times\Pi} \left|\frac{\partial X^r}{\partial\hat{\theta}}(t,\hat{\theta})\right| + 2 \sup_{(t,\hat{\theta})\in\mathbb{R}\times\Pi} \left|\frac{\partial u^r}{\partial\hat{\theta}}(t,\hat{\theta})\right| (1+\hat{D}M_K M_B),$$
$$\left(\left(M_{\partial K/\partial\hat{\theta}} + M_K|A_i|\bar{D}\right) M_B + |A_i|M_K\right)(2\bar{D}+2\bar{D}M_K M_B + M_K)\Big\}$$
$$\times\, \mathrm{e}^{(M_A+M_{A+BK})\bar{D}}. \tag{9.72}$$

Consequently, if we define

$$M_P' = \max_{1\le i\le p} \sup_{\hat{\theta}\in\Pi} \left|\frac{\partial P(\hat{\theta})}{\partial\hat{\theta}_i}\right|, \tag{9.73}$$

$$M_r = \max_{1\le i\le p} \sup_{\hat{\theta}\in\Pi, t\ge 0} \left|\frac{\partial X^r(t,\hat{\theta})}{\partial\hat{\theta}_i}\right|, \tag{9.74}$$

using (9.67)–(9.70) in (9.65), we obtain

$$\begin{aligned}\dot{V}(t) \le& -\frac{D}{2N(t)}\left(\lambda_{\min}(Q)|\tilde{X}|^2 + \frac{b}{D}\|w\|^2 + 2\frac{b}{D}w(0,t)^2\right)\\ &+ 2Db\gamma_1 M_0^2 \frac{|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2}{N(t)}\frac{|\tilde{X}(t)|^2+\|w(t)\|^2}{N(t)}\\ &+ D\gamma_2\sum_{i=1}^{p} M_i \frac{|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2+\|w(t)\|}{N(t)}\\ &\times\left(\frac{M_P'|\tilde{X}(t)|^2+M_r|\bar{P}||\tilde{X}(t)|}{N(t)} + 2bM_i\frac{|\tilde{X}(t)|^2+\|w(t)\|^2+\|w(t)\|}{N(t)}\right).\end{aligned} \tag{9.75}$$

Bounding the cubic and quadric terms with the help of $N(t)$, we arrive at

$$\begin{aligned}\dot{V}(t) \le& -\frac{D}{2N(t)}\left(\underline{\lambda}|\tilde{X}(t)|^2 + \frac{b}{D}\|w(t)\|^2 + 2\frac{b}{D}w(0,t)^2\right)\\ &+ \frac{2Db\gamma_1 M_0^2}{\min\{\underline{\lambda},b\}}\frac{|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2}{N(t)}\\ &+ D\gamma_2\sum_{i=1}^{p} M_i\left(M_P'\left(\frac{1}{\underline{\lambda}}+\frac{1}{2\min\{1,b\}}\right)\right.\\ &+ M_P M_r\left(\frac{1}{2}+\frac{1}{2\min\{1,\underline{\lambda}\}}\right)\\ &\left.+ 2bM_i\left(\frac{1}{\min\{\underline{\lambda},b\}}+\frac{1}{2\min\{1,b\}}+1\right)\right)\\ &\times\frac{|\tilde{X}(t)|^2+\|w(t)\|^2+w(0,t)^2}{N(t)}.\end{aligned} \tag{9.76}$$

Defining the following constants:

$$m = \frac{2\max\left\{bM_0^2, \sum_{i=1}^{p} M_i(M_P' + M_P M_r + 3bM_i)\right\}}{\min\{1, \underline{\lambda}, b\}}, \tag{9.77}$$

$$\gamma^* = \frac{\min\{\underline{\lambda}, b/D\}}{4bm}, \tag{9.78}$$

we finally obtain

$$\dot{V}(t) \leq -\frac{D}{2N(t)}\left(\min\left\{\underline{\lambda}, \frac{b}{D}\right\} - 2(\gamma_1+\gamma_2)m\right) \times (|\tilde{X}(t)|^2 + \|w(t)\|^2 + w(0,t)^2). \tag{9.79}$$

Consequently, by choosing $(\gamma_1, \gamma_2) \in [0;\gamma^*[^2$, we make $\dot{V}(t)$ negative semidefinite and hence

$$V(t) \leq V(0), \qquad \forall t \geq 0. \tag{9.80}$$

Starting from this result, we now prove the results stated in Theorem 9.1. From the transformation (9.43) and its inverse (9.50), we obtain these two inequalities:

$$\|w(t)\|^2 \leq r_1\|e(t)\|^2 + r_2|\tilde{X}(t)|^2, \tag{9.81}$$

$$\|e(t)\|^2 \leq s_1\|w(t)\|^2 + s_2|\tilde{X}(t)|^2, \tag{9.82}$$

where r_1, r_2, s_1, and s_2 are sufficiently large positive constants given by

$$r_1 = 3\left(1 + \bar{D}^2 M_K^2 \mathrm{e}^{2M_{A+BK}\bar{D}} M_B^2\right), \tag{9.83}$$

$$r_2 = 3M_K^2 \mathrm{e}^{2M_{A+BK}\bar{D}}, \tag{9.84}$$

$$s_1 = 3\left(1 + \bar{D}^2 M_K^2 \mathrm{e}^{2M_A\bar{D}} M_B^2\right), \tag{9.85}$$

$$s_2 = 3M_K^2 \mathrm{e}^{2M_A\bar{D}}. \tag{9.86}$$

Furthermore, from (9.63) and (9.82), it follows that

$$\tilde{D}(t)^2 + \tilde{\theta}(t)^T\tilde{\theta}(t) \leq \frac{\gamma_1+\gamma_2}{b}V(t), \tag{9.87}$$

$$\|\tilde{X}(t)\|^2 \leq \frac{1}{\underline{\lambda}}(\mathrm{e}^{V(t)/D} - 1), \tag{9.88}$$

$$\|e(t)\| \leq \frac{s_1}{b}(\mathrm{e}^{V(t)/D} - 1) + s_2\|\tilde{X}(t)\|. \tag{9.89}$$

Thus, from the definition of $\Upsilon(t)$, it is easy to show that

$$\Upsilon(t) \leq \left(\frac{1+s_2}{\underline{\lambda}} + \frac{s_1}{b} + \frac{(\gamma_1+\gamma_2)D}{b}\right)(\mathrm{e}^{V(t)/D} - 1). \tag{9.90}$$

Besides, using (9.81), we also obtain

$$V(0) \leq \left(D(\bar{\lambda}+s_2 b+2s_1 b)+b\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right)\right)\Upsilon(0). \tag{9.91}$$

Finally, if we define

$$R = \frac{1+s_2}{\underline{\lambda}}+\frac{s_1}{b}+\frac{(\gamma_1+\gamma_2)D}{b}, \tag{9.92}$$

$$\rho = \bar{\lambda}+s_2 b+2s_1 b+\frac{b}{D}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right), \tag{9.93}$$

we obtain the global stability result given in Theorem 9.1.

We now prove tracking. From (9.80), we obtain the uniform boundedness of $\|\tilde{X}(t)\|$, $\|w(t)\|$, $\hat{D}(t)$, and $\|\hat{\theta}(t)\|$. From (9.50), we obtain that $\|e(t)\|$ is also uniformly bounded in time. From (9.33), we get the uniform boundedness of $U(t)$ and consequently of $\tilde{U}(t)$ for $t \geq 0$. Thus, we get that $u(0,t)$ and $e(0,t)$ are uniformly bounded for $t \geq D$. Besides, from (9.35) and (9.60), we obtain the uniform boundedness of $\|\dot{\hat{\theta}}(t)\|$ for $t \geq D$. Finally, with (9.51), we obtain that $d\tilde{X}(t)^2/dt$ is uniformly bounded for $t \geq D$. As $|\tilde{X}(t)|$ is square integrable, from (9.79), we conclude from Barbalat's lemma that $\tilde{X}(t) \to 0$ when $t \to \infty$.

Additionally, from (9.79), we get the square integrability of $\|w(t)\|$. From (9.82), we obtain the square integrability of $\|e(t)\|$. Consequently, with (9.33), we obtain the square integrability of $\tilde{U}(t)$. Furthermore,

$$\frac{d\tilde{U}(t)^2}{dt} = 2\tilde{U}(t)\left(K(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}\dot{\tilde{X}}(t)+\dot{\hat{D}}(t)G_0(t)+\sum_{i=1}^{p}\dot{\hat{\theta}}_i(t)G_i(t)+\frac{\hat{D}}{D}H(t)\right), \tag{9.94}$$

with

$$\begin{aligned}G_0(t) = K(\hat{\theta})\Bigg[&A(\hat{\theta})\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}\tilde{X}(t)+\int_0^1\left(I+A(\hat{\theta})\hat{D}(t)(1-y)\right)\\ &\times \mathrm{e}^{A(\hat{\theta})\hat{D}(t)(1-y)}B(\hat{\theta})e(y,t)dy\Bigg],\end{aligned} \tag{9.95}$$

$$\begin{aligned}G_i(t) = \frac{\partial K}{\partial\hat{\theta}}(\hat{\theta})&\left[\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}\tilde{X}(t)+\hat{D}(t)\int_0^1 \mathrm{e}^{A(\hat{\theta})\hat{D}(t)(1-y)}B(\hat{\theta})e(y,t)dy\right]\\ &+K(\hat{\theta})\Big[A_i\hat{D}(t)\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}\tilde{X}(t)\\ &+\hat{D}(t)\int_0^1\Big[A_i\hat{D}(t)(1-y)\mathrm{e}^{A(\hat{\theta})\hat{D}(t)(1-y)}B(\hat{\theta})\\ &+\mathrm{e}^{A(\hat{\theta})\hat{D}(t)}B_i\Big]e(y,t)dy\Big],\end{aligned} \tag{9.96}$$

$$H(t) = K(\hat{\theta})\left[B(\hat{\theta})\tilde{U}(t) - \mathrm{e}^{A(\hat{\theta})\hat{D}(t)}B(\hat{\theta})e(0,t) + \int_0^1 A(\hat{\theta})\hat{D}(t)\mathrm{e}^{A(\hat{\theta})\hat{D}(t)(1-y)}B(\hat{\theta})e(y,t)dy\right]. \tag{9.97}$$

The signals $\dot{\hat{D}}$, $\dot{\hat{\theta}}_1, \dots, \dot{\hat{\theta}}_p$ 0are uniformly bounded over $t \geq 0$ and according to (9.34)–(9.35). By using the uniform boundedness of $\tilde{X}(t), \dot{\tilde{X}}(t), \|e(t)\|$, and $\tilde{U}(t)$ over $t \geq 0$ and of $e(0,t)$ for $t \geq D$ and the uniform boundedness of all the signals that are functions of $\hat{\theta}$ for $t \geq 0$, we obtain the uniform boundedness of $d\tilde{U}(t)^2/dt$ over $t \geq D$. Then, with Barbalat's lemma, we conclude that $\tilde{U}(t) \to 0$ when $t \to \infty$.

9.5 Notes and References

This chapter introduced a global adaptive tracking design for ODEs with unknown parameters and an arbitrarily long *unknown* actuator delay, with full-state feedback. In comparison with existing results by Ortega and Lozano [182], Niculescu and Annaswamy [171], and Evesque et al. [44], which deal with output feedback problems where the plant has unknown parameters but where the delay value is known, our design introduces a useful tool for combined tuning of both the delay parameter and the ODE parameters in the predictor-based feedback law.

The key difference between the regulation problem and the tracking problem is that tracking imposes a heavier restriction on the adaptation gain values. Going back through the proof, we can see that γ^* depends on m, which depends on M_i, which, in turn, depend on the size of both the reference trajectory and its time derivative. Hence, the larger and faster the reference, the more cautiously we have to adapt the parameters, at least according to the proof.

As we noted in Section 9.3, a persistently exciting reference results in perfect convergence of the plant parameter estimates, but not necessarily of the delay estimate. Even in the special case of known plant parameters, such as in Chapter 7, we are not able to provide a persistency of excitation condition for convergence of the delay estimate.

Part III
Nonlinear Systems

Chapter 10
Nonlinear Predictor Feedback

In this chapter and in Chapters 11 and 12 we develop predictor-based feedback laws for nonlinear systems. We consider general systems of the form

$$\dot{Z}(t) = f(Z(t), U(t-D)), \tag{10.1}$$

where $Z \in \mathbb{R}^n$ is the state vector of a nonlinear ODE, $U \in \mathbb{R}$ is the control input, and D is the actuator delay.

We consider only systems that are globally stabilizable in the absence of delay, namely, we assume that a function $\kappa(Z)$ is known such that

$$\dot{Z} = f(Z, \kappa(Z)) \tag{10.2}$$

is globally asymptotically stable at the origin.

Within the class of systems that are globally stabilizable at the origin, we differentiate between two classes of systems:

1. systems that are *forward-complete*, namely, systems such that

$$\dot{Z}(t) = f(Z(t), \Omega(t)) \tag{10.3}$$

   has bounded solutions (and a suitable continuous gain function) for any bounded input function $\Omega(t)$, i.e., systems that may be unstable but do not exhibit finite escape time;
2. system that are *not* forward-complete, namely, those where (10.3) may have finite escape for some initial conditions Z_0 and/or some bounded input function $\Omega(t)$ (or possibly when the input function is identically zero).

A scalar example of a system in category 1 is

$$\dot{Z}(t) = \sin(Z(t)) + U(t-D), \tag{10.4}$$

whereas a scalar example of a system in category 2 is

$$\dot{Z}(t) = Z^2(t) + U(t-D). \tag{10.5}$$

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_10,

The difference between the two categories of systems is that those in category 1 will lend themselves to global stabilization by predictor feedback in the presence of an arbitrarily long delay, whereas systems in category 2 may suffer finite escape before the control "kicks in" at $t = D$ and for this reason are not globally stabilizable (the achievable region of attraction will depend on D).

The system category 2 is clear—it includes many systems that have high-growth nonlinearities and where the control must act aggressively to prevent explosive instability. Such systems have been studied using the nonlinear "backstepping" design [112] for strict-feedback systems and other approaches.

The system category 1 includes the somewhat narrower class of strict-feedforward systems [217, 195, 102] as well as various mechanical systems that, while possibly unstable, have bounded solutions (independent of the size of the input, as long as it is bounded).

In this chapter we study a problem in category 2, whereas in Chapter 12 we study the systems in category 1. Our consideration in this chapter, for systems that are *not forward-complete*, is focused on the scalar example (10.5). We illustrate the issues associated with the possibility of finite escape before $t = D$ and show that the region of attraction of predictor feedback is essentially $\{Z(0) < 1/D\}$, namely, the system is stabilized from any initial condition that does not lead to a finite escape during the control input's "dead time." Similar analysis is possible for general systems in category 2, but, though it is not more complex conceptually, it is not as elegant and leads to more conservative estimates of the region of attraction.

Before we proceed with the design for example (10.5), we review the basics of the linear predictor feedback

$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right]. \tag{10.6}$$

This feedback law uses feedback of the form

$$U(t) = KP(t), \tag{10.7}$$

where $P(t)$ is the prediction of the state D seconds in the future,

$$P(t) = \mathrm{e}^{AD}X + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta, \tag{10.8}$$

with the current state $X(t)$ as the "initial condition" and the past D-second history of the control $U(t)$ as the input. Clearly, the linearity of the plant helps to obtain a predictor that is obtained as a superposition of the effects of the current state X and of the past inputs $U(\theta)$. Such a simple formula for the predictor, which can be used to directly compute the control input based on $X(t)$ and $U(\theta)$, cannot be expected with a nonlinear plant, because the predictor will necessarily be governed by the nonlinear model of the plant, for which we cannot write the solution in closed form (whether forward-complete or not).

10.1 Predictor Feedback Design for a Scalar Plant with a Quadratic Nonlinearity

Consider the scalar nonlinear control system

$$\frac{dZ(t)}{dt} = Z(t)^2 + U(t-D), \tag{10.9}$$

where U is the input, delayed by D time units, and the objective is stabilization to the origin. For $D = 0$, this is a trivial problem, solvable by many different feedback laws, the simplest of them being

$$U(t) = -Z(t)^2 - cZ(t), \quad c > 0. \tag{10.10}$$

To prepare for our presentation of a predictor feedback for a nonlinear scalar plant, it is helpful to recall that for a scalar linear plant

$$\dot{Z}(t) = Z(t) + U(t-D), \tag{10.11}$$

a stabilizing control law with delay compensation is

$$U(t) = -2\mathrm{e}^{D} Z(t) - 2\int_{t-D}^{t} \mathrm{e}^{t-\theta} U(\theta) d\theta, \tag{10.12}$$

which is a D-compensated version of the static feedback law

$$U(t) = -2Z(t) \tag{10.13}$$

that would achieve

$$\dot{Z}(t) = -Z(t) \tag{10.14}$$

for $D = 0$.

There are three important and obvious properties of the linear feedback law (10.12):

1. it is infinite-dimensional;
2. it is given explicitly in terms of the current state and past controls;
3. it is well defined for all values of the current state and past controls.

As we shall see, the nonlinear predictor feedback (10.9) will possess property 1 but not properties 2 and 3. It will be given by a formula that requires the solution of an integral equation, with the current state and past controls as inputs to the equation (so it is not explicit), and it will not be defined for very large values of the state and past controls.

We start by stating our design first, and then "explain" it in the next section:

$$U(t) = -P(t)^2 - cP(t), \tag{10.15}$$

$$P(t) = \int_{t-D}^{t} P(\theta)^2 d\theta + Z(t) + \int_{t-D}^{t} U(\theta) d\theta, \tag{10.16}$$

with an initial condition

$$P(\theta)=\int_{-D}^{\theta}P(\sigma)^2d\sigma+Z(0)+\int_{-D}^{\theta}U(\sigma)d\sigma \tag{10.17}$$

defined for $\theta\in[-D,0]$. Note that the function $P(t)$, the (D-seconds ahead) "*P*redictor" of $Z(t)$, is given implicitly, through the nonlinear integral equation (10.16). Note also that for $D=0$, this design specializes to the "nominal" design (10.10).

Alternative *implicit* implementations of (10.16) as a DDE, with the initial condition defined in (10.17), are

$$\dot{P}(t)=P(t)^2-P(t-D)^2+Z(t)^2+U(t) \tag{10.18}$$

$$=-cP(t)-P(t-D)^2+Z(t)^2\,. \tag{10.19}$$

As a first iteration in explaining the control design (10.15), (10.16), we point out that it will be proved in the next section that the feedback system (10.9), (10.15), (10.16) is equivalent to the system

$$\dot{Z}(t)=-cZ(t)+W(t-D)\,, \tag{10.20}$$

$$W(t)\equiv 0\,,\qquad \text{for } t\geq 0\,, \tag{10.21}$$

where the function $W(\theta)$, which is possibly nonzero for $\theta\in[-D,0]$, is defined *implicitly* in terms of $U(\theta)$ (understood as the "delay state," not as control) and $Z(t)$ using the nonlinear "integral-operator" relation

$$W(\theta)=U(\theta)+P(\theta)^2+cP(\theta)\,, \tag{10.22}$$

where $-D\leq t-D\leq\theta\leq t$, and, most importantly,

$$P(\theta)=\int_{t-D}^{\theta}P(\sigma)^2+Z(t)+\int_{t-D}^{\theta}U(\sigma)d\sigma\,. \tag{10.23}$$

The inverse of (10.22) is also nonlinear and given *explicitly* as

$$\begin{aligned}U(\theta)=W(\theta)&\\ &-\left(\int_{t-D}^{\theta}\mathrm{e}^{-c(\theta-\sigma)}W(\sigma)d\sigma+\mathrm{e}^{-c(\theta-t+D)}Z(t)\right)^2\\ &-c\left(\int_{t-D}^{\theta}\mathrm{e}^{-c(\theta-\sigma)}W(\sigma)d\sigma+\mathrm{e}^{-c(\theta-t+D)}Z(t)\right)\,.\end{aligned} \tag{10.24}$$

The key outcome of our "backstepping" control synthesis, besides the control design (10.15), (10.16), will be the construction of the Lyapunov function,

$$Z(t)^2+c\int_{t-D}^{t}(1+\theta+D-t)W(\theta)^2d\theta\,, \tag{10.25}$$

and the derivation of the stability estimates for the L_2 norm of the system state,

$$Z(t)^2 + \int_{t-D}^{t} U(\theta)^2 d\theta \,. \tag{10.26}$$

[Note the difference between (10.25) and (10.26), with W appearing in the former and U appearing in the latter, and note that they are related through a nonlinear, infinite-dimensional transformation (10.24).]

10.2 Nonlinear Infinite-Dimensional "Backstepping Transformation" and Its Inverse

A convenient way to study the problem (10.9) is using the representation where the delay state is modeled using the first-order hyperbolic (transport) PDE

$$\dot{Z}(t) = Z(t)^2 + u(0,t), \tag{10.27}$$

$$u_t(x,t) = u_x(x,t)\,, \qquad x \in (0,D), \tag{10.28}$$

$$u(D,t) = U(t)\,. \tag{10.29}$$

Note that

$$u(x,t) = U(t+x-D)\,. \tag{10.30}$$

Consider the "spatially causal" (backstepping) state transformation

$$(u(x,t), Z(t)) \mapsto (\phi(x,t), Z(t)) \tag{10.31}$$

given by

$$\phi(x,t) = u(x,t) + p(x,t)^2, \tag{10.32}$$

$$p(x,t) = \int_0^x p(y,t)^2 dy + \int_0^x u(y,t) dy + Z(t)\,, \tag{10.33}$$

where the variable $p(x,t)$ is given implicitly in terms of $u(x,t)$ and $Z(t)$. It can be shown that the transformation (10.32), (10.33) converts the plant (10.27), (10.28) into

$$\dot{Z}(t) = \phi(0,t), \tag{10.34}$$

$$\phi_t(x,t) = \phi_x(x,t)\,, \qquad x \in (0,D)\,. \tag{10.35}$$

The boundary condition for $\phi(D,t)$ has yet to be designed (it is stated ahead).

The Z-equation (10.27) has now been linearized, but it is not asymptotically stabilized yet. To this end, we apply another transformation,

$$(\phi(x,t), Z(t)) \mapsto (w(x,t), Z(t)), \tag{10.36}$$

given by

$$w(x,t) = \phi(x,t) + c\left(\int_0^x \phi(y,t)dy + Z(t)\right), \tag{10.37}$$

where $c > 0$ will be used as a control gain. It can be shown that the transformation (10.37) converts the system (10.34), (10.35) into

$$\dot{Z}(t) = -cZ(t) + w(0,t), \tag{10.38}$$

$$w_t(x,t) = w_x(x,t), \qquad x \in (0,D), \tag{10.39}$$

with a boundary condition $w(D,t)$ to be specified ahead.

Remark 10.1. We apologize to the reader that the transformations (10.32), (10.33), and (10.37) may appear a bit like "magic." Their choice is guided by a general "backstepping" design procedure in [229, 230], which employs Volterra series in u as a function of x for *infinite-dimensional feedback linearization*, and which simplifies to the compact form here due to the special structure that exists in the case of transport equations, but not in the case of parabolic PDEs. The "predictor" ideas can also explain the choice of the controller (10.15), (10.16) but not the choice of the transformations (10.22), (10.24).

Having brought the system (10.27), (10.28) into the form (10.38), (10.39), it remains to ensure that $w(0,t)$ goes to zero and, in fact, that the entire infinite-dimensional state $w(x,t)$ goes to zero. This is achieved with

$$w(D,t) = 0, \qquad \forall t \geq 0, \tag{10.40}$$

namely, by ensuring that the "transport PDE" (10.39) for $w(x,t)$ is fed by a zero input at $x = D$, and thus that its entire state will be "emptied out" in D seconds. The condition (10.40) is met with

$$\phi(D,t) = -c\left(\int_0^D \phi(y,t)dy + Z(t)\right), \tag{10.41}$$

which, in turn, is satisfied with the control law

$$U(t) = u(D,t) = -p(D,t)^2 + \phi(D,t). \tag{10.42}$$

In summary, the control law in (10.42), (10.41), with the help of (10.32), (10.33), can be written as

$$U(t) = -p(D,t)^2 - cp(D,t), \tag{10.43}$$

$$p(D,t) = \int_0^D p(\eta,t)^2 d\eta + Z(t) + \int_0^D u(\eta,t)d\eta, \tag{10.44}$$

and it results in closed-loop behavior given by

$$\dot{Z}(t) = -cZ(t) + w(0,t), \tag{10.45}$$

$$w_t(x,t) = w_x(x,t)\,, \tag{10.46}$$
$$w(D,t) = 0\,. \tag{10.47}$$

Denoting

$$P(t) = p(D,t)\,, \tag{10.48}$$

we get (10.15), (10.16). From (10.33), for $t = 0$, we get (10.17).

Before we discuss the transformation cascade

$$u(x,t) \mapsto \phi(x,t) \mapsto w(x,t) \tag{10.49}$$

in (10.32), (10.33), and (10.37), we point out that its inverse is given by

$$u(x,t) = \phi(x,t) - \left(\int_0^x \phi(y,t)dy + Z(t)\right)^2, \tag{10.50}$$

$$\phi(x,t) = w(x,t) - c\left(\int_0^x \mathrm{e}^{-c(x-y)} w(y,t)dy + \mathrm{e}^{-cx}Z(t)\right), \tag{10.51}$$

which can be simplified to

$$\begin{aligned} u(x,t) = w(x,t) &\\ &- \left(\int_0^x \mathrm{e}^{-c(x-y)} w(y,t)dy + \mathrm{e}^{-cx}Z(t)\right)^2 \\ &- c\left(\int_0^x \mathrm{e}^{-c(x-y)} w(y,t)dy + \mathrm{e}^{-cx}Z(t)\right), \end{aligned} \tag{10.52}$$

and is explicit and globally well defined [if $w(x,t)$ and $Z(t)$ are bounded, $u(x,t)$ is bounded]. It is from (10.52) that one gets (10.24).

While the inverse backstepping transformation $w \mapsto u$ is well defined, the situation is not so simple for the direct transformation $u \mapsto w$ given by (10.32), (10.33), and (10.37). The nonlinear integral equation (10.33) for the p-system, with x as the running argument and $\int_0^x u(y,t)dy + Z(t)$ as the input, is unfortunately not solvable globally, i.e., not solvable for arbitrarily large values of $\int_0^x u(y,t)dy + Z(t)$. This failure is consistent with the fact that system (10.9) is not globally stabilizable, i.e., for large initial conditions $Z(0)$ and large positive initial values of the delay state $U(t), t \in [-D, 0]$. Hence, the lack of a global result is not a failure of the method but inherent to the problem.

10.3 Stability

From (10.45)–(10.47), it is clear that some form of exponential stability (less than global, but more than infinitesimally local) holds for the closed-loop system. In this section we provide an estimate of the region of attraction and an exponential stability bound. Since the gain c presents us with too many agonizing choices in computing the estimates, we provide a result simply for $c = 1$, in which those choices (and various forms of conservativeness associated with them) are eliminated.

Theorem 10.1. *Consider the system (10.9), (10.15), (10.16). If*

$$Z(0) + \sup_{\theta\in[-D,0]} \int_{-D}^{\theta} U(\sigma)d\sigma < \frac{1}{D}, \tag{10.53}$$

then the following holds:

$$L(t) \leq 4(1+D)\left(\Lambda + (1+D)\Lambda^2\right) e^{-t/(1+D)}, \tag{10.54}$$

$$|U(t)| \leq \sqrt{2(1+D)}\left(\sqrt{\Lambda} + \sqrt{2(1+D)}\Lambda\right) e^{-t/(2(1+D))} \tag{10.55}$$

for all $t \geq 0$, where

$$L(t) = Z(t)^2 + \int_{t-D}^{t} U(\theta)^2 d\theta, \tag{10.56}$$

$$\Lambda = \Lambda_0 + \frac{1+D}{(1-D\zeta)^4}\Lambda_0^2, \tag{10.57}$$

$$\Lambda_0 = 4(1+D)L(0), \tag{10.58}$$

$$\zeta = Z(0) + \sup_{\theta\in[-D,0]} \int_{-D}^{\theta} U(\sigma)d\sigma. \tag{10.59}$$

Proof. By applying Lemmas 10.1–10.7, in the exact order given next. □

In what follows we denote

$$Z_0 = Z(0), \tag{10.60}$$

$$w_0(x) = w(x,0), \tag{10.61}$$

etc. The following notation is used:

$$\|w(t)\|^2 = \int_0^D w(x,t)^2 dx. \tag{10.62}$$

Occasionally we will suppress the dependence on t.

First, we prove the exponential stability of the linear target system.

Lemma 10.1. *The following holds for (10.45)–(10.47):*

$$Z(t)^2 + \|w(t)\|^2 \leq (1+D)\left(Z_0^2 + \|w_0\|^2\right) e^{-t/(1+D)}. \tag{10.63}$$

Proof. Consider the Lyapunov functional

$$\Omega(t) = \frac{1}{2}\left(Z(t)^2 + \int_0^D (1+x)w(x,t)^2 dx\right). \tag{10.64}$$

Its derivative is

$$\begin{aligned}
\dot{\Omega} &= -Z^2 + Zw(0,t) + \int_0^D (1+x)w(x,t)w_x(x,t)dx \\
&= -Z^2 + Zw(0,t) + \frac{1}{2}\int_0^D (1+x)d\left(w(x,t)^2\right) \\
&= -Z^2 + Zw(0,t) - \frac{1}{2}w(0,t)^2 - \frac{1}{2}\int_0^D w(x,t)^2dx \\
&= -\frac{1}{2}Z^2 - \frac{1}{2}(Z-w(0,t))^2 - \frac{1}{2}\int_0^D w(x,t)^2dx \\
&\leq -\frac{1}{2}Z^2 - \frac{1}{2}\int_0^D w(x,t)^2dx \\
&\leq -\frac{1}{2}Z^2 - \frac{1}{2(1+D)}\int_0^D (1+x)w(x,t)^2dx \\
&\leq -\frac{1}{2(1+D)}Z^2 - \frac{1}{2(1+D)}\int_0^D (1+x)w(x,t)^2dx \\
&\leq -\frac{1}{1+D}\Omega\,,
\end{aligned} \tag{10.65}$$

so

$$\Omega(t) \leq \Omega(0)\mathrm{e}^{-t/(1+D)}\,. \tag{10.66}$$

Noting that

$$Z(t)^2 + \|w(t)\|^2 \leq 2\Omega(t) \tag{10.67}$$

and

$$\Omega(0) \leq \frac{1+D}{2}\left(Z_0^2 + \|w_0\|^2\right), \tag{10.68}$$

the lemma is proved. □

Next we bound the state of the original system by the state of the target system.

Lemma 10.2. *The following holds for (10.52):*

$$Z^2 + \|u\|^2 \leq 4\left[Z^2 + \|w\|^2 + \left(Z^2 + \|w\|^2\right)^2\right]. \tag{10.69}$$

Proof. We start by writing (10.52) as

$$u(x) = w(x) - \left(\psi(x) + \mathrm{e}^{-x}Z\right) - \left(\psi(x) + \mathrm{e}^{-x}Z\right)^2, \tag{10.70}$$

where

$$\psi(x) = \mathrm{e}^{-x} \star w(x) \tag{10.71}$$

and $\star$ denotes the convolution operator. By squaring up (10.70), applying Young's inequality, and integrating in x, we get

$$\begin{aligned}\|u\|^2 \le\ & 2\|w\|^2 + 2\|\psi\|^2 + 4\|\psi^2\|^2 \\ & + 2\left(\int_0^D \mathrm{e}^{-2x} dx\right) Z^2 + 4\left(\int_0^D \mathrm{e}^{-4x} dx\right) Z^4 \\ \le\ & 2\|w\|^2 + 2\|\psi\|^2 + 4\|\psi^2\|^2 \\ & + Z^2 + Z^4 .\end{aligned} \tag{10.72}$$

From [112, Theorem B.2(ii)], it follows that

$$\|\psi\|^2 \le \|w\|^2 . \tag{10.73}$$

It remains to consider the term $\|\psi^2\|^2$. We have

$$\|\psi^2\|^2 \le \|\psi\|^2 \sup_{x\in[0,D]} \psi(x)^2 . \tag{10.74}$$

Noting that

$$\psi' = -\psi + w , \tag{10.75}$$

and that this implies

$$(\psi^2)' \le -\psi^2 + w^2 , \tag{10.76}$$

it follows from Lemma C.3 that

$$\sup_{x\in[0,D]} \psi(x)^2 \le \|w\|^2 . \tag{10.77}$$

Hence,

$$\|\psi^2\|^2 \le \|\psi\|^2 \|w\|^2 \le \|w\|^4 , \tag{10.78}$$

and we get

$$\|u\|^2 \le 4\|w\|^2 + 4\|w\|^4 + Z^2 + Z^4 . \tag{10.79}$$

The (rather conservative) bound (10.69) follows. □

In the next lemma we bound the norm of the initial condition of the transport PDE in the target system.

Lemma 10.3. *The following holds for (10.37):*

$$\|w_0\|^2 \le 2(1+D)\|\phi_0\|^2 + 2DZ_0^2 . \tag{10.80}$$

Proof. Immediate, by noting that

$$\int_0^D \left(\int_0^x \phi_0(y) dy\right)^2 dx \le D\|\phi_0\|^2 , \tag{10.81}$$

which follows from the Cauchy–Schwartz inequality. □

The next lemma is one of the key steps in our proof. This is the link in which the lack of globality of the backstepping transformation manifests itself through an estimate that holds if initial conditions are not too large.

Lemma 10.4. *Denote*

$$\zeta = Z_0 + \sup_{x\in[0,D]} \int_0^x u_0(y)dy \tag{10.82}$$

and consider the transformation (10.33). Then

$$p_0(x)^2 \leq \frac{\zeta^2}{(1-\zeta x)^2}. \tag{10.83}$$

Proof. We start by noting that

$$p_0(x)^2 \leq \left(\int_0^x p_0(y)^2 dy + \zeta\right)^2, \tag{10.84}$$

which is true because $v_0(x)$ is defined as a nonnegative-valued function. We introduce the change of variable

$$r(x) = \int_0^x p_0(y)^2 dy, \tag{10.85}$$

or, equivalently,

$$r'(x) = p_0(x)^2, \tag{10.86}$$

which gives a nonlinear differential inequality

$$r' \leq (r+\zeta)^2, \tag{10.87}$$

with an initial condition

$$r(0) = 0. \tag{10.88}$$

By the comparison principle, it follows that

$$r(x) \leq \rho(x), \tag{10.89}$$

where $\rho(x)$ is the solution of the nonlinear differential equation

$$\rho' = (\rho+\zeta)^2, \tag{10.90}$$

$$\rho(0) = 0. \tag{10.91}$$

Since

$$\rho(x) = \frac{\zeta^2 x}{1-\zeta x}, \tag{10.92}$$

it follows that

$$\int_0^x p_0(y)^2 dy \leq \frac{\zeta^2 x}{1-\zeta x}. \tag{10.93}$$

By using the inequality (10.84), the result of the lemma follows. □

In the next lemma we provide an upper bound to a quantity that appears in the bound on the transport PDE in Lemma 10.3.

Lemma 10.5. *Consider the transformation*

$$(u_0(x), Z_0) \mapsto \phi_0(x) \tag{10.94}$$

defined by the expressions (10.32), (10.33). It satisfies the following bound:

$$\|\phi_0\|^2 \leq 2\left[\|u_0\|^2 + 4\frac{(1+D)^2}{(1-\zeta D)^2}\left(Z_0^2 + \|u_0\|^2\right)^2\right]. \tag{10.95}$$

Proof. We start by observing from (10.32) that

$$\|\phi_0\|^2 \leq 2\|u_0\|^2 + 2\|p_0^2\|^2. \tag{10.96}$$

Using the estimate in Lemma 10.4, we get

$$\|p_0^2\|^2 \leq D\frac{\zeta^4}{(1-\zeta D)^4}, \tag{10.97}$$

which yields

$$\|\phi_0\|^2 \leq 2\left(\|u_0\|^2 + D\frac{\zeta^4}{(1-\zeta D)^4}\right). \tag{10.98}$$

From (10.82), it follows that

$$\begin{aligned} \zeta^4 &\leq 4\left(Z_0^2 + D\|u_0\|^2\right)^2 \\ &\leq 4(1+D)^2\left(Z_0^2 + \|u_0\|^2\right)^2. \end{aligned} \tag{10.99}$$

Combining the last two inequalities, the lemma is proved. □

In the next lemma we complete a bound on the initial state of the target system in terms of a bound on the initial state of the system in the original variables.

Lemma 10.6. *Denote*

$$\Lambda_0 = 4(1+D)\left(Z_0^2 + \|u_0\|^2\right). \tag{10.100}$$

The following holds:

$$Z_0^2 + \|w_0\|^2 \leq \Lambda_0 + \frac{1+D}{(1-D\zeta)^4}\Lambda_0^2. \tag{10.101}$$

Proof. Immediate, by substituting the inequality of Lemma 10.5 into the inequality of Lemma 10.3, and by applying the inequality

$$2DZ_0^2 \leq 4(1+D)^2 Z_0^2 \tag{10.102}$$

(which is extremely conservative, but we use it for simplicity of expression in the result of the main theorem). □

In the next lemma we provide a bound on the control effort and a proof that the control signal converges to zero.

Lemma 10.7. *The following holds:*

$$|U(t)| \leq \left(\sqrt{2(1+D)}\sqrt{\Lambda} + 2(1+D)\Lambda\right) \mathrm{e}^{-t/(1+D)} . \tag{10.103}$$

Proof. With

$$w(D,t) \equiv 0 , \tag{10.104}$$

from (10.52) we get

$$u(D) = -\left(\psi(D) + \mathrm{e}^{-D}Z\right) - \left(\psi(D) + \mathrm{e}^{-D}Z\right)^2 , \tag{10.105}$$

where

$$\psi(x) = \mathrm{e}^{-x} \star w(x) , \tag{10.106}$$

and hence,

$$|u(D)| \leq |\psi(D)| + 2|\psi(D)|^2 + \mathrm{e}^{-D}|Z| + \mathrm{e}^{-2D}Z^2 . \tag{10.107}$$

As we noted in the proof of Lemma 10.2,

$$|\psi(D)| \leq \sup_{x\in[0,D]} |\psi(x)| \leq \|w\| . \tag{10.108}$$

Thus,

$$|u(D)| \leq \|w\| + 2\|w\|^2 + \mathrm{e}^{-D}|Z| + \mathrm{e}^{-2D}Z^2 , \tag{10.109}$$

which implies

$$|u(D)| \leq \sqrt{2}\sqrt{Z^2 + \|w\|^2} + 2\left(Z^2 + \|w\|^2\right) . \tag{10.110}$$

From Lemma 10.1, we get

$$\begin{aligned} |U(t)| = |u(D,t)| \leq \sqrt{2(1+D)}\sqrt{Z_0^2 + \|w_0\|^2}\mathrm{e}^{-t/(2(1+D))} \\ + 2(1+D)\left(Z_0^2 + \|w_0\|^2\right)\mathrm{e}^{-t/(1+D)} . \end{aligned} \tag{10.111}$$

Lemma 10.6 completes the proof. □

10.4 Failure of the Uncompensated Controller

In the previous section we showed that the nonlinear predictor feedback essentially achieves "global"-like stabilization within the set of initial conditions that do not lead to finite escape during the "dead-time" period of the actuator. It would be interesting to ask a question regarding the capability of the uncompensated feedback,

$$U(t) = -Z(t)^2 - cZ(t), \quad c > 0, \tag{10.112}$$

for the same set of initial conditions. In other words, if the system survives the "dead-time" period of the first D seconds, is it "home-free" thereafter under the uncompensated feedback?

This question is relevant in light of the result [221, Theorem 3 and Remark 2], which guarantees that in the absence of delay compensation, one can expect, in general, stability to be robust to actuator delay when D is sufficiently small, with the region of attraction that is "proportional" in an appropriate sense to $1/D$ [furthermore, the region of attraction becomes infinite if $f(\cdot,\cdot)$ and $\kappa(\cdot)$ are globally Lipschitz, $\dot{Z} = f(Z, \kappa(Z))$ is globally exponentially stable, and D is, again, sufficiently small].

With the next theorem, we provide a negative answer to the above question, namely, even if the system survives the first D dead-time seconds, it may still experience finite escape time subsequently, despite the application of uncompensated feedback.

Theorem 10.2. *Consider the plant (10.9) under the nominal controller (10.10). For a given $D > 0$, there exist initial conditions $Z(0)$ satisfying condition (10.53), i.e., not causing finite escape before $t = D$ in an open loop and being within the region of attraction in a closedloop with the compensated controller (10.15), for which the solution of the uncompensated closed-loop system (10.9), (10.10) escapes to infinity before $t = 3D/2$.*

Proof. Take

$$U(\theta) = 0, \qquad \forall \theta \in [-D, 0], \tag{10.113}$$

and denote $Z_0 = Z(0)$. During the time interval $[0, D]$, the solution is

$$Z(t) = \frac{Z_0}{1 - Z_0 t}. \tag{10.114}$$

Over the interval $[D, 2D]$, the system is governed by

$$Z(t) = \int_D^t Z(\tau)^2 d\tau + \frac{Z_0}{1 - Z_0 D} + \int_D^t U(\tau - D) d\tau, \tag{10.115}$$

where

$$U(t-D) = -\left(\frac{Z_0}{1 - Z_0(t-D)}\right)^2 - \frac{Z_0}{1 - Z_0(t-D)}. \tag{10.116}$$

It can be easily shown that

$$\int_D^t U(\tau - D) d\tau = Z_0 - \frac{Z_0}{1 - Z_0(t-D)} + \ln\left(1 - Z_0(t-D)\right). \tag{10.117}$$

It then follows that

$$Z(t)^2 \geq \left(\int_D^t Z(\tau)^2 d\tau + \gamma\right)^2, \tag{10.118}$$

where

$$\gamma = \frac{Z_0}{1-Z_0D} + Z_0 - \frac{Z_0}{1-Z_0D/2} + \ln\left(1-Z_0D/2\right). \tag{10.119}$$

The system will have finite escape before

$$t = \frac{3}{2}D \tag{10.120}$$

if

$$\gamma > \frac{2}{D}. \tag{10.121}$$

Denote

$$\varepsilon = 1 - Z_0D. \tag{10.122}$$

After some calculations, the condition $\gamma > 2/D$ can be written as

$$\frac{1}{2\varepsilon} > \frac{2}{1+\varepsilon} + \frac{\varepsilon}{2} + \frac{D}{2}\ln\frac{2}{1+\varepsilon}, \tag{10.123}$$

where as Z_0 increases toward $1/D$, the left side goes to infinity, while the right side goes toward $2 + (D/2)\ln 2$, where D is fixed. Hence, the condition $\gamma > 2/D$ is satisfied and $Z(t)$ escapes to infinity before $t = 3D/2$. □

To further help the understanding of the importance of delay compensation, we consider the plant (10.9) under the predictor-based controller (10.15), (10.16), with $U(\theta) = 0, \forall\theta \in [-D, 0]$, and $Z_0 < 1/D$. The closed-loop solution is

$$Z(t) = \begin{cases} \dfrac{Z_0}{1-Z_0t}, & 0 \le t < D, \\ \dfrac{Z_0}{1-Z_0D}\mathrm{e}^{-(t-D)}, & t \ge D, \end{cases} \tag{10.124}$$

whereas the control is

$$U(t) = -\left(\frac{Z_0}{1-Z_0D}\mathrm{e}^{-t}\right)^2 - \frac{Z_0}{1-Z_0D}\mathrm{e}^{-t}. \tag{10.125}$$

The expressions (10.124) and (10.125) are so clear that we find it unnecessary to show them graphically. The initial condition $Z_0 \in (0, 1/D)$ is a particularly interesting case to study. According to (10.124), $Z(t)$ grows aggressively until $t = D$ and then decays exponentially to zero. The control starts with a large negative value

$$U(0) = -\frac{Z_0^2}{(1-Z_0D)^2} - \frac{Z_0}{1-Z_0D}, \tag{10.126}$$

anticipating that it will need to bring $Z(t)$ down from a large value at $t = D$, and then decays exponentially. In contrast, the nominal controller, studied in Theorem 10.2, starts with a much more “modest”

$$U(0) = -Z_0^2 - Z_0\,, \tag{10.127}$$

not anticipating the size of $Z(D)$, and even though $U(t)$ grows over $[0,D]$, this growth is "too little, too late" to prevent finite escape.

10.5 What Would the Nonlinear Version of the Standard "Smith Predictor" Be?

As we discussed in the Notes and References section of Chapter 2, there exist two basic predictor-based controllers for linear systems, the original one by Otto Smith [201] and the so-called modified Smith predictor, which is essentially the method of "finite spectrum assignment" [135]. The original Smith predictor structure compensates only for the predicted effect of the control input $U(t)$, D seconds in the future, without accounting for the future evolution of the system state/output $Z(t)$, and would be given in this nonlinear problem as

$$U(t) = -(Z(t)+\Upsilon(t))^2 - c(Z(t)+\Upsilon(t))\,, \tag{10.128}$$

where

$$\Upsilon(t) = \int_0^t \Upsilon(\theta)^2 d\theta + \int_0^t U(\theta)d\theta - \int_{-D}^{t-D} U(\theta)d\theta\,. \tag{10.129}$$

It is also of interest to elucidate the connection with the "reduction" transformation [8, Eq. (5.2) for $B_0 = 0$]. In the nonlinear case, we would define this transformation as (10.16):

$$P(t) = \int_{t-D}^t P(\theta)^2 d\theta + Z(t) + \int_{t-D}^t U(\theta)d\theta\,, \tag{10.130}$$

and we would then perform control design on (10.18):

$$\dot{P}(t) = P(t)^2 + U(t)\,, \tag{10.131}$$

arriving at the controller (10.15) and the closed-loop system (10.19):

$$\dot{P}(t) = -cP(t)\,. \tag{10.132}$$

The study of the closed-loop behavior would then proceed by studying the behavior of $P(t)$ for $t \in [0,D]$ from (10.130) and for $t \geq D$ from the ODE $\dot{P} = -cP$, then inferring the properties of $U(t)$ from (10.15), $U(t) = -P(t)^2 - cP(t)$, and finally deducing the properties of $Z(t)$ from (10.130), written as

$$Z(t) = -P(t) - \int_{t-D}^t P(\theta)^2 d\theta - \int_{t-D}^t U(\theta)d\theta\,, \tag{10.133}$$

missing the benefits of the complete Lyapunov function (10.25).

10.6 Notes and References

Input unmodeled dynamics have been considered notoriously challenging in control of nonlinear systems [5, 6, 76, 77, 78, 99, 113, 119]. Relative degree-altering and/or nonminimum phase input dynamics are considered to be particularly challenging. Input delay is considered an extreme form of input dynamics.

The robustness of stability of nonlinear control systems to sufficiently small input delay D, with a region of attraction "proportional" to $1/D$, was established by Teel [221, Theorem 3].

Control designs for nonlinear systems have been proposed by Jankovic [69, 70, 72, 71] and Mazenc and coauthors [139, 142, 143], and Pepe and coauthors [187, 188, 240]. These designs address systems in both strict-feedback and strict-feedforward forms, with delays appearing at various locations in the system. However, they do not deal with a long delay at the input of an unstable system, except [143], which deals with a class of forward-complete systems. The design in [143] deals with a long input delay using a low-gain idea (executed through nested saturations), rather than using *compensation* of input delay.

Even though we employed a feedback-linearizing nominal controller in this chapter, one should not assume that the predictor feedback design requires feedback linearization, which is potentially wasteful in control effort or nonrobust in terms of cancellation of nonlinearities. The control (10.15) can be replaced by the less "wasteful" feedback law

$$U(t) = -P(t)^2 - P(t)\sqrt{P(t)^2+1}\,, \tag{10.134}$$

which doesn't linearize the Z-system, and the qualitative result in this chapter would still hold. The main element in the analysis that would change is that not only would the direct backstepping transformation (10.22) be implicit, but the inverse transformation (10.24) would also become implicit.

Chapter 11
Forward-Complete Systems

As we have seen in Chapter 10, the predictor feedback has no chance of achieving global stability for systems that are prone to finite escape in an open loop, which is due to the absence of control during the actuator's "dead time." For this reason, in this chapter we focus on the class of *forward-complete* systems, which are guaranteed to have solutions that remain bounded (despite a possible exponential instability) for all finite time, as long as the input remains finite. This is not a small class of systems. It includes all linear systems—both stable and unstable. It also includes various nonlinear systems that have linearly bounded nonlinearities, such as systems in mechanics that contain trigonometric nonlinearities, as a result of rotational motions.

We start in Section 11.1 with a general predictor feedback design for nonlinear systems, and in Section 11.2 we discuss the classes of systems for which the predictor design yields global stabilization in the presence of input delay. In Section 11.3 we introduce a PDE form of the predictor feedback and the infinite-dimensional nonlinear backstepping transformation as well as the so-called target system, which is driven only by an unforced transport PDE. In Section 11.4 we overview the stability properties of the transport PDE in various norms and present the related Lyapunov functions. These facts are important because in order to construct a Lyapunov functional for the broadest class of forward-complete systems under predictor feedback, we need some nonstandard norms and Lyapunov functions for the transport PDE. Then in Section 11.5 we present a global stability analysis and a Lyapunov function construction for the general class of forward-complete systems. Finally, in Section 11.6 we present an alternative proof of global asymptotic stability, where we exploit the finite-time convergence of the delay subsystem and generate non-Lyapunov estimates over the time intervals $[0,D]$ and $[D,\infty)$.

11.1 Predictor Feedback for General Nonlinear Systems

Consider the system

$$\dot{Z}(t) = f(Z(t), U(t-D)), \tag{11.1}$$

where $Z \in \mathbb{R}^n$ is the state vector and U is a scalar control input, as given in Fig. 11.1.

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_11,

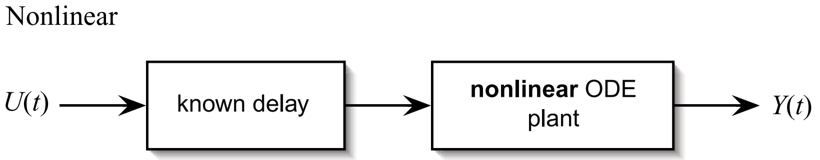

Fig. 11.1 Nonlinear system with input delay.

Assume that a continuous function $\kappa(Z)$ is known such that

$$\dot{Z} = f(Z, \kappa(Z)) \tag{11.2}$$

is globally asymptotically stable at the origin $Z = 0$.

We define our delay-compensating nonlinear predictor-based controller as

$$U(t) = \kappa(P(t)), \tag{11.3}$$

$$P(t) = \int_{t-D}^{t} f(P(\theta), U(\theta))d\theta + Z(t), \tag{11.4}$$

where the initial condition for the integral equation for $P(t)$ is defined as

$$P(\theta) = \int_{-D}^{\theta} f(P(\sigma), U(\sigma))d\sigma + Z(0), \quad \theta \in [-D, 0]. \tag{11.5}$$

The backstepping transformation that we associate with this control law and its inverse transformation are given by

$$W(t) = U(t) - \kappa(P(t)), \tag{11.6}$$

$$U(t) = W(t) + \kappa(\Pi(t)), \tag{11.7}$$

where $P(t)$ is defined above and $\Pi(t)$ is defined via the integral equation

$$\Pi(t) = \int_{t-D}^{t} f(\Pi(\theta), \kappa(\Pi(\theta)) + W(\theta))d\theta + Z(t) \tag{11.8}$$

with initial condition

$$\Pi(\theta) = \int_{-D}^{\theta} f(\Pi(\sigma), \kappa(\Pi(\sigma)) + W(\sigma))d\sigma + Z(0), \quad \theta \in [-D, 0]. \tag{11.9}$$

The purpose of this backstepping transformation is that it results in a closed-loop system (*target system*) of the form

$$\dot{Z}(t) = f(Z(t), \kappa(Z(t)) + W(t-D)), \tag{11.10}$$

$$W(t) \equiv 0, \qquad \text{for} \quad t \geq 0, \tag{11.11}$$

whereas $W(t)$ for $t \in [-D,0]$ is defined by (11.6), (11.5). Clearly, the nonzero values of $W(t)$, which occur only over the interval $[-D,0]$, depend only on the initial condition $Z(0)$ and the initial actuator state, $U(\sigma), \sigma \in [-D,0]$.

Note that

$$P = \Pi\,. \tag{11.12}$$

However, they play different roles because they are driven by different inputs (U versus W). The mapping (11.6) represents the direct backstepping transformation $U \mapsto W$, whereas (11.7) represents the inverse backstepping transformation $W \mapsto U$. Both transformations are nonlinear and infinite-dimensional.

11.2 A Categorization of Systems That Are Globally Stabilizable Under Predictor Feedback

As we shall see, to obtain global closed-loop stability under the nonlinear predictor feedback (11.3)–(11.4), we will require that the system

$$\dot{Z} = f(Z,\omega), \tag{11.13}$$

with $\omega(t)$ as an input signal, be forward-complete and that the system

$$\dot{Z} = f(Z,\kappa(Z)+\omega) \tag{11.14}$$

be input-to-state stable.

We remind the reader that a system is said to be *forward-complete* if, for every initial condition and every measurable locally essentially bounded input signal, the corresponding solution is defined for all $t \geq 0$; i.e., the maximal interval of existence is $T_{\max} = +\infty$.

The input-to-state stabilizability condition is rather mild and easily satisfiable; however, the forward-completeness condition is restrictive. As we have already observed in Chapter 10, many stabilizable systems are not forward-complete—it is only under stabilizing feedback that they become forward-complete. Even the scalar system $\dot{Z}(t) = Z^2(t) + U(t-D)$ fails to be globally stabilizable for $D > 0$ because it is not forward-complete. Many systems within the popular classes such as feedback-linearizable systems, or strict-feedback systems given by

$$\dot{Z}_1(t) = Z_2(t) + \varphi_1(Z_1(t)), \tag{11.15}$$
$$\dot{Z}_2(t) = Z_3(t) + \varphi_2(Z_1(t),Z_2(t)), \tag{11.16}$$
$$\vdots$$
$$\dot{Z}_{n-1}(t) = Z_n(t) + \varphi_{n-1}(Z_1(t),\ldots,Z_{n-1}(t)), \tag{11.17}$$
$$\dot{Z}_n(t) = U(t-D) + \varphi_n(Z_1(t),\ldots,Z_n(t))\,, \tag{11.18}$$

are not globally stabilizable for $D > 0$ because they are not forward-complete.

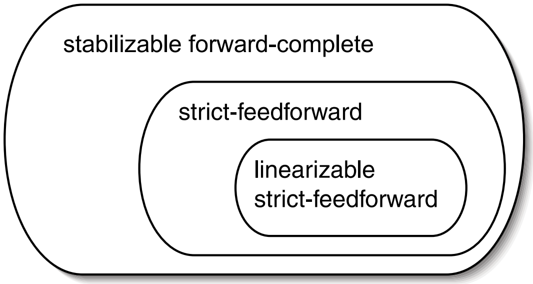

Fig. 11.2 Relation among forward-complete systems, strict-feedforward systems, and strict-feedforward systems that are feedback-linearizable. For all these systems, predictor feedback achieves global stabilization.

Hence, we look for globally stabilizable nonlinear systems with a long input delay among forward-complete systems.

One set of systems that are forward-complete includes many mechanical systems. Their nonlinearities are trigonometric or polynomial and, for finite initial conditions and finite input signals, their solutions typically remain bounded. This is the case also with electric machines, models of vehicles, and various other physical and engineering systems. Hence, the class of systems for which we will be able to achieve global stabilization by predictor feedback will be quite substantial, despite the fact that this class does not include all feedback-linearizable or strict-feedback systems.

A class of systems that plays a special role among systems that are globally stabilizable by predictor feedback is the class of *strict-feedforward* systems. Not only do these systems belong to the stabilizable subclass of forward-complete systems, but for them the predictor feedback can be written in explicit form.

An even more special subclass of forward-complete systems, for which not only the predictor feedback law can be found explicitly, but even the solutions of the closed-loop nonlinear infinite-dimensional systems can be found in closed form, are *linearizable strict-feedforward systems*. Figure 11.2 displays the relationship among the three classes of systems for which we develop globally stabilizing predictor feedbacks in this book.

Figure 11.3 displays the relation of the forward complete and strict-feedforward systems, for which we are able to develop globally stabilizing predictor feedback laws, and the relevant class of strict-feedback systems, for which global stabilization in the presence of input delays is not feasible in general because of the possibility of finite escape time.

The infinite-dimensional nonlinear systems through which we introduced backstepping and inverse backstepping transformations in Section 11.1, namely, the $P(t)$-system and the $\Pi(t)$-system, play crucial roles in determining whether or not a closed-loop system under predictor feedback is globally stable.

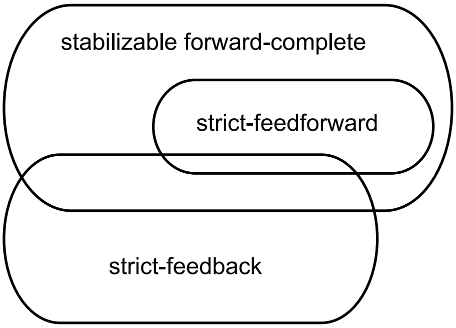

Fig. 11.3 Relation between forward-complete systems, strict-feedforward systems, and strict-feedback systems. The subclass of strict-feedback systems that is outside of the class of forward complete systems is not globally stabilizable in the presence of input delays due to the possibility of finite escape time.

We remind the reader of the forms of these two systems:

$$P(t) = \int_{t-D}^{t} f(P(\theta), U(\theta))d\theta + Z(t), \tag{11.19}$$

$$\Pi(t) = \int_{t-D}^{t} f(\Pi(\theta), \kappa(\Pi(\theta)) + W(\theta))d\theta + Z(t). \tag{11.20}$$

We refer to them as the *plant-predictor* system and the *target-predictor* system, respectively.

If the plant is forward-complete, then the plant-predictor system is globally well defined, and so is the direct backstepping transformation $W = U - \kappa(P[U,Z])$. If the plant is input-to-state stabilizable, then the target-predictor system is globally well defined, and so is the inverse backstepping transformation $U = W + \kappa(\Pi[W,Z])$.

For global stabilization via predictor feedback, we require all of the following three ingredients:

1. the target system is globally asymptotically stable;
2. the direct backstepping transformation is globally well defined;
3. the inverse backstepping transformation is globally well defined.

Ingredients 1 and 3 are almost automatically satisfied by the existence of a globally stabilizing feedback in the absence of input delay ($D = 0$). As for ingredient 2, we remind the reader that this ingredient was missing from the scalar example in Chapter 10, which was not forward-complete.

To summarize our conclusions, which at this point are not supposed to be obvious but should be helpful in guiding the reader through the coming sections and chapters:

- For general systems that are globally stabilizable in the absence of input delay, including feedback-linearizable systems and systems in the strict-feedback form, the target-predictor system and the inverse backstepping transformation will be

globally well defined, but this is not necessarily the case for the plant-predictor system and the direct backstepping transformation. Consequently, predictor feedback will not be globally (but only regionally) stabilizing within this broad class of systems.
- For forward-complete systems that are globally stabilizable in the absence of input delay, both the plant-predictor and target-predictor systems, and both the direct and inverse backstepping transformations, will be globally well defined. Consequently, predictor feedback will be globally stabilizing within this class, including its subclass of strict-feedforward systems.

11.3 The Nonlinear Backstepping Transformation of the Actuator State

To prepare for our subsequent stability analysis (Section 11.5), we introduce the representations of the plant and of the target system as, respectively,

$$\dot{Z}(t) = f(Z(t), u(t,0)), \tag{11.21}$$

$$u_t(x,t) = u_x(x,t), \tag{11.22}$$

$$u(D,t) = U(t), \tag{11.23}$$

and

$$\dot{Z}(t) = f(Z(t), \kappa(Z(t)) + w(t,0)), \tag{11.24}$$

$$w_t(x,t) = w_x(x,t), \tag{11.25}$$

$$w(D,t) = 0. \tag{11.26}$$

The predictor variables are represented by the following integral equations:

$$p(x,t) = \int_0^x f(p(\xi,t)u(\xi,t))d\xi + Z(t) \tag{11.27}$$

and

$$\pi(x,t) = \int_0^x f(\pi(\xi,t)\kappa(\pi(x,t)) + w(\xi,t))d\xi + Z(t), \tag{11.28}$$

where $p : [0,D] \times \mathbb{R}_+ \to \mathbb{R}^n$. It should be noted that in the above equations t acts as a parameter. It is helpful *not* to view it in its role as a time variable when thinking about solutions of these two nonlinear integral equations.

The alternative form of these integral equations is as differential equations, with appropriate initial conditions, given as

$$p_x(x,t) = f(p(x,t), u(x,t)), \tag{11.29}$$

$$p(0,t) = Z(t), \tag{11.30}$$

and

$$\pi_x(x,t) = f(\pi(x,t), \kappa(\pi(x,t)) + w(x,t)), \tag{11.31}$$
$$\pi(0,t) = Z(t), \tag{11.32}$$

where we reiterate that these are equations in only one independent variable, x, so they are not PDEs but ODEs, despite our use of partial derivative notation.

The backstepping transformations (direct and inverse) are defined by

$$w(x,t) = u(x,t) - \kappa(p(x,t)), \tag{11.33}$$
$$u(x,t) = w(x,t) + \kappa(\pi(x,t)), \tag{11.34}$$

with $x \in [0,D]$.

Before we state our first technical lemma, we remind the reader of the ultimate purpose of the p-system and the π-system. They are used to generate the plant-predictor and the target predictor, respectively, in the following manner:

$$P(t) = p(D,t), \tag{11.35}$$
$$\Pi(t) = \pi(D,t). \tag{11.36}$$

Now we proceed with a series of technical lemmas, in order to arrive at our main stability result at the end of this section.

Lemma 11.1. *The functions $(Z(t), u(x,t))$ satisfy Eqs. (11.21), (11.22) if and only if the functions $(Z(t), w(x,t))$ satisfy Eqs. (11.24), (11.25), where the three functions $Z(t), u(x,t)$, and $w(x,t)$ are related through (11.27)–(11.34).*

Proof. This result is immediate by noting that $u(x,t)$ and $w(x,t)$ are functions of only one variable, $x+t$, and therefore so are $p(x,t)$ and $\pi(x,t)$ based on the ODEs (11.29)–(11.32). This implies that

$$p_t(x,t) = p_x(x,t), \tag{11.37}$$
$$\pi_t(x,t) = \pi_x(x,t). \tag{11.38}$$

From this observation it follows that

$$\begin{aligned} w_t(x,t) - w_x(x,t) = u_t(x,t) - \frac{\partial \kappa(p(x,t))}{\partial p} p_t(x,t) \\ - \left(u_x(x,t) - \frac{\partial \kappa(p(x,t))}{\partial p} p_x(x,t) \right) = 0, \end{aligned} \tag{11.39}$$

$$\begin{aligned} u_t(x,t) - u_x(x,t) = w_t(x,t) + \frac{\partial \kappa(\pi(x,t))}{\partial \pi} \pi_t(x,t) \\ - \left(w_x(x,t) + \frac{\partial \kappa(\pi(x,t))}{\partial \pi} \pi_x(x,t) \right) = 0, \end{aligned} \tag{11.40}$$

which completes the proof. □

From (11.33) and (11.26), the backstepping control law is given by

$$U(t) = u(D,t) = \kappa(p(D,t)). \tag{11.41}$$

11.4 Lyapunov Functions for the Autonomous Transport PDE

Consider now the autonomous transport PDE (i.e., a transport PDE with a zero input at $x = D$)

$$w_t(x,t) = w_x(x,t), \tag{11.42}$$

$$w(D,t) = 0, \tag{11.43}$$

where

$$w_0(x) = w(x,0) \tag{11.44}$$

denotes the initial condition.

The following results on stability and Lyapunov functions for this system are useful in our subsequent construction of a Lyapunov function for the overall closed-loop system under predictor feedback or are simply interesting in their own right.

Theorem 11.1. *The following holds for system (11.42), (11.43):*

$$\int_0^D \delta(|w(x,t)|)dx \le \mathrm{e}^{g(D-t)} \int_0^D \delta(|w_0(x)|)dx, \qquad \forall t \ge 0, \tag{11.45}$$

for any $g > 0$ and any function δ in class $\mathscr{K}$.

Proof. Take the Lyapunov function

$$V(t) = \int_0^D \mathrm{e}^{gx}\delta(|w(x,t)|)dx. \tag{11.46}$$

Its derivative is

$$\begin{aligned}
\dot V(t) &= \int_0^D \mathrm{e}^{gx}\delta'(|w(x,t)|)\mathrm{sgn}\{w(x,t)\}w_t(x,t)dx \\
&= \int_0^D \mathrm{e}^{gx}\delta'(|w(x,t)|)\mathrm{sgn}\{w(x,t)\}w_x(x,t)dx \\
&= \int_0^D \mathrm{e}^{gx}\delta'(|w(x,t)|)\mathrm{sgn}\{w(x,t)\}dw(x,t) \\
&= \int_0^D \mathrm{e}^{gx}\delta'(|w(x,t)|)d|w(x,t)| \\
&= \int_0^D \mathrm{e}^{gx}d\delta(|w(x,t)|)
\end{aligned}$$

$$\begin{aligned}
&= \mathrm{e}^{gx}\delta(|w(x,t)|)|_0^D - g\int_0^D \mathrm{e}^{gx}\delta(|w(x,t)|)dx \\
&= -\delta(|w(0,t)|) - g\int_0^D \mathrm{e}^{gx}\delta(|w(x,t)|)dx \\
&= -\delta(|w(0,t)|) - gV(t)\,. \qquad (11.47)
\end{aligned}$$

Hence, we get

$$V(t) \le V_0\mathrm{e}^{-gt}\,. \qquad (11.48)$$

Next, we observe that

$$\int_0^D \delta(|w(x,t)|)dx \le V(t) \le \mathrm{e}^{gD}\int_0^D \delta(|w(x,t)|)dx\,. \qquad (11.49)$$

Combining the last two inequalities, we obtain the conclusion of the theorem. □

Taking

$$\delta(r) = r^p \qquad (11.50)$$

and

$$g = bp\,, \qquad (11.51)$$

for $p, b > 0$, we obtain the following corollary.

Corollary 11.1. *The following holds for system (11.42), (11.43):*

$$\|w(t)\|_{L_p[0,D]} \le \mathrm{e}^{b(D-t)}\|w_0\|_{L_p[0,D]}\,, \qquad \forall t \ge 0, \qquad (11.52)$$

for any $b > 0$ and any $p \in [1,\infty)$.

This corollary does not cover the case $p = \infty$, in which we are also interested. This result is proved separately.

Theorem 11.2. *The following holds for system (11.42), (11.43):*

$$\|w(t)\|_\infty \le \mathrm{e}^{c(D-t)}\|w_0\|_\infty \qquad (11.53)$$

for any $c \ge 0$, where $\|\cdot\|_\infty$ denotes $\sup_{x\in[0,D]}|w(x,t)|$.

Proof. Take the Lyapunov function

$$V(t) = \|w(t)\|_{c,\infty}, \qquad (11.54)$$

where $\|w(t)\|_{c,\infty}$ denotes the following "spatially weighted norm":

$$\|w(t)\|_{c,\infty} = \sup_{x\in[0,D]}|\mathrm{e}^{cx}w(x,t)| = \lim_{n\to\infty}\left(\int_0^D \mathrm{e}^{2ncx}w(x,t)^{2n}dx\right)^{\frac{1}{2n}}, \qquad (11.55)$$

where n is a positive integer. The derivative of $V(t)$ is

$$
\begin{aligned}
\dot{V}(t) &= \lim_{n\to\infty} \frac{d}{dt}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}} \\
&= \lim_{n\to\infty} \frac{1}{2n}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}-1} \int_0^D \mathrm{e}^{2ncx} \frac{d}{dt} w(x,t)^{2n} dx \\
&= \lim_{n\to\infty} \frac{1}{2n}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}-1} \int_0^D \mathrm{e}^{2ncx} 2n w(x,t)^{2n-1} w_t(x,t) dx \\
&= \lim_{n\to\infty} \frac{1}{2n}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}-1} \int_0^D \mathrm{e}^{2ncx} 2n w(x,t)^{2n-1} w_x(x,t) dx \\
&= \lim_{n\to\infty} \frac{1}{2n}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}-1} \int_0^D \mathrm{e}^{2ncx} 2n w(x,t)^{2n-1} dw(x,t) \\
&= \lim_{n\to\infty} \frac{1}{2n}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}-1} \int_0^D \mathrm{e}^{2ncx} dw(x,t)^{2n} . \qquad (11.56)
\end{aligned}
$$

With integration by parts, we get

$$
\begin{aligned}
\dot{V}(t) = \lim_{n\to\infty} \frac{1}{2n}&\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}-1} \\
&\times \left(\mathrm{e}^{2ncx} w(x,t)^{2n}\big|_0^D - 2nc\int_0^D \mathrm{e}^{2ncx} dw(x,t)^{2n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{2n}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}-1} \left(-w(0,t)^{2n} - 2nc\int_0^D \mathrm{e}^{2ncx} dw(x,t)^{2n}\right) \\
&= -\lim_{n\to\infty} w(0,t)^{2n} \frac{1}{2n}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}-1} \\
&\quad - c\lim_{n\to\infty}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}} \\
&= -\lim_{n\to\infty} w(0,t)^{2n} \frac{1}{2n}\left(\int_0^D \mathrm{e}^{2ncx} w(x,t)^{2n} dx\right)^{\frac{1}{2n}-1} - cV(t), \qquad (11.57)
\end{aligned}
$$

which yields

$$\dot{V}(t) \leq -cV(t) \qquad (11.58)$$

and finally

$$V(t) \leq V_0 \mathrm{e}^{-ct} . \qquad (11.59)$$

Then one gets (11.53) as follows:

$$\|w(t)\|_\infty \le \|w(t)\|_{c,\infty} \le \|w_0\|_{c,\infty} \mathrm{e}^{-ct} \le \|w_0\|_\infty \mathrm{e}^{cD} \mathrm{e}^{-ct}\,. \tag{11.60}$$

□

The following Lyapunov function fact is also useful.

Lemma 11.2. *For any $h \in \mathscr{K}$ with $h'(0) < \infty$ and any $c > 0$, the following holds for the system (11.42), (11.43):*

$$V(t) = \int_0^{\|w(t)\|_{c,\infty}} \frac{h(r)}{r} dr \tag{11.61}$$

$$\Downarrow$$

$$\dot V(t) \le -ch(\|w(t)\|_{c,\infty})\,. \tag{11.62}$$

Proof. Immediate using (11.58). □

11.5 Lyapunov-Based Stability Analysis for Forward-Complete Nonlinear Systems

We start with a discussion of our key assumption that the plant $\dot Z = f(Z,\omega)$ is forward-complete. It was proved in [4] that if a system is forward-complete, then there exist a nonnegative-valued, radially unbounded, smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a class-$\mathscr{K}_\infty$ function σ such that

$$\frac{\partial V(x)}{\partial x} f(x,u) \le V(x) + \sigma(|u|) \tag{11.63}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}$.

We make a slightly stronger assumption here.

Assumption 11.1. *The system $\dot Z = f(Z,\omega)$ has a right-hand side that satisfies*

$$f(0,0) = 0, \tag{11.64}$$

and there exist a smooth function $R : \mathbb{R}^n \to \mathbb{R}_+$ and class-$\mathscr{K}_\infty$ functions α_1, α_2, and σ such that

$$\alpha_1(|Z|) \le R(Z) \le \alpha_2(|Z|), \tag{11.65}$$

$$\frac{\partial R(Z)}{\partial Z} f(Z,\omega) \le R(Z) + \sigma(|\omega|) \tag{11.66}$$

for all $Z \in \mathbb{R}^n$ and all $\omega \in \mathbb{R}$.

The difference between Assumption 11.1 and the results proved in [4] is that [4, Corollary 2.11] guarantees that $V(x)$ is nonnegative and radially unbounded, whereas we also assume that it is positive definite. Our assumption is justified by the fact that we make the additional assumption that $f(0,0) = 0$, which allows us to establish [4, Corollary 2.3] with a less conservative estimate,

$$|Z(t)| \leq \nu(t)\psi\left(|Z(0))| + \sup_{\theta\in[0,t]} |\omega(\theta)|\right), \tag{11.67}$$

with a continuous positive-valued monotonically increasing function $\nu(\cdot)$ and a function $\psi(\cdot)$ in class $\mathscr{K}$, which actually follows from [79, Lemma 3.5] for $R = 0$. Proving [4, Corollary 2.11] with a positive-definite $V(x)$ unfortunately requires completely reworking many of the steps in [4] and is beyond the scope of our consideration here. So we proceed simply with Assumption 11.1.

The assumption $f(0,0) = 0$ is not conservative. If the equilibrium that is stabilized in the absence of input delay is different than $Z = 0$, then we would perform a shift of both the state variable and the control variable to obtain a vector field that is vanishing at $(0,0)$. So, without loss of generality, we proceed with the assumption that $f(0,0) = 0$ and

$$\kappa(0) = 0\,. \tag{11.68}$$

Now take the Lyapunov-like function $R(p(x,t))$. We have

$$\frac{\partial R(p(x,t))}{\partial p} f(p(x,t),u(x,t)) \leq R(p(x,t)) + \sigma(|u(x,t)|)\,. \tag{11.69}$$

It then follows that

$$\begin{aligned} R(p(x,t)) &\leq \mathrm{e}^x R(p(0,t)) + \int_0^x \mathrm{e}^{x-\xi}\sigma(|u(\xi,t)|)d\xi \\ &= \mathrm{e}^x R(Z(t)) + \int_0^x \mathrm{e}^{x-\xi}\sigma(|u(\xi,t)|)d\xi \\ &\leq \mathrm{e}^x R(Z(t)) + (\mathrm{e}^x - 1)\sup_{0\leq\xi\leq x}\sigma(|u(\xi,t)|)\,. \end{aligned} \tag{11.70}$$

Using the relations (11.65), we get

$$\alpha_1(|p(x,t)|) \leq \mathrm{e}^x\alpha_2(|Z(t)|) + (\mathrm{e}^x - 1)\sup_{0\leq\xi\leq x}\sigma(|u(\xi,t)|), \tag{11.71}$$

which yields

$$\begin{aligned} |p(x,t)| &\leq \alpha_1^{-1}\left(\mathrm{e}^x\alpha_2(|Z(t)|) + (\mathrm{e}^x - 1)\sup_{0\leq\xi\leq x}\sigma(|u(\xi,t)|)\right) \\ &\leq \alpha_1^{-1}\left(\mathrm{e}^D\alpha_2(|Z(t)|) + \left(\mathrm{e}^D - 1\right)\sigma\left(\sup_{0\leq\xi\leq x}|u(\xi,t)|\right)\right). \end{aligned} \tag{11.72}$$

With standard properties of class-$\mathscr{K}_\infty$ functions, we get that there exists a function $\rho_1 \in \mathscr{K}_\infty$ such that

$$|p(x,t)| \leq \rho_1 \left(|Z(t)| + \sup_{0\leq\xi\leq x} |u(\xi,t)| \right) . \tag{11.73}$$

To proceed, we will need to introduce some notation. For a vector-valued function of two variables,

$$p(x,t) = \begin{bmatrix} p_1(x,t) \\ \vdots \\ p_n(x,t) \end{bmatrix} , \tag{11.74}$$

let $\|\cdot\|_{L_\infty[0,D]}$ denote a supremum norm in x, namely,

$$\|p(t)\|_{L_\infty[0,D]} = \sup_{0\leq x\leq D} \left(p_1^2(x,t) + \cdots + p_n^2(x,t) \right)^{1/2} . \tag{11.75}$$

Returning to (11.73), taking a supremum of both sides, we get the following lemma.

Lemma 11.3. *Let system (11.29), (11.30) satisfy Assumption 11.1. Then there exists a function $\rho_1 \in \mathscr{K}_\infty$ such that*

$$\|p(t)\|_{L_\infty[0,D]} \leq \rho_1 \left(|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \right) . \tag{11.76}$$

Due to the continuity of $\kappa(\cdot)$, there exists $\rho_2 \in \mathscr{K}_\infty$ such that

$$|\kappa(p)| \leq \rho_2(|p|) . \tag{11.77}$$

With (11.33), we obtain the following lemma.

Lemma 11.4. *Let system (11.29), (11.30) satisfy Assumption 11.1 and consider (11.33) as its output map. Then there exists a function $\rho_3 \in \mathscr{K}_\infty$ such that*

$$|Z(t)| + \|w(t)\|_{L_\infty[0,D]} \leq \rho_3 \left(|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \right) . \tag{11.78}$$

Now we turn our attention to the π-system (11.31), (11.32) and make the following assumption about $\kappa(\cdot)$.

Assumption 11.2. *The system $\dot{Z} = f(Z, \kappa(Z) + \omega)$ is input-to-state stable.*

Under Assumption 11.2, there exist a class-$\mathscr{KL}$ function $\beta_1(\cdot,\cdot)$ and a class-$\mathscr{K}$ function $\gamma_1(\cdot)$ such that

$$\begin{aligned} |\pi(x,t)| &\leq \beta_1(|\pi(0,t)|, x) + \gamma_1 \left(\sup_{0\leq\xi\leq x} |w(x,t)| \right) \\ &\leq \beta_1(|Z(t)|, x) + \gamma_1 \left(\sup_{0\leq\xi\leq x} |w(x,t)| \right) . \end{aligned} \tag{11.79}$$

Taking a supremum of both sides, we get the following result.

Lemma 11.5. *Let system (11.31), (11.32) satisfy Assumption 11.2. Then*

$$\|\pi(t)\|_{L_\infty[0,D]} \leq \beta_1(|Z(t)|,0) + \gamma_1\left(\|w(t)\|_{L_\infty[0,D]}\right). \tag{11.80}$$

Then, with (11.77) and (11.34), we obtain the following lemma.

Lemma 11.6. *Let system (11.31), (11.32) satisfy Assumption 11.2 and consider (11.34) as its output map. Then there exists a function $\rho_4 \in \mathscr{K}_\infty$ such that*

$$|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \leq \rho_4\left(|Z(t)| + \|w(t)\|_{L_\infty[0,D]}\right). \tag{11.81}$$

Now we turn our attention to the full target system (11.24)–(11.26). Based on Theorem C.4, there exist a smooth function $S : \mathbb{R}^n \to \mathbb{R}_+$ and class-$\mathscr{K}_\infty$ functions $\alpha_3, \alpha_4, \alpha_5$, and α_6 such that

$$\alpha_3(|Z(t)|) \leq S(Z(t)) \leq \alpha_4(|Z(t)|), \tag{11.82}$$

$$\frac{\partial S(Z(t))}{\partial Z} f(Z(t), \kappa(Z(t)) + w(0,t)) \leq -\alpha_5(|Z(t)|) + \alpha_6(|w(0,t)|) \tag{11.83}$$

for all $Z \in \mathbb{R}^n$ and all $\omega \in \mathbb{R}$. Suppose that $\alpha_6(r)/r$ is a class-$\mathscr{K}$ function or that α_6 has been appropriately majorized so this is true (with no generality lost). Take a Lyapunov function

$$V(t) = S(Z(t)) + \frac{2}{c}\int_0^{\|w(t)\|_{c,\infty}} \frac{\alpha_6(r)}{r} dr, \tag{11.84}$$

where $c > 0$. This Lyapunov function is positive definite and radially unbounded (due to the assumption on α_6). From Lemma 11.2, we get

$$\begin{aligned}
\dot{V}(t) &\leq -\alpha_5(|Z(t)|) + \alpha_6(|w(0,t)|) - 2\alpha_6(\|w(t)\|_{c,\infty}) \\
&\leq -\alpha_5(|Z(t)|) + \alpha_6\left(\sup_{x\in[0,D]} |w(x,t)|\right) - 2\alpha_6(\|w(t)\|_{c,\infty}) \\
&\leq -\alpha_5(|Z(t)|) + \alpha_6\left(\sup_{x\in[0,D]} |\mathrm{e}^{cx} w(x,t)|\right) - 2\alpha_6(\|w(t)\|_{c,\infty}) \\
&\leq -\alpha_5(|Z(t)|) - \alpha_6(\|w(t)\|_{c,\infty}).
\end{aligned} \tag{11.85}$$

It follows then, with the help of (11.82), that there exists a class-$\mathscr{K}$ function α_7 so that

$$\dot{V}(t) \leq -\alpha_7(V(t)), \tag{11.86}$$

and then there exists a class-$\mathscr{KL}$ function $\beta_2(\cdot,\cdot)$ such that

$$V(t) \leq \beta_2(V(0),t), \qquad \forall t \geq 0. \tag{11.87}$$

With additional routine class-$\mathscr{K}$ calculations, one can show that there exists a function $\beta_3 \in \mathscr{KL}$ such that

$$|Z(t)| + \|w(t)\|_{c,\infty} \leq \beta_3\left(|Z(0)| + \|w(0)\|_{c\infty}, t\right). \tag{11.88}$$

From (11.60), we get

$$\|w(t)\|_{L_\infty[0,D]} \leq \|w(t)\|_{c,\infty}, \tag{11.89}$$

$$\|w(0)\|_{c,\infty} \leq \mathrm{e}^{cD}\|w(0)\|_{L_\infty[0,D]}. \tag{11.90}$$

Hence,

$$|Z(t)| + \|w(t)\|_{L_\infty[0,D]} \leq \beta_3\left(|Z(0)| + \mathrm{e}^{cD}\|w(0)\|_{L_\infty[0,D]}, t\right), \tag{11.91}$$

and we arrive at the following result.

Lemma 11.7. *Let system (11.24)–(11.26) satisfy Assumption 11.2. Then there exists a function $\beta_4 \in \mathscr{KL}$ such that*

$$|Z(t)| + \|w(t)\|_{L_\infty[0,D]} \leq \beta_4\left(|Z(0)| + \|w(0)\|_{L_\infty[0,D]}, t\right). \tag{11.92}$$

By combining Lemmas 11.4, 11.6, and 11.7, we get

$$|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \leq \rho_4\left(\beta_4\left(\rho_3\left(|Z(0)| + \|u(0)\|_{L_\infty[0,D]}\right), t\right)\right). \tag{11.93}$$

To summarize, we obtain the following main result.

Theorem 11.3. *Let Assumptions 11.1 and 11.2 hold. Then there exists a function $\beta_5 \in \mathscr{KL}$ such that*

$$|Z(t)| + \|u(t)\|_{L_\infty[0,D]} \leq \beta_5\left(|Z(0)| + \|u(0)\|_{L_\infty[0,D]}, t\right). \tag{11.94}$$

A slightly different and relevant way to state the same global asymptotic stability result is as follows.

Corollary 11.2. *Let Assumptions 11.1 and 11.2 hold. Then*

$$|Z(t)| + \sup_{t-D\leq\theta\leq t} |U(\theta)| \leq \beta_5\left(|Z(0)| + \sup_{-D\leq\theta\leq 0} |U(\theta)|, t\right). \tag{11.95}$$

The norm on the delay state used in Theorem 11.3 and Corollary 11.2 is a somewhat nonstandard norm in the delay system literature. Stability in the sense of other norms is also possible. Take a Lyapunov function

$$V(t) = S(Z(t)) + \int_0^D \mathrm{e}^{gx}\delta(|w(x,t)|)dx, \tag{11.96}$$

where $g > 0$ and $\delta \in \mathscr{K}_\infty$. With the help of (11.47) and (11.83), its derivative is

$$\dot{V}(t) \leq -\alpha_5(|Z(t)|) + \alpha_6(|w(0,t)|) \\ - \delta(|w(0,t))) - g\int_0^D \mathrm{e}^{gx}\delta(|w(x,t)|)dx. \tag{11.97}$$

With some routine class-$\mathscr{K}$ majorizations, the following result is obtained.

Theorem 11.4. *Let system (11.24)–(11.26) satisfy Assumptions 11.1 and 11.2. Then, for any class-$\mathscr{K}_\infty$ function δ such that*

$$\delta(r) \geq \alpha_6(r), \qquad \forall r \geq 0, \tag{11.98}$$

there exists a function $\beta_6 \in \mathscr{KL}$ such that

$$|Z(t)|^2 + \int_{t-D}^t \delta(|U(\theta)|)d\theta \leq \beta_6\left(|Z(0)|^2 + \int_{-D}^0 \delta(|U(\theta)|)d\theta, t\right). \tag{11.99}$$

Note that $\delta(\cdot)$ allows a significant degree of freedom in terms of relative (functional) weighting of the ODE state and the actuator state; however, this extra freedom is "paid for" through $\beta_6(\cdot,\cdot)$.

The following example illustrates the nonlinear predictor feedback for a system that is forward-complete.

Example 11.1. Consider the system

$$\dot{Z}_1(t) = Z_2(t), \tag{11.100}$$
$$\dot{Z}_2(t) = \sin(Z_1(t)) + U(t-D), \tag{11.101}$$

which is motivated by the pendulum problem with torque control. A predictor-based feedback law for stabilization at the origin is given by

$$U(t) = -\sin(P_1(t)) - P_1(t) - P_2(t), \tag{11.102}$$
$$P_1(t) = \int_{t-D}^t P_2(\theta)d\theta + Z_1(t), \tag{11.103}$$
$$P_2(t) = \int_{t-D}^t [\sin(P_1(\theta)) + U(\theta)]\,d\theta + Z_2(t), \tag{11.104}$$

with an appropriate initial condition on $P(\theta)$. The closed-loop system can be shown to be globally exponentially stable in terms of the norm

$$\left(Z_1^2(t) + Z_2^2(t) + \int_{t-D}^t U^2(\theta)d\theta\right)^{1/2} \tag{11.105}$$

by employing quadratic choices for $S, R, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$, and δ.

11.6 Stability Proof Without a Lyapunov Function

As we commented in Sections 2.6 and 2.8, for systems with a (known) input delay, more than one approach exists to proving stability—a Lyapunov function-based approach and an approach that uses an explicit bound on the solution growth over the interval $[0,D]$ and then exploits the fact that the delay is perfectly compensated over $t \geq 0$. The latter approach may not offer the designer the advantages to study robustness, inverse optimality, adaptive design, and other problems, but it may be more compact and even possibly less conservative than the Lyapunov function-based approach.

In this section we explore an alternative approach to proving Theorem 11.3 and Corollary 11.2, where we do not build a Lyapunov function and where we relax Assumptions 11.1 and 11.2.

Theorem 11.5. *Consider the closed-loop system*

$$\dot{Z}(t) = f(Z(t), U(t-D)), \tag{11.106}$$

$$U(t) = \kappa(P(t)), \tag{11.107}$$

$$P(t) = \int_{t-D}^{t} f(P(\theta), U(\theta))d\theta + Z(t), \quad t \geq 0, \tag{11.108}$$

$$P(\theta) = \int_{-D}^{\theta} f(P(\sigma), U(\sigma))d\sigma + Z(0), \quad \theta \in [-D, 0], \tag{11.109}$$

with an initial condition $Z_0 = Z(0)$ and $U_0(\theta) = U(\theta), \theta \in [-D, 0]$, where

$$f(0,0) = 0, \tag{11.110}$$

$$\kappa(0) = 0, \tag{11.111}$$

and let $\dot{Z} = f(Z, \omega)$ be forward-complete and $\dot{Z} = f(Z, \kappa(Z))$ be globally asymptotically stable at $Z = 0$. Then there exists a function $\hat{\beta} \in \mathcal{KL}$ such that

$$|Z(t)| + \|U\|_{L_\infty[t-D,t]} \leq \hat{\beta}\left(|Z(0)| + \|U_0\|_{L_\infty[-D,0]}, t\right) \tag{11.112}$$

for all $(Z_0, U_0) \in \mathbb{R}^n \times L_\infty[-D, 0]$ and all $t \geq 0$.

Proof. From the forward completeness of $\dot{Z} = f(Z, \omega)$, from [79, Lemma 3.5], using the fact that $f(0,0) = 0$, which allows us to set $R = 0$, we get

$$|Z(t)| \leq \nu(t)\psi\left(|Z(0)| + \sup_{\theta \in [-D, t-D]} |U(\theta)|\right), \tag{11.113}$$

with a continuous, positive-valued, monotonically increasing function $\nu(\cdot)$ and a function $\psi(\cdot)$ in class $\mathcal{K}$. It follows that

$$|Z(t)| \leq \nu(D)\psi\left(|Z(0)| + \sup_{\theta \in [-D, 0]} |U(\theta)|\right), \quad t \in [0, D]. \tag{11.114}$$

Using the fact that

$$U(t) = \kappa(P(t)) = \kappa(Z(t+D)), \quad \forall t \geq 0, \tag{11.115}$$

and using the fact that $\dot{Z} = f(Z, \kappa(Z))$ is globally asymptotically stable at the origin, there exists a class-$\mathcal{KL}$ function $\hat{\sigma}$ such that

$$|Z(t)| \leq \hat{\sigma}\left(|Z(D)|, t-D\right), \quad \forall t \geq D. \tag{11.116}$$

It follows that

$$|Z(t)| \leq \hat{\sigma}\left(\nu(D)\psi\left(|Z(0)| + \sup_{\theta\in[-D,0]} |U(\theta)|\right), \max\{0, t-D\}\right), \quad \forall t \geq 0, \tag{11.117}$$

where we have used the fact that

$$\hat{\sigma}(s, 0) \geq s, \quad \forall s \geq 0. \tag{11.118}$$

Recalling (11.77), (11.115), and (11.116), we get

$$\begin{aligned} |U(t)| &\leq \rho_2\left(\hat{\sigma}\left(|Z(D)|, t\right)\right) \\ &\leq \rho_2\left(\hat{\sigma}\left(\nu(D)\psi\left(|Z(0)| + \sup_{\theta\in[-D,0]} |U(\theta)|\right), t\right)\right), \quad \forall t \geq 0, \end{aligned} \tag{11.119}$$

which also implies that

$$\begin{aligned} &\sup_{\theta\in[t-D,t]} |U(t)| \\ &\quad \leq \rho_2\left(\hat{\sigma}\left(\nu(D)\psi\left(|Z(0)| + \sup_{\theta\in[-D,0]} |U(\theta)|\right), t-D\right)\right), \quad \forall t \geq D. \end{aligned} \tag{11.120}$$

Now we turn our attention to estimating $\sup_{\theta\in[t-D,t]} |U(\theta)|$ over $t \in [0, D]$. We split the interval $[t-D, t]$ in the following manner:

$$\begin{aligned} \sup_{\theta\in[t-D,t]} |U(\theta)| &\leq \sup_{\theta\in[t-D,0]} |U(\theta)| + \sup_{\theta\in[0,t]} |U(\theta)| \\ &\leq \sup_{\theta\in[-D,0]} |U(\theta)| + \sup_{\theta\in[0,t]} |U(\theta)| \\ &\leq \sup_{\theta\in[-D,0]} |U(\theta)| \\ &\quad + \rho_2\left(\hat{\sigma}\left(\nu(D)\psi\left(|Z(0)| + \sup_{\theta\in[-D,0]} |U(\theta)|\right), 0\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\leq |Z(0)| + \sup_{\theta\in[-D,0]} |U(\theta)| \\
&\quad + \rho_2\left(\hat{\sigma}\left(\nu(D)\psi\left(|Z(0)| + \sup_{\theta\in[-D,0]}|U(\theta)|\right),0\right)\right) \\
&= \hat{\zeta}\left(|Z(0)| + \sup_{\theta\in[-D,0]}|U(\theta)|\right), \qquad \forall t\in[0,D], \qquad (11.121)
\end{aligned}
$$

where

$$
\hat{\zeta}(s) = s + \rho_2\left(\hat{\sigma}\left(\nu(D)\psi(s),0\right)\right). \tag{11.122}
$$

Let us now consider the function $\eta:\mathbb{R}_+^2\to\mathbb{R}_+$:

$$
\hat{\eta}(r,t) = \begin{cases} \hat{\zeta}(r), & t\in[0,D], \\ \rho_2\left(\hat{\sigma}\left(\nu(D)\psi(r),t-D\right)\right), & t\geq D. \end{cases} \tag{11.123}
$$

Since $\hat{\sigma}$ is a class-$\mathscr{KL}$ function and $\hat{\zeta},\rho_2$, and ψ are class $\mathscr{K}$, there exists a class-$\mathscr{KL}$ function $\hat{\xi}$ such that

$$
\hat{\xi}(r,t) \geq \hat{\eta}(r,t), \quad \forall (r,t)\in\mathbb{R}_+^2. \tag{11.124}
$$

For example, the function $\hat{\xi}(r,t)$ can be chosen as

$$
\hat{\xi}(r,t) = \begin{cases} \hat{\zeta}(r), & t\in[0,D], \\ \hat{\zeta}(r)\dfrac{\rho_2\left(\hat{\sigma}\left(\nu(D)\psi(r),t-D\right)\right)}{\rho_2\left(\hat{\sigma}\left(\nu(D)\psi(r),0\right)\right)}, & t>D, r>0, \\ 0, & t>D, r=0. \end{cases} \tag{11.125}
$$

Hence,

$$
\sup_{\theta\in[t-D,t]}|U(\theta)| \leq \hat{\xi}\left(|Z(0)| + \sup_{\theta\in[-D,0]}|U(\theta)|, t\right), \qquad \forall t\geq 0. \tag{11.126}
$$

Now adding the bound (11.117), we get

$$
\begin{aligned}
&|Z(t)| + \sup_{\theta\in[t-D,t]}|U(\theta)| \\
&\quad \leq \hat{\sigma}\left(\nu(D)\psi\left(|Z(0)| + \sup_{\theta\in[-D,0]}|U(\theta)|\right), \max\{0,t-D\}\right) \\
&\qquad + \hat{\xi}\left(|Z(0)| + \sup_{\theta\in[-D,0]}|U(\theta)|, t\right), \qquad \forall t\geq 0. \qquad (11.127)
\end{aligned}
$$

Denoting

$$\hat{\beta}(r,t) = \hat{\sigma}\left(\nu(D)\psi(r), \max\{0, t-D\}\right) + \hat{\xi}(r,t), \tag{11.128}$$

we complete the proof of the theorem. □

11.7 Notes and References

The general predictor design that we presented in this chapter is applicable to any nonlinear system that is globally stabilizable in the absence of input delay. If the system is forward-complete, then global stability is achieved in the presence of a delay of any length.

We had to use some uncommon norms in this chapter to construct a Lyapunov function for the entire class of forward-complete systems. We produced two global asymptotic stability results, one in Corollary 11.2 that uses a supremum norm on the delay state and another in Theorem 11.4 that uses an integral norm on the delay state. Since all linear plants are forward-complete, it is appropriate to ask how these results specialize to the linear case. In the linear case the function $\delta(\cdot)$ in Theorem 11.4 can be taken as quadratic, so one recovers the standard result in Chapter 2. In addition, the general result of Corollary 11.2 also applies to the linear case, so we obtain stability in a supremum norm of the input state. Finally, with the help of Corollary 11.1, one can obtain stability in any L_p norm ($p \geq 1$) of the input state, but the norm would have to include a power other than two on $|Z(t)|$ when $p \neq 2$.

In Section 11.6 we proved Corollary 11.2 without using a Lyapunov function, without imposing an input-to-state stability property on the target system, and without using a Lyapunov characterization of forward completeness. The linear version of Theorem 11.5 was proved in Theorem 2.2.

The key tool we relied on in the stability analysis in this chapter is the Lyapunov characterization of forward completeness from the paper by Angeli and Sontag [4] and the related result by Karafyllis [79].

Chapter 12
Strict-Feedforward Systems

We now focus on a special subclass of the class of forward-complete systems—the strict-feedforward systems. Their general structure, in the absence of delay, is

$$\dot{Z}_1(t) = Z_2(t) + \psi_1(Z_2(t), Z_3(t), \ldots, Z_n(t)) + \phi_1(Z_2(t), Z_3(t), \ldots, Z_n(t))u(0,t)\,, \tag{12.1}$$

$$\vdots$$

$$\dot{Z}_{n-2}(t) = Z_{n-1}(t) + \psi_{n-2}(Z_{n-1}(t), Z_n(t)) + \phi_{n-2}(Z_{n-1}(t), Z_n(t))u(0,t)\,, \tag{12.2}$$

$$\dot{Z}_{n-1}(t) = Z_n(t) + \phi_{n-1}(Z_n(t))u(0,t)\,, \tag{12.3}$$

$$\dot{Z}_n(t) = U(t)\,. \tag{12.4}$$

The similarity in name between forward-complete and strict-feedforward systems is a pure coincidence. For *forward*-complete systems, "forward" refers to the direction of time. Such systems have finite solutions for all finite *positive* time. With feed *forward* systems, the word "forward" refers to the absence of feedback in the structure of the system. The system consists of a particular cascade of scalar systems.

While forward-complete systems yield global stability when predictor feedback is applied to them, the strict-feedforward systems have the additional property that, despite being nonlinear, they can be solved explicitly. Consequently, the predictor state can be defined explicitly. Related to this, the direct infinite-dimensional backstepping transformation can be explicitly constructed.

A special subclass of strict-feedforward systems exists that are linearizable by coordinate change. For these systems, not only is the predictor state explicitly defined, but the closed-loop solutions can be found explicitly.

We introduce these ideas first through an example in Section 12.1. Then we move on to the general class of strict-feedforward systems by first reviewing the integrator forwarding design in the absence of input delay (Section 12.2), which we then follow by the construction of the state predictor, which we derive in an explicit form (Section 12.3). A Lyapunov-based proof of stability is presented in Section 12.4. A worked example of a predictor feedback for a third-order strict-feedforward

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system that is not (feedback-)linearizable is shown in Section 12.5. In section 12.6 we recall the design based on nested saturations, which is one of the nominal feedback choices that can be combined with a predictor-based compensator. Finally, in section 12.7 we present an extension to a time-varying input delay.

12.1 Example: A Second-Order Strict-Feedforward Nonlinear System

Consider the second-order system (see Example G.2 in Appendix G for additional discussion),

$$\dot{Z}_1(t) = Z_2(t) - Z_2^2(t)U(t-D)\,, \tag{12.5}$$
$$\dot{Z}_2(t) = U(t-D)\,. \tag{12.6}$$

This system is the simplest "interesting" example of a strict-feedforward system. The nominal ($D = 0$) controller is given by

$$U(t) = -Z_1(t) - 2Z_2(t) - \frac{1}{3}Z_2^3(t) \tag{12.7}$$

and results in the closed-loop system

$$\dot{\zeta}_1(t) = \zeta_2(t)\,, \tag{12.8}$$
$$\dot{\zeta}_2(t) = -\zeta_1(t) - \zeta_2(t)\,, \tag{12.9}$$

where $\zeta(t)$ is defined by the diffeomorphic transformation

$$\zeta_1(t) = Z_1(t) + Z_2(t) + \frac{1}{3}Z_2^3(t)\,, \tag{12.10}$$
$$\zeta_2(t) = Z_2(t)\,. \tag{12.11}$$

The predictor is found by explicitly solving the nonlinear ODE

$$p_x(x,t) = f(p(x,t),u(x,t))\,, \quad p(0,t) = Z(t)\,, \tag{12.12}$$

which for this example is given by

$$\frac{\partial}{\partial x}p_1(x,t) = p_2(x,t) - p_2^2(x,t)u(x,t)\,, \tag{12.13}$$
$$\frac{\partial}{\partial x}p_2(x,t) = u(x,t)\,, \tag{12.14}$$

with initial conditions

$$p_1(0,t) = Z_1(t)\,, \tag{12.15}$$
$$p_2(0,t) = Z_2(t)\,. \tag{12.16}$$

The control is given by

$$U(t) = -P_1(t) - 2P_2(t) - \frac{1}{3}P_2^3(t)\,, \tag{12.17}$$

where $P_1(t) = p_1(D,t)$ and $P_2(t) = p_2(D,t)$ are given by

$$\begin{aligned} P_1(t) = {} & Z_1(t) + DZ_2(t) + \int_{t-D}^{t} (t-\theta)U(\theta)d\theta \\ & - Z_2^2(t)\int_{t-D}^{t} U(\theta)d\theta - Z_2(t)\left(\int_{t-D}^{t} U(\theta)d\theta\right)^2 \\ & - \frac{1}{3}\left(\int_{t-D}^{t} U(\theta)d\theta\right)^3, \end{aligned} \tag{12.18}$$

$$P_2(t) = Z_2(t) + \int_{t-D}^{t} U(\theta)d\theta\,. \tag{12.19}$$

The control law is a nonlinear infinite-dimensional operator, but it is given explicitly. The backstepping transformation is also given explicitly:

$$w(x,t) = u(x,t) + p_1(x,t) + 2p_2(x,t) + \frac{1}{3}p_2^3(x,t)\,, \tag{12.20}$$

$$\begin{aligned} p_1(x,t) = {} & Z_1(t) + xZ_2(t) + \int_0^x (x-y)u(y,t)dy \\ & - Z_2^2(t)\int_0^x u(y,t)dy - Z_2(t)\left(\int_0^x u(y,t)dy\right)^2 \\ & - \frac{1}{3}\left(\int_0^x u(y,t)dy\right)^3, \end{aligned} \tag{12.21}$$

$$p_2(x,t) = Z_2(t) + \int_0^x u(y,t)dy\,. \tag{12.22}$$

Now we derive the inverse transformation. This transformation is given by

$$u(x,t) = w(x,t) - \pi_1(x,t) - 2\pi_2(x,t) - \frac{1}{2}\pi_2^3\,, \tag{12.23}$$

where $\pi_1(x,t)$ and $\pi_2(x,t)$ are the solutions of the ODEs

$$\frac{\partial}{\partial x}\pi_1(x,t) = \pi_2(x,t) + \pi_2^2(x,t)\left(\pi_1(x,t) + 2\pi_2(x,t) + \frac{1}{2}\pi_2^3 - w(x,t)\right), \tag{12.24}$$

$$\frac{\partial}{\partial x}\pi_2(x,t) = -\pi_1(x,t) - 2\pi_2(x,t) - \frac{1}{2}\pi_2^3 + w(x,t)\,, \tag{12.25}$$

with initial conditions

$$\pi_1(0,t) = Z_1(t)\,, \tag{12.26}$$

$$\pi_2(0,t) = Z_2(t)\,. \tag{12.27}$$

However, it is hard to imagine that one could solve these ODEs for $\pi_1(x,t)$ and $\pi_2(x,t)$ directly.

Fortunately, this system is linearizable by a change of variable. Indeed, the plant (12.5), (12.6) can be converted by a change of variable

$$h_1(t) = Z_1(t) + \frac{1}{3}Z_2^3(t), \tag{12.28}$$

$$h_2(t) = Z_2(t) \tag{12.29}$$

into

$$\dot{h}_1(t) = h_2(t), \tag{12.30}$$

$$\dot{h}_2(t) = U(t-D). \tag{12.31}$$

Then the inverse backstepping transformation is given by

$$u(x,t) = w(x,t) - \eta_1(x,t) - 2\eta_2(x,t), \tag{12.32}$$

where the functions $\eta_1(x,t)$ and $\eta_2(x,t)$ are defined through the ODEs

$$\frac{\partial}{\partial x}\eta_1(x,t) = \eta_2(x,t), \tag{12.33}$$

$$\frac{\partial}{\partial x}\eta_2(x,t) = -\eta_1(x,t) - 2\eta_2(x,t) + w(x,t), \tag{12.34}$$

with initial conditions

$$\eta_1(0,t) = h_1(t) = Z_1(t) + \frac{1}{3}Z_2^3(t), \tag{12.35}$$

$$\eta_2(0,t) = h_2(t) = Z_2(t). \tag{12.36}$$

This ODE is linear and we will solve it explicitly. For this, we need the matrix exponential

$$e^{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}x} = e^{-x}\begin{bmatrix} 1+x & x \\ -x & 1-x \end{bmatrix}. \tag{12.37}$$

With the help of this matrix exponential, we find the solution

$$\eta_1(x,t) = e^{-x}\left[(1+x)Z_1(t) + (1+x)\frac{1}{3}Z_2^3(t) + xZ_2(t)\right] + \int_0^x (x-y)e^{-(x-y)}w(y,t)dy, \tag{12.38}$$

$$\eta_2(x,t) = e^{-x}\left[-xZ_1(t) - x\frac{1}{3}Z_2^3(t) + (1-x)Z_2(t)\right] + \int_0^x (1-x+y)e^{-(x-y)}w(y,t)dy, \tag{12.39}$$

with which we obtain an explicit definition of the inverse backstepping transformation (12.32). Like the direct one, this transformation is nonlinear and infinite-dimensional.

To summarize, for the example nonlinear plant (12.5), (12.6), we have obtained the transformation

$$(Z(t),u(x,t)) \mapsto (Z(t),w(x,t)) \tag{12.40}$$

and

$$(Z(t),w(x,t)) \mapsto (Z(t),u(x,t)) \tag{12.41}$$

explicitly.

Now we discuss the target system. It is given by

$$\dot{\zeta}_1(t) = -\zeta_2(t) + w(0,t)\,, \tag{12.42}$$

$$\dot{\zeta}_2(t) = -\zeta_1(t) - \zeta_2(t) + w(0,t)\,, \tag{12.43}$$

$$w_t(x,t) = w_x(x,t)\,, \tag{12.44}$$

$$w(D,t) = 0\,, \tag{12.45}$$

where the variables (ζ_1,ζ_2) are defined as in (12.10), (12.11). This target system is a cascade of the exponentially stable transport PDE for $w(x,t)$ and the linear exponentially stable ODE for $\zeta(t)$.

To analyze the stability of the closed-loop system, we would first perform the stability analysis of the (ζ,w)-system, using a standard Lyapunov–Krasovskii functional as in Chapter 2. This would yield a stability estimate in terms of the norm

$$\left(\zeta_1^2(t) + \zeta_2^2(t) + \int_0^D w^2(x,t)dx\right)^{1/2}. \tag{12.46}$$

Then we would turn to the direct backstepping transformation (12.20)–(12.22) and to the forwarding transformation (12.10), (12.11) to obtain a bound on the initial value of the norm

$$\left(Z_1^2(t) + Z_2^2(t) + \int_0^D u^2(x,t)dx\right)^{1/2} \tag{12.47}$$

in terms of the initial value of the norm (12.46). Note that the relation between these norms would be nonlinear. Finally, we would invoke the direct backstepping transformation (12.32) with (12.38), (12.39), as well as the inverse of the forwarding transformation (12.10), (12.11), to bound (12.47) in terms of (12.46).

This would yield the result that there exists a class-$\mathscr{KL}$ function $\beta(\cdot,\cdot)$ such that

$$Z_1^2(t) + Z_2^2(t) + \int_0^D u^2(x,t)dx \leq \beta\left(Z_1^2(0) + Z_2^2(0) + \int_0^D u^2(x,0)dx, t\right) \tag{12.48}$$

for all $t \geq 0$. In addition, due to the facts that the target system (ζ,w) is exponentially stable and that all the transformations and inverse transformations, though nonlinear, have a locally linear component, we would obtain that $\beta(\cdot,\cdot)$ is locally linear in the first argument and exponentially decaying in t for sufficiently large t. Hence, we obtain the following result.

Theorem 12.1. *Consider the plant (12.5), (12.6) in a closed loop with the controller (12.17)–(12.19). Its equilibrium at the origin* $(Z,u) \equiv 0$ *is globally asymptotically stable and locally exponentially stable in terms of the norm (12.47).*

Remark 12.1. When we plug the predictor equations (12.18), (12.19) into the control law (12.17), we obtain the feedback

$$U(t) = -Z_1(t) - (2+D)Z_2(t) - \frac{1}{3}Z_2^3(t) - \int_{t-D}^{t}(2+t-\theta)U(\theta)d\theta\,. \tag{12.49}$$

Compared with the nominal controller (12.7), this appears to be an amazing simplification of (12.17). This kind of a simplification won't be possible for strict-feedforward systems in general, but only for the subclass of strict-feedforward systems that are linearizable by a diffeomorphic change of coordinates. This simplified feedback could have been obtained by starting directly from the linearized system (12.30), (12.31) and by applying the linear predictor design from Chapter 2.

Remark 12.2. The linearizability of this example by coordinate change allows us to perform the stability analysis in a particularly transparent way. Both (ζ_1,ζ_2) and (h_1,h_2) are linearizing coordinates. Let us next discuss the relation between the open-loop system in the variables (Z,u) and the closed-loop system in the variables (h,w). The open-loop system is given by

$$\dot{Z}_1(t) = Z_2(t) - Z_2^2(t)u(0,t)\,, \tag{12.50}$$
$$\dot{Z}_2(t) = u(0,t)\,, \tag{12.51}$$
$$u_t(x,t) = u_x(x,t)\,, \tag{12.52}$$
$$u(D,t) = U(t)\,, \tag{12.53}$$

whereas the closed-loop system is

$$\dot{h}_1(t) = h_2(t)\,, \tag{12.54}$$
$$\dot{h}_2(t) = -h_1(t) - 2h_2(t) + w(0,t)\,, \tag{12.55}$$
$$w_t(x,t) = w_x(x,t)\,, \tag{12.56}$$
$$w(D,t) = 0\,. \tag{12.57}$$

The transformation

$$(Z,u) \mapsto (h,w) \tag{12.58}$$

is defined by

$$h_1(t) = Z_1(t) + \frac{1}{3}Z_2^3(t)\,, \tag{12.59}$$
$$h_2(t) = Z_2(t)\,, \tag{12.60}$$

$$w(x,t) = u(x,t) + \int_0^x (2+x-y)u(y,t)dy$$
$$+ Z_1(t) + (2+x)Z_2(t) + \frac{1}{3}Z_2^3(t)\,, \tag{12.61}$$

whereas the transformation

$$(h,w) \mapsto (Z,u) \tag{12.62}$$

is defined by

$$Z_1(t) = h_1(t) - \frac{1}{3}h_2^3(t)\,, \tag{12.63}$$
$$Z_2(t) = h_2(t)\,, \tag{12.64}$$
$$u(x,t) = w(x,t) - \int_0^x (2-x+y)\mathrm{e}^{-(x-y)}w(y,t)dy$$
$$- \mathrm{e}^{-x}\left[(1-x)h_1(t) + (2-x)h_2(t)\right]. \tag{12.65}$$

With these simple transformations, one can not only get an easy estimate of the function $\beta(\cdot,\cdot)$ in (12.48), but one can find explicit solutions of the closed-loop system consisting of the nonlinear plant (12.50)–(12.53) and the infinite-dimensional controller (12.49). This is done by solving the linear ODE-PDE system (12.54)–(12.57), for an initial condition defined with the help of the transformation (12.59)–(12.61), and finally substituted into the transformation (12.63)–(12.65) to obtain the explicit solution for $(Z(t),u(x,t))$.

12.2 General Strict-Feedforward Nonlinear Systems: Integrator Forwarding

Consider the class of *strict-feedforward systems*

$$\dot{Z}_1(t) = Z_2(t) + \psi_1(Z_2(t), Z_3(t), \ldots, Z_n(t))$$
$$+ \phi_1(Z_2(t), Z_3(t), \ldots, Z_n(t))u(0,t)\,, \tag{12.66}$$
$$\dot{Z}_2(t) = Z_3(t) + \psi_2(Z_3(t), \ldots, Z_n(t)) + \phi_2(Z_3(t), \ldots, Z_n(t))u(0,t)\,, \tag{12.67}$$
$$\vdots$$
$$\dot{Z}_{n-2}(t) = Z_{n-1}(t) + \psi_{n-2}(Z_{n-1}(t), Z_n(t)) + \phi_{n-2}(Z_{n-1}(t), Z_n(t))u(0,t)\,, \tag{12.68}$$
$$\dot{Z}_{n-1}(t) = Z_n(t) + \phi_{n-1}(Z_n(t))u(0,t)\,, \tag{12.69}$$
$$\dot{Z}_n(t) = u(0,t)\,, \tag{12.70}$$

with input delay

$$u_t(x,t) = u_x(x,t)\,, \tag{12.71}$$
$$u(D,t) = U(t)\,, \tag{12.72}$$

$$U(t-D) = u(0,t)\,, \tag{12.73}$$

or, for short,

$$\dot{Z}_i(t) = Z_{i+1}(t) + \psi_i(\underline{Z}_{i+1}(t)) + \phi_i(\underline{Z}_{i+1}(t))U(t-D)\,, \qquad i = 1,2,\ldots,n\,, \tag{12.74}$$

where

$$\underline{Z}_j = \begin{bmatrix} Z_j \\ Z_{j+1} \\ \vdots \\ Z_n \end{bmatrix}\,, \tag{12.75}$$

$$Z_{n+1}(t) = U(t-D)\,, \tag{12.76}$$

$$\phi_n = 1\,, \tag{12.77}$$

$$\phi_i(0) = 0\,, \tag{12.78}$$

$$\psi_i(Z_{i+1},0,\ldots,0) \equiv 0\,, \tag{12.79}$$

$$\frac{\partial \psi_i(0)}{\partial Z_j} = 0 \tag{12.80}$$

for $i = 1,2,\ldots,n-1$, $j = i+1,\ldots,n$.

In Appendices F, G, and H we provide an extensive overview of design procedures based on "integrator forwarding," a dual of "integrator backstepping." In Appendix F we overview the basic designs for the general class of strict-feedforward systems, and then in Appendix G we present explicit design procedures for two subclasses of strict-feedforward systems that are linearizable by coordinate change. Finally, in Appendix H we present explicit designs for some classes of feedforward and feedforward-like[1] systems.

The nominal *integrator forwarding* control design ($D = 0$) for the class of systems (12.74) is given by the following recursive procedure. Let

$$\vartheta_{n+1} = 0\,, \tag{12.81}$$

$$\alpha_{n+1} = 0\,. \tag{12.82}$$

For $i = n, n-1, \ldots, 2, 1$, the designer needs to symbolically (preferably) or numerically calculate

$$h_i(Z_i,\ldots,Z_n) = Z_i - \vartheta_{i+1}\left(Z_{i+1},\ldots,Z_n\right)\,, \tag{12.83}$$

$$\omega_i(\underline{Z}_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial \vartheta_{i+1}}{\partial Z_j}\phi_j - \frac{\partial \vartheta_{i+1}}{\partial Z_n}\,, \tag{12.84}$$

$$\alpha_i(\underline{Z}_i) = \alpha_{i+1} - \omega_i h_i\,, \tag{12.85}$$

[1] Block-feedforward and interlaced feedforward-feedback systems.

$$\vartheta_i(\underline{Z}_i) = -\int_0^\infty \left[\xi_i^{[i]}(\tau,\underline{Z}_i) + \psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau,\underline{Z}_i)\right) + \phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau,\underline{Z}_i)\right)\alpha_i\left(\underline{\xi}_i^{[i]}(\tau,\underline{Z}_i)\right)\right] d\tau\,, \tag{12.86}$$

where the notation in the integrand of (12.86) refers to the solutions of the (sub) system(s)

$$\frac{d}{d\tau}\xi_j^{[i]} = \xi_{j+1}^{[i]} + \psi_j\left(\underline{\xi}_{j+1}^{[i]}\right) + \phi_j\left(\underline{\xi}_{j+1}^{[i]}\right)\alpha_i\left(\underline{\xi}_i^{[i]}\right) \tag{12.87}$$

for $j = i, i+1, \ldots, n$, at time τ, starting from the initial condition $\underline{X}_i$.

The control law for $D = 0$ is given by

$$U(t) = \alpha_1(Z(t))\,. \tag{12.88}$$

It is important to understand the meaning of the integral in (12.86). Clearly, the solution $\underline{\xi}_i(\tau,\underline{Z}_i)$ is impossible to obtain analytically in general but, when possible, will lead to an implementable control law. Note that the last of the ϑ_i's that need to be computed is ϑ_2 (ϑ_1 is not defined).

12.3 Predictor for Strict-Feedforward Systems

As in the case of general nonlinear systems (Section 11.1), the predictor-based feedback law is obtained from (12.88) as

$$U(t) = \alpha_1(P(t)) = \alpha_1(p(D,t))\,, \tag{12.89}$$

where the predictor variable $p(D,t) = P(t)$ is defined next. Consider the ODE (in x) given by

$$\frac{\partial}{\partial x}p_1(x,t) = p_2(x,t) + \psi_1(p_2(x,t), p_3(x,t), \ldots, p_n(x,t)) + \phi_1(p_2(x,t), p_3(x,t), \ldots, p_n(x,t))u(x,t)\,, \tag{12.90}$$

$$\frac{\partial}{\partial x}p_2(x,t) = p_3(x,t) + \psi_2(p_3(x,t), \ldots, p_n(x,t)) + \phi_2(p_3(x,t), \ldots, p_n(x,t))u(x,t)\,, \tag{12.91}$$

$$\vdots$$

$$\frac{\partial}{\partial x}p_{n-2}(x,t) = p_{n-1}(x,t) + \psi_{n-2}(p_{n-1}(x,t), p_n(x,t)) + \phi_{n-2}(p_{n-1}(x,t), p_n(x,t))u(x,t)\,, \tag{12.92}$$

$$\frac{\partial}{\partial x}p_{n-1}(x,t) = p_n(x,t) + \phi_{n-1}(p_n(x,t))u(x,t)\,, \tag{12.93}$$

$$\frac{\partial}{\partial x} p_n(x,t) = u(x,t)\,, \tag{12.94}$$

with an initial condition

$$p_i(0,t) = Z_i(t)\,, \qquad i = 1,\ldots,n\,. \tag{12.95}$$

The set of equations for $p_i(x,t)$ can be solved explicitly, starting from the bottom,

$$p_n(x,t) = Z_n(t) + \int_0^x u(y,t)dy\,, \tag{12.96}$$

continuing on to

$$\begin{aligned} p_{n-1}(x,t) &= Z_{n-1}(t) + \int_0^x \left[p_n(y,t) + \phi_{n-1}(p_n(y,t))u(y,t)\right] dy \\ &= Z_{n-1}(t) + xZ_n(t) + \int_0^x (x-y)u(y,t)dy \\ &\quad + \int_0^x \phi_{n-1}\left(Z_n(t) + \int_0^y u(\sigma,t)ds\right) u(y,t)dy\,, \end{aligned} \tag{12.97}$$

and so on. For a general i, the predictor solution is given recursively as

$$\begin{aligned} p_i(x,t) = Z_i(t) + \int_0^x &\left[p_{i+1}(y,t) + \psi_i(p_{i+1}(y,t),\ldots,p_n(y,t))\right. \\ &\left. + \phi_i(p_{i+1}(y,t),\ldots,p_n(y,t))u(y,t)\right] dy\,. \end{aligned} \tag{12.98}$$

Clearly, this procedure involves only the computation of integrals with no implicit problems to solve (such as differential or integral equations).

Hence, the predictor state

$$p(D,t) = P(t)\,, \tag{12.99}$$

where

$$P_1(t) = \text{function of } (Z_1(t), Z_2(t), \ldots, Z_n(t)) \text{ and } U(\theta), \theta \in [0,D]\,, \tag{12.100}$$

$$P_2(t) = \text{function of } (Z_2(t), \ldots, Z_n(t)) \text{ and } U(\theta), \theta \in [0,D]\,, \tag{12.101}$$

$$\vdots$$

$$P_{n-1}(t) = \text{function of } (Z_{n-1}(t), Z_n(t)) \text{ and } U(\theta), \theta \in [0,D]\,, \tag{12.102}$$

$$P_n(t) = \text{function of } Z_n(t) \text{ and } U(\theta), \theta \in [0,D]\,, \tag{12.103}$$

is obtainable explicitly due to the strict-feedforward structure of the class of systems.

An example of an explicit design of a nonlinear infinite-dimensional predictor for a third-order system is presented in (12.181)–(12.183).

12.4 General Strict-Feedforward Nonlinear Systems: Stability Analysis

Before we start, we need to define the π-subsystem, which is used in the inverse backstepping transformation:

$$\frac{\partial}{\partial x}\pi_1(x,t) = \pi_2(x,t) + \psi_1(\pi_2(x,t),\dots,\pi_n(x,t)) \\ + \phi_1(\pi_2(x,t),\dots,\pi_n(x,t))\left(\alpha_1(\pi(x,t)) + w(x,t)\right), \tag{12.104}$$

$$\frac{\partial}{\partial x}\pi_2(x,t) = \pi_3(x,t) + \psi_2(\pi_3(x,t),\dots,\pi_n(x,t)) \\ + \phi_2(\pi_3(x,t),\dots,\pi_n(x,t))\left(\alpha_1(\pi(x,t)) + w(x,t)\right), \tag{12.105}$$

$$\vdots$$

$$\frac{\partial}{\partial x}\pi_{n-2}(x,t) = \pi_{n-1}(x,t) + \psi_{n-2}(\pi_{n-1}(x,t),\pi_n(x,t)) \\ + \phi_{n-2}(\pi_{n-1}(x,t),\pi_n(x,t))\left(\alpha_1(\pi(x,t)) + w(x,t)\right), \tag{12.106}$$

$$\frac{\partial}{\partial x}\pi_{n-1}(x,t) = \pi_n(x,t) + \phi_{n-1}(\pi_n(x,t))\left(\alpha_1(\pi(x,t)) + w(x,t)\right), \tag{12.107}$$

$$\frac{\partial}{\partial x}\pi_n(x,t) = \left(\alpha_1(\pi(x,t)) + w(x,t)\right), \tag{12.108}$$

with an initial condition

$$\pi_i(0,t) = Z_i(t), \qquad i = 1,\dots,n. \tag{12.109}$$

In our analysis we will have to employ both the norm defined by (11.75) as well as the standard $L_2[0,D]$ norm, which we denote simply as

$$\|u(t)\| = \left(\int_0^D u^2(x,t)dx\right)^{1/2}. \tag{12.110}$$

The following result follows immediately from (11.33) and (11.34).

Lemma 12.1. *If the mapping* $\kappa : \mathbb{R}^n \to \mathbb{R}$ *in (11.33), (11.34) is continuous and* $\kappa(0) = 0$, *there exists a class-*$\mathcal{K}_\infty$ *function* ρ_1 *such that*

$$\|w(t)\|^2 \le 2\|u(t)\|^2 + D\rho_1\left(\|p(t)\|_{L_\infty[0,D]}\right), \tag{12.111}$$

$$\|u(t)\|^2 \le 2\|w(t)\|^2 + D\rho_1\left(\|\pi(t)\|_{L_\infty[0,D]}\right). \tag{12.112}$$

Next, we prove the following lemma.

Lemma 12.2. *For the system*

$$\frac{\partial}{\partial x}p_i(x,t) = p_{i+1}(x,t) + \psi_i(p_{i+1}(x,t),\dots,p_n(x,t)) + \phi_i(p_{i+1}(x,t),\dots,p_n(x,t))u(x,t)\,, \qquad i=1,\dots,n-1\,, \tag{12.113}$$

$$\frac{\partial}{\partial x}p_n(x,t) = u(x,t)\,, \tag{12.114}$$

with initial condition $p(0,t)=Z(t)$, the following bounds hold:

$$|p_i(x,t)| \le |Z_i(t)| + \nu_i\left(\|p_{i+1}(t)\|_{L_\infty[0,D]}\right)(1+\|u(t)\|)\,, \quad i=1,\dots,n-1\,, \tag{12.115}$$

$$|p_n(x,t)| = |Z_n(t)| + D\|u(t)\| \tag{12.116}$$

for all $x\in[0,D]$ and all $t\ge 0$, where $\nu_i(\cdot)$ are class-$\mathcal{K}$ functions.

Proof. First, we note that

$$p_n(x,t) = Z_n(t) + \int_0^x u(y,t)dy\,, \tag{12.117}$$

from which (12.116) follows. Then we write

$$p_i(x,t) = Z_i(t) + \int_0^x \psi_i(p_{i+1}(y,t),\dots,p_n(y,t))dy + \int_0^x \phi_i(p_{i+1}(y,t),\dots,p_n(y,t))u(y,t)dy\,. \tag{12.118}$$

With the Cauchy–Schwartz inequality, we get

$$|p_i(x,t)| \le |Z_i(t)| + \int_0^x |\psi_i(p_{i+1}(y,t),\dots,p_n(y,t))|dy + \left(\int_0^D \phi_i^2(p_{i+1}(y,t),\dots,p_n(y,t))dy\right)^{1/2}\|u(t)\|\,. \tag{12.119}$$

Then, with a suitably chosen class-$\mathcal{K}$ function λ_i, we get

$$\begin{aligned}
|p_i(x,t)| &\le |Z_i(t)| + \int_0^D \lambda_i\left(\left|\underline{p}_{i+1}(y,t)\right|\right)dy \\
&\quad + \left(\int_0^D \lambda_i\left(\left|\underline{p}_{i+1}(y,t)\right|\right)dy\right)^{1/2}\|u(t)\| \\
&\le |Z_i(t)| + \int_0^D \lambda_i\left(\left\|\underline{p}_{i+1}(t)\right\|_{L_\infty[0,D]}\right)dy \\
&\quad + \left(\int_0^D \lambda_i\left(\left\|\underline{p}_{i+1}(t)\right\|_{L_\infty[0,D]}\right)dy\right)^{1/2}\|u(t)\|
\end{aligned}$$

$$\leq |Z_i(t)| + D\lambda_i \left(\left\| \underline{p}_{i+1}(t) \right\|_{L_\infty[0,D]} \right) + \left(D\lambda_i \left(\left\| \underline{p}_{i+1}(t) \right\|_{L_\infty[0,D]} \right) \right)^{1/2} \|u(t)\| . \tag{12.120}$$

Taking $\nu_i(\cdot) = \max\left\{ D\lambda_i(\cdot), \sqrt{D\lambda_i(\cdot)} \right\}$, we complete the proof of the lemma. □

By successive application of Lemma 12.2, in the order $i = n-1, n-2, \ldots, 2, 1$, we obtain the following lemma.

Lemma 12.3. *There exist class-$\mathcal{K}$ functions $\sigma_i(\cdot)$ such that*

$$|p_i(x,t)| \leq \sigma_i \left(|\underline{Z}_i(t)| + \|u(t)\| \right), \qquad i = 1, \ldots, n, \tag{12.121}$$

for all $x \in [0,D]$ and all $t \geq 0$.

From Lemma 12.3, the following result is immediate.

Lemma 12.4. *There exists a class-$\mathcal{K}$ function $\sigma^*(\cdot)$ such that*

$$\|p(t)\|_{L_\infty[0,D]} \leq \sigma^* \left(|Z(t)| + \|u(t)\| \right) \tag{12.122}$$

for all $t \geq 0$.

With the help of Lemmas 12.1 and 12.4, we obtain the following result.

Lemma 12.5. *There exists a class-$\mathcal{K}$ function $\bar{\sigma}(\cdot)$ such that*

$$|Z(t)| + \|w(t)\| \leq \bar{\sigma} \left(|Z(t)| + \|u(t)\| \right) \tag{12.123}$$

for all $t \geq 0$.

This is an important upper bound on the transformation $(Z,u) \mapsto (Z,w)$, which we will use soon. However, we also need to derive a bound on the inverse of that transformation.

Toward that end, we first prove the following result.

Lemma 12.6. *There exists a class-$\mathcal{K}$ function $\tau^*(\cdot)$ such that*

$$\|\pi(t)\|_{L_\infty[0,D]} \leq \tau^* \left(|Z(t)| + \|w(t)\| \right) \tag{12.124}$$

for all $t \geq 0$.

Proof. Consider the system (12.104)–(12.108) along with the diffeomorphic transformation

$$\zeta(t) = H(Z(t)) \tag{12.125}$$

defined by (12.81)–(12.87) as

$$H(Z(t)) = \begin{bmatrix} h_1(Z_1, Z_2, \dots, Z_n) \\ h_2(Z_2, \dots, Z_n) \\ \vdots \\ h_{n-1}(Z_{n-1}, Z_n) \\ h_n(Z_n) \end{bmatrix}. \tag{12.126}$$

Denote a transformed variable for the π-system,

$$\varepsilon(x,t) = H(\pi(x,t)). \tag{12.127}$$

With the observation that

$$Z_{i+1} + \psi_i + \phi_i \alpha_{i+1} = \sum_{j=i+1}^{n} \frac{\partial \vartheta_{i+1}}{\partial Z_j} \left(Z_{j+1} + \psi_j + \phi_j \alpha_{i+1} \right), \tag{12.128}$$

it is easy to verify that

$$\frac{\partial}{\partial x} \varepsilon_i(x,t) = \omega_i \left(\alpha_i + w(x,t) + \sum_{j=i+1}^{n} \omega_j \varepsilon_j \right). \tag{12.129}$$

Noting from (12.88) and (12.85) that

$$\alpha_1 = -\sum_{i=1}^{n} \omega_i \varepsilon_i, \tag{12.130}$$

we get

$$\frac{\partial}{\partial x} \varepsilon_i = -\omega_i^2 \varepsilon_i - \sum_{j=1}^{i-1} \omega_i \omega_j \varepsilon_j + \omega_i w(x,t) \tag{12.131}$$

[note that this notation implies that $\partial \varepsilon_1 / \partial x = -\omega_1^2 \varepsilon_1 + \omega_1 w(x,t)$]. Taking the Lyapunov function

$$\mathscr{S}(x,t) = \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i^2(x,t), \tag{12.132}$$

one obtains

$$\begin{aligned} \mathscr{S}_x(x,t) &= -\frac{1}{2} \sum_{i=1}^{n} \omega_i^2 \varepsilon_i^2 - \frac{1}{2} \left(\sum_{i=1}^{n} \varepsilon_i \omega_i \right)^2 + w(x,t) \sum_{i=1}^{n} \omega_i \varepsilon_i \\ &\le -\frac{1}{4} \sum_{i=1}^{n} \omega_i^2 \varepsilon_i^2 - \frac{1}{2} \left(\sum_{i=1}^{n} \varepsilon_i \omega_i \right)^2 + n w^2(x,t) \\ &\le -\frac{1}{4} \sum_{i=1}^{n} \omega_i^2 \varepsilon_i^2 + n w^2(x,t). \end{aligned} \tag{12.133}$$

Noting that t is being treated only as a parameter here, we obtain, by integrating in x, the following bound:

$$\mathscr{S}(x,t) \leq \mathscr{S}(0,t) + n\int_0^x w^2(y,t)dy - \frac{1}{4}\sum_{i=1}^n \int_0^x \omega_i^2 \varepsilon_i^2 dy. \tag{12.134}$$

Since

$$\mathscr{S}(0,t) = \frac{1}{2}\sum_{i=1}^n \varepsilon_i^2(0,t) = \frac{1}{2}|\varepsilon(0,t)|^2 = \frac{1}{2}|H(\pi(0,t))|^2 = \frac{1}{2}|H(Z(t))|^2, \tag{12.135}$$

we get

$$\frac{1}{2}|H(\pi(x,t))|^2 \leq \frac{1}{2}|H(Z(t))|^2 + n\int_0^D w^2(y,t)dy. \tag{12.136}$$

Due to the fact that $H(\cdot)$ is a diffeomorphism, there exists a class-$\mathscr{K}$ function $\tau^*(\cdot)$ such that

$$|\pi(x,t)| \leq \tau^*\left(|Z(t)| + \|w(t)\|\right) \tag{12.137}$$

for all $x \in [0,D]$ and all $t \geq 0$, from which the result of the lemma follows by taking a supremum in x. □

With the help of Lemmas 12.1 and 12.6, we obtain the following result.

Lemma 12.7. *There exists a class-$\mathscr{K}_\infty$ function $\bar{\sigma}(\cdot)$ such that*

$$\underline{\sigma}\left(|Z(t)| + \|u(t)\|\right) \leq |Z(t)| + \|w(t)\| \tag{12.138}$$

for all $t \geq 0$.

Now we turn our attention to the target system (Z,w) and prove the following result.

Lemma 12.8. *There exists a function $\beta_1 \in \mathscr{KL}$ such that*

$$|Z(t)| + \|w(t)\| \leq \beta_1\left(|Z(0)| + \|w(0)\|, t\right). \tag{12.139}$$

Proof. Taking a Lyapunov function

$$S(t) = \frac{1}{2}\sum_{i=1}^n \zeta_i^2(t) = \frac{1}{2}|H(Z)|^2, \tag{12.140}$$

we have

$$\dot{S}(t) \leq -\frac{1}{4}\sum_{i=1}^n \omega_i^2 \zeta_i^2 + nw^2(0,t), \tag{12.141}$$

which we proved in (12.133) using the "predictor-equivalent" of the system model

$$\dot{\zeta}_i = -\omega_i^2 \zeta_i - \sum_{j=1}^{i-1} \omega_i \omega_j \zeta_j + \omega_i w(0,t). \tag{12.142}$$

Now we introduce an overall Lyapunov function

$$V(t) = S(t) + n\int_0^D \mathrm{e}^{gx} w^2(x,t)dx\,, \tag{12.143}$$

where $g > 0$. Using (11.47), we get

$$\dot{V}(t) \le -\frac{1}{4}\sum_{i=1}^{n} \omega_i^2 \zeta_i^2 - g\int_0^D \mathrm{e}^{gx} w^2(x,t)dx\,. \tag{12.144}$$

Since the function $\sum_{i=1}^{n} \omega_i^2 \zeta_i^2$ is positive definite (though not necessarily radially unbounded) in $Z(t)$, there exists a class-$\mathscr{K}$ function $\alpha_1(\cdot)$ such that

$$\dot{V}(t) \le -\alpha_1(V(t))\,. \tag{12.145}$$

Then there exists a class-$\mathscr{KL}$ function $\beta_2(\cdot,\cdot)$ such that

$$V(t) \le \beta_2(V(0),t)\,, \qquad \forall t \ge 0\,. \tag{12.146}$$

With additional routine class-$\mathscr{K}$ calculations, one finds β_1, which completes the proof of the lemma. □

By combining Lemmas 12.5, 12.7, and 12.8, we get the following main result.

Theorem 12.2. *Consider the closed-loop system consisting of the plant (12.66)–(12.72) and controller (12.89)–(12.95). There exists a function* $\beta_3 \in \mathscr{KL}$ *such that*

$$|Z(t)| + \|u(t)\| \le \beta_3\left(|Z(0)| + \|u(0)\|, t\right)\,. \tag{12.147}$$

A slightly different and relevant way to state the same global asymptotic stability result is as follows.

Corollary 12.1. *Consider the closed-loop system consisting of the plant (12.66)–(12.72) and controller (12.89)–(12.95). Then*

$$|Z(t)| + \left(\int_{t-D}^{t} U^2(\theta)d\theta\right)^{1/2} \le \beta_3\left(|Z(0)| + \left(\int_{-D}^{0} U^2(\theta)d\theta\right)^{1/2}, t\right)\,. \tag{12.148}$$

The following result is also true.

Theorem 12.3. *The closed-loop system (12.66)–(12.72), (12.89)–(12.95) is locally exponentially stable in the sense of the norm* $|Z(t)| + \|u(t)\|$.

We leave this result without a proof as it is very much to be expected, since the linearized plant is a chain of integrators, with delay at the input, and the linearized feedback is predictor feedback of the standard form

$$U(t) = K\left(\mathrm{e}^{AD}Z(t) + \int_{t-D}^{t} \mathrm{e}^{A(t-\theta)}BU(\theta)d\theta\right)\,, \tag{12.149}$$

where $B = [0, \ldots, 0\ , 1]^T$,

$$K = [k_1,\ k_2, \ldots, k_n]\,, \tag{12.150}$$

$$k_i = -\binom{n}{i-1}\,, \tag{12.151}$$

and

$$A = \left\{a_{i,j}\right\}\,, \tag{12.152}$$

$$a_{i,j} = \begin{cases} 1, & j = i+1, \\ 0, & \text{else}. \end{cases} \tag{12.153}$$

The spectrum of the nominal system matrix, $A + BK$, is $\{-1,\ -1, \ldots, -1\}$.

12.5 Example of Predictor Design for a Third-Order System That Is Not Linearizable

To illustrate the construction of a nominal forwarding design in Section H.1, we consider the following example:

$$\dot{Z}_1 = Z_2 + Z_3^2\,, \tag{12.154}$$

$$\dot{Z}_2 = Z_3 + Z_3 U\,, \tag{12.155}$$

$$\dot{Z}_3 = U\,. \tag{12.156}$$

Then, to illustrate the predictor feedback in Section 12.3, we consider the same system with input delay, $U(t-D)$.

The second-order (Z_2, Z_3)-subsystem is linearizable (and is of both Type I and Type II, as defined in Section G.2). Like Teel's "benchmark problem" [217],

$$\dot{Z}_1 = Z_2 + Z_3^2\,, \tag{12.157}$$

$$\dot{Z}_2 = Z_3\,, \tag{12.158}$$

$$\dot{Z}_3 = U\,, \tag{12.159}$$

the overall system (12.154)–(12.156) is *not* linearizable.

While the benchmark system (12.157)–(12.159) requires only two steps of "forwarding" design because the (Z_2, Z_3)-subsystem is linear, the system (12.154)–(12.156) requires three steps. The first two steps are already precomputed in Lemma G.1, yielding

$$\xi_3 = \left(Z_3 - \tau\left(Z_2 + Z_3 - \frac{Z_3^2}{2}\right)\right)\mathrm{e}^{-\tau}\,, \tag{12.160}$$

$$\xi_2 = \left((1+\tau)\left(Z_2+Z_3-\frac{Z_3^2}{2}\right)-Z_3\right)\mathrm{e}^{-\tau} + \frac{1}{2}\left(Z_3-\tau\left(Z_2+Z_3-\frac{Z_3^2}{2}\right)\right)^2 \mathrm{e}^{-2\tau}, \tag{12.161}$$

which are then employed in

$$\tilde{\alpha}_2 = -\xi_2 - \xi_3 + \frac{\xi_3^2}{2}. \tag{12.162}$$

The third step of forwarding is about calculating (H.7),

$$\vartheta_2 = -2Z_2 - Z_3 + \frac{5}{8}Z_3^2 - \frac{3}{8}\left(Z_2 - \frac{Z_3^2}{2}\right)^2, \tag{12.163}$$

(H.8),

$$\omega_1 = 1 + \frac{3}{4}Z_3, \tag{12.164}$$

and the final control law

$$\alpha_1 = -\omega_1(Z_1 - \vartheta_2) - \left(Z_2 + Z_3 - \frac{Z_3^2}{2}\right) - Z_3, \tag{12.165}$$

i.e.,

$$U = \alpha_1 = -Z_1 - 3Z_2 - 3Z_3 - \frac{3}{8}Z_2^2 + \frac{3}{4}Z_3\left(-Z_1 - 2Z_2 + \frac{1}{2}Z_3 + \frac{Z_2Z_3}{2} + \frac{5}{8}Z_3^2 - \frac{1}{4}Z_3^3 - \frac{3}{8}\left(Z_2 - \frac{Z_3^2}{2}\right)^2\right). \tag{12.166}$$

Now we consider the same plant but with input delay:

$$\dot{Z}_1(t) = Z_2(t) + Z_3^2(t), \tag{12.167}$$

$$\dot{Z}_2(t) = Z_3(t) + Z_3(t)U(t-D), \tag{12.168}$$

$$\dot{Z}_3(t) = U(t-D). \tag{12.169}$$

The predictor feedback is obtained as

$$U(t) = \alpha_1(P(t)) = -P_1(t) - 3P_2(t) - 3P_3(t) - \frac{3}{8}P_2^2(t) + \frac{3}{4}P_3(t)\left(-P_1(t) - 2P_2(t) + \frac{1}{2}P_3(t) + \frac{P_2(t)P_3(t)}{2} + \frac{5}{8}P_3^2(t) - \frac{1}{4}P_3^3(t) - \frac{3}{8}\left(P_2(t) - \frac{P_3^2(t)}{2}\right)^2\right), \tag{12.170}$$

where the predictor $P(t)$ is determined as

$$P(t) = p(D,t) \tag{12.171}$$

from the ODE system

$$\frac{\partial}{\partial x} p_1(x,t) = p_2(x,t) + p_3^2(x,t)\,, \tag{12.172}$$

$$\frac{\partial}{\partial x} p_2(x,t) = p_3(x,t) + p_3(x,t)u(x,t)\,, \tag{12.173}$$

$$\frac{\partial}{\partial x} p_3(x,t) = u(x,t)\,, \tag{12.174}$$

with initial condition

$$p_1(0,t) = Z_1(t), \qquad p_2(0,t) = Z_2(t), \qquad p_3(0,t) = Z_3(t)\,. \tag{12.175}$$

We start the solution process from $p_3(x,t)$, obtaining

$$p_3(x,t) = Z_3(t) + \int_0^x u(y,t)dy\,. \tag{12.176}$$

Then, substituting this solution into the ODE for $p_2(x,t)$, we obtain

$$\begin{aligned} p_2(x,t) &= Z_2(t) + \int_0^x p_3(y,t)\,(1+u(y,t))\,dy \\ &= Z_2(t) + \int_0^x \left(Z_3(t) + \int_0^y u(s,t)ds\right)(1+u(y,t))\,dy \\ &= Z_2(t) + xZ_3(t) + Z_3(t)\int_0^x u(y,t)dy \\ &\quad + \int_0^x \int_0^y u(s,t)ds\,(1+u(y,t))\,dy \\ &= Z_2(t) + xZ_3(t) + Z_3(t)\int_0^x u(y,t)dy + \int_0^x (x-y)u(y,t)dy \\ &\quad + \int_0^x \int_0^y u(s,t)dsu(y,t)dy \\ &= Z_2(t) + xZ_3(t) + Z_3(t)\int_0^x u(y,t)dy + \int_0^x (x-y)u(y,t)dy \\ &\quad + \int_0^x \left(\int_0^y u(s,t)ds\right) d\left(\int_0^y u(s,t)ds\right)\,, \end{aligned} \tag{12.177}$$

finally obtaining

$$\begin{aligned} p_2(x,t) &= Z_2(t) + xZ_3(t) + Z_3(t)\int_0^x u(y,t)dy + \int_0^x (x-y)u(y,t)dy \\ &\quad + \frac{1}{2}\left(\int_0^x u(y,t)dy\right)^2. \end{aligned} \tag{12.178}$$

In the last step of the predictor derivation we calculate

$$
\begin{aligned}
p_1(x,t) &= Z_1(t) + \int_0^x p_2(y,t)dy + \int_0^x p_3^2(y,t)dy \\
&= Z_1(t) + xZ_2(t) + \frac{1}{2}x^2 Z_3(t) + Z_3(t)\int_0^x (x-y)u(y,t)dy \\
&\quad + \frac{1}{2}\int_0^x (x-y)^2 u(y,t)dy + \frac{1}{2}\int_0^x \left(\int_0^y u(s,t)ds\right)^2 dy \\
&\quad + \int_0^x \left(Z_3(t) + \int_0^y u(s,t)ds\right)^2 dy,
\end{aligned}
\tag{12.179}
$$

obtaining

$$
\begin{aligned}
p_1(x,t) = {} & Z_1(t) + xZ_2(t) + \frac{1}{2}x^2 Z_3(t) + xZ_3^2(t) \\
& + 3Z_3(t)\int_0^x (x-y)u(y,t)dy \\
& + \frac{1}{2}\int_0^x (x-y)^2 u(y,t)dy + \frac{3}{2}\int_0^x \left(\int_0^y u(s,t)ds\right)^2 dy.
\end{aligned}
\tag{12.180}
$$

From the explicit formulas for $p_1(x,t), p_2(x,t)$, and $p_3(x,t)$, we obtain explicit formulas for $P_1(t) = p_1(D,t), P_2(t) = p_2(D,t)$, and $P_3(t) = p_3(D,t)$ as

$$
\begin{aligned}
P_1(t) = {} & Z_1(t) + DZ_2(t) + \frac{1}{2}D^2 Z_3(t) + DZ_3^2(t) \\
& + 3Z_3(t)\int_{t-D}^t (t-\theta)U(\theta)d\theta \\
& + \frac{1}{2}\int_{t-D}^t (t-\theta)^2 U(\theta)d\theta + \frac{3}{2}\int_{t-D}^t \left(\int_{t-D}^\theta U(\sigma)d\sigma\right)^2 d\theta,
\end{aligned}
\tag{12.181}
$$

$$
\begin{aligned}
P_2(t) = {} & Z_2(t) + DZ_3(t) + Z_3(t)\int_{t-D}^t U(\theta)d\theta + \int_{t-D}^t (t-\theta)U(\theta)d\theta \\
& + \frac{1}{2}\left(\int_{t-D}^t U(\theta)d\theta\right)^2,
\end{aligned}
\tag{12.182}
$$

$$
P_3(t) = Z_3(t) + \int_{t-D}^t U(\theta)d\theta. \tag{12.183}
$$

Hence, the *explicit* infinite-dimensional nonlinear controller (12.170), (12.181)–(12.183) achieves global asymptotic stability of the nonlinearizable strict-feedforward system (12.167)–(12.169).

12.6 An Alternative: A Design with Nested Saturations

The predictor given in Section 12.3 can be combined with any stabilizing feedback for strict-feedforward systems. Thus, the nominal design based on integrator forwarding in Section 12.2 is not the designer's only option.

The design alternatives include the nested saturation designs [217, 219] and other designs for feedforward systems [145]. We remind the reader that a nested saturation design in the absence of delay would have the form

$$U(t) = N(Z(t)), \tag{12.184}$$

$$\begin{aligned} N(Z(t)) = -b_n\sigma\Bigg(& a_{nn}Z_n(t) \\ & + b_{n-1}\sigma\Bigg(a_{n-1,n-1}Z_{n-1}(t) + a_{n-1,n}Z_n(t) \\ & + \cdots \\ & + b_2\sigma\Bigg(a_{22}Z_2(t) + a_{2n}Z_n(t) \\ & + b_1\sigma\Bigg(a_1Z_1(t) + a_{1n}Z_n(t)\Bigg)\Bigg)\Bigg)\Bigg), \end{aligned} \tag{12.185}$$

where b_i and a_{ij} are positive constants and $\sigma(\cdot)$ is the standard unit saturation function.

It is quite crucial that the alternative stabilizing designs guarantee not only stability but also a particular form of robustness to *input disturbances* that are L_2-bounded [220], as was the case with our design in this chapter, which allowed it to be robust to the predictor error disturbance, $w(0,t)$.

It is due to this $L_2[0,+\infty)$-robustness property that the alternative designs for feedforward systems can be combined with the predictors constructed in this chapter. The predictor design is independent of the method employed to develop the nominal stabilizing controller, and it is given explicitly in Section 12.3. Thus, for example, for the nested saturation approach, the predictor-compensated feedback would be

$$U(t) = N(p(D,t)). \tag{12.186}$$

A design based on nested saturations was developed by Mazenc et al. [143] for the stabilization of strict-feedforward systems in the presence of input delay of arbitrary size. This design employs no predictor of any kind in the feedback law. Rather than compensating for the delay, this design cleverly exploits the inherent robustness to delay in the particular structure of the feedback law and the plant.

The nested saturation design in [143] has an advantage over the design in this chapter of achieving robustness to input delay without any increase of the dynamic order of the controller, while our design clearly employs an infinite-dimensional compensator.

On the other hand, the advantage of our design is that the nominal design, integrator forwarding, goes after achieving a quantifiable closed-loop performance, rather than just stability, which the predictor-based compensator maintains, whereas the nested saturation design is highly "restrained" in applying a control effort toward achieving stabilization.

Thus, a clear trade-off exists between the design in this chapter and the design in [143]. It is to be expected that the predictor-compensated integrator forwarding design here would be capable of achieving higher performance than the nested saturation design, but at the expense of higher complexity.

12.7 Extension to Nonlinear Systems with Time-Varying Input Delay

Given the result in Chapter 6 for linear systems with a time-varying input delay, the extension to the nonlinear case is fairly straightforward.

For the general class of nonlinear systems

$$\dot{Z}(t) = f(Z(t), U(\phi(t))), \tag{12.187}$$

the predictor-based control law is given by

$$U(t) = \kappa(P(t)), \tag{12.188}$$

where the predictor is defined by the integral equation

$$P(t) = \int_{\phi(t)}^{t} \frac{1}{\phi'(\phi^{-1}(\theta))} f(P(\theta), U(\theta)) d\theta + Z(t). \tag{12.189}$$

The entire theory from Chapter 11 for forward-complete systems extends to the case of time-varying delays under Assumptions 6.1 and 6.2 on $\phi(t)$, namely, under the assumption that the delay is strictly positive and uniformly bounded from above, and that the delay *rate* is strictly smaller than 1 and uniformly bounded from below. This extension is developed with the help of

$$p(x,t) = P\left(\phi\left(t + x\left(\phi^{-1}(t) - t\right)\right)\right), \tag{12.190}$$

where $x \in [0,1]$ and where $p(x,t)$ satisfies

$$p(x,t) = \int_0^{x\left(\phi^{-1}(t)-t\right)} f(p(\xi,t)u(\xi,t))\left(\phi^{-1}(t) - t\right) d\xi + Z(t) \tag{12.191}$$

and

$$p_t(x,t) = \pi(x,t) p_x(x,t), \tag{12.192}$$

$$p(1,t) = P(t), \tag{12.193}$$

where

$$\pi(x,t) = \frac{1 + x\left(\frac{d\left(\phi^{-1}(t)\right)}{dt} - 1\right)}{\phi^{-1}(t) - t}. \tag{12.194}$$

Now we turn our attention to strict-feedforward systems with time-varying input delay:

$$\begin{aligned}
\dot{Z}_1(t) &= Z_2(t) + \psi_1(Z_2(t), Z_3(t), \ldots, Z_n(t)) \\
&\quad + \phi_1(Z_2(t), Z_3(t), \ldots, Z_n(t))U(\phi(t)), && (12.195)\\
\dot{Z}_2(t) &= Z_3(t) + \psi_2(Z_3(t), \ldots, Z_n(t)) \\
&\quad + \phi_2(Z_3(t), \ldots, Z_n(t))U(\phi(t)), && (12.196)\\
&\vdots \\
\dot{Z}_{n-2}(t) &= Z_{n-1}(t) + \psi_{n-2}(Z_{n-1}(t), Z_n(t)) \\
&\quad + \phi_{n-2}(Z_{n-1}(t), Z_n(t))U(\phi(t)), && (12.197)\\
\dot{Z}_{n-1}(t) &= Z_n(t) + \phi_{n-1}(Z_n(t))U(\phi(t)), && (12.198)\\
\dot{Z}_n(t) &= U(\phi(t)). && (12.199)
\end{aligned}$$

We employ a predictor version of the integrator forwarding or the nested saturation feedback law,

$$U(t) = \kappa(P(t)), \tag{12.200}$$

so the only question is deriving the formulas for the predictor state, in the presence of time-varying delay. For a general nonlinear system, the predictor state is defined by the vector integral equation (12.189). For the strict-feedforward class, the predictor integral equations are given by

$$\begin{aligned}
P_1(t) &= \int_{\phi(t)}^{t} \frac{1}{\phi'(\phi^{-1}(\theta))} \Big[P_2(\theta) + \psi_1(P_2(\theta), P_3(\theta), \ldots, P_n(\theta)) \\
&\quad + \phi_1(P_2(\theta), P_3(\theta), \ldots, P_n(\theta))U(\theta)\Big] d\theta + Z_1(t), && (12.201)\\
P_2(t) &= \int_{\phi(t)}^{t} \frac{1}{\phi'(\phi^{-1}(\theta))} \Big[P_3(\theta) + \psi_2(P_3(\theta), \ldots, P_n(\theta)) \\
&\quad + \phi_2(P_3(\theta), \ldots, P_n(\theta))U(\theta)\Big] d\theta + Z_2(t), && (12.202)\\
&\vdots \\
P_{n-2}(t) &= \int_{\phi(t)}^{t} \frac{1}{\phi'(\phi^{-1}(\theta))} \Big[P_{n-1}(\theta) + \psi_{n-2}(P_{n-1}(\theta), P_n(\theta)) \\
&\quad + \phi_{n-2}(P_{n-1}(\theta), P_n(\theta))U(\theta)\Big] d\theta + Z_{n-2}(t), && (12.203)
\end{aligned}$$

$$P_{n-1}(t) = \int_{\phi(t)}^{t} \frac{1}{\phi'\left(\phi^{-1}(\theta)\right)} \Big[P_n(\theta) + \phi_{n-1}(P_n(\theta))U(\theta)\Big] d\theta + Z_{n-1}(t)\,, \quad (12.204)$$

$$P_n(t) = \int_{\phi(t)}^{t} \frac{1}{\phi'\left(\phi^{-1}(\theta)\right)} U(\theta) d\theta + Z_n(t)\,. \quad (12.205)$$

The predictor variables $P_1(t), P_2(t), \ldots, P_n(t)$ can be expressed explicitly in terms of $Z_1(t), Z_2(t), \ldots, Z_n(t)$ and $U(\theta), \theta \in [\phi(t), t]$, in the following manner. First, one substitutes the solution for $P_n(\theta)$ from (12.205) into (12.204); then one substitutes the solutions for $P_{n-1}(\theta)$ and $P_n(\theta)$ from (12.204) and (12.205) into (12.203), and so on.

12.8 Notes and References

In the world of recursive control designs for nonlinear systems, two basic classes of systems are the most easily recognizable—the systems with (strict-)feed*back* structure and the systems with (strict-)feed*forward* structure. The strict-feedback systems, which occupied the attention of the nonlinear control community in the first half of the 1990s, are controlled using *backstepping*, a method that employs aggressive controls[2] necessary to suppress finite-escape instabilities inherent (in an open loop) to strict-feedback systems. In contrast, the strict-feedforward systems, which were studied intensively in the mid- and late-1990s, can be only marginally unstable in an open loop[3] and permit (and in many cases call for) cautious controllers.

The theoretical foundation of how to exercise "caution" in the control design for feedforward systems was laid by Teel in his 1992 dissertation [217], where he introduced the technique of nested saturations whose parameters are carefully selected to essentially achieve robustness of linear controllers to nonlinearities (of superlinear and other types). Soon after this first design, Teel [219] developed a series of results that, among other things, interpreted and generalized [217] in the light of nonlinear small-gain techniques that he developed in [219].

The next major spurt of progress on feedforward systems came with the paper of Mazenc and Praly [145], which introduced a Lyapunov approach for stabilization of feedforward systems. This approach, initially conceived in March 1993, has roots that go farther back to Praly's 1991–1992 designs for adaptive nonlinear control [189] and output feedback stabilization [190], where he was designing forwarding-like coordinate changes involving a stable manifold that can be written as a graph of a function. A related idea was used by Sontag and Sussmann [208] for stabilization of linear systems with saturated controls. Subsequently, Praly, Ortega, and Kaliora [192] relaxed the conditions under which such manifolds can be found.

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[2] As measured by the growth of their nonlinearities.

[3] With solutions growing only polynomially in time—if the nonlinearities are polynomial—or up to exponentially in time, if the nonlinearities are exponential.

Jankovic, Sepulchre, and Kokotovic [74] developed a different Lyapunov solution to the problem of forwarding (and stabilization of a broad class of cascade systems), which, rather than a coordinate change or domination of (certain) “cross terms” (as Mazenc and Praly), employs an exact cross term in the Lyapunov function. In [195] they presented an algorithmic, inverse optimal design for a class of feedforward systems and provided a detailed insight into the structure of the target system in the forwarding recursion.

Further developments on feedforward systems have gone in several directions. The nested saturation ideas have been expanded upon by Lin and Li [124], Arcak, Teel, and Kokotovic [7], Marconi and Isidori [136], and Xudong [241]. Implicit (or explicit) in the first three papers are robustness results with respect to certain classes of unmodeled dynamics. The Lyapunov approach has been developed further by Sepulchre, Jankovic, and Kokotovic [195, 196], Mazenc, Sepulchre, and Jankovic [148], and Mazenc and Praly [147]. Lin and Qian [125] proposed designs for systems satisfying certain growth conditions.

In [220] Teel designed L_2-stabilizing controllers for feedforward systems (L_∞-disturbance attenuation, while impossible in general, remains a problem of interest for subclasses of feedforward systems). Trajectory tracking, while hard to achieve for arbitrary trajectories, has been solved under reasonable conditions by Mazenc and Praly [146] and Mazenc and Bowong [141]. Extensions to nonlinear integrator chains have been proposed by Mazenc [138] and Tsinias and Tzamtzi [222]. Even a generalization to feedforward systems with exponentially unstable linearizations has been reported by Grognard, Sepulchre, and Bastin [57]. Discrete-time feedforward systems have also been studied in a recent paper by Mazenc and Nijmeijer [144]. The linear low-gain semiglobal stabilization of feedforward systems was proposed by Grognard, Bastin, Sepulchre, and Praly [58]. An output feedback problem for feedforward systems was recently solved by Mazenc and Vivalda [149]. Feedforward systems do not lend themselves easily to adaptive control—one related result is by Jankovic, Sepulchre, and Kokotovic [75]. Nonparametric robust control, i.e., disturbance attenuation in the style of [109] (for example) with disturbances entering through a nonlinear vector field, has so far remained intractable (except when the vector field is constant).

Starting with Teel’s original interest in the ball-and-beam problem [217] and Mazenc and Praly’s design for the pendulum-cart problem [145], the research on forwarding has continuously been driven by applications. The following papers on forwarding are fully (or almost fully) dedicated to applications: Spong and Praly [209] (pole-cart), Barbu, Sepulchre, Lin, and Kokotovic [16] (ball-and-beam), Albouy and Praly [2] (spherical inverted pendulum), Praly, Ortega, and Kaliora [192] (inverted pendulum with disk inertia), Mazenc and Bowong [140] (pendulum-cart), and Praly [191] (satellite orbit transfer with weak but continuous thrust).

The differential geometric characterization of feedforward systems had eluded researchers until Tall and Respondek [211] solved this problem.

As we have seen in this chapter, strict-feedforward systems occupy a special place among nonlinear systems stabilizable by predictor feedback because the predictor can be written explicitly. When the nonlinearities in the strict-feedforward plant are polynomial, then the predictor $P(t)$ and the control $U(t)$ can both be written as finite Volterra series in $U(\theta), \theta \in [t-D,t]$, with polynomial dependence on $(Z_1(t), Z_2(t), \ldots, Z_n(t))$.

Chapter 13
Linearizable Strict-Feedforward Systems

Most strict-feedforward systems are *not* feedback linearizable; however, a small class of strict-feedforward systems is linearizable, and, in fact, it is linearizable by coordinate change alone, without the use of feedback.

In this chapter we review the conditions for the linearizability of strict-feedforward systems, present a control algorithm that results in explicit formulas for control laws, present formulas for predictor feedbacks that compensate for actuator delays (which happen to be nonlinear in the ODE state but linear in the distributed actuator state), derive formulas for closed-loop solutions in the presence of actuator delay, and, finally, present a few examples of third-order linearizable strict-feedforward systems.

We start the chapter by introducing a characterization of linearizable strict-feedforward systems in Section 13.1. This is a special subclass of strict-feedforward systems; it is a rather narrow subclass but is useful for pedagogical purposes because it lends itself to analytical treatment due to linearizability.

In Section 13.2 we specialize the integrator forwarding algorithm to linearizable strict-feedforward systems. For this subclass, the forwarding feedback laws are obtained explicitly.

In Section 13.3 we introduce two sets of linearizing coordinate changes for linearizable strict-feedforward systems. One coordinate change takes the system into the classical Brunovsky form and the other into a special form that we refer to as the "Teel form," a lower-triangular linear form with -1s along the diagonal.

In Section 13.4 we derive the formulas for the predictor variables for linearizable strict-feedforward systems. These formulas are nonlinear in the finite-dimensional plant state but are linear in the distributed state of the actuator.

In Section 13.5 we present explicit solutions to the closed-loop system with predictor feedback for linearizable strict-feedforward systems.

Finally, in Section 13.6 we discuss examples of linearizable strict-feedforward systems, focusing on the three-dimensional case.

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13.1 Linearizable Strict-Feedforward Systems

In Section G.1 we explain that a strict-feedforward system (without delay),

$$\dot{Z}_i = Z_{i+1} + \psi_i(\underline{Z}_{i+1}) + \phi_i(\underline{Z}_{i+1})U\,, \qquad i = 1,2,\ldots,n\,, \tag{13.1}$$

is linearizable provided the following assumption is satisfied.

Assumption 13.1. *The functions $\psi_i(\underline{Z}_{i+1}), \phi_i(\underline{Z}_{i+1})$ can be written as*

$$\phi_{n-1}(Z_n) = \theta_n'(Z_n)\,, \tag{13.2}$$

$$\psi_{n-1}(Z_n) = 0\,, \tag{13.3}$$

and

$$\phi_i(\underline{Z}_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(\underline{Z}_{i+1})}{\partial Z_j} \phi_j(\underline{Z}_{j+1}) + \frac{\partial \theta_{i+1}(\underline{Z}_{i+1})}{\partial Z_n}\,, \tag{13.4}$$

$$\psi_i(\underline{Z}_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(\underline{Z}_{i+1})}{\partial Z_j} \left(Z_{j+1} + \psi_j(\underline{Z}_{j+1})\right) - \theta_{i+2}(\underline{Z}_{i+2}) \tag{13.5}$$

for $i = n-2,\ldots,1$, using some C^1 scalar-valued functions $\theta_i(\underline{Z}_i)$ satisfying

$$\theta_i(0) = \frac{\partial \theta_i(0)}{\partial Z_j} = 0\,, \qquad i = 2,\ldots,n, \quad j = i,\ldots,n\,. \tag{13.6}$$

If Assumption 13.1 is satisfied, then the functions $\theta_i(\underline{Z}_i)$ are used in the diffeomorphism

$$h_i = Z_i - \theta_{i+1}(\underline{Z}_{i+1})\,, \qquad i = 1,\ldots,n-1\,, \tag{13.7}$$

$$h_n = Z_n\,, \tag{13.8}$$

for transforming the strict-feedforward system (13.1) into a system of the "chain of integrators" form

$$\dot{h}_i = h_{i+1}\,, \qquad i = 1,2,\ldots,n-1\,, \tag{13.9}$$

$$\dot{h}_n = U\,. \tag{13.10}$$

13.2 Integrator Forwarding (SJK) Algorithm Applied to Linearizable Strict-Feedforward Systems

The general control design algorithm for linearizable strict-feedforward systems starts with

$$\vartheta_{n+1} = 0\,, \tag{13.11}$$

$$\alpha_{n+1} = 0 \tag{13.12}$$

and continues recursively, for $i = n, n-1, \ldots, 2, 1$, as

$$\alpha_i(\underline{Z}_i) = -\sum_{j=i}^{n} \left(Z_j - \vartheta_{j+1}(\underline{Z}_{j+1})\right)\,, \tag{13.13}$$

$$\xi_n^{[i]}(\tau, \underline{Z}_i) = \mathrm{e}^{-\tau} \sum_{k=0}^{n-i} \frac{(-\tau)^k}{k!} \left(Z_{n-k} - \vartheta_{n-k+1}(\underline{Z}_{n-k+1})\right)\,, \tag{13.14}$$

$$\xi_j^{[i]}(\tau, \underline{Z}_i) = \mathrm{e}^{-\tau} \sum_{k=0}^{j-i} \frac{(-\tau)^k}{k!} \left(Z_{j-k} - \vartheta_{j-k+1}(\underline{Z}_{j-k+1})\right) + \vartheta_{j+1}\left(\xi_{j+1}^{[i]}(\tau, \underline{Z}_i)\right)\,,$$

$$j = n-1, \ldots, i+1, i\,, \tag{13.15}$$

$$\begin{aligned} \vartheta_i(\underline{Z}_i) = -\int_0^\infty \Big[& \xi_i^{[i]}(\tau, \underline{Z}_i) + \psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{Z}_i)\right) \\ & + \phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{Z}_i)\right) \alpha_i\left(\underline{\xi}_i^{[i]}(\tau, \underline{Z}_i)\right) \Big] d\tau\,. \end{aligned} \tag{13.16}$$

The control law is

$$U = \alpha_1(Z)\,. \tag{13.17}$$

13.3 Two Sets of Linearizing Coordinates

There are two sets of linearizing coordinates, one given by

$$\zeta_i = Z_i - \vartheta_{i+1}\left(\underline{Z}_{i+1}\right)\,, \tag{13.18}$$

which, with the control law

$$U = \alpha_1(Z) = -\zeta_1 - \zeta_2 - \cdots - \zeta_n\,, \tag{13.19}$$

yields the closed-loop system (in the "Teel form" [218])

$$\dot{\zeta} = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & & \vdots \\ \vdots & -1 & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & -1 \end{bmatrix} \zeta\,, \tag{13.20}$$

and the other set of coordinates given by

$$h_i = \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} \zeta_j \,, \tag{13.21}$$

which, with the control law

$$U = \alpha_1(Z) = -\sum_{i=1}^{n} \binom{n}{i-1} h_i \,, \tag{13.22}$$

yields the closed-loop system (in the Brunovsky form)

$$\dot{h}_i = h_{i+1} \,, \qquad i = 1, 2, \ldots, n-1 \,, \tag{13.23}$$

$$\dot{h}_n = -\sum_{i=1}^{n} \binom{n}{i-1} h_i \,. \tag{13.24}$$

Both the ζ-coordinates and the h-coordinates have a useful purpose, as we shall see when we study the system in the presence of actuator delay.

Two specific classes of linearizable strict-feedforward systems (Types I and II) are identified in Section G.2.

13.4 Predictor Feedback for Linearizable Strict-Feedforward Systems

Now we consider the system with actuator delay,

$$\dot{Z}_i(t) = Z_{i+1}(t) + \psi_i(\underline{Z}_{i+1}(t)) + \phi_i(\underline{Z}_{i+1}(t))u(0,t) \,, \quad i = 1, 2, \ldots, n \,, \tag{13.25}$$

$$u_t(x,t) = u_x(x,t) \,, \tag{13.26}$$

$$u(D,t) = U(t) \,. \tag{13.27}$$

With the diffeomorphic transformation $G : Z \mapsto \zeta \mapsto h$, i.e.,

$$h = G(Z) \,, \tag{13.28}$$

which is recursively given by (the initial step)

$$h_n = Z_n, \tag{13.29}$$

and (the subsequent iterates)

$$h_i = \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} \left[Z_j - \vartheta_{j+1}\left(\underline{Z}_{j+1}\right) \right] \,, \qquad i = n-1, \ldots, 2, 1 \,, \tag{13.30}$$

we get the system

$$\dot{h}_i(t) = h_{i+1}(t)\,, \qquad i = 1,2,\ldots,n-1\,, \tag{13.31}$$
$$\dot{h}_n(t) = u(0,t)\,, \tag{13.32}$$
$$u_t(x,t) = u_x(x,t)\,, \tag{13.33}$$
$$u(D,t) = U(t)\,, \tag{13.34}$$

which is a cascade of a delay line and a chain of integrators. The predictor feedback design for this system is easy.

Denote by $\eta(x,t)$ the state of the system

$$\frac{\partial}{\partial x}\eta_i(x,t) = \eta_{i+1}(x,t)\,, \qquad i = 1,2,\ldots,n-1\,, \tag{13.35}$$
$$\frac{\partial}{\partial x}\eta_i(x,t) = u(x,t)\,, \tag{13.36}$$

with initial condition

$$\eta(0,t) = h(t)\,. \tag{13.37}$$

The predictor feedback is given as

$$\begin{aligned} U(t) &= \alpha_1\left(G^{-1}(\eta(D,t))\right) \\ &= -\sum_{i=1}^{n}\binom{n}{i-1}\eta_i(D,t)\,. \end{aligned} \tag{13.38}$$

Fortunately, the η-system can be solved explicitly. Its solution is

$$\eta_i(x,t) = \sum_{j=i}^{n}\frac{x^{j-i}}{(j-i)!}h_j(t) + \int_0^x \frac{(x-y)^{n+1-i}}{(n+1-i)!}u(y,t)dy\,, \tag{13.39}$$

so the “predictor” $\eta(D,t)$ is obtained as

$$\eta_i(D,t) = \sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}h_j(t) + \int_0^D \frac{(D-y)^{n+1-i}}{(n+1-i)!}u(y,t)dy\,. \tag{13.40}$$

Substituting the transformation $G : Z \mapsto \zeta \mapsto h$, we get the predictor

$$\begin{aligned} \eta_i(D,t) = {}& \sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}\sum_{l=j}^{n}\binom{n-j}{l-j}(-1)^{l-j}\left(Z_l(t) - \vartheta_{l+1}\left(\underline{Z}_{l+1}(t)\right)\right) \\ & + \int_0^D \frac{(D-y)^{n+1-i}}{(n+1-i)!}u(y,t)dy\,. \end{aligned} \tag{13.41}$$

Plugging this predictor into the predictor feedback law, we get the feedback law explicitly:

$$
\begin{aligned}
U(t) &= \alpha_1\left(G^{-1}(\eta(D,t))\right) \\
&= -\sum_{i=1}^{n}\begin{pmatrix} n \\ i-1 \end{pmatrix}\left[\sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}\sum_{l=j}^{n}\begin{pmatrix} n-j \\ l-j \end{pmatrix}(-1)^{l-j}\left(Z_l(t)-\vartheta_{l+1}\left(\underline{Z}_{l+1}(t)\right)\right)\right. \\
&\quad \left. + \int_0^D \frac{(D-y)^{n+1-i}}{(n+1-i)!}u(y,t)dy\right] . \qquad (13.42)
\end{aligned}
$$

Replacing $u(y,t)$ by $U(t+y-D)$, we finally get

$$
\begin{aligned}
U(t) &= \alpha_1\left(G^{-1}(\eta(D,t))\right) \\
&= -\sum_{i=1}^{n}\begin{pmatrix} n \\ i-1 \end{pmatrix}\left[\sum_{j=i}^{n}\frac{D^{j-i}}{(j-i)!}\sum_{l=j}^{n}\begin{pmatrix} n-j \\ l-j \end{pmatrix}(-1)^{l-j}\left(Z_l(t)-\vartheta_{l+1}\left(\underline{Z}_{l+1}(t)\right)\right)\right. \\
&\quad \left. + \int_{t-D}^{t} \frac{(t-\theta)^{n+1-i}}{(n+1-i)!}U(\theta)d\theta\right] . \qquad (13.43)
\end{aligned}
$$

This feedback law is linear in the infinite-dimensional delay state $U(\theta)$, but non-linear in the ODE plant state $Z(t)$.

The infinite-dimensional backstepping transformation is

$$w(x,t) = u(x,t) - \alpha_1\left(G^{-1}(\eta(x,t))\right) . \qquad (13.44)$$

The inverse transformation is defined as

$$u(x,t) = w(x,t) + \alpha_1\left(G^{-1}(\varpi(x,t))\right) , \qquad (13.45)$$

where

$$\frac{\partial}{\partial x}\varpi_1(x,t) = -\varpi_1(x,t) + w(x,t), \qquad (13.46)$$

$$\frac{\partial}{\partial x}\varpi_i(x,t) = -\sum_{j=1}^{i}\varpi_j(x,t) + w(x,t), \qquad i = 2,\dots,n, \qquad (13.47)$$

with initial condition

$$\varpi(0,t) = \zeta(t) . \qquad (13.48)$$

Remark 13.1. In addition to finding the control law explicitly, the closed-loop solutions can also be found explicitly. This is done by noting that, with the predictor feedback, the closed-loop system in the (ζ, w) variables is

$$\dot{\zeta}_1(t) = -\zeta_1(t) + w(0,t) , \qquad (13.49)$$

$$\dot{\zeta}_i(t) = -\sum_{j=1}^{i}\zeta_j(t) + w(0,t) , \qquad i = 2,\dots,n, \qquad (13.50)$$

$$w_t(x,t) = w_x(x,t)\,, \tag{13.51}$$

$$w(D,t) = 0\,. \tag{13.52}$$

This system is solved explicitly in the following order:

$$w(x,t) \to \zeta_1(t) \to \zeta_2(t) \to \cdots \to \zeta_n(t)\,. \tag{13.53}$$

Before using this solution, one converts the initial condition (Z_0, u_0) into (ζ_0, w_0), while going through the system ϖ. Once the solution (ζ, w) is obtained, it is converted back to the (Z, u) variables using the transformations presented above, while going through the system η.

The proof of stability for the general design in this section for linearizable strict-feedforward systems proceeds in a similar manner as for general strict-feedforward systems, except that a few of the steps can be completed explicitly or more directly. In the end, the following result is obtained.

Theorem 13.1. *Consider the closed-loop system consisting of the plant (13.25)–(13.27) under Assumption 13.1 and controller (13.43). There exists a class-$\mathscr{KL}$ function $\beta_4(\cdot,\cdot)$ such that*

$$|Z(t)|^2 + \int_{t-D}^{t} U^2(\theta)d\theta \le \beta_4\left(|Z(0)|^2 + \int_{-D}^{0} U^2(\theta)d\theta, t\right)\,. \tag{13.54}$$

13.5 Explicit Closed-Loop Solutions for Linearizable Strict-Feedforward Systems

For linearizable strict-feedforward systems, one can find the closed-loop solutions, and one can do so in a manner that is even simpler than the general idea using the linearizing transformations in Remark 13.1.

Over the time interval $t \in [0, D]$, one would use the "open-loop" linear model

$$\dot h_i(t) = h_{i+1}(t)\,, \qquad i = 1, 2, \ldots, n-1\,, \tag{13.55}$$

$$\dot h_n(t) = U(t-D)\,, \tag{13.56}$$

whereas over the time interval $t \ge D$, one would use the "closed-loop" model

$$\dot h_i(t) = h_{i+1}(t)\,, \qquad i = 1, 2, \ldots, n-1\,, \tag{13.57}$$

$$\dot h_n(t) = -\sum_{i=1}^{n} \binom{n}{i-1} h_i(t)\,, \tag{13.58}$$

where the delay has been completely compensated.

For the time period $t \in [0, D]$, we obtain

$$h(t) = \sum_{j=i}^{n} \frac{t^{j-i}}{(j-i)!} \sum_{l=j}^{n} \binom{n-j}{l-j} (-1)^{l-j} \left(Z_l(0) - \vartheta_{l+1}\left(\underline{Z}_{l+1}(0)\right)\right) + \int_{-D}^{t-D} \frac{(t-D-\theta)^{n+1-i}}{(n+1-i)!} U(\theta) d\theta \,, \tag{13.59}$$

whereas for the time period $t \geq D$, we get

$$h(t) = \mathrm{e}^{(\bar{A}+\bar{B}\bar{K})(t-D)} h(D) \,, \tag{13.60}$$

where $\bar{B} = [0, \ldots, 0\ , 1]^T$,

$$\begin{aligned} &\bar{K} = [\bar{k}_1,\, \bar{k}_2, \ldots, \bar{k}_n] \,, && \bar{A} = \left\{a_{i,j}\right\} \,, \\ &\bar{k}_i = -\binom{n}{i-1} \,, && \bar{a}_{i,j} = \begin{cases} 1, & j = i+1 \,, \\ 0, & \text{else} \,, \end{cases} \end{aligned} \tag{13.61}$$

and

$$h(D) = \sum_{j=i}^{n} \frac{D^{j-i}}{(j-i)!} \sum_{l=j}^{n} \binom{n-j}{l-j} (-1)^{l-j} \left(Z_l(0) - \vartheta_{l+1}\left(\underline{Z}_{l+1}(0)\right)\right) + \int_{-D}^{0} \frac{(-\theta)^{n+1-i}}{(n+1-i)!} U(\theta) d\theta \,. \tag{13.62}$$

To summarize our construction, we state the following result.

Theorem 13.2. *Consider the closed-loop system consisting of the plant (13.25)–(13.27) under Assumption 13.1 and controller (13.43). The closed-loop solution is given by*

$$Z(t) = G^{-1}(h(t)) \,, \tag{13.63}$$

where $h(t)$ is given by (13.59) for $t \in [0, D]$ and by (13.60), (13.62) for $t \geq D$.

To illustrate our construction of closed-loop solutions, we present the next example.

Example 13.1. We return to the example

$$\dot{Z}_1(t) = Z_2(t) - Z_2^2(t) U(t-D) \,, \tag{13.64}$$

$$\dot{Z}_2(t) = U(t-D) \tag{13.65}$$

from Section 12.1. We will now calculate the explicit solution for this system in a closed loop with feedback

$$U(t) = -Z_1(t) - (2+D) Z_2(t) - \frac{1}{3} Z_2^3(t) - \int_{t-D}^{t} (2+t-\theta) U(\theta) d\theta \,. \tag{13.66}$$

For simplicity of calculations, we will assume that the initial actuator state is zero, namely, $U(\theta)=0, \theta \in [-D,0]$. Over the time interval $t \in [0,D]$, the solution $Z(t)$ is given by

$$Z_1(t) = Z_1(0) + tZ_2(0)\,, \tag{13.67}$$

$$Z_2(t) = Z_2(0)\,. \tag{13.68}$$

To find the solution for $t \geq D$, we recall the linearizing transformation for this example:

$$h_1(t) = Z_1(t) + \frac{1}{3}Z_2^3(t)\,, \tag{13.69}$$

$$h_2(t) = Z_2(t)\,. \tag{13.70}$$

The resulting equations for $t \geq D$,

$$\dot{h}_1 = h_2\,, \tag{13.71}$$

$$\dot{h}_2 = -h_1 - h_2\,, \tag{13.72}$$

can be solved as

$$h_1(t) = \mathrm{e}^{-(t-D)}\left[(1+t-D)h_1(D) + (t-D)h_2(D)\right]\,, \tag{13.73}$$

$$h_2(t) = \mathrm{e}^{-(t-D)}\left[-(t-D)h_1(D) + (1-t+D)h_2(D)\right]\,. \tag{13.74}$$

Using the linearizing transformation, $h(D)$ is obtained as

$$h_1(D) = Z_1(0) + DZ_2(0) + \frac{1}{3}Z_2^3(0)\,, \tag{13.75}$$

$$h_2(D) = Z_2(0)\,. \tag{13.76}$$

To find the solution $Z(t)$ for $t \geq 0$, we need the inverse of the linearizing transformation:

$$Z_1(t) = h_1(t) - \frac{1}{3}h_2^3(t)\,, \tag{13.77}$$

$$Z_2(t) = h_2(t)\,. \tag{13.78}$$

By substituting $h(D)$ into $h(t)$ and then into $Z(t)$, we obtain the closed-loop solutions explicitly as

$$\begin{aligned} Z_1(t) = \mathrm{e}^{-(t-D)}\Bigg[&(1+t-D)\left(Z_1(0) + DZ_2(0) + \frac{1}{3}Z_2^3(0)\right) \\ &+ (t-D)Z_2(D)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{3}\mathrm{e}^{-3(t-D)}\left[-(t-D)\left(Z_1(0)+DZ_2(0)+\frac{1}{3}Z_2^3(0)\right)\right.\\
&\left.+(1-t+D)Z_2(D)\right]^3, \qquad (13.79)
\end{aligned}
$$

$$
\begin{aligned}
Z_2(t)=\mathrm{e}^{-(t-D)}&\left[-(t-D)\left(Z_1(0)+DZ_2(0)+\frac{1}{3}Z_2^3(0)\right)\right.\\
&\left.+(1-t+D)Z_2(D)\right] \qquad (13.80)
\end{aligned}
$$

for $t \geq D$. The closed-loop control signal is

$$
U(t)=-h_1(t+D)-2h_2(t+D)\,, \quad t\geq 0\,, \qquad (13.81)
$$

which gives

$$
U(t)=-\mathrm{e}^{-(t-D)}\left[(1-t+D)h_1(D)+(2-t+D)h_2(D)\right], \qquad (13.82)
$$

and in its final form becomes

$$
\begin{aligned}
U(t)=-\mathrm{e}^{-(t-D)}&\left[\left(Z_1(0)+(2+D)Z_2(0)+\frac{1}{3}Z_2^3(0)\right)\right.\\
&\left.-(t-D)\left(Z_1(0)+(1+D)Z_2(0)+\frac{1}{3}Z_2^3(0)\right)\right], \quad t\geq 0. \qquad (13.83)
\end{aligned}
$$

For example, if we take the initial conditions as

$$
Z_1(0)=0\,, \qquad (13.84)
$$

$$
Z_2(0)=1\,, \qquad (13.85)
$$

we obtain the closed-loop solution for $Z_1(t)$ as

$$
Z_1(t)=t\,, \qquad 0\leq t\leq D\,, \qquad (13.86)
$$

$$
\begin{aligned}
Z_1(t)=\mathrm{e}^{-(t-D)}&\left[\frac{1}{3}+D+(t-D)\left(\frac{4}{3}+D\right)\right]\\
&-\frac{1}{3}\mathrm{e}^{-3(t-D)}\left[1-(t-D)\left(\frac{4}{3}+D\right)\right]^3, \qquad t\geq D\,, \qquad (13.87)
\end{aligned}
$$

for $Z_2(t)$ as

$$
Z_2(t)=1\,, \qquad 0\leq t\leq D\,, \qquad (13.88)
$$

$$
Z_2(t)=\mathrm{e}^{-(t-D)}\left[1-(t-D)\left(\frac{4}{3}+D\right)\right], \qquad t\geq D\,, \qquad (13.89)
$$

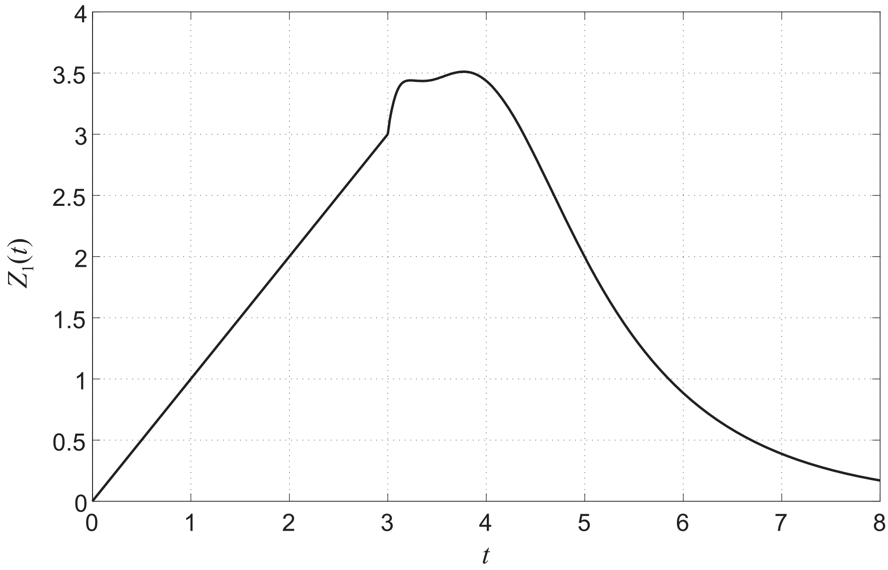

Fig. 13.1 The graph of $Z_1(t)$ from Example 13.1 for $D = 3$. Note the nonlinear transient after $t = 3$ s.

and for $U(t)$ as

$$U(t) = -\mathrm{e}^{-(t-D)}\left[\frac{7}{3} + D - (t-D)\left(\frac{4}{3} + D\right)\right], \qquad t \geq 0. \tag{13.90}$$

The plots of these solutions are given in Figs. 13.1, 13.2, and 13.3.

13.6 Examples with Linearizable Strict-Feedforward Systems

As we explain in Theorem G.7, all second-order strict-feedforward systems are linearizable. So it should be no surprise that this was the case with Example 13.1.

Hence, the lowest order in which one can discuss differences between strict-feedforward systems that are linearizable and those that are not is the third order. Since linearizability is determined through the conditions stated in Assumption 13.1, we start with the following third-order example:

$$\dot{Z}_1 = Z_2 + \cosh(Z_3) - 1, \tag{13.91}$$

$$\dot{Z}_2 = Z_3 + \sinh(Z_3)u, \tag{13.92}$$

$$\dot{Z}_3 = U, \tag{13.93}$$

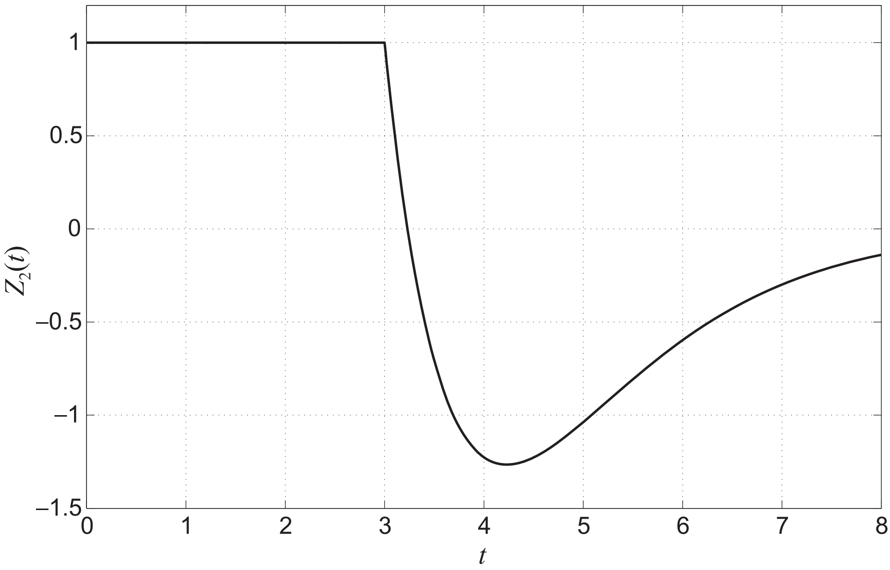

Fig. 13.2 The graph of $Z_2(t)$ from Example 13.1 for $D = 3$.

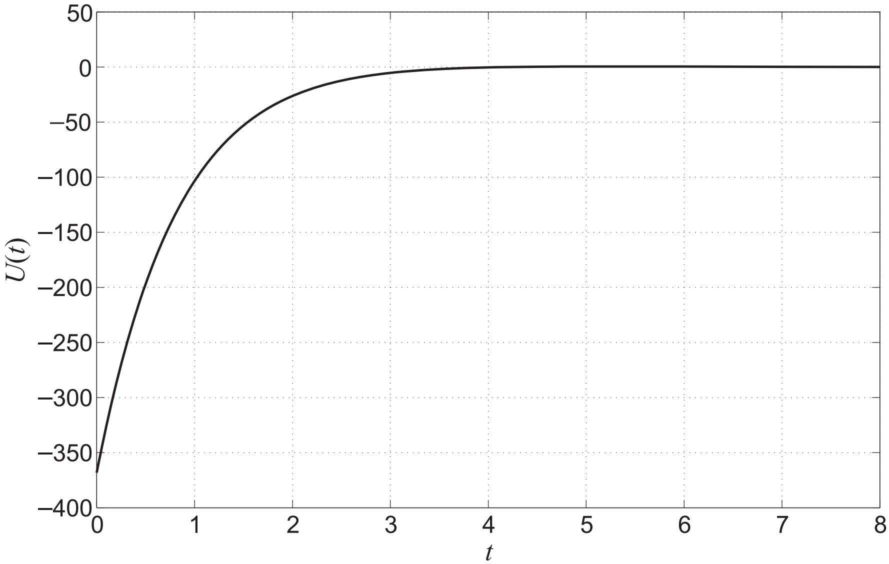

Fig. 13.3 The graph of $U(t)$ from Example 13.1. The size of the control input is due to the need to compensate for a long input delay, $D = 3$.

where

$$\theta_2(Z_2, Z_3) \equiv 0 \tag{13.94}$$

and

$$\theta_3(Z_3) = \cosh(Z_3) - 1\,, \tag{13.95}$$

which is locally quadratic. This system is linearizable using the particularly simple coordinate change

$$h_1 = Z_1\,, \tag{13.96}$$

$$h_2 = Z_2 + \cosh(Z_3) - 1\,, \tag{13.97}$$

$$h_3 = Z_3\,. \tag{13.98}$$

The following systems are also linearizable. The system

$$\dot{Z}_1 = Z_2 + Z_3^2 u\,, \tag{13.99}$$

$$\dot{Z}_2 = Z_3\,, \tag{13.100}$$

$$\dot{Z}_3 = U \tag{13.101}$$

is linearizable, as it is of both Type I and Type II (see Section G.2). The system

$$\dot{Z}_1 = Z_2 + Z_2^2 Z_3\,, \tag{13.102}$$

$$\dot{Z}_2 = Z_3\,, \tag{13.103}$$

$$\dot{Z}_3 = U \tag{13.104}$$

is of Type I and therefore linearizable.

Other such systems exist, outside Types I or II, that are linearizable. For example,

$$\dot{Z}_1 = Z_2 + Z_3^2 + Z_2 U\,, \tag{13.105}$$

$$\dot{Z}_2 = Z_3\,, \tag{13.106}$$

$$\dot{Z}_3 = U \tag{13.107}$$

(which is temptingly close in appearance to Type I but is not in that class) is linearizable using the coordinate change

$$h_1 = Z_1 - Z_2 Z_3\,, \tag{13.108}$$

$$h_2 = Z_2\,, \tag{13.109}$$

$$h_3 = Z_3\,. \tag{13.110}$$

The above examples all had the last two equations actually linear. The neither-Type-I-nor-II feedforward system

$$\dot{Z}_1 = Z_2 + Z_2^2 Z_3 + \frac{Z_3^2}{3} - Z_2^2 Z_3^2 U\,, \tag{13.111}$$

$$\dot{Z}_2 = Z_3 - Z_3^2 U\,, \tag{13.112}$$

$$\dot{Z}_3 = U\,, \tag{13.113}$$

which includes nonlinearities in both of the first two equations, is linearizable using

$$h_1 = Z_1 - \frac{Z_2^3}{3}\,, \tag{13.114}$$

$$h_2 = Z_2 + \frac{Z_3^3}{3}\,, \tag{13.115}$$

$$h_3 = Z_3\,. \tag{13.116}$$

Clearly, since the systems (13.105)–(13.107) and (13.111)–(13.113) are neither of Type I nor II, the coordinate changes (13.108)–(13.110) and (13.114)–(13.116) cannot be obtained from the explicit formulas in Section G.2. However, they can be obtained following the procedure in Section 13.4, which, we remind the reader, avoids the requirement to solve the nonlinear ODEs (12.87).

As a contrast to all of the other third-order systems in this section, we mention the celebrated "benchmark problem"

$$\dot{Z}_1 = Z_2 + Z_3^2\,, \tag{13.117}$$

$$\dot{Z}_2 = Z_3\,, \tag{13.118}$$

$$\dot{Z}_3 = U \tag{13.119}$$

first solved by Teel [217] using his method of nested saturations. The system (13.117)–(13.119) is not feedback linearizable, nor is the system

$$\dot{Z}_1 = Z_2 + (Z_2 - Z_3)^2\,, \tag{13.120}$$

$$\dot{Z}_2 = Z_3\,, \tag{13.121}$$

$$\dot{Z}_3 = U\,, \tag{13.122}$$

which was considered in [195].

13.7 Notes and References

Strict-feedforward systems were always considered a class of generically nonlinearizable systems (which they indeed are), so it was no small surprise when it was revealed in [102] that some substantial subclasses of these systems are actually linearizable by coordinate change, with a precise characterization of linearizable strict-feedforward systems and explicit feedback laws for all such systems. Subsequently, Tall and Respondek [213, 214] and Tall [212] developed a systematic way for finding the coordinate changes in Assumption 13.1.

	Direct bkst. transf.		Inverse bkst. transf.	
	Global	Explicit	Global	Explicit
All stabilizable systems			✓	
Fbk.-linearizable and strict-feedback			✓	✓
Stabilizable forward-complete systems	✓		✓	
Strict-feedforward systems	✓	✓	✓	
Linearizable strict-feedforward systems	✓	✓	✓	✓

Table 13.1 Properties of different classes of nonlinear systems considered in the book for predictor feedback design.

One should observe that the significance of linearizability of strict-feedforward systems, which is related to explicit feedback design, goes beyond the limited class that are exactly linearizable. Explicit feedback designs for a number of classes of strict-feedforward and nonstrict-feedforward systems are reviewed in Appendix H, where the linearizability of subclasses of strict-feedforward systems is the key enabling step for explicit feedback design for systems that are not linearizable. These designs include, among other things, block-feedforward systems (Section H.2) and interlaced feedforward-feedback systems (Section H.3).

In Table 13.1 we summarize the properties of various systems that we have considered in the past four chapters. The direct and inverse backstepping transformations have enabled our stability analysis. The explicit form of the direct backstepping transformation was important because it implies that the predictor feedback can be obtained as an explicit formula, i.e., no online solving of integral equations is needed.

Part IV
PDE-ODE Cascades

Chapter 14
ODEs with General Transport-Like Actuator Dynamics

In this chapter we start the development of feedback laws that compensate actuator (or sensor) dynamics of a more complex type than the pure delay. Having dealt with the pure delay, i.e., the transport PDE in Chapter 2, in this chapter we expand our scope to general first-order hyperbolic PDEs in one dimension.

We first focus on first-order hyperbolic PDEs alone, without a cascade with an ODE. First-order hyperbolic PDEs serve as a model for such physical phenomena as traffic flows, chemical reactors, and heat exchangers. We design controllers using the backstepping method—with the integral transformation and boundary feedback, the unstable PDE is converted into a "delay line" system that converges to zero in finite time.

We then show that the proposed method can be used for boundary control of a Korteweg–de Vries-like (KdV) third-order PDE and illustrate the design with simulations. The equation we deal with consists of a first-order hyperbolic PDE coupled with a second-order (in space) ODE. The classical KdV equations (see [11] and references therein) describe shallow-water waves and ion acoustic waves in plasma.

Finally, the central design in this chapter is for a cascade of a general first-order hyperbolic partial integro-differential equation with an arbitrary, possibly unstable but stabilizable, linear ODE. At the end we address a special case of a pure "advection-reaction" PDE that may induce attenuation or amplification of the control signal, in addition to transport.

14.1 First-Order Hyperbolic Partial Integro-Differential Equations

We first consider a general class of one-dimensional first-order hyperbolic partial integro-differential equations. The transport PDE is the simplest member of this class.

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_14,

We consider the plant

$$v_t(x,t) = v_x(x,t) + \lambda(x)v(x,t) + \bar{g}(x)v(0,t) + \int_0^x \bar{f}(x,y)v(y,t)\,dy \tag{14.1}$$

for $0 < x < 1$ with initial condition $v(x,0) = v_0(x)$ and boundary condition

$$v(1,t) = \bar{U}(t)\,, \tag{14.2}$$

where $\bar{U}(t)$ is a control input. We assume that functions λ, $\bar{g}$, and $\bar{f}$ are continuous. Our objective is to stabilize the zero equilibrium of this system with the boundary control $\bar{U}(t)$ [when $\bar{U}(t) \equiv 0$, this system is unstable for large positive $\bar{g}$ and $\bar{f}$].

We start by applying the state transformation

$$v(x,t) = \mathrm{e}^{-\int_0^x \lambda(\xi)\,d\xi} u(x,t)\,, \tag{14.3}$$

which results in the following plant:

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) + \int_0^x f(x,y)u(y,t)dy, \tag{14.4}$$

$$u(1,t) = U(t)\,, \tag{14.5}$$

where

$$U(t) = \bar{U}(t)\mathrm{e}^{\int_0^1 \lambda(\xi)\,d\xi} \tag{14.6}$$

and

$$g(x) = \bar{g}(x)\mathrm{e}^{\int_0^x \lambda(\xi)\,d\xi}, \tag{14.7}$$

$$f(x,y) = \bar{f}(x,y)\mathrm{e}^{\int_y^x \lambda(\xi)\,d\xi}\,. \tag{14.8}$$

Following the backstepping approach [202], we use the transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\,dy \tag{14.9}$$

along with the feedback

$$u(1,t) = \int_0^1 k(1,y)u(y,t)\,dy \tag{14.10}$$

to convert the plant (14.4) into the target system

$$w_t(x,t) = w_x(x,t), \tag{14.11}$$

$$w(1,t) = 0. \tag{14.12}$$

This system is a delay line with unit delay, output

$$w(0,t) = w(1,t-1)\,, \tag{14.13}$$

and zero input at $w(1,t)$. Its solution is

$$w(x,t) = \begin{cases} w_0(t+x), & 0 \le x+t < 1, \\ 0, & x+t \ge 1, \end{cases} \tag{14.14}$$

where $w_0(x)$ is the initial condition. We see that this solution converges to zero in finite time.

To derive the condition that $k(x,y)$ should satisfy, we compute

$$w_x(x,t) = u_x(x,t) - k(x,x)u_x(x,t) - \int_0^x k_x(x,y)u(y,t)\,dy \tag{14.15}$$

and

$$\begin{aligned} w_t(x,t) = {} & u_t(x,t) - \int_0^x k(x,y)\left(u_x(y,t) + g(y)u(0,t)\right)dy \\ & - \int_0^x k(x,y)\int_0^y f(y,\xi)u(\xi,t)\,d\xi\,dy \\ = {} & u_x(x,t) + u(0,t)\left(g(x) - \int_0^x k(x,y)g(y)\,dy\right) \\ & + \int_0^x u(y,t)\left(f(x,y) - \int_y^x k(x,\xi)f(\xi,y)d\xi\right)dy \\ & - k(x,x)u(x,t) \\ & + k(x,0)u(0,t) + \int_0^x k_y(x,y)u(y)\,dy\,. \end{aligned} \tag{14.16}$$

Subtracting (14.15) from (14.16) and using (14.11), we obtain the following set of conditions on $k(x,y)$:

$$k_x(x,y) + k_y(x,y) = \int_y^x k(x,\xi)f(\xi,y)d\xi - f(x,y), \tag{14.17}$$

$$k(x,0) = \int_0^x k(x,y)g(y)dy - g(x)\,. \tag{14.18}$$

The following theorem establishes the well-posedness of the PDE (14.17), (14.18).

Theorem 14.1. *The PDE* (14.17)*,* (14.18) *has a unique* $C^1([0,1]\times[0,1])$ *solution with a bound*

$$|k(x,y)| \le (\bar{g}+\bar{f})\mathrm{e}^{(\bar{g}+\bar{f})(x-y)}\,, \tag{14.19}$$

where

$$\bar{g} = \max_{x\in[0,1]} g(x), \tag{14.20}$$

$$\bar{f} = \max_{(x,y)\in[0,1]\times[0,1]} f(x,y)\,. \tag{14.21}$$

Proof. It is easy to show that $k(x,y)$ satisfies the following integral equation:

$$k(x,y) = F_0(x,y) + F[k](x,y), \tag{14.22}$$

where

$$F_0(x,y) = -g(x-y) - \int_0^y f(x-y+\xi,\xi)\,d\xi, \tag{14.23}$$

$$\begin{aligned} F[k](x,y) = &\int_0^y \int_0^{x-y} k(x-y+\eta,\xi+\eta) f(\xi+\eta,\eta)\,d\xi\,d\eta \\ &+ \int_0^{x-y} k(x-y,\xi) g(\xi)\,d\xi\,. \end{aligned} \tag{14.24}$$

Let us solve this equation using the method of successive approximations. Set

$$k^0(x,y) = F_0(x,y), \tag{14.25}$$

$$k^{n+1}(x,y) = F_0(x,y) + F[k^n](x,y) \tag{14.26}$$

for $n = 0,1,\ldots$, and consider the differences

$$\Delta k^{n+1} = k^{n+1} - k^n \tag{14.27}$$

with

$$\Delta k^0 = F_0\,. \tag{14.28}$$

It is easy to see that the Δk^n satisfy

$$\Delta k^{n+1}(x,y) = F[\Delta k^n](x,y)\,. \tag{14.29}$$

Let us assume that

$$|\Delta k^n(x,y)| \le \frac{(\bar{g}+\bar{f})^{n+1}(x-y)^n}{n!}\,. \tag{14.30}$$

Then from (14.24) and (14.29), we get

$$\begin{aligned} |\Delta k^{n+1}| \le \bar{g}&\frac{(\bar{g}+\bar{f})^{n+1}(x-y)^{n+1}}{(n+1)!} \\ &+ \bar{f}(\bar{g}+\bar{f})^{n+1} \int_0^y \int_0^{x-y} \frac{(x-y-\xi)^n}{n!}\,d\xi\,d\eta \\ \le &\frac{(\bar{g}+\bar{f})^{n+2}(x-y)^{n+1}}{(n+1)!}\,. \end{aligned} \tag{14.31}$$

By induction, (14.30) is proved. Therefore, the series

$$k(x,y) = \sum_{n=0}^{\infty} \Delta k^n(x,y) \tag{14.32}$$

uniformly converges to the solution of (14.26) with $n \to \infty$ and the bound (14.19). The fact that this solution satisfies the PDE (14.17), (14.18) is checked by simple differentiation. To show the uniqueness of this solution, consider the difference between two solutions k_1 and k_2:

$$\delta k = k_1 - k_2 \,. \tag{14.33}$$

For δk, we obtain the homogeneous integral equation

$$\delta k(x,y) = F[\delta k](x,y) \,. \tag{14.34}$$

It is now easy to show by repeating the above calculations that

$$|\delta k(x,y)| \leq \frac{(\bar{g}+\bar{f})^{n+1}(x-y)^n}{n!} \tag{14.35}$$

for any n, which implies that

$$\delta k(x,y) \equiv 0 \tag{14.36}$$

or

$$k_1 \equiv k_2 \,. \tag{14.37}$$

□

We are ready to state the main result of this section.

Theorem 14.2. *For any initial condition*

$$u(x,0) = u_0 \in H = \left\{ f | f \in H^1(0,1),\ f(1) = \int_0^1 k(1,y) f(y) dy \right\}, \tag{14.38}$$

the closed-loop system (14.4), (14.10) *with* $k(x,y)$ *given by* (14.17), (14.18) *has a unique solution* $u \in C([0,\infty),H) \cap C^1([0,\infty),L^2(0,1))$ *that becomes zero in finite time.*

Proof. From the transformation (14.9), we see that the initial condition of the target system $w_0 \in \bar{H} = \{f | f \in H^1(0,1), f(1) = 0\}$, and it therefore immediately follows from (14.14) that $w \in C([0,\infty),\bar{H}) \cap C^1([0,\infty),L^2(0,1))$. One can show that the transformation, inverse to (14.9), has the form

$$u(x,t) = w(x,t) + \int_0^x l(x,y) w(y,t)\, dy, \tag{14.39}$$

where $l(x,y)$ satisfies the following PDE:

$$l_x(x,y) + l_y(x,y) = -\int_y^x f(x,\xi) l(\xi,y)\, d\xi - f(x,y), \tag{14.40}$$

$$l(x,0) = -g(x) \,. \tag{14.41}$$

This PDE is very similar to the PDE (14.17), (14.18); repeating the arguments in the proof of Theorem 14.1, one can show that (14.40), (14.41) has a unique continuously differentiable solution and, furthermore, that

$$|l(x,y)| \leq (\bar{g}+\bar{f})\mathrm{e}^{(\bar{g}+\bar{f})(x-y)}, \tag{14.42}$$

Therefore, from (14.39), we obtain $u \in C([0,\infty),H) \cap C^1([0,\infty),L^2(0,1))$. The explicit form of the solution is obtained using (14.14) and transformations (14.9), (14.39):

$$\begin{aligned} u(x,t) = u_0(x+t) - \int_0^x u_0(y+t)\Big[\ & k(x+t,y+t) - l(x,y) \\ & + \int_y^x l(x,\xi)k(\xi+t,y+t)\,d\xi\Big]\,dy, \quad x+t<1, \end{aligned} \tag{14.43}$$

and $u(x,t) \equiv 0$ for $x+t \geq 1$, so that the control objective is achieved for all $t \geq 1$. The uniqueness of this solution follows from the well-known uniqueness of the weak solution to (14.11), (14.12) (see, e.g., [27]). □

Remark 14.1. When $u_0 \in L^2(0,1)$ (without the compatibility condition), the solution (14.43) belongs to $C([0,\infty),L^2(0,1))$.

Convergence to zero in finite time is an important result; however, we also state a proper exponential stability result.

Theorem 14.3. *The solutions of the closed-loop system* (14.4)*,* (14.10) *with* $k(x,y)$ *given by* (14.17)*,* (14.18) *satisfy the (conservative) exponential stability bound*

$$\|u(t)\| \leq 4(1+\beta^2)\|u_0\|\mathrm{e}^{-t/2}, \tag{14.44}$$

where β *denotes*

$$\beta = \left(\bar{g}+\bar{f}\right)\mathrm{e}^{\bar{g}+\bar{f}} \tag{14.45}$$

and $\|u(t)\|$ *denotes*

$$\|u(t)\| = \left(\int_0^1 u^2(x,t)dx\right)^{1/2}. \tag{14.46}$$

Proof. We start by considering the system (14.11), (14.12) along with the Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 (1+x)w^2(x,t)dx. \tag{14.47}$$

The derivative of this Lyapunov function is

$$\begin{aligned} \dot{V}(t) &= \frac{1}{2}\int_0^1 (1+x)w(x,t)w_t(x,t)dx \\ &= \frac{1}{2}\int_0^1 (1+x)w(x,t)w_x(x,t)dx \\ &= \int_0^1 (1+x)dw(x,t) \end{aligned}$$

$$
\begin{aligned}
&= (1+x)w^2(x,t)|_0^1 - \|w(t)\|^2 \\
&\leq -\frac{1}{2}\int_0^1 (1+2)w^2(x,t)dx \\
&\leq -V(t)\,.
\end{aligned} \tag{14.48}
$$

Hence,

$$V(t) \leq \mathrm{e}^{-t} V(0)\,, \qquad \forall t \geq 0\,. \tag{14.49}$$

Next, we observe that

$$\frac{1}{2}\|w(t)\|^2 \leq V(t) \leq \|w(t)\|^2\,. \tag{14.50}$$

It is easy to show that

$$\|u(t)\|^2 \leq 2\left(1 + \max_{0\leq y\leq x} l^2(x,y)\right)\|w(t)\|^2, \tag{14.51}$$

$$\|w(t)\|^2 \leq 2\left(1 + \max_{0\leq y\leq x} k^2(x,y)\right)\|u(t)\|^2. \tag{14.52}$$

Noting from Theorem 14.1 that

$$\max_{0\leq y\leq x} |l(x,y)| \leq \beta, \tag{14.53}$$

$$\max_{0\leq y\leq x} |k(x,y)| \leq \beta\,, \tag{14.54}$$

we get

$$\|u(t)\|^2 \leq 2\left(1+\beta^2\right)\|w(t)\|^2, \tag{14.55}$$

$$\|w(t)\|^2 \leq 2\left(1+\beta^2\right)\|u(t)\|^2. \tag{14.56}$$

To summarize, we have shown that

$$\frac{1}{4(1+\beta^2)}\|u(t)\|^2 \leq V(t) \leq 2(1+\beta^2)\|u(t)\|^2\,. \tag{14.57}$$

Hence, we get

$$\|u(t)\|^2 \leq 4(1+\beta^2)V(0)\mathrm{e}^{-t}, \tag{14.58}$$

$$V(0) \leq 2(1+\beta^2)\|u_0\|^2\,, \tag{14.59}$$

which leads to the result of the theorem. □

Since we have established stabilizability of the class (14.1) with boundary feedback, it may be natural to expect that this class of systems would be controllable in an appropriate sense. The null controllability for $T \geq 1$ of the special case of system (14.67) for $f = 1, \lambda = 0$ is established in [27], and a similar result may very well hold for the entire class (14.1).

14.2 Examples of Explicit Design

We now illustrate the design with two examples for which explicit feedback laws can be obtained.

Example 14.1. Consider the plant

$$u_t(x,t) = u_x(x,t) + g e^{bx} u(0,t)\,, \tag{14.60}$$

where g and b are constants. Equation (14.17) becomes

$$k_x(x,y) + k_y(x,y) = 0\,, \tag{14.61}$$

which has a general solution

$$k(x,y) = \phi(x-y)\,. \tag{14.62}$$

If we substitute this solution into (14.18), we get the integral equation for $\phi(x)$:

$$\phi(x) = \int_0^x g e^{by} \phi(x-y) dy - g e^{bx}. \tag{14.63}$$

The solution to this equation can be easily obtained by applying the Laplace transform in x to both sides of (14.63). We get

$$\hat{\phi}(s) = -\frac{g}{s-b-g} \tag{14.64}$$

and, after taking the inverse Laplace transform,

$$\phi(x) = -g e^{(b+g)x}. \tag{14.65}$$

Therefore, the solution to the kernel PDE is

$$k(x,y) = -g e^{(b+g)(x-y)}, \tag{14.66}$$

and the controller is given by (14.10).

Example 14.2. Consider the plant

$$u_t(x,t) = u_x(x,t) + \int_0^x f e^{\lambda(x-y)} u(y,t)\, dy\,, \tag{14.67}$$

where f and λ are constants. The kernel PDE (14.17), (14.18) takes the form

$$k_x(x,y) + k_y(x,y) = \int_y^x k(x,\xi) f e^{\lambda(\xi-y)} d\xi - f e^{\lambda(x-y)}, \tag{14.68}$$

$$k(x,0) = 0\,. \tag{14.69}$$

After we differentiate (14.68) with respect to y, the integral term is eliminated:

$$k_{xy}(x,y)+k_{yy}(x,y)=-fk(x,y)-\lambda k_x(x,y)-\lambda k_y(x,y)\,. \tag{14.70}$$

Since we now increased the order of the equation, we need an extra boundary condition. We get it by setting $y=x$ in (14.68):

$$\frac{d}{dx}k(x,x)=k_x(x,x)+k_y(x,x)=-f\,, \tag{14.71}$$

which, after integration, becomes

$$k(x,x)=-fx\,. \tag{14.72}$$

Introducing the change of variables

$$k(x,y)=p(z,y)\mathrm{e}^{\lambda(z-y)/2}, \tag{14.73}$$
$$z=2x-y\,, \tag{14.74}$$

we get the following PDE for $p(z,y)$:

$$p_{zz}(z,y)-p_{yy}(z,y)=fp(z,y), \tag{14.75}$$
$$p(z,0)=0, \tag{14.76}$$
$$p(z,z)=-fz\,. \tag{14.77}$$

This PDE has the following solution [202]:

$$p(z,y)=-2fy\frac{I_1\left(\sqrt{f(z^2-y^2)}\right)}{\sqrt{f(z^2-y^2)}}\,, \tag{14.78}$$

where I_1 is the modified Bessel function. In the original variables we obtain

$$k(x,y)=-f\mathrm{e}^{\lambda(x-y)}y\frac{I_1\left(2\sqrt{fx(x-y)}\right)}{\sqrt{fx(x-y)}}\,, \tag{14.79}$$

and the controller is given by

$$u(1,t)=-\int_0^1 f\mathrm{e}^{\lambda(1-y)}y\frac{I_1\left(2\sqrt{f(1-y)}\right)}{\sqrt{f(1-y)}}u(y,t)\,dy\,. \tag{14.80}$$

14.3 Korteweg–de Vries-like Equation

To further illustrate the physical relevance of developing a method for first-order hyperbolic partial *integro*-differential equations, we consider the following coupled system of a first-order hyperbolic PDE with a second-order ODE in space:

$$\varepsilon u_t(x,t) = u_x(x,t) - v(x,t), \tag{14.81}$$
$$0 = \varepsilon v_{xx}(x,t) + a(-v(x,t) + \gamma u_x(x,t)), \tag{14.82}$$

where $\varepsilon > 0$, $a > 0$. The boundary conditions are

$$v_x(0,t) = 0, \tag{14.83}$$
$$u(1,t) = U_1(t), \tag{14.84}$$
$$v(1,t) = U_2(t), \tag{14.85}$$

where U_1 and U_2 are control inputs.

The motivation for considering the system (14.81), (14.82) comes from the fact that it can be viewed as a third-order PDE

$$u_t(x,t) - \nu u_{txx}(x,t) + \delta u_{xxx}(x,t) + \lambda u_x(x,t) = 0, \tag{14.86}$$

which is obtained by differentiating (14.81) with respect to x twice, substituting the result into (14.82), and denoting

$$\delta = \frac{1}{a}, \tag{14.87}$$
$$\nu = \frac{\varepsilon}{a}, \tag{14.88}$$
$$\lambda = \frac{\gamma - 1}{\varepsilon}. \tag{14.89}$$

The PDE (14.86) resembles a linearized Korteweg–de Vries equation that serves as a model of shallow-water waves and ion acoustic waves in plasma. Compared to the traditional form of the Korteweg–de Vries equation, it has an additional term $-\nu u_{txx}$, which is small when ε/a is small in the original system (14.81), (14.82). In fact, this term appears in the derivation of the KdV equation but is then dropped as small compared to u_t [97]. The PDE (14.86) is unstable when λ/δ is positive and large. Besides being related to the Korteweg–de Vries PDE, Eq. (14.86) can be obtained as an approximation of the linearized Boussinesq PDE system modeling complex water waves such as tidal bores [48].

To apply the backstepping design to the system (14.81)–(14.82), we first solve (14.82) with respect to v:

$$v(x,t) = \cosh(bx)v(0,t) - \gamma b \int_0^x \sinh(b(x-y))u_y(y,t)\,dy, \tag{14.90}$$

where

$$b = \sqrt{\frac{a}{\varepsilon}}. \tag{14.91}$$

Setting $x = 1$ in (14.90), we express $v(0,t)$ in terms of $v(1,t)$:

$$v(0,t) = \frac{1}{\cosh b}\left[v(1,t) - \gamma b \sinh(b)u(0,t) + \gamma b^2 \int_0^1 \cosh(b(1-y))u(y,t)\,dy\right]. \tag{14.92}$$

The integral in (14.92) has the limits from 0 to 1 and is not in the class of PDEs (14.4). Therefore, we select the first boundary control to be

$$v(1,t) = \gamma b \sinh(b) u(0,t) - \gamma b^2 \int_0^1 \cosh(b(1-y)) u(y,t)\, dy, \tag{14.93}$$

which guarantees that

$$v(0,t) = 0. \tag{14.94}$$

Substituting (14.90) into (14.81), we get

$$\begin{aligned} \varepsilon u_t(x,t) = u_x(x,t) - \gamma b \sinh(bx) u(0,t) \\ + \gamma b^2 \int_0^x \cosh(b(x-y)) u(y,t)\, dy. \end{aligned} \tag{14.95}$$

Note that this PDE is exactly of the form (14.4). We can now use the design developed in Section 14.1. The second control law is

$$u(1,t) = \int_0^1 k(1,y) u(y,t)\, dy, \tag{14.96}$$

where the control kernel $k(x,y)$ is found from the PDE

$$k_x(x,y) + k_y(x,y) = \gamma b^2 \int_y^x k(x,\xi) \cosh(b(\xi - y))\, d\xi - \gamma b^2 \cosh(b(x-y)) \tag{14.97}$$

with the boundary condition

$$k(x,0) = \gamma b \sinh(bx) - \gamma b \int_0^x k(x,y) \sinh(by)\, dy. \tag{14.98}$$

Using Theorem 14.2, we obtain the following result.

Theorem 14.4. *For any initial condition $u_0 \in H$, the system* (14.81)–(14.82) *with the controllers* (14.93)*,* (14.96) *has a unique solution $u \in C([0,\infty), H) \cap C^1([0,\infty),$ $L^2(0,1))$ that becomes zero in finite time.*

Furthermore, we obtain a stability result.

Theorem 14.5. *There exists a positive number L independent of the initial conditions such that the solutions of the system* (14.81)–(14.82) *with the controllers* (14.93)*,* (14.96) *satisfy the bound*

$$\|u(t)\| \le L \|u_0\| \mathrm{e}^{-t/2}, \qquad \forall t \ge 0. \tag{14.99}$$

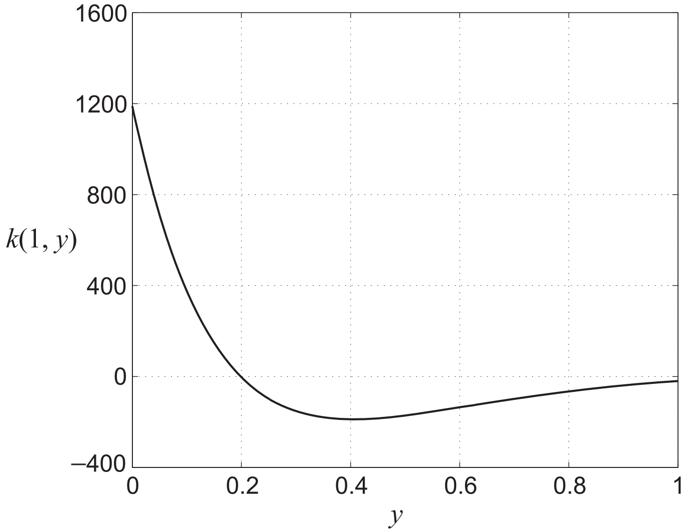

Fig. 14.1 Control gain for the Korteweg–de Vries equation.

14.4 Simulation Example

For the system from Section 14.3, the simulation results for $a = 1$, $\varepsilon = 0.2$, and $\gamma = 4$ are presented in Figs. 14.1 and 14.2. The control gain (Fig. 14.1) is obtained by discretizing (14.97), (14.98) using the implicit Euler finite-difference scheme [an alternative is to use the series (14.26)]. We can see that the open-loop plant (14.81)–(14.82) is unstable and the controller stabilizes the system.

14.5 ODE with Actuator Dynamics Given by a General First-Order Hyperbolic PIDE

Now we address the main problem of this chapter, a cascade of general first-order hyperbolic partial integro-differential equation (PIDE) actuator dynamics and a general LTI, possibly unstable, ODE:

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{14.100}$$

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) + \int_0^x f(x,y)u(y,t)dy, \tag{14.101}$$

$$u(1,t) = U(t). \tag{14.102}$$

Consider the backstepping transformation

$$w(x) = u(x) - \int_0^x q(x,y)u(y)dy - \gamma(x)^T X, \tag{14.103}$$

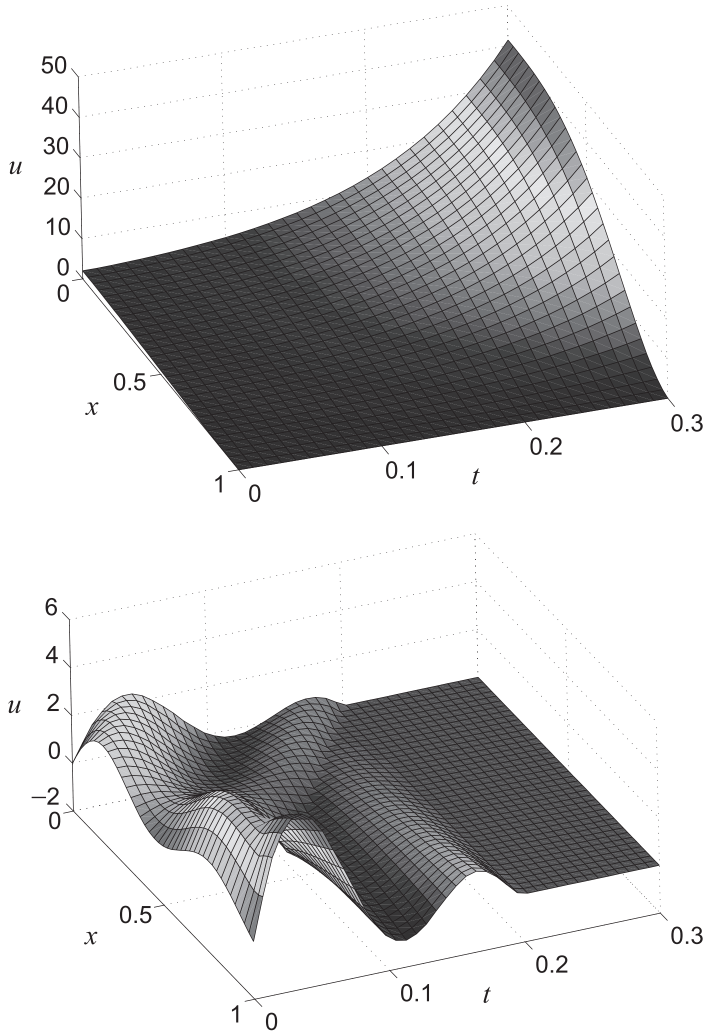

Fig. 14.2 The open-loop (left) and the closed-loop (right) responses of the Korteweg–de Vries-like plant (14.81)–(14.82) with backstepping controllers (14.93), (14.96)–(14.98).

which maps the plant into the target system

$$\dot{X} = (A+BK)X + Bw(0), \tag{14.104}$$

$$w_t = w_x, \tag{14.105}$$

$$w(D) = 0\,. \tag{14.106}$$

First, let us set $x = 0$ in (14.103), which gives

$$w(0) = u(0) - \gamma(0)^T X\,. \tag{14.107}$$

Substituting this expression into (14.104), we get

$$\dot{X} = AX + Bu(0) + B\left(K - \gamma(0)^T\right)X\,. \tag{14.108}$$

We pick

$$\gamma(0) = K^T\,, \tag{14.109}$$

where K is such that $A + BK$ is Hurwitz; i.e., there exists a matrix $P = P^T > 0$ that is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^T P = -Q \tag{14.110}$$

for some $Q = Q^T > 0$.

Let us now calculate the time and spatial derivatives of the transformation (14.103):

$$w_x = u_x - q(x,x)u(x) - \int_0^x q_x(x,y)u(y)dy - \gamma'(x)^T X, \tag{14.111}$$

$$\begin{aligned}
w_t &= u_t - \int_0^x q(x,y)u_t(y)dy - \gamma(x)^T\left[AX + Bu(0)\right] \\
&= u_x + g(x)u(0) + \int_0^x f(x,y)u(y)dy - q(x,x)u(x) + q(x,0)u(0) \\
&\quad - \left(\int_0^x q(x,y)g(y)dy\right)u(0) - \int_0^x q(x,y)\int_0^y f(y,s)u(s)dsdy \\
&\quad - \int_0^x q_y(x,y)u(y)dy - \gamma(x)^T\left[AX + Bu(0)\right] \\
&= u_x + g(x)u(0) + \int_0^x f(x,y)u(y)dy - q(x,x)u(x) + q(x,0)u(0) \\
&\quad - \left(\int_0^x q(x,y)g(y)dy\right)u(0) - \int_0^x u(y)\int_y^x q(x,s)f(s,y)dsdy \\
&\quad - \int_0^x q_y(x,y)u(y)dy - \gamma(x)^T\left[AX + Bu(0)\right].
\end{aligned} \tag{14.112}$$

Subtracting (14.111) from (14.112), we get

$$\begin{aligned}
&\int_0^x \left(q_x(x,y) + q_y(x,y) - \int_y^x k(x,s)f(s,y)ds + f(x,y)\right)u(y)dy \\
&\quad + \left[q(x,0) - \int_0^x q(x,y)g(y)dy + g(x) - \gamma(x)^T B\right]u(0) \\
&\quad + \left[\gamma'(x)^T - \gamma(x)^T A\right]X = 0\,.
\end{aligned} \tag{14.113}$$

This equation should be valid for all u and X, so we get conditions:

$$q_x(x,y) + q_y(x,y) = \int_y^x k(x,s) f(s,y) ds - f(x,y), \tag{14.114}$$

$$q(x,0) = \int_0^x q(x,y) g(y) dy - g(x) + \gamma(x)^T B, \tag{14.115}$$

$$\gamma'(x) = A^T \gamma(x), \tag{14.116}$$

$$\gamma(0) = K^T, \tag{14.117}$$

where the last condition was obtained earlier. The first two conditions form a first-order hyperbolic PDE, whereas the last two form an ODE. The solution to the ODE (14.116) is $\gamma(x) = \mathrm{e}^{A^T x} K^T$, which gives

$$\gamma(x)^T = K\mathrm{e}^{Ax}. \tag{14.118}$$

Hence, our PDE for $q(x,y)$ becomes

$$q_x(x,y) + q_y(x,y) = \int_y^x k(x,s) f(s,y) ds - f(x,y), \tag{14.119}$$

$$q(x,0) = \int_0^x q(x,y) g(y) dy - g(x) + K\mathrm{e}^{Ax} B. \tag{14.120}$$

The PDE for $q(x,y)$ cannot be solved explicitly, but the following result can be established.

Theorem 14.6. *The PDE* (14.119)*,* (14.120) *has a unique* $C^1([0,1]\times[0,1])$ *solution with a uniform bound that is a continuous function of* $\|K\|$, $\|A\|$, $\|B\|$, $\bar{g}$, *and* $\bar{f}$.

With the gain functions computable, the control law is obtained from the condition $w(1,t) = 0$ as

$$U(t) = K\mathrm{e}^{A} X(t) + \int_0^1 q(1,y) u(y,t) dy. \tag{14.121}$$

This control law has resemblance with the classical predictor feedback, except that the gain kernel under the integral, given by $q(1,y)$, is more complex than the simple $K\mathrm{e}^{A(1-y)}B$ in the predictor feedback.

Theorem 14.7. *Consider the feedback system*

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{14.122}$$

$$u_t(x,t) = u_x(x,t) + g(x)u(0,t) + \int_0^x f(x,y) u(y,t) dy, \tag{14.123}$$

$$u(1,t) = K\mathrm{e}^{A} X(t) + \int_0^1 q(1,y) u(y,t) dy, \tag{14.124}$$

where $q(1,y)$ is defined as a solution to the PDE (14.119), (14.120). *There exists a positive number L independent of the initial conditions such that the system's solutions satisfy the bound*

$$\left(|X(t)|^2 + \|u(t)\|^2\right)^{1/2} \leq L\left(|X_0|^2 + \|u_0\|^2\right)^{1/2} e^{-t/2}, \qquad \forall t \geq 0. \tag{14.125}$$

Now we recall that our approach is also equipped to handle a system of the more general form, with a "reaction" term included:

$$\dot{X}(t) = AX(t) + Bv(0,t), \tag{14.126}$$

$$v_t(x,t) = v_x(x,t) + \lambda(x)v(x,t) + \bar{g}(x)v(0,t) + \int_0^x \bar{f}(x,y)v(y,t)\,dy, \tag{14.127}$$

$$v(1,t) = \bar{U}(t), \tag{14.128}$$

where $\bar{U}(t)$ is the control input. We derive the control law as

$$\bar{U}(t) = e^{-\int_0^1 \lambda(\xi)d\xi} U(t) \tag{14.129}$$

$$= e^{-\int_0^1 \lambda(\xi)d\xi}\left(Ke^{A}X(t) + \int_0^1 q(1,y)u(y,t)dy\right) \tag{14.130}$$

$$= e^{-\int_0^1 \lambda(\xi)d\xi}\left(Ke^{A}X(t) + \int_0^1 q(1,y)e^{\int_0^y \lambda(\xi)d\xi}v(y,t)dy\right), \tag{14.131}$$

obtaining in the end

$$\bar{U}(t) = Ke^{-\int_0^1 \lambda(\xi)d\xi}e^{A}X(t) + \int_0^1 q(1,y)e^{-\int_y^1 \lambda(\xi)d\xi}v(y,t)dy, \tag{14.132}$$

where $q(1,y)$ is obtained from the PDE (14.119), (14.120) with

$$g(x) = \bar{g}(x)e^{\int_0^x \lambda(\xi)d\xi}, \tag{14.133}$$

$$f(x,y) = \bar{f}(x,y)e^{\int_y^x \lambda(\xi)d\xi}. \tag{14.134}$$

14.6 An ODE with Pure Advection-Reaction Actuator Dynamics

Let us consider the system

$$\dot{X}(t) = AX(t) + Bv(0,t), \tag{14.135}$$

$$v_t(x,t) = v_x(x,t) + \lambda v(x,t), \tag{14.136}$$

$$v(1,t) = \bar{U}(t), \tag{14.137}$$

where λ is constant. The resulting controller is

$$\bar{U}(t) = K\mathrm{e}^{-\lambda}\left(\mathrm{e}^{A}X(t) + \int_0^1 \mathrm{e}^{A(1-y)}B\mathrm{e}^{\lambda y}v(y,t)dy\right). \tag{14.138}$$

We observe the reduction in the overall gain in this feedback law when $\lambda > 0$. This is needed because the "advection-reaction" PDE $v_t(x,t) = v_x(x,t) + \lambda v(x,t)$ has the property that the input signal is not only transported from $x = 1$ to $x = 0$, but the signal actually grows. So, to compensate for this growth, the control law uses the reduction in gain in the form of the factor $\mathrm{e}^{-\lambda}$.

Conversely, if $\lambda < 0$, the overall gain increases in order to compensate for the damping (attenuation) of the control signal through the advection-reaction medium.

Theorem 14.8. *Consider the feedback system*

$$\dot{X}(t) = AX(t) + Bv(0,t), \tag{14.139}$$

$$v_t(x,t) = v_x(x,t) + \lambda v(x,t), \tag{14.140}$$

$$v(1,t) = K\mathrm{e}^{-\lambda}\left(\mathrm{e}^{A}X(t) + \int_0^1 \mathrm{e}^{A(1-y)}B\mathrm{e}^{\lambda y}v(y,t)dy\right). \tag{14.141}$$

There exists a positive number L independent of the initial conditions such that the system's solutions satisfy the bound

$$\left(|X(t)|^2 + \|v(t)\|^2\right)^{1/2} \le L\left(|X_0|^2 + \|v_0\|^2\right)^{1/2}\mathrm{e}^{-t/2}, \qquad \forall t \ge 0. \tag{14.142}$$

14.7 Notes and References

The existing results on feedback control of first-order hyperbolic PDEs include [25, 26, 28, 120, 194, 246]. Boundary controllability, including null controllability, of these systems is studied in [3, 21, 27, 89]. The focus in the field of control of first-order hyperbolic PDEs is on coupled systems of conservation laws, including nonlinear conservation laws. As *conservation* laws, such systems are typically neutrally stable, but with the possibility of infinitely many eigenvalues on the imaginary axis (in the case of *coupled* first-order hyperbolic PDEs). Such systems are stabilizable by static output feedbacks in the form of simple boundary conditions. However, the construction of strict Lyapunov functions is a delicate matter, and so is proving the local stability of the nonlinear closed-loop PDE system (versus the easier problem of proving stability of the linearization).

The first-order hyperbolic PIDEs that we consider here can be open-loop unstable, and this is the main challenge in the design. The KdV-like equation that we considered can be unstable, but the particular method we presented here is not applicable to the classical KdV ($\nu = \varepsilon = 0$). For this equation, a Gramian-based method for not only stabilization but for "rapid stabilization" (assignment of an arbitrarily high level of damping) was developed by Eduardo Cerpa.

Chapter 15
ODEs with Heat PDE Actuator Dynamics

In this chapter we use the backstepping approach from Chapter 2 to expand the scope of predictor feedback and build a much broader paradigm for the design of control laws for systems with infinite-dimensional actuator dynamics, as well as for observer design for systems with infinite-dimensional sensor dynamics.

In this chapter we address the problems of compensating for the actuator and sensor dynamics dominated by *diffusion*, i.e., modeled by the heat equation. Purely convective/first-order hyperbolic PDE dynamics (i.e., transport equation or, simply, delay) and diffusive/parabolic PDE dynamics (i.e., heat equation) introduce different problems with respect to controllability and stabilization. On the elementary level, the convective dynamics have a constant-magnitude response at all frequencies but are limited by a finite speed of propagation. The diffusive dynamics, when control enters through one boundary of a 1D domain and exits (to feed the ODE) through the other, are not limited in the speed of propagation but introduce an infinite relative degree, with the associated significant roll-off of the magnitude response at high frequencies.

The key difference in our design for diffusive input dynamics in this chapter relative to the convective input dynamics in Chapter 2 is in the transformation kernel functions (and the associated ODEs and PDEs, which need to be solved). While in Chapter 2 the kernel ODEs and PDEs were first-order, here they are second-order. To be more precise, the design PDEs for control gains arising in delay problems were first-order hyperbolic, whereas with diffusion problems they are second-order hyperbolic. As we did in Chapter 2, we solve them explicitly.

We start in Section 15.1 with an actuator compensation design with full-state feedback. With a simple design we achieve closed-loop stability. We follow this with a more complex design that also endows the closed-loop system with an arbitrarily fast decay rate. In Section 15.2 we illustrate the general design through an example. In Section 15.3 we approach the question of the robustness of our infinite-dimensional feedback law with respect to uncertainty in the diffusion coefficient. This question is rather nontrivial for actuator delays. We resolved it positively for small delay perturbations in Chapter 5, and we resolve it positively here for small perturbations in the diffusion coefficient. In Section 15.4 we cast the compensator

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_15,

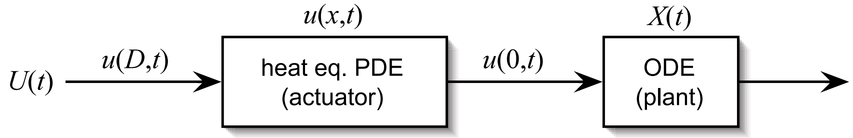

Fig. 15.1 The cascade of the heat equation PDE dynamics of the actuator with the ODE dynamics of the plant.

in terms of the input signal, rather than in terms of the state of the heat equation. This is an easy task when the input dynamics are of the pure delay type, but it is somewhat more involved when the input dynamics are governed by the heat PDE. In Section 15.5 we discuss some specific differences between the control laws compensating the delay dynamics and heat PDE dynamics.

15.1 Stabilization with Full-State Feedback

We consider the cascade of a heat equation and an LTI finite-dimensional system given by

$$\dot{X}(t) = AX(t) + Bu(0,t)\,, \tag{15.1}$$

$$u_t(x,t) = u_{xx}(x,t)\,, \tag{15.2}$$

$$u_x(0,t) = 0\,, \tag{15.3}$$

$$u(D,t) = U(t)\,, \tag{15.4}$$

where $X \in \mathbb{R}^n$ is the ODE state, U is the scalar input to the entire system, and $u(x,t)$ is the state of the PDE dynamics of the diffusive actuator. The cascade system is depicted in Fig. 15.1.

The length of the PDE domain, D, is arbitrary. Thus, we take the diffusion coefficient to be unity without loss of generality. We assume that the pair (A,B) is stabilizable and take K to be a known vector such that $A+BK$ is Hurwitz.

We recall from Chapter 2 that if (15.2), (15.3) are replaced by the delay/transport equation

$$u_t(x,t) = u_x(x,t)\,, \tag{15.5}$$

then the predictor-based control law

$$U(t) = K\left[\mathrm{e}^{AD}X(t) + \int_0^D \mathrm{e}^{A(D-y)}Bu(y,t)dy\right] \tag{15.6}$$

achieves perfect compensation of the actuator delay and achieves exponential stability at $u \equiv 0, X = 0$.

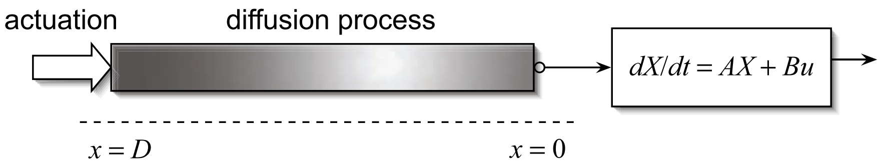

Fig. 15.2 An arbitrary ODE controlled through a diffusion process.

Before we start, we remark that the PDE system (15.2)–(15.4) has a transfer function representation

$$u(0,t) = \frac{1}{\cosh(D\sqrt{s})}[U(t)]. \tag{15.7}$$

This means that while the ODE plant with input delay has a transfer function representation

$$X(s) = (sI-A)^{-1}B\mathrm{e}^{-sD}U(s), \tag{15.8}$$

the ODE plant with a heat PDE at the input has a transfer function representation

$$X(s) = (sI-A)^{-1}B\frac{1}{\cosh(D\sqrt{s})}U(s). \tag{15.9}$$

Next we state a new controller that compensates for the *diffusive* actuator dynamics and prove the exponential stability of the resulting closed-loop system (see Fig. 15.2).

Theorem 15.1 (Stabilization). *Consider a closed-loop system consisting of the plant (15.1)–(15.4) and the control law*

$$U(t) = K\begin{bmatrix} I & 0 \end{bmatrix}\left\{ \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} D}\begin{bmatrix} I \\ 0 \end{bmatrix}X(t) + \int_0^D \left(\int_0^{D-y} \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}\xi} d\xi\right)\begin{bmatrix} I \\ 0 \end{bmatrix} Bu(y,t)dy \right\}. \tag{15.10}$$

For any initial condition such that $u(x,0)$ is square integrable in x and compatible with the control law (15.10), the closed-loop system has a unique classical solution and is exponentially stable in the sense of the norm

$$\left(|X(t)|^2 + \int_0^D u(x,t)^2 dx\right)^{1/2}. \tag{15.11}$$

Proof. We start by formulating an infinite-dimensional transformation of the form

$$w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)dy - \gamma(x)X(t), \tag{15.12}$$

with kernels $q(x,y)$ and $\gamma(x)$ to be derived, which should transform the plant (15.1)–(15.4), along with the control law (15.10), into the "target system"

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t)\,, \tag{15.13}$$

$$w_t(x,t) = w_{xx}(x,t)\,, \tag{15.14}$$

$$w_x(0,t) = 0\,, \tag{15.15}$$

$$w(D,t) = 0\,. \tag{15.16}$$

We first derive the kernels $q(x,y)$ and $\gamma(x)$ and then show that the target system is exponentially stable. The first two derivatives with respect to x of $w(x,t)$, as defined in (15.12), are given by

$$\begin{aligned} w_x(x,t) &= u_x(x,t) - q(x,x)u(x,t) - \int_0^x q_x(x,y)u(y,t)dy \\ &\quad - \gamma'(x)X(t)\,, \end{aligned} \tag{15.17}$$

$$\begin{aligned} w_{xx}(x,t) &= u_{xx}(x,t) - \big(q(x,x)\big)'u(x,t) - q(x,x)u_x(x,t) \\ &\quad - q_x(x,x)u(x,t) - \int_0^x q_{xx}(x,y)u(y,t)dy \\ &\quad - \gamma''(x)X(t)\,. \end{aligned} \tag{15.18}$$

The first derivative of $w(x,t)$ with respect to t is

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \int_0^x q(x,y)u_t(y,t)dy - \gamma(x)\left(AX(t) + Bu(0,t)\right) \\ &= u_{xx}(x,t) - \int_0^x q(x,y)u_{xx}(y,t)dy - \gamma(x)\left(AX(t) + Bu(0,t)\right) \\ &= u_{xx}(x,t) - q(x,x)u_x(x,t) + q(x,0)u_x(0,t) \\ &\quad + q_y(x,x)u(x,t) - q_y(x,0)u(0,t) \\ &\quad - \int_0^x q_{yy}(x,y)u(y,t)dy - \gamma(x)\left(AX(t) + Bu(0,t)\right). \end{aligned} \tag{15.19}$$

Let us now examine the expressions

$$w(0,t) = u(0,t) - \gamma(0)X(t)\,, \tag{15.20}$$

$$w_x(0,t) = -q(0,0)u(0,t) - \gamma'(0)X(t)\,, \tag{15.21}$$

$$\begin{aligned} w_t(x,t) - w_{xx}(x,t) &= 2\big(q(x,x)\big)'u(x,t) \\ &\quad + \big(\gamma''(x) - \gamma(x)A\big)X(t) \\ &\quad - \left(q_y(x,0) + \gamma(x)B\right)u(0,t) \\ &\quad + \int_0^x \left(q_{xx}(x,y) - q_{yy}(x,y)\right)u(y,t)dy\,, \end{aligned} \tag{15.22}$$

where we have employed the fact that $u_x(0,t) = 0$. A sufficient condition for (15.13)–(15.15) to hold for any continuous functions $u(x,t)$ and $X(t)$ is that $\gamma(x)$ and $q(x,y)$ satisfy

$$\gamma''(x) = \gamma(x)A\,, \tag{15.23}$$

$$\gamma(0) = K\,, \tag{15.24}$$

$$\gamma'(0) = 0\,, \tag{15.25}$$

which happens to represent a second-order ODE in x, and

$$q_{xx}(x,y) = q_{yy}(x,y)\,, \tag{15.26}$$

$$q(x,x) = 0\,, \tag{15.27}$$

$$q_y(x,0) = -\gamma(x)B\,, \tag{15.28}$$

which is a second-order hyperbolic PDE of the Goursat type. We then proceed to solve this cascade system explicitly. The explicit solution to the ODE (15.23)–(15.25) is readily found as

$$\gamma(x) = \begin{bmatrix} K & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{15.29}$$

and the explicit solution to the PDE (15.26)–(15.28) is

$$q(x,y) = \int_0^{x-y} \gamma(\sigma)B d\sigma\,. \tag{15.30}$$

This explicit solution is obtained by postulating the solution of (15.26)–(15.28) in the form

$$q(x,y) = \varphi(x-y) + \varsigma(x+y)\,, \tag{15.31}$$

which, using the boundary conditions, yields

$$\varsigma(2x) \equiv 0\,, \tag{15.32}$$

$$-\varphi'(x) = -\gamma(x)B\,. \tag{15.33}$$

Integrating the second equation, one gets the expression for $\varphi(x)$ and thus for $q(x,y)$. In a similar manner to finding the kernels $q(x,y)$ and $\gamma(x)$ of the direct transformation, the inverse of the transformation (15.12) can be found. To summarize and to introduce a compact notation for further use in the proof, the direct and inverse backstapping transformations are given by

$$w(x,t) = u(x,t) - \int_0^x m(x-y)u(y,t)dy - KM(x)X(t)\,, \tag{15.34}$$

$$u(x,t) = w(x,t) + \int_0^x n(x-y)w(y,t)dy + KN(x)X(t)\,, \tag{15.35}$$

where

$$m(s) = \int_0^s KM(\xi)Bd\xi\,, \tag{15.36}$$

$$n(s) = \int_0^s KN(\xi)Bd\xi\,, \tag{15.37}$$

$$M(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{15.38}$$

$$N(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A+BK \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{15.39}$$

Now we proceed to prove exponential stability. Consider the Lyapunov function

$$V = X^T PX + \frac{a}{2}\|w\|^2, \tag{15.40}$$

where $\|w(t)\|^2$ is a compact notation for $\int_0^D w(x,t)^2\,dx$, the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^T P = -Q \tag{15.41}$$

for some $Q = Q^T > 0$, and the parameter $a > 0$ is to be chosen later. It is easy to show, using (15.34) and (15.35), that

$$\|w\|^2 \leq \alpha_1 \|u\|^2 + \alpha_2 |X|^2, \tag{15.42}$$

$$\|u\|^2 \leq \beta_1 \|w\|^2 + \beta_2 |X|^2, \tag{15.43}$$

where

$$\alpha_1 = 3\left(1 + D\|m\|^2\right), \tag{15.44}$$

$$\alpha_2 = 3\|KM\|^2, \tag{15.45}$$

$$\beta_1 = 3\left(1 + D\|n\|^2\right), \tag{15.46}$$

$$\beta_2 = 3\|KN\|^2. \tag{15.47}$$

Hence,

$$\underline{\delta}\left(|X|^2 + \|u\|^2\right) \leq V \leq \bar{\delta}\left(|X|^2 + \|u\|^2\right), \tag{15.48}$$

where

$$\underline{\delta} = \frac{\min\left\{\frac{a}{2}, \lambda_{\min}(P)\right\}}{\max\{\beta_1, \beta_2 + 1\}}, \tag{15.49}$$

$$\bar{\delta} = \max\left\{\frac{a}{2}\alpha_1, \frac{a}{2}\alpha_2 + \lambda_{\max}(P)\right\}. \tag{15.50}$$

Taking a derivative of the Lyapunov function along the solutions of the PDE-ODE system (15.13)–(15.16), we get

$$\begin{aligned}\dot{V} &= -X^T QX + 2X^T PBw(0,t) - a\|w_x\|^2 \\ &\leq -\frac{\lambda_{\min}(Q)}{2}|X|^2 + \frac{2|PB|^2}{\lambda_{\min}(Q)}w(0,t)^2 - a\|w_x\|^2 \\ &\leq -\frac{\lambda_{\min}(Q)}{2}|X|^2 - \left(a - \frac{8D|PB|^2}{\lambda_{\min}(Q)}\right)\|w_x\|^2, \end{aligned} \tag{15.51}$$

where the last line is obtained by using Agmon's inequality. [To see this, note that, with $w(D,t) = 0$, one obtains

$$\int_0^D w^2(x,t)dx \leq 4D^2 \int_0^D w_x^2(x,t)dx, \tag{15.52}$$

$$w^2(0,t) \leq \max_{0\leq x\leq D} w^2(x,t) \leq 2\|w(t)\|\|w_x(t)\|, \tag{15.53}$$

where the first inequality is Poincaré's and the second is Agmon's. From these two inequalities, one obtains

$$w^2(0,t) \leq 4D \int_0^D w_x^2(x,t)dx, \tag{15.54}$$

which yields (15.51).] Now, taking

$$a > \frac{8D|PB|^2}{\lambda_{\min}(Q)}, \tag{15.55}$$

and using Poincaré's inequality, we get

$$\dot{V} \leq -bV, \tag{15.56}$$

where

$$b = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{1}{2} - \frac{4D|PB|^2}{a\lambda_{\min}(Q)}\right\} > 0. \tag{15.57}$$

Hence,

$$|X(t)|^2 + \|u(t)\|^2 \leq \frac{\bar{\delta}}{\underline{\delta}}\mathrm{e}^{-bt}\left(|X_0|^2 + \|u_0\|^2\right) \tag{15.58}$$

for all $t \geq 0$, which completes the proof. □

The convergence rate to zero for the closed-loop system is determined by the eigenvalues of the PDE-ODE system (15.13)–(15.16). These eigenvalues are the union of the eigenvalues of $A + BK$, which are placed at desirable locations by the control vector K, and of the eigenvalues of the heat equation with a Neumann boundary condition on one end and a Dirichlet boundary condition on the other end.

While exponentially stable, the heat equation PDE need not necessarily have a fast decay. Its decay rate is limited by its first eigenvalue, $-\pi^2/\left(4D^2\right)$.

Fortunately, the compensated actuator dynamics, i.e., the w-dynamics in (15.15)–(15.16), can be sped up arbitrarily by a modified controller.

Theorem 15.2 (Performance improvement). *Consider a closed-loop system consisting of the plant (15.1)–(15.4) and the control law*

$$U(t) = \phi(D)X(t) + \int_0^D \psi(D,y)u(y,t)dy\,, \tag{15.59}$$

where

$$\phi(x) = KM(x) - \int_0^x \kappa(x,y)KM(y)dy\,, \tag{15.60}$$

$$\begin{aligned}\psi(x,y) = \kappa(x,y) + \int_0^{x-y} KM(\xi)Bd\xi \\ - \int_y^x \kappa(x,\xi)\int_0^{\xi-y} KM(\eta)Bd\eta d\xi\,,\end{aligned} \tag{15.61}$$

$$\kappa(x,y) = -cx\frac{I_1\left(\sqrt{c\left(x^2-y^2\right)}\right)}{\sqrt{c\left(x^2-y^2\right)}}\,, \quad c>0\,, \tag{15.62}$$

and I_1 denotes the appropriate Bessel function. For any initial condition such that $u(x,0)$ is square integrable in x and compatible with the control law (15.59), the closed-loop system has a unique classical solution and its eigenvalues are given by the set

$$\operatorname{eig}\{A+BK\} \cup \left\{-c - \frac{\pi^2}{D^2}\left(n+\frac{1}{2}\right)^2,\ n=0,1,2,\ldots\right\}. \tag{15.63}$$

Proof. Consider the new (invertible) state transformation

$$z(x,t) = w(x,t) - \int_0^x \kappa(x,y)w(y,t)dy\,. \tag{15.64}$$

By direct substitution of the transformation

$$w(x,t) = u(x,t) - \int_0^x\int_0^{x-y} KM(\xi)Bd\xi u(y,t)dy - KM(x)X(t) \tag{15.65}$$

into (15.64), and by changing the order of integration, one obtains

$$z(x,t) = u(x,t) - \int_0^x \psi(x,y)u(y,t)dy - \phi(x)X(t)\,, \tag{15.66}$$

where the functions $\psi(x,y)$ and $\phi(x)$ are as defined in the statement of the theorem. It was shown in [202, Sections VIII.A and VIII.B] that the function $\kappa(x,y)$ satisfies the PDE

$$\kappa_{xx}(x,y) = \kappa_{yy}(x,y) + c\kappa(x,y)\,, \tag{15.67}$$

$$\kappa_y(x,0) = 0\,, \tag{15.68}$$

$$\kappa(x,x) = -\frac{c}{2}x\,. \tag{15.69}$$

Using these relations and (15.64), a direct verification yields the transformed closed-loop system

$$\dot{X}(t) = (A+BK)X(t) + Bz(0,t)\,, \tag{15.70}$$

$$z_t(x,t) = z_{xx}(x,t) - cz(x,t)\,, \tag{15.71}$$

$$z_x(0,t) = 0\,, \tag{15.72}$$

$$z(D,t) = 0\,. \tag{15.73}$$

With an elementary calculation of the eigenvalues of the z-system, the result of the theorem follows. □

15.2 Example: Heat PDE Actuator Dynamics

We consider a system consisting of a scalar unstable ODE with a heat equation at its input. This system is given by

$$\dot{X}(t) = X(t) + u(0,t)\,, \tag{15.74}$$

$$u_t(x,t) = u_{xx}(x,t)\,, \tag{15.75}$$

$$u_x(0,t) = 0\,, \tag{15.76}$$

$$u(D,t) = U(t)\,. \tag{15.77}$$

Our task is to design a controller in the form

$$U(t) = K\begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} D} \begin{bmatrix} I \\ 0 \end{bmatrix} X(t) + \int_0^D K\begin{bmatrix} I & 0 \end{bmatrix} \left(\int_0^{D-y} \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} \xi} d\xi \right) \begin{bmatrix} I \\ 0 \end{bmatrix} Bu(y,t)dy\,. \tag{15.78}$$

In our problem

$$A = 1\,, \tag{15.79}$$

$$B = 1\,, \tag{15.80}$$

$$K = -(1+h)\,, \qquad h > 0\,. \tag{15.81}$$

First, we observe that

$$e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} x} = e^{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x} = \begin{bmatrix} \sinh x & \cosh x \\ \cosh x & \sinh x \end{bmatrix} \tag{15.82}$$

and

$$\int_0^x e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} \xi} d\xi = \begin{bmatrix} \cosh x - 1 & \sinh x \\ \sinh x & \cosh x - 1 \end{bmatrix}. \tag{15.83}$$

Then we note that

$$K \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} D} \begin{bmatrix} I \\ 0 \end{bmatrix} = -(1+h)\sinh D \tag{15.84}$$

and

$$K \begin{bmatrix} I & 0 \end{bmatrix} \left(\int_0^{D-y} e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} \xi} d\xi \right) \begin{bmatrix} I \\ 0 \end{bmatrix} B = -(1+h)\left(\cosh(D-y) - 1\right). \tag{15.85}$$

Hence, our controller is

$$U(t) = -(1+h)\left[\sinh(D)X(t) + \int_0^D \left(\cosh(D-y) - 1\right) u(y,t)dy\right]. \tag{15.86}$$

This example illustrates the requirement that the controller use high gain to overcome the diffusive actuator dynamics when the domain length D is large. It also illustrates that the gain on the actuator state $u(y,t)$ is the highest on the far end from the input ($y = 0$) and the lowest near the actual input ($y = D$), where by the "actual input," we are referring to $U(t)$.

15.3 Robustness to Diffusion Coefficient Uncertainty

We now study the robustness of the feedback law (15.10) to a perturbation in the diffusion coefficient of the actuator dynamics; i.e., we study the stability robustness of the closed-loop system

$$\dot{X}(t) = AX(t) + Bu(0,t)\,, \tag{15.87}$$

$$u_t(x,t) = (1+\varepsilon)u_{xx}(x,t)\,, \tag{15.88}$$

$$u_x(0,t) = 0\,, \tag{15.89}$$

$$u(D,t) = \int_0^D m(D-y)u(y,t)dy + KM(D)X(t) \tag{15.90}$$

to the perturbation parameter ε, which we allow to be either positive or negative but small.

Theorem 15.3 (Robustness to diffusion uncertainty). *Consider a closed-loop system (15.87)–(15.90). There exists a sufficiently small $\varepsilon^* > 0$ such that for all $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$, the closed-loop system has a unique classical solution (under feedback-compatible initial data in L_2) and is exponentially stable in the sense of the norm $\left(|X(t)|^2 + \int_0^D u(x,t)^2 dx\right)^{1/2}$.*

Proof. It can be readily verified that

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t)\,, \tag{15.91}$$

$$\begin{aligned} w_t(x,t) &= (1+\varepsilon)w_{xx}(x,t) \\ &\quad + \varepsilon KM(x)\left((A+BK)X(t) + Bw(0,t)\right), \end{aligned} \tag{15.92}$$

$$w_x(0,t) = 0\,, \tag{15.93}$$

$$w(D,t) = 0\,. \tag{15.94}$$

Along the solutions of this system, the derivative of the Lyapunov function (15.40) is

$$\begin{aligned} \dot{V} \leq& -\frac{\lambda_{\min}(Q)}{2}|X|^2 - \left(a - \frac{8D|PB|^2}{\lambda_{\min}(Q)} - |\varepsilon|a\right)\|w_x\|^2 \\ &+ a\varepsilon\int_0^D w(x)KM(x)dx\left((A+BK)X(t) + Bw(0,t)\right) \\ \leq& -\frac{\lambda_{\min}(Q)}{4}|X|^2 - \left(a - \frac{8D|PB|^2}{\lambda_{\min}(Q)}\right)\|w_x\|^2 \\ &+ |\varepsilon|a\left(1 + 4\|\mu_1\| + |\varepsilon|a\frac{4\|\mu_2\|^2}{\lambda_{\min}(Q)}\right)\|w_x\|^2\,, \end{aligned} \tag{15.95}$$

where

$$\mu_1(x) = KM(x)B\,, \tag{15.96}$$

$$\mu_2(x) = |KM(x)|\,. \tag{15.97}$$

In the second inequality we have employed Young's and Agmon's inequalities. Choosing now, for example,

$$a = \frac{16D|PB|^2}{\lambda_{\min}(Q)}\,, \tag{15.98}$$

it is possible to select $|\varepsilon|$ sufficiently small to achieve negative definiteness of $\dot{V}$. □

15.4 Expressing the Compensator in Terms of Input Signal Rather Than Heat Equation State

In this section we return to the controller

$$U(t) = K\left[I\ 0\right]\left\{ e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} D} \begin{bmatrix} I \\ 0 \end{bmatrix} X(t) + \int_0^D \left(\int_0^{D-y} e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} \xi} d\xi \right) \begin{bmatrix} I \\ 0 \end{bmatrix} Bu(y,t)dy \right\}, \tag{15.99}$$

which we write more compactly as

$$U(t) = KM(D)X(t) + K\int_0^D \left(\int_0^{D-y} M(\xi)d\xi \right) Bu(y,t)dy, \tag{15.100}$$

and develop a form of this compensator where the integral term is expressed in terms of $U(t)$ rather than $u(y,t)$.

We start by noting that with the initial condition $u(x,0) = u_0(x)$ set to zero, the solution $u(x,t)$ is given by

$$u(x,t) = \frac{\cosh(x\sqrt{s})}{\cosh(D\sqrt{s})}[U(t)]. \tag{15.101}$$

Note that we are not carelessly mixing the time-domain and frequency-domain notations. The transfer function $\cosh(x\sqrt{s})/\cosh(D\sqrt{s})$ is to be understood as an operator acting on the signal $U(t)$ in the brackets $[\cdot]$.

The control law is now obtained as

$$U(t) = KM(D)X(t) + H(s)[U(t)], \tag{15.102}$$

where the transfer function $H(s)$ is given by

$$H(s) = K\int_0^D \left(\int_0^{D-y} M(\xi)d\xi \right) B\frac{\cosh(y\sqrt{s})}{\cosh(D\sqrt{s})}dy. \tag{15.103}$$

We return to the question of the form of this compensator in an example in the next section; see (15.116) and (15.118).

15.5 On Differences Between Compensation of Delay Dynamics and Diffusion Dynamics

It is of interest to try to elucidate the difference between predictor feedback for systems with input delay, which is given by

$$U(t) = K\mathrm{e}^{AD}X(t) \\ + K\int_0^D \mathrm{e}^{A(D-y)}Bu(y,t)dy\,, \tag{15.104}$$

and feedback that compensates the diffusive input dynamics, which is given by

$$U(t) = K\begin{bmatrix} I & 0 \end{bmatrix}\mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}D}\begin{bmatrix} I \\ 0 \end{bmatrix}X(t) \\ + \int_0^D K\begin{bmatrix} I & 0 \end{bmatrix}\left(\int_0^{D-y}\mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}\xi}d\xi\right)\begin{bmatrix} I \\ 0 \end{bmatrix}Bu(y,t)dy\,. \tag{15.105}$$

While the feedback (15.104) for the delay case is "obvious" in retrospect, since it is based on D-seconds-ahead "prediction," the controller (15.105) is a rather non-obvious choice. The controller (15.59)–(15.62) with an arbitrarily fast decay rate is an even less obvious choice.

The example in Section 15.2 is useful in illustrating our design as well as in illustrating the difference between the designs for the delay/predictor problems in Chapter 2 and the designs in this chapter. For the sake of contrast, consider the problem

$$\dot{X}(t) = X(t) + U(t-D)\,, \tag{15.106}$$

which can be equivalently represented as

$$\dot{X}(t) = X(t) + u(0,t)\,, \tag{15.107}$$

$$u_t(x,t) = u_x(x,t)\,, \tag{15.108}$$

$$u(D,t) = U(t)\,. \tag{15.109}$$

For this system, the predictor design is

$$U(t) = -(1+h)\left[\mathrm{e}^{D}X(t) + \int_0^D \mathrm{e}^{D-y}u(y,t)dy\right], \tag{15.110}$$

whereas for the diffusion problem in Section 15.2, the design is (we repeat it here for convenience)

$$U(t) = -(1+h)\left[\sinh(D)X(t) + \int_0^D (\cosh(D-y)-1)u(y,t)dy\right]. \tag{15.111}$$

Both designs compensate for the actuator dynamics through the use of exponentials. The difference is subtle (straight exponential versus hyperbolic sine and cosine).

The reader should also note that $u(y,t)$ is not the same signal in (15.110) and (15.111) since it comes from two different systems (transport PDE and heat PDE, respectively). For the case of input delay, the compensator can be written as

$$U(t) = -(1+h)\mathrm{e}^{D}X(t) - (1+h)\frac{1-\mathrm{e}^{D(1-s)}}{s-1}[U(t)]\,, \tag{15.112}$$

where we have used the fact that when the initial conditions of the input dynamics are zero, the state $u(x,t)$ is given by

$$u(x,t) = \mathrm{e}^{(x-D)s}[U(t)]\,. \tag{15.113}$$

For the case of input dynamics governed by the heat equation, we first recall that

$$u(x,t) = \frac{\cosh(x\sqrt{s})}{\cosh(D\sqrt{s})}[U(t)] \tag{15.114}$$

and obtain the controller as

$$\begin{aligned} U(t) &= -(1+h)\sinh(D)X(t) \\ &\quad + (1+h)\left(\int_0^D (1-\cosh(D-y))\frac{\cosh(y\sqrt{s})}{\cosh(D\sqrt{s})}dy\right)[U(t)]\,. \end{aligned} \tag{15.115}$$

We write this control law compactly as

$$U(t) = -(1+h)\sinh(D)X(t) + (1+h)G(s)[U(t)]\,, \tag{15.116}$$

where

$$G(s) = \int_0^D (1-\cosh(D-y))\frac{\cosh(y\sqrt{s})}{\cosh(D\sqrt{s})}dy\,. \tag{15.117}$$

The integral in $G(s)$ can be computed explicitly. The resulting expression is

$$\begin{aligned} G(s) &= \frac{\sinh(D\sqrt{s})}{\sqrt{s}\cosh(D\sqrt{s})} \\ &\quad - \frac{(\sqrt{s}+1)(\sinh(D\sqrt{s})-\sinh(D)) + (\sqrt{s}-1)(\sinh(D\sqrt{s})+\sinh(D))}{2(s-1)\cosh(D\sqrt{s})}\,. \end{aligned} \tag{15.118}$$

Comparing the input compensator term $-(1-\mathrm{e}^{D(1-s)})/(s-1)$ in (15.112), which is obtained in place of $G(s)$ for the delay case, with $G(s)$ used as the input compensator in the heat equation case, we observe the marked increase in the complexity of compensating for the diffusive input dynamics relative to the pure transport input dynamics. Another difference is that the compensator term $-(1-\mathrm{e}^{D(1-s)})/(s-1)$ employs only the values of $U(\theta)$ for the time window $[t-D,t]$, whereas it is not clear if this is the case with the compensator $G(s)$.

15.6 Notes and References

Though various finite-dimensional forms of actuator dynamics (consisting of linear and nonlinear integrators) have been successfully tackled in the context of the backstepping methods, realistic forms of infinite-dimensional actuator and sensor dynamics different than pure delays have not received attention. In this chapter we developed explicit formulas for full-state control laws and observers in the presence of diffusion-governed actuator and sensor dynamics.

Since using the control law (15.10), we have established the stabilizability of system (15.1)–(15.4), other control designs should be possible—both heuristic control designs for *some* pairs (A,B) and other systematic PDE-based control designs for all stabilizable pairs (A,B). For example, an optimal control problem could be formulated with quadratic penalties on $X(t)$ and $U(t)$, as well as an L_2 penalty (in x) on $u(x,t)$, yielding an operator Riccati equation-based control law. This alternative would lack the explicit character of the control law (15.10).

Though we focused on purely diffusion-based actuator dynamics $u_t(x,t) = u_{xx}(x,t)$, as in Chapter 14 where the design was extended from the transport PDE actuator dynamics to a whole class of first-order hyperbolic PDEs, there is no obstacle to extending the results in this chapter to a broader class of parabolic PDEs, for example, to diffusion-advection actuator dynamics

$$u_t(x,t) = u_{xx}(x,t) + bu_x(x,t)\,, \tag{15.119}$$

where b can have any value (a particularly interesting case would be $b < 0$ with $|b|$ large), or to reaction-diffusion dynamics

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t)\,, \tag{15.120}$$

which can have many unstable eigenvalues (for $\lambda > 0$ and large), or to much more complex dynamics governed by partial integro-differential equations of the parabolic type.

Another interesting problem would be to consider the problem of an ODE with diffusive input dynamics with an unknown diffusion coefficient, namely,

$$\dot{X}(t) = AX(t) + Bu(0,t)\,, \tag{15.121}$$

$$u_t(x,t) = \rho u_{xx}(x,t)\,, \tag{15.122}$$

$$u_x(0,t) = 0\,, \tag{15.123}$$

$$u(1,t) = U(t)\,, \tag{15.124}$$

where ρ is completely unknown (but positive). This problem is similar to the adaptive control problem in Chapter 7. To approach it, one would bring to bear the adaptive design techniques from [117].

Chapter 16
ODEs with Wave PDE Actuator Dynamics

In Chapter 15 we provided a first extension of the predictor feedback concept to systems whose actuator dynamics are infinite-dimensional and more complex than a simple pure delay. We provided a design for actuator dynamics governed by the heat PDE.

In this chapter we consider actuator dynamics governed by the wave PDE, a rather more challenging problem than that in Chapter 15.

To imagine the physical meaning of having a wave PDE in the actuation path, one can think of having to stabilize a system to whose input one has access through a *string*. The challenges of overcoming string/wave dynamics in the actuation path include their infinite dimension, the finite (limited) propagation speed of the control signal (large control doesn't help), and the fact that all of their (infinitely many) eigenvalues are on the imaginary axis.

The problem studied here is more challenging than that in Chapter 15 due to another difficulty—the PDE system is second-order in time, which means that the state is "doubly infinite-dimensional" (distributed displacement and distributed velocity). This is not so much of a problem dimensionally as it is a problem in constructing the state transformations for compensating the PDE dynamics. One has to deal with the coupling of two infinite-dimensional states.

As in Chapters 2 and 15, we design feedback laws that are given by explicit formulas. We start in Section 16.1 with an actuator compensation design with full-state feedback and present a stability proof in Section 16.2. In Section 16.3 we approach the question of robustness of our infinite-dimensional feedback law with respect to a small uncertainty in the wave propagation speed and provide an affirmative answer. While in Section 16.1 we present a design that uses the Neumann boundary actuation in the wave equation, in Section 16.4 we present a rather different alternative design that employs Dirichlet actuation, which is quite uncommon in the area of boundary control of wave PDEs. In Section 16.5 we cast the compensator in terms of the input signal rather than in terms of the state of the heat equation. We provide examples of controller design in Section 16.6. Finally, in Section 16.7 we specialize our feedback laws to the case of an undamped wave PDE alone and provide feedback laws for its stabilization.

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Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_16,

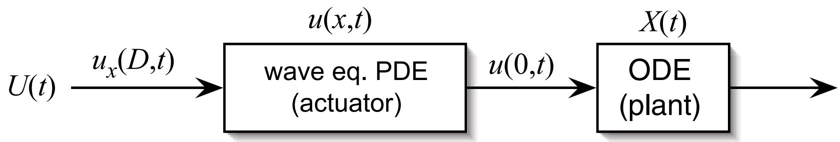

Fig. 16.1 The cascade of the wave equation PDE dynamics of the actuator with the ODE dynamics of the plant.

We return to problems involving ODE and wave PDE dynamics in Chapter 17. In Section 17.4 we develop a dual of our actuator dynamics compensator in this chapter and design an infinite-dimensional observer that compensates the wave PDE dynamics of the sensor. In Section 17.6 we combine an ODE observer with the full-state feedback compensator of the wave PDE actuator dynamics in this chapter and establish a form of a separation principle, where the observer-based compensator is stabilizing for the overall systems consisting of the ODE plant, ODE observer, PDE actuator dynamics, and PDE observer. The observer for the wave PDE actuator dynamics is designed in a particular way to ensure convergence of the infinite-dimensional estimation error state, since the wave PDE actuator dynamics are only neutrally stable (not exponentially stable), so a simple copy of the PDE model does not suffice as a choice for the observer (output injection is needed).

16.1 Control Design for Wave PDE Compensation with Neumann Actuation

We consider the cascade of a wave (string) equation and an LTI finite-dimensional system given by

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{16.1}$$

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{16.2}$$

$$u_x(0,t) = 0, \tag{16.3}$$

$$u_x(D,t) = U(t), \tag{16.4}$$

where $X \in \mathbb{R}^n$ is the ODE state, U is the scalar input to the entire system, and $u(x,t)$ is the state of the PDE dynamics of the actuator governed by a wave equation. The cascade system is depicted in Fig. 16.1.

The Neumann actuation choice $u_x(D,t) = U(t)$ is pursued because this is a natural physical choice since $u_x(D,t)$ corresponds to a force on the string's boundary. In Section 16.4 we address the case of an alternative actuation choice, Dirichlet actuation via $u(D,t) = U(t)$.

The length of the PDE domain (see Fig. 16.2), D, is arbitrary. Thus, we take the wave propagation speed to be unity without a loss of generality. We assume that

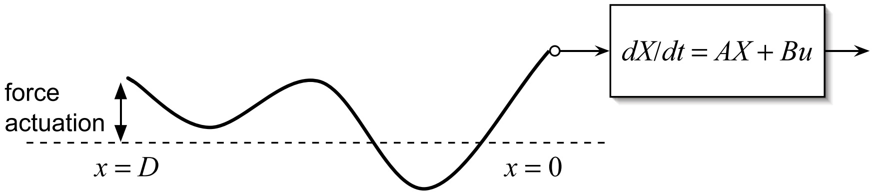

Fig. 16.2 An arbitrary ODE controlled through a string.

the pair (A,B) is stabilizable and take K to be a known vector such that $A+BK$ is Hurwitz.

Before we start, we remark that the PDE system (16.2)–(16.4) has a transfer function representation

$$u(0,t) = \frac{1}{\sinh(Ds)}[U(t)]\,. \tag{16.5}$$

This means that while the ODE plant with input delay has a transfer function representation

$$X(s) = (sI-A)^{-1}B\mathrm{e}^{-sD}U(s)\,, \tag{16.6}$$

the ODE plant with a wave PDE at the input has a transfer function representation

$$X(s) = (sI-A)^{-1}B\frac{1}{\sinh(Ds)}U(s)\,. \tag{16.7}$$

Now we start with our design. We seek an invertible transformation

$$(X,u,u_t) \mapsto (X,v,v_t) \tag{16.8}$$

that converts (16.1)–(16.3) into

$$\dot{X}(t) = (A+BK)X(t) + Bv(0,t), \tag{16.9}$$

$$v_{tt}(x,t) = v_{xx}(x,t), \tag{16.10}$$

$$v_x(0,t) = 0\,, \tag{16.11}$$

where the v-system is an undamped string equation with a free end at $x=0$, and then another transformation

$$(X,v,v_t) \mapsto (X,w,w_t) \tag{16.12}$$

that converts (16.9)–(16.11) into

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \tag{16.13}$$

$$w_{tt}(x,t) = w_{xx}(x,t), \tag{16.14}$$

$$w_x(0,t) = c_0 w(0,t)\,, \qquad c_0 > 0\,, \tag{16.15}$$

which is an undamped string equation with a “spring/stiffness” boundary condition at the end $x = 0$.

We also seek a feedback law that achieves

$$w_x(D,t) = -c_1 w_t(D,t)\,, \qquad c_1 > 0\,, \tag{16.16}$$

which represents boundary damping at the end $x = D$, which has a damping effect on the entire domain [due to a nonobvious fact that the operator from $w_x(D,t)$ to $w_t(D,t)$ is passive and the w-system is zero-state-observable].

The system (16.13)–(16.16) is exponentially stable, as we shall see. With the invertibility of the composite transformation

$$(X, u, u_t) \mapsto (X, w, w_t)\,, \tag{16.17}$$

we will achieve exponential stability of the closed-loop system in the original variables (X, u, u_t).

We postulate the transformation $(X, u, u_t) \mapsto (X, v, v_t)$ in the form

$$\begin{aligned} v(x,t) = u(x,t) &- \int_0^x k(x,y)u(y,t)dy - \int_0^x l(x,y)u_t(y,t)dy \\ &- \gamma(x)X(t)\,, \end{aligned} \tag{16.18}$$

where the kernel functions $k(x,y)$, $l(x,y)$, and $\gamma(x)$ are to be found. We will need the expressions for $v_{tt}(x,t)$, $v_{xx}(x,t)$, $v_x(0,t)$, and $v(0,t)$ to derive the conditions on the kernels that result in the target system (16.9)–(16.11). In deriving the expression for $v_{tt}(x,t)$, we will need the expression for $\ddot{X}(t)$, which is given by

$$\begin{aligned} \ddot{X}(t) &= A\dot{X}(t) + Bu_t(0,t) \\ &= A^2X(t) + ABu(0,t) + Bu_t(0,t)\,. \end{aligned} \tag{16.19}$$

Then, differentiating (16.18) twice with respect to t, substituting $v_{tt}(x,t) = v_{xx}(x,t)$, and integrating twice by parts with respect to y, we obtain

$$\begin{aligned} v_{tt}(x,t) = u_{xx}(x,t) &- k(x,x)u_x(x,t) + k_y(x,x)u(x,t) \\ &- k_y(x,0)u(0,t) - \int_0^x k_{yy}(x,y)u(y,t)dy \\ &- l(x,x)u_{xt}(x,t) + l_y(x,x)u_t(x,t) \\ &- l_y(x,0)u_t(0,t) - \int_0^x l_{yy}(x,y)u_t(y,t)dy \\ &- \gamma(x)A^2X(t) - \gamma(x)ABu(0,t) - \gamma(x)Bu_t(0,t)\,. \end{aligned} \tag{16.20}$$

Differentiating (16.18) once and twice with respect to x, we get, respectively,

$$\begin{aligned}v_x(x,t) = u_x(x,t) - k(x,x)u(x,t) - \int_0^x k_x(x,y)u(y,t)dy,\\ - l(x,x)u_t(x,t) - \int_0^x l_x(x,y)u_t(y,t)dy\\ - \gamma'(x)X(t)\end{aligned} \tag{16.21}$$

and

$$\begin{aligned}v_{xx}(x,t) = u_{xx}(x,t) - (k(x,x))'u(x,t) - k(x,x)u_x(x,t)\\ - k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)dy\\ - (l(x,x))'u_t(x,t) - l(x,x)u_{xt}(x,t)\\ - l_x(x,x)u_t(x,t) - \int_0^x l_{xx}(x,y)u_t(y,t)dy\\ - \gamma''(x)X(t).\end{aligned} \tag{16.22}$$

Then we obtain

$$\dot{X}(t) = X(t) + B\big(v(0,t) + \gamma(0)X(t)\big), \tag{16.23}$$

$$\begin{aligned}v_{tt}(x,t) - v_{xx}(x,t) = \Big(\big(k(x,x)\big)' + k_x(x,x) + k_y(x,x)\Big)u(x,t)\\ + \Big(\big(l(x,x)\big)' + l_x(x,x) + l_y(x,x)\Big)u_t(x,t)\\ - \Big(k_y(x,0) + \gamma(x)AB\Big)u(0,t)\\ - \Big(l_y(x,0) + \gamma(x)B\Big)u_t(0,t)\\ + \Big(\gamma''(x) - \gamma(x)A^2\Big)X(t)\\ + \int_0^x \Big(k_{xx}(x,y) - k_{yy}(x,y)\Big)u(y,t)dy\\ + \int_0^x \Big(l_{xx}(x,y) - l_{yy}(x,y)\Big)u_t(y,t)dy,\end{aligned} \tag{16.24}$$

$$\begin{aligned}v_x(0,t) = -k(0,0)u(0,t) - l(0,0)u_t(0,t)\\ - \gamma'(0)X(t),\end{aligned} \tag{16.25}$$

By matching this system with (16.9)–(16.11), we get the following conditions on the kernels:

$$\gamma''(x) = \gamma(x)A^2, \tag{16.26}$$
$$\gamma(0) = K, \tag{16.27}$$
$$\gamma'(0) = 0, \tag{16.28}$$

$$l_{xx}(x,y) = l_{yy}(x,y), \tag{16.29}$$
$$l(x,x) = 0, \tag{16.30}$$
$$l_y(x,0) = -\gamma(x)B, \tag{16.31}$$

$$k_{xx}(x,y) = k_{yy}(x,y), \tag{16.32}$$
$$k(x,x) = 0, \tag{16.33}$$
$$k_y(x,0) = -\gamma(x)AB. \tag{16.34}$$

These differential equations can be solved explicitly. The solutions are

$$\gamma(x) = KM(x), \tag{16.35}$$
$$l(x,y) = m(x-y), \tag{16.36}$$
$$k(x,y) = \mu(x-y), \tag{16.37}$$
$$M(x) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{16.38}$$
$$m(s) = \int_0^s \gamma(\xi) B d\xi, \tag{16.39}$$
$$\mu(s) = \int_0^s \gamma(\xi) AB d\xi. \tag{16.40}$$

Thus, the transformation $(X, u, u_t) \mapsto (X, v, v_t)$ is defined as

$$\begin{aligned} v(x,t) = u(x,t) &- \int_0^x \mu(x-y)u(y,t)dy \\ &- \int_0^x m(x-y)u_t(y,t)dy - \gamma(x)X(t), \end{aligned} \tag{16.41}$$
$$\begin{aligned} v_t(x,t) = u_t(x,t) - KBu(x,t) &- \int_0^x \mu(x-y)u_t(y,t)dy \\ &- \int_0^x m''(x-y)u(y,t)dy - \gamma(x)AX(t). \end{aligned} \tag{16.42}$$

With similar derivations, one can show that the inverse of the transformation $(X, u, u_t) \mapsto (X, v, v_t)$ is defined as

$$\begin{aligned} u(x,t) = v(x,t) &- \int_0^x \sigma(x-y)v(y,t)dy \\ &- \int_0^x n(x-y)v_t(y,t)dy - \rho(x)X(t), \end{aligned} \tag{16.43}$$

$$u_t(x,t) = v_t(x,t) + KBv(x,t) - \int_0^x \sigma(x-y)v_t(y,t)dy - \int_0^x n''(x-y)v(y,t)dy - \rho(x)AX(t)\,, \tag{16.44}$$

where

$$\rho(x) = -KN(x), \tag{16.45}$$

$$N(x) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & (A+BK)^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{16.46}$$

$$n(s) = \int_0^s \rho(\xi)Bd\xi, \tag{16.47}$$

$$\sigma(s) = \int_0^s \rho(\xi)ABd\xi\,. \tag{16.48}$$

The transformation $(X,v,v_t) \mapsto (X,w,w_t)$ is simpler and given by

$$w(x,t) = v(x,t) + c_0 \int_0^x v(y,t)dy, \tag{16.49}$$

$$w_t(x,t) = v_t(x,t) + c_0 \int_0^x v_t(y,t)dy\,, \tag{16.50}$$

whereas its inverse is

$$v(x,t) = w(x,t) - c_0 \int_0^x \mathrm{e}^{-c_0(x-y)} w(y,t)dy, \tag{16.51}$$

$$v_t(x,t) = w_t(x,t) - c_0 \int_0^x \mathrm{e}^{-c_0(x-y)} w_t(y,t)dy\,. \tag{16.52}$$

The composite transformation $(X,u,u_t) \mapsto (X,w,w_t)$ is

$$\begin{aligned} w(x,t) = {} & u(x,t) \\ & + \int_0^x \left(c_0 - \mu(x-y) - c_0 \int_0^{x-y} \mu(\xi)d\xi \right) u(y,t)dy \\ & - \int_0^x \left(m(x-y) + c_0 \int_0^{x-y} m(\xi)d\xi \right) u_t(y,t)dy \\ & - \left(\gamma(x) + c_0 \int_0^x \gamma(\xi)d\xi \right) X(t), \end{aligned} \tag{16.53}$$

$$\begin{aligned} w_t(x,t) = {} & u_t(x,t) - KBu(x,t) \\ & - \int_0^x \left(c_0 m'(x-y) + m''(x-y) \right) u(y,t)dy \end{aligned}$$

$$+\int_0^x \left(c_0-\mu(x-y)-c_0\int_0^{x-y}\mu(\xi)d\xi\right)u_t(y,t)dy$$
$$-\left(\gamma(x)+c_0\int_0^x\gamma(\xi)d\xi\right)AX(t), \tag{16.54}$$

and its inverse is

$$\begin{aligned}u(x,t)=w(x,t)&\\ &-\int_0^x\left(c_0\mathrm{e}^{-c_0(x-y)}+\sigma(x-y)\right.\\ &\left.-c_0\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}\sigma(\xi)d\xi\right)w(y,t)dy\\ &-\int_0^x\left(n(x-y)-c_0\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}n(\xi)d\xi\right)w_t(y,t)dy\\ &-\rho(x)X(t),\end{aligned} \tag{16.55}$$

$$\begin{aligned}u_t(x,t)=w_t(x,t)+KBw(x,t)&\\ &+\int_0^x\left(n''(x-y)-c_0n'(x-y)+c_0^2n(x-y)\right.\\ &\left.-c_0^3\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}n(\xi)d\xi\right)w(y,t)dy\\ &-\int_0^x\left(c_0\mathrm{e}^{-c_0(x-y)}+\sigma(x-y)\right.\\ &\left.-c_0\int_0^{x-y}\mathrm{e}^{-c_0(x-y-\xi)}\sigma(\xi)d\xi\right)w_t(y,t)dy\\ &-\rho(x)AX(t).\end{aligned} \tag{16.56}$$

Next, we design a controller that satisfies the boundary condition (16.16). First, from (16.53), we get

$$\begin{aligned}w_x(x,t)=u_x(x,t)+c_0u(x,t)&\\ &-\int_0^x\left(\mu'(x-y)+c_0\mu(x-y)\right)u(y,t)dy\\ &-\int_0^x\left(m'(x-y)+c_0m(x-y)\right)u_t(y,t)dy\\ &-\left(\gamma'(x)+c_0\gamma(x)\right)X(t).\end{aligned} \tag{16.57}$$

Then the control law is

$$\begin{aligned}U(t)=(-c_0+c_1KB)u(D,t)-c_1u_t(D,t)&\\ &+\int_0^D p(D-y)u(y,t)dy+\int_0^D q(D-y)u_t(y,t)dy\\ &+\pi(D)X(t),\end{aligned} \tag{16.58}$$

where

$$p(s) = \mu'(s) + c_0\mu(s) + c_1\left(m''(s) + c_0 m'(s)\right), \tag{16.59}$$

$$q(s) = m'(s) + c_0 m(s) + c_1\left(\mu(s) + c_0\int_0^s \mu(\xi)d\xi - c_0\right), \tag{16.60}$$

$$\pi(x) = \gamma'(x) + \gamma(x)(c_0 I + c_1 A) + c_1 c_0\int_0^x \gamma(\xi)d\xi A. \tag{16.61}$$

16.2 Stability of the Closed-Loop System

Now we state a stability result for our controller that compensates the *wave PDE* actuator dynamics.

Theorem 16.1 (Stability). *Consider a closed-loop system consisting of the plant (16.1)–(16.4) and the control law (16.58). For any initial condition such that* $u(\cdot,0) \in H^1(0,D)$ *and* $u_t(\cdot,0) \in L^2(0,D)$*, the closed-loop system has a unique solution*

$$\left(X(t), u(\cdot,t), u_t(\cdot,t)\right) \in C([0,\infty), \mathbb{R}^n \times H^1(0,D) \times L^2(0,D)) \tag{16.62}$$

and is exponentially stable in the sense of the norm

$$\left(|X(t)|^2 + u(0,t)^2 + \int_0^D u_x(x,t)^2 dx + \int_0^D u_t(x,t)^2 dx\right)^{1/2}. \tag{16.63}$$

Moreover, if the initial condition $\left(u(\cdot,0), u_t(\cdot,0)\right)$ *is compatible with the control law (16.58) and belongs to* $H^2(0,D) \times H^1(0,D)$*, then*

$$\left(X(t), u(\cdot,t), u_t(\cdot,t)\right) \in C^1([0,\infty), \mathbb{R}^n \times H^1(0,D) \times L^2(0,D)) \tag{16.64}$$

is the classical solution of the closed-loop system.

Proof. We will use the system norms

$$\Omega(t) = u(0,t)^2 + \|u_x(t)\|^2 + \|u_t(t)\|^2 + |X(t)|^2, \tag{16.65}$$

$$\Xi(t) = w(0,t)^2 + \|w_x(t)\|^2 + \|w_t(t)\|^2 + |X(t)|^2, \tag{16.66}$$

where $\|u(t)\|^2$ is a compact notation for $\int_0^D u(x,t)^2\,dx$. In addition, we employ a Lyapunov function

$$V(t) = X(t)^T P X(t) + aE(t), \tag{16.67}$$

where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^T P = -Q \tag{16.68}$$

for some $Q = Q^T > 0$, the parameter $a > 0$ is to be chosen later, and the function $E(t)$ is defined by

$$E(t) = \frac{1}{2}\left(c_0 w(0,t)^2 + \|w_x(t)\|^2 + \|w_t(t)\|^2\right) + \delta\int_0^D (1+y)w_x(y,t)w_t(y,t)dy, \tag{16.69}$$

where $\delta > 0$ is an analysis parameter to be chosen later. First, we observe that

$$\theta_1 \Xi \le V \le \theta_2 \Xi, \tag{16.70}$$

where

$$\theta_1 = \min\left\{\lambda_{\min}(P),\ \frac{ac_0}{2}, \frac{a}{2}(1-\delta(1+D))\right\}, \tag{16.71}$$

$$\theta_2 = \max\left\{\lambda_{\max}(P),\ \frac{ac_0}{2},\ \frac{a}{2}(1+\delta(1+D))\right\}, \tag{16.72}$$

where we choose

$$0 < \delta < \frac{1}{1+D} \tag{16.73}$$

to ensure that $\theta_1 > 0$ as well as to exploit a positive δ in the subsequent Lyapunov analysis. By using (16.54), (16.56), (16.57), and

$$\begin{aligned} u_x(x,t) = {} & w_x(x,t) - c_0 w(x,t) \\ & - \int_0^x \Big(-c_0^2 \mathrm{e}^{-c_0(x-y)} + \sigma'(x-y) - c_0\sigma(x-y) \\ & + c_0^2\int_0^{x-y} \mathrm{e}^{-c_0(x-y-\xi)}\sigma(\xi)d\xi\Big) w(y,t)dy \\ & - \int_0^x \Big(n'(x-y) - c_0 n(x-y) \\ & + c_0^2\int_0^{x-y} \mathrm{e}^{-c_0(x-y-\xi)}n(\xi)d\xi\Big) w_t(y,t)dy \\ & - \rho'(x)X(t), \end{aligned} \tag{16.74}$$

$$u(0,t) = w(0,t) + KX(t), \tag{16.75}$$

and by using Poincaré's inequality, we show next that there exist positive constants θ_3, θ_4 such that

$$\theta_3 \Xi \le \Omega \le \theta_4 \Xi. \tag{16.76}$$

To show this fact, we first write (16.54), (16.56), (16.57), and (16.74) as

$$\begin{aligned} w_x(x) = {} & u_x(x) + c_0 u(x) \\ & + a_1(x) \star u(x) + a_2(x) \star u_t(x) + a_3(x)X, \end{aligned} \tag{16.77}$$

$$\begin{aligned} w_t(x) &= u_t(x) - KBu(x) \\ &\quad + b_1(x) \star u(x) + b_2(x) \star u_t(x) + b_3(x)X, \end{aligned} \tag{16.78}$$

$$\begin{aligned} u_x(x) &= w_x(x) - c_0 w(x) \\ &\quad + \alpha_1(x) \star w(x) + \alpha_2(x) \star w_t(x) + \alpha_3(x)X, \end{aligned} \tag{16.79}$$

$$\begin{aligned} u_t(x) &= w_t(x) + KBw(x) \\ &\quad + \beta_1(x) \star w(x) + \beta_2(x) \star w_t(x) + \beta_3(x)X\,, \end{aligned} \tag{16.80}$$

where

$$a_1(x) = -\left(\mu'(x) + c_0\mu(x)\right), \tag{16.81}$$

$$a_2(x) = -\left(m'(x) + c_0 m(x)\right), \tag{16.82}$$

$$a_3(x) = -\left(\gamma'(x) + c_0\gamma(x)\right), \tag{16.83}$$

$$b_1(x) = -\left(m''(x) + c_0 m'(x)\right), \tag{16.84}$$

$$b_2(x) = c_0 - \left(\mu(x) + c_0 \int_0^x \mu(\xi)d\xi\right), \tag{16.85}$$

$$b_3(x) = -\left(\gamma(x) + c_0 \int_0^x \gamma(\xi)d\xi\right), \tag{16.86}$$

and

$$\alpha_1(x) = c_0^2 \mathrm{e}^{-c_0 x} - \sigma'(x) + c_0^2\sigma(x) - c_0^2 \mathrm{e}^{-c_0 x} \star \sigma(x), \tag{16.87}$$

$$\alpha_2(x) = -n'(x) + c_0 n(x) - c_0^2 \mathrm{e}^{-c_0 x} \star n(x), \tag{16.88}$$

$$\alpha_3(x) = -\rho'(x), \tag{16.89}$$

$$\beta_1(x) = n''(x) - c_0 n'(x) + c_0^2 n(x) - c_0^3 \mathrm{e}^{-c_0 x} \star n(x), \tag{16.90}$$

$$\beta_2(x) = -c_0 \mathrm{e}^{-c_0 x} - \sigma(x) + c_0 \mathrm{e}^{-c_0 x} \star \sigma(x), \tag{16.91}$$

$$\beta_3(x) = -\rho(x)A\,. \tag{16.92}$$

With the Young and Cauchy–Schwartz inequalities, we get the inequalities

$$\begin{aligned} \|w_x\|^2 \le 5\bigg(&\|u_x\|^2 + \left(c_0^2 + D\|a_1\|^2\right)\|u\|^2 \\ &+ D\|a_2\|^2\|u_t\|^2 + \|a_3\|^2|X|^2 \bigg), \end{aligned} \tag{16.93}$$

$$\begin{aligned} \|w_t\|^2 \le 5\bigg(&\|u_t\|^2 + \left((KB)^2 + D\|b_1\|^2\right)\|u\|^2 \\ &+ D\|b_2\|^2\|u_t\|^2 + \|b_3\|^2|X|^2 \bigg), \end{aligned} \tag{16.94}$$

$$\|u_x\|^2 \le 5\Big(\|w_x\|^2 + \left(c_0^2 + D\|\alpha_1\|^2\right) \|w\|^2 + D\|\alpha_2\|^2\|w_t\|^2 + \|\alpha_3\|^2|X|^2 \Big), \tag{16.95}$$

$$\|u_t\|^2 \le 5\Big(\|w_t\|^2 + \left((KB)^2 + D\|\beta_1\|^2\right) \|w\|^2 + D\|\beta_2\|^2\|w_t\|^2 + \|\beta_3\|^2|X|^2 \Big). \tag{16.96}$$

With these inequalities, we get

$$\begin{aligned}\Xi(t) \le\ & 2u^2(0,t) + 5\|u_x(t)\|^2 \\ & + 5\left(c_0^2 + (KB)^2 + \|a_1\|^2 + \|b_1\|^2 \right) \|u(t)\|^2 \\ & + 5\left(1 + \|a_2\|^2 + \|b_2\|^2 \right) \|u_t(t)\|^2 \\ & + \left(1 + 2(KB)^2 + 5\left(\|a_2\|^2 + \|b_2\|^2\right) \right) |X(t)|^2 \end{aligned} \tag{16.97}$$

and

$$\begin{aligned}\Omega(t) \le\ & 2w^2(0,t) + 5\|w_x(t)\|^2 \\ & + 5\left(c_0^2 + (KB)^2 + \|\alpha_1\|^2 + \|\beta_1\|^2 \right) \|w(t)\|^2 \\ & + 5\left(1 + \|\alpha_2\|^2 + \|\beta_2\|^2 \right) \|w_t(t)\|^2 \\ & + \left(1 + 2(KB)^2 + 5\left(\|\alpha_2\|^2 + \|\beta_2\|^2\right) \right) |X(t)|^2 . \end{aligned} \tag{16.98}$$

Now we recall that, on a nonunity interval, the Poincaré inequality is given as

$$\|u(t)\| \le 2u^2(0,t) + 4D^2\|u_x(t)\|^2, \tag{16.99}$$

$$\|w(t)\| \le 2w^2(0,t) + 4D^2\|w_x(t)\|^2 . \tag{16.100}$$

Substituting these inequalities into the inequalities for $\Xi(t)$ and $\Omega(t)$, we get

$$\begin{aligned}\Xi(t) \le\ & 2\left[1 + 5\left(c_0^2 + (KB)^2 + \|a_1\|^2 + \|b_1\|^2 \right) \right] u^2(0,t) \\ & + 5\left[1 + 4D^2\left(c_0^2 + (KB)^2 + \|a_1\|^2 + \|b_1\|^2 \right) \right] \|u_x(t)\|^2 \\ & + 5\left(1 + \|a_2\|^2 + \|b_2\|^2 \right) \|u_t(t)\|^2 \\ & + \left(1 + 2(KB)^2 + 5\left(\|a_2\|^2 + \|b_2\|^2\right) \right) |X(t)|^2 \end{aligned} \tag{16.101}$$

and

$$
\begin{aligned}
\Omega(t) \le\, & 2\left[1+5\left(c_0^2+(KB)^2+\|\alpha_1\|^2+\|\beta_1\|^2\right)\right]w^2(0,t) \\
& +5\left[1+4D^2\left(c_0^2+(KB)^2+\|\alpha_1\|^2+\|\beta_1\|^2\right)\right]\|w_x(t)\|^2 \\
& +5\left(1+\|\alpha_2\|^2+\|\beta_2\|^2\right)\|w_t(t)\|^2 \\
& +\left(1+2(KB)^2+5\left(\|\alpha_2\|^2+\|\beta_2\|^2\right)\right)|X(t)|^2 .
\end{aligned} \tag{16.102}
$$

These finally yield

$$
\begin{aligned}
\theta_3^{-1} = \max\Bigg\{ & 2\left[1+5\left(c_0^2+(KB)^2+\|a_1\|^2+\|b_1\|^2\right)\right], \\
& 5\left[1+4D^2\left(c_0^2+(KB)^2+\|a_1\|^2+\|b_1\|^2\right)\right], \\
& 5\left(1+\|a_2\|^2+\|b_2\|^2\right)\|u_t(t)\|^2 \\
& +\left(1+2(KB)^2+5\left(\|a_2\|^2+\|b_2\|^2\right)\right)\Bigg\}
\end{aligned} \tag{16.103}
$$

and

$$
\begin{aligned}
\theta_4 = \max\Bigg\{ & 2\left[1+5\left(c_0^2+(KB)^2+\|\alpha_1\|^2+\|\beta_1\|^2\right)\right], \\
& 5\left[1+4D^2\left(c_0^2+(KB)^2+\|\alpha_1\|^2+\|\beta_1\|^2\right)\right], \\
& 5\left(1+\|\alpha_2\|^2+\|\beta_2\|^2\right), \\
& \left(1+2(KB)^2+5\left(\|\alpha_2\|^2+\|\beta_2\|^2\right)\right)\Bigg\} .
\end{aligned} \tag{16.104}
$$

Now we turn our attention to the Lyapunov functions $E(t)$ and $V(t)$. First, the derivative of $E(t)$ is calculated as

$$
\begin{aligned}
\dot{E}(t) = \, & c_0 w_t(0,t)w(0,t) \\
& +\int_0^D w_{tx}(y,t)w_x(y,t)dy+\int_0^D w_{xx}(y,t)w_t(y,t)dy \\
& +\delta\int_0^D (1+y)\big(w_{xt}(y,t)w_t(y,t)+w_x(y,t)w_{xx}(y,t)\big)dy
\end{aligned}
$$

$$
\begin{aligned}
&= w_x(0,t)w_t(0,t) + w_x(y,t)w_t(y,t)\Big|_{y=0}^{y=D} \\
&\quad + \frac{\delta}{2}(1+y)\left(w_t(y,t)^2 + w_x(y,t)^2\right)\Big|_{y=0}^{y=D} \\
&\quad - \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right) \\
&= w_x(D,t)w_t(D,t) + \frac{\delta}{2}(1+D)\left(w_t(D,t)^2 + w_x(D,t)^2\right) \\
&\quad - \frac{\delta}{2}\left(w_t(0,t)^2 + w_x(0,t)^2\right) \\
&\quad - \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right). \qquad (16.105)
\end{aligned}
$$

Substituting $w_x(0,t) = c_0 w(0,t)$ and $w_x(D,t) = -c_1 w_t(D,t)$, we get

$$
\begin{aligned}
\dot{E}(t) = &-\left(c_1 - \delta\frac{1+D}{2}\left(1+c_1^2\right)\right) w_t(D,t)^2 \\
&- \frac{\delta}{2}\left(w_t(0,t)^2 + c_0^2 w(0,t)^2\right) \\
&- \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right). \qquad (16.106)
\end{aligned}
$$

Taking the time derivative of $V(t)$, we obtain

$$
\begin{aligned}
\dot{V}(t) = &-X(t)^T QX(t) + 2X(t)^T PBw(0,t) \\
&- a\Bigg[\left(c_1 - \delta\frac{1+D}{2}\left(1+c_1^2\right)\right) w_t(D,t)^2 \\
&+ \frac{\delta}{2}\left(w_t(0,t)^2 + c_0^2 w(0,t)^2\right) \\
&+ \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right)\Bigg] \\
\leq &-\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 + \frac{2|PB|^2}{\lambda_{\min}(Q)} w^2(0,t) \\
&- a\Bigg[\left(c_1 - \delta\frac{1+D}{2}\left(1+c_1^2\right)\right) w_t(D,t)^2 \\
&+ \frac{\delta}{2}\left(w_t(0,t)^2 + c_0^2 w(0,t)^2\right) \\
&+ \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right)\Bigg]. \qquad (16.107)
\end{aligned}
$$

Then, by choosing

$$a \geq \frac{8|PB|^2}{\delta c_0^2 \lambda_{\min}(Q)}, \tag{16.108}$$

we get

$$\begin{aligned}\dot{V}(t) \leq & -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \frac{2|PB|^2}{\lambda_{\min}(Q)} w^2(0,t) \\ & - a\left[\left(c_1 - \delta \frac{1+D}{2}\left(1+c_1^2\right)\right) w_t(D,t)^2 + \frac{\delta}{2} w_t(0,t)^2 \right. \\ & \left. + \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right)\right]. \end{aligned} \tag{16.109}$$

Now choosing

$$\delta \leq \frac{2c_1}{(1+D)\left(1+c_1^2\right)}, \tag{16.110}$$

we arrive at

$$\begin{aligned}\dot{V}(t) & \leq -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2 - \frac{2|PB|^2}{\lambda_{\min}(Q)} w^2(0,t) - \frac{a\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right) \\ & \leq -\min\left\{\frac{\lambda_{\min}(Q)}{2}, \frac{2|PB|^2}{\lambda_{\min}(Q)}, \frac{a\delta}{2}\right\} \Xi(t). \end{aligned} \tag{16.111}$$

Then

$$\dot{V} \leq -\eta V, \tag{16.112}$$

where

$$\eta = \frac{1}{\theta_2} \min\left\{\frac{\lambda_{\min}(Q)}{2}, \frac{2|PB|^2}{\lambda_{\min}(Q)}, \frac{a\delta}{2}\right\}. \tag{16.113}$$

From (16.70), (16.76), and (16.112), it follows that

$$\Omega(t) \leq \frac{\theta_1 \theta_3}{\theta_2 \theta_4} \Omega(0) \mathrm{e}^{-\eta t}. \tag{16.114}$$

The rest of the argument is almost identical to [110]. □

16.3 Robustness to Uncertainty in the Wave Propagation Speed

We now study the robustness of the feedback law (16.58) to a small perturbation of the propagation speed in the actuator dynamics; i.e., we study the stability robustness of the closed-loop system

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{16.115}$$

$$u_{tt}(x,t) = (1+\varepsilon)u_{xx}(x,t), \tag{16.116}$$
$$u_x(0,t) = 0, \tag{16.117}$$
$$\begin{aligned} u_x(D,t) = {} & (-c_0 + c_1 KB)u(D,t) - c_1 u_t(D,t) \\ & + \int_0^D p(D-y)u(y,t)dy + \int_0^D q(D-y)u_t(y,t)dy \\ & + \pi(D)X(t) \end{aligned} \tag{16.118}$$

to the perturbation parameter ε, which we allow to be either positive or negative but small.

The following robustness result with respect to a small ε holds.

Theorem 16.2 (Robustness to small error in wave propagation speed). *Consider the closed-loop system (16.115)–(16.118). There exists a sufficiently small $\varepsilon^* > 0$ such that for all $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$, the closed-loop system has a unique solution*

$$\big(X(t), u(\cdot,t), u_t(\cdot,t)\big) \in C([0,\infty), \mathbb{R}^n \times H^1(0,D) \times L^2(0,D)) \tag{16.119}$$

for any initial condition such that $u(\cdot,0) \in H^1(0,D)$ and $u_t(\cdot,0) \in L^2(0,D)$, and the system is exponentially stable in the sense of the norm

$$\left(|X(t)|^2 + u(0,t)^2 + \int_0^D u_x(x,t)^2 dx + \int_0^D u_t(x,t)^2 dx\right)^{1/2}. \tag{16.120}$$

Moreover, if the initial condition $\big(u(\cdot,0), u_t(\cdot,0)\big)$ is compatible with the control law (16.118) and belongs to $H^2(0,D) \times H^1(0,D)$, then

$$\big(X(t), u(\cdot,t), u_t(\cdot,t)\big) \in C^1([0,\infty), \mathbb{R}^n \times H^1(0,D) \times L^2(0,D)) \tag{16.121}$$

is the classical solution of the closed-loop system.

Proof. With a very long calculation, we arrive at the representation of the system (16.115)–(16.118) in the w-variable:

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \tag{16.122}$$
$$w_{tt}(x,t) = (1+\varepsilon)w_{xx}(x,t) + \varepsilon \Pi(x,t), \tag{16.123}$$
$$w_x(0,t) = c_0 w(0,t), \tag{16.124}$$
$$w_x(D,t) = -c_1 w_t(D,t), \tag{16.125}$$

where

$$\begin{aligned} \Pi(x,t) = {} & \varsigma(x)w(0,t) \\ & + (KB)^2 w(x,t) + \int_0^x g(x-y)w(y,t)dy \end{aligned}$$

$$+KBw_t(x,t)+\int_0^x h(x-y)w_t(y,t)dy$$
$$+\vartheta(x)X(t) \tag{16.126}$$

and

$$\omega(x)=m''(x)+c_0m'(x), \tag{16.127}$$

$$g(x)=KB\phi(x)+KB\omega(x)+\int_0^x \omega(x-y)\phi(y)dy, \tag{16.128}$$

$$h(x)=KB\psi(x)+\omega(x)+\int_0^x \omega(x-y)\psi(y)dy, \tag{16.129}$$

$$\phi(x)=-n''(x)+c_0n'(x)-c_0^2n(x)+c_0^3\int_0^x \mathrm{e}^{-c_0(x-y)}n(y)dy, \tag{16.130}$$

$$\psi(x)=-c_0\mathrm{e}^{-c_0x}-\sigma(x)+c_0\int_0^x \mathrm{e}^{-c_0(x-y)}\sigma(y)dy, \tag{16.131}$$

$$\varsigma(x)=\left(\gamma(x)+c_0\int_0^x \gamma(\xi)d\xi\right)B, \tag{16.132}$$

$$\vartheta(x)=\left(\gamma(x)+c_0\int_0^x \gamma(\xi)d\xi\right)(A+BK)$$
$$-\left(KB\rho(x)+\int_0^x \omega(x-\xi)\rho(\xi)d\xi\right)A\,. \tag{16.133}$$

The state perturbation $\Pi(x,t)$ is very complicated in appearance, but $\int_0^D \Pi(x,t)^2dx$ can be bounded in terms of $\Xi(t)$ as defined in (16.66), and hence also in terms of $V(t)$ as defined in (16.67). Consequently, the same kind of Lyapunov analysis can be conducted as in the proof of Theorem 16.1, with a slightly modified Lyapunov function

$$E(t)=\frac{1}{2}\left\{(1+\varepsilon)\left[c_0w(0,t)^2+\|w_x(t)\|^2\right]+\|w_t(t)\|^2\right\}$$
$$+\delta\int_0^D (1+y)w_x(y,t)w_t(y,t)dy\,. \tag{16.134}$$

We start with

$$\dot{E}(t)=(1+\varepsilon)c_0w_t(0,t)w(0,t)$$
$$+(1+\varepsilon)\int_0^D w_{tx}(y,t)w_x(y,t)dy+\int_0^D (1+\varepsilon)w_{xx}(y,t)w_t(y,t)dy$$
$$+\varepsilon\int_0^D w_t(y,t)\Pi(y,t)dy$$
$$+\delta\int_0^D (1+y)\big(w_{xt}(y,t)w_t(y,t)+w_x(y,t)w_{xx}(y,t)\big)dy$$

$$
\begin{aligned}
&= (1+\varepsilon)\left(w_x(0,t)w_t(0,t) + w_x(y,t)w_t(y,t)\Big|_{y=0}^{y=D} \right) \\
&\quad + \varepsilon \int_0^D w_t(y,t)\Pi(y,t)dy \\
&\quad + \frac{\delta}{2}(1+y)\left(w_t(y,t)^2 + w_x(y,t)^2 \right)\Big|_{y=0}^{y=D} \\
&\quad - \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2 \right) \\
&= (1+\varepsilon)w_x(D,t)w_t(D,t) + \frac{\delta}{2}(1+D)\left(w_t(D,t)^2 + w_x(D,t)^2 \right) \\
&\quad + \varepsilon \int_0^D w_t(y,t)\Pi(y,t)dy \\
&\quad - \frac{\delta}{2}\left(w_t(0,t)^2 + w_x(0,t)^2 \right) \\
&\quad - \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2 \right).
\end{aligned} \tag{16.135}
$$

Substituting $w_x(0,t) = c_0 w(0,t)$ and $w_x(D,t) = -c_1 w_t(D,t)$, we get

$$
\begin{aligned}
\dot{E}(t) = &-\left(c_1(1+\varepsilon) - \delta\frac{1+D}{2}\left(1+c_1^2\right) \right) w_t(D,t)^2 \\
&- \frac{\delta}{2}\left(w_t(0,t)^2 + c_0^2 w(0,t)^2 \right) \\
&- \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2 \right) \\
&+ \varepsilon \int_0^D w_t(y,t)\Pi(y,t)dy.
\end{aligned} \tag{16.136}
$$

With Young's inequality, we get

$$
\begin{aligned}
\dot{E}(t) \leq &-\left(c_1(1+\varepsilon) - \delta\frac{1+D}{2}\left(1+c_1^2\right) \right) w_t(D,t)^2 \\
&- \frac{\delta}{2}\left(w_t(0,t)^2 + c_0^2 w(0,t)^2 \right) \\
&- \frac{\delta}{2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2 \right) \\
&+ \frac{\delta}{4}\|w_t(t)\|^2 + \frac{\varepsilon^2}{\delta}\|\Pi(t)\|^2
\end{aligned}
$$

$$\leq -\left(c_1(1-|\varepsilon|) - \delta\frac{1+D}{2}\left(1+c_1^2\right)\right) w_t(D,t)^2$$
$$- \frac{\delta c_0^2}{2} w(0,t)^2 - \frac{\delta}{4}\|w_x(t)\|^2 - \frac{\delta}{2}\|w_t(t)\|^2$$
$$+ \frac{\varepsilon^2}{\delta}\|\Pi(t)\|^2. \tag{16.137}$$

Now we calculate an estimate of $\|\Pi(t)\|^2$ as

$$\|\Pi(t)\|^2 \leq 7\Bigg(\|\sigma\|^2 w^2(0,t) + \left((KB)^4 + D\|g\|^2\right)\|w(t)\|^2$$
$$+ \left((KB)^2 + D\|h\|^2\right)\|w_t(t)\|^2 + \|\vartheta\|^2 |X(t)|^2 \Bigg). \tag{16.138}$$

With the Poincaré inequality $\|w(t)\| \leq 2w^2(0,t) + 4D^2\|w_x(t)\|^2$, we obtain

$$\|\Pi(t)\|^2 \leq 7\left(\|\sigma\|^2 + 2\left((KB)^4 + D\|g\|^2\right)\right) w^2(0,t)$$
$$+ 28D^2\left((KB)^4 + D\|g\|^2\right)\|w_x(t)\|^2$$
$$+ 7\left((KB)^2 + D\|h\|^2\right)\|w_t(t)\|^2$$
$$+ 7\|\vartheta\|^2 |X(t)|^2. \tag{16.139}$$

Substituting this into $\dot{E}(t)$, we get

$$\dot{E}(t) \leq -\left(c_1(1-|\varepsilon|) - \delta\frac{1+D}{2}\left(1+c_1^2\right)\right) w_t(D,t)^2$$
$$- \left(\frac{\delta c_0^2}{2} - \frac{7\varepsilon^2}{\delta}\left(\|\sigma\|^2 + 2\left((KB)^4 + D\|g\|^2\right)\right)\right) w^2(0,t)$$
$$- \left(\frac{\delta}{4} - \frac{28\varepsilon^2}{\delta} D^2 \left((KB)^4 + D\|g\|^2\right)\right)\|w_x(t)\|^2$$
$$- \left(\frac{\delta}{2} - \frac{7\varepsilon^2}{\delta}\left((KB)^2 + D\|h\|^2\right)\right)\|w_t(t)\|^2$$
$$+ \frac{7\varepsilon^2}{\delta}\|\vartheta\|^2 |X(t)|^2. \tag{16.140}$$

Now we choose $|\varepsilon| \le \varepsilon_1$, where

$$\varepsilon_1 = \min\left\{\frac{1}{2},\ \frac{\delta^2 c_0^2}{28\left(\|\sigma\|^2 + 2\left((KB)^4 + D\|g\|^2\right)\right)},\right.$$
$$\frac{\delta^2}{224D^2\left((KB)^4 + D\|g\|^2\right)},$$
$$\left.\frac{\delta^2}{28\left((KB)^2 + D\|h\|^2\right)}\right\}, \tag{16.141}$$

and obtain

$$\dot{E}(t) \le -\left(\frac{c_1}{2} - \delta\frac{1+D}{2}\left(1+c_1^2\right)\right) w_t(D,t)^2$$
$$-\frac{\delta c_0^2}{4} w^2(0,t) - \frac{\delta}{8}\|w_x(t)\|^2 - \frac{\delta}{4}\|w_t(t)\|^2$$
$$+\frac{7\varepsilon^2}{\delta}\|\vartheta\|^2 |X(t)|^2 . \tag{16.142}$$

Now we pick

$$0 < \delta < \min\left\{\frac{1}{4(1+D)},\ \frac{c_1}{(1+D)(1+c_1^2)}\right\} \tag{16.143}$$

and get

$$\dot{E}(t) \le -\frac{\delta c_0^2}{4} w^2(0,t) - \frac{\delta}{8}\|w_x(t)\|^2 - \frac{\delta}{4}\|w_t(t)\|^2$$
$$+\frac{7\varepsilon^2}{\delta}\|\vartheta\|^2 |X(t)|^2 . \tag{16.144}$$

Now we introduce the overall Lyapunov function

$$V(t) = X(t)^T P X(t) + aE(t) . \tag{16.145}$$

The derivative of this Lyapunov function is bounded by

$$\dot{V}(t) \le -a\left(\frac{\delta c_0^2}{4} w^2(0,t) + \frac{\delta}{8}\|w_x(t)\|^2 + \frac{\delta}{4}\|w_t(t)\|^2\right)$$
$$+a\frac{7\varepsilon^2}{\delta}\|\vartheta\|^2 |X(t)|^2$$
$$-X(t)^T Q X(t) + 2X(t)^T P B w(0,t)$$

$$\begin{aligned}
&\leq -a\left(\frac{\delta c_0^2}{4}w^2(0,t)+\frac{\delta}{8}\|w_x(t)\|^2+\frac{\delta}{4}\|w_t(t)\|^2\right)\\
&\quad +a\frac{7\varepsilon^2}{\delta}\|\vartheta\|^2|X(t)|^2\\
&\quad -\frac{\lambda_{\min}(Q)}{2}|X(t)|^2+\frac{2|PB|^2}{\lambda_{\min}(Q)}w^2(0,t)\\
&\leq -\left(a\frac{\delta c_0^2}{4}-\frac{2|PB|^2}{\lambda_{\min}(Q)}\right)w^2(0,t)\\
&\quad -a\left(\frac{\delta}{8}\|w_x(t)\|^2+\frac{\delta}{4}\|w_t(t)\|^2\right)\\
&\quad -\left(\frac{\lambda_{\min}(Q)}{2}-a\frac{7\varepsilon^2}{\delta}\|\vartheta\|^2\right)|X(t)|^2. \qquad (16.146)
\end{aligned}$$

Now we choose

$$a \geq \frac{16|PB|^2}{\delta c_0^2\lambda_{\min}(Q)} \qquad (16.147)$$

and restrict ε to

$$|\varepsilon| \leq \min\left\{\varepsilon_1,\ \sqrt{\frac{\delta\lambda_{\min}(Q)}{28a}}\right\}. \qquad (16.148)$$

This results in

$$\begin{aligned}
\dot V(t) &\leq -\frac{a\delta}{8}\left(c_0^2w^2(0,t)+\|w_x(t)\|^2+2\|w_t(t)\|^2\right)\\
&\quad -\frac{\lambda_{\min}(Q)}{4}|X(t)|^2\\
&\leq -\min\left\{\frac{a\delta}{8}c_0^2,\ \frac{a\delta}{8},\ \frac{\lambda_{\min}(Q)}{4}\right\}\Xi(t), \qquad (16.149)
\end{aligned}$$

where we recall that $\Xi(t)$ denotes

$$\Xi(t) = w(0,t)^2+\|w_x(t)\|^2+\|w_t(t)\|^2+|X(t)|^2. \qquad (16.150)$$

Now, from the definitions of V and E, we observe that

$$\theta_1\Xi \leq V \leq \theta_2\Xi, \qquad (16.151)$$

where

$$\theta_1 = \min\left\{\lambda_{\min}(P),\ \frac{ac_0(1-|\varepsilon|)}{2},\frac{a}{2}\left(1-|\varepsilon|-\delta(1+D)\right)\right\}, \qquad (16.152)$$

$$\theta_2 = \max\left\{\lambda_{\max}(P),\ \frac{ac_0(1+|\varepsilon|)}{2},\ \frac{a}{2}(1+|\varepsilon|+\delta(1+D))\right\}. \tag{16.153}$$

Obviously, θ_2 is positive, but we have to ascertain the same for θ_1. Since $|\varepsilon|$ was restricted to be no greater than $1/2$ in (16.141), it follows that

$$\theta_1 \geq \min\left\{\lambda_{\min}(P),\ \frac{ac_0}{4}, \frac{a}{2}\left(\frac{1}{2}-\delta(1+D)\right)\right\}. \tag{16.154}$$

Recalling condition (16.143), where $\delta < 1/4(1+D)$, we get

$$\theta_1 \geq \min\left\{\lambda_{\min}(P),\ \frac{ac_0}{4}, \frac{a}{8}\right\} > 0. \tag{16.155}$$

Returning to the Lyapunov inequality, we get

$$\dot{V}(t) \leq -\eta V(t), \tag{16.156}$$

where

$$\eta = \frac{1}{\theta_2}\min\left\{\frac{a\delta}{8}c_0^2,\ \frac{a\delta}{8},\ \frac{\lambda_{\min}(Q)}{4}\right\}. \tag{16.157}$$

With (16.76), it follows that

$$\Omega(t) \leq \frac{\theta_1\theta_3}{\theta_2\theta_4}\Omega(0)\mathrm{e}^{-\eta t}, \tag{16.158}$$

where

$$\Omega(t) = u(0,t)^2 + \|u_x(t)\|^2 + \|u_t(t)\|^2 + |X(t)|^2, \tag{16.159}$$

and θ_3 and θ_4 are given by (16.103) and (16.104), respectively. Hence, we have completed the proof of the theorem. □

16.4 An Alternative Design with Dirichlet Actuation

We now return to the same problem as in Section 16.1, but with an alternative choice for the actuated variable in the wave PDE.

We consider the system

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{16.160}$$

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{16.161}$$

$$u_x(0,t) = 0, \tag{16.162}$$

$$u(D,t) = U(t), \tag{16.163}$$

where instead of the Neumann actuation choice, $u_x(D,t) = U(t)$, we consider Dirichlet actuation, $u(D,t) = U(t)$. We note that the actuator transfer function in this case is

$$u(0,t) = \frac{1}{\cosh(Ds)}[U(t)], \tag{16.164}$$

as opposed to $1/\sinh(Ds)$ in the case of Neumann actuation.

Instead of a target system of the form

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t), \tag{16.165}$$
$$w_{tt}(x,t) = w_{xx}(x,t), \tag{16.166}$$
$$w_x(0,t) = c_0 w(0,t), \qquad c_0 > 0, \tag{16.167}$$
$$w(D,t) = 0, \tag{16.168}$$

where the w-system is an undamped string equation with a "spring/stiffness" boundary condition at the end $x = 0$ and pinned at the end $x = D$, we pursue a target system of the form

$$\dot{X}(t) = (A+BK)X(t) + B\varpi(0,t), \tag{16.169}$$
$$\varpi_{tt}(x,t) = \varpi_{xx}(x,t), \tag{16.170}$$
$$\varpi_x(0,t) = c\varpi_t(0,t), \qquad c > 0, \tag{16.171}$$
$$\varpi(D,t) = 0. \tag{16.172}$$

The ϖ-system is a string equation, pinned at the end $x = D$, and with boundary damping at the uncontrolled end, $x = 0$. This target system is well known to be exponentially stable. However, why do we choose a target system in this particular form? The fact that we employ Dirichlet actuation at $x = D$ prevents us from applying damping at this end; hence, we induce boundary damping at the opposite end.

The question now is, how do we induce boundary damping at the uncontrolled end? We first recall that the transformation $(X,u,u_t) \mapsto (X,v,v_t)$, which is defined as

$$v(x,t) = u(x,t) - \mu(x) \star u(x,t) - m(x) \star u_t(x,t) - \gamma(x)X(t), \tag{16.173}$$
$$v_t(x,t) = u_t(x,t) - KBu(x,t) - \mu(x) \star u_t(x,t) - m''(x) \star u(x,t) - \gamma(x)AX(t), \tag{16.174}$$

where $\star$ denotes the convolution operation, converts the (X,u,u_t)-system into the target system

$$\dot{X}(t) = (A+BK)X(t) + Bv(0,t), \tag{16.175}$$
$$v_{tt}(x,t) = v_{xx}(x,t), \tag{16.176}$$
$$v_x(0,t) = 0. \tag{16.177}$$

Now we introduce a second transformation

$$(X, v, v_t) \mapsto (X, \varpi, \varpi_t) \tag{16.178}$$

given by

$$\varpi(x,t) = v(x,t) + c\int_0^x v_t(y,t)dy\,, \tag{16.179}$$

and alternatively represented as

$$\varpi_t(x,t) = v_t(x,t) + cv_x(x,t), \tag{16.180}$$
$$\varpi_x(x,t) = v_x(x,t) + cv_t(x,t)\,. \tag{16.181}$$

It can be verified that this transformation converts (16.176), (16.177) into (16.170), (16.171). In addition, the transformation (16.180), (16.181) is invertible whenever

$$c \neq 0\,. \tag{16.182}$$

The boundary condition (16.172) is obtained with the control law

$$v(D,t) = -c\int_0^x v_t(y,t)dy\,. \tag{16.183}$$

We will return in a moment to expressing this control law in terms of the input $u(D,t) = U(t)$.

The composition of transformations $(X, u, u_t) \mapsto (X, v, v_t) \mapsto (X, \varpi, \varpi_t)$ results in

$$\begin{aligned}\varpi(x,t) &= v(x,t) + c\int_0^x v_t(y,t)dy \\ &= u(x,t) - \mu(x)\star u(x,t) - m(x)\star u_t(x,t) - \gamma(x)X(t) \\ &\quad + c\int_0^x \Big(u_t(y,t) - KBu(y,t) - \mu(y)\star u_t(y,t) \\ &\quad - m''(y)\star u(y,t) - \gamma(y)AX(t)\Big)dy\,.\end{aligned} \tag{16.184}$$

With a few rearrangements and substitutions, we obtain

$$\begin{aligned}\varpi(x,t) = u(x,t) &- \Big(\mu(x) + cK(I + M(x))B\Big)\star u(x,t) \\ &- \left(m(x) - c + c\int_0^x \mu(y)dy\right)\star u_t(x,t) \\ &- \left(\gamma(x) + c\int_0^x \gamma(y)Ady\right)X(t)\,.\end{aligned} \tag{16.185}$$

With the convolution operation written out explicitly, the transformation assumes the form

$$\begin{aligned}\varpi(x,t) = u(x,t) &- \int_0^x \Big(\mu(x-y) + cK(I+M(x-y))B\Big) u(y,t)dy \\ &- \int_0^x \left(m(x-y) - c + c\int_0^{x-y} \mu(s)ds\right) u_t(y,t)dy \\ &- \left(\gamma(x) + c\int_0^x \gamma(y)Ady\right) X(t)\,. \end{aligned} \tag{16.186}$$

Finally, to achieve the boundary condition $\varpi(D,t) = 0$, we pick the control law $u(D,t) = U(t)$ as

$$\begin{aligned}U(t) = &\int_0^D \Big(\mu(D-y) + cK(I+M(D-y))B\Big) u(y,t)dy \\ &+ \int_0^D \left(m(D-y) - c + c\int_0^{D-y} \mu(s)ds\right) u_t(y,t)dy \\ &+ K\left(M(D) + c\int_0^D M(y)Ady\right) X(t)\,. \end{aligned} \tag{16.187}$$

While the role of the first two lines in this control law is relatively difficult to appreciate, the third line is just a modification of the nominal control law, $U(t) = KX(t)$, with the matrix

$$M(D) + c\int_0^D M(y)Ady = \begin{bmatrix} I & cA \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} D} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{16.188}$$

inserted between the gain vector K and the measured state $X(t)$, to compensate the wave PDE actuator dynamics.

A similar stability theorem (with a slightly different system norm) can be proved as for the case of Neumann actuation.

Theorem 16.3 (Dirichlet actuation). *Consider a closed-loop system consisting of the plant (16.161)–(16.163) and the control law (16.187), with the control gain chosen such that*

$$c \in (0,1) \cup (1,\infty)\,. \tag{16.189}$$

For any initial condition such that $u(\cdot,0) \in H^1(0,D)$ and $u_t(\cdot,0) \in L^2(0,D)$, the closed-loop system has a unique solution

$$\big(X(t), u(\cdot,t), u_t(\cdot,t)\big) \in C([0,\infty), \mathbb{R}^n \times H^1(0,D) \times L^2(0,D)) \tag{16.190}$$

and is exponentially stable in the sense of the norm

$$\left(|X(t)|^2 + \int_0^D u_x(x,t)^2 dx + \int_0^D u_t(x,t)^2 dx\right)^{1/2}. \tag{16.191}$$

Moreover, if the initial condition $\big(u(\cdot,0),u_t(\cdot,0)\big)$ *is compatible with the control law (16.187) and belongs to* $H^2(0,D)\times H^1(0,D)$*, then*

$$\big(X(t),u(\cdot,t),u_t(\cdot,t)\big)\in C^1([0,\infty),\mathbb{R}^n\times H^1(0,D)\times L^2(0,D)) \tag{16.192}$$

is the classical solution of the closed-loop system.

This theorem established exponential stability, but with a very conservative decay rate. In the next proposition we make a statement about the closed-loop eigenvalues, which indicates that an arbitrarily fast decay rate is achievable.

Proposition 16.1. *The spectrum of the system (16.161)–(16.163), (16.187) is given by*

$$\operatorname{eig}\{A+BK\}\cup\left\{-\frac{1}{2}\ln\left|\frac{1+c}{1-c}\right|+j\frac{\pi}{D}\begin{cases}n+\frac{1}{2}, & 0\le c<1\\ n, & c>1\end{cases}\right\}, \tag{16.193}$$

where $n\in\mathbb{Z}$.

16.5 Expressing the Compensator in Terms of Input Signal Rather Than Wave Equation State

In this section we focus on the problem with Dirichlet actuation in Section 16.4. A similar result can be obtained for the problem with Neumann actuation in Section 16.1, but with some more effort.

Before starting with our developments, we write the controller (16.187) in the following compact form:

$$\begin{aligned}U(t)&=K\Sigma(D,c)X(t)\\&\quad+\int_0^D\varphi(D-y)u(y,t)dy+\int_0^D\psi(D-y)u_t(y,t)dy,\end{aligned} \tag{16.194}$$

where

$$\varphi(\tau)=\mu(\tau)+cK(I+M(\tau))B, \tag{16.195}$$

$$\psi(\tau)=m(\tau)-c+c\int_0^\tau\mu(\eta)d\eta, \tag{16.196}$$

$$\Sigma(D,c)=M(D)+c\int_0^D M(y)Ady. \tag{16.197}$$

When the initial state of the actuator dynamics is zero, namely, when

$$u(x,0)=u_0(x)\equiv 0, \tag{16.198}$$

$$u_t(x,0)=u_1(x)\equiv 0, \tag{16.199}$$

the state is given explicitly as

$$u(x,t) = \frac{\cosh(xs)}{\cosh(Ds)}[U(t)]. \tag{16.200}$$

Furthermore, from this relation, one obtains

$$u(x,t) = \frac{1}{1+e^{-2Ds}}[U(t+x-D)+U(t-x-D)], \tag{16.201}$$

$$u_t(x,t) = \frac{1}{1+e^{-2Ds}}\left[\dot{U}(t+x-D)-\dot{U}(t-x-D)\right]. \tag{16.202}$$

Substituting these expressions into the control law, we obtain

$$\begin{aligned} U(t) = K\Sigma(D,c)X(t) \\ &+ \frac{1}{1+e^{-2Ds}}\left[\int_0^D \varphi(D-y)U(t+y-D)dy\right. \\ &+ \int_0^D \varphi(D-y)U(t-y-D)dy \\ &+ \int_0^D \psi(D-y)\dot{U}(t+y-D)dy \\ &\left. - \int_0^D \psi(D-y)\dot{U}(t-y-D)dy\right]. \end{aligned} \tag{16.203}$$

Changing the independent variable of integration, we change the integration in space to integration in time:

$$\begin{aligned} U(t) = K\Sigma(D,c)X(t) \\ &+ \frac{1}{1+e^{-2Ds}}\left[\int_{t-D}^{t} \varphi(t-\theta)U(\theta)d\theta\right. \\ &- \int_{t-2D}^{t-D} \varphi(\sigma-t+2D)U(\sigma)d\sigma \\ &+ \int_{t-D}^{t} \psi(t-\theta)\dot{U}(\theta)d\theta \\ &\left. + \int_{t-2D}^{t-D} \psi(\sigma-t+2D)\dot{U}(\sigma)d\sigma\right]. \end{aligned} \tag{16.204}$$

Now we deal with the $\dot{U}$ terms under the integrals by integrating by parts:

$$\begin{aligned} \int_{t-D}^{t} \psi(t-\theta)\dot{U}(\theta)d\theta = \psi(0)U(t) - \psi(D)U(t-D) \\ &+ \int_{t-D}^{t} \psi'(t-\theta)U(\theta)d\theta, \end{aligned} \tag{16.205}$$

$$-\int_{t-2D}^{t-D} \psi(\sigma - t + 2D)\dot{U}(\sigma)d\sigma = -\psi(0)U(t-2D) + \psi(D)U(t-D) + \int_{t-2D}^{t-D} \psi'(\sigma - t + 2D)U(\sigma)d\sigma . \quad (16.206)$$

Noting that $m(0) = 0$ and hence that $\psi(0) = -c$, we get

$$\begin{aligned}\int_{t-D}^{t} \psi(t-\theta)\dot{U}(\theta)d\theta + \int_{t-2D}^{t-D} \psi(\sigma - t + 2D)\dot{U}(\sigma)d\sigma \\ = -c\big(U(t) - U(t-2D)\big) \\ + \int_{t-D}^{t} \psi'(t-\theta)U(\theta)d\theta \\ - \int_{t-2D}^{t-D} \psi'(\sigma - t + 2D)U(\sigma)d\sigma . \end{aligned} \quad (16.207)$$

Substituting this expression into the control law, we obtain

$$\begin{aligned}U(t) = K\Sigma(D,c)X(t) \\ + \frac{1}{1+\mathrm{e}^{-2Ds}}\Bigg[-c\big(U(t) - U(t-2D)\big) \\ + \int_{t-D}^{t} \rho(t-\theta)U(\theta)d\theta \\ - \int_{t-2D}^{t-D} \rho(\sigma - t + 2D)U(\sigma)d\sigma \Bigg], \end{aligned} \quad (16.208)$$

where

$$\rho(\tau) = \varphi(\tau) + \psi'(\tau) . \quad (16.209)$$

Taking a derivative of $\psi(\tau)$ and noting that

$$m'(\tau) = \gamma(\tau)B = KM(\tau)B , \quad (16.210)$$

we get

$$\rho(\tau) = (1+c)\mu(\tau) + K\big(cI + (1+c)M(\tau)\big)B . \quad (16.211)$$

Solving for $U(t)$ in (16.208), we finally get the control law

$$\begin{aligned}U(t) = \frac{1}{1+c\tanh(Ds)}[K\Sigma(D,c)X(t)] \\ + \frac{1}{1+\mathrm{e}^{-2Ds} + c(1-\mathrm{e}^{-2Ds})}[\mathscr{Q}(t)] , \end{aligned} \quad (16.212)$$

where

$$\mathscr{Q}(t) = \int_{t-D}^{t} \rho(t-\theta)U(\theta)d\theta - \int_{t-2D}^{t-D} \rho(\sigma - t + 2D)U(\sigma)d\sigma. \tag{16.213}$$

16.6 Examples: Wave PDE Actuator Dynamics

Example 16.1. We return to the plant studied in Example 2.1, but with delay dynamics at the input replaced by the wave PDE dynamics:

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{16.214}$$

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{16.215}$$

$$u_x(0,t) = 0, \tag{16.216}$$

$$u_x(D,t) = U(t), \tag{16.217}$$

where

$$X = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{16.218}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{16.219}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{16.220}$$

We want to design a control law of the form

$$\begin{aligned} U(t) = {} & (-c_0 + c_1 KB - c_1 \partial_t) u(D,t) \\ & + \int_0^D \left(p(D-y) + q(D-y)\partial_t\right) u(y,t)dy, \\ & + \pi(D)X(t), \end{aligned} \tag{16.221}$$

where

$$p(s) = \mu'(s) + c_0\mu(s) + c_1\left(m''(s) + c_0 m'(s)\right), \tag{16.222}$$

$$q(s) = m'(s) + c_0 m(s) + c_1\left(\mu(s) + c_0\int_0^s \mu(\xi)d\xi - c_0\right), \tag{16.223}$$

$$\pi(x) = \gamma'(x) + \gamma(x)(c_0 I + c_1 A) + c_1 c_0 \int_0^x \gamma(\xi)d\xi A, \tag{16.224}$$

and

$$\gamma(x) = KM(x), \tag{16.225}$$

$$M(x) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{16.226}$$

$$m(s) = \int_0^s \gamma(\xi) B d\xi, \tag{16.227}$$

$$\mu(s) = \int_0^s \gamma(\xi) AB d\xi. \tag{16.228}$$

We start by deriving $M(x)$. To do this, we first observe that

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I. \tag{16.229}$$

With a lengthy calculation, we determine that

$$\mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} = \begin{bmatrix} \cos x & 0 & -\sin x & 0 \\ 0 & \cos x & 0 & -\sin x \\ \sin x & 0 & \cos x & 0 \\ 0 & \sin x & 0 & \cos x \end{bmatrix}. \tag{16.230}$$

Then

$$M(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cos x. \tag{16.231}$$

We take the nominal gain vector in the same way as in Example 2.1:

$$K = [0 \quad -h], \qquad h > 0. \tag{16.232}$$

Then we obtain $\gamma(x) = KM(x)$ as

$$\gamma(x) = [0 \quad -h\cos x]. \tag{16.233}$$

Since

$$\gamma(x)B = -h\cos x, \tag{16.234}$$

$$\gamma(x)AB = 0, \tag{16.235}$$

we get

$$m(x) = -h\sin x, \tag{16.236}$$

$$\mu(x) = 0. \tag{16.237}$$

For the formulas (16.222)–(16.224), we need the following expressions:

$$m'(x) = -h\cos x, \tag{16.238}$$

$$m''(x) = h\sin x, \tag{16.239}$$

$$\gamma'(x) = [0 \quad h\sin x], \tag{16.240}$$

$$\gamma(x)A = [h\cos x \quad 0], \tag{16.241}$$

$$\int_0^x \gamma(\xi)d\xi A = [h\sin x \quad 0]. \tag{16.242}$$

Now we get

$$p(x) = hc_1\left(\sin x - c_0\cos x\right), \tag{16.243}$$

$$q(x) = -h\left(\cos x + c_0\sin x\right) - c_0c_1, \tag{16.244}$$

$$\pi(x) = h\left[c_1(\cos x + c_0\sin x), \quad (\sin x - c_0\cos x)\right]. \tag{16.245}$$

To implement the controller (16.221), we need the gains

$$-c_0 + c_1KB - c_1\partial_t = -(c_0 + hc_1 + c_1\partial_t), \tag{16.246}$$

$$\begin{aligned} p(D-y) + q(D-y)\partial_t = {} & hc_1\left(\sin(D-y) - c_0\cos(D-y)\right) \\ & - \left[h\left(\cos(D-y) + c_0\sin(D-y)\right) + c_0c_1\right]\partial_t, \end{aligned} \tag{16.247}$$

$$\pi(D) = h\left[c_1(\cos D + c_0\sin D), \quad (\sin D - c_0\cos D)\right]. \tag{16.248}$$

In summary, the controller (16.221) is

$$\begin{aligned} u_x(D,t) &= U(t) \\ &= -(c_0 + hc_1)u(D,t) - c_1u_t(D,t) \\ &\quad + \int_0^D hc_1\left(\sin(D-y) - c_0\cos(D-y)\right)u(y,t)dy \\ &\quad - \int_0^D \left[h\left(\cos(D-y) + c_0\sin(D-y)\right) + c_0c_1\right]u_t(y,t)dy \\ &\quad + hc_1(\cos D + c_0\sin D)\xi_1(t) \\ &\quad + h(\sin D - c_0\cos D)\xi_2(t). \end{aligned} \tag{16.249}$$

It is worth commenting on the physical meaning of the controller. The plant

$$\dot{\xi}_1(t) = \xi_2(t), \tag{16.250}$$

$$\dot{\xi}_2(t) = -\xi_1(t) + u(0,t) \tag{16.251}$$

needs damping added to it, namely, the "velocity" feedback of $\xi_2 = \dot{\xi}_1$. Likewise, the actuator

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{16.252}$$

$$u_x(0,t) = 0, \tag{16.253}$$

$$u_x(D,t) = U(t), \tag{16.254}$$

which is an undamped wave equation, needs damping to it. Clearly, in the controller (16.249) the terms

$$-c_1 u_t(D,t), \tag{16.255}$$

$$-\int_0^D \left[h\left(\cos(D-y)+c_0\sin(D-y)\right)+c_0c_1\right] u_t(y,t)dy, \tag{16.256}$$

$$+h(\sin D - c_0\cos D)\xi_2(t) \tag{16.257}$$

are velocity feedback terms and represent the addition of damping.

The term (16.255) adds boundary damping, the term (16.256) adds distributed damping, and the term (16.257) adds plant damping. On the other hand, the position feedback terms

$$\begin{aligned}
&-(c_0+hc_1)u(D,t),\\
&+\int_0^D hc_1\left(\sin(D-y)-c_0\cos(D-y)\right)u(y,t)dy,\\
&+hc_1(\cos D + c_0\sin D)\xi_1(t)
\end{aligned} \tag{16.258}$$

in the controller (16.249) are also needed because the wave equation has “unpinned” boundaries.

Example 16.2. We return to the plant studied in Example 16.1, but with Dirichlet actuation rather than Neumann actuation:

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{16.259}$$

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{16.260}$$

$$u_x(0,t) = 0, \tag{16.261}$$

$$u(D,t) = U(t), \tag{16.262}$$

where X, A, B, as well as the nominal control gain vector K, are defined in Example 16.1. The control design presented in Section 16.4, and summarized as the feedback law (16.187), is obtained as

$$\begin{aligned}
U(t) = &-ch\int_0^D \left(1+\cos(D-y)\right)u(y,t)dy\\
&-\int_0^D \left(h\sin(D-y)+c\right)u_t(y,t)dy\\
&+hc\sin(D)\xi_1(t) - h\cos(D)\xi_2(t).
\end{aligned} \tag{16.263}$$

For $D=0$, this controller reduces to the nominal controller:

$$U(t) = -h\xi_2(t) = -h\dot{\xi}_1(t). \tag{16.264}$$

One can also observe that for $h=0$, the resulting controller (16.263) is a stabilizing controller for the wave equation alone. Its formula is

$$U(t) = -c\int_0^D u_t(y,t)dy. \tag{16.265}$$

Since the resulting target system for the wave equation is

$$\varpi_{tt}(x,t) = \varpi_{xx}(x,t), \tag{16.266}$$
$$\varpi_x(0,t) = c\varpi_t(0,t), \qquad c > 0, \tag{16.267}$$
$$\varpi(D,t) = 0, \tag{16.268}$$

the closed-loop eigenvalues of the wave equation $u_{tt}(x,t) = u_{xx}(x,t)$, $u_x(0,t) = 0$, and $u(D,t) = U(t)$ under the feedback (16.265) are

$$\sigma_n = -\frac{1}{2}\ln\left|\frac{1+c}{1-c}\right| + j\frac{\pi}{D}\begin{cases} n+\frac{1}{2}, & 0 \le c < 1, \\ n, & c > 1, \end{cases} \tag{16.269}$$

where $n \in \mathbb{Z}$.

We now recall, from Section 16.5, the control law

$$U(t) = \frac{1}{1+c\tanh(Ds)}[K\Sigma(D,c)X(t)] + \frac{1}{1+\mathrm{e}^{-2Ds}+c(1-\mathrm{e}^{-2Ds})}[\mathscr{Q}(t)], \tag{16.270}$$

where

$$\mathscr{Q}(t) = \int_{t-D}^{t}\rho(t-\theta)U(\theta)d\theta - \int_{t-2D}^{t-D}\rho(\sigma-t+2D)U(\sigma)d\sigma, \tag{16.271}$$
$$\rho(\tau) = (1+c)\mu(\tau) + K\big(cI + (1+c)M(\tau)\big)B. \tag{16.272}$$

For the problem in the present example, we have

$$K\Sigma(D,c)X(t) = hc\sin(D)\xi_1(t) - h\cos(D)\xi_2(t) \tag{16.273}$$

and

$$\rho(\tau) = -h(c+(1+c)\cos(\tau)). \tag{16.274}$$

Finally, we recall from (2.91) that when the input dynamics are of a pure delay type, the controller is

$$U(t) = h\sin(D)\xi_1(t) - h\cos(D)\xi_2(t) - h\int_{t-D}^{t}\cos(t-\theta)U(\theta)d\theta. \tag{16.275}$$

By comparing (16.263) with (16.275), as well as by comparing (16.270)–(16.272) with (16.275), several similarities and differences can be observed between the delay compensator and the wave equation compensator.

16.7 On the Stabilization of the Wave PDE Alone by Neumann and Dirichlet Actuation

In (16.265) we noted that a special case of the control law for stabilization of the PDE-ODE cascade in Example 16.2, where the nominal feedback gain on the ODE is zero, is obtained as $U(t) = -c\int_0^D u_t(y,t)dy$. Such a result can be obtained in general, for both Neumann actuation and Dirichlet actuation, to arrive at a boundary feedback law for stabilization of the undamped wave PDE, which is a result that is of interest in its own right.

We consider two separate cases of boundary actuation—Neumann first and Dirichlet second.

Consider the feedback law (16.58) but with

$$K = 0 \tag{16.276}$$

in Eqs. (16.59)–(16.61), which define the gain functions p, q, and π. One obtains the feedback law

$$U(t) = -c_0u(D,t) - c_1u_t(D,t) - c_0c_1\int_0^D u_t(y,t)dy \tag{16.277}$$

and arrives at the following result.

Theorem 16.4. *Consider the closed-loop system*

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{16.278}$$

$$u_x(0,t) = 0, \tag{16.279}$$

$$u_x(D,t) = -c_0u(D,t) - c_1u_t(D,t) - c_0c_1\int_0^D u_t(y,t)dy. \tag{16.280}$$

For any initial condition such that $u(\cdot,0) \in H^1(0,D)$ and $u_t(\cdot,0) \in L^2(0,D)$, the closed-loop system has a unique solution

$$\big(u(\cdot,t), u_t(\cdot,t)\big) \in C([0,\infty), H^1(0,D)\times L^2(0,D)) \tag{16.281}$$

and is exponentially stable in the sense of the norm

$$\left(u(0,t)^2 + \int_0^D u_x(x,t)^2dx + \int_0^D u_t(x,t)^2dx\right)^{1/2}. \tag{16.282}$$

Moreover, if the initial condition $\big(u(\cdot,0), u_t(\cdot,0)\big)$ is compatible with the control law (16.280) and belongs to $H^2(0,D)\times H^1(0,D)$, then

$$\big(u(\cdot,t), u_t(\cdot,t)\big) \in C^1([0,\infty), H^1(0,D)\times L^2(0,D)) \tag{16.283}$$

is the classical solution of the closed-loop system.

Now we consider the case of Dirichlet actuation. Setting $K = 0$ in the controller (16.187), we obtain a simple derivative full-state feedback law

$$U(t) = -c\int_0^D u_t(y,t)dy, \tag{16.284}$$

which yields the following result.

Theorem 16.5. *Consider a closed-loop system*

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{16.285}$$

$$u_x(0,t) = 0, \tag{16.286}$$

$$u(D,t) = -c\int_0^D u_t(y,t)dy, \tag{16.287}$$

with the control gain chosen such that

$$c \in (0,1)\cup(1,\infty). \tag{16.288}$$

For any initial condition such that $u(\cdot,0) \in H^1(0,D)$ and $u_t(\cdot,0) \in L^2(0,D)$, the closed-loop system has a unique solution

$$\big(u(\cdot,t),u_t(\cdot,t)\big) \in C([0,\infty),H^1(0,D)\times L^2(0,D)) \tag{16.289}$$

and is exponentially stable in the sense of the norm

$$\left(\int_0^D u_x(x,t)^2dx + \int_0^D u_t(x,t)^2dx\right)^{1/2}. \tag{16.290}$$

Moreover, if the initial condition $\big(u(\cdot,0),u_t(\cdot,0)\big)$ is compatible with the control law (16.287) and belongs to $H^2(0,D)\times H^1(0,D)$, then

$$\big(u(\cdot,t),u_t(\cdot,t)\big) \in C^1([0,\infty),H^1(0,D)\times L^2(0,D)) \tag{16.291}$$

is the classical solution of the closed-loop system.

No counterpart of the results from this section is provided for the case of a heat equation in Chapter 15, since the uncontrolled heat equation is already exponentially stable, unlike the wave equation.

We return to the problem from this section in Chapter 19, where we consider a Dirichlet-actuated wave equation with input delay. We extend the feedback law (16.284), which is given here for the simple undamped wave equation, to a wave equation with boundary antidamping on the uncontrolled boundary, and then design a feedback law that compensates the input delay.

16.8 Notes and References

The transformation (16.49)–(16.52) is inspired by [110], whereas the transformation (16.41)–(16.44) is inspired by [207].

With additional design effort, the wave PDE actuator dynamics can be allowed to be much more complex than the plain wave equation:

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{16.292}$$

$$u_{tt}(x,t) = u_{xx}(x,t) + \lambda_0(x)u(x,t) + \lambda_1(x)u_t(x,t) + b(x)u_x(x,t), \tag{16.293}$$

$$u_x(0,t) = -q_0u(0,t) - q_1u_t(0,t), \tag{16.294}$$

$$u(D,t) = U(t). \tag{16.295}$$

Finally, while in both this chapter and Chapter 15 we have considered only Neumann-type boundary conditions at $x = 0$, we can also allow Dirichlet-type boundary conditions at $x = 0$, namely, we can design compensators for systems

$$\dot{X}(t) = AX(t) + Bu_x(0,t), \tag{16.296}$$

$$u_t(x,t) = u_{xx}(x,t), \tag{16.297}$$

$$u(0,t) = 0, \tag{16.298}$$

$$u(D,t) = U(t), \tag{16.299}$$

and

$$\dot{X}(t) = AX(t) + Bu_x(0,t), \tag{16.300}$$

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{16.301}$$

$$u(0,t) = 0, \tag{16.302}$$

$$u_x(D,t) = U(t). \tag{16.303}$$

Chapter 17
Observers for ODEs Involving PDE Sensor and Actuator Dynamics

This chapter parallels the development in Chapter 3 but for the more challenging cases where the sensor dynamics are not of a pure delay type but instead are modeled by heat or wave PDEs (see Fig. 17.1). The chapter consists of two distinct halves, the first half dealing with the heat PDE case in Sections 17.1, 17.2, and 17.3, and the second half dealing with the heat PDE case in Sections 17.4, 17.5, and 17.6.

In Section 17.1 we develop a dual of our actuator dynamics compensator in Chapter 15 and design an infinite-dimensional observer that compensates the diffusion dynamics of the sensor. In Section 17.3 we combine an ODE observer with the full-state feedback compensator of the heat PDE actuator dynamics in Chapter 15 and establish a form of a separation principle, where the observer-based compensator is stabilizing for the overall systems consisting of the ODE plant, ODE observer, heat PDE actuator dynamics, and heat PDE observer. The heat PDE observer is a simple copy of the system since the heat PDE dynamics are exponentially stable, so the observer error for that part of the system is exponentially convergent.

In Section 17.4 we develop a dual of our actuator dynamics compensator in Chapter 16 and design an infinite-dimensional observer that compensates the wave PDE dynamics of the sensor. In Section 17.6 we combine an ODE observer with the full-state feedback compensator of the wave PDE actuator dynamics in Chapter 16 and establish a form of a separation principle, where the observer-based compensator is stabilizing for the overall systems consisting of the ODE plant, ODE observer, wave PDE actuator dynamics, and wave PDE observer. The observer for the wave PDE actuator dynamics is designed in a particular way to ensure convergence of the infinite-dimensional estimation error state, since the wave PDE actuator dynamics are only neutrally stable (not exponentially stable), so a simple copy of the PDE model does not suffice as a choice for the observer (output injection is needed).

In Sections 17.2 and 17.5 we present observer designs for ODEs with PDE sensor dynamics of the heat and wave types, respectively.

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_17,

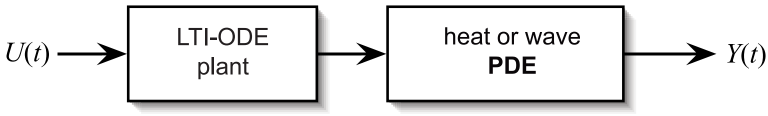

Fig. 17.1 Observer problem for ODEs with sensor dynamics modeled by the heat PDE or wave PDE.

17.1 Observer for ODE with Heat PDE Sensor Dynamics

Consider the LTI-ODE system in cascade with diffusive sensor dynamics at the output (as depicted in Figs. 17.2 and 17.3),

$$Y(t) = u(0,t), \tag{17.1}$$
$$u_t(x,t) = u_{xx}(x,t), \tag{17.2}$$
$$u_x(0,t) = 0, \tag{17.3}$$
$$u(D,t) = CX(t), \tag{17.4}$$
$$\dot{X}(t) = AX(t) + BU(t). \tag{17.5}$$

The sensor dynamics are thus given by the transfer function

$$Y(t) = \frac{1}{\cosh(D\sqrt{s})}[CX(t)]. \tag{17.6}$$

We recall from Chapter 3 that if (17.2), (17.3) are replaced by the delay/transport equation $u_t(x,t) = u_x(x,t)$, then the predictor-based observer

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) + Ce^{Ax}L(Y(t) - \hat{u}(0,t)), \tag{17.7}$$
$$\hat{u}(D,t) = C\hat{X}(t), \tag{17.8}$$
$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + e^{AD}L(Y(t) - \hat{u}(0,t)) \tag{17.9}$$

achieves perfect compensation of the observer delay and achieves exponential stability at $u - \hat{u} \equiv 0, X - \hat{X} = 0$.

Next we state a new observer that compensates the *diffusive* sensor dynamics and prove exponential convergence of the resulting observer error system.

Theorem 17.1 (Observer convergence—heat PDE sensor dynamics). *The observer*

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + CM(x)L(Y(t) - \hat{u}(0,t)), \tag{17.10}$$
$$\hat{u}_x(0,t) = 0, \tag{17.11}$$
$$\hat{u}(D,t) = C\hat{X}(t), \tag{17.12}$$
$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + M(D)L(Y(t) - \hat{u}(0,t)), \tag{17.13}$$

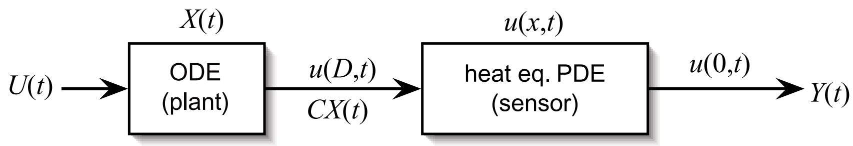

Fig. 17.2 The cascade of the ODE dynamics of the plant with the heat equation PDE dynamics of the sensor.

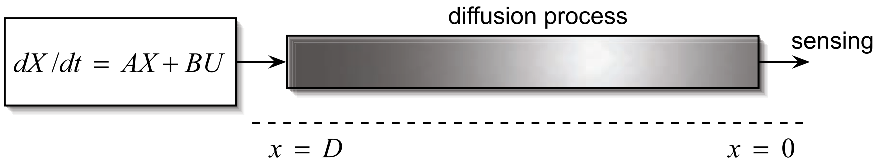

Fig. 17.3 An arbitrary ODE whose output is measured through a diffusion process.

where L is chosen such that $A - LC$ is Hurwitz and $M(x)$ stands for

$$M(x) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{17.14}$$

guarantees that $\hat{X}$, $\hat{u}$ exponentially converge to X, u, i.e., more specifically, that the observer error system is exponentially stable in the sense of the norm

$$\left(|X(t) - \hat{X}(t)|^2 + \int_0^D (u(x,t) - \hat{u}(x,t))^2 \, dx \right)^{1/2}.$$

Proof. Introducing the error variables

$$\tilde{X} = X - \hat{X}, \tag{17.15}$$

$$\tilde{u} = u - \hat{u}, \tag{17.16}$$

we obtain

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) - CM(x)L\tilde{u}(0,t), \tag{17.17}$$

$$\tilde{u}_x(0,t) = 0, \tag{17.18}$$

$$\tilde{u}(D,t) = C\tilde{X}(t), \tag{17.19}$$

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - M(D)L\tilde{u}(0,t). \tag{17.20}$$

Consider the transformation

$$\tilde{w}(x) = \tilde{u}(x) - CM(x)M(D)^{-1}\tilde{X}. \tag{17.21}$$

Its derivatives in x and t are

$$\tilde{w}_x(x,t) = \tilde{u}_x(x,t) - CM'(x)M(D)^{-1}\tilde{X}(t)\,, \tag{17.22}$$

$$\tilde{w}_{xx}(x,t) = \tilde{u}_{xx}(x,t) - CM''(x)M(D)^{-1}\tilde{X}(t)\,, \tag{17.23}$$

$$\tilde{w}_t(x,t) = \tilde{u}_t(x,t) - CM(x)M(D)^{-1}\left(A\tilde{X}(t) - M(D)L\tilde{u}(0,t)\right)\,, \tag{17.24}$$

and, furthermore,

$$\tilde{w}(0,t) = \tilde{u}(0,t) - CM(D)^{-1}\tilde{X}(t)\,, \tag{17.25}$$

where we have used the fact that $M(0) = I$. Then, using the facts that $M'(0) = 0$, that $M(D)^{-1}$ commutes with A [since $M(x)$ commutes with A for any x], that $M''(x) = M(x)A$, and that $\tilde{u}_x(0,t) = 0$, we obtain

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t)\,, \tag{17.26}$$

$$\tilde{w}_x(0,t) = 0\,, \tag{17.27}$$

$$\tilde{w}(D,t) = 0\,, \tag{17.28}$$

$$\dot{\tilde{X}}(t) = \left(A - M(D)LCM(D)^{-1}\right)\tilde{X} - M(D)L\tilde{w}(0,t)\,. \tag{17.29}$$

The matrix $A - M(D)LCM(D)^{-1}$ is Hurwitz, which can be easily seen by using a similarity transformation $M(D)$, which commutes with A.

With a Lyapunov function

$$V = \tilde{X}^T M(D)^{-T} PM(D)^{-1}\tilde{X} + \frac{a}{2}\int_0^D \tilde{w}(x)^2\,dx\,, \tag{17.30}$$

where $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A - LC) + (A - LC)^T P = -Q \tag{17.31}$$

for some $Q = Q^T > 0$, one gets

$$\begin{aligned}\dot{V} = &-\tilde{X}^T M(D)^{-T} QM(D)^{-1}\tilde{X}\\ &- 2\tilde{X}^T M(D)^{-T} PL\tilde{w}(0,t) - a\|\tilde{w}_x\|^2\,.\end{aligned} \tag{17.32}$$

Applying Young's and Agmon's inequalities, taking a as sufficiently large, and then applying Poincaré's inequality, one can show that

$$\dot{V} \le -\mu V \tag{17.33}$$

for some $\mu > 0$; i.e., the $(\tilde{X}, \tilde{w})$-system is exponentially stable at the origin. From (17.21), we get exponential stability in the sense of $(|\tilde{X}(t)|^2 + \int_0^D \tilde{u}(x,t)^2 dx)^{1/2}$. □

The convergence rate of the observer is limited by the first eigenvalue of the heat equation (17.26)–(17.28), i.e., by $-\pi^2/(4D^2)$. A similar observer redesign, as applied for the full-state control design in Theorem 15.2, can be applied to speed up the observer convergence.

17.2 Example: Heat PDE Sensor Dynamics

We again consider a scalar plant as in Section 15.2, but this time with heat equation-type sensor dynamics:

$$Y(t) = u(0,t)\,, \tag{17.34}$$
$$u_t(x,t) = u_{xx}(x,t)\,, \tag{17.35}$$
$$u_x(0,t) = 0\,, \tag{17.36}$$
$$u(D,t) = X(t)\,, \tag{17.37}$$
$$\dot{X}(t) = X(t) + U(t)\,. \tag{17.38}$$

We want to construct an observer in the form

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + CM(x)L\left(Y(t) - \hat{u}(0,t)\right)\,, \tag{17.39}$$
$$\hat{u}_x(0,t) = 0\,, \tag{17.40}$$
$$\hat{u}(D,t) = C\hat{X}(t)\,, \tag{17.41}$$
$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + M(D)L\left(Y(t) - \hat{u}(0,t)\right), \tag{17.42}$$

where

$$A = B = C = 1\,. \tag{17.43}$$

The nominal ($D = 0$) observer error system is governed by the system "matrix"

$$A - CL = 1 - L, \tag{17.44}$$

so we choose the observer gain as

$$L = 1 + g\,, \qquad g > 0\,. \tag{17.45}$$

To design the observer, we need to first derive $M(x)$, which is given in general by

$$M(x) = \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix} = \sinh x\,. \tag{17.46}$$

Then we have

$$CM(x)L = (1+g)\sinh x\,, \tag{17.47}$$
$$M(D)L = (1+g)\sinh D\,. \tag{17.48}$$

In summary, our observer is given by

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + (1+g)\sinh(x)\left(Y(t) - \hat{u}(0,t)\right)\,, \tag{17.49}$$
$$\hat{u}_x(0,t) = 0\,, \tag{17.50}$$

$$\hat{u}(D,t) = \hat{X}(t)\,, \tag{17.51}$$

$$\dot{\hat{X}}(t) = \hat{X}(t) + U(t) + (1+g)\sinh(D)\,(Y(t) - \hat{u}(0,t))\,. \tag{17.52}$$

Again, we note that the observer gain grows with D and that the gain on the sensor state is the highest on the part of the sensor state that is the farthest away from the sensor location ($y = 0$).

17.3 Observer-Based Controller for ODEs with Heat PDE Actuator Dynamics

Now we return to the same problem as in Section 3.4, namely, the problem where we combined a full-state predictor feedback with a finite-dimensional observer for the ODE. However, here we deal with actuator dynamics of a heat PDE type.

We consider the plant

$$\dot{X}(t) = AX(t) + Bu(0,t)\,, \tag{17.53}$$

$$u_t(x,t) = u_{xx}(x,t)\,, \tag{17.54}$$

$$u_x(0,t) = 0\,, \tag{17.55}$$

$$u(D,t) = U(t)\,, \tag{17.56}$$

$$Y(t) = CX(t)\,, \tag{17.57}$$

as in Section (15.1), but with the output map (17.57) added. Let a vector L be chosen so that the matrix $A - LC$ is Hurwitz.

Then we introduce an observer for the entire plant (ODE and PDE) and an observer-based controller, namely,

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}(0,t) + L\left(Y(t) - C\hat{X}(t)\right)\,, \tag{17.58}$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t)\,, \tag{17.59}$$

$$\hat{u}_x(0,t) = 0\,, \tag{17.60}$$

$$\hat{u}(D,t) = U(t)\,, \tag{17.61}$$

$$U(t) = KM(D)\hat{X}(t) + K\int_0^D \left(\int_0^{D-y} M(\xi)d\xi\right) B\hat{u}(y,t)dy\,, \tag{17.62}$$

where $M(\xi)$ is defined in (15.38), and we use it to compactly denote the certainty-equivalence version of the controller (15.10).

Note that while the observer for the ODE (17.58) uses output injection, the observer for the PDE (17.59)–(17.61) is a trivial copy of the system. We are able to resort to such a simple choice because the PDE actuator dynamics are modeled by the heat equation, so they are exponentially stable.

The following stability result holds for the closed-loop system (17.53)–(17.56), (17.58)–(17.62).

Theorem 17.2. *The closed-loop system (17.53)–(17.56), (17.58)–(17.62) is exponentially stable in the sense of the norm*

$$\left(|X(t)|^2 + |\hat{X}(t)|^2 + \int_0^D u^2(x,t)dx + \int_0^D \hat{u}^2(x,t)dx\right)^{1/2}. \tag{17.63}$$

Furthermore, its eigenvalues are given by

$$\mathrm{eig}\,\{A+BK\} \cup \left\{-\frac{\pi^2}{D^2}\left(n+\frac{1}{2}\right)^2,\ n=0,1,2,\dots\right\}. \tag{17.64}$$

Proof. The observer error system is defined with

$$\tilde{X}(t) = X(t) - \hat{X}(t), \tag{17.65}$$

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t) \tag{17.66}$$

and given by

$$\dot{\tilde{X}}(t) = (A-LC)\tilde{X}(t) + B\tilde{u}(0,t), \tag{17.67}$$

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t), \tag{17.68}$$

$$\tilde{u}_x(0,t) = 0, \tag{17.69}$$

$$\tilde{u}(D,t) = 0. \tag{17.70}$$

The stability of the overall system is analyzed by studying the observer error system (17.67)–(17.70) and the observer system with the feedback substituted:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}(0,t) + LC\tilde{X}(t), \tag{17.71}$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t), \tag{17.72}$$

$$\hat{u}_x(0,t) = 0, \tag{17.73}$$

$$\hat{u}(D,t) = KM(D)\hat{X}(t) + K\int_0^D \left(\int_0^{D-y} M(\xi)d\xi\right) B\hat{u}(y,t)dy. \tag{17.74}$$

The overall closed-loop system (17.67)–(17.70), (17.71)–(17.74) is a four-component system, $\left(\tilde{X}, \hat{X}, \tilde{u}, \hat{u}\right)$. The $\tilde{u}$-component is autonomous and exponentially stable, and it feeds into the exponentially stable $\tilde{X}$-subsystem, which then feeds into the exponentially stable $\left(\hat{X}, \hat{u}\right)$-subsystem. So the cascade structure is

$$\tilde{u} \longrightarrow \tilde{X} \longrightarrow \left(\hat{X}, \hat{u}\right). \tag{17.75}$$

The overall system is not in the form in which it is ready for a Lyapunov stability analysis. We do have a Lyapunov function for the $\left(\tilde{X}, \tilde{u}\right)$-subsystem, and it is given by

$$\alpha_1 \tilde{X}(t)^T \Pi \tilde{X}(t) + \frac{\alpha_2}{2}\int_0^D \tilde{u}^2(x,t)dx, \tag{17.76}$$

where Π is a solution to the Lyapunov equation

$$\Pi(A-LC)+(A-LC)^T\Pi=-Q_2 \tag{17.77}$$

for some $Q_2=Q_2^T>0$; however, we have yet to cast the $\left(\hat{X},\hat{u}\right)$-subsystem in a form in which a Lyapunov function is available.

We consider the transformation

$$\hat{w}(x,t)=\hat{u}(x,t)-\int_0^x m(x-y)\hat{u}(y,t)dy-KM(x)\hat{X}(t)\,, \tag{17.78}$$

where the kernel $m(x)$ is defined by (15.36). This transformation converts the system (17.71)–(17.74) into the form

$$\dot{\hat{X}}(t)=(A+BK)\hat{X}(t)+B\hat{w}(0,t)+LC\tilde{X}(t)\,, \tag{17.79}$$

$$\hat{w}_t(x,t)=\hat{w}_{xx}(x,t)-KM(x)LC\tilde{X}(t)\,, \tag{17.80}$$

$$\hat{w}_x(0,t)=0\,, \tag{17.81}$$

$$\hat{w}(D,t)=0\,. \tag{17.82}$$

So the structure of the system whose stability we want to analyze is

$$\tilde{u}\longrightarrow\tilde{X}\longrightarrow\left(\hat{X},\hat{w}\right)\,. \tag{17.83}$$

The Lyapunov function for the system (17.79)–(17.82) can be chosen as

$$\hat{X}(t)^TP\hat{X}(t)+\frac{a}{2}\int_0^D\hat{w}^2(x,t)dx\,, \tag{17.84}$$

where P is the solution to the Lyapunov equation

$$P(A+BK)+(A+BK)^TP=-Q_1 \tag{17.85}$$

for some $Q_1=Q_1^T>0$.

The overall Lyapunov function is then chosen as

$$\begin{aligned}V(t)=\hat{X}(t)^TP\hat{X}(t)+\frac{a}{2}\int_0^D\hat{w}^2(x,t)dx\\+\alpha_1\tilde{X}(t)^T\Pi\tilde{X}(t)+\frac{\alpha_2}{2}\int_0^D\tilde{u}^2(x,t)dx\,,\end{aligned} \tag{17.86}$$

where the values of the positive constants a,α_1, and α_2 are to be chosen in the analysis.

Differentiating $V(t)$ along the solutions of (17.67)–(17.70) and (17.79)–(17.82), and using integration by parts, we get

$$\begin{aligned}\dot{V}(t) = & -\hat{X}(t)^T Q_1 \hat{X}(t) + 2\hat{X}(t)^T PB\hat{w}(0,t) + 2\hat{X}(t)^T PLC\tilde{X}(t) \\ & - a\int_0^D \hat{w}_x^2(x,t)dx + a\int_0^D \hat{w}(x,t)KM(x)LC\tilde{X}(t)dx \\ & - \alpha_1 \tilde{X}(t)^T Q_2 \tilde{X}(t) + 2\alpha_1 \tilde{X}(t)^T \Pi B\tilde{u}(0,t) \\ & - \alpha_2 \int_0^D \tilde{u}_x^2(x,t)dx. \end{aligned} \tag{17.87}$$

With Young's inequality, we get

$$\begin{aligned}\dot{V}(t) \le & -\lambda_{\max}(Q_1)|\hat{X}(t)|^2 + \frac{\lambda_{\max}(Q_1)}{4}|\hat{X}(t)|^2 + \frac{4|PB|^2}{\lambda_{\max}(Q_1)}|\hat{w}(0,t)|^2 \\ & + \frac{\lambda_{\max}(Q_1)}{4}|\hat{X}(t)|^2 + \frac{4|PLC|^2}{\lambda_{\max}(Q_1)}|\tilde{X}(t)|^2 \\ & - a\int_0^D \hat{w}_x^2(x,t)dx + a\int_0^D \hat{w}(x,t)KM(x)LC\tilde{X}(t)dx \\ & - \alpha_1 \lambda_{\max}(Q_2)|\tilde{X}(t)|^2 + 2\alpha_1 \tilde{X}(t)^T \Pi B\tilde{u}(0,t) \\ & - \alpha_2 \int_0^D \tilde{u}_x^2(x,t)dx. \end{aligned} \tag{17.88}$$

With the Agmon inequality in the form $\hat{w}^2(0,t) \le 4D\int_0^D \hat{w}_x^2(x,t)dx$, we obtain

$$\begin{aligned}\dot{V}(t) \le & -\frac{\lambda_{\max}(Q_1)}{2}|\hat{X}(t)|^2 \\ & - \left(a - \frac{16D|PB|^2}{\lambda_{\max}(Q_1)}\right)\int_0^D \hat{w}_x^2(x,t)dx + a\int_0^D \hat{w}(x,t)KM(x)LC\tilde{X}(t)dx \\ & - \left(\alpha_1 \lambda_{\max}(Q_2) - \frac{4|PLC|^2}{\lambda_{\max}(Q_1)}\right)|\tilde{X}(t)|^2 + 2\alpha_1 \tilde{X}(t)^T \Pi B\tilde{u}(0,t) \\ & - \alpha_2 \int_0^D \tilde{u}_x^2(x,t)dx. \end{aligned} \tag{17.89}$$

Now we consider the term $\int_0^D \hat{w}(x,t)KM(x)LC\tilde{X}(t)dx$ and compute

$$\begin{aligned}\int_0^D \hat{w}(x,t)KM(x)LC\tilde{X}(t)dx & \le \|\hat{w}(t)\| \|KMLC\| |\tilde{X}(t)| \\ & \le \|\hat{w}_x(t)\| 2\sqrt{D} \|KMLC\| |\tilde{X}(t)| \\ & \le \frac{1}{2}\|\hat{w}_x(t)\|^2 + 2D\|KMLC\|^2 |\tilde{X}(t)|^2, \end{aligned} \tag{17.90}$$

where the norm $\|\cdot\|$ is to be understood as $\|\cdot\|_{[0,D]}$. Substituting this inequality into $\dot{V}(t)$, we get

$$
\begin{aligned}
\dot{V}(t) \leq & -\frac{\lambda_{\max}(Q_1)}{2}|\hat{X}(t)|^2 \\
& - \left(\frac{a}{2} - \frac{16D|PB|^2}{\lambda_{\max}(Q_1)}\right)\int_0^D \hat{w}_x^2(x,t)dx \\
& - \left(\alpha_1 \lambda_{\max}(Q_2) - \frac{4|PLC|^2}{\lambda_{\max}(Q_1)} - 2D\|KMLC\|^2\right)|\tilde{X}(t)|^2 \\
& + 2\alpha_1 \tilde{X}(t)^T \Pi B\tilde{u}(0,t) - \alpha_2 \int_0^D \tilde{u}_x^2(x,t)dx\,.
\end{aligned} \tag{17.91}
$$

We bound the term $2\tilde{X}(t)^T \Pi B\tilde{u}(0,t)$ as

$$
2\tilde{X}(t)^T \Pi B\tilde{u}(0,t) \leq \frac{\lambda_{\max}(Q_2)}{2}|\tilde{X}(t)|^2 + \frac{2|\Pi B|^2}{\lambda_{\max}(Q_2)}\tilde{u}^2(0,t)\,, \tag{17.92}
$$

which yields

$$
\begin{aligned}
\dot{V}(t) \leq & -\frac{\lambda_{\max}(Q_1)}{2}|\hat{X}(t)|^2 \\
& - \left(\frac{a}{2} - \frac{16D|PB|^2}{\lambda_{\max}(Q_1)}\right)\int_0^D \hat{w}_x^2(x,t)dx \\
& - \left(\alpha_1 \frac{\lambda_{\max}(Q_2)}{2} - \frac{4|PLC|^2}{\lambda_{\max}(Q_1)} - 2D\|KMLC\|^2\right)|\tilde{X}(t)|^2 \\
& - \left(\alpha_2 - \alpha_1 \frac{2|\Pi B|^2}{\lambda_{\max}(Q_2)}\right)\int_0^D \tilde{u}_x^2(x,t)dx\,.
\end{aligned} \tag{17.93}
$$

Now we pick

$$
a = \frac{64D|PB|^2}{\lambda_{\max}(Q_1)}\,, \tag{17.94}
$$

$$
\alpha_1 = \frac{8}{\lambda_{\max}(Q_2)}\left(\frac{2|PLC|^2}{\lambda_{\max}(Q_1)} + D\|KMLC\|^2\right), \tag{17.95}
$$

$$
\alpha_2 = \alpha_1 \frac{4|\Pi B|^2}{\lambda_{\max}(Q_2)} \tag{17.96}
$$

and obtain

$$
\begin{aligned}
\dot{V}(t) \leq & -\frac{\lambda_{\max}(Q_1)}{2}|\hat{X}(t)|^2 - \frac{a}{4}\int_0^D \hat{w}_x^2(x,t)dx \\
& - \alpha_1 \frac{\lambda_{\max}(Q_2)}{4}|\tilde{X}(t)|^2 - \frac{\alpha_2}{2}\int_0^D \tilde{u}_x^2(x,t)dx\,.
\end{aligned} \tag{17.97}
$$

With a slight majorization and a rearrangement, performed for the sake of clarity, we get

$$\dot{V}(t) \leq -\lambda_{\max}(Q_1)\frac{1}{2}|\hat{X}(t)|^2 - \frac{a}{4}\int_0^D \hat{w}_x^2(x,t)dx - \frac{\lambda_{\max}(Q_2)}{2}\frac{\alpha_1}{2}|\tilde{X}(t)|^2 - \frac{\alpha_2}{4}\int_0^D \tilde{u}_x^2(x,t)dx. \tag{17.98}$$

Defining

$$\mu = \frac{1}{2}\min\left\{1\,,\ \frac{\lambda_{\max}(Q_2)}{\lambda_{\min}(P)}\,,\ \frac{\lambda_{\max}(Q_1)}{2\lambda_{\min}(\Pi)}\right\}, \tag{17.99}$$

we arrive at

$$\dot{V}(t) \leq -\mu V(t). \tag{17.100}$$

Since

$$\underline{\rho}\,\Xi(t) \leq V(t) \leq \bar{\rho}\,\Xi(t), \tag{17.101}$$

where

$$\Xi(t) = |\tilde{X}(t)|^2 + |\hat{X}(t)|^2 + \int_0^D \tilde{u}^2(x,t)dx + \int_0^D \hat{w}^2(x,t)dx, \tag{17.102}$$

and

$$\underline{\rho} = \min\left\{\lambda_{\min}(P)\,,\ \frac{a}{2}\,,\ \alpha_1\lambda_{\min}(\Pi)\,,\ \frac{\alpha_2}{2}\right\}, \tag{17.103}$$

$$\bar{\rho} = \max\left\{\lambda_{\max}(P)\,,\ \frac{a}{2}\,,\ \alpha_1\lambda_{\max}(\Pi)\,,\ \frac{\alpha_2}{2}\right\}, \tag{17.104}$$

we get an exponential stability estimate:

$$\Xi(t) \leq \frac{\bar{\rho}}{\underline{\rho}}\Xi(0)\mathrm{e}^{-\mu t}, \quad t \geq 0. \tag{17.105}$$

From the proof of Theorem 15.1, we also have

$$\|\hat{w}(t)\|^2 \leq \alpha_1\|\hat{u}(t)\|^2 + \alpha_2|\hat{X}(t)|^2, \tag{17.106}$$

$$\|\hat{u}(t)\|^2 \leq \beta_1\|\hat{w}(t)\|^2 + \beta_2|\hat{X}(t)|^2, \tag{17.107}$$

where

$$\alpha_1 = 3\left(1 + D\|m\|^2\right), \tag{17.108}$$

$$\alpha_2 = 3\|KM\|^2, \tag{17.109}$$

$$\beta_1 = 3\left(1 + D\|n\|^2\right), \tag{17.110}$$

$$\beta_2 = 3\|KN\|^2. \tag{17.111}$$

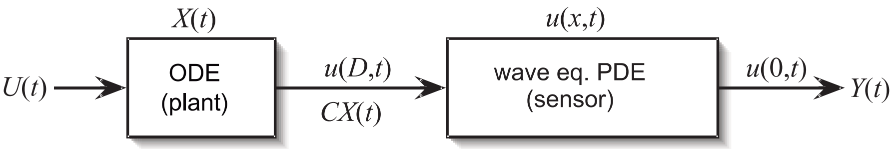

Fig. 17.4 The cascade of the ODE dynamics of the plant with the heat equation PDE dynamics of the sensor.

Therefore,

$$\underline{\delta}\Upsilon(t) \leq \Xi(t) \leq \bar{\delta}\Upsilon(t), \tag{17.112}$$

where

$$\Upsilon(t) = |\tilde{X}(t)|^2 + |\hat{X}(t)|^2 + \int_0^D \tilde{u}^2(x,t)dx + \int_0^D \hat{u}^2(x,t)dx \tag{17.113}$$

and

$$\underline{\delta} = \frac{1}{\max\{1, \alpha_1, 1+\alpha_2\}}, \tag{17.114}$$

$$\bar{\delta} = \max\{1, \beta_1, 1+\beta_2\}. \tag{17.115}$$

So we get an exponential stability estimate:

$$\Upsilon(t) \leq \frac{\bar{\rho}\bar{\delta}}{\underline{\rho}\underline{\delta}}\Upsilon(0)\mathrm{e}^{-\mu t}, \quad t \geq 0. \tag{17.116}$$

Finally, recalling that

$$X = \tilde{X} + \hat{X}, \tag{17.117}$$

$$u = \tilde{u} + \hat{u}, \tag{17.118}$$

we establish the result of the theorem. □

17.4 Observer for ODE with Wave PDE Sensor Dynamics

Consider the LTI-ODE system in a cascade with a wave PDE in the sensing path (as depicted in Figs. 17.4 and 17.5):

$$Y(t) = u(0,t), \tag{17.119}$$

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{17.120}$$

$$u_x(0,t) = 0, \tag{17.121}$$

$$u(D,t) = CX(t), \tag{17.122}$$

$$\dot{X}(t) = AX(t) + BU(t). \tag{17.123}$$

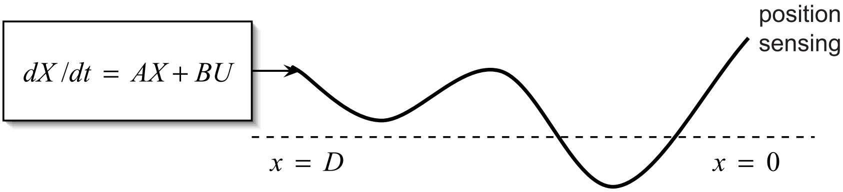

Fig. 17.5 An arbitrary ODE whose output is measured through string dynamics.

The sensor transfer function is given by

$$Y(t) = \frac{1}{\cosh(Ds)}[CX(t)]\,. \tag{17.124}$$

We recall from Chapter 3 that if (17.120), (17.121) are replaced by the delay/transport equation $u_t(x,t) = u_x(x,t)$, then the predictor-based observer

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) + C\mathrm{e}^{Ax}L\left(Y(t) - \hat{u}(0,t)\right), \tag{17.125}$$

$$\hat{u}(D,t) = C\hat{X}(t), \tag{17.126}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \mathrm{e}^{AD}L\left(Y(t) - \hat{u}(0,t)\right) \tag{17.127}$$

achieves perfect compensation of the observer delay and achieves exponential stability at $u - \hat{u} \equiv 0, X - \hat{X} = 0$.

We are seeking an observer of the form

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) + \alpha(x)\left(Y(t) - \hat{u}(0,t)\right) + \beta(x)\left(\dot{Y}(t) - \hat{u}_t(0,t)\right), \tag{17.128}$$

$$\hat{u}_x(0,t) = -a\left(Y(t) - \hat{u}(0,t)\right) - b\left(\dot{Y}(t) - \hat{u}_t(0,t)\right), \tag{17.129}$$

$$\hat{u}(D,t) = C\hat{X}(t), \tag{17.130}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \Lambda\left(Y(t) - \hat{u}(0,t)\right), \tag{17.131}$$

where the functions $\alpha(x), \beta(x)$, the scalars a, b, and the vector Λ are to be determined, to achieve exponential stability of the observer error system

$$\tilde{u}_{tt}(x,t) = \tilde{u}_{xx}(x,t) - \alpha(x)\tilde{u}(0,t) - \beta(x)\tilde{u}_t(0,t), \tag{17.132}$$

$$\tilde{u}_x(0,t) = a\tilde{u}(0,t) + b\tilde{u}_t(0,t), \tag{17.133}$$

$$\tilde{u}(D,t) = C\tilde{X}(t), \tag{17.134}$$

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - \Lambda\tilde{u}(0,t), \tag{17.135}$$

where

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t), \tag{17.136}$$

$$\tilde{X}(t) = X(t) - \hat{X}(t). \tag{17.137}$$

We consider the transformation

$$\tilde{w}(x) = \tilde{u}(x) - \Gamma(x)\tilde{X} \tag{17.138}$$

and try to find $\Gamma(x)$, along with $\alpha(x), \beta(x), a, b$, and Λ, that convert (17.132)–(17.135) into the exponentially stable system

$$\tilde{w}_{tt}(x,t) = \tilde{w}_{xx}(x,t), \tag{17.139}$$
$$\tilde{w}_x(0,t) = c_0 \tilde{w}_t(0,t), \tag{17.140}$$
$$\tilde{w}(D,t) = 0, \tag{17.141}$$
$$\dot{\tilde{X}}(t) = (A - \Lambda\Gamma(0))\tilde{X} - \Lambda\tilde{w}(0,t), \tag{17.142}$$

where $c_0 > 0$ and $A - \Lambda\Gamma(0)$ is a Hurwitz matrix.

By matching the systems (17.132)–(17.135) and (17.139)–(17.142), we obtain the conditions

$$\Gamma''(x) = \Gamma(x)A^2, \tag{17.143}$$
$$\Gamma'(0) = c_0\Gamma(0)A, \tag{17.144}$$
$$\Gamma(D) = C, \tag{17.145}$$

as well as

$$\alpha(x) = \Gamma(x)A\Lambda, \tag{17.146}$$
$$\beta(x) = \Gamma(x)\Lambda, \tag{17.147}$$
$$a = c_0\Gamma(0)\Lambda, \tag{17.148}$$
$$b = c_0. \tag{17.149}$$

Solving the linear ODE two-point boundary-value problem (17.143)–(17.145), we obtain

$$\Gamma(x) = \Gamma(0)G(x), \tag{17.150}$$

where

$$\Gamma(0) = CG(D)^{-1}, \tag{17.151}$$
$$G(x) = \begin{bmatrix} I & c_0A \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{17.152}$$

Thus, we have determined all the quantities needed to implement the observer (17.128)–(17.131) except Λ, which needs to be chosen so that the matrix $A - \Lambda\Gamma(0)$ is Hurwitz. We pick Λ as

$$\Lambda = G(D)L, \tag{17.153}$$

where L is chosen so that the matrix $A-LC$ is Hurwitz. Since A and $G(D)$ commute, using $G(D)$ as a similarity transformation for the matrix

$$A-\Lambda\Gamma(0)=A-G(D)LCG(D)^{-1}, \tag{17.154}$$

we get that the matrices $A-LC$ and $A-\Lambda\Gamma(0)$ have the same eigenvalues, so the latter matrix is Hurwitz.

So the system (17.139)–(17.142) is a cascade of a wave equation (17.139)–(17.141), which is exponentially stable due to the "damping" boundary condition (17.140), and of the exponentially stable ODE (17.142). Thus, the entire observer error system is exponentially stable.

Theorem 17.3 (Observer convergence—wave PDE sensor dynamics). *Assume that the matrix $G(D)$ is nonsingular. The observer (17.128)–(17.131), with gains defined through (17.146)–(17.153), guarantees that $\hat{X}$, $\hat{u}$ exponentially converge to X, u, i.e., more precisely, that the observer error system is exponentially stable in the sense of the norm*

$$\Bigg(|X(t)-\hat{X}(t)|^2 + \int_0^D (u_x(x,t)-\hat{u}_x(x,t))^2\,dx + \int_0^D (u_t(x,t)-\hat{u}_t(x,t))^2\,dx\Bigg)^{1/2}. \tag{17.155}$$

Proof. Very similar to the proof of Theorem 16.1, with a Lyapunov function

$$V(t)=\tilde{X}(t)^T G(D)^{-T}PG(D)^{-1}\tilde{X}(t)+aE(t), \tag{17.156}$$

where $P=P^T>0$ is the solution to the Lyapunov equation

$$P(A-LC)+(A-LC)^TP=-Q \tag{17.157}$$

for some $Q=Q^T>0$, and with

$$E(t)=\frac{1}{2}\left(\|\tilde{w}_x(t)\|^2+\|\tilde{w}_t(t)\|^2\right) + \delta\int_0^D(-1-D+y)\tilde{w}_x(y,t)\tilde{w}_t(y,t)dy. \tag{17.158}$$

The system norms are simpler:

$$\Omega(t)=\|\tilde{u}_x(t)\|^2+\|\tilde{u}_t(t)\|^2+|\tilde{X}(t)|^2, \tag{17.159}$$

$$\Xi(t)=\|\tilde{w}_x(t)\|^2+\|\tilde{w}_t(t)\|^2+|\tilde{X}(t)|^2, \tag{17.160}$$

and the system transformations are much simpler:

$$\tilde{w}_x(x,t)=\tilde{u}_x(x,t)-\Gamma'(x)\tilde{X}(t), \tag{17.161}$$

$$\tilde{w}_t(x,t) = \tilde{u}_t(x,t) - \Gamma(x)A\tilde{X}(t) + \Gamma(x)\Lambda\tilde{u}(0,t)\,, \tag{17.162}$$

$$\begin{aligned}\tilde{u}_t(x,t) = \tilde{w}_t(x,t) &+ \Gamma(x)(A - \Lambda\Gamma(0))\tilde{X}(t)\\ &- \Gamma(x)\Lambda\tilde{w}(0,t)\,.\end{aligned} \tag{17.163}$$

One obtains the inequalities (16.76), (16.70) with the help of Agmon's inequality, or, with the help of Poincaré's inequality and the alternative representation of the state transformation,

$$\begin{aligned}\tilde{w}_t(x,t) = \tilde{u}_t(x,t) + \Gamma(x)\Lambda\tilde{u}(x,t) - \Gamma(x)\Lambda\int_0^D \tilde{u}_x(y,t)dy\\ - \Gamma(x)A\tilde{X}(t)\,,\end{aligned} \tag{17.164}$$

$$\begin{aligned}\tilde{u}_t(x,t) = \tilde{w}_t(x,t) - \Gamma(x)\Lambda\tilde{w}(x,t) + \Gamma(x)\Lambda\int_0^D \tilde{w}_x(y,t)dy\\ + \Gamma(x)(A - \Lambda\Gamma(0))\tilde{X}(t)\,.\end{aligned} \tag{17.165}$$

Then, one obtains (16.114), which completes the proof. □

17.5 Example: Wave PDE Sensor Dynamics

We return to the example in Section 3.2, but with the sensor delay replaced by a wave PDE governing the sensor dynamics:

$$Y(t) = u(0,t)\,, \tag{17.166}$$

$$u_{tt}(x,t) = u_{xx}(x,t)\,, \tag{17.167}$$

$$u_x(0,t) = 0\,, \tag{17.168}$$

$$u(D,t) = CX(t)\,, \tag{17.169}$$

$$\dot{X}(t) = AX(t) + BU(t)\,, \tag{17.170}$$

where

$$X = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{17.171}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{17.172}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{17.173}$$

$$C = [1 \;\; 0]\,. \tag{17.174}$$

We are seeking an observer of the form

$$\begin{aligned}\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) &+ \alpha(x)\,(Y(t) - \hat{u}(0,t))\\ &+ \beta(x)\left(\dot{Y}(t) - \hat{u}_t(0,t)\right)\,,\end{aligned} \tag{17.175}$$

$$\hat{u}_x(0,t) = -a\,(Y(t) - \hat{u}(0,t)) - b\left(\dot{Y}(t) - \hat{u}_t(0,t)\right)\,, \tag{17.176}$$

$$\hat{u}(D,t) = C\hat{X}(t)\,, \tag{17.177}$$

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \Lambda\,(Y(t) - \hat{u}(0,t))\,, \tag{17.178}$$

where

$$\alpha(x) = CG(D)^{-1}G(x)AG(D)L\,, \tag{17.179}$$

$$\beta(x) = CG(D)^{-1}G(x)G(D)L\,, \tag{17.180}$$

$$a = c_0 CL\,, \tag{17.181}$$

$$b = c_0\,, \tag{17.182}$$

$$\Lambda = G(D)L\,, \tag{17.183}$$

$$G(x) = \begin{bmatrix} I & c_0 A \end{bmatrix} \mathrm{e}^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{17.184}$$

We take the nominal observer gain vector as

$$L = \begin{bmatrix} g \\ 0 \end{bmatrix}, \qquad g > 0\,, \tag{17.185}$$

so that the nominal (undelayed) observer error system is governed by the system matrix

$$A - LC = \begin{bmatrix} -g & 1 \\ -1 & 0 \end{bmatrix}, \tag{17.186}$$

which is Hurwitz. For $G(x)$, we obtain

$$G(x) = \begin{bmatrix} \cos x & c_0 \sin x \\ -c_0 \sin x & \cos x \end{bmatrix}. \tag{17.187}$$

With $CL = g$, we get

$$\alpha(x) = CG(x)AL = -gc_0 \sin x\,, \tag{17.188}$$

$$\beta(x) = CG(x)L = g\cos x\,, \tag{17.189}$$

$$a = c_0 g\,, \tag{17.190}$$

$$b = c_0\,, \tag{17.191}$$

$$\Lambda = g \begin{bmatrix} \cos D \\ -c_0 \sin D \end{bmatrix}. \tag{17.192}$$

Hence, the observer is

$$\begin{aligned} \hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) - gc_0 \sin(x)\,(Y(t) - \hat{u}(0,t)) \\ + g\cos(x)\left(\dot{Y}(t) - \hat{u}_t(0,t)\right)\,, \end{aligned} \tag{17.193}$$

$$\hat{u}_x(0,t) = -gc_0\,(Y(t) - \hat{u}(0,t)) - c_0\left(\dot{Y}(t) - \hat{u}_t(0,t)\right)\,, \tag{17.194}$$

$$\hat{u}(D,t) = \hat{\xi}_1(t), \tag{17.195}$$

$$\dot{\hat{\xi}}_1(t) = \hat{\xi}_2(t) + g\cos(D)\left(Y(t) - \hat{u}(0,t)\right), \tag{17.196}$$

$$\dot{\hat{\xi}}_2(t) = -\hat{\xi}_1(t) + U(t) - gc_0\sin(D)\left(Y(t) - \hat{u}(0,t)\right). \tag{17.197}$$

For the sake of comparison with the observer in the example in Section 3.2, we give that observer here:

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) - g\cos(x)\left(Y(t) - \hat{u}(0,t)\right), \tag{17.198}$$

$$\hat{u}(D,t) = \hat{\xi}_1(t), \tag{17.199}$$

$$\dot{\hat{\xi}}_1(t) = \hat{\xi}_2(t) + g\cos(D)\left(Y(t) - \hat{Y}(t)\right), \tag{17.200}$$

$$\dot{\hat{\xi}}_2(t) = -\hat{\xi}_1(t) + U(t) - g\sin(D)\left(Y(t) - \hat{Y}(t)\right). \tag{17.201}$$

The reader should note that when $D = 0$, both the observer (17.196), (17.197) and the observer (17.200), (17.201) reduce to

$$\dot{\hat{\xi}}_1(t) = \hat{\xi}_2(t) + g\left(Y(t) - \hat{\xi}_1(t)\right), \tag{17.202}$$

$$\dot{\hat{\xi}}_2(t) = -\hat{\xi}_1(t) + U(t), \tag{17.203}$$

which is an exponentially convergent observer for the system

$$\dot{\xi}_1(t) = \xi_2(t), \tag{17.204}$$

$$\dot{\xi}_2(t) = -\xi_1(t) + U(t), \tag{17.205}$$

$$Y(t) = \xi_1(t). \tag{17.206}$$

17.6 Observer-Based Controller for ODEs with Wave PDE Actuator Dynamics

Now we return to the same problem as in Section 3.4, namely, the problem where we combined a full-state predictor feedback with a finite-dimensional observer for the ODE. However, here we deal with actuator dynamics of a wave PDE type.

We consider the plant

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{17.207}$$

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{17.208}$$

$$u_x(0,t) = 0, \tag{17.209}$$

$$u_x(D,t) = U(t), \tag{17.210}$$

$$Y(t) = CX(t), \tag{17.211}$$

$$\mathscr{Y}(t) = u(D,t), \tag{17.212}$$

as in Section (16.2), but with the output maps (17.211) and (17.212) added. Note the crucial difference between the problem formulation here for a wave equation and the problem formulation in Section 17.3 for the heat equation. Here we employ an additional measurement $\mathscr{Y}(t)$, which is the wave equation displacement $u(D,t)$ and is collocated with the input force $u_x(D,t)$.

The additional measurement $\mathscr{Y}(t)$ is employed because the actuator dynamics are not exponentially stable, as is the case with the heat equation and the delay dynamics, so we employ this additional measurement in order to stabilize the observer error system via output injection. Several other output choices are possible, collocated or noncollocated with the input. We make this collocated choice because it seems more physically reasonable than, say, a measurement of $u(0,t)$ at the actual input to the plant, which would actually be technically easier.

Let a vector L be chosen so that the matrix $A-LC$ is Hurwitz. Then we introduce an observer for the entire plant (ODE and PDE) and an observer-based controller, namely,

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}(0,t) + L\left(Y(t) - C\hat{X}(t)\right), \tag{17.213}$$

$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) + d_0 d_1\left(\dot{\mathscr{Y}}(t) - \hat{u}_t(D,t)\right), \tag{17.214}$$

$$\hat{u}_x(0,t) = 0, \tag{17.215}$$

$$\hat{u}_x(D,t) = U(t) + d_0\left(\mathscr{Y}(t) - \hat{u}(D,t)\right) + d_1\left(\dot{\mathscr{Y}}(t) - \hat{u}_t(D,t)\right), \tag{17.216}$$

$$\begin{aligned} U(t) = {} & (-c_0 + c_1 KB)\hat{u}(D,t) - c_1\hat{u}_t(D,t) \\ & + \int_0^D p(D-y)\hat{u}(y,t)dy + \int_0^D q(D-y)\hat{u}_t(y,t)dy \\ & + \pi(D)\hat{X}(t), \end{aligned} \tag{17.217}$$

where $p(s), q(s)$, and $\pi(s)$ are defined in (16.59), (16.60), and (16.61), respectively, and (17.217) is the certainty-equivalence version of the controller (16.58).

We note that the observer equations (17.214) and (17.216) are more complex than the respective observer equations (17.59) and (17.61) for the heat equation—we employ output injection here, with positive gains $d_0,\ d_1$, to stabilize the wave equation observer error system.

The following stability result holds for the closed-loop system (17.207)–(17.210), (17.213)–(17.217).

Theorem 17.4. *The closed-loop system (17.207)–(17.210), (17.213)–(17.217) is exponentially stable in the sense of the norm*

$$\begin{aligned} \Bigg(& |X(t)|^2 + |\hat{X}(t)|^2 \\ & + u^2(0,t) + \int_0^D u_x^2(x,t)dx + \int_0^D u_t^2(x,t)dx \\ & + \hat{u}^2(0,t) + \int_0^D \hat{u}_x^2(x,t)dx + \int_0^D \hat{u}_t^2(x,t)dx \Bigg)^{1/2}. \end{aligned} \tag{17.218}$$

Proof. The observer error system is now defined with

$$\tilde{X}(t) = X(t) - \hat{X}(t)\,, \tag{17.219}$$
$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t) \tag{17.220}$$

and given by

$$\dot{\tilde{X}}(t) = (A - LC)\tilde{X}(t) + B\tilde{u}(0,t)\,, \tag{17.221}$$
$$\tilde{u}_{tt}(x,t) = \tilde{u}_{xx}(x,t) - d_0 d_1 \tilde{u}_t(D,t)\,, \tag{17.222}$$
$$\tilde{u}_x(0,t) = 0\,, \tag{17.223}$$
$$\tilde{u}_x(D,t) = -d_0\tilde{u}(D,t) - d_1\tilde{u}_t(D,t)\,. \tag{17.224}$$

The stability of the overall system is analyzed by studying the observer error system (17.221)–(17.224) and the observer system with the feedback substituted:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B\hat{u}(0,t) + LC\tilde{X}(t)\,, \tag{17.225}$$
$$\hat{u}_{tt}(x,t) = \hat{u}_{xx}(x,t) + d_0 d_1 \tilde{u}_t(D,t)\,, \tag{17.226}$$
$$\hat{u}_x(0,t) = 0\,, \tag{17.227}$$
$$\begin{aligned}\hat{u}_x(D,t) = {} & (-c_0 + c_1 KB)\hat{u}(D,t) - c_1\hat{u}_t(D,t) \\ & + \int_0^D p(D-y)\hat{u}(y,t)dy + \int_0^D q(D-y)\hat{u}_t(y,t)dy \\ & + \pi(D)\hat{X}(t) \\ & + d_0\tilde{u}(D,t) + d_1\tilde{u}_t(D,t)\,. \end{aligned} \tag{17.228}$$

The overall closed-loop system (17.221)–(17.224), (17.225)–(17.228) is a four-component system, $\left(\tilde{X},\hat{X},\tilde{u},\hat{u}\right)$, but its structure is more complex than for the respective system in Section 17.3. The $\tilde{u}$-component is autonomous and exponentially stable. Its exponential stability is not obvious, but we will explain it in a second. The $\tilde{u}$-system feeds into the exponentially stable $\tilde{X}$-subsystem. Then both components of the $(\tilde{X},\tilde{u})$-system feed into the respective components of the $\left(\hat{X},\hat{u}\right)$-subsystem. So the cascade structure is

$$\begin{array}{ccc} \tilde{X} & \rightarrow & \hat{X} \\ \uparrow & & \downarrow\uparrow \\ \tilde{u} & \rightarrow & \hat{u} \end{array} \tag{17.229}$$

Let us now turn our attention to the $\tilde{u}$-system (17.222)–(17.224) and its exponential stability. Consider the backstepping transformation

$$\tilde{\varepsilon}(x,t) = \tilde{u}(x,t) - d_0 \int_x^D \mathrm{e}^{d_0(x-y)}\tilde{u}(y,t)dy\,, \tag{17.230}$$
$$\tilde{u}(x,t) = \tilde{\varepsilon}(x,t) + d_0 \int_x^D \tilde{\varepsilon}(y,t)dy\,. \tag{17.231}$$

Note the limits of the integrals, which are from x to D, rather than the usual from 0 to x. With this transformation, a routine calculation shows that the new state variable of the observer error system $\tilde{\varepsilon}(x,t)$ satisfies the wave equation PDE

$$\tilde{\varepsilon}_{tt}(x,t) = \tilde{\varepsilon}_{xx}(x,t)\,, \tag{17.232}$$

$$\tilde{\varepsilon}_x(0,t) = d_0\tilde{\varepsilon}(0,t)\,, \tag{17.233}$$

$$\tilde{\varepsilon}_x(D,t) = -d_1\tilde{\varepsilon}_t(D,t)\,. \tag{17.234}$$

With

$$E_2(t) = \frac{1}{2}\left(d_0\tilde{\varepsilon}(0,t)^2 + \|\tilde{\varepsilon}_x(t)\|^2 + \|\tilde{\varepsilon}_t(t)\|^2\right) + \delta_2\int_0^D (1+y)\tilde{\varepsilon}_x(y,t)\tilde{\varepsilon}_t(y,t)dy\,, \tag{17.235}$$

we get

$$\dot{E}_2(t) = -\left(d_1 - \delta_2\frac{1+D}{2}\left(1+d_1^2\right)\right)\tilde{\varepsilon}_t(D,t)^2 - \frac{\delta_2}{2}\left(\tilde{\varepsilon}_t(0,t)^2 + c_0^2\tilde{\varepsilon}(0,t)^2\right) - \frac{\delta_2}{2}\left(\|\tilde{\varepsilon}_x(t)\|^2 + \|\tilde{\varepsilon}_t(t)\|^2\right)\,, \tag{17.236}$$

so with a sufficiently small δ, we establish the exponential stability of the $\tilde{\varepsilon}$-system and hence that of the $\tilde{u}$-system.

Now let $P = P^T > 0$ and $\Pi = \Pi^T > 0$ denote, respectively, the solutions to the Lyapunov equations

$$P(A+BK) + (A+BK)^T P = -Q_1\,, \tag{17.237}$$

$$\Pi(A-LC) + (A-LC)^T\Pi = -Q_2 \tag{17.238}$$

for some $Q_1 = Q_1^T > 0$ and $Q_2 = Q_2^T > 0$. The exponential stability of the $(\tilde{X},\tilde{u})$-system is obtained through the study of the $(\tilde{X},\tilde{\varepsilon})$-system

$$\dot{\tilde{X}}(t) = (A-LC)\tilde{X}(t) + B\left(\tilde{\varepsilon}(0,t) + d_0\int_0^D \tilde{\varepsilon}(y,t)dy\right)\,, \tag{17.239}$$

$$\tilde{\varepsilon}_{tt}(x,t) = \tilde{\varepsilon}_{xx}(x,t)\,, \tag{17.240}$$

$$\tilde{\varepsilon}_x(0,t) = d_0\tilde{\varepsilon}(0,t)\,, \tag{17.241}$$

$$\tilde{\varepsilon}_x(D,t) = -d_1\tilde{\varepsilon}_t(D,t)\,, \tag{17.242}$$

using the same Lyapunov function $E(t)$ augmented by a Lyapunov function for the $\tilde{X}$-subsystem, i.e.,

$$V_2(t) = \alpha_1\tilde{X}(t)^T\Pi\tilde{X}(t) + \alpha_2 E_2(t)\,, \tag{17.243}$$

where α_1 and α_2 are positive constants.

Now let us consider the $(\hat{X},\hat{u})$-subsystem. For $\tilde{X}\equiv 0, \tilde{u}\equiv 0$, this system was shown to be exponentially stable in Theorem 16.1. Here we introduce the transformation

$$\begin{aligned}\hat{w}(x,t) = \hat{u}(x,t)\\ &+\int_0^x \left(c_0-\mu(x-y)-c_0\int_0^{x-y}\mu(\xi)d\xi\right)\hat{u}(y,t)dy\\ &-\int_0^x \left(m(x-y)+c_0\int_0^{x-y}m(\xi)d\xi\right)\hat{u}_t(y,t)dy\\ &-\left(\gamma(x)+c_0\int_0^x \gamma(\xi)d\xi\right)\hat{X}(t)\end{aligned} \tag{17.244}$$

and derive a PDE for the $\hat{w}$-system, which incorporates the input terms $\tilde{X}(t)$, $\tilde{u}(D,t)$, and $\tilde{u}_t(D,t)$. We forego this derivation here due to its length. For the $(\hat{X},\hat{w})$-system, we introduce the Lyapunov function

$$V_1(t)=\hat{X}(t)^T P\hat{X}(t)+aE_1(t)\,, \tag{17.245}$$

where the positive constant a is to be determined, and where

$$\begin{aligned}E_1(t) = &\frac{1}{2}\left(c_0\hat{w}(0,t)^2+\|\hat{w}_x(t)\|^2+\|\hat{w}_t(t)\|^2\right)\\ &+\delta_1\int_0^D (1+y)\hat{w}_x(y,t)\hat{w}_t(y,t)dy\,.\end{aligned} \tag{17.246}$$

The Lyapunov function of the overall $(\tilde{X},\tilde{u},\hat{X},\hat{u})$-system is then introduced as

$$V(t)=V_1(t)+V_2(t)\,. \tag{17.247}$$

For the $(\tilde{X},\tilde{u})$-subsystem, we have

$$\dot{V}_2(t)\le -\mu_2V_2(t)-\alpha_2\left(d_1-\delta_2\frac{1+D}{2}\left(1+d_1^2\right)\right)\tilde{\varepsilon}_t(D,t)^2 \tag{17.248}$$

for some positive μ_2. For the $(\hat{X},\hat{u})$-subsystem, we can establish that

$$\begin{aligned}\dot{V}_1(t)\le &-\mu_1V_1(t)-a\left(c_1-\delta_1\frac{1+D}{2}\left(1+c_1^2\right)\right)\hat{w}(D,t)^2\\ &+\eta_1\left(|\tilde{X}(t)|^2+\tilde{\varepsilon}(D,t)^2+\tilde{\varepsilon}_t(D,t)^2\right)\end{aligned} \tag{17.249}$$

for some positive η_1. Choosing δ_1 and δ_2 sufficiently small, and picking α_1 and α_2 in V_2 sufficiently large, we get

$$\dot{V}(t)\le -\mu V(t) \tag{17.250}$$

for some positive μ. We point out that in [110, Lemma 3] we studied stability of the $(\hat{u},\tilde{u})$-system alone, namely, of what would remain of our overall $(\tilde{X},\tilde{u},\hat{X},\hat{u})$-system here if one were to set $K=0$. That analysis dealt with the key issue of establishing exponential stability of the cascade $\tilde{u}\to\hat{u}$, where the interconnection involves a boundary value $\tilde{u}(D,t)$, and which enters a boundary condition for $\hat{u}_x(D,t)$. Hence, we obtain

$$V(t)\le V(0)\mathrm{e}^{-\mu t}\,,\quad t\ge 0\,. \tag{17.251}$$

Then we relate $V(t)$ with

$$\begin{aligned}\Xi(t)=&|\tilde{X}(t)|^2+\tilde{u}(0,t)^2+\|\tilde{u}_x(t)\|^2+\|\tilde{u}_t(t)\|^2\\&+|\hat{X}(t)|^2+\hat{w}(0,t)^2+\|\hat{w}_x(t)\|^2+\|\hat{w}_t(t)\|^2\end{aligned} \tag{17.252}$$

and with

$$\begin{aligned}\Omega(t)=&|\tilde{X}(t)|^2+\tilde{u}(0,t)^2+\|\tilde{u}_x(t)\|^2+\|\tilde{u}_t(t)\|^2\\&+|\hat{X}(t)|^2+\hat{u}(0,t)^2+\|\hat{u}_x(t)\|^2+\|\hat{u}_t(t)\|^2\end{aligned} \tag{17.253}$$

in a similar manner as we did in the proof of Theorem 16.1. Recalling that

$$X=\tilde{X}+\hat{X}\,, \tag{17.254}$$

$$u=\tilde{u}+\hat{u}\,, \tag{17.255}$$

we arrive at the result of the theorem. □

17.7 Notes and References

The observer designs in this chapter are inspired by observer designs for PDEs with boundary sensors in [203].

Part V
Delay-PDE and PDE-PDE Cascades

Chapter 18
Unstable Reaction-Diffusion PDE with Input Delay

In this chapter and in Chapter 20 we introduce the problems of stabilization of PDE-PDE cascades. First, we deal with PDEs with input delays (the reaction-diffusion PDE in this chapter, and the antistable wave PDE in Chapter 20). Then we deal with cascades of unstable heat and wave PDEs (in either order) in Chapter 20.

Stability analysis for cascades of stable PDEs from different classes, when interconnected through a boundary, virtually explodes in complexity despite the seemingly simple structure where one PDE is autonomous and exponentially stable and feeds into the other PDE. The difficulty arises for two reasons. One is that the connectivity through the boundary gives rise to an unbounded input operator in the interconnection. The second reason is that the two subsystems are from different PDE classes, with different numbers of derivatives in space or time (or both). This requires delicate combinations of norms in the Lyapunov functions for the overall systems.

We start with a presentation of the control design for the unstable reaction-diffusion equation with input delay in Section 18.1. For clarity and historical context, we specialize this design to the problem without input delay in Section 18.2. We conduct the stability analysis in the following four sections, starting with a development of the inverse backstepping transformation in Section 18.3, a proof of stability of the target system in Section 18.4, a proof of stability of the system in the original variables in Section 18.5, and a derivation of the estimates on the transformation kernels in Section 18.6, which complete the overall stability proof. In Section 18.7 we derive explicit formulas for the control gains, and in Section 18.8 we also find the explicit formulas for the solutions of the closed-loop system.

18.1 Control Design for the Unstable Reaction-Diffusion PDE with Input Delay

Consider the system

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t)\,, \tag{18.1}$$

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_18,

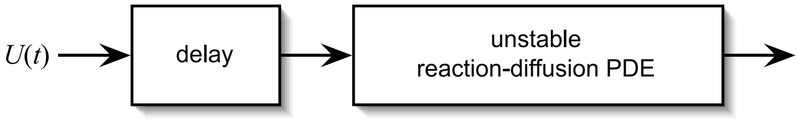

Fig. 18.1 Reaction-diffusion PDE system with input delay.

$$u(0,t) = 0\,, \tag{18.2}$$
$$u(1,t) = U(t-D)\,, \tag{18.3}$$

which is depicted in Fig. 18.1, or, in an alternative representation,

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t)\,, \quad x \in (0,1)\,, \tag{18.4}$$
$$u(0,t) = 0\,, \tag{18.5}$$
$$u(1,t) = v(1,t)\,, \tag{18.6}$$

$$v_t(x,t) = v_x(x,t)\,, \quad x \in [1,1+D)\,, \tag{18.7}$$
$$v(1+D,t) = U(t)\,, \tag{18.8}$$

where $U(t)$ is the input and (u,v) is the state. As we have observed on many earlier occasions throughout the book,

$$v(x,t) = U(t+x-1-D)\,, \quad x \in [1,1+D]\,. \tag{18.9}$$

We consider the backstepping transformation of the form

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy\,, \quad x \in [0,1]\,, \tag{18.10}$$
$$\begin{aligned} z(x,t) = v(x,t) &- \int_1^x p(x-y)v(y,t)dy \\ &- \int_0^1 \gamma(x,y)u(y,t)dy\,, \quad x \in [1,1+D]\,, \end{aligned} \tag{18.11}$$

where the kernels k, p, and γ need to be chosen to transform the cascade PDE system into the target system

$$w_t(x,t) = w_{xx}(x,t)\,, \quad x \in (0,1)\,, \tag{18.12}$$
$$w(0,t) = 0\,, \tag{18.13}$$
$$w(1,t) = z(1,t)\,, \tag{18.14}$$

$$z_t(x,t) = z_x(x,t)\,, \quad x \in [1,1+D)\,, \tag{18.15}$$
$$z(1+D,t) = 0\,, \tag{18.16}$$

with the control

$$U(t)=\int_1^{1+D}p(1+D-y)v(y,t)dy+\int_0^1\gamma(1+D,y)u(y,t)dy. \tag{18.17}$$

The cascade connection

$$z\to w \tag{18.18}$$

is a cascade of an exponentially stable autonomous transport PDE for $z(x,t)$, feeding into the exponentially stable heat PDE for $w(x,t)$.

The change of variables

$$(u,v)\mapsto(w,z) \tag{18.19}$$

is defined through the three integral operator kernels, $k(x,y)$, $\gamma(x,y)$, and $p(x)$. With a lengthy calculation, we show that the kernel $k(x,y)$ has to satisfy the PDE

$$k_{xx}(x,y)-k_{yy}(x,y)=\lambda k(x,y),\quad 0\le y\le x\le 1, \tag{18.20}$$

$$k(x,0)=0, \tag{18.21}$$

$$k(x,x)=-\frac{\lambda}{2}x, \tag{18.22}$$

for which the solution was found explicitly in [202] as

$$k(x,y)=-\lambda y\frac{I_1\left(\sqrt{\lambda\,(x^2-y^2)}\right)}{\sqrt{\lambda\,(x^2-y^2)}}, \tag{18.23}$$

where $I_1(\cdot)$ denotes the appropriate modified Bessel function. The kernel γ is found to be governed by the reaction-diffusion PDE

$$\gamma_x(x,y)=\gamma_{yy}(x,y)+\lambda\gamma(x,y),\qquad (x,y)\in[1,1+D]\times(0,1), \tag{18.24}$$

$$\gamma(x,0)=0, \tag{18.25}$$

$$\gamma(x,1)=0, \tag{18.26}$$

where $x\in[1,1+D]$ should be viewed as the time variable and $y\in(0,1)$ as the space variable, and where the initial condition is given by

$$\gamma(1,y)=k(1,y). \tag{18.27}$$

After solving for $\gamma(x,y)$, the kernel p is obtained as

$$p(s)=-\gamma_y(1+s,1),\qquad s\in[0,D]. \tag{18.28}$$

Before we proceed, we make the following observation about the target system.

Proposition 18.1. *The spectrum of the system (18.12)–(18.16) is given by*

$$\sigma_n=-\pi^2n^2,\qquad n=1,2,\ldots,+\infty. \tag{18.29}$$

The following theorem establishes an exponential stability result in the appropriate norm for the cascade system of two PDEs that are interconnected through a boundary.

Theorem 18.1. *Consider the closed-loop system consisting of the plant (18.4)–(18.8) and the control law*

$$U(t) = \int_0^1 \gamma(D,y)u(y,t)dy - \int_1^{1+D} \gamma_y(1+D-y,1)v(y,t)dy. \tag{18.30}$$

If the initial conditions are such that $(u_0, v_0) \in L_2(0,1) \times H_1[1,1+D)$*, then the system has a unique solution* $(u(\cdot,t), v(\cdot,t)) \in C([0,\infty), L_2(0,1) \times H_1[1,1+D))$ *and there exists a positive continuous function* $M : \mathbb{R}^2 \to \mathbb{R}_+$ *such that*

$$\Upsilon(t) \le M(D,\lambda)\mathrm{e}^{cD}\Upsilon(0)\mathrm{e}^{-\min\{2,c\}t}, \qquad \forall t \ge 0, \tag{18.31}$$

for any $c > 0$*, where*

$$\Upsilon(t) = \int_0^1 u^2(x,t)dx + \int_1^{1+D} \left(v^2(x,t) + v_x^2(x,t)\right) dx. \tag{18.32}$$

This result has a rather lengthy proof, through a series of lemmas presented in Sections 18.3, 18.4, 18.5, and 18.6.

18.2 The Baseline Design ($D = 0$) for the Unstable Reaction-Diffusion PDE

Before we continue with the analysis of the delay-compensating design developed in Section 18.3, we present the feedback law that is obtained in the absence of delay. This special case is of interest in its own right since it achieves stabilization of the unstable reaction-diffusion equation and explains the origins of the design pursued in this chapter.

Setting $D = 0$ in the control law (18.30), we obtain the feedback law

$$U(t) = -\int_0^1 \lambda x \frac{I_1\left(\sqrt{\lambda\,(1-x^2)}\right)}{\sqrt{\lambda\,(1-x^2)}} u(x,t)dx. \tag{18.33}$$

This feedback law was designed in [202]. The following stability result holds.

Theorem 18.2. *Consider a closed-loop system*

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \tag{18.34}$$

$$u(0,t) = 0, \tag{18.35}$$

$$u(1,t) = -\int_0^1 \lambda x \frac{I_1\left(\sqrt{\lambda\,(1-x^2)}\right)}{\sqrt{\lambda\,(1-x^2)}} u(x,t)dx. \tag{18.36}$$

If the initial condition is such that $u_0 \in L_2(0,1)$, then the system has a unique solution $u(\cdot,t) \in C([0,\infty), L_2(0,1))$ and there exists a positive constant G such that

$$\Upsilon(t) \leq G\Upsilon(0)\mathrm{e}^{-2t}, \qquad \forall t \geq 0, \tag{18.37}$$

where

$$\Upsilon(t) = \int_0^1 u^2(x,t)dx. \tag{18.38}$$

The significance of the control law (18.36) is that it provides a compact formula for stabilization of the PDE system (18.34), (18.35). With this feedback law, all of the eigenvalues of the open-loop system,

$$\sigma_n = \lambda - \pi^2 n^2, \qquad n = 1,2,\ldots,+\infty, \tag{18.39}$$

are shifted leftward by exactly λ, placing them at the eigenvalue locations of the heat equation, (18.29), without a complicated procedure that places individual eigenvalues. Theorem 18.2 is proved using the backstepping transformation

$$w(x,t) = u(x,t) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda\,(x^2-y^2)}\right)}{\sqrt{\lambda\,(x^2-y^2)}} u(y,t)dy, \tag{18.40}$$

which transforms the system (18.34)–(18.36) into a target system governed by the heat equation:

$$w_t(x,t) = w_{xx}(x,t), \quad x \in (0,1), \tag{18.41}$$

$$w(0,t) = 0, \tag{18.42}$$

$$w(1,t) = 0. \tag{18.43}$$

18.3 Inverse Backstepping Transformations

Now we return to the analysis of the general design presented in Section 18.1 and the proof of Theorem 18.1.

First, we seek the inverse transformation $(u,v) \mapsto (w,z)$. We postulate it in the form

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy, \qquad x \in [0,1], \tag{18.44}$$

$$v(x,t) = z(x,t) + \int_1^x q(x-y)z(y,t)dy + \int_0^1 \delta(x,y)w(y,t)dy, \qquad x \in [1,1+D]. \tag{18.45}$$

With a lengthy calculation, we show that the kernel $l(x,y)$ has to satisfy the PDE

$$l_{xx}(x,y) - l_{yy}(x,y) = -\lambda l(x,y), \quad 0 \le y \le x \le 1, \tag{18.46}$$
$$l(x,0) = 0, \tag{18.47}$$
$$l(x,x) = -\frac{\lambda}{2}x, \tag{18.48}$$

for which the solution was found explicitly in [202] as

$$l(x,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda\,(x^2-y^2)}\right)}{\sqrt{\lambda\,(x^2-y^2)}}, \tag{18.49}$$

with $J_1(\cdot)$ again being the appropriate Bessel function. The kernel δ is found to be governed by the heat PDE:

$$\delta_x(x,y) = \delta_{yy}(x,y), \qquad (x,y) \in [1, 1+D] \times (0,1), \tag{18.50}$$
$$\delta(x,0) = 0, \tag{18.51}$$
$$\delta(x,1) = 0, \tag{18.52}$$

where $x \in [1, 1+D]$ should be viewed as the time variable and $y \in (0,1)$ as the space variable, and where the initial condition is given by

$$\delta(1,y) = l(1,y). \tag{18.53}$$

After solving for $\delta(x,y)$, the kernel q is obtained as

$$q(s) = -\delta_y(1+s,1), \qquad s \in [0,D]. \tag{18.54}$$

18.4 Stability of the Target System (w,z)

We now prove Theorem 18.1 through a sequence of lemmas given in this section and in Section 18.5.

Lemma 18.1. *Consider the change of variable*

$$\omega(x,t) = w(x,t) - xz(1,t) \tag{18.55}$$

and the resulting cascade system of PDEs:

$$\omega_t(x,t) = \omega_{xx}(x,t) - xz_t(1,t), \tag{18.56}$$
$$\omega(0,t) = 0, \tag{18.57}$$
$$\omega(1,t) = 0, \tag{18.58}$$

$$z_t(x,t) = z_x(x,t)\,, \tag{18.59}$$

$$z(1+D,t) = 0\,. \tag{18.60}$$

Then the following is true:

$$\Pi(t) \le \Pi(0)\mathrm{e}^{-\min\{2,c\}t}\,, \qquad \forall t \ge 0\,, \tag{18.61}$$

where

$$\Pi(t) = \frac{1}{2}\int_0^1 \omega^2(x,t)dx + \frac{1}{12}\int_1^{1+D} \mathrm{e}^{c(x-1)} z_x^2(x,t)dx\,. \tag{18.62}$$

Proof. First, we note that $z_t(1,t) = z_x(1,t)$ and

$$z_{xt}(x,t) = z_{xx}(x,t)\,, \qquad x \in [1, D+1)\,, \tag{18.63}$$

$$z_x(1+D,t) = 0\,. \tag{18.64}$$

It is easy to verify that

$$\frac{d}{dt}\int_1^{1+D} \mathrm{e}^{c(x-1)} z_x^2(x,t)dx = -z_x(1,t)^2 - c\int_1^{1+D} \mathrm{e}^{c(x-1)} z_x^2(x,t)dx\,. \tag{18.65}$$

For the ω-subsystem, we have

$$\frac{d}{dt}\frac{1}{2}\|\omega(t)\|^2 = -\|\omega_x(t)\|^2 - z_x(1,t)\int_0^1 x\omega(x,t)dx\,. \tag{18.66}$$

Based on Lemma A.2,

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\omega(t)\|^2 &\le -\frac{\pi^2}{4}\|\omega(t)\|^2 - z_x(1,t)\int_0^1 x\omega(x,t)dx \\ &\le -2\|\omega(t)\|^2 - z_x(1,t)\int_0^1 x\omega(x,t)dx\,.\end{aligned} \tag{18.67}$$

Then, with Young's inequality,

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\omega(t)\|^2 &\le -\|\omega(t)\|^2 + z_x^2(1,t)\int_0^1 \frac{x^2}{4}dx \\ &\le -\|\omega(t)\|^2 + \frac{1}{12}z_x^2(1,t)\,.\end{aligned} \tag{18.68}$$

Combining (18.65) and (18.68), we get

$$\begin{aligned}\dot{\Pi}(t) &\le -\|\omega(t)\|^2 - \frac{c}{12}\int_1^{1+D} \mathrm{e}^{c(x-1)} z_x^2(x,t)dx \\ &\le -\min\{2,c\}\Pi(t)\,,\end{aligned} \tag{18.69}$$

from which we obtain the result of the lemma. □

So far we have introduced three system representations:

$$(u,v)\,, \quad (w,z)\,, \quad (\omega,z)\,. \tag{18.70}$$

Our analysis of stability was completed for the (ω,z) representation. We will have to establish the relations among the norms of the three different representations so that we can get a stability estimate in the norm of the original system (u,v).

In the next two lemmas we relate the Lyapunov function $\Pi(t)$ with the norm of the transformed system, $\|w(t)\|^2+\|z_x(t)\|^2$.

Lemma 18.2.

$$\Pi(t)\le\frac{3}{2}\mathrm{e}^{cD}\left(\int_0^1 w^2(x,t)dx+\int_1^{1+D} z_x^2(x,t)dx\right). \tag{18.71}$$

Proof. We start with

$$\begin{aligned}\Pi(t)&=\frac{1}{2}\int_0^1\left(w^2(x,t)-2xw(x,t)z(1,t)+x^2z^2(1,t)\right)dx\\&\quad+\frac{1}{12}\int_1^{1+D}\mathrm{e}^{c(x-1)}z_x^2(x,t)dx\\&=\frac{1}{2}\|w(t)\|^2+\frac{1}{6}z^2(1,t)-z(1,t)\int_0^1 xw(x,t)dx\\&\quad+\frac{1}{12}\int_1^{1+D}\mathrm{e}^{c(x-1)}z_x^2(x,t)dx\,.\end{aligned} \tag{18.72}$$

With the Cauchy–Schwartz inequality, we get

$$\begin{aligned}\Pi(t)&\le\frac{1}{2}\|w(t)\|^2+\frac{1}{6}z^2(1,t)-|z(1,t)|\|w(t)\|\left(\int_0^1 x^2dx\right)^{1/2}\\&\quad+\frac{1}{12}\int_1^{1+D}\mathrm{e}^{c(x-1)}z_x^2(x,t)dx\\&\le\|w(t)\|^2+\frac{1}{3}z^2(1,t)+\frac{1}{12}\int_1^{1+D}\mathrm{e}^{c(x-1)}z_x^2(x,t)dx\,.\end{aligned} \tag{18.73}$$

With Agmon's inequality, we get

$$\Pi(t)\le\|w(t)\|^2+\frac{4}{3}\int_1^{1+D} z_x^2(x,t)+\frac{1}{12}\int_1^{1+D}\mathrm{e}^{c(x-1)}z_x^2(x,t)dx\,. \tag{18.74}$$

Finally,

$$\Pi(t)\le\|w(t)\|^2+\left(\frac{4}{3}+\frac{\mathrm{e}^{cD}}{12}\right)\int_1^{1+D} z_x^2(x,t)dx\,, \tag{18.75}$$

which yields the result of the lemma. □

Lemma 18.3.

$$\int_0^1 w^2(x,t)dx + \int_1^{1+D} z_x^2(x,t)dx \le 48\Pi(t)\,. \tag{18.76}$$

Proof. We start with

$$\begin{aligned}\|w(t)\|^2 &= \|\omega(t)\|^2 + 2z(1,t)\int_0^1 x\omega(x,t)dx|z^2(1,t)\int_0^1 x^2 dx\\ &\le \|\omega(t)\|^2 + \frac{2}{\sqrt{3}}|z(1,t)|\|\omega(t)\| + \frac{1}{3}z^2(1,t)\,, \end{aligned}\tag{18.77}$$

where we used the Cauchy–Schwartz inequality. With Young's and Agmon's inequalities, we get

$$\begin{aligned}\|w(t)\|^2 &\le 2\|\omega(t)\|^2 + \frac{2}{3}z^2(1,t)\\ &\le 2\|\omega(t)\|^2 + \frac{8}{3}\int_1^{1+D} z_x^2(x,t)dx\\ &\le 3\left(\|\omega(t)\|^2 + \int_1^{1+D} z_x^2(x,t)dx\right). \end{aligned}\tag{18.78}$$

Since

$$\|\omega(t)\|^2 + \int_1^{1+D} z_x^2(x,t)dx \le 12\Pi(t)\,, \tag{18.79}$$

we arrive at the result of the lemma. □

Now we obtain a stability result in terms of the state of the (w,z)-system.

Lemma 18.4.

$$\Xi(t) \le 72\mathrm{e}^{cD}\Xi(0)\mathrm{e}^{-\min\{2,c\}t}\,, \qquad \forall t \ge 0\,, \tag{18.80}$$

where

$$\Xi(t) = \int_0^1 w^2(x,t)dx + \int_1^{1+D}\left(z^2(x,t) + z_x^2(x,t)\right)dx\,. \tag{18.81}$$

Proof. The result of this lemma follows immediately from the last three lemmas and from the fact that

$$\frac{d}{dt}\int_1^{1+D} \mathrm{e}^{c(x-1)}z^2(x,t)dx = -z(1,t)^2 - c\int_1^{1+D}\mathrm{e}^{c(x-1)}z^2(x,t)dx\,. \tag{18.82}$$

□

18.5 Stability of the System in the Original Variables (u,v)

With several applications of the Cauchy–Schwartz inequality, we get the following lemma.

Lemma 18.5.

$$\Xi(t) \leq \alpha_1 \Upsilon(t)\,, \tag{18.83}$$

$$\Upsilon(t) \leq \alpha_2 \Xi(t)\,, \tag{18.84}$$

where

$$\begin{aligned}
\alpha_1 = {} & 2\left(1+\int_0^1\int_0^x k^2(x,y)dydx\right)+3\int_1^{1+D}\int_0^1 \gamma^2(x,y)dydx \\
& +4\int_1^{1+D}\int_0^1 \gamma_x^2(x,y)dydx \\
& +3\left(1+D\int_1^{1+D}\gamma_y^2(x,1)dx\right) \\
& +4\left(\gamma_y^2(1,1)+D\int_1^{1+D}\gamma_{xy}^2(x,1)dx\right)+4\,,
\end{aligned} \tag{18.85}$$

$$\begin{aligned}
\alpha_2 = {} & 2\left(1+\int_0^1\int_0^x l^2(x,y)dydx\right)+3\int_1^{1+D}\int_0^1 \delta^2(x,y)dydx \\
& +4\int_1^{1+D}\int_0^1 \delta_x^2(x,y)dydx \\
& +3\left(1+D\int_1^{1+D}\delta_y^2(x,1)dx\right) \\
& +4\left(\delta_y^2(1,1)+D\int_1^{1+D}\delta_{xy}^2(x,1)dx\right)+4\,.
\end{aligned} \tag{18.86}$$

Proof. We just highlight several steps in the proof of the first half of the lemma. The proof of the second half is identical. First, from $w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy$, we can obtain

$$\|w(t)\|^2 \leq 2\left(1+\int_0^1\int_0^x k^2(x,y)dydx\right)\|u(t)\|^2\,. \tag{18.87}$$

Then, from $z(x,t) = v(x,t) + \int_1^x \gamma_y(1+x-y,1)v(y,t)dy - \int_0^1 \gamma(x,y)u(y,t)dy$, we get

$$\begin{aligned}
\int_1^{1+D} z^2(x,t)dx \leq {} & 3\left(1+\int_1^{1+D}\int_0^{x-1}\gamma_y^2(1+s,1)dsdx\right)\int_1^{1+D} v^2(x,t)dx \\
& +3\left(\int_1^{1+D}\int_0^1 \gamma^2(x,y)dydx\right)\|u(t)\|^2\,.
\end{aligned} \tag{18.88}$$

Next, from $z(x,t) = v(x,t) + \int_1^x \gamma_y(1+x-y,1)v(y,t)dy - \int_0^1 \gamma(x,y)u(y,t)dy$, we derive

$$z_x(x,t) = v_x(x,t) + \gamma_y(1,1)v(x,t) + \int_1^x \gamma_{xy}(1+x-y,1)v(y,t)dy$$
$$- \int_0^1 \gamma_x(x,y)u(y,t)dy, \qquad (18.89)$$

which yields

$$\int_1^{1+D} z_x^2(x,t)dx \leq 4\int_1^{1+D} v_x^2(x,t)dx$$
$$+4\left(\gamma_y^2(1,1) + \int_1^{1+D}\int_0^{x-1} \gamma_{xy}^2(1+s,1)dsdx\right)\int_1^{1+D} v^2(x,t)dx$$
$$+4\left(\int_1^{1+D}\int_0^1 \gamma_x^2(x,y)dydx\right)\|u(t)\|^2. \qquad (18.90)$$

Combining the above steps, along with the fact that

$$\int_1^{1+D}\int_0^{x-1} \gamma_y^2(1+s,1)dsdx = \int_1^{1+D} (1+D-x)\gamma_y^2(x,1)dx$$
$$\leq D\int_1^{1+D} \gamma_y^2(x,1)dx, \qquad (18.91)$$
$$\int_1^{1+D}\int_0^{x-1} \gamma_{xy}^2(1+s,1)dsdx = \int_1^{1+D} (1+D-x)\gamma_{xy}^2(x,1)dx$$
$$\leq D\int_1^{1+D} \gamma_{xy}^2(x,1)dx, \qquad (18.92)$$

we obtain the first half of the lemma. In a similar manner we also prove the second half. □

The constants α_1 and α_2 in Lemma 18.5 are used in Theorem 18.1 to provide an estimate of the constant M in the overshoot coefficient:

$$M(\lambda,D) = 72\alpha_1\alpha_2. \qquad (18.93)$$

18.6 Estimates for the Transformation Kernels

We need to provide estimates of the constants α_1 and α_2 in Lemma 18.5 since they provide an estimate of the overshoot coefficient in Theorem 18.1. The constants α_1 and α_2 should be expected to be finite since the γ-system and the δ-system are parabolic PDEs that generate analytic semigroups, whereas their respective initial conditions $\gamma(1,y) = k(1,y)$ and $\delta(1,y) = l(1,y)$ are C^∞ in y. However, we don't stop at this observation but actually compute bounds for α_1 and α_2 in this section.

To start with, some parts of α_1 and α_2 can actually be calculated analytically, as given by the next lemma.

Lemma 18.6.

$$\gamma_y(1,1) = \frac{\lambda^2}{8} - \frac{\lambda}{2}, \tag{18.94}$$

$$\delta_y(1,1) = -\frac{\lambda^2}{8} - \frac{\lambda}{2}. \tag{18.95}$$

Proof. By calculating

$$k_y(1,y) = -\lambda \frac{I_1\left(\sqrt{\lambda\,(1-y^2)}\right)}{\sqrt{\lambda\,(1-y^2)}} + \lambda^2 y^2 \frac{I_2\left(\sqrt{\lambda\,(1-y^2)}\right)}{\lambda\,(1-y^2)}, \tag{18.96}$$

$$l_y(1,y) = -\lambda \frac{J_1\left(\sqrt{\lambda\,(1-y^2)}\right)}{\sqrt{\lambda\,(1-y^2)}} - \lambda^2 y^2 \frac{J_2\left(\sqrt{\lambda\,(1-y^2)}\right)}{\lambda\,(1-y^2)} \tag{18.97}$$

and using the facts that

$$\lim_{\xi\to 0} \frac{I_n(\xi)}{\xi^n} = \frac{1}{2^n n!}, \tag{18.98}$$

$$\lim_{\xi\to 0} \frac{J_n(\xi)}{\xi^n} = \frac{1}{2^n n!} \tag{18.99}$$

for all $n \in \mathbb{N}$. □

For other parts of α_1 and α_2, a bound can easily be calculated, as given in the next lemma.

Lemma 18.7. *The following hold:*

$$\int_1^{1+D}\int_0^1 \gamma^2(x,y)dydx \le \frac{1}{2|\lambda| - \pi^2/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D} - 1\right)\int_0^1 k^2(1,y)dy, \tag{18.100}$$

$$\int_1^{1+D}\int_0^1 \gamma_x^2(x,y)dydx \le \frac{1}{|\lambda| - \pi^2/4}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D} - 1\right) \times\left(\int_0^1 k_{yy}^2(1,y)dy + |\lambda|\int_0^1 k^2(1,y)dy\right), \tag{18.101}$$

$$\int_1^{1+D}\int_0^1 \delta^2(x,y)dydx \le \frac{2}{\pi^2}\left(1 - \mathrm{e}^{D\pi^2/2}\right)\int_0^1 l^2(1,y)dy, \tag{18.102}$$

$$\int_1^{1+D}\int_0^1 \delta_x^2(x,y)dydx \le \frac{2}{\pi^2}\left(1 - \mathrm{e}^{D\pi^2/2}\right)\int_0^1 l_{yy}^2(1,y)dy, \tag{18.103}$$

where

$$k(1,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda\,(1-y^2)}\right)}{\sqrt{\lambda\,(1-y^2)}}, \tag{18.104}$$

$$k_{yy}(1,y) = 3\lambda^2 y \frac{I_2\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\lambda\left(1-y^2\right)} - \lambda^3 y^3 \frac{I_3\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\left(\sqrt{\lambda\left(1-y^2\right)}\right)^3}, \tag{18.105}$$

$$l(1,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\sqrt{\lambda\left(1-y^2\right)}}, \tag{18.106}$$

$$l_{yy}(1,y) = -3\lambda^2 y \frac{J_2\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\lambda\left(1-y^2\right)} - \lambda^3 y^3 \frac{J_3\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\left(\sqrt{\lambda\left(1-y^2\right)}\right)^3} \tag{18.107}$$

are continuous functions with

$$k(1,1) = -\lambda, \tag{18.108}$$

$$k_{yy}(1,1) = -\frac{\lambda^3}{48} + \frac{3\lambda^2}{8}, \tag{18.109}$$

$$l(1,1) = -\lambda, \tag{18.110}$$

$$l_{yy}(1,1) = -\frac{\lambda^3}{48} - \frac{3\lambda^2}{8}. \tag{18.111}$$

Proof. We prove the results only for $\gamma(x,y)$. The results for $\delta(x,y)$ are similar. We start from the fact that

$$\frac{d}{dx}\frac{1}{2}\|\gamma(x)\|^2 = -\|\gamma_y(x)\|^2 + \lambda\|\gamma(x)\|^2 \tag{18.112}$$

$$\leq \left(|\lambda| - \frac{\pi^2}{4}\right)\|\gamma(x)\|^2, \tag{18.113}$$

where we have used the Wirtinger inequality and the norm $\|\cdot\|$ is taken with respect to y. Then we get

$$\|\gamma(x)\|^2 \leq \mathrm{e}^{(2|\lambda|-\pi^2/2)(x-1)}\|\gamma(1)\|^2. \tag{18.114}$$

Since $\gamma(1,y) = k(1,y)$, we get

$$\|\gamma(x)\|^2 \leq \mathrm{e}^{(2|\lambda|-\pi^2/2)(x-1)}\int_0^1 k^2(1,y)dy. \tag{18.115}$$

Finally,

$$\int_1^{1+D}\int_0^1 \gamma^2(x,y)dydx = \int_1^{1+D}\|\gamma(x)\|^2 dx$$
$$\leq \frac{1}{2|\lambda| - \pi^2/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D} - 1\right)\int_0^1 k^2(1,y)dy. \tag{18.116}$$

This proves the first inequality in the lemma. To prove the second inequality, we use the fact that $\gamma_x = \gamma_{yy} + \lambda\gamma$. With the boundary conditions $\gamma(x,0) = \gamma(x,1) \equiv 0$, we get the system

$$\gamma_{yyx} = \gamma_{yyyy} + \lambda\gamma_{yy}\,, \tag{18.117}$$

$$\gamma_{yy}(x,0) = 0\,, \tag{18.118}$$

$$\gamma_{yy}(x,1) = 0\,. \tag{18.119}$$

Then, using a similar calculation as for obtaining the first inequality, we get

$$\|\gamma_{yy}(x)\|^2 \le \mathrm{e}^{(2|\lambda|-\pi^2/2)(x-1)} \int_0^1 k_{yy}^2(1,y)dy\,, \tag{18.120}$$

and, finally,

$$\int_1^{1+D}\int_0^1 \gamma_{yy}^2(x,y)dydx \le \frac{1}{2|\lambda|-\pi^2/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right)\int_0^1 k_{yy}^2(1,y)dy\,. \tag{18.121}$$

By combining the above results for γ_{yy} and γ, we get the second inequality in the lemma, for γ_x. The proof of the inequalities for δ mimic those for γ. □

Remark 18.1. Alternative bounds can be derived that do not involve L_2 bounds on $k_{yy}(1,y)$ and $l_{yy}(1,y)$ but only on $k_y(1,y)$ and $l_y(1,y)$. First, one would integrate (18.112) in x and obtain

$$\int_1^{1+D}\int_0^1 \gamma_y^2(x,y)dydx \le \lambda\int_1^{1+D}\int_0^1 \gamma^2(x,y)dydx + \frac{1}{2}\int_0^1 \gamma^2(1,y)dy\,. \tag{18.122}$$

Then one would consider the system

$$\gamma_{yx} = \gamma_{yyy} + \lambda\gamma_y\,, \tag{18.123}$$

$$\gamma_{yy}(x,0) = 0\,, \tag{18.124}$$

$$\gamma_{yy}(x,1) = 0 \tag{18.125}$$

and obtain

$$\frac{d}{dx}\frac{1}{2}\|\gamma_y(x)\|^2 = -\|\gamma_{yy}(x)\|^2 + \lambda\|\gamma_y(x)\|^2\,, \tag{18.126}$$

which, upon integration in x, yields

$$\int_1^{1+D}\int_0^1 \gamma_{yy}^2(x,y)dydx \le \lambda\int_1^{1+D}\int_0^1 \gamma_y^2(x,y)dydx + \frac{1}{2}\int_0^1 \gamma_y^2(1,y)dy\,. \tag{18.127}$$

Substituting (18.122) into (18.122), we get

$$\begin{aligned}\int_1^{1+D}\int_0^1 \gamma_{yy}^2(x,y)dydx &\le \lambda^2\int_1^{1+D}\int_0^1 \gamma^2(x,y)dydx\\ &\quad+\frac{\lambda}{2}\int_0^1 \gamma^2(1,y)dy+\frac{1}{2}\int_0^1 \gamma_y^2(1,y)dy\\ &\le\left[\frac{\lambda^2}{2|\lambda|-\pi^2/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right)+\frac{\lambda}{2}\right]\int_0^1 k^2(1,y)dy\\ &\quad+\frac{1}{2}\int_0^1 k_y^2(1,y)dy. \end{aligned}\tag{18.128}$$

Finally, using $\gamma_x=\gamma_{yy}+\lambda\gamma$, we get

$$\begin{aligned}&\int_1^{1+D}\int_0^1 \gamma_x^2(x,y)dydx\\ &\quad\le\left[\frac{2\lambda^2}{|\lambda|-\pi^4/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right)+\lambda\right]\int_0^1 k^2(1,y)dy+\int_0^1 k_y^2(1,y)dy. \end{aligned}\tag{18.129}$$

For δ_x, we get

$$\int_1^{1+D}\int_0^1 \delta_x^2(x,y)dydx\le\frac{1}{2}\int_0^1 l_y^2(1,y)dy.\tag{18.130}$$

Finally, we need to provide estimates for the norms $\int_1^{1+D}\gamma_y^2(x,1)dx$, $\int_1^{1+D}\gamma_{xy}^2(x,1)dx$, $\int_1^{1+D}\delta_y^2(x,1)dx$, and $\int_1^{1+D}\delta_{xy}^2(x,1)dx$. First, we do it for the latter two quantities, as they are easier to obtain.

Lemma 18.8. *The following are true:*

$$\int_1^{1+D}\delta_y^2(x,1)dx\le\int_0^1 l^2(1,y)dy+\frac{1}{2}\int_0^1 l_y^2(1,y)dy,\tag{18.131}$$

$$\int_1^{1+D}\delta_{yx}^2(x,1)dx\le\int_0^1 l_{yy}^2(1,y)dy+\frac{1}{2}\int_0^1 l_{yyy}^2(1,y)dy,\tag{18.132}$$

where

$$\begin{aligned}l_{yyy}(1,y)&=-3\lambda^2\frac{J_2\left(\sqrt{\lambda\,(1-y^2)}\right)}{\lambda\,(1-y^2)}-6\lambda^3y^2\frac{J_3\left(\sqrt{\lambda\,(1-y^2)}\right)}{\left(\lambda\,(1-y^2)\right)^{3/2}}\\ &\quad-\lambda^4y^4\frac{J_4\left(\sqrt{\lambda\,(1-y^2)}\right)}{\left(\lambda\,(1-y^2)\right)^2}\end{aligned}\tag{18.133}$$

is a continuous function with

$$l_{yy}(1,1) = -\frac{\lambda^4}{384} - \frac{\lambda^3}{8} - \frac{3\lambda^2}{8}. \tag{18.134}$$

Proof. We start with the PDEs $\delta_x = \delta_{yy}$ and $\delta_{xx} = \delta_{xyy}$ and multiply them, respectively, by $2y\delta_y(x,y)$ and $2y\delta_{xy}(x,y)$, obtaining

$$2y\delta_x(x,y)\delta_y(x,y) = 2y\delta_y(x,y)\delta_{yy}(x,y), \tag{18.135}$$

$$2y\delta_{xx}(x,y)\delta_{xy}(x,y) = 2y\delta_{xy}(x,y)\delta_{xyy}(x,y). \tag{18.136}$$

Integrating both sides in y and integrating by parts on the right side, we get

$$2\int_0^1 y\delta_x(x,y)\delta_y(x,y)dy = \delta_y^2(x,1) - \int_0^1 \delta_y^2(x,y)dy, \tag{18.137}$$

$$2\int_0^1 y\delta_{xx}(x,y)\delta_{xy}(x,y)dy = \delta_{xy}^2(x,1) - \int_0^1 \delta_{xy}^2(x,y)dy. \tag{18.138}$$

Applying Young's inequality, we obtain

$$\delta_y^2(x,1) \le 2\int_0^1 \delta_y^2(x,y)dy + \int_0^1 \delta_x^2(x,y)dy, \tag{18.139}$$

$$\delta_{xy}^2(x,1) \le 2\int_0^1 \delta_{xy}^2(x,y)dy + \int_0^1 \delta_{xyy}^2(x,y)dy. \tag{18.140}$$

The four quantities on the right-hand sides of the two inequalities are bounded by

$$\int_1^{1+D}\int_0^1 \delta_y^2(x,y)dydx \le \frac{1}{2}\int_0^1 l^2(1,y)dy, \tag{18.141}$$

$$\int_1^{1+D}\int_0^1 \delta_x^2(x,y)dydx \le \frac{1}{2}\int_0^1 l_y^2(1,y)dy, \tag{18.142}$$

$$\int_1^{1+D}\int_0^1 \delta_{xy}^2(x,y)dydx \le \frac{1}{2}\int_0^1 l_{yy}^2(1,y)dy, \tag{18.143}$$

$$\int_1^{1+D}\int_0^1 \delta_{xyy}^2(x,y)dydx \le \frac{1}{2}\int_0^1 l_{yyy}^2(1,y)dy. \tag{18.144}$$

We don't prove all of them, but only the last one. From the PDE $\delta_{yxx} = \delta_{yyyx}$ with boundary conditions $\delta_{yyx}(x,0) = \delta_{yyx}(x,1) \equiv 0$, we get

$$\frac{d}{dx}\frac{1}{2}\int_0^1 \delta_{yx}^2(x,y)dy = -\int_0^1 \delta_{yyx}^2(x,y)dy. \tag{18.145}$$

Integrating this equation in x, we get

$$\int_1^{1+D}\int_0^1 \delta_{xyy}^2(x,y)dydx \le \frac{1}{2}\int_0^1 \delta_{yyy}^2(1,y)dy, \tag{18.146}$$

yielding (18.144) with the initial condition $\delta_{yyy}(1,y) = l_{yyy}(1,y)$. Integrating the inequalities (18.139), (18.140) and substituting (18.141)–(18.144), we complete the proof of the lemma. □

Finally, we provide estimates for the norms $\int_1^{1+D} \gamma_y^2(x,1)dx$ and $\int_1^{1+D} \gamma_{xy}^2(x,1)dx$.

Lemma 18.9. *The following are true:*

$$\int_1^{1+D} \gamma_y^2(x,1)dx \leq \left((4\lambda^2+\lambda)\frac{\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1}{2|\lambda|-\pi^2/2} + \lambda + 1 \right) \int_0^1 k^2(1,y)dy + \int_0^1 k_y^2(1,y)dy, \tag{18.147}$$

$$\begin{aligned} \int_1^{1+D} \gamma_{yx}^2(x,1)dx \leq{}& 4\lambda^2(\lambda+1)\left[\frac{2\lambda}{|\lambda|-\pi^4/2}\left(\mathrm{e}^{(2|\lambda|-\pi^2/2)D}-1\right)+\frac{1}{2}\right]\int_0^1 k^2(1,y)dy \\ &+(2\lambda^2+1)\int_0^1 k_y^2(1,y)dy \\ &+2\left((\lambda+1)\int_0^1 k_{yy}^2(1,y)dy + \int_0^1 k_{yyy}^2(1,y)dy\right), \end{aligned} \tag{18.148}$$

where

$$\begin{aligned} k_{yyy}(1,y) ={}& 3\lambda^2 \frac{I_2\left(\sqrt{\lambda\,(1-y^2)}\right)}{\lambda\,(1-y^2)} - 6\lambda^3 y^2 \frac{I_3\left(\sqrt{\lambda\,(1-y^2)}\right)}{(\lambda\,(1-y^2))^{3/2}} \\ &-\lambda^4 y^4 \frac{I_4\left(\sqrt{\lambda\,(1-y^2)}\right)}{(\lambda\,(1-y^2))^2} \end{aligned} \tag{18.149}$$

is a continuous function with

$$k_{yy}(1,1) = \frac{\lambda^4}{384} - \frac{\lambda^3}{8} - \frac{3\lambda^2}{8}. \tag{18.150}$$

Proof. We start with the same steps as at the beginning of the proof of Lemma 18.8 and obtain

$$\begin{aligned} \gamma_y^2(x,1) ={}& \int_0^1 \gamma_y^2(x,y)dy + \lambda \int_0^1 \gamma^2(x,y)dy \\ &-\lambda\gamma^2(x,1) + 2\int_0^1 y\gamma_x(x,y)\gamma_y(x,y)dy, \end{aligned} \tag{18.151}$$

$$\begin{aligned} \gamma_{xy}^2(x,1) ={}& \int_0^1 \gamma_{xy}^2(x,y)dy + \lambda \int_0^1 \gamma_x^2(x,y)dy \\ &-\lambda\gamma_x^2(x,1) + 2\int_0^1 y\gamma_{xx}(x,y)\gamma_{xy}(x,y)dy. \end{aligned} \tag{18.152}$$

From (18.151), we obtain

$$\gamma_y^2(x,1) \le \lambda \int_0^1 \gamma^2(x,y)dy + 2\int_0^1 \gamma_y^2(x,y)dy + \int_0^1 \gamma_x^2(x,y)dy. \tag{18.153}$$

By integrating both sides in x and substituting a long sequence of various inequalities that we have derived so far, we obtain (18.147). From (18.152), we get

$$\gamma_{xy}^2(x,1) \le \lambda \int_0^1 \gamma_x^2(x,y)dy + 2\int_0^1 \gamma_{xy}^2(x,y)dy + \int_0^1 \gamma_{xx}^2(x,y)dy. \tag{18.154}$$

Of the three terms on the right, for the first one, $\int_0^1 \gamma_x^2(x,y)dy$, we have computed an integral bound in (18.129). For the second term, we have

$$\begin{aligned}\int_1^{1+D}\int_0^1 \gamma_{xy}^2(x,y)dydx &\le \lambda \int_1^{1+D}\int_0^1 \gamma_x^2(x,y)dydx + \frac{1}{2}\int_0^1 \gamma_x^2(1,y)dy \\ &\le \lambda \int_1^{1+D}\int_0^1 \gamma_x^2(x,y)dydx \\ &\quad + \lambda^2 \int_0^1 k^2(1,y)dy + \int_0^1 k_{yy}^2(1,y)dy. \end{aligned} \tag{18.155}$$

Again, $\int_0^1 \gamma_x^2(x,y)dy$ is bounded by (18.129). Finally, we estimate the integral in x of the term $\int_0^1 \gamma_{xx}^2(x,y)dy$. First, from the PDE $\gamma_{xxy} = \gamma_{xyyy} + \lambda xy$ with boundary conditions $\gamma_{xyy}(x,0) = \gamma_{xyy}(x,1) \equiv 0$, we get

$$\int_1^{1+D}\int_0^1 \gamma_{xyy}^2(x,y)dydx \le \lambda \int_1^{1+D}\int_0^1 \gamma_{xy}^2(x,y)dydx + \frac{1}{2}\int_0^1 \gamma_{xy}^2(1,y)dy. \tag{18.156}$$

Then, with a chain of inequalities, whose details we omit, we obtain

$$\begin{aligned}\int_1^{1+D}\int_0^1 \gamma_{xx}^2(x,y)dydx \le 4\lambda^2 &\int_1^{1+D}\int_0^1 \gamma_x^2(x,y)dydx \\ &+ 2\left(\lambda^3 \int_0^1 k^2(1,y)dy + \lambda^2 \int_0^1 k_y^2(1,y)dy \right. \\ &\left. + \lambda \int_0^1 k_{yy}^2(1,y)dy + \int_0^1 k_{yyy}^2(1,y)dy\right). \end{aligned} \tag{18.157}$$

Collecting the results from all of the above inequalities, we obtain

$$\begin{aligned}\int_1^{1+D} \gamma_{yx}^2(x,1)dx \le (4\lambda^2 + 3\lambda) &\int_1^{1+D}\int_0^1 \gamma_x^2(x,y)dydx \\ &+ 2\left((\lambda^3+\lambda^2) \int_0^1 k^2(1,y)dy + \lambda^2 \int_0^1 k_y^2(1,y)dy \right. \\ &\left. + (\lambda+1) \int_0^1 k_{yy}^2(1,y)dy + \int_0^1 k_{yyy}^2(1,y)dy\right). \end{aligned} \tag{18.158}$$

With a substitution of (18.129), we arrive at the result (18.148) of the lemma. □

With all the lemmas in this section, we prove Theorem 18.1 with explicit expressions for α_1 and α_2 in $M(\lambda, D) = 72\alpha_1\alpha_2$.

18.7 Explicit Solutions for the Control Gains

In this section we determine the explicit closed-loop solutions.

Lemma 18.10. *The solution of the equation for $\gamma(x,y)$ is given explicitly as*

$$\begin{aligned}\gamma(x,y) &= 2\sum_{n=1}^{\infty} \mathrm{e}^{\left(\lambda-\pi^2n^2\right)(x-1)} \sin(\pi n y)\int_0^1 \sin(\pi n\xi)k(1,\xi)d\xi \\ &= -2\sum_{n=1}^{\infty} \mathrm{e}^{\left(\lambda-\pi^2n^2\right)(x-1)} \sin(\pi n y)\int_0^1 \sin(\pi n\xi)\lambda\xi\frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}d\xi\end{aligned} \tag{18.159}$$

and yields the following expression for $\gamma_y(x,1)$:

$$\begin{aligned}\gamma_y(x,1) &= 2\sum_{n=1}^{\infty} \mathrm{e}^{\left(\lambda-\pi^2n^2\right)(x-1)} \pi n(-1)^n\int_0^1 \sin(\pi n\xi)k(1,\xi)d\xi \\ &= -2\sum_{n=1}^{\infty} \mathrm{e}^{\left(\lambda-\pi^2n^2\right)(x-1)} \pi n(-1)^n\int_0^1 \sin(\pi n\xi)\lambda\xi\frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}d\xi\,.\end{aligned} \tag{18.160}$$

Substituting the gain functions $\gamma(1+D,y)$ and $\gamma_y(1+D-\eta,1)$ into the feedback law

$$U(t) = \int_0^1 \gamma(D,y)u(y,t)dy - \int_{t-D}^{t} \gamma_y(t-\theta,1)U(\theta)d\theta\,, \tag{18.161}$$

we obtain

$$\begin{aligned}U(t) = 2\sum_{n=1}^{\infty}\int_0^1 &\sin(\pi n\xi)\lambda\xi\frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}d\xi \\ &\times\left(-\mathrm{e}^{\left(\lambda-\pi^2n^2\right)D}\int_0^1 \sin(\pi n y)u(y,t)dy\right. \\ &\left.+\pi n(-1)^n\int_{t-D}^{t}\mathrm{e}^{\left(\lambda-\pi^2n^2\right)(t-\theta)}U(\theta)d\theta\right)\,.\end{aligned} \tag{18.162}$$

18.8 Explicit Solutions of the Closed-Loop System

By explicitly determining the kernel functions $k(x,y)$, $\gamma(x,y)$, and $p(x)$, we have not only found the control law explicitly, but we have also found the transformation $(u,v) \mapsto (w,z)$ explicitly. Now we seek $l(x,y)$, $\delta(x,y)$, and $q(x)$ explicitly, so we can find the transformation $(w,z) \mapsto (u,v)$ explicitly.

Lemma 18.11. *The solution of the equation for $\delta(x,y)$ is given explicitly as*

$$\begin{aligned}\delta(x,y) &= 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 (x-1)} \sin(\pi n y) \int_0^1 \sin(\pi n \xi) l(1,\xi) d\xi \\ &= -2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 (x-1)} \sin(\pi n y) \int_0^1 \sin(\pi n \xi) \lambda \xi \frac{J_1\left(\sqrt{\lambda(1-\xi^2)}\right)}{\sqrt{\lambda(1-\xi^2)}} d\xi\end{aligned} \tag{18.163}$$

and yields the following expression for $\delta_y(x,1)$:

$$\begin{aligned}\delta_y(x,1) &= 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 (x-1)} \pi n (-1)^n \int_0^1 \sin(\pi n \xi) l(1,\xi) d\xi \\ &= -2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 (x-1)} \pi n (-1)^n \int_0^1 \sin(\pi n \xi) \lambda \xi \frac{J_1\left(\sqrt{\lambda(1-\xi^2)}\right)}{\sqrt{\lambda(1-\xi^2)}} d\xi\,.\end{aligned} \tag{18.164}$$

Lemma 18.12. *The explicit solutions of the system*

$$w_t(x,t) = w_{xx}(x,t)\,, \tag{18.165}$$

$$w(0,t) = 0\,, \tag{18.166}$$

$$w(1,t) = z(1,t)\,, \tag{18.167}$$

$$z_t(x,t) = z_x(x,t)\,, \tag{18.168}$$

$$z(1+D,t) = 0 \tag{18.169}$$

from the initial conditions (w_0, z_0) are given by

$$z(x,t) = \begin{cases} z_0(t+x)\,, & t\in[0,D]\,,\\ 0\,, & t>D\,,\end{cases} \tag{18.170}$$

and

$$\begin{aligned}w(x,t) = 2\sum_{n=1}^{\infty} &\mathrm{e}^{-\pi^2 n^2 t} \sin(\pi n x) \\ &\times \left(\int_0^1 \sin(\pi n y) w_0(y) dy\right. \\ &\left.+ \int_0^1 \sin(\pi n y) \pi^2 n^2 y dy \left(\int_0^t \mathrm{e}^{\pi^2 n^2 \tau} z_0(1+\tau) d\tau\right)\right)\end{aligned} \tag{18.171}$$

for $t \in [0,D]$ *and*

$$w(x,t) = 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n y) w(y,D)\,dy \tag{18.172}$$

for $t > D$.

Proof. First, we observe that

$$z_t(1,t) = \begin{cases} z_0'(1+t), & t \in [0,D], \\ 0, & t > D. \end{cases} \tag{18.173}$$

Then, from (18.56)–(18.58), we get

$$\begin{aligned} \omega(x,t) &= 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n y) \omega_0(y)\,dy \\ &\quad - 2\sum_{n=1}^{\infty} \int_0^t \mathrm{e}^{-\pi^2 n^2 (t-\tau)} \sin(\pi n x) \int_0^1 \sin(\pi n y) y\,dy\, z_t(1,\tau)\,d\tau \\ &= 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n y) \omega_0(y)\,dy \\ &\quad - 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n y) y\,dy \\ &\quad \times \left(\mathrm{e}^{\pi^2 n^2 t} z(1,t) - z(1,0) - \pi^2 n^2 \int_0^t \mathrm{e}^{\pi^2 n^2 \tau} z(1,\tau)\,d\tau \right) \\ &= 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n y) \omega_0(y)\,dy \\ &\quad + 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n y) y\,dy \\ &\quad \times \left(z(1,0) + \pi^2 n^2 \int_0^t \mathrm{e}^{\pi^2 n^2 \tau} z(1,\tau)\,d\tau \right) \\ &\quad - \left(2\sum_{n=1}^{\infty} \sin(\pi n x) \int_0^1 \sin(\pi n y) y\,dy \right) z(1,t). \end{aligned} \tag{18.174}$$

Using the Fourier series representation of x on $[0,1]$, we get

$$\begin{aligned} \omega(x,t) &= 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n y) \omega_0(y)\,dy \\ &\quad + 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n y) y\,dy \end{aligned}$$

$$\times\left(z(1,0)+\pi^2n^2\int_0^t e^{\pi^2n^2\tau}z(1,\tau)d\tau\right)$$
$$-xz(1,t)\,. \tag{18.175}$$

Using (18.55), i.e., the fact that $w(x,t)=\omega(x,t)+xz(1,t)$, we obtain

$$\begin{aligned}w(x,t)=2\sum_{n=1}^{\infty}e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1\sin(\pi ny)\omega_0(y)dy\\+2\sum_{n=1}^{\infty}e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1\sin(\pi ny)ydy\\\times\left(z(1,0)+\pi^2n^2\int_0^t e^{\pi^2n^2\tau}z(1,\tau)d\tau\right)\,.\end{aligned} \tag{18.176}$$

Further, with $\omega_0(y)+yz(1,0)=w_0(y)$, we get

$$\begin{aligned}w(x,t)=2\sum_{n=1}^{\infty}e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1\sin(\pi ny)w_0(y)dy\\+2\sum_{n=1}^{\infty}e^{-\pi^2n^2t}\sin(\pi nx)\int_0^1\sin(\pi ny)ydy\pi^2n^2\int_0^t e^{\pi^2n^2\tau}z(1,\tau)d\tau\,.\end{aligned} \tag{18.177}$$

Recalling that $z(1,t)=z_0(1+t)$ for $t\in[0,D]$, we complete the proof of (18.171). Finally, to obtain (18.172), we observe that for $t>D$, the w-system is just the heat equation with homogeneous boundary conditions, which completes the proof of the lemma. □

We have thus established the following.

Proposition 18.2. *The closed-loop system consisting of the plant*

$$u_t(x,t)=u_{xx}(x,t)+\lambda u(x,t)\,, \tag{18.178}$$
$$u(0,t)=0\,, \tag{18.179}$$
$$u(1,t)=v(1,t)\,, \tag{18.180}$$

$$v_t(x,t)=v_x(x,t)\,, \tag{18.181}$$
$$v(1+D,t)=U(t)\,, \tag{18.182}$$

and the control law

$$U(t)=2\sum_{n=1}^{\infty}\int_0^1\sin(\pi n\xi)\lambda\xi\frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}d\xi$$
$$\times\left(-e^{(\lambda-\pi^2n^2)D}\int_0^1\sin(\pi ny)u(y,t)dy\right.$$

$$+\pi n(-1)^n \int_1^{1+D} e^{\left(\lambda-\pi^2 n^2\right)(1+D-y)} v(y,t)dy\Bigg), \tag{18.183}$$

and starting from the initial condition $(u_0(x), v_0(x))$, *has solutions given by*

$$u(x,t) = w(x,t) - \int_0^x \lambda y \frac{J_1\left(\sqrt{\lambda\,(x^2-y^2)}\right)}{\sqrt{\lambda\,(x^2-y^2)}} w(y,t)dy, \tag{18.184}$$

$$\begin{aligned} v(x,t) = z(x,t) + 2 &\left(\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{J_1\left(\sqrt{\lambda\,(1-\xi^2)}\right)}{\sqrt{\lambda\,(1-\xi^2)}} d\xi\right) \\ &\times \sum_{n=1}^{\infty} \left(\pi n(-1)^n \int_1^x e^{-\pi^2 n^2 (x-y)} z(y,t)dy \right. \\ &\left. - e^{-\pi^2 n^2 (x-1)} \int_0^1 \sin(\pi n y) w(y,t) dy\right), \end{aligned} \tag{18.185}$$

where $(w(x,t), z(x,t))$ *are given by (18.170)–(18.172) with initial conditions*

$$w_0(x) = u_0(x) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda\,(x^2-y^2)}\right)}{\sqrt{\lambda\,(x^2-y^2)}} u_0(y)dy, \tag{18.186}$$

$$\begin{aligned} z_0(x) = v_0(x) - 2 &\left(\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{I_1\left(\sqrt{\lambda\,(1-\xi^2)}\right)}{\sqrt{\lambda\,(1-\xi^2)}} d\xi\right) \\ &\times \sum_{n=1}^{\infty} \left(\pi n(-1)^n \int_1^x e^{\left(\lambda-\pi^2 n^2\right)(x-y)} v_0(y)dy \right. \\ &\left. - e^{\left(\lambda-\pi^2 n^2\right)(x-1)} \int_0^1 \sin(\pi n y) u_0(y) dy\right). \end{aligned} \tag{18.187}$$

It can be noted that $v(x,t)$ can be written in a form that is even more direct, using the orthogonality of the basis functions $\sin(\pi n y)$. We get that for $t \in [0,D]$,

$$\begin{aligned} v(x,t) = z_0(x+t) - 2 &\left(\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{J_1\left(\sqrt{\lambda\,(1-\xi^2)}\right)}{\sqrt{\lambda\,(1-\xi^2)}} d\xi\right) \\ &\times \sum_{n=1}^{\infty} \left(\pi n(-1)^{n+1} \int_1^x e^{-\pi^2 n^2 (x-y)} z_0(y+t)dy \right. \\ &+ \int_0^1 \sin(\pi n y)\pi^2 n^2 y dy \int_0^t e^{-\pi^2 n^2 (t-\tau+x-1)} z_0(1+\tau) d\tau \\ &\left. + e^{-\pi^2 n^2 (t+x-1)} \int_0^1 \sin(\pi n y) w_0(y) dy\right), \end{aligned} \tag{18.188}$$

whereas for $t > D$,

$$v(x,t) = -2\left(\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{J_1\left(\sqrt{\lambda(1-\xi^2)}\right)}{\sqrt{\lambda(1-\xi^2)}} d\xi\right) \times \sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2 (t+x-1)} \int_0^1 \sin(\pi n y) w(y,D) dy. \tag{18.189}$$

18.9 Notes and References

After our lengthy stability analysis, it is fair to ask what the crucial difference is between the result for a delay-PDE cascade in this chapter and the general delay-ODE cascade result in Chapter 2. We recall that the stability result for delay-ODE systems of the form

$$\dot{X}(t) = AX(t) + BU(t-D) \tag{18.190}$$

is

$$|X(t)|^2 + \int_{t-D}^{t} U^2(\theta)d\theta \leq \frac{\phi_2\psi_2}{\phi_1\psi_1}\mathrm{e}^{-\mu t}\left(|X(0)|^2 + \int_{-D}^{0} U^2(\theta)d\theta\right), \tag{18.191}$$

where

$$\frac{\phi_2}{\phi_1} = \max\left\{3\left(1 + D\|m\|^2\right), 1 + 3\|KM\|^2\right\} \max\left\{3\left(1 + D\|n\|^2\right), 1 + 3\|KN\|^2\right\}, \tag{18.192}$$

$$\frac{\psi_2}{\psi_1} = \frac{\max\left\{\lambda_{\max}(P), a\right\}}{\min\left\{\lambda_{\min}(P), \frac{a(1+D)}{2}\right\}}, \tag{18.193}$$

$$m(x) = K\mathrm{e}^{Ax}B, \tag{18.194}$$

$$n(x) = K\mathrm{e}^{(A+BK)x}B, \tag{18.195}$$

$$M(x) = \mathrm{e}^{Ax}, \tag{18.196}$$

$$N(x) = \mathrm{e}^{(A+BK)x}. \tag{18.197}$$

The critical portion of these estimates is the dependence on $\|m\|$ and $\|n\|$. The functions $m(x)$ and $n(x)$ depend on the input vector B. In the boundary-controlled PDE problems, such as the one in this chapter, the input operator B is unbounded, so a much more delicate analysis, with involvement of the H_1 norm of the delay state in the system norm, namely $\int_{t-D}^{t} \dot{U}^2(\theta)d\theta$, was needed in this chapter.

To a reader who has navigated the details of Sections 18.4, 18.5, and 18.6, the analysis may seem a daunting maze of PDEs. To a reader who has successfully

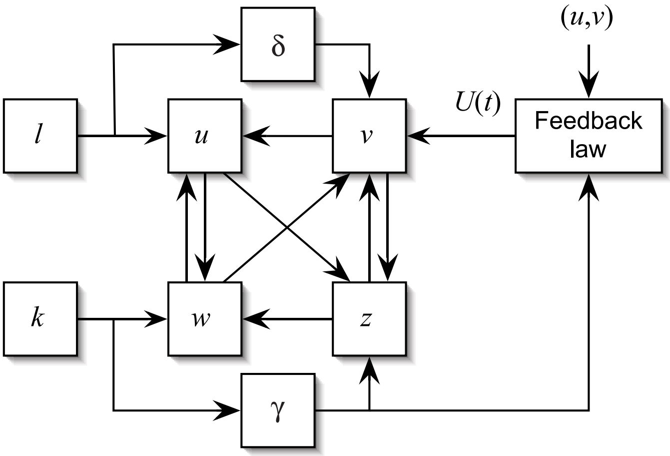

Fig. 18.2 Interconnection of various PDEs in the analysis of the feedback system for a reaction-diffusion PDE with input delay.

digested this analysis, this interconnection of various PDEs may seem fascinating. We show a diagram of the interconnections in Fig. 18.2. For example, the k-PDE is autonomous, is a second-order hyperbolic PDE in Goursat form, and has an explicit solution via Bessel functions. The solution to the k-PDE acts as an initial condition to the γ-PDE, which is of the parabolic type. A similar relation exists between the l-PDE and δ-PDE. The k-, l-, γ-, and δ-PDEs appear as kernels (i.e., multiplicatively) in the transformations between the (u,v) and (w,z) PDE systems.

It should be clear that we focused on the "simple" plant (18.4)–(18.8) only for notational simplicity. It is straightforward to extend the result of this chapter to the system

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) + b(x)u_x(x,t) + \lambda_1(x)u(x,t) + g_1(x)u(0,t) \\
&\quad + \int_0^x f_1(x,y)u(y,t)dy, \quad x \in (0,1), && (18.198) \\
u(0,t) &= 0, && (18.199) \\
u(1,t) &= v(1,t), && (18.200)
\end{aligned}
$$

$$
\begin{aligned}
v_t(x,t) &= v_x(x,t) + \lambda_2(x)v(x,t) + g_2(x)v(0,t) \\
&\quad + \int_0^x f_2(x,y)v(y,t)dy, \quad x \in [1,1+D), && (18.201) \\
v(1+D,t) &= U(t), && (18.202)
\end{aligned}
$$

where $b(x), \lambda_1(x), \lambda_2(x), g_1(x), g_2(x), f_1(x,y)$, and $f_2(x,y)$ are arbitrary continuous functions. The tools used in this extension are those in [202] and in Chapter 14.

Chapter 19
Antistable Wave PDE with Input Delay

In this chapter we continue with the designs for delay-PDE cascades as in Chapter 18. Here we deal with an antistable wave PDE, which has all of its infinitely many eigenvalues in the right half-plane (all located on a vertical line).

The wave PDE problem with input delay is much more complex than the delay-heat cascade in Chapter 18. The primary reason is the second-order-in-time character of the wave equation, though the "antistability" of the plant also creates a challenge.

Due to the extra complexity of the wave PDE, in this chapter we forego the derivation of explicit closed-loop solutions such as those that we derived in Section 18.8. However, we do derive the explicit expressions for the control gains and present a stability analysis.

We present the design for an antistable wave PDE with input delay in Section 19.1 and explain its origins in the baseline delay for an antistable wave PDE without delay in Section 19.2. The explicit solutions for the controller's gain kernels are derived in Section 19.3. The stability analysis is presented in two steps, first for the target system in Section 19.4, and then for the system in the original variables in Section 19.5.

19.1 Control Design for Antistable Wave PDE with Input Delay

Having presented the delay compensation design for the delay-heat cascade in Chapter 18, we now consider the delay-wave cascade system (see Fig. 19.1)

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad x \in (0,1), \tag{19.1}$$

$$u_x(0,t) = -q u_t(0,t), \tag{19.2}$$

$$u(1,t) = U(t-D), \tag{19.3}$$

where for $q > 0$ and $U(t) \equiv 0$, we have an "antistable" wave equation, which has all of its eigenvalues in the right half-plane. The eigenvalues are given by

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*,
Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_19,

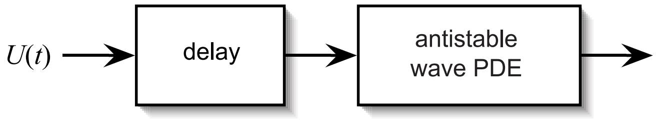

Fig. 19.1 Antistable wave PDE system with input delay.

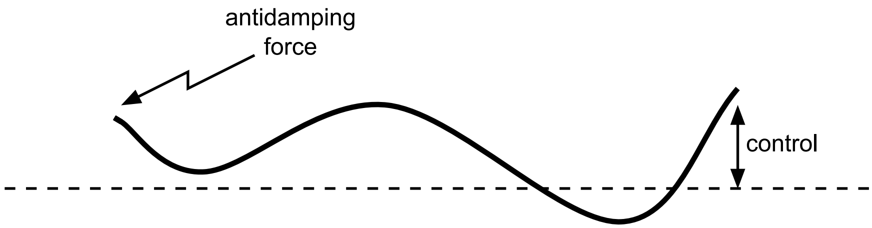

Fig. 19.2 A diagram of a string with control applied at boundary $x = 1$ and an antidamping force acting at the boundary $x = 0$.

$$\sigma_n = \frac{1}{2}\ln\left|\frac{1+q}{1-q}\right| + j\pi\begin{cases} n+\frac{1}{2}, & 0 \le q < 1, \\ n, & q > 1, \end{cases} \tag{19.4}$$

where $n \in \mathbb{N}$. Figures 19.3 and 19.4 show graphically the distribution of the eigenvalues and their dependence on q and n. As q grows from 0 to $+1$, the eigenvalues move rightward, all the way to $+\infty$. As q further grows from $+1$ to $+\infty$, the eigenvalues move leftward from $+\infty$ towards the imaginary axis, whereas their imaginary parts drop down by $\pi/2$. For a fixed q, the eigenvalues are always distributed on a vertical line. They depend linearly on n, namely, they are equidistant along the vertical line. The system is not well posed when $q = 1$. Hence, we assume that

$$q \in (0,1) \cup (1,\infty). \tag{19.5}$$

The structure of the system is such that the destabilizing force acts on the opposite boundary from the input, as represented in Fig. 19.2.

Remark 19.1. The unstable spectrum (19.4) of the antistable wave equation looks so extreme that one may wonder if the phenomenon of "antidamping" can occur in any physical system. One example that is reasonably related to the problem we are considering, though not exactly the same, is that of electrically amplified stringed instruments (for example, electric guitar). Such instruments employ an electromagnetic pickup (see Fig. 19.5), where the voltage at the terminals of the pickup is proportional, according to Faraday's law of induction, to the velocity of the string above it. The pickup's voltage is then amplified using an electric amplifier. The loudspeaker of a high-gain amplifier, when played at high volume, is capable of producing an acoustic excitation of such intensity that its force acts to mechanically excite the string. This is a positive-feedback loop, where the string velocity is

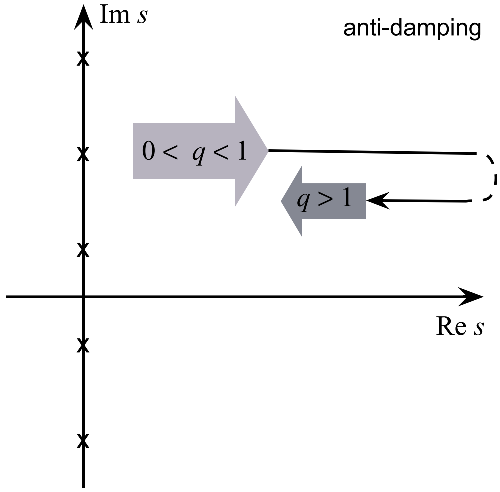

Fig. 19.3 Eigenvalues of the wave PDE with boundary anti-damping. As q grows from 0 to $+1$, the eigenvalues move rightward, all the way to $+\infty$. As q further grows from $+1$ to $+\infty$, the eigenvalues move leftward from $+\infty$ towards the imaginary axis, whereas their imaginary parts drop down by $\pi/2$.

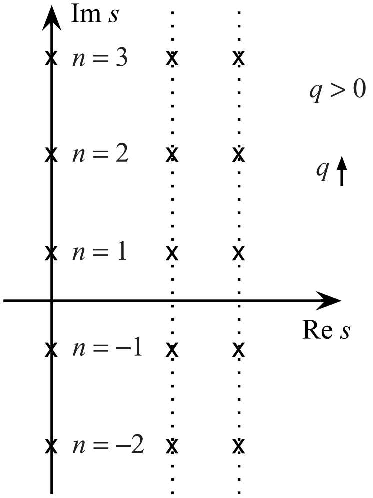

Fig. 19.4 Eigenvalues of the wave PDE with boundary anti-damping, continued from Figure 19.3. For a fixed q, the eigenvalues are always distributed on a vertical line. They depend linearly on n, namely, they are equidistant along the vertical line.

converted to voltage, multiplied by high gain, and then applied back as a force on the string. The phenomenon is not exactly the same as our antidamping at the boundary, but it appears in the form $u_{tt}(x,t) = u_{xx}(x,t) + gu_t(p,t)$, where g is the gain and $p \in (0,1)$ is the location of the pickup along the length of the string. The instability described here manifests itself as a loud, sustaining tone, even when the string is

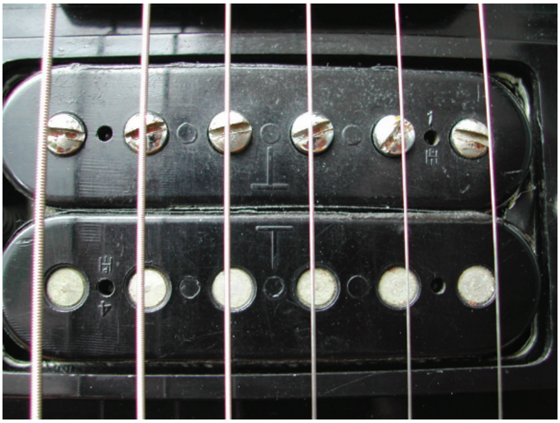

Fig. 19.5 A pickup on an electric guitar. Based on Faraday's law of induction, it converts the string velocity into voltage. Connected into a high-gain amplifier, this system results in (domain-wide) "antidamping," which manifests itself as a "swell" in volume, up to a saturation of the amplifier, which guitarists refer to simply as *feedback*.

not being played, though when used in a control manner, it can be employed musically (leading to "swells" of sound that the musician can induce on chosen notes). Among electric guitarists, this phenomenon is referred to simply as *feedback*. The antidamping is not located at the boundary in this system, nor is there an intent to attenuate the sound by boundary (or any other) control in this application.

Now we return to our control design. The delay-wave system is alternatively written as

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad x \in (0,1), \tag{19.6}$$

$$u_x(0,t) = -qu_t(0,t), \tag{19.7}$$

$$u(1,t) = v(1,t), \tag{19.8}$$

$$v_t(x,t) = v_x(x,t), \qquad x \in [1, 1+D), \tag{19.9}$$

$$v(1+D,t) = U(t), \tag{19.10}$$

where $U(t)$ is the overall system input and (u, u_t, v) is the state.

We consider the backstepping transformation

$$w(x,t) = u(x,t) - q\frac{q+c}{1+qc}u(0,t) + \frac{q+c}{1+qc}\int_0^x u_t(y,t)dy, \tag{19.11}$$

$$\begin{aligned} z(x,t) = v(x,t) - \int_1^x p(x-y)v(y,t)dy - \theta(x)u(0,t) \\ - \int_0^1 \gamma(x,y)u(y,t)dy - \int_0^1 \rho(x,y)u_t(y,t)dy, \end{aligned} \tag{19.12}$$

where the kernels k, θ, γ, and ρ need to be chosen to transform the cascade PDE system into the target system

$$w_{tt}(x,t) = w_{xx}(x,t), \tag{19.13}$$
$$w_x(0,t) = cw_t(0,t), \tag{19.14}$$
$$w(1,t) = z(1,t), \tag{19.15}$$

$$z_t(x,t) = z_x(x,t), \tag{19.16}$$
$$z(1+D,t) = 0, \tag{19.17}$$

with the control

$$U(t) = \int_1^{1+D} p(1+D-y)v(y,t)dy + \theta(1+D)u(0,t) + \int_0^1 \gamma(1+D,y)u(y,t)dy + \int_0^1 \rho(1+D,y)u_t(y,t)dy. \tag{19.18}$$

The target system (w,z), which is a transport-wave cascade interconnected through a boundary, is exponentially stable in an appropriate norm.

The kernel ρ is governed by the PDE

$$\rho_{xx}(x,y) = \rho_{yy}(x,y), \tag{19.19}$$
$$\rho_y(x,0) = -q\rho_x(x,0), \tag{19.20}$$
$$\rho(x,1) = 0, \tag{19.21}$$

where $x \in [1,1+D]$ should be viewed as the time variable and $y \in (0,1)$ as the space variable. The initial condition of this PDE is

$$\rho(1,y) = -\frac{q+c}{1+qc}, \tag{19.22}$$
$$\rho_x(1,y) = 0. \tag{19.23}$$

After solving for $\rho(x,y)$, the kernels p, θ, and γ are obtained as

$$p(s) = -\rho_y(1+s,1), \qquad s \in [0,D], \tag{19.24}$$
$$\theta(x) = -q\rho(x,0), \tag{19.25}$$
$$\gamma(x,y) = \rho_x(x,y). \tag{19.26}$$

We will present a detailed Lyapunov stability analysis in Sections 19.4 and 19.5; however, we first state a result on closed-loop eigenvalues [for the target system (w,z)].

Proposition 19.1. *The spectrum of the system (19.13)–(19.17) is given by*

$$\sigma_n = -\frac{1}{2}\ln\left|\frac{1+c}{1-c}\right| + j\pi \begin{cases} n+\frac{1}{2}, & 0 \le c < 1, \\ n, & c > 1, \end{cases} \tag{19.27}$$

where $n \in \mathbb{Z}$.

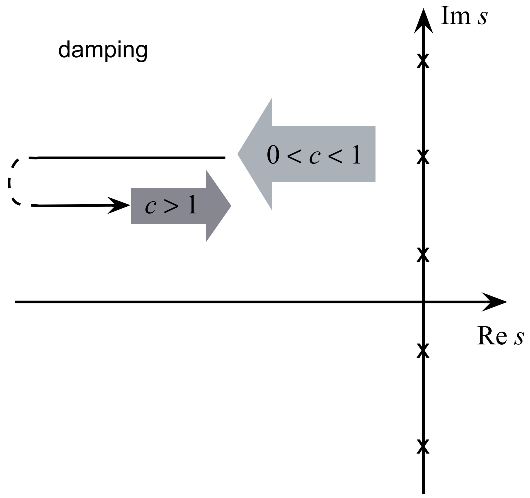

Fig. 19.6 Eigenvalues of the wave PDE with boundary anti-damping. As c grows from 0 to $+1$, the eigenvalues move leftward, all the way to $-\infty$. As c further grows from $+1$ to $+\infty$, the eigenvalues move rightward from $-\infty$ towards the imaginary axis, whereas their imaginary parts drop down by $\pi/2$.

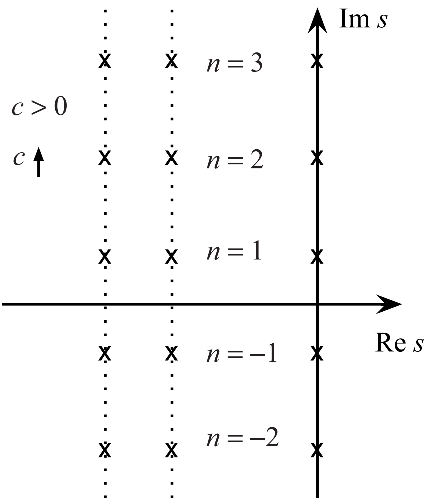

Fig. 19.7 Eigenvalues of the wave PDE with boundary anti-damping, continued from Figure 19.6. For a fixed c, the eigenvalues are always distributed on a vertical line. They depend linearly on n, namely, they are equidistant along the vertical line.

Figures 19.6 and 19.7 show graphically the distribution of the closed-loop eigenvalues and their dependence on c and n. As gain c grows from 0 to $+1$, the eigenvalues move leftward, all the way to $-\infty$. As c further grows from $+1$ to $+\infty$, the eigenvalues move rightward from $-\infty$ towards the imaginary axis, whereas their imaginary parts drop down by $\pi/2$. For a fixed c, the eigenvalues are always distributed on a vertical line. They depend linearly on n, namely, they are equidistant along the vertical line. The system is not well posed when $q = 1$. Hence, we take the gain c as

$$c \in (0,1) \cup (1,\infty). \tag{19.28}$$

The inverse backstepping transformation is given by

$$u(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)} \left(w(x,t) - c\frac{q+c}{1+qc} w(0,t) - \frac{q+c}{1+qc} \int_0^x w_t(y,t)dy \right), \tag{19.29}$$

$$v(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)} z(x,t) - \int_1^x \pi(x-y)z(y,t)dy - \eta(x)w(0,t) - \int_0^1 \delta(x,y)w(y,t)dy - \int_0^1 \mu(x,y)w_t(y,t)dy, \tag{19.30}$$

where the kernel μ is governed by the PDE

$$\mu_{xx}(x,y) = \mu_{yy}(x,y), \tag{19.31}$$

$$\mu_y(x,0) = c\mu_x(x,0), \tag{19.32}$$

$$\mu(x,1) = 0, \tag{19.33}$$

where $x \in [1, 1+D]$ should be viewed as the time variable and $y \in (0,1)$ as the space variable. The initial condition of this PDE is

$$\mu(1,y) = \frac{(q+c)(1+qc)}{(1-q^2)(1-c^2)}, \tag{19.34}$$

$$\mu_x(1,y) = 0. \tag{19.35}$$

After solving for $\mu(x,y)$, the kernels π, η, and δ are obtained as

$$\pi(s) = -\mu_y(1+s,1), \qquad s \in [0,D], \tag{19.36}$$

$$\eta(x) = c\mu(x,0), \tag{19.37}$$

$$\delta(x,y) = \mu_x(x,y). \tag{19.38}$$

19.2 The Baseline Design ($D = 0$) for the Antistable Wave PDE

Before we continue with the analysis of the delay-compensating design developed in Section 19.1, we present the feedback law that is obtained in the absence of delay. This special case is of interest in its own right since it achieves stabilization of the antistable wave equation.

Setting $D = 0$ in the control law (19.18), we obtain the feedback law

$$U(t) = \frac{q(q+c)}{1+qc}u(0,t) - \frac{q+c}{1+qc}\int_0^1 u_t(x,t)dx. \tag{19.39}$$

This feedback law was designed in [207]. The following stability result holds.

Theorem 19.1. *Consider the closed-loop system*

$$u_{tt}(x,t) = u_{xx}(x,t), \tag{19.40}$$
$$u_x(0,t) = -qu_t(0,t), \tag{19.41}$$
$$u(1,t) = \frac{q(q+c)}{1+qc}u(0,t) - \frac{q+c}{1+qc}\int_0^1 u_t(x,t)dx, \tag{19.42}$$

with $q \in (0,1)\cup(1,\infty)$ and with the control gain chosen such that

$$c \in (0,1)\cup(1,\infty). \tag{19.43}$$

For any initial condition such that $u(\cdot,0) \in H^1(0,1)$ and $u_t(\cdot,0) \in L^2(0,1)$, the closed-loop system has a unique solution

$$\big(u(\cdot,t),u_t(\cdot,t)\big) \in C([0,\infty),H^1(0,1)\times L^2(0,1)) \tag{19.44}$$

and is exponentially stable in the sense of the norm

$$\left(\int_0^1 u_x(x,t)^2dx + \int_0^1 u_t(x,t)^2dx\right)^{1/2}. \tag{19.45}$$

Moreover, if the initial condition $\big(u(\cdot,0),u_t(\cdot,0)\big)$ is compatible with the control law (19.42) and belongs to $H^2(0,1)\times H^1(0,1)$, then

$$\big(u(\cdot,t),u_t(\cdot,t)\big) \in C^1([0,\infty),H^1(0,1)\times L^2(0,1)) \tag{19.46}$$

is the classical solution of the closed-loop system.

The feedback law (19.39) is an extension of the feedback law (16.284) for the undamped wave equation from Section 16.7 to the antidamped wave equation considered in this chapter.

In establishing Theorem 19.1, we use the fact that

$$w_x(x,t) = u_x(x,t) + \frac{q+c}{1+qc}u_t(x,t), \tag{19.47}$$
$$w_t(x,t) = \frac{q+c}{1+qc}u_x(x,t) + u_t(x,t), \tag{19.48}$$

and

$$u_x(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)}\left(w_x(x,t) - \frac{q+c}{1+qc}w_t(x,t)\right), \tag{19.49}$$

$$u_t(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)} \left(-\frac{q+c}{1+qc} w_x(x,t) + w_t(x,t) \right), \tag{19.50}$$

along with

$$w(0,t) = \frac{1-q^2}{1+qc} u(0,t). \tag{19.51}$$

19.3 Explicit Gain Functions

Now we return to the design from Section 19.1. In the standard predictor feedback form, the controller (19.18) is written as

$$\begin{aligned} U(t) = &- q\rho(1+D,0)u(0,t) + \int_0^1 \rho_x(1+D,y)u(y,t)dy \\ &+ \int_0^1 \rho(1+D,y)u_t(y,t)dy - \int_{t-D}^t \rho_y(t-\theta,1)U(\theta)d\theta. \end{aligned} \tag{19.52}$$

So our task is to find the solution $\rho(x,y)$ and its first derivatives with respect to both x and y.

We seek the solution of (19.19)–(19.23) by seeking the solution of the PDE system

$$\varsigma_{tt}(x,t) = \varsigma_{xx}(x,t), \tag{19.53}$$

$$\varsigma(0,t) = 0, \tag{19.54}$$

$$\varsigma_x(1,t) = q\varsigma_t(1,t), \tag{19.55}$$

with initial conditions

$$\varsigma(x,0) = -\frac{q+c}{1+qc}, \tag{19.56}$$

$$\varsigma_t(x,0) = 0, \tag{19.57}$$

from which we shall obtain

$$\rho(x,y) = \varsigma(1-y, x-1), \quad y \in [0,1] \;\; x \geq 1. \tag{19.58}$$

We present the construction of $\varsigma(x,t)$ through a series of lemmas. The first lemma introduces a backstepping-style transformation that moves the antidamping effect from the boundary condition $\varsigma_x(1,t) = q\varsigma_t(1,t)$ into the domain, where it can be handled (for the purpose of solving the PDE) more easily.

Most lemmas that we state are reasonably straightforward to prove, or the proofs can be obtained by direct (albeit possibly lengthy) verification. Hence, we omit most proofs.

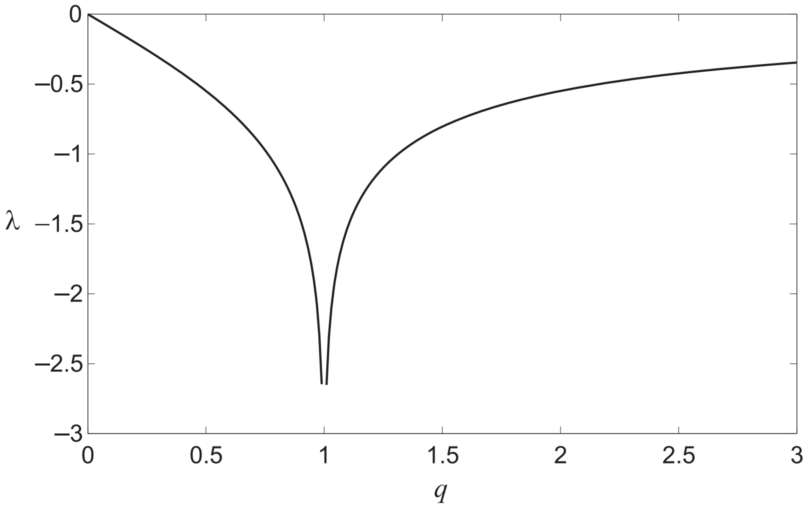

Fig. 19.8 The graph of the function $\lambda(q)$.

Lemma 19.1. *Consider the system (19.53)–(19.55) and the transformations*

$$\vartheta(x,t) = \cosh(\lambda x)\varsigma(x,t) + \int_0^x \sinh(\lambda y)\left(\varsigma_t(y,t) - \lambda\varsigma(y,t)\right)dy, \tag{19.59}$$

$$\varsigma(x,t) = \cosh(\lambda x)\vartheta(x,t) - \int_0^x \sinh(\lambda y)\left(\vartheta_t(y,t) + 2\lambda\vartheta(y,t)\right)dy, \tag{19.60}$$

where λ denotes (see Fig. 19.8)

$$\lambda(q) = \begin{cases} -\tanh^{-1}(q), & q \in [0,1), \\ -\coth^{-1}(q), & q > 1. \end{cases} \tag{19.61}$$

For $q \in [0,1)$, the function $\varsigma(x,t)$ satisfies (19.53)–(19.55) if and only if the function $\vartheta(x,t)$ satisfies

$$\vartheta_{tt}(x,t) + 2\lambda\vartheta_t(x,t) + \lambda^2\vartheta(x,t) = \vartheta_{xx}(x,t), \tag{19.62}$$

$$\vartheta(0,t) = 0, \tag{19.63}$$

$$\vartheta_x(1,t) = 0. \tag{19.64}$$

For $q > 1$, the function $\varsigma(x,t)$ satisfies (19.53)–(19.55) if and only if the function $\vartheta(x,t)$ satisfies

$$\vartheta_{tt}(x,t) + 2\lambda\vartheta_t(x,t) + \lambda^2\vartheta(x,t) = \vartheta_{xx}(x,t), \tag{19.65}$$

$$\vartheta(0,t) = 0, \tag{19.66}$$

$$\vartheta_t(1,t) + \lambda\vartheta(1,t) = 0. \tag{19.67}$$

Proof. By deriving and using the facts that the transformation $u \mapsto w$ yields

$$\vartheta_x(x,t) = \cosh(\lambda x)\varsigma_x(x,t) + \sinh(\lambda x)\varsigma_t(x,t), \tag{19.68}$$
$$\vartheta_t(x,t) + \lambda\vartheta(x,t) = \sinh(\lambda x)\varsigma_x(x,t) + \cosh(\lambda x)\varsigma_t(x,t) \tag{19.69}$$

and that the transformation $w \mapsto u$ yields

$$\varsigma_x(x,t) = \cosh(\lambda x)\vartheta_x(x,t) - \sinh(\lambda x)\left(\vartheta_t(x,t) + \lambda\vartheta(x,t)\right), \tag{19.70}$$
$$\varsigma_t(x,t) = -\sinh(\lambda x)\vartheta_x(x,t) + \cosh(\lambda x)\left(\vartheta_t(x,t) + \lambda\vartheta(x,t)\right). \tag{19.71}$$

□

The reader should note that the relationship between q and a, which is given by (19.61), yields

$$\sinh(\lambda x) = \frac{1}{2}\left(\left|\frac{1-q}{1+q}\right|^{x/2} - \left|\frac{1+q}{1-q}\right|^{x/2}\right), \tag{19.72}$$
$$\cosh(\lambda x) = \frac{1}{2}\left(\left|\frac{1-q}{1+q}\right|^{x/2} + \left|\frac{1+q}{1-q}\right|^{x/2}\right). \tag{19.73}$$

To make the ς-system easily solvable, we introduce another transformation, given in the next lemma.

Lemma 19.2. *Let* $\varpi(x,t) = \mathrm{e}^{\lambda t}\vartheta(x,t)$, *i.e.,*

$$\vartheta(x,t) = \left|\frac{1+q}{1-q}\right|^{t/2}\varpi(x,t). \tag{19.74}$$

For $q \in [0,1)$, *the function* $\vartheta(x,t)$ *satisfies (19.62)–(19.64) if and only if the function* $\varpi(x,t)$ *satisfies*

$$\varpi_{tt}(x,t) = \varpi_{xx}(x,t), \tag{19.75}$$
$$\varpi(0,t) = 0, \tag{19.76}$$
$$\varpi_x(1,t) = 0. \tag{19.77}$$

For $q > 1$, *the function* $\vartheta(x,t)$ *satisfies (19.65)–(19.67) if and only if the function* $\varpi(x,t)$ *satisfies*

$$\varpi_{tt}(x,t) = \varpi_{xx}(x,t), \tag{19.78}$$
$$\varpi(0,t) = 0, \tag{19.79}$$
$$\varpi_t(1,t) = 0. \tag{19.80}$$

The ϖ-systems are readily solvable in explicit form. Their solutions, for arbitrary initial conditions, are stated in the next two lemmas.

Lemma 19.3. *The solution of the system (19.75)–(19.77) with arbitrary initial conditions*

$$\varpi(x,0) = \varpi_0(x), \tag{19.81}$$

$$\varpi_t(x,0) = \varpi_1(x) \tag{19.82}$$

is

$$\begin{aligned}\varpi(x,t) = 2\sum_{n=0}^{\infty} \sin\left((2n+1)\frac{\pi}{2}x\right) \\ \times\left[\int_0^1 \sin\left((2n+1)\frac{\pi}{2}y\right)\pi_0(y)dy\cos\left((2n+1)\frac{\pi}{2}t\right)\right. \\ \left.+\frac{2}{(2n+1)\pi}\int_0^1 \sin\left((2n+1)\frac{\pi}{2}y\right)\pi_1(y)dy\sin\left((2n+1)\frac{\pi}{2}t\right)\right].\end{aligned} \tag{19.83}$$

Lemma 19.4. *The solution of the system (19.78)–(19.80) with arbitrary initial conditions*

$$\varpi(x,0) = \varpi_0(x), \tag{19.84}$$

$$\varpi_t(x,0) = \varpi_1(x) \tag{19.85}$$

is

$$\begin{aligned}\varpi(x,t) = 2\sum_{n=1}^{\infty} \sin(n\pi x)\left[\int_0^1 \sin(n\pi y)\pi_0(y)dy\cos(n\pi t)\right. \\ \left.+\frac{1}{n\pi}\int_0^1 \sin(n\pi y)\pi_1(y)dy\sin(n\pi t)\right].\end{aligned} \tag{19.86}$$

While the last two lemmas characterize the solutions of the ϖ-system with a general initial condition, our interest is in finding the control gain functions that are obtained for the initial conditions of the ς-system that are given by (19.56) and (19.57). We first need the following easily verifiable result that gives the initial conditions for the ϖ-system.

Lemma 19.5. *If the initial conditions of the system (19.53)–(19.55) are given by (19.56) and (19.57), then the initial conditions for the ϖ-system are given by*

$$\varpi_0(x) = -\frac{q+c}{1+qc}, \tag{19.87}$$

$$\varpi_1(x) = 0. \tag{19.88}$$

For these initial conditions, we obtain the explicit solutions $\varpi(x,t)$ as follows.

Lemma 19.6. *The solution of the system (19.75)–(19.77) with initial conditions given by (19.87) and (19.88) is*

$$\begin{aligned}\varpi(x,t) &= -\frac{q+c}{1+qc}2\sum_{n=0}^{\infty}\frac{\sin\left((2n+1)\frac{\pi}{2}x\right)\cos\left((2n+1)\frac{\pi}{2}t\right)}{(2n+1)\frac{\pi}{2}}\\ &= -\frac{q+c}{1+qc}\sum_{n=0}^{\infty}\frac{\sin\left((2n+1)\frac{\pi}{2}(x+t)\right)-\sin\left((2n+1)\frac{\pi}{2}(t-x)\right)}{(2n+1)\frac{\pi}{2}}.\end{aligned} \tag{19.89}$$

Lemma 19.7. *The solution of the system (19.78)–(19.80) with initial conditions given by (19.87) and (19.88) is*

$$\begin{aligned}\varpi(x,t) &= -\frac{q+c}{1+qc}2\sum_{m=1}^{\infty}\frac{\sin(2m\pi x)\cos(2m\pi t)}{m\pi}\\ &= -\frac{q+c}{1+qc}\sum_{m=1}^{\infty}\frac{\sin(2m\pi(x+t))-\sin(2m\pi(t-x))}{m\pi}.\end{aligned} \tag{19.90}$$

Now we return to the gain formula (19.58) and, using all of the above lemmas, obtain the following explicit expression for the gain kernel.

Proposition 19.2. *The solution of the system (19.19)–(19.23) is given by*

$$\begin{aligned}\rho(x,y) = \mathrm{e}^{-\lambda(x-1)}\Bigg[&\cosh(\lambda(1-y))\varpi(1-y,x-1)\\ &-\int_0^{1-y}\sinh(\lambda s)\left(\varpi_t(s,x-1)+a\varpi(s,x-1)\right)ds\Bigg],\end{aligned} \tag{19.91}$$

where, for $q\in[0,1)$, the function $\varpi(\cdot,\cdot)$ is given by (19.83) and for $q>1$, the function $\varpi(\cdot,\cdot)$ is given by (19.86).

Finally, we note that the control law (19.52) also requires the functions $\rho_x(1+D,y)$ and $\rho_y(t-\theta,1)$. They are given, for completeness, by the next proposition.

Proposition 19.3. *The partial derivatives of (19.91) are*

$$\begin{aligned}\rho_x(x,y) = -\lambda\rho(x,y)+\mathrm{e}^{-\lambda(x-1)}\Bigg[&\cosh(\lambda(1-y))\varpi_t(1-y,x-1)\\ &-\int_0^{1-y}\sinh(\lambda s)\left(\varpi_{tt}(s,x-1)+a\varpi_t(s,x-1)\right)ds\Bigg],\end{aligned} \tag{19.92}$$

$$\begin{aligned}\rho_y(x,y) = \mathrm{e}^{-\lambda(x-1)}[&-\lambda\sinh(\lambda(1-y))\varpi(1-y,x-1)\\ &-cosh(\lambda(1-y))\varpi_x(1-y,x-1)\\ &-\sinh(\lambda(1-y))\left(\varpi_t(1-y,x-1)+a\varpi(1-y,x-1)\right)],\end{aligned} \tag{19.93}$$

where, for $q \in [0,1)$, the functions $\varpi_x(x,t), \varpi_t(x,t)$, and $\varpi_{tt}(x,t)$ are given by

$$\varpi_x(x,t) = -\frac{q+c}{1+qc} 2 \sum_{n=0}^{\infty} \cos\left((2n+1)\frac{\pi}{2}x\right) \cos\left((2n+1)\frac{\pi}{2}t\right), \tag{19.94}$$

$$\varpi_t(x,t) = \frac{q+c}{1+qc} 2 \sum_{n=0}^{\infty} \sin\left((2n+1)\frac{\pi}{2}x\right) \sin\left((2n+1)\frac{\pi}{2}t\right), \tag{19.95}$$

$$\varpi_{tt}(x,t) = \frac{q+c}{1+qc} 2 \sum_{n=0}^{\infty} (2n+1)\frac{\pi}{2} \sin\left((2n+1)\frac{\pi}{2}x\right) \cos\left((2n+1)\frac{\pi}{2}t\right), \tag{19.96}$$

and for $q > 1$, the functions $\varpi_x(x,t), \varpi_t(x,t)$, and $\varpi_{tt}(x,t)$ are given by

$$\varpi_x(x,t) = -\frac{q+c}{1+qc} 2 \sum_{m=1}^{\infty} \cos(2m\pi x), \cos(2m\pi t), \tag{19.97}$$

$$\varpi_t(x,t) = \frac{q+c}{1+qc} 2 \sum_{m=1}^{\infty} \sin(2m\pi x) \sin(2m\pi t), \tag{19.98}$$

$$\varpi_{tt}(x,t) = \frac{q+c}{1+qc} 2 \sum_{m=1}^{\infty} m\pi \sin(2m\pi x) \cos(2m\pi t). \tag{19.99}$$

Having completed the derivation of explicit expressions for the control gains, we turn our attention in the next section to the stability analysis for the closed-loop system.

19.4 Stability of the Target System (w,z)

We now return to the target system

$$w_{tt}(x,t) = w_{xx}(x,t), \quad x \in (0,1), \tag{19.100}$$
$$w_x(0,t) = cw_t(0,t), \tag{19.101}$$
$$w(1,t) = z(1,t), \tag{19.102}$$

$$z_t(x,t) = z_x(x,t), \quad x \in [1, 1+D), \tag{19.103}$$
$$z(1+D,t) = 0, \tag{19.104}$$

and study its exponential stability.

We first denote

$$a(c) = \begin{cases} \tanh^{-1}(c), & c \in [0,1), \\ \coth^{-1}(c), & c > 1, \end{cases} \tag{19.105}$$

and then introduce the transformations

$$\varphi_x(x,t) = \cosh(a(1-x))w_x(x,t) - \sinh(a(1-x))w_t(x,t), \tag{19.106}$$

$$\varphi_t(x,t)+a\varphi(x,t) = -\sinh(a(1-x))w_x(x,t)+\cosh(a(1-x))w_t(x,t), \quad (19.107)$$
$$\varphi(1,t) = z(1,t), \quad (19.108)$$

and

$$w_x(x,t) = \cosh(a(1-x))\varphi_x(x,t)+\sinh(a(1-x))\,(\varphi_t(x,t)+a\varphi(x,t)), \quad (19.109)$$
$$w_t(x,t) = \sinh(a(1-x))\varphi_x(x,t)+\cosh(a(1-x))\,(\varphi_t(x,t)+a\varphi(x,t)), \quad (19.110)$$
$$w(1,t) = z(1,t). \quad (19.111)$$

By integrating in x, these transformations are also written as

$$\varphi(x,t) = \cosh(a(1-x))w(x,t)+\int_x^1 \sinh(a(1-y))\,(w_t(y,t)-aw(y,t))\,dy, \quad (19.112)$$

$$w(x,t) = \cosh(a(1-x))\varphi(x,t)-\int_x^1 \sinh(a(1-y))\,(\varphi_t(y,t)+2a\varphi(y,t))\,dy. \quad (19.113)$$

For $c \in [0,1)$, the transformation converts the w-system into

$$\varphi_{tt}(x,t)+2a\varphi_t(x,t)+a^2\varphi(x,t) = \varphi_{xx}(x,t), \quad (19.114)$$
$$\varphi_x(0,t) = 0, \quad (19.115)$$
$$\varphi(1,t) = z(1,t). \quad (19.116)$$

For $c>1$, the transformation converts the w-system into

$$\varphi_{tt}(x,t)+2a\varphi_t(x,t)+a^2\varphi(x,t) = \varphi_{xx}(x,t), \quad (19.117)$$
$$\varphi_t(0,t)+a\varphi(0,t) = 0, \quad (19.118)$$
$$\varphi(1,t) = z(1,t). \quad (19.119)$$

Even though the z-system is the exponentially stable transport equation, $z_t(x,t) = z_x(x,t)$, $z(1,t)=0$, the stability analysis for the φ-system cannot proceed in this form because of $z(1,t)$ entering the φ-system through a boundary condition, which makes the resulting input operator unbounded and the resulting gain from $z(1,t)$ to $\varphi(x,t)$ (in any suitable norm) unbounded.

So we first perform a transformation that shifts $z(1,t)$ into the interior of the domain $(0,1)$. The next lemma introduces this transformation and presents a Lyapunov function for the resulting system.

Lemma 19.8. *Consider the change of variable*

$$\psi(x,t) = \varphi(x,t)-x^2z(1,t) \quad (19.120)$$

and the resulting system, which for $c \in [0,1)$ *is*

$$\psi_{tt}(x,t)+2a\psi_t(x,t)+a^2\psi(x,t) = \psi_{xx}(x,t)+g(x,t), \quad (19.121)$$

$$\psi_x(0,t) = 0, \tag{19.122}$$
$$\psi(1,t) = 0, \tag{19.123}$$

and for $c > 1$ *is*

$$\psi_{tt}(x,t) + 2a\psi_t(x,t) + a^2\psi(x,t) = \psi_{xx}(x,t) + g(x,t), \tag{19.124}$$
$$\psi_t(0,t) + a\psi(0,t) = 0, \tag{19.125}$$
$$\psi(1,t) = 0\,, \tag{19.126}$$

and where

$$g(x,t) = 2z(1,t) - x^2\left(z_{xx}(1,t) + 2z_x(1,t) + a^2 z(1,t)\right)\,. \tag{19.127}$$

Then the following is true:

$$\dot{V}(t) = -aV(t) + \int_0^1 (\psi_t(x,t) + a\psi(x,t))\, g(x,t)dx\,, \tag{19.128}$$

where

$$V(t) = \frac{1}{2}\int_0^1 \left((\psi_t(x,t) + a\psi(x,t))^2 + \psi_x^2(x,t)\right) dx\,. \tag{19.129}$$

Proof. Most of this lemma is obtained by direct verification. The expression for $g(x,t)$ involves $z_x(1,t)$ and $z_{xx}(1,t)$, which are obtained, respectively, from the equations $z_t = z_x$ and

$$z_{tt} = z_{xt} = z_{xx}\,, \tag{19.130}$$

as $z_t(1,t) = z_x(1,t)$ and $z_{tt}(1,t) = z_{xx}(1,t)$. The derivative of the Lyapunov function is obtained using integration by parts as

$$\begin{aligned}\dot{V}(t) = {} & -aV(t) + (\psi_t(x,t) + a\psi(x,t))\,\psi_x(x,t)\big|_0^1 \\ & + \int_0^1 (\psi_t(x,t) + a\psi(x,t))\, g(x,t)dx\end{aligned} \tag{19.131}$$

and by substituting either of the boundary conditions (19.122) or (19.125). □

The following lemma is readily verifiable.

Lemma 19.9.

$$\dot{V}(t) \leq -\frac{a}{2}V(t) + \frac{1}{2a}\|g(t)\|^2, \tag{19.132}$$
$$\|g(t)\|^2 \leq 3\left((5 + a^2)z^2(1,t) + a^2 z_x^2(1,t) + \frac{1}{5}z_{xx}^2(1,t)\right)\,. \tag{19.133}$$

Next, we turn our attention to the z-system (19.103), (19.104).

Lemma 19.10. *The following are true:*

$$\frac{d}{dt}\int_1^{1+D} \mathrm{e}^{b(x-1)} z^2(x,t)dx = -z(1,t)^2 - b\int_1^{1+D} \mathrm{e}^{b(x-1)} z^2(x,t)dx, \tag{19.134}$$

$$\frac{d}{dt}\int_1^{1+D} \mathrm{e}^{b(x-1)} z_x^2(x,t)dx = -z_x(1,t)^2 - b\int_1^{1+D} \mathrm{e}^{b(x-1)} z_x^2(x,t)dx, \tag{19.135}$$

$$\frac{d}{dt}\int_1^{1+D} \mathrm{e}^{b(x-1)} z_{xx}^2(x,t)dx = -z_{xx}(1,t)^2 - b\int_1^{1+D} \mathrm{e}^{b(x-1)} z_{xx}^2(x,t)dx \tag{19.136}$$

for any $b > 0$.

With Lemmas 19.9 and 19.10, we obtain the following result.

Lemma 19.11. *The following holds:*

$$\Omega(t) \leq \Omega_0 \mathrm{e}^{-\min\left\{\frac{a}{2},b\right\}t}, \quad \forall t \geq 0, \tag{19.137}$$

for all $b > 0$*, where*

$$\Omega(t) = V(t) + \frac{3(5+a^2)}{2a}\int_1^{1+D} \mathrm{e}^{b(x-1)} \left(z^2(x,t) + z_x^2(x,t) + z_{xx}^2(x,t)\right) dx\,. \tag{19.138}$$

Proof. With Lemmas 19.9 and 19.10, we obtain

$$\begin{aligned}
\dot{\Omega}(t) \leq {} & -\frac{a}{2}V(t) + \frac{3}{2a}\left((5+a^2)z^2(1,t) + a^2 z_x^2(1,t) + \frac{1}{5}z_{xx}^2(1,t)\right) \\
& - \frac{3(5+a^2)}{2a}\left(z^2(1,t) + z_x^2(1,t) + z_{xx}^2(1,t)\right) \\
& - \frac{3(5+a^2)}{2a}b\int_1^{1+D} \mathrm{e}^{b(x-1)} \left(z^2(x,t) + z_x^2(x,t) + z_{xx}^2(x,t)\right) dx \\
\leq {} & -\frac{a}{2}V(t) - \frac{3(5+a^2)}{2a}b\int_1^{1+D} \mathrm{e}^{b(x-1)} \left(z^2(x,t) + z_x^2(x,t) + z_{xx}^2(x,t)\right) dx\,,
\end{aligned} \tag{19.139}$$

which yields

$$\dot{\Omega}(t) \leq -\min\left\{\frac{a}{2}, b\right\}\Omega(t) \tag{19.140}$$

and leads to the result of the lemma. □

Even though we have obtained exponential stability in the (ψ,z) variables, we have yet to establish exponential stability in the (w,z) variables. We have to consider the chain of transformations

$$(w,z) \mapsto (\varphi,z) \mapsto (\psi,z)\,, \tag{19.141}$$

as well as their inverses, to establish stability of the (w,z)-system.

The following lemma, which establishes the equivalence between the Lyapunov function $\Omega(t)$ and the appropriate norm of the (w,z)-system, is the key to establishing exponential stability in the (w,z) variables.

Lemma 19.12. *The following holds:*

$$\alpha_1 \Xi(t) \le \Omega(t) \le \alpha_2 \Xi(t)\,, \tag{19.142}$$

where

$$\begin{aligned}\Xi(t) = \quad & \int_0^1 \left(w_x^2(x,t) + w_t^2(x,t)\right) dx \\ & + \int_1^{1+D} \left(z^2(x,t) + z_x^2(x,t) + z_{xx}^2(x,t)\right) dx \end{aligned} \tag{19.143}$$

and

$$\alpha_1 = \min\left\{ \frac{3(5+a^2)}{2a}, \frac{1}{8\cosh(2a)\max\{1,a\}} \right\}, \tag{19.144}$$

$$\alpha_2 = \max\left\{ 2\cosh(2a), \frac{8}{5}(1+a^2) + \frac{3(5+a^2)}{2a} \mathrm{e}^{bD} \right\}. \tag{19.145}$$

Proof. To save on notation, in this proof we use the symbol $\|\cdot\|$ to mean both $\|\cdot\|_{L_2[0,1]}$ and $\|\cdot\|_{L_2[1,1+D]}$. We first consider the transformations $w \mapsto \varphi$ and $\varphi \mapsto w$. Squaring up (19.106), we get

$$\begin{aligned} w_x^2(x,t) \le{} & 2\cosh^2(a(1-x))\varphi_x^2(x,t) \\ & + 2\sinh^2(a(1-x))\left(\varphi_t(x,t) + a\varphi(x,t)\right)^2 . \end{aligned} \tag{19.146}$$

Doing the same with (19.107), (19.109), and (19.110), integrating from 0 to 1, and majorizing $\cosh^2(a(1-x))$ and $\sinh^2(a(1-x))$ over $[0,1]$ under the integrals as

$$\cosh^2(a(1-x)) \le \cosh^2(a), \tag{19.147}$$

$$\sinh^2(a(1-x)) \le \sinh^2(a)\,, \tag{19.148}$$

we get

$$\|w_x(t)\|^2 + \|w_t(t)\|^2 \le 2\left(\cosh^2(a) + \sinh^2(a)\right)\left(\|\varphi_x(t)\|^2 + \|\varphi_t(t) + a\varphi(t)\|^2\right) \tag{19.149}$$

and

$$\|\varphi_x(t)\|^2 + \|\varphi_t(t) + a\varphi(t)\|^2 \le 2\left(\cosh^2(a) + \sinh^2(a)\right)\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right). \tag{19.150}$$

From (19.120), we get

$$\varphi_t(x,t) + a\varphi(x,t) = \psi_t(x,t) + a\psi(x,t) + x^2\left(z_x(1,t) + az(1,t)\right), \tag{19.151}$$
$$\varphi_x(x,t) = \psi_x(x,t) + 2xz(1,t), \tag{19.152}$$

where we have used the fact that $z_t(1,t) = z_x(1,t)$. Taking the L_2 norm of both sides of both equations, we obtain

$$\|\varphi_t(t) + a\varphi(t)\|^2 \le 2\|\psi_t(t) + a\psi(t)\|^2 + \frac{2}{5}\left(z_x(1,t) + az(1,t)\right)^2, \tag{19.153}$$
$$\|\varphi_x(t)\|^2 \le 2\|\psi_x(t)\|^2 + \frac{2}{3}z^2(1,t), \tag{19.154}$$

as well as

$$\|\psi_t(t) + a\psi(t)\|^2 \le 2\|\varphi_t(t) + a\varphi(t)\|^2 + \frac{2}{5}\left(z_x(1,t) + az(1,t)\right)^2, \tag{19.155}$$
$$\|\psi_x(t)\|^2 \le 2\|\varphi_x(t)\|^2 + \frac{2}{3}z^2(1,t). \tag{19.156}$$

Using the facts that $z(1,t) \equiv 0$ and $z_x(1,t) \equiv 0$, where the latter follows from the fact that $z_t(1,t) \equiv 0$, with Agmon's inequality, we get

$$\|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2 \le 2\left(\|\psi_t(t) + a\psi(t)\|^2 + \|\psi_x(t)\|^2\right) + \frac{16}{5}\|z_{xx}(t)\|^2 + 4\left(\frac{4}{5}a^2 + \frac{2}{3}\right)\|z_x(t)\|^2, \tag{19.157}$$
$$\|\psi_t(t) + a\psi(t)\|^2 + \|\psi_x(t)\|^2 \le 2\left(\|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2\right) + \frac{16}{5}\|z_{xx}(t)\|^2 + 4\left(\frac{4}{5}a^2 + \frac{2}{3}\right)\|z_x(t)\|^2. \tag{19.158}$$

With further majorizations, we achieve simplifications of expressions:

$$\|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2 \le 4V(t) + \frac{16}{5}(1+a^2)\left(\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2\right) \tag{19.159}$$

and

$$V(t) \le \|\varphi_t(t) + a\varphi(t)\|^2 + \|\varphi_x(t)\|^2 + \frac{8}{5}(1+a^2)\left(\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2\right). \tag{19.160}$$

Now we first focus on (19.159):

$$\begin{aligned}
&\|\varphi_t(t)+a\varphi(t)\|^2+\|\varphi_x(t)\|^2\\
&\quad\le 4\left(V(t)+\frac{4}{5}(1+a^2)\left(\|z(t)\|^2+\|z_x(t)\|^2+\|z_{xx}(t)\|^2\right)\right)\\
&\quad\le 4\left(V(t)+\frac{4}{5}(1+a^2)\int_1^{1+D}\mathrm{e}^{b(x-1)}\left(z^2(x,t)+z_x^2(x,t)+z_{xx}^2(x,t)\right)dx\right).
\end{aligned} \tag{19.161}$$

Invoking (19.138), we get

$$\|\varphi_t(t)+a\varphi(t)\|^2+\|\varphi_x(t)\|^2\le 4\max\left\{1,\frac{\frac{4}{5}(1+a^2)}{\frac{3(5+a^2)}{2a}}\right\}\Omega(t). \tag{19.162}$$

With a few steps of majorization, it is easy to see that

$$\frac{\frac{4}{5}(1+a^2)}{\frac{3(5+a^2)}{2a}}\le\frac{8a}{15}<a. \tag{19.163}$$

Hence,

$$\|\varphi_t(t)+a\varphi(t)\|^2+\|\varphi_x(t)\|^2\le 4\max\{1,a\}\,\Omega(t). \tag{19.164}$$

Recalling (19.149), we get

$$\begin{aligned}
\|w_x(t)\|^2+\|w_t(t)\|^2&\le 8\left(\cosh^2(a)+\sinh^2(a)\right)\max\{1,a\}\,\Omega(t)\\
&=8\cosh(2a)\max\{1,a\}\,\Omega(t),
\end{aligned} \tag{19.165}$$

where we have used the fact that $\cosh^2(a)+\sinh^2(a)=\cosh(2a)$. Furthermore, from (19.138), we get

$$\|z(t)\|^2+\|z_x(t)\|^2+\|z_{xx}(t)\|^2\le\frac{2a}{3(5+a^2)}\Omega(t). \tag{19.166}$$

With (19.165) and (19.166), we obtain the left side of the inequality (19.142) with α_1 given by (19.144). Now we turn our attention to (19.160) and to proving the right-hand side of the inequality (19.142). From (19.160) and (19.150), we get

$$\begin{aligned}
V(t)&\le 2\left(\cosh^2(a)+\sinh^2(a)\right)\left(\|w_x(t)\|^2+\|w_t(t)\|^2\right)\\
&\quad+\frac{8}{5}(1+a^2)\left(\|z(t)\|^2+\|z_x(t)\|^2+\|z_{xx}(t)\|^2\right)\\
&=2\cosh(2a)\left(\|w_x(t)\|^2+\|w_t(t)\|^2\right)\\
&\quad+\frac{8}{5}(1+a^2)\left(\|z(t)\|^2+\|z_x(t)\|^2+\|z_{xx}(t)\|^2\right).
\end{aligned} \tag{19.167}$$

Then, with (19.138) and (19.167), we obtain

$$\begin{aligned}\Omega(t) &\leq 2\cosh(2a)\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right) \\ &\quad + \frac{8}{5}(1+a^2)\left(\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2\right) \\ &\quad + \frac{3(5+a^2)}{2a}\int_1^{1+D} \mathrm{e}^{b(x-1)}\left(z^2(x,t) + z_x^2(x,t) + z_{xx}^2(x,t)\right)dx \\ &\leq 2\cosh(2a)\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right) \\ &\quad + \left(\frac{8}{5}(1+a^2) + \frac{3(5+a^2)}{2a}\mathrm{e}^{bD}\right)\left(\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2\right).\end{aligned} \tag{19.168}$$

This completes the proof of the right side of the inequality (19.142) with α_2 given by (19.145). □

With Lemmas 19.11 and 19.12, we prove the following result on the exponential stability of the target system (w, z).

Proposition 19.4. *For the system (19.100)–(19.104), the following holds for all $b > 0$:*

$$\Xi(t) \leq \frac{\alpha_2}{\alpha_1}\Xi_0 \mathrm{e}^{-\min\left\{\frac{a}{2}, b\right\}t}, \quad \forall t \geq 0. \tag{19.169}$$

19.5 Stability in the Original Plant Variables (u, v)

We now return to the backstepping transformations $(u, v) \mapsto (w, z)$ and $(w, z) \mapsto (u, v)$ in Section 19.1. After the substitution of the gain kernels expressed in terms of $\rho(x, y)$ and $\mu(x, y)$, the backstepping transformation is written as

$$w(x,t) = u(x,t) - \frac{q(q+c)}{1+qc}u(0,t) + \frac{q+c}{1+qc}\int_0^x u_t(y,t)dy, \tag{19.170}$$

$$\begin{aligned}z(x,t) &= v(x,t) + \int_1^x \rho_y(1+x-y,1)v(y,t)dy + q\rho(x,0)u(0,t) \\ &\quad - \int_0^1 \rho_x(x,y)u(y,t)dy - \int_0^1 \rho(x,y)u_t(y,t)dy,\end{aligned} \tag{19.171}$$

and the inverse backstepping transformation is written as

$$\begin{aligned}u(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)}\Bigg(& w(x,t) - \frac{q(q+c)}{1+qc}w(0,t) \\ & - \frac{q+c}{1+qc}\int_0^x w_t(y,t)dy\Bigg),\end{aligned} \tag{19.172}$$

$$v(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)} z(x,t) + \int_1^x \mu_y(1+x-y,1) z(y,t)dy - c\mu(x,0)w(0,t) - \int_0^1 \mu_x(x,y)w(y,t)dy - \int_0^1 \mu(x,y)w_t(y,t)dy. \qquad (19.173)$$

Since the stability result in Proposition 19.4 is given in terms of

$$\Xi(t) = \|w_x(t)\|^2 + \|w_t(t)\|^2 + \|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2, \qquad (19.174)$$

and we want to establish stability in terms of

$$\Upsilon(t) = \|u_x(t)\|^2 + \|u_t(t)\|^2 + \|v(t)\|^2 + \|v_x(t)\|^2 + \|v_{xx}(t)\|^2, \qquad (19.175)$$

we need to derive the expressions for all of the five normed quantities appearing in $\Xi(t)$ and all of the five normed quantities appearing in $\Upsilon(t)$.

The normed quantities appearing in $\Xi(t)$ are given by

$$w_x(x,t) = u_x(x,t) + \frac{q+c}{1+qc} u_t(x,t), \qquad (19.176)$$

$$w_t(x,t) = \frac{q+c}{1+qc} u_x(x,t) + u_t(x,t), \qquad (19.177)$$

$$z(x,t) = v(x,t) + \int_1^x \rho_y(1+x-y,1)v(y,t)dy + q\rho(x,0)u(0,t) - \int_0^1 \rho_x(x,y)u(y,t)dy - \int_0^1 \rho(x,y)u_t(y,t)dy, \qquad (19.178)$$

$$z_x(x,t) = v_x(x,t) + \rho_y(1,1)v(x,t) + \int_1^x \rho_{xy}(1+x-y,1)v(y,t)dy + q\rho_x(x,0)u(0,t) - \int_0^1 \rho_{xx}(x,y)u(y,t)dy - \int_0^1 \rho_x(x,y)u_t(y,t)dy, \qquad (19.179)$$

$$z_{xx}(x,t) = v_{xx}(x,t) + \rho_y(1,1)v_x(x,t) + \rho_{xy}(1,1)v(x,t) + \int_1^x \rho_{xxy}(1+x-y,1)v(y,t)dy + q\rho_{xx}(x,0)u(0,t) - \int_0^1 \rho_{xxx}(x,y)u(y,t)dy - \int_0^1 \rho_{xx}(x,y)u_t(y,t)dy. \qquad (19.180)$$

The normed quantities appearing in $\Upsilon(t)$ are given by

$$u_x(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)}\left(w_x(x,t) - \frac{q+c}{1+qc}w_t(x,t)\right), \tag{19.181}$$

$$u_t(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)}\left(-\frac{q+c}{1+qc}w_x(x,t) + w_t(x,t)\right), \tag{19.182}$$

$$\begin{aligned} v(x,t) = {} & \frac{(1+qc)^2}{(1-q^2)(1-c^2)}z(x,t) + \int_1^x \mu_y(1+x-y,1)z(y,t)dy \\ & - c\mu(x,0)w(0,t) - \int_0^1 \mu_x(x,y)w(y,t)dy \\ & - \int_0^1 \mu(x,y)w_t(y,t)dy, \end{aligned} \tag{19.183}$$

$$\begin{aligned} v_x(x,t) = {} & \frac{(1+qc)^2}{(1-q^2)(1-c^2)}z_x(x,t) + \mu_y(1,1)z(x,t) \\ & + \int_1^x \mu_{xy}(1+x-y,1)z(y,t)dy \\ & - c\mu_x(x,0)w(0,t) - \int_0^1 \mu_{xx}(x,y)w(y,t)dy, \\ & - \int_0^1 \mu_x(x,y)w_t(y,t)dy, \end{aligned} \tag{19.184}$$

$$\begin{aligned} v_{xx}(x,t) = {} & \frac{(1+qc)^2}{(1-q^2)(1-c^2)}z_{xx}(x,t) \\ & + \mu_y(1,1)z_x(x,t) + \mu_{xy}(1,1)z(x,t) \\ & + \int_1^x \mu_{xxy}(1+x-y,1)z(y,t)dy \\ & - c\mu_{xx}(x,0)w(0,t) - \int_0^1 \mu_{xxx}(x,y)w(y,t)dy \\ & - \int_0^1 \mu_{xx}(x,y)w_t(y,t)dy\,. \end{aligned} \tag{19.185}$$

In establishing a relation between $\Upsilon(t)$ and $\Xi(t)$, first we establish a relation between $\|u_x(t)\|^2 + \|u_t(t)\|^2$ and $\|w_x(t)\|^2 + \|w_t(t)\|^2$.

Lemma 19.13. *The following are true:*

$$\begin{aligned} \|u_x(t)\|^2 + \|u_t(t)\|^2 & \le 2\frac{1+\eta^2}{(1-\eta^2)^2}\left(\|w_x(t)\|^2 + \|w_t(t)\|^2\right) \\ & \le 2\frac{1+\eta^2}{(1-\eta^2)^2}\Xi(t), \end{aligned} \tag{19.186}$$

$$\|w_x(t)\|^2 + \|w_t(t)\|^2 \leq 2\left(1+\eta^2\right)\left(\|u_x(t)\|^2 + \|u_t(t)\|^2\right)$$
$$\leq 2\left(1+\eta^2\right)\Upsilon(t), \tag{19.187}$$

where

$$\eta = \frac{q+c}{1+qc}. \tag{19.188}$$

In establishing this result, we have used the fact that

$$\frac{(1+qc)^2}{(1-q^2)(1-c^2)} = \frac{1}{1-\eta^2}. \tag{19.189}$$

Next, we focus on relating $\|v(t)\|^2 + \|v_x(t)\|^2 + \|v_{xx}(t)\|^2$ to $\Xi(t)$.

Lemma 19.14. *The following are true:*

$$\|v(t)\|^2 \leq \gamma_0 \Xi(t), \tag{19.190}$$
$$\|v_x(t)\|^2 \leq \gamma_1 \Xi(t), \tag{19.191}$$
$$\|v_{xx}(t)\|^2 \leq \gamma_2 \Xi(t), \tag{19.192}$$

where

$$\gamma_0 = 5\max\left\{\frac{1}{(1-\eta^2)^2} + D\int_1^{1+D} \mu_y^2(x,1)dx,\right.$$
$$4\left(c\int_1^{1+D} \mu^2(x,0)dx + \int_1^{1+D}\int_0^1 \mu_x^2(x,y)dydx\right),$$
$$\left.\int_1^{1+D}\int_0^1 \mu^2(x,y)dydx\right\}, \tag{19.193}$$

$$\gamma_1 = 6\max\left\{\frac{1}{(1-\eta^2)^2},\right.$$
$$\mu_y^2(1,1) + D\int_1^{1+D} \mu_{xy}^2(x,1)dx,$$
$$4\left(c\int_1^{1+D} \mu_x^2(x,0)dx + \int_1^{1+D}\int_0^1 \mu_{xx}^2(x,y)dydx\right),$$
$$\left.\int_1^{1+D}\int_0^1 \mu_x^2(x,y)dydx\right\}, \tag{19.194}$$

$$\gamma_2 = 7\max\left\{\frac{1}{(1-\eta^2)^2},\right.$$
$$\mu_y^2(1,1),$$

$$\mu_{xy}^2(1,1)+D\int_1^{1+D}\mu_{xxy}^2(x,1)dx,$$
$$4\left(c\int_1^{1+D}\mu_{xx}^2(x,0)dx+\int_1^{1+D}\int_0^1\mu_{xxx}^2(x,y)dydx\right),$$
$$\left.\int_1^{1+D}\int_0^1\mu_{xx}^2(x,y)dydx\right\}. \tag{19.195}$$

Proof. We only prove inequality (19.192). All of the other inequalities are easier to prove. Starting from (19.185), we get

$$\begin{aligned}\|v_{xx}(t)\|^2 \le 7\Bigg(&\frac{1}{(1-\eta^2)^2}\|z_{xx}(t)\|^2\\ &+\mu_y^2(1,1)\|z_x(t)\|^2+\mu_{xy}^2(1,1)\|z(t)\|^2\\ &+D\int_1^{1+D}\mu_{xxy}^2(x,1)dx\|z(t)\|^2\\ &+c\int_1^{1+D}\mu_{xx}^2(x,0)dxw^2(0,t)\\ &+\int_1^{1+D}\int_0^1\mu_{xxx}^2(x,y)dydx\|w(t)\|^2\\ &+\int_1^{1+D}\int_0^1\mu_{xx}^2(x,y)dydx\|w_t(t)\|^2\Bigg),\end{aligned} \tag{19.196}$$

where we have used the fact that

$$\begin{aligned}\int_1^{1+D}\int_0^{x-1}\mu_{xxy}^2(1+s,1)dsdx &= \int_1^{1+D}(1+D-x)\mu_{xxy}^2(x,1)dx\\ &\le D\int_1^{1+D}\mu_{xxy}^2(x,1)dx.\end{aligned} \tag{19.197}$$

Employing Agmon's inequality, we get

$$\begin{aligned}\|v_{xx}(t)\|^2 \le 7\Bigg[&\frac{1}{(1-\eta^2)^2}\|z_{xx}(t)\|^2+\mu_y^2(1,1)\|z_x(t)\|^2\\ &+\left(\mu_{xy}^2(1,1)+D\int_1^{1+D}\mu_{xxy}^2(x,1)dx\right)\|z(t)\|^2\\ &+4\left(c\int_1^{1+D}\mu_{xx}^2(x,0)dx+\int_1^{1+D}\int_0^1\mu_{xxx}^2(x,y)dydx\right)\|w_(t)\|^2\\ &+\int_1^{1+D}\int_0^1\mu_{xx}^2(x,y)dydx\|w_t(t)\|^2\Bigg],\end{aligned} \tag{19.198}$$

from which (19.192) follows with (19.195). □

Next, we relate $\|z(t)\|^2 + \|z_x(t)\|^2 + \|z_{xx}(t)\|^2$ to $\Upsilon(t)$.

Lemma 19.15. *The following are true:*

$$\|z(t)\|^2 \le \delta_0 \Upsilon(t), \tag{19.199}$$
$$\|z_x(t)\|^2 \le \delta_1 \Upsilon(t), \tag{19.200}$$
$$\|z_{xx}(t)\|^2 \le \delta_2 \Upsilon(t), \tag{19.201}$$

where

$$\begin{aligned}\delta_0 = 5\max\Bigg\{ & 1 + D\int_1^{1+D} \rho_y^2(x,1)dx, \\ & 4\left(q\int_1^{1+D} \rho^2(x,0)dx + \int_1^{1+D}\int_0^1 \rho_x^2(x,y)dydx\right), \\ & \int_1^{1+D}\int_0^1 \rho^2(x,y)dydx\Bigg\},\end{aligned} \tag{19.202}$$

$$\begin{aligned}\delta_1 = 6\max\Bigg\{ & 1, \rho_y^2(1,1) + D\int_1^{1+D} \rho_{xy}^2(x,1)dx, \\ & 4\left(q\int_1^{1+D} \rho_x^2(x,0)dx + \int_1^{1+D}\int_0^1 \rho_{xx}^2(x,y)dydx\right), \\ & \int_1^{1+D}\int_0^1 \rho_x^2(x,y)dydx\Bigg\},\end{aligned} \tag{19.203}$$

$$\begin{aligned}\delta_2 = 7\max\Bigg\{ & 1, \rho_y^2(1,1), \rho_{xy}^2(1,1) + D\int_1^{1+D} \rho_{xxy}^2(x,1)dx, \\ & 4\left(q\int_1^{1+D} \rho_{xx}^2(x,0)dx + \int_1^{1+D}\int_0^1 \rho_{xxx}^2(x,y)dydx\right), \\ & \int_1^{1+D}\int_0^1 \rho_{xx}^2(x,y)dydx\Bigg\}.\end{aligned} \tag{19.204}$$

With Lemmas 19.13, 19.14, and 19.15, we get the following relation between $\Upsilon(t)$ and $\Xi(t)$.

Lemma 19.16. *The following is true:*

$$\alpha_3 \Upsilon(t) \le \Xi(t) \le \alpha_4 \Upsilon(t), \tag{19.205}$$

where

$$\alpha_3 = 1/\left[2(1+\eta^2) + \gamma_0 + \gamma_1 + \gamma_3\right], \tag{19.206}$$
$$\alpha_4 = 2(1+\eta^2) + \delta_0 + \delta_1 + \delta_3. \tag{19.207}$$

Finally, we obtain our main result on the exponential stability of the (u,v)-system.

Theorem 19.2. *Consider the closed-loop system consisting of the plant (19.6)–(19.10) and the control law (19.52), for*

$$q,c \in (0,1) \cup (1,\infty). \tag{19.208}$$

The following holds for all $b > 0$:

$$\Upsilon(t) \leq \frac{\alpha_2 \alpha_4}{\alpha_1 \alpha_3} \Upsilon_0 \mathrm{e}^{-\min\{\frac{a}{2},b\}t}, \quad \forall t \geq 0, \tag{19.209}$$

where the system norm $\Upsilon(t)$ is defined in (19.175) and a is defined in (19.105).

Since the coefficient b is arbitrary, we can choose it as

$$b = \frac{a}{2}. \tag{19.210}$$

Then we obtain the following corollary.

Corollary 19.1. *Consider the closed-loop system consisting of the plant (19.6)–(19.10) and the control law (19.52), under the condition (19.208) on q and c. The following holds:*

$$\Upsilon(t) \leq \frac{\alpha_2 \alpha_4}{\alpha_1 \alpha_3} \Upsilon_0 \left| \frac{1-c}{1+c} \right|^{t/4}, \quad \forall t \geq 0. \tag{19.211}$$

It should be noted that all of the coefficients $\alpha_1, \alpha_2, \alpha_3$, and α_4 depend on $a(c)$. In addition, α_3 and α_4 depend on q. Finally, it is important to observe that α_2 as well as α_3 and α_4 are nondecreasing functions of D.

19.6 Notes and References

The baseline design (the design for $D = 0$), which is used in this chapter for stabilization of the antistable wave equation, was proposed by Smyshlyaev, in [207].

Control of PDEs with input or output delays is an interesting area that is just opening up for research. An example of a relevant effort includes the stabilization of a beam equation with output delay by Guo and Chang [61]. Efforts in delay-PDE cascades are motivated by the interest to address the lack of delay robustness identified by Datko [32, 33, 34, 35, 36, 37].

Datko et al. [32] showed that standard feedback laws for wave equations have a zero robustness margin to the introduction of a delay in the feedback loop—an arbitrarily small measurement delay or input delay results in closed-loop instability. Such a result does not arise with finite-dimensional plants, nor with parabolic PDEs.

The result of this chapter resolves the Datko problem in the sense that it stabilizes the wave equation system in the presence of delay. In fact, our design stabilizes the wave equation in the presence of an arbitrarily long delay, not only in the presence of a small delay. Furthermore, we achieve stabilization for a wave equation plant that has all of its infinitely many open-loop eigenvalues in the open right half-plane.

Some very interesting open problems arise from the considerations in this chapter. First, one would want to consider the problem of robustness to small errors in D. It is not obvious that a Datko-type loss of robustness margin to delay occurs when a delay compensator is present.

Second, the foremost problem would be to consider the problem of adaptive stabilization of the antistable wave equation with unknown delay, namely, of the system

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad x \in (0,1), \tag{19.212}$$
$$u_x(0,t) = -qu_t(0,t), \tag{19.213}$$
$$u(1,t) = v(1,t), \tag{19.214}$$

$$Dv_t(x,t) = v_x(x,t), \qquad x \in [1,2), \tag{19.215}$$
$$v(2,t) = U(t), \tag{19.216}$$

where D and q are unknown. An adaptive design for this system would indirectly address the Datko [33] question, but in a more challenging setting where D is not small, but it is large and has a large uncertainty, and where the wave equation plant is not neutrally stable but antistable. The q-adaptive problem for the antistable wave equation with $D = 0$ is solved in [107].

Chapter 20
Other PDE-PDE Cascades

In this chapter we deal with cascades of parabolic and second-order hyperbolic PDEs. These are example problems. The parabolic-hyperbolic cascade is represented by a heat equation at the input of an antistable wave equation. The hyperbolic-parabolic cascade is represented by a wave equation at the input of an unstable reaction-diffusion equation.

The topic of PDE-PDE cascades is in its infancy. Its comprehensive coverage is beyond the scope of this book. Unlike the previous chapters in this book, our presentation in this chapter is fairly informal. We do derive the feedback laws and make statements of closed-loop eigenvalues; however, we forego a detailed Lyapunov stability analysis and the associated estimates for the transformations between the plant and the target system.

20.1 Antistable Wave Equation with Heat Equation at Its Input

Now we consider the heat-wave cascade system (depicted in Fig. 20.1)

$$u_{tt}(x,t) = u_{xx}(x,t)\,, \qquad x \in (0,1)\,, \tag{20.1}$$

$$u_x(0,t) = -q u_t(0,t)\,, \tag{20.2}$$

$$u(1,t) = v(1,t)\,, \tag{20.3}$$

$$v_t(x,t) = v_{xx}(x,t)\,, \qquad x \in (1,1+D)\,, \tag{20.4}$$

$$v_x(1,t) = 0\,, \tag{20.5}$$

$$v(1+D,t) = U(t)\,, \tag{20.6}$$

where $U(t)$ is the input and (u, u_t, v) is the state.

We consider the backstepping transformation

$$w(x,t) = u(x,t) - \frac{q(q+c)}{1+qc} u(0,t) + \frac{q+c}{1+qc} \int_0^x u_t(y,t)dy\,, \tag{20.7}$$

M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Systems & Control: Foundations & Applications, DOI 10.1007/978-0-8176-4877-0_20,

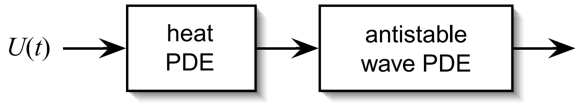

Fig. 20.1 A cascade of heat PDE with antistable wave PDE.

$$z(x,t) = v(x,t) - \int_1^x p(x-y)v(y,t)dy - \theta(x)u(0,t) - \int_0^1 \gamma(x,y)u(y,t)dy - \int_0^1 \rho(x,y)u_t(y,t)dy\,, \tag{20.8}$$

where the kernels p, θ, γ, and ρ need to be chosen to transform the cascade PDE system into the target system

$$w_{tt}(x,t) = w_{xx}(x,t)\,, \tag{20.9}$$
$$w_x(0,t) = cw_t(0,t)\,, \tag{20.10}$$
$$w(1,t) = z(1,t)\,, \tag{20.11}$$

$$z_t(x,t) = z_{xx}(x,t)\,, \tag{20.12}$$
$$z_x(1,t) = 0\,, \tag{20.13}$$
$$z(1+D,t) = 0\,, \tag{20.14}$$

with the control

$$U(t) = \int_1^{1+D} p(1+D-y)v(y,t)dy + \theta(1+D)u(0,t) + \int_0^1 \gamma(1+D,y)u(y,t)dy + \int_0^1 \rho(1+D,y)u_t(y,t)dy\,. \tag{20.15}$$

The kernel ρ is governed by the PDE

$$\rho_{xxxx}(x,y) = \rho_{yy}(x,y)\,, \tag{20.16}$$
$$\rho_y(x,0) = q\rho_{xx}(x,0)\,, \tag{20.17}$$
$$\rho(x,1) = 0\,, \tag{20.18}$$

where $x \in [1, 1+D]$ should be viewed as the time variable and $y \in (0,1)$ as the space variable. The initial condition of this PDE is

$$\rho(1,y) = -\frac{q+c}{1+qc}\,, \tag{20.19}$$
$$\rho_x(1,y) = 0\,, \tag{20.20}$$
$$\rho_{xx}(1,y) = 0\,, \tag{20.21}$$
$$\rho_{xxx}(1,y) = 0\,. \tag{20.22}$$

After solving for $\rho(x,y)$, the kernels p, θ, and γ are obtained as

$$p(s) = -\int_1^{1+s} \rho_y(\xi,1)d\xi\,, \quad s \in [0,D]\,, \tag{20.23}$$

$$\theta(x) = -q\rho(x,0)\,, \tag{20.24}$$

$$\gamma(x,y) = \rho_{xx}(x,y)\,. \tag{20.25}$$

The ρ-PDE is an unusual equation. It is not clear whether this equation is well posed and whether it possesses the regularity properties needed for the kernels of the feedback law and of the backstepping (direct and inverse) transformations to be well defined.

The stability of the closed-loop target system is characterized by the following proposition.

Proposition 20.1. *The spectrum of the system (20.9)–(20.14) is given by*

$$\sigma_n = -\frac{1}{2}\ln\left|\frac{1+c}{1-c}\right| + j\pi\begin{cases} n+\frac{1}{2}\,, & 0 \le c < 1\,,\\ n\,, & c > 1\,,\end{cases} \tag{20.26}$$

$$\sigma_m = -\frac{\pi^2}{D^2}m^2\,, \tag{20.27}$$

where $n \in \mathbb{Z}$ and $m \in \mathbb{N}_+$.

The inverse backstepping transformation is given by

$$u(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)}\left(w(x,t) - \frac{c(q+c)}{1+qc}w(0,t) - \frac{q+c}{1+qc}\int_0^x w_t(y,t)dy\right), \tag{20.28}$$

$$v(x,t) = \frac{(1+qc)^2}{(1-q^2)(1-c^2)}z(x,t) - \int_1^x \pi(x-y)z(y,t)dy - \eta(x)w(0,t) - \int_0^1 \delta(x,y)w(y,t)dy - \int_0^1 \mu(x,y)w_t(y,t)dy\,, \tag{20.29}$$

where the kernel μ is governed by the PDE

$$\mu_{xxxx}(x,y) = \mu_{yy}(x,y)\,, \tag{20.30}$$

$$\mu_y(x,0) = -c\mu_{xx}(x,0)\,, \tag{20.31}$$

$$\mu(x,1) = 0\,, \tag{20.32}$$

where $x \in [1, 1+D]$ should be viewed as the time variable and $y \in (0,1)$ as the space variable. The initial condition of this PDE is

$$\mu(1,y) = \frac{(q+c)(1+qc)}{(1-q^2)(1-c^2)}\,, \tag{20.33}$$

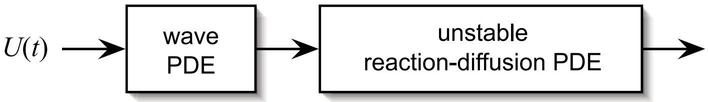

Fig. 20.2 A cascade of a wave PDE with an unstable reaction-diffusion PDE.

$$\mu_x(1,y) = 0\,, \tag{20.34}$$
$$\mu_{xx}(1,y) = 0\,, \tag{20.35}$$
$$\mu_{xxx}(1,y) = 0\,. \tag{20.36}$$

After solving for $\mu(x,y)$, the kernels π, η, and δ are obtained as

$$\pi(s) = -\int_1^{1+s} \mu_y(\xi,1)d\xi\,, \quad s \in [0,D]\,, \tag{20.37}$$
$$\eta(x) = c\mu(x,0)\,, \tag{20.38}$$
$$\delta(x,y) = \mu_{xx}(x,y)\,. \tag{20.39}$$

20.2 Unstable Reaction-Diffusion Equation with a Wave Equation at Its Input

Finally, we consider the wave-heat cascade system

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t)\,, \quad x \in (0,1)\,, \tag{20.40}$$
$$u(0,t) = 0\,, \tag{20.41}$$
$$u(1,t) = v(1,t)\,, \tag{20.42}$$

$$v_{tt}(x,t) = v_{xx}(x,t)\,, \quad x \in (1,1+D)\,, \tag{20.43}$$
$$v_x(1,t) = 0\,, \tag{20.44}$$
$$v(1+D,t) = U(t)\,, \tag{20.45}$$

where $U(t)$ is the input and (u, v, v_t) is the state.

We consider the backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy\,, \tag{20.46}$$
$$z(x,t) = v(x,t) - \int_1^x p(x-y)v(y,t)dy - \int_1^x r(x-y)v_t(y,t)dy - \int_0^1 \gamma(x,y)u(y,t)dy\,, \tag{20.47}$$

where the kernels p, θ, γ, and ρ need to be chosen to transform the cascade PDE system into the target system

$$w_t(x,t) = w_{xx}(x,t)\,, \tag{20.48}$$
$$w(0,t) = 0\,, \tag{20.49}$$
$$w(1,t) = z(1,t)\,, \tag{20.50}$$

$$z_{tt}(x,t) = z_{xx}(x,t)\,, \tag{20.51}$$
$$z_x(1,t) = cz_t(1,t)\,, \tag{20.52}$$
$$z(1+D,t) = 0\,, \tag{20.53}$$

with the control

$$U(t) = \int_1^{1+D} p(1+D-y)v(y,t)dy + \int_1^{1+D} r(1+D-y)v_t(y,t)dy + \int_0^1 \gamma(1+D,y)u(y,t)dy\,. \tag{20.54}$$

The kernel k is given by

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda\,(x^2-y^2)}\right)}{\sqrt{\lambda\,(x^2-y^2)}}\,. \tag{20.55}$$

The kernel γ is governed by the PDE

$$\gamma_{xx}(x,y) = \gamma_{yyyy}(x,y) + \lambda\gamma_{yy}(x,y)\,, \tag{20.56}$$
$$\gamma(x,0) = 0\,, \tag{20.57}$$
$$\gamma_{yy}(x,0) = 0\,, \tag{20.58}$$
$$\gamma(x,1) = 0\,, \tag{20.59}$$
$$\gamma_{yy}(x,1) = 0\,, \tag{20.60}$$

where $x \in [1, 1+D]$ should be viewed as the time variable and $y \in (0,1)$ as the space variable. The initial condition of this PDE is

$$\gamma(1,y) = k(1,y)\,, \tag{20.61}$$
$$\gamma_x(1,y) = c\left(k_{yy}(1,y) + \lambda k(1,y)\right)\,. \tag{20.62}$$

After solving for $\gamma(x,y)$, the kernels p, r are obtained as

$$p(s) = -ck_y(1,1) - \int_1^{1+s} \left(\gamma_{yyy}(\xi,1) + \lambda\gamma_y(\xi,1)\right)d\xi\,, \tag{20.63}$$
$$r(s) = -c - \int_1^{1+s} \gamma_y(\xi,1)d\xi\,, \tag{20.64}$$

where $s \in [0,D]$.

As with the heat-wave cascade, it is not clear if the γ-PDE cascade is well posed and possesses the needed regularity properties for control implementation and stability analysis.

The stability of the closed-loop target system is characterized by the following proposition.

Proposition 20.2. *The spectrum of the system (20.48)–(20.53) is given by*

$$\sigma_m = -\pi^2 m^2 \,, \tag{20.65}$$

$$\sigma_n = -\frac{1}{2}\ln\left|\frac{1+c}{1-c}\right| + j\frac{\pi}{D}\begin{cases} n+\frac{1}{2}\,, & 0 \le c < 1\,, \\ n\,, & c > 1\,, \end{cases} \tag{20.66}$$

where $m \in \mathbb{N}_+$ and $n \in \mathbb{Z}$.

The inverse backstepping transformation is given by

$$u(x,t) = w(x,t) - \int_0^x l(x,y)w(y,t)dy\,, \tag{20.67}$$

$$\begin{aligned} v(x,t) = z(x,t) - \int_1^x \pi(x-y)z(y,t)dy - \int_1^x \rho(x-y)z_t(y,t)dy \\ - \int_0^1 \delta(x,y)w(y,t)dy\,. \end{aligned} \tag{20.68}$$

The kernel l is given by

$$l(x,y) = \lambda y \frac{J_1\left(\sqrt{\lambda\,(x^2-y^2)}\right)}{\sqrt{\lambda\,(x^2-y^2)}}\,. \tag{20.69}$$

The kernel δ is governed by the PDE

$$\delta_{xx}(x,y) = \delta_{yyyy}(x,y)\,, \tag{20.70}$$

$$\delta(x,0) = 0\,, \tag{20.71}$$

$$\delta_{yy}(x,0) = 0\,, \tag{20.72}$$

$$\delta(x,1) = 0\,, \tag{20.73}$$

$$\delta_{yy}(x,1) = 0\,, \tag{20.74}$$

where $x \in [1,1+D]$ should be viewed as the time variable and $y \in (0,1)$ as the space variable. The initial condition of this PDE is

$$\delta(1,y) = l(1,y)\,, \tag{20.75}$$

$$\delta_x(1,y) = -c l_{yy}(1,y)\,. \tag{20.76}$$

After solving for $\delta(x,y)$, the kernels π,ρ are obtained as

$$\pi(s) = c l_y(1,1) - \int_1^{1+s} \delta_{yyy}(\xi,1)d\xi\,, \tag{20.77}$$

$$\rho(s) = c - \int_1^{1+s} \delta_y(\xi,1)d\xi\,, \tag{20.78}$$

where $s \in [0,D]$.

20.3 Notes and References

The control of PDE cascades or interconnected PDEs is an exciting and challenging problem that has become active over the last 10 to 15 years. An example of such an effort is stability and controllability analysis of a heat-wave system by Zhang and Zuazua [248]. The effort in PDE cascades is motivated partly by problems in fluid-structure interaction and other interactive physical processes.

The problem dealt with in [248] differs significantly from our consideration in Section 20.1. In [248] the heat and wave equations are *coupled* through two boundary conditions, one equating the boundary values of the two PDEs' state variables and the other equating their respective first derivatives, i.e.,

$$u_{tt}(x,t) = u_{xx}(x,t)\,, \qquad x \in (0,1)\,, \tag{20.79}$$
$$u(0,t) = 0\,, \tag{20.80}$$
$$u_x(1,t) = v_x(1,t)\,, \tag{20.81}$$

$$v_t(x,t) = v_{xx}(x,t)\,, \qquad x \in (1,1+D)\,, \tag{20.82}$$
$$v(1,t) = u(1,t)\,, \tag{20.83}$$
$$v(1+D,t) = U(t)\,. \tag{20.84}$$

In contrast, our two PDEs are in a *cascade* connection—one feeding into the other. In addition, the difference is that the authors in [248] consider the standard, undamped wave equation with one pinned end, whereas we consider the antistable wave equation (with boundary antidamping).

A key question for future research that is related to this chapter is that of the well-posedness of the PDEs such as

$$\rho_{xxxx}(x,y) = \rho_{yy}(x,y)\,, \quad y \in (0,1)\,, \tag{20.85}$$
$$\rho_y(x,0) - q\rho_{xx}(x,0) = \rho(x,1) = 0\,, \quad x > 0\,, \tag{20.86}$$

in Section 20.1 and

$$\gamma_{xx}(x,y) = \gamma_{yyyy}(x,y) + \lambda\gamma_{yy}(x,y)\,, \quad y \in (0,1)\,, \tag{20.87}$$
$$\gamma(x,0) = \gamma_{yy}(x,0) = \gamma(x,1) = \gamma_{yy}(x,1) = 0\,, \quad x > 0\,, \tag{20.88}$$

in Section 20.2. Both of these equations look like potentially challenging problems, but they are not hopeless. For example, the γ-system is of a Boussinesq-like form.

Appendix A
Poincaré, Agmon, and Other Basic Inequalities

In this appendix we review a few inequalities for basic Sobolev spaces of functions of one variable.

Let us first recall two elementary well-known inequalities:

Young's inequality (most elementary version)

$$\boxed{ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2} \qquad \forall \gamma > 0\,. \tag{A.1}$$

Cauchy–Schwartz inequality

$$\boxed{\int_0^1 u(x)w(x)\,dx \leq \left(\int_0^1 u(x)^2\,dx\right)^{1/2}\left(\int_0^1 w(x)^2\,dx\right)^{1/2}.} \tag{A.2}$$

The following lemma establishes the relationship between the L_2 norms of w and w_x.

Lemma A.1 (Poincaré inequality). *For any w, continuously differentiable on* $[0,1]$,

$$\boxed{\begin{aligned} \int_0^1 w(x)^2\,dx &\leq 2w^2(1) + 4\int_0^1 w_x(x)^2\,dx\,,\\ \int_0^1 w(x)^2\,dx &\leq 2w^2(0) + 4\int_0^1 w_x(x)^2\,dx\,. \end{aligned}} \tag{A.3}$$

Proof. We start with the L_2 norm,

$$\begin{aligned} \int_0^1 w^2\,dx &= xw^2|_0^1 - 2\int_0^1 xww_x\,dx \quad \text{(integration by parts)}\\ &= w^2(1) - 2\int_0^1 xww_x\,dx\\ &\leq w^2(1) + \frac{1}{2}\int_0^1 w^2 dx + 2\int_0^1 x^2 w_x^2\,dx\,. \end{aligned}$$

Subtracting the second term from both sides of the inequality, we get the first inequality in (A.3):

$$\frac{1}{2}\int_0^1 w^2 dx \le w^2(1) + 2\int_0^1 x^2 w_x^2 dx$$
$$\le w^2(1) + 2\int_0^1 w_x^2 dx. \tag{A.4}$$

The second inequality in (A.3) is obtained in a similar fashion. □

The inequalities (A.3) are conservative. The tight version of (A.3) is given next, which is sometimes called "a variation of Wirtinger's inequality" [64].

Lemma A.2.

$$\int_0^1 (w(x) - w(0))^2 \, dx \le \frac{4}{\pi^2}\int_0^1 w_x^2(x) \, dx. \tag{A.5}$$

Equality holds only for $w(x) = A + B\sin\frac{\pi x}{2}$.

The proof of (A.5) is far more complicated than the proof of (A.3).

Now we turn to reviewing the basic relationships between the L_2 and H_1 Sobolev norms and the maximum norm. The H_1 norm can be defined in more than one way. We define it as

$$\|w\|_{H_1}^2 := \int_0^1 w^2 dx + \int_0^1 w_x^2 dx. \tag{A.6}$$

Note also that by using the Poincaré inequality, it is possible to drop the first integral in (A.6) whenever the function is zero at at least one of the boundaries.

Lemma A.3 (Agmon's inequality). *For a function* $w \in H_1$*, the following inequalities hold:*

$$\boxed{\begin{aligned} \max_{x\in[0,1]} |w(x)|^2 &\le w(0)^2 + 2\|w\|\|w_x\|, \\ \max_{x\in[0,1]} |w(x)|^2 &\le w(1)^2 + 2\|w\|\|w_x\|. \end{aligned}} \tag{A.7}$$

Proof. We begin with

$$\int_0^x w w_x dx = \int_0^x d\frac{1}{2}w^2$$
$$= \frac{1}{2}w^2|_0^x$$
$$= \frac{1}{2}w(x)^2 - \frac{1}{2}w(0)^2, \tag{A.8}$$

which gives

$$\frac{1}{2}|w(x)^2| \le \int_0^x |w||w_x| dx + \frac{1}{2}w(0)^2. \tag{A.9}$$

Using the fact that an integral of a positive function is an increasing function of its upper limit, we can rewrite the last inequality as

$$|w(x)|^2 \leq w(0)^2 + 2\int_0^1 |w(x)||w_x(x)|dx. \tag{A.10}$$

The right-hand side of this inequality does not depend on x; therefore,

$$\max_{x\in[0,1]} |w(x)|^2 \leq w(0)^2 + 2\int_0^1 |w(x)||w_x(x)|dx. \tag{A.11}$$

Using the Cauchy–Schwartz inequality, we get the first inequality of (A.7). The second inequality is obtained in a similar fashion. □

Appendix B
Input–Output Lemmas for LTI and LTV Systems

In addition to a review of basic input–output stability results, we give several technical lemmas used in the book.

For a function $x : \mathbb{R}_+ \to \mathbb{R}^n$, we define the L_p norm, $p \in [1,\infty]$, as

$$\|x\|_p = \begin{cases} \left(\int_0^\infty |x(t)|^p dt \right)^{1/p}, & p \in [1,\infty), \\ \sup_{t \geq 0} |x(t)|, & p = \infty, \end{cases} \tag{B.1}$$

and the $L_{p,e}$ norm (truncated L_p norm) as

$$\|x_t\|_p = \begin{cases} \left(\int_0^t |x(\tau)|^p d\tau \right)^{1/p}, & p \in [1,\infty), \\ \sup_{\tau \in [0,t]} |x(\tau)|, & p = \infty. \end{cases} \tag{B.2}$$

Lemma B.1 (Hölder's inequality). *If $p,q \in [1,\infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\|(fg)_t\|_1 \leq \|f_t\|_p \|g_t\|_q, \qquad \forall t \geq 0. \tag{B.3}$$

We consider an LTI causal system described by the convolution

$$y(t) = h \star u = \int_0^t h(t-\tau)u(\tau)d\tau, \tag{B.4}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is the input, $y : \mathbb{R}_+ \to \mathbb{R}$ is the output, and $h : \mathbb{R} \to \mathbb{R}$ is the system's impulse response, which is defined to be zero for negative values of its argument.

Theorem B.1 (Young's convolution theorem). *If $h \in L_{1,e}$, then*

$$\|(h \star u)_t\|_p \leq \|h_t\|_1 \|u_t\|_p, \qquad p \in [1,\infty]. \tag{B.5}$$

Proof. Let $y = h \star u$. Then, for $p \in [1,\infty)$, we have

$$
\begin{aligned}
|y(t)| &\leq \int_0^t |h(t-\tau)|\; |u(\tau)| d\tau \\
&= \int_0^t |h(t-\tau)|^{\frac{p-1}{p}}\; |h(t-\tau)|^{\frac{1}{p}}\; |u(\tau)| d\tau \\
&\leq \left(\int_0^t |h(t-\tau)| d\tau \right)^{\frac{p-1}{p}} \left(\int_0^t |h(t-\tau)|\; |u(\tau)|^p d\tau \right)^{\frac{1}{p}} \\
&= \|h_t\|_1^{\frac{p-1}{p}} \left(\int_0^t |h(t-\tau)|\; |u(\tau)|^p d\tau \right)^{\frac{1}{p}}, \qquad \text{(B.6)}
\end{aligned}
$$

where the second inequality is obtained by applying Hölder's inequality. Raising (B.6) to power p and integrating from 0 to t, we get

$$
\begin{aligned}
\|y_t\|_p^p &\leq \int_0^t \|h_t\|_1^{p-1} \left(\int_0^\tau |h(\tau-s)|\; |u(s)|^p ds \right) d\tau \\
&= \|h_t\|_1^{p-1} \int_0^t \left(\int_s^t |h(\tau-s)|\; |u(s)|^p d\tau \right) ds \\
&= \|h_t\|_1^{p-1} \int_0^t \left(\int_0^t |h(\tau-s)|\; |u(s)|^p d\tau \right) ds \\
&= \|h_t\|_1^{p-1} \int_0^t |u(s)|^p \left(\int_0^t |h(\tau-s)| d\tau \right) ds \\
&\leq \|h_t\|_1^{p-1} \int_0^t |u(s)|^p \left(\int_0^t |h(\tau)| d\tau \right) ds \\
&\leq \|h_t\|_1^{p-1} \|h\|_1^1 \|u_t\|_1^p \\
&\leq \|h_t\|_1^p \|u_t\|_p^p, \qquad \text{(B.7)}
\end{aligned}
$$

where the second line is obtained by changing the sequence of integration, and the third line by using the causality of h. The proof for the case $p = \infty$ is immediate by taking a supremum of u over $[0,t]$ in the convolution. □

Lemma B.2. *Let v and ρ be real-valued functions defined on $\mathbb{R}_+$, and let b and c be positive constants. If they satisfy the differential inequality*

$$\dot{v} \leq -cv + b\rho(t)^2, \qquad v(0) \geq 0, \tag{B.8}$$

(i) then the following integral inequality holds:

$$v(t) \leq v(0)\mathrm{e}^{-ct} + b \int_0^t \mathrm{e}^{-c(t-\tau)} \rho(\tau)^2 d\tau. \tag{B.9}$$

(ii) If, in addition, $\rho \in L_2$, then $v \in L_1$ and

$$\|v\|_1 \leq \frac{1}{c} \left(v(0) + b\|\rho\|_2^2 \right). \tag{B.10}$$

Proof. *(i)* Upon multiplication of (B.8) by e^{ct}, it becomes

$$\frac{d}{dt}\left(v(t)e^{ct}\right) \leq b\rho(t)^2 e^{ct}\,. \tag{B.11}$$

Integrating (B.11) over $[0,t]$, we arrive at (B.9).

(ii) By integrating (B.9) over $[0,t]$, we get

$$\begin{aligned}\int_0^t v(\tau)d\tau &\leq \int_0^t v(0)e^{-c\tau}d\tau + b\int_0^t \left[\int_0^\tau e^{-c(\tau-s)}\rho(s)^2 ds\right]d\tau \\ &\leq \frac{1}{c}v(0) + b\int_0^t \left[\int_0^\tau e^{-c(\tau-s)}\rho(s)^2 ds\right]d\tau\,. \end{aligned}\tag{B.12}$$

Noting that the second term is $b\|(h\star\rho^2)_t\|_1$, where

$$h(t) = e^{-ct}\,, \qquad t \geq 0\,, \tag{B.13}$$

we apply Theorem B.1. Since

$$\|h\|_1 = \frac{1}{c}\,, \tag{B.14}$$

we obtain (B.10). □

Lemma B.3. *Let v, l_1, and l_2 be real-valued functions defined on $\mathbb{R}_+$, and let c be a positive constant. If l_1 and l_2 are nonnegative and in L_1 and satisfy the differential inequality*

$$\dot{v} \leq -cv + l_1(t)v + l_2(t)\,, \qquad v(0) \geq 0\,, \tag{B.15}$$

then $v \in L_\infty \cap L_1$ and

$$v(t) \leq \left(v(0)e^{-ct} + \|l_2\|_1\right)e^{\|l_1\|_1}\,, \tag{B.16}$$

$$\|v\|_1 \leq \frac{1}{c}\left(v(0) + \|l_2\|_1\right)e^{\|l_1\|_1}\,. \tag{B.17}$$

Proof. Using the facts that

$$v(t) \leq w(t)\,, \tag{B.18}$$

$$\dot{w} = -cw + l_1(t)w + l_2(t)\,, \tag{B.19}$$

$$w(0) = v(0) \tag{B.20}$$

(the comparison principle), and applying the variation-of-constants formula, the differential inequality (B.15) is rewritten as

$$\begin{aligned} v(t) &\leq v(0)e^{\int_0^t[-c+l_1(s)]ds} + \int_0^t e^{\int_\tau^t[-c+l_1(s)]ds} l_2(\tau)d\tau \\ &\leq v(0)e^{-ct}e^{\int_0^\infty l_1(s)ds} + \int_0^t e^{-c(t-\tau)}l_2(\tau)d\tau e^{\int_0^\infty l_1(s)ds} \\ &\leq \left[v(0)e^{-ct} + \int_0^t e^{-c(t-\tau)}l_2(\tau)d\tau\right]e^{\|l_1\|_1}\,. \end{aligned}\tag{B.21}$$

By taking a supremum of $e^{-c(t-\tau)}$ over $[0,\infty]$, we obtain (B.16). Integrating (B.21) over $[0,\infty]$, we get

$$\int_0^t v(\tau)d\tau \leq \left(\frac{1}{c}v(0) + \int_0^t \left[\int_0^\tau e^{-c(\tau-s)}l_2(s)ds\right]d\tau\right)e^{\|l_1\|_1}. \tag{B.22}$$

Applying Theorem B.1 to the double integral, we arrive at (B.17). □

Remark B.1. An alternative proof that $v \in L_\infty \cap L_1$ in Lemma B.3 is using the Gronwall lemma (Lemma B.5). However, with the Gronwall lemma, the estimates of the bounds (B.16) and (B.17) are more conservative:

$$v(t) \leq \left(v(0)e^{-ct} + \|l_2\|_1\right)\left(1 + \|l_1\|_1 e^{\|l_1\|_1}\right), \tag{B.23}$$

$$\|v\|_1 \leq \frac{1}{c}\left(v(0) + \|l_2\|_1\right)\left(1 + \|l_1\|_1 e^{\|l_1\|_1}\right), \tag{B.24}$$

because

$$e^x < (1 + xe^x), \qquad \forall x > 0. \tag{B.25}$$

Note that the ratio between the bounds (B.23) and (B.16) and that between the bounds (B.24) and (B.17) are of the order $\|l_1\|_1$ when $\|l_1\|_1 \to \infty$. ◇

For cases where l_1 and l_2 are functions of time that converge to zero but are not in L_p for any $p \in [1,\infty)$, we have the following lemma.

Lemma B.4. *Consider the differential inequality*

$$\dot{v} \leq -[c - \beta_1(r_0,t)]v + \beta_2(r_0,t) + \rho, \qquad v(0) = v_0 \geq 0, \tag{B.26}$$

where $c > 0$ and $r_0 \geq 0$ are constants, and β_1 and β_2 are class-$\mathcal{KL}$ functions. Then there exist a class-$\mathcal{KL}$ function β_v and a class-$\mathcal{K}$ function γ_v such that

$$v(t) \leq \beta_v(v_0 + r_0, t) + \gamma_v(\rho), \qquad \forall t \geq 0. \tag{B.27}$$

Moreover, if

$$\beta_i(r,t) = \alpha_i(r)e^{-\sigma_i t}, \quad i = 1,2, \tag{B.28}$$

where $\alpha_i \in \mathcal{K}$ and $\sigma_i > 0$, then there exist $\alpha_v \in \mathcal{K}$ and $\sigma_v > 0$ such that

$$\beta_v(r,t) = \alpha_v(r)e^{-\sigma_v t}. \tag{B.29}$$

Proof. We start by introducing

$$\tilde{v} = v - \frac{\rho}{c} \tag{B.30}$$

and rewriting (B.26) as

$$\dot{\tilde{v}} \leq -[c - \beta_1(r_0,t)]\tilde{v} + \frac{\rho}{c}\beta_1(r_0,t) + \beta_2(r_0,t). \tag{B.31}$$

It then follows that

$$v(t) \le v_0 \mathrm{e}^{\int_0^t [\beta_1(r_0,s)-c]ds} + \int_0^t \left[\frac{\rho}{c}\beta_1(r_0,\tau) + \beta_2(r_0,\tau)\right] \mathrm{e}^{\int_\tau^t [\beta_1(r_0,s)-c]ds} d\tau + \frac{\rho}{c}. \tag{B.32}$$

We note that

$$\mathrm{e}^{\int_\tau^t [\beta_1(r_0,s)-c]ds} \le k(r_0)\mathrm{e}^{-\frac{c}{2}(t-\tau)}, \qquad \forall \tau \in [0,t], \tag{B.33}$$

where k is a positive, continuous, increasing function. To get an estimate of the overshoot coefficient $k(r_0)$, we provide a proof of (B.33). For each c, there exists a class-$\mathscr{K}$ function $T_c : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\beta_1(r_0,s) \le \frac{c}{2}, \qquad \forall s \ge T_c(r_0). \tag{B.34}$$

Therefore, for $0 \le \tau \le T_c(r_0) \le t$, we have

$$\begin{aligned}
\int_\tau^t [\beta_1(r_0,s) - c]\, ds &\le \int_\tau^{T_c(r_0)} [\beta_1(r_0,s) - c]\, ds + \int_{T_c(r_0)}^t \left(-\frac{c}{2}\right) ds \\
&\le (\beta_1(r_0,0) - c)(T_c(r_0) - \tau) - \frac{c}{2}(t - T_c(r_0)) \\
&\le T_c(r_0)\beta_1(r_0,0) - \frac{c}{2}(t-\tau),
\end{aligned} \tag{B.35}$$

so the overshoot coefficient in (B.33) is given by

$$k(r_0) \stackrel{\triangle}{=} \mathrm{e}^{T_c(r_0)\beta_1(r_0,0)}. \tag{B.36}$$

For the other two cases, $t \le T_c(r_0)$ and $T_c(r_0) \le \tau$, getting (B.33) with $k(r_0)$ as in (B.36) is immediate. Now substituting (B.33) into (B.32), we get

$$v(t) \le v_0 k(r_0)\mathrm{e}^{-\frac{c}{2}t} + k(r_0)\int_0^t \left[\frac{\rho}{c}\beta_1(r_0,\tau) + \beta_2(r_0,\tau)\right] \mathrm{e}^{-\frac{c}{2}(t-\tau)} d\tau + \frac{\rho}{c}. \tag{B.37}$$

To complete the proof, we show that a class-$\mathscr{KL}$ function β convolved with an exponentially decaying kernel is bounded by another class-$\mathscr{KL}$ function:

$$\begin{aligned}
\int_0^t \mathrm{e}^{-\frac{c}{2}(t-\tau)}\beta(r_0,\tau)d\tau &= \int_0^{t/2} \mathrm{e}^{-\frac{c}{2}(t-\tau)}\beta(r_0,\tau)d\tau + \int_{t/2}^t \mathrm{e}^{-\frac{c}{2}(t-\tau)}\beta(r_0,\tau)d\tau \\
&\le \beta(r_0,0)\int_0^{t/2} \mathrm{e}^{-\frac{c}{2}(t-\tau)}d\tau + \beta(r_0,t/2)\int_{t/2}^t \mathrm{e}^{-\frac{c}{2}(t-\tau)}d\tau \\
&\le \frac{2}{c}\left[\beta(r_0,0)\mathrm{e}^{-\frac{c}{4}t} + \beta(r_0,t/2)\right].
\end{aligned} \tag{B.38}$$

Thus, (B.37) becomes

$$v(t) \le k(r_0)\left\{\left[v_0 + \frac{2\rho}{c^2}\beta_1(r_0,0) + \frac{2}{c}\beta_2(r_0,0)\right] e^{-\frac{c}{4}t} + \frac{2\rho}{c^2}\beta_1(r_0,t/2) + \frac{2}{c}\beta_2(r_0,t/2)\right\} + \frac{\rho}{c}\,. \tag{B.39}$$

By applying Young's inequality to the terms

$$k(r_0)\frac{2\rho}{c^2}\beta_1(r_0,0)e^{-\frac{c}{4}t} \tag{B.40}$$

and

$$k(r_0)\frac{2\rho}{c^2}\beta_1(r_0,t/2)\,, \tag{B.41}$$

we obtain (B.27) with

$$\beta_v(r,t) = k(r)\left\{\left[r + \frac{k(r)}{c^2}\beta_1(r,0)^2 + \frac{2}{c}\beta_2(r,0)\right] e^{-\frac{c}{4}t} + \frac{k(r)}{c^2}\beta_1(r,t/2)^2 + \frac{2}{c}\beta_2(r,t/2)\right\}, \tag{B.42}$$

$$\gamma_v(r) = \frac{r}{c} + \frac{r^2}{c^2}\,. \tag{B.43}$$

The last statement of the lemma is immediate by substitution into (B.42). □

Now we give a version of Gronwall's lemma.

Lemma B.5 (Gronwall). *Consider the continuous functions* $\lambda : \mathbb{R}_+ \to \mathbb{R}$, $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, *and* $\nu : \mathbb{R}_+ \to \mathbb{R}_+$, *where* μ *and* ν *are also nonnegative. If a continuous function* $y : \mathbb{R}_+ \to \mathbb{R}$ *satisfies the inequality*

$$y(t) \le \lambda(t) + \mu(t)\int_{t_0}^{t} \nu(s)y(s)ds, \qquad \forall t \ge t_0 \ge 0\,, \tag{B.44}$$

then

$$y(t) \le \lambda(t) + \mu(t)\int_{t_0}^{t} \lambda(s)\nu(s)e^{\int_s^t \mu(\tau)\nu(\tau)d\tau}ds, \qquad \forall t \ge t_0 \ge 0\,. \tag{B.45}$$

In particular, if $\lambda(t) \equiv \lambda$ *is a constant and* $\mu(t) \equiv 1$, *then*

$$y(t) \le \lambda e^{\int_{t_0}^{t} \nu(\tau)d\tau}, \qquad \forall t \ge t_0 \ge 0\,. \tag{B.46}$$

Appendix C
Lyapunov Stability and ISS for Nonlinear ODEs

C.1 Lyapunov Stability and Class-$\mathscr{K}$ Functions

Consider the nonautonomous ODE system

$$\dot{x} = f(x,t), \tag{C.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is locally Lipschitz in x and piecewise continuous in t.

Definition C.1. The origin $x = 0$ is the equilibrium point for (C.1) if

$$f(0,t) = 0, \qquad \forall t \geq 0. \tag{C.2}$$

Scalar *comparison functions* are important stability tools.

Definition C.2. A continuous function $\gamma : [0,a) \to \mathbb{R}_+$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\gamma(0) = 0$. It is said to belong to class $\mathscr{K}_\infty$ if $a = \infty$ and $\gamma(r) \to \infty$ as $r \to \infty$.

Definition C.3. A continuous function $\beta : [0,a) \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathscr{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathscr{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$. It is said to belong to class $\mathscr{KL}_\infty$ if, in addition, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathscr{K}_\infty$ with respect to r.

The main list of stability definitions for ODE systems is given next.

Definition C.4 (Stability). The equilibrium point $x = 0$ of (C.1) is

- *uniformly stable* if there exist a class-$\mathscr{K}$ function $\gamma(\cdot)$ and a positive constant c, independent of t_0, such that

$$|x(t)| \leq \gamma(|x(t_0)|), \quad \forall t \geq t_0 \geq 0, \ \forall x(t_0) \ \text{s.t.} \ |x(t_0)| < c; \tag{C.3}$$

- *uniformly asymptotically stable* if there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a positive constant c, independent of t_0, such that

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0), \quad \forall t \geq t_0 \geq 0, \ \forall x(t_0) \text{ s.t. } |x(t_0)| < c; \tag{C.4}$$

- *exponentially stable* if (C.4) is satisfied with $\beta(r,s) = kr\mathrm{e}^{-\alpha s}$, $k > 0, \ \alpha > 0$;
- *globally uniformly stable* if (C.3) is satisfied with $\gamma \in \mathcal{K}_\infty$ for any initial state $x(t_0)$;
- *globally uniformly asymptotically stable* if (C.4) is satisfied with $\beta \in \mathcal{KL}_\infty$ for any initial state $x(t_0)$; and
- *globally exponentially stable* if (C.4) is satisfied for any initial state $x(t_0)$ and with $\beta(r,s) = kr\mathrm{e}^{-\alpha s}$, $k > 0, \ \alpha > 0$.

The main Lyapunov stability theorem is then formulated as follows.

Theorem C.1 (Lyapunov theorem). *Let $x = 0$ be an equilibrium point of (C.1) and $D = \{x \in \mathbb{R}^n \mid |x| < r\}$. Let $V : D \times \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable function such that $\forall t \geq 0$, $\forall x \in D$,*

$$\gamma_1(|x|) \leq V(x,t) \leq \gamma_2(|x|), \tag{C.5}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -\gamma_3(|x|). \tag{C.6}$$

Then the equilibrium $x = 0$ is

- uniformly stable *if γ_1 and γ_2 are class-$\mathcal{K}$ functions on $[0,r)$ and $\gamma_3(\cdot) \geq 0$ on $[0,r)$;*
- uniformly asymptotically stable *if γ_1, γ_2,* and *γ_3 are class-$\mathcal{K}$ functions on $[0,r)$;*
- exponentially stable *if $\gamma_i(\rho) = k_i \rho^\alpha$ on $[0,r)$, $k_i > 0$, $\alpha > 0$, $i = 1,2,3$;*
- globally uniformly stable *if $D = \mathbb{R}^n$, γ_1 and γ_2 are class-$\mathcal{K}_\infty$ functions, and $\gamma_3(\cdot) \geq 0$ on $\mathbb{R}_+$;*
- globally uniformly asymptotically stable *if $D = \mathbb{R}^n$, γ_1 and γ_2 are class-$\mathcal{K}_\infty$ functions, and γ_3 is a class-$\mathcal{K}$ function on $\mathbb{R}_+$; and*
- globally exponentially stable *if $D = \mathbb{R}^n$ and $\gamma_i(\rho) = k_i \rho^\alpha$ on $\mathbb{R}_+$, $k_i > 0$, $\alpha > 0$, $i = 1,2,3$.*

In adaptive control our goal is to achieve convergence to a set. For time-invariant systems, the main convergence tool is LaSalle's invariance theorem. For time-varying systems, a more refined tool is the LaSalle–Yoshizawa theorem. For pedagogical reasons, we introduce it via a technical lemma due to Barbalat. These key results and their proofs are of importance in guaranteeing that an adaptive system will fulfill its tracking task.

Lemma C.1 (Barbalat). *Consider the function $\phi : \mathbb{R}_+ \to \mathbb{R}$. If ϕ is uniformly continuous and $\lim_{t\to\infty} \int_0^\infty \phi(\tau) d\tau$ exists and is finite, then*

$$\lim_{t\to\infty} \phi(t) = 0. \tag{C.7}$$

Proof. Suppose that (C.7) does not hold; that is, either the limit does not exist or it is not equal to zero. Then there exists $\varepsilon > 0$ such that for every $T > 0$, one can find $t_1 \geq T$ with $|\phi(t_1)| > \varepsilon$. Since ϕ is uniformly continuous, there is a positive constant $\delta(\varepsilon)$ such that $|\phi(t) - \phi(t_1)| < \varepsilon/2$ for all $t_1 \geq 0$ and all t such that $|t - t_1| \leq \delta(\varepsilon)$. Hence, for all $t \in [t_1, t_1 + \delta(\varepsilon)]$, we have

$$\begin{aligned} |\phi(t)| &= |\phi(t) - \phi(t_1) + \phi(t_1)| \\ &\geq |\phi(t_1)| - |\phi(t) - \phi(t_1)| \\ &> \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}, \end{aligned} \tag{C.8}$$

which implies that

$$\left| \int_{t_1}^{t_1+\delta(\varepsilon)} \phi(\tau) d\tau \right| = \int_{t_1}^{t_1+\delta(\varepsilon)} |\phi(\tau)| d\tau > \frac{\varepsilon\delta(\varepsilon)}{2}, \tag{C.9}$$

where the first equality holds since $\phi(t)$ does not change sign on $[t_1, t_1 + \delta(\varepsilon)]$. Noting that $\int_0^{t_1+\delta(\varepsilon)} \phi(\tau)d\tau = \int_0^{t_1} \phi(\tau)d\tau + \int_{t_1}^{t_1+\delta(\varepsilon)} \phi(\tau)d\tau$, we conclude that $\int_0^t \phi(\tau)d\tau$ cannot converge to a finite limit as $t \to \infty$, which contradicts the assumption of the lemma. Thus, $\lim_{t\to\infty} \phi(t) = 0$. □

Corollary C.1. *Consider the function* $\phi : \mathbb{R}_+ \to \mathbb{R}$. *If* $\phi, \dot{\phi} \in \mathscr{L}_\infty$, *and* $\phi \in \mathscr{L}_p$ *for some* $p \in [1, \infty)$, *then*

$$\lim_{t\to\infty} \phi(t) = 0. \tag{C.10}$$

Theorem C.2 (LaSalle–Yoshizawa). *Let* $x = 0$ *be an equilibrium point of (C.1) and suppose* f *is locally Lipschitz in* x *uniformly in* t. *Let* $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuously differentiable function such that*

$$\gamma_1(|x|) \leq V(x,t) \leq \gamma_2(|x|), \tag{C.11}$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -W(x) \leq 0, \tag{C.12}$$

$\forall t \geq 0$, $\forall x \in \mathbb{R}^n$, *where* γ_1 *and* γ_2 *are class-*$\mathscr{K}_\infty$ *functions and* W *is a continuous function. Then all solutions of (C.1) are globally uniformly bounded and satisfy*

$$\lim_{t\to\infty} W(x(t)) = 0. \tag{C.13}$$

In addition, if $W(x)$ *is positive definite, then the equilibrium* $x = 0$ *is globally uniformly asymptotically stable.*

Proof. Since $\dot{V} \leq 0$, V is nonincreasing. Thus, in view of the first inequality in (C.11), we conclude that x is globally uniformly bounded, that is, $|x(t)| \leq B$, $\forall t \geq 0$. Since $V(x(t),t)$ is nonincreasing and bounded from below by zero, we conclude that it has a limit V_∞ as $t \to \infty$. Integrating (C.12), we have

$$
\begin{aligned}
\lim_{t\to\infty}\int_{t_0}^{t} W(x(\tau))d\tau &\le -\lim_{t\to\infty}\int_{t_0}^{t} \dot{V}(x(\tau),\tau)d\tau \\
&= \lim_{t\to\infty}\{V(x(t_0),t_0)-V(x(t),t)\} \\
&= V(x(t_0),t_0)-V_\infty\,, \qquad \text{(C.14)}
\end{aligned}
$$

which means that $\int_{t_0}^{\infty} W(x(\tau))d\tau$ exists and is finite. Now we show that $W(x(t))$ is also uniformly continuous. Since $|x(t)| \le B$ and f is locally Lipschitz in x uniformly in t, we see that for any $t \ge t_0 \ge 0$,

$$
\begin{aligned}
|x(t)-x(t_0)| = \left|\int_{t_0}^{t} f(x(\tau),\tau)d\tau\right| &\le L\int_{t_0}^{t}|x(\tau)|d\tau \\
&\le LB|t-t_0|\,, \qquad \text{(C.15)}
\end{aligned}
$$

where L is the Lipschitz constant of f on $\{|x| \le B\}$. Choosing $\delta(\varepsilon) = \varepsilon/LB$, we have

$$
|x(t)-x(t_0)| < \varepsilon, \qquad \forall\ |t-t_0| \le \delta(\varepsilon)\,, \qquad \text{(C.16)}
$$

which means that $x(t)$ is uniformly continuous. Since W is continuous, it is uniformly continuous on the compact set $\{|x| \le B\}$. From the uniform continuity of $W(x)$ and $x(t)$, we conclude that $W(x(t))$ is uniformly continuous. Hence, it satisfies the conditions of Lemma C.1, which then guarantees that $W(x(t)) \to 0$ as $t \to \infty$.

If, in addition, $W(x)$ is positive definite, there exists a class-$\mathcal{K}$ function $\gamma_3(\cdot)$ such that $W(x) \ge \gamma_3(|x|)$. Using Theorem C.1, we conclude that $x = 0$ is globally uniformly asymptotically stable. □

In applications, we usually have $W(x) = x^{\mathrm{T}}Qx$, where Q is a symmetric positive-semidefinite matrix. For this case, the proof of Theorem C.2 simplifies using Corollary C.1 with $p = 1$.

C.2 Input-to-State Stability

Input-to-state stability introduced by Sontag plays a crucial role in the analysis of nonlinear predictor feedback design.

Definition C.5 (ISS). The system

$$
\dot{x} = f(t,x,u)\,, \qquad \text{(C.17)}
$$

where f is piecewise continuous in t and locally Lipschitz in x and u, is said to be *input-to-state stable* (ISS) if there exist a class-$\mathcal{KL}$ function β and a class-$\mathcal{K}$ function γ such that for any $x(0)$ and for any input $u(\cdot)$ continuous and bounded on $[0,\infty)$, the solution exists for all $t \ge 0$ and satisfies

$$|x(t)| \le \beta(|x(t_0)|, t-t_0) + \gamma\left(\sup_{t_0 \le \tau \le t} |u(\tau)|\right) \tag{C.18}$$

for all t_0 and t such that $0 \le t_0 \le t$.

The following theorem establishes the connection between the existence of a Lyapunov-like function and the input-to-state stability.

Theorem C.3. *Suppose that for the system (C.17), there exists a C^1 function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,*

$$\gamma_1(|x|) \le V(t,x) \le \gamma_2(|x|), \tag{C.19}$$

$$|x| \ge \rho(|u|) \quad \Rightarrow \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \le -\gamma_3(|x|), \tag{C.20}$$

where γ_1, γ_2, and ρ are class-$\mathcal{K}_\infty$ functions and γ_3 is a class-$\mathcal{K}$ function. Then system (C.17) is ISS with $\gamma = \gamma_1^{-1} \circ \gamma_2 \circ \rho$.

Proof. (Outline) If $x(t_0)$ is in the set

$$R_{t_0} = \left\{ x \in \mathbb{R}^n \,\middle|\, |x| \le \rho\left(\sup_{\tau \ge t_0} |u(\tau)|\right) \right\}, \tag{C.21}$$

then $x(t)$ remains within the set

$$S_{t_0} = \left\{ x \in \mathbb{R}^n \,\middle|\, |x| \le \gamma_1^{-1} \circ \gamma_2 \circ \rho\left(\sup_{\tau \ge t_0} |u(\tau)|\right) \right\} \tag{C.22}$$

for all $t \ge t_0$. Define $B = [t_0, T)$ as the time interval before $x(t)$ enters R_{t_0} for the first time. In view of the definition of R_{t_0}, we have

$$\dot{V} \le -\gamma_3 \circ \gamma_2^{-1}(V), \qquad \forall t \in B. \tag{C.23}$$

Then there exists a class-$\mathcal{KL}$ function β_V such that

$$V(t) \le \beta_V(V(t_0), t-t_0), \qquad \forall t \in B, \tag{C.24}$$

which implies

$$|x(t)| \le \gamma_1^{-1}(\beta_V(\gamma_2(|x(t_0)|), t-t_0)) \stackrel{\triangle}{=} \beta(|x(t_0)|, t-t_0), \qquad \forall t \in B. \tag{C.25}$$

On the other hand, by (C.22), we conclude that

$$|x(t)| \le \gamma_1^{-1} \circ \gamma_2 \circ \rho\left(\sup_{\tau \ge t_0} |u(\tau)|\right) \stackrel{\triangle}{=} \gamma\left(\sup_{\tau \ge t_0} |u(\tau)|\right), \quad \forall t \in [t_0, \infty] \setminus B. \tag{C.26}$$

Then, by (C.25) and (C.26),

$$|x(t)| \leq \beta(|x(t_0)|, t-t_0) + \gamma\left(\sup_{\tau \geq t_0} |u(\tau)|\right), \qquad \forall t \geq t_0 \geq 0. \tag{C.27}$$

By causality, it follows that

$$|x(t)| \leq \beta(|x(t_0)|, t-t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} |u(\tau)|\right), \qquad \forall t \geq t_0 \geq 0. \tag{C.28}$$

□

A function V satisfying the conditions of Theorem C.3 is called an *ISS–Lyapunov function*. The inverse of Theorem C.3 is introduced next (stated here only for the time-invariant case for notational compactness and because we don't need the time-varying case in this book), and an equivalent dissipativity-type characterization of ISS is also introduced.

Theorem C.4 (Lyapunov characterization of ISS). *For the system*

$$\dot{x} = f(x,u),$$

the following properties are equivalent:

1. *the system is ISS;*
2. *there exists a smooth ISS–Lyapunov function;*
3. *there exist a smooth, positive-definite, radially unbounded function V and class-$\mathcal{K}_\infty$ functions ρ_1 and ρ_2 such that the following* dissipativity *inequality is satisfied:*
$$\frac{\partial V}{\partial x} f(x,u) \leq -\rho_1(|x|) + \rho_2(|u|).$$

The following lemma establishes a useful property that a cascade of two ISS systems is itself ISS.

Lemma C.2. *Suppose that in the system*

$$\dot{x}_1 = f_1(t, x_1, x_2, u), \tag{C.29}$$

$$\dot{x}_2 = f_2(t, x_2, u), \tag{C.30}$$

the x_1-subsystem is ISS with respect to x_2 and u, and the x_2-subsystem is ISS with respect to u; that is,

$$|x_1(t)| \leq \beta_1(|x_1(s)|, t-s) + \gamma_1\left(\sup_{s \leq \tau \leq t} \{|x_2(\tau)| + |u(\tau)|\}\right), \tag{C.31}$$

$$|x_2(t)| \leq \beta_2(|x_2(s)|, t-s) + \gamma_2\left(\sup_{s \leq \tau \leq t} |u(\tau)|\right), \tag{C.32}$$

where β_1 *and* β_2 *are class-*$\mathscr{KL}$ *functions and* γ_1 *and* γ_2 *are class-*$\mathscr{K}$ *functions. Then the complete* $x = (x_1, x_2)$*-system is ISS with*

$$|x(t)| \le \beta(|x(s)|, t-s) + \gamma\left(\sup_{s\le\tau\le t} |u(\tau)|\right), \tag{C.33}$$

where

$$\begin{aligned}\beta(r,t) = {} & \beta_1(2\beta_1(r,t/2) + 2\gamma_1(2\beta_2(r,0)), t/2) \\ & + \gamma_1(2\beta_2(r,t/2)) + \beta_2(r,t),\end{aligned} \tag{C.34}$$

$$\gamma(r) = \beta_1(2\gamma_1(2\gamma_2(r) + 2r), 0) + \gamma_1(2\gamma_2(r) + 2r) + \gamma_2(r). \tag{C.35}$$

Proof. With $(s,t) = (t/2, t)$, (C.31) is rewritten as

$$|x_1(t)| \le \beta_1(|x_1(t/2)|, t/2) + \gamma_1\left(\sup_{t/2\le\tau\le t} \{|x_2(\tau)| + |u(\tau)|\}\right). \tag{C.36}$$

From (C.32), we have

$$\begin{aligned}\sup_{t/2\le\tau\le t} |x_2(\tau)| & \le \sup_{t/2\le\tau\le t} \left\{\beta_2(|x_2(0)|, \tau) + \gamma_2\left(\sup_{0\le\sigma\le\tau} |u(\sigma)|\right)\right\} \\ & \le \beta_2(|x_2(0)|, t/2) + \gamma_2\left(\sup_{0\le\tau\le t} |u(\tau)|\right),\end{aligned} \tag{C.37}$$

and from (C.31), we obtain

$$\begin{aligned}|x_1(t/2)| & \le \beta_1(|x_1(0)|, t/2) + \gamma_1\left(\sup_{0\le\tau\le t/2} \{|x_2(\tau)| + |u(\tau)|\}\right) \\ & \le \beta_1(|x_1(0)|, t/2) \\ & \quad + \gamma_1\left(\sup_{0\le\tau\le t/2} \left\{\beta_2(|x_2(0)|, \tau) + \gamma_2\left(\sup_{0\le\sigma\le\tau} |u(\sigma)|\right) + |u(\tau)|\right\}\right) \\ & \le \beta_1(|x_1(0)|, t/2) \\ & \quad + \gamma_1\left(\beta_2(|x_2(0)|, 0) + \sup_{0\le\tau\le t/2} \{\gamma_2(|u(\tau)|) + |u(\tau)|\}\right) \\ & \le \beta_1(|x_1(0)|, t/2) + \gamma_1(2\beta_2(|x_2(0)|, 0)) \\ & \quad + \gamma_1\left(2 \sup_{0\le\tau\le t/2} \{\gamma_2(|u(\tau)|) + |u(\tau)|\}\right),\end{aligned} \tag{C.38}$$

where in the last inequality we have used the fact that $\delta(a+b) \le \delta(2a) + \delta(2b)$ for any class-$\mathscr{K}$ function δ and any nonnegative a and b. Then, substituting (C.37) and (C.38) into (C.36), we get

$$
\begin{aligned}
|x_1(t)| \le \beta_1 &(\beta_1(|x_1(0)|, t/2) + \gamma_1(2\beta_2(|x_2(0)|, 0)) \\
&+ \gamma_1\left(2 \sup_{0\le\tau\le t/2} \{\gamma_2(|u(\tau)|) + |u(\tau)|\}\right)\Bigg) \\
&+ \gamma_1\left(\beta_2(|x_2(0)|, t/2) + \gamma_2\left(\sup_{0\le\tau\le t} |u(\tau)|\right) + \sup_{t/2\le\tau\le t} \{|u(\tau)|\}\right) \\
\le\ & \beta_1(2\beta_1(|x_1(0)|, t/2) + 2\gamma_1(2\beta_2(|x_2(0)|, 0)), t/2) \\
&+ \gamma_1(2\beta_2(|x_2(0)|, t/2)) \\
&+ \beta_1\left(2\gamma_1\left(2 \sup_{0\le\tau\le t} \{\gamma_2(|u(\tau)|) + |u(\tau)|\}\right), 0\right) \\
&+ \gamma_1\left(2 \sup_{0\le\tau\le t} \{\gamma_2(|u(\tau)|) + |u(\tau)|\}\right). \qquad \text{(C.39)}
\end{aligned}
$$

Combining (C.39) and (C.32), we arrive at (C.33) with (C.34)–(C.35). □

Since (C.29) and (C.30) are ISS, then there exist ISS–Lyapunov functions V_1 and V_2 and class-$\mathcal{K}_\infty$ functions $\alpha_1, \rho_1, \alpha_2$, and ρ_2 such that

$$\frac{\partial V_1}{\partial x_1} f_1(t, x_1, x_2, u) \le -\alpha_1(|x_1|) + \rho_1(|x_2|) + \rho_1(|u|), \qquad \text{(C.40)}$$

$$\frac{\partial V_2}{\partial x_2} f_2(t, x_2, u) \le -\alpha_2(|x_2|) + \rho_2(|u|). \qquad \text{(C.41)}$$

The functions $V_1, V_2, \alpha_1, \rho_1, \alpha_2$, and ρ_2 can *always* be found such that

$$\rho_1 = \alpha_2/2. \qquad \text{(C.42)}$$

Then the ISS–Lyapunov function for the complete system (C.29)–(C.30) can be defined as

$$V(x) = V_1(x_1) + V_2(x_2), \qquad \text{(C.43)}$$

and its derivative

$$\dot{V} \le -\alpha_1(|x_1|) - \frac{1}{2}\alpha_2(|x_2|) + \rho_1(|u|) + \rho_2(|u|) \qquad \text{(C.44)}$$

establishes the ISS property of (C.29)–(C.30) by part 3 of Theorem C.4.

In some applications of input-to-state stability, the following lemma is useful, as it is much simpler than Theorem C.3.

Lemma C.3. *Let v and ρ be real-valued functions defined on $\mathbb{R}_+$, and let b and c be positive constants. If they satisfy the differential inequality*

$$\dot{v} \le -cv + b\rho(t)^2, \qquad v(0) \ge 0, \qquad \text{(C.45)}$$

then the following hold:

(i) If $\rho \in \mathscr{L}_\infty$, then $v \in \mathscr{L}_\infty$ and

$$v(t) \leq v(0)\mathrm{e}^{-ct} + \frac{b}{c}\|\rho\|_\infty^2. \tag{C.46}$$

(ii) If $\rho \in \mathscr{L}_2$, then $v \in \mathscr{L}_\infty$ and

$$v(t) \leq v(0)\mathrm{e}^{-ct} + b\|\rho\|_2^2. \tag{C.47}$$

Proof. (i) From Lemma B.2, we have

$$\begin{aligned} v(t) &\leq v(0)\mathrm{e}^{-ct} + b\int_0^t \mathrm{e}^{-c(t-\tau)}\rho(\tau)^2 d\tau \\ &\leq v(0)\mathrm{e}^{-ct} + b\sup_{\tau\in[0,t]}\{\rho(\tau)^2\}\int_0^t \mathrm{e}^{-c(t-\tau)}d\tau \\ &\leq v(0)\mathrm{e}^{-ct} + b\|\rho\|_\infty^2\frac{1}{c}\left(1-\mathrm{e}^{-ct}\right) \\ &\leq v(0)\mathrm{e}^{-ct} + \frac{b}{c}\|\rho\|_\infty^2. \end{aligned} \tag{C.48}$$

(ii) From (B.9), we have

$$\begin{aligned} v(t) &\leq v(0)\mathrm{e}^{-ct} + b\sup_{\tau\in[0,t]}\left\{\mathrm{e}^{-c(t-\tau)}\right\}\int_0^t \rho(\tau)^2 d\tau \\ &= v(0)\mathrm{e}^{-ct} + b\|\rho\|_2^2. \end{aligned} \tag{C.49}$$

□

Remark C.1. From Lemma C.3, it follows that if

$$\dot{v} \leq -cv + b_1\rho_1(t)^2 + b_2\rho_2(t)^2, \qquad v(0) \geq 0, \tag{C.50}$$

and $\rho_1 \in \mathscr{L}_\infty$ and $\rho_2 \in \mathscr{L}_2$, then $v \in \mathscr{L}_\infty$ and

$$v(t) \leq v(0)\mathrm{e}^{-ct} + \frac{b_1}{c}\|\rho_1\|_\infty^2 + b_2\|\rho_2\|_2^2. \tag{C.51}$$

This, in particular, implies the input-to-state stability with respect to two inputs: ρ_1 and $\|\rho_2\|_2$. ◇

In this book we study feedback design for *forward-complete* systems with input delay.

Definition C.6 (Forward completeness). A system

$$\dot{x} = f(x,u) \tag{C.52}$$

with a locally Lipschitz vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is said to be *forward complete* if, for every initial condition $x(0) = \xi$ and every measurable locally essentially bounded input signal $u : \mathbb{R}_+ \to \mathbb{R}$, the corresponding solution is defined for all $t \geq 0$; i.e., the maximal interval of existence of solutions is $T_{\max} = +\infty$.

The following Lyapunov characterization of forward completeness was proved in [4].

Theorem C.5. *System (C.52) is forward complete if and only if there exist a nonnegative-valued, radially unbounded, smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a class-$\mathscr{K}_\infty$ function σ such that*

$$\frac{\partial V(x)}{\partial x} f(x,u) \leq V(x) + \sigma(|u|) \tag{C.53}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}$.

Appendix D
Bessel Functions

We review the definitions, basic properties, and graphical forms of Bessel functions.

D.1 Bessel Function J_n

The function (depicted in Fig. D.1)

$$y(x) = J_n(x) \tag{D.1}$$

is a solution to the following ODE:

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{D.2}$$

Series representation

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(m+n)!} \tag{D.3}$$

Properties

$$2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x)) \tag{D.4}$$

$$J_n(-x) = (-1)^n J_n(x) \tag{D.5}$$

Differentiation

$$\frac{d}{dx}J_n(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) = \frac{n}{x}J_n(x) - J_{n+1}(x) \tag{D.6}$$

$$\frac{d}{dx}(x^nJ_n(x)) = x^nJ_{n-1}, \qquad \frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1} \tag{D.7}$$

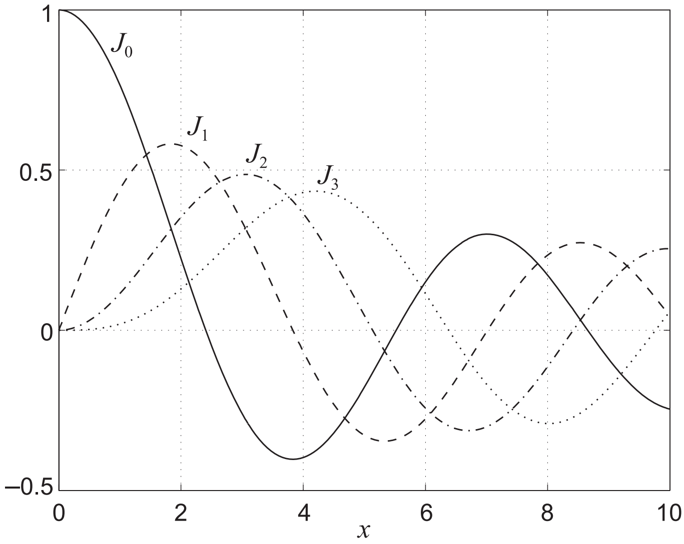

Fig. D.1 Bessel functions J_n.

Asymptotic properties

$$J_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \to 0 \tag{D.8}$$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right), \qquad x \to \infty \tag{D.9}$$

D.2 Modified Bessel Function I_n

The function (depicted in Fig. D.2)

$$y(x) = I_n(x) \tag{D.10}$$

is a solution to the following ODE:

$$x^2 y'' + xy' - (x^2 + n^2)y = 0. \tag{D.11}$$

Series representation

$$I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(m+n)!} \tag{D.12}$$

Relationship with $J_n(x)$

$$I_n(x) = i^{-n} J_n(ix), \qquad I_n(ix) = i^n J_n(x) \tag{D.13}$$

Properties

$$2nI_n(x) = x(I_{n-1}(x) - I_{n+1}(x)) \tag{D.14}$$

$$I_n(-x) = (-1)^n I_n(x) \tag{D.15}$$

Differentiation

$$\frac{d}{dx} I_n(x) = \frac{1}{2}(I_{n-1}(x) + I_{n+1}(x)) = \frac{n}{x} I_n(x) + I_{n+1}(x) \tag{D.16}$$

$$\frac{d}{dx}(x^n I_n(x)) = x^n I_{n-1}, \qquad \frac{d}{dx}(x^{-n} I_n(x)) = x^{-n} I_{n+1} \tag{D.17}$$

Asymptotic properties

$$I_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \to 0 \tag{D.18}$$

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \qquad x \to \infty \tag{D.19}$$

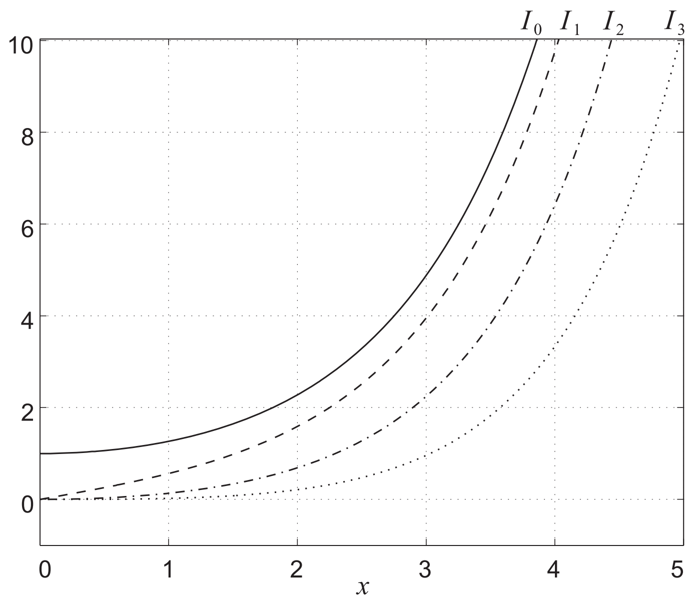

Fig. D.2 Modified Bessel functions I_n.

Appendix E
Parameter Projection

Our adaptive designs rely on the use of parameter projection in our identifiers. We provide a treatment of projection for a general convex parameter set. The treatment for some of our designs where projection is used for only a scalar estimate $\hat{D}$ is easily deduced from the general case.

Let us define the following convex set:

$$\Pi = \left\{ \hat{\theta} \in \mathbb{R}^p \,\middle|\, \mathscr{P}(\hat{\theta}) \leq 0 \right\}, \tag{E.1}$$

where by assuming that the convex function $\mathscr{P} : \mathbb{R}^p \to \mathbb{R}$ is smooth, we ensure that the boundary $\partial\Pi$ of Π is smooth. Let us denote the interior of Π by $\overset{\circ}{\Pi}$ and observe that $\nabla_{\hat{\theta}}\mathscr{P}$ represents an outward normal vector at $\hat{\theta} \in \partial\Pi$. The standard projection operator is

$$\mathrm{Proj}\{\tau\} = \begin{cases} \tau, & \hat{\theta} \in \overset{\circ}{\Pi} \text{ or } \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau \leq 0, \\ \left(I - \Gamma \dfrac{\nabla_{\hat{\theta}}\mathscr{P}\, \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}}{\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\Gamma\nabla_{\hat{\theta}}\mathscr{P}} \right)\tau, & \hat{\theta} \in \partial\Pi \text{ and } \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau > 0, \end{cases} \tag{E.2}$$

where Γ belongs to the set $\mathscr{G}$ of all positive-definite symmetric $p \times p$ matrices. Although Proj is a function of three arguments, τ, $\hat{\theta}$ and Γ, for compactness of notation, we write only $\mathrm{Proj}\{\tau\}$.

The meaning of (E.2) is that when $\hat{\theta}$ is in the interior of Π or at the boundary with τ pointing inward, then $\mathrm{Proj}\{\tau\} = \tau$. When $\hat{\theta}$ is at the boundary with τ pointing outward, then Proj projects τ on the hyperplane tangent to $\partial\Pi$ at $\hat{\theta}$.

In general, the mapping (E.2) is discontinuous. This is undesirable for two reasons. First, the discontinuity represents a difficulty for implementation in continuous time. Second, since the Lipschitz continuity is violated, we cannot use standard theorems for the existence of solutions. Therefore, we sometimes want to smooth the projection operator. Let us consider the following convex set:

$$\Pi_\varepsilon = \left\{ \hat{\theta} \in \mathbb{R}^p \,\middle|\, \mathscr{P}(\hat{\theta}) \leq \varepsilon \right\}, \tag{E.3}$$

which is a union of the set Π and an $O(\varepsilon)$-boundary layer around it. We now modify (E.2) to achieve continuity of the transition from the vector field τ on the boundary of Π to the vector field $\left(I - \Gamma \frac{\nabla_{\hat{\theta}}\mathscr{P}\,\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}}{\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\Gamma\nabla_{\hat{\theta}}\mathscr{P}}\right)\tau$ on the boundary of Π_ε:

$$\mathrm{Proj}\{\tau\} = \begin{cases} \tau, & \hat{\theta} \in \overset{\circ}{\Pi} \text{ or } \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau \leq 0, \\ \left(I - c(\hat{\theta})\Gamma \dfrac{\nabla_{\hat{\theta}}\mathscr{P}\,\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}}{\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\Gamma\nabla_{\hat{\theta}}\mathscr{P}}\right)\tau, & \hat{\theta} \in \Pi_\varepsilon \setminus \overset{\circ}{\Pi} \text{ and } \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau > 0, \end{cases} \tag{E.4}$$

$$c(\hat{\theta}) = \min\left\{1, \frac{\mathscr{P}(\hat{\theta})}{\varepsilon}\right\}. \tag{E.5}$$

It is helpful to note that $c(\partial\Pi) = 0$ and $c(\partial\Pi_\varepsilon) = 1$.

In our proofs of stability of adaptive systems, we use the following technical properties of the projection operator (E.4).

Lemma E.1 (Projection operator). *The following are the properties of the projection operator (E.4):*

(i) *The mapping* $\mathrm{Proj} : \mathbb{R}^p \times \Pi_\varepsilon \times \mathscr{G} \to \mathbb{R}^p$ *is locally Lipschitz in its arguments* $\tau, \hat{\theta}$, *and* Γ.
(ii) $\mathrm{Proj}\{\tau\}^{\mathrm{T}}\Gamma^{-1}\mathrm{Proj}\{\tau\} \leq \tau^{\mathrm{T}}\Gamma^{-1}\tau, \forall\hat{\theta} \in \Pi_\varepsilon$.
(iii) *Let* $\Gamma(t), \tau(t)$ *be continuously differentiable and*

$$\dot{\hat{\theta}} = \mathrm{Proj}\{\tau\}, \qquad \hat{\theta}(0) \in \Pi_\varepsilon.$$

Then, on its domain of definition, the solution $\hat{\theta}(t)$ *remains in* Π_ε.
(iv) $-\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\mathrm{Proj}\{\tau\} \leq -\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tau, \forall\hat{\theta} \in \Pi_\varepsilon, \theta \in \Pi$.

Proof. (i) The proof of this point is lengthy but straightforward and is omitted here.

(ii) For $\hat{\theta} \in \overset{\circ}{\Pi}$ or $\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau \leq 0$, we have $\mathrm{Proj}\{\tau\} = \tau$ and (ii) trivially holds with equality. Otherwise, a direct computation gives

$$\begin{aligned} \mathrm{Proj}\{\tau\}^{\mathrm{T}}\Gamma^{-1}\mathrm{Proj}\{\tau\} &= \tau^{\mathrm{T}}\Gamma^{-1}\tau - 2c(\hat{\theta})\frac{\left(\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau\right)^2}{\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\Gamma\nabla_{\hat{\theta}}\mathscr{P}} + c(\hat{\theta})^2\frac{\left|\nabla_{\hat{\theta}}\mathscr{P}\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau\right|_\Gamma^2}{\left(\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\Gamma\nabla_{\hat{\theta}}\mathscr{P}\right)^2} \\ &= \tau^{\mathrm{T}}\Gamma^{-1}\tau - c(\hat{\theta})\left(2 - c(\hat{\theta})\right)\frac{\left(\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau\right)^2}{\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\Gamma\nabla_{\hat{\theta}}\mathscr{P}} \\ &\leq \tau^{\mathrm{T}}\Gamma^{-1}\tau, \end{aligned} \tag{E.6}$$

where the last inequality follows by noting that $c(\hat{\theta}) \in [0,1]$ for $\hat{\theta} \in \Pi_\varepsilon \setminus \overset{\circ}{\Pi}$.

(iii) Using the definition of the Proj operator, we get

$$\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\mathrm{Proj}\{\tau\} = \begin{cases} \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau, & \hat{\theta} \in \overset{\circ}{\Pi} \text{ or } \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau \leq 0, \\ \left(1-c(\hat{\theta})\right)\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau, & \hat{\theta} \in \Pi_{\varepsilon}\setminus \overset{\circ}{\Pi} \text{ and } \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau > 0, \end{cases} \tag{E.7}$$

which, in view of the fact that $c(\hat{\theta}) \in [0,1]$ for $\hat{\theta} \in \Pi_{\varepsilon}\setminus \overset{\circ}{\Pi}$, implies that

$$\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\mathrm{Proj}\{\tau\} \leq 0 \;\text{ whenever }\; \hat{\theta} \in \partial\Pi_{\varepsilon}\,; \tag{E.8}$$

that is, the vector $\mathrm{Proj}\{\tau\}$ either points inside Π_{ε} or is tangential to the hyperplane of $\partial\Pi_{\varepsilon}$ at $\hat{\theta}$. Since $\hat{\theta}(0) \in \Pi_{\varepsilon}$, it follows that $\hat{\theta}(t) \in \Pi_{\varepsilon}$ as long as the solution exists.

(iv) For $\hat{\theta} \in \overset{\circ}{\Pi}$, (iv) trivially holds with equality. For $\hat{\theta} \in \Pi_{\varepsilon}\setminus \overset{\circ}{\Pi}$, since $\theta \in \Pi$ and $\mathscr{P}$ is a convex function, we have

$$(\theta - \hat{\theta})^{\mathrm{T}}\nabla_{\hat{\theta}}\mathscr{P} \leq 0 \;\text{ whenever }\; \hat{\theta} \in \Pi_{\varepsilon}\setminus \overset{\circ}{\Pi}\,. \tag{E.9}$$

With (E.9), we now calculate

$$\begin{aligned} -\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\mathrm{Proj}\{\tau\} &= -\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tau \\ &\quad + \begin{cases} 0, & \hat{\theta} \in \overset{\circ}{\Pi} \text{ or } \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau \leq 0 \\ c(\hat{\theta})\dfrac{\left(\tilde{\theta}^{\mathrm{T}}\nabla_{\hat{\theta}}\mathscr{P}\right)\left(\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau\right)}{\nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\Gamma\nabla_{\hat{\theta}}\mathscr{P}}, & \hat{\theta} \in \Pi_{\varepsilon}\setminus \overset{\circ}{\Pi} \text{ and } \\ & \nabla_{\hat{\theta}}\mathscr{P}^{\mathrm{T}}\tau > 0 \end{cases} \\ &\leq -\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tau\,, \end{aligned} \tag{E.10}$$

which completes the proof. □

Appendix F
Strict-Feedforward Systems: A General Design

In this appendix and the next two appendices we give an extensive review of tools for the design of explicitly computable feedback laws for the stabilization of strict-feedforward systems. The emphasis is on a subclass of strict-feedforward systems that are feedback linearizable. This property yields a closed-form expression for the inverse backstepping transformation for the infinite-dimensional actuator state in predictor-based feedback laws. In addition, for *all* strict-feedforward systems, the direct backstepping transformation is obtainable in closed form, making them the most interesting class of systems from the point of view of predictor-based feedback design.

F.1 The Class of Systems

Consider the class of *strict-feedforward systems*

$$\begin{aligned}
\dot{x}_1 &= x_2 + \psi_1(x_2, x_3, \ldots, x_n) + \phi_1(x_2, x_3, \ldots, x_n)u, \\
\dot{x}_2 &= x_3 + \psi_2(x_3, \ldots, x_n) + \phi_2(x_3, \ldots, x_n)u, \\
&\vdots \\
\dot{x}_{n-2} &= x_{n-1} + \psi_{n-2}(x_{n-1}, x_n) + \phi_{n-2}(x_{n-1}, x_n)u, \\
\dot{x}_{n-1} &= x_n + \phi_{n-1}(x_n)u, \\
\dot{x}_n &= u,
\end{aligned} \tag{F.1}$$

or, for short,

$$\dot{x}_i = x_{i+1} + \psi_i(\underline{x}_{i+1}) + \phi_i(\underline{x}_{i+1})u, \qquad i = 1, 2, \ldots, n, \tag{F.2}$$

where

$$\underline{x}_j = [x_j,\ x_{j+1}, \ldots, x_n]^T, \tag{F.3}$$

$$x_{n+1} = u, \tag{F.4}$$

$$\phi_n = 1, \tag{F.5}$$

$$\phi_i(0) = 0, \tag{F.6}$$

$$\psi_i(x_{i+1}, 0, \ldots, 0) \equiv 0, \tag{F.7}$$

$$\frac{\partial \psi_i(0)}{\partial x_j} = 0 \tag{F.8}$$

for $i = 1, 2, \ldots, n-1$, $j = i+1, \ldots, n$.

Relative to the class of systems in [195], we make a trade of generality for conceptual clarity by requiring that the drift term be of the form $x_{i+1} + \psi_i(\underline{x}_{i+1})$, where the ψ_i's, in addition to being higher-order, vanish whenever $x_{i+2}, \ldots, x_n$ vanish. At the end of Section H.1 we show that this restriction can be relaxed in some cases; however, we keep it throughout most of the present appendix and the next two appendices for notational and conceptual convenience. We note that condition (F.7) means, in particular, that $\psi_n = 0$ and $\psi_{n-1}(x_n) \equiv 0$.

F.2 The Sepulchre–Jankovic–Kokotovic Algorithm

The control law for this class of systems is designed as follows. Let

$$\beta_{n+1} = 0, \tag{F.9}$$

$$\alpha_{n+1} = 0\,. \tag{F.10}$$

For $i = n, n-1, \ldots, 2, 1$, the designer needs to symbolically (preferably) or numerically calculate

$$z_i = x_i - \beta_{i+1}, \tag{F.11}$$

$$w_i(\underline{x}_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial \beta_{i+1}}{\partial x_j}\phi_j - \frac{\partial \beta_{i+1}}{\partial x_n}, \tag{F.12}$$

$$\alpha_i(\underline{x}_i) = \alpha_{i+1} - w_i z_i, \tag{F.13}$$

$$\begin{aligned}\beta_i(\underline{x}_i) = -\int_0^\infty &\Big[\xi_i^{[i]}(\tau, \underline{x}_i) + \psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right) \\ &+ \phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right)\alpha_i\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right)\Big]\,d\tau,\end{aligned} \tag{F.14}$$

where the notation in the integrand of (F.14) refers to the solutions of the (sub)system(s)

$$\frac{d}{d\tau}\xi_j^{[i]} = \xi_{j+1}^{[i]} + \psi_j\left(\underline{\xi}_{j+1}^{[i]}\right) + \phi_j\left(\underline{\xi}_{j+1}^{[i]}\right)\alpha_i\left(\underline{\xi}_i^{[i]}\right) \tag{F.15}$$

for $j = i, i+1, \ldots, n$, at time τ, starting from the initial condition $\underline{x}_i$. The control law is

$$u = \alpha_1 \,. \tag{F.16}$$

It is important to first understand the meaning of the integral in (F.14). Clearly, the solution $\underline{\xi}_i(\tau, \underline{x}_i)$ is impossible to obtain analytically in general but, when possible, will lead to an implementable control law. Note that the last of the β_i's that needs to be computed is β_2 (β_1 is not defined).

The stability analysis of the closed-loop system is straightforward. Starting with the observation that

$$x_{i+1} + \psi_i + \phi_i \alpha_{i+1} = \sum_{j=i+1}^{n} \frac{\partial \beta_{i+1}}{\partial x_j} \left(x_{j+1} + \psi_j + \phi_j \alpha_{i+1} \right), \tag{F.17}$$

it is easy to verify that

$$\dot{z}_i = w_i \left(u + \sum_{j=i+1}^{n} w_j z_j \right) . \tag{F.18}$$

Noting from (F.16) and (F.13) that

$$u = -\sum_{i=1}^{n} w_i z_i \,, \tag{F.19}$$

we get

$$\dot{z}_i = -w_i^2 z_i - \sum_{j=1}^{i-1} w_i w_j z_j \tag{F.20}$$

(note that this notation implies that $\dot{z}_1 = -w_1^2 z_1$). Taking the Lyapunov function

$$V = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \,, \tag{F.21}$$

one obtains

$$\dot{V} = -\frac{1}{2} \sum_{i=1}^{n} w_i^2 z_i^2 - \frac{1}{2} \left(\sum_{i=1}^{n} z_i w_i \right)^2 . \tag{F.22}$$

Theorem F.1 ([195]). *The feedback system (F.2), (F.16) is globally asymptotically stable at the origin.*

Proof. Although the proof of this theorem is available in [195], we provide some of its elements here for two reasons—one is to ease a nonexpert reader into the topic of forwarding, and the other is that some of our further arguments mimic those used in the proof of this theorem (and we shall not repeat them). First, a careful inspection of the design algorithm reveals that

$$\beta_i(0) = 0 \,, \tag{F.23}$$

which means that the triangular coordinate transformation $z(x)$ is a global diffeomorphism with

$$z(0) = 0\,. \tag{F.24}$$

From (F.22), it then follows that the equilibrium $x = 0$ is globally stable. LaSalle's theorem guarantees that $z_i w_i \to 0$ as $t \to \infty$. Since $w_n \equiv 1$ and $z_n = x_n$, it follows that $x_n(t) \to \infty$. One can verify recursively that $w_i(0) = 1$ for all i [this is a consequence of the fact that $x_{n+1} = u$ and of the presence of the linear term $\xi_i^{[i]}$ in (F.14)]. Thus, it follows that $w_{n-1}(x_n(t)) \to 1$, which, along with $\beta_n(0) = 0$, implies that $x_{n-1}(t) \to \infty$. Continuing in this fashion, one recursively shows that $w_i(t) \to 1, \beta_{i+1}(t) \to 0$ for each i and, thus, that $x(t) \to 0$ as $t \to \infty$. □

Appendix G
Strict-Feedforward Systems: A Linearizable Class

In this appendix we focus on a linearizable subclass of strict-feedforward systems and specialize the SJK algorithm to this class. For linearizable systems, explicit formulas for the control laws are obtained.

G.1 Linearizability of Feedforward Systems

The main interest from the application's point of view is making the forwarding control law explicit, namely, making the closed-form computation of the integral in (F.14) tractable. Toward that end, let us start by noting that system (F.15), which needs to be solved analytically, can be written in the z-coordinates[1] as

$$\frac{d}{d\tau}\zeta_j^{[i]} = -w_j^2\zeta_j^{[i]} - \sum_{l=1}^{j-1} w_j w_l \zeta_l^{[i]}, \qquad j = i, i+1, \ldots, n, \tag{G.1}$$

which is obtained with $\dot{\zeta}_j^{[i]} = w_j \alpha_i$.

Suppose now that (somehow) all of the w_l's were equal to 1 [for all values of their arguments, rather than just $w_l(0) = 1$]. We would have a lower-triangular linear system

$$\frac{d}{d\tau}\zeta_j^{[i]} = -\zeta_j^{[i]} - \sum_{l=1}^{j-1} \zeta_l^{[i]}, \qquad j = i, i+1, \ldots, n, \tag{G.2}$$

which is easily solvable in closed form. Then the only difficulty remaining would be the integration with respect to τ of the integral (F.14) (using an appropriate coordinate change from $\underline{\zeta}_i^{[i]}$ to $\underline{\xi}_i^{[i]}$). Calculating the integral is by no means trivial, but

[1] We point out that, analogous to (F.15), we use ζ, a Greek version of z, to denote the solution of the $\underline{z}_i$-subsystem, under the control α_i, starting from initial condition $\underline{z}_i$. It should also be understood that w_j stands for $w_j\left(\underline{\xi}_{j+1}^{[i]}\right)$, where $\xi_k^{[i]} = \zeta_k^{[i]} + \beta_{k+1}\left(\underline{\xi}_{k+1}^{[i]}\right)$, and so on (i.e., expressing w_j as a function of $\underline{\zeta}_{j+1}^{[i]}$).

it is a much easier task than solving the nonlinear ODE (F.15) *and* calculating the integral.

Before we start exploring the conditions under which one would get

$$w_i(\underline{x}_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial \beta_{i+1}}{\partial x_j}\phi_j - \frac{\partial \beta_{i+1}}{\partial x_n} = 1\,, \tag{G.3}$$

let us note another consequence of this. In this case the coordinate change, before applying the feedback, would yield

$$\dot{z} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & & \vdots \\ \vdots & 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u\,. \tag{G.4}$$

We refer to this as the Teel [218] canonical form. This is a completely controllable linear system. Hence, the systems that satisfy condition (G.3) are linearizable (into this linear form and, ultimately, into the Brunovsky canonical form).

Thus, the exploration of analytical computability of control laws for strict-feedforward systems amounts, to a large extent, to a study of linearizability. Clearly, merely checking the coordinate-free conditions for linearizability [67] won't get us any closer to actually finding the control laws. Such a test would lead to conditions on the ϕ_i's in the form of partial differential equations that they have to satisfy (these conditions would arise from the involutivity test).

Until now we have used the word "linearizable" somewhat loosely. In the next definition we make this notion precise.

Definition G.1. If there exists a diffeomorphism

$$y_i = x_i - \theta_{i+1}(\underline{x}_{i+1})\,, \qquad i = 1,\ldots,n-1\,, \tag{G.5}$$

$$y_n = x_n\,, \tag{G.6}$$

where

$$\theta_i(0) = \frac{\partial \theta_i(0)}{\partial x_j} = 0\,, \qquad i = 2,\ldots,n, \quad j = i,\ldots,n\,, \tag{G.7}$$

transforming the strict-feedforward system (F.2)–(F.7) into a system of the form

$$\dot{y}_i = y_{i+1}\,, \qquad i = 1,2,\ldots,n-1\,, \tag{G.8}$$

$$\dot{y}_n = u\,, \tag{G.9}$$

then system (F.2)–(F.7) is said to be *diffeomorphically equivalent to a chain of integrators (DECI).*

We point out that the term "DECI" does not reflect that (G.5), (G.6) restrict the class of admissible diffeomorphisms to a "triangular" form. In the next theorem we give sufficient conditions for characterizing DECI strict-feedforward systems.

Theorem G.1. *All strict-feedforward systems (F.2)–(F.7) with $\psi_i(\underline{x}_{i+1}), \phi_i(\underline{x}_{i+1})$ that can be written as*

$$\phi_{n-1}(x_n) = \theta_n'(x_n)\,, \tag{G.10}$$

$$\psi_{n-1}(x_n) = 0\,, \tag{G.11}$$

and

$$\phi_i(\underline{x}_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(\underline{x}_{i+1})}{\partial x_j} \phi_j(\underline{x}_{j+1}) + \frac{\partial \theta_{i+1}(\underline{x}_{i+1})}{\partial x_n}\,, \tag{G.12}$$

$$\psi_i(\underline{x}_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(\underline{x}_{i+1})}{\partial x_j} \left(x_{j+1} + \psi_j(\underline{x}_{j+1})\right) - \theta_{i+2}(\underline{x}_{i+2}) \tag{G.13}$$

for $i = n-2, \ldots, 1$, using some C^1 scalar-valued functions $\theta_i(\underline{x}_i)$ satisfying (G.7), are DECI.

Proof. Straightforward to verify using (G.5), (G.6). □

Theorem G.1 is not a substitute for a geometric test of linearizability, nor is it a control design tool. It is just a *parametrization* of a subclass of strict-feedforward systems that are DECI.

For instance, all third-order strict-feedforward systems of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + \frac{\partial \theta_2(x_2,x_3)}{\partial x_2} x_3 - \theta_3(x_3) \\ &\quad + \left(\frac{\partial \theta_2(x_2,x_3)}{\partial x_2} \theta_3'(x_3) + \frac{\partial \theta_2(x_2,x_3)}{\partial x_3} \right) u\,, \end{aligned} \tag{G.14}$$

$$\dot{x}_2 = x_3 + \theta_3'(x_3) u\,, \tag{G.15}$$

$$\dot{x}_3 = u \tag{G.16}$$

are linearizable, where any two locally quadratic C^1 functions $\theta_2(x_2,x_3)$ and $\theta_3(x_3)$ are the "parameters."

In the next section we show that the SJK procedure greatly simplifies for DECI strict-feedforward systems and, in particular, directly leads to (13.96) for (13.91) without having to solve nonlinear ODEs of the form (F.15).

G.2 Algorithms for Linearizable Feedforward Systems

General Algorithm

For linearizable strict-feedforward systems, we present the following design algorithm, which eliminates the requirement to solve the ODEs (F.15) and reduces the problem to calculating a set of integrals with respect to time. Let

$$\beta_{n+1} = 0\,, \tag{G.17}$$

$$\alpha_{n+1} = 0\,. \tag{G.18}$$

For $i = n, n-1, \ldots, 2, 1$,

$$\alpha_i(\underline{x}_i) = -\sum_{j=i}^{n} \left(x_j - \beta_{j+1}(\underline{x}_{j+1})\right)\,, \tag{G.19}$$

$$\xi_n^{[i]}(\tau, \underline{x}_i) = \mathrm{e}^{-\tau} \sum_{k=0}^{n-i} \frac{(-\tau)^k}{k!} \left(x_{n-k} - \beta_{n-k+1}(\underline{x}_{n-k+1})\right)\,, \tag{G.20}$$

$$\xi_j^{[i]}(\tau, \underline{x}_i) = \mathrm{e}^{-\tau} \sum_{k=0}^{j-i} \frac{(-\tau)^k}{k!} \left(x_{j-k} - \beta_{j-k+1}(\underline{x}_{j-k+1})\right) + \beta_{j+1}\left(\xi_{j+1}^{[i]}(\tau, \underline{x}_i)\right)\,,$$

$$j = n-1, \ldots, i+1, i\,, \tag{G.21}$$

$$\beta_i(\underline{x}_i) = -\int_0^\infty \Big[\xi_i^{[i]}(\tau, \underline{x}_i) + \psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right) \tag{G.22}$$

$$+ \phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right) \alpha_i\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i)\right)\Big]\, d\tau\,. \tag{G.23}$$

The control law is

$$u = \alpha_1\,. \tag{G.24}$$

We stress that, due to linearizability, the ODEs (F.15) are solved in closed form, and the only calculation remaining is the integrals (G.23), which can be obtained with symbolic software (coded in Mathematica or Maple/MATLAB). This calculation is particularly straightforward (and can be done, in principle, by hand) when the nonlinearities $\psi_i(\cdot), \phi_i(\cdot)$ are polynomial. In that case, the following identity is useful in calculating (G.23):

$$\int_0^\infty \tau^p \mathrm{e}^{-q\tau} d\tau = \frac{p!}{q^{p+1}}\,, \qquad \forall p, q \in N\,. \tag{G.25}$$

Theorem G.2. *If the strict-feedforward plant (F.2)–(F.7) is DECI, then the feedback system (F.2), (G.24) is globally asymptotically stable at the origin.*

Proof. One can verify that in the coordinates

$$z_i = x_i - \beta_{i+1}(\underline{x}_{i+1}) \tag{G.26}$$

the control system becomes (G.4), and under the feedback control (G.24), the resulting system is

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & & \vdots \\ \vdots & -1 & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ -1 & \cdots & \cdots & -1 & -1 \end{bmatrix} z. \tag{G.27}$$

The rest of the proof is as in Theorem F.1. □

As we indicated in Section G.1, checking the geometric conditions for linearizability is easy, whereas actually constructing the linearizing coordinates is not. The algorithm (G.19)–(G.23) constructs the coordinate change into the (non-Brunovsky) Teel canonical form (G.4). The next theorem gives the coordinate change into the Brunovsky/chain-of-integrators form.

Theorem G.3. *If the strict-feedforward plant (F.2)–(F.7) is DECI, it has a relative degree*[2] *n with respect to the output*

$$y_1 = \sum_{j=1}^{n} \binom{n-1}{j-1} (-1)^{j-1} \left(x_j - \beta_{j+1}(\underline{x}_{j+1})\right). \tag{G.28}$$

Furthermore, the coordinate change (G.19)–(G.23), (G.26), and

$$y_i = \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} z_j, \qquad i = 1, 2, \ldots, n, \tag{G.29}$$

converts system (F.2) into the chain of integrators (G.8)–(G.9).

Proof. By verification. □

Inverse optimality, proved for the general case in [195], becomes particularly meaningful in the linearizable case.

Theorem G.4 (Inverse optimality). *The control law*

$$u^* = 2\alpha_1(x) = -2 \sum_{j=1}^{n} \left(x_j - \beta_{j+1}(\underline{x}_{j+1})\right), \tag{G.30}$$

where $\alpha_1(x)$ is defined via (G.19)–(G.23), minimizes the cost functional

$$J = \int_0^\infty \left(l(x(t)) + u(t)^2\right) dt \tag{G.31}$$

[2] As defined in [67].

along the solutions of (F.2), where

$$l(x) = \sum_{j=1}^{n} \left(x_j - \beta_{j+1}(\underline{x}_{j+1})\right)^2 + \left(\sum_{j=1}^{n} \left(x_j - \beta_{j+1}(\underline{x}_{j+1})\right)\right)^2 \tag{G.32}$$

is a positive-definite, radially unbounded function. Furthermore, the control law (G.30) remains globally asymptotically stabilizing at the origin in the presence of input-unmodeled dynamics of the form

$$a(I + \mathscr{P}), \tag{G.33}$$

where $a \geq 1/2$ is a constant, $\mathscr{P}u$ is the output of any strictly passive nonlinear system[3] with u as its input, and I denotes the identity operator.

Proof. It follows from Theorem 2.8, Theorem 2.17, and Corollary 2.18 in [109]. □

The main result of this section was a control algorithm that eliminates the requirement to solve the ODEs (F.15) and reduces the problem to calculating only the integrals (G.23). In the next two sections we present algorithms that eliminate even the need to calculate the integrals (G.23) for two subclasses of DECI strict-feedforward systems.

Linearizable Feedforward Systems of Type I

Consider the class of strict-feedforward systems given by

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u\,, \tag{G.34}$$

$$\dot{x}_i = x_{i+1}\,, \qquad i = 2, \ldots, n-1\,, \tag{G.35}$$

$$\dot{x}_n = u\,, \tag{G.36}$$

where $\pi_j(0) = 0$. Any system in this class is DECI.

Theorem G.5. *The diffeomorphic transformation*

$$y_1 = x_1 - \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s)ds\,, \tag{G.37}$$

$$y_i = x_i\,, \qquad i = 2, \ldots, n\,, \tag{G.38}$$

converts the strict-feedforward system (G.34)–(G.36) into the chain of integrators (G.8)–(G.9). The feedback law

[3] With possibly nonzero initial conditions.

$$u = \alpha_1(x) = -\sum_{i=1}^{n} \binom{n}{i-1} y_i \tag{G.39}$$

globally asymptotically stabilizes the origin of (G.34)–(G.36).

Proof. The first part is by verification. In the second part we note that the y-system has n closed-loop poles at -1 and use that fact that the coordinate change is diffeomorphic. □

We note that in the design (G.37), (G.38), and (G.39) we have completely circumvented the SJK procedure. It is therefore worth noting that, following the SJK procedure, one would have obtained

$$\alpha_i(\underline{x}_i) = -\sum_{j=i}^{n} \binom{n-i+1}{j-i} x_j + \delta_{i,1} \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s)ds\,, \tag{G.40}$$

$$w_i = 1\,, \tag{G.41}$$

where $\delta_{i,1}$ denotes the Kronecker delta.[4] However, the most important product of the SJK procedure is the coordinate shift β_i (from x to z), which is given in the context of the following result.

Corollary G.1. *The control law (G.30), with $\alpha_1(x)$ defined in (G.39), applied to the plant (G.34)–(G.36) achieves the result of Theorem G.4 with*

$$\beta_{i+1}(\underline{x}_{i+1}) = -\sum_{j=i+1}^{n} \binom{n-i}{j-i} x_j + \delta_{i,1} \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s)ds \tag{G.42}$$

for $i = 1, \ldots, n-1$.

While in Section G.2 we showed that one can avoid having to solve the nonlinear ODEs (F.15), in Theorem G.5 we showed that for the feedforward subclass (G.34)–(G.36), one can also avoid having to calculate the integrals (G.23). In the next result we go even further and show that not only does one have a closed-form formula for the control law (G.39), but one can even get a closed-form formula for the *solutions* of the system under that control law. This is not just an aesthetically pleasing result—it will allow us, in Section H.1, to extend the constructive methodology to a class of strict-feedforward systems that are *not linearizable*.

To prevent confusion about the notation in the theorem, before its statement we emphasize that x, which denotes the initial condition, is constant. This notation is important for a seamless use of the theorem in subsequent results. We also point out that, relative to the notation in Sections F.2 and G.1, $\xi(\tau,x)$ and $\zeta(\tau,z)$ should be understood, respectively, as $\xi^{[1]}(\tau,x)$ and $\zeta^{[1]}(\tau,z)$.

Lemma G.1. *Starting from the initial condition denoted by x, the solution $\xi_i(\tau,x)$ of the feedback system (G.34)–(G.36), (G.37)–(G.39) at time τ is*

[4] Note that (G.40) for $i = 1$ is the same as (G.39).

$$\xi_i(\tau,x) = \mathrm{e}^{-\tau}\left[\sum_{j=i}^{n}\binom{n-i}{j-i}(-1)^{j-i}\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}x_l\right.$$
$$\left. + (-1)^i\sum_{j=i}^{n}\binom{n-i}{j-i}\frac{\tau^{j-1}}{(j-1)!}\left(\sum_{m=2}^{n}\int_0^{x_m}\pi_m(s)ds\right)\right],$$
$$i = 2,\ldots,n, \tag{G.43}$$

for $i = 2,\ldots,n$, and

$$\xi_1(\tau,x) = \mathrm{e}^{-\tau}\left[\sum_{j=1}^{n}\binom{n-1}{j-1}(-1)^{j-1}\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}x_l\right.$$
$$\left. - \sum_{j=1}^{n}\binom{n-1}{j-1}\frac{\tau^{j-1}}{(j-1)!}\left(\sum_{m=2}^{n}\int_0^{x_m}\pi_m(s)ds\right)\right]$$
$$+ \sum_{j=2}^{n}\int_0^{\xi_j(\tau,x)}\pi_j(s)ds, \tag{G.44}$$

whereas the control signal is

$$u = \tilde{\alpha}_1(\tau,x) = -\mathrm{e}^{-\tau}\sum_{i=1}^{n}\binom{n}{i-1}\left[\sum_{j=i}^{n}\binom{n-i}{j-i}(-1)^{j-i}\right.$$
$$\times\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}x_l$$
$$\left. + (-1)^i\sum_{j=i}^{n}\binom{n-i}{j-i}\frac{\tau^{j-1}}{(j-1)!}\left(\sum_{m=2}^{n}\int_0^{x_m}\pi_m(s)ds\right)\right]. \tag{G.45}$$

Proof. By using (G.37), (G.38), their inverse,

$$x_1 = y_1 - \sum_{j=2}^{n}\int_0^{y_j}\pi_j(s)ds, \tag{G.46}$$

$$x_i = y_i, \qquad i = 2,\ldots,n, \tag{G.47}$$

the transformation (G.29), and its inverse,

$$z_i = \sum_{j=i}^{n}\binom{n-i}{j-i}y_j, \qquad i = 1,2,\ldots,n, \tag{G.48}$$

the explicit form of the solution of (G.27),

$$\zeta_j(\tau,z) = \sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\mathrm{e}^{-\tau}z_{j-k}, \tag{G.49}$$

and (G.39). □

Linearizable Feedforward Systems of Type II

Consider the subclass of the strict-feedforward systems (F.2) given by

$$\dot{x}_i = x_{i+1} + \phi_i(\underline{x}_{i+1})u\,, \qquad i = 1,\ldots,n-1\,, \tag{G.50}$$

$$\dot{x}_n = u\,, \tag{G.51}$$

where $\phi_i(0) = 0$. In this section we construct control laws for a linearizable *subclass* of (G.50), (G.51).

To characterize the linearizable subclass, let us consider the functions $\phi_{n-1}(x_n)$ and $\phi_i(0,\ldots,0,x_n)$, $i = 1,\ldots,n-2$, as given and introduce the following sequence of functions:

$$\mu_n(x_n) = \frac{\int_0^{x_n}\phi_{n-1}(s)ds}{x_n}\,, \tag{G.52}$$

$$\mu_i(x_n) = \frac{1}{x_n}\int_0^{x_n}\left[\phi_{i-1}(0,\ldots,0,s) - \sum_{j=i+1}^{n}\mu_j(s)\phi_{i+n-j}(0,\ldots,0,s)\right]ds \tag{G.53}$$

for $i = n-1, n-2, \ldots, 2$, and

$$\gamma_1(x_n) = \mu_n'(x_n)\,, \tag{G.54}$$

$$\gamma_k(x_n) = \sum_{l=1}^{k-1}\gamma_l(x_n)\mu_{l+n+1-k}(x_n) + \frac{d\mu_{n+1-k}(x_n)}{dx_n} \tag{G.55}$$

for $k = 2,\ldots,n-2$.

Theorem G.6. *If*

$$\phi_i(\underline{x}_{i+1}) = \sum_{j=i+1}^{n-1}\gamma_{j-i}(x_n)x_j + \phi_i(0,\ldots,0,x_n)\,, \tag{G.56}$$

$\forall x, i = 1,\ldots,n-2$*, then the diffeomorphic transformation*

$$y_i = x_i - \sum_{j=i+1}^{n}\mu_{i+1+n-j}(x_n)x_j\,, \qquad i = 1,\ldots,n-1\,, \tag{G.57}$$

$$y_n = x_n \tag{G.58}$$

converts the strict-feedforward system (G.50)–(G.51) into the chain of integrators (G.8)–(G.9). The feedback law

$$u = \alpha_1(x) = -\sum_{i=1}^{n}\binom{n}{i-1}y_i \tag{G.59}$$

globally asymptotically stabilizes the origin of (G.50)–(G.51).

Proof. The first part is by (lengthy) verification. The rest is as in the proof of Theorem G.5. □

As in Section G.2, we point out that, following the SJK procedure, one would have obtained

$$\alpha_i(\underline{x}_i) = -x_i - \sum_{m=i+1}^{n} x_m \left[\binom{n-i+1}{m-i} - \sum_{j=i}^{m} \binom{n-i+1}{j-i} \mu_{j+1+n-m}(x_n) \right], \tag{G.60}$$

$$w_i = 1, \tag{G.61}$$

and the coordinate shift β_i is given in the context of the following result.

Corollary G.2. *The control law (G.30), with $\alpha_1(x)$ defined in (G.59), applied to the plant (G.50)–(G.51), (G.52), (G.53), (G.54), (G.55), (G.56), achieves the result of Theorem G.4 with*

$$\beta_{i+1}(\underline{x}_{i+1}) = -\sum_{m=i+1}^{n} x_m \left[\binom{n-i}{m-i} - \sum_{j=i}^{m} \binom{n-i}{j-i} \mu_{j+1+n-m}(x_n) \right], \tag{G.62}$$
$$i = 1, \ldots, n-1.$$

Example G.1. To illustrate the above concepts (and notation), let us consider a fourth-order example of a Type II feedforward system:

$$\dot{x}_1 = x_2 + \left(\frac{x_2}{2} - \frac{x_3 x_4}{12} \right) u, \tag{G.63}$$

$$\dot{x}_2 = x_3 + \frac{x_3}{2} u, \tag{G.64}$$

$$\dot{x}_3 = x_4 + x_4 u, \tag{G.65}$$

$$\dot{x}_4 = u. \tag{G.66}$$

The control law

$$u = -y_1 - 4y_2 - 6y_3 - 4y_4 \tag{G.67}$$

$$= -z_1 - z_2 - z_3 - z_4, \tag{G.68}$$

where

$$y_1 = x_1 - \frac{x_4 x_2}{2} + \frac{x_4^2 x_3}{6} - \frac{x_4^4}{24}, \tag{G.69}$$

$$y_2 = x_2 - \frac{x_4 x_3}{2} + \frac{x_4^3}{6}, \tag{G.70}$$

$$y_3 = x_3 - \frac{x_4^2}{2}, \tag{G.71}$$

$$y_4 = x_4, \tag{G.72}$$

which is obtained with

$$\mu_2 = \frac{x_4^3}{24}, \tag{G.73}$$

$$\mu_3 = -\frac{x_4^2}{6}, \tag{G.74}$$

$$\mu_4 = \frac{x_4}{2}, \tag{G.75}$$

and

$$z_i = x_i - \beta_{i+1}, \tag{G.76}$$

with

$$\beta_4 = \left(\frac{x_4}{2} - 1\right) x_4, \tag{G.77}$$

$$\beta_3 = \left(\frac{x_4}{2} - 2\right) x_3 - x_4 + x_4^2 - \frac{x_4^3}{6}, \tag{G.78}$$

$$\begin{aligned} \beta_2 = {} & \left(\frac{x_4}{2} - 3\right) x_2 + \left(-3 + \frac{3}{2}x_4 - \frac{x_4^2}{6}\right) x_3 \\ & - x_4 - \frac{3}{2}x_4^2 + \frac{1}{2}x_4^3 - \frac{1}{24}x_4^4, \end{aligned} \tag{G.79}$$

achieves (G.4) for $n = 4$,

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} z \tag{G.80}$$

and

$$(s+1)^4 y_1(s) = 0. \tag{G.81}$$

◇

To use the results of this section for control designs beyond the Type II class of systems, we need the inverse of the coordinate transformation (G.57). The explicit form of the inverse transformation is given in the following theorem.

Lemma G.2. *Consider the series of functions*

$$\lambda_n(x_n) = \mu_n(x_n), \tag{G.82}$$

$$\lambda_i(x_n) = \frac{1}{x_n}\int_0^{x_n} \left(s \sum_{l=i+1}^{n} \gamma_{l-i}(s)\lambda_l(s) + \phi_{i-1}(0,\ldots,0,s) \right) ds \tag{G.83}$$

for $i = n-1,\dots,2$. *The inverse of the diffeomorphic transformation (G.57) is*

$$x_i = y_i + \sum_{j=i+1}^{n} \lambda_{i+1+n-j}(y_n) y_j\,, \qquad i = 1,\dots,n-1\,, \tag{G.84}$$

$$x_n = y_n\,. \tag{G.85}$$

Proof. By induction, using the intermediate step that

$$\gamma_{n-i+1}(x_n) = -\sum_{m=1}^{n-i} \gamma_m(x_n)\lambda_{m+i}(x_n) + \lambda_i'(x_n) \tag{G.86}$$

for $i = n-1,\dots,3$. □

As in Lemma G.1, in the next result we give a closed-form formula for the *solutions* of the feedback system from Theorem G.6, which will allow us, in Section H.1, to extend the constructive methodology to a class of strict-feedforward systems that are *not linearizable*.

Lemma G.3. *Starting from the initial condition x, the solution of the feedback system (G.50)–(G.56), (G.59) at time* τ *is*

$$\begin{aligned}\xi_i(\tau,x) = \mathrm{e}^{-\tau}\Bigg[&\sum_{j=i}^{n}\binom{n-i}{j-i}(-1)^{j-i}\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\\ &\times\binom{n-j+k}{l-j+k}\left(x_l - \sum_{m=l+1}^{n}\mu_{l+1+n-m}(x_n)x_m\right)\\ &+\sum_{p=i+1}^{n}\lambda_{i+1+n-p}\left(\mathrm{e}^{-\tau}\sum_{k=0}^{n-1}\frac{(-\tau)^k}{k!}\sum_{l=n-k}^{n}\right.\\ &\left.\times\binom{k}{l-n+k}\left(x_l - \sum_{m=l+1}^{n}\mu_{l+1+n-m}(x_n)x_m\right)\right)\\ &\times\sum_{j=p}^{n}\binom{n-p}{j-p}(-1)^{j-p}\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}\\ &\times\left(x_l - \sum_{m=l+1}^{n}\mu_{l+1+n-m}(x_n)x_m\right)\Bigg]\,,\end{aligned} \tag{G.87}$$

where $i = 1,\dots,n$, *and the control signal is*

$$\begin{aligned}u = \tilde{\alpha}_1(\tau,x) = -\mathrm{e}^{-\tau}&\sum_{i=1}^{n}\binom{n}{i-1}\sum_{j=i}^{n}\binom{n-i}{j-i}(-1)^{j-i}\sum_{k=0}^{j-1}\frac{(-\tau)^k}{k!}\\ &\times\sum_{l=j-k}^{n}\binom{n-j+k}{l-j+k}\left(x_l - \sum_{m=l+1}^{n}\mu_{l+1+n-m}(x_n)x_m\right)\,.\end{aligned} \tag{G.88}$$

Proof. Analogous to the proof of Lemma G.1, employing also Lemma G.2. □

Type I and II Systems in Dimensions Two and Three

We start by pointing out that in dimension two all strict-feedforward systems are simultaneously of Types I and II. This implies that all second-order strict-feedforward systems are linearizable.

Theorem G.7. *Consider the system*

$$\dot{x}_1 = x_2 + \phi_1(x_2)u\,, \tag{G.89}$$

$$\dot{x}_2 = u\,, \tag{G.90}$$

where $\phi_1(x_1)$ is continuous and

$$\phi_1(0) = 0\,. \tag{G.91}$$

The control law

$$u = -x_1 - 2x_2 + \int_0^{x_2} \phi_1(s)ds \tag{G.92}$$

ensures the global asymptotic stability of the origin.

Proof. By verification that

$$\dot{z}_1 = x_2 + u\,, \tag{G.93}$$

$$\dot{x}_2 = u\,, \tag{G.94}$$

where

$$z_1 = x_1 - \beta_2(x_2)\,, \tag{G.95}$$

$$\beta_2(x_2) = -x_2 + \int_0^{x_2} \phi_1(s)ds\,, \tag{G.96}$$

and

$$u = -z_1 - x_2\,. \tag{G.97}$$

□

Example G.2. Let us now consider an example with

$$\phi_1(x_2) = -x_2^2\,. \tag{G.98}$$

This example was worked out in [196]. In this case the formula (G.92) gives[5]

$$u = -x_1 - 2x_2 - \frac{x_2^3}{3}\,. \tag{G.99}$$

[5] A reader checking the details in [196] will notice that this control law differs from (6.2.12) in [196]. This is due to an extra "x_3^2" term that has crept into the calculations in [196], in Eq. (6.2.7).

One should recognize that the "$-x_1 - 2x_2$" portion of the control law (G.99) is responsible for the exponential stabilization of the linearized system. To see that this linear controller is not sufficient for global stabilization, we plug it back into the plant and obtain a closed-loop system, written in the form of a second-order equation, as

$$\ddot{x}_2 + (2 - x_2^2)\dot{x}_2 + x_2 = 0\,. \tag{G.100}$$

This is a Van der Pol equation with an unstable limit cycle, which exhibits a finite escape instability. Hence, the nonlinear term "$-x_2^3/3$," designed to accommodate the input nonlinearity $\phi_1(x_2) = -x_2^2$, is crucial for global stabilization. ◇

The possibilities, as well as the limits, of Type I/II linearizability for strict-feedforward systems are best understood in dimension three. For the following class of systems, which represents a union of all three-dimensional Type I and Type II feedforward systems, a linearizing coordinate change and a stabilizing control law are designed in the next theorem.

Theorem G.8. *Consider the class of systems*

$$\dot{x}_1 = x_2 + \pi_2(x_2)x_3 + \left(\frac{x_3\phi_2(x_3) - \int_0^{x_3}\phi_2(s)ds}{x_3^2}x_2 + \pi_3(x_3)\right)u\,, \tag{G.101}$$

$$\dot{x}_2 = x_3 + \phi_2(x_3)u\,, \tag{G.102}$$

$$\dot{x}_3 = u\,, \tag{G.103}$$

where $\pi_2(\cdot), \pi_3(\cdot) \in C^0$, *and* $\phi_2(\cdot) \in C^1$ *vanish at the origin and*

$$\pi_2(x_2)\phi_2(x_3) \equiv 0\,. \tag{G.104}$$

Then the control law

$$u = -y_1 - 3y_2 - 3y_3\,, \tag{G.105}$$

where

$$\begin{aligned} y_1 = x_1 &- \int_0^{x_2}\pi_2(s)ds - \mu_3(x_3)x_2 - \int_0^{x_3}\pi_3(s)ds \\ &+ \frac{1}{2}x_3\left(\mu_3(x_3)\right)^2 + \frac{1}{2}\int_0^{x_3}\left(\mu_3(s)\right)^2 ds\,, \end{aligned} \tag{G.106}$$

$$y_2 = x_2 - \int_0^{x_3}\phi_2(s)ds\,, \tag{G.107}$$

$$y_3 = x_3\,, \tag{G.108}$$

and

$$\mu_3(x_3) = \frac{\int_0^{x_3}\phi_2(s)ds}{x_3}\,, \tag{G.109}$$

achieves global asymptotic stability of the origin.

Proof. One can verify that

$$\dddot{y}_1 + 3\ddot{y}_1 + 3\dot{y}_1 + y_1 = 0 \tag{G.110}$$

and that

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix} z\,. \tag{G.111}$$

□

A Type II example of a system from this class is

$$\dot{x}_1 = x_2 + \left(\frac{\mathbf{1}}{\mathbf{2}}x_2 + x_3 \sin x_3\right) u\,, \tag{G.112}$$

$$\dot{x}_2 = x_3 + x_3 u\,, \tag{G.113}$$

$$\dot{x}_3 = u\,, \tag{G.114}$$

which is stabilized (and feedback-linearized) using

$$\begin{aligned} u = {} & -x_1 - 3x_2 - 3x_3 + \frac{x_2 x_3}{2} + \frac{3}{2}x_3^2 - \frac{1}{6}x_3^3 \\ & + x_3 \sin x_3 + \cos x_3 - 1\,. \end{aligned} \tag{G.115}$$

We point out that the key restriction in this example is the boldfaced 1/2. If this value were anything else (say, 1 or 0), this system would not be linearizable. It would, however, be stabilizable using the procedure we present in Section H.1.

Appendix H
Strict-Feedforward Systems: Not Linearizable

In this appendix we review three major extensions to the algorithms for linearizable strict-feedforward systems in Appendix H. The first extension is for certain strict-feedforward systems that are not linearizable. The second extension is for a class of systems referred to as the "block-feedforward" systems. The third extension combines forwarding and backstepping for an interlaced feedforward-feedback class of nonlinear systems.

H.1 Algorithms for Nonlinearizable Feedforward Systems

In this section we expand upon the Type I and II feedforward systems, to develop algorithms for feedforward systems that are not linearizable. Two classes of systems that we consider consist of a linearizable subsystem $[x_1,\ldots,x_n]^T$ and a scalar equation x_0 that is (possibly) not linearizable. This, structure belongs to the class of nonflat *Liouvillian systems* of defect equal to one; see Chelouah [24] (especially Example 2).

Consider the following extension of the Type I strict-feedforward systems:

$$\dot{x}_0 = x_1 + \psi_0(x) + \phi_0(x)u, \tag{H.1}$$

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u, \tag{H.2}$$

$$\dot{x}_i = x_{i+1}\,, \qquad i = 2,\ldots,n-1, \tag{H.3}$$

$$\dot{x}_n = u\,, \tag{H.4}$$

where x denotes $[x_1,\ldots,x_n]^T$ (i.e., x_0 is not included in x),

$$\psi_0(0) = \phi_0(0) = \pi_j(0) = 0\,, \quad j = 2,\ldots,n\,, \tag{H.5}$$

and

$$\frac{\partial \psi_0(0)}{\partial x_i} = 0, \qquad i = 1, \ldots, n. \tag{H.6}$$

The subsystem (H.2)–(H.4) is linearizable. This makes it possible to develop a closed-form formula for a globally stabilizing SJK-type control law.

We propose the following design algorithm. Start by computing the expressions in Lemma G.1. Then calculate

$$\beta_1(x) = -\int_0^\infty \left[\xi_1(\tau,x) + \psi_0(\xi(\tau,x)) + \phi_0(\xi(\tau,x))\,\tilde{\alpha}_1(\tau,x)\right] d\tau, \tag{H.7}$$

$$w_0(x) = \phi_0(x) - \frac{\partial \beta_1(x)}{\partial x_1}\pi_n(x_n) - \frac{\partial \beta_1(x)}{\partial x_n}, \tag{H.8}$$

and

$$u = \alpha_0(x_0,x) = -w_0(x)(x_0 - \beta_1(x)) - \sum_{i=1}^n \binom{n}{i-1} x_i + \sum_{i=2}^n \int_0^{x_i} \pi_i(s)ds. \tag{H.9}$$

Theorem H.1. *The feedback system (H.1)–(H.4), (H.9) is globally asymptotically stable at the origin.*

Proof. Lengthy calculations verify that

$$\frac{d}{dt}\sum_{i=0}^n z_i^2 = -w_0^2 z_0^2 - \sum_{i=1}^n z_i^2 - \left(w_0 z_0 + \sum_{i=1}^n z_i\right)^2, \tag{H.10}$$

where $w_0(0) = 1$ and

$$z_0 = x_0 - \beta_1, \tag{H.11}$$

$$z_i = \sum_{j=i}^n \binom{n-i}{j-i} x_j - \delta_{i,1} \sum_{j=2}^n \int_0^{x_j} \pi_j(s)ds \tag{H.12}$$

for $i = 1, \ldots, n$. □

Next, consider the following extension of the Type II strict-feedforward systems:

$$\dot{x}_0 = x_1 + \psi_0(x) + \phi_0(x)u, \tag{H.13}$$

$$\dot{x}_1 = x_2 + \phi_i(\underline{x}_{i+1})u, \qquad i = 1, \ldots, n-2, \tag{H.14}$$

$$\dot{x}_{n-1} = x_n + \phi_{n-1}(x_n)u, \tag{H.15}$$

$$\dot{x}_n = u, \tag{H.16}$$

where the ϕ_i's satisfy the conditions of Theorem G.6.

We propose the following design algorithm. Start by computing the expressions in Theorem G.3. Then calculate

$$\beta_1(x) = -\int_0^\infty \left[\xi_1(\tau,x) + \psi_0(\xi(\tau,x)) + \phi_0(\xi(\tau,x))\,\tilde{\alpha}_1(\tau,x)\right] d\tau, \tag{H.17}$$

$$w_0(x) = \phi_0(x) - \sum_{i=1}^{n-1} \frac{\partial \beta_1(x)}{\partial x_i} \phi_i(\underline{x}_{i+1}) - \frac{\partial \beta_1(x)}{\partial x_n}, \tag{H.18}$$

and

$$\begin{aligned} u = \alpha_0(x_0, x) = & -w_0(x)(x_0 - \beta_1(x)) \\ & - x_1 - \sum_{m=2}^{n} x_m \left[\binom{n}{m-1} - \sum_{j=1}^{m} \binom{n}{j-1} \mu_{j+1+n-m}(x_n) \right]. \end{aligned} \tag{H.19}$$

Theorem H.2. *The feedback system (H.13)–(H.16), (H.19) is globally asymptotically stable at the origin.*

Proof. The same as the proof of Theorem H.1, except that

$$z_i = x_i + \sum_{m=i+1}^{n} x_m \left[\binom{n-i}{m-i} - \sum_{j=i}^{m} \binom{n-i}{j-i} \mu_{j+1+n-m}(x_n) \right] \tag{H.20}$$

for $i = 1, \ldots, n-1$. □

The restriction (F.7) can be lifted in some cases, to expand the class of strict-feedback systems that are not linearizable but for which an explicit feedback law can be developed.

Consider the example

$$\dot{x}_1 = x_2 + x_3^2, \tag{H.21}$$
$$\dot{x}_2 = \sinh x_3 + x_3 u, \tag{H.22}$$
$$\dot{x}_3 = u, \tag{H.23}$$

which, although only a slight variation from (12.154)–(12.156), is not represented in the class (H.1)–(H.4). The difference in (H.22) is easily accommodated by the coordinate/prefeedback change

$$X_3 = \sinh x_3, \tag{H.24}$$
$$v = \sqrt{1 + (\sinh x_3)^2} u, \tag{H.25}$$

which converts (H.21) into

$$\dot{x}_1 = x_2 + \left(\sinh^{-1}(X_3)\right)^2, \tag{H.26}$$
$$\dot{x}_2 = X_3 + \frac{\sinh^{-1}(X_3)}{\sqrt{1 + X_3^2}} v, \tag{H.27}$$
$$\dot{X}_3 = v. \tag{H.28}$$

This system fits the forms in Section H.1.

However, the system

$$\dot{x}_i = \sin(x_{i+1}), \qquad i = 1,\ldots,n-1, \tag{H.29}$$
$$\dot{x}_n = u, \tag{H.30}$$

suggested to us by Teel, (very) remotely motivated by the ball-and-beam problem [217], cannot be brought into those forms, except in the case $n = 2$, where the resulting control law is

$$u = -x_2 - \frac{\sin x_2}{x_2}\left(x_1 - \int_0^{x_2} \frac{\sin\xi}{\xi} d\xi\right). \tag{H.31}$$

H.2 Block-Forwarding

In this section we extend the class of systems to which the SJK forwarding procedure is applicable. Then we present our explicit controller formulas for this class of systems.

Consider the class of *block-strict-feedforward* systems[1]

$$\dot{x}_i = x_{i+1} + \psi_i\left(\underline{x}_{i+1}, \underline{q}_{i+1}\right) + \phi_i\left(\underline{x}_{i+1}, \underline{q}_{i+1}\right) u, \tag{H.35}$$
$$\dot{q}_i = A_i q_i + \omega_i\left(\underline{x}_i, \underline{q}_{i+1}\right), \tag{H.36}$$

where $i = 1,2,\ldots,n$, each x_i is scalar-valued, each q_i is r_i-vector-valued,

$$\underline{x}_i = [x_i,\ x_{i+1},\ldots,x_n]^T, \tag{H.37}$$
$$\underline{q}_i = [q_i^T,\ q_{i+1}^T,\ldots,q_n^T]^T, \tag{H.38}$$

A_i is a Hurwitz matrix for all $i = 1,2,\ldots,n$,

$$x_{n+1} = u, \tag{H.39}$$
$$q_{n+1} = 0, \tag{H.40}$$
$$\phi_n = 0, \tag{H.41}$$

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[1] The blocks considered here are less general than those in [219, 145, 74]. We can generalize the idea we are presenting (even somewhat beyond the classes considered [219, 145, 74]) to include blocks q_i that are merely input-to-state stable with respect to $\left(\underline{x}_i, \underline{q}_{i+1}\right)$, rather than being linear in q_i. A simple example is the system

$$\dot{q} = -q^3 + x_2, \tag{H.32}$$
$$\dot{x}_1 = x_2 + qu, \tag{H.33}$$
$$\dot{x}_2 = u. \tag{H.34}$$

This generalization would, however, preclude closed-form solvability of the problem; the result would only be an extension of [195].

and

$$\frac{\partial \psi_i(0)}{\partial x_j} = 0, \tag{H.42}$$
$$\phi_i(0) = 0, \tag{H.43}$$
$$\omega_i(0) = 0 \tag{H.44}$$

for $i = 1, 2, \ldots, n-1$, $j = i+1, \ldots, n$. This class of systems should be understood as a dual of the *block-strict-feedback* systems in Section 4.5.2 of [112].

The control law for this class of systems is designed as follows. Let

$$\beta_{n+1} = 0, \tag{H.45}$$
$$\alpha_{n+1} = 0\,. \tag{H.46}$$

For $i = n, n-1, \ldots, 2, 1$,

$$z_i = x_i - \beta_{i+1}, \tag{H.47}$$
$$w_i(\underline{x}_{i+1}, \underline{q}_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial \beta_{i+1}}{\partial x_j}\phi_j - \frac{\partial \beta_{i+1}}{\partial x_n}, \tag{H.48}$$
$$\alpha_i(\underline{x}_i, \underline{q}_{i+1}) = \alpha_{i+1} - w_i z_i, \tag{H.49}$$
$$\begin{aligned}\beta_i(\underline{x}_i, \underline{q}_i) = -\int_0^{\infty} \Big[& \xi_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i) + \psi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i), \underline{\eta}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i)\right) \\ & + \phi_{i-1}\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i), \underline{\eta}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i)\right) \\ & \times \alpha_i\left(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i)\underline{\eta}_{i+1}^{[i]}(\tau, \underline{x}_i, \underline{q}_i)\right)\Big] d\tau,\end{aligned} \tag{H.50}$$

where the notation in the integrand of (H.50) refers to the solutions of the (sub)system(s)

$$\begin{aligned}\frac{d}{d\tau}\xi_j^{[i]} = \xi_{j+1}^{[i]} + \psi_j\left(\underline{\xi}_{j+1}^{[i]}, \underline{\eta}_{j+1}^{[i]}\right) \\ + \phi_j\left(\underline{\xi}_{j+1}^{[i]}, \underline{\eta}_{j+1}^{[i]}\right)\alpha_i\left(\underline{\xi}_i^{[i]}, \underline{\eta}_{i+1}^{[i]}\right),\end{aligned} \tag{H.51}$$
$$\frac{d}{d\tau}\eta_j^{[i]} = A_j \eta_j^{[i]} + \omega_j\left(\underline{\xi}_j^{[i]}, \underline{\eta}_{j+1}^{[i]}\right) \tag{H.52}$$

for $j = i-1, i, \ldots, n$, at time τ, starting from the initial condition $(\underline{x}_i, \underline{q}_i)$. The control law is

$$u = \alpha_1\,. \tag{H.53}$$

Theorem H.3. *The feedback system (H.35), (H.36), (H.53) is globally asymptotically stable at the origin.*

Proof. As in the proof of Theorem F.1, the Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^{n} z_i^2 \tag{H.54}$$

has a negative-definite derivative:

$$\dot{V} = -\frac{1}{2}\sum_{i=1}^{n} w_i^2 z_i^2 - \frac{1}{2}\left(\sum_{i=2}^{n} z_i w_i\right)^2 . \tag{H.55}$$

This implies that $x_n(t)$ converges to zero. Since $\omega_n(0) = 0$, we have that $\omega_n(x_n(t))$ converges to zero. Because A_n is Hurwitz, $q_n(t)$ converges to zero. One can show recursively that $w_i(0) = 1$ and $\beta_i(0) = 0$. It then follows that $w_{n-1}(x_n(t), q_n(t))$ converges to one. Since (H.55) guarantees that $w_{n-1}z_{n-1}$ goes to zero, $z_{n-1}(t)$ also goes to zero. Hence,

$$x_{n-1}(t) = z_{n-1}(t) + \beta_n(x_n(t), q_n(t)) \tag{H.56}$$

converges to zero. Continuing in the same fashion, one shows that $x(t), q(t) \to 0$ as $t \to \infty$. This establishes that the equilibrium $x = 0, q = 0$ is (uniformly) attractive. Global stability is argued in a similar, recursive fashion, using (H.55) and the fact that the subsystems (H.36) are input-to-state stable. In conclusion, the origin is globally asymptotically stable. □

As in Section F.2, the solution $(\underline{\xi}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i), \underline{\eta}_i^{[i]}(\tau, \underline{x}_i, \underline{q}_i))$, needed in the integral (H.50), is impossible to obtain analytically in general. For this reason, we consider two classes of block-feedforward systems, inspired by feedforward systems of Types I and II, for which a closed-form controller can be obtained.

Consider the class of systems we refer to as *Type I block-feedforward* systems:

$$\dot{x}_0 = x_1 + \psi_0(x,q) + \phi_0(x,q)u, \tag{H.57}$$

$$\dot{q}_0 = A_0 q_0 + \omega_0(x_0, x, q), \tag{H.58}$$

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u, \tag{H.59}$$

$$\dot{q}_1 = A_1 q_1 + \omega_1\left(x, \underline{q}_2\right), \tag{H.60}$$

$$\dot{x}_i = x_{i+1}, \qquad i = 2, \ldots, n-1, \tag{H.61}$$

$$\dot{q}_i = A_i q_i + \omega_i\left(\underline{x}_i, \underline{q}_{i+1}\right), \tag{H.62}$$

$$\dot{x}_n = u, \tag{H.63}$$

$$\dot{q}_n = A_n q_n + \omega_n(x_n), \tag{H.64}$$

where x denotes $[x_1, \ldots, x_n]^T$, q denotes $[q_1^T, \ldots, q_n^T]^T$ (i.e., it does not include q_0),

$$\psi_0(0) = \phi_0(0) = \omega_0(0) = \omega_1(0) = \omega_j(0) = \pi_j(0) = 0, \quad j = 2, \ldots, n, \tag{H.65}$$

and

$$\frac{\partial \psi_0(0)}{\partial x_i} = 0, \qquad i = 1, \ldots, n. \tag{H.66}$$

The subsystem $(x_1, \ldots, x_n)$ is linearizable, which makes it possible to develop a closed-form formula. The first step in the design algorithm is to compute the expressions in Lemma G.1. It is worth noting that $\xi(\tau, x)$ and $\tilde{\alpha}_1(\tau, x)$ are both independent of q. Then, for $i = n, n-1, \ldots, 2$, we calculate

$$\eta_i\left(\tau, \underline{q}_i, x\right) = \quad e^{A_i \tau} q_i + \int_0^{\tau} e^{A_i(\tau-\sigma)} \omega_i \left(\underline{\xi}_i(\sigma, x), \underline{\eta}_{i+1}\left(\sigma, \underline{q}_{i+1}, x\right)\right) d\sigma, \tag{H.67}$$

followed by

$$\begin{aligned} \beta_1(x,q) = -\int_0^{\infty} &[\xi_1(\tau,x) + \psi_0(\xi(\tau,x), \eta(\tau,q,x)) \\ &+ \phi_0(\xi(\tau,x), \eta(\tau,q,x))\,\tilde{\alpha}_1(\tau,x)]\,d\tau, \end{aligned} \tag{H.68}$$

$$w_0(x,q) = \phi_0(x) - \frac{\partial \beta_1(x,q)}{\partial x_1}\pi_n(x_n) - \frac{\partial \beta_1(x,q)}{\partial x_n}, \tag{H.69}$$

and

$$\begin{aligned} u = \alpha_0(x_0, x, q) = &-w_0(x,q)(x_0 - \beta_1(x,q)) \\ &- \sum_{i=1}^{n} \binom{n}{i-1} x_i + \sum_{i=2}^{n} \int_0^{x_i} \pi_i(s)\,ds. \end{aligned} \tag{H.70}$$

Theorem H.4. *The feedback system (H.57)–(H.64), (H.70) is globally asymptotically stable at the origin.*

Proof. Lengthy calculations verify that the same expressions hold as in the proof of Theorem H.1. In the present proof, however, z_0 depends not only on x_0, x but also on $q_n, q_{n-1}, \ldots, q_1$. Thus, convergence to the origin is proved in the following order: $x_n, x_{n-1}, \ldots, x_1, q_n, q_{n-1}, \ldots, q_1, x_0, q_0$. Global stability is argued similarly. Hence, the equilibrium $x_0 = q_0 = 0, x = 0, q = 0$ is globally asymptotically stable. □

Finally, consider the class of systems we refer to as *Type II block-feedforward* systems:

$$\dot{x}_0 = x_1 + \psi_0(x,q) + \phi_0(x,q)u, \tag{H.71}$$

$$\dot{q}_0 = A_0 q_0 + \omega_0(x_0, x, q), \tag{H.72}$$

$$\dot{x}_i = x_{i+1} + \phi_i(x_{i+1})\,u, \qquad i = 1, \ldots, n-1, \tag{H.73}$$

$$\dot{q}_i = A_i q_i + \omega_i\left(\underline{x}_i, \underline{q}_{i+1}\right), \tag{H.74}$$

$$\dot{x}_n = u, \tag{H.75}$$

$$\dot{q}_n = A_n q_n + \omega_n(x_n), \tag{H.76}$$

where the ϕ_i's satisfy the conditions of Theorem G.6. With $\xi(\tau,x)$ and $\tilde{\alpha}_1(\tau,x)$ calculated as in Theorem G.3, and the η_i's and β_1 calculated as in (H.67), (H.68), respectively, the algorithm's final step is to calculate

$$w_0(x,q) = \phi_0(x) - \sum_{i=1}^{n-1} \frac{\partial\beta_1(x,q)}{\partial x_i}\phi_i(\underline{x}_{i+1}) - \frac{\partial\beta_1(x,q)}{\partial x_n} \tag{H.77}$$

and

$$\begin{aligned} u &= \alpha_0(x_0,x,q) \\ &= -w_0(x,q)(x_0-\beta_1(x,q)) - x_1 \\ &\quad - \sum_{m=2}^{n} x_m \left[\binom{n}{m-1} - \sum_{j=1}^{m} \binom{n}{j-1} \mu_{j+1+n-m}(x_n) \right]. \end{aligned} \tag{H.78}$$

Theorem H.5. *The feedback system (H.71)–(H.76), (H.78) is globally asymptotically stable at the origin.*

Proof. Analogous to the proof of Theorem H.4. □

H.3 Interlaced Feedforward-Feedback Systems

General Design

The ability to stabilize systems that are neither in the strict-feedback form nor in the strict-feedforward form is nicely illustrated in [196]. In this section we present designs for two classes of systems obtained by interlacing strict-feedback systems [112] with feedforward systems of Types I and II.

First, consider the class of *interlaced systems of Type I*:

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u, \tag{H.79}$$

$$\dot{x}_i = x_{i+1}\,, \qquad i = 2,\ldots,n, \tag{H.80}$$

$$\dot{x}_{n+1} = x_{n+2} + f_1(\underline{x}_1, x_{n+1}), \tag{H.81}$$

$$\dot{x}_{n+j} = x_{n+j+1} + f_j(\underline{x}_1, \bar{x}_{n+j})\,, \qquad j = 2,\ldots,N, \tag{H.82}$$

where $x_{n+N+1} = u$. In this system $\bar{x}_{n+j}$ denotes $[x_{n+1},\ldots,x_{n+j}]^T$, and, as before, $\underline{x}_j$ denotes $[x_j,\ x_{j+1},\ldots,x_n]^T$ (which means, in particular, that $\underline{x}_1 = [x_1,\ldots,x_n]^T$). It is clear from the above notation that the overall system order is $n+N$, where the feedforward part (top) is of order n and the feedback part (bottom) is of order N. We assume that $\pi_i(0) = 0\,, i = 2,\ldots,n$, and $f_i(0) = 0\,, i = 1,\ldots,N$. The control synthesis for this system is given in the following theorem.

Theorem H.6. *The control law given by*

$$z_i = x_i + \sum_{j=i+1}^{n} \binom{n-i}{j-i} x_j - \delta_{i,1} \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s) ds, \tag{H.83}$$

$$\alpha_1(\underline{z}_1) = -\sum_{i=1}^{n} z_i \tag{H.84}$$

for $i = 1, \ldots, n$,

$$z_{n+1} = x_{n+1} - \alpha_1, \tag{H.85}$$

$$\alpha_{n+1}(\underline{z}_1, z_{n+1}) = -(n+1)z_{n+1} + \sum_{l=1}^{n} (n-l) z_l - f_1(\underline{x}_1, x_{n+1}), \tag{H.86}$$

$$z_{n+j} = x_{n+j} - \alpha_{n+j-1}(\underline{z}_1, z_{n+j-1}), \tag{H.87}$$

$$\begin{aligned} \alpha_{n+j} = & -z_{n+j-1} - z_{n+j} - f_j(\underline{x}_1, \bar{x}_{n+j}) \\ & + \sum_{l=1}^{n} \frac{\partial \alpha_{n+j-1}}{\partial z_l} \left(-\sum_{k=1}^{i} z_k + z_{n+1} \right) + \frac{\partial \alpha_{n+j-1}}{\partial z_{n+1}} \left(-\sum_{k=1}^{n+1} z_k + z_{n+2} \right) \\ & + \sum_{l=2}^{j-1} \frac{\partial \alpha_{n+j-1}}{\partial z_{n+l}} (-z_{n+l} + z_{n+l+1}) \end{aligned} \tag{H.88}$$

for $j = 2, \ldots, n$, *and*

$$u = \alpha_{n+N} \tag{H.89}$$

globally asymptotically stabilizes the system (H.79)–(H.82) at the origin.

Proof. It can be verified that the closed-loop system in the z-coordinates is

$$\dot{z}_i = -\sum_{k=1}^{i} z_k + z_{n+1}, \qquad i = 1, \ldots, n, \tag{H.90}$$

$$\dot{z}_{n+1} = -\sum_{k=1}^{n} z_k - z_{n+1} + z_{n+2}, \tag{H.91}$$

$$\dot{z}_{n+j} = -z_{n+j} + z_{n+j+1}, \qquad j = 2, \ldots, N, \tag{H.92}$$

where $z_{n+N+1} = 0$. The Lyapunov function

$$V = \sum_{i=1}^{n+N} z_i^2 \tag{H.93}$$

satisfies

$$\dot{V} = -\sum_{i=1}^{n+N} z_i^2 - \sum_{i=n+2}^{n+N} (z_i - z_{i-1})^2 - \left(\sum_{i=1}^{n} z_i \right)^2, \tag{H.94}$$

which proves the result. □

Next, consider the class of *interlaced systems of Type II*:

$$\dot{x}_1 = x_2 + \phi_i(\underline{x}_{i+1})u\,, \qquad i = 1,\ldots,n-2, \tag{H.95}$$

$$\dot{x}_{n-1} = x_n + \phi_{n-1}(x_n)u, \tag{H.96}$$

$$\dot{x}_n = x_{n+1}, \tag{H.97}$$

$$\dot{x}_{n+1} = x_{n+2} + f_1(\underline{x}_1, x_{n+1}), \tag{H.98}$$

$$\dot{x}_{n+j} = x_{n+j+1} + f_j(\underline{x}_1, \bar{x}_{n+j})\,, \qquad j = 2,\ldots,N, \tag{H.99}$$

where $x_{n+N+1} = u$. We assume that

$$\phi_i(0) = f_j(0) = 0, \tag{H.100}$$

and the ϕ_i's satisfy the conditions of Theorem G.6.

Theorem H.7. *The control law given by*

$$z_i = x_i + \sum_{m=i+1}^{n} x_m \left[\binom{n-i}{m-i} - \sum_{j=i}^{m}\binom{n-i}{j-i}\mu_{j+1+n-m}(x_n)\right], \qquad i = 1,\ldots,n-1, \tag{H.101}$$

$$z_n = x_n, \tag{H.102}$$

and (H.84)–(H.89) globally asymptotically stabilizes the system (H.95)–(H.99) at the origin.

Proof. The same as Theorem H.6. □

Since the interlaced systems of both Types I and II are feedback linearizable, one does not have to necessarily commit to the integrator forwarding + integrator backstepping design procedure. It suffices to define an output with respect to which one has a relative degree equal to the order of the system, with which one can pursue full-state feedback linearization by conversion to the Brunovsky canonical form. This is spelled out in the next theorem.

Theorem H.8. *The systems (H.79)–(H.82) and (H.95)–(H.99) are of relative degree $n+N$ from u to the respective outputs*

$$y_1 = x_1 - \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s)ds \tag{H.103}$$

and

$$y_1 = x_1 - \sum_{j=2}^{n} \mu_{2+n-j}(x_n)x_j\,. \tag{H.104}$$

Example: Combining Block-Backstepping and Block-Forwarding

In this section we show that *block*-backstepping and *block*-forwarding can be combined in a similar manner on an example that is outside the forms considered in Section H.3 (and also outside those in [196]):

$$\dot{q} = -2q + x_2^2, \tag{H.105}$$
$$\dot{x}_1 = x_2 + qx_3, \tag{H.106}$$
$$\dot{x}_2 = x_3 + q, \tag{H.107}$$
$$\dot{x}_3 = u + qx_1 . \tag{H.108}$$

This system is neither in the block-strict-feedforward form (because of qx_1 in the x_3-equation) nor in the block-strict-feedback form (because of qx_3 in the x_1-equation). However, the x_1, x_2, q-subsystem is block-strict-feedforward if one views x_3 as control, and the x_2, x_3, q-subsystem is block-strict-feedback with u as control. Hence, we will derive a controller for this system using one step of forwarding followed by one step of backstepping.

Following the design from Section H.2, we first calculate

$$\xi_2^{[2]}(\tau, x_2) = x_2 \mathrm{e}^{-\tau} \tag{H.109}$$

and

$$\eta^{[2]}(\tau, x_2, q) = (q + \tau x_2^2)\mathrm{e}^{-\tau}. \tag{H.110}$$

Then we derive

$$\beta_2(x_2, q) = -x_2 + \frac{qx_2}{3} + \frac{qx_2^2}{8} + \frac{q^2}{4} + \frac{x_2^3}{9} + \frac{x_2^4}{32}, \tag{H.111}$$
$$w_1(x_2, q) = 1 + \frac{2}{3}q - \frac{qx_2}{4} - \frac{x_2^2}{3} - \frac{x_2^3}{8} . \tag{H.112}$$

The system is converted from the x_1, x_2, x_3-coordinates into z_1, x_2, z_3 (note that x_2 is unaltered), where

$$z_1 = x_1 - \beta_1, \tag{H.113}$$
$$z_3 = x_3 + q + w_1 z_1 + x_2 . \tag{H.114}$$

Note that (H.113) corresponds to one step of forwarding, resulting in a "virtual control" $-q - w_1 z_1 - x_2$ for x_3 as a control input, whereas (H.114) corresponds to one step of backstepping. The control law

$$\begin{aligned} u = &-z_3 - x_2 - w_1 z_1 - x_1 q + 2q - x_2^2 - w_1^2(x_3 + q + x_2) \\ &- (x_3 + q) + z_1 \left[\left(\frac{x_2}{4} - \frac{2}{3} \right) \left(-2q + x_2^2 \right) \right. \\ &\left. + \left(\frac{q}{4} - \frac{2}{3}x_2 + \frac{3}{8}x_2^2 \right) (x_3 + q) \right] \end{aligned} \tag{H.115}$$

results in the system being transformed into

$$\dot{z}_1 = -w_1^2 z_1 + w_1 z_3, \tag{H.116}$$

$$\dot{x}_2 = -w_1 z_1 - x_2 + z_3, \tag{H.117}$$

$$\dot{z}_3 = -w_1 z_1 - x_2 - z_3\,. \tag{H.118}$$

The stability of this system follows from the Lyapunov function

$$V(x,q) = z_1(x_1,x_2,q)^2 + x_2^2 + z_3(x_1,x_2,x_3,q)^2 \tag{H.119}$$

because

$$\dot{V} = -w_1^2 z_1^2 - x_2^2 - (w_1 z_1 + x_2)^2 - 2z_3^2\,. \tag{H.120}$$

The convergence to zero can be seen in the following order: x_2 [from (H.120)], q [from (H.105)], x_1 [from (H.113) and (H.111], x_3 [from (H.114)].

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