



**KTH Electrical Engineering**

# **Theory of transformation optics and invisibility cloak design**

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# Abstract

Research on metamaterials has been growing ever since the first experimental realization of double negative media. The theory of transformation optics provides people with a perfect tool to make use of vast possibilities of the constitutive parameters for metamaterials. A lot of fascinating designs have been brought to us by transformation optics, with invisibility cloaks being the most intensely studied. The present thesis aims to develop the basic theory of transformation optics, and utilize it to design invisibility cloaks for various applications.

After the background description of this field, the theory of transformation optics is first introduced. Formulas of transformation medium parameters and transformed fields are derived with every detail explained, so that the working knowledge of transformation optics can be grasped with minimal prerequisite mathematics. Proof of form invariance of full Maxwell's equations with sources is presented. Design procedure of transformation optics is then demonstrated by creating perfect invisibility cloaks. The introduction to basic theory is followed by discussions on our works included in our published papers.

As our first application, a method of designing two-dimensional reduced cloaks of complex shapes is proposed to relieve the difficulty of singularity occurring in perfect cloaks. The simple and intuitive method is the first way to design two-dimensional reduced cloaks of shapes other than cylindrical. Elliptical and bowtie shaped reduced cloaks are presented to verify the effectiveness of the method. Prominent scattering reduction is observed for both examples.

Considering the practical realization, transformations continuous in the whole space must be the identity operation outside certain volume, and thus they can only manipulate fields locally. Discontinuous transformations are naturally considered to break the limitation. We study the possible reflections from such a transformation medium due to a discontinuous transformation by a new concept of inverse transformation. This way, the reflection falls into the framework of transformation optics as well. A necessary and sufficient condition for no reflection is derived as a special case.

Unlike the invisibility realized by perfect cloaks, cloaking an object over a dielectric half-space has advantages in some particular applications. Starting from a perfect cloak, a half-space cloak is designed to achieve this. In our design, two matching strips embedded in the dielectric ground are used to induce proper reflection in the upper air space, so that the reflected field is the same as that from the bare dielectric ground.

Cloaks obtained from singular transformations and even reduced models all have null principal value in their material parameters, making invisibility inherently very narrowband. In contrast, a carpet cloak designed by only coordinate deformation does not have the narrowband issue, and can perform well in a broad spectrum. The invisibility accomplished by the carpet cloak is also for the half-space case as our previous design. In this part, we extend the original version of a carpet cloak above a PEC sheet to a general dielectric ground.

**Key words:** Transformation optics, Maxwell's equations, invisibility cloak, half-space cloak, reduced cloak, inverse transformation, carpet cloak.

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# Table of the Contents

Abstract .....	III
Acknowledgments.....	V
List of publications .....	IX
Acronyms.....	XI
1. Introduction .....	1
1.1 Background .....	1
1.2 Thesis Outline.....	3
References.....	4
2. Theory of Transformation Optics .....	9
2.1 Elements of Coordinate Transformation .....	9
2.2 Form Invariance of Maxwell's Equations.....	13
2.3 Transformation Optics with Orthogonal Coordinates .....	20
2.4 Spherical and Cylindrical Invisibility Cloaks.....	22
References.....	26
3. Reduced Two-dimensional Cloaks of Arbitrary Shapes .....	29
3.1 Reduced Cylindrical Cloaks.....	29
3.2 Reduced Two-dimensional Cloaks of Arbitrary Shapes .....	30
References.....	35
4. Inverse Transformation Optics and its Applications .....	37
4.1 Inverse Transformation Optics .....	37
4.2 Reflection Analysis for 2D Finite-Embedded Coordinate Transformation.....	38
References.....	43
5. Half-space Cloaking on a Dielectric Ground .....	45
5.1 Introduction to Half-space Cloaking.....	45
5.2 Semi-cylindrical Half-space Cloak.....	46
5.3 Carpet Cloaking on a Dielectric Half-space .....	50
References.....	53
6. Conclusion and Future Work.....	55
7. Summary of Contributions .....	57





# List of publications

## List of publications included in the thesis

- A. **Pu Zhang**, Yi Jin, and Sailing He, “Cloaking an object on a dielectric half-space,” *Opt. Express*, **16**, 3161 (2008).
- B. **Pu Zhang**, Yi Jin, and Sailing He, “Obtaining a nonsingular two-dimensional cloak of complex shape from a perfect three-dimensional cloak,” *Appl. Phys. Lett.*, **93**, 243502 (2008).
- C. **Pu Zhang**, Yi Jin, and Sailing He, “Inverse Transformation Optics and Reflection Analysis for Two-Dimensional Finite-Embedded Coordinate Transformation,” *IEEE J. Sel. Top. Quant. Electron.*, **16**, 427 (2010).
- D. **Pu Zhang**, Michaël Lobet, and Sailing He, “Carpet cloaking on a dielectric half-space,” *Opt. Express*, **18**, 18158 (2010).

## List of publications not included in the thesis

- E. Yi Jin, **Pu Zhang**, and Sailing He, “Abnormal enhancement of electric field inside a thin permittivity-near-zero object in free space,” *Phys. Rev. B*, **82**, 075118 (2010).
- F. Yi Jin, **Pu Zhang**, and Sailing He, “Squeezing electromagnetic energy with a dielectric split ring inside a permeability-near-zero metamaterial,” *Phys. Rev. B*, **81**, 085117 (2010).
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- I. Zhili Lin, Jiechen Ding, and **Pu Zhang**, “Aberration-free two-thin-lens systems based on negative-index materials,” *Chin. Phys. B*, **17**, 954 (2008).
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- K. Xin Hu, **Pu Zhang**, and Sailing He, “Dual structure of composite right/left-handed transmission line,” *J. Zhejiang Univ. Sci. B*, **7**, 1777 (2006).
- L. Xin Hu and **Pu Zhang**, “A novel dual-band balun based on the dual structure of composite right/left handed transmission line,” International Symposium on Biophotonics, Nanophotonics and Metamaterials 2006, Hangzhou.



# Acronyms

2D	Two-dimensional
3D	Three-dimensional
CP	Constitutive Parameters
CS	Coordinate System
EM	Electromagnetic
FECT	Finite Embedded Coordinate Transformation
FEM	Finite Element Method
LHS	Left-hand Side
PEC	Perfect Electric Conductor
RHS	Right-hand Side
TE	Transverse Electric
TM	Transverse Magnetic
TrM	Transformation Medium or Transformation Media
TrO	Transformation Optics



# 1. Introduction

## 1.1 Background

The method of Transformation Optics (TrO) was invented and began to widely receive attention when two groundbreaking papers [1, 2] were published in the same issue of Science magazine. Two eye-catching devices of invisibility cloaks were proposed in these two papers. Invisibility cloaks can make illuminating light flow around a closed concealment volume, where objects are hidden, and retain the propagating features of light as if nothing exists in space, so that observers outside the device will not sense any electromagnetic/light disturbance. To achieve this unusual illusion, both J. B. Pendry and U. Leonhardt made use of the form invariance of governing equations, Maxwell's equations of electromagnetic (EM) theory and Helmholtz equation of geometrical optics, respectively. Thereafter the equivalence between constitutive parameters (CP) and geometry for light was established. The techniques were later recognized as transformation optics in general. The form invariance of Maxwell's equations and invisible bodies being not new in EM research, it is the marriage of the ingredients of TrO and invisibility cloak that contributes to the significance of the two papers.

As an essential property of Maxwell's equations (we will focus on the EM theory version of TrO onwards), the form invariance under transformation of Coordinate System (CS) has already been studied in several related papers [3-5] in the 1920s. In the early 1960s, Dolin [6], Post [7] and Lax-Nelso [8] discovered the fundamental idea of TrO. These results, however, have been forgotten for decades until they were rediscovered in the mid 1990s [9-11]. More detailed historical review can be found in reference [12]. Inspired by those previous works, the form invariance is innovatively applied in the design of invisibility cloaks and plenty of follow-up designs. In most cases, the desired geometry for light (characteristic of the propagation of light) is introduced by some specifically conceived coordinate transformations from a space with known light propagation behaviors. The geometry is then embodied by transformation media (TrM) according to the equivalence between CP and geometry. There are also instances [13-16] where the geometry is described by metrics, borrowing the knowledge from general relativity. Formally, metric formulation includes transformation formulation only as a special case.

Invisibility is commonly known as a popular and fancy factor in science fictions rather than a serious science or engineering subject. However, there were in fact scattered research works on invisibility for some particular situations, like zero scattering of small ellipsoids [17] and conductivity invisibility with respect to voltage and current measurements [18, 19]. The latter one utilized the idea of change of variables, sharing the same idea with TrO. Compared with previous results, the invisibility cloak based on TrO is in the framework of exact EM theory, and hence in principle can be called the perfect invisibility cloak for all spectra of EM waves, including microwave and visible frequencies. The hope for building a true invisibility cloak has been ignited

since the publication of these works.

According to the formulation of TrO, the CP of the obtained media generally turn out to have high anisotropy and assume a much broader range of values than exhibited by natural materials. The experimental realization of TrO generated devices therefore relies heavily on artificial materials, *i.e.*, metamaterials [20]. The first demonstration of invisibility cloak [21] was in fact constructed with split ring resonators, which are the most used unit cells of metamaterials. On the other hand, until the invention of TrO, a systematic method for designing metamaterial based devices is lacking. Now people can exploit the greatest potential of metamaterials by using TrO to create novel devices otherwise impossible. The mutual dependence makes TrO and metamaterials closely related to each other.

Months after the theoretical paper, a demonstration of invisibility cloak comprised of metamaterials was reported by D. R. Smith *et.al* [21]. This experiment triggered the explosion of research on TrO and invisibility cloaks. The amazing properties of the perfect cloaks attracted great attention from researchers around the world. A lot of analytical and numerical analysis was conducted to reveal the interesting characteristics, such as perfection of invisibility [22], EM scattering in case of imperfection [23], dispersion behavior [24], *etc.* For a perfect cloak, the material parameters have very extreme values. The implementation is not possible for natural materials and quite difficult even when metamaterials are used. Many simplified or reduced models [25, 26] were proposed as a compromise between experimental feasibility and performance. In addition, the working bandwidth is a serious concern for cloaking operations [27-29]. As shown in the first experimental demonstration, reduced models can be fabricated with metamaterials for a particular polarized EM excitation. Considerable efforts in experimental realization were made towards higher frequencies, easier fabrication, and better performance [30-34]. Apart from the cloaks with a closed concealment volume, the curiosity in invisibility leads people to find other schemes for cloaking. The following ones are also designed with TrO. The carpet cloak [35] was much investigated due to its easier fabrication and potential wideband behavior. Geometrical optical cloaking [2, 36, 37] based on optical conformal mapping method enables invisibility in the sense of only light rays, but not in the sense of phase unless some negative index media are involved [38]. External cloaking [39] was achieved by optical cancelling with complementary media. Techniques other than TrO are used as well. Plasmonic coating induces transparency of particles whose sizes are small compared with wavelength [40]. The easy realization with metals in optical spectra facilitates possible applications [41, 42] in nanophotonics. Microwave transmission line network cloaks [43] have simple structures, low scattering, and broad operating bands, however, the objects must be small enough to fit into network grids for hiding. Besides passive ones, active cloaking strategy was proposed [44]. The incident wave (must be known in advance) is cancelled by deploying active sources accordingly.

Along with the rapid development of invisibility cloaks, people have taken a growing interest in the theory of TrO. Various applications have been proposed, such as perfect imaging [45-47],

waveguide transition [48, 49], beam control [50, 51], source manipulation (including antennas) [52-54], optical illusion [55], singularity transmutation [56] *etc.* The integration of TrO with other branches of photonics brings about fruitful results. Transformational plasmonics [57, 58] can help create surface plasmonic devices which are almost free from inelastic scattering. In the theoretical aspect, TrO was generalized to include a time variable as well as non-reciprocal and indefinite media [59-61]. The mathematical formulation implies that TrO can not be a physical phenomenon only for electromagnetic waves. The form invariance is actually ubiquitous for all wave phenomena. Parallel transformation theories were established for acoustic [62], elastic [63] and matter waves [64].

## 1.2 Thesis Outline

The present thesis consists of two parts. In the first part, the background and fundamentals of TrO and invisibility cloaks are elaborated. In the second part, the author's research works on TrO and invisibility cloaks are described in detail. Time convention  $\exp(-i\omega t)$  is used when working in frequency domain.

In the first part, the theory of TrO is developed after the background introduction. The essential idea of TrO is revealed, and working formulas are derived from Maxwell's equations and coordinate transformation. The perfect cloak design is treated at last to show how to apply the theory, and also as a starting point for the invisibility cloak study.

In the second part, the works from our four research papers are discussed. Detailed description is provided for both the results and the technical methods:

1. A method of designing two-dimensional (2D) reduced cloaks of complex shapes is proposed. The effectiveness is validated by Finite Element Method (FEM) based numerical analysis of an elliptic and a bowtie-shaped cloak.
2. The concept of inverse transformation optics is introduced to analyze reflection at an interface of a discontinuous transformation medium. A necessary and sufficient condition is obtained for the special situation of no reflection.
3. A half-space cloak is proposed to achieve cloaking above a dielectric ground. Two carefully designed strips are inserted underneath for reflection coefficient matching.
4. Carpet cloaking is extended to operation on a dielectric ground. A conformal mapping generated by a complex function is utilized to calculate the CP.

Finally, the conclusions of the thesis and future work in this field, followed by a summary of the author's contributions to the published papers, are given at the end of the thesis.

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## 2. Theory of Transformation Optics

The fundamental theory of TrO is developed, starting from Maxwell's equations and a coordinate transformation between two spaces. Multivariable calculus is enough for the derivation of all formulas. Necessary concepts from differential geometry will be clearly defined and explained. The theoretical model of EM media discussed in the present thesis is limited to be linear and anisotropic. Time dependent and magnetoelectric responses of materials are not considered here, but can be found in generalized formulations (see Section 1.1 Background). The derivation of TrO has been treated in literature [1-3]. It is, however, still helpful to formulate the theory with consistent notations, so that the thesis is self-contained as a complete work. The last section will describe the design procedure of TrO and show how the geometrical features of the transformation are embodied for light by TrM. Perfect cloaks are taken for example.

### 2.1 Elements of Coordinate Transformation

In the mathematical language, the theory of TrO discusses the relation between quantities in two spaces, whose coordinate variables are connected by a coordinate transformation. The CSs used in the two spaces do not depend on what the transformation is, and can be chosen for convenience. Cartesian CS is adopted for both spaces in current development, as it is the simplest CS. Results for other orthogonal CSs are obtained in Section 2.3 by linking to the current Cartesian coordinate case. Quantities in the spaces can be given physical meanings when they satisfy corresponding governing equations. Although the two spaces are mathematically identical, we would like to distinguish them by names of virtual space and physical space according to their roles in TrO. Quantities in the physical space are denoted by notations with prime, while no prime is used for virtual space quantities.

Assume the coordinate variables of the two spaces are related by the transformation function

$$\mathbf{r}' = \mathbf{r}'(\mathbf{r}), \quad (2.1)$$

where  $\mathbf{r}' = p^i \mathbf{x}'_i$  and  $\mathbf{r} = p^i \mathbf{x}_i$  are radius vectors expanded with vector bases of Cartesian CSs. The transformation expressed in component form looks like

$$p'^i = p^i(p^1, p^2, p^3).$$

Before moving on we would like to state some conventions of index usage at an early stage to avoid any possible confusion caused by index notations. Single (stand alone) indices are usually meant to represent a whole set of quantities by taking all possible index values (1, 2 and 3). A same index occurring twice, one subscript and one superscript, implies a summation with the index running over all possible values (Einstein summation convention). Whether subscript or superscript should be used follows the rules of tensor notation, which are explained later in this

section. These conventions ensure that indices in a physical formula are always balanced, *i.e.*, both sides of the equation have the same indices at the same positions.

The two CSs have the same role in the coordinate transformation. Equivalent to Eq.(2.1), the coordinate variables in the virtual space are also functions of those in the physical space. For most applications, transformations are in good form that the function system is invertible. We note that the set of coordinate variables in one space, say the physical space, can be interpreted as another CS (usually curvilinear) in the other space, the virtual space. Curved coordinate grid lines can be drawn to represent the transformation geometrically as illustrated in Fig. 2.1(a) (the third dimension can not be visualized on a sheet of paper). Quantities in the virtual space are then also functions of the curvilinear coordinate variables.

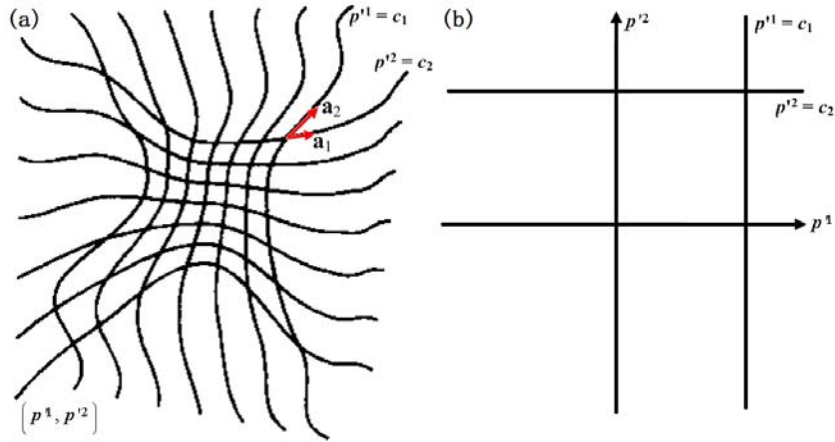


Figure 2.1: Coordinate grid lines of  $(p^1, p^2)$  in the virtual space (a) and the physical space (b).

Let's start with studying the curvilinear CS defined by  $p^i$  in the virtual space. More systematic discussions on curvilinear CSs can be found in Ref. [4, 5]. Radius vector  $\mathbf{r}$  is a vector field and functions of  $p^i$ . Its partial derivatives with respect to  $p^i$  are defined as *unitary vectors*,

$$\mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial p^i}. \quad (2.2)$$

Their geometrical features are straightforward from the definition of partial derivative and shown in Fig. 2.1(a). At any point, a unitary vector is parallel to the tangential of the corresponding coordinate grid line and in the direction of increasing the coordinate variable. They form a vector basis in the space. The parallelepiped spanned by the three unitary vectors has volume

$$V = \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = \mathbf{a}_3 \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = \mathbf{a}_2 \cdot (\mathbf{a}_3 \times \mathbf{a}_1). \quad (2.3)$$

Thanks to Eq. (2.3), one can define *reciprocal unitary vectors* explicitly

$$\mathbf{a}^1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{V}, \mathbf{a}^2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{V}, \mathbf{a}^3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{V}. \quad (2.4)$$

They form another vector basis in the space. From the definition, reciprocal unitary vectors and unitary vectors have the following obvious orthogonality property

$$\mathbf{a}^i \cdot \mathbf{a}_j = \delta_j^i, \quad (2.5)$$

where subscript and superscript indicate unitary and reciprocal unitary vectors respectively, and  $\delta_j^i$  is the Kronecker delta symbol. The total differential change in radius vector can be expressed as the sum of all partial differential changes,

$$d\mathbf{r} = \mathbf{a}_i dp^i. \quad (2.6)$$

Eq. (2.6) is for the most general case (with differential changes in all coordinate variables). If there is only differential change in one coordinate variable, one term in the equation suffices to represent the differential change in  $\mathbf{r}$ . Reciprocal unitary vector basis can also be used to expand radius vector

$$d\mathbf{r} = \mathbf{a}^i dp'_i, \quad (2.7)$$

which acts as the definition of the components  $dp'_i$ .

The curvilinear CS  $p^i$  and the Cartesian CS  $p^i$  in the virtual space are related to each other by the coordinate transformation in Eq. (2.1). Quantities in the space may have different components under the two CSs. The connection between the components is usually revealed through some quantities independent of CSs. Radius vector  $\mathbf{r}$  is a good example. The connection between the unitary vectors of the two CSs are found by using the chain rule of derivative,

$$\mathbf{x}_i = \frac{\partial \mathbf{r}}{\partial p^i} = \frac{\partial \mathbf{r}}{\partial p'^j} \frac{\partial p'^j}{\partial p^i} = \frac{\partial p'^j}{\partial p^i} \mathbf{a}_j. \quad (2.8)$$

We define the set of derivatives, relating the two sets of unitary vectors, as the *Jacobian matrix*,

$$\mathbf{\Lambda} = \left( \Lambda_i^{j'} \right) = \left( \frac{\partial p'^j}{\partial p^i} \right). \quad (2.9)$$

According to this definition, it is easy to verify that the inverse of  $\mathbf{\Lambda}$  is

$$\mathbf{\Lambda}^{-1} = \left( \Lambda_j^i \right) = \left( \frac{\partial p^i}{\partial p'^j} \right). \quad (2.10)$$

Using the inverse, we have

$$\mathbf{a}_i = \Lambda_i^{j'} \mathbf{x}_j. \quad (2.11)$$

The total differential change in  $\mathbf{r}$  is independent of CSs too. For the Cartesian CS,

$$d\mathbf{r} = \mathbf{x}_j dp^j = \Lambda_j^{i'} \mathbf{a}_i dp^j = \mathbf{a}_i \Lambda_j^{i'} dp^j.$$

Comparing with Eq. (2.6) shows

$$dp^i = \Lambda_j^{i'} dp^j. \quad (2.12)$$

Again using the inverse Eq. (2.10), we have

$$\Lambda_j^i dp'^j = \Lambda_j^i \Lambda_k^{j'} dp^k = \delta_k^i dp^k = dp^i. \quad (2.13)$$

Eqs. (2.11) and (2.12) present two different ways of transformation from the Cartesian CS to the curvilinear CS. Quantities that transform like  $\mathbf{a}_i$  are *covariant*, and quantities that transform like  $dp^i$  are *contravariant*. The positions of indices are designed to manifest the distinct ways of transformation. Subscript and superscript indices indicate covariant and contravariant, respectively. More than one index may be required to denote components of a complex quantity. In general, a quantity, whose components follow the two transformation rules with respect to all the indices, is called a *tensor*. A quantity without index and invariant of CSs is a scalar, the zero-th order tensor. To see if a quantity is a tensor, we should identify how the components transform by using its physical or geometrical properties and check against the two rules. Indices for all the tensors we have encountered are placed properly according to their tensorial nature.

For a tensor, the Cartesian components are usually easy to calculate. Therefore one can determine the curvilinear components by transforming from the Cartesian components. Consider the *line element*  $ds$ , the length of  $d\mathbf{r}$ ,

$$(ds)^2 = \mathbf{a}_i \cdot \mathbf{a}_j dp^i dp^j = \mathbf{a}^i \cdot \mathbf{a}^j dp'_i dp'_j, \quad (2.14)$$

where Eqs. (2.6) and (2.7) are substituted. The scalar products of (reciprocal) unitary vectors are defined as *metric coefficients*, characterizing the distance between points in the space

$$(ds)^2 = g_{ij} dp^i dp^j = g^{ij} dp'_i dp'_j. \quad (2.15)$$

In Eq.(2.15), all quantities except metric coefficients are tensors to our knowledge. This implies that the metric coefficients themselves are also tensors. The metric tensors under the Cartesian CS are simply  $\delta_{ij}$  and  $\delta^{ij}$ . Then under the curvilinear CS the metric tensors are determined through transformation

$$g_{ij} = \Lambda_i^k \Lambda_j^l \delta_{kl}, \quad g^{ij} = \Lambda_k^i \Lambda_l^j \delta^{kl}. \quad (2.16)$$

The summations in Eq. (2.16) remind us of matrix products, *e.g.*,

$$\begin{aligned} k_{ij} = l_{ik} m_{kj} &\Rightarrow \mathbf{K} = \mathbf{L}\mathbf{M}, \\ k_{ij} = l_{ki} m_{kj} &\Rightarrow \mathbf{K} = \mathbf{L}^T \mathbf{M}, \\ k_{ij} = l_{ik} m_{jk} &\Rightarrow \mathbf{K} = \mathbf{L}\mathbf{M}^T. \end{aligned}$$

Eq. (2.16) is then expressed as matrix products

$$\begin{aligned} \mathbf{g} = (g_{ij}) &= (\Lambda^{-1})^T \mathbf{I} \Lambda^{-1} = (\Lambda^{-1})^T \Lambda^{-1}, \\ \mathbf{g}^{-1} = (g^{ij}) &= \Lambda \mathbf{I} \Lambda^T = \Lambda \Lambda^T. \end{aligned} \quad (2.17)$$

It should be noted that matrix formulation is just a method of grouping components for representation clarity. Which components that a matrix is holding can not be told by the matrix itself, and additional information must be provided. Observation of the RHSs in Eq. (2.17) shows



that the two metric tensors are inverse matrices to each other. Another very useful property of metric tensors is its function of raising and lowering an index. Comparison of the following equations

$$\begin{aligned}\mathbf{a}_i \cdot d\mathbf{r} &= \mathbf{a}_i \cdot \mathbf{a}_j dp'^j = g_{ij} dp'^j, \\ \mathbf{a}_i \cdot d\mathbf{r} &= \mathbf{a}_i \cdot \mathbf{a}^j dp'_j = \delta_i^j dp'_j = dp'_i, \\ \mathbf{a}^i \cdot d\mathbf{r} &= \mathbf{a}^i \cdot \mathbf{a}_j dp'^j = \delta_j^i dp'^j = dp'^i, \\ \mathbf{a}^i \cdot d\mathbf{r} &= \mathbf{a}^i \cdot \mathbf{a}^j dp'_j = g^{ij} dp'_j,\end{aligned}$$

proves to us

$$g_{ij} dp'^j = dp'_i, g^{ij} dp'_j = dp'^i. \quad (2.18)$$

Unitary vectors, having metric tensors as their scalar products, generally do not have unit lengths. Their corresponding unit vectors are obtained by normalization

$$\mathbf{u}_i = \frac{\mathbf{a}_i}{\sqrt{\mathbf{a}_i \cdot \mathbf{a}_i}} = \frac{\mathbf{a}_i}{\sqrt{g_{ii}}}, \quad \mathbf{u}^i = \frac{\mathbf{a}^i}{\sqrt{\mathbf{a}^i \cdot \mathbf{a}^i}} = \frac{\mathbf{a}^i}{\sqrt{g^{ii}}}.$$

Considering the unity magnitudes of these vectors, it is sometimes convenient to use them as bases for vector expansion. However, the unit vectors and the corresponding expansion components do not transform like tensors.

The volume in Eq. (2.3) is important when deciding differential volume element, and will be frequently used in the next section. By applying the similar technique for calculating metric tensors, we get a useful formula for the volume. Substitute Eq. (2.11) into Eq. (2.3),

$$V = \Lambda_1^i \mathbf{x}_i \cdot (\Lambda_2^j \mathbf{x}_j \times \Lambda_3^k \mathbf{x}_k) = \Lambda_1^i \Lambda_2^j \Lambda_3^k \mathbf{x}_i \cdot (\mathbf{x}_j \times \mathbf{x}_k). \quad (2.19)$$

The triple scalar products of the Cartesian unitary vectors take nonvanishing values  $\pm 1$  when  $[i, j, k]$  is a permutation of set  $\{1, 2, 3\}$ , and the sign depends on the signature of the permutation. The RHS of Eq. (2.19) is equivalent to the definition of matrix determinant, *i.e.*,

$$V = \det(\Lambda^{-1}) = \text{sgn}(V) \sqrt{g}. \quad (2.20)$$

The last step results from the fact  $g = \det(\mathbf{g}) = [\det(\Lambda^{-1})]^2$  implied by Eq. (2.17). The sign can be easily decided according to Eq. (2.3), positive for a right-hand CS and negative for a left-hand CS.

## 2.2 Form Invariance of Maxwell's Equations

Equipped with the toolbox of coordinate transformation developed in Section 2.1, we are now ready to treat the core part of TrO, namely, form invariance of Maxwell's equations. We are moving on to work with the two spaces, the virtual space and the physical space, introduced in Section 2.1. The two spaces are equivalently related by the coordinate transformation Eq. (2.1), but play different

roles when EM fields are introduced. The virtual space is supposed to be filled with anisotropic media with CP ( $\boldsymbol{\epsilon}$ ,  $\boldsymbol{\mu}$ ) and have known EM fields characterized by ( $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{D}$ ,  $\mathbf{B}$ ) in it. They satisfy the following Maxwell's equations and the constitutive relations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.21)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \quad (2.22)$$

$$\nabla \cdot \mathbf{D} = \rho, \quad (2.23)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.24)$$

$$\mathbf{D} = \boldsymbol{\epsilon} \mathbf{E}, \quad \mathbf{B} = \boldsymbol{\mu} \mathbf{H}. \quad (2.25)$$

These physical quantities, including fields, sources and CP, are independent of CSs, but may have different components when expressed in terms of different vector bases, *e.g.*,

$$\mathbf{D} = d^i \mathbf{a}_i = d_i \mathbf{a}^i = D^i \mathbf{x}_i. \quad (2.26)$$

By form invariance under a coordinate transformation, we attempt to construct a new set of EM fields, sources and CP in the physical space using those in the virtual space, so that in the physical space the form of Maxwell's equations is conserved for the new set. The media specified in the new set in the physical space are called *transformation media*. In practice, it is the TrM that are fabricated to realize desired EM properties. Therefore this space is called the *physical space*, while the other space is called the *virtual space*.

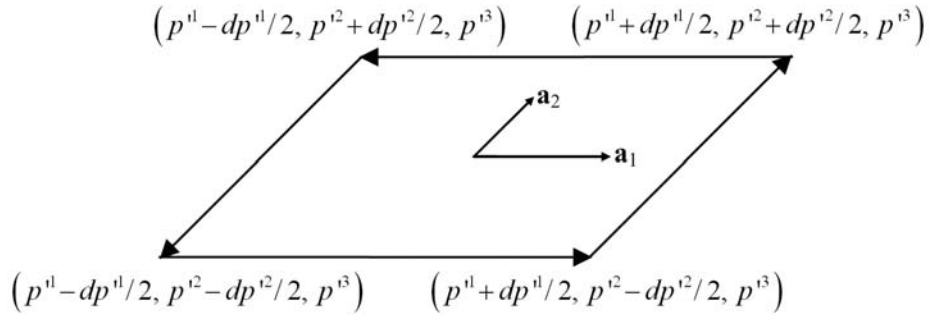


Figure 2.2: A differential integration element perpendicular to  $\mathbf{a}^3$ .

To construct such a set, it is natural to start with formulating Maxwell's equations in the curvilinear CS. The key point is to express the differential operators in the curvilinear CS. For the curl in Eq. (2.22), this can be achieved with the help of the integral identity

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \nabla \times \mathbf{H} \cdot d\mathbf{a}. \quad (2.27)$$

Let us consider the integration over the differential element shown in Fig. 2.2. The LHS of Eq. (2.27) becomes

$$\begin{aligned}
 \oint \mathbf{H} \cdot d\mathbf{r} &= (\mathbf{H} \cdot \mathbf{a}_1 dp^1)_{p^2-dp^2/2} + (\mathbf{H} \cdot \mathbf{a}_2 dp^2)_{p^1+dp^1/2} \\
 &\quad - (\mathbf{H} \cdot \mathbf{a}_1 dp^1)_{p^2+dp^2/2} - (\mathbf{H} \cdot \mathbf{a}_2 dp^2)_{p^1-dp^1/2} \\
 &= \left[ \mathbf{H} \cdot \mathbf{a}_1 dp^1 - \frac{\partial(\mathbf{H} \cdot \mathbf{a}_1 dp^1)}{\partial p^2} \frac{dp^2}{2} \right] + \left[ \mathbf{H} \cdot \mathbf{a}_2 dp^2 + \frac{\partial(\mathbf{H} \cdot \mathbf{a}_2 dp^2)}{\partial p^1} \frac{dp^1}{2} \right] \\
 &\quad - \left[ \mathbf{H} \cdot \mathbf{a}_1 dp^1 + \frac{\partial(\mathbf{H} \cdot \mathbf{a}_1 dp^1)}{\partial p^2} \frac{dp^2}{2} \right] - \left[ \mathbf{H} \cdot \mathbf{a}_2 dp^2 - \frac{\partial(\mathbf{H} \cdot \mathbf{a}_2 dp^2)}{\partial p^1} \frac{dp^1}{2} \right] \\
 &= \left( h_1 dp^1 - \frac{1}{2} \frac{\partial h_1}{\partial p^2} dp^1 dp^2 \right) + \left( h_2 dp^2 + \frac{1}{2} \frac{\partial h_2}{\partial p^1} dp^1 dp^2 \right) \\
 &\quad - \left( h_1 dp^1 + \frac{1}{2} \frac{\partial h_1}{\partial p^2} dp^1 dp^2 \right) - \left( h_2 dp^2 - \frac{1}{2} \frac{\partial h_2}{\partial p^1} dp^1 dp^2 \right) \\
 &= \left( \frac{\partial h_2}{\partial p^1} - \frac{\partial h_1}{\partial p^2} \right) dp^1 dp^2, \tag{2.28}
 \end{aligned}$$

where  $h_i$  are covariant components of the magnetic field. In Eq. (2.28), the scalar product along every one of the four laterals is obtained by adding a variational part to that evaluated at the element center  $p^i$ . The RHS of Eq. (2.27) is

$$\int \nabla \times \mathbf{H} \cdot d\mathbf{a} = (\nabla \times \mathbf{H}) \cdot (\mathbf{a}_1 dp^1 \times \mathbf{a}_2 dp^2).$$

Eq. (2.27) is then simplified to

$$(\nabla \times \mathbf{H}) \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = \frac{\partial h_2}{\partial p^1} - \frac{\partial h_1}{\partial p^2}. \tag{2.29}$$

Note that  $p^i$  is the Cartesian CS in the physical space. Thus the RHS of Eq. (2.29) is the 3<sup>rd</sup> Cartesian component of the curl operator in the physical space.  $h_i$  are accordingly recognized as the Cartesian components of the magnetic field in the physical space,

$$H'_i = h_i. \tag{2.30}$$

The LHS and the RHS being from different spaces, Eq. (2.30) is only a numerical equality without physical implication. In a Cartesian CS, all vector bases (covariant or contravariant, unit or unitary) coincide into a constant basis. For the Cartesian components  $H'_i$ , the index positions do not make any difference. Here subscript is used for the sake of formal index balance. Substitute the definition Eq. (2.30) into Eq. (2.29),

$$(\nabla \times \mathbf{H}) \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = \frac{\partial H'_2}{\partial p^1} - \frac{\partial H'_1}{\partial p^2} = (\nabla' \times \mathbf{H}')^3. \tag{2.31}$$

Parallel studies for Eq. (2.21) will result in the counterpart definition of the electric field in the physical space

$$E'_i = e_i, \tag{2.32}$$

where  $e_i$  are the covariant components of the electric field in the virtual space. As for the LHS of Eq. (2.29), substitution of Eq. (2.22) leads to

$$(\nabla \times \mathbf{H}) \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = \left( \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) \cdot (\mathbf{a}_1 \times \mathbf{a}_2).$$

Expand  $\mathbf{D}$  and  $\mathbf{J}$  with vector basis  $\mathbf{a}_i$  and only one term survives according to Eqs. (2.3) and (2.20),

$$\begin{aligned} (\nabla \times \mathbf{H}) \cdot (\mathbf{a}_1 \times \mathbf{a}_2) &= \left( \frac{\partial d^i}{\partial t} + j^i \right) \mathbf{a}_i \cdot (\mathbf{a}_1 \times \mathbf{a}_2) \\ &= \left( \frac{\partial d^3}{\partial t} + j^3 \right) \mathbf{a}_3 \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = \left( \frac{\partial d^3}{\partial t} + j^3 \right) \det(\Lambda^{-1}). \end{aligned}$$

Eliminate  $d^3$  with the constitutive relation  $\mathbf{D} = \boldsymbol{\varepsilon} \mathbf{E}$ ,

$$d^3 = \Lambda_i^{3'} D^i = \Lambda_i^{3'} \varepsilon_j^i E^j = \Lambda_i^{3'} \varepsilon_j^i \Lambda_k^j e^k = \Lambda_i^{3'} \varepsilon_j^i \Lambda_k^j g^{kl} e_l.$$

Here we use the Cartesian components of the permittivity tensor. The contravariant component  $d^3$  is first transformed to Cartesian components  $D^i$  by using Eqs. (2.11) and (2.26), and the Cartesian components of the electric field are transformed back to curvilinear components afterwards. Put  $d^3$  back and replace  $e_l$  with  $E'_l$  (Eq. (2.32)),

$$(\nabla \times \mathbf{H}) \cdot (\mathbf{a}_1 \times \mathbf{a}_2) = \Lambda_i^{3'} \varepsilon_j^i \Lambda_k^j g^{kl} \det(\Lambda^{-1}) \frac{\partial E'_l}{\partial t} + \det(\Lambda^{-1}) j^3. \quad (2.33)$$

Combine Eqs. (2.31) and (2.33),

$$(\nabla \times \mathbf{H}')^3 = \Lambda_i^{3'} \varepsilon_j^i \Lambda_k^j g^{kl} \det(\Lambda^{-1}) \frac{\partial E'_l}{\partial t} + \det(\Lambda^{-1}) j^3. \quad (2.34)$$

Eq. (2.34) infers

$$(\nabla \times \mathbf{H}')^m = \Lambda_i^{m'} \varepsilon_j^i \Lambda_k^j g^{kl} \det(\Lambda^{-1}) \frac{\partial E'_l}{\partial t} + \det(\Lambda^{-1}) j^m. \quad (2.35)$$

In order to establish the form invariance, we define the electric current density by its Cartesian components

$$J^m = \det(\Lambda^{-1}) j^m. \quad (2.36)$$

Eq. (2.35) then can be written in matrix form

$$\begin{aligned} \nabla \times \mathbf{H}' &= \boldsymbol{\Lambda} \boldsymbol{\varepsilon} \boldsymbol{\Lambda}^{-1} \mathbf{g}^{-1} \det(\Lambda^{-1}) \frac{\partial \mathbf{E}'}{\partial t} + \mathbf{J}' \\ &= \boldsymbol{\Lambda} \boldsymbol{\varepsilon} \boldsymbol{\Lambda}^T \det(\Lambda^{-1}) \frac{\partial \mathbf{E}'}{\partial t} + \mathbf{J}'. \end{aligned} \quad (2.37)$$

Eq. (2.37) assumes the form of Eq. (2.22) as long as the following permittivity tensor for the TrM is introduced,

$$\boldsymbol{\varepsilon}' = \boldsymbol{\Lambda} \boldsymbol{\varepsilon} \boldsymbol{\Lambda}^T \det(\Lambda^{-1}). \quad (2.38)$$

The magnetic permeability tensor can be found similarly by studying the other curl equation,

$$\boldsymbol{\mu}' = \boldsymbol{\Lambda} \boldsymbol{\mu} \boldsymbol{\Lambda}^T \det(\boldsymbol{\Lambda}^{-1}). \quad (2.39)$$

Eq. (2.37) finally turns out to be,

$$\nabla' \times \mathbf{H}' = \boldsymbol{\varepsilon}' \frac{\partial \mathbf{E}'}{\partial t} + \mathbf{J}'. \quad (2.40)$$

In Eqs. (2.30), (2.32) and (2.36), curvilinear components are used for definition. In practice, it is more convenient to use Cartesian components, which are independent of the curvilinear CS. Using Eqs. (2.11) and (2.12), the curvilinear components are transformed to Cartesian,

$$E'_i = e_i = \Lambda_i^j E_j, H'_i = h_i = \Lambda_i^j E_j, J'^i = j^i \det(\boldsymbol{\Lambda}^{-1}) = \Lambda_j^i J^j \det(\boldsymbol{\Lambda}^{-1}).$$

They are further put in matrix form,

$$\mathbf{E}' = (\boldsymbol{\Lambda}^{-1})^T \mathbf{E}, \mathbf{H}' = (\boldsymbol{\Lambda}^{-1})^T \mathbf{H}, \mathbf{J}' = \boldsymbol{\Lambda} \det(\boldsymbol{\Lambda}^{-1}) \mathbf{J}. \quad (2.41)$$

The same results as above can be derived by another similar but shorter method. Starting with Eq. (2.31), expand  $\nabla \times \mathbf{H}$  into

$$\nabla \times \mathbf{H} = (\nabla \times \mathbf{h})^i \mathbf{a}_i = (\nabla \times \mathbf{H})^i \mathbf{x}_i, \quad (2.42)$$

where the notation  $(\nabla \times \mathbf{h})^i$  is coined to denote the curvilinear components of  $\nabla \times \mathbf{H}$ , and does not mean a curl operation. The two sets of components are related by

$$(\nabla \times \mathbf{h})^i = \Lambda_j^i (\nabla \times \mathbf{H})^j. \quad (2.43)$$

Inserting Eqs. (2.42) and (2.43), Eq. (2.31) becomes

$$\begin{aligned} (\nabla' \times \mathbf{H}')^3 &= (\nabla \times \mathbf{h})^i \mathbf{a}_i \cdot (\mathbf{a}_1 \times \mathbf{a}_2) \\ &= \det(\boldsymbol{\Lambda}^{-1}) (\nabla \times \mathbf{h})^3 = \det(\boldsymbol{\Lambda}^{-1}) \Lambda_j^3 (\nabla \times \mathbf{H})^j. \end{aligned}$$

It can be inferred

$$(\nabla' \times \mathbf{H}')^i = \det(\boldsymbol{\Lambda}^{-1}) \Lambda_j^i (\nabla \times \mathbf{H})^j.$$

Put it in matrix form,

$$\nabla' \times \mathbf{H}' = \det(\boldsymbol{\Lambda}^{-1}) \boldsymbol{\Lambda} \nabla \times \mathbf{H}. \quad (2.44)$$

Substitute Eq. (2.22) into Eq. (2.44),

$$\begin{aligned} \nabla' \times \mathbf{H}' &= \det(\boldsymbol{\Lambda}^{-1}) \boldsymbol{\Lambda} \left( \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) = \det(\boldsymbol{\Lambda}^{-1}) \boldsymbol{\Lambda} \left( \frac{\partial \boldsymbol{\varepsilon} \mathbf{E}}{\partial t} + \mathbf{J} \right) \\ &= \det(\boldsymbol{\Lambda}^{-1}) \boldsymbol{\Lambda} \left( \boldsymbol{\varepsilon} \boldsymbol{\Lambda}^T \frac{\partial \mathbf{E}'}{\partial t} + \mathbf{J} \right) \\ &= \det(\boldsymbol{\Lambda}^{-1}) \boldsymbol{\Lambda} \boldsymbol{\varepsilon} \boldsymbol{\Lambda}^T \frac{\partial \mathbf{E}'}{\partial t} + \det(\boldsymbol{\Lambda}^{-1}) \boldsymbol{\Lambda} \mathbf{J}. \end{aligned}$$

From the above equation, we found the permittivity tensor of the TrM and the current density in the physical space are the same as the results obtained previously.

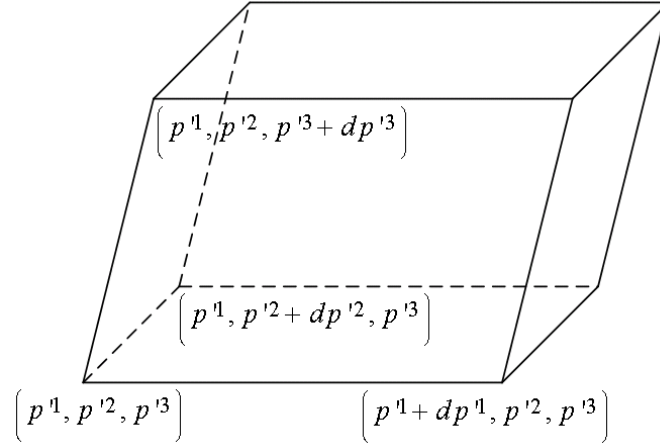


Figure 2.3: Three-dimensional (3D) differential integration element.

The form invariance of the divergence equations in Maxwell's equations can be treated with the same procedure. Not much explanation as that for curl equations will be presented during the derivation here. The divergence operator in the curvilinear CS is found by making use of the integral identity

$$\oint \mathbf{D} \cdot d\mathbf{a} = \int \nabla \cdot \mathbf{D} dv. \quad (2.45)$$

Consider the integration over the three-dimensional differential element shown in Fig. 2.3. The LHS of Eq. (2.45) becomes

$$\begin{aligned} \oint \mathbf{D} \cdot d\mathbf{a} &= dp^3 \frac{\partial}{\partial p^3} [\mathbf{D} \cdot (\mathbf{a}_1 dp^1 \times \mathbf{a}_2 dp^2)] \\ &+ dp^1 \frac{\partial}{\partial p^1} [\mathbf{D} \cdot (\mathbf{a}_2 dp^2 \times \mathbf{a}_3 dp^3)] + dp^2 \frac{\partial}{\partial p^2} [\mathbf{D} \cdot (\mathbf{a}_3 dp^3 \times \mathbf{a}_1 dp^1)] \\ &= \left[ \frac{\partial \mathbf{D} \cdot (\mathbf{a}_1 \times \mathbf{a}_2)}{\partial p^3} + \frac{\partial \mathbf{D} \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}{\partial p^1} + \frac{\partial \mathbf{D} \cdot (\mathbf{a}_3 \times \mathbf{a}_1)}{\partial p^2} \right] dp^1 dp^2 dp^3 \\ &= \left[ \frac{\partial d^3 \mathbf{a}_3 \cdot (\mathbf{a}_1 \times \mathbf{a}_2)}{\partial p^3} + \frac{\partial d^1 \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)}{\partial p^1} + \frac{\partial d^2 \mathbf{a}_2 \cdot (\mathbf{a}_3 \times \mathbf{a}_1)}{\partial p^2} \right] dp^1 dp^2 dp^3 \\ &= \left[ \frac{\partial \det(\boldsymbol{\Lambda}^{-1}) d^3}{\partial p^3} + \frac{\partial \det(\boldsymbol{\Lambda}^{-1}) d^1}{\partial p^1} + \frac{\partial \det(\boldsymbol{\Lambda}^{-1}) d^2}{\partial p^2} \right] dp^1 dp^2 dp^3. \end{aligned}$$

The RHS is

$$\begin{aligned} \int \nabla \cdot \mathbf{D} dv &= \mathbf{a}_1 dp^1 \cdot (\mathbf{a}_2 dp^2 \times \mathbf{a}_3 dp^3) \nabla \cdot \mathbf{D} = \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3) \nabla \cdot \mathbf{D} dp^1 dp^2 dp^3 \\ &= \nabla \cdot \mathbf{D} \det(\boldsymbol{\Lambda}^{-1}) dp^1 dp^2 dp^3. \end{aligned}$$

Then Eq. (2.45) reduces to

$$\nabla \cdot \mathbf{D} \det(\boldsymbol{\Lambda}^{-1}) = \frac{\partial \det(\boldsymbol{\Lambda}^{-1}) d^3}{\partial p^3} + \frac{\partial \det(\boldsymbol{\Lambda}^{-1}) d^1}{\partial p^1} + \frac{\partial \det(\boldsymbol{\Lambda}^{-1}) d^2}{\partial p^2}. \quad (2.46)$$

If we define the electric flux density in the physical space by specifying its Cartesian components

$$D^i = \det(\Lambda^{-1}) d^i, \quad (2.47)$$

Eq. (2.46) gives

$$\nabla' \cdot \mathbf{D}' = \det(\Lambda^{-1}) \nabla \cdot \mathbf{D}. \quad (2.48)$$

Substitute (2.23) into (2.48),

$$\nabla' \cdot \mathbf{D}' = \det(\Lambda^{-1}) \rho. \quad (2.49)$$

The electric charge density in the physical space is

$$\rho' = \det(\Lambda^{-1}) \rho. \quad (2.50)$$

The definition Eq. (2.47) is changed to matrix form by transforming the curvilinear components to Cartesian components,

$$D^i = \det(\Lambda^{-1}) d^i = \det(\Lambda^{-1}) \Lambda_j^i D^j.$$

Write it in matrix form

$$\mathbf{D}' = \det(\Lambda^{-1}) \Lambda \mathbf{D}. \quad (2.51)$$

Study of the other divergence equation will result in the definition of the magnetic flux density in the physical space

$$\mathbf{B}' = \det(\Lambda^{-1}) \Lambda \mathbf{B}. \quad (2.52)$$

So far, we have constructed the set of EM fields, sources and TrM in the physical space using the form invariance of Maxwell's equations. The complete list is given below for quick reference

$$\mathbf{E}' = (\Lambda^{-1})^T \mathbf{E}, \quad \mathbf{H}' = (\Lambda^{-1})^T \mathbf{H}, \quad (2.41)$$

$$\mathbf{D}' = \det(\Lambda^{-1}) \Lambda \mathbf{D}, \quad (2.51)$$

$$\mathbf{B}' = \det(\Lambda^{-1}) \Lambda \mathbf{B}, \quad (2.52)$$

$$\mathbf{J}' = \Lambda \det(\Lambda^{-1}) \mathbf{J}, \quad (2.41)$$

$$\rho' = \det(\Lambda^{-1}) \rho, \quad (2.50)$$

$$\boldsymbol{\varepsilon}' = \Lambda \boldsymbol{\varepsilon} \Lambda^T \det(\Lambda^{-1}), \quad (2.38)$$

$$\boldsymbol{\mu}' = \Lambda \boldsymbol{\mu} \Lambda^T \det(\Lambda^{-1}). \quad (2.39)$$

Besides, the current continuity equation can be easily proven form invariant. From Eqs. (2.41), (2.51) and (2.52), we note that current density and flux densities follow the same transformation rule, and Eq. (2.48) also applies for current density,

$$\nabla' \cdot \mathbf{J}' = \det(\Lambda^{-1}) \nabla \cdot \mathbf{J}. \quad (2.53)$$

The form invariance is then clearly seen with Eqs. (2.50) and (2.53). Considering the duality of EM theory, it is not surprising to get the same form of the results pertinent to electric and magnetic fields. From this section on, we will only present discussions for one case for brevity.

### 2.3 Transformation Optics with Orthogonal Coordinates

In the previous section, the fundamental results of TrO have been derived. In the formulation, the relation of the two spaces is characterized by a transformation between Cartesian CSs and the results are expressed with Cartesian components. However in many situations, other orthogonal CSs (*e.g.*, cylindrical and spherical CSs) are very convenient for coordinate transformation and geometrical shape description. The Cartesian formulation of TrO becomes awkward in those situations. In order to ameliorate the problem, this section is devoted to build a formulation of TrO suitable for general orthogonal CSs.

First of all, we introduce two general orthogonal CSs in the two spaces,  $q^i$  and  $q^j$ . Note that the two CSs are not necessarily the same kind, although most of the time they are. Orthogonal CSs are also curvilinear CSs, just like the curvilinear CS  $p^i$  we have thoroughly investigated in Section 2.1. The properties of the curvilinear CS  $p^i$  apply for  $q^i$  and  $q^j$  as well. Apart from the general properties for curvilinear CSs, unitary vectors of an orthogonal CS, say  $q^i$ , are orthogonal to each other. The metric tensor is thus diagonal,

$$\tilde{g}_{ij} = \mathbf{b}_i \cdot \mathbf{b}_j = (h_i)^2 \delta_{ij} \Rightarrow \tilde{\mathbf{g}} = [\text{diag}(h_1, h_2, h_3)]^2, \quad (2.54)$$

where  $\mathbf{b}_i$  are unitary vectors.  $h_i$  in above equations are called *scale factors*. They are not components of a tensor, and the indices are not counted with regards to index balance and summation.

The relation between the two spaces is specified by a transformation between the orthogonal CSs,

$$q^i = q^i(q^1, q^2, q^3).$$

The Jacobian matrix is

$$\boldsymbol{\lambda} = (\lambda_j^i) = \left( \frac{\partial q^i}{\partial q^j} \right).$$

The Jacobian matrix utilized in the Cartesian formulation of TrO is for the transformation between the Cartesian CSs. It can be calculated as

$$\Lambda_j^{i'} = \frac{\partial p^{i'}}{\partial p^j} = \frac{\partial p^{i'}}{\partial q^{nk}} \frac{\partial q^{nk}}{\partial p^j} = \frac{\partial p^{i'}}{\partial q^{nk}} \frac{\partial q^{nk}}{\partial q^l} \frac{\partial q^l}{\partial p^j}. \quad (2.55)$$

If we define the Jacobian matrix of the orthogonal CS  $q^i$  with respect to the Cartesian CS  $p^j$



$$\mathbf{T} = \left( \frac{\partial q^i}{\partial p^j} \right),$$

Eq. (2.55) can be written in matrix form as

$$\mathbf{\Lambda} = \mathbf{T}^{-1} \boldsymbol{\lambda} \mathbf{T}. \quad (2.56)$$

In orthogonal CSs, the constitutive relation for electric flux density and electric field becomes

$$\tilde{d}^i = \varepsilon_j^i \tilde{e}^j \quad (2.57)$$

in component form ( $\tilde{d}^i$  and  $\tilde{e}^j$  are their contravariant components) or

$$\tilde{\mathbf{d}} = \boldsymbol{\varepsilon} \tilde{\mathbf{e}} \quad (2.58)$$

in matrix form. The permittivity tensor in orthogonal components can be transformed to that in Cartesian components according to the tensor transformation rules,

$$\varepsilon_j^i = \frac{\partial p^i}{\partial q^k} \frac{\partial q^l}{\partial p^j} \varepsilon_l^k,$$

which is equivalent to

$$\boldsymbol{\varepsilon} = \mathbf{T}^{-1} \boldsymbol{\varepsilon} \mathbf{T}. \quad (2.59)$$

Now we are prepared to derive the permittivity tensor of the TrM in orthogonal CSs by transforming it into the Cartesian formulation. Substitute Eqs. (2.56) and (2.59) into Eq. (2.38),

$$\begin{aligned} \mathbf{T}^{-1} \boldsymbol{\varepsilon}' \mathbf{T} &= \boldsymbol{\Lambda} \mathbf{T}^{-1} \boldsymbol{\varepsilon} \mathbf{T} \boldsymbol{\Lambda}^T \det(\boldsymbol{\Lambda}^{-1}) \\ &= \mathbf{T}^{-1} \boldsymbol{\lambda} \mathbf{T} \mathbf{T}^{-1} \boldsymbol{\varepsilon} \mathbf{T} \mathbf{T}^T \boldsymbol{\lambda}^T (\mathbf{T}^{iT})^{-1} \det(\boldsymbol{\Lambda}^{-1}) \\ &= \mathbf{T}^{-1} \boldsymbol{\lambda} \boldsymbol{\varepsilon} \mathbf{T} \mathbf{T}^T \boldsymbol{\lambda}^T (\mathbf{T}^{iT})^{-1} \det(\boldsymbol{\Lambda}^{-1}). \end{aligned}$$

The permittivity tensor in orthogonal CSs turns out

$$\begin{aligned} \boldsymbol{\varepsilon}' &= \boldsymbol{\lambda} \boldsymbol{\varepsilon} \mathbf{T} \mathbf{T}^T \boldsymbol{\lambda}^T (\mathbf{T}' \mathbf{T}'^T)^{-1} \det(\boldsymbol{\Lambda}^{-1}) \\ &= \boldsymbol{\lambda} \boldsymbol{\varepsilon} \tilde{\mathbf{g}}^{-1} \boldsymbol{\lambda}^T \tilde{\mathbf{g}}' \det(\boldsymbol{\Lambda}^{-1}). \end{aligned} \quad (2.60)$$

Eq. (2.17) is used in the last step. This permittivity tensor describes the linear relation between contravariant components of the electric flux density and electric field (see Eq. (2.57)). Since the unitary vectors don't have unit length, the contravariant components alone don't reflect the physical value. Consequently, we also need the metric tensor together with the permittivity tensor to find the absolute ratio of flux density and field. To overcome this unpleasant problem, we utilize the unit vectors of the orthogonal CS, instead of unitary vectors, when introducing a new permittivity tensor. Since unit vectors usually don't transform like tensors, the new permittivity tensor is actually not a tensor. Despite that, we will call it permittivity tensor as in most literature. To avoid ambiguity, the previous one will be referred to as covariant permittivity tensor.

Electric flux density and electric field are expanded with unit vectors  $\mathbf{v}_i$  as

$$\mathbf{E} = \tilde{E}^i \mathbf{v}_i, \mathbf{D} = \tilde{D}^i \mathbf{v}_i.$$

These components are related to the contravariant components by scale factors,

$$\tilde{E}^i = h_i \tilde{e}^i, \tilde{D}^i = h_i \tilde{d}^i. \quad (2.61)$$

Notice that the metric tensor has a diagonal matrix form with squared scale factors on the diagonal (Eq. (2.54)). Then Eq. (2.61) can be written as matrix products

$$\tilde{\mathbf{E}} = \tilde{\mathbf{g}}^{1/2} \tilde{\mathbf{e}}, \tilde{\mathbf{D}} = \tilde{\mathbf{g}}^{1/2} \tilde{\mathbf{d}}. \quad (2.62)$$

The permittivity tensor is defined as

$$\tilde{\mathbf{D}} = \tilde{\boldsymbol{\epsilon}} \tilde{\mathbf{E}}, \quad (2.63)$$

whose relation with the covariant permittivity tensor is established by inserting Eqs. (2.62),

$$\tilde{\mathbf{g}}^{1/2} \tilde{\mathbf{d}} = \tilde{\boldsymbol{\epsilon}} \tilde{\mathbf{g}}^{1/2} \tilde{\mathbf{e}}.$$

A comparison with Eq. (2.58) shows that

$$\boldsymbol{\epsilon} = \tilde{\mathbf{g}}^{-1/2} \tilde{\boldsymbol{\epsilon}} \tilde{\mathbf{g}}^{1/2}, \tilde{\boldsymbol{\epsilon}} = \tilde{\mathbf{g}}^{1/2} \boldsymbol{\epsilon} \tilde{\mathbf{g}}^{-1/2}. \quad (2.64)$$

With Eq. (2.64), the covariant permittivity tensor of the TrM in Eq. (2.60) is thus converted to

$$\tilde{\boldsymbol{\epsilon}}' = \mathbf{S} \tilde{\boldsymbol{\epsilon}} \mathbf{S}^T / \det(\mathbf{S}), \quad (2.65)$$

where  $\mathbf{S} = \tilde{\mathbf{g}}^{1/2} \boldsymbol{\lambda} \tilde{\mathbf{g}}^{-1/2}$ . The determinant  $\det(\mathbf{S})$  is obtained by using Eqs. (2.17) and (2.56).

Orthogonal CSs bring us convenience for calculation. On the other hand, Cartesian components are usually required for numerical simulation or fabrication at the final step of design. Combining the transformations in Eq. (2.59) and Eq. (2.64) leads to the permittivity tensor in Cartesian components

$$\boldsymbol{\epsilon} = \mathbf{U}^{-1} \tilde{\boldsymbol{\epsilon}} \mathbf{U},$$

with  $\mathbf{U} = \tilde{\mathbf{g}}^{1/2} \mathbf{T}$ . A little more observation of Eq. (2.17) shows us  $\mathbf{U}$  is an orthogonal matrix,  $\mathbf{U}^{-1} = \mathbf{U}^T$ . The above equation becomes easier for evaluation

$$\boldsymbol{\epsilon} = \mathbf{U}^T \tilde{\boldsymbol{\epsilon}} \mathbf{U}. \quad (2.66)$$

## 2.4 Spherical and Cylindrical Invisibility Cloaks

Within our assumed range of anisotropic media, the theoretical formulation of TrO has been fully established in the previous three sections. In this section, the designs of a spherical and a cylindrical invisibility cloak [6, 7] are taken as examples to demonstrate the application of TrO. A 2D FEM simulation is performed to illustrate the invisibility of the cylindrical cloak. The results of the CP also allow us to find out important characteristics of the invisibility cloaks.

The essence of the transformation for invisibility cloak design is to expand a scattering free element, like a point or a line, in a vacuous virtual space to a closed volume in the physical space. At the same time, the transformation outside a closed boundary is the identity operator. Assuming an EM wave propagates in the vacuous virtual space, then there is a transformed EM wave in the physical space according to TrO. Outside the boundary the identity transformation makes the EM wave identical to that in the virtual space, indicating no scattering in the physical space. Furthermore, the closed volume does not have a corresponding volume in the virtual space. Objects can be put inside it without causing any disturbance for the EM wave outside. They become invisible to external observers. To be specific, let's take closer looks at spherical and cylindrical cloaks, the most studied ones.

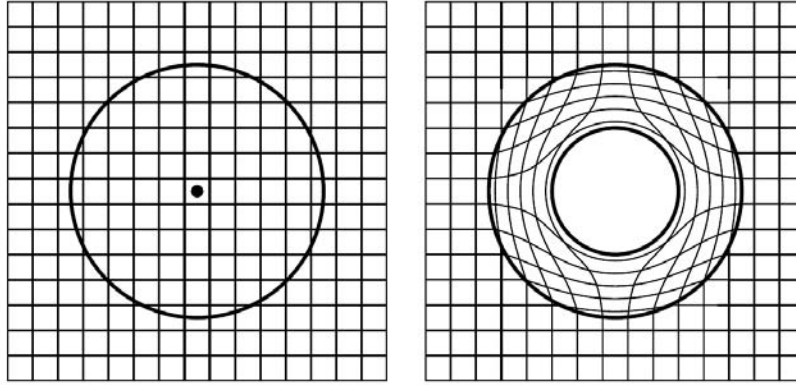


Figure 2.4: A transformation for invisibility cloak design. A point in the virtual space (left) is expanded to an inner volume in the physical space (right).

We will consider the 3D spherical case first, and the results for a 2D cylindrical cloak are presented after this. The transformation we need is as depicted in Fig. 2.4. A point in the virtual space is expanded to an inner volume in the physical space, while the region outside a concentric sphere is kept intact. The following transformation works well for our intention,

$$r' = \frac{R_2 - R_1}{R_2} r + R_1, 0 < r < R_2 \quad (2.67)$$

$$\theta' = \theta, \phi' = \phi,$$

where  $R_1$  and  $R_2$  are the radii of the inner and the outer spheres, respectively. Since the transformation outside the larger sphere is the identity operator, the TrM there is vacuum. The only nontrivial region, *i.e.*, the cloaking structure, is the spherical shell ( $R_1 < r' < R_2$ ) obtained by the transformation in Eq. (2.67). Its CP can be calculated with the orthogonal formulation of TrO (Eq. (2.65)). To proceed further, the Jacobian matrices of spherical CS and the transformation Eq. (2.67) must first be found. Spherical CS and Cartesian CS are related as

$$r = \sqrt{x^2 + y^2 + z^2}, \cos \theta = z / r, \tan \phi = y / x, \quad (2.68)$$

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta.$$

Evaluating the differentiations, the Jacobian matrix of the spherical CS is

$$\mathbf{T} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\frac{\sin \theta}{r} \\ r & r & r \\ -\frac{\sin \phi}{r \sin \theta} & \frac{\cos \phi}{r \sin \theta} & 0 \end{pmatrix}, \quad (2.69)$$

and the metric tensor is also obtained thanks to the relation in Eq. (2.17),

$$\tilde{\mathbf{g}}^{-1} = \mathbf{T}\mathbf{T}^T = \text{diag} \left( 1 \quad \frac{1}{r^2} \quad \frac{1}{r^2 \sin^2 \theta} \right). \quad (2.70)$$

For the transformation Eq. (2.67), the Jacobian matrix is

$$\boldsymbol{\lambda} = \text{diag} \left( \frac{R_2 - R_1}{R_2} \quad 1 \quad 1 \right). \quad (2.71)$$

Substitution of Eqs. (2.70) and (2.71) into Eq. (2.65) gives

$$\mathbf{S} = \tilde{\mathbf{g}}^{1/2} \boldsymbol{\lambda} \tilde{\mathbf{g}}^{-1/2} = \text{diag} \left( \frac{R_2 - R_1}{R_2} \quad \frac{r'}{r} \quad \frac{r'}{r} \right), \quad (2.72)$$

and the CP of the spherical cloak

$$\frac{\tilde{\boldsymbol{\epsilon}}'}{\epsilon_0} = \frac{R_2}{R_2 - R_1} \text{diag} \left( \left( \frac{R_2 - R_1}{R_2} \frac{r}{r'} \right)^2 \quad 1 \quad 1 \right).$$

The final result should only contain coordinate variables in the physical space. Eliminating the radial variable  $r$  of the virtual space, we obtain

$$\frac{\tilde{\boldsymbol{\epsilon}}'}{\epsilon_0} = \frac{R_2}{R_2 - R_1} \text{diag} \left( \left( \frac{r' - R_1}{r'} \right)^2 \quad 1 \quad 1 \right). \quad (2.73)$$

For the 2D cylindrical case, using cylindrical coordinates  $(r, \theta, z)$ , the transformation used for the design is essentially unchanged

$$\begin{aligned} r' &= \frac{R_2 - R_1}{R_2} r + R_1, \quad 0 < r < R_2 \\ \theta' &= \theta, \quad z' = z. \end{aligned} \quad (2.74)$$

The Jacobian matrix and the metric tensor become

$$\mathbf{T} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.75)$$

$$\tilde{\mathbf{g}}^{-1} = \mathbf{T}\mathbf{T}^T = \text{diag}\left(1 \quad \frac{1}{r^2} \quad 1\right). \quad (2.76)$$

The Jacobian matrix of the transformation Eq. (2.74) is the same as Eq. (2.71). Insert Eq. (2.71) and Eq. (2.76) into Eq. (2.65),

$$\mathbf{S} = \tilde{\mathbf{g}}'^{1/2} \lambda \tilde{\mathbf{g}}'^{-1/2} = \text{diag}\left(\frac{R_2 - R_1}{R_2} \quad \frac{r'}{r} \quad 1\right) \quad (2.77)$$

and

$$\frac{\tilde{\boldsymbol{\epsilon}}'}{\varepsilon_0} = \frac{R_2}{R_2 - R_1} \frac{r}{r'} \text{diag}\left(\left(\frac{R_2 - R_1}{R_2}\right)^2 \quad \left(\frac{r'}{r}\right)^2 \quad 1\right).$$

Eliminating the radial variable  $r$ , we get

$$\frac{\tilde{\boldsymbol{\epsilon}}'}{\varepsilon_0} = \text{diag}\left(\frac{r' - R_1}{r'} \quad \frac{r'}{r' - R_1} \quad \left(\frac{R_2}{R_2 - R_1}\right)^2 \frac{r' - R_1}{r'}\right). \quad (2.78)$$

Besides experimental verification, numerical simulation is a widely used and well accepted technique to test theoretical ideas and designs. We would like to use numerical simulation to illustrate the invisibility effect. The CP of the 2D cylindrical cloak enable the decoupling of Transverse Electric (TE) and Transverse Magnetic (TM) excitations. In the 2D simulation, let's take the TM case as an example, whose field components include  $H_z$ ,  $E_x$  and  $E_y$ . Accordingly, only the axial component of the magnetic permeability and the transverse components of the electric permittivity are required in simulation. The FEM based commercial software COMSOL [8] is applied as the simulation tool throughout the present thesis. In COMSOL, cylindrical components of the permittivity tensor can not be directly put in for simulation yet. We need the Cartesian components. The Cartesian permittivity tensor is transformed from that in cylindrical components Eq. (2.78) with the formula in Eq. (2.66),

$$\boldsymbol{\epsilon}' = \mathbf{U}^T \tilde{\boldsymbol{\epsilon}}' \mathbf{U}' = \begin{pmatrix} \frac{r' - R_1}{r'} \cos^2 \theta' + \frac{r'}{r' - R_1} \sin^2 \theta' & \left(\frac{r' - R_1}{r'} - \frac{r'}{r' - R_1}\right) \sin \theta' \cos \theta' & 0 \\ \left(\frac{r' - R_1}{r'} - \frac{r'}{r' - R_1}\right) \sin \theta' \cos \theta' & \frac{r' - R_1}{r'} \sin^2 \theta' + \frac{r'}{r' - R_1} \cos^2 \theta' & 0 \\ 0 & 0 & \left(\frac{R_2}{R_2 - R_1}\right)^2 \frac{r' - R_1}{r'} \end{pmatrix}.$$

In Fig. 2.5, the numerical implementation of the Cartesian permittivity tensor provides the total magnetic field distribution with a plane wave illumination. The invisibility effect of the cylindrical cloak is clearly illustrated.

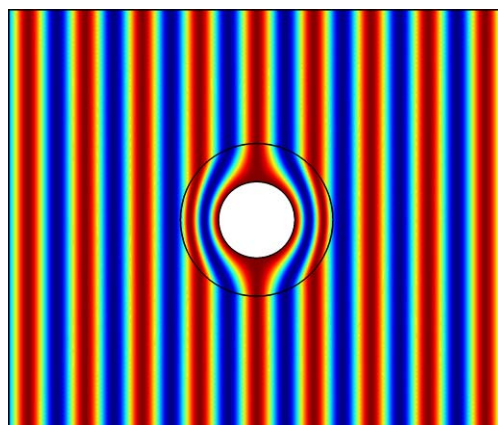


Figure 2.5: An EM plane wave propagates through a cylindrical invisibility cloak.

With the help of the orthogonal formulation of TrO, the CP of a spherical and a cylindrical cloak are calculated in Eqs. (2.73) and (2.78), from which we can also learn some significant characteristics of invisibility cloaks. The CP are highly anisotropic and inhomogeneous, and take a very extreme range of value at the inner boundaries ( $r' = R_1$ ). Values smaller than one and down to zero are required for both cloaks. The cylindrical cloak even has infinitely large azimuthal components. These singularities in CP are actually predictable considering the special functions the cloaks perform. The cloaks prevent EM waves from penetrating into the inner volume, which is usually enabled by hard boundaries with infinite parameters. In the cloaks, EM waves propagate longer paths around the inner volume. To keep the phase the same as that outside the cloaks, CP smaller than one are necessary to provide a phase velocity faster than light speed. The formidable complexity and singularity in CP make the realization of perfect cloaks almost impossible. As a compromise, reduced models (especially for 2D cloaks) with pretty good performance and moderate CP achievable with metamaterials are widely studied. The above features displayed by the CP are not particular for spherical and cylindrical cloaks, but general for 2D and 3D cloaks.

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### 3. Reduced Two-dimensional Cloaks of Arbitrary Shapes

The idea of a reduced model for 2D cloaks was mentioned in the previous chapter. It is crucial for the practical realization and application of invisibility cloaks. In this chapter we will first give a brief introduction to reduced models and how they are designed. Thereafter a method of designing reduced 2D cloaks of arbitrary shapes is proposed. Numerical simulations for an elliptic and a bow-tie shaped cloak are carried out to demonstrate its effectiveness.

#### 3.1 Reduced Cylindrical Cloaks

In the cylindrical cloak example in Section 2.4, we have already witnessed the singularity in its CP. Near the inner boundary the radial and axial parameters are smaller than one, and become zero at the boundary, while azimuthal components even become infinite at the inner boundary. These singular behaviors make the realization of 2D perfect cloaks impossible. Taking advantage of metamaterials, parameters less than one are feasible, but infinitely large parameters are beyond reach. Without infinities in the CP, 3D cloaks appear to be easier to realize. The anisotropy and inhomogeneity in 3D structures pose huge difficulties for fabrication. As a compromise between performance and the convenience of realization, reduced models for 2D cloaks free from infinities in the CP can produce a good but not perfect invisibility effect. The earliest experimental demonstration of a cloak [1] was actually constructed as a reduced 2D cloak.

In view of the high symmetry, the cylindrical cloak is the most studied 2D cloak. Almost all studies of reduced models are about cylindrical cloaks. The general design strategy for most of them is also specific to cylindrical cloaks. Considering TE polarized excitations, the EM wave behaviors are completely described by the electric field component  $E_z$ , which follows the wave equation

$$\frac{1}{\varepsilon_z r} \left[ \frac{\partial}{\partial r} \left( \frac{r}{\mu_\theta} \frac{\partial E_z}{\partial r} \right) \right] + \frac{1}{\varepsilon_z r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\mu_r} \frac{\partial E_z}{\partial \theta} \right) + k_0^2 E_z = 0. \quad (3.1)$$

In the design strategy, we make the wrong assumption of radially invariant  $\mu_\theta$  whereby Eq. (3.1) becomes

$$\frac{1}{\varepsilon_z \mu_\theta} \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{\varepsilon_z \mu_\theta} \frac{1}{r} \frac{\partial E_z}{\partial r} + \frac{1}{\varepsilon_z \mu_r} \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \theta^2} + k_0^2 E_z = 0. \quad (3.2)$$

Though not accurate, Eq. (3.2) does provide a guide for designing reduced cloaks in some sense. Products  $\varepsilon_z \mu_\theta$  and  $\varepsilon_z \mu_r$  can specify Eq. (3.2) completely. As long as the products for reduced cloaks are kept the same as those for perfect cloaks, we can expect similar EM wave behaviors as in the case of perfect invisibility. The two products for the perfect cylindrical cloak designed in Section 2.4 are

$$\frac{\varepsilon_z \mu_\theta}{\varepsilon_0 \mu_0} = \left( \frac{R_2}{R_2 - R_1} \right)^2, \quad \frac{\varepsilon_z \mu_r}{\varepsilon_0 \mu_0} = \left( \frac{R_2}{R_2 - R_1} \right)^2 \left( \frac{r - R_1}{r} \right)^2. \quad (3.3)$$

Manipulation with the three parameters helps to avoid the singularity in the CP and keep the products unchanged at the same time. In [1], the first reduced model has the CP

$$\frac{\varepsilon_z}{\varepsilon_0} = \left( \frac{R_2}{R_2 - R_1} \right)^2, \quad \frac{\mu_r}{\mu_0} = \left( \frac{r - R_1}{r} \right)^2, \quad \frac{\mu_\theta}{\mu_0} = 1, \quad (3.4)$$

which fulfill Eq. (3.3) and are free of infinite parameters. For the TM polarization case, another set of products similar to that in Eq. (3.3) is used. In [2], a non-magnetic cloak, supposed to work at visible frequencies has the CP

$$\frac{\mu_z}{\mu_0} = 1, \quad \frac{\varepsilon_r}{\varepsilon_0} = \left( \frac{R_2}{R_2 - R_1} \right)^2 \left( \frac{r - R_1}{r} \right)^2, \quad \frac{\varepsilon_\theta}{\varepsilon_0} = \left( \frac{R_2}{R_2 - R_1} \right)^2. \quad (3.5)$$

Later an impedance matched reduced model [3] was proposed to suppress the strong scattering due to outer boundary reflection,

$$\frac{\varepsilon_z}{\varepsilon_0} = \frac{R_2}{R_2 - R_1}, \quad \frac{\mu_r}{\mu_0} = \left( \frac{r - R_1}{r} \right)^2 \frac{R_2}{R_2 - R_1}, \quad \frac{\mu_\theta}{\mu_0} = \frac{R_2}{R_2 - R_1}. \quad (3.6)$$

It also has the same products as in Eq. (3.3).

Apart from this mostly used strategy, another more systematic study on designing reduced cylindrical cloaks was reported in [4]. The method introduces a series expansion to represent  $\mu_\theta$  of a perfect cloak. Then reduced models are obtained when taking different truncation orders in the expansion. The lowest order result is exactly the same as the previously discussed strategy. It will get closer to the perfect cloak when more terms are included.

## 3.2 Reduced Two-dimensional Cloaks of Arbitrary Shapes

Although reduced models for 2D cloaks are widely investigated as discussed in the introduction, almost all research concentrates on cylindrical cloaks. For arbitrarily shaped cloaks, only perfect models are developed with TrO in the literature. The experimental realization, however, must be assisted by corresponding reduced models. In this section, we propose a general method to design reduced models for 2D cloaks of arbitrary shapes. The description of the general guideline is followed by two concrete examples of reduced models, an elliptic and a bow-tie shaped cloak. Prominent cloaking effects are observed in numerical demonstrations.

As noticed in reduced cylindrical cloaks, the singularity alleviated by reduced models is the infinity in the CP. In this sense, 3D perfect cloaks are non-singular. It is thus natural to take advantage of this non-singular nature of 3D perfect cloaks to construct CP for reduced 2D cloaks. Consider a 3D perfect cloak with mirror symmetry. When this cloak is illuminated by a plane wave with its wave vector parallel to the symmetry plane, the plane wave has the Poynting vector on the symmetry plane confined in the plane and flows around the inner volume without scattering. If

focused on the EM wave in the symmetry plane, it's like in a perfect 2D cloak. So we can take the CP of the 3D perfect cloak on the symmetry plane as those of a reduced 2D cloak, and expect similarly good cloaking effects. Due to the mirror symmetry, the CP in the symmetry plane don't induce any correlation between transverse and axial field components, and enable TE and TM polarization separation. A constraint as weak as mirror symmetry does not cause any loss of generality for our purpose. The generated reduced models can have arbitrary shapes. Readers may notice that the shapes of the two examples both have symmetry axes. That is not an implication from the method, but a choice for the convenience of constructing the 3D cloaks by rotation. As this method springs largely from intuition, its effectiveness must be checked with examples. In the following, an elliptic and a bow-tie shaped reduced cloak are designed with this method. The effectiveness of the method is demonstrated by numerical simulation results.

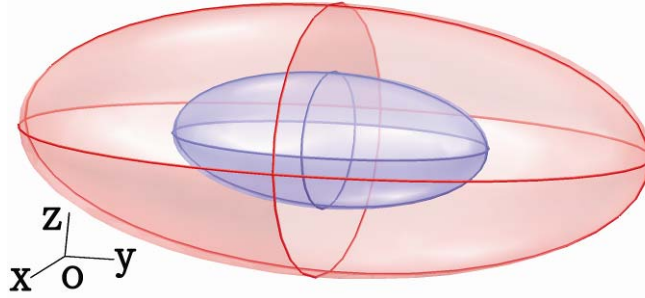


Figure 3.1: A 3D ellipsoidal cloak used for a reduced 2D elliptic cloak design.

First let's design a reduced 2D cloak of elliptic shape. This shape can be taken from a symmetry plane (e.g.,  $x - y$  plane) of the perfect 3D ellipsoidal cloak as shown in Fig. 3.1. For the description of the 3D cloak, ellipsoidal CS

$$\xi = \sqrt{(x/l)^2 + (y/m)^2 + (z/n)^2}, \tan \alpha = \frac{ly}{mx}, \sin \beta = \frac{z}{n\xi}, \quad (3.7)$$

where  $l$ ,  $m$  and  $n$  are positive constants, is introduced. The inner and outer boundaries of the ellipsoidal cloak are specified as  $\xi = a$  and  $\xi = b$ . This 3D cloak is designed with the transformation

$$\xi = \frac{b-a}{b} \xi' + a, \alpha = \alpha', \beta = \beta', \quad (3.8)$$

mapping the origin in the virtual space ( $\xi'$ ,  $\alpha'$ ,  $\beta'$ ) to the inner boundary, and the outer boundary to itself. Hereafter, we will start using primes to indicate virtual spaces, while not for real spaces. The Cartesian CP of the perfect ellipsoidal cloak are calculated with Eq. (2.38),

$$\frac{\boldsymbol{\varepsilon}_p}{\varepsilon_0} = \frac{\boldsymbol{\mu}_p}{\mu_0} = \frac{b}{b-a} \mathbf{A}, \quad (3.9)$$

where the nonzero elements of  $\mathbf{A}$  are

$$\begin{aligned}
 A_{11} &= 1 - \frac{ax^2}{\xi^3} \frac{2}{l^2} + \frac{a^2x^2}{\xi^6} \left( \frac{x^2}{l^4} + \frac{y^2}{m^4} + \frac{z^2}{n^4} \right), \\
 A_{22} &= 1 - \frac{ay^2}{\xi^3} \frac{2}{m^2} + \frac{a^2y^2}{\xi^6} \left( \frac{x^2}{l^4} + \frac{y^2}{m^4} + \frac{z^2}{n^4} \right), \\
 A_{33} &= 1 - \frac{az^2}{\xi^3} \frac{2}{n^2} + \frac{a^2z^2}{\xi^6} \left( \frac{x^2}{l^4} + \frac{y^2}{m^4} + \frac{z^2}{n^4} \right), \\
 A_{12} = A_{21} &= -\frac{axy}{\xi^3} \left( \frac{1}{l^2} + \frac{1}{m^2} \right) + \frac{a^2xy}{\xi^6} \left( \frac{x^2}{l^4} + \frac{y^2}{m^4} + \frac{z^2}{n^4} \right), \\
 A_{23} = A_{32} &= -\frac{ayz}{\xi^3} \left( \frac{1}{m^2} + \frac{1}{n^2} \right) + \frac{a^2yz}{\xi^6} \left( \frac{x^2}{l^4} + \frac{y^2}{m^4} + \frac{z^2}{n^4} \right), \\
 A_{31} = A_{13} &= -\frac{azx}{\xi^3} \left( \frac{1}{n^2} + \frac{1}{l^2} \right) + \frac{a^2zx}{\xi^6} \left( \frac{x^2}{l^4} + \frac{y^2}{m^4} + \frac{z^2}{n^4} \right).
 \end{aligned}$$

Taking the aforementioned design procedure, the CP for the 2D reduced elliptic cloak are obtained by simply setting  $z = 0$  in Eq. (3.9),

$$\frac{\boldsymbol{\varepsilon}_e}{\boldsymbol{\varepsilon}_0} = \frac{\boldsymbol{\mu}_e}{\mu_0} = \frac{b}{b-a} \mathbf{B}, \quad (3.10)$$

where the nonzero elements of  $\mathbf{B}$  are  $B_{33} = 1$  and

$$\begin{aligned}
 B_{11} &= 1 - \frac{ax^2}{\xi^3} \frac{2}{l^2} + \frac{a^2x^2}{\xi^6} \left( \frac{x^2}{l^4} + \frac{y^2}{m^4} \right), \\
 B_{22} &= 1 - \frac{ay^2}{\xi^3} \frac{2}{m^2} + \frac{a^2y^2}{\xi^6} \left( \frac{x^2}{l^4} + \frac{y^2}{m^4} \right), \\
 B_{12} = B_{21} &= -\frac{axy}{\xi^3} \left( \frac{1}{l^2} + \frac{1}{m^2} \right) + \frac{a^2xy}{\xi^6} \left( \frac{x^2}{l^4} + \frac{y^2}{m^4} \right).
 \end{aligned}$$

It is clear that the CP in Eq. (3.10) do not have singularities at the inner boundary. When  $l = m = 1$ , the elliptic cloak becomes cylindrical. The CP are shown to coincide with the impedance matched reduced model in Eq. (3.6). Actually, the reduced model generated by this method is inherently impedance matched at the outer boundary. The CP in Eq. (3.10) are also the limiting result of Eq. (3.9) when the parameter  $n$  in Eq. (3.7) approaches infinity. When the CP of the ellipsoidal cloak are impedance matched, then so are those of the reduced model.

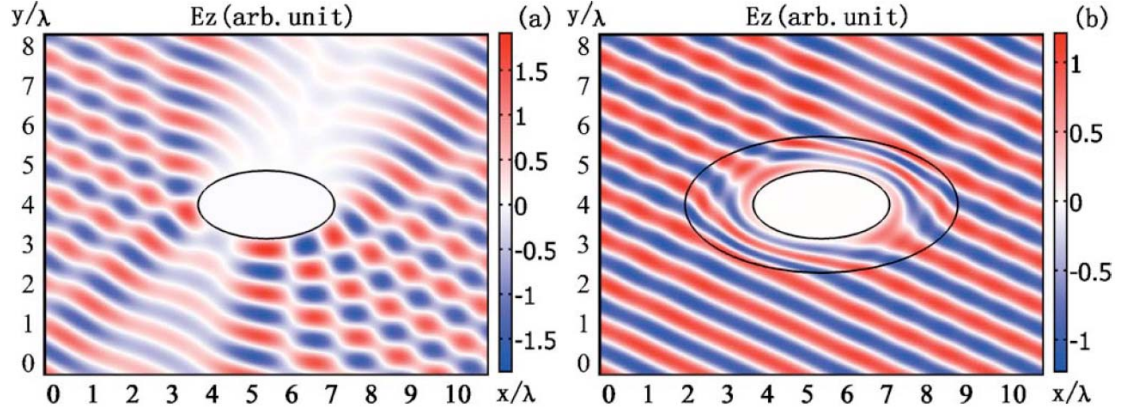


Figure 3.2: Snapshots of the total electric fields when a TE-polarized incident wave is scattered by a conducting object without (a) and with (b) the reduced elliptic cloak applied.

The cloaking effect of the reduced model is evaluated by running an FEM based simulation. The design method works for both TE and TM polarizations. Only the TE polarization case is presented, as the TM case gives similar results. To make it general, an arbitrarily oblique plane wave with the wave vector  $\mathbf{k} = k_0 (\cos 65^\circ \mathbf{x} + \sin 65^\circ \mathbf{y})$  is applied as the excitation source in the simulation. A reduced elliptic cloak with its inner and outer boundaries specified by semiaxes  $(5\lambda/3, 5\lambda/6)$  and  $(10\lambda/3, 5\lambda/3)$  is deployed. The inner boundary is set to perfect electric conductor (PEC) to represent a conducting object conformal to it. The scatterings of the conducting object with and without the reduced cloak are modeled for comparison. Snapshots of the total electric fields are shown in Fig. 3.2. The incident plane wave is strongly scattered by the bare conducting object. In comparison, the propagation of the plane wave is well conserved by the reduced cloak. Thanks to the impedance matched outer boundary, little backscattering is observed.

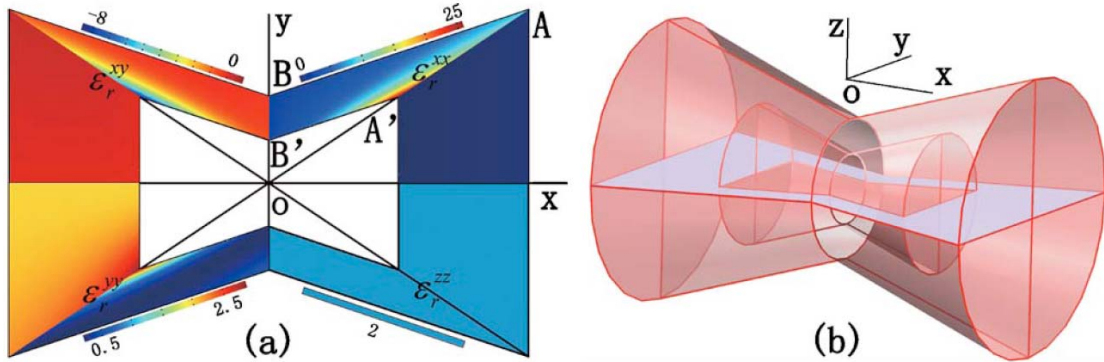


Figure 3.3: Schematic for a 2D reduced bow-tie cloak (a) and the corresponding 3D perfect cloak (b). The color map in (a) shows how the CP of the reduced cloak vary spatially.

The method is shown to perform well for a reduced elliptic cloak. We would like to evaluate the validity further with a cloak in a more complex shape (as shown in Fig. 3.3(a)) with concave parts and sharp corners. The profile and the size of the cloak is determined by the coordinates of point  $A(t, h_1)$  and point  $B(0, h_2)$  and shrinking factor  $\alpha = |A'A| / |OA|$ . The corresponding 3D perfect

cloak from which the reduced model is derived is the result of rotating the bow-tie shape about  $x$ -axis as shown in Fig. 3.3(b). The transformation for this 3D perfect cloak design should map the origin in the virtual space  $(x', y', z')$  to the inner boundary and keep the outer boundary unchanged. Due to the polygonal bow-tie shape, the transformation takes a piecewise form. In the solid region whose projection on  $x$ - $y$  plane is the triangular area OAB in Fig. 3.3(a), the transformation can be expressed as

$$x/x' = y/y' = z/z' = \alpha - (1-\alpha)w/(ux' + v\rho'), \quad (3.11)$$

where the line AB is described by  $ux + vy + w = 0$  ( $w < 0$ ) and  $\rho' = y'[1 + (z'/y')^2]^{1/2}$ . In constructing this expression for transformation, the same approach as that for the spherical cloak design in Eq. (2.67) is adopted. When doing so, a distance from the origin to a point in space needs to be defined, so that any two points on a surface swept out by a line parallel to AB have the same distance. Such a distance can be defined as

$$d(x, \rho) = \frac{ux + v\rho}{\sqrt{u^2 + v^2}},$$

with  $\rho = y[1 + (z/y)^2]^{1/2}$ . The expressions of the transformation in other regions can be written down in the same way with different values of  $u$ ,  $v$  and  $w$ . Then the CP of the 3D perfect cloak are found with Eq. (2.38), and setting  $z = 0$  leads to the CP of the reduced bow-tie cloak. While the expressions for the CP are not provided for brevity, their distributions are compactly plotted in Fig. 3.3(a) via symmetry consideration. We see that a negative value appear in the non-diagonal component. That does not mean negative material is required for the reduced cloak, since only the elements of the diagonalized permittivity tensor have physical significance.

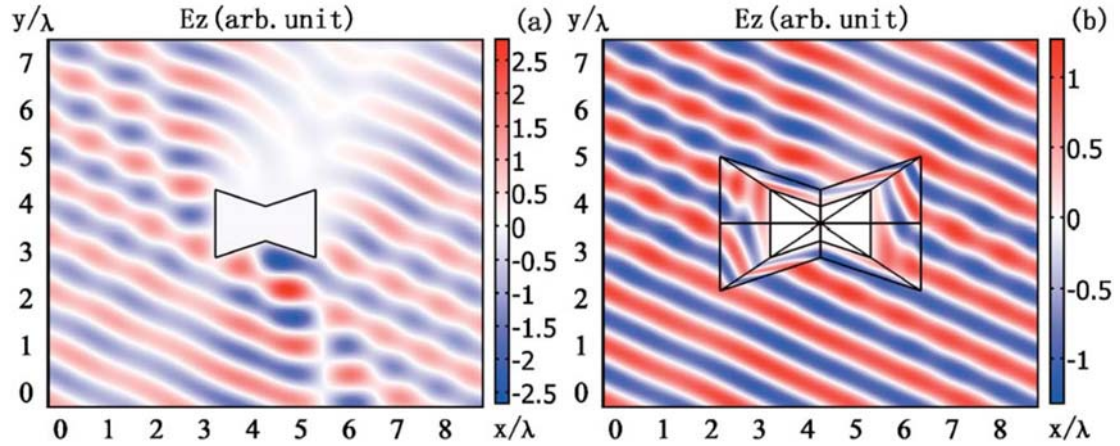


Figure 3.4: Snapshots of the total electric fields when a TE-polarized plane wave is scattered by a conducting bow-tie shaped object without (a) and with (b) the reduced bow-tie cloak applied.

The invisibility performance of the reduced bow-tie cloak is tested by numerical simulations as well. The same excitation source is launched in the modeling. The size parameters are  $t = 2\lambda$ ,  $h_1 = 4\lambda/3$ ,  $h_2 = 2\lambda/3$ , and  $\alpha = 0.5$ . Fig. 3.4 shows snapshots of the total electric fields when the incident plane wave is scattered by a conducting object without and with the reduced cloak applied. Rather

good cloaking performance of the reduced model is observed, compared with the strong scattering caused by the bare conducting object. The method is thus proven robust even for the design of such an irregularly shaped reduced cloak.

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# 4. Inverse Transformation Optics and its Applications

In this chapter, we introduce a technique for reversing the procedure of TrO. In this sense, we call it inverse transformation optics hereafter. The general idea of inverse TrO is discussed in the first section. Then the technique is applied to analyze the reflection for 2D finite-embedded coordinate transformations.

## 4.1 Inverse Transformation Optics

When TrO is applied to a device design, given a virtual space with known material specifications (usually vacuous or isotropic) and a transformation to the real space, the TrM or the designed device can then be determined. As seen from the formulae in Eqs. (2.38) and (2.39), TrM are generally anisotropic and inhomogeneous, much more complex than the original media before transformation. What do we have if the procedure is reversed? Given an anisotropic and perhaps inhomogeneous medium distribution, we need to find a transformation and transform the anisotropic medium back to an isotropic medium in a virtual space. Since isotropic media are usually much easier to deal with, this inverse TrO can be utilized to help treat problems involving anisotropic and inhomogeneous medium distribution. Once a proper transformation relation is established, the problems are transformed to simpler ones in the virtual space containing isotropic media. The simplified problems can then hopefully be solved a smaller effort. The results for the original problems are finally obtained from the results for the simplified problems by standard TrO procedures.

To explore the feasibility of the rough idea, take a look at the CP of the TrM in Eq. (2.38)

$$\boldsymbol{\varepsilon} = \boldsymbol{\Lambda} \boldsymbol{\varepsilon}' \boldsymbol{\Lambda}^T \det(\boldsymbol{\Lambda}^{-1}), \quad (4.1)$$

where the permittivity of the original medium in the virtual space is assumed to be isotropic, represented by a scalar  $\varepsilon'$ . For the inverse TrO, the anisotropic and inhomogeneous medium specified by  $\boldsymbol{\varepsilon}$  is known. We are looking for some transformation characteristic of its Jacobian matrix  $\boldsymbol{\Lambda}$ , so that an isotropic medium is obtained

$$\boldsymbol{\varepsilon}' = \boldsymbol{\Lambda}^{-1} \boldsymbol{\varepsilon} (\boldsymbol{\Lambda}^{-1})^T \det(\boldsymbol{\Lambda}). \quad (4.2)$$

For an ordinary lossless medium, the permittivity tensor is Hermitian [1] and positive definite. Suppose a simple case of a real tensor  $\boldsymbol{\varepsilon}$  for the current development. The scalar  $\varepsilon'$  is just an identity matrix. As long as the general symmetric matrix  $\boldsymbol{\varepsilon}$  can be decomposed as

$$\boldsymbol{\varepsilon} = \mathbf{Q} \text{diag}(\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3) \mathbf{Q}^T, \quad (4.3)$$

the Jacobian matrix is formally found to be

$$\Lambda = \mathbf{Q} \text{diag} \left( \sqrt{\varepsilon_1} \quad \sqrt{\varepsilon_2} \quad \sqrt{\varepsilon_3} \right), \quad (4.4)$$

and the isotropic medium in Eq. (4.2) turns out

$$\varepsilon' = \sqrt{\varepsilon_1 \varepsilon_2 \varepsilon_3} \det(\mathbf{Q}). \quad (4.5)$$

Actually Eq. (4.3) is simply the eigendecomposition of  $\boldsymbol{\varepsilon}$  [2], which is available for any symmetric matrix. Thus, in principle inverse TrO is applicable for any anisotropic medium. One may note that the diagonalization process is essentially finding the principal axes of the anisotropic medium. However, the significant scheme of TrO enables the treatment of the spatially inhomogeneous medium distribution at the same time. The decomposition in Eq. (4.3) is not unique, meaning that we have additional flexibility to play with. Auxiliary conditions can be imposed on transformations for special applications, as will be seen in the reflection analysis example.

In TrO, electric and magnetic fields transform in the same way, and so do permittivities and permeabilities (Eqs. (2.38) and (2.39)). An inverse transformation giving isotropy in permittivity is not guaranteed to simultaneously give isotropy in permeability. It is therefore not possible for inverse TrO to produce an isotropic medium from a general anisotropic medium with anisotropy in both permittivity and permeability. Fortunately, such kind of general anisotropic media is rare in real applications. For 2D problems, *e.g.*, the example we are about to discuss, two polarization states can be considered separately. This issue of correlation does not exist.

## 4.2 Reflection Analysis for 2D Finite-Embedded Coordinate Transformation

In TrO, the transformation between two spaces is assumed to apply for the whole domain in spaces. The TrM specification is also for the whole physical space. If the transformation is nontrivial even when extended into infinity, the realization of such a TrM would be an impossible mission. For the case of invisibility cloaks, the transformation is the identity operator outside a certain finite region. The TrM is different from vacuum only in the finite region. According to this argument, the transformation has to be the identity outside a finite region so that the TrM can be realized. In consequence, the TrM can only allow for local manipulation of EM waves. They are all invisible. To break this severe limitation, the functional and nontrivial transformation is assumed to be embedded in a finite region for device design [3]. TrO is only applied in the finite region, *i.e.*, within the device. This scheme is called Finite Embedded Coordinate Transformation (FECT). In Ref. [3] where FECT is proposed, a beam shifter and a beam splitter are presented. After that, ensuing research on device design with FECT has been extensively conducted [4-8].

In the theoretical aspect, FECT is not totally consistent with TrO, where whole domain transformation is assumed. The performance of a device designed by FECT can not be completely described by TrO, especially with consideration to the boundary. The non-transformed outer region can be perceived alternatively as transformed with identity transformation. Then the transformation

across the boundary of the finite region may be discontinuous. Since a discontinuous transformation is not included in the theory of TrO, the behavior of EM waves at the boundary can not be predicted by TrO. Reflection is generally observed at the boundary, causing power loss and EM wave disturbance. For most device designs, being reflectionless is an advantageous feature. Investigations into the reflectionless conditions on FECT [9, 10] are carried out to provide a guide for both design and evaluation. In this thesis, the particular perspective of inverse TrO is taken to study the reflection at the boundary of a device designed with FECT, so that the reflection is understood in the framework of TrO. A reflectionless condition is also derived.

In view of the fact that 2D transformations are widely used in most of present device designs based on TrO, we will focus on 2D inverse TrO. Consider a general anisotropic medium to be inversely transformed to an isotropic one. For a specific polarization state, say TE, the anisotropic medium has the pertinent CP components:  $\varepsilon$  (zz-component of the permittivity tensor) and the transverse components of the permeability tensor

$$\boldsymbol{\mu}_t = \begin{pmatrix} \mu_{xx} & \mu_{xy} \\ \mu_{xy} & \mu_{yy} \end{pmatrix}.$$

And assume the original isotropic medium in a virtual space is characterized by  $\varepsilon'$  and  $\mu'$ , and the transformation is

$$(x, y, z) = [X(x', y'), Y(x', y'), Z(z')]. \quad (4.6)$$

The original medium in the virtual space and the transformation are then supposed to generate the anisotropic medium through the standard TrO formulae Eqs. (2.38) and (2.39). For the 2D case, it is not hard to solve the equations and find out the elements of the Jacobian matrix  $\mathbf{J}$  for the transformation

$$\begin{aligned} \frac{\partial X}{\partial x'} &= A \cos \alpha, & \frac{\partial Y}{\partial x'} &= B \cos \beta, \\ \frac{\partial X}{\partial y'} &= A \sin \alpha, & \frac{\partial Y}{\partial y'} &= B \sin \beta, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} A &= \frac{\sqrt{\frac{\varepsilon' \mu'}{\varepsilon \mu_{yy}}}}{\sin(\beta - \alpha)}, & B &= \frac{\sqrt{\frac{\varepsilon' \mu'}{\varepsilon \mu_{xx}}}}{\sin(\beta - \alpha)}, \\ \frac{\partial Z}{\partial z'} &= \frac{\mu' \sqrt{\mu_{xx} \mu_{yy}}}{\sin(\beta - \alpha)}, & \cos(\beta - \alpha) &= \frac{\mu_{xy}}{\sqrt{\mu_{xx} \mu_{yy}}}. \end{aligned}$$

We can formally denote the results of inverse TrO for the anisotropic medium with the pair  $[(\varepsilon', \mu'), \mathbf{J}^{-1}]$  of the original isotropic medium and the transformation. From Eq. (4.7) one can see that the solution is far from determined. The original medium can be chosen freely and there is also a degree of freedom for the parameters  $\alpha$  and  $\beta$ . The nonuniqueness of the solution is actually very reasonable.

What we have specified is only the anisotropic medium, while the original medium and the transformation are not subject to any additional limitation.

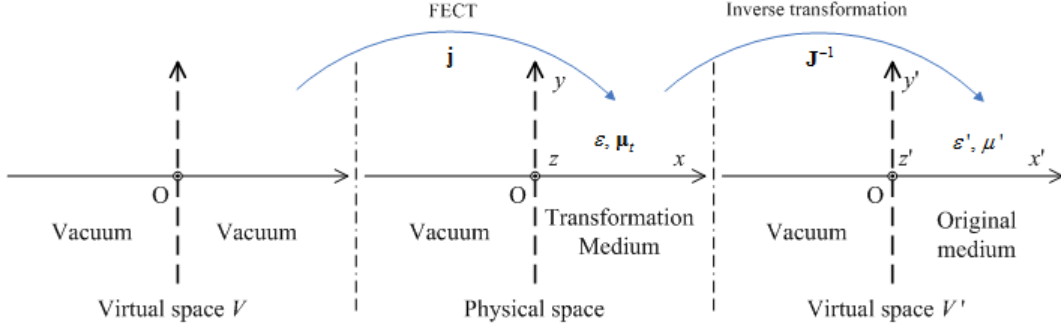


Figure 4.1: Transformation configurations for reflection analysis near the local boundary of a device designed with FECT.

The boundary of a device designed with FECT may be curved and inhomogeneous, and can not be easily described in a general form, which is also not necessary. Instead, it is convenient to use a local view to treat reflection at any local area of the boundary. Under the assumption of the local view, linear approximation is enough to characterize transformations and consequently the media become homogeneous. Fig. 4.1 shows transformation configurations near the local boundary of a device designed with FECT. The boundary is denoted by the vertical dashed axis in the physical space. On the right side is the TrM in the device, obtained with FECT from a virtual space  $V$  in the leftmost panel of the figure. On the left side, corresponding to the outside of the device, is a vacuum, which can be considered as transformed with identity transformation. In most cases, the FECT and identity transformation are not continuously connected at the boundary, hence there is no global transformation from the virtual space  $V$  to the physical space. The rightmost panel illustrates the inverse transformation to a virtual space  $V'$  with an isotropic original medium. As remarked in the comment of Eq. (4.7), the inverse transformation is not unique. We need to pick out a specific one so that it is continuous at the boundary, and a global transformation between the virtual space  $V'$  and the physical space is established. A linear transformation continuous with identity transformation at  $x = 0$  satisfies  $X(0, y') = 0$ ,  $Y(0, y') = y'$ , and  $Z(z) = z'$ . The Jacobian matrix elements follow

$$\frac{\partial X}{\partial y'} = 0, \quad \frac{\partial Y}{\partial y'} = \frac{\partial Z}{\partial z'} = 1. \quad (4.8)$$

Substitution of Eq. (4.7) into Eq. (4.8) leads to the unique transformation continuous at the boundary and the corresponding original medium,

$$\begin{aligned} \sin \alpha = 0, \quad \cos \beta &= \frac{\mu_{xy}}{\sqrt{\mu_{xx}\mu_{yy}}}, \quad \varepsilon' = \sqrt{\frac{\mu_{xx}}{\mu_{yy}}} \frac{\varepsilon}{\sin \beta}, \\ \mu' &= \sqrt{\mu_{xx}\mu_{yy}} \sin \beta. \end{aligned} \quad (4.9)$$

Now the whole physical space can be interpreted in the framework of TrO with the continuous inverse transformation to the virtual space  $V'$ . Suppose a plane wave

$$E_{inc} = e^{ik_0(x \cos \theta + y \sin \theta) - i\omega t}$$

incident upon the boundary from the vacuum side. The reflected wave is

$$E_R = \text{Re}^{ik_0(-x \cos \theta + y \sin \theta) - i\omega t}.$$

Due to the identity transformation on the left side, the incident and reflected waves in the virtual space  $V'$  are identical to those in the physical space. In the virtual space  $V'$ , the reflection coefficient  $R$  from the isotropic medium is available from the Fresnel formula

$$R = \frac{1-p}{1+p}, \quad (4.10)$$

where  $p = \sqrt{\varepsilon' \mu'} \cos \gamma / \cos \theta$  and the refraction angle is determined by  $\sin \gamma = \sin \theta / \sqrt{\varepsilon' \mu'}$ . Although the transmitted wave on the TrM side is not our concern, it can be calculated by transforming from the virtual space  $V'$ .

The TrM has been described by  $\varepsilon$  and  $\boldsymbol{\mu}$ . Considering that the device is designed with FECT, they are expressible with the Jacobian matrix  $\mathbf{j}$  of the FECT by

$$\begin{aligned} \varepsilon &= \frac{j_{zz}^2}{\det(\mathbf{j})}, \quad \mu_{xx} = \frac{j_{xx}^2 + j_{xy}^2}{\det(\mathbf{j})}, \\ \mu_{xy} &= \frac{j_{xx}j_{yx} + j_{xy}j_{yy}}{\det(\mathbf{j})}, \quad \mu_{yy} = \frac{j_{yx}^2 + j_{yy}^2}{\det(\mathbf{j})}. \end{aligned} \quad (4.11)$$

Insertion of Eq. (4.11) into Eq. (4.9) gives

$$\begin{aligned} \sin \beta &= \frac{j_{xx}j_{yy} - j_{xy}j_{yx}}{\sqrt{(j_{xx}^2 + j_{xy}^2)(j_{yx}^2 + j_{yy}^2)}}, \\ \varepsilon' &= \frac{j_{zz}(j_{xx}^2 + j_{xy}^2)}{(j_{xx}j_{yy} - j_{xy}j_{yx})^2}, \quad \mu' = \frac{1}{j_{zz}}. \end{aligned} \quad (4.12)$$

In the virtual space  $V'$  the reflectionless condition is straightforward. The necessary and sufficient condition is that the original medium itself is a vacuum, *i.e.*,  $\varepsilon' = \mu' = 1$ . Then the reflectionless condition in terms of the Jacobian matrix of the FECT is obtained from Eq. (4.12)

$$\frac{j_{xx}^2 + j_{xy}^2}{(j_{xx}j_{yy} - j_{xy}j_{yx})^2} = j_{zz} = 1. \quad (4.13)$$

The configuration in Fig. 4.1 does not have symmetry in coordinates  $x$  and  $y$ , neither has the reflectionless condition in Eq. (4.13).

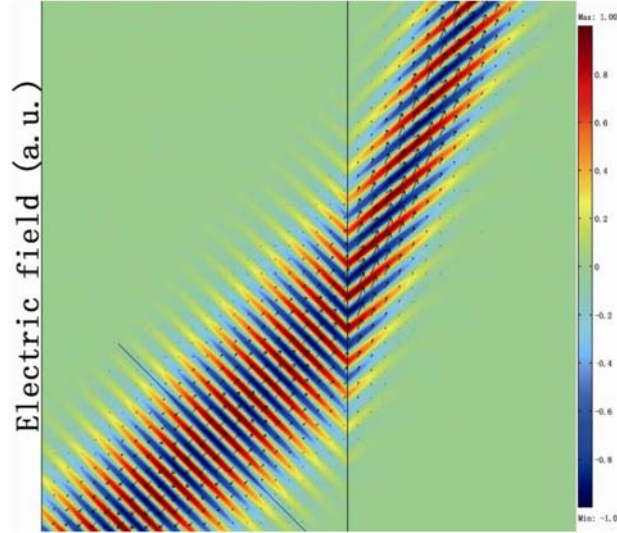


Figure 4.2: Distribution of the electric field for the interaction at the boundary of the TrM (with  $\delta = 0$ ) for a Gaussian beam.

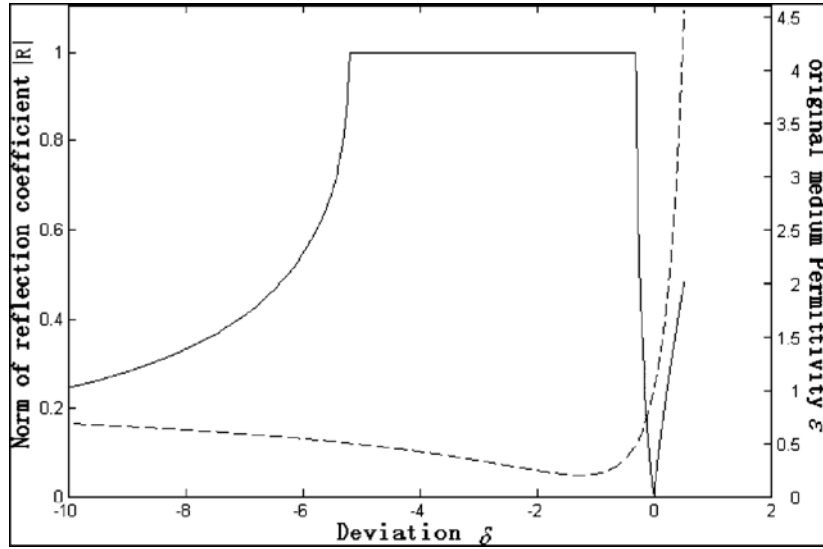


Figure 4.3: The permittivity of the original medium (dashed line) and the norm of the reflection coefficient (solid line) at the boundary of the TrM for a Gaussian beam with an incidence angle of  $45^\circ$ .

In order to demonstrate the effectiveness of inverse TrO, FEM based numerical simulations are performed. Take a FECT with the Jacobian matrix

$$\mathbf{j} = \begin{pmatrix} 1 & 3/4 + \delta & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as an example. A 2D Gaussian beam illuminates the boundary at an incidence angle of  $45^\circ$ . When  $\delta = 0$ , the Jacobian matrix satisfies the reflectionless condition Eq. (4.13). The electric field distribution for the interaction of the TrM (with  $\delta = 0$ ) with the incident Gaussian beam is shown in Fig. 4.2. As expected, no reflection occurs. The arrows indicating power flow also manifest the total

transmission. For situations with nonzero reflection, the original medium and the reflection coefficient are accessible with Eqs. (4.12) and (4.10). They are plotted in Fig. 4.3 with  $\delta$  in the range of  $[-10, 5/4]$  (the permeability is constant 1). The reflection coefficient can be well understood with the help of the permittivity of the original medium.

Besides the present application instance, we notice that in Ref. [11] the authors essentially utilize the same idea of inverse TrO to transform a reduced cloak (anisotropic) to an isotropic original medium in a virtual space. The properties of reduced cloaks are thus simplified to those of isotropic structures.

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## 5. Half-space Cloaking on a Dielectric Ground

The invisibility cloaks considered in the previous chapters of this thesis are supposed to make objects invisible for EM waves coming from any direction. This kind of cloaking is called full-space cloaking. In many applications, however, we want to cloak objects resting on a ground from probing EM waves that only come from above the ground. We call it half-space cloaking in comparison with full-space cloaking. This chapter is dedicated to studies on half-space cloaking. Following a background introduction, two different half-space cloak designs are presented and discussed.

### 5.1 Introduction to Half-space Cloaking

The most fascinating fact about invisibility cloaks is that the device and objects hidden inside can not be observed externally regardless from which angle they are viewed. This is also the reason why it captures so much attention. On the other hand, in most applications the cloaked objects are situated on some supporting plane or ground, and EM illuminations come from the half-space above the ground, similar to the stealth of military targets from radar detection. Full-space cloaks are not convenient for such configurations, as objects must be levitated above the ground. In this case, we need a cloaking cover structure with an open concealment volume. Different from the stealth technology, the scattering from objects is not merely removed to make the objects “black”. The scattering is restored to such as if there is nothing on the ground, so that true half-space invisibility is achieved. As the first attempt at constructing a half-space cloaking cover, half of a perfect cylindrical cloak padded with matching strips is proposed by us in order to realize half-space cloaking. This design is elaborated in the next section.

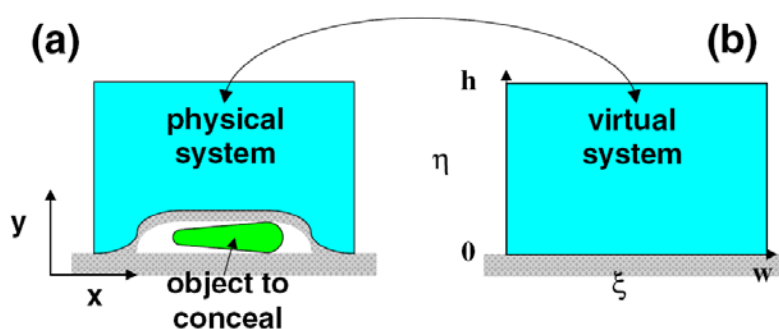


Figure 5.1: The transform configuration of carpet cloak [1].

Shortly after our design of the half-space cloaking cover, a carpet cloak was proposed by J. Li and J. B. Pendry to cloak an object on a PEC ground [1]. The transform configuration of carpet cloaks is shown in Fig. 5.1. The planar PEC ground is deformed and becomes convex by a

transformation to create some internal space to conceal the object. Outside the rectangular region (light blue in Fig. 5.1), identity transformation makes sure that the external field is the same as that when there is only the PEC ground, *i.e.*, half-space invisibility. Inside the cloak, the transformation only deforms a boundary, and is obviously not singular as that for full-space cloaks. The CP of carpet cloaks are thus nonsingular. In Ref. [1], numerical optimization is applied to make the transformation quasi-conformal. Only one nonsingular material parameter distribution suffices to characterize the carpet cloak, making the experimental realization much more feasible than full-space cloaks. Although the parameter still takes values less than one, it can also be scaled up to larger than one by considering non-vacuum background material, promising broadband behaviors. The greatly reduced difficulty in experimental fabrication makes carpet cloaks advantageous over other cloak designs based on TrO.

The feasibility of carpet cloaks drew huge interest of experiment research soon after the invention. A microwave broadband carpet cloak consisting of metamaterials was first demonstrated [2]. Later carpet cloaks working at optical frequencies were fabricated with dielectric structures [3-8]. Carpet cloaks were also realized for surface plasmon waves by integration of TrO and plasmonics [9, 10]. More recently, homogeneous and anisotropic natural crystals are utilized to construct macroscopic carpet cloaks for visible light [11, 12]. Many theoretical studies on carpet cloaks have also been carried out [13-17]. The carpet cloak enables experimental demonstration of invisibility. There are still some obstacles to overcome before it can become a practical application. Several limitations are put forward with theoretical analysis [18, 19]. Using non-resonant media to build carpet cloaks with moderate thickness and the capability of hiding relatively large objects is a tough challenge. The design of a real 3D carpet cloak is crucial for application as well. A few works towards this aim have been done [20, 21].

The original carpet cloak is on a PEC ground, and this assumption has been kept in all subsequent research published in the literature up to now. In practical applications, the supporting ground is in general dielectric and the carpet cloak needs to be generalized to be on a dielectric ground. In another of our half-space cloaking designs, which will be presented in Section 5.3, the carpet cloak is extended so that it can work on a dielectric ground by introducing an absorbing layer.

### 5.2 Semi-cylindrical Half-space Cloak

As the first attempt at half-space cloaking, it is natural to utilize half of a 2D perfect cloak in hopes of taking advantage of its perfect invisibility property. Let's consider a semi-cylindrical cover, *i.e.*, half of a perfect cylindrical cloak, resting on a dielectric ground characterized by relative permittivity  $\epsilon_d$  and relative permeability  $\mu_d$ . We can not expect the semi-cylindrical cover alone to perform as well as a half-space cloak on an indefinite dielectric ground. This is also confirmed by numerical simulation of EM wave scattering from the semi-cylindrical cover on a dielectric ground (see below in the simulation part). Relatively weak reflection at the ground interface right below the semi-cylindrical cover is observed. To achieve half-space invisibility, the reflection coefficient at

the ground interface must be uniform everywhere. Two vertical matching strips are inserted below the cover to induce proper reflection strength, as shown in Fig. 5.2. The properties of the matching strips will be determined analytically in the following study.

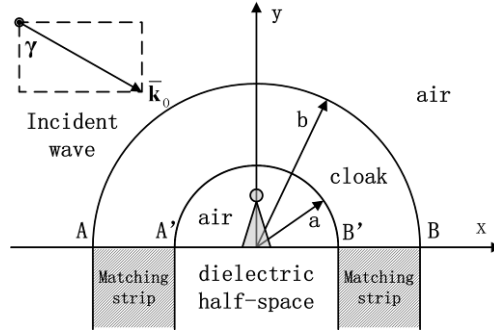


Figure 5.2: Schematic diagram for a semi-cylindrical cover with vertical matching strips underneath interfaces AA' and BB'.

Specifically, assume that the semi-cylindrical cover is half of the cylindrical perfect cloak designed in Section 2.4, whose transformation function is given by Eqs. (2.74), except that now the inner and outer radii are  $a$  and  $b$ ,

$$\begin{aligned} r &= \frac{b-a}{b} r' + a, \quad 0 < r < b \\ \theta &= \theta', \quad z = z'. \end{aligned} \quad (5.1)$$

Suppose an arbitrary plane wave of TE polarization is incident upon the semi-cylindrical cover padded with matching strips (see Fig. 5.2). TM polarization case can be treated similarly. The electric field of the incident wave is

$$E_{i,z} = E_0 \exp[ik_0(x \sin \gamma - y \cos \gamma)] \exp(-i\omega t), \quad (5.2)$$

where  $\gamma$  is the incident angle. For a perfect cloak, EM waves enter the shell structure without any reflection at the outer boundary and propagate inside still as a well defined wave, which is easily described by using Eq. (2.41) in TrO,

$$E_{c,z} = E_0 \exp[ik_0(x' \sin \gamma - y' \cos \gamma)] \exp(-i\omega t). \quad (5.3)$$

Although the current configuration consists of only half of a perfect cloak, the incident wave inside the semi-cylindrical cover should propagate in the same way as in the perfect cylindrical cloak until it hits the ground interface, *i.e.*, the electric field of the incident wave inside the semi-cylindrical cover can be represented by Eq. (5.3). The contour of the phase term  $x' \sin \gamma - y' \cos \gamma$  ( $x'$  and  $y'$  are variables in the virtual space) determines the curved phase front of the inside incident wave. To analyze the reflection of such an inhomogeneous wave as that in Eq. (5.3), we need to introduce a local reflection coefficient by treating it as a local plane wave near any particular point  $P(x_p, 0)$  on the ground interface. Expanding the phase in Eq. (5.3) into

power series of  $x$  and  $y$  around the point P, the local plane wave near P is expressed as

$$E_{c,z} = E_0 \exp^{ik_0 \{ (x-x_p) \sin \gamma [b/(b-a)] - y \cos \gamma (r'_p/r_p) + x'_p \sin \gamma \}} \exp^{-i\omega t}. \quad (5.4)$$

Locally around the point P, the inhomogeneity in the CP of the semi-cylindrical cover can be omitted when calculating the local reflection coefficient. Fresnel formula can be applied to find it

$$R_c = \frac{\mu_m \cos \gamma - \sqrt{\varepsilon_m \mu_m [(b-a)/b]^2 - \sin^2 \gamma}}{\mu_m \cos \gamma + \sqrt{\varepsilon_m \mu_m [(b-a)/b]^2 - \sin^2 \gamma}}, \quad (5.5)$$

where  $\varepsilon_m$  and  $\mu_m$  are the relative permittivity and permeability of the matching strips. It is surprising to note that  $R_c$  is independent of the position of the point P. This interesting result is critical for the design of the proposed half-space cloak. Outside the cover, the reflection coefficient of the incident plane wave from the ground is simply

$$R_d = \frac{\mu_d \cos \gamma - \sqrt{\varepsilon_d \mu_d - \sin^2 \gamma}}{\mu_d \cos \gamma + \sqrt{\varepsilon_d \mu_d - \sin^2 \gamma}}. \quad (5.6)$$

In order to make the semi-cylindrical half-space cloak invisible, the local coefficient inside the cover should be matched with that outside,  $R_c = R_d$ . Substitutions of Eqs. (5.5) and (5.6) lead to

$$\left( \varepsilon_m \mu_m [(b-a)/b]^2 - \sin^2 \gamma \right) / \mu_m^2 = \left( \varepsilon_d \mu_d - \sin^2 \gamma \right) / \mu_d^2. \quad (5.7)$$

Among all possible solutions of  $(\varepsilon_m, \mu_m)$  to Eq. (5.7), there is one solution independent of the incident angle  $\gamma$ ,

$$\varepsilon_m = [b/(b-a)]^2 \varepsilon_d, \mu_m = \mu_d. \quad (5.8)$$

Thanks to the uniform local coefficient and the matching material independent of the incident angle, the semi-cylindrical cover padded with the matching strips should be able to realize half-space invisibility for plane waves with any incident angle, thus any form of incident waves. The excellent performance is confirmed and illustrated with numerical simulations.

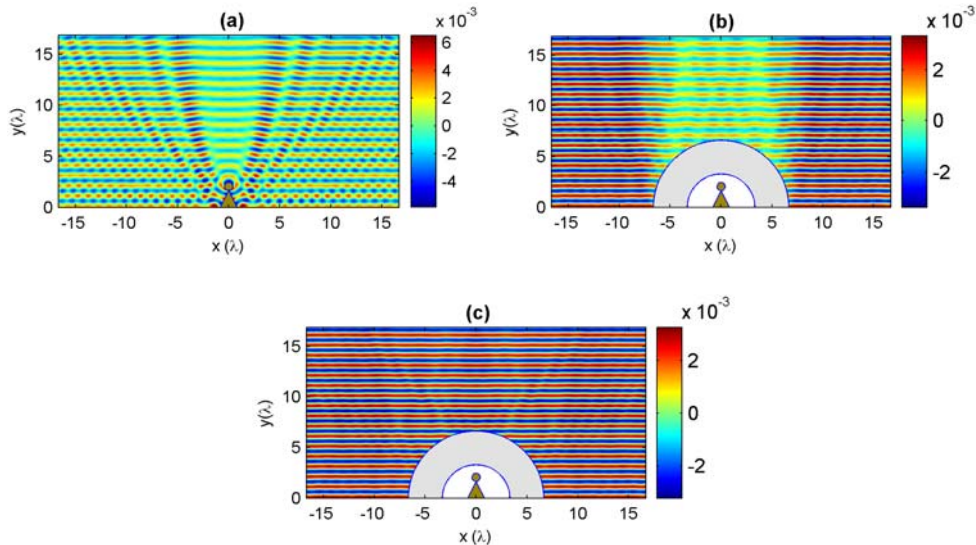


Figure 5.3: Snapshots of the scattered electric fields for a normally incident plane wave on (a) a bare scatterer, (b) a scatterer covered by a semi-cylindrical cover without matching strips, and (c) a scatterer covered by a semi-cylindrical cover with two matching strips.

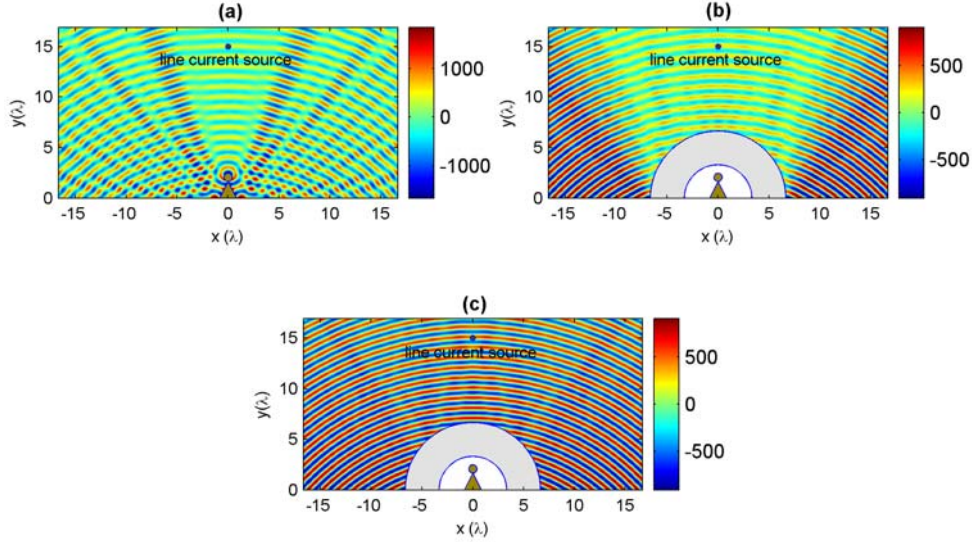


Figure 5.4: Snapshots of the scattered electric fields of a line current source for the case of (a) a bare scatterer, (b) a scatterer covered by a semi-cylindrical cover without matching strips, and (c) a scatterer covered by a semi-cylindrical cover with two matching strips. The black dots indicate the position of the line current source.

FEM based numerical simulations are performed to validate and illustrate the half-space invisibility of the semi-cylindrical half-space cloak. Detailed settings of the model include:  $a = 20\lambda_0 / 3$ ,  $b = 10\lambda_0 / 3$ ,  $\varepsilon_d = 10$ ,  $\mu_d = 1$ ,  $\varepsilon_m = 40$ ,  $\mu_m = 1$ . The CP of the semi-cylindrical cover are calculated with Eqs. (2.38) and (2.39). A PEC scatterer is placed on the dielectric ground for comparison. As excitation sources, a normally incident plane wave and a line current source are applied. The simulation results are shown in Figs. 5.3 and 5.4. When only the scatterer is placed on the ground, strong scattering is seen induced in (a). If the semi-cylindrical cover alone is used, partial cloaking effect is observed in (b), and the weak reflection right above the cover makes them exposed. The weak reflection can be compensated by inserting the matching strips under the cover. In (c), our proposed semi-cylindrical half-space cloak is proven able to restore the scattered field to the same as that from the bare dielectric ground. The cylindrical wave has a continuous spectrum of plane waves. Fig. 5.4(c) confirms that the semi-cylindrical half-space cloak works for plane waves with any incident angle. In the above design and numerical simulations, the matching strips are supposed to extend downwards infinitely. In practice, they should be terminated at a finite length. The termination will cause additional disturbance in the scattered field. To ameliorate the performance, a small amount of loss can be added to the matching strips, without changing the wave impedance much. The improvement can be demonstrated by numerical simulations.  $\varepsilon_m = 40 + 4i$  and one vacuum wavelength long matching strips are used. The results are compactly shown in Fig. 5.5

taking advantage of the symmetry of the configuration. In the left panel, the terminated matching strips are lossless. Standing waves and perturbation in the reflected field are observed in the strips and outside the cover, respectively. When loss is introduced, the transmitted field in the matching strips gets attenuated, and the excellent cloaking effect is recovered.

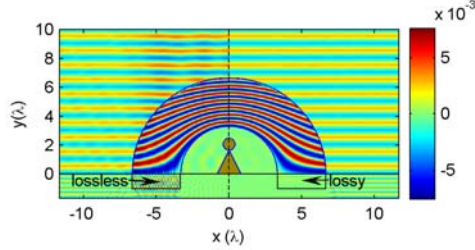


Figure 5.5: A snapshot of the electric field distributions for a normally incident plane wave when the length of the two matching strips is finite. The matching strips are lossy in the right part and lossless in the left part (to clearly show the perturbation outside the cover, the electric field outside the cover is for the reflected wave, while the plotted field in the cover, strips and dielectric ground is for the total field).

### 5.3 Carpet Cloaking on a Dielectric Half-space

The semi-cylindrical half-space cloak discussed in the last section could produce excellent half-space invisibility, whereas the semi-cylindrical cover from a perfect cylindrical cloak has strong singularity in the CP. It can be realized only if a perfect cylindrical cloak is constructed in the first place, which is rather difficult. Considering the good features of the carpet cloak, this section turns to carpet cloaks for half-space invisibility on a dielectric ground. As mentioned in the introduction section, all carpet cloaks follow the original paper, and rest on a PEC or a highly reflective ground. In the present design, the extension to carpet cloaking on a dielectric ground is made.

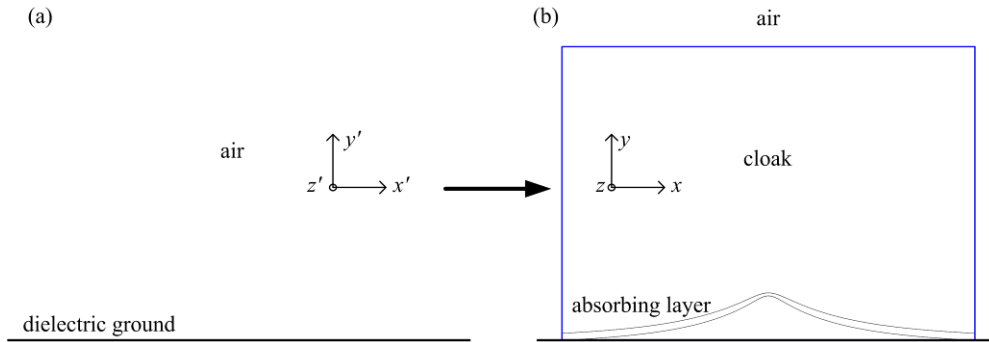


Figure 5.6: Schematic diagram of a carpet cloak on a dielectric ground: Air and a layer of the dielectric ground in a virtual space (a) are transformed to the carpet cloak in the physical space (b).

Instead of a PEC ground for previous carpet cloaks, here we consider a general lossy dielectric ground with the permittivity of  $\epsilon_g = \epsilon_r + \epsilon_i * i$ . To apply TrO, suppose a virtual space as shown in Fig. 5.6(a) with only the dielectric ground. An EM wave incident upon the ground gets partially reflected

and some transmits into the ground. The transmitted wave, however, can not propagate far, and is almost totally attenuated within a finite layer of the lossy material (with the lower bound indicated by the dashed line in Fig. 5.6(a)). Essentially, EM waves only exist in an upper half of the virtual space, just as for the case with a PEC ground. Then a nonsingular transformation can be similarly applied to the upper half-space to design a carpet cloak on the dielectric ground. The lossy layer is transformed from planar in the virtual space to convex in the physical space in Fig. 5.6(b). Under the curved absorbing layer a concealment volume is formed. The transformation must be approximated by identity transformation outside a finite transformation region, so that the carpet cloak has a finite size and the external reflected field is the same as that from the bare dielectric ground. Such a region is enclosed by the blue lines in Fig. 5.6(b). In 2D TrO designs, conformal mappings can simplify the CP of the TrM greatly. The conformal mapping here is introduced by a simple analytical function  $z = z' - 1 / z'$ , where  $z' = x' + y' * i$  and  $z = x + y * i$ . In this design, arbitrary units are used unless specifically stated. The lossy layer is bounded by  $y' = 1.3$  and  $y' = 1.4$ . It is obvious that  $z$  approaches  $z'$  as  $|z'|$  goes to infinity. Thus the transformation is properly approximated by identity transformation outside a transformation region, whose upper boundary is  $y' = 5.8$ , and vertical boundaries are at positions where the curved absorbing layer touches the ground. Separation of the real and imaginary parts of the complex variables in the analytical function gives the transformation between the two spaces,

$$x = x' \left( 1 - \frac{1}{x'^2 + y'^2} \right), y = y' \left( 1 + \frac{1}{x'^2 + y'^2} \right), z = z', \quad (5.9)$$

and

$$x' = \frac{x+p}{2}, y' = \frac{y+q}{2}, z' = z, \quad (5.10)$$

where  $p = \text{sgn}(x) \sqrt{(s + \sqrt{s^2 + t^2}) / 2}$ ,  $q = \sqrt{(-s + \sqrt{s^2 + t^2}) / 2}$ ,  $s = x^2 - y^2 + 4$ , and  $t = 2xy$ . Substituting Eq. (5.9) into Eq. (2.38), the CP of the carpet cloak are

$$\eta \equiv \frac{\varepsilon_l}{\varepsilon_g} = \frac{\varepsilon_c}{\varepsilon_0} = \left[ 1 + \frac{2(x'^2 - y'^2) + 1}{(x'^2 + y'^2)^2} \right]^{-1} \quad (5.11)$$

for the TE polarization case.  $\varepsilon_l$  and  $\varepsilon_c$  represent the permittivities of the absorbing layer and the upper part of the carpet cloak.

The capability of half-space cloaking of the design is tested by FEM numerical simulations. The configuration for simulation is the same as that in Fig. 5.6. An obliquely incident 2D Gaussian beam with the wavelength of 0.12 is used as an excitation. The ground is described by  $\varepsilon_g / \varepsilon_0 = 5 - 5 * i$ . The CP of the carpet cloak are implemented with Eqs. (5.10) and (5.11). Three different settings are simulated for comparison. In Fig. 5.7(a), the incident Gaussian beam is strongly scattered by a PEC bump on the ground. When the carpet cloak is covered on the bump in Fig.



5.7(b), the scattered field of the Gaussian beam is the same as the reflected beam from the bare ground shown in Fig. 5.7(c) in the sense of both direction and intensity.

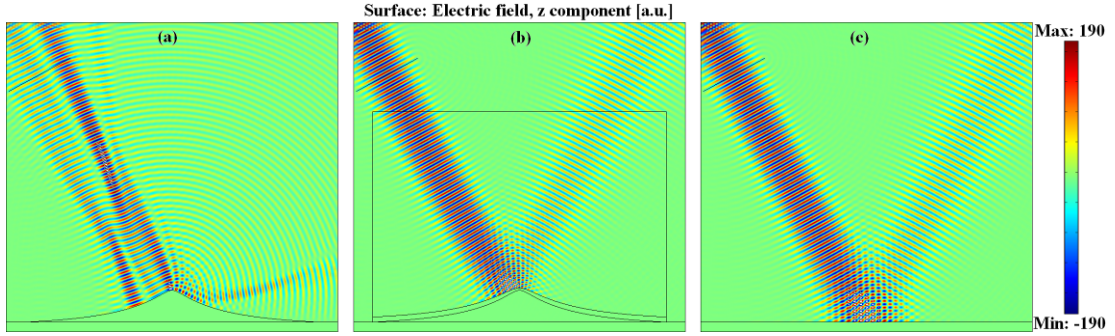


Figure 5.7: Snapshots of the electric field distributions for a Gaussian beam impinging upon (a) a PEC bump on a dielectric ground characterized by  $\epsilon_g / \epsilon_0 = 5 + 5 * i$ , (b) the PEC bump covered by a carpet cloak, and (c) the bare dielectric ground.

The nonsingular CP of carpet cloak may be scaled up to larger than one by utilizing a nonvacuum background, so that broadband invisibility is possible. This trick has been exploited in experimental demonstration of broadband cloaking. The current carpet cloak can also be manipulated in this way to realize broadband cloaking on a dielectric ground, which is illustrated by simulation. To be specific, meter is chosen as the length unit. Numerical simulations at three frequencies for the carpet cloak are performed and the results are shown in Fig. 5.8. Clear reflected Gaussian beams are restored in all cases.

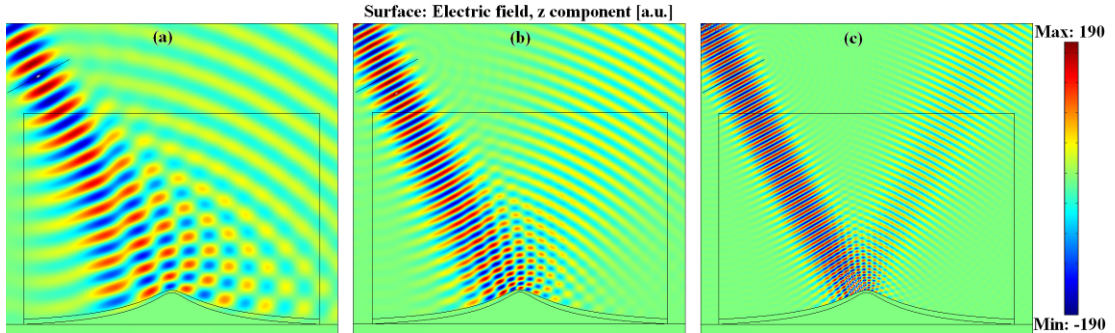


Figure 5.8: Snapshots of the electric field distributions for a Gaussian beam impinging upon a PEC bump covered by a carpet cloak on a dielectric ground at (a) 0.5 GHz; (b) 1 GHz; and (c) 2 GHz.

The dielectric ground has been assumed to be considerably lossy until now. Yet there are also cases where the dielectric ground has very low loss, so that the lossy layer in the design does not have a reasonable thickness. Fictitious loss must be added to the ground so that we can introduce the lossy layer as before. On the other hand, to keep the wave impedance of the ground the fictitious loss should have a gradient distribution, increasing from 0 at the interface of the ground. An example for a lossless ground with  $\epsilon_g = 2$  is simulated to confirm the method of adding fictitious loss. The permittivity is artificially modified lossy as  $\epsilon_g [1 + 150 (1.4 - y) 2 * i]$ , and the lossy layer is still bounded by  $y' = 1.3$  and  $y' = 1.4$ . With the same simulation settings, the results are shown in Fig. 5.9. The reflected beam is again restored with correct direction and intensity.



The transmitted wave is seen to be efficiently absorbed in the absorbing layer.

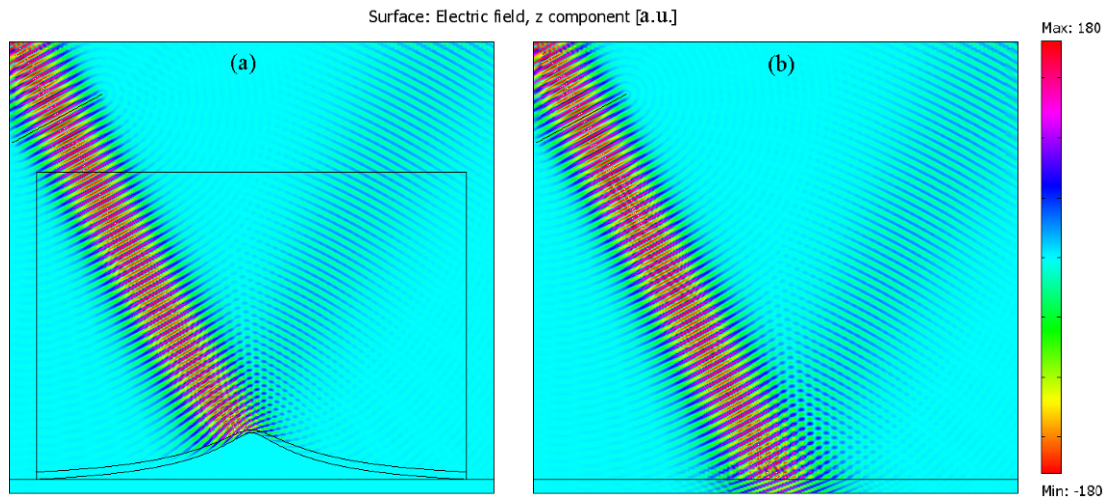


Figure 5.9: Snapshots of the electric field distributions for a Gaussian beam impinging upon (a) a carpet cloak covering a PEC bump on a lossless dielectric ground ( $\epsilon_g / \epsilon_0 = 2$ ), and (b) the bare dielectric ground.

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## 6. Conclusion and Future Work

In this thesis, we focus on the theory of TrO and invisibility cloak designs as the application of TrO. The essence of TrO is elaborated. The concept of inverse TrO is established and used to analyze reflection for FECT media. A novel design method of reduced 2D cloaks of arbitrary shapes is introduced and validated. We propose two designs for cloaking objects on a dielectric ground as well.

By taking CP of perfect 3D cloaks of mirror symmetry on the symmetry plane, an intuitive and effective method is introduced to design reduced 2D cloaks of arbitrary shapes. Two examples, an elliptic and a bow-tie shaped reduced cloak, are designed and numerically simulated to validate and illustrate the effectiveness of the method.

As a complementary concept to TrO, we consider a general idea of inverse TrO. 2D inverse TrO is specifically developed to study reflection at the boundary of a FECT medium in the framework of TrO. Different from the FECT, a whole space mapping between the physical space and a virtual space is realized by an inverse transformation continuous across the boundary. A necessary and sufficient condition of reflectionlessness is derived as a special result.

For the convenience of particular applications, we investigate half-space cloaking on dielectric grounds. For the first time, we propose a semi-cylindrical half-space cloak to achieve invisibility on a dielectric ground. Two carefully designed matching strips are inserted underneath half of a perfect cylindrical cloak to induce proper reflection. The local reflection coefficient is calculated analytically and matched to that from the bare ground. Thus the half-space cloak can give rather good half-space invisibility.

The carpet cloak is an important idea for the accomplishment of half-space cloaking. We extend the original design of the carpet cloak on a PEC ground to a dielectric ground. Taking advantage of a lossy layer in a virtual space, a carpet cloak with an additional absorbing layer is obtained to cloak objects on a dielectric ground. Through numerical simulations, both direction and intensity of reflected Gaussian beams are restored to the same state as in the case of the bare dielectric ground.

For future work, there are several hot ongoing topics in the field of TrO and invisibility cloaks. Realization of invisibility cloaks of practical usage is studied a lot. 3D full-space invisibility cloaks, carpet cloaks and geometrical optical cloaks have all been investigated in recent research. Quasi-conformal mapping is also pushed to 3D for device designs with isotropic media. The theory of TrO is generalized further and further to include more EM media in the framework.



## 7. Summary of Contributions

### **Paper A.** Cloaking an object on a dielectric half-space

**Pu Zhang**, Yi Jin, and Sailing He, “Cloaking an object on a dielectric half-space,” *Opt. Express*, 16, 3161 (2008).

*Contribution of the author:* Design and analysis of the half-space cloak. Numerical simulation of the proposed device. The first draft of the manuscript.

### **Paper B.** Obtaining a nonsingular two-dimensional cloak of complex shape from a perfect three-dimensional cloak

**Pu Zhang**, Yi Jin, and Sailing He, “Obtaining a nonsingular two-dimensional cloak of complex shape from a perfect three-dimensional cloak,” *Appl. Phys. Lett.*, 93, 243502 (2008).

*Contribution of the author:* Introduction of the design method. Calculations of material parameters for the two examples. Numerical simulation of the reduced cloaks. The first draft of the manuscript.

### **Paper C.** Inverse Transformation Optics and Reflection Analysis for Two-Dimensional Finite-Embedded Coordinate Transformation

**Pu Zhang**, Yi Jin, and Sailing He, “Inverse Transformation Optics and Reflection Analysis for Two-Dimensional Finite-Embedded Coordinate Transformation,” *IEEE J. Sel. Top. Quant. Electron.*, 16, 427 (2010).

*Contribution of the author:* Proposal of the concept of inverse TrO. Calculation of the reflection from FECT media and reflectionless condition. Numerical simulation of examples. The first draft of the manuscript.

### **Paper D.** Carpet cloaking on a dielectric half-space

**Pu Zhang**, Michaël Lobet, and Sailing He, “Carpet cloaking on a dielectric half-space,” *Opt. Express*, 18, 18158 (2010).

*Contribution of the author:* Propose the design of carpet cloak on a dielectric ground. Analytical and numerical analysis of the device. The first draft of the manuscript.