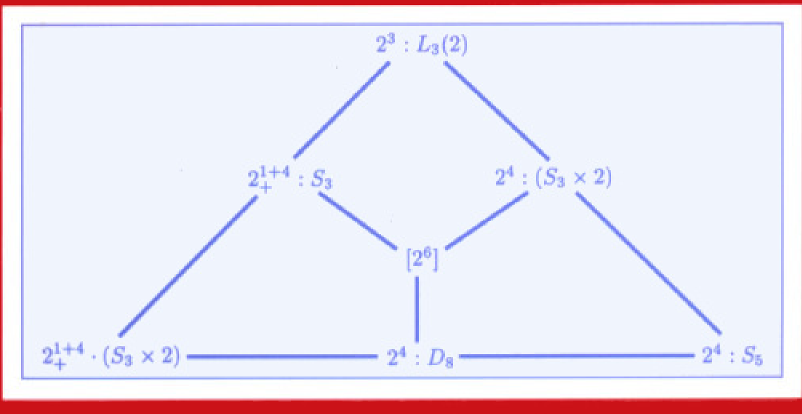
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS 91

GEOMETRY OF SPORADIC GROUPS II
REPRESENTATIONS AND AMALGAMS

A. A. IVANOV and S. V. SHPECTOROV



ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

FOUNDED BY G.-C. ROTA

Volume 91

Geometry of Sporadic Groups II
Representations and amalgams

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

4 W. Miller, Jr. *Symmetry and separation of variables*
6 H. Minc *Permanents*
11 W. B. Jones and W. J. Thron *Continued fractions*
12 N. F. G. Martin and J. W. England *Mathematical theory of entropy*
18 H. O. Fattorini *The Cauchy problem*
19 G. G. Lorentz, K. Jetter and S. D. Riemenschneider *Birkhoff interpolation*
21 W. T. Tutte *Graph theory*
22 J. R. Bastida *Field extensions and Galois theory*
23 J. R. Cannon *The one-dimensional heat equation*
25 A. Salomaa *Computation and automata*
26 N. White (ed.) *Theory of matroids*
27 N. H. Bingham, C. M. Goldie and J. L. Teugels *Regular variation*
28 P. P. Petrushev and V. A. Popov *Rational approximation of real functions*
29 N. White (ed.) *Combinatorial geometrics*
30 M. Pohst and H. Zassenhaus *Algorithmic algebraic number theory*
31 J. Aczel and J. Dhombres *Functional equations containing several variables*
32 M. Kuczma, B. Chozewski and R. Ger *Iterative functional equations*
33 R. V. Ambartzumian *Factorization calculus and geometric probability*
34 G. Gripenberg, S.-O. Londen and O. Staffans *Volterra integral and functional equations*
35 G. Gasper and M. Rahman *Basic hypergeometric series*
36 E. Torgersen *Comparison of statistical experiments*
37 A. Neumaier *Intervals methods for systems of equations*
38 N. Korneichuk *Exact constants in approximation theory*
39 R. A. Brualdi and H. J. Ryser *Combinatorial matrix theory*
40 N. White (ed.) *Matroid applications*
41 S. Sakai *Operator algebras in dynamical systems*
42 W. Hodges *Model theory*
43 H. Stahl and V. Totik *General orthogonal polynomials*
44 R. Schneider *Convex bodies*
45 G. Da Prato and J. Zabczyk *Stochastic equations in infinite dimensions*
46 A. Bjorner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler *Oriented matroids*
47 E. A. Edgar and L. Sucheston *Stopping times and directed processes*
48 C. Sims *Computation with finitely presented groups*
49 T. Palmer *Banach algebras and the general theory of *-algebras*
50 F. Borceux *Handbook of categorical algebra I*
51 F. Borceux *Handbook of categorical algebra II*
52 F. Borceux *Handbook of categorical algebra III*
54 A. Katok and B. Hassleblatt *Introduction to the modern theory of dynamical systems*
55 V. N. Sachkov *Combinatorial methods in discrete mathematics*
56 V. N. Sachkov *Probabilistic methods in discrete mathematics*
57 P. M. Cohn *Skew Fields*
58 Richard J. Gardner *Geometric tomography*
59 George A. Baker, Jr. and Peter Graves-Morris *Padé approximants*
60 Jan Krajicek *Bounded arithmetic, propositional logic, and complex theory*
61 H. Gromer *Geometric applications of Fourier series and spherical harmonics*
62 H. O. Fattorini *Infinite dimensional optimization and control theory*
63 A. C. Thompson *Minkowski geometry*
64 R. B. Bapat and T. E. S. Raghavan *Nonnegative matrices and applications*
65 K. Engel *Sperner theory*
66 D. Cvetkovic, P. Rowlinson and S. Simic *Eigenspaces of graphs*
67 F. Bergeron, G. Labelle and P. Leroux *Combinatorial species and tree-like structures*
68 R. Goodman and N. Wallach *Representations of the classical groups*
69 T. Beth, D. Jungnickel and H. Lenz *Design theory volume I 2 ed.*
70 A. Pietsch and J. Wenzel *Orthonormal systems and Banach space geometry*
71 George E. Andrews, Richard Askey and Ranjan Roy *Special functions*
72 R. Ticciati *Quantum field theory for mathematicians*
76 A. A. Ivanov *Geometry of sporadic groups I*
78 T. Beth, D. Jungnickel and H. Lenz *Design theory volume II 2 ed.*
80 O. Stormark *Lie's structural approach to PDE systems*
81 C. F. Dunkl and Y. Xu *Orthogonal polynomials of several variables*
82 J. P. Mayberry *The foundations of mathematics in the theory of sets*

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Geometry of Sporadic Groups II
Representations and Amalgams

A. A. IVANOV

Imperial College, London

S. V. SHPECTOROV

Bowling Green State University, Ohio

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge, CB2 2RU, UK
40 West 20th Street, New York, NY 10011–4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2002

Printed in the United Kingdom at the University Press, Cambridge

Typeface Monotype Times 10/12pt *System* LaTeX [UPH]

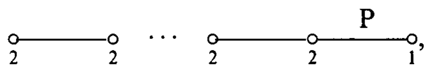
A catalogue record of this book is available from the British Library

ISBN 0 521 62349 9 hardback

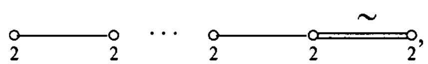
Contents

Preface

This is the second volume of the two-volume series which contains the proof of the classification of the flag-transitive P- and T-geometries. A P-geometry (Petersen geometry) has diagram



where $\underset{2}{\circ}\overset{P}{\rule[0.5ex]{2em}{0.4pt}}\underset{1}{\circ}$ denotes the geometry of 15 edges and 10 vertices of the Petersen graph. A T-geometry (Tilde geometry) has diagram



where $\underset{2}{\circ}\overset{\sim}{=\!=\!=}\underset{2}{\circ}$ denotes the 3-fold cover of the generalized quadrangle of order $(2,2)$, associated with the non-split extension $3 \cdot S_4(2) \cong 3 \cdot Sym_6$.

The final result of the classification, as announced in [ISh94b], is the following (we write $\mathcal{G}(G)$ for the P- or T-geometry admitting G as a flag-transitive automorphism group).

Theorem 1 *Let $\mathcal{G}$ be a flag-transitive P- or T-geometry and G be a flag-transitive automorphism group of $\mathcal{G}$. Then $\mathcal{G}$ is isomorphic to a geometry $\mathcal{H}$ in Table I or II and G is isomorphic to a group H in the row corresponding to $\mathcal{H}$.*

In the first volume [Iv99] and in [IMe99] for the case $\mathcal{G}(J_4)$ the following has been established (for the difference between coverings and 2-coverings cf. Section 1.2).

Theorem 2 *Let $\mathcal{H}$ be a geometry from Table I or II of rank at least* 3 *and H be a group in the row corresponding to $\mathcal{H}$. Then*

(i) *$\mathcal{H}$ exists and is of correct type (i.e., P- or T-geometry);*
(ii) *H is a flag-transitive automorphism group of $\mathcal{H}$;*
(iii) *suppose that $\widetilde{\mathcal{H}}$ is a P- or T-geometry, $\widetilde{H}$ is a flag-transitive automorphism group of $\widetilde{\mathcal{H}}$, $\varphi : \widetilde{\mathcal{H}} \to \mathcal{H}$ is a 2-covering which commutes with the action of $\widetilde{H}$ and the induced action of $\widetilde{H}$ on $\mathcal{H}$ coincides with H, then either φ is an isomorphism or one of the following holds:*
   (a) *$\widetilde{\mathcal{H}} \cong \mathcal{G}(3 \cdot M_{22})$, $\mathcal{H} \cong \mathcal{G}(M_{22})$, $\widetilde{H} \cong 3 \cdot M_{22}$ or $3 \cdot \mathrm{Aut}\, M_{22}$ and φ is a covering;*
   (b) *$\widetilde{\mathcal{H}} \cong \mathcal{G}(3^{23} \cdot Co_2)$, $\mathcal{H} \cong \mathcal{G}(Co_2)$, $\widetilde{H} \cong 3^{23} \cdot Co_2$ and φ is not a covering;*
   (c) *$\widetilde{\mathcal{H}} \cong \mathcal{G}(3^{4371} \cdot BM)$, $\mathcal{H} \cong \mathcal{G}(BM)$, $\widetilde{H} \cong 3^{4371} \cdot BM$ and φ is not a covering, in particular,*
(iv) *either $\mathcal{H}$ is simply connected or $\mathcal{H} \cong \mathcal{G}(M_{22})$ and the universal cover of $\mathcal{H}$ is $\mathcal{G}(3 \cdot M_{22})$.*

Table I. Flag-transitive *P*-geometries

Rank	Geometry $\mathcal{H}$	Flag-transitive automorphism groups H
2	$\mathcal{G}(Alt_5)$	Alt_5, Sym_5
3	$\mathcal{G}(M_{22})$	M_{22}, $\mathrm{Aut}\, M_{22}$
	$\mathcal{G}(3 \cdot M_{22})$	$3 \cdot M_{22}$, $3 \cdot \mathrm{Aut}\, M_{22}$
4	$\mathcal{G}(M_{23})$	M_{23}
	$\mathcal{G}(Co_2)$	Co_2
	$\mathcal{G}(3^{23} \cdot Co_2)$	$3^{23} \cdot Co_2$
	$\mathcal{G}(J_4)$	J_4
5	$\mathcal{G}(BM)$	BM
	$\mathcal{G}(3^{4371} \cdot BM)$	$3^{4371} \cdot BM$

If $\mathcal{F}$ is a geometry and F is a flag-transitive automorphism group of $\mathcal{F}$ then $\mathcal{A}(F, \mathcal{F})$ denotes the amalgam of maximal parabolics associated with the action of F on $\mathscr{F}$. In these terms the main result of this second volume can be stated as follows:

Theorem 3 *Let $\mathscr{G}$ be a flag-transitive P- or T-geometry of rank at least 3 and G be a flag-transitive automorphism group of $\mathscr{G}$. Then for a geometry $\mathscr{H}$ and its automorphism group H from Table I or II we have the following:*

$$\mathscr{A}(G,\mathscr{G}) \cong \mathscr{A}(H,\mathscr{H}).$$

In the above theorem we can assume that $\mathscr{H}$ is simply connected. Then by Theorem 1.4.5, $\mathscr{H}$ is the universal cover of $\mathscr{G}$ and H is the universal completion of $\mathscr{A}(G,\mathscr{G})$.

Notice that Theorem 3 immediately implies the following

Corollary 4 *Let $\mathscr{H}$ be a geometry from Table I or II and let H be a flag-transitive automorphism group of $\mathscr{H}$. Then H is one of the groups in the row corresponding to $\mathscr{H}$ and either*

(i) *H is the full automorphism group of $\mathscr{H}$, or*
(ii) *$\mathscr{H} \cong \mathscr{G}(M_{22})$ or $\mathscr{G}(3 \cdot M_{22})$ and $H \cong M_{22}$ or $3 \cdot M_{22}$, respectively (so that H is the unique self-centralized subgroup of index 2 in the automorphism group of $\mathscr{H}$).*

Table II. Flag-transitive T-geometries

Rank	Geometry $\mathscr{H}$	Flag-transitive automorphism groups H
2	$\mathscr{G}(3 \cdot S_4(2))$	$3 \cdot Alt_6$, $3 \cdot S_4(2) \cong 3 \cdot Sym_6$
3	$\mathscr{G}(M_{24})$ $\mathscr{G}(He)$	M_{24} He
4	$\mathscr{G}(Co_1)$	Co_1
5	$\mathscr{G}(M)$	M
n	$\mathscr{G}(3^{[\frac{n}{2}]_2} \cdot S_{2n}(2))$	$3^{[\frac{n}{2}]_2} \cdot S_{2n}(2)$

Now in order to deduce Theorem 1 from Theorems 2 and 3 it is sufficient to prove the following

Proposition 5 *Let $\mathscr{H}$ be a geometry from Table I or II of rank at least 3 and let H be a group in the row corresponding to $\mathscr{H}$. Suppose that $\sigma : \mathscr{H} \to \overline{\mathscr{H}}$ is a covering of geometries which commutes with the action of H and let $\overline{H}$ denote the action induced by H on $\overline{\mathscr{H}}$. Then the pair $(\overline{\mathscr{H}}, \overline{H})$ is also from Table I or II, respectively.*

Proof. Suppose first that $\mathscr{H}$ is not a P-geometry of rank 3. Then by Theorem 2 (iv) and Corollary 4 $\mathscr{H}$ is simply connected and H is the only flag-transitive automorphism group of $\mathscr{H}$, in particular H is the group of all liftings of elements of $\overline{H}$ to automorphisms of $\mathscr{H}$. Let N be the kernel of the homomorphism of H onto $\overline{H}$. Then N is the deck group of σ and hence N acts regularly on each of the fibers of σ. So $N = 1$ if and only if σ is an isomorphism. It follows from the structure of H that $H/O_3(H)$ is a non-abelian simple group and $O_3(H)$, if non-trivial, is an irreducible $GF(3)$-module for $H/O_3(H)$. Hence either $N = 1$ or $N = O_3(H)$. In the latter case $\overline{\mathscr{H}} = \mathscr{G}(H/O_3(H))$ and by Theorem 2 (iii) the mapping $\mathscr{H} \to \overline{\mathscr{H}}$ is not a covering. Hence $N = 1$. The situation when $\mathscr{H}$ is a P-geometry of rank 3 (i.e., $\mathscr{G}(M_{22})$ or $\mathscr{G}(3 \cdot M_{22})$) can treated in a similar way with a few extra possibilities to be considered. □

Below we outline our main strategy for proving Theorem 3. Let $\mathscr{G}$ be a P- or T-geometry of rank $n \geq 3$, G be a flag-transitive automorphism group of G and

$$\mathscr{A} = \mathscr{A}(G, \mathscr{G}) = \{G_i \mid 1 \leq i \leq n\}$$

be the amalgam of maximal parabolics associated with the action of G on $\mathscr{G}$ (here $G_i = G(x_i)$ is the stabilizer in G of the element x_i of type i in a maximal flag $\Phi = \{x_1, ..., x_n\}$ in $\mathscr{G}$). Our goal is to identify $\mathscr{A}$ up to isomorphism or, more specifically, to show that $\mathscr{A}$ is isomorphic to the amalgam $\mathscr{A}(H, \mathscr{H})$ for a geometry $\mathscr{H}$ and a group H from Table I or II. In fact, it is sufficient to show that given the type of $\mathscr{G}$ and its rank there are at most as many possibilities for the isomorphism type of $\mathscr{A}$ as there are corresponding pairs in Tables I and II.

We proceed by induction on the rank n and assume that all the flag-transitive P- and T-geometries of rank up to $n-1$ (along with their flag-transitive automorphism groups) are known (as in the tables). Then we can assume that for every $1 \leq i \leq n$ the residue $\mathrm{res}_{\mathscr{G}}(x_i)$ and the action $\overline{G_i}$ of G_i on this residue are known. The kernel K_i of this action is a subgroup in the Borel subgroup $B = \cap_{i=1}^{n} G_i$ which in all the cases turns out to be a 2-group.

The induction hypothesis can be used further since certain normal factors of K_i resemble the structure of the residue $\text{res}_{\mathcal{G}}(x_i)$. The most important case is that the action of K_1 on the set of points collinear to x_1 is a quotient of the universal representation module of the residue $\text{res}_{\mathcal{G}}(x_1)$, which is a P- or T-geometry.

Thus, in order to accomplish the identification of the amalgams of maximal parabolics it would be helpful (and essential within our approach) to determine the universal representations of the known P- and T-geometries. Recall that if $\mathcal{H}$ is a geometry (or rather a point-line incidence system) with three points per line, then the universal representation module $V(\mathcal{H})$ is a group generated by pairwise commuting involutions indexed by the points of $\mathcal{H}$ and subject to the relations that the product of the three involutions corresponding to a line is the identity. It is immediate from the definition that $V(\mathcal{H})$ is an elementary abelian 2-group (possibly trivial).

Table III. Natural representations of P-geometries

Rank	Geometry $\mathcal{H}$	dim $V(\mathcal{H})$	$R(\mathcal{H})$
2	$\mathcal{G}(Alt_5)$	6	infinite
3	$\mathcal{G}(M_{22})$	11	$\overline{\mathcal{C}}_{11}$
	$\mathcal{G}(3 \cdot M_{22})$	23	?
4	$\mathcal{G}(M_{23})$	0	1
	$\mathcal{G}(Co_2)$	23	$\overline{\Lambda}^{(23)}$
	$\mathcal{G}(3^{23} \cdot Co_2)$	23	?
	$\mathcal{G}(J_4)$	0	J_4
5	$\mathcal{G}(BM)$	0	$2 \cdot BM$
	$\mathcal{G}(3^{4371} \cdot BM)$	0	?

For the geometries $\mathcal{G}(J_4)$, $\mathcal{G}(BM)$, $\mathcal{G}(M)$ of large sporadic simple groups the universal representation modules are trivial and this is the reason why these geometries do not appear as residues in flag-transitive P- and T-geometries of higher ranks. On the other hand, if $\mathcal{G}$ is one of the above three geometries and G is the automorphism group of $\mathcal{G}$, then the points and lines of $\mathcal{G}$ are certain elementary abelian subgroups in G of order 2 and 2^2, respectively, so that the incidence relation is via inclusion. This means that G is a quotient of the universal representation group $R(\mathcal{G})$ of $\mathcal{G}$. The definition of $R(\mathcal{G})$ is that of $V(\mathcal{G})$ with the wording 'pairwise commuting' removed. Since $V(\mathcal{G})$ is the quotient of $R(\mathcal{G})$ over the commutator subgroup of $R(\mathcal{G})$, sometimes it turns out to be easier to show that $R(\mathcal{G})$ is perfect rather than showing the triviality of $V(\mathcal{G})$ directly. In Part I we calculate the modules $V(\mathcal{G})$ for all flag-transitive P- and T-geometries and the groups $R(\mathcal{G})$ for most of them. These results are summarized in Tables III and IV. The determination problem for $R(\mathcal{G})$ for various geometries $\mathcal{G}$ (including the P- and T-geometries) is of an independent interest, since, in particular, representations control the c-extensions of geometries.

Table IV. Natural representations of T-geometries

Rank	Geometry $\mathcal{H}$	dim $V(\mathcal{H})$	$R(\mathcal{H})$
2	$\mathcal{G}(3 \cdot S_4(2))$	11	infinite
3	$\mathcal{G}(M_{24})$	11	$\overline{\mathcal{C}}_{11}$
4	$\mathcal{G}(Co_1)$	24	$\overline{\Lambda}^{(24)}$
5	$\mathcal{G}(M)$	0	M
n	$\mathcal{G}(3^{[{n \atop 2}]_2} \cdot S_{2n}(2))$	$(2n+1)+2^n(2^n-1)$	infinite

The knowledge of the module $V(\mathcal{H})$ for known geometries $\mathcal{H}$ forms a strong background for the classification of the amalgams $\mathcal{A}(G,\mathcal{G})$ for the flag-transitive automorphism groups G of a P- or T-geometry $\mathcal{G}$. This classification is presented in Part II of this second volume. As an immediate outcome we have the following.

Proposition 6 *Let $\mathcal{G}$ be a P- or T-geometry and G be a flag-transitive automorphism group of $\mathcal{G}$. Let p be a point (an element of type 1) in $\mathcal{G}$, $\mathcal{F} = \mathrm{res}_{\mathcal{G}}(p)$, $F = G(p)$ be the stabilizer of p in G and $\overline{F}$ be the action induced by F on $\mathcal{F}$. Then $(\mathcal{F}, \overline{F})$ is not one of the following pairs:*

$$(\mathcal{G}(M_{23}), M_{23}), \quad (\mathcal{G}(BM), BM), \quad (\mathcal{G}(3^{4371} \cdot BM), 3^{4371} \cdot BM), \quad (\mathcal{G}(M), M).$$

Proof. We apply (1.5.2). Suppose that $(\mathcal{F}, \overline{F})$ is one of the above four pairs. The condition (i) in (1.5.2) follows from Tables III and IV. If (p, l, π) is a flag of rank 3 in $\mathcal{G}$ consisting of a point p, line l and plane π, then the structure of the maximal parabolics associated with the action of $\overline{F}$ on $\mathcal{F}$ (cf. pp. 114, 224, 210 and 234 in [Iv99]) shows that in each case $\overline{F}(\pi)$ induces Sym_3 on the set of lines incident to p and π (so that (ii) in (1.5.2) holds) and that $\overline{F}(l)$ is isomorphic respectively to

$$M_{22}, \quad 2_+^{1+22}.Co_2, \quad (2_+^{1+22} \times 3^{23}).Co_2, \quad 2_+^{1+24}.Co_1.$$

Since none of these groups contains a subgroup of index 2 the proof follows. □

Notice that in the case $(\mathcal{F}, \overline{F}) = (\mathcal{G}(J_4), J_4)$ the subgroup $\overline{F}(l) \cong 2_+^{1+12} \cdot 3 \cdot \mathrm{Aut}\, M_{22}$ does contain a subgroup of index two, so this case requires a further analysis to be eliminated (this will be accomplished in Section 11.6).

The knowledge of universal representations groups enables us to construct and prove simple connectedness of so-called affine c-extensions $\mathcal{AF}(\mathcal{G}, R(\mathcal{G}))$ of the known P- and T-geometries $\mathcal{G}$ (cf. Section 2.7). These extensions have diagrams

or

depending on whether $\mathcal{G}$ is a P- or T-geometry.

We formulate here the results on both simple connectedness and the full automorphisms groups.

Proposition 7 *The following assertions hold:*

(i) $\mathcal{AF}(\mathcal{G}(M_{22}), \overline{\mathcal{C}}_{11})$ *is simply connected with the automorphism group* $2^{11} : \mathrm{Aut}\, M_{22}$*;*

(ii) $\mathcal{G}(M_{23})$ *does not possess flag-transitive affine c-extensions;*

(iii) $\mathcal{AF}(\mathcal{G}(Co_2), \overline{\Lambda}^{(23)})$ *is simply connected with the automorphism group* $2^{23} : Co_2$;

(iv) $\mathcal{AF}(\mathcal{G}(J_4), J_4)$ *is simply connected with the automorphism group* $J_4 \wr 2$;

(v) $\mathcal{AF}(\mathcal{G}(BM), 2 \cdot BM)$ *is simply connected with the automorphism group* $(2 \cdot BM * 2 \cdot BM).2$;

(vi) $\mathcal{AF}(\mathcal{G}(M_{24}), \overline{\mathcal{C}}_{11})$ *is simply connected with the automorphism group* $2^{11} : M_{24}$;

(vii) $\mathcal{AF}(\mathcal{G}(Co_1), \overline{\Lambda}^{(24)})$ *is simply connected with the automorphism group* $2^{24} : Co_1$;

(viii) $\mathcal{AF}(\mathcal{G}(M), M)$ *is simply connected with the automorphism group* $M \wr 2$ *(the Bimonster).* □

The analysis of the amalgam $\mathcal{A}$ is via consideration of the normal factors of the parabolics G_1 and G_n. This analysis brings us to a restricted number of possibilities for the normal factors.

We proceed by accomplishing the following sequence of steps (we follow notation as introduced at the end of Section 1.1). First we reconstruct up to isomorphism the point stabilizer G_1. Our approach is inductive so we assume that the action $\overline{G}_1 = G_1/K_1$ of G_1 on $\mathrm{res}_{\mathcal{G}}(x_1)$ is one of the known actions in Table I or II. Then we turn to G_2, or more precisely to the subamalgam $\mathcal{B} = \{G_1, G_2\}$ in $\mathcal{A}$. The subgroup G_2 is the stabilizer of the line x_2 and it induces Sym_3 on the triple of points incident to x_2 (of course x_1 is in this triple). Hence $G_{12} = G_1 \cap G_2$ contains a subgroup K_2^- of index 2 (the pointwise stabilizer of x_2), which is normal in G_2 and $G_2/K_2^- \cong Sym_3$. Therefore we identify K_2^- as a subgroup of G_1, determine the automorphism group of K_2^- and then classify the extensions of K_2^- by automorphisms forming Sym_3. In this step we can refine the choice of the isomorphism type of G_1, since within the wrong choice K_2^- might not possess the required automorphisms.

A glance at Tables I and II gives the following.

Proposition 8 *Let $\mathcal{F}$ be the residue of a point in a (known) P- or T-geometry of rank $n \geq 2$ (so that either $n \geq 3$ and $\mathcal{F}$ is itself a P- or T-geometry or $n = 2$ and $\mathcal{F}$ is of rank 1 with 2 or 3 points, respectively) and let F be a flag-transitive automorphism group of $\mathcal{F}$. Then* $|\mathrm{Aut}\,\mathcal{F} : F| \leq 2$. □

This immediately gives the following

Proposition 9 *In the above terms $\overline{G}_2 = G_2/K_2$ is isomorphic to a subgroup of index at most 2 in the direct product*

$$G_2/K_2^- \times G_2/K_2^+,$$

where $G_2/K_2^- \cong Sym_3$ and G_2/K_2^+ is a flag-transitive automorphism group of $\mathrm{res}^+_{\mathcal{G}}(x_2)$. In particular the centre of $O^2(G_2/K_2)$ contains a subgroup X which permutes transitively the points incident to x_2. □

By Proposition 9 the automorphisms of K_2^- that we were talking about can always be chosen to commute with $O^2(K_2^-/K_2)$.

Next we extend $\mathcal{B}$ to the rank 3 amalgam $\mathcal{C} = \{G_1, G_2, G_3\}$. Towards this end we first identify $\mathcal{D} = \{G_{13}, G_{23}\}$ as a subamalgam in $\mathcal{B}$. Since the action of G_1 on $\mathrm{res}_{\mathcal{G}}(x_1)$ is known, G_{13} and G_{123} are specified uniquely up to conjugation in G_1. By Proposition 9, $G_{23} = \langle G_{123}, Y\rangle$, where Y maps onto the subgroup X as in that proposition. Since K_2 is a 2-group, we can choose Y to be a Sylow 3-subgroup (of order 3) in K_2^+.

Thus we obtain the amalgam $\widetilde{\mathcal{C}} = \{G_1, G_2, \widetilde{G}_3\}$, where $\widetilde{G}_3$ is the universal completion (free amalgamated product) of the subamalgam $\mathcal{D}$ in $\mathcal{B}$. In order to get the amalgam $\mathcal{C}$ we have to identify in $\widetilde{G}_3$ the normal subgroup N such that $G_3 = \widetilde{G}_3/N$. The subgroup K_3^- can be specified as the largest subgroup in G_{123} which is normal in both G_{13} and G_{23}. Then

$$G_3/K_3^- \cong L_3(2), \quad G_{13}/K_3^- \cong G_{23}/K_3^- \cong Sym_4$$

and the latter two quotients are maximal parabolics in the former one. In all cases the parabolics are 2-constrained and the images of both G_{13} and G_{23} in $\mathrm{Out}\, K_3^-$ are isomorphic to Sym_4. These two images must generate in $\mathrm{Out}\, K_3^-$ the group $L_3(2)$ (otherwise there is no way to extend $\mathcal{B}$ to a correct $\mathcal{C}$). Hence we may assume that

$$\widetilde{G}_3/(K_3^- C_{\widetilde{G}_3}(K_3^-)) \cong L_3(2).$$

Since $\widetilde{G}_3/K_3^- N$ is also $L_3(2)$, we see that N must be a subgroup in the centralizer of K_3^- in $\widetilde{G}_3$, which trivially intersects K_3^- and such that

$$K_3^- N = K_3^- C_{\widetilde{G}_3}(K_3^-).$$

The easiest situation is when the centre of K_3^- is trivial in which case we are forced to put $N = C_{\widetilde{G}_3}(K_3^-)$, so that N is uniquely determined (8.5.1). In fact the uniqueness of N can be proved under a weaker assumption: the centre of K_3^- does not contain 8-dimensional composition factors with respect to $\widetilde{G}_3/K_3^- C_{\widetilde{G}_3}(K_3^-) \cong L_3(2)$ (8.5.3). The following property of the known P- and T-geometries (which can easily be checked by inspection using information contained in [Iv99] and [IMe99]) shows that (8.5.3) always applies when $\mathscr{B}$ is isomorphic to the amalgam from a known example.

Proposition 10 *Let $(\mathscr{H}, H)$ be a pair from Table I or II and suppose that the rank of $\mathscr{H}$ is at least 3. Let π be a plane in $\mathscr{H}$ (an element of type 3), $H(\pi)$ be the stabilizer of π in H and $K^-(\pi)$ be the kernel of the action of $H(\pi)$ on the set of points and lines incident to π (these points and lines form a projective plane of order 2). Then every chief factor of $H(\pi)$ inside $Z(K^-(\pi))$ is an elementary abelian 2-group which is either 1- or 3-dimensional module for $H(\pi)/K^-(\pi) \cong L_3(2)$.* □

After $\mathscr{C}$ is reconstructed, the structure of the whole amalgam $\mathscr{A}$ is pretty much forced. Indeed G_4 is a completion of the subamalgam $\mathscr{E} = \{G_{i4} \mid 1 \le i \le 3\}$ in $\mathscr{C}$. It turns out that this subamalgam is always uniquely determined in $\mathscr{C}$ (up to conjugation). On the other hand, the residue $\mathrm{res}^-_{\mathscr{G}}(x_4)$ is the rank 3 projective $GF(2)$-geometry, which is simply connected. By the fundamental principle (1.4.6) this implies that G_4 is the universal completion of $\mathscr{E}$. Hence there is a unique way to extend $\mathscr{C}$ to the rank 4 amalgam and to carry on in the same manner to get the whole amalgam $\mathscr{A}$ of maximal parabolics.

We would like to thank our colleagues, and especially D.V. Pasechnik, A. Pasini and C. Wiedorn, for their support and help while we were writing this book. We dedicate the book to the memory of A.I. Kostrikin, without whose encouragement the book would not have been written.

1
Preliminaries

In this introductory chapter after recalling the main notions and notation concerning diagram geometries and their flag-transitive automorphism groups we prove the fundamental principle (Theorem 1.4.5), which relates the universal cover of a geometry $\mathcal{G}$ and the universal completion of the amalgam $\mathcal{A}$ of maximal parabolics in a flag-transitive automorphism group G of $\mathcal{G}$. This principle lies in the foundation of our approach to the classification of flag-transitive geometries in terms of their diagrams. In the last section of the chapter we recall what is meant by a representation of geometry. The importance of representations for our classification approach is explained in Proposition 1.5.1, which shows that under certain natural assumptions one of the chief factors of the stabilizer of a point in a flag-transitive automorphism group carries a representation of the residue of the point (this result is generalized in Proposition 9.4.1 for other maximal parabolics).

1.1 Geometries and diagrams

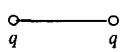
In this section we recall the main terminology and notations concerning diagram geometries (cf. Introduction in [Iv99] and references therein).

An *incidence system* of *rank* n is a set $\mathcal{G}$ of *elements* that is a disjoint union of subsets $\mathcal{G}^{\alpha_1}, ..., \mathcal{G}^{\alpha_n}$ (where $\mathcal{G}^{\alpha_i}$ is the set of elements of type α_i in $\mathcal{G}$) and a binary reflexive symmetric *incidence relation* on $\mathcal{G}$, with respect to which no two distinct elements of the same type are incident. We can identify $\mathcal{G}$ with its *incidence graph* $\Gamma = \Gamma(\mathcal{G})$ having $\mathcal{G}$ as the set of vertices, in which two distinct elements are adjacent if they are incident. A *flag* in $\mathcal{G}$ is a set Φ of pairwise incident elements (the vertex-set of a complete subgraph in the incidence graph). The *type* (respectively *cotype*) of Φ is the set of types in $\mathcal{G}$ present (respectively not present) in Φ. The sizes of these sets are the *rank* and the *corank* of Φ. By the definition a flag contains at most one element of any given type. If Φ is a flag in $\mathcal{G}$, then the *residue* $\mathrm{res}_{\mathcal{G}}(\Phi)$ of Φ in $\mathcal{G}$ is an incidence system whose elements are those from $\mathcal{G} \setminus \Phi$ incident to every element in Φ with respect to the induced type function and incidence relation.

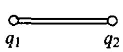
An incidence system $\mathcal{G}$ of rank n is called a *geometry* if for every flag Φ (possibly empty) of corank at least 2 and every $\alpha_i \neq \alpha_j$ from the cotype of Φ the subgraph in the incidence graph induced by $\mathcal{G}^{\alpha_i} \cap \mathcal{G}^{\alpha_j} \cap \mathrm{res}_{\mathcal{G}}(\Phi)$ is non-empty and connected (this implies that a maximal flag contains elements of all types). Clearly the residue of a geometry is again a geometry.

In what follows, unless stated otherwise, the set of types in a geometry of rank n is taken to be $\{1, 2, ..., n\}$. A diagram of a geometry $\mathcal{G}$ is a graph with labeled edges on the set of types in $\mathcal{G}$ in which the edge (or absence of such) joining i and j symbolizes the class of geometries appearing as residues of flags of cotype $\{i, j\}$ in $\mathcal{G}$. Under the node i it is common to write the number q_i such that every flag of cotype i in $\mathcal{G}$ is contained in exactly $q_i + 1$ maximal flags. We will mainly deal with the following rank 2 residues:

- generalized digon: any two elements of different types are incident, the incidence graph is complete bipartite with parts of size $q_1 + 1$ and $q_2 + 1$;



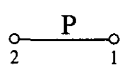
- projective plane $pg(2, q)$ of order q;



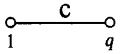
- generalized quadrangle $gq(q_1, q_2)$ of order (q_1, q_2);

- the generalized quadrangle $\mathcal{G}(S_4(2))$ of order $(2, 2)$, whose elements are the 2-element subsets of a 6-set and the partitions of the 6-set three 2-element subsets (equivalently the 1-subspaces and totally isotropic 2-subspaces in a 4-dimensional symplectic $GF(2)$-space) with the natural incidence relation; the automorphism group is $S_4(2) \cong Sym_6$ and the outer automorphism of this group induces a diagram automorphism of $\mathcal{G}(S_4(2))$;

- the triple cover $\mathcal{G}(3 \cdot S_4(2))$ of $\mathcal{G}(S_4(2))$ associated with the non-split extension $3 \cdot S_4(2) \cong 3 \cdot Sym_6$;



- the geometry $\mathcal{G}(Alt_5)$ of edges and vertices of the Petersen graph; the vertices of the Petersen graph are the 2-element subsets of a 5-set and two such subsets are adjacent if they are disjoint;



- the geometry of 1- and 2-element subsets of a $(q+2)$-set with the incidence relation defined by inclusion; when $q = 2$ this is the affine plane of order 2.

If Φ is a flag in $\mathcal{G}$, then the diagram of $\mathrm{res}_{\mathcal{G}}(\Phi)$ is the subdiagram in the diagram of $\mathcal{G}$ induced by the cotype of Φ.

The notation we are about to introduce can be applied to any rank n geometry $\mathcal{G}$, but it is particularly useful when $\mathcal{G}$ belongs to a string diagram, i.e., when the residue of a flag of cotype $\{i, j\}$ is a generalized digon whenever $|i - j| \geq 2$.

For an element x_i of type i, where $1 \leq i \leq n$, we denote by $\mathrm{res}^+_{\mathcal{G}}(x_i)$ and $\mathrm{res}^-_{\mathcal{G}}(x_i)$ the set of elements of types larger than i and less than i, respectively, that are incident to x_i. When $\mathcal{G}$ belongs to a string diagram they are residues of a flag of type $\{1, ..., i\}$ containing x_i and of a flag of type $\{i, ..., n\}$ containing x_i, respectively. If G is an automorphism group of $\mathcal{G}$ (often assumed to be flag-transitive), then $G(x_i)$ is the stabilizer of x_i in G, $K(x_i)$, $K^+(x_i)$ and $K^-(x_i)$ are the kernels of the actions of $G(x_i)$ on $\mathrm{res}_{\mathcal{G}}(x_i)$, $\mathrm{res}^+_{\mathcal{G}}(x_i)$ and $\mathrm{res}^-_{\mathcal{G}}(x_i)$, respectively. By $L(x_i)$ we denote the kernel of the action of $G(x_i)$ on the set of elements y_i of type i in $\mathcal{G}$ such that there exists a premaximal flag Ψ of cotype i such that both $\Psi \cup \{x_i\}$ and $\Psi \cup \{y_i\}$ are maximal flags.

When we deal with a fixed maximal flag $\Phi = \{x_1, ..., x_n\}$ in $\mathcal{G}$, we write G_i instead of $G(x_i)$, K_i instead of $K(x_i)$, etc. If $J \subseteq \{1, 2, ..., n\}$, then

$$G_J = \bigcap_{j \in J} G_j$$

and we write, for instance, G_{12} instead of $G_{\{1,2\}}$, and similar. The subgroups G_J are called *parabolic subgroups* or simply *parabolics*. The subgroups G_i are *maximal parabolics*. Most of our geometries are 2-local, so that the parabolics are 2-local subgroups and we put $Q(x_i) = O_2(G(x_i))$ (which can also be written simply as Q_i). Notice that if $\mathcal{G}$ belongs to a string diagram and x_1 is a point then L_1 is the elementwise stabilizer in G_1 of the set of points collinear to x_1.

1.2 Coverings of geometries

Let $\mathcal{H}$ and $\mathcal{G}$ be geometries (or more generally incidence systems). A *morphism* of geometries is a mapping $\varphi : \mathcal{H} \to \mathcal{G}$ of the element set of $\mathcal{H}$ the element set of $\mathcal{G}$ which maps incident pairs of elements onto incident pairs and preserves the type function. A bijective morphism, whose inverse is also a morphism is called an *isomorphism*.

A surjective morphism $\varphi : \mathcal{H} \to \mathcal{G}$ is said to be a *covering* of $\mathcal{G}$ if for every non-empty flag Φ of $\mathcal{H}$ the restriction of φ to the residue $\mathrm{res}_{\mathcal{H}}(\Phi)$ is an isomorphism onto $\mathrm{res}_{\mathcal{G}}(\varphi(\Phi))$. In this case $\mathcal{H}$ is a *cover* of $\mathcal{G}$ and $\mathcal{G}$ is a *quotient* of $\mathcal{H}$. If every covering of $\mathcal{G}$ is an isomorphism then $\mathcal{G}$ is said to be *simply connected.* Clearly a morphism is a covering if its restriction to the residue of every element (considered as a flag of rank 1) is an isomorphism. If $\psi : \widetilde{\mathcal{G}} \to \mathcal{G}$ is a covering and $\widetilde{\mathcal{G}}$ is simply connected, then ψ is the *universal covering* and $\widetilde{\mathcal{G}}$ is the *universal cover* of $\mathcal{G}$. The universal cover of a geometry exists and it is uniquely determined up to isomorphism. If $\varphi : \mathcal{H} \to \mathcal{G}$ is any covering then there exists a covering $\chi : \widetilde{\mathcal{G}} \to \mathcal{H}$ such that ψ is the composition of χ and φ.

A morphism $\varphi : \mathcal{H} \to \mathcal{G}$ of arbitrary incidence systems is called an *s-covering* if it is an isomorphism when restricted to every residue of rank s or more. This means that if Φ is a flag whose corank is less than or equal to s, then the restriction of φ to $\mathrm{res}_{\mathcal{H}}(\Phi)$ is an isomorphism. An incidence system, every s-cover of which is an isomorphism, is said to be *s-simply connected.* It is clear that when $s = n - 1$'s-covering' and 'covering' mean the same thing.

An isomorphism of a geometry onto itself is called an *automorphism.* By the definition an isomorphism preserves the types. Sometimes we will need a more general type of automorphisms which permute types. We will refer to them as *diagram automorphisms.*

The set of all automorphisms of a geometry $\mathcal{G}$ forms a group called the *automorphism group* of $\mathcal{G}$ and denoted by $\mathrm{Aut}\,\mathcal{G}$. An automorphism group G of $\mathcal{G}$ (that is a subgroup of $\mathrm{Aut}\,\mathcal{G}$) is said to be *flag-transitive* if any two flags Φ_1 and Φ_2 in $\mathcal{G}$ of the same type are in the same G-orbit. Clearly an automorphism group is flag-transitive if and only if it acts transitively on the set of maximal flags in $\mathcal{G}$. A geometry $\mathcal{G}$ possessing a flag-transitive automorphism group is said to be *flag-transitive.*

Let $\varphi : \mathcal{H} \to \mathcal{G}$ be a covering and H be a group of automorphisms of $\mathcal{H}$. We say that H commutes with φ if for every $h \in H$ whenever $\varphi(x) = \varphi(y)$, for $x, y \in \mathcal{H}$, the equality $\varphi(x^h) = \varphi(y^h)$ holds. In this case we can define the action of h on $\mathcal{G}$ via $\varphi(x)^h = \varphi(x^h)$. Let the induced action be denoted by $\bar{H}$. The kernel of the action is called the subgroup of *deck transformation* in H with respect to φ.

The following observation is quite important.

Lemma 1.2.1 *Let $\varphi : \mathcal{H} \to \mathcal{G}$ be a covering of geometries and H be a flag-transitive automorphism group of $\mathcal{H}$ commuting with φ. Then the action $\bar{H}$ induced by H on $\mathcal{G}$ is flag-transitive.* □

Let $\mathscr{G}$ be a geometry (or rather an incidence system) of rank n and N be a group of automorphisms of $\mathscr{G}$. Then the *quotient of* $\mathscr{G}$ *over* N is an incidence system $\bar{\mathscr{G}}$ whose elements of type i are the orbits of N on $\mathscr{G}^i$ and two N-orbits, say Ω and Δ, are incident if some $\omega \in \Omega$ is incident to some $\delta \in \Delta$ in $\mathscr{G}$. If the mapping $\varphi : \mathscr{G} \to \bar{\mathscr{G}}$ that sends every element $x \in \mathscr{G}$ onto its N-orbit, is a covering and N is normal in H then it is easy to see that H commutes with φ.

1.3 Amalgams of groups

Our approach for classifying P- and T-geometry is based on the method of group amalgams. This method can be applied to the classification of other geometries in terms of their diagrams and already has been proved to be adequate, for instance within the classification of c-extensions of classical dual polar spaces [Iv97], [Iv98].

Let us recall the definition of amalgam and related notions briefly introduced in volume 1 [Iv99]. Here we make our notation slightly more explicit and general.

Definition 1.3.1 *An amalgam* $\mathscr{A}$ *of finite type and rank* $n \geq 2$ *is a set such that for every* $1 \leq i \leq n$ *there is a subset* A_i *in* $\mathscr{A}$ *and a binary operation* $\star_i$ *on* A_i *such that the following conditions hold:*

- (A1) $(A_i, \star_i)$ *is a group for* $1 \leq i \leq n$;
- (A2) $\mathscr{A} = \cup_{i=1}^n A_i$;
- (A3) $|A_i \cap A_j|$ *is finite if* $i \neq j$ *and* $\cap_{i=1}^n A_i \neq \emptyset$;
- (A4) $(A_i \cap A_j, \star_i)$ *is a subgroup in* $(A_i, \star_i)$ *for all* $1 \leq i, j \leq n$;
- (A5) *if* $x, y \in A_i \cap A_j$ *then* $x \star_i y = x \star_j y$.

Abusing the notation we often write $\mathscr{A} = \{A_i \mid 1 \leq i \leq n\}$ in order to indicate explicitly which groups constitute $\mathscr{A}$. In what follows, unless explicitly stated otherwise, all amalgams under consideration will be of finite type.

Let $\mathscr{A} = \{A_i \mid 1 \leq i \leq n\}$ be an amalgam. A *completion* of $\mathscr{A}$ is a pair (G, φ) where G is a group and φ is a mapping of $\mathscr{A}$ into G such that

- (C1) G is generated by the image of φ;
- (C2) for every i the restriction of φ to A_i is a homomorphism, i.e.,

$$\varphi(x \star_i y) = \varphi(x) \cdot \varphi(y) \quad \text{for all} \quad x, y \in A_i$$

(here '·' stands for the group multiplication in G).

If (G_1, φ_1) and (G_2, φ_2) are two completions of the same amalgam $\mathscr{A}$ then a homomorphism χ of G_1 onto G_2 is said to be a *homomorphism of completions* if φ_2 is the composition of φ_1 and χ, i.e., if $\varphi_2(x) = \chi(\varphi_1(x))$ for all $x \in \mathscr{A}$. If K is the kernel of χ then (G_2, φ_2) is called the *quotient* of (G_1, φ_1) over K. Since G_2 is isomorphic to G_1/K via isomorphism $\varphi_2(x) = \varphi_1(x)K$ for $x \in \mathscr{A}$, the completion (G_2, φ_2) is determined by (G_1, φ_1) and K.

When the mapping φ is irrelevant or clear from the context we will talk about a completion G of $\mathscr{A}$. The completion (G, φ) is said to be *faithful* if φ is injective.

Two elements $x, y \in \mathscr{A}$ are said to be *conjugate* in $\mathscr{A}$ if there is a sequence $x_0 = x, x_1, ..., x_m = y$ of elements of $\mathscr{A}$ such that for every $1 \leq j \leq m$ the elements x_{j-1} and x_j are contained in A_i (where i might depend on j) and are conjugate in A_i (in the sense that $x_i = z^{-1}x_{i-1}z$ for some $z \in A_i$). It is easy to see that if (G, φ) is a completion of $\mathscr{A}$ then $\varphi(x)$ and $\varphi(y)$ are conjugate in G whenever x and y are conjugate in $\mathscr{A}$.

For an amalgam $\mathscr{A} = \{A_i \mid 1 \leq i \leq n\}$ let $U(\mathscr{A})$ be the group defined by the following presentation:

$$U(\mathscr{A}) = \langle u_x, x \in \mathscr{A} \mid u_x u_y = u_z \text{ if } x, y, z \in A_i \text{ for some } i \text{ and } x \star_i y = z \rangle.$$

Thus the generators of $U(\mathscr{A})$ are indexed by the elements of $\mathscr{A}$ and the relations are all the equalities that can be seen in the groups constituting the amalgam.

Lemma 1.3.2 *In the above terms let v be the mapping of $\mathscr{A}$ into $U(\mathscr{A})$ defined by*

$$v : x \to u_x$$

for all $x \in \mathscr{A}$. Then $(U(\mathscr{A}), v)$ is a completion of $\mathscr{A}$, which is universal in the sense that every completion of $\mathscr{A}$ is a quotient of $(U(\mathscr{A}), v)$.

Proof. The fact that $(U(\mathscr{A}), v)$ is a completion follows directly from the definition. Let (G, φ) be any completion of $\mathscr{A}$. Define ψ to be a mapping which sends u_x onto $\varphi(x)$ for every $x \in \mathscr{A}$. We claim that ψ extends uniquely to a homomorphism of $U(\mathscr{A})$ onto G. By (C1) ψ maps a generating set of $U(\mathscr{A})$ onto a generating set of G which implies the uniqueness. Now consider a defining relation $u_x u_y = u_z$ of $U(\mathscr{A})$. Then $x, y, z \in A_i$ for some i and $x \star_i y = z$. Since (G, φ) is a representation, we have

$$\psi(u_x)\psi(u_y) = \varphi(x)\varphi(y) = \varphi(z) = \psi(u_z).$$

Hence ψ extends to a homomorphism. □

Thus there is a natural bijection between the completions of $\mathscr{A}$ and the normal subgroups of the universal completion (group) $U(\mathscr{A})$. If N is a normal subgroup in $U(\mathscr{A})$ then the corresponding completion is the quotient of $(U(\mathscr{A}), v)$ over N. The following result is rather obvious.

Lemma 1.3.3 *An amalgam $\mathscr{A}$ possesses a faithful completion if and only if its universal completion is faithful.* □

The subgroup $B := \cap_{i=1}^{n} A_i$ is called the *Borel subgroup* of $\mathscr{A}$. By (A3) and (A5), B is a finite group in which the group operation coincides with the restriction of $\star_i$ for every $1 \leq i \leq n$. In particular, the identity element of B is the identity element of every $(A_i, \star_i)$. The following result can be easily deduced from Section 35 in [Kur60].

Proposition 1.3.4 *Let $\mathscr{A} = \{A_i \mid 1 \leq i \leq n\}$ be an amalgam of rank $n \geq 2$ with Borel subgroup B. Suppose that $B = A_i \cap A_j$ for all $1 \leq i < j \leq n$ (which always holds when $n = 2$) and $\mathscr{A} \nsubseteq A_i$ for every $1 \leq i \leq n$. Then the universal completion of $\mathscr{A}$ is faithful and $U(\mathscr{A})$ is the free amalgamated product of the groups A_i over the subgroup B, in particular, it is infinite.* □

One should not confuse the set of all amalgams and their very special class covered by (1.3.4). For an amalgam $\mathscr{A}$ of rank $n \geq 3$ the universal completions might or might not be faithful and might be infinite or finite (or even trivial). In general it is very difficult to decide what $U(\mathscr{A})$ is and this problem is clearly equivalent to the identification problem of a group defined by generators and relations.

A subgroup M of B which is normal in $(A_i, \star_i)$ for every $1 \leq i \leq n$ is said to be a *normal subgroup* of the amalgam $\mathscr{A}$. The largest normal subgroup in $\mathscr{A}$ is called the *core* of $\mathscr{A}$ and the amalgam is said to be *simple* if its core is trivial (the identity subgroup of B). Notice that if M is normal in $\mathscr{A}$ then $\varphi(M)$ is a normal subgroup in G for every completion (G, φ) of $\mathscr{A}$, but even when $\mathscr{A}$ is a simple amalgam, a completion group G is not necessarily simple.

1.4 Simple connectedness via universal completion

Let $\mathscr{G}$ be a geometry of rank n, G be a flag-transitive automorphism group of $\mathscr{G}$ and $\Phi = \{x_1, ..., x_n\}$ be a maximal flag in $\mathscr{G}$, where x_i is of type i. Let $G_i = G(x_i)$ be the stabilizer of x_i in G (the maximal parabolic of type i associated with the action of G on $\mathcal{G}$) and

$$\mathcal{A} := \mathcal{A}(G, \mathcal{G}) = \{G_i \mid 1 \leq i \leq n\}$$

be the amalgam of the maximal parabolics.

We define the *coset geometry* $\mathcal{C} = \mathcal{C}(G, \mathcal{A})$ in the following way (it might not be completely obvious at this stage that $\mathcal{C}$ is a geometry rather than just an incidence system). The elements of type i in $\mathcal{C}$ are the right cosets of the subgroup G_i in G, so that

$$\mathcal{C}^i = \{G_i g \mid g \in G\} \quad \text{and}$$

$$\mathcal{C} = \bigcup_{1 \leq i \leq n} \mathcal{C}^i \text{ (disjoint union).}$$

Two different cosets are incident if and only if they have an element in common:

$$G_i h \sim G_j k \iff G_i h \cap G_j k \neq \emptyset.$$

Lemma 1.4.1 *Let ϱ be the mapping which sends the coset $G_i g$ from $\mathcal{C}^i$ onto the image x_i^g of x_i under $g \in G$:*

$$\varrho : G_i g \mapsto x_i^g.$$

Then ϱ is an isomorphism of $\mathcal{C}$ onto $\mathcal{G}$.

Proof. First notice that ϱ is well defined, since if $g' \in G_i g$, say $g' = fg$ for $f \in G_i$, then we have

$$x_i^{g'} = x_i^{fg} = (x_i^f)^g = x_i^g.$$

This also shows that for $y_i \in \mathcal{G}^i$ the set $\varrho^{-1}(y_i)$ consists of the elements of G which map x_i onto y_i.

Next we check that ϱ preserves the incidence relation. Suppose first that $G_i h$ and $G_j k$ are incident in $\mathcal{C}$, which means that they contain an element g in common. Then $G_i h = G_i g$, $G_j k = G_j g$ and

$$\{\varrho(G_i h), \varrho(G_j k)\} = \{x_i^g, x_j^g\}.$$

Since x_i and x_j are incident and g is an automorphism of $\mathcal{G}$, x_i^g and x_j^g are also incident. On the other hand, suppose that $y_i = \varrho(G_i h)$ and $y_j = \varrho(G_j k)$ are incident elements of types i and j in $\mathcal{G}$. Since G acts flag-transitively on $\mathcal{G}$, there is a $g \in G$ such that $\{y_i, y_j\} = \{x_i^g, x_j^g\}$. By the above observation $g \in G_i h \cap G_j k$, which means that $G_i h$ and $G_j k$ are incident in $\mathcal{C}$. □

In the above terms, for $1 \le i \le n$ the maximal parabolic G_i acts flag-transitively on the residue $\mathrm{res}_{\mathcal{G}}(x_i)$ of x_i in $\mathcal{G}$. By (1.4.1) we have the following.

Corollary 1.4.2 *The residue* $\mathrm{res}_{\mathcal{G}}(x_i)$ *is isomorphic to the coset geometry* $\mathcal{C}(G_i, \mathcal{A}_i)$, *where*

$$\mathcal{A}_i = \{G_i \cap G_j \mid 1 \le j \le n, j \ne i\}.$$

□

By the above corollary the isomorphism types of the residues in $\mathcal{G}$ are completely determined by the amalgam $\mathcal{A}$ of maximal parabolics in a flag-transitive automorphism group. Next we discuss up to what extent the amalgam $\mathcal{A}$ determines the structure of the whole of $\mathcal{G}$.

Let $\mathcal{G}$ and $\mathcal{G}'$ be geometries of rank n with flag-transitive automorphism groups G and G', amalgams $\mathcal{A}$ and $\mathcal{A}'$ of maximal parabolics associated with maximal flags $\Phi = \{x_1, ..., x_n\}$ and $\Phi' = \{x'_1, ..., x'_n\}$, respectively. Suppose there is an isomorphism $\tau_{\mathcal{A}}$ of $\mathcal{A}'$ onto $\mathcal{A}$ (which maps $G'_i = G'(x'_i)$ onto $G_i = G(x_i)$). Suppose first that $\tau_{\mathcal{A}}$ is a restriction to $\mathcal{A}'$ of a homomorphism τ_G of G' onto G. Then τ_G induces a mapping $\tau_{\mathcal{C}}$ of $\mathcal{C}' = \mathcal{C}(G', \mathcal{A}')$ (isomorphic to $\mathcal{G}'$) onto $\mathcal{C} = \mathcal{C}(G, \mathcal{A})$ (isomorphic to $\mathcal{G}$):

$$\tau_{\mathcal{C}} : G'_i g' \mapsto G_i \tau_G(g')$$

for all $1 \le i \le n$ and $g' \in G'$.

Lemma 1.4.3 *The mapping* $\tau_{\mathcal{C}}$ *is a covering of geometries.*

Proof. By the definition $\tau_{\mathcal{C}}$ preserves the type function. If $G'_i h'$ and $G'_j k'$ are incident (contain a common element g', say) then their images both contain the element $\tau_G(g')$ and hence they are incident as well. Thus $\tau_{\mathcal{C}}$ is a morphism of geometries. By (1.4.2) and the flag-transitivity of G', $\tau_{\mathcal{C}}$ maps the residue of x' in $\mathcal{G}'$ onto the residue of $\tau_{\mathcal{C}}(x')$ in $\mathcal{G}$ and the proof follows. □

In the above terms G and G' are two completions of the same amalgam $\mathcal{A} \cong \mathcal{A}'$. In general one cannot guarantee that one of the completions is a homomorphic image of the other. But this can be guaranteed if one of the completions is universal.

With G and $\mathcal{A}$ as above, let $\widetilde{G} = U(\widetilde{\mathcal{A}})$ be the universal completion of an amalgam $\widetilde{\mathcal{A}} = \{\widetilde{G}_i \mid 1 \le i \le n\}$ and suppose that $\widetilde{\mathcal{A}}$ possesses an isomorphism $\widetilde{\tau}_{\mathcal{A}}$ onto $\mathcal{A}$. Since $\widetilde{G}$ is a universal completion of $\widetilde{\mathcal{A}}$

by (1.4.3) the geometry $\widetilde{\mathscr{G}} := \mathscr{C}(\widetilde{G}, \widetilde{\mathscr{A}})$ possesses a covering $\widetilde{\tau}_{\mathscr{G}}$ onto $\mathscr{G} = \mathscr{C}(G, \mathscr{A})$. We formulate this in the following lemma.

Lemma 1.4.4 *Let G be a faithful completion of the amalgam $\mathscr{A}$. Then there is a covering of $\widetilde{\mathscr{G}} = \mathscr{C}(\widetilde{G}, \widetilde{\mathscr{A}})$ onto $\mathscr{C}(G, \mathscr{A})$.* □

The following result was established independently in [Pasi85], [Ti86] and in an unpublished manuscript by the second author of the present book (who claims that the first author lost it) dated around 1984.

Theorem 1.4.5 *The covering $\widetilde{\tau}_{\mathscr{G}}$ is universal.*

Proof. Let

$$\widehat{\tau} : \widehat{\mathscr{G}} \to \mathscr{G}$$

be the universal covering. Let $\widehat{\Phi} = \{\widehat{x}_1, ..., \widehat{x}_n\}$ be a maximal flag in $\widehat{\mathscr{G}}$ being mapped under $\widehat{\tau}$ onto the maximal flag $\Phi = \{x_1, ..., x_n\}$ in $\mathscr{G}$ (i.e., $\widehat{\tau}(\widehat{x}_i) = x_i$ for $1 \leq i \leq n$).

For $g \in G_i$ let us define an automorphism $\widehat{g} = \widehat{g}^{(i)}$ of $\widehat{\mathscr{G}}$ as follows. First $\widehat{x_i^g} = \widehat{x}_i$. Next, if $\widehat{x} \in \widehat{\mathscr{G}}$ is arbitrary, in order to define $\widehat{x^g}$ we proceed in the following way. Consider a path

$$\widehat{\gamma} = (\widehat{y}_0 = \widehat{x}_i, \widehat{y}_1, ..., \widehat{y}_m = \widehat{x})$$

in $\widehat{\mathscr{G}}$ joining $\widehat{x}_i$ with $\widehat{x}$ (such a path exists since $\widehat{\mathscr{G}}$ is connected). Let

$$\gamma = (y_0 = x_i, y_1, ..., y_m)$$

be the image of $\widehat{\gamma}$ under $\widehat{\tau}$ (i.e., $y_j = \widehat{\tau}(\widehat{y}_j)$ for $0 \leq j \leq m$) and let

$$\gamma^g = (y_0^g = y_0 = x_i, y_1^g, ..., y_m^g)$$

be the image of γ under the element g. Then, since γ^g is a path starting at x_i, there is a unique path

$$\widehat{\gamma^g} = (\widehat{y_0^g} = \widehat{y}_0 = \widehat{x}_i, \widehat{y_1^g}, ..., \widehat{y_m^g})$$

in $\widehat{\mathscr{G}}$ starting at $\widehat{x}_i$ and being mapped onto γ^g under $\widehat{\tau}$. We define $\widehat{x^g}$ to be the end term of $\widehat{\gamma^g}$ (i.e., $\widehat{y_m^g}$ in the above terms). First we show that $\widehat{g}$ is well defined, which means it is independent on the particular choice of the path $\widehat{\gamma}$ joining $\widehat{x}_i$ and $\widehat{x}$. Suppose that $\widehat{\gamma}$ and $\widehat{\delta}$ are paths both starting at $\widehat{x}_i$ and ending at $\widehat{x}$. Then, by a theorem from algebraic topology [Sp66], since $\widehat{\tau}$ is universal, the corresponding images γ and δ are homotopic. Since g is an automorphism of $\mathscr{G}$, it maps the pairs of homotopic paths onto the pairs of homotopic paths. Hence γ^g and δ^g

are homotopic, which means that the end terms of their liftings $\widehat{\gamma^g}$ and $\widehat{\delta^g}$ coincide. Thus $\widehat{g}$ is well defined. Finally it is easy to see from the definition that $\widehat{g}$ is an automorphism of $\widehat{\mathscr{G}}$.

Let

$$\widehat{G}_i = \{\widehat{g} = \widehat{g}^{(i)} \mid g \in G_i\}.$$

It is easy to check that $\widehat{g_1 g_2} = \widehat{g}_1 \widehat{g}_2$ and $\widehat{g^{-1}} = \widehat{g}^{-1}$. So $\widehat{G}_i$ is a group and $\lambda_i : g \mapsto \widehat{g}^{(i)}$ is a surjective homomorphism. It is also clear that for $\widehat{g} \in \widehat{G}_i$ the preimage $\lambda_i^{-1}(\widehat{g})$ is a uniquely determined element of G_i, so λ_i is an isomorphism of G_i onto $\widehat{G}_i$. Let $\widehat{\mathscr{A}} = \{\widehat{G}_i \mid 1 \leq i \leq n\}$ be the amalgam formed by the subgroups $\widehat{G}_i$ and λ be the mapping of $\mathscr{A}$ onto $\widehat{\mathscr{A}}$ whose restriction to G_i coincides with λ_i for every $1 \leq i \leq n$. We claim that λ is an isomorphism of amalgams. Since the λ_i are group isomorphisms, in order to achieve this, it is sufficient to show that λ is well defined. Namely for $g \in G_i \cap G_j$ we have to show that $\widehat{g}^{(i)} = \widehat{g}^{(j)}$. Let $\widehat{x} \in \widehat{\mathscr{G}}$ and suppose that $\widehat{\gamma} = (\widehat{x}_i = \widehat{y}_0, \widehat{y}_1, ..., \widehat{y}_m = \widehat{x})$ is a path used to define the image of $\widehat{x}$ under $\widehat{g}^{(i)}$. Swapping i and j if necessary, we assume that $\widehat{y}_1 \neq \widehat{x}_j$. Then the path $\widehat{\delta} = (\widehat{x}_j, \widehat{y}_0, ..., \widehat{y}_m = \widehat{x})$ can be used to define the image of $\widehat{x}$ under $\widehat{g}^{(j)}$. Since g fixes the path (x_j, x_i) it is quite clear that the lifted paths $\widehat{\gamma^g}$ and $\widehat{\delta^g}$ have the same end term. Hence the images of $\widehat{x}$ under $\widehat{g}^{(i)}$ and $\widehat{g}^{(j)}$ coincide. Since the element $\widehat{x}$ was arbitrary, we conclude that $\widehat{g}^{(i)} = \widehat{g}^{(j)}$.

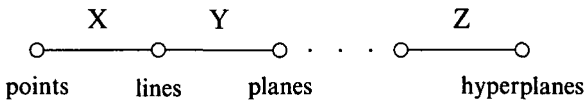
Thus $\mu := \lambda^{-1}$ is an isomorphism of $\widehat{\mathscr{A}}$ onto $\mathscr{A}$. Let $\widehat{G}$ be the subgroup in the automorphism group of $\widehat{\mathscr{G}}$ generated by $\widehat{\mathscr{A}}$. Then clearly μ induces a homomorphism of $\widehat{G}$ onto G that commutes with the covering $\widehat{\tau}$. Since G_i is the stabilizer of x_i in G and $\widehat{G}_i$ maps isomorphically onto G_i under μ, we conclude that $\widehat{G}_i$ is the stabilizer of $\widehat{x}_i$ in $\widehat{G}$. Now by (1.4.1) we observe that $\widehat{\mathscr{G}}$ is isomorphic to $\mathscr{C}(\widehat{G}, \widehat{\mathscr{A}})$ and since we have proved that $\widehat{\mathscr{A}}$ is isomorphic to $\mathscr{A} \cong \widetilde{\mathscr{A}}$, by (1.4.3) there must be a covering $\overline{\tau}$ of $\widetilde{\mathscr{G}}$ onto $\widehat{\mathscr{G}}$. Since $\widehat{\tau}$ is universal, $\overline{\tau}$ must be an isomorphism and hence $\widetilde{\tau}_{\mathscr{G}}$ is also universal. □

The following direct consequence of Theorem 1.4.5 is very useful.

Corollary 1.4.6 *Suppose that a geometry $\mathscr{G}$ of rank $n \geq 3$ is simply connected and G is a group acting flag-transitively (and possibly unfaithfully) on $\mathscr{G}$. Then G is the universal completion of the amalgam $\mathscr{A}(G, \mathscr{G})$.* □

1.5 Representations of geometries

We say that a geometry $\mathscr{G}$ of rank n belongs to a *string diagram* if all rank 2 residues of type $\{i, j\}$ for $|i - j| > 1$ are generalized digons. In this case the types on the diagram usually increase rightward from 1 to n. The elements which correspond, respectively, to the leftmost, the second left, the third left, and the rightmost nodes on the diagram will be called *points*, *lines*, *planes*, and *hyperplanes*:



The graph $\Gamma = \Gamma(\mathcal{G})$ on the set of points of $\mathcal{G}$ in which two points are adjacent if and only if they are incident to a common line, is called the *collinearity graph* of $\mathcal{G}$.

Given such a geometry $\mathcal{G}$ and a vector space V, we can ask is it possible to define a mapping φ from the element set of $\mathcal{G}$ onto the set of proper subspaces of V, such that dim $\varphi(x)$ is uniquely determined by the type of x and whenever x and y are incident, either $\varphi(x) \leq \varphi(y)$ or $\varphi(y) \leq \varphi(x)$? This question leads to a very important and profound theory of presheaves on geometries which was introduced and developed in [RSm86] and [RSm89]. A special class of the presheaves, described below, has played a crucial rôle in the classification of P- and T-geometries.

Let $\mathcal{G}$ be a geometry with elements of one type called points and elements of some other type called lines. Unless stated otherwise, if $\mathcal{G}$ has a string diagram the points and lines are as defined above. Suppose that $\mathcal{G}$ is of $GF(2)$-type, which means that every line is incident to exactly three points. Let Π and L denote, respectively, the point set and the line set of $\mathcal{G}$. In order to simplify the notation we will assume that every line is uniquely determined by the triple of points to which it is incident. Let V be a vector space over $GF(2)$. A *natural representation* of (the point–line incidence system associated with) $\mathcal{G}$ is a mapping φ of $\Pi \cup L$ into the set of subspaces of V such that

(i) V is generated by Im φ,
(ii) dim $\varphi(p) = 1$ for $p \in \Pi$ and dim $\varphi(l) = 2$ for $l \in L$,
(iii) if $l \in L$ and $\{p, q, r\}$ is the set of points incident to l, then $\{\varphi(p), \varphi(q), \varphi(r)\}$ is the set of 1-dimensional subspaces in $\varphi(l)$.

If $\mathcal{G}$ possesses a natural representation then it possesses the *universal abelian representation* φ_a such that any other natural representation is a composition of φ_a and a linear mapping. The $GF(2)$-vector space underlying the universal natural representation (considered as an abstract group with additive notation for the group operation) has the presentation

$$V(\mathcal{G}) = \langle v_p,\ p \in P \mid v_p + v_p = 0;\ \ v_p + v_q = v_q + v_p \text{ for } p,q \in P;$$

$$v_p + v_q + v_r = 0 \text{ if } \{p,q,r\} = l \in L\rangle$$

and the universal abelian representation itself is defined by

$$\varphi_a : p \mapsto v_p \text{ for } p \in P$$

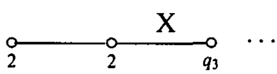
and

$$\varphi_a : l \mapsto \langle v_p, v_q, v_r\rangle \text{ for } \{p,q,r\} = l \in L.$$

In this case $V(\mathcal{G})$ will be called *the universal representation module of* $\mathcal{G}$. Notice that $V(\mathcal{G})$ can be defined for any geometry with three points on a line.

Natural representations of geometries usually provide a nice model for geometries and 'natural' modules for their automorphism groups. Besides that, in a certain sense natural representations control extensions of geometries. Below we explain this claim.

Let $\mathcal{G}$ be a geometry of rank at least 3 with a string diagram such that the residue of a flag of cotype $\{1,2\}$ is a projective plane of order 2, so that the diagram of $\mathcal{G}$ has the following form:



Let G be a flag-transitive automorphism group of $\mathcal{G}$. Let p be a point of $\mathcal{G}$ (an element of type 1), $G_1 = G(p)$ and $\mathcal{H} = \mathrm{res}_{\mathcal{G}}(p)$. Then the points and lines of $\mathcal{H}$ are the lines and planes of $\mathcal{G}$ incident to p. Let K be the kernel of the (flag-transitive) action of G_1 on $\mathcal{H}$, let U be the action induced by K on the set of points collinear to p and suppose that $U \neq 1$. Let $l = \{p,q,r\}$ be a line containing p. Since every $k \in K$ stabilizes the flag $\{p,l\}$ it either fixes q and r or swaps these two points. Furthermore, since $U \neq 1$ and G_1 acts transitively on the point-set of $\mathcal{H}$, some elements of K must swap q and r. Hence U is a non-identity elementary abelian 2-group (which can be treated as a $GF(2)$-vector space). The set of elements in U which fix l pointwise is a hyperplane $U(l)$ in U. Let U^* be the dual space of U and $U^*(l)$ be the 1-subspace in U^* corresponding to $U(l)$. Then we have a mapping

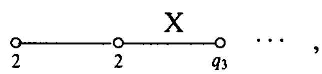
$$\varphi : l \mapsto U^*(l)$$

from the point-set of $\mathscr{H}$ into the set of 1-spaces in U^*. We claim that φ defines a natural representation of $\mathscr{H}$. For this purpose consider a plane π in $\mathscr{G}$ containing l. By the diagram the set $\mathscr{F} = \text{res}^-_{\mathscr{G}}(\pi)$ of points and lines in $\mathscr{G}$ incident to π forms a projective plane $pg(2,2)$ of order 2. By the flag-transitivity of G the subgroup $G_3 = G(\pi)$ acts flag-transitively on $\mathscr{F}$. The subgroup K is contained in G_3 and since $U \neq 1$, K induces on $\mathscr{F}$ a non-trivial action (whose order is a power of 2). Since $pg(2,2)$ possesses only one flag-transitive automorphism group of even order, we conclude that G_3 induces on $\mathscr{F}$ the group $L_3(2)$. Then $G_1 \cap G_3$ induces $Sym_4 \cong 2^2.Sym_3$ on $\mathscr{F}$ and since K is a normal 2-subgroup in G_1 contained in G_3, we observe that the action of U on $\mathscr{F}$ is of order 2^2. Let $l_1 = l$, l_2, and l_3 be the lines incident to both p and π. Then the $U(l_i)$ are pairwise different hyperplanes for $1 \leq i \leq 3$ and $U(l_i) \cap U(l_j)$ is the kernel of the action of U on $\mathscr{F}$ (having codimension 2 in U) for all $1 \leq i < j \leq 3$. In dual terms this means that the $U^*(l_i)$ are pairwise different 1-spaces and

$$\langle U^*(l_i) \mid 1 \leq i \leq 3 \rangle$$

is 2-dimensional. Hence φ is a natural representation and we have the following.

Proposition 1.5.1 *Let $\mathscr{G}$ be a geometry with diagram of the form*



let G be a flag-transitive automorphism group of $\mathscr{G}$, let p be a point in $\mathscr{G}$ (an element of the leftmost type on the diagram), let $K(p)$ be the kernel of the action of $G(p)$ on $\mathscr{H} = \text{res}_{\mathscr{G}}(p)$, let U be the action which $K(p)$ induces on the set of points collinear to p and suppose that $U \neq 1$. Then U is an elementary abelian 2-group, whose dual U^* supports a natural $G(p)/K(p)$-admissible representation of $\text{res}_{\mathscr{G}}(p)$, *in particular, U^* is a quotient of $V(\mathscr{H})$.* □

When we follow an inductive approach to the classification of geometries, we can assume that $\mathscr{H}$ and its flag-transitive automorphism groups are known and we are interested in geometries $\mathscr{G}$ that are extensions of $\mathscr{H}$ by the projective plane edge in the diagram. Then the section U is either trivial or related to a natural representation of $\mathscr{H}$. In particular, this section is trivial if $\mathscr{H}$ does not possess a natural representation. In practice it often happens that in this case there are no extensions of $\mathscr{H}$ at all. One of the reasons for this is the following result.

Proposition 1.5.2 *In the hypothesis of* (1.5.1) *let* $\overline{H}$ *be the action induced by* $G(p)$ *on* $\mathcal{H}$ *(so that* $\overline{H} \cong G(p)/K(p)$*). Let* l *and* π *be a line and a plane in* $\mathcal{G}$ *incident to* p *(which are a point and a line in* $\mathcal{H}$*). Suppose further that*

(i) $U = 1$ *(which always holds when* $V(\mathcal{H})$ *is trivial);*

(ii) $\overline{H}(\pi)$ *induces* Sym_3 *on the set of lines incident to both* p *and* π.

Then $\overline{H}(l)$ *contains a subgroup of index* 2.

Proof. The stabilizer $G(\pi)$ of π in G induces a flag-transitive action X of the residual projective plane of order 2 formed by the points and lines in $\mathcal{G}$ incident to π. Hence by [Sei73] $X \cong L_3(2)$ or $X \cong F_7^3$. By (ii) the latter case is impossible. Hence $G(l)$ induces Sym_3 on the point-set of l (we can see this action already in $G(l) \cap G(\pi)$ assuming that l and π are incident). Hence the pointwise stabilizer of l has index 2 in $G(p) \cap G(l)$. Finally, by (i), $K(p)$ fixes every point collinear to p and hence the index 2 subgroup contains $K(p)$. □

For various reasons it is convenient to consider also a non-abelian version of natural representations. The *universal representation group* of a geometry $\mathcal{G}$ with 3 points on every line has the following definition in terms of generators and relations:

$$R(\mathcal{G}) = \langle z_p,\ p \in \Pi \mid z_p^2 = 1,\ z_p z_q z_r = 1 \text{ if } \{p,q,r\} = l \in L\rangle.$$

It is easy to observe that $V(\mathcal{G}) = R(\mathcal{G})/[R(\mathcal{G}), R(\mathcal{G})]$. Notice that generators z_p and z_q of $R(\mathcal{G})$ commute whenever p and q are collinear. There are geometries whose universal representation groups are perfect. In particular, the geometries $\mathcal{G}(J_4)$, $\mathcal{G}(BM)$ and $\mathcal{G}(M)$ have non-trivial representation groups while their representation modules are trivial.

We had originally introduced the notion of non-abelian representations in order to simplify and to make more conceptional the non-existence proofs for abelian representations, which are important for the classification of amalgams of maximal parabolics. But this notion eventually led to a completely new research area in the theory of groups and geometries [Iv01]. It turned out that the knowledge of these representations is crucial to the construction of affine and c-extensions of geometries. More recently the calculation of the universal representation group of $\mathcal{G}(M)$ has been used in a new identification of the famous Y_{555}-group with the Bimonster (cf. Section 8.6 in [Iv99]).

Part I

Representations

2
General features

In this chapter we present some techniques for calculating representations of geometries of $GF(2)$-type, i.e., with three points on a line. In the last two sections we discuss some applications of the representations for construction of c-extensions of geometries and non-split extensions of groups and modules.

2.1 Terminology and notation

Let $\mathcal{S} = (\Pi, L)$ be a point-line incidence system with 3 points on every line. This simply means that Π is a finite set and L is a set of 3-element subsets of Π. We define the *universal representation group* of $\mathcal{S}$ by the following generators and relations:

$$R(\mathcal{S}) = \langle z_p, p \in \Pi \mid z_p^2 = 1, z_p z_q z_r = 1 \text{ if } \{p, q, r\} = l \in L \rangle.$$

So the generators of $R(\mathcal{S})$ are indexed by the points from Π subject to the following relations: the square of every generator is the identity; the product (in any order) of three generators corresponding to the point-set of a line is the identity. The *universal representation* of $\mathcal{S}$ is the pair $(R(\mathcal{S}), \varphi_u)$ where φ_u is the mapping of Π into $R(\mathcal{S})$ defined by

$$\varphi_u : p \mapsto z_p \text{ for } p \in \Pi.$$

Let $\psi : R(\mathcal{S}) \to R$ be a surjective homomorphism and φ be the composition of φ_u and ψ (i.e., $\varphi(p) = \psi(\varphi_u(p))$ for every $p \in \Pi$). Then (R, φ) is a *representation* of $\mathcal{S}$. Thus a representation of $\mathcal{S}$ is a pair (R, φ) where R is a group and φ is a mapping of Π into R such that

(R1) R is generated by the image of φ;
(R2) $\varphi(p)^2 = 1$ for every $p \in \Pi$;
(R3) whenever $\{p, q, r\}$ is a line, the equality $\varphi(p)\varphi(q)\varphi(r) = 1$ holds.

If in addition R is abelian, i.e.,

(R4) $[\varphi(p), \varphi(q)] = 1$ for all $p, q \in \Pi$,

then the representation is said to be *abelian*. The *order* of a representation (R, φ) is the order of R.

Let $V(\mathcal{S})$ be the largest abelian factor group of $R(\mathcal{S})$ (i.e., the quotient of $R(\mathcal{S})$ over its commutator subgroup), ψ be the corresponding homomorphism and φ_a be the composition of φ_u and ψ. Then $(V(\mathcal{S}), \varphi_a)$ is the *universal abelian representation* and $V(\mathcal{S})$ is the *universal representation module* of $\mathcal{S}$.

Let G be an automorphism group of $\mathcal{S}$. Then the action

$$(z_p)^g = z_{p^g} \text{ for } p \in \Pi \text{ and } g \in G$$

defines a homomorphism χ of G into the automorphism group of $R(\mathcal{S})$. Let (R, φ) be an arbitrary representation and N be the kernel of the homomorphism of $R(\mathcal{S})$ onto R. Then (R, φ) is said to be *G-admissible* if and only if N is $\chi(G)$-invariant. In this case the action $\varphi(p)^g = \varphi(p^g)$ defines a homomorphism of G into the automorphism group of R. The universal representation is clearly $\mathrm{Aut}\,\mathcal{S}$-admissible and so is a representation for which the kernel of the homomorphism ψ is a characteristic subgroup in $R(\mathcal{S})$. In particular $(V(\mathcal{S}), \varphi_a)$ is $\mathrm{Aut}\,\mathcal{S}$-admissible.

Let $\mathcal{G}$ be a geometry, in which Π is the set of points and L is the set of lines, and every line is incident to exactly three points. Then by a representation of $\mathcal{G}$ we understand a representation of its point-line incidence system $\mathcal{S} = (\Pi, L)$ (which is a truncation of $\mathcal{G}$). We denote by $(R(\mathcal{G}), \varphi_u)$ and by $(V(\mathcal{G}), \varphi_a)$ the universal and the universal abelian such representations.

The group $V(\mathcal{S})$ is abelian generated by elements of order at most 2. Hence it is an elementary abelian 2-group and can be treated as a $GF(2)$-vector space. In these terms $V(\mathcal{S})$ is the quotient of the power space 2^{Π} of Π (the set of all subsets of Π with addition performed by the symmetric difference operator) over the image of 2^L with respect to the incidence map that sends a line $l \in L$ onto its point-set (which is an element of 2^{Π}).

Then the $GF(2)$-dimension of $V(\mathcal{S})$ is the number of points minus the $GF(2)$-rank of the *incidence matrix* whose rows are indexed by the lines in L, and columns are indexed by the points in Π, and the (l, p)-entry is 1 if $p \in l$ and 0 otherwise (notice that every row contains exactly three non-zero entries equal to 1). Thus the dimension of the universal representation module can (at least in principle) be found by means of linear algebra over $GF(2)$.

The universal representation module $V(\mathcal{S})$ is a $GF(2)$-module for the automorphism group $\mathrm{Aut}\,\mathcal{S}$ and there is a natural bijection between the $\mathrm{Aut}\,\mathcal{S}$-admissible abelian representations and G-submodules in $V(\mathcal{S})$. The following lemma easily shows that in the point-transitive case $V(\mathcal{S})$ does not contain codimension 1 submodules.

Lemma 2.1.1 *Let $\mathcal{S} = (\Pi, L)$ be a point-line incidence system with 3 points on every line, G be a group of automorphisms of $\mathcal{S}$ which acts transitively on Π and suppose that there is at least one line. Then there are no G-admissible representations of order 2.*

Proof. Suppose that (R, φ) is a G-admissible representation of order 2, say $R = \{1, f\}$. Since R is generated by the image of φ, the representation is G-admissible and G is point-transitive, $\varphi(p) = f$ for every $p \in \Pi$. Then if $l = \{p, q, r\}$ is a line, we have

$$\varphi(p)\varphi(q)\varphi(r) = f^3 = f \neq 1,$$

which is contrary to the assumption that φ is a representation. □

Let (R, φ) be a representation of $\mathcal{S} = (\Pi, L)$ and Λ be a subset of Π. Put

$$R[\Lambda] = \langle \varphi(y) \mid y \in \Lambda \rangle$$

(the subgroup in R generated by the elements $\varphi(y)$ taken for all $y \in \Lambda$).

If φ_Λ is the restriction of φ to Λ and $L(\Lambda)$ is the set of lines from L contained in Λ, then we have the following

Lemma 2.1.2 *$(R[\Lambda], \varphi_\Lambda)$ is a representation of $(\Lambda, L(\Lambda))$.* □

If the representation (R, φ) in the above lemma is G-admissible for an automorphism group G of $\mathcal{S}$, H is the stabilizer of Λ and $\overline{H}$ is the action induced by H on Λ, then clearly $(R[\Lambda], \varphi_\Lambda)$ is $\overline{H}$-admissible.

Now let Δ be a subset of Λ and suppose that $R[\Delta]$ is normal in $R[\Lambda]$ (this is always the case when R is abelian). Then $(R[\Lambda]/R[\Delta], \chi)$ is a representation of $(\Lambda, L(\Lambda))$ (where χ is the composition of φ_Λ and the homomorphism of $R[\Lambda]$ onto $R[\Lambda]/R[\Delta]$). The following observation is rather useful.

Lemma 2.1.3 *Let $\{p, q, r\}$ be a line in $L(\Lambda)$ such that $p \in \Delta$. Then $\chi(q) = \chi(r)$.* □

The following result is quite obvious.

Lemma 2.1.4 *Let* (R_i, φ_i) *be representations of* $\mathcal{S} = (\Pi, L)$ *for* $1 \leq i \leq m$. *Let*

$$R = R_1 \times ... \times R_m = \{(r_1, ..., r_m) \mid r_i \in R_i\}$$

be the direct product of the representation groups R_i *and* φ *be the mapping which sends* $p \in \Pi$ *onto* $(\varphi_1(p), ..., \varphi_m(p)) \in R$. *Then* $(\mathrm{Im}\, \varphi, \varphi)$ *is a representation of* $\mathcal{S}$. □

The representation $(\mathrm{Im}\, \varphi, \varphi)$ in the above lemma will be called the *product* of the representations (R_i, φ_i) and we will write

$$(\mathrm{Im}\, \varphi, \varphi) = (R_1, \varphi_1) \times ... \times (R_m, \varphi_m).$$

Notice that the representation group of the product is not always the direct product of the R_i but rather a sub-direct product.

For the remainder of the chapter $\mathcal{S} = (\Pi, L)$ is a point-line incidence system with three points on every line and this system might or might not be a truncation of a geometry of rank 3 or more.

2.2 Collinearity graph

Let Γ be the collinearity graph of the point-line incidence system $\mathcal{S} = (\Pi, L)$ which is a graph on the set of points in which two points are adjacent if they are incident to a common line. For $x, y \in \Pi$ by $d_\Gamma(x, y)$ we denote the distance from x to y in the natural metric of Γ. Notice that the set of points incident to a line is a triangle. For a vertex x of Γ, as usual $\Gamma_i(x)$ denotes the set of vertices at distance i from x in Γ and $\Gamma(x) = \Gamma_1(x)$.

For a vertex x of Γ and $0 \leq i \leq d$ put

$$R_i(x) = \langle \varphi(y) \mid d_\Gamma(x, y) \leq i \rangle,$$

or equivalently

$$R_i(x) = R[\{x\} \cup \Gamma_1(x) \cup ... \cup \Gamma_i(x)].$$

If for some $i \geq 1$ the subgroup $R_{i-1}(x)$ is a normal subgroup in $R_i(x)$ (of course this is always the case when R is abelian), we put

$$\overline{R}_i(x) = R_i(x)/R_{i-1}(x).$$

Notice that $R_0(x)$ is in the centre of $R_1(x)$, so that $\overline{R}_1(x)$ is always defined.

We introduce a certain invariant of Γ which will be used to obtain upper bounds on dimensions of $V(\mathcal{S})$. Let $\Sigma_i(x)$ be a graph on the set

$\Gamma_i(x)$ in which two vertices $\{u, v\}$ are adjacent if there is a line containing u, v and intersecting $\Gamma_{i-1}(x)$ (here $1 \leq i \leq d$ where d is the diameter of Γ). Notice that $\Sigma_i(x)$ is a subgraph of Γ but not necessarily the subgraph induced by $\Gamma_i(x)$ (the latter subgraph might contain more edges than $\Sigma_i(x)$). Let $c(\Sigma_i(x))$ be the number of connected components of $\Sigma_i(x)$ and put

$$\beta(\Gamma) = 1 + \min_{x \in \Pi} \left(\sum_{i=1}^{d} c(\Sigma_i(x)) \right).$$

Notice that in general $\beta(\Gamma)$ depends not only on the graph Γ but also on the line set L, but if $\mathcal{S} = (\Pi, L)$ is flag-transitive (in fact point-transitivity is enough), then $c(\Sigma_i(x)) = c(\Sigma_i(y))$ for any $x, y \in \Pi$.

Lemma 2.2.1 dim $V(\mathcal{S}) \leq \beta(\Gamma)$.

Proof. Let (W, φ) be an abelian representation of $\mathcal{S}$ and $x \in \Gamma$. Then

$$\dim W = 1 + \sum_{i=1}^{d} \dim \overline{W}_i(x).$$

Let $u, v \in \Gamma_i(x)$ be adjacent in $\Sigma_i(x)$ and l be a line containing u, v and intersecting $\Gamma_{i-1}(x)$ in a point w, say. Then by (2.1.3)

$$\langle \varphi(u), W_{i-1}(x) \rangle = \langle \varphi(v), W_{i-1}(x) \rangle.$$

If $u_1, u_2, ..., u_m$ is a path in $\Sigma_i(x)$ then by the above statement $\langle \varphi(u_j), W_{i-1} \rangle$ is independent on the choice of $1 \leq j \leq m$. Hence all the points in a connected component of Σ_i have the same image in $\overline{W}_i(x)$ and the proof follows. □

Lemma 2.2.2 *Let* $C = (y_0, y_1, ..., y_m = y_0)$ *be a cycle in the collinearity graph* Γ *of* $\mathcal{S}$ *and suppose that* z_i, $0 \leq i \leq (m-1)$, *are points such that* $\{y_i, y_{i+1}, z_i\} \in L$. *Then for every representation* (R, φ) *of* $\mathcal{S}$ *we have*

$$\varphi(z_0)\varphi(z_1)...\varphi(z_{m-1}) = 1.$$

Proof. By (R2) we have $\varphi(x)\varphi(x) = 1$ for every point x, hence

$$\varphi(y_0)\varphi(y_1)\varphi(y_1)...\varphi(y_{m-1})\varphi(y_{m-1})\varphi(y_0) = 1.$$

On the other hand, since (R, φ) is a representation, we have $\varphi(z_i) = \varphi(y_i)\varphi(y_{i+1})$, which immediately gives the result. □

Lemma 2.2.3 *Suppose that $\overline{R}_1(x) = R_1(x)/R_0(x)$ is abelian for every $x \in \Pi$. If $u, v \in \Pi$ with $d_\Gamma(u,v) \leq 2$, one of the following holds:*

(i) $[\varphi(u), \varphi(v)] = 1$;

(ii) $d_\Gamma(u,v) = 2$, $\Gamma(u) \cap \Gamma(v)$ *consists of a unique vertex w, say, and* $[\varphi(u), \varphi(v)] = \varphi(w)$.

In particular, $\varphi(u)$ and $\varphi(v)$ commute if $d_\Gamma(u,v) = 1$ or if $d_\Gamma(u,v) = 2$ and there are more than one paths of length 2 in Γ joining u and v.

Proof. If u and v are adjacent then $\varphi(u)\varphi(v) = \varphi(t)$ where $\{u, v, t\}$ is a line and hence $[\varphi(u), \varphi(v)] = 1$. If $R_1(x)$ is abelian for every $x \in \Pi$ then again $\varphi(u)$ and $\varphi(v)$ commute. If $R_1(x)$ is non-abelian, then its commutator is $R_0(x)$ and the latter contains at most one non-identity element, which is $\varphi(x)$. Now the result is immediate. □

2.3 Geometric hyperplanes

A *geometric hyperplane* H in $\mathcal{S}$ is a proper subset of points such that every line is either entirely contained in H or intersects it at exactly one point. The *complement* of H is the subgraph in the collinearity graph of $\mathcal{S}$ induced by $\Pi \setminus H$. The following result is quite obvious.

Lemma 2.3.1 *Let $\chi : \widetilde{\mathcal{S}} \to \mathcal{S}$ be a covering of geometries and H be a geometric hyperplane in $\mathcal{S}$. Then $\chi^{-1}(H)$ is a geometric hyperplane in $\widetilde{\mathcal{S}}$.* □

The following result shows that when every line is incident to exactly 3 points the geometric hyperplanes correspond to vectors in the dual of the universal representation module of the geometry. In particular, the universal representation module of a point-line incidence system $\mathcal{S}$ with 3 points per line is trivial if and only if $\mathcal{S}$ has no geometric hyperplanes.the following lemma establishes a natural bijection between the geometric hyperplane and the codimension 1 subspaces in the universal representation module.

Lemma 2.3.2 *Let (V, φ_a) be the universal abelian representation of $\mathcal{S}$, W be a codimension 1 subspace in V and H be a geometric hyperplane in $\mathcal{S}$. Let*

$$\chi(W) = \{x \in \Pi \mid \varphi_a(x) \in W\}.$$

Let $Z_H = \{0, 1\}$ be a group of order 2 and φ_H be a mapping of Π into Z_H such that $\varphi_H = 0$ if $x \in H$ and $\varphi_H = 1$ otherwise. Then

(i) $\chi(W)$ *is a geometric hyperplane in* $\mathcal{S}$;
(ii) (Z_H, φ_H) *is a representation of* $\mathcal{S}$;
(iii) *the kernel* U_H *of the representation homomorphism of* V *onto* Z_H *has codimension* 1 *in* V;
(iv) $\chi(U_H) = H$.

itemize

Proof. Consider the mapping φ_W of Π into V/W defined by

$$\varphi_W : p \mapsto \varphi_a(p)W.$$

Clearly $(V/W, \varphi_W)$ is a representation of $\mathcal{S}$. Since V/W is of order 2, for every line from L either for all or for exactly one point the image under φ_W is 0 and since φ_W is surjective, the latter possibility occurs. Hence (i) follows. The assertions (ii) to (iv) are rather obvious. □

By the above lemma the universal abelian representation can be reconstructed from the geometric hyperplanes in the following way. Let $H_1, ..., H_m$ be the set of geometric hyperplanes in $\mathcal{S}$, $Z(H_i) = \{0, 1\}$ be a group of order 2 and $\varphi_{H_i} : \Pi \to Z(H_i)$ be the mapping, such that $\varphi_{H_i}(p) = 0$ if $p \in H_i$ and $\varphi_{H_i}(p) = 1$ otherwise.

Lemma 2.3.3 *The universal abelian representation* (V_a, φ_a) *of* $\mathcal{S}$ *is isomorphic to the product of the representations* $(Z(H_i), \varphi_{H_i})$ *taken for all the geometric hyperplanes* H_i *in* $\mathcal{S}$.

Proof. Let $V_1, ..., V_m$ be the set of all subgroups of index 2 in V_a and suppose that $\chi(V_i) = H_i$ in terms of (2.3.2). Define a mapping ψ from V_a into the direct product of $Z(H_1) \times ... \times Z(H_m)$ by $\psi(v) = (\alpha_1(v), ..., \alpha_m(v))$, where $\alpha_i(v) = 0$ if $v \in V_i$ and $\alpha_i(v) = 1$ otherwise. It is easy to see that ψ is a representation homomorphism of (V_a, φ_a) onto the product of the $(Z(H_i), \varphi_{H_i})$, which proves the universality of the product. □

Corollary 2.3.4 *If* (V, φ) *is a representation of* $\mathcal{S}$ *such that* V *is generated by the images under* φ *of the points from a geometric hyperplane* H *in* $\mathcal{S}$. *Then the product* $(V, \varphi) \times (Z(H), \varphi_H)$ *possesses a proper representation homomorphism onto* (V, φ), *in particular the latter is not universal.* □

The next lemma generalizes this observation for the case of non-abelian representations.

Lemma 2.3.5 *Let* (R, φ) *be a representation of* $\mathcal{S}$. *Suppose that* H *is a geometric hyperplane in* $\mathcal{S}$ *such that the elements* $\varphi(x)$ *taken for all*

$x \in H$ generate the whole of R. Then the representation group of the product $(R, \varphi) \times (Z(H), \varphi_H)$ is the direct product of R and the group $Z(H)$ of order 2. □

The following result describes a situation when the universal representation group is infinite.

Lemma 2.3.6 *Suppose that H is a geometric hyperplane in $\mathcal{S}$ whose complement consists of m connected components. Then $R(\mathcal{S})$ possesses a homomorphism onto a group, freely generated by m involutions. In particular, $R(\mathcal{S})$ is infinite if $m \geq 2$.*

Proof. Let $A_1, ..., A_m$ be the connected components of the complement of H. Let D be a group freely generated by m involutions $a_1, ..., a_m$. Let ψ be the mapping from Π into D, such that $\psi(x) = a_i$ if $x \in A_i$, $1 \leq i \leq m$, and $\psi(x)$ is the identity element of D if $x \in H$. It is easy to check that (D, ψ) is a representation of $\mathcal{S}$ and the proof follows. □

Lemma 2.3.7 *Suppose that for every point $x \in \Pi$ there is a partition $\Pi = A(x) \cup B(x)$ of Π into disjoint subsets $A(x)$ and $B(x)$ such that the following conditions are satisfied:*

(i) *the graph Ξ on Π with the edge set $E(\Xi) = \{(x, y) \mid y \in B(x)\}$ is connected and undirected (the latter means that $x \in B(y)$ whenever $y \in B(x)$);*

(ii) *for every $x \in \Pi$ the graph Σ^x on $B(x)$ with the edge set $E(\Sigma^x) = \{\{u, v\} \mid \{u, v, w\} \in L$ for some $w \in A(x)\}$ is connected.*

Suppose that (R, φ) is a representation of $\mathcal{S}$ such that $[\varphi(x), \varphi(y)] = 1$ whenever $y \in A(x)$. Then the commutator subgroup of R has order at most 2.

Proof. For $x, y \in \Pi$ let $c_{xy} = [\varphi(x), \varphi(y)]$ and C_{xy} be the subgroup in R generated by c_{xy}. Then by the assumption $c_{xy} = 1$ if $y \in A(x)$. Let $\{u, v, w\}$ be a line in L such that $\{u, v\}$ is an edge in Σ^x and $w \in A(x)$. Since $\varphi(u) = \varphi(w)\varphi(v)$ by definition of the representation and $[\varphi(x), \varphi(w)] = 1$, we have

$$c_{xu} = [\varphi(x), \varphi(u)] = [\varphi(x), \varphi(w)\varphi(v)]$$
$$= [\varphi(x), \varphi(v)][\varphi(x), \varphi(w)]^{\varphi(v)} = [\varphi(x), \varphi(v)] = c_{xv}.$$

This calculation together with the connectivity of Σ^x implies that C_{xu} is independent on the particular choice of $u \in B(x)$ and will be denoted by

C_x. Since

$$c_{xy} = [\varphi(x), \varphi(y)] = [\varphi(y), \varphi(x)]^{-1} = c_{yx}^{-1},$$

we also have $C_{xy} = C_{yx}$, which means that $C_x = C_y$ whenever $y \in B(x)$, i.e., whenever x and y are adjacent in the graph Ξ as in (i). Since Ξ is undirected and connected, C_x is independent on the choice of x and will be denoted by C. By the definition, $\varphi(x)^{-1}c_{xy}\varphi(x) = c_{xy}^{-1}$, which means that C is inverted by the element $\varphi(x)$ for every $x \in \Pi$. Now if $\{x, y, z\} \in L$ then $\varphi(x) = \varphi(y)\varphi(z)$ and hence $\varphi(x)$ also centralizes C, which means that the order of C is at most 2. Since R is generated by the elements $\varphi(x)$, taken for all $x \in \Pi$, we also observe that C is in the centre of R, in particular, it is normal in R. Since the images of $\varphi(x)$ and $\varphi(y)$ in R/C commute for all $x, y \in \Pi$ we conclude that the order of the commutator subgroup of R is at most the order of C and the proof follows. □

Suppose that the conditions in (2.3.7) are satisfied and R is non-abelian. Then $C = R'$ is of order 2 generated by an element c, say. One can see from the proof of (2.3.7) that in the considered situation $[\varphi(x), \varphi(y)] = c$ whenever $y \in B(x)$, in particular $x \in A(x)$ and we have the following

Corollary 2.3.8 *Suppose that the conditions in* (2.3.7) *are satisfied and* R *is non-abelian. Let* (V, ψ) *be the abelian representation where* $V = R/R'$ *and* $\psi(x) = \varphi(x)R'/R'$. *Then the mapping* $\chi : V \times V \to GF(2)$, *such that* $\chi(\varphi(x), \varphi(y)) = 0$ *if* $y \in A(x)$ *and* $\chi(\varphi(x), \varphi(y)) = 1$ *if* $y \in B(x)$, *is a non-zero bilinear symplectic form. In particular,* $A(x)$ *is a hyperplane for every* $x \in \Pi$. □

Corollary 2.3.9 *Suppose that in the conditions of* (2.3.8) *the representation* (R, φ) *is* G*-admissible for a group* G *(which is the case, for instance, if* (R, φ) *is the universal representation and* $G = \mathrm{Aut}\,\mathcal{S}$*). Then the mapping* χ *is* G*-invariant.* □

2.4 Odd order subgroups

Let G be a flag-transitive automorphism group of $\mathcal{S} = (\Pi, L)$ and suppose that E is a normal subgroup in G of odd order. Let $\overline{\mathcal{S}} = (\overline{\Pi}, \overline{L})$ be the quotient of $\mathcal{S}$ with respect to E, so that G commutes with the natural morphism

$$\chi : \mathcal{S} \to \overline{\mathcal{S}}$$

and induces a flag-transitive automorphism group of $\overline{\mathcal{S}}$. Suppose further that the situation a non-degenerate in the sense that every line from $\overline{L}$ is still incident to three points from $\overline{\Pi}$.

Let (V, φ) be the universal abelian representation of $\mathcal{S}$. Let $V^z = C_V(E)$ and $V^c = [V, E]$ so that $V = V^z \oplus V^c$ and let φ^z and φ^c be the mappings of the point set of $\mathcal{S}$ into V^z and V^c, respectively, such that $\varphi(x) = \varphi^z(x) + \varphi^c(x)$ for every $x \in \Pi$.

Lemma 2.4.1 *In the above notation (V^z, φ^z) is the universal abelian representation of $\overline{\mathcal{S}}$.*

Proof. Since the mapping φ^z is constant on every E-orbit on the set of points of $\mathcal{S}$, it is easy to see that (V^z, φ^z) is a representation of $\overline{\mathcal{S}}$. Let (W, ψ) be the universal representation of $\overline{\mathcal{S}}$ and χ be the natural morphism of $\mathcal{S}$ onto $\overline{\mathcal{S}}$. Then it is easy to see that $(W, \psi\chi)$ is a representation of $\mathcal{S}$ and in the induced action of G on W the subgroup E is in the kernel. Since V is the universal representation module of $\mathcal{S}$, W is a quotient of V. Furthermore, if U is the kernel of the homomorphism of V onto W then U contains V^c. This shows that $U = V^c$ and $W \cong V^z$. □

Lemma 2.4.2 *Let V_1 and V_2 be $GF(2)$-vector spaces and $\mathcal{S} = (\Pi, L)$ be the point-line incidence system such that $\Pi = V_1^\# \times V_2^\#$ and whose lines are the triples $\{(a,x),(b,x),(a+b,x)\}$ and the triples $\{(a,x),(a,y),(a,x+y)\}$, for all $a, b \in V_1^\#$, $x, y \in V_2^\#$. Then the universal representation group of $\mathcal{S}$ is abelian, isomorphic to the tensor product $V_1 \otimes V_2$.*

Proof. Let (R, φ) be the universal representation of $\mathcal{S}$. Then the following sequence of equalities for $a, b \in V_1^\#, x, y \in V_2^\#$ imply the commutativity of R:

$$\varphi(a,x)\varphi(b,y) = \varphi(a+b,x)\varphi(b,x)\varphi(b,y)$$

$$= \varphi(a+b,x)\varphi(b,x+y) = \varphi(a+b,y)\varphi(a+b,x+y)\varphi(b,x+y)$$

$$= \varphi(b,y)\varphi(a,y)\varphi(a,x+y) = \varphi(b,y)\varphi(a,x).$$

The structure of R now follows from the definition of the tensor product. □

Suppose now that $\mathcal{S} = (\Pi, L)$ possesses an automorphism group E of order 3 which acts fixed-point freely on the set Π of points. Then every orbit of E on Π is of size 3 and we can adjoin these orbits to the set L of lines. The point-line incidence system obtained in this way will be called the *enrichment* of $\mathcal{S}$ associated with E. We will denote this enriched system by $\mathcal{S}^{\bullet}$.

Lemma 2.4.3 *In terms introduced before* (2.4.1) *if* $|E| = 3$ *then* (V^c, φ^c) *is the universal abelian representation of* $\mathcal{S}^{\bullet}$.

Proof. Since E acts fixed-point freely on V^c, $\varphi^c(x)+\varphi^c(x^z)+\varphi^c(x^{z^2}) = 0$ for any $x \in \Pi$ and a generator z of E. □

Lemma 2.4.4 *Let* $\mathcal{S}^{\bullet}$ *be the enrichment of* $\mathcal{S}$ *associated with a fixed-point free subgroup* E *of order* 3 *and* $(R^{\bullet}, \varphi)$ *be a representation of* $\mathcal{S}^{\bullet}$. *Let* $x \in \Pi$ *and* y *be an image under* E *of a point collinear to* x. *Then* $[\varphi(x), \varphi(y)] = 1$.

Proof. Let $\{x_0^0 = x, x_0^1, x_0^2\}$ be a line containing x. Let e be a generator of E and let x_i^j be the image of x_0^j under the i-th power of e, $0 \le i \le 2$. Let $\Phi = \{x_i^j \mid 0 \le i \le 2, 0 \le j \le 2\}$ (we assume that $y \in \Phi$). Then

$$\Lambda = \{\{x_i^0, x_i^1, x_i^2\} \mid 0 \le i \le 2\} \cup \{\{x_0^j, x_1^j, x_2^j\} \mid 0 \le j \le 2\}$$

is a set of lines of $\mathcal{S}^{\bullet}$ contained in Φ. Then the conditions of (2.4.2) are satisfied for (Φ, Λ) with

$$V_1 = \langle \varphi(x_i^0) \mid 0 \le i \le 2 \rangle, \quad V_2 = \langle \varphi(x_0^j) \mid 0 \le j \le 2 \rangle$$

and hence the elements $\varphi(z)$ taken for all $z \in \Phi$ generate in $R^{\bullet}$ an abelian subgroup of order at most 16. □

The technique presented in the remainder of the section was introduced in [Sh95] to determine the universal representation modules of the geometries $\mathcal{G}(3^{[\binom{n}{2}]_2} \cdot S_{2n}(2))$ for $n \ge 3$ and $\mathcal{G}(3^{23} \cdot Co_2)$.

In terms introduced at the beginning of the section assume that E is an elementary abelian 3-group normal in G so that E is a $GF(3)$-module for $\overline{G} = G/E$ and that $V^c \neq 0$. Since the characteristic of V^c is 2, by Maschke's theorem V^c is a direct sum of irreducible E-modules. Let U be one of these irreducibles. Since $V^c = [V, E]$, U is non-trivial, hence it is 2-dimensional and E induces on U an action of order 3. The kernel of this action is an index 3 subgroup in E. A subgroup Y of index 3 in E is said to be *represented* if $V_Y^c := C_{V^c}(Y) \neq 0$. Let Ξ be the set of all represented subgroups (of index 3) in E. Then we have a decomposition

$$V^c = \bigoplus_{Y \in \Xi} V^c_Y,$$

which is clearly G-invariant with respect to the action $(V^c_Y)^g = V^c_{Y^g}$ for $g \in G$.

Let $x \in \Pi$ be a point and let $E(x) = E \cap G(x)$ be the stabilizer of x in E. Since E is abelian, $E(x)$ depends only on the image $\overline{x}$ of x in $\overline{\Pi}$, so we can put $E(\overline{x}) = E(x)$. Thus for every point x we obtain a subgroup $E(\overline{x})$ in E normalized by $G(\overline{x})$. Put $\widehat{E} = E/E(\overline{x})$ and adopt the hat convention for subgroups in $\widehat{E}$. We will assume that the following condition is satisfied.

(M) The elementary abelian 3-group $\widehat{E}$ is generated by a $G(x)$-invariant set $\mathscr{B} = \{B_i \mid i \in I\}$ of distinct subgroups of order 3. Let Σ be the graph on the index set I such that $\{i, j\} \subseteq I$ is an edge of Σ if and only if there is a line $\{x, u, w\} \in L$ containing x such that the intersections of $B_{ij} := \langle B_i, B_j \rangle$ with $\widehat{E}(\overline{u})$ and $\widehat{E}(\overline{w})$ together with B_i and B_j form the complete set of subgroups of order 3 in the group B_{ij} (which is elementary abelian of order 9).

For a point $x \in \Pi$ and a represented subgroup $Y \in \Xi$ let $v_{x,Y}$ be the projection of $\varphi^c(x)$ into V^c_Y and put

$$S(\overline{x}) = \{Y \in \Xi \mid v_{x,Y} \neq 0\}$$

(notice that $S(\overline{x})$ does not depend on the particular choice of the preimage x of $\overline{x}$ in Π). For a represented subgroup $Y \in \Xi$ put

$$\Omega_Y = \{\overline{x} \in \overline{\Pi} \mid Y \notin S(\overline{x})\} = \{\overline{x} \in \overline{\Pi} \mid v_{x,Y} = 0\}.$$

Notice that if $\overline{x} \notin \Omega_Y$ then $E(\overline{x}) \leq Y$.

Proposition 2.4.5 *If* **(M)** *holds then* Ω_Y *is a geometric hyperplane in* $\overline{\mathscr{S}}$ *for every* $Y \in \Xi$.

Proof. Choose $Y \in \Xi$. Since V^c is generated by the vectors $\varphi^c(y)$ taken for all $y \in \Pi$, there is $x \in \Pi$ such that $v_{x,Y} \neq 0$ and hence there is $\overline{x} \in \overline{\Pi}$ outside Ω_Y and so the latter is a proper subset of $\overline{\Pi}$. If $l = \{x, u, w\} \in L$ then since (V^c, φ^c) is a representation, $v_{x,Y} + v_{u,Y} + v_{w,Y} = 0$, which shows that every line from $\overline{L}$ intersects Ω_Y in 0, 1 or 3 points and all we have to show is that the intersection is never empty.

Suppose to the contrary that both $\overline{u}$ and $\overline{w}$ are not in Ω_Y (where $\{\overline{x}, \overline{u}, \overline{w}\}$ is a line in $\overline{L}$). Consider $\widehat{E} = E/E(\overline{x})$. Since $\overline{x} \notin \Omega_Y$, we have $E(\overline{x}) \leq Y$, which shows that the image $\widehat{Y}$ of Y in $\widehat{E}$ is a proper hyperplane in $\widehat{E}$. Consider the generating set $\mathscr{B}$ from **(M)**. By the flag-transitivity, $G(x)$ acts transitively on the set of lines passing through x. This together with **(M)** implies that there is an edge $\{i, j\}$ of Σ such that B_{ij} is generated by its intersections with $E(\overline{u})$ and $E(\overline{w})$. Since both $\overline{u}$ and $\overline{w}$ are not in Ω_Y we have $B_i, B_j \leq \widehat{Y}$. Let $k \in I \setminus \{j\}$ be adjacent to i in Σ and $\{\overline{x}, \overline{u}', \overline{w}'\}$ be a line in $\overline{L}$ such that the intersections of B_{ik} with $\widehat{E}(\overline{u}')$ and $\widehat{E}(\overline{v}')$ are of order 3 distinct from each other and also from B_i and B_k. Since at least one of $\overline{u}'$ and $\overline{w}'$ is not contained in Ω_Y, the corresponding intersection is contained in $\widehat{Y}$, because we know already that $B_i \leq \widehat{Y}$ this gives $B_k \leq \widehat{Y}$. Finally, since Σ is connected we obtain $\widehat{Y} = \widehat{E}$, a contradiction. □

The above proof also suggests how we can reconstruct Y from Ω_Y. For a geometric hyperplane Ω in $\overline{\mathscr{S}}$ put

$$Y(\Omega) = \langle E(\overline{x}) \mid \overline{x} \notin \Omega \rangle.$$

Lemma 2.4.6 *Suppose that* **(M)** *holds and* Ω *is a geometric hyperplane in* $\overline{\mathscr{S}}$. *Then*

(i) *the index of* $Y(\Omega)$ *in* E *is at most* 3;
(ii) *if* $Y \in \Xi$ *is represented, then* $Y = Y(\Omega_Y)$.

Proof. Let $\overline{x} \in \overline{\Pi} \setminus \Omega$. Then by the definition $E(\overline{x}) \leq Y(\Omega)$. Consider $\widehat{E} = E/E(\overline{x})$. Let $\{i, j\}$ be an edge of Σ. Then there is a line $\{x, u, w\}$ such that among the four subgroups in B_{ij} one is contained in $\widehat{E}(\overline{u})$ and one is in $\widehat{E}(\overline{w})$. Since one of the points $\overline{u}$ and $\overline{w}$ is contained in Ω, a subgroup of order 3 in B_{ij} is contained in $\widehat{Y(\Omega)}$. Hence the images in $\widehat{E}/\widehat{Y(\Omega)}$ of B_i and B_j coincide. Since $\{i, j\}$ was an arbitrary edge of Σ and the latter is connected, we obtain (i). In the proof of (2.4.5) we observed that $E(\overline{x}) \leq Y$ whenever $\overline{x} \notin \Omega_Y$. Hence (ii) follows from (i) and (2.4.5). □

A geometric hyperplane Ω in $\overline{\mathscr{S}}$ is said to be *acceptable* if

$$Y(\Omega) \neq E.$$

By (2.4.6) every Ω_Y is acceptable. Thus the number of represented subgroups in E (the cardinality of Ξ) is at most the number of acceptable hyperplanes in $\overline{\mathscr{S}}$.

Now in order to bound the dimension of V^c it is sufficient to bound the dimension of V_Y^c for a represented subgroup Y in E. Notice that a line which is not in Ω_Y has exactly two of its points outside Ω_Y. Hence all such lines define in a natural way a structure of a graph on the complement of Ω_Y. Let n_Y be the number of connected components of this graph.

Lemma 2.4.7 *Suppose that* (M) *holds. Then* dim $V_Y^c \leq 2n_Y$.

Proof. Let T be the complement of Ω_Y. It is clear that V_Y^c is spanned by the vectors $v_{x,Y}$ taken for all points $\overline{x} \in T$. For a fixed x its image $\overline{x}$ in $\overline{\mathcal{S}}$ is the E-orbit containing x. Hence the vectors $v_{u,Y}$ taken for all $u \in \overline{x}$ generate a 2-dimensional irreducible E-submodule (in fact any E-orbit on the non-zero elements of V_Y^c spans a 2-dimensional irreducible E-submodule). Let $\overline{x}, \overline{u}$ be collinear points in T. Then by (2.4.5) there exists a line $\overline{l} = \{\overline{x}, \overline{u}, \overline{w}\}$ in $\overline{\mathcal{S}}$ such that $\overline{w} \in \Omega_Y$. Choose a line $l = \{x, u, w\}$ of $\mathcal{S}$ which is a preimage of $\overline{l}$. Since

$$v_{x,Y} + v_{u,Y} + v_{w,Y} = 0 \text{ and } v_{w,Y} = 0,$$

we obtain $v_{x,Y} = v_{u,Y}$. Hence the points in every connected component of T correspond to the same 2-dimensional E-submodule of V_Y^c and the proof follows. □

By (2.3.6) the existence of a geometric hyperplane whose complement induces a disconnected subgraph in the collinearity graph forces the universal representation group to be infinite. In view of (2.4.7) this observation implies the following.

Corollary 2.4.8 *Suppose that* (M) *holds and the universal representation group of* $\overline{\mathcal{S}}$ *is finite. Then the dimension of* V_Y^c *is either* 0 *or* 2. □

2.5 Cayley graphs

In some circumstances calculation of the universal representation of a point-line incidence system can be reduced to calculation of the universal cover of a certain Cayley graph with respect to a class of triangles.

Let $\mathcal{S} = (\Pi, L)$ be a point-line incidence system with 3 points on a line, (Q, ψ) be a representation of $\mathcal{S}$, and suppose that ψ is injective. Then

$$\psi(\Pi) := \{\psi(x) \mid x \in \Pi\}$$

is a generating set of Q and we can consider the Cayley graph $\Theta := Cay(Q, \psi(\Pi))$ of Q with respect to this generating set. This means that the vertices of Θ are the elements of Q and two such elements q and p are adjacent if $qp^{-1} \in \psi(\Pi)$. Since $\psi(\Pi)$ consists of involutions, Θ is undirected. If e is the identity element of Q (considered as a vertex of Θ) then ψ establishes a bijection of Π onto $\Theta(e) = \psi(\Pi)$. A triangle $T = \{p, q, r\}$ in Θ will be called *geometric* if $\{pq^{-1}, qr^{-1}, rp^{-1}\}$ is a line from L. If $\{x, y, z\} \in L$ then $\{e, \psi(x), \psi(y)\}$ is a geometric triangle and all geometric triangles containing e are of this form.

Let $(\widetilde{Q}, \widetilde{\psi})$ be another representation of $\mathcal{S}$ such that there is a representation homomorphism $\chi : \widetilde{Q} \to Q$. Since χ is a representation homomorphism, it maps vertices adjacent in $\widetilde{\Theta} := Cay(\widetilde{Q}, \widetilde{\psi}(\Pi))$ onto vertices adjacent in Θ. Since in addition the valencies of both $\widetilde{\Theta}$ and Θ are equal to $|\Pi|$, χ induces a covering of $\widetilde{\Theta}$ onto Θ (we denote this covering also by χ). Furthermore one can easily see that a connected component of the preimage under χ of a geometric triangle in Θ is a geometric triangle in $\widetilde{\Theta}$, which shows that the geometric triangles in Θ are contractible with respect to χ.

Lemma 2.5.1 *In the above terms let (R, φ) be the universal representation of $\mathcal{S}$ and $\sigma : R \to Q$ be the corresponding homomorphism of representations. Then the induced covering*

$$\sigma : Cay(R, \varphi(\Pi)) \to \Theta$$

is universal among the covers with respect to which the geometric triangles are contractible.

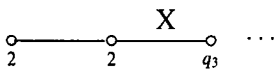
Proof. Let $\delta : \widehat{\Theta} \to \Theta$ be the universal cover with respect to the geometric triangles in Θ. By the universality property the group of deck transformations acts regularly on every fiber and since Q acts regularly on Θ, the group $\widehat{Q}$ of all liftings of elements of Q to automorphisms of $\widehat{\Theta}$ acts regularly on the vertex set of $\widehat{\Theta}$. This means that $\widehat{\Theta}$ is a Cayley graph of $\widehat{Q}$. Let $\widehat{e}$ be a preimage of e in $\widehat{\Theta}$. Then a vertex $\widehat{f} \in \widehat{\Theta}$ can be identified with the unique element in $\widehat{Q}$ which maps $\widehat{e}$ onto $\widehat{f}$ and under this identification δ is a homomorphism of $\widehat{Q}$ onto Q. Since δ is a covering of graphs, it induces a bijection β of $\widehat{\Theta}(\widehat{e})$ onto $\Theta(e)$ and since ψ is a bijection of Π onto $\Theta(e)$ the mapping $\varphi := \beta^{-1}\psi$ is a bijection of Π onto $\widehat{\Theta}(\widehat{e})$. We claim that $\varphi(x)$ is an involution for every $x \in \Pi$. The claim follows from the fact that δ is a covering of graphs, $\delta(\varphi(x)) = \psi(x)$ is an involution and $\widehat{Q}$ acts regularly on $\widehat{\Theta}$. Let $\{x, y, z\} \in L$. Since the geometric triangles are contractible with respect to δ, $\varphi(x)$ and $\varphi(y)$ are adjacent in $\widehat{\Theta}$, which means that the element $\alpha := \varphi(x)\varphi(y)$ belongs to the set $\varphi(\Pi)$ of generators. Since $\delta(\alpha) = \psi(z)$ we have $\alpha = \varphi(z)$ and hence $\widehat{Q}$ is a representation group of $\mathcal{S}$. The universality of δ implies that $\widehat{Q}$ is the universal representation group, i.e., $\widehat{Q} = R$. □

2.6 Higher ranks

Let $\mathcal{S} = (\Pi, L)$ be as above, (R, φ) be a representation of $\mathcal{S}$, Λ be a subset of Π and $L(\Lambda)$ be the set of lines contained in Λ. Let $\varphi[\Lambda]$ be the subgroup in R generated by the elements $\varphi(x)$ taken for all $x \in \Lambda$ and φ_Λ be the restriction of φ to Λ. The following result is quite obvious.

Lemma 2.6.1 *The pair* $(\varphi[\Lambda], \varphi_\Lambda)$ *is a representation of* $(\Lambda, L(\Lambda))$. □

Suppose now that $\mathcal{S}$ is the point-line incidence system of a geometry $\mathcal{G}$ of rank $n \geq 3$ with its diagram of the form



so that (R, φ) is also a representation of $\mathcal{G}$. For an element $u \in \mathcal{G}$ define $\varphi^*(u)$ to be the subgroup in R generated by the elements $\varphi(x)$ taken for all points x incident to u. In this way for a point x the element $\varphi(x)$ is identified with the subgroup $\varphi^*(x)$ it generates in R. For u as above let φ_u be the restriction of φ to the set of points in $\mathcal{G}$ incident to u. Then by (2.6.1) $(\varphi^*(u), \varphi_u)$ is a representation of the point-line incidence system with the point-set $\Pi \cap \mathrm{res}_{\mathcal{G}}(u)$ and whose lines are those of $\mathcal{G}$ incident to u. In particular if u is a plane of $\mathcal{G}$ then $(\varphi^*(u), \varphi_u)$ is a representation of the projective plane $pg(2,2)$ of order 2 formed by the points and lines of $\mathcal{G}$ incident to u, in particular $\varphi^*(u)$ is abelian of order at most 2^3.

Let x be a point in $\mathcal{G}$ and $\mathcal{S}_x = (\Pi_x, L_x)$ be the point-line system of $\mathrm{res}_{\mathcal{G}}(x)$, which means that Π_x and L_x are the lines and planes in $\mathcal{G}$ incident to x.

Lemma 2.6.2 *In the above terms let* (R, φ) *be a representation of* $\mathcal{G}$, x *be a point of* $\mathcal{G}$, $R_1(x)$ *be the subgroup in* R *generated by the elements* $\varphi(y)$ *taken for all points* y *collinear to* x, $\overline{R}_1(x) = R_1(x)/\varphi(x)$. *Let*

$$\varphi_x : u \mapsto \varphi^*(u)/\varphi^*(x)$$

for $u \in \Pi_x$. *Then* $(\overline{R}_1(x), \varphi_x)$ *is a representation of* $\mathrm{res}_{\mathcal{G}}(x)$. *Furthermore, let* G *be an automorphism group of* $\mathcal{G}$ *such that* (R, φ) *is* G*-admissible and let* $\overline{G}(x)$ *be the action which* $G(x)$ *induces on* $\mathrm{res}_{\mathcal{G}}(x)$, *then* $(\overline{R}_1(x), \varphi_x)$ *is* $\overline{G}(x)$*-admissible.*

Proof. For $y \in \Pi_x$ the order of $\varphi^*(y)/\varphi^*(x)$ is at most 2 and hence the condition (R2) (cf. Section 2.1) is satisfied. Let $\pi \in L_x$ (a plane in $\mathcal{G}$ containing x), l_1, l_2, l_3 be the lines in $\mathcal{G}$ incident to both x and π, and $y_i \in l_i \setminus \{x\}$ for $1 \leq i \leq 3$ be such points that $\{y_1, y_2, y_3\}$ is a line of $\mathcal{G}$, then $\varphi(y_1)\varphi(y_2)\varphi(y_3) = 1$, which implies (R3). □

The above result possesses the following reformulation in terms of the collinearity graph Γ of $\mathcal{G}$.

Lemma 2.6.3 *Let (R, φ) be a representation of $\mathcal{G}$ which is G-admissible for an automorphism group G of $\mathcal{G}$, let Γ be the collinearity graph of $\mathcal{G}$, let x be a point and $\overline{G}(x)$ be the action induced by $G(x)$ on* $\text{res}_{\mathcal{G}}(x)$,

$$R_1(x) = \langle \varphi(y) \mid y \in \Gamma(x) \rangle,$$

$R_0(x) = \langle \varphi(x) \rangle$, $\overline{R}_1(x) = R_1(x)/R_0(x)$. *Then $\overline{R}_1(x)$ is a $\overline{G}(x)$-admissible representation group of* $\text{res}_{\mathcal{G}}(x)$. □

Let us repeat the definition of the mapping φ_x that turns $\overline{R}_1(x)$ into a representation group:

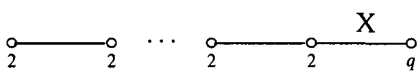
$$\varphi_x : l \mapsto \varphi(y_1)R_0(x) = \varphi(y_2)R_0(x),$$

where $l = \{x, y_1, y_2\}$ is a point of $\text{res}_{\mathcal{G}}(x)$ which is a line of $\mathcal{G}$ containing x.

Suppose that $\mathcal{G}$ belongs to a string diagram and the residue of an element of type n (the rightmost on the diagram) is the projective space $pg(n-1, 2)$ of rank $n-1$ over $GF(2)$ (this is the case when $\mathcal{G}$ is a P- or T-geometry) and G is a flag-transitive automorphism group of $\mathcal{G}$. If (R, φ) is a non-trivial G-admissible representation (i.e., $R \neq 1$) then $\varphi(u)$ is abelian of order 2^i whenever u is an element of type i in $\mathcal{G}$.

2.7 c-extensions

Let $\mathcal{G}$ be a geometry of rank $n \geq 2$ with its diagram of the form



(in particular, $\mathcal{G}$ can be a P- or T-geometry), and let G be a flag-transitive automorphism group of $\mathcal{G}$. Let (R, φ) be a G-admissible representation of $\mathcal{G}$. Suppose that the representation is non-trivial in the sense that the order of R is not 1. Then it follows from the flag-transitivity (already from the point-transitivity) that φ maps the point-set of $\mathcal{G}$ into the set of involutions in R. Let us extend φ to a mapping φ^* from the element-set of $\mathcal{G}$ into the set of subgroups in R as we did in Section 2.6 (i.e., for $x \in \mathcal{G}$ define $\varphi^*(x)$ to be the subgroup generated by the involutions $\varphi(p)$ taken for all points p incident to x). Since (R, φ) is G-admissible, for an element x of type $1 < i \leq n$ in $\mathcal{G}$ the pair $(\varphi^*(x), \varphi_x)$ is a $G(x)$-admissible representation of $\mathrm{res}^-_{\mathcal{G}}(x)$, where φ_x is the restriction of φ to the set of points incident to x. Since $\mathrm{res}^-_{\mathcal{G}}(x)$ is the $GF(2)$-projective geometry of rank $i-1$, it is clear (compare (3.1.2)) that $\varphi^*(x)$ is elementary abelian of order 2^i.

Definition 2.7.1 *In the above terms the representation* (R, φ) *is* separable *if* $\varphi^*(x) = \varphi^*(y)$ *implies* $x = y$ *for all* $x, y \in \mathcal{G}$.

Suppose that the representation (R, φ) is separable. Then we can identify every element $x \in \mathcal{G}$ with its image $\varphi^*(x)$ so that the incidence relation is via inclusion. Define a geometry $\mathcal{AF}(\mathcal{G}, R)$ of rank $n+1$ by the following rule. The elements of type 1 are the elements of R (also considered as the right cosets of the identity subgroup) and for $j > 1$ the elements of type j are all the right cosets of the subgroups $\varphi^*(x)$ for all elements x of type $j-1$ in $\mathcal{G}$; the incidence relation is via inclusion.

Proposition 2.7.2 *The following assertions hold:*

(i) *$\mathcal{AF}(\mathcal{G}, R)$ is a geometry with the diagram*

(ii) *the residue of an element of type 1 in $\mathcal{AF}(\mathcal{G}, R)$ is isomorphic to $\mathcal{G}$;*

(iii) *the semidirect product $H := R : G$ with respect to the natural action (by multiplication from the right) is a flag-transitive automorphism group of $\mathcal{AF}(\mathcal{G}, R)$;*

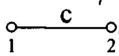
(iv) *if $(\widetilde{R}, \widetilde{\varphi})$ is another representation of $\mathcal{G}$ and*

$$\chi : \widetilde{R} \to R$$

is a representation homomorphism, then χ induces a 2-covering

$$\psi : \mathcal{AF}(\mathcal{G}, \widetilde{R}) \to \mathcal{AF}(\mathcal{G}, R).$$

Proof. Let α be the element of type 1 in $\mathcal{H} = \mathcal{AF}(\mathcal{G}, R)$ which is the identity element of R. Then the elements of $\mathcal{H}$ incident to α are exactly the subgroups $\varphi(x)^*$ representing the elements of $\mathcal{G}$. Since (R, φ) is separable, this shows that $\mathrm{res}_{\mathcal{H}}(\alpha) \cong \mathcal{G}$. Clearly $R : G$ (and even R) acts transitively on the set of elements of type 1 in $\mathcal{H}$ and hence (ii) follows.

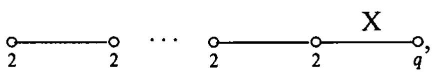
It follows from the definition that if X_i and X_j are incident elements in $\mathscr{H}$ of type i and j, respectively, with $i < j$ then $X_i \subset X_j$. This shows that every maximal flag contains an element of type 1 and also that $\mathscr{H}$ belongs to a string diagram. Let γ be an element of type 3 in $\mathscr{H}$ (without loss of generality we assume that $\gamma = \varphi^*(l)$ where l is a line in $\mathscr{G}$.) Since (R, φ) is separable, γ is elementary abelian of order 4. Now the elements of type 1 and 2 in $\mathscr{H}$ incident to γ are the elements of $\varphi^*(l)$ and the cosets of the subgroups of order 2 in $\varphi^*(l)$, respectively. Clearly this is the geometry with the diagram , so (i) follows. And (iii) follows directly from the definition of $\mathscr{H}$. For a homomorphism χ as that in (iv) define a morphism ψ of $\widetilde{\mathscr{H}} = \mathscr{AF}(\mathscr{G}, \widetilde{R})$ onto $\mathscr{H}$ by

$$\psi(\widetilde{\varphi}^*(x)\widetilde{r}) = \varphi^*(x)\,\chi(\widetilde{r}),$$

where $x \in \mathscr{G}$ and $\widetilde{r} \in \widetilde{R}$. Then it is easy to see from the above that ψ is a 2-covering (furthermore, ψ is an isomorphism when restricted to the residue of an element of type 1). □

A geometry with the same diagram as that in (2.7.2 (i)) in which the residue of an element of type 1 is isomorphic to $\mathscr{G}$ will be called a *c-extension* of $\mathscr{G}$; the geometry $\mathscr{AF}(\mathscr{G}, R)$ will be said to be an *affine c-extension* of $\mathscr{G}$.

Proposition 2.7.3 *Let $\mathscr{G}$ be a geometry with the diagram*



such that

(i) *the number of lines passing through a point is odd.*

Let $\mathscr{H}$ be a c-extension of $\mathscr{G}$ and H be a flag-transitive automorphism group of $\mathscr{H}$ such that

(ii) *any two elements of type 1 in $\mathscr{H}$ are incident to at most one common element of type 2;*

(iii) *H contains a normal subgroup R which acts regularly on the set of elements of type 1 in $\mathscr{H}$;*

(iv) *if $\{x_1, y_1\}$ is a pair of elements of type 1 in $\mathscr{H}$ incident to an element of type 2 then y_1 is the only element of type 1 incident with x_1 to a common element of type 2, which is stabilized by $H(x_1) \cap H(y_1)$.*

Then R is a representation group of $\mathscr{G}$. If in addition R is separable then $\mathscr{H} \cong \mathscr{AF}(\mathscr{G}, R)$.

Proof. Let α be an element of type 1 in $\mathscr{H}$. Then by (ii) there is a bijection ν of the point-set of $\mathscr{G}$ onto the set of elements of type 1 in $\mathscr{H}$ incident with α to a common element of type 2. For a point p of $\mathscr{G}$ let r_p be the unique element in R which maps α onto $\nu(p)$.

Claim 1: r_p is an involution.

It is clear that $H(\alpha) \cap H(\nu(p))$ centralizes r_p and hence it fixes elementwise the orbit of α under r_p. By (iv) this means that the image of α under $(r_p)^{-1}$ must be $\nu(p)$. Since r_p acts regularly on the set of elements of type 1 in $\mathscr{H}$, the claim follows.

Let β denote the unique element of type 2 incident to both α and $\nu(p)$.

Claim 2: r_p fixes $\mathrm{res}^+_{\mathscr{H}}(\beta)$ elementwise.

By Claim 1, r_p is an involution which commutes with

$$H(\beta) = \langle H(\alpha) \cap H(\nu(p)), r_p \rangle,$$

while $H(\beta)$ acts transitively on the set Ξ of elements of type 3 in $\mathscr{H}$ incident to β. By (i) the number of elements in Ξ is odd and hence the claim follows.

Claim 3: If $\{p, s, t\}$ is the point-set of a line l in $\mathscr{G}$, then $r_p r_s r_t = 1$.

Let γ be the element of type 3 in $\mathscr{H}$ that corresponds to l. Then by Claim 2 $\langle r_p, r_s, r_t \rangle$ is contained in $H(\gamma)$ and clearly it induces an elementary abelian group of order 4 on the set of four elements of type 1 incident to γ. Hence $r_p r_s r_t$ fixes each of these four elements. By (iii) the claim follows.

Thus if we put $\varphi : p \mapsto r_p$ then, by the above presentation, (R, φ) is a representation of $\mathscr{G}$. The last sentence in the statement of the proposition is rather clear. □

In certain circumstances the geometry $\mathscr{AF}(\mathscr{G}, R)$ possesses some further automorphisms. Indeed, suppose that in terms of (2.7.2) the representation group R is a covering group of G, i.e., that

$$G \cong \overline{R} := R/Z(R).$$

Let $\nu : r \mapsto \overline{r}$ be the natural homomorphism of R onto $\overline{R}$. Then the group H as in (2.7.2 (iii)) possesses a subgroup other than R which also acts regularly on the point-set of $\mathscr{AF}(\mathscr{G}, R)$. Indeed, in the considered situation we have

$$H = \{(r_1, \overline{r}_2) \mid r_1, r_2 \in R\}$$

with the multiplication

$$(r_1, \bar{r}_2) \cdot (r'_1, \overline{r'_2}) = (r_1 r_2 r'_1 r_2^{-1}, \overline{r_2 r'_2})$$

and it is straightforward to check that

$$S = \{(r, \overline{r^{-1}}) \mid r \in R\}$$

is a normal subgroup in H, isomorphic to R. Furthermore, $S \cap R = Z(R)$, $[R, S] = 1$ and $RS = H$. This shows that H is the central product of R and S. Thus S acts regularly on the point-set of $\mathscr{AF}(\mathscr{G}, R)$ and the geometry can be described in terms of cosets of certain subgroups in S (compare (2.7.3)). In particular the automorphism of H which swaps the two central product factors R and S is an automorphism of $\mathscr{AF}(\mathscr{G}, R)$ and we obtain the following.

Lemma 2.7.4 *In terms of* (2.7.2) *suppose that* $G \cong \overline{R} := R/Z(R)$. *Then* $\widehat{H} = (R * R).2$ *(the central product of two copies of* R *extended by the automorphism which swaps the central factors) is an automorphism group of* $\mathscr{AF}(\mathscr{G}, R)$. □

The situation in (2.7.4) occurs when $\mathscr{G}$ is isomorphic to $\mathscr{G}(J_4)$, $\mathscr{G}(BM)$ or $\mathscr{G}(M)$ and R is the universal representation group of $\mathscr{G}$ isomorphic to J_4, $2 \cdot BM$ or M, respectively. It is not difficult to show that in each of the three cases $\widehat{H} = (R * R).2$ is the full automorphism group of $\mathscr{AF}(\mathscr{G}, R)$.

The following results were established in [FW99] and [StW01].

Proposition 2.7.5 *Let* $\mathscr{G}$ *be a flag-transitive* P*-geometry of rank* n*, such that either* $n = 3$ *and* $\mathscr{G} = \mathscr{G}(M_{22})$ *or* $n \geq 4$ *and every rank 3 residual* P*-geometry in* $\mathscr{G}$ *is isomorphic to* $\mathscr{G}(M_{22})$. *Let* $\mathscr{H}$ *be a non-affine flag-transitive, simply connected c-extension of* $\mathscr{G}$ *and* H *be the automorphism group of* $\mathscr{H}$. *Then one of the following holds:*

(i) $n = 3$ *and* $H \cong 2 \cdot U_6(2).2$;

(ii) $n = 3$ *and* $H \cong M_{24}$;

(iii) $n = 4$, $\mathscr{G} = \mathscr{G}(M_{23})$ *and* $H \cong M_{24}$;

(iv) $n = 4$, $\mathscr{G} = \mathscr{G}(Co_2)$ *and* $H \cong Co_1$;

(v) $n = 5$, $\mathscr{G} = \mathscr{G}(BM)$ *and* $H \cong M$. □

The geometry $\mathscr{H}$ in (2.7.5 (iii)) possesses the following description in terms of the $S(5, 8, 24)$-Steiner system $(\mathscr{P}, \mathscr{B})$ (where $\mathscr{T}$ is the set of trios) (cf. Subsection 1.1 in [StW01]):

$$\begin{aligned}
\mathscr{H}^1 &= \mathscr{P},\\
\mathscr{H}^2 &= \{\{p_1,p_2\} \mid p_1,p_2 \in \mathscr{P}\},\\
\mathscr{H}^3 &= \{\{p_1,p_2,p_3,p_4\} \mid p_i \in \mathscr{P}, p_i \neq p_j \text{ for } i \neq j\},\\
\mathscr{H}^4 &= \{(B_1,\{B_2,B_3\}) \mid \{B_1,B_2,B_3\} \in \mathscr{T}\},\\
\mathscr{H}^5 &= \mathscr{B}.
\end{aligned}$$

Incidences between elements of types 1, 2 and 3 are by inclusion. An element $p \in \mathscr{H}^1$ is incident to an element $(B_1,\{B_2,B_3\}) \in \mathscr{H}^4$ if $p \in B_1$ and to $B \in \mathscr{H}^5$ if $p \notin B$. Elements $x \in \mathscr{H}^2 \cup \mathscr{H}^3$ and $y \in \mathscr{H}^4 \cup \mathscr{H}^5$ are incident if all elements of x are incident to y. The elements of type 5 in $\mathrm{res}_{\mathscr{H}}(x)$ for $x = (B_1,\{B_2,B_3\}) \in \mathscr{H}^4$ are B_2 and B_3.

Proposition 2.7.6 *Let $\mathscr{G}$ be a flag-transitive T-geometry of rank n such that either $n = 3$ and $\mathscr{G} = \mathscr{G}(M_{24})$ or $n \geq 4$ and every rank 3 residual T-geometry in $\mathscr{G}$ is isomorphic to $\mathscr{G}(M_{24})$. Then every flag-transitive c-extension of $\mathscr{G}$ is affine.* □

2.8 Non-split extensions

In this section we show that certain extensions of a representation group by a group of order 2 lead to larger representation groups. Notice that if G is an automorphism group of a geometry $\mathscr{G}$ and (R,φ) is a G-admissible representation of $\mathscr{G}$ then the action of G on the point set Π defines a homomorphism of G into the automorphism group of R and if the action is faithful and φ is injective, then the homomorphism is also injective.

Lemma 2.8.1 *Let $\mathscr{S} = (\Pi, L)$ be a point-line incidence system with 3 points on a line, G be an automorphism group of $\mathscr{S}$ that acts transitively on Π and on L, and (R,φ) be a G-admissible representation of $\mathscr{S}$. Let $\widetilde{R}$ be a group, possessing a homomorphism χ onto R with kernel K of order 2. Let*

$$\Phi = \{r \in \widetilde{R} \mid \chi(r) = \varphi(x) \text{ for some } x \in \Pi\}$$

(so that $|\Phi| = 2|\varphi(\Pi)|$). Suppose that the following conditions hold:

(i) *there is a subgroup $\widetilde{G}$ in $\mathrm{Aut}\,\widetilde{R}$ which centralizes K and whose induced action on R coincides with G;*
(ii) *$\widetilde{G}$ has two orbits say Φ_1 and Φ_2 on Φ;*
(iii) *there are no $\widetilde{G}$-invariant complements to K in $\widetilde{R}$.*

For $i = 1,2$ let φ_i be the mapping of Π onto Φ_i such that $\varphi(x) = \chi(\varphi_i(x))$ for every $x \in \Pi$. Then for exactly one $i \in \{1,2\}$ the pair $(\tilde{R}, \varphi_i)$ is a representation of $\mathscr{S}$.

Proof. Let κ be the generator of K and for $i = 1,2$ let φ_i be as defined above. Then for every $x \in \Pi$ we have $\varphi_2(x) = \varphi_1(x)\kappa$. Let $l = \{x, y, z\}$ be a line from L and $\pi_i(l) = \varphi_i(x)\varphi_i(y)\varphi_i(z)$. Since (R, φ) is a representation of $\mathscr{S}$ and κ is the unique non-identity element in the kernel of the homomorphism of $\tilde{R}$ onto R, $\pi_i(l) \in \{1, \kappa\}$ and $\pi_2(l) = \pi_1(l)\kappa$. Since the action of $\tilde{G}$ (with K being the kernel) is transitive on the set of lines, $\pi_i(l)$ is independent of the choice of l. Finally by (iii) Φ_i generates $\tilde{R}$ for $i = 1,2$ and the proof follows. □

Notice that the condition (ii) in (2.8.1) always holds when the stabilizer in $\tilde{G}$ of a point from Π does not have subgroups of index 2. In view of this observation (2.8.1) can be used for calculating the first cohomology groups of certain modules. First recall a standard result (cf. Section 17 in [A86]).

Proposition 2.8.2 *Let G be a group, V be a $GF(2)$-module for G and V^* be the module dual to V. Let V^u be the largest indecomposable extension of V by trivial modules (i.e., such that $[G, V^u] \leq V$ and $C_{V^u}(G) = 0$) and V^d be the largest indecomposable extension of a trivial module by V (i.e., such that $[V^d, G] = V^d$ and $V^d/C_{V^d}(G) \cong V$). Then* $\dim V^u/V = H^1(G, V)$ *and* $\dim C_{V^d}(G) = H^1(G, V^*)$*, here $H^1(G, V)$ is the first cohomology group of the G-module V.* □

We illustrate the calculating method of the first cohomology by the following example (for further examples see (8.2.7)).

Lemma 2.8.3 *Let $P \sim 2_+^{1+20} : U_6(2)$ be a maximal parabolic subgroup associated with the action of the Lie type group ${}^2E_6(2)$ on its F_4-building. Let $U = P/O_2(P)$ and $W = O_2(P)/Z(P)$ so that W is a 20-dimensional $GF(2)$-module for $U \cong U_6(2)$. Then* $\dim H^1(U, W) = 2$.

Proof. The commutator mapping on $O_2(P)$ defines a bilinear form on W, invariant under U, so that W is self-dual. Hence by (2.8.2) $\dim H^1(U, W) = \dim H^1(U, W^*)$ is equal to the dimension of the centre of the largest indecomposable extension of a trivial module by W. By (3.7.7), W is a representation module of the dual polar space $\mathscr{D} = \mathscr{D}_4(3)$ of U, and by (3.7.5) the universal representation module of $\mathscr{D}$ is 22-dimensional. Hence the kernel of the homomorphism of $V(\mathscr{D})$ onto

W is 2-dimensional and clearly U acts trivially on this kernel. By (2.1.1) $V(\mathcal{D})$ does not possess 1-dimensional factor modules and hence $V(\mathcal{D})$ is an indecomposable extension of a trivial modules by W and $\dim H^1(U, W) \geq 2$. On the other hand, the stabilizer in U of a point from $\mathcal{D}$ (isomorphic to $2^9 : L_3(4)$) does not have subgroups of index 2. By (2.8.1) this means that whenever V is a U-admissible representation module of $\mathcal{D}$ and $\widetilde{V}$ is an indecomposable extension of a trivial module by V, then $\widetilde{V}$ is also a representation module of $\mathcal{D}$ (we consider a sequence $V_0 = V$, $V_1,...,V_m = \widetilde{V}$, where V_i is an indecomposable extension by V_{i-1} of the trivial 1-dimensional module for $1 \leq i \leq m$ and argue inductively). Hence the proof follows. □

3
Classical geometries

In this chapter we study representations of the classical geometries of $GF(2)$-type and of the tilde geometries of symplectic type (the representations of the latter geometries were originally calculated in [Sh95]). In Section 3.7 we discuss the recent results which led to the proof of Brouwer's conjecture on the universal abelian representations of the dual polar spaces of $GF(2)$-type.

3.1 Linear groups

Let $V = V_n(2)$ be an n-dimensional $GF(2)$-space, $n \geq 1$. Let $\mathcal{L} = \mathcal{G}(L_n(2))$ be the projective geometry of V: the elements of $\mathcal{L}$ are the proper subspaces of V, the type of a subspace is its dimension and the incidence relation is via inclusion. The rank of $\mathcal{L}$ is $n - 1$ and the diagram is

The isomorphism between V and the dual V^* of V which is the space of linear functions on V performs a diagram automorphism of $\mathcal{L}$. We identify a point of $\mathcal{L}$ (which is a 1-subspace in V) with the unique non-zero element it contains.

The following classical result (cf. [Sei73] or Theorem 1.6.5 in [Iv99]) is quite important.

Lemma 3.1.1 *Suppose that G is a flag-transitive automorphism group of $\mathcal{G}(L_n(2))$, $n \geq 3$. Then one of the following holds:*

(i) $G \cong L_n(2)$;

(ii) $n = 3$ *and* $G \cong Frob_7^3$ *(the Frobenius group of order 21);*

(iii) $n = 4$ *and* $G \cong Alt_7$.

In anyone of these cases the action of G on V is irreducible. □

Lemma 3.1.2 *If (R, φ) is the universal representation of $\mathscr{L}$, then $R \cong V$. Furthermore (R, φ) is the unique G-admissible representation for a flag-transitive automorphism group of $\mathscr{L}$.*

Proof. We turn R into a $GF(2)$-vector space by defining the addition $*$ via

$$\varphi(x) * \varphi(y) = \varphi(x + y)$$

for $x, y \in \mathscr{L}^1$. The last sentence follows from that in (3.1.1). □

There are further point-line incidence systems with three points on a line associated with $\mathscr{L}$. As usual let $\mathscr{L}^i$ be the set of elements of type i in $\mathscr{L}$ (the i-subspaces). Let x and y be incident elements of type k and l, respectively, where $0 \le k < i < l \le n$ (if $k = 0$ then x is assumed to be the zero subspace and if $l = n$ then y is assumed to be the whole space V). The set of elements in $\mathscr{L}^i$ incident to both x and y is said to be a (k, l)-flag in $\mathscr{L}^i$. Let $\Phi^i(k, l)$ be the set of all (k, l)-flags in $\mathscr{L}^i$. Clearly the size of a (k, l)-flag is equal to the number of $(i - k)$-subspaces in an $(l - k)$-space.

Thus an $(i-1, i+1)$-flag in $\mathscr{L}^i$ has size 3 and hence $(\mathscr{L}^i, \Phi^i(i-1, i+1))$ is a point-line incidence system with three points on a line. In these terms the point-line incidence system of $\mathscr{L}$ is just $(\mathscr{L}^1, \Phi^1(0, 2))$.

Lemma 3.1.3 *Let (R^i, φ) be the universal abelian representation of the point-line incidence system $(\mathscr{L}^i, \Phi^i(i - 1, i + 1))$. Then R^i is isomorphic to the i-th exterior power $\bigwedge^i V$ of V.*

Proof. We define a mapping ψ from the set of i-subsets of vectors in V onto R^i which sends a linearly dependent set onto zero, otherwise

$$\psi(\{x_1, ..., x_i\}) \mapsto \varphi(\langle x_1, ..., x_i \rangle).$$

Let $\{x_1, ..., x_{i-1}, x_i\}$ and $\{x_1, ..., x_{i-1}, x_i'\}$ be linearly independent i-subsets, where $x_i' \notin \langle x_1, ..., x_{i-1}, x_i \rangle$. Then $\langle x_1, ..., x_{i-1} \rangle$ and $\langle x_1, ..., x_{i-1}, x_i, x_i' \rangle$ are incident elements from $\mathscr{L}^{i-1}$ and $\mathscr{L}^{i+1}$, respectively. Hence

$$\varphi(\langle x_1, ..., x_{i-1}, x_i \rangle) + \varphi(\langle x_1, ..., x_{i-1}, x_i' \rangle) = \varphi(\langle x_1, ..., x_{i-1}, x_i + x_i' \rangle)$$

and this is all we need in order to define the exterior space structure on R^i. □

The above lemma is equivalent to the fact that the permutation module of $L_n(2)$ acting on the set of i-dimensional subspaces in the natural module V, factored over the subspace spanned by the lines from $(\mathscr{L}^i, \Phi^i(i - 1, i + 1))$, is isomorphic to $\bigwedge^i V$.

In what follows we will need some standard results on the $GF(2)$-permutation module of $PGL_3(4)$ acting on the set of 1-dimensional subspaces of the natural module $V_3(4)$ (cf. [BCN89]).

Lemma 3.1.4 *Let V be a 3-dimensional $GF(4)$-space, Ω be the set of 1-subspaces in V (so that Ω is of size 21) on which $GL(V)$ induces the doubly transitive action of $G \cong PGL(3,4)$. Let W be the power space of Ω (the $GF(2)$-permutation module of (G,Ω)). Then*

(i) *$W = W^1 \oplus W^e$, where $W^1 = \{\emptyset, \Omega\}$ and W^e consists of the even subsets of Ω;*

(ii) *W^e possesses a unique composition series*

$$0 < T_1 < T_2 < W^e,$$

where

(a) *T_1 is the 9-dimensional Golay code module for G (isomorphic to the module of Hermitian forms on V) and $T_1 \oplus W^1$ is generated by the 2-dimensional subspaces in V (considered as 5-element subsets of Ω);*

(b) *W^e/T_2 is dual to T_1;*

(c) *T_2/T_1 is 2-dimensional with kernel $G' \cong PSL(3,4)$.* □

3.2 The Grassmanian

The characterization (3.1.3) of the exterior powers of V can be placed into the following context.

Let $\mathscr{P}^i$ be the power space of $\mathscr{L}^i$ that also can be considered as the $GF(2)$-permutation module of $L_n(2)$ acting on the set $\mathscr{L}^i$ of i-subspaces in V.

For $0 \le j \le i \le n$ define the *incidence map*

$$\psi_{ij} : \mathscr{P}^i \to \mathscr{P}^j$$

by the following rule: if $w \in \mathscr{L}^i$ then $\psi_{ij}(w)$ is the set of j-subspaces contained in w and ψ_{ij} is extended on the whole $\mathscr{P}^i$ by linearity.

Lemma 3.2.1 *Let $0 \le j \le k \le i \le n$. Then ψ_{ij} is the composition of ψ_{ik} and ψ_{kj}.*

Proof. Let $w \in \mathscr{L}^i$ and $u \in \mathscr{L}^j$. Then $u \in \psi_{ij}(w)$ if and only if there is a k-subspace t containing u and contained in w (i.e., $u \in \psi_{kj}(t)$ and

$t \in \psi_{ik}(w)$). If the number of such subspaces t is non-zero, it equals to 1 modulo 2. Hence the proof. □

The above lemma implies the following inclusions:

$$\mathscr{P}^j = \operatorname{Im}\psi_{jj} \geq \operatorname{Im}\psi_{j+1j} \geq \dots \geq \operatorname{Im}\psi_{nj} = \{\emptyset, \mathscr{L}^j\}$$

and we can consider the mapping

$$\Psi_{ij} : \mathscr{P}^i \to \mathscr{P}^j / \operatorname{Im}\psi_{i+1j}$$

induced by ψ_{ij} (here we assume that $1 \leq j \leq i \leq n-1$).

Lemma 3.2.2 *If $\Delta \in \Phi^i(i-j, i+1)$, then $\Delta \in \ker \Psi_{ij}$.*

Proof. We have to show that $\psi_{ij}(\Delta) \in \operatorname{Im}\psi_{i+1,j}$. Let (x, y) be the $(i-j, i+1)$-flag in $\mathscr{L}$ such that

$$\Delta = \{z \mid z \in \mathscr{L}^i, x \leq z \leq y\}.$$

We claim that $\psi_{ij}(\Delta) = \psi_{i+1,j}(y)$. If $u \in \psi_{ij}(\Delta)$, then u is contained in some $w \in \Delta$, hence u is also contained in y and belongs to $\psi_{i+1,j}(y)$. On the other hand suppose that $u \in \psi_{i+1,j}(y)$, which means that u is a j-subspace in y. Let v be the subspace in y generated by u and x. Then

$$i - j \leq \dim v \leq \dim u + \dim x = i.$$

Since the number of i-subspaces from Δ containing v is odd, $u \in \psi_{ij}(\Delta)$ and the proof follows. □

In 1996 at a conference in Montreal the first author posed the following conjecture.

Conjecture 3.2.3 *If $1 \leq j \leq i \leq n-1$ then the flags from $\Phi^i(i-j, i+1)$ generate the kernel of Ψ_{ij}.*

Let $\mathscr{P}^i(j)$ be the quotient of $\mathscr{P}^i$ over the subspace generated by the flags from $\Phi^i(i-j, i+1)$. The following observation can be easily deduced from (3.2.1).

Lemma 3.2.4 *For a given j the conjecture* (3.2.3) *is equivalent to the equality*

$$\sum_{i=j}^{n} \dim \mathscr{P}^i(j) = \dim \mathscr{P}^j$$

(where $\dim \mathscr{P}^j$ is $\left[{n \atop j}\right]_2$). □

Lemma 3.2.5 *The conjecture* (3.2.3) *holds for* $j = 1$.

Proof. By (3.1.3) $\mathscr{P}^i(1)$ is the i-th exterior power of V which has dimension $\binom{n}{i}$. Since

$$\sum_{i=1}^{n} \binom{n}{i} = 2^n - 1 = \dim \mathscr{P}^1,$$

the proof follows from (3.2.4). □

The next case turned out to be much more complicated. It was accomplished in [Li01] (using some results and methods from [McC00]) and implies Brouwer's conjecture discussed in Section 3.7.

Proposition 3.2.6 *The conjecture* (3.2.3) *holds for* $j = 2$. □

In Part II of this volume we will make use of the submodule structure of $\mathscr{P}^1$ and of the information on the first and second degree cohomologies of modules $\bigwedge^i V$.

Recall that $\mathscr{P}^1$ is the $GF(2)$-permutation module of $L_n(2)$ on the set of the 1-dimensional submodules in V. Let $\mathscr{P}^1_c = \operatorname{Im} \psi_{n1} = \{\emptyset, \mathscr{L}^1\}$ be the subspace of constant functions, $\mathscr{P}^1_e$ be the subspace of functions with even support and put

$$\mathscr{X}(i) = \mathscr{P}^1_e \cap \operatorname{Im} \psi_{i1}$$

for $1 \leq i \leq n$. Then $\mathscr{X}(i)/\mathscr{X}(i+1) \cong \mathscr{P}^i(1)$ is isomorphic to $\bigwedge^i V$ (cf. (3.1.3) and (3.2.5)) for $1 \leq i \leq n-1$.

We summarize this in the following

Lemma 3.2.7 *The following assertions hold:*

(i) $\mathscr{P}^1 = \mathscr{P}^1_c \oplus \mathscr{P}^1_e$ *as a module for* $L_n(2)$;

(ii) $\mathscr{P}^1_e = \mathscr{X}(1) > \mathscr{X}(2) > \ldots > \mathscr{X}(n-1) > \mathscr{X}(n) = 0$ *is a composition series for* $\mathscr{P}^1_e$;

(iii) $\mathscr{X}(i)/\mathscr{X}(i+1) \cong \bigwedge^i V$, $1 \leq i \leq n-1$ *are the composition factors of* $\mathscr{P}^1_e$. □

In the next section we show that the composition series in (3.2.7 (ii)) is the unique one.

3.3 $\mathscr{P}^1_e$ is uniserial

We analyze the subspace in $\mathscr{P}^1_e$ formed by the vectors fixed by a Sylow 2-subgroup B of $L_n(2)$. As above we identify every 1-subspace from $\mathscr{L}^1$

with the unique non-zero vector of V it contains and treat $\mathscr{P}^1$ as the power space of $\mathscr{L}^1$ with addition performed by the symmetric difference operator. Then $\mathscr{P}^1_e$ consists of the subsets of even size.

Since B is a Borel subgroup associated with the action of $L_n(2)$ on the projective geometry $\mathscr{G}(L_n(2))$ of V the subgroup B is the stabilizer of a uniquely determined maximal flag Φ:

$$0 = V_0 < V_1 ... < V_{n-1} < V_n = V,$$

where $\dim V_i = i$ for $0 \le i \le n$. The orbits of B on $\mathscr{L}^1$ are the sets $O_i = V_i \setminus V_{i-1}$, $1 \le i \le n$. Furthermore, $| O_i |= 2^{i-1}$, so that all the orbits except for O_1 (which is of size 1) have even length. This gives the following

Lemma 3.3.1

$$C_{\mathscr{P}^1}(B) = \{F(J) \mid J \subseteq \{1, 2, ..., n\}\},$$

where $F(J) = \bigcup_{i \in J} O_i$ *and* $F(J) \in \mathscr{P}^1_e$ *if and only if* $1 \notin J$. *In particular*

$$\dim C_{\mathscr{P}^1}(B) = n \quad \text{and} \quad \dim C_{\mathscr{P}^1_e}(B) = n - 1.$$

□

Lemma 3.3.2 *Let* W *be an* $L_n(2)$*-submodule in* $\mathscr{P}^1$, *which contains* $F(J)$ *for some* $J \subseteq \{1, 2, ..., n\}$. *If* $i \in J$ *and* $i < n$, *then* W *contains*

$$F(J \cup \{i + 1\}).$$

Proof. We can certainly assume that $i + 1 \notin J$. Let V_i, $U_i^{(1)}$ and $U_i^{(2)}$ be the distinct i-subspaces containing V_{i-1} and contained in V_{i+1}. Then $O_i \cup O_{i+1} = V_{i+1} \setminus V_{i-1}$ is the disjoint union of

$$O_i = V_i \setminus V_{i-1}, \quad U_i^{(1)} \setminus V_{i-1} \quad \text{and} \quad U_i^{(2)} \setminus V_{i-1}.$$

For $\alpha = 1$ or 2 let $g^{(\alpha)}$ be an element in $L_n(2)$ which stabilizes the premaximal flag $\Phi \setminus V_i$ and maps V_i onto $U_i^{(\alpha)}$ (such an element can be found in the minimal parabolic of type i). Then

$$F(J) \cup F(J)^{g^{(1)}} \cup F(J)^{g^{(2)}} = F(J \cup \{i + 1\})$$

and the result follows. □

Lemma 3.3.3 *Let* $\emptyset \neq J \subseteq \{1, 2, ..., n\}$ *and* $i = \min J$. *Then* $\mathscr{X}(i - 1)$ *is the minimal* $L_n(2)$*-submodule in* $\mathscr{P}^1_e$ *containing* $F(J)$ *and*

$$C_{\mathscr{X}(i-1)}(B) = \{F(K) \mid K \subseteq \{1, 2, ..., n\}, \ \min K \ge i\},$$

in particular $\dim C_{\mathscr{X}(i-1)}(B) = n - i - 1$.

Proof. By (3.3.2) a submodule which contains $F(J)$ also contains $F(J_i)$, where $J_i = \{i, i+1, ..., n\}$. We claim that $\mathscr{X}(i-1)$ is the minimal $L_n(2)$-submodule in $\mathscr{P}_e^1$ which contains $F(J_i)$. Indeed, by the definition $\operatorname{Im}\psi_{i-1,1}$ is generated by the $(i-1)$-subspaces in V (treated as subsets of $\mathscr{L}^1$). Since $\operatorname{Im}\psi_{i-1,1}$ contains $\operatorname{Im}\psi_{n1} = \{\emptyset, \mathscr{L}^1\}$, $\mathscr{X}(i-1)$ is generated by the complements of the $(i-1)$-subspaces, i.e., by the images under $L_n(2)$ of $V \setminus V_{i-1} = F(J_i)$. Hence the claim follows. Since $\mathscr{X}(i-1)$ contains $\mathscr{X}(j-1)$ for every $j \geq i$, $\mathscr{X}(i-1)$ contains $F(J_j)$ for these j, in particular it contains $F(K)$ for all $K \subseteq \{1, 2, ..., n\}$ with $\min K \geq i$. Since $\mathscr{X}(i-1)$ does not contain $\mathscr{X}(j-1)$ for $j < i$, the proof follows. □

We need the following standard result from the representation theory of groups of Lie type in their own characteristic [Cur70], which can also be proved by elementary methods.

Lemma 3.3.4 *The centralizer of B in $\bigwedge^i V$ is 1-dimensional for every* $1 \leq i \leq n-1$. □

Now we are ready to prove the main result of this section.

Proposition 3.3.5 *The only composition series of $\mathscr{P}_e^1$, as a module for $L_n(2)$, is the one in* (3.2.7(*ii*)).

Proof. Let

$$\mathscr{P}_e^1 = W(1) > W(2) > ... > W(m-1) > W(m) = 0$$

be a composition series of $\mathscr{P}_e^1$. Then by (3.3.5) and the Jordan–Hölder theorem $m = n$ and $W(i)/W(i+1) \cong \bigwedge^{\sigma(i)} V$ for a permutation σ of $\{1, 2, ..., n\}$. By (3.3.4) the centralizer of B in each composition factor is 1-dimensional and hence $\dim C_{W(i)}(B) \leq n - i$. Since $\dim C_{\mathscr{P}_e^1}(B) = n-1$ by (3.3.1), we have

$$\dim C_{W(i)}(B) = n - i \quad \text{for} \quad 1 \leq i \leq n.$$

In particular $W(i-1) \setminus W(i)$ contains a vector fixed by B. Let j be the minimal index, such that $W(k) = \mathscr{X}(k)$ for all $k > j$ and suppose that $j \geq 2$. Then by (3.3.3) $W(j) \setminus W(j+1)$ contains a vector $F(J)$ such that $\min J \leq j+1$. By (3.3.2) $W(j)$ contains $F(J_l)$ for some $l \leq j+1$. Hence by (3.3.3) $W(j)$ contains $\mathscr{X}(l)$ for some $l \leq j$. Since $W(j)/W(j+1) \cong W(j)/\mathscr{X}(j+1)$ is irreducible, this gives $W(j) = \mathscr{X}(j)$ contrary to the minimality assumption on j. Hence the result follows. □

3.4 $\mathcal{G}(S_4(2))$

In this section we start by calculating the universal representation module of $\mathcal{G}(S_4(2))$, which turns out to be the universal representation group of this geometry. The treatment is very elementary and we only present it here in order to illustrate the technique we use.

First recall some results from Section 2.5 in [Iv99]. So let $\mathcal{S} = (\Pi, L)$ be the generalized quadrangle $\mathcal{G}(S_4(2))$ of order $(2,2)$. Then Π is the set of 2-subsets in a set Ω of size 6, L is the set of partitions of Ω into three 2-subsets and the incidence relation is via inclusion. Let 2^Ω be the power space of Ω and let $\mathcal{P}(\Omega)^+$ be the codimension 1 subspace in 2^Ω, formed by the subsets of even size. Let

$$\varphi : p \mapsto \Omega \setminus p$$

be the mapping of Π into $\mathcal{P}(\Omega)^+$ (where p is treated as a 2-subset of Ω).

Lemma 3.4.1 *$(\mathcal{P}(\Omega)^+, \varphi)$ is an abelian representation of $\mathcal{S} = \mathcal{G}(S_4(2))$.*

Proof. It is clear that $\mathcal{P}(\Omega)^+$ is generated by $\varphi(\Pi)$ (the set of 4-subsets in Ω). If $\Omega = p_1 \cup p_2 \cup p_3$ is a line in $\mathcal{S}$ then

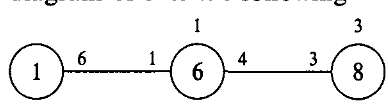
$$\varphi(p_1) = p_2 \cup p_3, \ \varphi(p_2) = p_1 \cup p_3, \ \varphi(p_3) = p_1 \cup p_2$$

and since the addition is performed by the symmetric difference operator,

$$\varphi(p_1) + \varphi(p_2) + \varphi(p_3) = 0$$

and the proof follows. □

Let Γ be the collinearity graph of $\mathcal{S}$, so that Γ is the graph on the set of 2-subsets of Ω in which two such subsets are adjacent if they are disjoint. The suborbit diagram of Γ is the following



Lemma 3.4.2 $\beta(\Gamma) = 5$.

Proof. Since $\mathcal{S}$ is a generalized polygon with lines of size 3, every edge in Γ is contained in a unique triangle that is the point-set of a line and for a point x and a triangle T there is a unique point in T which is nearest to x. In view of these it suffices to notice that the subgraph induced by $\Gamma_1(x)$ is the union of 3 disjoint edges, so $c(\Gamma_1(x)) = 3$ and the subgraph induced by $\Gamma_2(x)$ is connected (isomorphic to the 3-dimensional cube), so that $c(\Gamma_2(x)) = 1$. □

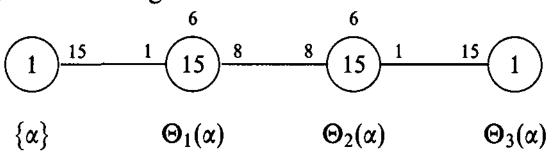
Combining (2.2.1), (3.4.1) and (3.4.2) we obtain the following.

Lemma 3.4.3 *The representation* $(\mathscr{P}(\Omega)^+, \varphi)$ *in* (3.4.1) *is the universal abelian representation.* □

But in fact the following holds.

Lemma 3.4.4 *The representation* $(\mathscr{P}(\Omega)^+, \varphi)$ *is universal.*

Proof. Let $\Theta = Cay(\mathscr{P}(\Omega)^+, \varphi(\Pi))$. Then Θ is a Taylor graph with the following suborbit diagram:



By (2.5.1) our representation is universal if and only if the fundamental group of Θ is generated by the geometric triangles. One can easily see from the above suborbit diagram that every triangle in Θ is a geometric triangle. Thus we have to show that every cycle in Θ is triangulable. Of course it is sufficient to consider non-degenerate cycles and in Θ they are of lengths 4, 5 and 6. To check the triangulability is an elementary exercise. □

In view of (2.3.1) and (2.3.2), by (3.4.3) there are 31 proper geometric hyperplanes in $\mathscr{S}$. These hyperplanes possess a uniform description. If Δ is a subset of Ω then

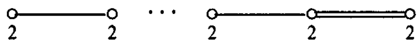
$$H(\Delta) := \{v \mid v \in \Pi,\ |\Delta| = |\Delta \cap v| \ (mod\ 2)\}$$

is a geometric hyperplane in $\mathscr{S}$. Since clearly $H(\Delta) = H(\Omega \setminus \Delta)$, in this we obtain all 31 geometric hyperplanes.

If $|\Delta| = 2$ then $H(\Delta)$ are the points at distance at most 1 from Δ (treated as a point) in the collinearity graph. If $|\Delta| = 1$ then $H(\Delta)$ is stabilized by $Sym_5 \cong O_4^-(2)$ while if $|\Delta| = 3$ then $H(\Delta)$ is stabilized by $Sym_3 \wr Sym_2 \cong O_4^+(2)$.

3.5 Symplectic groups

Let V be a $2n$-dimensional ($n \geq 2$) $GF(2)$-space with a non-singular symplectic form Ψ, then if $\{v_1^1, ..., v_n^1, v_1^2, ..., v_n^2\}$ is a (symplectic) basis, we can put $\Psi(v_i^k, v_j^l) = 1$ if $i = j$ and $k \neq l$ and $\Psi(v_i^k, v_j^l) = 0$ otherwise. The symplectic geometry $\mathcal{G} = \mathcal{G}(S_{2n}(2))$ is the set of all non-zero *totally singular* subspaces U in V with respect to Ψ (i.e., such that $\Psi(u, v) = 0$ for all $u, v \in U$). The type of an element is its dimension and the incidence is via inclusion. The automorphism group $G \cong S_{2n}(2)$ of $\mathcal{G}$ is the group of all linear transformations of V preserving Ψ. The diagram of $\mathcal{G}$ is



Since the points and lines of $\mathcal{G}$ are realized by certain 1- and 2-subspaces in V with the incidence relation via inclusion, we observe that V supports a natural representation of $\mathcal{G}$. We will see below that the universal representation group is abelian twice larger than V.

Let v be a point (a 1-subspace in V which we identify with the only non-zero vector it contains) and

$$v^{\perp} = \{u \in V^{\#} \mid \Psi(v, u) = 0\}$$

be the orthogonal complement of v with respect to Ψ.

The form Ψ induces on $v^{\perp}/v$ (which is a $(2n-2)$-dimensional $GF(2)$-space) a non-singular symplectic form and the totally singular subspaces in $v^{\perp}/v$ constitute the residue $\mathrm{res}_{\mathcal{G}}(v) \cong \mathcal{G}(S_{2n-2}(2))$. The stabilizer $G(v)$ induces $S_{2n-2}(2)$ on $v^{\perp}/v$. The kernel $K(v)$ of this action is an elementary abelian group of order 2^{2n-1}. The kernel $R(v)$ of the action of $G(v)$ on $v^{\perp}$ (on the set of points collinear to v) is of order 2 and its unique non-trivial element is the symplectic transvection

$$\tau(v) : u \mapsto u + \Psi(v, u)v.$$

The quotient $K(v)/R(v)$ is the natural symplectic module of $G(v)/K(v) \cong S_{2n-2}(2)$ and $\mathrm{res}_{\mathcal{G}}(v)$ possesses a representation in this quotient by (1.5.1). But in fact $\mathrm{res}_{\mathcal{G}}(v)$ possesses a representation in the whole of $K(v)$ and this representation is universal. We start with the following.

Lemma 3.5.1 *Let (W, φ_a) be the universal abelian representation of $\mathcal{G} \cong \mathcal{G}(S_{2n}(2))$. Then* $\dim W \leq 2n+1$ *and for a point x the dimension of $\overline{W}_2(x)$ is at most* 1.

Proof. We proceed by induction on n. By (3.4.4) the result holds for $n = 2$. Suppose that $n \geq 3$ and that the universal abelian representation of $\mathcal{G}(S_{2n-2}(2))$ is $(2n-1)$-dimensional. Consider the collinearity graph Γ of $\mathcal{G}$ and let v be a vertex. Then $W_0(v)$ is 1-dimensional, $\overline{W}_1(v)$ is at most $(2n-1)$-dimensional by (2.6.3) and the induction hypothesis (recall that

$\text{res}_{\mathcal{G}}(v) \cong \mathcal{G}(S_{2n-2}(2))$). Finally $\overline{W}_2(v)$ is at most 1-dimensional since the subgraph in Γ induced by $\Gamma_2(v)$ is connected (this is a well-known fact and can be established as an easy exercise). Since the diameter of Γ is 2, we have finished. □

By (2.3.2) $|W| - 1$ is equal to the number of geometric hyperplanes in $\mathcal{G}$. Thus we can establish the equality $\dim W = 2n + 1$ by producing a sufficient number of geometric hyperplanes.

So let us discuss the geometric hyperplanes in $\mathcal{G}$. Let q be a point of $\mathcal{G}$. Then for every line l of $\mathcal{G}$ the point q is collinear either to all three or to exactly one point on l. Therefore the set $H(q)$ consisting of q and all the points collinear to q forms a geometric hyperplane in $\mathcal{G}$ (in other terms $H(q) = q^{\perp}$). Since Ψ is non-singular, different points give rise to different hyperplanes and we obtain a family

$$\mathcal{H}^p = \{q^{\perp} \mid q \text{ is a point of } \mathcal{G}\}$$

of $2^{2n} - 1$ (which is the number of points in $\mathcal{G}$) geometric hyperplanes. Clearly G acts transitively on $\mathcal{H}^p$.

The remaining hyperplanes come from the quadratic forms on V associated with Ψ. Recall that a quadratic form f on V is said to be associated with Ψ if

$$\Psi(u, v) = f(u) + f(v) + f(u + v).$$

for all $u, v \in V$. Let $\mathcal{Q}$ denote the set of all quadratic forms on V associated with Ψ. The following result is standard [Tay92].

Lemma 3.5.2 *The group $G \cong S_{2n}(2)$ acting on $\mathcal{Q}$ has two orbits $\mathcal{Q}^+$ and $\mathcal{Q}^-$ with lengths $2^{n-1}(2^n + 1)$ and $2^{n-1}(2^n - 1)$, with stabilizers isomorphic to $O_{2n}^+(2) \cong \Omega_{2n}^+(2).2$ and $O_{2n}^-(2) \cong \Omega_{2n}^-(2).2$, respectively. The action on either of these orbits is doubly transitive.* □

A subspace U in V is said to be totally singular with respect to a quadratic form f (associated with Ψ) if $f(u) = 0$ for all $u \in U$ (in this case it is clearly totally singular with respect to Ψ). Thus the dimension of a totally singular subspace with respect to f (the Witt index $w(f)$) is at most n. In fact $w(f) = n$ if f is of plus type (i.e., if $f \in \mathcal{Q}^+$) and $w(f) = n - 1$ if f is of minus type (i.e., if $f \in \mathcal{Q}^-$).

Lemma 3.5.3 *Let f be a quadratic form on V associated with Ψ and $H(f)$ be the set of non-zero singular vectors with respect to f:*

$$H(f) = \{v \in V^{\#} \mid f(v) = 0\}.$$

Then $H(f)$ (considered as a subset of the point-set) is a geometric hyperplane in $\mathcal{G}(S_{2n}(2))$.

Proof. Let $T = \{x, y, z\}$ be a line in $\mathcal{G}$ (the non-zero vectors of a totally singular 2-subspace). Since $\Psi(x, y) = 0$, $x + y + z = 0$ and f is associated with Ψ, we have

$$f(z) + f(x) + f(y) = 0$$

and hence $|T \cap H(f)|$ is of size 1 or 3 and the proof follows. □

Put

$$\mathcal{H}^{\varepsilon} = \{H(f) \mid f \in \mathcal{Q}^{\varepsilon}\}$$

for $\varepsilon = +$ or $-$. In view of (3.5.2) so far we have seen

$$2^{2n+1} = (2^{2n} - 1) + 2^{n-1}(2^n + 1) + 2^{n-1}(2^n - 1) = |\mathcal{H}^p| + |\mathcal{H}^+| + |\mathcal{H}^-|$$

geometric hyperplanes in $\mathcal{G}$.

Lemma 3.5.4 *In the above terms the following assertions hold.*

(i) $\dim W = 2n + 1$;
(ii) $\{\mathcal{H}^p, \mathcal{H}^+, \mathcal{H}^-\}$ *is the complete set of orbits of* $G \cong S_{2n}(2)$ *on the set of geometric hyperplanes in* $\mathcal{G}$ *with stabilizers isomorphic to* $2^{2n-1} : S_{2n-2}(2)$, $O_{2n}^+(2)$, $O_{2n}^-(2)$, *respectively;*
(iii) $\dim \overline{W}_2(x) = 2$.

Proof. The assertions (i) and (ii) follow from (3.5.1) and the paragraph preceding this lemma. Then (iii) is immediate from the proof of (3.5.1). □

The universal representation module of $\mathcal{G}(S_{2n}(2))$ is the so-called *orthogonal module* of $S_{2n}(2) \cong \Omega_{2n+1}(2)$. Our final result of this section is the following.

Proposition 3.5.5 *The universal representation of $\mathcal{G}(S_{2n}(2))$ is abelian.*

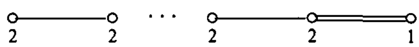
Proof. Let Γ be the collinearity graph of $\mathcal{G} = \mathcal{G}(S_{2n}(2))$, $x, y \in \Gamma$ and (V, φ_u) be the universal representation of $\mathcal{G}$. We have to show that $\varphi_u(x)$ and $\varphi_u(y)$ commute. If x and y are collinear this is clear. Otherwise $d_\Gamma(x, y) = 2$ and there is a vertex, say z, collinear to them both. Again proceeding by induction on n we assume that $\overline{R}_1(z)$ is abelian. Then $[\varphi_u(x), \varphi_u(y)] \in R_0(z)$ since $\overline{R}_1(z) = R_1(z)/R_0(z)$ is abelian by our induction hypothesis. But since two vertices at distance 2 in Γ have more than one common neighbour (this is easy to check), the commutator must be trivial. □

3.6 Orthogonal groups

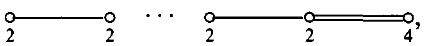
In view of the isomorphism $S_{2n}(2) \cong \Omega_{2n+1}(2)$, the results (3.5.1) and (3.5.5) describe the universal representation of the polar space $\mathcal{P}(\Omega_{2n+1}(2))$ of the odd dimensional orthogonal group over $GF(2)$. In this section we establish similar result in the even dimensional case.

Let V be a $2n$-dimensional $GF(2)$-space, where $n \geq 2$ and f be a non-singular orthogonal form on V. Then the Witt index (the dimension of a maximal totally isotropic subspace) is either n or $n-1$, so that f is of plus or minus type, respectively. The commutator subgroup of the group of linear transformations of V preserving f is $\Omega_{2n}^{+}(2)$ or $\Omega_{2n}^{-}(2)$ depending on whether f is of plus or minus type.

Let $\varepsilon = +$ or $-$ denote the type of f. The corresponding polar space $\mathcal{P} = \mathcal{P}(\Omega_{2n}^{\varepsilon}(2))$ is the geometry whose elements are the subspaces of V that are totally singular with respect to f; the type of an element is its dimension and the incidence relation is via inclusion. Then the rank of $\mathcal{P}$ is the Witt index of f (i.e., n or $n-1$) and the diagram of $\mathcal{P}$ is



or



respectively.

By the definition if φ is the identity mapping then (V, φ) is an abelian representation of $\mathcal{P}$.

Lemma 3.6.1 *The representation (V, φ) is universal.*

Proof. Probably the easiest way to proceed is to follow the strategy of the proof of (3.4.4). So we consider the graph $\Theta = Cay(V, \mathrm{Im}\varphi)$. Then again Θ is a strongly regular graph (in particular, it is of diameter 2) that is locally the collinearity graph of $\mathcal{P}$. Every triangle turns out to be geometric and it is an easy combinatorial exercise to check that Θ is triangulable. □

We summarize the results in this and the previous sections in the following.

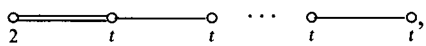
Proposition 3.6.2 *Let V be an m-dimensional $GF(2)$-space and f be a non-singular orthogonal form on X. Let $\mathcal{P}$ be the polar space associated with V and f, and Γ be the collinearity graph of $\mathcal{P}$, and suppose that the rank of $\mathcal{P}$ is at least 2. Then*

(i) (V, φ) *is the universal representation (where φ is the identity mapping);*
(ii) Γ *is of diameter* 2;
(iii) *if p is a point then $\overline{V}_2(p)$ has order* 2. □

3.7 Brouwer's conjecture

In this section we discuss representations of the dual polar spaces with 3 points per line. The question about representations of such dual polar spaces in itself interesting and is also important for the classification of extended dual polar spaces (cf. Theorem 1.13.6 in [Iv99]).

Let $\mathscr{D}_t(n)$ denote the classical dual polar space of rank $n \geq 2$ with 3 points per line and D be the simple subgroup in the automorphism group of $\mathscr{D}_t(n)$. Then $\mathscr{D}_t(n)$ belongs to the diagram



where $t = 2$ or 4 and D is isomorphic to $S_{2n}(2)$ or $U_{2n}(2)$, respectively. If X is the natural module of D (a $2n$-dimensional $GF(t)$-space) then the elements of $\mathscr{D}_t(n)$ are the non-zero subspaces of X that are totally singular with respect to the non-singular bilinear form Ψ on X preserved by D; the type of a subspace of dimension k is $n - k + 1$ and the incidence relation is via inclusion. In particular the points of $\mathscr{D}_t(n)$ are the maximal (i.e., n-dimensional) totally singular subspaces. Below we summarize some basic properties of $\mathscr{D}_t(n)$ (cf. [BCN89] and Section 6.3 in [Iv99]).

Let Γ be the collinearity graph of $\mathscr{D}_t(n)$ and $x \in \Gamma$. Then $\mathrm{res}_{\mathscr{D}_t(n)}(x)$ is the dual of the projective geometry of the proper subspaces of x. The stabilizer $D(x)$ of x induces $L_n(t)$ on this residue with $Q(x) = O_2(D(x))$ being the kernel. The subgroup $Q(x)$ is an elementary abelian 2-group which (as a $GF(2)$-module for $D(x)/Q(x)$) is isomorphic to the $n(n+1)/2$-dimensional module of quadratic forms on X if $t = 2$ and to the n^2-dimensional module of the Hermitian forms on X if $t = 4$. The action of $Q(x)$ on $\Gamma_n(x)$ is regular.

The graph Γ is a near n-gon which means that on every line there is a unique element which is nearest to x in Γ. Let $y \in \Gamma_i(x)$ for $1 \leq i \leq n-1$. Then $x \cap y$ is the unique element of type $n - i$ incident to both x and y. The vertices of Γ (treated as subspaces in X) that contain $x \cap y$ induce in Γ a strongly geodetically closed subgraph isomorphic to the collinearity graph of

$$\mathscr{D}_t(i) \cong \mathrm{res}_{\mathscr{D}_t(n)}(x \cap y).$$

If $y_1, y_2 \in \Gamma_i(x)$ for $1 \leq i \leq n$, then y_1 and y_2 are in the same connected component of the subgraph induced by $\Gamma_i(x)$ if and only if $x \cap y_1 = x \cap y_2$. In particular the subgraph induced by $\Gamma_n(x)$ is connected. Thus $D(x)$ acts on the set of connected components of the subgraph induced by $\Gamma_i(x)$ as it acts on the set of $(n-i)$-dimensional subspaces in x, in particular $Q(x)$ is the kernel of the action.

Let us turn to the representations of $\mathscr{D}_t(n)$. The rank 2 case has already been done.

Lemma 3.7.1 *The universal representation group of $\mathscr{D}_t(2)$ is elementary abelian of orders 2^5 and 2^6, for $t = 2$ and 4, respectively.*

Proof. Because of the isomorphisms $S_4(2) \cong \Omega_5(2)$ and $U_4(2) \cong \Omega_6^-(2)$, the dual polar spaces under consideration are isomorphic to the polar spaces of the corresponding orthogonal groups, so (3.6.2) applies. □

Lemma 3.7.2 *The dimension $d_t(n)$ of the universal representation module of $\mathscr{D}_t(n)$ is greater than or equal to $m_t(n)$, where*

$$m_2(n) = 1 + \begin{bmatrix} n \\ 1 \end{bmatrix}_2 + \begin{bmatrix} n \\ 2 \end{bmatrix}_2$$

and

$$m_4(n) = 1 + \begin{bmatrix} n \\ 1 \end{bmatrix}_4.$$

Proof. Let N be the incidence matrix of the point-line incidence system of $\mathscr{D}_t(n)$. This means that the rows of N are indexed by the points in $\mathscr{D}_t(n)$, the columns are indexed by the lines in $\mathscr{D}_t(n)$ and the (p, l)-entry is 1 if $p \in l$ and 0 otherwise. Then $d_t(n)$ is the number of points in Π minus the $GF(2)$-rank $\mathrm{rk}_2\, N$ of N. The latter rank is at most the rank $\mathrm{rk}\, N$ of N over the real numbers. By elementary linear algebra we have the following:

$$\mathrm{rk}\, N = \mathrm{rk}\, NN^T \text{ and } NN^T = A + \begin{bmatrix} n \\ 1 \end{bmatrix}_t I,$$

where A is the adjacency matrix of the collinearity graph Γ of $\mathscr{D}_t(n)$ and $[{n \atop 1}]_t$ is the number of lines incident to a given point. This shows that $d_t(n)$ is at least the multiplicity of $-[{n \atop 1}]_t$ as an eigenvalue of A. It is known (cf. Section 8.4 in [BCN89]) that this multiplicity is exactly $m_t(n)$. □

The above result for the case $t = 2$ was established in an unpublished work of A.E. Brouwer in 1990 (cf. [BB00]). Brouwer has also checked that the bound is exact for $n \leq 4$ and posed the following.

Conjecture 3.7.3 *The dimension of the universal representation module of $\mathcal{D}_2(n)$ is precisely $m_2(n)$.*

This conjecture (known as Brouwer's conjecture) has attracted the attention of a number of mathematicians during the 1990s. It was proved for $n = 3$ in [Yos92] and [CS97], for $n = 4, 5$ in [Coo97], for $n = 6, 7$ in [BI97].

Lemma 3.7.4 *Let (V, φ_a) be the universal abelian representation of $\mathcal{D}_t(n)$ and let the sections $\overline{V}_i(x)$, $1 \leq i \leq n$, be defined with respect to a vertex x of the collinearity graph Γ of $\mathcal{D}_t(n)$. Let $\mathcal{L}$ be the projective geometry of the dual of x, so that $\mathcal{L}^i$ is the set of $(n-i)$-dimensional subspaces in x, and let $\mathcal{P}^i$ be the power space of $\mathcal{L}^i$. Then*

(i) *$V_0(x)$ and $\overline{V}_n(x)$ are 1-dimensional;*
(ii) *for $1 \leq i \leq n-1$ there is a mapping*

$$\chi : \mathcal{P}^i \to \overline{V}_i(x)$$

which is a surjective homomorphism of $D(x)$-modules;
(iii) *$\overline{V}_{n-1}(x)$ is isomorphic to a factor module of $Q(x)$;*
(iv) *if $t = 2$ and $2 \leq i \leq n-1$ then the flags from the dual of x, contained in the set $\Phi^i(i-2, i+1)$ are in the kernel of the homomorphism χ as in* (ii).

Proof. (i) is obvious. Since the connected components of the subgraph induced by $\Gamma_i(x)$ are indexed by the elements of $\mathcal{L}^i$, (ii) follows from the proof of (2.2.1). Let $u \in \Gamma_n(x)$, let $\{z_1, ..., z_k\} = \Gamma(u) \cap \Gamma_n(x)$ and let y_i be the vertex in $\Gamma_{n-1}(x)$ such that $\{u, z_i, y_i\}$ is a line, $1 \leq i \leq k = [{n \atop 1}]_t$. Then it is easy to check that the vertices y_i are in pairwise different connected components of the subgraph induced by $\Gamma_{n-1}(x)$. On the other hand $Q(x)$ acts regularly on $\Gamma_n(x)$, which means that the subgraph induced by this set is a Cayley graph of $Q(x)$. This shows that $Q(x)$ possesses a generating set $\{q_1, ..., q_k\}$ where q_i maps u onto z_i. Let $\overline{y}_i$ be the image of $\varphi_a(y_i)$ in $\overline{V}_{n-1}(x)$ and put

$$\nu : q_i \mapsto \overline{y}_i$$

for $1 \leq i \leq k$. We claim that ν induces a homomorphism of $Q(x)$ onto $\overline{V}_{n-1}(x)$. In order to prove the claim we have to show that whenever

$q_{i_1}q_{i_2}...q_{i_m} = 1$ we have $\overline{y}_{i_1}\overline{y}_{i_2}...\overline{y}_{i_m} = 1$. Assuming the former equality put $u_0 = u$ and for $1 \le j \le m$ let u_j be the image of u_{j-1} under q_{i_j}. Since $Q(x)$ acts regularly on $\Gamma_n(x)$, $(u_0, u_1, ..., u_m)$ is a cycle and if v_j is such that $\{u_{j-1}, v_j, u_j\}$ is a line then it is easy to check that $\overline{v}_j = \overline{y}_j$ and the claim follows from (2.2.2).

Notice that if $t = 2$ and $n = 3$, then $Q(x)$ is of order 2^6, and in this case $\overline{V}_2(x)$ is generated by seven pairwise commuting involutions indexed by the connected components of the subgraph induced by $\Gamma_2(x)$. The product of these involutions is the identity element.

In order to prove (iv) let $y \in \Gamma_{i+1}(x)$ and $z \in \Gamma_{i-2}(x)$ be such that $d_\Gamma(z, y) = 3$. Then $x \cap y$ is an $(n-i-1)$-dimensional subspace contained in $x \cap z$ that is $(n-i+2)$-dimensional. Let Δ be the subgraph in Γ induced by the vertices which contain $z \cap y$. Then Δ is isomorphic to the collinearity graph of $\mathscr{D}_2(3)$. Let $u_1, ..., u_7$ be representatives of the connected components of the subgraph induced by $\Gamma_i(x)$ which intersect Δ. Then $T := \{u_j \cap x \mid 1 \le j \le 7\}$ is the set of $(n-i)$-subspaces in x containing $x \cap y$ and contained in $x \cap z$. In other terms $T \in \Phi^i(i-j, i+1)$. Let $\overline{u}_j$ be the image of $\varphi_a(u_j)$ in $\overline{V}_i(x)$. Then by (iii) and the previous paragraph we have

$$\overline{u}_1\overline{u}_2....\overline{u}_7 = 1$$

and (iv) is proved. □

We apply (3.7.4 (i), (ii) and (iii)) and (3.7.2) to the rank 3 case to obtain the following result originally proved in [Yos92], [CS97] and [Yos94].

Lemma 3.7.5 *The universal representation module for $\mathscr{D}_t(3)$ has dimension $m_2(3) = 15$ for $t = 2$ and $m_4(3) = 22$ for $t = 4$.*

Proof. Suppose first that $t = 2$. Then x is contained in exactly 7 lines, so that $\dim \overline{V}_1(x) \le 7$. This, together with the bounds in (3.7.4) gives

$$\dim V = \dim V_0 + \dim \overline{V}_1 + \dim \overline{V}_2 + \dim \overline{V}_3 \le 1 + 7 + 6 + 1 = 15,$$

which meets the lower bound in (3.7.2).

If $t = 4$ then $\overline{V}_1(x)$ is a quotient of the permutation modules of $(D(x)/Q(x))^\infty \cong PSL(3,4)$ on 1-subspaces of a natural module (which is a 3-dimensional $GF(4)$-space). Let Σ be a quad in Γ, isomorphic to the Schläfli graph. Let $w_1, ..., w_5$ be a maximal set of pairwise non-collinear points from $\Sigma(x)$ (one point from every line on x contained in Σ). It is easy to deduce from the fact that the universal representation module of $\mathscr{P}(\Omega_6^-(2))$ is 6-dimensional (cf. (3.6.1)) that the image in $\overline{V}_1(x)$ of the product

$$\varphi_a(w_1) \cdot ... \cdot \varphi_a(w_5)$$

is the identity. Hence $\dim \overline{V}_1(x) \leq 11$ by (3.1.4 (ii) (a)). This together with the bounds (3.7.4) gives

$$\dim V = \dim V_0 + \dim \overline{V}_1 + \dim \overline{V}_2 + \dim \overline{V}_3 \leq 1 + 11 + 9 + 1 = 22,$$

which again meets the lower bound in (3.7.2). □

By (3.7.4 (iv)) the main result (3.2.6) of [Li01] implies Brouwer's conjecture for all $n \geq 2$. An alternative independent proof of this conjecture was established in [BB00]. Very recently Paul Li applied his technique to prove in [Li00] the natural analogue of Brouwer's conjecture for the unitary dual polar spaces (i.e., for the case $t = 4$) [Li00]. Thus we have the following final result:

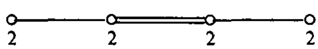
Theorem 3.7.6 *The dimension of the universal representation module of $\mathscr{D}_t(n)$ is equal to the number $m_t(n)$ defined in* (3.7.2). □

In the rank 3 case the question about the universal representation group can also be answered completely.

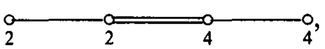
Lemma 3.7.7 *Let R be the universal representation group of $\mathscr{D}_t(3)$. Then*

(i) *R is non-abelian;*
(ii) *the commutator subgroup of R is of order* 2.

Proof. Let F be the Lie type group $F_4(2)$ or ${}^2E_6(2)$, $\mathscr{F}$ be the F_4-building associated with F and Ξ be the collinearity graph of $\mathscr{F}$. Then the diagram of $\mathscr{F}$ is



or

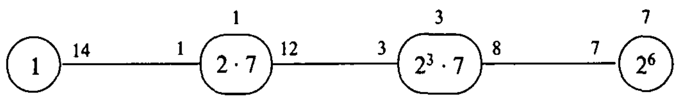


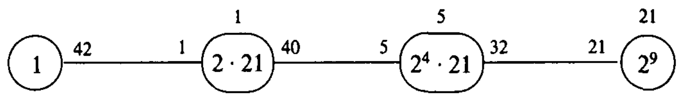
and if x is a point of $\mathscr{F}$ then $\mathrm{res}_{\mathscr{F}}(x)$ is isomorphic to $\mathscr{D}_t(3)$ for $t = 2$ or 4, respectively, and the suborbit diagram of Ξ with respect to the action of F can be found in Section 5.5 of [Iv99]. If $F(x)$ is the stabilizer of x in F and $Q(x) = O_2(F(x))$, then $F(x) \cong 2^{1+6+8} : S_6(2)$ if $F \cong F_4(2)$ and $F(x) \cong 2_+^{1+20} : U_6(2)$ if $F \cong {}^2E_6(2)$; $Q(x)$ is non-abelian (with commutator subgroup of order 2) and acts regularly on the set $\Xi_3(x)$ of vertices at distance 3 from x in Ξ.

Let Θ denote the subgraph in Ξ induced by the set $\Xi_3(x)$ and let $y \in \Theta$. The $F(x,y)$ is a complement to $Q(x)$ in $F(x)$, in particular it acts transitively on $\mathrm{res}_{\mathscr{F}}(x)$. In addition $\Xi_3(x)$ is a complement of a geometric hyperplane in $\mathscr{F}$. Hence there is bijection between the point set of $\mathrm{res}_{\mathscr{F}}(x) \cong \mathscr{D}_t(3)$ and the set $\Theta(y)$ of neighbours of y in Θ. Let $q \in Q(x)$ map y onto $z \in \Theta(y)$. By considering the orbit lengths of $Q(x) = O_2(F(x))$ on Ξ or using the geometrical properties of the F_4-building we can check that $q^2 = 1$, which means that Θ is the Cayley graph of $Q(x)$ with respect to a generating set indexed by the point set of $\mathscr{D}_t(3)$.

Since Θ is a subgraph in the collinearity graph of $\mathscr{F}$, it is clear that the geometric triangles are present and in view of the discussions in Section 2.5, we observe that $Q(x)$ is a representation group of $\mathscr{D}$ and hence (i) follows.

The suborbit diagrams of the collinearity graph of $\mathscr{D}_t(3)$ (when $t = 2$ and 4, respectively) are given below.





We apply (2.3.7) for $B(x) = \Gamma_3(x)$. The conditions in (2.3.7) follow from the above mentioned basic properties of $\mathscr{D}_t(3)$. □

The remainder of this section concerns the universal representation group of $\mathscr{D}_2(n)$, in particular for $n = 4$. We will use the following elementary result.

Lemma 3.7.8 *Let x and y be points of $\mathscr{D}_2(4)$ such that $d_\Gamma(x,y) = 4$ (so that x and y are disjoint maximal totally isotropic subspaces in X with respect to the symplectic form Ψ). Define a $GF(2)$-valued function $f = f^{[x,y]}$ on X by the following rule: $f(v) = 1$ if and only if $v = v_x + v_y$ for $v_x \in x$, $v_y \in y$ and $\Psi(v_x, v_y) = 1$. Then f is a quadratic form of plus type associated with Ψ.*

Proof. Since $D \cong S_8(2)$ acts transitively on the pairs of disjoint maximal totally isotropic subspaces, there is a quadratic form g of plus type associated with Ψ such that both x and y are totally isotropic with respect to g. Let H be the stabilizer of x and y in $O(f) \cong O_8^+(2)$. Then

$H \cong L_4(2) \cong \Omega_8^+(2)$, H induces non-equivalent doubly transitive actions on $x^\#$ and $y^\#$, and these actions are conjugate in $\mathrm{Aut}\, H \cong O_8^+(2)$. This shows that H has exactly four orbits $T_1, ..., T_4$ on $X^\#$, where

$$T_1 = x^\#, \quad T_2 = y^\#, \quad T_3 = \{v_x + v_y \mid v_x \in x^\#, v_y \in y^\#, \Psi(v_x, v_y) = 0\},$$

$$T_4 = \{v_x + v_y \mid v_x \in x, v_y \in y, \Psi(v_x, v_y) = 1\}.$$

The lengths of these orbits are 15, 15, 105 and 120, respectively. Since exactly 120 vectors in X are non-isotropic with respect to g, the equality $g = f$ holds. □

We formulate yet another useful property of $\mathscr{D}_2(n)$ which can be deduced directly from the definitions.

Lemma 3.7.9 *Let $\mathscr{D} = \mathscr{D}_2(n)$, Γ be the collinearity graph of $\mathscr{D}$, v be an element of type n in $\mathscr{D}$ (a 1-subspace in the natural module X) and $\Delta = \Delta(v)$ be the subgraph in Γ induced by the vertices containing v. Then*

(i) *Δ is isomorphic to the collinearity graph of* $\mathrm{res}_{\mathscr{D}}(v) \cong \mathscr{D}_2(n-1)$*;*

(ii) *if $x \in \Gamma \setminus \Delta$ then x is adjacent in Γ to a unique vertex from Δ which we denote by $\pi_\Delta(x)$;*

(iii) *if $l = \{x, y, z\}$ is a line in $\mathscr{D}$ then either $l \subset \Delta$, or $|l \cap \Delta| = 1$ or $l \subset \Gamma \setminus \Delta$;*

(iv) *if $l \subset \Gamma \setminus \Delta$ then $\{\pi_\Delta(x), \pi_\Delta(y), \pi_\Delta(z)\}$ is a line of $\mathscr{D}$.* □

Lemma 3.7.10 *In terms of* (3.7.9) *let R be a group and $\varphi : \Delta \to R$ be a mapping such that (R, φ) is a representation of* $\mathrm{res}_{\mathscr{D}}(v) \cong \mathscr{D}_2(n-1)$. *Define a mapping $\psi : \Gamma \to R$ by the following rule:*

$$\psi(x) = \begin{cases} 1 & \textit{if } x \in \Delta; \\ \varphi(\pi_\Delta(x)) & \textit{otherwise.} \end{cases}$$

Then (R, ψ) is a representation of $\mathscr{D}$.

Proof. Easily follows from (3.7.9). □

Let $\Pi = \{v_1, v_2, ..., v_k\}$ be the set of elements of type n in $\mathscr{D}_2(n)$, where $k = 2^{2n} - 1$. For $1 \leq i \leq k$ let (R_i, φ_i) be the universal representation of $\mathrm{res}_{\mathscr{D}_2(n)}(v_i)$ and (R_i, ψ_i) be the representation of $\mathscr{D}_2(n)$ obtained from (R_i, φ_i) as in (3.7.10). Let

$$(T, \psi) = (R_1, \psi_1) \times ... \times (R_k, \psi_k)$$

be the product of the representations (R_i, ψ_i). By the general result (2.1.4) we obtain the following lemma.

Lemma 3.7.11 *(T, ψ) is a representation of $\mathcal{D}_2(n)$.* □

Let us estimate the order of the commutator subgroup T' of T when $n = 4$. Let z_i be the unique non-identity element in the commutator subgroup R_i' of R_i (R_i' is of order 2 by (3.7.7)). Then the commutator subgroup T_0' of the direct product $T_0 := R_1 \times ... \times R_k$ is of order 2^k consisting of the elements

$$(z_1^{\varepsilon_1}, z_2^{\varepsilon_2}, ..., z_k^{\varepsilon_k}),$$

where $\varepsilon_i \in \{0, 1\}$. Thus T_0' is isomorphic to the power space of the set Π (the set of elements of type 4 in $\mathcal{D}_2(4)$). By (3.7.10) and the proof of (3.7.7 (ii)) we have the following.

Lemma 3.7.12 *For $x, y \in \Gamma$ we have*

$$[\psi_i(x), \psi_i(y)] = z_i$$

if and only if $\{x, y\} \cap \Delta(v_i) = \emptyset$ and $d_\Gamma(\pi_{\Delta(v_i)}(x), \pi_{\Delta(v_i)}(y)) = 3$. □

Lemma 3.7.13 *For $x, y \in \Gamma$ let*

$$[\psi(x), \psi(y)] = (z_1^{\varepsilon_1(x,y)}, z_2^{\varepsilon_2(x,y)}, ..., z_k^{\varepsilon_k(x,y)}).$$

Then

(i) *if $d_\Gamma(x, y) \leq 2$ then $\varepsilon_i(x, y) = 0$ for all $1 \leq i \leq k$;*
(ii) *if $d_\Gamma(x, y) = 3$ then $\varepsilon_i(x, y) = 1$ if and only if $\Psi(x \cap y, v_i) = 1$;*
(iii) *if $d_\Gamma(x, y) = 4$ then $\varepsilon_i(x, y) = 1$ if and only if $f^{[x,y]}(v_i) = 1$, where $f^{[x,y]}$ in the quadratic form defined in* (3.7.8).

Proof. If $d_\Gamma(x, y) \leq 2$ then x and y are contained in a common quad and by (3.7.1) their images commute even in the universal representation group of $\mathcal{D}_2(4)$, which gives (i). If $d_\Gamma(x, y) = 3$ then $u := x \cap y$ is 1-dimensional. Hence the intersection $(v_i^\perp \cap x) \cap (v_i^\perp \cap y)$ if non-empty can only be u and u is in the intersection if and only if $\Psi(u, v_i) = 0$, hence (ii) follows. If $d_\Gamma(x, y) = 4$ then $x \cap y = 0$ and every $v_i \in X$ possesses a unique presentation $v_i = v_x + v_y$, where $v_x \in x$, $v_y \in y$. If $v_x = 0$ or $v_y = 0$ then $v_i \in y$ or $v_i \in x$, respectively and $\varepsilon_i(x, y) = 0$ in both cases. Suppose that $v_x \neq 0$, $v_y \neq 0$ and $\Psi(v_x, v_y) = 0$. Then $\{v_i, v_x, v_u\}$ is the set of non-zero vectors in a totally isotropic 2-subspace contained in

$$I := \langle v_i^\perp \cap x, v_i \rangle \cap \langle v_i^\perp \cap y, v_i \rangle,$$

and hence again $\varepsilon_i(x,y) = 0$. On the other hand, suppose that I contain a non-zero vector, say w. Then

$$w = w_x + \alpha v_i = w_y + \beta v_i,$$

where $w_x \in v_i^\perp \cap x$, $w_y \in v_i^\perp \cap y$, $\alpha, \beta \in GF(2)$. Since $v_i \notin x \cup y$, $\{\alpha, \beta\} = \{0, 1\}$. Assume without loss of generality that $\alpha = 1$, $\beta = 0$. Then $v_i = w_x + w_y$ and

$$\Psi(w_x, w_y) = \Psi(w + v_i, w_y) = \Psi(w, w_y) + \Psi(v_i, w_y) = 0.$$

This completes the proof. □

It is easy to see that the vectors as in (3.7.13 (ii) and (iii)) generate the 9-dimensional submodule whose non-zero vectors are the complements of the geometric hyperplanes in $\mathcal{G}(S_8(2))$. Thus we have the following.

Proposition 3.7.14 *The commutator subgroup of the universal representation group of $\mathcal{D}_2(4)$ possesses an elementary abelian quotient, isomorphic to the dual of the orthogonal module of $S_8(2) \cong \Omega_9(2)$.* □

3.8 $\mathcal{G}(3 \cdot S_4(2))$

Let $\mathcal{G} = \mathcal{G}(3 \cdot S_4(2))$, $G = \mathrm{Aut}\,\mathcal{G} \cong 3 \cdot S_4(2)$, $E = O_3(G)$ and (V, φ) be the universal abelian representation of $\mathcal{G}$. Let $V^z = C_V(E)$, $V^c = [V, E]$. Then by the previous subsection and (2.4.1) V^z is the 5-dimensional natural module for $O_5(2) \cong S_4(2)$. From the basic properties of the action of M_{24} on $\mathcal{G}(M_{24})$ we observe that the hexacode module V_h is a representation module for $\mathcal{G}$. Since E acts on V_h fixed-point freely, V_h is a quotient of V^c.

Lemma 3.8.1 $V^c = V_h$.

Proof. The fixed-point free action of E on V^c turns the latter into a $GF(4)$-module for G. If $\overline{x} = \{x, y, z\}$ is an orbit of E on the point set of $\mathcal{G}$ then $\varphi^c(\overline{x}) := \langle \varphi^c(x), \varphi^c(y), \varphi^c(z) \rangle$ is a 1-dimensional $GF(4)$ subspaces of V^c. On the other hand, $\overline{x}$ is a point of $\overline{\mathcal{G}} = \mathcal{G}(S_4(2))$. Hence we can consider the mapping $\chi : \overline{x} \mapsto \varphi^c(\overline{x})$ of the point-set into the set of 1-dimensional subspaces of V^c (recall that we are speaking of $GF(4)$-subspaces). Arguing as in the proof of (2.2.1) and in view of (3.4.2) it is easy to show that the $GF(4)$-dimension of V^c is at most 5. Let U be the kernel of the homomorphism of V^c onto V_h. Then the $GF(4)$-dimension of U is at most 2 and the action of E on U is fixed-point free. Since

$G' \cong 3 \cdot Alt_6$ does not split over E, unless U is trivial, U must be a faithful $GF(4)$-module for G'. By the order consideration we observe that G' is not a subgroup of $GL(2,4)$, hence U is trivial and the result follows. □

Let $\mathcal{G}^*$ be the enrichment of $\mathcal{G} = \mathcal{G}(3 \cdot S_4(2))$. Recall that the points of $\mathcal{G}^*$ are those of $\mathcal{G}$ while the lines of $\mathcal{G}^*$ are the lines of $\mathcal{G}$ together with the orbits of E on the set of points (thus $\mathcal{G}$ has 45 points and 60 lines).

Lemma 3.8.2 $R(\mathcal{G}^*(3 \cdot S_4(2))) \cong V_h$.

Proof. By (3.8.1) we only have to show that $R^* := R(\mathcal{G}^*(3 \cdot S_4(2)))$ is abelian. For this we apply (2.3.7). Consider the collinearity graph Γ of $\mathcal{G} = \mathcal{G}(3 \cdot S_4(2))$ the suborbit diagram of which is given in Section 2.6 in [Iv99] and let φ be the mapping which turns R^* into the representation group of $\mathcal{G}^*$. Let

$$B(x) = \Gamma_2(x), \quad A(x) = \Pi \setminus B(x).$$

We claim that the conditions in (2.3.7) are satisfied. Since $\{x\} \cup \Gamma_4(x)$ is the only non-trivial imprimitivity block of $3 \cdot S_4(2)$ on Γ containing x, it is clear that the graph Ξ defined as in (2.3.7) (i) is connected. We claim that the graph Σ^x defined as in (2.3.7 (ii)) is connected. From the suborbit diagram of Γ we observe that Σ^x is regular of valency 3 on 24 vertices. Since Γ is distance-transitive, Σ^x is vertex-transitive and hence the size of a connected component of Σ^x divides 24. Finally, the girth of Σ^x is at least the girth of Γ which is 5 and hence a connected component of Σ^x contains at least $1 + 3 + 3 \cdot 2 = 10$ vertices, and the claim follows. Let $y \in \Gamma_i(x)$ for $i = 0, 1, 3$ or 4. If $i = 0, 1$ or 4 then x and y are equal or adjacent in $\mathcal{G}^*$ and hence $[\varphi(x), \varphi(y)] = 1$. If $i = 3$ then $\varphi(x)$ and $\varphi(y)$ commute by (2.4.4). Thus by (2.3.7) the commutator subgroup of R^* is of order at most 2. By (3.8.1) and (2.4.3) $R^*/(R^*)' \cong V_h$ and since $3 \cdot S_4(2)$ does not preserve a non-zero symplectic form on V_h, R^* is abelian by (2.3.8) and (2.3.9). □

In what follows we will make use of the following property of the hexacode module, which can be checked directly.

Lemma 3.8.3 *Let (R^*, φ) be the universal representation of $\mathcal{G}^*(3 \cdot S_4(2))$, where R^* is isomorphic to the hexacode module V_h. Let x be a point and $R_1^*(x)$ be the subgroup in R^* generated by the elements $\varphi(y)$ taken for the points y collinear to x in $\mathcal{G}(3 \cdot S_4(2))$ (there are six such points). Then $R_1^*(x)$ is of order 2^3.* □

Lemma 3.8.4 *The universal representation group of $\mathcal{G}(3 \cdot S_4(2))$ is infinite.*

Proof. By (2.3.6) it is sufficient to show that $\mathcal{G} = \mathcal{G}(3 \cdot S_4(2))$ contains a hyperplane with a disconnected complement. Let $\overline{\mathcal{G}} = \mathcal{G}(S_4(2))$ and χ be the covering of $\mathcal{G}$ onto $\overline{\mathcal{G}}$. Let Ω be a set of size 6 so that the points of $\overline{\mathcal{G}}$ are the transpositions in $\overline{G} = Sym(\Omega)$. Then the lines of $\overline{\mathcal{G}}$ are maximal sets of pairwise commuting transpositions. Notice that the points of $\mathcal{G}$ are the involutions of G which map onto transpositions under the homomorphism of $\mathcal{G}$ onto $\overline{\mathcal{G}}$ and the lines of $\mathcal{G}$ are maximal sets of such involutions which commute. Let α be an element of Ω and $\overline{H}$ be the set of transpositions which do not stabilize α. Then $|\overline{H}| = 5$ and it is easy to see that $\overline{H}$ is a geometric hyperplane. The complement of $\overline{H}$ consists of 10 transpositions in the stabilizer of α in $\overline{G}$, which form a Petersen subgraph. Let $H = \chi^{-1}(\overline{H})$, so that H is a hyperplane in $\mathcal{G}$ by (2.3.1). Let S be the preimage in G of the stabilizer of α in $\overline{G}$. Then $A := S^{\infty} \cong Alt_5$ and $S/A \cong Sym_3$. It is easy to see that the points in the complement of H (considered as involutions in G) map surjectively into the set of involutions in S/A. Since two points in the collinearity graph of $\mathcal{G}$ are adjacent if they commute, the preimage in the complement of H of an involution from S/A is a connected component (isomorphic to the Petersen graph). □

In Section 10.2 we will make use of the following property of the universal representation module of $\mathcal{G}(3 \cdot S_4(2))$, which can be checked by direct calculation.

Lemma 3.8.5 *Let (W, ψ) be an abelian representation of $\mathcal{G} = \mathcal{G}(3 \cdot S_4(2))$. Let l be a line of $\mathcal{G}$ and Ξ be the set of points of $\mathcal{G}$ collinear to at least one point in l (so that $|\ \Xi\ |= 15$) and*

$$d_l(W) = \dim \langle \psi(x) \mid x \in \Xi \rangle.$$

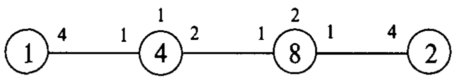
Then

(i) *if $W = V$ is the universal abelian representation module of $\mathcal{G}$ (so that $\dim W = 11$) then $d_l(W) = 8$;*

(ii) *if $W = V^z$ is the 5-dimensional orthogonal module, then $d_l(W) = \dim W = 5$.* □

3.9 $\mathcal{G}(Alt_5)$

Recall that the points and lines of $\mathcal{G} = \mathcal{G}(Alt_5)$ are the edges and vertices of the Petersen graph with the natural incidence relation. The collinearity graph Γ of $\mathcal{G}$ is a triple antipodal covering of the complete graph on 5 vertices with the following intersection diagram.



Thus every edge is contained in a unique antipodal block of size 3 called an *antipodal triple.* The following result is an easy combinatorial exercise.

Lemma 3.9.1 *Let $\mathcal{G}^*$ be the point-line incidence system whose points are the points of $\mathcal{G}(Alt_5)$ and whose lines are the lines of $\mathcal{G}(Alt_5)$ together with the antipodal triples. Then $\mathcal{G}^* \cong \mathcal{G}(S_4(2))$.* □

By the above lemma the universal representation group $V_5(2)$ of $\mathcal{G}(S_4(2))$ is a representation group of $\mathcal{G}$ and it is the largest one with the property that the product of the images of points in an antipodal triple is the identity. The next result shows that the universal representation module of $\mathcal{G}$ is related to $\mathcal{G}(3 \cdot S_4(2))$.

Lemma 3.9.2 *The universal representation module $V(\mathcal{G}(Alt_5))$ has dimension six and is isomorphic to the hexacode module restricted to a subgroup Sym_5 in $3 \cdot S_4(2)$.*

Proof. Let $\mathcal{H} = \mathcal{G}(3 \cdot S_4(2))$, $\overline{\mathcal{H}} = \mathcal{G}(S_4(2))$, $H = 3 \cdot S_4(2)$, $\overline{H} = S_4(2)$. Let $G \cong Sym_5$ be a subgroup in H, whose (isomorphic) image in $\overline{H}$ acts transitively on the point set of $\overline{\mathcal{H}}$. Then G has two orbits, Π_1 and Π_2 on the point set of $\mathcal{H}$ with lengths 15 and 30, respectively. The points in Π_1 together with the lines contained in Π_1 form a subgeometry in $\mathcal{H}$ isomorphic to $\mathcal{G}$ and the image of Π_1 in the hexacode module forms a spanning set. These facts can be checked by a direct calculation in the hexacode module and also follow from (4.2.6) and (4.3.2) below. It remains to prove that V_h is universal. This can easily be achieved by calculating the $GF(2)$-rank of the point-line incidence matrix of $\mathcal{G}(Alt_5)$. □

The next lemma shows that the universal representation *group* of $\mathcal{G}$ is infinite.

Lemma 3.9.3 *The universal representation group of $\mathcal{G}(Alt_5)$ is infinite.*

Proof. The points and lines of $\mathcal{G} = \mathcal{G}(Alt_5)$ are the edges and vertices of the Petersen graph with the natural incidence relation. Take the standard picture of the Petersen graph and let H be the set of 5 edges which join the external pentagon with the internal star. Then it is easy to see that H is a geometric hyperplane whose complement consists of two connected components - the pentagon and the star. Now the proof is immediate from (2.3.6). □

Recall that if $\mathcal{G}$ is a P-geometry of rank $n \geq 2$ then the *derived graph* $\Delta = \Delta(\mathcal{G})$ has $\mathcal{G}^n$ as the set of vertices and two such vertices are adjacent if they are incident in $\mathcal{G}$ to a common element of type $n-1$ (the derived graph explains the term *vertices* for the elements of type n and the term *links* for the elements of type $n-1$). The vertices and links incident to a given element u of type $n-2$ in $\mathcal{G}$ form a Petersen subgraph $\Delta[u]$ in Δ. The *derived system* $\mathcal{D} = \mathcal{D}(\mathcal{G})$ of $\mathcal{G}$ is the point-line incidence system (Π, L) whose points are the elements of type n (the vertices) and a triple of such elements form a line if they are incident to a common element u of type $n-2$ and are the neighbours of a vertex in the Petersen subgraph $\Delta[u]$. A representation group of $\mathcal{D}$ is called a *derived group* of $\mathcal{G}$. In the case of $\mathcal{G} = \mathcal{G}(Alt_5)$ the points of $\mathcal{D}$ are the vertices of the Petersen graph Δ and the lines are the sets $\Delta(x)$ taken for all the vertices x in Δ. Let V_o be the orthogonal module of $O_4^-(2) \cong Sym_5$ which is also the heart of the permutation $GF(2)$-module on a set Σ of size 5. Then V_o is the 4-dimensional irreducible module for Sym_5 called the *orthogonal module.* The group Sym_5 acts on the set of non-zero vectors in V_o with two orbits of length 5 and 10 indexed by 1- and 2-element subsets of Σ. Let ψ be the mapping from the set of 2-element subsets of Σ (the points of $\mathcal{D}$) into V_o which commutes with the action of Sym_5. It is easy to check that (V_o, ψ) is the universal representation of $\mathcal{D}$ that gives the following.

Lemma 3.9.4 *The universal representation group of $\mathcal{D}(\mathcal{G}(Alt_5))$ is the orthogonal module V_o for Sym_5.* □

3.10 $\mathcal{G}(3^{\left[{n \atop 2}\right]_2} \cdot S_{2n}(2))$

Let $\widetilde{\mathcal{G}} = \mathcal{G}(3^{\left[{n \atop 2}\right]_2} \cdot S_{2n}(2))$, $n \geq 3$, so that $\widetilde{\mathcal{G}}$ is a T-geometry of rank n with the automorphism group $\widetilde{G} \cong 3^{\left[{n \atop 2}\right]_2} \cdot S_{2n}(2)$. Let $\chi : \widetilde{\mathcal{G}} \to \mathcal{G}$ be the morphism of geometries where $\mathcal{G} = \mathcal{G}(S_{2n}(2))$. We can identify the elements of $\mathcal{G}$ with the E-orbits on $\widetilde{\mathcal{G}}$, where $E = O_3(\widetilde{G})$ and then χ sends an element of $\widetilde{\mathcal{G}}$ onto the E-orbit containing this element. Clearly the morphism χ commutes with the action of $\widetilde{G}$ and $G \cong S_{2n}(2)$ is the action induced by $\widetilde{G}$ on $\mathcal{G}$ (which is the full automorphism group of $\mathcal{G}$).

Let (U, φ_a) be the universal abelian representation of $\widetilde{\mathcal{G}}$. Then

$$U = U^z \oplus U^c = C_U(E) \oplus [U, E].$$

By (2.4.1) and (3.6.2) U^z is the $(2n+1)$-dimensional orthogonal module for $G \cong S_{2n}(2) \cong \Omega_{2n+1}(2)$. In this section we prove the following.

Proposition 3.10.1 *In the above terms U^c, as a $GF(2)$-module for $\widetilde{G}$, is induced from the unique 2-dimensional irreducible $GF(2)$-module of*

$$3^{[\frac{n}{2}]_2}.\Omega_{2n}^-(2).2 < \widetilde{G}.$$

In particular $\dim U^c = 2^n(2^n - 1)$.

Within the proof of the above proposition we will see that the universal representation group of $\widetilde{\mathscr{G}}$ is infinite.

Let us recall some basic properties of $\widetilde{\mathscr{G}}$ and $\mathscr{G}$ (cf. Chapter 6 in [Iv99]). Concerning $\mathscr{G}$ we follow the notation introduced in Section 3.5, so that V is the natural symplectic module of G, Ψ is the symplectic form on V preserved by G and $\mathscr{Q} = \mathscr{Q}^+ \cup \mathscr{Q}^-$ is the set of quadratic forms on V associated with Ψ. For $f \in \mathscr{Q}^\varepsilon$ (where $\varepsilon \in \{+,-\}$) let $O(f) \cong O_{2n}^\varepsilon(2)$ be the stabilizer of f in G and $\Omega(f) \cong \Omega_{2n}^\varepsilon(2)$ be the commutator subgroup of $O(f)$.

Let v be a point of $\mathscr{G}$ (which is a 1-dimensional subspace of V identified with its unique non-zero vector). Let $G(v) \cong 2^{2n-1} : S_{2n-2}(2)$ be the stabilizer of v in G, $K(v) = O_2(G(v))$ be the kernel of the action of $G(v)$ on $\mathrm{res}_{\mathscr{G}}(v)$ and $R(v)$ be the centre of $G(v)$ which is the kernel of the action of $G(v)$ on the set of points collinear to v. The subgroup $R(v)$ is of order 2 generated by the element

$$\tau(v) : u \mapsto u + \Psi(u,v)v,$$

which is the transvection of V with centre v and axis $v^\perp$ (the orthogonal complement of v with respect to Ψ). The following result is rather standard.

Lemma 3.10.2 *Let v be a point of $\mathscr{G}$ and $f \in \mathscr{Q}$. Then the following assertions hold:*

(i) $C_V(\tau(v)) = v^\perp$;
(ii) *if* $f(v) = 0$ *then* $\tau(v) \notin O(f)$;
(iii) *if* $f(v) = 1$ *then* $\tau(v) \in O(f) \setminus \Omega(f)$.

Proof. The group G induces a rank 3 action on the point-set of $\mathscr{G}$. Since $\tau(v)$ is in the centre of $G(v)$ and fixes every point in $v^\perp$, it must act fixed-point freely on $V \setminus v^{\perp}$ and hence we have (i). Let u be a point of $\mathcal{G}$. If $u \in v^{\perp}$ then $u^{\tau(v)} = u$ and hence $f(u^{\tau(v)}) = f(u)$; if $u \in V \setminus v^{\perp}$, then

$$f(u^{\tau(v)}) = f(u+v) = f(u) + f(v) + \Psi(v,u).$$

Since in this case $\Psi(v,u) = 1$, the equality $f(u^{\tau(v)}) = f(u)$ holds if and only if $f(v) = 1$. By (i) $\dim C_V(\tau(v)) = 2n+1$ (which is an odd number) but we know (cf. p. xii in [CCNPW]) that an element $g \in O(f)$ is contained in $\Omega(f)$ if and only if $\dim C_V(g)$ is even. Hence we have (ii) and (iii). □

Let $\widetilde{v}$ be a point of $\widetilde{\mathcal{G}}$ such that $\chi(\widetilde{v}) = v$ and $\widetilde{G}(\widetilde{v})$ be the stabilizer of $\widetilde{v}$ in $\widetilde{G}$. Then $\widetilde{G}(\widetilde{v})$ induces the full automorphism group of $\mathrm{res}_{\widetilde{\mathcal{G}}}(\widetilde{v})$ and $\widetilde{K}(\widetilde{v}) = O_2(\widetilde{G}(\widetilde{v}))$ is the kernel of the action. The natural homomorphism of $\widetilde{G}$ onto G induced by the morphism χ maps $\widetilde{K}(\widetilde{v})$ isomorphically onto $K(v)$. In particular the centre of $\widetilde{K}(\widetilde{v})$ is generated by the unique element $\widetilde{\tau}(\widetilde{v})$. Let $\widetilde{R}(\widetilde{v})$ be the subgroup in $\widetilde{G}$ generated by $\widetilde{\tau}(\widetilde{v})$.

Lemma 3.10.3 *The following assertion holds:*

(i) *$\widetilde{R}(\widetilde{v})$ is the kernel of the action of $\widetilde{G}(\widetilde{v})$ on the set of points collinear to $\widetilde{v}$;*

(ii) *if $\widetilde{u}$ is a point collinear to $\widetilde{v}$ then $[\widetilde{\tau}(\widetilde{v}), \widetilde{\tau}(\widetilde{u})] = 1$.*

Proof. Since $\widetilde{K}(\widetilde{v})$ stabilizes every line incident to $\widetilde{v}$, the morphism χ commutes with the action of $\widetilde{G}$ and $\tau(v)$ fixes every point collinear to v, (i) follows. Since $\widetilde{R}(\widetilde{u})$ is a characteristic subgroup of $\widetilde{G}(\widetilde{v})$, (ii) follows from (i). □

Recall that $\widetilde{G}$ is a subgroup in the semidirect product $\widehat{G} = W : G$, where W is an elementary abelian 3-group which (as a $GF(3)$-module for G) is induced from a non-trivial 1-dimensional module W_f of the subgroup $O(f)$ of G, where $f \in \mathcal{Q}^-$. This means that the elements of $\Omega(f)$ centralize W_f and the elements from $O(f) \setminus \Omega(f)$ act by negation. Thus W possesses a direct sum decomposition

$$W = \bigoplus_{f \in \mathcal{Q}^-} W_f.$$

The group G permutes the direct summands in the natural (doubly transitive) way.

For a form $f \in \mathcal{Q}^-$ let $\widetilde{O}(f)$ be the full preimage of $O(f)$ in $\widetilde{G}$ (with respect to the natural homomorphism). Let $\widehat{O}(f) = W : O(f)$ be a subgroup of $\widehat{G}$ (where $O(f)$ is treated as a subgroup of the complement

G to W). It is clear that

$$W\lfloor f \rfloor := \bigoplus_{g \in \mathcal{Q}^-, g \neq f} W_g$$

is a subgroup of index 3 in W normalized by $O(f)$ while $\widehat{N}\lfloor f \rfloor = W\lfloor f \rfloor$: $\Omega(f)$ is a normal subgroup of index 6 in $\widehat{O}(f)$ and the corresponding factor group is isomorphic to Sym_3. The next statement follows directly from the definitions.

Lemma 3.10.4 *$\widetilde{N}\lfloor f \rfloor := \widehat{N}\lfloor f \rfloor \cap \widetilde{O}(f)$ is a normal subgroup in $\widetilde{O}(f)$ and*

$$D\lfloor f \rfloor := \widetilde{O}(f)/\widetilde{N}\lfloor f \rfloor \cong Sym_3.$$

□

Let ξ denote the natural homomorphism of $\widetilde{O}(f)$ onto $D\lfloor f \rfloor$. Let e be the identity element and i_1, i_2, i_3 be the involutions in $D\lfloor f \rfloor$. We define a mapping ϱ of the point-set of $\widetilde{\mathcal{G}}$ onto $\{e, i_1, i_2, i_3\}$ by the following rule

$$\varrho(\widetilde{v}) = \begin{cases} e & \text{if } \widetilde{\tau}(\widetilde{v}) \notin \widetilde{O}(f); \\ \xi(\widetilde{\tau}(\widetilde{v})) & \text{otherwise.} \end{cases}$$

Lemma 3.10.5 *The following assertions hold:*

(i) *$\varrho^{-1}(e)$ is a geometric hyperplane $\widetilde{H}(f)$ in $\widetilde{\mathcal{G}}$;*
(ii) *for $\alpha \in \{1, 2, 3\}$ the set $\varrho^{-1}(i_\alpha)$ is a union of connected components of the subgraph in the collinearity graph of $\widetilde{\mathcal{G}}$ induced by the complement of $\widetilde{H}(f)$.*

Proof. Notice first that by (3.10.2) if $\tau(\widetilde{v}) \in \widetilde{O}(f)$ we have $\xi(\widetilde{\tau}(\widetilde{v})) = i_\alpha$ for $\alpha \in \{1, 2, 3\}$. Let $\widetilde{l} = \{\widetilde{v}, \widetilde{u}, \widetilde{w}\}$ be a line in $\widetilde{\mathcal{G}}$ and $l = \{v, u, w\}$ be its image under χ. Then $\{0, v, u, w\}$ is an isotropic subspace in V. Hence f is zero on exactly one or on all three points in l. In the former case $\widetilde{\tau}(\widetilde{p}) \notin \widetilde{O}(f)$ for every $\widetilde{p} \in \widetilde{l}$ and $\widetilde{l}$ is in $\varrho^{-1}(e)$. In the latter case exactly one of the points of $\widetilde{l}$ (say $\widetilde{v}$) is in $\varrho^{-1}(e)$ and hence (i) follows. Also in the latter case we have $\xi(\widetilde{\tau}(\widetilde{u})) = i_\alpha$ and $\xi(\widetilde{\tau}(\widetilde{w})) = i_\beta$. Since $[\widetilde{\tau}(\widetilde{u}), \widetilde{\tau}(\widetilde{w})] = 1$ by (3.10.3 (ii)), we have $\alpha = \beta$, which gives (ii). □

Now by (2.3.6) and (3.10.5) we have the following

Lemma 3.10.6 *Let $F = F(f)$ be the group freely generated by the involutions i_1, i_2 and i_3 and let e be the identity element of F. Then (F, ϱ) is a $\widetilde{O}(f)$-admissible representation of $\widetilde{\mathcal{G}}$, in particular the universal representation group of $\widetilde{\mathcal{G}}$ is infinite.* □

Let $\overline{F}$ be the quotient of F as (3.10.6) over the commutator subgroup of F. Then $\overline{F}$ is elementary abelian of order 2^3 and is a quotient of the universal representation module U of $\widetilde{\mathcal{G}}$. Furthermore $C_{\overline{F}}(E)$ is of order 2 and is a quotient of U^z, while

$$U\lfloor f\rfloor := \overline{F}(f)/C_{\overline{F}(f)}(E)$$

is a 2-dimensional quotient of U^c.

Lemma 3.10.7 *Let U^0 be the direct sum of the representation modules $U\lfloor f\rfloor$ taken for all $f \in \mathcal{Q}^-$. Then U^0 is a representation module of $\widetilde{\mathcal{G}}$ of dimension $2^n(2^n - 1)$.*

Proof. We can define a mapping ϱ^0 from the point-set of $\widetilde{\mathcal{G}}$ into U^0 applying the construction similar to that after the proof of (2.3.2), so that the line relations hold. It is easy to see that the kernels of E acting on the $U\lfloor f\rfloor$ are pairwise different, which implies that U^0 is an irreducible $\widetilde{G}$-module. Hence U^0 is generated by the image of ϱ^0. □

The above lemma gives a lower bound on the dimension of U^c. We complete the proof of (3.10.1) by establishing the upper bound using the technique of Section 2.4. We are going to show that in the considered situation the condition **(M)** from Section 2.4 holds and describe the acceptable hyperplanes in $\mathcal{G}$. Towards this end we need a better understanding of the structure of E as a $GF(3)$-module for $G(v) \cong 2^{2n-1} : S_{2n-2}(2)$.

As above let $\widetilde{v}$ be a point of $\widetilde{\mathcal{G}}$ such that $\chi(\widetilde{v}) = v$ and $E(\widetilde{v})$ be the stabilizer of $\widetilde{v}$ in E. The next lemma summarizes what we have observed above.

Lemma 3.10.8 *The following assertions hold:*

(i) *the subgroup $E(\widetilde{v})$ is independent on the particular choice of $\widetilde{v} \in \chi^{-1}(v)$ (and hence will be denoted by $E(v)$);*
(ii) *the subgroup $E(v)$ is of order $3^{[\frac{n-1}{2}]_2}$ and it coincides with O_3 of the action of $\widetilde{G}(\widetilde{v})$ on $\mathrm{res}_{\widetilde{\mathcal{G}}}(\widetilde{v}) \cong \mathcal{G}(3^{[\frac{n-1}{2}]_2} \cdot S_{2n-2}(2))$;*
(iii) $E(v) \leq C_E(K(v))$. □

We have observed in Section 3.5 that $K(v) = O_2(G(v))$ is elementary abelian isomorphic to the orthogonal module of $G(v)/K(v) \cong S_{2n-2}(2) \cong \Omega_{2n-1}(2)$. Hence (3.5.4 (ii)) implies that $G(v)$ has three orbits, $\mathcal{H}^p$, $\mathcal{H}^+$ and $\mathcal{H}^-$ on the set $\mathcal{H}$ of hyperplanes (subgroups of index 2) in $K(v)$

with lengths

$$2^{2n-2}-1,\quad 2^{n-2}(2^{n-1}+1),\quad 2^{n-2}(2^{n-1}-1),$$

respectively.

On the other hand, since $K(v)$ is a 2-group,

$$E = C_E(K(v)) \oplus [E, K(v)]$$

and every non-trivial irreducible $K(v)$-submodule in E is 1-dimensional contained in $[E, K(v)]$ with kernel being a hyperplane in $K(v)$. Let $\mathcal{E}_H$ be the sum of the irreducibles for which H is the kernel. It is clear that $\dim \mathcal{E}_H$ is independent on the choice of H from its $G(v)$-orbit and we have the following decomposition

$$[E, K(v)] = \bigoplus_{H \in \mathcal{H}} \mathcal{E}_H.$$

Since $\dim C_E(K(v)) \geq \dim E(v) = [{}^{n-1}_{\ 2}]_2$ by (3.10.8 (iii)), we conclude that $\dim [E, K(v)]$ is at most $[{}^{n}_{2}]_2 - [{}^{n-1}_{\ 2}]_2 = 2^{n-2}(2^{n-1}-1)$, which is exactly the length of the shortest $G(v)$-orbit on $\mathcal{H}$. This gives the following.

Lemma 3.10.9 *The following assertions hold:*

(i) $E(v) = C_E(K(v))$;

(ii) $[E, H] = [E, K(v)]$ *for all* $H \in \mathcal{H} \setminus \mathcal{H}^-$;

(iii) $[E, K(v)]$ *possesses the direct sum decomposition*

$$[E, K(v)] = \bigoplus_{H \in \mathcal{H}^-} \mathcal{E}_H,$$

where $\mathcal{H}^-$ *is the* $G(v)$*-orbit on the hyperplanes in* $K(v)$ *indexed by the quadratic forms of minus type and* $\dim \mathcal{E}_H = 1$;

(iv) $G(v)$ *induces on the set of direct summands in (ii) the doubly transitive action of* $G(v)/K(v) \cong S_{2n-2}(2)$ *on the cosets of* $O^-_{2n-2}(2)$;

(v) *the element* $\tau(v)$ *negates* $\mathcal{E}_H$ *for every* $H \in \mathcal{H}^-$, *so that* $E(v) = C_E(\tau(v))$.

Proof. The assertions (i) to (iv) follow from the equality of upper and lower bounds on $\dim [E, K(v)]$ deduced before the lemma. Since $\tau(v)$ is in the centre of $G(v)$ and the latter acts transitively on $\mathcal{H}^-$, it is clear that $\tau(v)$ acts on all the $\mathcal{E}_H$ in the same way. Since $\tau(v)$ can not centralize the whole E, (v) follows. □

In order to establish the condition **(M)** we need the following lemma.

Lemma 3.10.10 *Let $\{v,u,w\}$ be a line in $\mathscr{G}$. Then*
(i) *the images of $\tau(u)$ and $\tau(w)$ in $G(v)/K(v)$ are non-trivial and equal;*
(ii) $E(u) \cap E(w) \leq E(v)$.

Proof. It is immediate from (3.10.3 (ii)) that $[\tau(v), \tau(u)] = 1$ and hence $\tau(u) \in G(v)$ (similarly for w). It is easy to deduce directly from the definition of the transvections $\tau(u)$ and $\tau(w)$ that they induce the same non-trivial action on $\mathrm{res}_{\mathscr{G}}(v)$, which gives (i). By (3.10.9 (v)) $E(v) \cap E(u) = C_{E(v)}(\tau(u))$ and $E(v) \cap E(w) = C_{E(v)}(\tau(w))$. Since $K(v)$ commutes with $E(v)$, in view of (i), we have $E(v) \cap E(u) = E(v) \cap E(w)$. By the obvious symmetry, the intersections are also equal to $E(u) \cap E(w)$ and hence (ii) follows. □

Lemma 3.10.11 *In the considered situation the condition* **(M)** *holds.*

Proof. Put $I = \mathscr{H}^-$ (which is the $G(v)$-orbit on the set of hyperplanes in $K(v)$ indexed by the quadratic forms of minus type) and for $i \in I$ let B_i be the image in $E/E(v)$ of the subspace $\mathscr{E}_i$ as in (3.10.9 (iii)). Then the B_i are 1-dimensional and $G(v)$ permutes them doubly transitively by (3.10.9 (iv)). Thus in order to show that the graph Σ in the condition **(M)** is connected, it is sufficient to show that it has at least one edge. Let $\{v,u,w\}$ be a line in $\mathscr{G}$. Then by (3.10.9 (v)) and (3.10.10) $E(u) \neq E(v)$ and therefore $\tau(u)$ has on I an orbit $\{i,j\}$ of length 2. By (3.10.10 (i)) the action of $\tau(w)$ on I coincides with that of $\tau(u)$ and hence $\{i,j\}$ is also a $\tau(w)$-orbit. Put $B_{ij} = \langle B_i, B_j \rangle$ and let B_u and B_w be the centralizers in B_{ij} of $\tau(u)$ and $\tau(w)$, respectively. Then B_u and B_w are contained in the images in $E/E(v)$ of $E(u)$ and $E(w)$, respectively, and $B_u \neq B_w$ by (3.10.10 (ii)). Since clearly $\{B_u, B_w\} \cap \{B_i, B_j\} = \emptyset$, **(M)** holds. □

Now we are going to complete the proof of (3.10.1) by showing that U^0 as in (3.10.7) is the whole U^c. Since the condition **(M)** holds by (3.10.11) we have to bound the number of acceptable hyperplanes. First of all since U^0 is a non-trivial quotient of U^c, there are acceptable hyperplanes. By noticing that the dimension of U^0 is twice the length of the shortest G-orbit on the set of geometric hyperplanes in $\mathscr{G}$, we conclude that (in the notation of (3.5.3)) the hyperplanes $H(f)$ for $f \in \mathscr{Q}^-$ are acceptable. Since the universal representation group of $\mathscr{G}$ is finite by (3.5.5), (2.4.8) applies and shows that $\dim U^c$ is at most twice the number of acceptable hyperplanes in $\mathscr{G}$. Hence it remains to prove the following.

Lemma 3.10.12 *Let H be a geometric hyperplane in $\mathscr{G}$, such that either $H = H(f)$ for $f \in \mathscr{Q}^+$ or $H = H(v)$ for a point v on $\mathscr{G}$. Then H is not acceptable.*

Proof. Suppose that H is acceptable. Then by (2.4.6 (i)) the subgroups $E(u)$ taken for all points u of $\mathscr{G}$ outside H generate a subgroup $Y(H)$ of index 3 in E. It is clear that $Y(H)$ is normalized by the stabilizer $G(H)$ of H in G. We know by Lemma 6.7.3 in [Iv99] that E (as a $GF(3)$-module for G) is self-dual. Hence $G(H)$ must normalize in E a 1-dimensional subspace (which is the dual of $Y(H)$).

Let x be an element of type n in $\mathscr{G}$, so that x is a maximal totally isotropic (which means n-dimensional) subspace in V. Its stabilizer $G(x) \cong 2^{n(n+1)/2} : L_n(2)$ acts monomially on E (cf. Lemma 6.8.1 in [Iv99]). More specifically $O_2(G(x))$ preserves the direct sum decomposition

$$E = \bigoplus_{\alpha \in \mathscr{L}^2} T_\alpha.$$

Here $\mathscr{L}^2$ is the set of 2-dimensional subspaces of x and every T_α is a 1-dimensional non-trivial module for $O_2(G(x))$. The factor group $G(x)/O_2(G(x)) \cong L_n(2)$ permutes the direct summands in the natural way (in particular the action is primitive). The kernels of the action of $O_2(G(x))$ on different T_α are pairwise different, in particular $G(x)$ acts irreducibly on E. We are going to show that $G(H,x) := G(H) \cap G(x)$ does not normalize 1-subspaces in E.

Let $H = H(f)$ for $f \in \mathscr{Q}^+$ and assume that x is totally singular with respect to f, in which case $G(H,x) \cong 2^{n(n-1)/2} : L_n(2)$. Since $G(H,x)O_2(G(x)) = G(x)$, we conclude that $G(H,x)$ acts primitively on the set of direct summands T_α. Hence the kernels of the action of $O_2(H,x)$ on different T_α are different and $G(H,x)$ still acts irreducibly on E, particularly it does not normalize 1-subspaces in E.

Finally let $H = H(v)$ where v is a point and we assume that v is contained in x. Then $G(H,x) = G(v) \cap G(x)$ contains $O_2(G(x))$ and has two orbits $\mathscr{L}_1^2(v)$ and $\mathscr{L}_2^2(v)$ on $\mathscr{L}^2$ with lengths $[{n-1 \atop 1}]_2$ and $[{n \atop 2}]_2 - [{n-1 \atop 1}]_2$ consisting of the 2-subspaces in x containing v and disjoint from v, respectively. Since $n \geq 3$, each orbit contains more than one element. Then E, as a module for $G(H,x)$, is the direct sum of two irreducible submodules

$$\bigoplus_{\alpha \in \mathscr{L}_1^2(v)} T_\alpha \quad \text{and} \quad \bigoplus_{\alpha \in \mathscr{L}_2^2(v)} T_\alpha,$$

each of dimension more than 1 and hence again $G(H,x)$ does not normalize 1-subspaces in E. □

4

Mathieu groups and Held group

Let $(\mathcal{P}, \mathcal{B})$ be a Steiner system of type $S(5,8,24)$, where $\mathcal{P}$ is a set of 24 elements and $\mathcal{B}$ is a set of 759 8-element subsets of $\mathcal{P}$ called *octads* such that every 5-element subset of $\mathcal{P}$ is in a unique octad. Such a system is unique up to isomorphism and its automorphism group is the sporadic Mathieu group M_{24}. The octads from $\mathcal{B}$ generate in the power space $2^{\mathcal{P}}$ of $\mathcal{P}$ a 12-dimensional subspace $\mathcal{C}_{12}$ called the *Golay code* . The empty set and the whole set $\mathcal{P}$ form a 1-dimensional subspace in $\mathcal{C}_{12}$ and the corresponding quotient $\mathcal{C}_{11}$ is an irreducible $GF(2)$-module for M_{24}. The quotient $\overline{\mathcal{C}}_{12} = 2^{\mathcal{P}}/\mathcal{C}_{12}$ (equivalently the dual of $\mathcal{C}_{12}$) is the *Todd module* . It contains a codimension 1 submodule $\overline{\mathcal{C}}_{11}$ which is dual to $\mathcal{C}_{11}$ ($\overline{\mathcal{C}}_{11}$ is called the *irreducible Todd module*). The stabilizer in M_{24} of an element $p \in \mathcal{P}$ is the Mathieu group M_{23} and the stabilizer of an ordered pair (p,q) of such points is the Mathieu group M_{22}. The setwise stabilizer of $\{p,q\}$ is the automorphism group $\mathrm{Aut}\, M_{22}$ of M_{22}. The irreducible Todd module $\overline{\mathcal{C}}_{11}$ restricted to $\mathrm{Aut}\, M_{22}$ is an indecomposable extension of a 10-dimensional Todd module $\overline{\mathcal{C}}_{10}$ for $\mathrm{Aut}\, M_{22}$ by a 1-dimensional submodule. Recall that a *trio* is a partition of $\mathcal{P}$ into three octads and a *sextet* is a partition of $\mathcal{P}$ into six 4-element subsets (called *tetrads*) such that the union of any two such tetrads is an octad.

4.1 $\mathcal{G}(M_{23})$

For the rank 4 P-geometry $\mathcal{G} = \mathcal{G}(M_{23})$ the universal representation group is trivial. Indeed, the point set of $\mathcal{G}$ is $\mathcal{P} \setminus \{p\}$ for an element $p \in \mathcal{P}$ and the automorphism group $G \cong M_{23}$ of $\mathcal{G}$ acts triply transitively on the set of points. Hence every 3-element subset of points is a line, which immediately implies the following result.

Proposition 4.1.1 *The universal representation group of $\mathcal{G}(M_{23})$ is trivial.* □

Since the representation group of $\mathcal{G}(M_{23})$ is trivial, the geometry does not possess flag-transitive affine c-extensions but there exists a non-affine flag-transitive c-extension having M_{24} as the automorphism group (2.7.5).

4.2 $\mathcal{G}(M_{22})$

If $\{p,q\}$ is a 2-element subset of $\mathcal{P}$ then the points of the rank 3 P-geometry $\mathcal{G} = \mathcal{G}(M_{22})$ are the sextets which contain p and q in the same tetrad. Since every tetrad is contained in a unique sextet the set of points can be identified with the set of 2-element subsets of $\mathcal{Q} := \mathcal{P} \setminus \{p,q\}$. If B is an octad containing $\{p,q\}$ then the 6-element subset $H := B \setminus \{p,q\}$ is called a *hexad*. There are 77 hexads which define on $\mathcal{Q}$ the structure of a Steiner system $S(3,6,22)$, in particular, every 3-element subset of $\mathcal{Q}$ is in a unique hexad. In these terms a triple of points of $\mathcal{G}$ is a line if and only if the union of these points is a hexad. Then the automorphism group $G \cong \mathrm{Aut}\, M_{22}$ of $\mathcal{G}$ is the setwise stabilizer of $\{p,q\}$ in the automorphism group of $(\mathcal{P},\mathcal{B})$ isomorphic to M_{24}. The octads from $\mathcal{B}$ disjoint from $\{p,q\}$ are called *octets*. The octets are the elements of type 3 in $\mathcal{G}(M_{22})$.

From the action of Co_2 on the rank 4 P-geometry $\mathcal{G}(Co_2)$ containing $\mathcal{G}(M_{22})$ as a point residue, we observe that the 10-dimensional Todd module is a representation module of $\mathcal{G}$. Let $x = \{a,b\}$ be a 2-element subset of $\mathcal{Q}$ (a point of $\mathcal{G}$) and $\psi(x)$ be the image in $\overline{\mathcal{C}}_{11}$ of the subset $\{p,q,a,b\}$ of $\mathcal{P}$.

Lemma 4.2.1 *$(\overline{\mathcal{C}}_{11},\psi)$ is an abelian representation of $\mathcal{G}(M_{22})$.*

Proof. Let $\{x_1,x_2,x_3\}$ be a line in $\mathcal{G}$, where $x_i = \{a_i,b_i\}$ for $1 \le i \le 3$. Then $\psi(x_1)+\psi(x_2)+\psi(x_3)$ is the image in $\overline{\mathcal{C}}_{11}$ of the set $\{p,q,a_1,b_1,a_2,b_2,a_3,b_3\}$ which is an octad and hence the image is zero. □

We will show that $(\overline{\mathcal{C}}_{11},\psi)$ is the universal representation of $\mathcal{G}$. First we show that if (V,χ) is the universal abelian representation of $\mathcal{G}$, then the dimension of V is at most 11.

Let H be a hexad. It follows directly from the definitions that the points and lines contained in H form a subgeometry $\mathcal{S}$ in $\mathcal{G}$ isomorphic to $\mathcal{G}(S_4(2))$ (cf. Lemma 3.4.4 in [Iv99]).

Lemma 4.2.2 *Let (V,χ) be the universal abelian representation of $\mathcal{G}(M_{22})$ and H be a hexad. Then*

(i) *the subspace $V[H]$ in V generated by the vectors $\chi(x)$ taken for all points x contained in H is a quotient of the universal representation module $V(\mathscr{G}(S_4(2))$ of $\mathscr{G}(S_4(2))$;*

(ii) *for every element $r \in H$ the vectors $\chi(\{r,s\})$ taken for all $s \in H \setminus \{r\}$ generate in $V[H]$ a subspace of codimension at most 1 and*

$$\sum_{s \in H \setminus \{r\}} \chi(\{r,s\}) = 0.$$

Proof. (i) follows from (2.1.2), (3.4.4) and the paragraph before the lemma while (ii) is implied by a property of $V(\mathscr{G}(S_4(2))$. □

Notice that in $\overline{\mathscr{C}}_{11}$ the images of all the pairs contained in a hexad generate a 5-dimensional subspace.

Let $r \in \mathscr{Q}$, $\mathscr{R} = \mathscr{Q} \setminus \{r\}$ and $\mathscr{L}$ be the set of hexads containing r (equivalently the octads containing $\{p,q,r\}$). Then by the basic property of the Steiner system $S(5,8,24)$ we observe that with respect to the natural incidence relation $\Pi = (\mathscr{R}, \mathscr{L})$ is a projective plane over $GF(4)$.

Lemma 4.2.3 *Let $V\lfloor r \rfloor$ be the submodule in V generated by the vectors $\chi(\{r,s\})$ taken for all $s \in \mathscr{R}$. Then the dimension of $V\lfloor r \rfloor$ is at most* 11.

Proof. Let L be the stabilizer of r in G. Then $L \cong P\Sigma L_3(4)$ acts doubly transitively on $\mathscr{R}$. Thus $V\lfloor r \rfloor$ is a quotient of the $GF(2)$-permutation module of L acting on the set of points of Π. Furthermore by (4.2.2 (ii)) the sum of points on a line is zero. Now the proof follows from the structure of the permutation module given in (3.1.4). □

Proposition 4.2.4 *In the above terms $V = V\lfloor r \rfloor$, in particular,* $\dim V = 11$ *and $V \cong \overline{\mathscr{C}}_{11}$.*

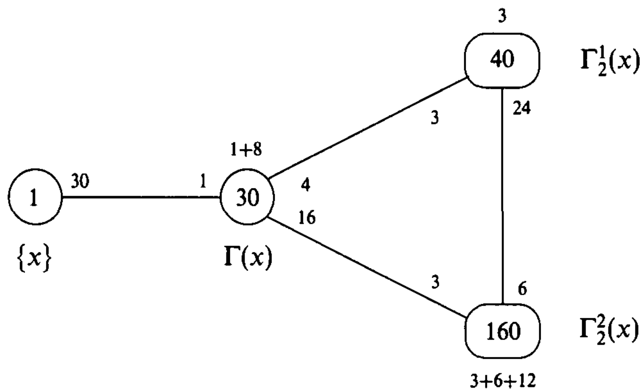
Proof. Suppose that $V \neq V\lfloor r \rfloor$ and put $\overline{V} = V/V\lfloor r \rfloor$. Since every point of $\mathscr{G}$ is contained in a hexad containing r, $V(H)$ is not contained in $V\lfloor r \rfloor$ for some hexad H containing r. Since $V\lfloor r \rfloor$ is normalized by L and L acts (doubly) transitively on the set $\mathscr{L}$ of hexads containing r, the image of $V(H)$ in $\overline{V}$ is non-trivial for every hexad H containing r. By (4.2.2 (ii)) this image is 1-dimensional. Let $i(H)$ denote the unique non-zero vector in this image. By considering a hexad which does not contain r we can find a triple H_1, H_2, H_3 of hexads containing r such that

$$i(H_1) + i(H_2) + i(H_3) = 0.$$

Since L acts doubly transitively on the set of 21 hexads containing r, this implies that $\overline{V}^{\#} = \{i(H) \mid r \in H\}$, which is not possible since 21 is not a power of 2 minus one. □

Proposition 4.2.5 *$(\overline{\mathscr{C}}_{11}, \psi)$ is the universal representation of $\mathcal{G}(M_{22})$.*

Proof. By (4.2.4) all we have to show is that if (R, χ) is the universal representation of $\mathcal{G}$ then R is abelian. Let Γ be the collinearity graph of $\mathcal{G} = \mathcal{G}(M_{22})$ whose suborbit diagram is the following:



Recall that $y \in \Gamma \setminus \{x\}$ is contained, respectively, in $\Gamma(x)$, $\Gamma_2^1(x)$ and $\Gamma_2^2(x)$ if x, y are disjoint contained in a hexad, intersect in one element, and are disjoint, not in a hexad (here x and y are considered as 2-element subsets of $\mathcal{Q}$).

We apply (2.3.7) for $B(x) = \Gamma_2^2(x)$ and $A(x) = \Gamma \setminus B(x)$. Since the action of $G \cong \operatorname{Aut} M_{22}$ on Γ is vertex-transitive and

$$|\Gamma_2^2(x)| > 1 + |\Gamma(x)| + |\Gamma_2^1(x)|,$$

Ξ is connected. Let us show that Σ^x is connected. For a hexad H let $\Gamma[H]$ be the subgraph in Γ induced by the points contained in H. Then $\Gamma[H]$ is the collinearity graph of $\mathcal{G}(S_4(2))$. Let $x = \{a, b\}$ and H be a hexad which contains a and does not contain b. Since any two hexads intersect in at most 2 elements, it is easy to see that the intersection $\Gamma[H] \cap \Gamma_2^2(x)$ is of size 10 (the pairs contained in H and disjoint from a) and the subgraph in Σ^x induced by the intersection is isomorphic to the Petersen graph. Since the hexads form the Steiner system $S(3, 6, 22)$, for every $z \in \Gamma_2^2$ there is a unique hexad which contains a and z (this hexad does not contain b). Hence the subgraphs induced by the subsets $\Gamma[H] \cap \Gamma_2^2(x)$ taken for all hexads containing a and not containing b form a partition of $\Gamma_2^2(x)$ into 16 disjoint Petersen subgraphs. In a similar way the hexads containing b and not containing a define another partition of $\Gamma_2^2(x)$ into 16 disjoint Petersen subgraphs. Furthermore, two Petersen subgraphs from different partitions intersect in at most one point. Hence every connected component of Σ^x contains at least $100 = 10 \times 10$ vertices. Since $G(x)$ acts transitively on $\Gamma_2^2(x)$ and a connected component is an imprimitivity block, we conclude that Σ^x is connected. By (3.4.4) for a hexad H the points contained in H generate in R an abelian group (of order at most 2^5). Since whenever $y \in A(x)$ there is a hexad containing x and y, all the assumptions of (2.3.7) are satisfied and the commutator subgroup of R has order at most 2. By (4.2.4), (2.3.8) and (2.3.9) if R is non-abelian, G acting on $\overline{\mathscr{C}}_{11}$ preserves a non-zero symplectic form. On the other hand, $\overline{\mathscr{C}}_{11}$ as a module for G is indecomposable with irreducible factors of dimension 1 and 10 (cf. Lemma 2.15.3 in [Iv99]), which shows that there is no such form. Hence R is abelian and the result follows. □

Let (V, φ) be the universal representation group of $\mathscr{G}$, so that $V \cong \overline{\mathscr{C}}_{11}$. Let x be a point of $\mathscr{G}$. We will need some information of the structure of V as a module for $G(x) \cong 2^5.Sym_5$. Put $\overline{G}(x) = G(x)/O_2(G(x)) \cong Sym_5$. and follow the notation introduced in Section 2.2.

Lemma 4.2.6 *The following assertions hold:*

(i) $\overline{V}_1(x)$ *is the universal representation module for* $\mathrm{res}_{\mathscr{G}}(x) \cong \mathscr{G}(Alt_5)$, *in particular* $\dim \overline{V}_1(x) = 6$;

(ii) $V_2(x) = V[\Gamma_2^1(x)] = V[\Gamma_2^2(x)]$;

(iii) $\overline{V}_2(x)$ *is the* 4*-dimensional orthogonal module for* $\overline{G}(x)$ *(with orbits on non-zero vectors of lengths* 5 *and* 10*).*

Proof. Let $V^* = \langle \varphi(y) \mid y \in \Gamma \setminus \Gamma_2^2(x) \rangle$. We claim that $V^* = V$. By the last paragraph of the proof of (4.2.5) the graph Σ^* on $\Gamma_2^2(x)$ is connected, hence

$$\dim\ V/V^* \leq 1.$$

If the equality holds, then V^* is a hyperplane in V stabilized by $G(x)$ which contradicts the fact that $V \cong \overline{\mathscr{C}}_{11}$ is not self-dual. Hence the claim follows and implies (ii).

Let $H_1, \dots, H_5$ be the hexads containing x. Then the intersections $\Gamma[H_i] \cap \Gamma_2^1(x)$, $1 \leq i \leq 5$ form a partition of $\Gamma_2^1(x)$. By (3.5.4) the image of $V[\Gamma[H]]$ in $\overline{V}_2(x)$ is at most 1-dimensional. In view of (ii) this implies that $\overline{V}_2(x)$ is at most 5-dimensional. Now let H be a hexad intersecting $x = \{a, b\}$ in exactly one point (as in the proof of (4.2.5)). Then (compare the diagram on p. 138 in [Iv99]) the stabilizer of H in $G(x)$ is a complement to $O_2(G(x))$, isomorphic to Sym_5 and

$$| \Gamma[H] \cap \Gamma_2^1(x) |= 5.$$

This implies that $| \Gamma[H] \cap \Gamma[H_i] |= 1$ for $1 \leq i \leq 5$ and hence $V[\Gamma[H]]V_1(x) = V$. On the other hand, $\Gamma[H] \cap \Gamma_2^1(x)$ is a geometric hyperplane in the $\mathcal{G}(S_4(2))$-geometry associated with H. Hence $\varphi(\Gamma[H] \cap \Gamma_2^1(x))$ span in V a 4-dimensional subspace. By (3.9.2) $\overline{V}_1(x)$ is at most 6-dimensional and so by the dimension consideration we obtain (i) and (iii). □

Notice that in (4.2.6) the module $\overline{V}_2(x)$ is isomorphic to the derived group of $\mathrm{res}_{\mathcal{G}}(x) \cong \mathcal{G}(Alt_5)$ (compare (3.9.4)).

4.3 $\mathcal{G}(M_{24})$

Considering the action of Co_1 on the rank 4 T-geometry $\mathcal{G}(Co_1)$ we observe that $\overline{\mathcal{C}}_{11}$ is a representation group of $\mathcal{G} = \mathcal{G}(M_{24})$. Let (R, φ) be the universal representation of $\mathcal{G}$. Recall that the points of $\mathcal{G}$ are the sextets. For a 2-element subset $\{p, q\}$ of $\mathcal{P}$ the sextets containing $\{p, q\}$ in a tetrad induce a subgeometry $\mathcal{F}(p, q)$ isomorphic to $\mathcal{G}(M_{22})$ (the lines and planes in the subgeometry are those of $\mathcal{G}$ contained in the point set of $\mathcal{F}(p, q)$). By (4.2.5) the image $\varphi(\mathcal{F}(p, q))$ of the points from the subgeometry $\mathcal{F}(p, q)$ in R is abelian of order at most 2^{11} isomorphic to a quotient of $\overline{\mathcal{C}}_{11}$. Let $S = \{p, q, r\}$ be a 3-element subset of $\mathcal{P}$. Then the intersection $\mathcal{F}(p, q) \cap \mathcal{F}(q, r)$ is of size 21 consisting of the sextets containing S in a tetrad. By (4.2.4) $\varphi(\mathcal{F}(p, q))$ is generated by $\varphi(\mathcal{F}(p, q) \cap \mathcal{F}(q, r))$, which immediately shows that $\varphi(\mathcal{F}(p, q)) = \varphi(\mathcal{F}(q, r))$. Since the graph on the set of 2-element subsets of $\mathcal{P}$, in which two such subsets are adjacent if their union is a 3-element subset, is connected, we conclude that $R = \varphi(\mathcal{F}(p, q))$, which gives the following.

Proposition 4.3.1 *The group $R(\mathcal{G}(M_{24}))$ is abelian isomorphic to the irreducible Todd module $\overline{\mathcal{C}}_{11}$.* □

We will need some further properties of $\overline{\mathcal{C}}_{11}$ as a representation group of $\mathcal{G}$. Let $x \in \Pi$, $G(x) \cong 2^6.3 \cdot S_4(2)$ be the stabilizer of x in $G \cong M_{24}$

and let Γ be the collinearity graph of $\mathcal{G}$. Let (V, φ) be the universal representation of $\mathcal{G}$ (where $V \cong \overline{\mathcal{C}}_{11}$ by (4.3.1)). The following result is immediate from the structure of V as a module for $G(x)$ (cf. Section 3.8 in [Iv99]).

Lemma 4.3.2 *Let* $\overline{G}(x) = G(x)/O_2(G(x)) \cong 3 \cdot S_4(2)$. *Then the following assertions hold:*

(i) $\overline{V}_1(x)$ *is isomorphic to the hexacode module for* $\overline{G}(x)$*;*
(ii) $V_2(x) = V[\Gamma_2^2(x)] = V[\Gamma_2^1(x)]$ *and* $\overline{V}_2(x)$ *is isomorphic to the 4-dimensional symplectic module of* $\overline{G}(x)/O_3(\overline{G}(x)) \cong S_4(2)$. □

4.4 $\mathcal{G}(3 \cdot M_{22})$

Let $\mathcal{G} = \mathcal{G}(3 \cdot M_{22})$, $G = \operatorname{Aut}\mathcal{G} \cong 3 \cdot \operatorname{Aut} M_{22}$, $E = O_3(G)$, $\mathcal{S} = (\Pi, L)$ be the point-line incidence system of $\mathcal{G}$ and $\mathcal{S}^*$ be the enrichment of $\mathcal{S}$ with respect to E. Recall that the quotient $\overline{\mathcal{G}}$ of $\mathcal{G}$ with respect to the action of E is isomorphic to $\mathcal{G}(M_{22})$. The point set of $\overline{\mathcal{G}}$ is the set of 2-element subsets of $\mathcal{Q} = \mathcal{P} \setminus \{p, q\}$. In this subsection we determine the universal representation module of $\mathcal{G}$ and the universal representation group of $\mathcal{S}^*$. We do not know what the universal representation group of $\mathcal{G}$ is and even whether or not it is finite.

Let (V, φ) be the universal abelian representation of $\mathcal{G}$. In terms of Subsection 2.4, $V = V^z \oplus V^c$. By (2.4.1) V^z is the universal representation module of $\mathcal{G}(M_{22})$ (isomorphic to $\overline{\mathcal{C}}_{11}$ by (4.2.4)) and by (2.4.3) V^c is the universal representation module of $\mathcal{S}^*$. Hence to achieve our goal it is sufficient to calculate the universal representation group R^* of $\mathcal{S}^*$, since V^c is the quotient of R^* over its commutator subgroup.

Lemma 4.4.1 *The group* R^* *possesses a G-invariant factor group isomorphic to* $Q \cong 2_+^{1+12}$.

Proof. Consider the action of $J \cong J_4$ on the rank 4 P-geometry $\mathcal{J} = \mathcal{G}(J_4)$. Let x be a point of $\mathcal{J}$, $J(x)$ be the stabilizer of x in J and $Q = O_2(J(x))$. Then $\operatorname{res}_{\mathcal{J}}(x) \cong \mathcal{G}$, Q is the kernel of the action of $J(x)$ on $\operatorname{res}_{\mathcal{J}}(x)$, $Q \cong 2_+^{1+12}$ and $J(x)/Q \cong G \cong 3 \cdot \operatorname{Aut} M_{22}$. Furthermore $Z(Q)$ is the kernel of the action of $J(x)$ on the set of points collinear to x. By (1.5.1) there is a $J(x)$-invariant mapping χ of the point set Π of $\mathcal{G} \cong \operatorname{res}_{\mathcal{J}}(x)$ into $\overline{Q} = Q/Z(Q)$ such that $(\overline{Q}, \chi)$ is an abelian representation of $\mathcal{G}$.

Let E be a Sylow 3-subgroup of $O_{2,3}(J(x))$. Then $EQ/Q = O_3(G)$ and E acts fixed-point freely on $\overline{Q}$, which implies that $(\overline{Q}, \chi)$ is a representation of the enriched point-line incidence system of $\mathcal{G}$. Let $\widetilde{G} = N_{J(x)}(E)$. Then $\widetilde{G}/Z(G) \cong G$ (in fact $\widetilde{G}$ does not split over $Z(Q)$). Let $\overline{\Phi} = \mathrm{Im}(\varphi)$ and Φ be the preimage of $\overline{\Phi}$ in Q. We claim that (a) Φ consists of involutions and (b) $\widetilde{G}$ acting on Φ has two orbits. Let D be the point set of a plane in $\mathcal{G}$. Then $\chi(D)$ is the set of 7 non-identity elements of an elementary abelian subgroup $\overline{A}$ of order 2^3 in $\overline{Q}$. Then normalizer of $\overline{A}$ in G induces on $\overline{A}$ the natural action of $L_3(2)$, in particular this action does preserve a non-trivial quadratic form on $\overline{A}$. Hence the preimage A of $\overline{A}$ in Q is elementary abelian and (a) follows. Now let T be an E-orbit on the point set Π of $\mathcal{G}$. Then $\chi(T)$ is the set of non-identity elements of an elementary abelian subgroup $\overline{B}$ in $\overline{Q}$ of order 2^2. By (a) the preimage B of $\overline{B}$ in Q is elementary abelian of order 2^3. Clearly E, acting on $B^{\#}$ has two orbits, say B_1 and B_2 each of length 3. It is easy to see that $B = \langle B_i \rangle$ for exactly one $i \in \{1, 2\}$. This means that the images of B_1 and B_2 under $\widetilde{G}$ form two different orbits of $\widetilde{G}$ on Φ and the claim follows. Applying (2.8.1) we obtain the proof. □

Proposition 4.4.2 *The universal representation module V^c of the enriched point-line incidence system $\mathcal{S}^*$ is 12-dimensional isomorphic to $\overline{Q} = Q/Z(Q)$.*

Proof. (A few lemmas will be formulated within the proof.) The fixed-point free action of E on V^c turns the latter into a $GF(4)$-vector space, so that the representation of $\mathcal{S}^*$ in V^c induces a mapping ν of the point set $\overline{\Pi}$ of $\overline{\mathcal{G}}$ into the set of 1-dimensional $GF(4)$-subspaces in V^c. Throughout the proof the dimensions of V^c and its subspaces are $GF(4)$-dimensions. By (4.4.1) all we have to show is that $\dim V^c \leq 6$. If H is a hexad, then the preimages of the points from $\overline{\Pi}$ contained in H form in $\mathcal{G}$ a subgeometry isomorphic to $\mathcal{G}(3 \cdot S_4(2))$ and hence by (3.8.1) and the fact that $3 \cdot S_4(2)$ acts irreducibly on the hexacode module, we obtain the following, where $V^c(H)$ is the subspace in V^c generated by the images under ν of the points contained in H.

Lemma 4.4.3 $\dim V^c(H) = 3$. □

Notice that the set of 1-dimensional subspaces $\nu(\overline{x})$ for $\overline{x} \in H$ are equal to the set of 15 points outside a hyperoval in the projective $GF(4)$-

space associated with $V^c(H)$. From the basic properties of the projective $GF(4)$-space (cf. Section 2.7 in [Iv99]) we deduce the following.

Lemma 4.4.4 *Let $\overline{x}$, $\overline{y}$ be different points contained in a hexad H and let W be the 2-dimensional subspace of $V^c(H)$ generated by $v(\overline{x})$ and $v(\overline{y})$. Let m be the number of 1-dimensional subspaces in W of the form $v(\overline{z})$ for $\overline{z} \in H$. Then $m = 5$ if $|\overline{x} \cap \overline{y}| = 1$ and $m = 3$ if $\overline{x}$ and $\overline{y}$ are disjoint.* □

Let $r \in \mathcal{Q}$ and $V^c\lfloor r \rfloor$ be the subspace in V^c generated by the images under v of the 2-element subsets in $\mathcal{Q}$ (points of $\overline{\mathcal{G}}$) from the set

$$\Delta = \{\{r, s\} \mid s \in \mathcal{Q} \setminus \{r\}\}.$$

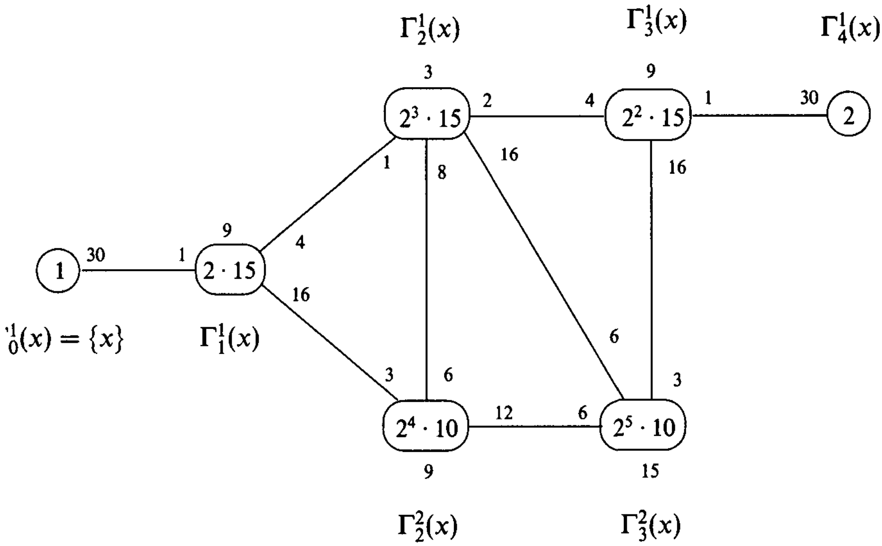
Let $\overline{x}, \overline{y}$ be different points from Δ and H be the unique hexad containing $\overline{x}$ and $\overline{y}$. Since $\overline{x} \cap \overline{y} = \{r\}$, by (4.4.4) every 1-dimensional subspace in the 2-dimensional subspace of $V^c(H)$ generated by $v(\overline{x})$ and $v(\overline{y})$ is of the form $v(\overline{z})$ for some $\overline{z} \in \Delta$. Hence every 1-dimensional subspace in $V^c\lfloor r \rfloor$ is of the form $v(\overline{z})$ for some $\overline{z} \in \Delta$ and we have the following.

Lemma 4.4.5 $\dim V^c\lfloor r \rfloor = 3$. □

Let $\overline{V}^c = V^c / V^c\lfloor r \rfloor$. For a hexad H containing r the image $\overline{V}^c(H)$ in $\overline{V}^c$ of $V^c(H)$ is 1-dimensional and it is easy to see that for $s \in \mathcal{Q} \setminus \{r\}$ the image in $\overline{V}^c$ of $V^c\lfloor s \rfloor$ is 2-dimensional, and every 1-dimensional subspace in this image is of the form $\overline{V}^c(H)$ for a hexad containing r and s (there are exactly 5 such hexads). Since the stabilizer of r in $\overline{G}$ acts doubly transitively on the 21 subspaces $\overline{V}^c(H)$ taken for all the hexads H containing r, we conclude that these are all 1-dimensional subspaces in $\overline{V}^c$. Hence $\dim \overline{V}^c = 3$ and in view of (4.4.5) this completes the proof of Proposition 4.4.2. □

Proposition 4.4.6 *Let (R^*, φ) be the universal representation of the enriched point-line incidence system of $\mathcal{G}(3 \cdot M_{22})$. Then $R^* \cong 2_+^{1+12}$.*

Proof. By (4.4.1) and (4.4.2) all we have to show is that the commutator subgroup of R^* has order at most 2. We apply (2.3.7). The suborbit diagram of the collinearity graph Γ of $\mathcal{G}(3 \cdot M_{22})$ with respect to $3 \cdot \mathrm{Aut}\, M_{22}$ (calculated by D.V. Pasechnik on a computer) is the following



We put $B(x) = \Gamma_3^2(x)$ and $A(x) = \Pi \setminus B(x)$. Let $x_0 = x$, $\{x_1, x_2\} = \Gamma_4^1(x)$. Since $\{x_0, x_1, x_2\}$ is the only imprimitivity block of G on the vertex set of Γ which contains x, the graph Ξ is connected. Our next goal is to show that Σ^x is connected. It is easy to see from the above suborbit diagram that

$$\Gamma_3^2(x) = \Gamma_2^2(x_1) \cup \Gamma_2^2(x_2).$$

Furthermore, E permutes the sets $\Gamma_2^2(x_i)$ for $i = 0, 1, 2$ fixed-point freely. Hence there is a line $\{z_0, z_1, z_2\}$ in $\mathcal{S}^*$ (an orbit of E), such that $z_i \in \Gamma_2^2(x_i)$, $0 \le i \le 2$. Thus it is sufficient to show that the subgraph in Σ^x induced by $\Gamma_2^2(x_1)$ is connected. For a hexad H the set $\Omega(H)$ of the preimages in $\mathcal{G}$ of the points from $\overline{G}$ contained in H induces a subgeometry isomorphic to $\mathcal{G}(3 \cdot S_4(2))$. By (3.8.2) and (4.4.3) the elements $\varphi(y)$ taken for all $y \in \Omega(H)$ generate in R^* an elementary abelian subgroup of order 2^6 isomorphic to the hexacode module for

$$G[\Omega(H)]/O_2(G[\Omega(H)]) \cong 3 \cdot S_4(2).$$

Let $\overline{x} = \{a, b\} \subset \mathcal{Q}$ be the image of x in $\overline{\mathcal{G}}$. If H is a hexad which contains a and does not contain b then comparing the proofs of (4.2.5) and (3.8.4) we can see that $\Omega(H) \cap \Gamma_2^1(x)$ is of size 15 while for every $0 \le i \le 2$ the intersection $\Omega(H) \cap \Gamma_2^2(x_i)$ is of size 10 and induces a Petersen subgraph. Now arguing as in the proof of (4.2.5) we conclude that the subgraph in Σ^x induced by $\Gamma_2^2(x_1)$ is connected.

Let us show that $\varphi(x)$ commutes with $\varphi(y)$ for every $y \in A(x)$. If $y \in \Gamma_i^1(x)$ for $0 \le i \le 4$ then there is a hexad H such that $x, y \in \Omega(H)$ and in this case the conclusion follows from the previous paragraph. Let $R_1^*(x)$ be the subgroup generated by the elements $\varphi(u)$ taken for all $u \in \Gamma_1^1(x)$ and $\overline{R}_1^*(x) = R_1^*(x)/\varphi(x)$. We claim that $\overline{R}_1^*(x)$ is abelian. By (2.6.2) $\overline{R}_1^*(x)$ is a representation group of $\mathscr{H} = \mathrm{res}_{\mathscr{G}}(x) \cong \mathscr{G}(Alt_5)$. Since the representation group of $\mathscr{H}$ is infinite, we need some additional conditions. Recall that the points of $\mathscr{H}$ are the edges of the Petersen graph and two such edges are collinear if they have a common vertex. If H is a hexad containing $\bar{x}$ then the lines of $\mathscr{G}$ contained in $\Omega(H)$ and containing x correspond to a triple of antipodal edges in the Petersen graph associated with $\mathscr{H}$. By (3.8.3) the product of images in $\overline{R}_1^*(x)$ of these antipodal edges is the identity. On the other hand, if we adjoin to the line set of $\mathscr{H}$ the five antipodal triple of edges, we obtain the geometry $\mathscr{G}(S_4(2))$. Thus $\overline{R}_1^*(x)$ is a representation group of $\mathscr{G}(S_4(2))$ and it is abelian by (3.4.4), so the claim follows. The suborbit diagram shows that there are 3 paths of length 2 joining a vertex $y \in \Gamma_2^2(x)$ with x. Since $\overline{R}_1^*(x)$ is abelian, by (2.2.3) $[\varphi(x), \varphi(y)] = 1$, which completes the proof. □

As an immediate consequence of the above proof we have the following.

Corollary 4.4.7 *Let (R^*, φ) be the universal representation of the enriched point-line system of $\mathscr{G} = \mathscr{G}(3 \cdot M_{22})$ (where $R^* \cong 2_+^{1+12}$) and r be the non-identity element in the centre of R^*. Then for points x, y of $\mathscr{G}$ we have $[\varphi(x), \varphi(y)] = r$ if $y \in \Gamma_3^2(x)$ and $[\varphi(x), \varphi(y)] = 1$ otherwise.* □

Let x be a point of $\mathscr{G}$. We will need some information of the structure of V^c as a module for $G(x) \cong 2^5.Sym_5$. Put $\overline{G}(x) = G(x)/O_2(G(x)) \cong Sym_5$.

Lemma 4.4.8 *The module V^c possesses a unique composition series of $G(x)$-submodules:*

$$V^{(1)} < V^{(2)} < V^{(3)} < V^{(4)} < V^{(5)} < V^c,$$

where $V^{(1)} = \varphi^c(x)$; $V^{(2)} = V^c[\Gamma_4^1(x)]$; $V^{(3)} = V^c[\Gamma_1^1(x)]$; $V^{(4)} = V^c[\Gamma_2^1(x)] = V^c[\Gamma_3^1(x)]$; $V^{(5)} = V^c[\Gamma_2^2(x)]$. Furthermore

(i) *$V^{(1)}$, $V^{(2)}/V^{(1)}$, $V^{(5)}/V^{(4)}$ and $V^c/V^{(5)}$ are 1-dimensional;*
(ii) *$V^{(3)}/V^{(2)}$ and $V^{(4)}/V^{(3)}$ are isomorphic to the natural (4-dimensional irreducible) module for $\overline{G}(x)$;*
(iii) *$V^{(5)}/V^{(3)}$ is isomorphic to the indecomposable extension of the natural module by the 1-dimensional trivial module and it is dual to $V^{(3)}/V^{(1)}$.*

Proof. Since $V^c = R^\bullet/Z(R^\bullet)$, where $R^\bullet \cong 2_+^{1+12}$, the square map on $R^\bullet$ induces a non-singular quadratic form q of plus type on V^c. Let H be a hexad. We know that $V^c[\Omega(H)]$ is isomorphic to the hexacode module V_h for $S/O_2(S) \cong 3 \cdot S_4(2)$ where S is the stabilizer of $\Omega(H)$ in G. Since $S/O_2(S)$ does not preserve a non-zero quadratic form on V_h, $V^c[\Omega(H)]$ is a maximal isotropic subspace with respect to q. Let f be the bilinear form associated with q (so that f is induced by the commutator map on $R^\bullet$). The proof of (4.4.6) in view of (2.3.8) shows that for $y \in \Pi$ we have $f(\varphi^c(x), \varphi^c(y)) \neq 0$ if and only if $y \in \Gamma_3^2(x)$. Put $V_\bullet^c(x) = V^c[\Gamma \setminus \Gamma_3^2(x)]$. Then $V_\bullet^c$ is in the orthogonal complement of $\varphi^c(x)$ and hence $\dim V^c/V_\bullet^c(x) \geq 1$. By the proof of (4.4.6) $\dim V^c/V_\bullet^c \leq 1$. Hence the equality holds and $V_\bullet^c(x)$ is the orthogonal complement of $\varphi^c(x)$ with respect to f. Let $\{x_1, x_2\} = \Gamma_4^1(x)$. Then

$$\Gamma_3^2(x_1) \cup \Gamma_3^2(x_2) = \Gamma_2^2(x) \cup \Gamma_3^2(x)$$

and $\langle \varphi^c(x), \varphi^c(x_1), \varphi^c(x_2) \rangle$ is 2-dimensional. Since f is non-singular, this implies that $V^c[\cup_{i=0}^4 \Gamma_i^1(x)]$ has codimension 2 in V^c and by the above exposition this is the orthogonal complement of $V^c[\{x\} \cup \Gamma_4^1(x)]$. By the proof of (4.4.6) $V^c[\Gamma_1^1(x)]$ has dimension at most 6. If the dimension is 5 then

$$V^c[\Gamma_1^1(x)]^\# = \{\varphi^c(y) \mid y \in \{x\} \cup \Gamma_1^1(x)\}$$

which is certainly impossible. Let us prove that $V^{(2)} < V^{(3)}$. We have just shown that $V^c[\Gamma_1^1(x)]$ is a maximal totally isotropic subspace of V^c, with respect to the form f. However by (2.4.4) $V^c[\Gamma_1^1(x)]$ is contained in the orthogonal complement of $V^c[\{x\} \cup \Gamma_4^1(x)]$ and the latter is an isotropic line. Hence

$$V^c[\{x\} \cup \Gamma_4^1(x)] < V^c[\Gamma_1^1(x)]$$

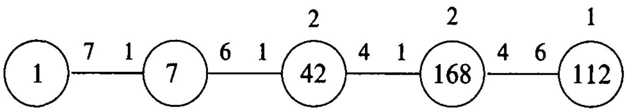
by the maximality of the totally isotropic subspace $V^c[\Gamma_1^1(x)]$. Now the remaining assertions are straightforward. □

The following information can be found in [J76] or deduced directly.

Lemma 4.4.9 *Let $(\overline{Q}, \varphi_a)$ be the universal abelian representation of the enriched point-line incidence system of $\mathcal{G}(3 \cdot M_{22})$ as in* (4.4.2). *Then $G =$ $3 \cdot \mathrm{Aut}\, M_{22}$ has exactly three orbits, Q_1, Q_2, and Q_3 on the set of non-identity elements of $\overline{Q}$, where $Q_1 = \mathrm{Im}\ \varphi$ is of size* 693, *Q_2 is of size* 1386 *and Q_3 is of size* 2016. *In particular, a Sylow 2-subgroup of G fixes a unique non-zero vector in $\overline{Q}$ and this fixed vector is in Q_1.* □

4.5 $\mathscr{D}(M_{22})$

Let $G = M_{22}$, $\mathscr{G} = \mathscr{G}(G)$ be the P-geometry of M_{22}, and $\Delta = \Delta(\mathscr{G})$ be the derived graph of $\mathscr{G}$. Then the action of M_{22} on Δ is distance-transitive and the intersection diagram is the following:



Let $\mathscr{D} = \mathscr{D}(M_{22})$ be the derived system of $\mathscr{G}$. Recall that the points of $\mathscr{D}$ are the vertices of Δ and a triple $\{u, v, w\}$ of such vertices is a line if there is a Petersen subgraph Σ in Δ (an element of type 2 in $\mathscr{G}$) and a vertex $x \in \Sigma$ such that $\{u, v, w\} = \Sigma(x)$ (the set of neighbours of x in Σ).

Let (D, δ) be the universal representation of $\mathscr{D}$. As usual for a subset Λ of the vertex set of Δ

$$D[\Lambda] = \langle \delta(z) \mid z \in \Lambda \rangle.$$

Lemma 4.5.1 *Let $\mathscr{C}_{10}$ be the 10-dimensional Golay code module (which is an irreducible $GF(2)$-module for M_{22}). Then $(\mathscr{C}_{10}, \chi)$ is a representation of $\mathscr{D}$ for a suitable mapping χ.*

Proof. The vertices of Δ (which are the points of $\mathscr{D}$) are the octets (the octads of the $S(5, 8, 24)$-Steiner system disjoint from the pair $\{p, q\}$ of points involved in the definition of $\mathscr{G}(M_{22})$) and two octets are adjacent if they are disjoint. The module $\mathscr{C}_{10}$ can be defined as the subspace in the power space of $\mathscr{P} \setminus \{p, q\}$ generated by the octets. Let $S = \{T_1, T_2, ..., T_6\}$ be a sextet such that $\{p, q\} \in T_1$. Then for $2 \le i < j \le 6$ the union $T_i \cup T_j$ is an octet and all the 10 octets arising in this way induce in Δ a Petersen subgraph Σ. Let x be a vertex of Σ, say $x = T_2 \cup T_3$. Then

$$u = T_4 \cup T_5, \; v = T_4 \cup T_6, \; w = T_5 \cup T_6$$

are the neighbours of x in Σ. Since $\mathscr{C}_{10}$ is a subspace in the power space, the addition is performed by the symmetric difference operator and hence

$$u + v + w = 0,$$

which means that $\mathscr{C}_{10}$ is a representation group of $\mathscr{D}$. □

We are going to show that $\mathscr{C}_{10}$ is the universal representation group of $\mathscr{D}$. First we recall some known properties of Δ. If Σ is a Petersen subgraph in Δ and $x \in \Delta$ then the *type of Σ with respect to x* is the sequence $(t_0, t_1, t_2, t_3, t_4)$, where $t_j = |\Sigma \cap \Delta_j(x)|$ for $0 \le j \le 4$. The next two lemmas are easy to deduce from the diagram on p. 137 in [Iv99].

Lemma 4.5.2 *For $x \in \Delta$ the subgroup $G(x) \cong 2^3 : L_3(2)$ acts transitively on the set of Petersen subgraphs in Δ of a given type with respect to x. Furthermore, if Σ is a Petersen subgraph and $\mathcal{O}$ is the orbit of Σ under $G(x)$, then one of the following holds:*

(i) *Σ is of type $(1,3,6,0,0)$ and $|\mathcal{O}| = 7$;*
(ii) *Σ is of type $(0,1,3,6,0)$ and $|\mathcal{O}| = 28$;*
(iii) *Σ is of type $(0,0,2,4,4)$, $|\mathcal{O}| = 84$ and a vertex from $\Sigma \cap \Delta_4(x)$ is adjacent to 2 vertices from $\Sigma \cap \Delta_3(x)$;*
(iv) *Σ is of type $(0,0,0,6,4)$, $|\mathcal{O}| = 112$, a vertex from $\Sigma \cap \Delta_4(x)$ is adjacent to 3 vertices from $\Sigma \cap \Delta_3(x)$, and a vertex from $\Sigma \cap \Delta_3(x)$ is adjacent to 2 vertices from $\Sigma \cap \Delta_4(x)$.* □

Lemma 4.5.3 *Let H be a hexad and $\mathscr{S} = (\Pi, L)$ be the incidence system, such that Π consists of the edges $\{x, y\}$ of Δ such that the sum of x and y in $\mathscr{C}_{10}$ is the complement of H and L is the set of non-empty intersections of Π with Petersen subgraphs in Δ. Then*

(i) *every line in L is of size 3 and forms an antipodal triple of edges in a Petersen subgraph;*
(ii) *$\mathscr{S}$ is isomorphic to the generalized quadrangle $\mathscr{G}(S_4(2))$ of order $(2,2)$;*
(iii) *for an edge $\{x, y\} \in \Pi$ the set Π contains 6 edges in $\Delta_2(x) \cap \Delta_2(y)$ and 8 edges in $\Delta_4(x) \cap \Delta_4(y)$.* □

Lemma 4.5.4 *Let Σ be a Petersen subgraph in Δ, let*

$$\{\{x_i, y_i\} \mid 1 \le i \le 3\}$$

be an antipodal triple of edges in Σ and Ξ be the set of vertices on these three edges. Then $D[\Xi]$ is elementary abelian of order 2^3 and the product $\delta(x_i)\delta(y_i)$ is independent of the choice of $i \in \{1, 2, 3\}$.

Proof. The statement can be deduced from (3.9.4) by means of elementary calculations. □

Lemma 4.5.5 *For $x \in \Delta$ the equality $D[\Delta_3(x)] = D[\Delta_4(x)]$ holds.*

Proof. Let Σ be a Petersen subgraph of type $(0,0,0,6,4)$ and $u \in \Sigma \cap \Delta_3(x)$. By (4.5.2 (iv)) u is adjacent to 2 vertices in $\Sigma \cap \Delta_4(x)$, say to y and z. Then if v is the unique vertex from $\Sigma \cap \Delta_3(x)$ adjacent to u then

$\delta(v) = \delta(y)\delta(z)$, which implies the inclusion $D[\Delta_3(x)] \leq D[\Delta_4(x)]$. The inverse inclusion can be established similarly by considering a Petersen subgraph of type $(0,0,2,4,4)$. □

Lemma 4.5.6 *In the notation of* (4.5.3) *let* Θ *be the set of* 30 *vertices incident to the edges from* Π. *Then* $D[\Theta]$ *is abelian of order at most* 2^6.

Proof. Let $d = \delta(x)\delta(y)$ for an edge $\{x, y\} \in \Pi$. Then by (4.5.4) and (4.5.3) d is independent of the particular choice of the edge. Let

$$\varepsilon : \{x, y\} \mapsto \langle \delta(x), \delta(y) \rangle / \langle d \rangle.$$

By the definition and (4.5.4) $(D[\Theta]/\langle d \rangle])$ is a representation of $\mathscr{S}$. By (3.4.4) $D[\Theta]/\langle d \rangle$ is elementary abelian of order at most 2^5. Hence the commutator subgroup of $D[\Theta]$ is contained in $\langle d \rangle$. We claim that the commutator subgroup is trivial. Indeed, consider the representation $(\mathscr{C}_{10}, \chi)$ as in (4.5.1) and let ψ be the homomorphism of D onto $\mathscr{C}_{10}$ such that χ is the composition of δ and ψ. Since $\mathscr{C}_{10}$ is abelian, in order to prove the claim it is sufficient to show that $\psi(d)$ is not the identity. But this is clear since the images under χ of two adjacent vertices are different. Hence the proof. □

Lemma 4.5.7 *D is abelian.*

Proof. For $x, y \in \Delta$ we have to show that $\delta(x)$ and $\delta(y)$ commute. If $d_\Delta(x, y) \leq 2$ then x and y are in a common Petersen subgraph and the commutativity follows from (3.9.4); if $d_\Delta(x, y) = 4$ then by (4.5.3) x and y are contained in a set Θ as in (4.5.6) and the commutativity follows from that lemma. Finally by (4.5.5) we have $D[\Delta_3(x)] \leq D[\Delta_4(x)]$, which completes the proof. □

Now we are ready to prove the main result of the section. As usual for a vertex $x \in \Delta$ and $0 \leq i \leq 4$ put

$$D_i(x) = \langle \delta(y) \mid d_\Delta(s, y) \leq i \rangle,$$

$$\overline{D}_i(x) = D_i(x)/D_{i-1}(x) \text{ for } i \geq 1.$$

Proposition 4.5.8 *The universal representation group of the derived system of* $\mathscr{G}(M_{22})$ *is abelian of order* 2^{10} *isomorphic to the* M_{22}*-irreducible Golay code module* $\mathscr{C}_{10}$. □

Proof. In view of (4.5.1) it is sufficient to show that the order of D is at most 2^{10}. We fix $x \in \Delta$ and consider $\overline{D}_i(x)$ to be $GF(2)$-modules of $G(x) \cong 2^3 : L_3(2)$. Let π_x denote the residue $\text{res}_{\mathcal{G}}(x)$ which is the projective plane of order 2 whose points are the edges incident to x and whose lines are the Petersen subgraphs containing x.

Step 0. $\dim D_0(x) \leq 1$.

Step 1. $\dim \overline{D}_1(x) \leq 3$.

The set $\Delta(x)$ is of size 7 and the lines of $\mathcal{D}$ contained in this set turn it into the point set of the projective plane π_x. Now the result is immediate from (3.1.2).

Step 2. $\dim \overline{D}_2(x) \leq 3$.

For a Petersen subgraph Σ of type $(1,3,6,0,0)$ the image of $D[\Sigma]$ in $\overline{D}_2(x)$ is 1-dimensional. There are 7 subgraphs of this type and hence there are 7 such images which clearly generate the whole $\overline{D}_2(x)$ and are naturally permuted by $G(x)/Q(x) \cong L_3(2)$. Now let $\Sigma^{(i)}$, $1 \leq i \leq 3$ be the Petersen subgraphs of type $(1,3,6,0,0)$ containing a given edge $\{x,y\}$. Then

$$\{\Sigma^{(i)}(y) \mid 1 \leq i \leq 3\}$$

are the lines of the projective plane π_y of order 2 on $\Delta(y)$ containing a given point x. Let Σ be a Petersen subgraph of type $(0,1,3,6,0)$ containing y. Since any two lines in a projective plane intersect in a point, we can assume that $\Sigma(y) = \{v_1, v_2, v_3\}$, where $v_i = \Sigma \cap \Sigma^{(i)}(y)$ for $1 \leq i \leq 3$. Then $\delta(v_1) + \delta(v_2) + \delta(v_3) = 0$ which turns $\overline{D}_2(x)$ into a representation module of the dual of π. Hence the claim again follows from (3.1.2).

Step 3. $\dim \overline{D}_3(x) \leq 3$.

By the previous step we see that the image in $\overline{D}_3(x)$ of $D_2[y]$ for $y \in \Delta(x)$ is at most 1-dimensional and these images generate the whole section. Let Σ be a Petersen subgraph of type $(0,0,0,6,4)$ and $z \in \Sigma \cap \Delta_4(x)$ and $\Sigma(z) = \{u_1, u_2, u_3\}$. By (4.5.2 (iv)) $\Sigma(z) \subseteq \Delta_3(x)$. Then the equality $\delta(u_1) + \delta(u_2) + \delta(u_3) = 0$ turns $\overline{D}_3(x)$ into a representation module for a $G(x)/Q(x) \cong L_3(2)$-invariant triple system and we apply (3.1.2) once again.

Step 4. $D_4(x) \leq D_3(x)$.

This is an immediate consequence of (4.5.5). □

As a consequence of the proof of (4.5.8) we obtain the following.

Corollary 4.5.9 *Let* (D, δ) *be the universal representation group of the derived system of* $\mathcal{G}(M_{22})$ *and* $x \in \Delta$. *Then* $D = D_3(x)$ *while* $D_2(x)$ *is of order* 2^7. □

Since Sym_5 acts primitively on the vertex-set of the Petersen graph, it is easy to deduce from (4.5.8) the following.

Corollary 4.5.10 *Let* U *be a quotient of the* $GF(2)$*-permutation module of* M_{22} *acting on the* 330 *vertices of the derived graph* $\Delta(\mathcal{G}(M_{22}))$ *such that the vertices of the Petersen subgraph generate a* 4*-dimensional subspace. Then* U *is isomorphic to the* 10*-dimensional Golay code module* $\mathcal{C}_{10}$. □

4.6 $\mathcal{G}(He)$

It was shown in [MSm82] that the rank 3 T-geometry $\mathcal{G}(He)$ associated with the Held sporadic simple group possesses a natural representation in an irreducible 51-dimensional $GF(2)$-module for He (which is the restriction modulo 2 of an irreducible module over complex numbers for He). It has been checked by Brendan McKay (private communication) on a computer that dim $V(\mathcal{G}(He))$ is 52. Thus in view of (2.1.1) we have the following result.

Proposition 4.6.1 *The universal representation module* $V(\mathcal{G}(He))$, *as a* $GF(2)$*-module for* He, *is an indecomposable extension of a* 51*-dimensional irreducible* He*-module by a* 1*-dimensional submodule.* □

5
Conway groups

The tilde geometry $\mathcal{G}(Co_1)$ of the first Conway group, the Petersen geometry $\mathcal{G}(Co_2)$ of the second Conway group and the c-extended dual polar space $\mathcal{G}(3 \cdot U_4(3))$ possess representations in 24-, 23- and 12-dimensional sections of $\overline{\Lambda}^{(24)}$ (the Leech lattice taken modulo 2). We show that in the former two cases the representations are universal (cf. Propositions 5.2.3, 5.3.2, and 5.4.1). In the latter case the extension of the 12-dimensional representation module to an extraspecial group supports the universal representation of the enriched point-line system of $\mathcal{G}(3 \cdot U_4(3))$ (cf. Proposition 5.6.5, which was originally proved in [Rich99]). In Section 5.5 it is shown that $\mathcal{G}(3^{23} \cdot Co_2)$ does not possess faithful abelian representations (the question about non-abelian ones is still open).

5.1 Leech lattice

The rank 4 T-geometry $\mathcal{G}(Co_1)$ and its P-subgeometry $\mathcal{G}(Co_2)$ are best defined in terms of the Leech lattice Λ. In this section we recall some basic facts about Λ.

Let $(\mathcal{P}, \mathcal{B})$ be the Steiner system $S(5, 8, 24)$. This means that $\mathcal{P}$ is a set of 24 elements and $\mathcal{B}$ is a collection of 759 8-subsets of $\mathcal{P}$ (called octads) such that every 5-subset of $\mathcal{P}$ is in a unique octad. Such a system is unique up to isomorphism and its automorphism group is the Mathieu group M_{24}. Let $\mathcal{C}_{12}$ be the Golay code which is the (12-dimensional) subspace in the power space of $\mathcal{P}$ generated by the octads. Let $\mathbb{R}^{24}$ be the space of all functions from $\mathcal{P}$ into the real numbers (a 24-dimensional real vector space). For $\lambda \in \mathbb{R}^{24}$ and $a \in \mathcal{P}$ we denote by λ_a the value of λ on a. Let e_a be the characteristic function of a (equal to 1 on a and 0 everywhere else). Then $\mathcal{E} = \{e_a \mid a \in \mathcal{P}\}$ is a basis of $\mathbb{R}^{24}$ and $\{\lambda_a \mid a \in \mathcal{P}\}$ are the coordinates of $\lambda \in \mathbb{R}^{24}$ in this basis.

Let Λ be the set of vectors $\lambda = \{\lambda_a \mid a \in \mathcal{P}\}$ in $\mathcal{R}^{24}$, satisfying the following three conditions for $m = 0$ or 1.

(Λ1) $\lambda_a = m \bmod 2$ for every $a \in \mathcal{P}$;
(Λ2) $\{a \mid \lambda_a = m \bmod 4\} \in \mathcal{C}_{12}$;
(Λ3) $\sum_{a \in \mathcal{P}} \lambda_a = 4m \bmod 8$.

Define the inner product $(\, , \,)$ of $\lambda, \nu \in \Lambda$ to be

$$(\lambda, \nu) = \frac{1}{8} \sum_{a \in \mathcal{P}} \lambda_a \nu_a.$$

Then Λ is an even unimodular lattice of dimension 24 without roots (vectors of length 2). The lattice Λ is determined by these properties up to isomorphism and it is the *Leech lattice*. The automorphism group of Λ (preserving the origin) is $Co_0 \cong 2 \cdot Co_1$ which is the extension of the first sporadic group of Conway by its Schur multiplier.

It is common to denote by Λ_i the set of Leech vectors (vectors in Λ) of length $2i$:

$$\Lambda_i = \{\lambda \mid \lambda \in \Lambda, \frac{1}{16} \sum_{a \in \mathcal{P}} \lambda_a^2 = i\}.$$

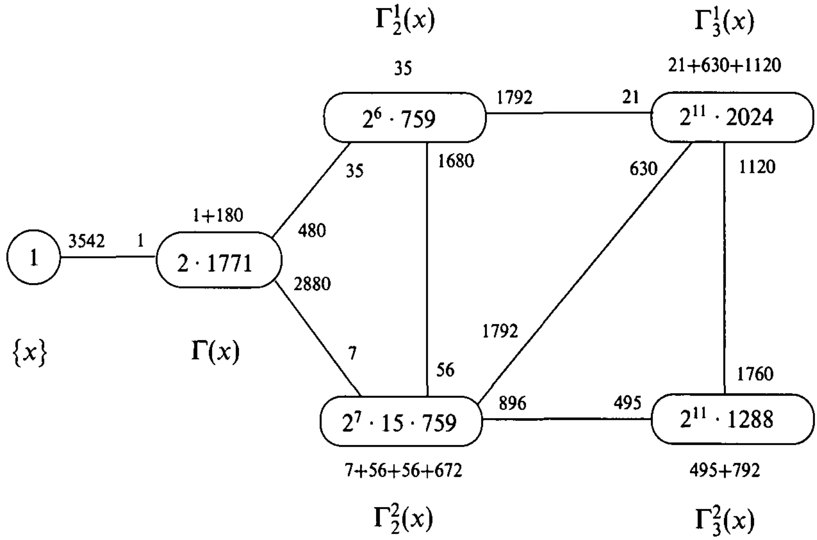
Then Λ_0 consists of the zero vector and Λ_1 is empty since there are no roots in Λ.

Let $\overline{\Lambda} = \Lambda / 2\Lambda$ be the Leech lattice modulo 2, which carries the structure of a 24-dimensional $GF(2)$-space. We sometimes write $\overline{\Lambda}^{(24)}$ for $\overline{\Lambda}$ to emphasize the dimension. The automorphism group of Λ induces on $\overline{\Lambda}$ the group $G \cong Co_1$. For a subset M of Λ by $\overline{M}$ we denote the image of M in $\overline{\Lambda}$. The following result is well known

Proposition 5.1.1 *The following assertions hold:*

(i) $\overline{\Lambda} = \overline{\Lambda}_0 \cup \overline{\Lambda}_2 \cup \overline{\Lambda}_3 \cup \overline{\Lambda}_4$ *(disjoint union);*
(ii) *if* $i = 2$ *or* 3 *then an element from* $\overline{\Lambda}_i$ *has exactly two preimages in* Λ_i *which differ by sign;*
(iii) *an element from* $\overline{\Lambda}_4$ *has exactly* 48 *preimages in* Λ_4*;*
(iv) $G \cong Co_1$ *acts transitively on* $\overline{\Lambda}_2$, $\overline{\Lambda}_3$ *and* $\overline{\Lambda}_4$ *with stabilizers isomorphic to* Co_2, Co_3 *and* $2^{11} : M_{24}$, *respectively;*
(v) *the* $GF(2)$*-valued function* θ *on* $\overline{\Lambda}$ *which is* 1 *on the elements from* Λ_3 *and* 0 *everywhere else is the only non-zero* G*-invariant quadratic form on* $\overline{\Lambda}$. □

Let Γ be the Leech graph that is the unique graph of valency $2 \cdot 1771$ on $\overline{\Lambda}_4$ invariant under the action of G on this set. Then the suborbit diagram of Γ is the following:



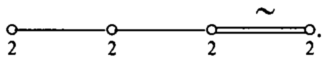
The graph Γ is the collinearity graph of the geometry $\mathcal{G}(Co_1)$. The lines can be defined as follows. If x is a vertex of Γ and $G(x) \cong 2^{11} : M_{24}$ is the stabilizer of x in G, then $\Gamma(x)$ is the union of the orbits of length 2 of $Q(x) = O_2(G(x))$ on Γ (this can be used for an alternative definition of Γ). If $\{y, z\}$ is such an orbit, then $T = \{x, y, z\}$ is a line (observe that every edge is contained in a unique line). If we treat the points in T as elements of $\overline{\Lambda}_4$, then the equality $x + y + z = 0$ holds (notice that not every triple $\{x, y, z\}$ of points with $x + y + z = 0$ is a line). Since $\overline{\Lambda}$ is generated by $\overline{\Lambda}_4$, we have the following

Lemma 5.1.2 *The pair $(\overline{\Lambda}, \varphi)$, (where φ is the identity mapping) is a representation of $\mathcal{G}(Co_1)$.* □

We will show below that the representation in the above lemma is universal.

In order to deal with representations of $\mathcal{G}(Co_1)$ we only need the point-line incidence system of the geometry but for the sake of completeness we recall how the remaining elements can be defined. A clique (complete subgraph) Ξ in Γ is said to be $*$-closed if together with every edge it contains the unique line containing this edge. Then lines are precisely the $*$-closed cliques of size 3; elements of type 3 in $\mathcal{G}(Co_1)$ are the $*$-closed cliques of size 7 and the set of elements of type 4 is one of the two

G-orbits on the set of $*$-closed cliques of size 15. The diagram of $\mathcal{G}(Co_1)$ is

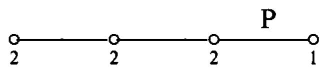


Let $u \in \overline{\Lambda}_2$, $F \cong Co_2$ be the stabilizer of u in G and for $j = 2, 3$ and 4 let

$$\Theta^{(j)} = \{x \in \overline{\Lambda}_4 \mid x + u \in \Lambda_j\}.$$

Lemma 5.1.3 *The sets $\Theta^{(j)}$, $j = 2, 3$ and 4 are the orbits of F on $\overline{\Lambda}_4$ (which is the vertex set of Γ) and the corresponding stabilizers are isomorphic to $2^{10} : \mathrm{Aut}\, M_{22}$, M_{23} and $2^5 : 2^4 : L_4(2)$, respectively.* □

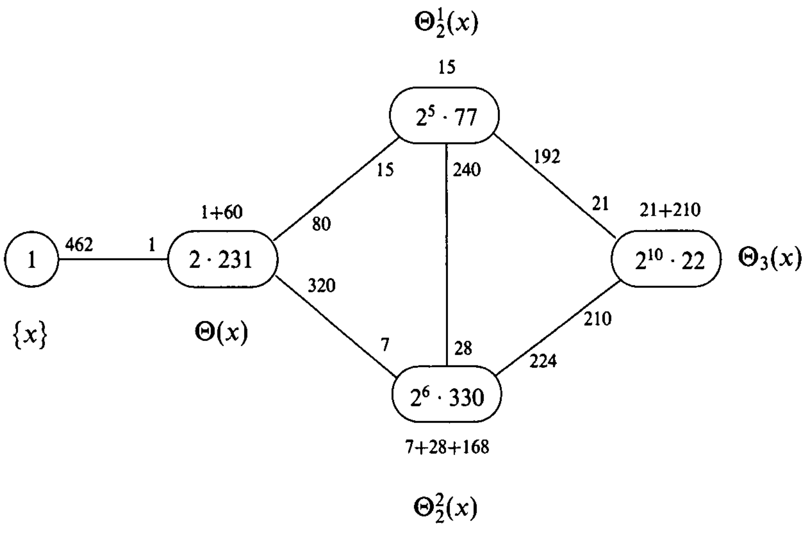
Let $\mathcal{F}$ be the subgeometry in $\mathcal{G} = \mathcal{G}(Co_1)$ formed by the elements contained in $\Theta^{(2)}$. Then $\mathcal{F} \cong \mathcal{G}(Co_2)$ is a geometry with the diagram



and F induces on $\mathcal{F}$ a flag-transitive action.

The points of $\mathcal{F}$ generate in $\overline{\Lambda}$ the orthogonal complement $u^\perp$ of the vector $u \in \overline{\Lambda}_2$ involved in the definition of $\mathcal{F}$ with respect to the Co_1-invariant quadratic form θ as in (5.1.1 (v)). Considered as a $GF(2)$-module for F the subspace $u^\perp$ of $\overline{\Lambda} = \overline{\Lambda}$ will be denoted by $\overline{\Lambda}^{(23)}$; it is an indecomposable extension of an irreducible 22-dimensional F-module $\overline{\Lambda}^{(22)}$ by a 1-dimensional submodule.

Let Θ denote the subgraph in Γ induced by $\Theta^{(2)}$. The suborbit diagram of Θ with respect to the action of F is the following:



5.2 $\mathcal{G}(Co_2)$

In this section we show that $\overline{\Lambda}^{(23)}$ is the universal representation module of $\mathcal{F} = \mathcal{G}(Co_2)$. We will make use of $\mathcal{G}(S_6(2))$-subgeometries in $\mathcal{F}$ described in the following lemma (compare Lemma 4.9.8 in [Iv99]).

Lemma 5.2.1 *Let $x \in \Theta$ and $y \in \Theta_2^1(x)$. Then x and y are contained in a unique subgraph Ξ in Θ isomorphic to the collinearity graph of the geometry $\mathcal{G}(S_6(2))$ which is a subgeometry in $\mathcal{G}$ formed by the elements contained in Ξ. The stabilizer of Ξ in $F \cong Co_2$ is of the form $2_+^{1+8}.S_6(2)$ and it contains $O_2(F(x))$.* □

We will also need the following result (where the vertices of Θ are treated as vectors from $\overline{\Lambda}$).

Lemma 5.2.2 *Let $x \in \Theta$, then*

(i) *the intersection of Θ with the orthogonal complement $x^\perp$ of x with respect to the Co_1-invariant quadratic form θ is $\Theta \setminus \Theta_3(x)$;*
(ii) *a line of $\mathcal{F}$ which intersects $\Theta_3(x)$ intersects it in exactly two points;*
(iii) *the subgraph in Θ induced by $\Theta_3(x)$ is connected.*

Proof. (i) follows from the definition of θ and the table on p. 176 in [Iv99]. Since a line is the set of non-zero vectors of a 2-subspace in $\overline{\Lambda}$, (ii) follows directly from (i). To establish (iii) recall that for $z \in \Theta_3(x)$ we have $F(x,z) \cong P\Sigma L_3(4)$. Suppose that the subgraph induced by $\Theta_3(x)$ is disconnected, let Υ be the connected component containing z and H be the setwise stabilizer of Υ in $F(x)$. Since $F(x)$ acts transitively on $\Theta_3(x)$, H acts transitively on Υ and

$$P\Sigma L_3(4) \cong F(x,z) < H < F(x) \cong 2^{10} : \mathrm{Aut}\, M_{22}.$$

Clearly $|\Upsilon| := [F(x) : H] = n_1 \cdot n_2$ where

$$n_1 = 2^{10}/|O_2(F(x)) \cap H|, \text{ and } n_2 = [F(x) : HO_2(F(x))].$$

Since $F(x,z)O_2(F(x))$ is a maximal subgroup in $F(x)$ of index 22 and $F(x)/O_2(F(x))$ acts irreducibly on $O_2(F(x))$ of order 2^{10}, we conclude that $[F(x) : H]$ is at least 22 and hence Υ contains at most $|\Theta_3(x)|/22 = 2^{10}$ vertices. On the other hand from the suborbit diagram of Θ we observe that (a) the valency of Υ is 231; (b) every edge of Υ is in at most 61 triangles and (c) any two vertices at distance 2 in Υ are joined by at most 15 paths of length 2. This shows that

$$|\Upsilon| \geq |\{z\}| + |\Upsilon(z)| + |\Upsilon_2(z)| \geq 1 + 231 + 231 \cdot (230 - 61)/15 > 2834,$$

which contradicts the upper bound we established earlier. □

Proposition 5.2.3 *Let (V, φ_a) be the universal abelian representation of $\mathcal{F} = \mathcal{G}(Co_2)$, $x \in \Theta$ be a point and $\mathcal{H} = \mathrm{res}_{\mathcal{G}}(x) \cong \mathcal{G}(M_{22})$. Then*

(i) $\dim V_0(x) = 1$;
(ii) $\overline{V}_1(x)$ *is either $V(\mathcal{H})$ (which is the 11-dimensional Todd module $\overline{\mathcal{C}}_{11}$) or the quotient of $V(\mathcal{H})$ over a 1-dimensional submodule;*
(iii) $\overline{V}_2(x)$ *is $V(\mathcal{D}(\mathcal{H}))$ (which is the 10-dimensional Golay code module $\mathcal{C}_{10}$);*
(iv) $\dim \overline{V}_3(x) \leq 1$;
(v) *V is isomorphic to the Co_2-submodule $\overline{\Lambda}^{(23)}$ in the Leech lattice taken modulo 2.*

Proof. We know that (V, φ) is non-trivial (of dimension at least 23) and F-admissible. Then (i) is obvious, (ii) follows from (2.6.3) and (4.2.4).

Now let us turn to $\overline{V}_2(x)$. In order to establish the statement we will prove three claims. Let $\overline{V}_2^j(x)$ be the subspace in $\overline{V}_2(x)$ generated by the cosets $\varphi_a(y)V_1(x)$ taken for all $y \in \Theta_2^j(x)$, where $j = 1$ or 2.

Claim 1. $\overline{V}_2(x) = \overline{V}_2^1(x) = \overline{V}_2^2(x)$.

Let $z \in \Theta(x)$, Υ be the collinearity graph of $\mathrm{res}_{\mathcal{F}}(z) \cong \mathcal{G}(M_{22})$ and let l_x denote the vertex of Υ containing x (this is the line of $\mathcal{F}$ containing x and z). Then $W := V_1(z)/V_0(z)$ is a quotient of the 11-dimensional Todd module $\overline{\mathcal{C}}_{11}$. The image of $V_1(z)$ in $\overline{V}_2(x)$ is a quotient of $\overline{W}_2(l_x)$ (where the latter is defined with respect to the graph Υ). Comparing the suborbit diagrams of Θ (in the previous section) and Υ (in Section 4.2), we observe that if $y \in \Theta_2^j(x)$, then the line l_y of $\mathcal{F}$ which contains z and y is in $\Upsilon_2^j(l_x)$ for $j = 1$ and 2. Hence the claim follows from (4.2.6 (ii)).

Claim 2. $O_2(F(x))$ centralizes $\overline{V}_2(x)$.

Let Ξ be a subgraph in Θ isomorphic to the collinearity graph of $\mathcal{G}(S_6(2))$ as in (5.2.1) which contains x. Then by (3.5.1) the image $\overline{V}_2[\Xi]$ of $V[\Xi]$ in $\overline{V}_2(x)$ is at most 1-dimensional and since $O_2(F(x))$ stabilizes Ξ, it centralizes $\overline{V}_2[\Xi]$. By (5.2.1) the images $\overline{V}_2[\Xi]$ taken for all such subgraphs Ξ containing x generate $\overline{V}_2^1(x)$ which is the whole $\overline{V}_2(x)$ by Claim 1.

Claim 3. $\overline{V}_2(x)$ is as in (iii).

By Claims 1 and 2, and in view of the suborbit diagram of Θ we observe that $\overline{V}_2(x) = \overline{V}_2^2(x)$ is generated by 330 elements indexed by the orbits of $O_2(F(x))$ on $\Theta_2^2(x)$. On the other hand, by Lemma 4.9.5 in [Iv99] these orbits are indexed by the octets of the Steiner system $S(3, 6, 22)$ in terms of which $\mathrm{res}_{\mathcal{F}}(x)$ is defined. Since (V, φ_a) is universal abelian, it is F-admissible and hence in view of the above exposition, $\overline{V}_2(x)$ is a quotient of the $GF(2)$-permutation module of $F(x)/O_2(F(x)) \cong \mathrm{Aut}\,M_{22}$ acting on the set of octets (the vertex set of the derived graph). As above let $z \in \Theta(x)$. Then in view of the diagram on p. 138 in [Iv99] we observe that $\Theta(z) \cap \Theta_2^2(x)$ intersects exactly 10 orbits of $O_2(F(x))$ on $\Theta_2^2(x)$ and these orbits correspond to the vertex-set of a Petersen subgraph in the derived graph of $\mathrm{res}_{\mathcal{F}}(x)$. By (4.2.6 (ii)) the 10 elements corresponding to these orbits generate in $\overline{V}_2(x)$ a quotient of a 4-dimensional submodule with respect to $F(x,z)$. Then (4.5.10) applies and gives the claim.

In view of (2.1.3), (iv) follows now from (5.2.2 (ii) and (iii)). Since the diameter of Θ is three, by the above exposition we observe that the dimension of V is at most 23. Since we know that $\mathcal{F}$ possesses a 23-dimensional representation in $\overline{\Lambda}^{(23)}$, (v) follows. □

Thus a Co_2-admissible representation module of $\mathcal{G}(Co_2)$ is isomorphic either to $\overline{\Lambda}^{(23)}$ or to $\overline{\Lambda}^{(22)}$.

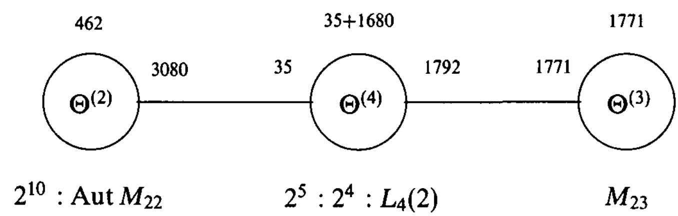
The Co_2-orbits on $\overline{\Lambda}^{(23)}$ are listed in [Wil89]. This list shows that the only orbit of odd length of the non-zero vectors in $\overline{\Lambda}^{(22)}$ is Im ψ where $(\overline{\Lambda}^{(22)}, \psi)$ is a representation of $\mathcal{G}(Co_2)$. The suborbit diagram of Θ shows that all the non-diagonal orbitals have even length, which gives the following.

Corollary 5.2.4 *A Sylow 2-subgroup of Co_2 fixes a unique non-zero vector v in $\overline{\Lambda}^{(22)}$ and a unique hyperplane which is the orthogonal complement of v with respect to the form induced by β. Furthermore v is the image of a point of $\mathcal{G}(Co_2)$ under the mapping which turns $\overline{\Lambda}^{(22)}$ into a representation module of the geometry.* □

5.3 $\mathcal{G}(Co_1)$

We look closer at the subgraphs induced in Γ by the orbits of $F \cong Co_2$ and at the adjacencies between vertices in different orbits.

Lemma 5.3.1 *The suborbit diagram of the Leech graph Γ with respect to the orbits of $F \cong Co_2$ is the following:*



Furthermore, a line of $\mathcal{G}(Co_1)$ which intersects $\Theta^{(3)}$ intersects it in exactly two points and the subgraph induced by $\Theta^{(3)}$ is connected.

Proof. For $x \in \Gamma$ let $\mathcal{S}(x)$ be the Steiner system of type $S(5,8,24)$ in terms of which the residue $\text{res}_{\mathcal{G}}(x) \cong \mathcal{G}(M_{24})$ is defined. In particular, the points of $\text{res}_{\mathcal{G}}(x)$ (which are the lines of $\mathcal{G}$ containing x) are the sextets of $\mathcal{S}(x)$. The stabilizer $G(x) \cong 2^{11} : M_{24}$ induces the automorphism group of $\mathcal{S}(x)$ with kernel $K(x) = O_2(G(x))$.

For $x_j \in \Theta^{(j)}$ we are interested in the orbits of $F(x_j)$ on $\Gamma(x_j)$ for $j = 2,3,4$. We know by (5.1.3) that $F(x_2) \cong 2^{10} : \text{Aut}\, M_{22}$. Then $F(x_2)Q(x_2)/Q(x_2)$ is the stabilizer in $M_{24} = \text{Aut}\,\mathcal{S}(x_2)$ of a pair of elements, say $\{p,q\}$. Then from the structure of a sextet stabilizer (cf. Lemma 2.10.2 in [Iv99]) we observe that $F(x_2)$ has two orbits on the set of lines containing x_2 with lengths 231 and 1540 corresponding to the sextets in which $\{p,q\}$ intersects one and two tetrads, respectively. Furthermore, $F(x_2) \cap Q(x_2)$ is a hyperplane in $Q(x_2)$ which is not the pointwise stabilizer of a line containing x_2. Hence $F(x_2)$ has two orbits on $\Gamma(x_2)$ with lengths 462 and 3080. From the suborbit diagram of Θ we see that the 462-orbit is in $\Theta^{(2)}$ and by the divisibility condition the 3080-orbit is in $\Theta^{(4)}$.

By Lemma 4.4.1 in [Iv99] $F(x_4)Q(x_4)/Q(x_4)$ is the stabilizer of an octad in $\text{Aut}\,\mathcal{S}(x_4)$. Hence by the diagram on p. 125 in [Iv99] the orbits of $F(x_4)$ on the sextets of $\mathcal{S}(x_4)$ are of length 35, 840 and 896. It is easy to see that $F(x_4) \cap Q(x_4)$ (which is of order 2^5) fixes pointwise exactly 35 lines through x_4. So the orbits of $F(x_4)$ on $\Gamma(x_4)$ are of lengths 35, 35, 1680 and 1792.

Finally $F(x_3) \cong M_{23}$ permutes transitively the 1771 lines through x_3. Since Γ is connected, in view of the above paragraph and the divisibility condition we conclude that every line though x_3 has one point in $\Theta^{(4)}$ and two in $\Theta^{(3)}$. Since $F(x_2) \cong M_{23}$ is a maximal subgroup in $F \cong Co_2$ (cf. [CCNPW] and references therein) the subgraph induced by $\Theta^{(3)}$ is connected. □

Proposition 5.3.2 *The Leech lattice $\overline{\Lambda} = \overline{\Lambda}^{(24)}$ taken modulo 2 is the universal representation module of $\mathcal{G}(Co_1)$.*

Proof. Let (V, φ_a) be the universal abelian representation of $\mathcal{G} = \mathcal{G}(Co_1)$. Since we know that $\mathcal{G}$ possesses a representation in $\overline{\Lambda}$, all we have to show is that V is at most 24-dimensional. We consider the decomposition of Γ into the orbits of $F \cong Co_2$. It follows from the definition of $\mathcal{F} \cong \mathcal{G}(Co_2)$ that $V[\Theta^{(2)}]$ supports a representation of $\mathscr{F}$ and hence it is at most 23-dimensional by (5.2.3 (v)). By (2.6.3) and (4.3.1) $\overline{V}_1(x_2)$ is a quotient of the 11-dimensional irreducible Todd module $\overline{\mathscr{C}}_{11}$. Comparing (4.3.1) with (4.2.5) or otherwise, one can see that the 231 vectors in $\overline{\mathscr{C}}_{11}$ corresponding to the octads containing a given pair of elements generate the whole $\overline{\mathscr{C}}_{11}$. Hence $V_1(x_2)$ is contained in $V[\Gamma(x_2) \cap \Theta^{(2)}]$. By (5.3.1) $\Gamma(x_2)$ contains vertices from $\Theta^{(4)}$ and hence $V[\Theta^{(4)}]$ is contained in $V[\Theta^{(2)}]$. Consider the quotient $\overline{V} = V/V[\Theta^{(2)}]$. By the above, $\overline{V}$ is generated by the images in this quotient of the elements $\varphi_a(y)$ for $y \in \Theta^{(3)}$. But it is immediate from the last sentence of (5.3.1) that all these images are the same, so $\overline{V}$ is at most 1-dimensional and the proof follows. □

By the proof of (5.3.2) and (5.2.3 (ii)) we have the following.

Corollary 5.3.3 *Let $(\overline{\Lambda}, \varphi_a)$ be the universal abelian representation of $\mathscr{G}(Co_1)$ and $x \in \Gamma$. Then the subspace in $\overline{\Lambda}$ generated by the elements $\varphi_a(y)$ taken for all $y \in \{x\} \cup \Gamma(x)$ is 12-dimensional.* □

It is well known that $\overline{\Lambda}_2$, $\overline{\Lambda}_3$ and $\overline{\Lambda}_4$ are the orbits of Co_1 on $\overline{\Lambda}^{\#}$ and only the latter of the orbits has odd length (cf. Lemma 4.5.5 in [Iv99]). Furthermore one can see from the suborbit diagram of the Leech graph Γ that all the non-diagonal orbitals have even length. This gives the following

Corollary 5.3.4 *A Sylow 2-subgroup of Co_1 fixes a unique non-zero vector v in $\overline{\Lambda}$ and a unique hyperplane which is the orthogonal complement of v with respect to β. Furthermore, $v \in \overline{\Lambda}_4 = \mathrm{Im}\ \varphi_a$.* □

5.4 Abelianization

In this section we complete determination of the universal representations of the geometries $\mathscr{G}(Co_2)$ and $\mathscr{G}(Co_1)$ by proving the following.

Proposition 5.4.1 *The universal representation groups of $\mathscr{G}(Co_2)$ and $\mathscr{G}(Co_1)$ are abelian and thus by (5.2.3) and (5.3.2) they are isomorphic to $\overline{\Lambda}^{(23)}$ and $\overline{\Lambda}^{(24)}$, respectively.*

The proof of the proposition will be achieved in a few steps. We start with the following.

Lemma 5.4.2 *Let (R, φ_u) be the universal representation of $\mathscr{G} = \mathscr{G}(Co_2)$. Then the order of the commutator subgroup of R is at most 2.*

Proof. As above, Θ denotes the collinearity graph of $\mathcal{G}$. We apply (2.3.7) for $B(x) = \Theta_3(x)$ and $A(x) = \Theta \setminus B(x)$. By (2.6.2) $\overline{R}_1(x)$ supports a representation of $\mathrm{res}_{\mathcal{G}}(x) \cong \mathcal{G}(M_{22})$, which is abelian by (4.2.5). Since any two points at distance 2 in Θ are joined by more than one (in fact at least 7) paths of length 2, $R_1(x)$ is abelian by (2.2.3). Since x can be any point of $\mathcal{G}$, we conclude that $[\varphi_u(x), \varphi(y)] = 1$ whenever $d_\Theta(x, y) \leq 2$ (i.e., whenever $y \in A(x)$). The set $B(x) = \Theta_3(x)$ is a non-trivial suborbit of the primitive action of Co_2 on the vertex set of Θ, hence the corresponding graph Ξ in (2.3.7 (i)) is connected. Finally, the connectivity of the graph Σ^x in (2.3.7 (ii)) holds by (5.2.2 (ii) and (iii)). □

We follow the notation of (5.4.2). Since the representation (R, φ_u) is universal it is F-admissible and hence there is an isomorphism χ of $F \cong Co_2$ into the automorphism group of R. Suppose that R is non-abelian. Then by (5.4.2) the commutator subgroup R' of R is of order 2 and by (5.2.3) there is an isomorphism of R/R' onto $\overline{\Lambda}^{(23)}$ which obviously commutes with the action of F (identified with its image under χ). In view of (2.3.8) and (2.3.9) the power and the commutator maps in R are the restrictions to $\overline{\Lambda}^{(23)}$ of the quadratic form θ as in (5.1.1 (v)) and the corresponding bilinear map β (we denote these restrictions by the same letters θ and β). This shows particularly that the centre $Z(R)$ of R is elementary abelian of order 2^2 and it is equal to the preimage of the radical of β. Clearly F acts trivially on $Z(R)$. Let K be a complement in $Z(R)$ to R' and $Q = R/K$. Then $Q \cong 2_+^{1+22}$ and we can consider the semidirect product C of Q and the image of F with respect to χ. Then

$$C \cong 2_+^{1+22}.Co_2$$

and the structure of C resembles that of the stabilizer B_1 of a point in the action of the Baby Monster group BM on its rank 5 P-geometry $\mathcal{G}(BM)$. But unlike C the point stabilizer B_1 does not split over $O_2(B_1)$ and this is where we will reach a contradiction. As we will see in Part II the chief factors of B_1 do not determine B_1 up to isomorphism but in either case the extension is non-split.

The proof of Proposition 5.7 in [IPS96] for the fact of non-splitness refers to the result established in Corollary 8.7 in [Wil87] that Co_2 is not a subgroup of the Baby Monster. Here we present a more direct argument suggested to us by G. Stroth.

Lemma 5.4.3 *Let $C \cong 2_+^{1+22}.Co_2$, $Q = O_2(C)$ and $\overline{C} = C/O_2(C)$, and suppose $\overline{C}$ acts on $Q/Z(Q)$ in the same way that it acts on the section $\overline{\Lambda}^{(22)}$ of the Leech lattice taken modulo 2. Then C does not split over Q.*

Proof. Since $\overline{C}$ preserves on $\overline{\Lambda}^{(22)}$ a unique non-zero quadratic, form the isomorphism type of $C/Z(Q)$ is uniquely determined as a subgroup in the automorphism group of Q (the automorphism group is isomorphic to $2^{22}.O^+_{22}(2)$). Thus $C/Z(Q)$ is isomorphic to the centralizer $\widetilde{C}$ of a central involution in the Baby Monster BM factored over the subgroup of order 2 generated by this involution. The centralizer of a $2D$-involution τ in $3 \cdot M(24)$ is of the form $U \cong 2^2 \cdot U_6(2).Sym_3$ and $V := O^\infty(U) \cong 2^2 \cdot U_6(2)$. If $3 \cdot M(24)$ is considered as the normalizer of a subgroup of order 3 in the Monster group M, then τ is a central involution in M and the full preimage of U in $C_M(\tau)$ is of the form $D \cong 2^{1+2+20+2}.U_6(2).Sym_3$. In particular, $D/O_2(U) \cong 2^{20+2} : U_6(2).Sym_3$ and $O_2(D/O_2(U))$ is an indecomposable $U_6(2)$-module. By (2.8.3) this implies that all subgroups in $D/O_2(U)$ isomorphic to $U_6(2)$ are conjugate and V is in the preimage of one of these complements. In $O_2(U)$ there are 3 involutions of the Baby Monster type in M and if we intersect D with the centralizer in M of one of these three involutions, say a, we obtain the intersection of D with $2 \cdot BM$. Factoring out the subgroup generated by a, we obtain a group E of the form $E \cong 2^{1+1+20+2}.U_6(2).2$ which is the preimage in $\widetilde{C}$ of a maximal subgroup in $\overline{C} \cong Co_2$ isomorphic to $U_6(2).2$. Since $U_6(2)$ does not split over $O_2(E)/Z(\widetilde{C})$, the proof follows. □

Thus the semidirect product of $Q \cong 2^{1+22}_+$ with the action of $Q/Z(Q)$ the same as on $\overline{\Lambda}^{(22)}$ does not exist. Thus the universal representation group of $\mathcal{G}(Co_2)$ is abelian and we have proved (5.4.1) for this geometry.

Now let (R, φ_u) be the universal representation of $\mathcal{G} = \mathcal{G}(Co_1)$. Since the universal representation group of $\mathcal{G}(Co_2)$ (which we treat as a subgeometry of $\mathcal{G}$) is proved to be abelian, we know that $[\varphi_u(x), \varphi_u(y)] = 1$ whenever x and y are in a common $\mathcal{G}(Co_2)$-subgeometry. Since

$$\Theta(x) \subset \Gamma(x),\ \Theta_2^1(x) \subset \Gamma_2^1(x),$$

$$\Theta_2^2(x) \subset \Gamma_2^2(x),\ \Theta_3(x) \subset \Gamma_3^1(x),$$

it remains to take $y \in \Gamma_3^2(x)$ and to show that $\varphi_u(y)$ commutes with $\varphi_u(x)$. Since $\Gamma \setminus \Gamma_3^1(x)$ is a geometric hyperplane in $\mathcal{G}$, the suborbit diagram of Γ shows that there is a line $\{a, b, y\}$ such that $a, b \in \Gamma_3^1(x)$. Since $\varphi_u(y) = \varphi_u(a)\varphi_u(b)$ the required commutativity is established and this completes the proof of (5.4.1).

5.5 $\mathcal{G}(3^{23} \cdot Co_2)$

In this section we prove the following

Proposition 5.5.1 *The universal abelian representation of* $\mathcal{G}(3^{23} \cdot Co_2)$ *is* $(\overline{\Lambda}^{(23)}, \nu)$, *where* ν *is the composition of the* 2*-covering*

$$\chi : \mathcal{G}(3^{23} \cdot Co_2) \to \mathcal{G}(Co_2)$$

and the universal (abelian) representation of $\mathcal{F}$ *as in* (5.2.3).

We apply the technique developed in Section 2.4. Our notation here slightly differs from that in the earlier sections of the chapter.

Let Λ be the Leech lattice, $G^* \cong Co_0 \cong 2 \cdot Co_1$ be the group of automorphisms of Λ preserving the origin. Put

$$\overline{\Lambda} = \Lambda/2\Lambda, \quad \widehat{\Lambda} = \Lambda/3\Lambda,$$

so that $\overline{\Lambda}$ and $\widehat{\Lambda}$ are irreducible 24-dimensional G^*-modules over $GF(2)$ and $GF(3)$, respectively. The group G^* induces $G \cong Co_1$ on $\overline{\Lambda}$ and acts faithfully on $\widehat{\Lambda}$ preserving the non-singular bilinear forms $\overline{\beta}$ and $\widehat{\beta}$ which are the inner product on Λ reduced modulo 2 and 3, respectively. For $\lambda \in \Lambda$ let $\overline{\lambda}$ and $\widehat{\lambda}$ be the images of λ in $\overline{\Lambda}$ and $\widehat{\Lambda}$, respectively. We identify $\overline{\lambda}$ and $\widehat{\lambda}$ with the 1-subspaces in $\overline{\Lambda}$ and $\widehat{\Lambda}$ they generate.

If $u \in \Lambda_2$ then the stabilizer $G^*(u)$ of u in G^* is $F \cong Co_2$ and it maps isomorphically onto $G(\overline{u})$. Let

$$\Theta = \{\{t, -t\} \mid t \in \Lambda_2, (u, t) = 0\}.$$

In what follows a pair $\{t, -t\} \in \Theta$ will be represented by a single vector t (or $-t$). The mapping $\varphi_a : \Theta \to \overline{\Lambda}_4$ defined by

$$\varphi_a : t \mapsto \overline{t} + \overline{u} = \overline{t + u}$$

is a bijection of Θ onto the set $\Theta^{(2)}$ defined in the paragraph preceding (5.1.3). Thus we can treat Θ as the point-set of $\mathcal{F} = \mathcal{G}(Co_2)$, so that $(\overline{\Lambda}^{(23)}, \varphi_a)$ is the universal (abelian) representation of $\mathcal{F}$, where $\overline{\Lambda}^{(23)}$ is the subspace in $\overline{\Lambda}$ generated by the image of φ_a. Notice that $\overline{\Lambda}^{(23)}$ is the orthogonal complement of $\overline{u}$ with respect to $\overline{\beta}$. By (2.3.2) the geometric hyperplanes in $\mathcal{F}$ are in a bijection with the index 2 subgroups in $\overline{\Lambda}^{(23)}$. In turn the index 2 subgroups correspond to the non-zero vectors of the module dual to $\overline{\Lambda}^{(23)}$ which is isomorphic to the quotient of $\overline{\Lambda}$ over $\overline{u}$. This gives the following.

Lemma 5.5.2 *Let* Ω *be a geometric hyperplane in* $\mathcal{F}$. *Then there is a vector* $x \in \Lambda$ *with* $\overline{x} \neq \overline{u}$, *such that* $\Omega = H_2(x)$, *where*

$$H_2(x) = \{t \mid t \in \Theta, (t + u, x) = 0 \bmod 2\}.$$

Furthermore, $H_2(x) = H_2(z)$ *if and only if* $\overline{x} = \overline{z} + \alpha \cdot \overline{u}$ *for* $\alpha \in \{0, 1\}$. □

Let $\widetilde{\mathcal{F}} = \mathcal{G}(3^{23} \cdot Co_2)$, $\widetilde{F} \cong 3^{23} \cdot Co_2$ be the automorphism group of $\widetilde{\mathcal{F}}$, $E = O_3(\widetilde{F})$ and $\chi : \widetilde{\mathcal{F}} \to \mathcal{F}$ be the corresponding 2-covering. Then the fibers of χ are the orbits of E on $\widetilde{\mathcal{F}}$. Thus we can treat the elements of $\mathcal{F}$ as E-orbits on $\widetilde{\mathcal{F}}$, so that χ sends an element onto its E-orbit.

The $GF(3)$-vector space $\widehat{\Lambda}$ as a module for $F = G^\bullet(u) \cong Co_2$ is a direct sum

$$\widehat{\Lambda} = \widehat{u} \oplus \widehat{\Lambda}^{(23)},$$

where $\widehat{\Lambda}^{(23)}$ is the orthogonal complement of $\widehat{u}$ with respect to $\widehat{\beta}$ and is generated by the 1-subspaces $\widehat{t}$ taken for all $t \in \Theta$. It was shown in [Sh92] (cf. Proposition 7.4.8 in [Iv99]) that $\widehat{\Lambda}^{(23)}$ is an irreducible F-module which is isomorphic to E. If we identify E and $\widehat{\Lambda}^{(23)}$ through this isomorphism then we have the following

Lemma 5.5.3 *Let $\widetilde{t}$ be a point of $\widetilde{\mathcal{F}}$, $t = \chi(\widetilde{t}) \in \Theta$. Then $E(\widetilde{t}) = \widehat{t}$. Thus $E(\widetilde{t})$ is cyclic of order* 3 *and it depends only on the E-orbit $t = \chi(\widetilde{t})$ containing $\widetilde{t}$.* □

Lemma 5.5.4 *Let $\Xi \subseteq \Theta$ and suppose that the elements $\widehat{t}$ taken for all $t \in \Xi$ generate in $\widehat{\Lambda}^{(23)}$ a proper subgroup. Then there is a vector $y \in \Lambda$ with $(y, u) = 0$ mod 3 such that $\Xi \subseteq H_3(y)$, where*

$$H_3(y) = \{t \mid t \in \Theta, (t, y) = 0 \text{ mod } 3\}.$$

Proof. By the assumption the set $\{\widehat{t} \mid t \in \Xi\}$ is contained in a maximal subgroup $\widehat{\Delta}$ (of index 3) in $\widehat{\Lambda}^{(23)}$. Since the restriction of $\widehat{\beta}$ to $\widehat{\Lambda}^{(23)}$ is non-singular, $\widehat{\Delta}$ is the orthogonal complement of a non-zero vector $\widehat{y} \in \widehat{\Lambda}^{(23)}$ with respect to $\widehat{\beta}$. Now the proof follows by considering a suitable preimage y of $\widehat{y}$ in Λ. □

In order to simplify the calculations we are going to perform, it is convenient to set $\mathcal{P} = \{1, 2, ..., 24\}$. Then $\mathcal{E} = (e_1, e_2, ..., e_{24})$ is a basis of $\mathbb{R}^{24}$ and for $\lambda \in \Lambda$ we have

$$\lambda = \lambda_1 e_1 + \lambda_2 e_2 + ... + \lambda_{24} e_{24},$$

where the coordinates λ_i satisfy the conditions (Λ1) to (Λ3) in Section 5.1.

Choose $u = 4e_1 - 4e_2$. Then $u \in \Lambda_2$ and $F = G^\bullet(u) \cong Co_2$. The vector $v = 4e_1 + 4e_2$ (strictly speaking the pair $\{v, -v\}$) belongs to Θ and it is characterized by the property that the stabilizer $F(v)$ acts monomially in the basis $\mathcal{E}$.

More specifically $F(v)$ is the semidirect product of $Q(v) \cong O_2(F(v))$ and $L(v) \cong \mathrm{Aut}\, M_{22}$. The subgroup $L(v)$ acts permutationally as the setwise stabilizer of $\{1,2\}$ in the automorphism group of the $S(5,8,24)$-Steiner system $(\mathcal{P},\mathcal{B})$. The elements of $Q(v)$ are indexed by the subsets from the Golay code $\mathcal{C}_{12}$ (associated with $(\mathcal{P},\mathcal{B})$) disjoint from $\{1,2\}$. If $Y \subseteq \mathcal{P} \setminus \{1,2\}$ is such a subset, then the corresponding element $\tau(Y) \in Q(v)$ stabilizes e_i if $i \notin Y$ and negates it if $i \in Y$. Recall that $Q(v)$ is the 10-dimensional Golay code module for $L(v)$.

In these terms the orbits of $F(v)$ on Θ are specified by the shapes of the vectors they contain (cf. Lemma 4.9.5 in [Iv99]). In particular

$$\Theta(v) = \{4e_i + \alpha 4e_j \mid 3 \le i < j \le 24, \alpha \in \{1,-1\}\}$$

so that

$$\{\{v, 4e_i + 4e_j, 4e_i - 4e_j\} \mid 3 \le i < j \le 24\}$$

is the set of lines containing v.

The structure of $\widehat{\Lambda}^{(23)}$ as a module for $F(v)$ easily follows from the above description of $F(v)$.

Lemma 5.5.5 *As a module for $F(v) \cong 2^{10} : \mathrm{Aut}\, M_{22}$ the module $\widehat{\Lambda}^{(23)}$ possesses the direct sum decomposition*

$$\widehat{\Lambda}^{(23)} = \widehat{v} \oplus \widehat{\Lambda}^{(22)},$$

where $\widehat{\Lambda}^{(22)} = [Q(v), \widehat{\Lambda}^{(23)}]$ is the orthogonal complement of $\widehat{v}$ with respect to $\widehat{\beta}$. As a module for $Q(v)$ the module $\widehat{\Lambda}^{(22)}$ possesses the direct sum decomposition

$$\widehat{\Lambda}^{(22)} = \bigoplus_{i=3}^{24} \widehat{T}_i,$$

where $\widehat{T}_i$ is generated by the image of the vector $8e_i \in \Lambda$ and $C_{Q(v)}(\widehat{T}_i)$ is a hyperplane in $Q(v)$ from the $L(v)$-orbit of length 22. In particular, $F(v)$ acts monomially and irreducibly on $\widehat{\Lambda}^{(22)}$. □

Now we proceed to the main proof of the section. Let W be the universal representation module of $\widetilde{\mathcal{F}} = \mathcal{G}(3^{23} \cdot Co_2)$. Then

$$W = W^z \oplus W^c, \text{ where } W^z = C_W(E),\ W^c = [W, E].$$

By (2.4.1), W^z is the universal representation module of $\mathcal{F}$ and thus $W^z \cong \overline{\Lambda}^{(23)}$ by (5.2.3). We are going to prove that W^c is trivial by showing that the condition **(M)** from Section 2.4 holds and that there are no acceptable geometric hyperplanes in $\mathcal{F}$.

Lemma 5.5.6 *The condition* **(M)** *from Section* 2.4 *holds.*

Proof. In terms of (5.5.5), $\widehat{\Lambda}^{(22)}$ is the complement to $\widehat{v} = E(v)$ in $E = \widehat{\Lambda}^{(23)}$, so it maps isomorphically onto its image in $E/E(v)$. Let B_i be the image of $\widehat{T}_i$ in $E/E(v)$ for $3 \leq i \leq 24$. By (5.5.5) and the above description of the lines in $\mathcal{F}$ passing through v the condition **(M)** follows. Notice that in this case the graph Σ in **(M)** is the complete graph on 22 vertices. □

Lemma 5.5.7 *There are no acceptable hyperplanes in* $\mathcal{F}$.

Proof. Suppose that Ω is an acceptable hyperplane in $\mathcal{F}$. Then, first of all, it is a hyperplane and by (5.5.2) there is a non-zero vector $x \in \Lambda$ such that $\Omega = H_2(x)$. On the other hand, Ω is acceptable which means that the subgroups $E(t) = \widehat{t}$ taken for all $t \in \Theta \setminus \Omega$ generate in $E = \widehat{\Lambda}^{(23)}$ a proper subgroup. By (5.5.4) this means that there is a vector $y \in \Lambda$ with $(y, u) = 0 \bmod 3$ such that $\Theta \setminus \Omega \subseteq H_3(y)$. Thus we must have

$$\Theta = H_2(x) \cup H_3(y)$$

and we will reach a contradiction by showing that this is not possible. Let $\widehat{\Delta}$ denote the subspace in $\widehat{\Lambda}^{(23)}$ generated by the elements $\widehat{t}$ taken for all $t \in H_3(y)$.

Since $H_2(x)$ is a proper subset of Θ (and F acts transitively on Θ) we can assume without loss of generality that $H_2(x)$ does not contain v. This of course means that $v \in H_3(y)$ and $\widehat{v} \in \widehat{\Delta}$, but also it means that

$$(u + v, x) = (8e_1, x) = \frac{1}{8}(8x_1)$$

is odd. Since x is a Leech vector, by (Λ1) we conclude that all the coordinates x_i of x (in the basis $\mathcal{E}$) are odd. For $r = 1$ or 3 let

$$C^{(r)} = \{i \mid 1 \leq i \leq 24, \quad x_i = r \bmod 4\}.$$

Then by (Λ2) the subsets $C^{(1)}$ and $C^{(3)}$ are contained in the Golay code $\mathcal{C}_{12}$. We will consider two cases separately.

Case 1: $(x, u) = 0 \bmod 2$.

In this case $t \in \Theta$ is in $H_2(x)$ if and only if (t, x) is even. Furthermore, $\{1, 2\}$ intersects both $C^{(1)}$ and $C^{(3)}$. Also for $3 \leq i < j \leq 24$ the point $e_i + e_j \in \Theta(v)$ is contained in $H_2(x)$ if and only if $\{i, j\}$ intersects both

$C^{(1)}$ and $C^{(3)}$. If $\{i,j,k\} \subseteq C^{(r)}$ for $r = 1$ or 3 then the points $e_i + e_j$, $e_i + e_k$ and $e_j + e_k$ are not in $H_2(x)$, hence they must be in $H_3(y)$. Since

$$(e_i + e_j) + (e_i + e_k) - (e_j + e_k) = 2e_i,$$

we conclude that $(y, 8e_i) = 0 \bmod 3$ and hence $\widehat{\Delta}$ contains the subgroup $\widehat{T}_i$ as in (5.5.5). The subsets $C^{(1)}$ and $C^{(3)}$ being non-empty subsets from the Golay code each contain at least 8 elements each, which shows that every $3 \le i \le 24$ is contained in a triple $\{i,j,k\}$ as above. Now (5.5.5) implies that $\widehat{\Delta} = \widehat{\Lambda}^{(23)}$, which is a contradiction.

Case 2: $(x,u) = 1 \bmod 2$.

In this case $t \in \Theta$ is in $H_2(x)$ if and only if (t,x) is odd. For $r = 1$ or 3 the subset $C^{(r)}$ is disjoint from $\{1,2\}$. Since the negation changes the residue modulo 4 we can apply to x the element $\sigma = \tau(C^{(r)})$ (where $\sigma(e_i) = -e_i$ if $e_i \in C^{(r)}$ and $\sigma(e_i) = e_i$ otherwise) to obtain a vector with all coordinates equal modulo 4. Then for $3 \le i < j \le 24$ the point $e_i + e_j$ is contained in $H_2(x)$, while $e_i - e_j$ is not and hence is contained in $H_3(y)$. This enables us to specify the coordinates of y modulo 3. Indeed, since $(y,u) = (y,v) = 0 \bmod 3$, we have $y_1 = y_2 = 0 \bmod 3$ and since $(y, e_i - e_j) = 0 \bmod 3$, the coordinates y_i for $3 \le i \le 24$ are all equal to the same number ε modulo 3. Clearly ε should not be 0, otherwise $\widehat{\Delta}$ will be the whole $\widehat{\Lambda}^{(23)}$.

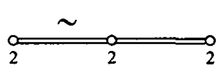
Thus the vector y is uniquely determined modulo 3Λ and hence $H_3(y)$ is also determined. In order to obtain the final contradiction let us assume that $\{3,4,...,10\}$ is an octad. Then the vectors $a = 2e_3+2e_3+2e_5+...+2e_{10}$ and $b = -2e_3 - 2e_4 + 2e_5 + ... + 2e_{10}$ are both in $\Theta_2^2(v)$ and direct calculations show that they are not in $H_3(y)$. Hence they must be in $H_2(x)$, i.e., $(x,a) = (x,b) = 1 \bmod 2$. But then

$$(x, a-b) = (x, 4e_3 + 4e_4) = 0 \bmod 2,$$

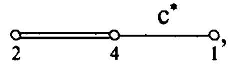
which means $4e_3 + 4e_4$ is not in $H_2(x)$. Since this contradicts what we have established in the previous paragraph, the proof is complete. □

5.6 $\mathcal{G}(3 \cdot U_4(3))$

As shown in Section 4.14 in [Iv99] the fixed vertices $\Phi = \Phi(X_s)$ in the Leech graph Γ of a particular subgroup X_s of order 3 are the point-set of two geometries $\mathcal{G}(3 \cdot U_4(3))$ and $\mathcal{E}(3 \cdot U_4(3))$ with diagrams

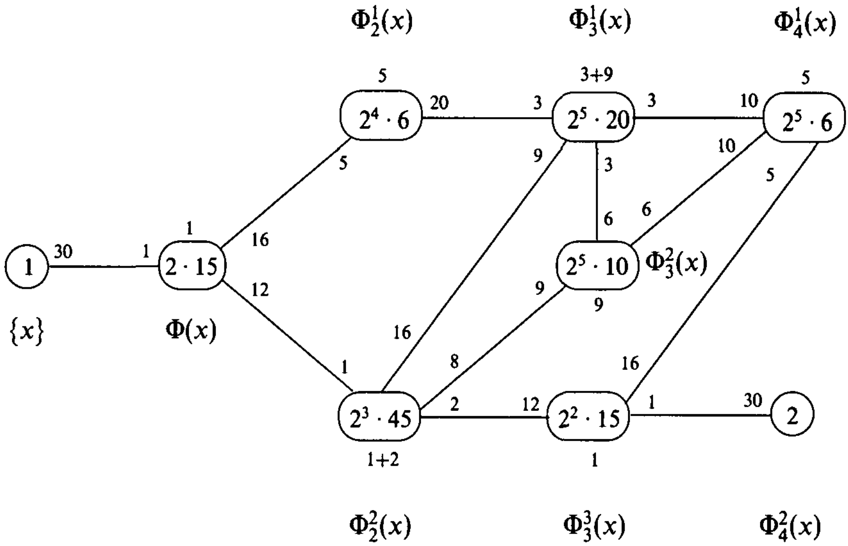


and



respectively. The group $U := C_G(X_s)/X_s \cong 3 \cdot U_4(3).2_2$, where $G \cong Co_1$, acts flag-transitively on both geometries. Notice that for $\mathscr{E}(3 \cdot U_4(3))$ our numbering of types is reverse to that in [Iv99].

The geometries $\mathscr{G}(3 \cdot U_4(3))$ and $\mathscr{E}(3 \cdot U_4(3))$ share the point-line incidence system $\mathscr{S} = (\Pi, L)$ and hence they also share the collinearity graph Φ whose suborbit diagram with respect to the action of U is given below.



If x is a vertex of Φ (which is also a vertex of the Leech graph Γ), then

$$\Phi(x) \cup \Phi_4^2(x) \subseteq \Gamma(x), \quad \Phi_2^1(x) \cup \Phi_3^3(x) \subseteq \Gamma_2^1(x),$$

$$\Phi_3^1(x) \cup \Phi_4^1(x) \subseteq \Gamma_3^1(x), \quad \Phi_3^2(x) \subseteq \Gamma_3^2(x), \quad \Phi_2^2(x) \subseteq \Gamma_2^2(x).$$

The subgroup $D = O_3(U)$ is of order 3, it acts fixed-point freely on Φ and the orbit containing x is $\{x\} \cup \Phi_4^2(x)$, so the above inclusions show that the subgraph in Γ induced by Φ is the collinearity graph $\Phi^\bullet$ of the enriched point-line incidence system $\mathscr{S}^\bullet$ of $\mathscr{S}$.

The planes in $\mathscr{G}(3 \cdot U_4(3))$ are the subgraphs in Φ isomorphic to the collinearity graph of the rank 2 tilde geometry. Such a subgraph containing x also contains 6, 24, 12 and 2 vertices from $\Phi(x)$, $\Phi_2^2(x)$, $\Phi_3^3(x)$ and $\Phi_4^2(x)$, respectively. The planes of $\mathscr{E}(3 \cdot U_4(3))$ are Schläfli subgraphs in Φ (isomorphic to the collinearity graph of $\mathscr{P}(\Omega_6^-(2))$). Such a subgraph containing x contains also 10 and 16 vertices from $\Phi(x)$ and $\Phi_2^1(x)$, respectively.

The vertices of Φ, treated as vectors in $\overline{\Lambda}_4$ generate a 12-dimensional irreducible U-submodule W in $\overline{\Lambda}$. The quadratic form θ as in (5.1.1 (v)) and the corresponding bilinear form β restricted to W are non-singular and by the above inclusions we have the following

Lemma 5.6.1 *If $x, y \in \Phi$, then $\beta(x, y) \neq 0$ if and only if $y \in \Phi_3^1(x) \cup \Phi_4^1(x)$.* □

By the above if φ is the identity mapping then (W, φ) is a representation of the enriched system $\mathcal{S}^\bullet$. The universal abelian representation of $\mathcal{E}(3 \cdot U_4(3))$ has been calculated in [Yos92].

Proposition 5.6.2 *The 12-dimensional representation (W, φ) of the enriched system $\mathcal{S}^\bullet$ is the universal abelian one.* □

Straightforward calculations in the Golay code and Todd modules give the following

Lemma 5.6.3 *If x is a point of Φ then $W[\{x\} \cup \Phi^\bullet(x)]$ is 6-dimensional.* □

Let $Q \cong 2_+^{1+12}$ be an extraspecial group in which the square and the commutator maps are determined by the (restrictions to W of the) forms θ and β via the isomorphism

$$Q/Z(Q) \to W.$$

We can embed the order 3 subgroup X_s into the parabolic $C \cong 2_+^{1+24}.Co_1$ of the Monster and put Q to be the centralizer of X_s in $O_2(C)$ (compare Lemma 5.6.1 in [Iv99]). Then arguing as in the proof of (4.4.1) we obtain

Lemma 5.6.4 *$Q \cong 2_+^{1+12}$ is a $3 \cdot U_4(3)$-admissible representation group of the enriched system $\mathcal{S}^\bullet$.* □

Proposition 5.6.5 *The group $Q \cong 2_+^{1+12}$ in (5.6.4) is the universal representation group of the enriched system $\mathcal{S}^\bullet$.* □

The above result was established in [Rich99] using a slight generalization of (2.3.7). The most complicated part of the proof was to show that the subgraph in Φ induced by $\Phi_3^2(x)$ is connected. This was achieved by cumbersome direct calculations in the graph treated as a subgraph in the Leech graph. We decided it is not practical to reproduce these arguments here (unfortunately we were unable to come up with an easier argument).

6
Involution geometries

In this chapter we consider a class of geometries that always possess non-trivial representations. Suppose that G is a group which contains a set $\mathscr{C}$ of involutions, such that $\mathscr{C}$ generates G, and let $\mathscr{K}$ be a set of elementary abelian subgroups of order four (Kleinian four-subgroups) in G, all the non-identity elements of which are contained in $\mathscr{C}$. If we identify a subgroup from $\mathscr{K}$ with the triple of involutions it contains, then $(\mathscr{C}, \mathscr{K})$ is a point-line incidence system with three points per line (the line-set might be empty). We denote this system by $\mathscr{I}(G, \mathscr{C}, \mathscr{K})$ and call it an *involution geometry* of G. It is clear from the definition that if i is the identity mapping, then (G, i) is a representation of $\mathscr{I}(G, \mathscr{C}, \mathscr{K})$. We are interested in the situation when this representation is universal.

6.1 General methods

Let $\mathscr{I}(G, \mathscr{C}, \mathscr{K})$ be an involution geometry of G. If $\mathscr{K}$ is the set of all $\mathscr{C}$-pure Kleinian four-subgroups in G (i.e., with all their involutions in $\mathscr{C}$) then instead of $\mathscr{I}(G, \mathscr{C}, \mathscr{K})$ we simply write $\mathscr{I}(G, \mathscr{C})$. If in addition $\mathscr{C}$ is the set of all involutions in G then we denote $\mathscr{I}(G, \mathscr{C})$ simply by $\mathscr{I}(G)$ and call it the involution geometry of G.

Lemma 6.1.1 *Let G be a group and $\mathscr{I} = \mathscr{I}(G, \mathscr{C}, \mathscr{K})$ be an involution geometry of G. Let $\widetilde{G}$ be a group possessing a homomorphism ψ onto G, such that the following conditions hold:*

(i) *the kernel $\widetilde{K}$ of ψ is of odd order;*
(ii) *$\widetilde{K}$ is in the centre of $\widetilde{G}$ (in particular, it is abelian);*
(iii) *if $\widetilde{H}$ is a subgroup in $\widetilde{G}$ such that $\psi(\widetilde{H}) = \widetilde{G}$, then $\widetilde{H} = \widetilde{G}$ (equivalently, for every $\widetilde{L} < \widetilde{K}$ there is no complement to $\widetilde{K}/\widetilde{L}$ in $\widetilde{G}/\widetilde{L}$).*

Then $\widetilde{G}$ is a representation group of $\mathscr{I}$.

Proof. Let $\tau \in \mathscr{C}$ be a point of $\mathscr{I}$ (an involution in G). Since by (i) and (ii), $\widetilde{K}$ is an odd order subgroup in the centre of $\widetilde{G}$, the full preimage of $\langle \tau \rangle$ in $\widetilde{G}$ is the direct product of $\widetilde{K}$ and a group of order 2. Thus $\psi^{-1}(\tau)$ contains a unique involution $\widetilde{\tau}$, say, and we put $\widetilde{\varphi}(\tau) = \widetilde{\tau}$. Let $\{\tau_1, \tau_2, \tau_3\}$ be a line in $\mathscr{I}$ (the set of involutions in a subgroup l of order 2^2 from $\mathscr{K}$). Then $\psi^{-1}(l)$ is the direct product of $\widetilde{K}$ and the Kleinian four-subgroup in $\widetilde{G}$, whose non-identity elements are the $\widetilde{\tau}_i$ for $1 \leq i \leq 3$. Finally, since G is generated by $\mathscr{C}$, the image of $\widetilde{\varphi}$ generates in $\widetilde{G}$ a subgroup which maps surjectively onto G. So we conclude from (iii), that $(\widetilde{G}, \widetilde{\varphi})$ is a representation of $\mathscr{I}$ and the proof follows. □

A special case of particular importance to us is when $\mathscr{C}$ is a conjugacy class in G. Let $\mathscr{I} = \mathscr{I}(G, \mathscr{C}, \mathscr{K})$ be such an involution geometry of G, and let (R, φ) be the universal representation of $\mathscr{I}$. Then, by the universality property, there is a homomorphism $\psi : R \to G$ such that

$$\psi(\varphi(\tau)) = \tau \text{ for every } \tau \in \mathscr{C}.$$

Lemma 6.1.2 *In the above terms suppose that $\varphi(\mathscr{C})$ is a conjugacy class of involutions in R. Then R possesses a homomorphism onto G, whose kernel $\widetilde{K}$ satisfies the conditions (ii) and (iii) in* (6.1.1).

Proof. Let $\tau_1, \tau_2 \in \mathscr{C}$. Since $\varphi(\mathscr{C})$ is a conjugacy class, we have

$$\varphi(\tau_1)\varphi(\tau_2)\varphi(\tau_1) = \varphi(\tau_3)$$

for some $\tau_3 \in \mathscr{C}$. Applying ψ to both sides of the above equality we see that

$$\tau_1\tau_2\tau_1 = \tau_3,$$

i.e., τ_3 is τ_2 conjugated by τ_1. We can then define the action of $\varphi(\tau_1)$ on $\mathscr{C}$ by the rule

$$\varphi(\tau_1) : \tau_2 \longmapsto \tau_3 \text{ where } \tau_3 = \tau_1\tau_2\tau_1.$$

Then $\varphi(\tau_1)$ acts exactly as τ_1 acts by conjugation. Hence the kernel $\widehat{K}$ of the homomorphism of R onto G is in the kernel of the action of R on $\varphi(\mathscr{C})$ by conjugation and since $\varphi(\mathscr{C})$ generates R, this means that $\widehat{K}$ is in the centre of $\widehat{G}$. Let $\widetilde{G}$ be the smallest subgroup in $\widehat{G}$ which maps surjectively onto G and $\overline{G} = R/\widetilde{G}$. Then $\overline{G}$ is abelian and the image of $\varphi(\mathscr{C})$ is a conjugacy class, generating $\overline{G}$. Hence $\overline{G}$ is of order 1 or 2. The latter case is impossible by (2.1.1). □

We will generally apply the following strategy. Given an involution geometry $\mathscr{I} = \mathscr{I}(G, \mathscr{C}, \mathscr{K})$ with the universal representation (R, φ), we try to prove that $\varphi(\mathscr{C})$ is a conjugacy class in R. When this is achieved, the structure of R becomes very restricted since by (6.1.2), R is a non-split central extension of G. Clearly $\varphi(\mathscr{C})$ is a conjugacy class in R if and only if for any $\tau_1, \tau_2 \in \mathscr{C}$ we have

$$\varphi(\tau_1)\varphi(\tau_2)\varphi(\tau_1) = \varphi(\tau_3),$$

where $\tau_3 = \tau_1\tau_2\tau_1 \in \mathscr{C}$. In particular, $\mathscr{C}$ must be a conjugacy class of G in the first place.

In all the examples we will deal with, $\mathscr{C}$ is a conjugacy class of involutions in G and $\mathscr{K}$ is the set of all $\mathscr{C}$-pure Kleinian four-subgroups (which is always non-empty). Then $\mathscr{I}(G, \mathscr{C}, \mathscr{K})$ is $\mathscr{I}(G, \mathscr{C})$ or even $\mathscr{I}(G)$ and G is a point-transitive automorphism group of $\mathscr{I} = \mathscr{I}(G, \mathscr{C}, \mathscr{K})$. Let (R, φ) be the universal representation of $\mathscr{I}$ (which is G-admissible). By considering the homomorphism of R onto G we observe the following.

Lemma 6.1.3 *Whenever* $\varphi(\tau)\varphi(\sigma)\varphi(\tau) \in \varphi(\mathscr{C})$ *for* $\tau, \sigma \in \mathscr{C}$, *the equality* $\varphi(\tau)\varphi(\sigma)\varphi(\tau) = \varphi(\tau\sigma\tau)$ *holds.* □

For $\tau \in \mathscr{C}$ put

$$\mathscr{N}(\tau) = \{\sigma \in \mathscr{C} \mid \varphi(\tau)\varphi(\sigma)\varphi(\tau) = \varphi(\tau\sigma\tau)\}.$$

We will gradually show for more and more points from $\mathscr{C}$ that they are contained in $\mathscr{N}(\tau)$, until we show that $\mathscr{C}$ contains all the points, which means (in view of point-transitivity) that $\varphi(\mathscr{C})$ is a conjugacy class in R and (6.1.2) applies. We will make use of the following result.

Lemma 6.1.4 *Let* $\mathscr{I} = \mathscr{I}(G, \mathscr{C}, \mathscr{K})$ *be an involution geometry of* G, *where* $\mathscr{C}$ *is a conjugacy class, and let* (R, φ) *be the universal representation of* $\mathscr{I}$. *Suppose that* $\tau, \sigma \in \mathscr{C}$ *are such that at least one of the following holds:*

(i) τ *and* σ *are contained in a common Kleinian four-subgroup from* $\mathscr{K}$;
(ii) *there is a subgroup* H *in* G *containing* τ *and* σ *which is generated by* $H \cap \mathscr{C}$ *and the universal representation* (Q, χ) *of* $\mathscr{J} := \mathscr{I}(H, H \cap \mathscr{C}, H \cap \mathscr{K})$ *is such that* $\chi(H \cap \mathscr{C})$ *is a conjugacy class in* Q;
(iii) *there is a subset* $\Delta \subset \mathscr{C}$ *containing* σ *such that the subgroup in* R *generated by the elements* $\varphi(\delta)$ *taken for all* $\delta \in \Delta$ *is also generated by such elements taken for all* $\delta \in \Delta \cap \mathscr{N}(\tau)$.

Then $\sigma \in \mathscr{N}(\tau)$.

Proof. In case (i) it is clear that the images of σ and τ in R commute and $\sigma \in \mathcal{N}(\tau)$. In case (ii) the restriction of φ to $H \cap \mathcal{C}$ induces a representation map for $\mathcal{J}$ and hence by the assumption we have $\varphi(\tau)\varphi(\sigma)\varphi(\tau) \in \varphi(H \cap \mathcal{C})$, which gives the proof. In case (iii) we have the equality

$$\langle \varphi(\tau)\varphi(\delta)\varphi(\tau) \mid \delta \in \Delta \rangle = \langle \varphi(\tau\delta\tau) \mid \delta \in \Delta \rangle.$$

Applying the homomorphism of R onto G it is easy to conclude that $\sigma \in \mathcal{N}(\tau)$. □

The following useful result is a special case of (6.1.4 (iii)).

Corollary 6.1.5 *If at least two points of a line from $\mathcal{K}$ are contained in $\mathcal{N}(\tau)$ then the whole line is in $\mathcal{N}(\tau)$.* □

The following lemma, whose proof is obvious, refines (6.1.4 (ii)).

Lemma 6.1.6 *Suppose that the hypothesis of* (6.1.4(*ii*)) *holds. For $\alpha, \beta \in H \cap \mathcal{C}$, let K_H and K_G be the conjugacy classes of H and G, respectively, containing the product $\alpha\beta$ (so that K_H fuses into K_G). Suppose that the natural action of G by conjugation on*

$$\Pi(K_G) = \{\{\tau, \sigma\} \mid \tau, \sigma \in \mathcal{C}, \tau\sigma \in K_G\}$$

is transitive. Then for $\{\tau, \sigma\} \in \Pi(K_G)$ we have $\sigma \in \mathcal{N}(\tau)$. □

The following lemma (which is rather an observation) has been used in our early studies of involution geometries and their representations. Although this lemma is not used within the present treatment, we decided to include it for the sake of completeness.

Lemma 6.1.7 *Let $Q \cong 2^{1+2n}_{\varepsilon}$ be the extraspecial group of type $\varepsilon \in \{+, -\}$ of order 2^{2n+1}, where $n \geq 2$ for $\varepsilon = +$ and $n \geq 3$ for $\varepsilon = -$. Let (R, φ) be the universal representation group of the involution geometry $\mathcal{I}$ of Q. Then $R \cong Q$.*

Proof. Let $\mathcal{C}$ and $\mathcal{K}$ be the set of involutions and the set of Kleinian four-subgroups in Q, so that $\mathcal{I} = \mathcal{I}(Q, \mathcal{C}, \mathcal{K})$. Let z be the unique non-identity element in the centre of Q, $\overline{Q} = Q/\langle z \rangle$. Let f be the quadratic form on $\overline{Q}$ induced by the power map on Q, $\overline{\mathcal{C}}$ be the image of $\mathcal{C}$ in $\overline{Q}$, and $\overline{\mathcal{K}}$ be the set of images in $\overline{Q}$ of the subgroups from $\mathcal{K}$ which do not contain z. Then f is non-singular, while $\overline{\mathscr{C}}$ and $\overline{\mathscr{K}}$ are the sets of 1- and 2-subspaces in $\overline{Q}$, isotropic with respect to f. By (3.6.2), $\overline{Q}$ is the universal representation group of $(\overline{\mathscr{C}}, \overline{\mathscr{K}})$. To complete the proof it is sufficient to notice that $\varphi(z)$ is in the centre of R and hence can be factored out. □

6.2 $\mathcal{I}(Alt_7)$

Let $A = Alt_7$ and $\mathcal{I} = \mathcal{I}(Alt_7, \mathscr{C}, \mathscr{K})$ be the involution geometry of A. Recall that according to our notation $\mathscr{C}$ and $\mathscr{K}$ are the set of all involutions and the set of all Kleinian four-subgroups in A. Every involution $\tau \in \mathscr{C}$ is a product of two disjoint transpositions. If $\tau = (a,b)(c,d)$ and $\sigma = (e,f)(h,g)$ are distinct involutions in A, then the product $\tau\sigma$ is an involution (equivalently $[\tau, \sigma] = 1$) if and only if one of the following holds:

(I) τ and σ have the same support, i.e., $\{a,b,c,d\} = \{e,f,h,g\}$;
(II) τ and σ share one transposition and the other transpositions are disjoint, for instance $\{a,b\} = \{e,f\}$ and $\{c,d\} \cap \{h,g\} = \emptyset$.

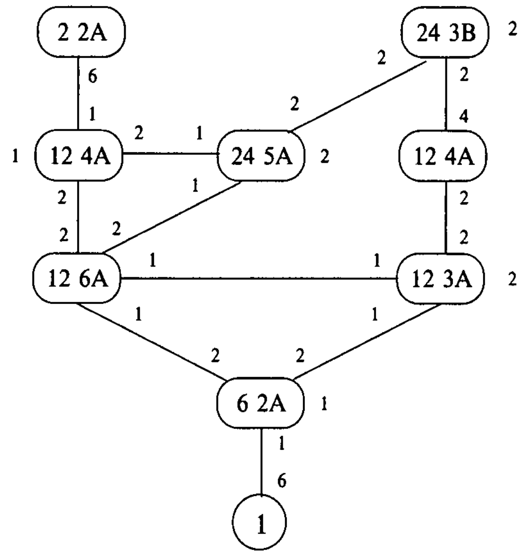
It is easy to see that if τ is an involution in A then $C_A(\tau) \cong (D_8 \times Sym_3)^+$ permutes transitively the pair of involutions of type (I) commuting with τ and the set of six involutions of type (II) commuting with τ.

The following result, of fundamental importance for the whole of our project, has been established by D.V. Pasechnik by means of computer calculations.

Proposition 6.2.1 *Let (R, φ) be the universal representation of the involution geometry $\mathcal{I}(Alt_7, \mathscr{C}, \mathscr{K})$ of Alt_7. Then $R \cong 3 \cdot Alt_7$, in particular, $\varphi(\mathscr{C})$ is a conjugacy class in R.* □

The fact that $3 \cdot Alt_7$ is a representation group of $\mathcal{I}(Alt_7)$ follows from the general principle (6.1.1) in view of the well-known fact that the Schur multiplier of Alt_7 is of order 6, but to conclude that it is the *universal* representation group would be highly non-trivial.

Below we present the suborbit diagram with respect to the action of A of the graph $\Sigma = \Sigma(Alt_7)$ on the set of involutions in A whose edges are the pairs of commuting involutions of type (II). This diagram plays an illustrative purpose in this section, but will be used more essentially in the subsequent sections.



Notice that the graph whose edges are the commuting pairs of type (I) is just the union of 35 disjoint triangles.

Let us explain the notation on this diagram and further diagrams in this chapter. Let τ be a fixed involution (a vertex of Σ), which corresponds to the orbit of length 1 on the diagram. Then $\Sigma(\tau, m, K)$ denotes an orbit which is of length m of $C_A(\tau)$ on Σ, and for every $\sigma \in \Sigma(\tau, m, K)$ the product $\tau\sigma$ belongs to the conjugacy class K of A. Such a suborbit will be said to be of type K. If K determines the length m uniquely, then we simply write $\Sigma(\tau, K)$ for such an orbit. On the other hand, if there are more than one suborbits of type K of a given length, then we use indexes l and r (l is for 'left' and r is for 'right') to indicate the suborbits on the left and on the right sides of the diagram, respectively. Thus on the above diagram of $\Sigma(Alt_7)$ we have two suborbits of type $4A$, which are $\Sigma_l(\tau, 12, 4A)$ and $\Sigma_r(\tau, 12, 4A)$.

We follow [CCNPW] for the names of conjugacy classes. Notice that the character tables of the groups whose involution geometries we are considering in this chapter (which are Alt_7, M_{22}, $U_4(3)$, Co_2 and Co_1) are given in [CCNPW] as well as in [GAP] in a computerised form. Using a standard routine we can deduce from these character tables the structure constants of multiplication of conjugacy classes in the relevant group G. Namely for any three conjugacy classes K, L and M in G we can calculate the value

$$m(K, L, M) = \#\{(k, l, m) \mid k \in K, l \in L, m \in M, kl = m\}.$$

Thus for a given class $\mathscr{C}$ of involutions in G and a conjugacy class K in G we can calculate the total lengths of suborbits of type K.

6.3 $\mathscr{I}(M_{22})$

In this section we study the involution geometry $\mathscr{I} = \mathscr{I}(M_{22})$ of the Mathieu group $H = M_{22}$. We know that H contains a single class $\mathscr{C}$ of involutions of size 1155. Let $\mathscr{H} = \mathscr{G}(M_{22})$ and $\Delta = \Delta(M_{22})$ be the derived graph of $\mathscr{H}$. By noticing that there are 1155 elements of type 2 in $\mathscr{H}$ (which are the edges of Δ), and the stabilizer in H of such an element is of the shape $2_+^{1+4}.(Sym_3 \times 2)$ and has the centre of order 2, we obtain the following.

Lemma 6.3.1 *There is a bijection ε, commuting with the action of H from the set $\mathscr{C}$ of involutions in H onto the set of edges of the derived graph Δ.* □

Below we present the suborbit diagram with respect to the action of H of the graph $\Sigma = \Sigma(M_{22})$ on $\mathscr{C}$ in which two distinct involutions τ and σ are adjacent if and only if the edges $\varepsilon(\tau)$ and $\varepsilon(\sigma)$ share a vertex of Δ. This means that Σ is the line graph of Δ. The diagram of Σ is deduced from the parameters of the centralizer algebra of the action of M_{22} on its involutions by conjugation, calculated by D.V. Pasechnik. It also follows from the parameters that the suborbits $\Sigma_l(\tau, 96, 4B)$ and $\Sigma_r(\tau, 96, 4B)$ are paired to each other.

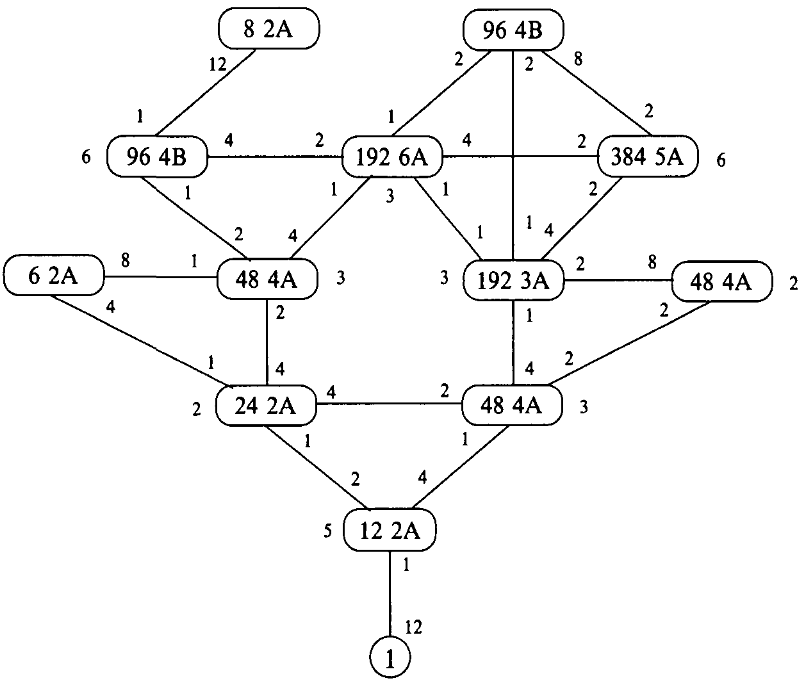
The next lemma provides us with a better understanding of the pairs of commuting involutions in H.

Lemma 6.3.2 *Let τ be an involution in $H = M_{22}$, let $\varepsilon(\tau) = \{v_1, v_2\}$ be the corresponding edge of Δ, let $\mathscr{S}$ be the $\mathscr{G}(S_4(2))$-subgeometry in $\mathscr{H} = \mathscr{G}(M_{22})$ containing $\varepsilon(\tau)$ and let Π_1, Π_2, Π_3 be the Petersen subgraphs in Δ containing $\varepsilon(\tau)$. Suppose that $\sigma \in \mathscr{C}$ commutes with τ and let m be the length of the orbit of $C_H(\tau)$ containing σ. Then either $\tau = \sigma$ or exactly one of the following holds:*

(i) *$\varepsilon(\sigma)$ contains v_i for some $i \in \{1,2\}$ and it is contained in Π_j for some $j \in \{1,2,3\}$, $\tau, \sigma \in O_2(H(v_i)) \cap O_2(H(\Pi_j))$ and $m = 12$;*
(ii) *$\varepsilon(\sigma)$ is contained in $\mathscr{S}$ and in Π_j for some $j \in \{1,2,3\}$, $\tau, \sigma \in O_2(H(\mathscr{S})) \cap O_2(H(\Pi_j))$ and $m = 6$;*
(iii) *$\varepsilon(\sigma)$ is contained in Π_j for some $j \in \{1,2,3\}$ but not in $\mathscr{S}$, $\tau, \sigma \in O_2(H(\Pi_j))$ and $m = 24$;*

(iv) $\varepsilon(\sigma)$ *is contained in* $\mathcal{S}$ *but not in* Π_j *for any* $j \in \{1,2,3\}$, $\tau, \sigma \in O_2(H(\mathcal{S}))$ *and* $m = 8$.

Proof. Recall that $H(v_i) \cong 2^3 : L_3(2)$, $H(\Pi_j) \cong 2^4 : Sym_5$ and $H(\mathcal{S}) \cong 2^4 : Alt_6$. It is easy to deduce from the basic properties of $\mathcal{G}(M_{22})$ and its derived graph that $\varepsilon(O_2(H(v_i))^{\#})$ is the set of 7 edges containing v_i, $\varepsilon(O_2(H(\Pi_j))^{\#})$ is the edge-set of Π_j and $\varepsilon(O_2(H(\mathcal{S}))^{\#})$ is the set of 15 edges contained in $\mathcal{S}$. In addition, $\Pi_j \cap \mathcal{S}$ is the antipodal triple in Π_j containing $\varepsilon(\tau)$. Finally, by the above suborbit diagram τ commutes with exactly 50 other involutions in M_{22}. Hence the proof. □



It is well known that $H = M_{22}$ contains two conjugacy classes of subgroups isomorphic to $A = Alt_7$ and these classes are fused in Aut M_{22}. The permutation character of H on the cosets of A given in [CCNPW] enables us to reconstruct the fusion pattern of the conjugacy classes of A into conjugacy classes of H. If K_A is a conjugacy class of A whose elements are products of pairs of involutions (these classes can be read from the suborbit diagram in Section 6.2) then the class of H containing K_A is shown in the table below.

Alt_7	$2A$	$3A$	$3B$	$4A$	$5A$	$6A$
M_{22}	$2A$	$3A$	$3A$	$4B$	$5A$	$6A$

Let us compare the table against the suborbit diagram $\Sigma(M_{22})$. In view of the above made remark that the suborbits $\Sigma_l(\tau, 96, 4B)$ and $\Sigma_r(\tau, 96, 4B)$ are paired and by (6.2.1), we obtain the following.

Lemma 6.3.3 *Let $\mathcal{I}(M_{22}) = \mathcal{I}(M_{22}, \mathcal{C}, \mathcal{K})$ be the involution geometry of M_{22} and $\tau, \sigma \in \mathcal{C}$. Then in terms of* (6.1.4(*ii*)) *and* (6.1.6) *if $\sigma \notin \Sigma(\tau, 48, 4A)$, there is a subgroup $A \cong Alt_7$ which contains both τ and σ, in particular, $\sigma \in \mathcal{N}(\tau)$.* □

Lemma 6.3.4 *If (R, φ) is the universal representation of $\mathcal{I}(M_{22})$. Then $\varphi(\mathcal{C})$ is a conjugacy class of R.*

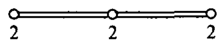
Proof. By (6.3.3) all we have to prove is that $\sigma \in \mathcal{N}(\tau)$ whenever $\sigma \in \Sigma(\tau, 48, 4A)$. Let us have a look at the suborbit diagram of $\Sigma(M_{22})$. Recall that two involutions α and β are adjacent in Σ if $\beta \in O_2(C_H(\alpha))$. Furthermore, such a pair $\{\alpha, \beta\}$ is in a unique line (contained in $O_2(C_H(\alpha))$). On the other hand, if $\sigma \in \Sigma(\tau, 48, 4A)$ then there are at least 9 (which is more than half the valency of Σ) vertices δ adjacent to σ such that $\delta \notin \Sigma(\tau, 48, 4A)$. Since such a vertex δ is in $\mathcal{N}(\tau)$ by (6.3.3), the proof is immediate from (6.1.5). □

Proposition 6.3.5 *The universal representation group of $\mathcal{I}(M_{22})$ is isomorphic to $3 \cdot M_{22}$.*

Proof. By (6.3.4) and (6.1.2) the representation group R of $\mathcal{I}(M_{22})$ is a non-split central extension of M_{22}. The Schur multiplier of M_{22} is cyclic of order 12 (cf. [CCNPW]). By (6.1.1) the non-split extension $3 \cdot M_{22}$ is a representation group of $\mathcal{I}(M_{22})$, so it only remains to show that the unique non-split extension $\widetilde{H} \cong 2 \cdot M_{22}$ is not an H-admissible representation group of $\mathcal{I}(M_{22})$. Calculating with the character table of $\widetilde{H}$ in the GAP package [GAP] we see that $\widetilde{H}$ has two classes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ of involutions which map onto $\mathcal{C}$ under the natural homomorphism of $\widetilde{H}$ onto H. Furthermore, (up to renumbering) for $i = 1$ or 2 an involution from $\widetilde{\mathcal{C}}_i$ commutes with 18 or 32 other involutions in $\widetilde{\mathcal{C}}_i$. Since an involution from $\mathcal{C}$ commutes with 50 other involutions from $\mathscr{C}$, this shows that $\widetilde{H}$ is not a representation group of $\mathscr{I}(M_{22})$ (the Kleinian four-subgroups are not lifted into a single class) and completes the proof. □

6.4 $\mathscr{I}(U_4(3))$

Let $\mathscr{U} = \mathscr{G}(U_4(3))$ be the GAB (geometry which is almost a building) associated with $U = U_4(3)$ (cf. Section 4.14 in [Iv99]). Then $\mathscr{U}$ belongs to the diagram

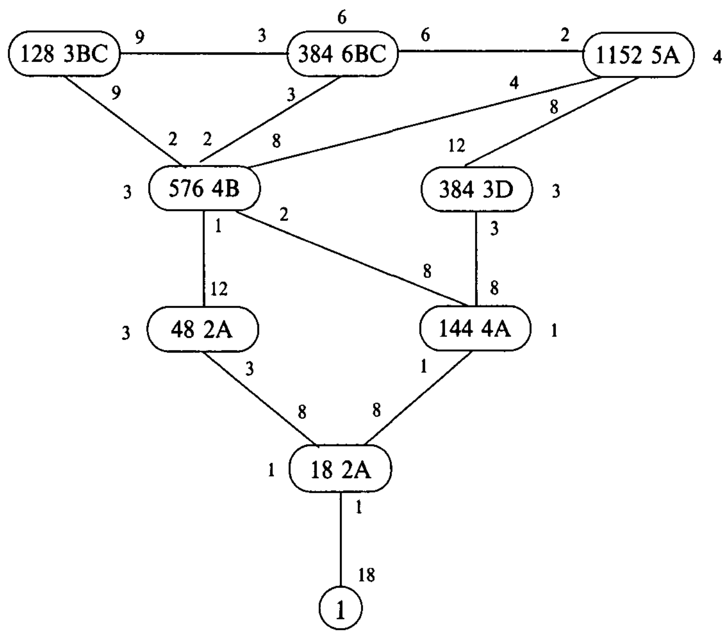


and admits a flag-transitive action of U. If $\{v_1, v_2, v_3\}$ is a maximal flag in $\mathscr{U}$ (where v_i is of type i), then $U(v_1) \cong U(v_3) \cong 2^4 : Alt_6$, $U(v_2) \cong 2^{1+4}_+ : (3 \times 3) : 4$. Since U contains a single conjugacy class $\mathscr{C}$ of involutions and $|C_U(\tau)| = 2^7 \cdot 3^2$ for $\tau \in \mathscr{C}$, we conclude that there is a bijection $\varepsilon : \mathscr{C} \to \mathscr{U}^2$ which commutes with the action of U.

Below is the suborbit diagram with respect to the action of $\mathrm{Aut}\, U \cong U_4(3).D_8$ of the graph $\Sigma = \Sigma(U_4(3))$ on $\mathscr{C}$ in which two distinct involutions τ, σ are adjacent if and only if

$$\mathrm{res}_{\mathscr{U}}(\varepsilon(\tau)) = \mathrm{res}_{\mathscr{U}}(\varepsilon(\sigma))$$

(notice that this equality holds exactly when $\sigma \in O_2(C_U(\tau))$).



It follows directly from the diagram of $\mathcal{U}$ that v_2 is incident to three elements of type 1 and three elements of type 3. Furthermore, there are 15 elements of type 2 incident to v_1, which are: v_2 itself; six elements incident with v_2 to a common element of type 3 and the remaining eight. In view of the fact that Aut U induces a diagram automorphism of $\mathcal{U}$, we obtain the following

Lemma 6.4.1 *If τ and σ are commuting involutions in U then there is $w \in \mathcal{U}^1 \cup \mathcal{U}^3$ such that $\tau, \sigma \in O_2(U(w))$ and both $\varepsilon(x)$ and $\varepsilon(y)$ are incident to w.* □

The group U contains four conjugacy classes of subgroups $A \cong Alt_7$ which are fused in Aut U. The permutation character of U acting on the cosets of A gives the following fusion pattern of the classes in A that are products of two involutions into conjugacy classes of U.

Alt_7	$2A$	$3A$	$3B$	$4A$	$5A$	$6A$
$U_4(3)$	$2A$	$3BC$	$3D$	$4B$	$5A$	$6BC$

Comparing the above table with the suborbits diagrams of $\Sigma(U_4(3))$ and $\Sigma(Alt_7)$, we obtain the following analogy of (6.3.3).

Lemma 6.4.2 *Let $\mathcal{I}(U_4(3)) = \mathcal{I}(U_4(3), \mathcal{C}, \mathcal{K})$ be the involution geometry of $U_4(3)$. In terms of (6.1.4(ii)) and (6.1.6) if $\tau, \sigma \in \mathcal{C}$ and $\sigma \notin \Sigma(\tau, 144, 4A)$, then there is a subgroup $A \cong Alt_7$ which contains both τ and σ, in particular, $\sigma \in \mathcal{N}(\tau)$.* □

It is absolutely clear from the suborbit diagram of $\Sigma(U_4(3))$ that there is a line $\{\sigma, \delta_1, \delta_2\}$ in $\mathcal{K}$ such that $\sigma \in \Sigma(\tau, 144, 4A)$ and $\delta_i \notin \Sigma(\tau, 144, 4A)$ for $i = 1, 2$ which gives the following analogy of (6.3.4).

Lemma 6.4.3 *If (R, φ) is the universal representation of $\mathcal{I}(U_4(3))$. Then $\varphi(\mathcal{C})$ is a conjugacy class of R.* □

Thus the universal representation R of $\mathcal{I}(U_4(3))$ is a non-split central extension of $U \cong U_4(3)$. The Schur multiplier of U is $3^2 \times 4$. By (6.1.1) $3^2 \cdot U_4(3)$ is a representation group of $\mathcal{I}(U_4(3))$. Let us have a look at $\widetilde{U} = 2 \cdot U_4(3)$. Calculations with GAP show that $\widetilde{U}$ has two classes $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$ of involutions outside the centre. Furthermore, an involution from $\widetilde{\mathcal{C}}_i$ commutes with 48 or 18 other involutions from $\widetilde{\mathcal{C}}_i$ where $i = 1$ or 2, respectively. Since an involution from $\mathcal{C}$ commutes with 64 other involutions from $\mathcal{C}$, similarly to the M_{22}-case we conclude that $\widetilde{U}$ is not a representation group of $\mathcal{I}(U_4(3))$ and we obtain the main result of the section.

Proposition 6.4.4 *The universal representation group of $\mathcal{I}(U_4(3))$ is isomorphic to $3^2 \cdot U_4(3)$.*

6.5 $\mathcal{I}(Co_2, 2B)$

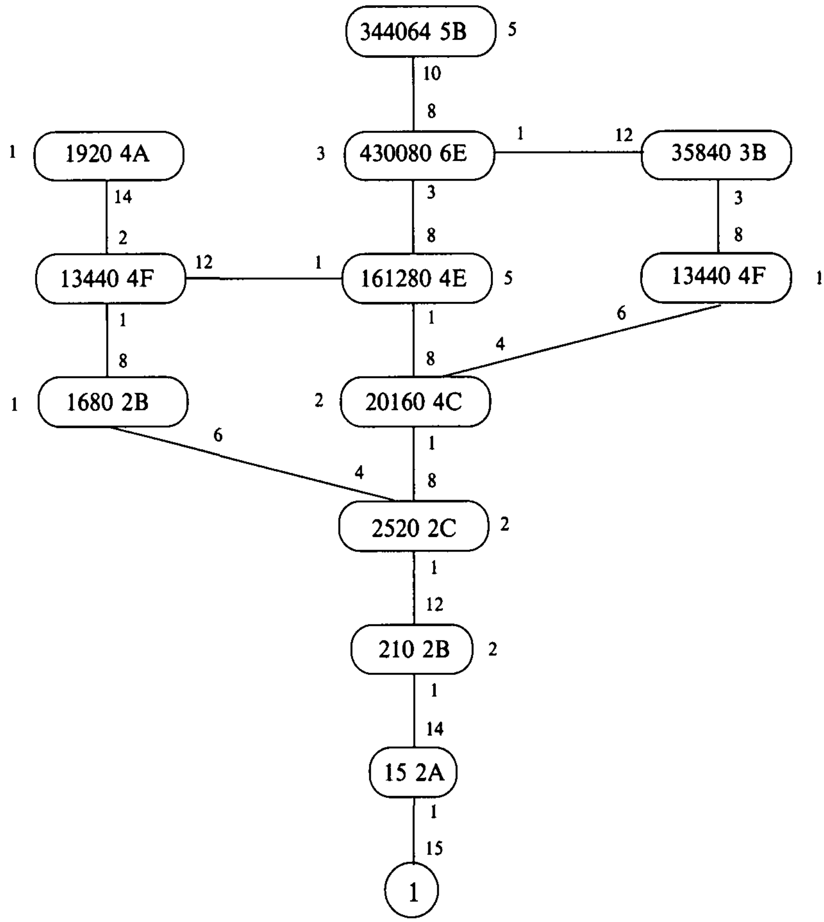
Let $\Sigma = \Sigma(Co_2)$ be the derived graph of $\mathcal{F} = \mathcal{G}(Co_2)$. The presented below suborbit diagram of this graph with respect to the action of $F \cong Co_2$ has been calculated by S.A. Linton. If v is a vertex of Σ (an element of type 4 in $\mathcal{F}$) then $F(v) \cong 2^{1+4+6}.L_4(2)$ coincides with the centralizer in F of a $2B$-involution in F from the conjugacy class $2B$. In this way we obtain a bijection ε from the conjugacy class $\mathcal{C}$ of $2B$-involutions in F onto the vertex-set of Δ.

Let u be an element of type 1 in $\mathcal{F}$ and $\Delta[u]$ be the set of vertices in Δ (which are elements of type 4 in $\mathcal{F}$) incident to u. Then $F(u) \cong 2^{10} : \operatorname{Aut} M_{22}$, and the subgraph in Σ induced by $\Sigma[u]$ is isomorphic to the 330-vertex derived graph of $\operatorname{res}_{\mathcal{F}}(u) \cong \mathcal{G}(M_{22})$ (cf. Section 4.5). Since $Q(u) := O_2(F(u))$ is the kernel of the action of $F(u)$ on $\operatorname{res}_{\mathcal{F}}(u)$, we conclude that

$$\{\varepsilon^{-1}(v) \mid v \in \Sigma[u]\}$$

is the orbit of length 330 of $F(u)/Q(u) \cong \operatorname{Aut} M_{22}$ on the set of non-identity elements in $Q(u)$. Since $Q(u)$ is the 10-dimensional Golay code module, by (4.5.1) we conclude that $Q(u)$ is a representation group of the derived system of $\operatorname{res}_{\mathcal{F}}(u) \cong \mathcal{G}(M_{22})$, which implies the following.

Lemma 6.5.1 *The pair (Co_2, ε^{-1}) is a representation of the derived system $\mathcal{D}(Co_2)$ of the geometry $\mathcal{G}(Co_2)$.* □



Comparing the suborbit diagram of $\Sigma(Co_2)$ and the suborbit diagram of the derived graph of $\mathscr{G}(M_{22})$ given in Section 4.5, we conclude the following result (the vertices of Σ are identified with the $2B$-involutions in $F \cong Co_2$ via the bijection ε).

Lemma 6.5.2 *Let* $\Sigma[u]$ *be the subgraph in* Σ *defined in the paragraph preceding* (6.5.1). *Suppose* $\tau \in \Sigma[u]$. *Then* $\Sigma[u]$ *consists of* τ, 7 *vertices from* $\Sigma(\tau, 15, 2A)$, 42 *vertices from* $\Sigma(\tau, 210, 2B)$, 168 *vertices from* $\Sigma(\tau, 2520, 2C)$ *and* 112 *vertices from* $\Sigma(\tau, 1680, 2B)$. □

Lemma 6.5.3 *Let* $\mathscr{I} = \mathscr{I}(Co_2, 2B)$ *and* (R, φ) *be the universal representation of* $\mathscr{I}$. *Then*

(i) *every line of* $\mathscr{I}$ *is contained in a conjugate of* $O_2(F(u))$;
(ii) *the elements* $\varphi(\alpha)$ *taken for all* $\alpha \in \Sigma[u]$ *generate in* R *a subgroup*

which maps isomorphically onto $O_2(F(u))$ under the natural homomorphism of R onto $F \cong Co_2$;

(iii) *(R, φ) is the universal representation of the derived system of $\mathscr{G}(Co_2)$.*

Proof. From the suborbit diagram of $\Sigma(Co_2)$ we observe that the line set $\mathscr{K}$ of $\mathscr{I}$ consists of two F-orbits, say $\mathscr{K}_1$ and $\mathscr{K}_2$, such that if $\{\tau, \sigma_1, \sigma_2\} \in \mathscr{K}_1$ then $\sigma_i \in \Sigma(\tau, 210, 2B)$ and $\sigma_i \in O_2(C_F(\tau))$ for $i = 1, 2$, and if $\{\tau, \sigma_1, \sigma_2\} \in \mathscr{K}_2$ then $\sigma_i \in \Sigma(\tau, 1680, 2B)$ and $\sigma_i \notin O_2(C_F(\tau))$ for $i = 1, 2$. By (6.5.2) we observe that $O_2(F(u))$ contains representatives of both the orbits, which gives (i). The assertion (ii) follows from (4.5.8). By (i) and (ii) the relations in R corresponding to the lines from $\mathscr{K}_1$ imply the relations corresponding to the lines from $\mathscr{K}_2$, which gives (iii). □

Let (R, φ) be the universal representation of $\mathscr{I}(Co_2, 2B)$ (which is also the universal representation of $\mathscr{D}(Co_2)$ by (6.5.3)). We are going to establish the isomorphism $R \cong Co_2$ by showing that $\varphi(\mathscr{C})$ is a conjugacy class of R. We follow the notation introduced in the paragraph subsequent to (6.1.3).

Lemma 6.5.4 *Let (R, φ) be the universal representation of $\mathscr{I}(Co_2, 2B)$ and let $\tau, \sigma \in \mathscr{C}$ (where $\mathscr{C}$ is the class of $2B$-involutions in $F \cong Co_2$) and let K be the conjugacy class of F containing the product $\tau\sigma$. Then $\sigma \in \mathscr{N}(\tau)$ whenever $K \in \{2A, 2B, 2C, 3B, 4C, 4E, 5B, 6E\}$.*

Proof. If $K \in \{2A, 2B, 2C\}$ then the claim follows from (6.5.3 (ii)). In the remaining cases we apply (6.4.4) together with the fact that Co_2 contains a subgroup isomorphic to $U_4(3)$. The relevant part of the fusion pattern of the classes obtained via GAP is presented below. This information gives the proof in view of (6.1.4 (ii)). □

$U_4(3)$	$2A$	$3BC$	$3D$	$4A$	$4B$	$5A$	$6BC$
Co_2	$2B$	$3B$	$3B$	$4C$	$4E$	$5B$	$6E$

In order to complete the proof that $\varphi(\mathscr{C})$ is a conjugacy class in R we apply a version of (6.1.4 (iii)). We use the following preliminary result (we continue to identify the vertex set of $\Sigma(Co_2)$ and the class of $2B$-involutions in Co_2 via the bijection ε).

Lemma 6.5.5 *In the notation of (6.5.4) suppose that δ is a vertex adjacent to σ in Σ such that at least 8 neighbours of δ are contained in $\mathscr{N}(\tau)$. Then $\sigma \in \mathscr{N}(\tau)$.*

Proof. Let R_δ be the subgroup generated by the elements $\varphi(\gamma)$ taken for all $\gamma \in \Sigma(\delta)$. We claim that R_δ is elementary abelian of order 2^4. Indeed, $\Sigma(\delta)$ (the set of 15 neighbours of δ in Σ) carries the structure of the point-set of a rank 3 projective $GF(2)$-geometry whose lines are those from $\mathcal{K}_1$ contained in this set. Hence the claim follows from (3.1.2). Since

$$\{\varphi(\gamma) \mid \gamma \in \Sigma(\delta)\}$$

is the set of non-identity elements of R_δ and a maximal subgroup in R_δ contains seven such elements, the proof follows. □

Lemma 6.5.6 *The $\varphi(\mathcal{C})$ is a conjugacy class in R.*

Proof. By (6.5.4) all we have to show is that $\sigma \in \mathcal{N}(\tau)$ whenever $\tau\sigma$ is in the class $4A$ or $4F$.

Let $\sigma \in \Sigma_l(\tau, 13440, 4F)$ and let $\delta \in \Sigma(\tau, 161280, 4E)$ be adjacent to σ. Then by (6.5.4) all the neighbours of δ are already in $\mathcal{N}(\tau)$ and hence so is σ by (6.5.5).

Let $\sigma \in \Sigma_r(\tau, 13440, 4F)$ and let δ be the unique neighbour of δ in the same orbit of $C_F(\tau)$. Then the remaining 14 neighbours of δ are in $\mathcal{N}(\tau)$ and (6.5.5) applies.

Finally if $\sigma \in \Sigma(\tau, 1920, 4A)$, then there is a neighbour δ of σ in the same orbits whose remaining 14 neighbours are in $\Sigma_l(\tau, 13440, 4F)$ and the latter orbit is already proved to be in $\mathcal{N}(\tau)$. □

Since the Schur multiplier of Co_2 is trivial, we deduce the following main result of the section from (6.1.2) and (6.5.6).

Proposition 6.5.7 *The universal representation (R, φ) of $\mathcal{I}(Co_2, 2B)$ is also the universal representation of the derived system of $\mathcal{G}(Co_2)$ and $R \cong Co_2$.* □

6.6 $\mathcal{I}(Co_1, 2A)$

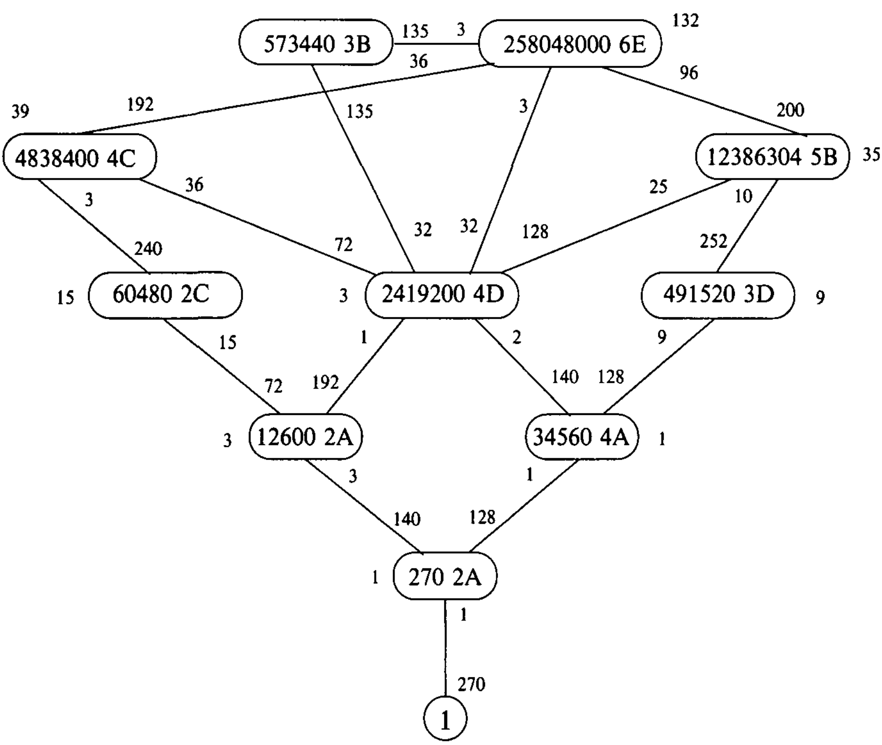
In this section $\mathcal{C}$ is the conjugacy class of central involutions ($2A$-involutions in terms of [CCNPW]) in $G \cong Co_1$ and $\Sigma = \Sigma(Co_1)$ is the graph on $\mathcal{C}$ in which two involutions $\tau, \sigma \in \mathcal{C}$ are adjacent if $\sigma \in O_2(C_G(\tau))$ (equivalently if $\tau \in O_2(C_G(\sigma))$). Notice

$$C_G(\tau) \cong 2_+^{1+8}.\Omega_8^+(2).$$

The suborbit diagram of Σ with respect to the action of G presented below is taken from [ILLSS] (the structure constants of the products of conjugacy classes are computed in GAP).

Notice that Σ is the collinearity graph of the dual of the maximal parabolic geometry $\mathscr{H}(Co_1)$ (cf. Lemma 4.9.1 in [Iv99]). Let p be a point of the tilde geometry $\mathscr{G}(Co_1)$ (which is also a point of the maximal parabolic geometry $\mathscr{H}(Co_1)$). Then $G(p) \cong 2^{11}.M_{24}$, $Q(p) := O_2(G(p))$ is the irreducible 11-dimensional Golay code module for $\overline{G}(p) = G(p)/Q(p) \cong M_{24}$. The intersection $Q(p) \cap \mathscr{C}$ contains exactly 759 involutions which naturally correspond to the octads of the $S(5,8,24)$-Steiner system associated with $\overline{G}(p)$. The subgraph in Σ induced by $Q(p) \cap \mathscr{C}$ is the octad graph (cf. Section 3.2 in [Iv99]). If $\tau \in Q(p) \cap \mathscr{C}$ then $Q(p) \cap \mathscr{C}$ contains 30 vertices from $\Sigma(\tau, 270, 2A)$, 280 vertices from $\Sigma(\tau, 12600, 2A)$, and 448 vertices from $\Sigma(\tau, 60480, 2C)$. In view of the above diagram, this gives the following.

Lemma 6.6.1 *Let $\mathscr{I}(Co_1, 2A) = \mathscr{I}(Co_1, \mathscr{C}, \mathscr{K})$ be an involution geometry of $G = Co_1$ (here $\mathscr{C}$ is the class of 2A-involutions and $\mathscr{K}$ is the set of all 2A-pure Kleinian four-subgroups in G). Then every line from $\mathscr{K}$ is contained in a conjugate of $Q(p)$.* □



The group Co_1 contains Co_2 as a subgroup. The fusion pattern of the relevant classes computed in GAP is presented below. Notice that the $2B$-involutions from Co_2 are fused to the class of $2A$-involutions in Co_1.

Co_2	$2A$	$2B$	$2C$	$3B$	$4A$	$4C$	$4E$	$4F$	$5B$	$6E$
Co_1	$2A$	$2A$	$2C$	$3B$	$4A$	$4C$	$4C$	$4D$	$5B$	$6E$

By (6.1.4), and comparing the above fusion pattern against the suborbit diagrams of $\Sigma(Co_1)$ and $\Sigma(Co_2)$, we obtain the following

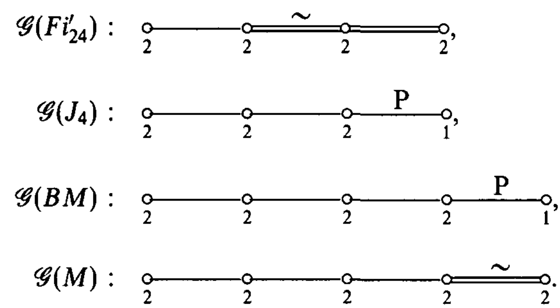
Lemma 6.6.2 *Let* (R, φ) *be the universal representation of* $\mathcal{I}(Co_1, 2A) = \mathcal{I}(Co_1, \mathscr{C}, \mathscr{K})$. *Then* $\varphi(\mathscr{C})$ *is a conjugacy class of* R. □

The Schur multiplier of Co_1 is of order 2 and the non-split central extension $2 \cdot Co_1$ is the automorphism group Co_0 of the Leech lattice preserving the origin. It can be checked, either by calculating the structure constants or by direct calculations in Co_0, that the latter is not a representation group of $\mathcal{I}(Co_1, 2A)$ and hence we have the following.

Proposition 6.6.3 Co_1 *is the universal representation group of the involution geometry* $\mathcal{I}(Co_1, 2A)$. □

7
Large sporadics

Let G be one of the following groups: F'_{24}, J_4, BM, and M, and $\mathcal{G}(G)$ be the corresponding 2-local parabolic geometry with the following respective diagrams:



As usual the first and second left nodes on the diagram correspond to points and lines, respectively. In this chapter we calculate the universal representations of these four geometries. Originally the calculations were accomplished in [Rich99] for Fi'_{24}, in [ISh97] for J_4 and in [IPS96] for BM and M. For the classification of the flag-transitive P- and T-geometries we only need to know that $\mathcal{G}(J_4)$, $\mathcal{G}(BM)$ and $\mathcal{G}(M)$ do not possess non-trivial abelian representations (cf. Proposition 10.4.3 and Section 10.5) and this becomes known as a consequence of Proposition 7.4.1, since the commutator subgroup of $\widetilde{Q}(p)$ is $\langle\widetilde{\varphi}(p)\rangle$.

7.1 Existence of the representations

The geometries $\mathcal{G} = \mathcal{G}(G)$ for $G = Fi'_{24}$, J_4, BM or M possess the following uniform description. The set $\mathcal{G}^1$ of points is the conjugacy class of central involutions in G. If p is a point, then $Q(p) := O_2(G(p))$

is an extraspecial 2-group of type 2_+^{1+2m} where $m = 6$, 6, 11 or 12, respectively, and $H := G(p)/Q(p)$ is a flag-transitive automorphism group of $\mathcal{H} := \mathrm{res}_{\mathcal{G}}(p)$ (sometimes we write H^p instead of H to indicate the point p explicitly). The latter residue is isomorphic to $\mathcal{G}(3 \cdot U_4(3))$, $\mathcal{G}(3 \cdot \mathrm{Aut}\, M_{22})$, $\mathcal{G}(Co_2)$, and $\mathcal{G}(Co_1)$, respectively. A triple $\{p_1, p_2, p_3\}$ of points is a line if and only if $p_1p_2p_3 = 1$ and $p_i \in Q(p_j)$ for all $1 \le i, j \le 3$. Since G is a simple group, it is generated by the points and hence we have the following.

Lemma 7.1.1 *If φ is the identity mapping, then (G, φ) is a representation of $\mathcal{G}$.* □

Next we show that in two of the four cases the universal representation group is larger than G.

Lemma 7.1.2 *With G as above let $\widetilde{G}$ be the extension of G by its Schur multiplier. Then $(\widetilde{G}, \widetilde{\varphi})$ is a representation of $\mathcal{G}$ for a suitable mapping $\widetilde{\varphi}$.*

Proof. The Schur multipliers of J_4 and M are trivial. The Schur multiplier of Fi'_{24} is of order 3 (an odd number), hence (6.1.1) applies. By the construction given in [Iv99] the geometry $\mathcal{G}(BM)$ is a subgeometry in $\mathcal{G}(M)$, which means that the points of $\mathcal{G}(BM)$ can be realized by some central involutions in M. These involutions generate in M a subgroup isomorphic to $2 \cdot BM$, which is the extension of BM by its Schur multiplier. □

The following theorem (which is the main result to be proved in this chapter) shows that the representation in (7.1.2) is universal.

Theorem 7.1.3 *Let $G = Fi'_{24}$, J_4, BM, or M, and $\mathcal{G} = \mathcal{G}(G)$ be the 2-local parabolic geometry of G. Then the universal representation group $R(\mathcal{G})$ of $\mathcal{G}$ is isomorphic to the extension of G by its Schur multiplier (i.e., to $3 \cdot Fi'_{24}$, J_4, $2 \cdot BM$, and M), respectively.*

In the remainder of this section we introduce some further notation. Let p be a point of $\mathcal{G}$ and $l = \{p, q, r\}$ be a line containing p. Let Υ be the collinearity graph of $\mathcal{H} = \mathrm{res}_{\mathcal{G}}(p)$ (so that l is a vertex of Υ). Let $\overline{Q}(p) = Q(p)/\langle p \rangle$ (an elementary abelian 2-group). For $q \in Q(p)$ and when $\overline{q}$ is the image of q in $\overline{Q}(p)$, let $\theta(\overline{q}) = 0$ if $q^2 = 1$ and $\theta(\overline{q}) = 1$ if $q^2 = p$. Then θ is a quadratic form on $\overline{Q}(p)$. In each of the four cases under consideration H acts irreducibly on $\overline{Q}(p)$ and θ is the only non-zero H-invariant quadratic form on $\overline{Q}(p)$ (viewed as vector space over $GF(2)$). Let β denote the bilinear form associated with θ:

$$\beta(x,y) = \theta(x) + \theta(y) + \theta(x+y).$$

Lemma 7.1.4 *Let $l = \{p,q,r\}$ and $l' = \{p,q',r'\}$ be two distinct lines containing p. Then*

(i) *$Q(p)$ induces on $l \cup l'$ an action of order 4;*
(ii) *the subgraph induced by $l \cup l'$ in the collinearity graph of $\mathcal{G}$ is either the union of two triangles sharing a vertex or the complete graph;*
(iii) *a point cannot be collinear to exactly two points on a line.*

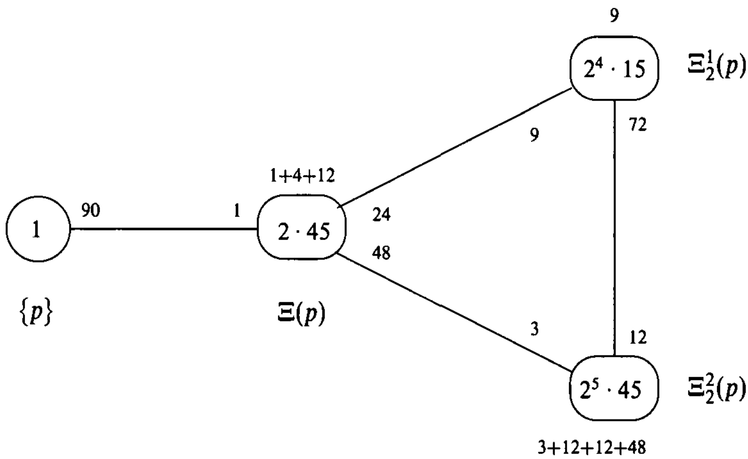
Proof. If $l'' = \{p,q'',r''\}$ is another line containing p then q'' commutes with l (where the latter is considered as a subgroup of order 2^2 in $Q(p)$) if and only if $\beta(l,l'') = 0$. Notice that if q'' does not commute with l, it exchanges the points q and r. Since β is non-singular we can find a point collinear to p which commutes with l but not with l'. In view of the obvious symmetry between l and l' we have (i). Now (ii) is immediate and implies (iii). □

Table V. Geometries of Large Sporadics

G	Fi'_{24}	J_4	BM	M
$Q(p)$	2_+^{1+12}	2_+^{1+12}	2_+^{1+22}	2_+^{1+24}
H	$3 \cdot U_4(3).2_2$	$3 \cdot \mathrm{Aut}\, M_{22}$	Co_2	Co_1
$H(l)$	$2^5.Alt_6$	$2^5.Sym_5$	$2^{10} : \mathrm{Aut}\, M_{22}$	$2^{11} : M_{24}$
$O_2(H(l))$	$2A_{15}D_6E_{10}$	$2A_{15}B_{10}C_6$	$2A_{77}B_{330}C_{616}$	$2A_{759}C_{1288}$

We summarize some of the above mentioned properties of the four geometries under consideration in Table V. The last row shows the intersections of $O_2(H(l))$ with the conjugacy classes of involutions in H (we follow the notation of [CCNPW] so that $2X_mY_n$... means that

$O_2(H(l))$ contains m elements from the class $2X$, n elements from the class $2Y$ etc.



Recall that the *sextet graph* Ξ is the collinearity graph of the rank 3 T-geometry $\mathcal{G}(M_{24})$. The vertices of Ξ are the sextets and two such sextets $\Sigma = \{S_1, ..., S_6\}$ and $\Sigma' = \{S'_1, ..., S'_6\}$ are adjacent if and only if $| S_i \cap S'_j |$ is even for every $1 \leq i, j \leq 6$. The suborbit diagram of Ξ with respect to the action of M_{24} is as above.

Lemma 7.1.5 *Let $G = Fi'_{24}$ or J_4 and let Γ be the collinearity graph of $\mathcal{G} = \mathcal{G}(G)$. Then Γ contains the sextet graph Ξ as a subgraph. The points, lines and planes of $\mathcal{G}$ contained in Ξ form a subgeometry $\mathcal{X} \cong \mathcal{G}(M_{24})$; if X is the stabilizer of Ξ in G, then $X \sim 2^{11}.M_{24}$, $O_2(X)$ is the irreducible Golay code module $\mathcal{C}_{11}$ (it is generated by the points in Ξ) and X contains $Q(p)$ for every $p \in \Xi$.*

Proof. For $G = Fi'_{24}$ the subgraph Ξ is induced by the points incident to an element x_4 of type 4 in $\mathcal{G}$ and $\mathcal{X} = \mathrm{res}_{\mathcal{G}}(x_4)$. For $G = J_4$ the subgeometry $\mathcal{X}$ is the one constructed as that in Lemma 7.1.7 in [Iv99]. □

Notice that X splits over $O_2(X)$ if $G = J_4$ and does not split if $G = Fi'_{24}$.

7.2 A reduction via simple connectedness

In the above notation let (R, φ_u) be the universal representation of $\mathcal{G}$. By (7.1.2) there is a homomorphism ψ of R onto $\widetilde{G}$ such that $\widetilde{\varphi}$ is the composition of φ_u and ψ and in order to prove (7.1.3) we have to show that ψ is an isomorphism. The group R acts on $\mathcal{G}$ inducing the group G with kernel being $\psi^{-1}(Z(\widetilde{G}))$. We are going to make use of the following fact.

Proposition 7.2.1 *The geometry $\mathcal{G}$ is simply connected.*

Proof. The simple connectedness of $\mathcal{G}(Fi_{24}')$ was established in [Iv95], of $\mathcal{G}(J_4)$ in [Iv92b] and again in [IMe99]. For the simple connectedness results for $\mathcal{G}(BM)$ and $\mathcal{G}(M)$ see Sections 5.11 and 5.15 in [Iv99] and references therein. □

By (1.4.6) and (7.2.1) if $\Phi = \{x_1, x_2, ..., x_n\}$ is a maximal flag in $\mathcal{G}$ (where n is the rank), then $\widetilde{G}$ is the universal completion of the amalgam

$$\mathscr{A}(\widetilde{G}, \mathcal{G}) = \{\widetilde{G}(x_i) \mid 1 \leq i \leq n\}.$$

Furthermore, since $\mathrm{res}_{\mathcal{G}}^{-}(x_j)$ is simply connected for $4 \leq j \leq n$ (this residue is the T-geometry $\mathcal{G}(M_{24})$ in the case $G = Fi_{24}'$, $j = 4$, and a projective $GF(2)$-geometry in the remaining cases). Hence $\widetilde{G}_j$ is the universal completion of the amalgam

$$\mathscr{E}_j = \{\widetilde{G}_j \cap \widetilde{G}_i \mid 1 \leq i \leq j-1\},$$

and we have the following refinement of (7.2.1).

Proposition 7.2.2 *Let p, l, π be a pairwise incident point, line and plane in $\mathcal{G}$. Then $\widetilde{G}$ is the universal completion of the amalgam*

$$\mathscr{B} = \{\widetilde{G}(p), \widetilde{G}(l), \widetilde{G}(\pi)\}.$$

□

Thus in order to prove (7.1.3) it would be sufficient to establish the following.

Lemma 7.2.3 *The universal representation group R of $\mathcal{G}$ contains a subamalgam $\mathscr{D} = \{R\lfloor p \rfloor, R\lfloor l \rfloor, R\lfloor \pi \rfloor\}$ which generates R and maps isomorphically onto the subamalgam $\mathscr{B}$ in $\widetilde{G}$ under the homomorphism ψ.*

We should be able to reconstruct the subgroups $R\lfloor \alpha \rfloor$ for $\alpha = p, l$ and π in terms of $\mathcal{G}$ and its representation in R. Towards this end we look at how the subgroups $\widetilde{G}(\alpha)$ can be reconstructed. It turns out that for $\alpha = p, l$ or π the subgroup $\widetilde{G}(\alpha)$ (which is the stabilizer of α in $\widetilde{G}$) is generated by the elements $\widetilde{\varphi}(q)$ it contains:

$$\widetilde{G}(\alpha) = \langle \widetilde{\varphi}(q) \mid q \in \mathcal{G}^1, \widetilde{\varphi}(q) \in \widetilde{G}(\alpha) \rangle.$$

Thus it is natural to define $R\lfloor\alpha\rfloor$ in the following way:

$$R\lfloor\alpha\rfloor = \langle\varphi_u(q) \mid q \in \mathcal{G}^1, \widetilde{\varphi}(q) \in \widetilde{G}(\alpha)\rangle.$$

Then we are sure at least that $R\lfloor\alpha\rfloor$ maps onto $\widetilde{G}(\alpha)$ under the homomorphism ψ.

By a number of reasons (of a technical nature) we prefer to deal with one type of parabolics, namely with the point stabilizers. So our goal is to prove the following.

Lemma 7.2.4 *For a point p in $\mathcal{G}$ define*

$$R\lfloor p\rfloor := \langle\varphi_u(q) \mid q \in \mathcal{G}^1, \widetilde{\varphi}(q) \in \widetilde{G}(p)\rangle.$$

Then

(i) *$R\lfloor p\rfloor$ maps isomorphically onto $\widetilde{G}(p)$ under the homomorphism $\psi : R \to \widetilde{G}$;*
(ii) *for a point r collinear to p the subgroup $R\lfloor p\rfloor \cap R\lfloor r\rfloor$ maps surjectively onto $\widetilde{G}(p) \cap \widetilde{G}(r)$.*

Since $\widetilde{G}(p)$ is the full preimage of the centralizer of p in G, we can redefine $R\lfloor p\rfloor$ as

$$R\lfloor p\rfloor = \langle\varphi_u(q) \mid q \in \mathcal{G}^1, [p,q] = 1\rangle.$$

Furthermore, it turns out that in each of the four cases under consideration if q commutes with p, then q is at distance at most 2 in the collinearity graph Γ of $\mathcal{G}$. Thus if we put

$$N(p) = \{q \mid q \in \mathcal{G}^1, [p,q] = 1, d_\Gamma(p,q) \leq 2\}$$

then $R\lfloor p\rfloor$ can be again redefined as

$$R\lfloor p\rfloor = R[N(p)].$$

We will use this definition (which involves only local properties of the collinearity graph Γ) and the fact that it is equivalent to the previous definitions will not be used.

We will establish (7.2.4 (i)) in Section 7.6 and after this is done, (7.2.4 (ii)) can be deduced from the following result (which is an internal property of $\widetilde{G}$) that will be established in Section 7.7.

Lemma 7.2.5 *If p and r are collinear points then $\widetilde{G}(p) \cap \widetilde{G}(r)$ is generated by the elements $\widetilde{\varphi}(q)$ taken for all $q \in N(p) \cap N(r)$.*

7.3 The structure of $N(p)$

In this section we describe the structure of the set $N(p)$ of vertices in the collinearity graph Γ of $\mathcal{G}$ which are at distance at most 2 from p and commute with p (considered as central involutions in G).

First we introduce some notation. Clearly $N(p)$ contains $\Gamma(p)$. Let $\Gamma_2^j(p)$, $1 \leq j \leq t = t(G)$, be the $G(p)$-orbits in $N(p) \cap \Gamma_2(p)$. Let 2^{α_j} be the length of a $Q(p)$-orbit in $\Gamma_2^j(p)$ (where $Q(p) = O_2(G(p))$) and let n_j be the number of such orbits, so that

$$|\Gamma_2^j(p)| = 2^{\alpha_j} \cdot n_j$$

(clearly the α_j and n_j depend on j and on G). We will see that for a given G the numbers α_j are pairwise different and we adopt the ordering for which $\alpha_1 < \alpha_2 < ... < \alpha_t$. Let b_1^j be the number of vertices in $\Gamma_2^j(p)$ adjacent in Γ to a given vertex from $\Gamma(p)$ and c_2^j be the number of vertices in $\Gamma(p)$ adjacent to a given vertex from $\Gamma_2^j(p)$. Then

$$|\Gamma_2^j(p)| = |\Gamma(p)| \cdot \frac{b_1^j}{c_2^j}.$$

Throughout this section (p, q, r) is a 2-path in Γ such that the lines $l = \{p, q, q'\}$ and $l' = \{q, r, r'\}$ are different. Then l and l' are different points of $\mathcal{H}^q = \mathrm{res}_{\mathcal{G}}(q)$. Let Υ be the collinearity graph of $\mathcal{H}^q$. The suborbit diagram of Υ with respect to the action of $H^q = G(q)/Q(q)$ can be found in Section 5.1 when $\mathcal{H}^q$ is $\mathcal{G}(Co_1)$ or $\mathcal{G}(Co_2)$, in Section 4.4 when $\mathcal{H}^q$ is $\mathcal{G}(3 \cdot M_{22})$ and in Section 5.6 when $\mathcal{H}^q$ is $\mathcal{G}(3 \cdot U_4(3))$.

In the cases $G = Fi_{24}'$ and $G = J_4$ the group H^q (isomorphic to $3 \cdot U_4(3).2_2$ and $3 \cdot \mathrm{Aut}\, M_{22}$, respectively) contains a normal subgroup D of order 3 which acts fixed-point freely on the point-set of $\mathcal{H}^q$. Let Υ^* denote the collinearity graph of the enriched point-line incidence system (whose lines are those of $\mathcal{H}^q$ together with the orbits of D on the point-set). In order to argue uniformly, for $G = BM$ and M we put $\Upsilon^* = \Upsilon$. Let $\mathcal{S}^*$ denote the point-line incidence system for which Υ^* is the collinearity graph.

Lemma 7.3.1 *Let A be the orbit of r under $Q(p)$ and B the orbit of l' under $O_2(H^q(l))$. Then*

(i) *$Q(p) \cap Q(q)$ is a maximal elementary abelian subgroup (of order 2^{m+1}) in $Q(q) \cong 2_+^{1+2m}$ and $Q(p) \cap G(q)$ maps surjectively onto $O_2(H^q(l))$;*
(ii) *$|A| = |B| = 2$ if $d_\Gamma(p, r) = 1$ and $|A| = 4 \cdot |B|$ if $d_\Gamma(p, r) = 2$;*
(iii) *$r \in N(p)$ if and only if $\beta(l, l') = 0$.*

Proof. Since the commutator subgroups of $Q(p)$ and $Q(q)$ are of order 2 generated by p and q, respectively, $Q(p) \cap Q(q)$ is elementary abelian and its image in $\overline{Q}(q)$ is totally singular with respect to θ. Hence the image is at most m-dimensional and $|Q(p) \cap Q(q)| \leq 2^{m+1}$. On the other hand, $Q(p) \cap G(q)$ has index 2 in $Q(p)$ and its image in H^q is contained in $O_2(H^q(l))$. We can see from Table V in Section 7.1 that $|O_2(H^q(l))| = 2^{m-1}$, which implies (i).

If r is adjacent to p then the $Q(p)$-orbit of r is of length 2 and clearly $|A| = |B| = 2$. Suppose that $d_\Gamma(p,r) = 2$. We claim that r and r' are in the same $Q(p)$-orbit. Indeed, otherwise l' (which is a subgroup of order 2^2 in $Q(q)$) commutes with $Q(p) \cap Q(q)$. But by (i), $Q(p) \cap Q(q)$ is a maximal abelian subgroup in $Q(q)$. Hence l' must be contained in $Q(p) \cap Q(q)$, but in this case $r \in l' \subseteq Q(p)$ and r is collinear to p by the definition of $\mathcal{G}$, contrary to our assumption. The image of r under an element from $Q(p) \setminus G(q)$ is not collinear to q. Hence the orbit of r under $Q(p)$ is twice longer than its orbit under $Q(p) \cap G(q)$ and (ii) follows. Finally (iii) is immediate from the definition of θ and β. □

Lemma 7.3.2 *The following three conditions are equivalent:*

(i) *p and r are adjacent in the collinearity graph Γ of $\mathcal{G}$;*
(ii) *$r \in Q(q) \cap Q(p)$;*
(iii) *l and l' are adjacent in $\Upsilon^\bullet$;*

Proof. First of all (i) and (ii) are equivalent by the definition of the collinearity in $\mathcal{G}$. By (7.3.1), p and r can be adjacent in Γ only if the orbit of l' under $O_2(H^q(l))$ has length at most 2. The orbit lengths of $O_2(H^q(l))$ can be read from the suborbit diagram of $\Upsilon^\bullet$. From these diagrams we see that p and r can be adjacent only if l and l' are adjacent in $\Upsilon^\bullet$. Hence (i) implies (iii). If l and l' are collinear in $\mathcal{H}^q$ then the union $l \cup l'$ is contained in a plane, in particular, it induces a complete subgraph in Γ. Suppose that l and l' are adjacent in $\Upsilon^\bullet$ but not in Υ. In this case $G = Fi_{24}'$ or $G = J_4$ and by (7.1.5), Γ contains the sextet graph Ξ as a subgraph. The suborbit diagram of Ξ shows that in the considered situation p and r are adjacent. This shows that (iii) implies (i) and completes the proof. □

As we have seen in the proof of (7.3.1), the image of $Q(p) \cap Q(q)$ in $\overline{Q}(q)$ is m-dimensional. We can alternatively deduce this fact from (7.3.2). Indeed, $\overline{Q}(q)$ supports the representation $(\overline{Q}(q), \varphi)$ of $\mathcal{S}^\bullet$ (compare (1.5.1)). In view of (5.6.2), (4.4.2), (5.3.2), and (5.2.3) this representation is universal when $G = Fi'_{24}$, J_4, or M and has codimension 1 in the universal when $G = BM$. Now by (5.6.3), (4.4.8 (i)), (5.2.3 (ii)) and (5.3.3) (for $G = Fi'_{24}$, J_4, BM, and M, respectively) we observe that the elements $\varphi(l')$ taken for all l' equal or adjacent to l in Υ^* generate in $\overline{Q}(p)$ subspaces of dimension m at least. Since for such an l' the subgroup $\varphi(l')$ is contained in the image of $Q(p) \cap Q(q)$ in $\overline{Q}(p)$, the dimension of the image is exactly m.

As a byproduct of this consideration we obtain the following useful consequence.

Corollary 7.3.3 *If p and q are adjacent vertices in Γ then $\widetilde{Q}(p) \cap \widetilde{Q}(q)$ is a maximal abelian subgroup of index 2^{m-1} in $\widetilde{Q}(p)$ (where $Q(p) \cong 2^{1+2m}_+$) and it is generated by the elements $\widetilde{\varphi}(r)$ taken for all*

$$r \in \{p,q\} \cup (\Gamma(p) \cap \Gamma(q)).$$

□

We will use the following straightforward principle.

Lemma 7.3.4 *Suppose that $r \in N(p) \cap \Gamma_2(p)$ and let $\Gamma_2^j(p)$ be the $G(p)$-orbit containing r. Let $\widehat{r}$ denote the image of r in $H^p = G(p)/Q(p)$. Then*

(i) $\widehat{r} \in O_2(H^p(l))$;
(i) *$\{\widehat{r} \mid r \in \Gamma_2^j(p)\}$ is a conjugacy class of involutions in H^p;*
(ii) *if r and s are in the same $Q(p)$-orbit then $\widehat{r} = \widehat{s}$;*
(iii) *the number n_j of $Q(p)$-orbits in $\Gamma_2^j(p)$ divides the size k_j of the conjugacy class of $\widehat{r}$ in H.*

Proof. (i) follows from (7.3.1 (i)), the rest is easy. □

Comparing (7.3.2) with the suborbit diagram of Υ^*, in view of (7.3.4) and Table V we obtain the following lemma (recall that $t = t(G)$ is the number of $G(p)$-orbits in $N(p) \cap \Gamma_2(p)$).

Lemma 7.3.5 *In terms of* (7.3.1) *and* (7.3.4) *we have the following:*

(i) *if $G \cong Fi'_{24}$ then $t = 4$; if $r \in \Gamma_2^j(p)$ then $l' \in \Upsilon_3^3(l)$, $\Upsilon_2^2(l)$, $\Upsilon_2^1(l)$ or $\Upsilon_3^2(l)$; the $Q(p)$-orbit of r has lengths 2^4, 2^5, 2^6 or 2^7; $\widehat{r}$ is in the H^p-conjugacy classes $2A$, $2A$, $2D$ or $2E$ for $j = 1,2,3$ or 4;*
(ii) *if $G = J_4$ then $t = 3$; if $r \in \Gamma_2^j(p)$ then $l' \in \Upsilon_3^1(l)$, $\Upsilon_2^1(l)$ or $\Upsilon_2^2(l)$; the $Q(p)$-orbit of r has lengths 2^4, 2^5 or 2^6; $\widehat{r}$ is in the H^p-conjugacy classes $2A$, $2A$ or $2B$ for $j = 1,2$ or 3;*

(iii) *if* $G = BM$ *then* $t = 2$*; if* $r \in \Gamma_2^j(p)$ *then* $l' \in \Upsilon_2^1(l)$ *or* $\Upsilon_2^2(l)$*; the* $Q(p)$*-orbit of* r *has lengths* 2^7 *or* 2^8*;* $\hat{r}$ *is in the* H^p*-conjugacy classes* $2A$ *or* $2B$ *for* $j = 1$ *or* 2*;*

(iv) *if* $G = M$ *then* $t = 3$*; if* $r \in \Gamma_2^j(p)$ *then* $l' \in \Upsilon_2^1(l)$, $\Upsilon_2^2(l)$ *or* $\Upsilon_3^2(l)$*; the* $Q(p)$*-orbit of* r *has lengths* 2^8, 2^9 *or* 2^{13}*;* $\hat{r}$ *is in the* H^p*-conjugacy classes* $2A$, $2A$ *or* $2C$ *for* $j = 1, 2$ *or* 3. □

By the above lemma for each G under consideration and every $1 \le j \le t$ we know that b_1^j is twice the length of the orbit of l' under $H^q(l)$ (assuming that $r \in \Gamma_2^j(p)$), the length 2^{α_j} of a $Q(p)$-orbit in $\Gamma_2^j(p)$ is also known and the number n_j of these orbits is divisible by the size k_j of the H^p-conjugacy class of $\hat{r}$ (which can be read from [CCNPW]). Thus in order to find the length of $\Gamma_2^j(p)$ we only have to calculate c_2^j. The above consideration gives the following upper bound on c_2^j.

Lemma 7.3.6 c_2^j *divides*

$$|\Gamma(p)| \cdot \frac{b_1^j}{2^{\alpha_j} \cdot k_j}.$$

□

A lower bound comes from the following rather general principle, which can be easily deduced from (7.1.4).

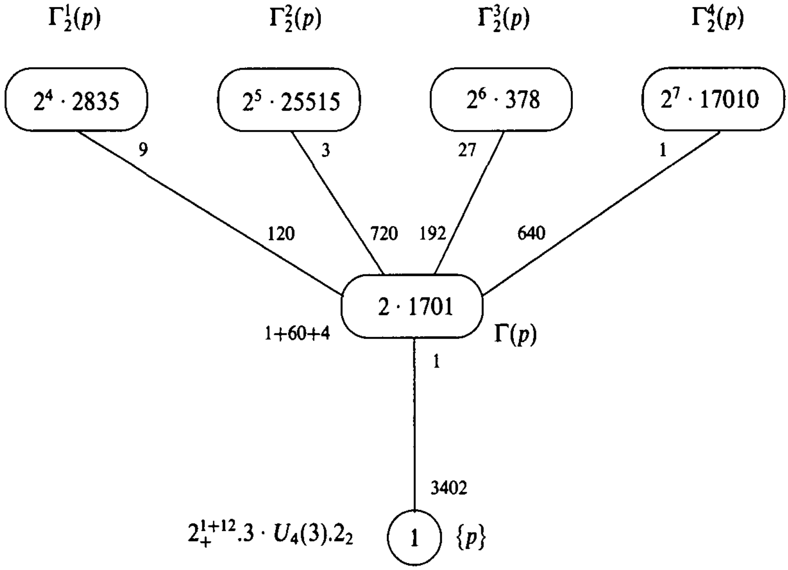
Lemma 7.3.7 *Suppose that* $r \in \Gamma_2^j(p)$*. Let* e *be the number of 2-paths in* Υ^* *joining* l *and* l'*, i.e.,*

$$e = |\Upsilon^*(l) \cap \Upsilon^*(l')|.$$

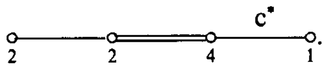
Then the subgraph in Γ *induced by* $\Gamma(p) \cap \Gamma(r)$ *has valency* $2 \cdot e$*, in particular,* $c_2^j \ge 1 + 2 \cdot e$. □

The next four lemmas deal with the individual cases. The diagrams given in these lemmas present fragments of the suborbit diagrams of Γ. These fragments show the orbits of $G(p)$ on $N(p)$ and the number of vertices in $\Gamma(p)$ adjacent to a vertex from such an orbit.

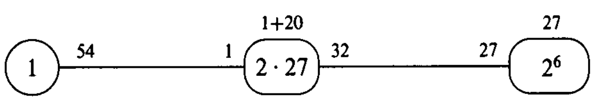
Lemma 7.3.8 *The structure of* $N(p)$ *when* $G = Fi_{24}'$ *is the same as that on the following diagram.*



Proof. The collinearity graph of $\mathcal{G} = \mathcal{G}(Fi'_{24})$ is also the collinearity graph of the extended dual polar space $\mathcal{E}(Fi'_{24})$ (cf. Lemma 5.6.6 in [Iv99]). The diagram of $\mathcal{E}(Fi'_{24})$ is



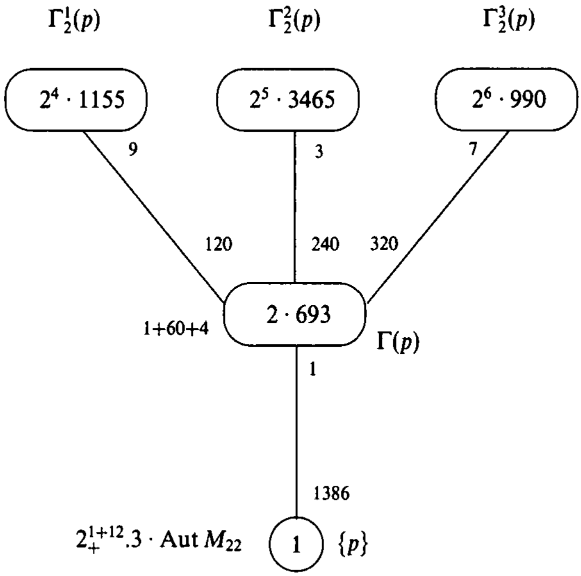
Let Θ be the subgraph in Γ induced by the vertices (points) incident to an element y of type 4 in $\mathcal{E}(Fi'_{24})$ (we assume that y is incident to p). Then Θ is the collinearity graph of the building $\mathcal{G}(\Omega_8^-(2))$ with the suborbit diagram



with respect to the action of $G(y) \cong 2^8 : \Omega_8^-(2).2$ and $G(y)$ contains $Q(p)$. Since $O_2(G(y))$ acts transitively on $\Theta_2(p)$ of size 2^6, we conclude that $\Theta_2(p) \subseteq \Gamma_2^3(p)$ and hence c_2^3 is at least 27. Since $k_3 = 378$, we obtain $c_2^3 = 27$.

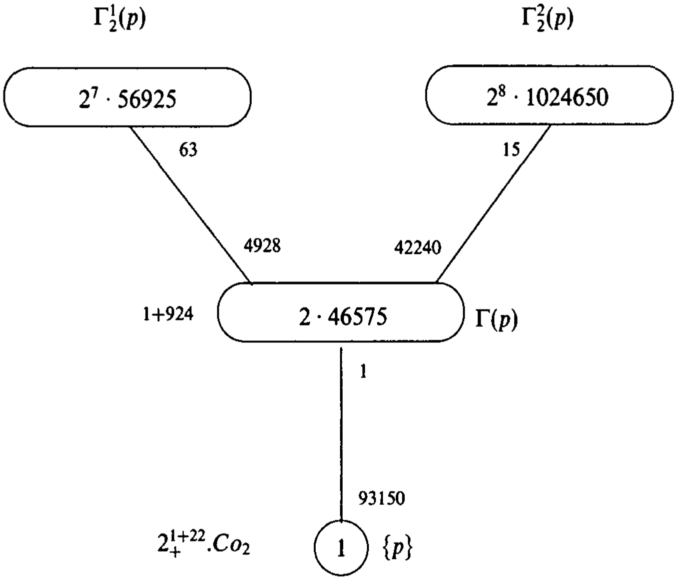
Now let Ξ be the subgraph as in (7.1.5) containing p and $X \cong 2^{11}.M_{24}$ be the stabilizer of Ξ in G. Since X contains $Q(p)$ and $O_2(X(p))$ acts on $\Xi_2^1(p)$ and $\Xi_2^2(p)$ with orbits of lengths 2^4 and 2^5, we conclude that $\Xi_2^1(p) \subseteq \Gamma_2^1(p)$ and $\Xi_2^1(p) \subseteq \Gamma_2^2(p)$, in particular when $c_2^1 \geq 9$ and $c_2^2 \geq 3$. Since $k_1 = k_2 = 2835$ we immediately conclude that $c_2^1 = 9$. A more detailed analysis shows that $c_2^2 = 3$. But since the particular value of c_2^2 will not be used in our subsequent arguments, we are not presenting this analysis here. Finally, since $k_4 = 17010$, direct calculation shows that $c_2^4 = 1$. □

Lemma 7.3.9 *The structure of $N(p)$ when $G = J_4$ is the same as that on the following diagram.*



Proof. By Propositions 1, 6, 9, and 15 in [J76] we see that $G(p) \setminus Q(p)$ contains involutions t', t_1, $\tilde{t}_1$ conjugate to p in G with centralizers in $Q(p)$ of orders 2^9, 2^8, 2^7, respectively. This shows that $t' \in \Gamma_2^1(p)$, $t_1 \in \Gamma_2^2(p)$, $\tilde{t}_1 \in \Gamma_2^3(p)$. Also by [J76] we know that $|C_{G(p)}(\tau)|$ is $2^{17} \cdot 3^2$, $2^{16} \cdot 3$, $2^{14} \cdot 3 \cdot 7$ for $\tau = t'$, t_1, $\tilde{t}_1$, respectively, and hence c_2^j are the same as those on the diagram. If Ξ is a subgraph from (7.1.5) containing p, then $\Xi_2^1(p) \subseteq \Gamma_2^1(p)$ and $\Xi_2^2(p) \subseteq \Gamma_2^2(p)$. Notice that $G(p)$ acts on the set of $Q(p)$-orbits in $\Gamma_2^3(p)$ as it does on the set of planes in $\mathrm{res}_{\mathcal{G}}(p) \cong \mathcal{G}(3 \cdot M_{22})$. □

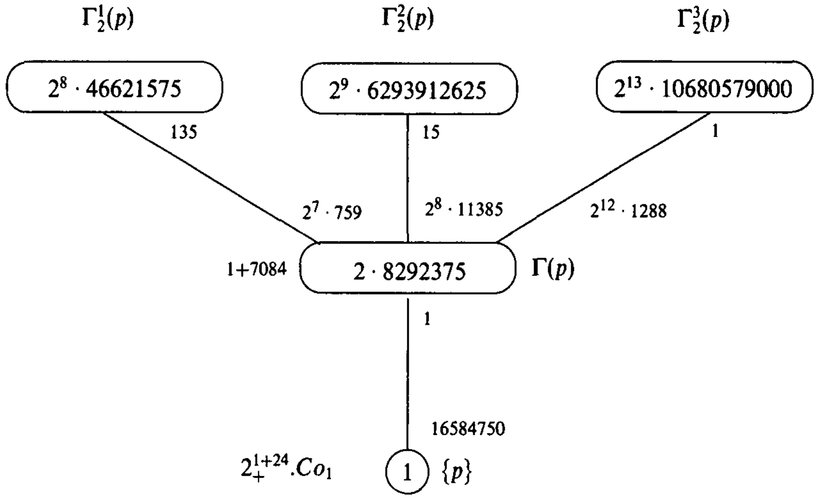
Lemma 7.3.10 *The structure of $N(p)$ in the case $G = BM$ is as on the following diagram.*



Proof. We have $k_1 = 56925$, $k_2 = 1024650$, so that c_2^1 divides 63 and c_2^2 divides 15. Let Σ be the subgraph induced by the vertices in a subgeometry $\mathcal{G}(S_8(2))$ in $\mathcal{G}$ as that in Lemma 5.4.5 in [Iv99]. Then it is easy to see that (assuming that $p \in \Sigma$) $\Sigma_2(p) \subseteq \Gamma_2^1(p)$ and $c_2^1 = 63$. By (7.3.7) we see that c_2^2 is at least 15. In view of the above it is exactly 15. □

Lemma 7.3.11 *The structure of $N(p)$ when $G = M$ is the same as that on the following diagram.*

Proof. Since $k_1 = k_2 = 46621575$ and $k_3 = 10680579000$ we conclude that $c_2^3 = 1$ and that c_2^1 divides 135. Let Ψ be the subgraph of valency 270 on 527 vertices introduced before Lemma 5.3.3 in [Iv99] and suppose that $p \in \Psi$. Then the stabilizer of Ψ in G contains $Q(p)$ and $|\Psi_2(p)| = 2^8$. Hence $\Psi_2(p) \subseteq \Gamma_2^1(p)$ and since Ψ contains 135 paths of length 2 joining a pair of vertices at distance 2, we have $c_2^1 = 135$. Proving the fact that $c_2^2 = 15$ is a bit more delicate, a proof of this equality can be found in [MSh01]. In the present work the particular value of c_2^2 does not play any role and we indicate it on the diagram only for the sake of completeness. □



7.4 Identifying $R_1(p)$

In this section we take a first step in establishing (7.2.4) by proving the following

Proposition 7.4.1 *The homomorphism $\psi : R \to \widetilde{G}$ restricted to*

$$R_1(p) = \langle \varphi_u(q) \mid d_\Gamma(p,q) \leq 1 \rangle$$

is an isomorphism onto $\widetilde{Q}(p) = O_2(\widetilde{G}(p))$.

Since it is clear that ψ maps $R_1(p)$ surjectively onto $\widetilde{Q}(p)$, in order to prove (7.4.1) it is sufficient to show that the order of $R_1(p)$ is at most that of $\widetilde{Q}(p)$ (which is 2^{13}, 2^{13}, 2^{24}, or 2^{25} for $G = Fi'_{24}$, J_4, BM, or M, respectively).

By (2.6.2) the mapping

$$\chi : l = \{p,q,r\} \mapsto \langle z_p, z_q, z_r \rangle / \langle z_p \rangle$$

turns $\overline{R}_1(p)$ into a representation group of $\mathscr{H} = \mathrm{res}_{\mathscr{G}}(p)$. If $G = BM$ or M then by (5.4.1) this immediately implies that $\overline{R}_1(p)$ is abelian of order at most 2^{23} or 2^{24}, respectively, and we have the following.

Lemma 7.4.2 *If $G = BM$ or M, then* (7.4.1) *holds.* □

For the remainder of this section we deal with the situation when $G = Fi'_{24}$ or J_4.

Lemma 7.4.3 *If $G = Fi'_{24}$ or J_4 then*

(i) $(\overline{R}_1(p), \chi)$ *is a representation of the enriched point-line incidence system $\mathcal{S}^\bullet$ of $\mathcal{H}$;*

(ii) $\overline{R}_1(p)$ *is a quotient of* $R(\mathcal{S}^\bullet) \cong 2_+^{1+12}$.

Proof. Let D be a Sylow 3-subgroup (of order 3) in $O_{2,3}(G(p))$ and let $\{l_1, l_2, l_3\}$ be a D-orbit on the set of lines in $\mathcal{G}$ containing p. Then the set $S = l_1 \cup l_2 \cup l_3$ is contained in a subgraph Ξ as in (7.1.5) stabilized by $X \sim 2^{11}.M_{24}$. Since Ξ generates $O_2(X)$ which is an irreducible Golay code module for $X/O_2(X) \cong M_{24}$ we can easily see that S is the set on non-identity elements of an elementary abelian subgroup of order 2^3 contained in $Q(s)$ for every $s \in S$. This shows (i). Now (ii) is by (4.4.6) and (5.6.5). □

By (7.4.3) we see that for $G = Fi'_{24}$ or J_4 the size of $R_1(p)$ is at most twice that of $\widetilde{Q}(p)$ (isomorphic to $Q(p)$ in the considered cases). The next lemma shows that this bound cannot be improved locally. Let $\mathcal{T} = (\Pi, L)$ be the point-line incidence system where $\Pi = \{p\} \cup \Gamma(p)$ and $L = L(\Pi)$ is the set of lines of $\mathcal{G}$ contained in Π.

Lemma 7.4.4 *If $G = Fi'_{24}$ or J_4 then* $R(\mathcal{T}) \cong Q(p) \times 2 \cong 2_+^{1+12} \times 2$.

Proof. Let $(Q(p), \varphi)$ be the representation of $\mathcal{T}$ where φ is the identity mapping. Let χ be the mapping of the point-set of $\mathcal{H} = \mathrm{res}_{\mathcal{G}}(p)$ into $Q(p)$ which turns the latter into a representation group of $\mathcal{H}$. Then χ can be constructed as follows.

Let D be a Sylow 3-subgroup of $O_{2,3}(G(p))$ and $C = C_{G(p)}(D)/\langle p \rangle$ (isomorphic to $3 \cdot U_4(3)$ or $3 \cdot M_{22}$). Then (compare the proof of (4.4.1)) C acts flag-transitively on $\mathcal{H}$ and has two orbits, say Φ_1 and Φ_2, on $\Gamma(p)$. Let χ_i be the mapping which sends a line l of $\mathcal{G}$ containing p onto its intersection with Φ_i. Then for exactly one $i \in \{1, 2\}$ the mapping χ_i is the required mapping χ. We claim that $\Phi := Im(\chi)$ is a geometric hyperplane in $\mathcal{T}$. It is clear from the above that every line containing p intersects Φ in exactly one point. Let $l \in L$ be a line disjoint from p. Let l_i, $1 \le i \le 3$, be the lines containing p and intersecting l and let $l_i = \{p, r_i, s_i\}$ where $r_i \in \Phi$ for $1 \le i \le 3$. Then l is one of the following four lines:

$$\{r_1, r_2, r_3\}, \ \{r_1, s_2, s_3\}, \ \{s_1, r_2, s_3\}, \ \{s_1, s_2, r_3\}.$$

Hence Φ is indeed a geometric hyperplane. Since $Q(p)$ is extraspecial, it is easy to see that it is generated by Φ. Now by (2.3.5), $\mathcal{T}$ possesses a representation in the direct product of $Q(p)$ and a group of order 2. On the other hand, arguing as in the proof of (7.4.3) we can see that the order of $R(\mathscr{T})$ is at most 2^{14} and the proof follows. □

Thus when $G = Fi'_{24}$ or J_4 we have the following two possibilities:

(P1) The restriction of ψ to $R_1(p)$ is an isomorphism onto $\widetilde{Q}(p)$.
(P2) The restriction of ψ to $R_1(p)$ is a homomorphism with kernel $Y(p)$ of order 2.

Suppose that (P2) holds and let Z be the normal closure in R of the subgroups $Y(p)$ taken for all points p. Then R/Z possesses a representation of $\mathscr{G}$ for which (P1) holds. Furthermore, R/Z is the universal representation group with this property in the sense that it possesses a homomorphism onto every representation group for which (P1) holds (for every point p). Below in this section we show that if (P2) holds then the kernel $Y(p)$ is independent of the particular choice of the point p. Hence Z is of order 2. In the subsequent sections of this chapter we show that the universal group R/Z for which (P1) holds is $\widetilde{G}$ (which is $3 \cdot Fi'_{24}$ or J_4). Since the Schur multiplier of $\widetilde{G}$ is trivial we must have

$$R \cong \widetilde{G} \times 2,$$

which is not possible by (2.1.1).

Thus in the remainder of this section we assume that (P2) holds and show that $Y(p)$ is independent on p and in the subsequent sections we show that the universal group satisfying (P1) is $\widetilde{G}$. In order to have uniform notation we denote this group by R instead of R/Z.

By (7.4.3) and (7.4.4) we have

$$R_1(p) \cong R(\mathscr{T}) \cong 2_+^{1+12} \times 2$$

and

$$\overline{R}_1(p) \cong R(\mathscr{S}^*) \cong 2_+^{1+12}.$$

This shows that the commutator subgroup of $R_1(p)$ is of order 2 and if c_p denotes the unique non-trivial element of this commutator subgroup then $c_p \neq z_p$ (where $z_p = \varphi_u(p)$) and $\langle c_p, z_p \rangle$ is the centre of $R_1(p)$. Under the homomorphism ψ both c_p and z_p map onto $\widetilde{\varphi}(p)$, which gives the following.

Lemma 7.4.5 *Let p and q be distinct collinear points of $\mathscr{G}$. Then the only possible equality among the elements z_p, z_q, c_p, c_q, z_pc_p and z_qc_q is the equality*

$$z_pc_p = z_qc_q.$$

□

We are going to show that the equality in the above lemma in fact holds for every pair of points. Since it is clear that $z_p c_p$ generates the kernel $Y(p)$ of the restriction of ψ to $R_1(p)$, by this we will accomplish the goal of this section.

Let $l = \{p, q, r\}$ be a line and let

$$\Gamma(l) = \{s \in \Gamma \mid d_\Gamma(s,t) \leq 1 \text{ for every } t \in l\}.$$

For $s \in \Gamma(p)$ let m be the line containing p and s. By (7.1.4) and (7.3.2) we know that $s \in \Gamma(l)$ if and only if l and m are either equal or adjacent in the collinearity graph Υ^* of the enriched point-line incidence system of $\mathcal{H} = \mathrm{res}_{\mathcal{G}}(p)$. Let $C(l)$ be the set of points s as above such that m is either equal or adjacent to l in Υ (i.e., m and l are equal or collinear in $\mathcal{H}$) and let $A(l)$ be the set of points s such that m is either equal or adjacent to l in Υ^* but not in Υ.

Lemma 7.4.6 *The following assertions hold:*

(i) *the pointwise stabilizer of l in G acts transitively both on $C(l)$ and on $A(l)$;*
(ii) *$\Gamma(l)$ is the disjoint union of l, $C(l) \setminus l$ and $A(l) \setminus l$ and this partition is independent on the particular choice of $p \in l$;*
(iii) *$R[\Gamma(l)]$ is elementary abelian of order at most 2^8;*
(iv) *$R[A(l)]$ has order 2^3 and $R[A(l)]^{\#} = \{z_s \mid s \in A(l)\}$;*
(v) *$R[C(l)]$ has order 2^7.*

Proof. (i) is easy to deduce from the suborbit diagram of Υ in view of (7.1.4 (i)). (ii) follows from (i) and (7.3.2). Since $\Gamma(l) = \Gamma(p) \cap \Gamma(q)$ (compare (7.1.4 (ii))), the commutator subgroups of $R_1(p)$ and $R_1(q)$ are generated by c_p and c_q, respectively, and $c_p \neq c_q$ by (7.4.5), $R[\Gamma(l)]$ is elementary abelian. Since $R_1(p)$ contains the extraspecial group 2_+^{1+12} with index 2, an abelian subgroup in $R_1(p)$ has order at most 2^8 and we obtain (iii). As we have seen in the proof of (7.4.3), $A(l)$ is the set of non-identity elements contained in $Q(s)$ for every $s \in A(l)$, which immediately gives (iv). Since $\overline{R}_1(p) \cong R(\mathcal{S}^*)$, (v) follows from (4.4.8 (i)) and (5.6.3). □

Lemma 7.4.7 *The following assertions hold:*

(i) *$R[C(l)]$ does not contain $R[A(l)]$;*
(ii) *$R[\Gamma(l)]$ is of order 2^8;*
(iii) *$c_p \in R[\Gamma(l)]$.*

Proof. Let $\Sigma = \{p, l\}$, $\mathscr{F} = \mathrm{res}_{\mathscr{G}}(\Sigma)$ and $\overline{M}$ be the action induced on $\mathscr{F}$ by $M := G(p) \cap G(l)$. Then $\mathscr{F} \cong \mathscr{G}(S_4(2))$ and $M \cong Alt_6$ if $G = Fi'_{24}$ and $\mathscr{F} \cong \mathscr{G}(Alt_5)$ and $M \cong Sym_5$ if $G \cong J_4$. Clearly M normalizes both $R[C(l)]$ and $R[A(l)]$. By (4.4.8 (i)) and (5.6.3) $\mathscr{Q}_5(l) := R[C(l)]/R[l]$ is a 5-dimensional representation module for $\mathscr{F}$ and as a module for $\overline{M}$ it contains a unique 1-dimensional submodule, which we denote by $\mathscr{Q}_1(l)$. By (7.4.6 (iv)) $R[A(l)]/R[l]$ is 1-dimensional. Suppose that $R[A(l)] \leq R[C(l)]$. Then $R[A(l)]/R[l] = \mathscr{Q}_1(l)$ and

$$\mathscr{Q}_4(l) := \mathscr{Q}_5(l)/\mathscr{Q}_1(l) = R[C(l)]/R[A(l)]$$

is the 4-dimensional irreducible representation module of $\mathscr{F}$ and $\overline{M}$ acts transitively on the set of non-identity elements of $\mathscr{Q}_4(l)$. Let

$$\lambda = \{l_1 = l, l_2, l_3\}$$

be the line of the enriched system of $\mathscr{H} = \mathrm{res}_{\mathscr{G}}(p)$ which is not a line of $\mathscr{H}$. This means that λ is an orbit of $D := O_3(G(p)/Q(p))$. Let

$$\mathscr{Q} = R[C(l_1) \cup C(l_2) \cup C(l_3)]/R[A(l)].$$

Then $\mathscr{Q}$ is generated by the elements of $\mathscr{Q}_4(l)$ and their images under D. Moreover, if $\pi \in \mathscr{Q}_4(l)^{\#}$ then $T := \langle \pi^d \mid d \in D \rangle$ is 2-dimensional. So the generators of $\mathscr{Q}$ are indexed by the pairs (a, x) for $a \in \mathscr{Q}_4(l)^{\#}$, $x \in T$ and the relations as in (2.4.2) hold. By the latter lemma in view of the irreducibility of $\overline{M}$ on $\mathscr{Q}_4(l)$ and of D on T we conclude that $\mathscr{Q}$ is elementary abelian of order 2^8 isomorphic to $\mathscr{Q}_4(l) \otimes T$. By (7.4.6 (iv)), $R[A(l)]$ does not contain c_p, which means that the full preimage of $\mathscr{Q}$ in $R_1(p)$ is abelian of order 2^{11}, which is impossible, since $R_1(p) \cong 2^{1+12}_+ \times 2$. This contradiction proves (i). Now (ii) follows from (i) in view of (7.4.6 (iii) to (v)). Since $R[\Gamma(l)]$ is a maximal abelian subgroup of $R_1(p)$, it contains the centre of $R_1(p)$, in particular it contains c_p and we have (iii). □

Lemma 7.4.8 *The subgroup*

$$R[l]^* = \langle z_s, c_s \mid s \in A(l) \rangle$$

is elementary abelian of order 2^4.

Proof. By (7.4.7) and its proof $\mathscr{Q}_1(l)$ is the unique 1-dimensional M-submodule in

$$R[\Gamma(l)]/R[A(l)] \cong R[C(l)]/R[l] \cong \mathscr{Q}_5(l).$$

Since $\langle R[A(l)], c_p\rangle / R[A(l)]$ is such a submodule, in view of the obvious symmetry we conclude that $R[l]^*$ is the full preimage of $\mathcal{Q}_1(l)$ in $R[\Gamma(l)]$ and the proof follows. □

Now we are ready to establish the final result of the section.

Proposition 7.4.9 *The subgroup $Y(p) = \langle z_p c_p\rangle$ is independent on the particular choice of p.*

Proof. By (7.4.8), $R[l]^*$ is elementary abelian of order 2^4. It contains seven elements z_s and seven elements c_s for $s \in A(l)$ which are all pairwise different by (7.4.5). Thus all the seven products $z_s c_s$ must be equal to the only remaining non-identity element in $R[l]^*$. Now the proof follows from the connectivity of Γ. □

7.5 $R_1(p)$ is normal in $R\lfloor p\rfloor$

In this section we assume (7.4.1) and prove the following.

Proposition 7.5.1 *$R_1(p)$ is a normal subgroup in $R\lfloor p\rfloor = R[N(p)]$.*

First of all by (7.3.3) we have the following

Lemma 7.5.2 *If q is a point collinear to p then $R_1(p) \cap R_1(q)$ is a maximal abelian subgroup of index 2^{m-1} in $R_1(p)$ (where $Q(p) \cong 2_+^{1+2m}$).* □

By (7.4.1) the group $\overline{R}_1(p)$ is abelian and hence by (2.2.3) we have the following

Lemma 7.5.3 *Let (p, q, r) be a 2-path in Γ. Then the commutator $[z_p, z_r]$ is either z_q or the identity.* □

Let $r \in N(p) \cap \Gamma_2(p)$. In order to show that z_r normalizes $R_1(p)$ it is sufficient to indicate a generating set of elements in $R_1(p)$, whose z_r-conjugates are also in $R_1(p)$. Using (7.5.3) we produce a family of such elements and then check under an appropriate choice of r that this is a generating family. Let

$$T_0(r) = \{p\}, \quad T_1(r) = \Gamma(p) \cap \Gamma(r), \quad T_2(r) = \bigcup_{q \in T_1(r)} \Gamma(p) \cap \Gamma(q),$$
$$T(r) = T_0(r) \cup T_1(r) \cup T_2(r).$$

Lemma 7.5.4 *If $s \in T(r)$ then $[z_r, z_s] \in R_1(p)$.*

Proof. If $s \in T_0(r) \cup T_1(r)$ then $[z_r, z_s] = 1$. Suppose that $s \in T_2(r)$ and q is a vertex in $T_1(r)$ adjacent to s. Then by (7.5.3) $[z_r, z_s] \in \langle z_q \rangle \leq R_1(p)$. □

Let $I_1(r)$ and $I(r)$ be the subgroups in $R_1(p)$ generated by the z_s for all s taken from $T_0(r) \cup T_1(r)$ and from $T(r)$, respectively. Clearly

$$\langle z_p \rangle \leq I_1(r) \leq I(r)$$

and we can put $\overline{I}_1(r)$ and $\overline{I}(r)$ to be the quotients over $\langle z_p \rangle$ of $I_1(r)$ and $I(r)$, respectively. These quotients are clearly subspaces in $\overline{R}_1(p)$ (when the latter is treated as a $GF(2)$-vector space).

Since the representation (R, φ_u) is universal, $\overline{R}_1(p)$ is a module for $H = G(p)/Q(p)$, which is isomorphic to $\widetilde{Q}(p)/\langle \widetilde{\varphi}(p) \rangle$ by (7.4.1). Put

$$H(r) = (G(p) \cap G(r))Q(p)/Q(p).$$

Directly by the definition we have the following

Lemma 7.5.5 *Both $\overline{I}_1(r)$ and $\overline{I}(r)$ are $H(r)$-submodules in $\overline{R}_1(p)$.* □

Let $(\overline{R}_1(p), \chi)$ be the representation of the (extended) point-line incidence system of $\mathscr{H} = \mathrm{res}_{\mathscr{G}}(p)$ as defined in the paragraph preceding (7.4.1). Let $J_1(r)$ and $J(r)$ be the sets of lines in $\mathscr{G}$ containing p and a point from $T_1(r)$ and $T(r)$, respectively. Since a point in $\mathscr{G}$ cannot be collinear to exactly two points on a line, we observe that

$$|J_1(r)| = |T_1(r)| \text{ and } |J(r)| = |T(r)|.$$

In these terms $\overline{I}_1(r)$ and $\overline{I}(r)$ are generated by the images under χ of the lines from $J_1(r)$ and $J(r)$, respectively.

Up to conjugation in H the submodule $\overline{I}(r)$ depends on the number j such that $r \in \Gamma_2^j(p)$. Since

$$|T_1(r)| = |\Gamma(p) \cap \Gamma(r)| = c_2^j$$

it is natural to expect that when c_2^j is larger, $\overline{I}(r)$ is more likely to be the whole of $\overline{R}_1(p)$. This informal expectation works, so we proceed according to it and put

$$c_2^\alpha = \max_{1 \leq j \leq t} c_2^j,$$

so that $\alpha = 3, 1, 1, 1$ and $c_2^\alpha = 27, 9, 63, 135$ for $G = Fi'_{24}$, J_4, BM, and M, respectively.

For the remainder of this section we assume that $r \in \Gamma_2^\alpha(p)$.

Lemma 7.5.6 *There is a subgraph Δ in Γ, such that*

(i) Δ *contains* p, r, $\Gamma(p) \cap \Gamma(r)$ *and the* $Q(p)$*-orbit of* r;
(ii) Δ *is isomorphic to the collinearity graph of the polar space* $\mathcal{P} = \mathcal{P}(\Omega)$ *of the classical orthogonal groups* Ω *isomorphic to* $\Omega_8^-(2)$, $\Omega_6^+(2)$, $\Omega_9(2) \cong S_8(2)$ *and* $\Omega_{10}^+(2)$ *for* $G = Fi'_{24}$, J_4, BM, *and* M, *respectively;*
(iii) *the lines of* $\mathcal{P}$ *are those of* $\mathcal{G}$ *contained in* Δ;
(iv) *the action induced on* Δ *by the stabilizer of* Δ *in* G *contains* Ω.

Proof. When $G = Fi'_{24}$ we take Δ to be the subgraph Θ as in the proof of (7.3.8).

When $G = J_4$ we first embed p and r in the sextet subgraph Ξ as in (7.1.5). Then p and r can be treated as sextets refining a unique octad B, say (compare Proposition 3.3.5). We take Δ to be the subgraph in Ξ induced by all the sextets refining B. Then the properties of Δ stated in the lemma follow from the basic properties of the $S(5,8,24)$ Steiner system.

When $G = BM$ or M we take Δ to be the subgraph Σ as in the proof of (7.3.10) or Ψ as in the proof of (7.3.11), respectively. □

Remark. We could also take $\alpha = 1$ when $G = Fi'_{24}$. Then proceeding as we did when $G = J_4$ we would produce a subgraph Δ which is the collinearity graph of $\mathcal{P}(\Omega_6^+(2))$.

It follows from the fundamental property of dual polar spaces that r is collinear to exactly one point on every line containing p, which gives the following

Lemma 7.5.7 $J_1(r)$ *is the set of lines in the polar space* $\mathcal{P}$ *as in* (7.5.6) *containing* p. □

Let μ be the restriction to Δ of the representation mapping φ_u and Y be the subgroup in R generated by the image of μ, so that (Y, μ) is a representation of $\mathcal{P}$.

Lemma 7.5.8 (Y, μ) *is the universal representation of* $\mathcal{P}(\Omega)$, *so that* Y *is elementary abelian, isomorphic to the natural orthogonal module of* Ω.

Proof. The result is obtained by comparing of the subgroup in $\widetilde{G}$ generated by the elements $\widetilde{\varphi}(x)$ taken for all $x \in \Delta$ with (3.6.2). □

Combining (7.5.7) and (7.5.8) we obtain our next result.

Lemma 7.5.9 *The following assertions hold:*

(i) $\overline{I}_1(r)$ *coincides with* $\overline{Y}_1(p) = Y_1(p)/Y_0(p)$;

(ii) $\overline{I}_1(r)$ *is isomorphic to the universal representation group (module) of* $\mathrm{res}_{\mathcal{P}}(p)$*;*

(iii) $\overline{I}_1(r)$ *is the natural (orthogonal) module of* $\Pi \cong \Omega_6^-(2)$, $\Omega_4^+(2)$, $\Omega_7(2)$ *or* $\Omega_8^+(2)$ *for* $G = Fi'_{24}$, J_4, BM, *or* M, *respectively;*

(iv) *the action induced by* $H(r)$ *on* $\overline{I}_1(r)$ *contains* Π. □

The square and the commutator maps on $R_1(p)$ induce on $\overline{R}_1(p)$ quadratic and related bilinear forms which are H-invariant. These forms will be denoted by the same letters θ and β as the forms introduced before (7.1.4). This should not cause any confusion in view of (7.4.1). Notice that if $G = Fi'_{24}$, J_4, or M then β is nonsingular (isomorphic to the corresponding form on $\overline{Q}(p)$) and if $G = BM$ then the radical of β is one dimensional and coincides with the kernel of the homomorphism

$$\overline{R}_1(p) \cong \overline{\Lambda}^{(23)} \to \overline{\Lambda}^{(22)} \cong \overline{Q}(p).$$

Since Y is abelian by (7.5.8), we have the following.

Lemma 7.5.10 *The submodule* $\overline{I}_1(r)$ *is totally singular with respect to* β *and contains the radical of* β. □

The following result is of crucial importance.

Lemma 7.5.11 *The orthogonal complement of* $\overline{I}_1(r)$ *with respect to* β *is the only maximal* $H(r)$*-submodule in* $\overline{R}_1(p)$ *containing* $\overline{I}_1(r)$.

Proof. If $G = Fi'_{24}$ then the result is immediate, since $\overline{I}_1(r)$ is a maximal totally singular subspace on which $H(r)$ acts irreducibly.

In the remaining three cases we make use of the fact that both c_2^α and n_α (which is the number of $Q(p)$-orbits in $\Gamma_2^\alpha(p)$, equivalently, the index of $H(r)$ in H) are odd numbers. This means that both $H(r)$ and the stabilizer in $H(r)$ of a line l from $J_1(r)$ contain a Sylow 2-subgroup S_2 of H. We claim that S_2 fixes a unique hyperplane in $\overline{R}_1(p)$ which contains the radical of β and that this hyperplane is the orthogonal complement of l with respect to β. This claim is true by (4.4.9), (5.2.4), and (5.3.4) for $G = J_4$, BM and M, respectively (notice that the hyperplanes in $\overline{\Lambda}^{(23)}$ containing the radical are in a natural bijection with the hyperplanes in $\overline{\Lambda}^{(22)}$). Hence an $H(r)$-submodule of $\overline{R}_1(p)$ containing $\overline{I}_1(r)$ must be contained in the intersection of the $P(l)$ taken for all $l \in J_1(r)$ and the proof follows. □

Now in order to establish the equality $\overline{I}(r) = \overline{R}_1(p)$ all we need to do is prove the following.

Lemma 7.5.12 *There is a line $l_1 \in J_1(r)$ and a line $l_2 \in J(r)$ such that $\beta(l_1, l_2) = 1$.*

Proof. As above in this chapter let Υ and Υ^* denote the collinearity graph of $\mathscr{H} = \mathrm{res}_{\mathscr{G}}(p)$ and of the enriched point-line incidence system of $\mathscr{H}$, respectively. Then $J_1(r)$ and $J(r)$ are subsets of the vertex set. Furthermore, $J(r)$ is the union of $J_1(r)$ and the set of vertices adjacent in Υ^* to a vertex from $J_1(r)$. Let $l_1 \in J_1(r)$. We have to show that there is a vertex in $J_1(r)$ adjacent in Υ^* to a vertex which is not perpendicular to l_1 with respect to β. By (7.5.6) and its proof we can easily identify $J_1(r)$.

If $G = Fi_{24}'$ then $J_1(r)$ induces the Schläfli subgraph (cf. Lemmas 4.14.9 and 4.14.10 in [Iv99]), which contains 10 vertices from $\Upsilon(l_1)$ and 16 vertices from $\Gamma_2^1(l_1)$. Since the vertices from $\Upsilon_3^1(l_1)$ are not perpendicular to l_1 with respect to β, the result is immediate from the suborbit diagram of Υ.

Let $G = J_4$. Then by (7.5.9 (iii)) the subgraph A in Υ^* induced by $J_1(r)$ is a 3×3 grid. Using the fact that in this case the subgraph Δ is contained in the sextet subgraph Ξ, it is easy to check that one of the parallel classes of triangles in A must be triangles from the enriched but not from the original point-line incidence systems. Hence $J_1(r)$ is the complete preimage of a triangle with respect to the covering

$$\Upsilon \cong \Gamma(\mathscr{G}(3 \cdot M_{22})) \to \Gamma(\mathscr{G}(M_{22})).$$

Hence $J_1(r)$ contains a vertex from $\Upsilon_3^1(l_1)$ and since the vertices in $\Gamma_3^2(x)$ are not perpendicular to l_1 the result is again immediate from the suborbit diagram of Υ.

If $G = BM$ then $J_1(r)$ is the point-set of a $\mathscr{G}(S_6(2))$-subgeometry in $\mathscr{H}$, it contains a vertex from $\Upsilon_2^1(l_1)$ which is adjacent to a vertex from $\Upsilon_3(l_1)$ and the latter is not perpendicular to l_1.

Finally, if $G = M$, then the result is immediate from the suborbit diagram since the vertices in $\Upsilon_3^1(l_1)$ are not perpendicular to l_1. □

The results (7.5.10) and (7.5.11) can be summarized in the following.

Proposition 7.5.13 *If $r \in \Gamma_2^{\alpha}(p)$, then z_r normalizes $R_1(p)$.* □

We are well prepared to prove the final result of the section.

Lemma 7.5.14 *Let $R\lfloor p\rfloor'$ be the subgroup in $R\lfloor p\rfloor$ generated by $R_1(p)$ and the elements z_r taken for all $r \in \Gamma_2^{\alpha}(p)$. Then*

(i) *$R\lfloor p\rfloor' = R\lfloor p\rfloor$ if $G = Fi_{24}'$, BM, or M;*

(ii) *$R\lfloor p\rfloor'$ has index 2 in $R\lfloor p\rfloor$ if $G = J_4$;*
(iii) (7.5.1) *holds, i.e., $R_1(p)$ is normal in $R\lfloor p\rfloor$.*

Proof. Let $q \in \Gamma(p)$. Then by (7.5.2) the quotient X of $R_1(q)$ over $R_1(p) \cap R_1(q)$ is elementary abelian of order 2^{m-1}. Furthermore the orbits of the action of $G(p) \cap G(q)$ on this quotient are described in Table V. By (7.3.5) the elements z_r for $r \in \Gamma_2^{\alpha}(p) \cap \Gamma(q)$ map onto the orbit O of lengths 6, 15, 77 and 759 for $G = Fi'_{24}$, J_4, BM, and M, respectively. In the first, third and fourth cases O generates the whole of X. Indeed, in the latter two cases X is irreducible and in the first case O is outside the unique proper submodule in X, so (i) follows.

Suppose that $G = J_4$. Then the elements $r \in \Gamma_2^{\alpha}(p)$ are contained in $O^2(G(p))$ (which has index 2 in $G(p)$) and hence the index of $R\lfloor p\rfloor'$ in $R\lfloor p\rfloor$ is at least 2. Let us show that it is exactly 2. The orbit O generates the unique codimension 1 submodule X' in X. On the other hand, by (7.3.5 (ii)) and (7.3.9) the orbit O_1 of length 10 formed by the images of the elements z_s for $s \in \Gamma_2^3(p) \cap \Gamma(p)$ generates the whole of X. Hence the set $E = \{z_s \mid s \in \Gamma_2^3(p)\}$ together with $R_1(p)$ generates the whole of $R\lfloor p\rfloor$. Let us say that two elements z_s and z_t from E are equivalent if $z_s = z_t y$ for some $y \in R\lfloor p\rfloor$. Since $[X : X'] = 2$ we conclude that two elements z_s and z_t are equivalent whenever s and t are adjacent to a common vertex in $\Gamma(p)$. Now it is very easy to see that all the elements in E are equivalent and (ii) is established.

By (i), (ii) and (7.5.13) in order to prove (iii) all we have to show is that when $G = J_4$ for every $s \in \Gamma_2^3(p)$ and $q \in \Gamma(p)$ we have $[z_s, z_q] \in R_1(p)$. But this is quite clear since by the above paragraph $z_s = z_t y$ for some t adjacent to q and $y \in R\lfloor p\rfloor'$. □

7.6 $R\lfloor p\rfloor$ is isomorphic to $\widetilde{G}(p)$

By (7.5.1) we can consider the factor-group

$$\overline{R}\lfloor p\rfloor = R\lfloor p\rfloor / R_1(p).$$

Since the elements $\widetilde{\varphi}(r)$ taken for all $r \in N(p)$ generate the stabilizer $\widetilde{G}(p)$ of p in $\widetilde{G}$, the homomorphism $\psi : R \to \widetilde{G}$ induces a homomorphism $\overline{\psi}$ of $\overline{R}\lfloor p\rfloor$ onto

$$\widetilde{H} := \widetilde{G}(p)/O_2(\widetilde{G}(p))$$

(isomorphic to $3^2 \cdot U_4(3).2_2$, $3 \cdot \mathrm{Aut}\, M_{22}$, Co_2, and Co_1 for $G = Fi'_{24}$, J_4, BM, and M).

In order to complete the proof of (7.2.4 (i)) it is sufficient to show that $\overline{\psi}$ is an isomorphism, which of course can be achieved by showing that the order of $\overline{R}\lfloor p \rfloor$ is at most that of $\widetilde{H}$.

Put $\delta = 1$, 1, 2, and 1 for $G = Fi'_{24}$, J_4, BM, and M, respectively. Let $\overline{Z}$ be the set of images in $\overline{R}\lfloor p \rfloor$ of the elements z_r taken for all $r \in \Gamma_2^{\delta}(p)$ and $\overline{R}\lfloor p \rfloor^{\bullet}$ be the subgroup in $\overline{R}\lfloor p \rfloor$ generated by $\overline{Z}$.

Lemma 7.6.1 *The following assertions hold:*

(i) $\overline{R}\lfloor p \rfloor^{\bullet} = \overline{R}\lfloor p \rfloor$ *if* $G = BM$ *or* M *and* $\overline{R}\lfloor p \rfloor^{\bullet}$ *has index* 2 *in* $\overline{R}\lfloor p \rfloor$ *if* $G = Fi'_{24}$ *or* J_4*;*
(ii) $\overline{\psi}(\overline{R}\lfloor p \rfloor^{\bullet}) = O^2(\widetilde{H})$*;*
(iii) $O_2(G(p))$ *is in the kernel of the action of* $G(p)$ *on* $\overline{R}\lfloor p \rfloor^{\bullet}$*;*
(iv) $\overline{\psi}$ *maps* $\overline{Z}$ *bijectively onto a conjugacy class* $\mathscr{X}$ *of involutions in* $O^2(\widetilde{H})$*;*
(v) $\mathscr{X}$ *is the class of* 2*A*, 2*A*, 2*B or* 2*A involutions in* $O^2(\widetilde{H})$ *for* $G = Fi'_{24}$, J_4, BM, *or* M, *respectively.*

Proof. (i) and (ii) follow from (7.5.14) and its proof. Recall that $\overline{R}\lfloor p \rfloor^{\bullet}$ is also generated by the images of the elements z_r taken for all $r \in \Gamma_2^1(p)$. Let Δ be the subgraph in Γ which is as in (7.5.6) for $G = J_4$, BM and M and as in the remark after that lemma for $G = Fi'_{24}$. Then by (3.6.2 (iii)) the images of the elements z_r for all $r \in \Delta \cap \Gamma_2^1(p)$ are the same. Since the stabilizer of Δ in $G(p)$ contains $Q(p)$, (iii) follows. Since $k_\delta = n_\delta$ in terms of Section 7.3, and the equality $sQ(p) = tQ(p)$ for $s, t \in \Gamma_2^{\delta}(p)$ holds if and only if s and t are in the same $Q(p)$-orbit, we obtain (iv). Finally (v) is by (7.3.5). □

Let $\mathscr{I}$ be the involution geometry of $O^2(\widetilde{H})/Z(O^2(\widetilde{H}))$, whose points are the $\mathscr{X}$-involutions (where $\mathscr{X}$ is as in (7.6.1 (v))) and whose lines are the $\mathscr{X}$-pure Kleinian four-subgroups. Then in the notation of the previous chapter $\mathscr{I}$ is $\mathscr{I}(U_4(3))$, $\mathscr{I}(M_{22})$, $\mathscr{I}(Co_2, 2B)$ or $\mathscr{I}(Co_1)$ for $G \cong Fi'_{24}$, J_4, BM or M, respectively. By (7.6.1 (iv)), $(\overline{\psi})^{-1}$ is a bijection of the point-set of $\mathscr{I}$ onto $\overline{Z}$, the latter being a generating set of involutions in $\overline{R}\lfloor p \rfloor^{\bullet}$. On the other hand, by (6.3.5), (6.4.4), (6.5.7) and (6.6.3), $O^2(\widetilde{H})$ is the universal representation group of $\mathscr{I}$. Thus in order to achieve the goal of this section it is sufficient to show that $(\overline{\psi})^{-1}$ maps every line of $\mathscr{I}$ onto a Kleinian four-subgroup (i.e., that $(\overline{R}\lfloor p \rfloor^{\bullet}, (\overline{\psi})^{-1})$ is a representation of $\mathscr{I}$). Towards this end we consider subgroups generated by various subsets of $\overline{Z}$.

Lemma 7.6.2 *Let* q *be a point collinear to* p, l *be the line of* $\mathscr{G}$ *containing* p *and* q *(so that* l *is a point of* $\mathscr{H} = \mathrm{res}_{\mathscr{G}}(p)$*). Let* $\overline{Z}_q$ *be the set of images* *in $\overline{Z}$ of the elements z_r taken for all $r \in \Gamma_2^\delta(p) \cap \Gamma(q)$. Then for $G = Fi_{24}'$, J_4, BM, and M the set $\overline{Z}_q$ is of sizes 15, 15, 330, and 759, respectively. The subgroup T_q in $\overline{R}\lfloor p \rfloor^\bullet$, generated by $\overline{Z}_q$ is elementary abelian of orders 2^4, 2^4, 2^{10} and 2^{11}, respectively, and it maps isomorphically onto $O_2(H(l))$.*

Proof. The result is immediate from (7.4.1) and (7.6.1 (iv)) in view of Table V. □

Lemma 7.6.3 *Let $G = Fi_{24}'$ or J_4 and Ξ be the sextet subgraph in the collinearity graph Γ of $\mathcal{G}$ as in (7.1.5), containing p. Let $\overline{Z}_\Xi$ be the set of images in $\overline{Z}$ of the elements z_r taken for all $r \in \Gamma_2^\delta(p) \cap \Xi$ and let T_Ξ be the subgroup in $\overline{R}\lfloor p \rfloor^\bullet$ generated by $\overline{Z}_\Xi$. Then $\overline{Z}_\Xi$ is of size 15, T_Ξ is elementary abelian of order 2^4 and*

(i) *if $G = Fi_{24}'$, then T_Ξ maps isomorphically onto $O_2(H(w))$, where w is an element of type 3 in $\mathcal{H}$;*
(ii) *if $G = J_4$, then T_Ξ maps isomorphically onto $O_2(H(\mathcal{S}))$, where $\mathcal{S}$ is a $\mathcal{G}(3 \cdot S_4(2))$-subgeometry in $\mathcal{H}$.*

Proof. By (4.3.1) the elements z_r taken for all $r \in \Xi$ generate in R an elementary abelian subgroup of order 2^{11} which maps isomorphically onto $O_2(X)$, where $X \sim 2^{11}.M_{24}$ is the stabilizer of Ξ in G. By (4.3.2) the image T_Ξ of this subgroup in $\overline{R}\lfloor p \rfloor^\bullet$ is elementary abelian of order 2^4. □

Finally we obtain the main result of this section.

Proposition 7.6.4 *The following assertions hold:*

(i) $(\overline{R}\lfloor p \rfloor^\bullet, (\overline{\psi})^{-1})$ *is a representation of the involution geometry $\mathcal{I}$;*
(ii) $\overline{R}\lfloor p \rfloor^\bullet \cong O^2(\widetilde{H})$;
(iii) $R\lfloor p \rfloor \cong \widetilde{G}(p)$.

Proof. The assertion (i) follows from (6.4.1), (6.3.2), (6.5.3 (i)) and (6.6.1) for $G = Fi_{24}'$, J_4, BM and M, respectively. We deduce (ii) from (i), applying respectively (6.4.4), (6.3.5), (6.5.7) and (6.6.3). Now (iii) follows from (i) and (ii) in view of (7.6.1 (i)). □

7.7 Generation of $\widetilde{G}(p) \cap \widetilde{G}(q)$

Let p and q be collinear points in $\mathcal{G}$ and let l be the line containing p and q. Let $\widetilde{K}^-(l)$, $\widetilde{K}^+(l)$ and $\widetilde{K}(l)$ be the kernels of the actions of $\widetilde{G}(l)$ on the point-set of l, on the set of elements of type 3 and more incident to

l and on $\mathrm{res}_{\mathcal{G}}(l)$, respectively. Then $\widetilde{K}(l) = O_2(G(l))$, $\widetilde{K}^+(l)/\widetilde{K}(l) \cong Sym_3$ and $\widetilde{K}^-(l)$ coincides with the subgroup

$$\widetilde{G}(p) \cap \widetilde{G}(q)$$

in which we are mainly interested in this section. Recall that $\widetilde{\varphi}$ is the mapping which turns $\widetilde{G}$ into a representation group of $\mathcal{G}$ and that $N(p)$ is the set of points in $\mathcal{G}$ which are at distance at most 2 from p in the collinearity graph Γ of $\mathcal{G}$ and which commute with p (as involutions in G). The goal of this section is to prove the following.

Proposition 7.7.1 *The elements $\widetilde{\varphi}(r)$ taken for all $r \in N(p) \cap N(q)$ generate $\widetilde{K}^-(l) = \widetilde{G}(p) \cap \widetilde{G}(q)$.*

The following statement is easy to deduce from the shape of the parabolic subgroups corresponding to the action of $\widetilde{G}$ on $\mathcal{G}$.

Lemma 7.7.2 *For $\widetilde{G} = 3{\cdot}Fi'_{24}$, J_4, $2{\cdot}BM$, and M, respectively, the following assertions hold:*

(i) *the kernels $\widetilde{K}(l)$ has orders 2^{17}, 2^{17}, 2^{33}, and 2^{35};*
(ii) *the quotient $\widetilde{K}^-(l)/\widetilde{K}(l) \cong \widetilde{G}(l)/\widetilde{K}^+(l)$ is isomorphic to $3 \cdot Alt_6$, Sym_5, $\mathrm{Aut}\, M_{22}$, and M_{24}.* □

Lemma 7.7.3 *The following assertions hold:*

(i) *the elements $\widetilde{\varphi}(r)$ taken for all $r \in \Gamma(p) \cup \Gamma(q)$ generate $\widetilde{K}^+(l)$;*
(ii) *the elements $\widetilde{\varphi}(r)$ taken for all $r \in (\Gamma(p) \cap N(q)) \cup (\Gamma(q) \cap N(p))$ generate $\widetilde{K}(l) = O_2(\widetilde{K}^-(l))$.*

Proof. It is clear (see for instance (7.4.1)) that the elements $\widetilde{\varphi}(r)$ taken for all $r \in \Gamma(p)$ generate $\widetilde{Q}(p)$. Then the result is by (7.3.3) and the order consideration. □

Let $\mathcal{Y}$ be the residue in $\mathcal{G}$ of the flag $\{p, l\}$ and Y be the flag-transitive automorphism group of $\mathcal{Y}$ induced by $\widetilde{K}^-(l)$. Then for $G = Fi'_{24}$, J_4, BM, and M, respectively, the geometry $\mathcal{Y}$ is isomorphic to $\mathcal{G}(S_4(2))$, $\mathcal{G}(Alt_5)$, $\mathcal{G}(M_{22})$ and $\mathcal{G}(M_{24})$ while $Y \cong Alt_6$, Sym_5, $\mathrm{Aut}\, M_{22}$, and M_{24}. The next result follows from the basic properties of $\mathcal{Y}$ and Y.

Lemma 7.7.4 *In the above terms the group Y is generated by the subgroups $O_2(Y(\pi))$ taken for all points π in $\mathcal{Y}$.* □

Notice that π in (7.7.4) is a plane in $\mathcal{G}$ incident to p and l. For such a plane π let s be a point incident to π but not to l. Then clearly every $r \in \Gamma(s)$ is at distance at most 2 from both p and q. We know that the elements $\widetilde{\varphi}(r)$ taken for all $r \in \Gamma(s)$ generate $\widetilde{Q}(s)$. The latter subgroup stabilizes π and induces on its point-set an action of order 4. It is easy to see that the kernel $\widetilde{Q}(s,\pi)$ of this action is generated by the elements $\widetilde{\varphi}(r)$ taken for all $r \in \Gamma(s) \cap N(p) \cap N(q)$.

Lemma 7.7.5 *The image of* $\widetilde{Q}(s,\pi)$ *in* $Y = \widetilde{K}^-(l)/\widetilde{K}(l)$ *coincides with* $O_2(Y(\pi))$.

Proof. The result is by the order consideration in view of (7.3.3). □

Now (7.7.1) is by (7.7.3 (ii)) and (7.7.5) in view of (7.7.4).

7.8 Reconstructing the rank 3 amalgam

In this section we use (7.2.4) in order to deduce (7.2.3). We know by (7.2.4 (i)) that the restriction of the homomorphism

$$\psi : R \to \widetilde{G}$$

to $R\lfloor p \rfloor := R[N(p)]$ (where $N(p)$ is the set of points commuting with p and at distance at most 2 from p in the collinearity graph of $\mathcal{G}$) is an isomorphism onto $\widetilde{G}(p)$ which is the stabilizer of p in the (possibly unfaithful) action of $\widetilde{G}$ on $\mathcal{G}$. Let ψ_p denote the restriction of ψ to $R\lfloor p \rfloor$. By (7.2.4 (ii)) if r is a point collinear to p then the restrictions of ψ_p and ψ_r to $R\lfloor p \rfloor \cap R\lfloor r \rfloor$ induce the same isomorphism (which we denote by ψ_{pr}) onto $\widetilde{G}(p) \cap \widetilde{G}(r)$.

We formulate explicitly an important property of $\mathcal{G}$.

Lemma 7.8.1 *For a point* p *of* $\mathcal{G}$ *the set* $\Gamma(p)$ *of points collinear to* p *(treated as central involutions in* G*) generate an extraspecial* 2*-group* $Q(p)$*. A line and plane containing* p *are elementary abelian subgroups in* $Q(p)$ *of orders* 2^2 *and* 2^3*, respectively. If* π *is a plane then its stabilizer* $G(\pi)$ *in* G *induces the natural action of* $L_3(2)$ *on the set of* 7 *points contained in* π. □

Let $l = \{p = p_1, p_2, p_3\}$ be a line containing p and $\widetilde{G}(l)$ be the stabilizer of l in $\widetilde{G}$. Then $\widetilde{G}(l)$ induces the group Sym_3 on the point-set of l. If $\widetilde{K}^-(l)$ is the kernel of this action then

$$\widetilde{K}^-(l) = \bigcap_{i=1}^{3} \widetilde{G}(p_i).$$

The images of the $\widetilde{G}(l) \cap \widetilde{G}(p_i)$ in the quotient $\widetilde{G}(l)/\widetilde{K}^-(l)$ for $i = 1, 2$, and 3 are of order 2 and they generate the whole quotient.

This observation suggests the way in which a preimage of $\widetilde{G}(l)$ in R can be defined. For $1 \leq i \leq 3$ put

$$R\lfloor p_i, l \rfloor = \psi_{p_i}^{-1}(\widetilde{G}(p_i) \cap \widetilde{G}(l))$$

and

$$R\lfloor l \rfloor = \langle R\lfloor p_i, l \rfloor \mid 1 \leq i \leq 3 \rangle.$$

Lemma 7.8.2 *The following assertions hold:*

(i) *the restriction of ψ to $R\lfloor l \rfloor$ is an isomorphism onto $\widetilde{G}(l)$ (we denote this isomorphism by ψ_l);*
(ii) *the restriction of ψ to $R\lfloor p \rfloor \cap R\lfloor l \rfloor$ is an isomorphism onto $\widetilde{G}(p) \cap \widetilde{G}(l)$.*

Proof. Since ψ_{p_i} is an isomorphism of $R\lfloor p_i \rfloor$, it is immediate from the definition that $R\lfloor l \rfloor$ maps surjectively onto $\widetilde{G}(l)$ and in order to establish (i) it is sufficient to show that the order of $R\lfloor l \rfloor$ is at most that of $\widetilde{G}(l)$. Let

$$R^-\lfloor l \rfloor = \psi_{p_i}^{-1}(\widetilde{K}^-(l)).$$

Then by (7.2.4 (i) and (ii)) $R^-\lfloor l \rfloor$ is independent of the particular choice of $i \in \{1, 2, 3\}$ and is of index 2 (in particular it is normal) in $R\lfloor p_i, l \rfloor$ for $1 \leq i \leq 3$. Hence $R^-\lfloor l \rfloor$ is a normal subgroup in $R\lfloor l \rfloor$ which maps isomorphically onto $\widetilde{K}^-(p)$. Hence to complete the proof of (i) it is sufficient to show that $\overline{R}\lfloor l \rfloor := R\lfloor l \rfloor / R^-\lfloor l \rfloor$ is isomorphic to Sym_3. Let $\overline{\tau}_i$ be the unique non-trivial element in the image of $R\lfloor p_i, l \rfloor$, where $1 \leq i \leq 3$. In order to identify $\overline{R}\lfloor l \rfloor$ with Sym_3 it is sufficient to find elements τ_i in R such that $\tau_i R^-\lfloor l \rfloor = \overline{\tau}_i$ and

$$\langle \tau_i \mid 1 \leq i \leq 3 \rangle R^-\lfloor l \rfloor / R^-\lfloor l \rfloor \cong Sym_3.$$

Towards this end let π be a plane containing l, and q a point in π but not in l. Since $Q(p)$ is extraspecial and π is an elementary abelian subgroup of order 2^3 in $Q(p)$, it is easy to see that there is an element $t_1 \in Q(p)$ which commutes with q and conjugates p_2 onto p_3. Then $t_1 \in G(q)$ and induces the transposition $(p_1)(p_2, p_3)$ on the point-set of l. In a similar way we can find elements t_2 and t_3 contained in $G(q) \cap Q(p_2)$ and $G(q) \cap Q(p_3)$, which induce on l the transpositions $(p_2)(p_1, p_3)$ and $(p_3)(p_1, p_2)$, respectively. Then

$$\langle t_i \mid 1 \leq i \leq 3 \rangle K^-(l)/K^-(l) \cong Sym_3.$$

Let $\widetilde{t}_i$ be a preimage of t_i in $\widetilde{G}(q)$, $1 \le i \le 3$, and $\tau_i = \psi_q^{-1}(\widetilde{t}_i)$. Since ψ_q is an isomorphism of $R\lfloor q \rfloor$ onto $\widetilde{G}(q)$ it is easy to see that the τ_i possess the required property and the proof of (i) is complete. Now (ii) is immediate from (i) and the definition of $R\lfloor l \rfloor$. □

Now let $\pi = \{p = p_1, p_2, ..., p_7\}$ be a plane containing l (and hence p as well). Then the stabilizer $\widetilde{G}(\pi)$ of π in $\widetilde{G}$ induces on the point-set of π the natural action of $L_3(2)$ (compare (7.8.1)) with kernel

$$\widetilde{K}^-(\pi) = \bigcap_{i=1}^{7} \widetilde{G}(p_i)$$

and the image of $\widetilde{G}(\pi) \cap \widetilde{G}(p_i)$ in $\widetilde{G}(\pi)/\widetilde{K}^-(\pi)$ is a maximal parabolic in $L_3(2)$ isomorphic to Sym_4. Put

$$R\lfloor p_i, \pi \rfloor = \psi_{p_i}^{-1}(\widetilde{G}(p_i) \cap \widetilde{G}(\pi))$$

and

$$R\lfloor \pi \rfloor = \langle R\lfloor p_i, \pi \rfloor \mid 1 \le i \le 7 \rangle.$$

Lemma 7.8.3 *The following assertions hold:*

(i) *the restriction of ψ to $R\lfloor \pi \rfloor$ is an isomorphism onto $\widetilde{G}(\pi)$ (we denote this isomorphism by ψ_π);*

(ii) *the restrictions of ψ to $R\lfloor p \rfloor \cap R\lfloor \pi \rfloor$ and to $R\lfloor l \rfloor \cap R\lfloor \pi \rfloor$ are isomorphisms onto $\widetilde{G}(p) \cap \widetilde{G}(\pi)$ and $\widetilde{G}(l) \cap \widetilde{G}(\pi)$, respectively.*

Proof. Again by the definition $R\lfloor \pi \rfloor$ maps surjectively onto $\widetilde{G}(\pi)$. Let

$$R^-\lfloor \pi \rfloor = \psi_{p_i}^{-1}(\widetilde{K}^-(\pi)).$$

By (7.2.4 (i) and (ii)) since the points in π are pairwise collinear, $R^-\lfloor \pi \rfloor$ is independent of the particular choice of $i \in \{1, ..., 7\}$ and is normal in each $R\lfloor p_i, \pi \rfloor$ and hence it is normal in $R\lfloor \pi \rfloor$. Put $\overline{R}\lfloor \pi \rfloor = R\lfloor \pi \rfloor / R^-\lfloor \pi \rfloor$. In order to prove (i) we have to show that $\overline{R}\lfloor \pi \rfloor \cong L_3(2)$. We use the fact that $L_3(2)$ is generated by the conjugacy class of its transvections.

Let $\widetilde{\tau}(q, m)$ be an element from $\widetilde{G}(\pi)$ which induces on $\mathrm{res}_{\widetilde{\mathcal{G}}}(\pi) \cong pg(2, 2)$ the transvection whose centre is q (which is a point) and whose axis is m (which is a line containing q). Let

$$t(q, m) = \psi_q^{-1}(\widetilde{\tau}(q, m))$$

and $\bar{t}(q, m)$ be the image of $t(q, m)$ in $\overline{R}\lfloor \pi \rfloor$. By (7.2.4) if r_1 and r_2 are any two points fixed by $\widetilde{\tau}(q, m)$ (i.e., $r_1, r_2 \in m$) then

$$\psi_{r_1}^{-1}(\widetilde{\tau}(q, m)) = \psi_{r_2}^{-1}(\widetilde{\tau}(q, m)),$$

which shows that $t(q,m)$ is contained in $R\lfloor r,\pi\rfloor$ for every $r \in m$. Hence $R\lfloor r,\pi\rfloor$ contains 9 elements $t(q,m)$ and the images of these elements in the quotient $\overline{R}\lfloor\pi\rfloor$ generate the whole image of $R\lfloor r,\pi\rfloor$ in the quotient (isomorphic to Sym_4). Hence $\overline{R}\lfloor\pi\rfloor$ is generated by the above defined 21 elements $\bar{t}(q,m)$. We claim that these elements form a conjugacy class in $\overline{R}\lfloor\pi\rfloor$. Towards this end we need to show that for any two flags (q_1,m_1) and (q_2,m_2) there is a flag (q_3,m_3) such that

$$\bar{t}(q_1,m_1)\bar{t}(q_2,m_2)\bar{t}(q_1,m_1) = \bar{t}(q_3,m_3).$$

The lines m_1 and m_2 always have a common point r, say. Then $\tilde{\tau}(q_1,m_1)$ and $\tilde{\tau}(q_2,m_2)$ are contained in $\widetilde{G}(r)$ and the conjugate $\tilde{\sigma}$ of $\tilde{\tau}(q_2,m_2)$ by $\tilde{\tau}(q_1,m_1)$ induces a transvection on π (the same as $\tilde{\tau}(q_3,m_3)$ for some flag (q_3,m_3)). Then the image of $\psi_r^{-1}(\tilde{\sigma})$ in $\overline{R}\lfloor\pi\rfloor$ coincides with $\bar{t}(q_3,m_3)$ and the claim follows. Since $\widetilde{G}(\pi)/\widetilde{K}^{-}(\pi) \cong L_3(2)$ is a homomorphic image of $\overline{R}\lfloor\pi\rfloor$, by (6.1.2) we have either $\overline{R}\lfloor\pi\rfloor \cong L_3(2)$ or $\overline{R}\lfloor\pi\rfloor \cong L_3(2)\times 2$. We can see inside the image of $R\lfloor p,\pi\rfloor$ in $\overline{R}\lfloor\pi\rfloor$ that if l_1,l_2,l_3 are the lines in π containing p, then

$$\bar{t}(p,l_1)\bar{t}(p,l_2)\bar{t}(p,l_3) = 1,$$

which excludes the latter possibility and completes the proof of (i). Now (ii) is immediate from (i) as is the fact that the relevant restrictions are surjective by the definition of $R\lfloor\pi\rfloor$. □

Now in order to complete the proof of (7.2.3) it is sufficient to show that $\mathscr{D}$ generates the whole of R. Let $\Phi = \{p,l,\pi\}$ be the flag associated with $\mathscr{D}$ and write $\mathscr{D}(\Phi)$ for $\mathscr{D}$ to emphasize the flag. Clearly it is sufficient to show that the subgroup in R generated by $\mathscr{D}(\Phi)$ contains the amalgam $\mathscr{D}(\Phi')$ for every flag Φ' of type $\{1,2,3\}$ in $\mathscr{G}$. Furthermore, since $\mathscr{G}$, being a geometry, satisfies the connectivity conditions, it is sufficient to consider the case when $|\Phi \cap \Phi'| = 2$.

In order to argue in a uniform way put $\Phi = \{\alpha_1,\alpha_2,\alpha_3\}$. Once again by the connectivity of $\mathscr{G}$ and the flag-transitivity of $\widetilde{G}$ we have

$$\widetilde{G}(\alpha_1) = \langle \widetilde{G}(\alpha_1)\cap\widetilde{G}(\alpha_2), \widetilde{G}(\alpha_1)\cap\widetilde{G}(\alpha_3)\rangle.$$

Since ψ is an isomorphism when restricted to $\mathscr{D}(\Phi)$, we have

$$R\lfloor\alpha_1\rfloor = \langle R\lfloor\alpha_1\rfloor \cap R\lfloor\alpha_2\rfloor, R\lfloor\alpha_1\rfloor\cap R\lfloor\alpha_3\rfloor\rangle.$$

Hence the subgroup in R generated by $\mathscr{D}(\{\alpha_1,\alpha_2,\alpha_3\})$ contains the amalgam $\mathscr{D}(\{\alpha_1',\alpha_2,\alpha_3\})$ for every α_1' of appropriate type incident to α_2 and α_3.

Thus (7.2.3) is proved and in view of (7.2.2) it implies (7.1.3).

7.9 $\mathcal{G}(3^{4371} \cdot BM)$

In this section we prove

Proposition 7.9.1 *The universal representation module of $\mathcal{G}(3^{4371} \cdot BM)$ is zero-dimensional.*

Let $\widetilde{\mathcal{G}} = \mathcal{G}(3^{4371} \cdot BM)$, $\mathcal{G} = \mathcal{G}(BM)$ and $\chi : \widetilde{\mathcal{G}} \to \mathcal{G}$ be the corresponding 2-covering. Let (R, φ_u) be the universal representation of $\mathcal{G}$, where $R \cong 2 \cdot BM$ (cf. (7.1.3)). If ν is the composition of χ and φ_u, then clearly (R, ν) is a representation of $\widetilde{\mathcal{G}}$. Let $\widetilde{x}$ be a point of $\widetilde{\mathcal{G}}$ and $x = \chi(\widetilde{x})$. Put $\widetilde{\mathcal{H}} = \mathrm{res}_{\widetilde{\mathcal{G}}}(\widetilde{x}) \cong \mathcal{G}(3^{23} \cdot Co_2)$, $\mathcal{H} = \mathrm{res}_{\mathcal{G}}(x) \cong \mathcal{G}(Co_2)$ and let μ denote the 2-covering of $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$ induced by χ. Let Γ be the collinearity graph of $\mathcal{G}$.

Lemma 7.9.2 *For the representation (R, ν) the following assertions hold:*

(i) *$R_1(\widetilde{x})$ is of order 2^{24} and the commutator subgroup of $R_1(\widetilde{x})$ is $\langle \nu(\widetilde{x}) \rangle$;*

(ii) *$\overline{R}_1(\widetilde{x}) \cong \overline{\Lambda}^{(23)}$ is the universal representation group of $\mathcal{H}$ and the universal representation module of $\widetilde{\mathcal{H}}$;*

(iii) *$R_1(\widetilde{x})$ is the universal representation group of the point-line incidence system $\mathcal{S} = (\Pi, L)$ where $\Pi = \{x\} \cup \Gamma(x)$ and L is the set of lines of $\mathcal{G}$ contained in Π.*

Proof. Since ν is the composition of χ and φ_u, (i) follows from (7.1.3), (7.1.2) and the definition of $\mathcal{G}$ in terms of central involutions in BM. And (ii) follows from (5.2.3) and (5.5.1). Since by (5.2.3), $\overline{\Lambda}^{(23)}$ is the universal representation of $\mathcal{H}$, (iii) follows from (7.4.1). □

Let $\widetilde{\Gamma}$ be the collinearity graph of $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{S}} = (\widetilde{\Pi}, \widetilde{L})$ be the point-line incidence system where $\widetilde{\Pi} = \{\widetilde{x}\} \cup \widetilde{\Gamma}(\widetilde{x})$ and $\widetilde{L}$ is the set of lines of $\widetilde{\mathcal{G}}$ contained in $\widetilde{\Pi}$. Notice that the 2-covering χ induces a morphism of $\widetilde{\mathcal{S}}$ onto the point-line system $\mathcal{S}$ as in (7.9.2 (iii)). Let (V, ψ) be the universal abelian representation of $\widetilde{\mathcal{G}}$. Then by (2.6.2) and (5.5.1) the section $\overline{V}_1(\widetilde{x})$ (defined with respect to $\widetilde{\Gamma}$ of course) is a quotient of $\overline{\Lambda}^{(23)}$, in particular, the representation of $\widetilde{\mathcal{H}}$ induced by ψ factored through the 2-covering $\mu : \widetilde{\mathcal{H}} \to \mathcal{H}$.

By the above paragraph we observe that for $\widetilde{y}, \widetilde{z} \in \widetilde{\Pi}$ we have $\psi(\widetilde{y}) = \psi(\widetilde{z})$ whenever $\chi(\widetilde{y}) = \chi(\widetilde{z})$. Thus the restriction of ψ to $\widetilde{\Pi}$ is a composition of the morphism of $\widetilde{\mathcal{S}}$ onto $\mathcal{S}$ induced by χ and an abelian representation of the point-line incidence system $\mathcal{S}$. Hence by (7.9.2 (iii)) $\psi(\widetilde{\Pi})$ is an abelian quotient of the group $R_1(\widetilde{x})$. By (7.9.2 (i)) the commutator subgroup of $R_1(\widetilde{x})$ is generated by the image of $\widetilde{x}$ under the corresponding representations. From this we conclude that $\psi(\widetilde{x}) = 0$ and since this holds for every point $\widetilde{x}$ the proof of (7.9.1) is complete.

Part II

Amalgams

8
Method of group amalgams

In this chapter we collect and develop some machinery for classifying the amalgams of maximal parabolics coming from flag-transitive actions on Petersen and tilde geometries.

8.1 General strategy

Let $\mathcal{G}$ be a P- or T-geometry of rank $n \geq 3$, let $\Phi = \{x_1, ..., x_n\}$ be a maximal flag in $\mathcal{G}$ (where x_i is of type i). Let G be a flag-transitive automorphism group of $\mathcal{G}$ and

$$\mathscr{A} = \{G_i \mid 1 \leq i \leq n\}$$

be the amalgam of maximal parabolics associated with this action and related to the flag Φ (i.e., $G_i = G(x_i)$ is the stabilizer of x_i in G). Then $\mathcal{G}$ can be identified with the coset geometry $\mathscr{C}(G, \mathscr{A})$ and it is a quotient of the coset geometry $\mathscr{C}(U(\mathscr{A}), \mathscr{A})$ associated with the universal completion $U(\mathscr{A})$ of $\mathscr{A}$. Our goal is to identify $\mathscr{A}$ up to isomorphism or more specifically to show that it is isomorphic to the amalgam associated with a known flag-transitive action.

Proceeding by induction on n we assume that

(a) the residue $\mathrm{res}_{\mathcal{G}}(x_1)$ is a known flag-transitive P- or T-geometry;
(b) the action $\overline{G}_1 = G_1/K_1$ is a known flag-transitive automorphism group of $\mathrm{res}_{\mathcal{G}}(x_1)$;
(c) if L_1 is the elementwise stabilizer of the set of points collinear to p, then K_1/L_1 is a known G_1/K_1-admissible representation module of $\mathscr{H}$ (which is the quotient of $V(\mathrm{res}_{\mathcal{G}}(x_1))$ over a G_1/K_1-invariant subgroup.)

We achieve the identification of $\mathscr{A}$ in a number of stages described below.

Stage 1. Bounding the order of G_n.

At this stage (see Chapter 9) we consider the action of G on the derived graph $\Delta = \Delta(\mathscr{G})$ of $\mathscr{G}$. Recall that the vertices of Δ are the elements of type n in $\mathscr{G}$ and two of them are adjacent whenever they are incident to a common element of type $n-1$. Then G_n is the stabilizer of the vertex x_n in this action. We assume that the residue $\text{res}_{\mathscr{G}}(x_1)$ is such that a so-called condition (*) (cf. Section 9.3) holds. Under this condition we are able to bound the number of chief factors in G_n and their orders.

Stage 2. The shape of $\{G_1, G_n\}$.

At this stage we match the structure of G_n against the possible structure of G_1 about which we know quite a lot by the assumptions (a) to (c). An inspection of the list of the known P- and T-geometries (which are candidates for the residue of a point in $\mathscr{G}$) and their flag-transitive automorphism groups shows that either the condition (*) holds (and hence G_n is bounded on stage 1) or the universal representation module is trivial. In the latter case we either exclude the possibility for the residue altogether by Proposition 6 (see the Preface) or bound the number of chief factors in G_1 and G_n. As a result of this stage (to be accomplished in Chapter 10) we obtain a limited number of possibilities for the chief factors of G_1 and G_n which satisfy certain consistency conditions. These possibilities (which we call *shapes*) are given in Table VIIIa and Table VIIIb. These shapes are named by the corresponding known examples if any.

Stage 3. Reconstructing a rank 2 subamalgam.

At this stage we start with a given shape from Table VIII and identify up to isomorphism the amalgam $\mathscr{B} = \{G_1, G_2\}$ or $\mathscr{X} = \{G_n, G_{n-1}\}$. In the former case we call our strategy *direct* and in the latter we call it *dual*. Let us first discuss the direct strategy. From stage 2 we know the chief factors of G_1. These factors normally leave us with a handful of possibilities for the isomorphism types of G_1 which depend on whether or not certain extensions split. We need to identify $\mathscr{B} = \{G_1, G_2\}$ up to isomorphism. First we determine the *type* of $\mathscr{B}$. By this we understand the identification of G_1 and G_2 up to isomorphism and the specification of $G_{12} = G_1 \cap G_2$ in G_1 and G_2 up to conjugation in the automorphism group of G_1 and G_2, respectively. Since the action $\overline{G}_1$ of G_1 on $\mathrm{res}_{\mathscr{G}}(x_1)$ is known by assumption (a), the subgroup G_{12} of G_1 is determined uniquely up to conjugation. Now for G_2 we should consider all the groups containing G_{12} as a subgroup of index 3. Towards this end we consider the kernel K_2^- of the action of G_2 on the point-set of x_2 (which is clearly the largest normal subgroup of G_2 contained in G_{12}). It can be shown that G_2/K_2^- is always isomorphic to Sym_3 and hence we should take for K_2^- a subgroup of index 2 in G_{12} (there is always a very limited number of such choices). Next we calculate the automorphism group of K_2^-. Often the existence of the required automorphisms (of order 3) of K_2^- imposes some further restrictions on the structure of G_1 which specify G_1 up to isomorphism. After the type of $\mathscr{B}$ is determined we apply Goldschmidt's theorem (8.3.2) to classify such amalgams up to isomorphism.

Within the dual strategy $K_{n-1}^+ = G_{n,n-1}$ is a uniquely determined (up to conjugation) subgroup of G_n and G_{n-1} contains K_{n-1}^+ with index 2.

Stage 4. Reconstructing the whole amalgam $\mathscr{A}$.

Here we start with the rank 2 subamalgam $\mathscr{B} = \{G_1, G_2\}$ or $\mathscr{X} = \{G_n, G_{n-1}\}$ reconstructed on stage 3 and identify up to isomorphism the whole amalgam $\mathscr{A}$. If we follow the direct strategy then as soon as we know that $\mathscr{B} = \{G_1, G_2\}$ is isomorphic to the similar amalgam coming from a known example, we have finished by (8.6.1). In the case of the dual (or mixed) strategy we apply *ad hoc* arguments based on (8.4.2), (8.4.3), (8.5.1), similar to those used in the proof of (8.6.1).

8.2 Some cohomologies

In this section we summarize the information on first and second cohomology groups to be used in the subsequent sections. If G is a group and V is a $GF(2)$-module for G, then $H^1(G, V)$ and $H^2(G, V)$ denote the first and the second cohomology groups of V (cf. Section 15.7 in [H59]). It is known that each of these groups carries a structure of a $GF(2)$-vector space, in particular it is an elementary abelian 2-group. The importance of these groups is due to the following two well known results (cf. (17.7) in [A86] and Theorem 15.8.1

in [H59], respectively). Another application of the first cohomology is (2.8.2).

Proposition 8.2.1 *If $S = V : G$ is the semidirect product of V and G with respect to the natural action, then the number of conjugacy classes of complements to V in S is equal to the order of $H^1(G, V)$. In particular all the complements are conjugate if and only if $H^1(G, V)$ is trivial.* □

Proposition 8.2.2 *The number of isomorphism types of groups S which contain a normal subgroup N, such that $S/N \cong G$ and N is isomorphic to V as a G-module, is equal to the order of $H^2(G, V)$. In particular every extension of V by G splits (isomorphic to the semidirect product of V and G) if and only if $H^2(G, V)$ is trivial.* □

Let us explain the notation used in Table VI. By V_n we denote the natural module of $SL_4(2) \cong Alt_5$ or $\Sigma L_4(2) \cong Sym_5$, considered as a $GF(2)$-module (notice that the action on the non-zero vectors is transitive). By V_o we denote the orthogonal module of $\Omega_4^-(2) \cong Alt_5$ or $O_4^-(2) \cong Sym_5$. The orthogonal module is also the heart of the $GF(2)$-permutation module on 5 points. By V_s we denote the natural 4-dimensional symplectic module for $Sym_6 \cong S_4(2)$ (or for $Alt_6 = S_4(2)'$) and of dimension 6 for $S_6(2)$. As usual $\mathcal{C}_{11}$ and $\overline{\mathcal{C}}_{11}$ denote the irreducible Golay code and Todd modules for M_{24} while $\mathcal{C}_{10}$ and $\overline{\mathcal{C}}_{10}$ denote the irreducible 10-dimensional Golay code and Todd modules for $\mathrm{Aut}\, M_{22}$ or M_{22}.

The dimensions of the first and second cohomology groups in Table VI were calculated by Derek Holt (whose cooperation is greatly appreciated) using his share package 'cohomolo' for GAP [GAP]. Most (if not all) of the dimensions were known in the literature. The first cohomologies of the modules V_n, V_o and V_s are given in [JP76] and in [Pol71]. The dimensions of $H^1(M_{24}, \mathcal{C}_{11})$ and $H^1(M_{24}, \overline{\mathcal{C}}_{11})$ have been calculated in Section 9 of [Gri74]. The first cohomology of $\overline{\mathcal{C}}_{10}$ is given in (22.7) in [A97]. The second cohomology of V_n and the non-triviality of $H^2(V_s, Sym_6)$ are Theorems 2 and 3 in [Gri73] (the latter theorem is attributed to J. McLaughlin). The triviality of $H^2(M_{24}, \mathcal{C}_{11})$ is stated in [Th79] (with a reference to the PhD Thesis of D. Jackson.) Since a maximal 2-local subgroup in the Fischer sporadic simple group Fi_{24}' is a non-split extension of $\overline{\mathcal{C}}_{11}$ by M_{24}, we know that $H^2(M_{24}, \overline{\mathcal{C}}_{11})$ must be non-trivial by (8.2.2).

Table VI. Cohomologies of some modules

G	V	$\dim V$	$\dim H^1(G, V)$	$\dim H^2(G, V)$
Alt_5	V_n	4	2	0
Sym_5	V_n	4	1	0
Alt_5	V_o	4	0	0
Sym_5	V_o	4	0	0
Alt_6	V_s	4	1	0
Sym_6	V_s	4	1	1
$S_6(2)$	V_s	6	1	1
M_{22}	$\mathcal{C}_{10}$	10	1	0
$\mathrm{Aut}\, M_{22}$	$\mathcal{C}_{10}$	10	1	1
M_{22}	$\overline{\mathcal{C}}_{10}$	10	0	0
$\mathrm{Aut}\, M_{22}$	$\overline{\mathcal{C}}_{10}$	10	0	0
M_{24}	$\mathcal{C}_{11}$	11	0	0
M_{24}	$\overline{\mathcal{C}}_{11}$	11	1	1

The situation described in the first and second rows of Table VI deserves further attention

Lemma 8.2.3 *Let $A \cong Alt_5 \cong SL_2(4)$ and $V = V_n$ be the natural module of A treated as a 4-dimensional $GF(2)$-module. Let $P = V : A$ be the semidirect product with respect to the natural action. Let S be a subgroup of* $\mathrm{Aut}\, P$ *containing* $\mathrm{Inn}\, P$ *(where the latter is identified with P). Then P is isomorphic to a maximal parabolic in $PSL_3(4)$ and*

(i) *P contains exactly four classes of complements to V and* $\mathrm{Out}\, P \cong Sym_4$ *acts faithfully on these classes;*

(ii) *if S/P is generated by a transposition then S is the semidirect product of V and Sym_5; S contains two classes of complements and it is isomorphic to a maximal parabolic in $P\Sigma L_3(4)$;*

(iii) *if S/P is generated by a fixed-point free involution then S is the semidirect product with A of an indecomposable extension $V^{(1)}$ of V by a 1-dimensional module; S contains two classes of complements to $V^{(1)}$;*

(iv) *if $S/P \cong 3$ then S is isomorphic to a maximal parabolic in $PGL_3(4)$;*

(v) *if S/P is the Kleinian four group then S is the semidirect product with A of an indecomposable extension $V^{(2)}$ of V by a 2-dimensional trivial module; S contains a single class of complements and the dual of $V^{(2)}$ is the universal representation module of $\mathcal{G}(Alt_5)$;*

(vi) *if $S/P \cong 2^2$ and contains a transposition then S is the semidirect product of $V^{(1)}$ and Sym_5 containing two classes of complements;*

(vii) *if $S/P \cong 4$ then S is a non-split extension of $V^{(1)}$ by Sym_5;*

(viii) *if $S/P \cong Sym_3$ then S is isomorphic to a maximal parabolic in $P\Gamma L_3(4)$;*

(ix) *if $S/P \cong D_8$ then S is the semidirect product of $V^{(2)}$ and Sym_5;*

(x) *if $S/P \cong Alt_4$ or Sym_4 then S is the semidirect product of $V^{(2)}$ (isomorphic to the hexacode module) and $Alt_5 \times 3$ or $(Alt_5 \times 3).2$ (considered as a subgroup of $3 \cdot Sym_6$).* □

Let $T \cong 3 \cdot Sym_6$ and V_h be the hexacode module of T. Since $Y = O_3(T)$ is of order 3 acting fixed-point freely on V_h, we immediately obtain the following.

Lemma 8.2.4 *$H^k(3 \cdot Sym_6, V_h)$ is trivial for $k = 1$ and 2.* □

The following result is deduced from Table I in [Bel78] (see also [Dem73]).

Proposition 8.2.5 *Let $d = \dim H^k(L_n(2), \bigwedge^i V)$, where $k = 1$ or 2, $1 \le i \le n-1$ and V is the natural module of $L_n(2)$. Then one of the following holds:*

(i) *$d = 0$;*

(ii) *$d = 1$ and the triple (n, i, k) is one of the following: $(3,1,1)$, $(3,2,1)$, $(3,1,2)$, $(3,2,2)$, $(4,2,1)$, $(4,1,2)$, $(4,3,2)$, $(5,1,2)$, $(5,4,2)$.* □

The standard reference for the next result is [JP76].

Lemma 8.2.6 *Let V_s be the natural $2n$-dimensional symplectic module of $S_{2n}(2)$. Then $\dim H^1(S_{2n}(2), V_s) = 1$.* □

Notice that the unique indecomposable extension of the trivial 1-dimensional module by V_s is the natural orthogonal module of $S_{2n}(2) \cong \Omega_{2n+1}(2)$.

Lemma 8.2.7 *The following assertions hold:*

(i) $H^1(Co_1, \overline{\Lambda}^{(24)})$ *is trivial;*
(ii) $H^1(Co_2, \overline{\Lambda}^{(22)})$ *is 1-dimensional.*

Proof. (i) Let $G = Co_1$ and $V = \overline{\Lambda}^{(24)}$. Since V is self-dual, by (2.8.2) we have $\dim H^1(G, V) = \dim C_{V^d}(G)$, where V^d is the largest indecomposable extension of a trivial module by V. Let $\widetilde{V}$ be an indecomposable extension of the 1-dimensional (trivial) module by V. Let φ be the mapping which turns V into a representation module of $\mathscr{G}(Co_1)$, Φ be the image of φ and $\widetilde{\Phi}$ be the preimage of Φ in $\widetilde{V}$. Since the stabilizer in G of a point from $\mathscr{G}(Co_1)$ (isomorphic to $2^{11} : M_{24}$) does not contain subgrous of index 2, G has two orbits in $\widetilde{\Phi}$. Then the hypothesis of (2.8.1) hold and $\widetilde{V}$ must be a representation module of $\mathscr{G}(Co_1)$, but since V is already universal by (5.3.2), (i) follows.

(ii) Since $\overline{\Lambda}^{(23)}$ is an indecomposable extension of the trivial module by $\overline{\Lambda}^{(22)}$, and $\overline{\Lambda}^{(22)}$ is self-dual $H^1(Co_2, \overline{\Lambda}^{(22)})$ is non-trivial. Put $V = \overline{\Lambda}^{(23)}$, $G = Co_2$ and let φ be the mapping which turns V into the universal representation module of $\mathscr{G}(Co_2)$ (compare 5.2.3 (v)) and let Φ be the image of φ. Let $\widetilde{V}$ be an indecomposable extension of the 1-dimensional module by V and $\widetilde{\Phi}$ be the preimage of Φ in $\widetilde{V}$. In this case the point stabilizer contains a subgroup of index 2, so in principal G could act transitively on $\widetilde{\Phi}$. Suppose this is the case. Then for $\widetilde{v} \in \widetilde{\Phi}$ we have $G(\widetilde{v}) \cong 2^{10} : M_{22}$. Let Ξ be the point-set of a $\mathscr{G}(S_6(2))$-subgeometry $\mathscr{S}$ in $\mathscr{G}(Co_2)$ so that $|\Xi| = 63$ and the setwise stabilizer S of Ξ in G is of the form $2^{1+8}_+.S_6(2)$ (compare (5.2.1)). We identify Ξ with its image under φ and let $\widetilde{\Xi}$ be the preimage of Ξ in $\widetilde{V}$. Let $\widetilde{v} \in \widetilde{\Xi}$, then on one hand

$$S(\widetilde{v}) \cong 2^{10}.2^4.Alt_6 < 2^{10}.M_{22},$$

is the stabilizer in $G(\widetilde{v})$ of a $\mathscr{G}(S_4(2))$-subgeometry in $\mathscr{G}(M_{22})$. On the other hand, $S(\widetilde{v})$ is a subgroup of index 2 in the stabilizer in S of a point from $\mathscr{S}$ and hence

$$S(\widetilde{v}) \cong 2^{1+8}_+.2^5.Alt_6,$$

which shows that $S(\widetilde{v})$ contains $O_2(S)$ and hence the latter is in the kernel of the action of S on $\widetilde{\Xi}$. Thus the submodule $\widetilde{W}$ in $\widetilde{V}$ generated by the vectors from $\widetilde{\Xi}$ is a module for $S_6(2) = S/O_2(S)$ with an orbit of length

$126 = |\tilde{\Xi}|$ on the non-zero vectors. On the other hand, it is easy to deduce from the proof of (5.2.3) that the submodule W in V generated by the vectors from Ξ is the universal (7-dimensional orthogonal) representation module of $\mathcal{S}$. By Table VI, W is the largest extension of a trivial module by the 6-dimensional symplectic module V_s for $S_6(2)$. Hence $\widetilde{W} = W \oplus U$ for a 1-dimensional module U and there are no S-orbits of length 126, which is a contradiction. Now arguing as in case (i) we complete the proof. □

We will widely use the following theorem due to Gaschütz (cf. Theorem 15.8.6 in [H59] or (10.4) in [A86]).

Theorem 8.2.8 *(Gaschütz' theorem) Let G be a group, p be a prime, V be an abelian normal p-subgroup in G, and S be a Sylow p-subgroup in G. Then G splits over V if and only if P splits over V.* □

In terms of cohomologies the above result states that $H^2(G/V, V)$ is trivial if and only if $H^2(P/V, V)$ is trivial. In fact this is an important consequence of Gaschütz' theorem which establishes an isomorphism between $H^2(G/V, V)$ and $H^2(P/V, V)$ (cf. Theorem 15.8.5 in [H59]).

Lemma 8.2.9 *Let G be a group and V be a $GF(2)$-module for G where the pair (G, V) is either from Table VI, except for (Alt_5, V_n), or one of the pairs $(Co_1, \overline{\Lambda}^{(24)})$, $(Co_2, \overline{\Lambda}^{(22)})$. Then the action of G on V is absolutely irreducible.*

Proof. This is all well known and easy to check. In fact, in each case there is a vector $v \in V^{\#}$ such that x is the only non-zero vector in V fixed by $G(x)$. □

Notice that Alt_5 preserves a $GF(4)$ structure on its natural module V_n.

8.3 Goldschmidt's theorem

In this section we discuss the conditions under which two rank 2 amalgams are isomorphic.

Let $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{A}' = \{A_1', A_2'\}$ be two amalgams, where $B = A_1 \cap A_2$ and $B' = A_1' \cap A_2'$; $*_i$ and $*_i'$ are the group product operations in A_i and A_i', respectively, for $i = 1$ and 2. Recall that an *isomorphism* of $\mathcal{A}$ onto $\mathcal{A}'$ is a bijection φ of

$$A_1 \cup A_2 \text{ onto } A_1' \cup A_2',$$

which maps A_i onto A'_i and such that the equality

$$\varphi(x *_i y) = \varphi(x) *'_i \varphi(y)$$

holds whenever $x, y \in A_i$ for $i = 1$ or 2. Equivalently, the restrictions φ_{A_1} and φ_{A_2} of φ to A_1 and A_2 are isomorphisms onto A'_1 and A'_2, respectively.

We say that the amalgams $\mathscr{A}$ and $\mathscr{A}'$ as described above have the same *type* if for $i = 1$ and 2 there is an isomorphism $\psi^{(i)}$ of A_i onto A'_i such that $\psi^{(i)}(B) = B'$. The pair $\pi = (\psi^{(1)}, \psi^{(2)})$ of such isomorphisms will be called the *type preserving pair*. Being of the same type is certainly an equivalence relation.

If φ is an isomorphism of $\mathscr{A}$ onto $\mathscr{A}'$ then clearly $(\varphi_{A_1}, \varphi_{A_2})$ is a type preserving pair. On the other hand, it is easy to see that the type of $\mathscr{A}$ is determined by

(1) the choice of A_1 and A_2 up to isomorphism, and
(2) the choice of B as a subgroup in A_1 and A_2 up to conjugation in the automorphism groups of A_1 and A_2, respectively.

As an illustration we present an example of a pair of non-isomorphic amalgams which are of the same type.

Let $P \cong Sym_8$ act as the automorphism group on the complete graph Γ on 8 vertices and let $\mathscr{P} = \{P_1, P_2\}$ be the amalgam formed by the stabilizers in P of two distinct (adjacent) vertices x and y. Then

$$P_1 \cong P_2 \cong Sym_7 \text{ and } B \cong Sym_6.$$

Let $P' \cong U_3(5) : 2$ act as the automorphism group on the Hoffman–Singleton graph Γ' (cf. [BCN89]) and let $\mathscr{P}' = \{P'_1, P'_2\}$ be the amalgam formed by the stabilizer in P' of two adjacent vertices x' and y' of Γ'. Then

$$P'_1 \cong P'_2 \cong Sym_7 \text{ and } B' \cong Sym_6.$$

Since the subgroups in Sym_7 isomorphic to Sym_6 form a single conjugacy class, it is clear that the amalgams $\mathscr{P}$ and $\mathscr{P}'$ have the same type. On the other hand, these amalgams are not isomorphic for the following reason.

Let $g \in P$ be an element which exchanges the vertices x and y and $g' \in P'$ be an element which exchanges x' and y'. Then g conjugates P_1 onto P_2 and vice versa while g' does the same with P'_1 and P'_2. Since the setwise stabilizer of $\{x, y\}$ in P is $Sym_6 \times 2$, g can be chosen to centralize B. On the other hand, the setwise stabilizer of $\{x', y'\}$ in P' is $\mathrm{Aut}\, Sym_6$, so g' always induces an outer automorphism of $B' \cong Sym_6$. Since Sym_7 has a unique faithful permutation representation of degree 7 the cyclic type of an element from Sym_7 is well defined (unlike the cyclic type of an element of Sym_6). By the above, a transposition from P_1, which is contained in B, is also a transposition in P_2 while a transposition from P_1', which is contained in B', is a product of three disjoint transpositions in P_2'. This shows that $\mathscr{P}$ and $\mathscr{P}'$ cannot possibly be isomorphic. (Here we have used the well-known fact that if we fix a degree 6 faithful permutation representation of Sym_6 then the image of a transposition under an outer automorphism is a product of three disjoint transpositions.)

It is clear (at least in principle) how to decide whether or not two amalgams have the same type. In the remainder of this section we discuss how to classify the amalgams of a given type up to isomorphism.

We may notice from the above example that the existence of non-isomorphic amalgams of the same type is somehow related to 'outer' automorphisms of the Borel subgroup B. We are going to formalize this observation.

Let $\mathscr{A} = \{A_1, A_2\}$ and $\mathscr{A}' = \{A_1', A_2'\}$ be two amalgams of the same type and let $\pi = (\psi^{(1)}, \psi^{(2)})$ be the corresponding type preserving pair. If the restrictions $\psi_B^{(1)}$ and $\psi_B^{(2)}$ of $\psi^{(1)}$ and $\psi^{(2)}$ to B coincide, then clearly there is an isomorphism φ of $\mathscr{A}$ onto $\mathscr{A}'$ such that $\psi^{(i)} = \varphi_{A_i}$ for $i = 1$ and 2. In general

$$\delta(\pi) = (\psi_B^{(2)})^{-1}\psi_B^{(1)}$$

is an element of $D = \mathrm{Aut}\, B$.

Let $\chi^{(1)}$ and $\chi^{(2)}$ be automorphisms of A_1 and A_2, respectively, that normalize B. Then

$$\pi' = (\psi^{(1)}\chi^{(1)}, \psi^{(2)}\chi^{(2)})$$

is another type preserving pair and

$$\delta(\pi') = (\chi_B^{(2)})^{-1}\delta(\pi)\chi_B^{(1)},$$

where $\chi_B^{(i)}$ (the restriction of $\chi^{(i)}$ to B) is an element of the subgroup D_i in D which is the image of the normalizer of B in $\mathrm{Aut}\, A_i$ (under the natural mapping). Notice that by the definition every element of D_i is of the form $\chi_B^{(i)}$ for a suitable $\chi^{(i)} \in N_{\mathrm{Aut}\, A_i}(B)$.

Lemma 8.3.1 *In the above terms $\mathscr{A}$ and $\mathscr{A}'$ are isomorphic if and only if $\delta(\pi) \in D_2D_1$.*

Proof. Suppose first that $\delta(\pi) = d_2d_1$, where $d_i \in D_i$ for $i = 1,2$. Choose $\chi^{(i)} \in N_{\mathrm{Aut}\, A_i}(B)$ so that $d_1^{-1} = \chi_B^{(1)}$ and $d_2 = \chi_B^{(2)}$. Then for the type preserving pair $\pi' = (\psi^{(1)}\chi^{(1)}, \psi^{(2)}\chi^{(2)})$ the automorphism $\delta(\pi')$ is trivial, which proves the required isomorphism between the amalgams.

Now if φ is an isomorphism of $\mathscr{A}$ onto $\mathscr{A}'$, then for the type preserving pair $\varepsilon = (\varphi_{A_1}, \varphi_{A_2})$ the automorphism $\delta(\varepsilon)$ is trivial. On the other hand, $\chi^{(i)} = (\psi^{(i)})^{-1}\varphi_{A_i}$ is an automorphism of A_i normalizing B and as we have seen above

$$\delta(\pi) = (\chi_B^{(2)})^{-1}\delta(\varepsilon)\chi_B^{(1)},$$

hence the proof. □

The next proposition which is a direct consequence of (8.3.1) is known as Goldschmidt's theorem (cf. (2.7) in [Gol80]).

Proposition 8.3.2 *(Goldschmidt's theorem) Let $\mathscr{A} = \{A_1, A_2\}$ be a rank two amalgam, where $B = A_1 \cap A_2$ is the Borel subgroup. Let $D = \mathrm{Aut}\, B$ and let D_i be the image in D of $N_{\mathrm{Aut}\, A_i}(B)$ for $i = 1$ and 2. Then a maximal set of pairwise non-isomorphic amalgams having the same type as $\mathscr{A}$ is in a natural bijection with the set of double cosets of the subgroups D_1 and D_2 in D.* □

Since both D_1 and D_2 contain the inner automorphisms of B the double cosets of D_1 and D_2 in D are in a bijection with the double cosets of O_1 and O_2 in O where $O = \mathrm{Out}\, B$ and O_i is the image of D_i in O for $i = 1$ and 2.

If $\mathscr{B} = \{Sym_7, Sym_7\}$ is the amalgam from the above example, then $O = \mathrm{Out}\, Sym_6$ is of order 2 while both O_1 and O_2 are trivial. Hence there are two double cosets and $\{\mathscr{B}, \mathscr{B}'\}$ is the complete list of pairwise non-isomorphic amalgams of the given type.

In fact (8.3.2) is a very general principle which classifies the ways to 'amalgamate' two algebraic or combinatorial systems of an arbitrary nature over isomorphic subobjects. Exactly the same argument works and gives the same result (compare [Th81] and [KL98]). Of course in the general case there is no such thing as an inner automorphism.

8.4 Factor amalgams

Let $\mathscr{A} = \{A_i \mid 1 \le i \le n\}$ be an amalgam of rank n and M be a normal subgroup in $\mathscr{A}$. This means that M is a subgroup in the Borel subgroup $B = \cap_{i=1}^n A_i$ which is normal in A_i for every $1 \le i \le n$. Then we can construct the *factor amalgam*

$$\overline{\mathscr{A}} = \mathscr{A}/M = \{A_i/M \mid 1 \le i \le n\}$$

whose elements are the cosets of M in A_i for all $1 \leq i \leq n$ and group operations are defined in the obvious way. Notice that the universal completion $(U(\overline{\mathscr{A}}), \overline{\nu})$ of $\overline{\mathscr{A}}$ is a completion of $\mathscr{A}$ which is the quotient of $(U(\mathscr{A}), \nu)$ over the subgroup $\nu(M)$. More generally, for every completion (G, φ) of $\mathscr{A}$ we can construct its quotient over $\varphi(M)$, which is a completion of $\overline{\mathscr{A}}$. We are interested in the following situation:

Hypothesis A. *Let $\mathscr{A} = \{A_i \mid 1 \leq i \leq n\}$ be an amalgam, M be a normal subgroup in $\mathscr{A}$ and $\overline{\mathscr{A}} = \mathscr{A}/M$ be the corresponding factor amalgam. Suppose further that $(\overline{G}, \overline{\varphi})$ is a faithful completion of $\overline{\mathscr{A}}$; (G_1, φ_1) and (G_2, φ_2) are faithful completions of $\mathscr{A}$ such that $(\overline{G}, \overline{\varphi})$ is the quotient of (G_1, φ_1) and (G_2, φ_2) over $\varphi_1(M)$ and $\varphi_2(M)$, respectively.*

We consider the above completions as being quotients of the universal completion $(U(\mathscr{A}), \nu)$ of $\mathscr{A}$. Since the (G_j, φ_j) are assumed to be faithful, the universal completion is faithful. In order to simplify the notation we identify M with $\nu(M)$. Let K_1, K_2 and K be the kernels of the natural homomorphisms of $U(\mathscr{A})$ onto G_1, G_2 and $\overline{G}$, respectively. Then (G_1, φ_1) and (G_2, φ_2) are isomorphic if and only if $K_1 = K_2$.

Lemma 8.4.1 *Under Hypothesis A we have*

(i) $K = K_1 M = K_2 M$;
(ii) $K_1 \cap M = K_2 \cap M = 1$.

Proof. (i) follows from the assumption that $(\overline{G}, \overline{\varphi})$ is a quotient of (G_j, φ_j) for $j = 1, 2$, while (ii) holds since the (G_j, φ_j) are faithful. □

Lemma 8.4.2 *Under Hypothesis A if the centre of M is trivial, then the completions (G_1, φ_1) and (G_2, φ_2) are isomorphic.*

Proof. By (8.4.1) for $j = 1$ and 2 the subgroups K_j and M are disjoint normal subgroups in $U(\mathscr{A})$, hence they centralize each other. Hence for $i = 1$ and 2 the subgroup K_i is a complement to $Z(M)$ in $C_K(M)$. If $Z(M) = 1$ then clearly $K_1 = C_K(M) = K_2$ and the proof follows. □

By the above lemma the centre $Z = Z(M)$ of M deserves a further study. In view of Hypothesis A we can define an action of $\overline{G}$ on Z which coincides with the action of G_j on the centre of $\varphi_j(M)$ (here $\varphi_j(M)$ identified with M) by conjugation for $j = 1$ and 2.

Suppose that $K_1 \neq K_2$, then $K_1/(K_1 \cap K_2)$ is isomorphic to a non-trivial subgroup N in Z which is normalized by the action of $\overline{G}$ on $Z(M)$. Let $(\widehat{G}, \widehat{\varphi})$ be the completion of $\mathscr{A}$ which is the quotient of $(U(\mathscr{A}), \nu)$ over the normal subgroup $(K_1 \cap K_2)M$. Then $(\widehat{G}, \widehat{\varphi})$ is a completion of $\overline{\mathscr{A}}$ and $(\overline{G}, \overline{\varphi})$ is its quotient over the subgroup $\widehat{N} = K/(K_1 \cap K_2)M$, isomorphic to N.

Lemma 8.4.3 *Under Hypothesis A either (G_1, φ_1) and (G_2, φ_2) are isomorphic or there is a non-trivial subgroup N in the centre of M normalized by the action of $\overline{G}$ and a completion $(\widehat{G}, \widehat{\nu})$ of $\overline{\mathscr{A}}$ such that there is a normal subgroup $\widehat{N}$ in $\widehat{G}$ isomorphic to N and the isomorphism commutes with the action of $\overline{G} = \widehat{G}/\widehat{N}$; $(\overline{G}, \overline{\varphi})$ is the quotient of $(\widehat{G}, \widehat{\varphi})$ over $\widehat{N}$.*

8.5 $L_3(2)$-lemma

In this section we apply the technique developed in the previous section to a particular situation, which is important for establishing uniqueness of the rank 3 amalgam $\mathscr{C} = \{G_1, G_2, G_3\}$ when the rank 2 amalgam $\mathscr{B} = \{G_1, G_2\}$ is given and satisfies certain properties.

When the amalgam $\mathscr{B}$ is given (usually it is isomorphic to the amalgam associated to a known example) we can indicate G_{13} and G_{23} inside G_1 and G_2, respectively, by considering the actions of G_1 and G_2 on the corresponding residues $\mathrm{res}_{\mathscr{G}}(x_1)$ and $\mathrm{res}_{\mathscr{G}}(x_2)$. The residue $\mathrm{res}^{-}_{\mathscr{G}}(x_3)$ is a projective plane of order 2 on which G_3 induces $L_3(2)$ with kernel K_3^- (so that K_3^- is the largest subgroup in G_{123} normal in both G_{13} and G_{23}). This enables us first to indicate K_3^- and then put $G_{i3} = N_{G_i}(K_3^-)$ for $i = 1$ and 2. Since G_{13} and G_{23} are the maximal parabolics associated with the action of G_3 on $\mathrm{res}_{\mathscr{G}}(x_3)$ we have

$$G_{13}/K_3^- \cong G_{23}/K_3^- \cong Sym_4.$$

Let $\mathscr{D} = \{G_{13}, G_{23}\}$, $\widetilde{G}_3$ be the universal completion of $\mathscr{D}$ and $\psi : \widetilde{G}_3 \to G_3$ be the natural homomorphism. In order to establish the uniqueness of $\mathscr{C}$ we need to show that the kernel K of ψ is uniquely determined. Since both K_3^- and K are normal subgroups in $\widetilde{G}_3$ and the restriction of ψ to K_3^- is an isomorphism, $K \leq C_{\widetilde{G}_3}(K_3^-)$.

Lemma 8.5.1 *Using the above terms suppose that $C_{G_{i3}}(K_3^-) = 1$ for $i = 1$ and 2. Then $K = C_{\widetilde{G}_3}(K_3^-)$, in particular, K is uniquely determined.*

Proof. The result follows from the observation that $L_3(2) \cong G_3/K_3^-$ is simple and hence by the hypothesis $C_{G_3}(K_3^-) = 1$. □

Now suppose that $Z = Z(K_3^-)$ is non-trivial. If there are two possible kernels K and K', say, we consider the group

$$\widehat{G}_3 = \widetilde{G}_3/(K \cap K')K_3^-,$$

which is generated by the image $\widehat{\mathscr{D}} = \{D_1, D_2\}$ of the amalgam $\mathscr{D}$ in $\widehat{G}_3$. Then $\widehat{\mathscr{D}}$ is the amalgam of maximal parabolics in $L_3(2)$ associated with its action on the projective plane of order 2. We formulate the uniqueness criterion in the follow proposition.

Proposition 8.5.2 *Let $\mathscr{B} = \{G_1, G_2\}$ be a rank 2 amalgam and K_3^- be a subgroup in $G_{12} = G_1 \cap G_2$. For $i = 1$ and 2 put $G_{i3} = N_{G_i}(K_3^-)$. Let D_i be the image in $\mathrm{Out}\, K_3^-$ of G_{i3} and $D = \langle D_1, D_2 \rangle$. Suppose that the following conditions (i) to (iv) hold.*

(i) $C_{G_{i3}}(K_3^-) \leq K_3^-$ *for $i = 1$ and 2;*
(ii) $D \cong L_3(2)$ *and $\widehat{\mathscr{D}} = \{D_1, D_2\}$ is the amalgam of maximal parabolics associated with the action of D on the projective plane of order 2;*
(iii) *the centre Z of K_3^- is a 2-group;*
(iv) *each chief factor of $\widehat{G}_3$ inside Z is either the trivial 1-dimensional or the 3-dimensional natural module for D (or its dual).*

Then there exists at most one homomorphism ψ of the universal completion $\widetilde{G}_3$ of $\{G_{13}, G_{23}\}$ such that the restriction of ψ to K_3^- is a bijection and $\psi(\widetilde{G}_3)/\psi(K_3^-) \cong L_3(2)$.

Proof. By (8.4.3) it is sufficient to show that the amalgam $\widehat{\mathscr{D}}$ does not possess a completion $\widehat{G}_3$ such that $\widehat{G}_3/O_2(\widehat{G}_3) \cong D \cong L_3(2)$ and $O_2(\widehat{G}_3)$ is isomorphic to a D-invariant subgroup Y in Z. Since $\widehat{\mathscr{D}}$ maps isomorphically onto its image in $\widehat{G}_3/O_2(\widehat{G}_3)$, such a group $\widehat{G}_3$ must split over $O_2(\widehat{G}_3)$ by (8.2.8) and hence it is isomorphic to a semidirect product of Y and $D \cong L_3(2)$. Thus it is sufficient to show that in such a semidirect product $Y : D$ every subamalgam that is isomorphic to $\widehat{\mathscr{D}}$ generates a complement to Y (isomorphic to $L_3(2)$). Furthermore, we may assume that Y is elementary abelian and irreducible as a module for D. Indeed, otherwise we take Y_1 to be the largest D-invariant subgroup in Y and consider the semidirect product $(Y/Y_1) : D$ which again must be a completion of $\widehat{\mathscr{D}}$. By (iv) up to isomorphism there are just two groups to be considered: $2 \times L_3(2)$ and $2^3 : L_3(2)$. These cases are dealt with in the next lemma (8.5.3). □

Lemma 8.5.3 *Let $D \cong L_3(2)$ and $\widehat{\mathscr{D}} = \{D_1, D_2\}$ be the amalgam of maximal parabolics associated with the action of D on the projective plane of order 2, so that $D_1 \cong D_2 \cong Sym_4$ and $D_1 \cap D_2$ is the dihedral group of order 8. Let $X = Y : D \cong 2^3 : L_3(2)$ be the semidirect product of D with its natural module Y. Then*

(i) *the universal completion of* $\widehat{\mathscr{D}}$ *does not possess non-trivial abelian factor-groups;*

(ii) $2 \times L_3(2)$ *is not a completion of* $\widehat{\mathscr{D}}$*;*

(iii) *every subamalgam in* X *isomorphic to* $\widehat{\mathscr{D}}$ *generates a complement to* Y *in* X*;*

(iv) X *is not a completion of* $\widehat{\mathscr{D}}$.

Proof. It is easy to see that all the involutions in $\widehat{\mathscr{D}}$ are conjugate, which immediately implies (i) and then of course (ii) follows.

Since $H^1(D, Y)$ is 1-dimensional by (8.2.5), X contains two classes of complements to Y. Every complement is generated by a subamalgam isomorphic to $\widehat{\mathscr{D}}$ and the subamalgams generating complements from different classes cannot be conjugate. Hence in order to prove (iii) it is sufficient to show that X (when it acts by conjugation) has on the set of the subamalgams in X isomorphic to $\widehat{\mathscr{D}}$ at most two orbits.

Let $\{D_1, D_2\}$ be a subamalgam in X isomorphic to $\widehat{\mathscr{D}}$. We assume without loss of generality that D_1 centralizes a 1-subspace in Y while D_2 normalizes a 2-subspace. Let $\{\widetilde{D}_1, \widetilde{D}_2\}$ be another subamalgam in X isomorphic to $\widehat{\mathscr{D}}$. Since we classify the subamalgams up to conjugation, we assume that $\{D_1, D_2\}$ and $\{\widetilde{D}_1, \widetilde{D}_2\}$ have the same image in the factor-group X/Y and also that D_2 and $\widetilde{D}_2$ share a subgroup T of order 3. Since $N_X(T) \cong D_{12}$, T is contained in exactly two subgroups isomorphic to Sym_3. Hence in order to prove that there are at most two X-orbits on the set of subamalgams isomorphic to $\widehat{\mathscr{D}}$ it is sufficient to show that the subamalgams under consideration are conjugate whenever D_2 and $\widetilde{D}_2$ share a subgroup Sym_3. Put $A = O_2(D_2)$ and $\widetilde{A} = O_2(\widetilde{D}_2)$. Then $\widetilde{A}$ is contained in the subgroup $C = [YA, T]$ which is an elementary abelian 2-group and if A and $\widetilde{A}$ are distinct, they are the only subgroups in C not contained in Y and invariant under $B := D_1 \cap D_2$. Hence there is an element in $C_Y(B)$ which conjugates A onto $\widetilde{A}$ and hence it conjugates $D_2 = AB$ onto $\widetilde{D}_2 = \widetilde{A}B$. This shows that D_2 and $\widetilde{D}_2$ are conjugate and so we assume that $D_2 = \widetilde{D}_2$.

Since D_2 maps isomorphically onto its image in X/Y, we have $D_1 \cap D_2 = \widetilde{D}_1 \cap \widetilde{D}_2$. Furthermore the intersection is a Sylow 2-subgroup in each of the four subgroups involved. This means $O_2(D_1) = O_2(\widetilde{D}_1)$. Since we also have $N_X(O_2(D_1)) \cong Sym_4 \times 2$, we must have $D_1 = \widetilde{D}_1$. Finally (iv) follows directly from (iii). □

The following lemma shows that the semidirect product of $D \cong L_3(2)$ with the 8-dimensional Steinberg module *is* a completion of $\widehat{\mathscr{D}}$.

Lemma 8.5.4 *Let $X = S : D \cong 2^8 : L_3(2)$ be the semidirect product of D with the irreducible 8-dimensional Steinberg module S for D with respect to the natural action. Let $\widehat{\mathscr{D}} = \{D_1, D_2\}$ be the subamalgam in D as in* (8.5.3). *Let z be the unique non-zero element in S centralized by $D_1 \cap D_2$. Then X is generated by the amalgam $\widehat{\mathscr{D}}' = \{D_1^z, D_2\}$ (which is isomorphic to $\widehat{\mathscr{D}}$).*

Proof. It is well known (cf. [JP76]) that $H^1(D, S)$ is trivial. Hence all the complements to S in X are conjugate. Furthermore, suppose that $\widehat{\mathscr{D}}^*$ is a subamalgam in X such that $\widehat{\mathscr{D}}$ and $\widehat{\mathscr{D}}^*$ have the same image under the natural homomorphism ψ of X onto $X/S \cong D$ and $\widehat{\mathscr{D}}^*$ generates a complement to S in X. Then $\widehat{\mathscr{D}}$ and $\widehat{\mathscr{D}}^*$ are conjugate in X. On the other hand, it is easy to check that $\psi(\widehat{\mathscr{D}}) = \psi(\widehat{\mathscr{D}}')$ but $\widehat{\mathscr{D}}$ and $\widehat{\mathscr{D}}'$ are not conjugate in X. □

Incidentally (8.5.3) resembles Lemma 13.4.7 in [FLM88].

8.6 Two parabolics are sufficient

In this section we prove the following.

Proposition 8.6.1 *Let $\mathscr{G}$ be a P- or T-geometry of rank $n \geq 3$, G be a flag-transitive automorphism group of $\mathscr{G}$ and let*

$$\mathscr{A}(G, \mathscr{G}) = \{G_i \mid 1 \leq i \leq n\}.$$

Let $(\mathscr{H}, H)$ be a pair from Table I or II and let

$$\mathscr{A}(H, \mathscr{H}) = \{H_i \mid 1 \leq i \leq n\}.$$

Suppose that $\mathscr{B} = \{G_1, G_2\}$ is isomorphic to $\{H_1, H_2\}$. Then

$$\mathscr{A}(G, \mathscr{G}) \cong \mathscr{A}(H, \mathscr{H}),$$

in particular $\mathscr{G}$ is a quotient of the universal cover of $\mathscr{H}$.

Proof. We first claim that the subamalgam $\mathscr{D} = \{G_{13}, G_{23}\}$ is uniquely specified in $\mathscr{B}$ up to conjugation by elements of G_{12}. Notice that $\mathscr{D}$ can be defined as the image of $\{H_{13}, H_{23}\}$ under an isomorphism of $\mathscr{A}(H, \mathscr{H})$ onto $\mathscr{A}(G, \mathscr{G})$. To establish the uniqueness, we observe that the subgroups G_{13} and G_{123} in G_1 are specified uniquely by the assumptions (a) and (b) in Section 8.1. Furthermore $G_{23} = \langle G_{123}, Y \rangle$, where Y is a Sylow 3-subgroup of K_2^+, so the claim follows. Notice that K_3^- is now also uniquely determined as the largest subgroup in G_{123} normal in both G_{13} and G_{23}. Now the conditions in (8.5.2) hold because of the isomorphism

$$\mathscr{B} \cong \{H_1, H_2\}$$

and by Proposition 10 in the Preface. Hence the isomorphism type of $\mathscr{C} = \{G_1, G_2, G_3\}$ is uniquely determined by (8.5.2) and coincides with that of $\{H_1, H_2, H_3\}$.

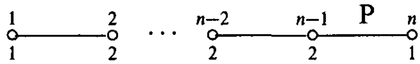
If $n = 3$ then we have finished, so suppose that $n \geq 4$. Since $\mathrm{res}_{\mathscr{G}}(x_4)$ is the projective $GF(2)$-space of rank 3 which is simply connected, by (1.4.6) G_4 is the universal completion of $\{G_{14}, G_{24}, G_{34}\}$. Thus there is a unique way to adjoin G_4 to $\mathscr{C}$. We carry on in a similar manner to adjoin all the remaining maximal parabolics. This effectively shows that the universal completions of $\mathscr{A}(G, \mathscr{G})$, $\mathscr{C}$, $\{H_1, H_2, H_3\}$ and $\mathscr{A}(H, \mathscr{H})$ are pairwise isomorphic. □

9
Action on the derived graph

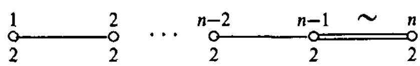
In this chapter we put the first crucial constraint on the structure of the maximal parabolics associated with a flag-transitive action on a Petersen or tilde geometry. The result comes by studying the action of the flag-transitive automorphism group on the derived graph of the corresponding geometry. The derived graph of a P- or T-geometry of rank n is on the set of elements of type n and two vertices are adjacent if they are incident to a common element of type $n-1$.

9.1 A graph theoretical setup

Let $\mathcal{G}$ be a P- or T-geometry of rank $n \geq 2$, so that the diagram of $\mathcal{G}$ is



(if $\mathcal{G}$ is a Petersen type geometry) and



(if $\mathcal{G}$ is a tilde type geometry).

On the diagrams, we indicate the type of the corresponding elements above the nodes. If x is an element of $\mathcal{G}$ then $t(x)$ denotes the type of x, where $1 \leq t(x) \leq n$. In this section it would probably be more convenient to work with the dual of $\mathcal{G}$ in which points, lines and planes are the elements of type n, $n-1$ and $n-2$. But since this might cause confusion with other parts of the book we decided to reserve the names points, lines and planes for elements of type 1, 2 and 3 and to introduce new names for elements of type n, $n-1$ and $n-2$. These elements will be called *vertices*, *links* and *quints*, respectively (the choice of the names will be justified below).

Let $\Delta = \Delta(\mathcal{G})$ be the *derived graph* of $\mathcal{G}$ which is the collinearity graph of the dual of $\mathcal{G}$. In our terms it can be defined in the following way. The vertices of Δ are the elements of type n in $\mathcal{G}$ (therefore we call such elements *vertices*) and two vertices are adjacent if they are joined by a link (incident to a common element of type $n-1$). As we will see shortly, in the case of Petersen type geometries links are the edges of Δ, while in the case of tilde type geometries they are 3-cliques. Since a link is incident to exactly two and three vertices for P- and T-geometries, respectively, it is clear that every link produces an edge or a 3-clique. In (9.1.1) below we will show that this mapping is bijective.

Every element x of $\mathcal{G}$ produces a subgraph $\Sigma[x]$ of Δ. If x is a vertex then $\Sigma[x]$ is the one-vertex subgraph x. For every other type $\Sigma[x]$ can be defined as the subgraph consisting of all the vertices incident to x in which edges are only those defined by the links incident to x. For example, if x is a link then $\Sigma[x]$ is an edge or a 3-clique depending on the type of the geometry. For higher types $\Sigma[x]$ may not be an induced subgraph of Δ, although in the known examples it is usually such. Recall that $\text{res}^+_{\mathcal{G}}(x)$ is the subgeometry of all those $y \in \text{res}_{\mathcal{G}}(x)$ with $t(y) > t(x)$. If $t(x) \leq n-2$ then $\text{res}^+_{\mathcal{G}}(x)$ is a P- or T-geometry of rank $n - t(x)$ and $\Sigma[x]$ is simply the derived graph of that geometry. In particular, it is always connected.

To finish with the basic terminology, the elements of type $n-2$ will be called *quints*. For a quint x, $\Sigma[x]$ is isomorphic to the Petersen graph or the tilde graph (which is the collinearity graph of the geometry $\mathcal{G}(3 \cdot S_4(2))$) depending on the type of the geometry. These subgraphs contain 5-cycles which are crucial for the subsequent arguments. This explains the terminology. Finally, let us note that if x is a vertex, link, or quint then we will apply the same name to the corresponding subgraph $\Sigma[x]$.

Now we are well prepared for our first lemma.

Lemma 9.1.1 *Two vertices are incident with at most one link.*

Proof. Suppose u and v are vertices, $u \neq v$, and suppose x and y are links incident to both u and v. Since $\text{res}_{\mathcal{G}}(u)$ is a projective space, it contains a quint q incident to both x and y. Furthermore, q is incident to v, since $\mathcal{G}$ has a string diagram. It follows that u, v, x and y are all contained in $\text{res}^+_{\mathcal{G}}(q)$, which is the geometry $\mathcal{G}(Alt_5)$ of the Petersen graph or the tilde graph. Hence $x = y$. □

Corollary 9.1.2 *The graph Δ has valency $2^n - 1$ if $\mathcal{G}$ is a Petersen type geometry, and $2(2^n - 1)$ if it is a tilde type geometry. In particular if $t(x) = i$, then*

(i) *the subgraph $\Sigma[x]$ has valency $2^{n-i} - 1$, if $\mathcal{G}$ is of Petersen type;*
(ii) *the subgraph $\Sigma[x]$ has valency $2(2^{n-i} - 1)$, if $\mathcal{G}$ is of tilde type.* □

We will now show that the geometry $\mathcal{G}$ can be recovered from the graph Δ and the set of all subgraphs $\Sigma[x]$, $x \in \mathcal{G}$.

Lemma 9.1.3 $\Sigma[x] \subseteq \Sigma[y]$ *if and only if x is incident to y and $t(x) \geq t(y)$.*

Proof. If $t(x) < t(y)$ then $\Sigma[x]$ cannot be a subgraph of $\Sigma[y]$ by (9.1.2). So without loss of generality we may assume that $t(x) \geq t(y)$. If x is a vertex then the claim follows by definition. If x is a link then the 'if' part follows by definition, while the 'only if' part follows from (9.1.1). Suppose that x is of type at most $n-2$. If x and y are incident then

$$\mathrm{res}^+_{\mathcal{G}}(x) \subseteq \mathrm{res}^+_{\mathcal{G}}(y)$$

and hence $\Sigma[x]$ is a subgraph of $\Sigma[y]$. Suppose now that $\Sigma[x]$ is contained in $\Sigma[y]$. Let v be a vertex of $\Sigma[x]$. Then both x and y are in $\mathrm{res}_{\mathcal{G}}(v)$. Furthermore, since $\Sigma[x]$ is a subgraph of $\Sigma[y]$, (9.1.1) implies that every link incident with x is also incident with y. Restricting this to those links that contain v, we obtain that x, as a subspace of the projective space $\mathrm{res}_{\mathcal{G}}(v)$, is fully contained in the subspace y. Hence x and y are incident. □

Let $\mathcal{S}$ be the set of all subgraphs $\Sigma[x]$, $x \in \mathcal{G}$. Let v be a vertex. Then $\mathrm{res}_{\mathcal{G}}(v)$ is a projective $GF(2)$-space of rank $(n-1)$. We can realize this residue by the set of all proper subspaces in an n-dimensional $GF(2)$-vector space $U = U(v)$ so that the type of an element is its dimension and the incidence is via inclusion. Let $\mathcal{S}(v)$ be the set of subgraphs in $\mathcal{S}$ containing v. Then by (9.1.3) the mapping

$$\sigma : x \mapsto \Sigma[x]$$

is a bijection which reverses the inclusion relation.

The following two lemmas record some of the properties of $\mathcal{S}$.

Lemma 9.1.4 *Suppose v is a vertex of both $\Sigma[x]$ and $\Sigma[y]$. Let $z \in \mathrm{res}_{\mathcal{G}}(v)$ correspond to the span of the subspaces x and y in $U(v)$ (we put $z = v$ if x and y span the whole $U(v)$). In other terms z has the smallest type among* *the elements incident to both x and y. Then the connected component of $\Sigma[x] \cap \Sigma[y]$ that contains v coincides with $\Sigma[z]$.*

Proof. Since $\mathrm{res}_{\mathscr{G}}(v)$ is a projective space, z (defined as in the statement of the lemma) is the unique element in $\mathrm{res}_{\mathscr{G}}(v)$ incident to both x and y, and with $t(z)$ minimal subject to $t(z) \geq \min(t(x), t(y))$. If $z = v$ then v is the entire connected component. So suppose $z \neq v$. Let u be a vertex that is adjacent to v in $\Sigma[x] \cap \Sigma[y]$. Then the link a through v and u (it is unique in view of (9.1.1)) is incident with both x and y. Furthermore, z is the unique element incident to a, x, and y of type minimal subject to $t(z) \geq \min(t(x), t(y))$. Symmetrically, we can now conclude that, in $\mathrm{res}_{\mathscr{G}}(u)$, z corresponds to the span of the subspaces x and y in $\mathrm{res}_{\mathscr{G}}(u)$. Thus, the neighbourhood of u in $\Sigma[x] \cap \Sigma[y]$ coincides with the neighbourhood of u in $\Sigma[z]$. Now the connectivity argument shows that $\Sigma[z]$ is the entire connected component of $\Sigma[x] \cap \Sigma[y]$. □

Lemma 9.1.5 *Every path in Δ of length k, $k \leq n-1$, is contained in $\Sigma[x]$ for some x of type $(n-k)$ or more.*

Proof. We will use induction on k. Clearly, the statement is true if $k = 0$. For the induction step, suppose the statement of the lemma holds for all $i < k$, where $k > 0$. Let $(v_0, v_1, \ldots, v_k)$ be a k-path. By the induction hypothesis, the $k-1$-path $(v_0, v_1, \ldots, v_{k-1})$ is contained in $\Sigma[y]$ for some y of type at least $n-k+1$. In $\mathrm{res}_{\mathscr{G}}(v_{k-1})$, y corresponds to a subspace of dimension at least $n-k+1$ and the link a through v_{k-1} and v_k corresponds to a hyperplane in $U(v_{k-1})$. Thus, both y and a are incident to an element $x \in \mathrm{res}_{\mathscr{G}}(v_{k-1})$ of type at least $n-k$ (the intersection of y and a). Clearly, $\Sigma[x]$ contains the entire path $(v_0, v_1, \ldots, v_{k-1}, v_k)$. □

Remark: It follows from (9.1.4) that there exists a unique element x of maximal type, such that $\Sigma[x]$ is of minimal valency and contains $(v_0, v_1, \ldots, v_{k-1}, v_k)$. Namely, $\Sigma[x]$ will be the connected component containing v_0 of the intersection of all those $\Sigma[y]$ that contain $(v_0, v_1, \ldots, v_{k-1}, v_k)$.

9.2 Normal series of the vertex stabilizer

We start by considering a flag-transitive action of a group G on $\mathscr{G}$. Clearly, G acts on the derived graph Δ. First we introduce some important notation associated with this action.

Let us fix a vertex v (i.e., a vertex of Δ) and let H be the stabilizer of v in G. Let Q be the kernel of H acting on $\mathrm{res}_{\mathscr{G}}(v)$ (recall that the latter is the $GF(2)$-projective space of rank $n-1$). Define a further series of normal subgroups in H as follows. Let $H_i = G_i(v)$, $i \geq 1$, be the joint stabilizer in H of all the vertices at distance at most i from v. (This set of vertices will be denoted by $\Delta_{\leq i}(v)$.) It is clear that in the considered situation we have

$$H_i \leq Q, \quad H_i \trianglelefteq H \quad \text{and} \quad H_{i+1} \leq H_i.$$

Let us explain the relationship between the introduced notation and the notation used throughout this book and introduced in Section 1.1. If $\Phi = \{x_1, ..., x_{n-1}, v = x_n\}$ is a maximal flag in $\mathcal{G}$ and $G_j = G(x_j)$ is the stabilizer of x_i in G for $1 \leq j \leq n$, then $H = G_n$, $Q = K_n$ and $H_1 = L_n$.

By (9.1.3) we know that different elements, say x and y in $\mathcal{G}$, are realized by different subgraphs $\Sigma[x]$ and $\Sigma[y]$. Hence an automorphism of $\mathcal{G}$ which fixes every vertex of Δ acts trivially on the whole $\mathcal{G}$ and hence must be the identity automorphism.

Lemma 9.2.1 *Suppose that a subgroup N is contained in $G_{jn} = G_j \cap G_n$ and normal in both G_j and $G_n = H$ for some $1 \leq j \leq n-1$. Then $N = 1$.*

Proof. Since $\mathcal{G}$ is a geometry and G acts on $\mathcal{G}$ flag-transitively, G_j and G_n generate the whole of G (compare Lemma 1.4.2 in [Iv99]). Hence N is normal in G and since $N \leq G_n$, N fixes the vertex x_n of Δ. Hence N fixes every vertex of Δ and must be trivial by the remark before the lemma. □

When considering more than one vertex at a time we will use the notation $G(v)$ for H, $G_i(v)$ for H_i, and $G_{\frac{1}{2}}(v)$ for Q.

We will first recall the properties of H_i when $\mathcal{G}$ is of rank two, that is, $\mathcal{G}$ is the Petersen graph geometry or the tilde geometry. Recall that if $\mathcal{G}$ is the Petersen graph geometry then $G \cong Sym_5$ or Alt_5, while if $\mathcal{G}$ is the tilde geometry then $G \cong 3 \cdot Sym_6$ or $3 \cdot Alt_6$. The properties summarized in the following lemma can be checked directly.

Lemma 9.2.2 *Suppose $\mathcal{G}$ is of rank two. Then*

(i) *$H/Q \cong Sym_3 \cong L_2(2)$;*
(ii) *Q/H_1 is trivial if $\mathcal{G}$ is the Petersen graph geometry, and it is isomorphic to 2^2 if $\mathcal{G}$ is the tilde geometry;*
(iii) *H_1 is trivial if $G \cong Alt_5$ or $3 \cdot Alt_6$; it has order two if $G \cong Sym_5$ or $3 \cdot Sym_6$;*
(iv) *if $H_1 \neq 1$ and $h \in H_1^{\#}$ then $h \notin G_{\frac{1}{2}}(u)$ for all vertices u adjacent to v;*

(v) *if* $u \in \Delta_2(v)$ *and* a *is a link on* u *then* $\Sigma[a]$ *contains a second (other than* u*) vertex at distance at most two from* v. □

Notice that in the above lemma $H_2 = 1$ in all cases.

Our approach to the classification of geometries $\mathcal{G}$ and their flag-transitive automorphism groups G will be via the study of the factors of the normal series

$$H \trianglerighteq Q \trianglerighteq H_1 \trianglerighteq \ldots \trianglerighteq H_i \trianglerighteq \ldots$$

We will have to bound the length of this series and identify its factors. Clearly, the top factor H/Q is the group induced by H on the $(n-1)$-dimensional projective space $\mathrm{res}_{\mathcal{G}}(v)$ defined over $GF(2)$. Because of the flag-transitivity of H/Q on this residue, by (3.1.1) we have the following

Lemma 9.2.3 *The group* H/Q *is a flag-transitive automorphism group of the projective space* $\mathrm{res}_{\mathcal{G}}(v)$. *In particular, either* $H/Q \cong L_n(2)$, *or* $Frob_7^3$ *(for* $n = 3$*), or* Alt_7 *(for* $n = 4$*).* □

The remaining factors of our series will be shown to be elementary abelian 2-groups, and so we will view them as $GF(2)$-modules for H. In what follows the *natural module* for H is provided by the action of H on the n-dimensional vector space $U = U(v)$ underlying the $(n-1)$-dimensional projective space $\mathrm{res}_{\mathcal{G}}(v)$. That means that the points in $\mathrm{res}_{\mathcal{G}}(v)$ correspond to the 1-subspaces of U while the links in $\mathrm{res}_{\mathcal{G}}(v)$ correspond to the hyperplanes in the natural module of H. Clearly, Q is the kernel of the action of H on its natural module U. Thus, we can also view U as an H/Q-module.

Let us now discuss the group Q/H_1.

Lemma 9.2.4 *Either* $Q = H_1$, *or* $\mathcal{G}$ *is of tilde type,* $Q/H_1 \cong 2^n$, *and, as a module for* H/Q, *the quotient* Q/H_1 *is isomorphic to the natural module* U.

Proof. If $\mathcal{G}$ is of Petersen type then by (9.1.1) the vertices adjacent to v bijectively correspond to the links on v. Hence $Q = H_1$ in this case. Now suppose that $\mathcal{G}$ is of tilde type and Q is strictly larger than H_1. Let $g \in Q$ and let a be a link on v. Since g is in Q, it must stabilize a, and hence it acts on the two points of a other than v. So g^2 fixes both of those points. Since a was arbitrary, $g^2 \in H_1$, which means that Q/H_1 is an elementary abelian 2-group. Consider $V = Q/H_1$ and its dual (as a $GF(2)$-vector space) V^*. By the transitivity of H on the links on v, Q

cannot fix every vertex on a (otherwise, $Q = H_1$). Hence the kernel of the action of Q on the points of a is a subgroup of index two in Q, and hence it corresponds to a non-zero vector v_a^* in V^*. Suppose a, b and c are three links on v, all of them incident to the same quint z. Suppose $g \in Q$ acts trivially on the points of a and b. It follows from (9.2.2 (ii)) that g also fixes all points on c. This means that the vectors v_a^*, v_b^* and v_c^* together with the zero vector form a 2-space in V^*, that is, we have a relation $v_a^* + v_b^* + v_c^* = 0$. It now follows from (3.1.2) that V^* is a quotient of the dual of the natural module U. Finally, since H/Q is transitive on the non-zero vectors of U, we have that U is irreducible, and hence $V \cong U$. □

At the moment, all we can say about the remaining factors, H_i/H_{i+1}, $i \geq 1$, is that they are elementary abelian 2-groups.

Lemma 9.2.5 *The factors H_i/H_{i+1} are elementary abelian 2-groups for all $i \geq 1$.*

Proof. Suppose $g \in H_i$ and $u \in \Delta_{i+1}(v)$ (so that u is at distance $i+1$ from v in Δ). Let w be a vertex at distance $i-1$ from v and at distance 2 from u. By (9.1.5), w and u are contained in $\Sigma[z]$ for a quint z. Since g fixes w and all its neighbours in Δ, we have that g stabilizes $\Sigma[z]$ as a set and hence it acts on it. By (9.2.2 (iii)), g^2 fixes $\Sigma[z]$ vertexwise; in particular, g^2 fixes u. Since u was arbitrary, $g^2 \in H_{i+1}$, and the claim follows. □

In the remainder of this section we will discuss the exceptional cases of H/Q and Q/H_1.

Lemma 9.2.6 *The following assertions hold:*

(i) *$H/Q \not\cong Frob_7^3$;*
(ii) *if $H/Q \cong Alt_7$ then $H_1 = 1$.*

Proof. Suppose first that $H/Q \cong Frob_7^3$. Then $n = 3$. Consider a quint x incident to v. By (9.2.2 (i)), the stabilizer of x in H induces on the three links incident to v and x the group Sym_3, which contradicts the fact that $H/Q \cong Frob_7^3$ (the latter group does not involve Sym_3). So (i) follows.

Now suppose $n = 4$ and $H/Q \cong Alt_7$. Let u be a vertex adjacent to v and let a be the link on v and u. Then the stabilizer of v and a induces on $\mathrm{res}_{\mathcal{G}}(v)$ the group $L_3(2)$. Since $G(v,u)$ is of index at most two in the stabilizer of v and a, $G(v,u)$ also induces on $\mathrm{res}_{\mathcal{G}}(v)$ the group $L_3(2)$. Symmetrically, $G(v,u)$ induces $L_3(2)$ on $\mathrm{res}_{\mathcal{G}}(u)$. Consider now the action of H_1 on $\text{res}_{\mathcal{G}}(u)$. Since H_2 acts on $\text{res}_{\mathcal{G}}(u)$ trivially, H_1 induces on $\text{res}_{\mathcal{G}}(u)$ a 2-group by (9.2.5). On the other hand, H_1 is normal in H, and hence in $G(v,u)$. Since $L_3(2)$ contains no non-trivial normal 2-group, this implies that $H_1 \leq G_{\frac{1}{2}}(u)$. We claim that in fact $H_1 \leq G_1(u)$. Indeed, let $w \in \Delta_1(u)$. By (9.1.5) there is a quint z such that $\Sigma[z]$ contains the path (v,u,w). By (9.2.2 (iv)), an element fixing all neighbours of v and all links on u must act trivially on $\Sigma[z]$. Hence it fixes every vertex $w \in \Delta_1(u)$.

We proved that $H_1 \leq G_1(u)$ for all $u \in \Delta_1(v)$. Hence $H_1 = H_2$, and by the vertex-transitivity of G on Δ, this implies that $H_1 = 1$. □

Thus we have the following.

Corollary 9.2.7 *If $H/Q \ncong L_n(2)$ then $n = 4$ and $H \cong Alt_7$, or $2^4.Alt_7$.* □

Let us conclude this section with a comment concerning the exceptional configuration for Q/H_1 (compare (9.2.4)). If $\mathcal{G}$ is a Petersen type geometry then, of course, Q must equal H_1. On the other hand, for tilde type geometries the generic case occurs when $Q/H_1 \cong 2^n$. Indeed, in view of (9.2.7), we may assume that $H/Q \cong L_n(2)$. Suppose $Q = H_1$. Let a be a link incident to v. Considering the action on $\Sigma[q]$ for a quint q incident to a and using (9.2.2 (ii)), we obtain that the stabilizer of a in H contains an element interchanging the two vertices in $\Sigma[a] \setminus \{v\}$. On the other hand, the stabilizer of a in $H/H_1 \cong L_n(2)$ has structure $2^{n-1} : L_{n-1}(2)$. If $n > 3$ then the latter has no subgroup of index two. So the stabilizer of a in H cannot act on $\Sigma[a] \setminus \{v\}$. This proves the following.

Lemma 9.2.8 *If $\mathcal{G}$ is of tilde type and $H/H_1 \cong L_n(2)$ then $n = 3$.* □

We will return to this exceptional configuration in Section 10.2 (cf. (10.2.2)).

9.3 Condition $(*_i)$

Throughout this section we assume $H/Q \cong L_n(2)$. We will investigate the impact of the following conditions on the structure of H.

$(*_i)$ If $\Sigma = \Sigma[x]$ for x of type $n-i$ (here $2 \leq i \leq n-1$) and if v is a vertex of Σ then the joint stabilizer R of all the vertices of Σ at distance (in Σ) at most $i-1$ from v induces on Σ an action of order at most two.

Notice that since R stabilizes v and all the links incident to both v and x, it must stabilize x and hence it indeed acts on Σ. Notice also that due to (9.2.2 (iii)) the property $(*_2)$ holds for all $\mathcal{G}$.

Let $V_i = H_i/H_{i+1}$, $i \geq 1$. By (9.2.5) V_i is an elementary abelian 2-group. So we can view it as a vector space over $GF(2)$ and as a module for H.

Lemma 9.3.1 *Suppose that* $(*_i)$ *holds. Then either:*

(i) $V_{i-1} = 1$, *or*

(ii) $\dim V_{i-1} = 1$, *or*

(iii) V_{i-1} *is isomorphic to the i-th exterior power* $\bigwedge^i U$ *of the natural module* U *of* H.

Proof. Put $\Sigma = \Sigma[x]$ for an arbitrary element $x \in \mathrm{res}_{\mathscr{G}}(v)$ of type $n-i$ (so that x is an $(n-i)$-subspace in the natural module U of H). By $(*_i)$ the group H_{i-1} induces on $\Sigma_i(v)$ (the set of vertices at distance i from v in Σ) an action of order at most two. If the action is trivial then the same is true for all $\Sigma' = \Sigma[y]$ for $y \in \mathrm{res}_{\mathscr{G}}(v)$ of type $n-i$ (because H is transitive on all such y). By (9.1.5) every vertex in $\Delta_i(v)$ is contained in some Σ' as above and hence $H_{i-1} = H_i$, which implies $V_{i-1} = 1$, and (i) holds.

So we can assume that H_{i-1} induces on each $\Sigma_i(v)$ a group of order exactly two. Let $V = V_{i-1}$ and V^* be the dual of V. Clearly, H_i acts trivially on Σ and hence the kernel of the action of H_{i-1} on Σ corresponds to a nonzero vector $v_x^* \in V^*$. Since every vertex from $\Delta_i(v)$ is contained in some $\Sigma = \Sigma[x]$, we have that the vectors v_x^* generate V^*. (In particular, this implies that Q centralizes V, as it fixes every x.) Consider now elements $x, y, z \in \mathrm{res}_{\mathscr{G}}(v)$ of type $n-i$ such that they are incident to common elements t and r of type $n-i+1$ and $n-i-1$ respectively. This means that

$$x \cap y \cap z = r, \langle x, y, z \rangle = t.$$

(If $i = n-1$ then we skip r.) Suppose $g \in H_{i-1}$ acts trivially on $\Sigma[x]$ and $\Sigma[y]$. We claim that g must also act trivially on $\Sigma[z]$. Suppose not, then g acts non-trivially on the neighbours in $\Sigma[z]$ of some vertex $u \in \Sigma[z]$ at distance $i-1$ from v. Let $h \in H$ take u to $u' = u^h \in \Sigma[t]$. Then $g' = g^h$ acts non-trivially on the neighbours of u' in $\Sigma[z]$. By $(*_i)$ the action of H_{i-1} on $\Sigma[z]$ is of order two. Hence g and g' induce the same action on $\Sigma[z]$. In particular, g acts non-trivially on the neighbourhood of u', and so we can assume that $u = u'$ is contained in $\Sigma[t]$.

Now in the projective geometry $\mathrm{res}_{\mathscr{G}}(u)$ the elements x and y are two different subspaces containing r with codimension 1 (two projective points if $i = n-1$). Since g acts trivially on both $\Sigma[x]$ and $\Sigma[y]$ it fixes every link containing u and contained in either of these subgraphs.

Hence it fixes every link contained in $\Sigma[r]$ (every link containing u if $i = n-1$). In particular, it fixes every link contained in $\Sigma[z]$, since z is yet another subspace containing r with codimension 1 and contained in t. This contradicts the fact that g acts non-trivially on the neighbours of u in $\Sigma[z]$.

We have shown that if g acts trivially on $\Sigma[x]$ and $\Sigma[y]$ then it also acts trivially on $\Sigma[z]$. This means that v_z^* is contained in the subspace generated by v_x^* and v_y^*. There are two cases. If this subspace is 1-dimensional then $v_x^* = v_y^* = v_z^*$. Since H acts flag-transitively on $\mathrm{res}_{\mathscr{G}}(v)$ it acts transitively on the set of all triples $\{x, y, z\}$ which are incident to common elements of type $n-i-1$ and $n-i+1$. This immediately implies that all vectors v_x^* are equal, and hence V^* is 1-dimensional and (ii) holds.

If the subspace spanned by v_x^* and v_y^* is 2-dimensional then the three vectors v_x^*, v_y^* and v_z^* are pairwise distinct, and this implies a relation $v_x^* + v_y^* + v_z^* = 0$. Again by flag-transitivity such a relation holds for every triple $\{x, y, z\}$ as above. It follows from (3.1.3) that V^* is a quotient of the $(n-i)$-th exterior power of the natural module U. Since $H/Q \cong L_n(2)$ is irreducible on the exterior powers, we finally conclude that V^* is in fact isomorphic to the $\bigwedge^{n-i} U$. Since the dual of $\bigwedge^{n-i} U$ is $\bigwedge^{i} U$, (iii) holds.□

If $V_{i-1} = 1$, then $H_{i-1} = H_i$. In view of the vertex-transitivity of G on Δ this implies that $H_{i-1} = 1$. Let us see that the length of the normal series can also be bounded when $\dim V_{i-1} = 1$.

Lemma 9.3.2 *If $|V_{i-1}| = 2$ then $H_i = 1$.*

Proof. Suppose $g \in H_i$ and let $u \in \Delta_1(v)$. Then g acts trivially on $\Delta_{i-1}(u)$. By our assumption the action of the pointwise stabilizer of $\Delta_{i-1}(u)$ on $\Delta_i(u)$ is of order two. Hence the action of g is either trivial on each $\Delta_{i-1}(w)$, $w \in \Delta_1(u)$, or it is non-trivial for all w. As the action is clearly trivial for $w = v$ we conclude that g acts trivially on $\Delta_i(u)$. Since u was arbitrary in $\Delta_1(v)$, it follows that $g \in H_{i+1}$, that is, $H_i = H_{i+1}$. Now the claim follows. □

Here is one more lemma bounding the length of the normal series.

Lemma 9.3.3 *If $(*_{n-1})$ holds then $H_n = 1$.*

Proof. Let $g \in H_n$ and suppose $u \in \Delta_{n+1}(v)$. Let w be a vertex in $\Delta_2(v) \cap \Delta_{n-1}(u)$. By (9.1.5), v and w are contained in some $\Theta = \Sigma[t]$ for a quint t, and similarly w and u are contained in some subgraph $\Sigma = \Sigma[r]$ for r being a point (an element of type 1). It follows from (9.1.4) that

Θ and Σ meet in $\Sigma[a]$ for a link a containing w. Now (9.2.2 (v)) implies that $\Sigma[a]$ contains a second vertex w' at distance at most two from v. Now observe that g fixes elementwise the set $\Sigma_{\leq n-2}(w)$. Because of the property $(*_{n-1})$, either g acts trivially on Σ, or it acts non-trivially on $\Sigma_{n-2}(t)$ for every $t \in \Sigma_1(w)$. Since the latter condition fails for $t = w'$ we conclude that g acts trivially on Σ. In particular, g fixes u. Since u was an arbitrary vertex in $\Delta_{n+1}(v)$, g is contained in H_{n+1}. Thus, $H_n = H_{n+1}$, and hence $H_n = 1$. □

Lemma 9.3.4 *Suppose that* $(*_{n-1})$ *holds. Then, as an* H*-module,* H_{n-1} *is isomorphic to a submodule of the* $GF(2)$*-permutation module on the vertices from* $\Delta_1(v)$.

Proof. By the preceding lemma we have that $H_n = 1$, so H_{n-1} acts faithfully on $\Delta_n(v)$. Let $u \in \Delta_1(v)$. We claim that H_{n-1} induces on $\Delta_{n-1}(u)$ an action of order two. It will be more convenient for us to prove the symmetric statement, namely, that $K = G_{n-1}(u)$ induces on $\Delta_{n-1}(v)$ an action of order two. Observe first that $V_{n-2} \cong \bigwedge^{n-1} U$. Indeed, according to (9.3.1), the only other possibilities are the trivial or 1-dimensional V_{n-2}, which would imply that $H_{n-1} = 1$ (cf. (9.3.2)). Notice that the $\bigwedge^{n-1} U \cong U^*$. Thus, V_{n-2} is the dual U^* of natural module U. The action induced by K on Δ_{n-1} is a subspace of V_{n-2} invariant under the subgroup $H \cap G(u)$. Modulo Q, the latter subgroup maps onto the full parabolic subgroup of $H/Q \cong L_n(2)$. Hence the action of K on $\Delta_{n-1}(v)$ is either the entire V_{n-2}, or it is 1-dimensional, or trivial. In the first case, $K = H_{n-1}$, which implies $H_{n-1} = 1$. Similarly, in the last case $K = H_n$, which again implies $H_{n-1} = 1$. So, as claimed, the action of K on $\Delta_{i-1}(v)$ is 1-dimensional, and symmetrically, the action of H_{n-1} on $\Delta_{n-1}(u)$ is also 1-dimensional.

Set $V = H_{n-1}$. By the previous paragraph, the kernel of the action of V on $\Delta_{n-1}(u)$ is a hyperplane of V, which corresponds to a 1-dimensional subspace $\langle v_u^* \rangle$ of V^*. Now observe that $\Delta_n(v)$ is contained in the union of the sets $\Delta_{n-1}(u)$ taken for $u \in \Delta_1(v)$. This shows that the vectors v_u^*, $u \in \Delta_1(v)$, span V^*. Hence V^* is a factor module of the permutation module on $\Delta_1(v)$. Equivalently, V is a submodule of the same permutation module. □

In quite a few cases H_{n-1} will be a trivial module for H. This situation is refined by the following lemma

Lemma 9.3.5 *In the hypothesis of* (9.3.4) *suppose that* H_{n-1} *is in the centre of* H. *Then* $|H_{n-1}| \leq 2$.

Proof. The result follows from the well-known fact that the centre of the permutation module of a transitive permutation group is 1-dimensional. □

9.4 Normal series of the point stabilizer

The variety of the possible structures of the vertex stabilizer $H = G_n = G(x_n)$ left by the results of the previous section can be further reduced if we play those results against the properties of other parabolics.

Let $1 \leq i \leq n-2$ if $\mathcal{G}$ is of P-type and $1 \leq i \leq n-1$ if $\mathcal{G}$ is of T-type. Let $G_i = G(x_i)$ be the stabilizer in G of the element x_i in the maximal flag $\Phi = \{x_1, ..., x_n\}$.

Recall that $\mathrm{res}^-_{\mathcal{G}}(x_i)$ is the subgeometry in $\mathcal{G}$ formed by the elements incident to x_i whose type is less than i. This residue is isomorphic to the projective $GF(2)$-space of rank $i-1$ (of course it is empty if $i = 1$). Let U_i^- denote the universal representation module of the dual of $\mathrm{res}_{\mathcal{G}}(x_i)$. Thus U_i^- is generated by pairwise commuting involutions indexed by the elements of type $i-1$ incident to x_i and the product of three such involutions corresponding to a, b and c is the identity whenever a, b and c are incident to a common element of type $i-2$ (this element is also incident to x_i).

Similarly $\mathrm{res}^+_{\mathcal{G}}(x_i)$ is the subgeometry formed by the elements in $\mathcal{G}$ incident to x_{i+1} whose type is greater than i. Since $i \leq n-2$, the residue $\mathrm{res}^+_{\mathcal{G}}(x_i)$ is a P- and T-geometry (depending on the type of $\mathcal{G}$) of rank $n-i$. Let U_i^+ be the universal representation module of $\mathrm{res}^+_{\mathcal{G}}(x_i)$ (whose points and lines are the elements of type $i+1$ and $i+2$ incident to x_i).

Let K_i be the kernel of the action of G_i on $\mathrm{res}_{\mathcal{G}}(x_i)$, so that $\overline{G}_i = G_i/K_i$ is a flag-transitive automorphism group of $\mathrm{res}_{\mathcal{G}}(x_i)$. Let $\mathcal{G}(x_i)$ be the set of elements y_i of type i in $\mathcal{G}$ such that there exists a premaximal flag Ψ of cotype i (depending on y_i) for which both

$$\Psi \cup \{x_i\} \quad \text{and} \quad \Psi \cup \{y_i\}$$

are maximal flags. Since $\mathcal{G}$ belongs to a string diagram $y_i \in \mathcal{G}(x_i)$ if and only if there is an element of type $i-1$ incident to both x_i and y_i and an element of type $i+1$ incident to both x_i and y_i. Let L_i be the kernel of the action of K_i on the set $\mathcal{G}(x_i)$.

Proposition 9.4.1 *In the above terms the quotient $E_i := K_i/L_i$ is an elementary abelian 2-group and as a module for $\overline{G}_i$ the dual E_i^* of E_i is isomorphic to a quotient of the tensor product $U_i^- \otimes U_i^+$.*

Proof. Without loss of generality we can assume that $E_i \neq 1$. If Ψ is a premaximal flag of cotype i in $\mathcal{G}$ incident to x_i (i.e., such that $\Psi \cup \{x_i\}$ is a maximal flag) then $\mathrm{res}_{\mathcal{G}}(\Psi)$ consists of three elements of type i, one of which is x_i. Let $g \in K_i$. Since K_i acts trivially on $\mathrm{res}_{\mathcal{G}}(x_i)$, g stabilizes every triple $\mathrm{res}_{\mathcal{G}}(\Psi)$ as above, fixing x_i as well. It follows that g^2 acts trivially on $\mathcal{G}(x_i)$ and hence $g^2 \in L_i$. This proves that E_i is an elementary abelian 2-group.

With Ψ as above consider the action of K_i on $\mathrm{res}_{\mathcal{G}}(\Psi)$ (of size 3). If this action is trivial for some Ψ then, because of the flag-transitivity of G_i on $\mathrm{res}_{\mathcal{G}}(x_i)$, the action is trivial for every such Ψ. Hence $K_i = L_i$ and $E_i = 1$, contradicting our assumption. Thus, the kernel of the action of K_i on $\mathrm{res}_{\mathcal{G}}(\Psi)$ is a subgroup of index 2, and it corresponds to a hyperplane in E_i, or, equivalently, a 1-subspace $\langle e_\Phi \rangle$ in the dual E_i^*.

Suppose j is a type in the diagram of $\mathcal{G}$, adjacent to i. That is, $j = i-1$ or $j = i+1$. Pick a flag Ξ in $\mathrm{res}_{\mathcal{G}}(x_i)$ of cotype j. (In the entire $\mathcal{G}$ the flag Ψ has cotype $\{i, j\}$.) Then $\mathrm{res}_{\mathcal{G}}(\{x_i\} \cup \Xi) = \{a, b, c\}$ for some elements a, b and c of type j. We claim that the following relation holds in E^*:

$$e_{\{a\}\cup\Xi} + e_{\{b\}\cup\Xi} + e_{\{c\}\cup\Xi} = 0.$$

Indeed, a group theoretic equivalent of this relation is that K_i induces on

$$\Omega := \mathrm{res}_{\mathcal{G}}(\{a\} \cup \Xi) \cup \mathrm{res}_{\mathcal{G}}(\{b\} \cup \Xi) \cup \mathrm{res}_{\mathcal{G}}(\{c\} \cup \Xi)$$

an action of order four. (Notice that if $e_{\{a\}\cup\Xi} = e_{\{b\}\cup\Xi}$ then also $e_{\{a\}\cup\Xi} = e_{\{c\}\cup\Xi}$ since the stabilizer in G_i of Ξ is transitive on $\{a, b, c\}$. Then the action on Ω is of order two.) Now observe that Ω is fully contained in $\mathrm{res}_{\mathcal{G}}(\Xi)$. If $\mathcal{G}$ is of tilde type, $i = n-1$ and $j = n$ then the fact that the action of K_i on Ω is of order four is recorded in (9.2.2 (ii)). In all other cases, $\mathrm{res}_{\mathcal{G}}(\Xi)$ is a projective plane of order two, and the desired property can be checked directly.

It remains for us to see that the relations we have just established do mean that E_i^* is a quotient of $U_i^- \otimes U_i^+$. First let $i = 1$. Notice that $\mathrm{res}^+_{\mathcal{G}}(x_1) = \mathrm{res}_{\mathcal{G}}(x_1)$ and $\mathrm{res}^-_{\mathcal{G}}(x_1) = \emptyset$. According to our definitions, the second factor in the tensor product is trivial (1-dimensional). So we need to show that E^* is a quotient of $U_i^+ = V(\mathrm{res}_{\mathcal{G}}(x_1))$. Observe that if Ψ and Ψ' are two maximal flags from $\mathrm{res}_{\mathcal{G}}(x_1)$ then $e_\Psi = e_{\Psi'}$ whenever Ψ and Ψ' contain the same element of type 2. So instead of e_Ψ we can write e_y, where y is the element of type 2 from Ψ. We only have to note that the elements of type 2 are the points of $\mathrm{res}_{\mathcal{G}}(x_i)$ and that the sets $\{a, b, c\} = \mathrm{res}_{\mathcal{G}}(\{x_i\} \cup \Xi)$ are the lines, where Ξ is a flag of $\mathrm{res}_{\mathcal{G}}(x_1)$

of cotype 2. So the relations we have established for $E^\bullet$ are exactly the same as the relations from the definition of $V(\mathrm{res}_{\mathcal{G}}(x_i))$.

Now let $i \geq 2$. Then $e_\Psi = e_{\Psi'}$ whenever Ψ and Ψ' contain the same elements y and z of types $i-1$ and $i+1$, respectively. So we can write e_{yz} in place of e_Ψ. With this notation the relations we established state that (1) $e_{ya} + e_{yb} + e_{yc} = 0$ for every line $\{a, b, c\}$ from $\mathrm{res}^+_{\mathcal{G}}(x)$, and (2) $e_{az} + e_{bz} + e_{cz} = 0$ for every line $\{a, b, c\}$ from $\mathrm{res}^-_{\mathcal{G}}(x)$. According to (2.4.2) these relations define $U_i^- \otimes U_i^+$. So $E_i^\bullet$ is a quotient of the latter module. □

The case $i = 1$ is of particular importance to us and we summarize this case in the following (notice that L_1 is the kernel of the action of K_1 on the set of points collinear to $x - 1$).

Corollary 9.4.2 *In the above terms the quotient K_1/L_1 is an elementary abelian 2-group and its dual is a $\overline{G}_1$-admissible representation module of* $\mathrm{res}_{\mathcal{G}}(x_1)$ *i.e., a quotient of the universal representation module* $V(\mathrm{res}_{\mathcal{G}}(x_1))$ *over a subgroup normalized by $\overline{G}_1$.* □

In the remainder of this section we deal only with the case $i = 1$. We will again be working with the derived graph Δ of $\mathcal{G}$. Let $\Sigma = \Sigma[x_1]$ (notice that the vertex $x_n$is contained in Σ).

Lemma 9.4.3 *The subgroup L_1 acts trivially on* $\mathrm{res}_{\mathcal{G}}(u)$ *for every vertex u of Σ.*

Proof. Let u be a vertex of Σ (which is an element of type n in $\mathcal{G}$) and let $y_1 \neq x_1$ be an element of $\mathrm{res}_{\mathcal{G}}(u)$ of type 1. Since $\mathrm{res}_{\mathcal{G}}(u)$ is a projective space, x_1 and y_1 are collinear points and hence they are both incident to an element z of type 2 (which is a line). Since $\mathcal{G}$ has a string diagram, $x_1, y_1 \in \mathrm{res}_{\mathcal{G}}(\Psi)$ for every flag Ψ cotype 1 that contains z. Hence L_1 stabilizes y_1. Since y_1 was arbitrary, L_1 stabilizes every point of the projective space $\mathrm{res}_{\mathcal{G}}(u)$ and so L_1 acts trivially on $\mathrm{res}_{\mathcal{G}}(u)$. □

Let N_1 be the joint stabilizer of all the vertices adjacent to $\Sigma = \Sigma[x_1]$ in Δ. Let us introduce the following property of $\mathcal{G}$ and G:

(**) $L_1 \neq N_1$.

Lemma 9.4.4 *If (**) holds then $\mathcal{G}$ is of tilde type and*

(i) *L_1/N_1 has order 2;*
(ii) *every $g \in L_1 \setminus N_1$ acts fixed-point freely on the set of vertices adjacent to Σ;*

(iii) $Q \neq H_1$;

(iv) *the property* (**) *holds for* $\text{res}_{\mathcal{G}}(x_1)$ *with respect to the action of* $\overline{G}_1$ *on it.*

Proof. The fact that $\mathcal{G}$ must be of tilde type follows from (9.4.3) and the definition of N_1. Suppose that $g \in L_1 \setminus N_1$. Suppose further that a is a link incident with a vertex u of $\Sigma = \Sigma[x_1]$ but not incident with x_n (notice that g fixes a by (9.4.2)). We claim that g permutes the two vertices of a other than u (since $\mathcal{G}$ is of tilde type every link consists of three vertices). Indeed, suppose g fixes all three vertices of a. Let $\Theta = \Sigma[z]$ be a quint containing $\Sigma[a]$. Let b be the link incident to both z and x. Then g acts trivially on both $\Sigma[a]$ and $\Sigma[b]$ and (9.2.2 (ii)) implies that g fixes all the neighbours of u in Θ. Furthermore, since g stabilizes all links incident to any vertex of $\Sigma[b]$, (9.2.2 (iv)) implies that g acts trivially on the entire Θ. Since Θ was arbitrary, g acts trivially on the set of neighbours of u in Δ. Also, observe that if u' is a neighbour of u in Σ then some Θ contains u' and a link a' incident with u' but not with x_1. Since g must fix the three vertices of $\Sigma[a']$ we can use the connectivity argument to deduce that g fixes every neighbour of Σ. So $g \in N_1$, which is a contradiction. Thus, g must act non-trivially on every $\Sigma[a]$ where a is a link incident to a point of Σ, but not incident to x. This proves (i) and (ii).

To prove (iii) observe that by (9.4.3) an element $g \in L_1 \setminus N_1$ is contained in Q, while (ii) implies that $g \notin H_1$.

For (iv), consider an element $y_1 \in \text{res}_{\mathcal{G}}(x_n)$ of type 1, $y_1 \neq x_1$. Let z be the element of type 2, that is incident with both x_1 and y_1, and let $g \in L_1 \setminus N_1$. Then in its action on $\Sigma[y_1]$ the element g fixes $\Sigma[z]$ vertexwise and it stabilizes all the links incident to the vertices of $\Sigma[z]$. On the other hand, by (ii), g acts non-trivially on the neighbours of $\Sigma[z]$ in $\Sigma[y]$. So $\Sigma[z]$ satisfies (**). □

Lemma 9.4.5 *If the property* $(*_i)$, *holds for every* $2 \leq i \leq k$ *(where* $k \leq n-1$*) then* N_1 *fixes all vertices at distance at most* k *from* $\Sigma[x_1]$.

Proof. We will prove the assertion by induction on the distance. If u is at distance one from $\Sigma = \Sigma[x_1]$ then N_1 fixes u by the definition. Now suppose it is known that all vertices at distance at most $i-1$ from Σ are fixed by N_1, where $2 \leq i \leq k$. Suppose u is at distance i from Σ. By (9.1.5) there exists an element y of type $n-i$ such that $\Sigma[y]$ contains u and a vertex w of Σ. By (9.1.4) both Σ and $\Sigma[y]$ contain $\Sigma[z]$ for some z of type $n-i+1$. In particular this means that Σ and $\Sigma[y]$ share some link

$\Sigma[a]$ containing w. Let $w' \in \Sigma[a]$ with $w' \neq w$. By (9.4.3), N_1 stabilizes y, and so it acts on $\Sigma[y]$. Since by the inductive assumption N_1 stabilizes all vertices at distance at most $i-1$ from either w or w', and since $(*_i)$ holds by the assumption of the lemma, we conclude from (9.4.3) that N_1 must act trivially on $\Sigma[y]$. In particular, N_1 fixes u. □

Lemma 9.4.6 *Suppose* $(*_i)$ *holds for every* $2 \leq i \leq n-1$. *Then* $|N_1| \leq 2$.

Proof. Suppose $N_1 \neq 1$ and let $g \in N_1^{\#}$. By (9.4.5) $g \in H_{n-1}$. In view of (9.3.3) the action of H_{n-1} on $\Delta_n(v)$ is faithful. Therefore, in order to prove that $|N_1| = 2$ it is sufficient to show that the action of g on $\Delta_n(v)$ is uniquely determined. Let $w \in \Delta_n(v)$ and let u be a neighbour of v such that the distance between u and w in Δ is $n-1$. By (9.1.5) the shortest path between u and w is contained in $\Sigma[y]$ for a point y (so that u and w are at distance $n-1$ in $\Sigma[y]$).

If $\Sigma[y]$ meets $\Sigma = \Sigma[x_1]$ in a vertex then $(*_{n-1})$ and (9.4.5) show that g fixes $\Sigma[y]$ vertexwise. So we only need to consider the case where y is not incident to the link a that is incident to both v and u. We claim that for such a y the action of g on $\Sigma[y]$ is non-trivial. In view of $(*_{n-1})$ the action of g on $\Sigma[y]$ is then unique and the lemma follows.

Thus it suffices to show that g acts on $\Sigma[y]$ non-trivially. Suppose *ad absurdum* that g fixes every vertex of $\Sigma[y]$. We will show that in this case g must act trivially on every $\Sigma[z]$, where $z \in \mathrm{res}_{\mathscr{G}}(u)$ is a point not incident to a. By (9.1.4) the intersection of $\Sigma[y]$ and $\Sigma[z]$ contains a link on u. Let t be a vertex of this link, $t \neq u$. Let $\Theta = \Sigma[q]$ be a quint containing the path (v, u, t) (compare (9.1.5)). It follows from (9.1.4) that Σ and Θ share a link on v. Let $v' \neq v$ be a vertex of that link that is at distance at most two from t (see (9.2.2 (v))) and let u' be the common neighbour in Θ of v' and t. Let a' be the link incident to v' and u'. If u' is in Σ then g fixes all vertices of $\Sigma[z]$ at distance at most $n-2$ from u or t. Then $(*_{n-1})$ implies that the action of g on $\Sigma[z]$ is trivial. So without loss of generality we may assume that $u' \notin \Sigma$. Finally, let $y' \in \mathrm{res}_{\mathscr{G}}(u')$ be a point incident to t, but not to v'.

Observe that g stabilizes in $\Sigma[y']$ all the vertices at distance at most $n-2$ from u'. Besides, it fixes all the vertices in the intersection of $\Sigma[y]$ and $\Sigma[y']$. By (9.1.4) the component of the intersection containing t coincides with $\Sigma[r]$ for some r of type 2. Observe that $\Sigma[r]$ cannot contain u' because it cannot contain the entire quint $\Sigma[q]$. Due to $(*_{n-2})$, g fixes $\Sigma[y']$ vertexwise. (Indeed, if X is the group induced on $\Sigma[y']$ by its stabilizer in G, then the stabilizer of u' in X acts transitively on the set of subgraphs $\Sigma[r']$ of $\Sigma[y']$ at distance one from u'. So by $(*_{n-1})$ if g acts trivially on one of them then it must act trivially on all of them.)

Symmetrically, since g acts trivially on $\Sigma[y']$, we can now show that it also acts trivially on $\Sigma[z]$. Since z was arbitrary, g fixes all vertices at distance n from v, that is, $g \in H_n = 1$, which is a contradiction. □

9.5 Pushing up

In this section we only consider the case where $\mathcal{G}$ is of Petersen type. We apply some pushing up techniques to reduce further the structure of H_{n-1} under the condition $(*_{n-1})$. First we recall some basic notions and results.

Suppose that T is a p-group for a prime number p. Then the *Thompson subgroup* $J(T)$ of T is generated by all elementary abelian subgroups A of T of maximal rank. Observe that $J(T) \neq 1$, if $T \neq 1$. The following is a further important property of the Thompson subgroup.

Lemma 9.5.1 *Let T be a p-group and $Q \leq T$. If $J(T) \leq Q$ then $J(T) = J(Q)$.* □

We denote by $\Omega_1(T)$ the subgroup in T generated by the elements of order p in T. For a group G, a faithful $GF(p^f)$-module V of G is said to be an FF-module (*failure-of-factorization module*) if for some elementary abelian subgroup $A \neq 1$ of G we have

$$|A| \geq |V/C_V(A)|.$$

A subgroup A with this property is called an *offending subgroup* (or just an offender).

Proposition 9.5.2 *Suppose that G is a group, Q is a normal p-subgroup of G, and T is a p-subgroup of G such that $Q \leq T$. Let $V = \Omega_1(Z(Q))$ and suppose $C_G(V) = Q$. Let $\overline{G} = G/Q$. Then one of the following holds:*

(i) *$J(T) = J(Q)$; or*

(ii) *V is an FF-module for $\overline{G}$ over $GF(p)$, and $\overline{T}$ contains an offending subgroup.*

Proof. Suppose A is an elementary abelian subgroup of T of maximal rank. If every such A is contained in Q then $J(T) = J(Q)$ and (i) holds. Thus, without loss of generality we may assume that $A \nleq Q$. Observe that $C_A(V) = A \cap Q$ and so $(A \cap Q)V$ is elementary abelian. Hence

$$|A| \geq |(A \cap Q)V| = \frac{|A \cap Q| \cdot |V|}{|(A \cap Q) \cap V|}.$$

Since $(A \cap Q) \cap V = A \cap V \leq C_V(A) = C_V(\overline{A})$, we finally obtain that $|\overline{A}| \geq |V/C_V(\overline{A})|$, that is, $\overline{A} \neq 1$ is an offending subgroup in $\overline{T}$ and so (ii) holds. □

We can now apply this proposition to reduce the structure of H_{n-1}.

Lemma 9.5.3 *Suppose that $\mathscr{G}$ is of Petersen type and $(*_{n-1})$ holds. Then $H_{n-1} = V_{n-1}$ is a submodule of the direct sum of the 1-dimensional module and dual natural module.*

Proof. It follows from (9.3.3) and (9.3.4) that $H_{n-1} = V_{n-1}$ is isomorphic, as an H-module, to a submodule of the permutation module $\mathscr{P}^1$ on points of the projective space $\Delta_1(v)$. (We will be using the notation introduced in Section 3.2.) The structure of this module is described in (3.2.7) and (3.3.5). In particular, unless the conclusion of the lemma holds, the submodule corresponding to H_{n-1} must contain $\mathscr{X}(n-2)$. That is, as an H-module, H_{n-1} must have at least two non-trivial composition factors: a composition factor W_1, isomorphic to the dual of the natural module, U^*, and another one, W_2, isomorphic to the second exterior power of the dual of the natural module, $\bigwedge^2 U^*$. In particular, $Q = C_H(H_{n-1})$.

We will apply (9.5.3) for $G = H = G(v)$. Let $T = O_2(G(v) \cap G(u))$, where $u \in \Delta_1(v)$. Also let $V = \Omega_1(Z(Q))$ and $\overline{H} = H/Q$. Clearly, $H_{n-1} \leq V$. In particular, $Q = C_H(V)$, because $Q = C_H(H_{n-1})$. According to (9.5.3), either $J(T) = J(Q)$, or V is an FF-module and $\overline{T}$ contains an offending subgroup. If $J(T) = J(Q)$ then $J(T)$ is normal in H, as well as in the stabilizer of the edge $\{v, u\}$. By (9.2.1) this means that $J(T)$ acts trivially on Δ, which is a contradiction. It remains for us to rule out the possibility that $\overline{T}$ contains an offending subgroup.

Suppose $\overline{A} \leq \overline{T}$ is an offending subgroup. If $\overline{x} \in \overline{A}^{\#}$ then $C_{W_1}(\overline{x})$ has index two in W_1, while the index of $C_{W_2}(\overline{x})$ in W_2 is 2^{n-2}. Therefore, $|\overline{A}| \geq |V/C_V(\overline{A})| \geq 2^{n-1} = |\overline{T}|$. Hence, $\overline{A} = \overline{T}$. However, the index of $C_{W_2}(\overline{T})$ in W_2 exceeds 2^{n-2}, which implies that the index of $C_V(\overline{T})$ in V exceeds $2^{n-1} = |\overline{T}|$. Thus, $\overline{T}$ cannot be an offending subgroup. □

10

Shapes of amalgams

As before we fix a vertex $v = x_n$ and a point x_1 incident to x_n. The parabolics $H = G_n$ and G_1 were defined as the stabilizers in G of $v = x_n$ and x_1, respectively. In Section 9.2 we introduced a normal series

$$G_n = H \trianglerighteq Q \trianglerighteq H_1 \trianglerighteq ... \trianglerighteq H_i \trianglerighteq ...$$

in which all the factors except for H/Q (which will be shown to be $L_n(2)$ in all the cases) are elementary abelian 2-groups and $H_n = 1$ provided, that the condition $(*_{n-1})$ holds (cf. Lemma 9.3.3). In Section 9.4 we have shown that G_1 possesses a normal series

$$G_1 \trianglerighteq K_1 \trianglerighteq L_1 \trianglerighteq N_1,$$

where the index of N_1 in L_1 is at most 2 by Lemma 9.4.4 (i) and if $(*_i)$ holds for every $2 \leq i \leq n-1$, then N_1 is itself of order at most 2 by Lemma 9.4.6. Finally $E = K_1/L_1$ is an elementary abelian 2-group whose dual E^* is a $\overline{G}_1$-admissible representation module of the point-residue $\mathrm{res}_{\mathcal{G}}(x_1)$ by Corollary 9.4.2. In this chapter we will compare the structures of G_n and G_1, which are related via $G_{1n} = G_1 \cap G_n$. This will allow us to compile a relatively short list of possible shapes (by which at present we only mean the information about the normal factors) of G_n and G_1 summarized in Tables VIIIa and VIIIb. In the next chapter some of these shapes will be shown to be impossible, and the others will lead to the actual examples.

10.1 The setting

Notice first that due to our inductive approach we assume that in the P- or T-geometry $\mathcal{G}$ of rank n under consideration the point residue

$res_{\mathcal{G}}(x_1)$ is a known P- or T-geometry, of rank $n-1$. In Tables VIIa and VIIb we record the structure of $H = G_n$ for the known examples. The information in these tables enables us to decide, in particular, in which cases the condition $(*_i)$ holds for the geometry $\mathcal{G}$ under consideration.

Table VIIa. Vertex stabilizers in the known P-geometries

rank	G	V_1	V_2	V_3	V_4
2	Alt_5				
2	Sym_5	2			
3	$(3\cdot)M_{22}$	2^3			
3	$(3\cdot)\mathrm{Aut}\,M_{22}$	2^3	2		
4	M_{23}				
4	$(3^{23}\cdot)Co_2$	2^6	2^4	2	
4	J_4	2^6	2^4	2^4	
5	$(3^{4371}\cdot)BM$	2^{10}	2^{10}	2^5	2^5

Table VIIb. Vertex stabilizers in the known T-geometries

rank	G	Q/H_1	V_1	V_2	V_3	V_4
2	$3 \cdot Alt_6$	2^2				
2	$3 \cdot Sym_6$	2^2	2			
3	M_{24}	2^3	2^3	2		
3	He	2^3	2^3	2		
4	Co_1	2^4	2^6	2^4	2	
5	M	2^5	2^{10}	2^{10}	2^5	2^6
n	$3^{[\frac{n}{2}]_2} \cdot Sp(2n,2)$	2^n	$2^{n(n-1)/2}$			

In the next section we start considering the concrete variants. Our method of comparing the structures of G_1 and G_n will be very simple. Given the normal factors of G_1 and G_n we can compute the chief factors of G_{1n} in two different ways and compare the results.

Notice that the kernel K_{1n} of the action of G_{1n} on $\text{res}_{\mathscr{G}}(\{x_1, x_n\})$ coincides with $O_2(G_{1n})$ and

$$\overline{G}_{1n} = G_{1n}/K_{1n} \cong L_{n-1}(2),$$

since $\overline{G}_n \cong L_n(2)$.

Let $m_i(F)$ be the number of chief factors of G_{1n} inside K_{1n}, isomorphic to F and calculated by restricting of the normal structure of G_i to G_{1n} (where $i = 1$ or n). We will use the following notation: T for the trivial 1-dimensional module; N for the natural module of $\overline{G}_{1n}$ (whose non-zero vectors are indexed by the elements of type 2 incident to x_1 and x_n); N^* for the dual natural module; X for any non-trivial module (in many cases $m_i(X) = m_i(N) + m_i(N^*)$) and others.

10.2 Rank three case

In this section we consider the case $n = 3$. The condition $(*_2)$ holds due to (9.2.2 (iii)). So (9.3.1), (9.3.3), (9.3.4), (9.4.5), and (9.4.6) apply. In particular, these results imply that $Q = K_3$ is a (finite) 2-group. It follows that $\overline{G}_{13} \cong Sym_3 \cong L_2(2)$ and every chief factor G_{13} inside K_3 is an elementary abelian 2-group of rank one (the trivial module T) or two (the natural module N).

First let $\mathscr{G}$ be a Petersen type geometry. Then $\text{res}_{\mathscr{G}}(x_1) \cong \mathscr{G}(Alt_5)$ and $\overline{G}_1 \cong Alt_5$ or Sym_5.

First suppose that $K_1 = L_1$. Then, since the image of G_{13} in $\overline{G}_1$ is Sym_3 or $Sym_3 \times 2$ in view of (9.4.5) and (9.4.6), we conclude that $m_1(N) = 0$. This is clearly impossible since the image of G_{13} in $\overline{G}_3$ is isomorphic to Sym_4, which implies that $m_3(N) \geq 1$. Thus $E := K_1/L_1$ is non-trivial and by (9.4.2) E^* is a $\overline{G}_1$-admissible representation module of $\mathscr{G}(Alt_5)$. By (3.9.2) and (8.2.3 (v)) we conclude that E, as a module for $O^2(\overline{G}_1) \cong Alt_5$, is an indecomposable extension of the (self-dual) 4-dimensional natural module by a trivial module of dimension 1 or 2. This means particularly that

$$m_1(N) = 2.$$

One of the 2-dimensional chief factors appears in the image of G_{13} in $\overline{G}_3$, which leaves just one 2-dimensional chief factor of G_{13} inside K_3.

Therefore, G_3 has a unique non-trivial chief factor in K_3. It now follows from (9.3.1) and (9.3.2) that $V_1 \cong 2^3$. Furthermore, since G_3 has a unique non-trivial chief factor in K_3, (9.3.4) and (9.3.5) imply that $|H_2| \leq 2$.

Now we are ready to prove the following.

Proposition 10.2.1 *Let $\mathcal{G}$ be a P-geometry of rank 3 and G be a flag-transitive automorphism group of $\mathcal{G}$. Then $\overline{G}_1 \cong Sym_5$, $\overline{G}_3 \cong L_3(2)$ and either*

(i) *K_1 is the natural 4-dimensional module for $\overline{G}_1$ and $K_3 \cong 2^3$ is the dual natural module of $\overline{G}_3$* (**M_{22}-shape**), *or*

(ii) *K_1 is the natural module of $\overline{G}_1$ indecomposably extended by the trivial 1-dimensional module and K_3 is an extension of the trivial 1-dimensional module by the dual natural module of $\overline{G}_3$* (Aut **M_{22}-shape.)**

Proof. Since $H_1 \neq H_2$, we must have $\overline{G}_1 \cong Sym_5$. Suppose first that $H_2 = 1$. Then $|K_1| = 2^4$, hence $L_1 = 1$ and K_1 is the (natural) module for $\overline{G}_1 \cong Sym_5$ and we are in case (i).

Now suppose that $|H_2| = 2$. Then $|K_1| = 2^5$. Observe that H_2 acts trivially on $\Sigma[x_1]$ (which is the Petersen graph of diameter 2) and hence $H_2 \leq K_1$. If $H_2 \leq L_1$ then H_2 is normal in both G_3 and G_1, which is impossible by (9.2.1). Hence $L_1 = 1$ and we are in case (ii). □

Now suppose that $\mathcal{G}$ is of tilde type. We first deal with the exceptional configuration from (9.2.8).

Proposition 10.2.2 *If $Q = H_1$, then $G_1 \cong 3 \cdot Alt_6$ and $G_3 \cong L_3(2)$ (Alt_7-*
shape.)

Proof. Since $Q = H_1$ we have $H/H_1 \cong L_3(2)$. Note that $G_3 = H$ acts transitively on the set of links incident to x_3 and that the stabilizer of such a link induces on the three vertices incident to the link a group Sym_3. This means, in particular, that G_3 is transitive on the 14 vertices from $\Delta_1(x_3)$. This uniquely specifies the action of $H/H_1 \cong L_3(2)$ on $\Delta_1(x_3)$ as on the cosets of a subgroup Alt_4. One of the properties of this action is that the stabilizer of x_1 in H/H_1 (isomorphic to Sym_4) acts faithfully on $\Sigma_1(x_3)$ (where as usual $\Sigma = \Sigma[x_1]$). It follows that the vertexwise stabilizer of $\Sigma_1(x_3)$ acts trivially on the entire $\Delta_1(x_3)$. In particular, K_1 acts trivially on $\Delta_1(x_3)$. Since G_1 acts transitively on the vertex set of Σ, we conclude that

$$K_1 = L_1 = N_1.$$

By (9.4.6) this means that $|K_1| \leq 2$. Therefore, $|G_{13}| \leq 2^5 \cdot 3$, which implies that $|H_1| \leq 4$. By (9.3.1) and (9.3.2), we now have that $H_2 = 1$ and $|H_1| \leq 2$. We claim that in fact $H_1 = 1$. Indeed, consider a vertex u adjacent to x_3 and the stabilizer $G(x_3, u) = H(u)$ of x_3 and u. Clearly, $H(u)$ induces on $\Delta_1(u)$ a group Alt_4. Since H_1 is normal in $H(u)$ and since Alt_4 has no normal subgroup of index two, H_1 must act trivially on $\Delta_1(u)$. Since u was arbitrary we have that $H_1 = H_2$ and hence, $H_1 = 1$. Thus, $G_3 = H \cong L_3(2)$ and, clearly, $G_1 \cong 3 \cdot Alt_6$ (since $|G_{13}| = 2^3 \cdot 3$). □

Now suppose $Q \neq H_1$ and hence $Q/H_1 \cong 2^3$ by (9.2.4). We will next discuss H_2. By (9.2.2), H_2 fixes Σ vertexwise. That is, $H_2 \leq K_1$.

Lemma 10.2.3 *The image of H_2 in $E = K_1/L_1$ has order at most 2^3.*

Proof. Let (E^*, φ) be the representation of $\mathrm{res}_{\mathscr{G}}(x_1)$ as in (9.4.2). Then φ is defined on the set of links contained in Σ and if y is such a link then $\varphi(y)$ is the subgroup of index 2 in E (a 1-subspace in E^*) which is the elementwise stabilizer of the pair $\{\Sigma_1, \Sigma_2\}$ of quints, other than Σ containing y.

An element $g \in H_2$ fixes every vertex at distance at most 2 from x_3 in the derived graph of $\mathscr{G}$. This means that g stabilizes every quint containing a vertex adjacent to x_3. Hence the image of g in E is contained in the intersection of the hyperplanes $\varphi(y)$ taken for all the links y contained in Σ and containing a vertex adjacent to x_3. By (the dual version of) (3.8.5 (i)) the intersection has dimension 3 and the proof follows. □

Since L_1 is centralized by $O^2(G_1)$, it is clearly centralized by $O^2(G_{13})$ and hence (10.2.3) immediately implies the following

Lemma 10.2.4 *G_{13} has at most one 2-dimensional chief factor inside H_2.* □

Lemma 10.2.5 $m_3(N) \leq 4$.

Proof. We estimate the number of chief factors of G_{13} treating it as a subgroup of G_3. One such factor is in $G_{13}/Q \cong Sym_4$, and one is inside $Q/H_1 \cong 2^3$. Since $| H_1/H_2 | \leq 2^3$ by (9.3.1), there is at most one factor in H_1/H_2 and finally we have at most one factor in H_2 by (10.2.4). □

Now we are in a position to further restrict the possibilities for $E = K_1/L_1$.

Lemma 10.2.6 *One of the following holds:*

(i) *E is (the dual of) the hexacode module V_h of $\overline{G}_1 \cong 3 \cdot S_4(2)$;*

(ii) *E is dual to the 5-dimensional orthogonal module V_o of $\overline{G}_1/O_3(\overline{G}_1) \cong O_5(2)$;*
(iii) *E is the (self-dual) 4-dimensional natural symplectic module of $\overline{G}_1/O_3(\overline{G}_1) \cong S_4(2)$;*
(iv) $E = 1$.

Proof. By (10.2.5) we have $m_1(N) \leq 4$. On the other hand, there is one 2-dimensional chief factor of G_{13} inside G_{13}/K_1 which leaves us with at most three such factors inside $E = K_1/L_1$. Recall that by (9.4.2) and (3.8.1) the dual of E is a quotient of the 11-dimensional universal representation module of $\mathcal{G}(3 \cdot S_4(2))$ and the universal module is the direct sum

$$V_o \oplus V_h,$$

where V_h is irreducible and V_o contains a unique proper submodule which is 1-dimensional. Under the natural action of G_{13}/K_1 each of the direct summands contains two 2-dimensional chief factors which gives the proof. □

Suppose first that we are in case (i) of (10.2.6). Then $E \cong V_h$ involves two 2-dimensional chief factors of G_{13} and hence $m_1(N) = 3$. Returning to G_3, we see that $H_1/H_2 \cong 2^3$ (the natural module of $\overline{G}_3 \cong L_3(2)$), while H_2 is a trivial module. It follows from (9.3.4) and (9.3.5) that $|H_2| \leq 2$. Since $H_1 \neq H_2$, we have that $\overline{G}_1$ is isomorphic to $3 \cdot Sym_6$ (rather than to $3 \cdot Alt_6$). Comparing now the orders of G_1 and G_3, we observe that $|H_2| = 2$ and $L_1 = 1$, which gives the following

Proposition 10.2.7 *Let $\mathcal{G}$ be a rank 3 tilde geometry, G be a flag-transitive automorphism group of $\mathcal{G}$. Suppose that $Q \neq H_1$ and $E = K_1/L_1$ is the hexacode module. Then $G_1 \sim 2^6.3 \cdot Sym_6$ and $G_3 \sim 2.2^3.2^3.L_3(2)$ (M_{24}-***shape.)** □

It remains for us to consider the case where $O_3(\overline{G}_1)$ acts trivially on E. This situation is handled in the next lemma.

Proposition 10.2.8 *Let $\mathcal{G}$ be a rank 3 tilde geometry, G be a flag-transitive automorphism group of $\mathcal{G}$. Suppose that $Q \neq H_1$ and $O_3(\overline{G}_1)$ acts trivially on $E = K_1/L_1$. Then $G_1 \sim 2^5.3 \cdot Sym_6$, $G_3 \sim 2^3.2^3.L_3(2)$, furthermore*

(i) *$N_1 = 1$ and $L_1 = Z(G_1)$ is of order 2;*
(ii) *$K_1 = O_2(G_1)$ and K_1/L_1 is the 4-dimensional symplectic module for $G_1/O_{2,3}(G_1) \cong S_4(2)$;*

(iii) *H_1 is the dual natural module for $\overline{G}_3 \cong L_3(2)$ and Q/H_1 is the natural module* **($S_6(2)$-shape.)**

Proof. By the hypothesis of the lemma we are in case (ii), (iii) or (iv) of (10.2.6). Since $Q/H_1 \cong 2^3$, $m_3(N)$ is at least two, so E cannot be trivial, i.e., the case (iv) does not occur. So E necessarily involves the 4-dimensional symplectic module, and hence $m_3(N) = 3$. From this we obtain that $H_1/H_2 \cong 2^3$ (the natural module) and that H_2 is a trivial module. In particular, $|H_2| \leq 2$. Arguing as in the proof of (10.2.3) but using (3.8.5 (ii)) instead of (3.8.5 (i)) we conclude that $H_2 \leq L_1$. We are going to show that in fact H_2 is trivial. Towards this end notice that $C_G(H_2) \geq G_1$ and also $C_G(H_2) \geq G_1^{\infty}$, since $|L_1| \leq 4$. Clearly,

$$\langle G_3, G_1^{\infty} \rangle = G,$$

which means that $H_2 = 1$. It remains for us to determine the normal factors of G_1. First of all, since $H_1 \neq H_2$ we have $\overline{G}_1 \cong 3{\cdot}Sym_6$. Therefore, $|K_1| = 2^5$. Suppose that $L_1 = 1$ and so $E \cong 2^5$. Then, as an $\overline{G}_1/O_3(\overline{G}_1)$-module, E is a non-split extension of a 4-dimensional irreducible module by a 1-dimensional one. In particular, L_1 is elementary abelian. We next notice that Q is also elementary abelian. Indeed, let C be the full preimage in Q of subgroup $\overline{C}$ of order two from $\overline{Q} = Q/H_1$. Clearly, C is abelian (since $H_1 \leq Z(Q)$). If it is not elementary abelian then the squares of the elements of C form a subgroup of order 2 in H_1, which is invariant under the stabilizer of $\overline{C}$ in $H/Q \cong L_3(2)$. This is impossible since Q/H_1 and H_1 are respectively the natural and the dual natural modules. Thus, C is elementary abelian. Since $\overline{C}$ was arbitrary, we conclude that Q is elementary abelian.

Set $Z = Q \cap K_1$. Clearly, $QK_1 = O_2(G_{13})$. Thus, $|Z| = 2^3$ and $Z = Z(O_2(G_{13}))$. Since G_{13} induces on Z a group Sym_3, it follows that Z contains a subgroup Z_1 of order 2 central in G_{13}. On the other hand, G_{13} acting on $E = K_1$ leaves invariant no 1-dimensional subspace. The contradiction proves that $L_1 \neq 1$. Hence $E \cong 2^4$ and $L_2 \cong 2$. Finally, since G_{13} leaves invariant no 1-dimensional subspace in H_2, $L_1 \not\leq H_2$. Hence L_1 acts non-trivially on $\Delta_1(x_3)$. Therefore, $|L_1/N_1| = 2$ and hence $N_1 = 1$. This completes the proof. □

10.3 Rank four case

Recall that we follow the inductive approach and assume that in the rank 4 P- or T-geometry $\mathcal{G}$ under consideration the point residue $\mathrm{res}_{\mathcal{G}}(x_1)$ is one of the known rank 3 geometries of appropriate type and $\overline{G}_1$ is a known flag-transitive automorphism group of the residue.

First we rule out the exceptional configuration from (9.2.7).

Lemma 10.3.1 *For every flag-transitive action on P- or T-geometry of rank $n \geq 3$ we have $H/Q \cong L_n(2)$.*

Proof. Suppose that $H/Q \not\cong L_n(2)$. Then by (9.2.7) we may assume that $n = 4$ and $H \cong Alt_7$ (with $Q = 1$) or $H \cong 2^4.Alt_7$ (with $Q \cong 2^4$). If $Q = 1$ then $H \cong Alt_7$ and hence $G_{14} \cong L_3(2)$, which immediately yields a contradiction with the structure of G_1 (compare (10.2.1)). So $\mathcal{G}$ is of tilde type and $Q \cong 2^4$. Then $G_{14} \cong 2^4.L_3(2)$ and again we run into a contradiction with the structure of G_1 (compare (10.2.2), (10.2.7), (10.2.8) and (12.1.1)). □

Since $\mathrm{res}_{\mathcal{G}}(x_1)$ is one of the known rank three Petersen type or tilde type geometries, we obtain from Tables VIIa and VIIb that $(*_3)$ holds along with $(*_2)$. This means that (9.3.1), (9.3.3), (9.3.4), (9.4.5), and (9.4.6) apply. In particular, $H_4 = 1$ and $|L_1| \leq 4$. Hence, Q and K_1 are (finite) 2-groups, and G_{14} is an extension of a 2-group by $L_3(2)$. As we will see below, every chief factor of G_{14} in $O_2(G_{14})$ is either the trivial 1-dimensional,the natural or the dual natural module for $\overline{G}_{14} \cong L_3(2)$ and we continue to use notation introduced at the end of Section 10.1.

We will again start with the case where $\mathcal{G}$ is a Petersen type geometry. Then $\mathrm{res}_{\mathcal{G}}(x_1)$ is isomorphic to either $\mathcal{G}(M_{22})$ or $\mathcal{G}(3 \cdot M_{22})$.

Proposition 10.3.2 *If $\mathcal{G}$ is a P-geometry of rank 4 and $|H_1/H_2| \leq 2$ then $G_4 \cong L_4(2)$ and G_1 is isomorphic to either M_{22} or $3 \cdot M_{22}$* **(M_{23}-shape.)**

Proof. By (9.3.2) we have $H_2 = 1$ and hence $|H_1| \leq 2$. If $H_1 = 1$ then $G_4 \cong L_4(2)$ and $|G_{14}| = 2^6 \cdot 3 \cdot 7$. Hence $K_1 = 1$ and $G_1 \cong M_{22}$ or $3 \cdot M_{22}$. So it only remains for us to show that $|H_1| \neq 2$. Suppose on the contrary that $H_1 \cong 2$. Then $H_1 = Z(G_4) = Z(G_{34})$. Since G_{34} is of index two in G_3, we obtain that H_1 is normal in both G_4 and G_3; by (9.2.1) this is a contradiction. □

Now assume that $|H_1/H_2| \geq 2$. Then by (9.3.1), $H_1/H_2 \cong 2^6$, the module being the second exterior power of the natural module for G_4. It follows that

$$m_4(X) = m_4(N) + m_4(N^*) \geq 3.$$

Since $O^2(G_{14}/K_1) \cong 2^3.L_3(2)$ involves exactly one 3-dimensional factor, at least two such factors are in K_1. Therefore, $E = K_1/L_1$ is non-trivial.

Recall that by (4.2.4) the universal representation module of $\mathcal{G}(M_{22})$ is isomorphic to the 11-dimensional Todd module $\overline{\mathcal{C}}_{11}$; as a module for M_{22} the latter module is an indecomposable extension of the 1-dimensional trivial module by the 10-dimensional Todd module $\overline{\mathcal{C}}_{10}$. By (4.4.6) the universal representation module for $\mathcal{G}(3 \cdot M_{22})$ is the direct sum

$$\overline{\mathcal{C}}_{11} \oplus T_{12},$$

where T_{12} is a 12-dimensional self-dual irreducible $3 \cdot \mathrm{Aut}\, M_{22}$-module on which the normal subgroup of order 3 acts fixed-point freely. Since E is non-trivial (as a module for $\overline{G}_1$), it involves either $\overline{\mathcal{C}}_{10}$, or T_{12}, or both. In either case, $m_1(X) \geq 4$. Returning to H, we obtain from (9.3.1) and (9.3.2) that $H_2/H_3 \cong 2^4$, the dual natural module. Now the branching starts. Let us consider the possibilities in turn.

Proposition 10.3.3 *Let $\mathcal{G}$ be a T-geometry of rank 4 and G be a flag-transitive automorphism group of $\mathcal{G}$. Suppose that E^* involves $\overline{\mathcal{C}}_{10}$. Then* $G_4 \sim 2.2^4.2^6.L_4(2)$,

$$G_1 \sim 2^{10}.\mathrm{Aut}\, M_{22} \quad \text{or} \quad 2^{10}.3 \cdot \mathrm{Aut}\, M_{22}.$$

Furthermore, $K_1 = O_2(G_1)$ is the irreducible Golay code module $\mathcal{C}_{10}$ for $G_1/O_{2,3}(G_1) \cong \mathrm{Aut}\, M_{22}$ (Co_2-**shape.**)

Proof. By the assumption and the paragraph before the lemma we know that E^* possesses a quotient isomorphic to $\overline{\mathcal{C}}_{10}$. Hence E contains a submodule U, isomorphic to $\mathcal{C}_{10}$. Let $\widehat{U}$ be the full preimage of that submodule (subgroup) in K_1. Since $|L_1| \leq 4$ and since $\mathcal{C}_{10}$ is not self-dual, we conclude that $\widehat{U}$ is an abelian group. Furthermore, since the only other possible non-1-dimensional chief factor of G_1 in K_1 is T_{12}, which has dimension 12 (rather than 10), the $\widehat{U}$ falls into $Z(K_1)$. It follows from [MSt90] and [MSt01] that $\mathcal{C}_{10}$ is not an FF-module for $\overline{G}_1$. So $J(S) = J(K_1)$ is normal in G_1, where $S \in Syl_2(G_{14})$. By (9.2.1), this means that $J(S)$ cannot be normal in G_4. Invoking (9.5.3), we conclude that $H_3 \cap Z(Q)$ is of index at most two in H_3 and $H_3 \cap Z(Q)$ is a submodule in the direct sum of a 1-dimensional module and the natural module of $\overline{G}_4$. In particular, $m_4(X) \leq 5$. Returning to E, we see that E^* cannot involve T_{12} along with $\overline{\mathcal{C}}_{10}$. Hence $E \cong \overline{\mathcal{C}}_{10}$ or $E \cong \overline{\mathcal{C}}_{11}$, so $m_1(X) = 4$.

By the above H_3 does not involve 3-dimensional chief factors for G_{14}, which implies by (9.3.4) and (9.3.5) that $|H_3| \leq 2$. Notice now that

$\overline{G}_1 \cong \mathrm{Aut}\, M_{22}$ or $3 \cdot \mathrm{Aut}\, M_{22}$, since H_2 induces a non-trivial action on $\Sigma_3(x_4)$. Considering G_{14} as a subgroup of G_4 we see that

$$|G_{14}| \le 2^{17} \cdot 3 \cdot 7.$$

On the other hand, considering G_{14} as a subgroup of G_1 we have

$$|G_{14}| \ge 2^{17} \cdot 3 \cdot 7.$$

Therefore, we have the equality in both cases. This implies the equalities

$$|H_3| = 2, \quad L_1 = 1 \text{ and } |E| = 2^{10}$$

and completes the proof. □

It remains for us to consider the case where E^* is non-trivial but does not involve $\overline{\mathscr{C}}_{10}$. In that case $E \cong E^* \cong T_{12}$ (since T_{12} is self-dual) and this situation is covered by the following lemma.

Lemma 10.3.4 *Let $\mathscr{G}$ be a P-geometry of rank 4 and G be a flag-transitive automorphism group of $\mathscr{G}$. Suppose that $E \cong T_{12}$. Then*

$$G_1 \sim 2.2^{12}.3 \cdot \mathrm{Aut}\, M_{22} \quad \textit{and} \quad G_4 \sim 2^4.2^4.2^6.L_4(2).$$

(J_4-shape.)

Proof. The hypothesis of the lemma immediately implies that $m_1(X) = 5$ and hence H_3 involves exactly one non-trivial composition factor. By (9.3.4) and (3.2.7) we obtain that $H_3 \cap Z(Q)$ has index at most two in H_3 and $H_3 \cap Z(Q)$ is either the natural module, or that plus a 1-dimensional module. In particular,

$$|G_{14}| \ge 2^{20} \cdot 3 \cdot 7,$$

which implies that $|L_1| \ge 2$. Since $|L_1| \le 4$, H_2 involves at most one 1-dimensional composition factor. By (9.5.3), $H_3 \le Z(Q)$. Suppose $H_3 \cong 2^5$ and let $\langle g \rangle$ be the 1-dimensional submodule of H_3 (so that $g \in Z(H)$). Observe that $g \in K_1$. Since $E^* \cong T_{12}$, E, as a G_{14}-module, contains no 1-dimensional composition factors. Thus, $g \in L_1$ and hence $C_G(g)$ contains G_1^∞, leading to a contradiction, since also $C_G(g) \ge H$. Thus, $H_3 \cong 2^4$ and $|L_1| = 2$. Finally, since $\mathscr{G}$ is of Petersen type, we have $L_1 = N_1$ and hence $|N_1| = 2$. This completes the proof. □

Thus we have completed the consideration of the case where $\mathscr{G}$ is rank 4 of Petersen type. Now suppose $\mathscr{G}$ is of tilde type.

By (10.3.1) and (9.2.4) we have $H/Q \cong L_4(2)$ and $Q/H_1 \cong 2^4$. By the induction hypothesis we also have that $\mathrm{res}_{\mathcal{G}}(x_1)$ is one of the three known geometries:

$$\mathcal{G}(M_{24}),\quad \mathcal{G}(He) \text{ and } \mathcal{G}(3^7 \cdot S_6(2)).$$

In each of the three cases $\overline{G}_1$ is determined uniquely (as M_{24}, He, or $3^7 \cdot S_6(2)$, respectively) by the condition that it acts flag-transitively on $\mathrm{res}_{\mathcal{G}}(x_1)$.

Proposition 10.3.5 *Let $\mathcal{G}$ be a T-geometry of rank 4 and G be a flag-transitive automorphism group of $\mathcal{G}$. Suppose that $|H_1/H_2| \le 2$. Then*

(i) *$H = G_4$ is a split extension of $Q \cong 2^4$ by $L_4(2)$;*
(ii) *G_1 is isomorphic to M_{24} or He* **(Truncated M_{24}-shape.)**

Proof. By the hypothesis we conclude that $m_4(X) = 2$, which means that G_1 has no non-trivial chief factors in K_1. This yields $K_1 = L_1$. We claim that H_1 must be trivial. Indeed, let $\Theta = \Sigma[x_2]$. Consider the action of H_1 on Θ. Observe that H_1 acts trivially on $\Delta_1(x_4)$ and H_1 is normal in H. According to Table VII, the vertexwise stabilizer in G_3 of $\Theta_1(x_4)$ induces on $\Theta_2(x_4)$ a group 2^3 which is irreducible under the action of G_{24} by (9.3.1). This implies that H_1 acts trivially on $\Theta_2(x_4)$. Since for x_2 we can take any quint containing x_4, H_1 acts trivially on $\Delta_2(x_4)$, i.e., $H_1 = 1$. Hence

$$|G_{14}| = 2^{10} \cdot 3 \cdot 7.$$

For G_1 this means that either $G_1 \cong M_{24}$ or He, or $G_1/K_1 \cong 3^7 \cdot S_6(2)$ and $|K_1| = 2$.

Now we are going to prove (i). The subgroup G_3 induces on $\mathrm{res}_{\mathcal{G}}(x_3)$ the group $\overline{G}_3 \cong Sym_3 \times L_3(2)$. Hence $|K_3| = 2^6$. Let g be an element of order three such that $\langle g \rangle$ maps onto the normal subgroup of order three in $\overline{G}_3$. Observe that G_{34} has two 3-dimensional chief factors in K_3. This implies that either g acts trivially on K_3, or it acts on K_3 fixed-point freely. In the former case one of the minimal parabolics is not 2-constraint. This yields a contradiction, since G_1 contains such a minimal parabolic. Hence g acts on K_1 fixed-point freely. It follows that

$$C_{G_3}(g) \cong 3 \times L_3(2).$$

Let $R = C_{G_3}(g)^{\infty}$. Observe that $(H_1 \cap Q)^g R$ is a complement to Q in G_{34}. It follows from Gaschütz' theorem (8.2.8) that H splits over Q and (i) follows.

Suppose that $\overline{G}_1 \cong 3^7 \cdot S_6(2)$. Set $R = O_2(G_{14})$. The subgroup K_1

is the unique normal subgroup of order two in G_{14}. Considering G_{14} as a subgroup of $G_4 \cong 2^4 : L_4(2)$, we see that, as a G_{14}/R-module, $R/K_1 \cong 2^6$ is a direct sum of the natural module and the module dual to the natural module. On the other hand, considering G_{14}/K_1 as a subgroup of $\overline{G}_1 \cong 3^7 \cdot S_6(2)$ and factoring out the normal subgroup 3^7, we obtain that the same R/K_1 is an indecomposable module, which is a contradiction that implies (ii). □

Proposition 10.3.6 *Let $\mathscr{G}$ be a T-geometry of rank 4 and G be a flag-transitive automorphism group of $\mathscr{G}$. Suppose that $|H_1/H_2| > 2$ and $\overline{G}_1 \ncong 3^7 \cdot S_6(2)$. Then $G_1 \sim 2^{11}.M_{24}$ and $K_1 = O_2(G_1)$ is the irreducible Golay code module $\mathscr{C}_{11}$ for $\overline{G}_1 \cong M_{24}$* (*$Co_1$*-**shape.**)

Proof. In view of (9.3.1), we have $H_1/H_2 \cong 2^6$. Consequently, $m_4(X) \geq 4$. Since in G_{14}/K_1 we only find two non-trivial chief factors, we conclude that $K_1 \neq L_1$. If $\text{res}_{\mathscr{G}}(x_1) \cong \mathscr{G}(He)$ then, according to (4.6.1), $\dim E^{\bullet}$ is at least 51. So the order of a Sylow 2-subgroup S of G_{14} is at least 2^{61}. On the other hand, taking into account (9.3.1), (9.3.3) and (9.3.4), we compute that $|S| \leq 2^{6+4+6+4+30} = 2^{50}$,which is a contradiction that rules out this case.

Thus we can assume that $\text{res}_{\mathscr{G}}(x) \cong \mathscr{G}(M_{24})$. Then according to (4.3.1), $E^{\bullet} \cong \overline{\mathscr{C}}_{11}$, the irreducible Todd module. Now we can compute that $m_1(X) = 5$. Therefore, $H_2/H_3 \cong 2^4$ (compare (9.3.1) and (9.3.2)). Furthermore, H has no non-1-dimensional chief factors in H_3. It follows from (9.3.4) and (9.3.5) that $|H_3| \leq 2$. Computing the order of G_{14} in two ways, we see that $|H_3| = 2$ and $L_1 = 1$. This completes the proof. □

We will deal with the possibility that $\text{res}_{\mathscr{G}}(x_1) \cong \mathscr{G}(3^7 \cdot S_6(2))$ in Section 10.6 where we will obtain an infinite series of configurations involving the symplectic groups. Notice that we have proved that $H_1/H_2 \cong 2^6$ even if $\text{res}_{\mathscr{G}}(x_1) \cong \mathscr{G}(3^7 \cdot S_6(2))$.

10.4 Rank five case

Here we split cases according to the isomorphism type of the point residue $\text{res}_{\mathscr{G}}(x_1)$. As usual we start with Petersen type geometries. The universal representation group of $\mathscr{G}(M_{23})$ is trivial and by Proposition 6 in the Preface we obtain the following.

Proposition 10.4.1 *$\mathscr{G}(M_{23})$ is not the residue of a point in a flag-transitive P-geometry of rank 5.* □

Now we look at the residue when it is the P-geometry $\mathcal{G}(Co_2)$ or its universal 2-cover $\mathcal{G}(3^{23} \cdot Co_2)$.

Proposition 10.4.2 *Let $\mathcal{G}$ be a P-geometry of rank 5, G be a flag-transitive automorphism group of $\mathcal{G}$. Suppose that* $\mathrm{res}_{\mathcal{G}}(x_1) \cong \mathcal{G}(Co_2)$ *or* $\mathcal{G}(3^{23} \cdot Co_2)$. *Then $\overline{G}_1 \cong Co_2$ or $3^{23} \cdot Co_2$, respectively, $|L_1| = 2$, $E = K_1/L_1$, as a module for $\overline{G}_1/O_{2,3}(\overline{G}_1)$, is isomorphic to the 22-dimensional section $\overline{\Lambda}^{(22)}$ of the Leech lattice modulo 2* **(BM-shape.)**

Proof. By Table VIIa in addition to $(*_2)$ and $(*_3)$ we also have $(*_4)$. So $H_5 = 1$. Considering the image of G_{15} in $\overline{G}_1$ (see Table VIIa once again), we determine that $m_1(X)$ (which is the number of non-trivial chief factors of G_{15} inside K_{15}) is at least 2. Hence $H_1/H_2 \cong 2^{10}$ by (9.3.1). In turn, this means that $m_5(X) \geq 3$, and hence $K_1 \neq L_1$. By (5.2.3 (v)) and the paragraph after the proof of that proposition we have

$$E \cong \overline{\Lambda}^{(23)} \quad \text{or} \quad \overline{\Lambda}^{(22)}.$$

From the structure of these modules we deduce that $m_1(X) = 5$. Now it follows that $H_2/H_3 \cong 2^{10}$, $H_3/H_4 \cong 2^5$ and H_4 contains, as an H-module, a unique non-trivial composition factor. Now (9.3.4) and (3.2.7) imply that H_4 is either the natural module or the direct sum of that with a 1-dimensional module. Suppose H_4 contains a 1-dimensional submodule, say $\langle g \rangle$. Then, clearly, g acts trivially on $\Sigma = \Sigma[x_1]$ and so $g \in K_1$. Furthermore, it follows from (5.2.4) that G_{15} acting on $E = K_1/L_1$ does not leave invariant a 1-space. Hence $g \in L_1$. However, this means that

$$C_G(g) \geq \langle G_5, G_1^{\infty} \rangle,$$

which is a contradiction. Hence $H_4 \cong 2^5$. It remains for us to determine whether $E \cong \overline{\Lambda}^{(22)}$ and $|L_1| = 2$ (since $\mathcal{G}$ is a Petersen type geometry, we have $L_1 = N_1$), or $E \cong \overline{\Lambda}^{(23)}$ and $L_1 = 1$. Suppose the latter holds. Then K_1 is an abelian group. Observe that $H_4 \leq K_1$. This means that $K_1 \leq C_H(H_4) = H_1$, i.e., $K_1 \leq H_1$. However, this means that K_1 acts trivially on $\mathrm{res}_{\mathcal{G}}(x_5)$. Since G_1 is transitive on the vertices of Σ, K_1 stabilizes every $\Sigma[y]$ where y is a point (an element of type 1) incident with a vertex of Σ. This yields $K_1 = L_1$, which is a contradiction. Hence $E = K_1/L_1 \cong \overline{\Lambda}^{(22)}$ and $|L_2| = 2$, which gives the proof. □

Proposition 10.4.3 *Let $\mathcal{G}$ be a P-geometry of rank 5 and G be a flag-transitive automorphism group of $\mathcal{G}$. Suppose that* $\mathrm{res}_{\mathcal{G}}(x_1) \cong \mathcal{G}(J_4)$. *Then $G_1 \cong J_4$ and $G_5 \sim 2^{10}.L_5(2)$* **(Truncated J_4-shape.)**

Proof. Notice that in the considered situation $(*_4)$ might not hold. So we need to use a different line of attack. First suppose that $|H_1| \leq 2$. Then $H_2 = 1$ and $m_5(X) = 1$, whereas, when we view G_{15}/K_1 as a subgroup of $\overline{G}_1 \cong J_4$, we find that $m_1(X) \geq 3$. The contradiction proves that $H_1/H_2 \cong 2^{10}$. (Since $(*_2)$ holds, (9.3.1) applies and $|S/H_2| = 2^{20}$, where $S \in Syl_2(G_{15})$.) We now turn to G_1. By (7.1.3) the universal representation module of $\mathcal{G}(J_4)$ is trivial and by (9.4.2) we have $K_1 = L_1$. Furthermore, by (9.4.4), $L_1 = N_1$. Since $(*_i)$ holds for $i = 2$ and 3, we obtain from (9.4.5) that $K_1 \leq H_3$. This gives $|S/H_3| \leq |S/K_1| = 2^{20}$. Therefore, $H_2 = H_3 = K_1 = 1$ and the proof follows. □

Now suppose $\mathcal{G}$ is of tilde type. The case $\mathrm{res}_{\mathcal{G}}(x) \cong \mathcal{G}(3^{35} \cdot S_8(2))$ will be considered in Section 10.6 along with other configurations involving the symplectic groups. So we have only one possibility to consider here.

Proposition 10.4.4 *Let $\mathcal{G}$ be a T-geometry of rank 5, G be a flag-transitive automorphism group of $\mathcal{G}$. Suppose that $\mathrm{res}_{\mathcal{G}}(x_1) \cong \mathcal{G}(Co_1)$. Then $G_1 \sim 2.2^{24}.Co_1$, where L_1 is of order 2, and K_1/L_1 is $\overline{\Lambda}^{(24)}$, the Leech lattice modulo* 2 (*M*-**shape.**)

Proof. In this proposition $(*_i)$ holds for $i = 2$, 3 and 4. In particular, $H_5 = 1$ and $|L_1| \leq 4$. By (5.3.2) we have that $E = \overline{\Lambda}^{(24)}$. Since the condition $(**)$ fails for $\mathcal{G}(Co_1)$, (9.4.4 (iv)) implies that $L_1 = N_1$. Thus, $|L_1| \leq 2$. We claim that K_1 is non-abelian, and hence $L_1 \neq 1$. If K_1 is abelian then $K_1 \leq C_H(H_4) \leq Q$, since $H_4 \leq K_1$. Therefore, K_1 acts trivially on $\mathrm{res}_{\mathcal{G}}(x_5)$ and, by the transitivity of G_1 on the vertices of Σ, it acts trivially on $\mathrm{res}_{\mathcal{G}}(w)$ for all vertices $w \in \Sigma$. However, this means that $K_1 = L_1$, which is a contradiction. Thus, $|L_1| = 2$. We can now compute that $m_1(X) = 8$ and that $|S| = 2^{46}$. This forces $H_1/H_2 \cong 2^{10}$, $H_2/H_3 \cong 2^{10}$, $H_3/H_4 \cong 2^5$, and also that H_4 has two composition factors: a 1-dimensional and a 5-dimensional. □

10.5 Rank six case

Suppose $n = 6$ and $\mathcal{G}$ is not of $S_{12}(2)$-shape. Then $\mathrm{res}_{\mathcal{G}}(x)$ is either $\mathcal{G}(BM)$, or $\mathcal{G}(3^{4371} \cdot BM)$, or $\mathcal{G}(M)$. In all three cases the universal representation module is trivial. This is the reason, in a sense, why none of these geometries appears as a point residue in a flag-transitive P- or T-geometry of rank 6 (cf. Proposition 6 in the Preface).

10.6 The symplectic shape

In this section we prove the following.

Proposition 10.6.1 *Let $\mathcal{G}$ be a T-geometry of rank $n \geq 4$ and G be an automorphism group of $\mathcal{G}$. Suppose that*

$$\mathrm{res}_{\mathcal{G}}(x_1) \cong \mathcal{G}(3^{[\frac{n-1}{2}]_2} \cdot S_{2n-2}(2)).$$

Then

$$G_1 \sim 2.2^{2n-2}.3^{[\frac{n-1}{2}]_2} \cdot S_{2n-2}(2);$$

$Z(G_1)$ is of order 2 and $K_1/Z(G_1)$ is the natural symplectic module for $\overline{G}_1/O_3(\overline{G}_1) \cong S_{2n-2}(2)$;

$$G_n \sim 2^{n(n-1)/2}.2^n.L_n(2);$$

L_n is the exterior square of the natural module of $\overline{G}_n \cong L_n(2)$ and K_n/L_n is the natural module for $\overline{G}_n$ (**$S_{2n}(2)$-shape.**)

Proof. First we claim that $H_2 = 1$. Indeed, let w be a vertex at distance three from x_n. By (9.1.5) without loss of generality we may assume that w is contained in $\Theta = \Sigma[x_{n-3}]$. According to Table VIIb, H_2 acts trivially on Θ, and hence H_2 fixes w. Since w was arbitrary, $H_2 = H_3$ and hence $H_2 = 1$. Next, we claim that $|H_1| > 2$. Indeed, if $n = 4$ then this was shown in Section 10.3. Thus, without loss of generality, we may assume that $n \geq 5$. We have $H/Q \cong L_n(2)$ and $Q/H_1 \cong 2^n$. Hence $m_n(X) = 2$, where as above $m_n(X)$ is the number of non-1-dimensional chief factors of G_{1n} in $O_2(G_{1n})$. Furthermore, the non-1-dimensional chief factors inside H/Q and inside Q/H_1 have dimension $n-1$. On the other hand, considering the image of G_{1n} in $\overline{G}_1$, we immediately obtain that G_{1n} has chief factors of dimensions $n-1$ and $(n-1)(n-2)/2$. The latter is clearly greater than $n-1$, which is a contradiction. Hence $|H_1| > 2$. As $(*_2)$ holds, we have that $H_1 \cong \bigwedge^2 U$, the second exterior power of the natural module U of $\overline{G}_1$. Since also $H/Q \cong L_n(2)$ and $Q/H_1 \cong 2^n$, we know the exact size of H and also that $m_n(X) = 4$. Now turning to G_1 we find that $\overline{G}_1 \cong 3^{[\frac{n-1}{2}]_2} \cdot S_{2n-2}(2)$ and $|K_1| = 2^{2n-1}$. Comparing this with the structure of the universal representation module of $\mathcal{G}(3^{[\frac{n-1}{2}]_2} \cdot S_{2n-2}(2))$ (compare the paragraph before (2.4.1), (3.5.1) and (3.10.1)), we see that the faithful component of that module is not present in E^*, where $E = K_1/L_1$. Therefore, $E \cong 2^{2n-2}$ and $|L_2| = 2$, or $E \cong 2^{2n-1}$ and $L_1 = 1$.

The second possibility can be ruled out by induction on n. By (10.2.8) it does not take place for $n = 3$. Suppose it does take place for $n = 4$. Then $E = K_1/L_1$ is the dual of the 7-dimensional orthogonal module for $\overline{G}_1/O_3(\overline{G}_1) \cong S_6(2)$ and by (10.2.8) $G_{12}/K_1 \cong 2^{1+4} : 3 \cdot S_4(2)$. Let us turn to G_2. By (9.4.1), K_2/L_2 is a tensor product of the 2-dimensional module of

$G_2/K_2^+ \cong Sym_3$ and a representation module U_2^+ of $\mathrm{res}^+_{\mathscr{G}}(x_2) \cong \mathscr{G}(3{\cdot}S_4(2))$. This representation module is 5-dimensional when considered as a section of E and 4-dimensional when considered as a section of $O_2(G_{12}/K_1)$, which is a contradiction. Similar argument works for larger n (see [ShSt94] for any missing details). □

10.7 Summary

In this section we present Tables VIIIa and VIIIb where we summarize the possible shapes of P- and T-geometries respectively (cf. (10.2.1), (10.2.2), (10.2.7), (10.2.8), (10.3.2), (10.3.3), (10.3.4), (10.3.5), (10.3.6), (10.4.2), (10.4.3), (10.4.4) and (10.6.1)). In the tables 'Tr' means 'truncated'.

Table VIIIa. Shapes of amalgams for P-geometries

rank	shape	G_1	G_n
3	M_{22}	$2^4.Sym_5$	$2^3.L_3(2)$
3	Aut M_{22}	$2^5.Sym_5$	$2.2^3.L_3(2)$
4	M_{23}	$(3\cdot)M_{22}$	$L_4(2)$
4	Co_2	$2^{10}.(3\cdot)\mathrm{Aut}\,M_{22}$	$2.2^4.2^6.L_4(2)$
4	J_4	$2.2^{12}.3\cdot\mathrm{Aut}\,M_{22}$	$2^4.2^4.2^6.L_4(2)$
5	Tr J_4	J_4	$2^{10}.L_5(2)$
5	BM	$2.2^{22}.(3^{23}\cdot)Co_2$	$2^5.2^5.2^{10}.2^{10}.L_5(2)$

Table VIIIb. Shapes of amalgams for T-geometries

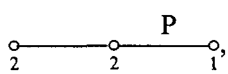
rank	shape	G_1	G_n
3	Alt_7	$3\cdot Alt_6$	$L_3(2)$
3	$S_6(2)$	$2^5.3\cdot Sym_6$	$2^3.2^3.L_3(2)$
3	M_{24}	$2^6.3\cdot Sym_6$	$2.2^3.2^3.L_3(2)$
4	Tr M_{24}	M_{24} or He	$2^4.L_4(2)$
4	Co_1	$2^{11}.M_{24}$	$2.2^4.2^6.2^4.L_4(2)$
5	M	$2.2^{24}.Co_1$	$2^6.2^5.2^{10}.2^{10}.2^5.L_5(2)$
n	$S_{2n}(2)$	$2.2^{2n-2}.3^{[\frac{n-1}{2}]_2}\cdot S_{2n-2}(2)$	$2^{n(n-1)/2}.2^n.L_n(2)$

11
Amalgams for P-geometries

In this chapter we consider the amalgam of maximal parabolics with shapes given in Table VIIIa. We consider the seven shapes one by one in the seven sections of the chapter. In Section 11.6 we show that an amalgam of truncated J_4-shape does not lead to a P-geometry. Originally this result was established in [Iv92b] and here we present a much shorter proof which makes an essential use of the classification of the flag-transitive T-geometries of rank 4. For the remaining shapes we prove that the isomorphism type of an amalgam is uniquely determined by that of $\overline{G}_1$. Thus there is a unique isomorphism type of amalgam for each of the shapes M_{22}, $\mathrm{Aut}\, M_{22}$ and J_4 and two types for the shapes M_{23}, Co_2 and BM. Let $\mathscr{A}$ be the amalgam of M_{23}-shape with $G_1 \cong 3 \cdot M_{22}$. If the universal completion of $\mathscr{A}$ is faithful, the corresponding coset geometry will be a 2-cover of $\mathscr{G}(M_{22})$. Since the latter geometry is 2-simply connected by Proposition 3.6.5 in [Iv99], there are no faithful completions. Thus up to isomorphism we obtain at most eight amalgams, which is exactly the number of amalgams coming from the known examples as in Table I. This proves Theorem 3 for P-geometries and in view of Proposition 4 and Theorem 2 completes the proof of Theorem 1 for P-geometries (see the Preface).

11.1 M_{22}-shape

In this section $\mathscr{G}$ is a rank 3 P-geometry with the diagram



G is a flag-transitive automorphism group of $\mathscr{G}$, such that

$$G_1 \sim 2^4.Sym_5, \quad G_3 \sim 2^3.L_3(2),$$

where $K_1 = O_2(G_1)$ is the natural module for $\overline{G}_1 \cong Sym_5$ and $K_3 = O_2(G_3)$ is the dual natural module for $\overline{G}_3 \cong L_3(2)$.

Lemma 11.1.1 *G_1 splits over K_1.*

Proof. Table VI in Section 8.2 shows that $H^2(\overline{G}_1, K_1)$ is trivial, hence the proof. □

Lemma 11.1.2 *G_3 splits over K_3.*

Proof. The subgroup G_1 induces the full automorphism group Sym_5 of the Petersen subgraph $\Sigma(x_1)$ with K_1 being the kernel. Hence by (11.1.1) G_{13} is the semidirect product of K_1 and a group $S \cong 2 \times Sym_3$. Let X be a Sylow 3-subgroup of S. Since K_1 is the natural module, the action of X on K_1 is fixed-point free. Hence $S = N_{G_1}(X)$. On the other hand, X is also a Sylow 3-subgroup of G_3 and $C_{K_3}(X)$ is of order 2. This shows that $K_3 = O_2(C_{G_{13}}(t))$ where t is the unique involution in $C_{G_{13}}(X)$. The action of X on K_1 turns the latter into a 2-dimensional $GF(4)$-vector space. Hence X normalizes 5 subgroups $T_1, ..., T_5$ of order 2^2 in K_1. It is clear that $K_1 \cap K_3$ is one of these subgroups. If σ is an involution in S which inverts X, then σ acts on $\mathcal{T} = \{T_1, ..., T_5\}$ as a transposition and hence normalizes a subgroup T from $\mathcal{T}$ other than $K_1 \cap K_3$. Then $\langle T, X, \sigma \rangle \cong Sym_4$ is a complement to K_3 in G_{13} and the proof is by Gaschütz theorem (8.2.8). □

Lemma 11.1.3 *The amalgam $\mathcal{D} = \{G_1, G_3\}$ is determined uniquely up to isomorphism.*

Proof. By (11.1.1), (11.1.2) and the proof of the latter lemma it is immediate that the type of $\mathcal{D}$ is uniquely determined. In order to apply Goldschmidt's theorem (8.3.2) we calculate the automorphism group of G_{13}. We claim that $\mathrm{Out}\, G_{13}$ is of order 2. Let τ be an automorphism of G_{13}. By Frattini argument we can assume that τ normalizes $S \cong Sym_3 \times 2$ (we follow notation introduced in the proof of (11.1.2)). Clearly $\mathrm{Out}\, S$ is of order 2. Thus it is sufficient to show that τ is inner whenever it centralizes S. The action of S on K_1 is faithful and we will identify S with its image in $\mathrm{Out}\, K_1 \cong L_4(2) \cong Alt_8$. It is an easy exercise to check that in the permutation action of Alt_8 on eight points the subgroup X is generated by a 3-cycle. From this it is easy to conclude that

$$C_{Alt_8}(S) = Z(S) = \langle t \rangle.$$

Thus the action of τ on K_1 is either trivial (and τ is the identity) or coincides with that of t. In the latter case τ is the inner automorphism induced by t.

Since $H^1(\overline{G}_3, K_3)$ is 1-dimensional, G_3 possesses an outer automorphism which permutes the classes of complements to K_3. Such an automorphism clearly does not centralize S and hence Goldschmidt's theorem (8.3.2) implies the uniqueness of $\mathscr{D}$. □

Let us turn to the parabolic G_2. Since K_3 is the dual natural module,

$$G_{23} = C_{G_3}(z) \sim 2^{1+4}.Sym_3,$$

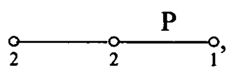
where z is an involution from K_3 and $K_2^- = O_2(G_{23})$. Since $[G_2 : G_{23}] = 2$, we observe that $G_2 \sim 2^{1+4}.(Sym_3 \times 2)$, which shows that $G_{12} = C_{G_1}(z)$ where $z \in K_1 \cap K_3$. Thus the subamalgam $\mathscr{F} = \{G_{12}, G_{23}\}$ is uniquely located inside $\mathscr{B}$ up to conjugation.

Proposition 11.1.4 *All the amalgams of* M_{22}*-shape are isomorphic to* $\mathscr{A}(M_{22}, \mathscr{G}(M_{22}))$ *and its universal completion is isomorphic to* $3 \cdot M_{22}$.

Proof. In view of the paragraph before the proposition all we have to show is that the universal completion $\widetilde{G}_2$ of $\mathscr{F}$ possesses at most one homomorphism ψ whose restriction to $K_2 \cong 2^{1+4}$ is an isomorphism and $\psi(\widetilde{G}_2)/\psi(K_2) \cong Sym_3 \times 2$. Since K_2 is extraspecial with centre of order 2, the kernel of ψ is of index 2 in $C_{\widetilde{G}_2}(K_2)$ disjoint from $Z(K_2)$. A direct application of (8.4.3) proves the uniqueness of ψ. The last sentence is by [Sh85] (see also Section 3.5 in [Iv99]). □

11.2 Aut M_{22}-shape

In this section $\mathscr{G}$ is a rank 3 P-geometry with the diagram



G is a flag-transitive automorphism group of $\mathscr{G}$ such that

$$G_1 \sim 2^5.Sym_5, \quad G_3 \sim 2^4.L_3(2),$$

where $K_1 = O_2(G_1)$ is the natural module for $\overline{G}_1 \cong Sym_5$, indecomposably extended by the trivial 1-dimensional module and $K_3 = O_2(G_3)$ is an extension of the trivial 1-dimensional module by the dual natural module of $\overline{G}_3 \cong L_3(2)$.

Lemma 11.2.1 G_3 *splits over* K_3.

Proof. Consider

$$G_{13} \sim 2^5.(Sym_3 \times 2) \sim 2^4.Sym_4$$

and let X be a Sylow 3-subgroup in G_{13} (which is also a Sylow 3-subgroup in G_1 and G_3). The structure of K_1 shows that $Y := O_2(C_{G_1}(X))$ is of order 2^2 and since $C_{G_3}(X) \leq K_3$, we conclude that $K_3 = C_{G_{13}}(Y)$. Considering the fixed-point free action of X on the codimension 1 submodule in K_1 we find (compare the proof of (11.1.1)) a subgroup T of order 2^2 in K_1 which is (a) disjoint from $K_1 \cap K_3$, (b) normalized by X, (c) normalized by an involution σ which inverts X. This produces a complement $\langle T, X, \sigma \rangle$ to K_3 in G_{13}. Since G_{13} contains a Sylow 2-subgroup of G_3, Gaschütz' theorem (8.2.8) completes the proof. □

Lemma 11.2.2 *K_3 is decomposable as a module for $\overline{G}_3 \cong L_3(2)$.*

Proof. Suppose to the contrary that K_3 is the indecomposable extension of the 1-dimensional submodule $Z(G_3)$ by the dual natural module. Then the orbits of G_3 on $K_3^\#$ are of length 1 and 14. This shows that whenever D is a Sylow 2-subgroup in G_3, the equality $Z(G_3) = Z(D)$ holds. We may assume that $D \leq G_{23}$. Since $[G_2 : G_{23}] = 2$, this implies that $Z(G_3)$ is normal in G_3 and in

$$G_2 = G_{23}N_{G_2}(D),$$

which is not possible by (9.2.1). Hence $G_3 \cong 2 \times 2^3 : L_3(2)$. □

Lemma 11.2.3 *G_1 splits over K_1.*

Proof. Denote by K_1' the codimension 1 submodule in K_1 and adopt the bar convention for the quotient of G_1 over K_1'. Since Sym_5 splits over its natural module K_1', it is sufficient to show that $\overline{G}_1 = 2 \times Sym_5$. In any case the centre of $\overline{G}_1$ is of order 2 and the quotient over the centre is Sym_5. If $\overline{G}_1$ is not as stated, it either contains $SL_2(5) \cong 2 \cdot Alt_5$ or is isomorphic to the semidirect product of Alt_5 and a cyclic group of order 4. In neither of these two cases is there a subgroup $\overline{G}_{13} \cong 2^2 \times Sym_3$. Hence the proof. □

Lemma 11.2.4 *The amalgam $\mathscr{D} = \{G_1, G_3\}$ is uniquely determined up to isomorphism.*

Proof. We claim that $\mathrm{Out}\, G_{13}$ is of order (at most) 4. Indeed, first it is easy to check that K_1 is the only elementary abelian 2-group of rank

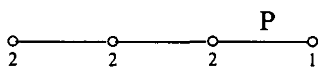
5 in G_{13} and hence it is characteristic. By Frattini argument without loss of generality we can assume that the automorphism τ, we consider, normalizes $N := N_{G_{13}}(X) \cong 2^2 \times Sym_3$. Since $|K_1 \cap N| = 2$, it is clear that N contains two classes of complements to K_1, which τ can permute. If $S \cong 2 \times Sym_3$ is one of the complements, then we know that $\mathrm{Out}\, S$ is of order 2 and hence the claim follows. By the proof of (11.1.4) we know that $\mathrm{Out}\, G_3$ is of order 2 and induces an outer automorphism σ_3 on G_{13}. By (8.2.3 (vi)), we know that $\mathrm{Out}\, G_1$ is also of order 2 and it induces an outer automorphism σ_1 of G_{13}. The automorphism σ_1 centralizes K_1 and hence it also centralizes the complement S modulo K_1. On the other hand, σ_3 centralizes K_3 and hence it normalizes a complement to K_1 in G_{13}. Thus σ_1 and σ_3 have different images in $\mathrm{Out}\, G_{13}$ and the proof follows by Goldschmidt's theorem (8.3.2). □

The final result of the section can be proved in a similar way to the proof of (11.1.4).

Proposition 11.2.5 *All the amalgams of* $\mathrm{Aut}\, M_{22}$*-shape are isomorphic to* $\mathscr{A}(\mathrm{Aut}\, M_{22}, \mathscr{G}(M_{22}))$ *and the universal completion of such an amalgam is isomorphic to* $3 \cdot \mathrm{Aut}\, M_{22}$. □

11.3 M_{23}-shape

In this section $\mathscr{G}$ is a rank 4 P-geometry with the diagram



and $G_4 \cong L_4(2)$. Then

$$G_{14} \cong 2^3 : L_3(2), \quad G_{24} \cong 2^4 : (Sym_3 \times Sym_3), \quad G_{34} \cong 2^3 : L_3(2)$$

are the maximal parabolics in G_4 associated with its action on $\mathrm{res}_{\mathscr{G}}(x_4)$ which is the rank 3 projective $GF(2)$-space.

We follow the dual strategy, so our first step is to classify up to isomorphism the amalgams $\mathscr{X} = \{G_4, G_3\}$ under the assumptions that $G_4 \cong L_4(2)$, $G_{34} \cong 2^3 : L_3(2)$ and $[G_3 : G_{34}] = 2$. Since G_{34} is normal in G_3, in order to determine the possible type of $\mathscr{X}$ we need the following.

Lemma 11.3.1 $\mathrm{Out}\, G_{34}$ *has order* 2.

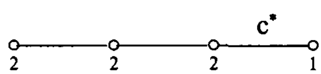
Proof. Since G_{34} is a maximal parabolic in $G_4 \cong L_4(2)$, we know that it is the semidirect product with respect to the natural action of $L \cong L_3(2)$

and $Q = O_2(G_{34})$ which is the natural module of L. If L' is another complement to Q in G_{34}, then clearly there is an automorphism of G_{34} which maps L onto L'. By (8.2.5) G_{34} contains exactly two conjugacy classes of such complements. Clearly an automorphism that sends L onto a complement which is not in the class of L is outer. Hence to complete the proof it is sufficient to show that an automorphism σ of G_{34} which preserves the classes of complements is inner. Adjusting σ by a suitable inner automorphism, we can assume that σ normalizes L. An outer automorphism of L exchanges the natural module with its dual. Since the dual module is not involved in Q, σ induces an inner automorphism of L and hence we can assume that σ centralizes L. In this case the action of σ on Q must centralize the action of L on Q. This immediately implies that σ acts trivially on Q. Hence σ is the identity automorphism and the proof follows. □

Lemma 11.3.2 *Let $\mathscr{X} = \{G_4, G_3\}$ be an amalgam such that $G_4 \cong L_4(2)$, $G_{34} \cong 2^3 : L_3(2)$ and $[G_3 : G_{34}] = 2$. Then $\mathscr{X}$ is isomorphic to one of two amalgams $\mathscr{X}^{(i)} = \{G_4^{(i)}, G_3^{(i)}\}$, $i = 1$ and 2, where $G_3^{(1)} \cong \mathrm{Aut}\, G_{34}$ and $G_3^{(2)} \cong G_{34} \times 2$.*

Proof. Since all subgroups in $G_4 \cong L_4(2)$ isomorphic to $2^3 : L_3(2)$ are conjugate in $\mathrm{Aut}\, G_4$ the type of $\mathscr{X}$ is determined by the isomorphism type of G_3. By (11.3.1) the type of $\mathscr{X}$ is that of $\mathscr{X}^{(1)}$ or $\mathscr{X}^{(2)}$. Since $\mathrm{Aut}\, G_3^{(i)} \cong \mathrm{Aut}\, G_{34}$ for both $i = 1$ and 2 and the centre of G_{34} is trivial, the type of $\mathscr{X}$ uniquely determines $\mathscr{X}$ up to isomorphism by (8.3.2). □

First let us show that the amalgam $\mathscr{X}^{(2)}$ does not lead to a P-geometry. Let $\mathscr{F}$ be the affine rank 4 geometry over $GF(2)$, which is formed by the cosets of the proper subspaces in a 4-dimensional $GF(2)$-space. The diagram of $\mathscr{F}$ is



and $A = AGL_4(2)$ is the flag-transitive automorphism group of $\mathscr{F}$. If A_i, $1 \le i \le 4$, are the maximal parabolics associated with the action of A on $\mathscr{F}$, then it is easy to see that $\{A_4, A_3\}$ is isomorphic to $\mathscr{X}^{(2)}$. An element of type 2 is incident to four elements of type 4 and its stabilizer A_2 induces Sym_4 on these four points with kernel $K_2^+ \cong 2^4 : Sym_3$. Furthermore, it is easy to check that the image of A_2 in $\mathrm{Out}\, K_2^+$ is Sym_3. Since A_2 is generated by A_{23} and A_{24}, the image is determined solely by the structure of $\{A_4, A_3\}$. Since no flag-transitive automorphism group of the Petersen graph possesses Sym_3 as a homomorphic image, the amalgam $\mathscr{X}^{(2)}$ indeed does not lead to a P-geometry.

Thus $\mathscr{X} = \{G_4, G_3\}$ is isomorphic to $\mathscr{X}^{(1)}$. Consider the action of $\overline{G} \cong M_{23}$ on $\overline{\mathscr{G}} = \mathscr{G}(M_{23})$ and let $\overline{G}_i$, $1 \le i \le 4$, be the maximal parabolics associated with this action. Then $\overline{\mathscr{X}} = \{\overline{G}_4, \overline{G}_3\}$ is also isomorphic to $\mathscr{X}^{(1)}$. Let $\overline{K}_2^+$ be the kernel of the action of $\overline{G}_2$ on $\mathrm{res}^+_{\mathscr{G}}(\overline{x}_2)$ (where $\overline{x}_2$ is the element of type 2 stabilized by $\overline{G}_2$). Then it is easy to deduce from the structure of $\overline{G}_2 \cong 2^4 : (3 \times Alt_5).2$ (compare p. 114 in [Iv99]) that $\overline{K}_2^+ \cong 2^4 : 3$ and the image of $\overline{G}_2$ in $\mathrm{Out}\,\overline{K}_2^+$ is isomorphic to Sym_5. Furthermore, an element of order 3 in $\overline{K}_2^+$ acts fixed-point freely on $O_2(\overline{K}_2^+)$, which implies that the centre of $\overline{K}_2^+$ is trivial and we have the following.

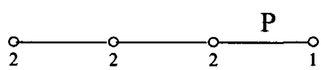
Lemma 11.3.3 *Let ψ be the natural homomorphism of the universal completion of $\mathscr{X} = \{G_4, G_3\}$ onto G and ψ_2 be the restriction of ψ to the subgroup $\widetilde{G}_2$ in the universal completion generated by the subgroups $G_{2i} = N_{G_i}(K_2^+)$ for $i = 3$ and 4. Then $\ker \psi_2 = C_{\widetilde{G}_2}(K_2^+)$.* □

By the above lemma the amalgam $\{G_2, G_3, G_4\}$ is isomorphic to the corresponding amalgam in $\overline{G} \cong M_{23}$. Furthermore the subamalgam $\mathscr{D} = \{G_{1i} \mid 2 \le i \le 4\}$ is uniquely determined and hence G_1 is either the universal completion of $\mathscr{D}$ (isomorphic to $3 \cdot M_{22}$) or the M_{22}-quotient of the universal completion. In the latter case $\mathscr{A} = \{G_i \mid 1 \le i \le 4\}$ is isomorphic to the amalgam of maximal parabolics in M_{23} while in the former case the coset geometry of the universal completion of $\mathscr{A}$ is the universal 2-cover of $\mathscr{G}(M_{23})$. By Proposition 3.6.5 in [Iv99] the geometry $\mathscr{G}(M_{23})$ is 2-simply connected, which gives the main result of the section.

Proposition 11.3.4 *All the amalgams of M_{23}-shape are isomorphic to $\mathscr{A}(M_{23}, \mathscr{G}(M_{23}))$ (in particular $G_1 \cong M_{22}$) and the universal completion of such an amalgam is M_{23}.* □

11.4 Co_2-shape

In this section $\mathscr{G}$ is a rank 4 P-geometry with the diagram



such that the residue of a point is isomorphic to either $\mathscr{G}(M_{22})$ or $\mathscr{G}(3 \cdot M_{22})$ and

$$G_1 \sim 2^{10}.\mathrm{Aut}\, M_{22} \quad \text{or} \quad G_1 \sim 2^{10}.3 \cdot \mathrm{Aut}\, M_{22}$$

with $K_1 = O_2(G_1)$ being the irreducible Golay code module $\mathscr{C}_{10}$ for $\overline{G}_1/O_3(\overline{G}_1) \cong \mathrm{Aut}\, M_{22}$ (where $\overline{G}_1 = G_1/K_1$ as usual.) We will assume that $\overline{G}_1 \cong \mathrm{Aut}\, M_{22}$, the arguments for the case when $\overline{G}_1 \cong 3 \cdot \mathrm{Aut}\, M_{22}$ are basically the same.

By Table VI in Section 8.2 the group $H^2(\mathrm{Aut}\, M_{22}, \mathscr{C}_{10})$ is non-trivial (1-dimensional), so a priori G_1 might or might not split over K_1. At this stage we can only say the following. Since $H^2(M_{22}, \mathscr{C}_{10})$ is trivial, the commutator subgroup G_1' of G_1 is the semidirect product of $\mathscr{C}_{10}$ and M_{22} with respect to the natural action. Since $H^1(M_{22}, \mathscr{C}_{10})$ is 1-dimensional, G_1' contains exactly two classes of complements to K_1. This shows that $O = \mathrm{Out}\, G_1'$ is elementary abelian of order 4 generated by the images of two automorphisms c and n, where c exchanges the classes of complements and commutes with $G_1'/K_1 \cong M_{22}$, while n normalizes one of the complements and induces on this complement an outer automorphism. Then the preimage in $\mathrm{Aut}\, G_1'$ of the subgroup $\langle cn \rangle$ of O is the unique non-split extension of $\mathscr{C}_{10}$ by $\mathrm{Aut}\, M_{22}$. Thus G_1 is isomorphic either to this extension or to the semidirect product of $\mathscr{C}_{10}$ and $\mathrm{Aut}\, M_{22}$ (the preimage in $\mathrm{Aut}\, G_1'$ of the subgroup $\langle n \rangle$). We will see in due course that the latter possibility holds.

We follow the direct strategy and reconstruct first the amalgam $\mathscr{B} = \{G_1, G_2\}$. The subgroup G_{12} is the preimage in G_1 of the stabilizer $\overline{S} \cong 2^5 : Sym_5$ in $\overline{G}_1$ of x_2 (which is a point in $\mathrm{res}_{\mathscr{G}}(x_1) \cong \mathscr{G}(M_{22})$). It follows from (4.2.6) that $\mathscr{C}_{10}$, as a module for $\overline{S}$, possesses the submodule series

$$1 < K_1^{(2)} < K_1^{(1)} < K_1,$$

where $K_1^{(2)} = C_{K_1}(O_2(\overline{S}))$ is the orthogonal module V_o of $\overline{S}/O_2(\overline{S}) \cong Sym_5$, $K_1^{(1)} = [K_1, O_2(\overline{S})]$ has codimension 1 in K_1 and $K_1^{(1)}/K_1^{(2)} \cong O_2(\overline{S})$ is the indecomposable extension of the natural module V_n of F by a trivial 1-dimensional module.

Recall that V_o is also the heart of the $GF(2)$-permutation module on 5 points. The orbits on the non-zero vectors in V_o have length 5 and 10 and V_o is the universal representation group of the derived system of $\mathscr{G}(Alt_5)$ (cf. (3.9.4)). The action of Sym_5 on the set of non-zero vectors in V_n is transitive. By (2.8.2) and Table VI in Section 8.2, $K_1^{(1)}/K_1^{(2)}$ is the largest extension V_n^u of V_n by trivial modules. We call V_n^u the *extended natural module* of Sym_5. The extended natural module is the dual of the universal representation module of $\mathscr{G}(Alt_5)$ factored over the 1-dimensional trivial Sym_5-submodule (notice that each of V_n and V_o is 4-dimensional and self-dual). The following result is similar to (12.6.2), and we follow its notation.

Lemma 11.4.1 *We have*

$$G_2 \sim 2^{4+8+2}.(Sym_5 \times Sym_3),$$

and furthermore

(i) $K_1 \cap K_2 = K_1^{(1)}$ *has index* 2 *in* K_1;
(ii) $K_2 = O_2(G_2)$ *and* K_2/L_2 *is the tensor product of the extended natural module of the* Sym_5*-direct factor of* $\overline{G}_2$ *and of the* 2*-dimensional module for the* Sym_3*-direct factor;*
(iii) $L_2 = \cap_{E \in \mathscr{E}} E$ *and* $L_2 \cong 2^4$ *is the orthogonal module for the* Sym_5*-direct factor of* $\overline{G}_2$;
(vi) *if* E *is an elementary abelian subgroup of order* 2^9 *in* K_2 *which is normal in* K_2^- *then* $E \in \mathscr{E}$. □

We know that at least G_1' splits over K_1 and hence $G_1' \cap G_{12}$ is a semidirect product of K_1 and a subgroup $T \cong 2^4 : Sym_5$, which maps isomorphically onto the stabilizer of x_2 in $\overline{G}_1' \cong M_{22}$. Since T is a maximal parabolic associated with the action of M_{22} on $\mathscr{G}(M_{22})$, we know that it splits over $O_2(T)$. Let $B \cong Sym_5$ be a complement to $O_2(T)$ in T,

$$C = \langle K_1, B\rangle \quad \text{and} \quad D = C \cap K_2^-.$$

Since K_1 induces on $\text{res}_{\mathscr{G}}^-(x_2)$ an action of order 2 with kernel $K_1^{(1)}$, we observe that D is an extension (split or non-split) of $K_1^{(1)}$ by Sym_5.

Lemma 11.4.2 *As a module for* $D/O_2(D) \cong Sym_5$, $K_1^{(1)}$ *possesses the direct sum decomposition:*

$$K_1^{(1)} = L_2 \oplus V_1,$$

where V_1 *maps isomorphically onto* $K_1^{(1)}/K_1^{(2)}$.

Proof. The result can be checked either by direct calculation in $\mathscr{C}_{10}$ or by noticing that L_2 being the orthogonal module is projective. □

Consider $\widetilde{D} = D/V_1$ which is an extension by Sym_5 of the orthogonal module $V_o \cong L_2$. Since $H^2(Sym_5, V_o)$ is trivial, $\widetilde{D}$ contains a complement $\widetilde{F} \cong Sym_5$ to $O_2(\widetilde{D})$.

Let F be the full preimage of $\widetilde{F}$ in D, so that F is an extension of V_1

(which is an elementary abelian subgroup of order 2^5) by Sym_5. Notice that by the construction we have

$$F < D < K_2^-.$$

Lemma 11.4.3 *Let t be a generator of a Sylow 3-subgroup of $O_{2,3}(G_2)$. Then*

(i) $F^t \leq G_1$ *and* $F^t \cap K_1 = 1$;
(ii) G_1 *splits over* K_1;
(iii) F *splits over* $O_2(F)$.

Proof. Since $F \leq K_2^-$ and K_2^- is normal in G_2, it is clear that $F^t \leq G_1$. Since $t \in G_2 \setminus G_{12}$, t permutes transitively the three subgroups constituting $\mathcal{E}$. Hence by (11.4.1 (iii)) we have

$$(K_1^{(1)})^t \cap K_1^{(1)} = L_2$$

and since $K_1^{(1)} = L_2 \oplus V_1$, where $V_1 = O_2(F)$, (i) follows. The image of F^t in $\overline{G}_1$ contains a Sylow 2-subgroup of $\overline{G}_1$ and hence (ii) follows from (i) and Gaschütz theorem. Finally, since F^t maps onto a maximal parabolic associated with the action of $\overline{G}_1 \cong \mathrm{Aut}\, M_{22}$ on $\mathcal{G}(M_{22})$, we know that it splits over its O_2, hence so does F. □

Thus G_1 is uniquely determined up to isomorphism and G_{12} is uniquely determined up to conjugation in G_1. The next lemma identifies K_2^- as a subgroup in G_{12} (recall that if P is a group, then P^∞ is the smallest normal subgroup in P such that P/P^∞ is solvable).

Lemma 11.4.4 *The following assertions hold:*

(i) K_2^- *is a semidirect product of* K_2 *and a subgroup* $X \cong Sym_5$;
(ii) L_2 *is the unique elementary abelian normal subgroup in* G_{12} *which is isomorphic to the orthogonal module for* X;
(iii) $O_2(G_{12}^\infty)/L_2$ *is the direct sum of two copies of the natural module for* X *and* $K_2 = C_{G_{12}}(O_2(G_{12}^\infty)/L_2)$;
(iv) *if* $Y = K_2/O_2(G_{12}^\infty)$ *then* Y *is elementary abelian of order* 2^2 *and* $K_2^- = C_{G_{12}}(Y)$.

Proof. (i) follows from (11.4.3 (iii)), the rest is an immediate consequence of (11.4.1). □

Our next objective is to calculate $\mathrm{Out}\, K_2^-$. Since the centre of K_2^- is trivial, G_2 is the preimage in $\mathrm{Aut}\, K_2^-$ of a Sym_3-subgroup in $\mathrm{Out}\, K_2^-$. We start with the following.

Lemma 11.4.5 *The group K_2^- contains exactly four classes of complements to $K_2 = O_2(K_2^-)$.*

Proof. By (11.4.4 (i)), X is one of the complements. Let $\mathscr{E} = \{E_i \mid 1 \leq i \leq 3\}$ and $E_1 = K_1^{(1)} = K_1 \cap K_2$. Then by (11.4.2), E_i as a module for X is the direct sum $L_2 \oplus V_i$, where L_2 is the orthogonal module and V_i is the extended natural module. It is easy to deduce from Table VI in Section 8.2 that $H^1(Sym_5, V_i)$ is one dimensional. Since $H^1(Sym_5, L_2)$ is trivial, by (8.2.1) we see that the group E_iX contains exactly two classes of complements with representatives $X_0 = X$ and X_i, where $1 \leq i \leq 3$. We claim that for $0 \leq i < j \leq 3$ the complements X_i and X_j are not conjugate in K_2^-. Let $X_i(j)$ denote the image of X_i in K_2^-/E_j. Clearly $X_0(j) = X_j(j)$, but for $k \neq j$ and $1 \leq k \leq 3$ the image E_kX in K_2^-/E_j is isomorphic to E_kX/L_2 and still contains two classes of complements, which shows that $X_0(j) \neq X_k(j)$ and proves the claim. In order to get an upper bound on the number of complements consider the normal series

$$L_2 < E_1 < K_2,$$

where L_2 is the orthogonal module while both E_1/L_2 and K_2/E_1 are isomorphic to the extended natural module V_1. We have seen already that all complements in L_2X are conjugate while V_1X contains two classes of complements. Hence altogether there are at most four classes of complements. □

Lemma 11.4.6 *The action of* Out K_2^- *on the set of four classes of complements to K_2 is faithful, in particular,* Out $K_2^- \leq Sym_4$.

Proof. Suppose that $\tau \in \mathrm{Aut}\,K_2^-$ stabilizes every class of complements as a whole. Then, adjusting τ by a suitable inner automorphism we can assume that τ normalizes $X_0 \cong Sym_5$ and since the latter group is complete, we can further assume that τ centralizes X_0. Consider the quotient $J = K_2^-/O_2(G_{12}^{\infty}) \cong 2^2 \times Sym_5$. Then the set of images in J of the complements X_i for $0 \leq i \leq 3$ forms the set of all Sym_5-subgroups in J, which shows that τ centralizes J. On the other hand, the images of the subgroups from $\mathscr{E}$ form the set of subgroups of order 2 in the centre of J. Hence τ normalizes every $E_i \in \mathscr{E}$. The action of τ on E_i must commute with the action of X on E_i. We know that E_i, as a module for X, is isomorphic to the direct sum of the orthogonal and the extended natural modules. Since these two modules do not have common composition factors, it is easy to conclude that τ must centralize E_i which shows that τ is the identity automorphism. □

Lemma 11.4.7 *Let $\widehat{G}_1$ be the semidirect product with respect to the natural action of the irreducible Golay code module $\mathscr{C}_{11}$ for M_{24} and* $\mathrm{Aut}\, M_{22}$ *(considered as a subgroup in M_{24}). Then*

(i) *$\widehat{G}_1$ contains G_1 with index 2;*
(ii) *$C_{\widehat{G}_1}(K_2^-)$ is trivial;*
(iii) *the image of $N_{\widehat{G}_1}(K_2^-)$ in* $\mathrm{Out}\, K_2^-$ *has order* 4.

Proof. (i) is immediate from (11.4.3 (ii)). It is easy to see that $\mathrm{Aut}\, M_{22}$ has three orbits on $\mathscr{C}_{11} \setminus \mathscr{C}_{10}$ with lengths 352, 616, 672 and with stabilizers Alt_7, $\mathrm{Aut}\, Sym_6$ and $PGL(2,11)$, respectively. This shows that $K_2^-/(K_2 \cap K_1) \cong 2^5 : Sym_5$ acts fixed-point freely on $\mathscr{C}_{11} \setminus \mathscr{C}_{10}$, which implies (ii), since we already know that the centre of K_2^- is trivial. It is clear that K_2^- has index 4 in its normalizer in $\widehat{G}_1$, so (ii) gives (iii). □

Lemma 11.4.8 $\mathrm{Out}\, K_2^- \cong Sym_4$.

Proof. By (11.4.6) all we have to do is present sufficient number of automorphisms. Since K_2^- is isomorphic to the corresponding subgroup associated with the action of Co_2 on $\mathscr{G}(Co_2)$, we know that $\mathrm{Out}\, K_2^-$ contains Sym_3. By (11.4.7) it also contains a subgroup of order 4, hence the proof. □

Proposition 11.4.9 *The amalgam $\mathscr{B} = \{G_1, G_2\}$ is uniquely determined up to isomorphism.*

Proof. Since all Sym_3-subgroups in Sym_4 are conjugate, by (11.4.3 (ii)), (11.4.4 (iv)) and (11.4.8) the type of $\mathscr{B}$ is uniquely determined and it only remains to apply Goldschmidt's theorem. Since the centralizer of K_2^- in G_{12} is trivial, it is easy to see that $\mathrm{Aut}\, G_{12}$ coincides with the normalizer of G_{12} in $\mathrm{Aut}\, K_2^-$. So $\mathrm{Out}\, G_{12}$ has order 2. On the other hand, by (11.4.7 (iii)) the image of $N_{\mathrm{Aut}\, G_1}(G_{12})$ in $\mathrm{Out}\, G_{12}$ is also of order 2. Hence the type of $\mathscr{B}$ determines $\mathscr{B}$ up to isomorphism. □

Now (8.6.1) applies and gives the following.

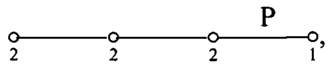
Proposition 11.4.10 *An amalgam $\mathscr{A}$ of Co_2-shape is isomorphic to either*

$$\mathscr{A}(Co_2, \mathscr{G}(Co_2)) \quad \text{or} \quad \mathscr{A}(3^{23} \cdot Co_2, \mathscr{G}(3^{23} \cdot Co_2))$$

and the universal completion of $\mathscr{A}$ is isomorphic to either Co_2 or $3^{23} \cdot Co_2$, respectively. □

11.5 J_4-shape

In this section $\mathcal{G}$ is a P-geometry of rank 4 with the diagram



the residue of a point is isomorphic to $\mathcal{G}(3 \cdot M_{22})$,

$$G_1 \sim 2.2^{12}.3 \cdot \mathrm{Aut}\, M_{22}, \quad G_4 \sim 2^4.2^4.2^6.L_4(2),$$

where L_1 is of order 2 and K_1/L_1 is the universal representation module of the extended system of $\mathcal{G}(3 \cdot M_{22})$. We start with the following.

Lemma 11.5.1 *$K_1 = O_2(G_1)$ is extraspecial of plus type, so that $G_1 \sim 2_+^{1+12}.3 \cdot \mathrm{Aut}\, M_{22}$.*

Proof. Since L_1 is of order 2 and K_1/L_1 is isomorphic to the universal representation module of the extended system of $\mathcal{G}(3 \cdot M_{22})$ on which $\overline{G}_1 \cong 3 \cdot \mathrm{Aut}\, M_{22}$ acts irreducibly, preserving a unique quadratic form of plus type, all we have to show is that K_1 is non-abelian.

We consider the action of G on the derived graph Δ of G and follow the notation in Chapter 9. The subgroup K_1 is the vertexwise stabilizer of the subgraph $\Sigma = \Sigma[x_1]$ induced by the vertices (the elements of type 4) incident to x_1. Since K_1/L_1 is non-trivial, K_1 acts non-trivially on $\Delta(x_4)$, which means that its image in $H/H_1 \cong L_4(2)$ is non-trivial. On the other hand, $H_3 \cong 2^4$ fixes every vertex whose distance from x_4 is at most 3 and since the action of $\overline{G}_1 \cong 3 \cdot M_{22}$ on Σ satisfies the $(*_3)$-condition, H_3 fixes Σ vertexwise and hence $H_3 \leq K_1$. Since H/H_1 acts faithfully on H_3, K_1 is non-abelian. □

Clearly G_{12} is the full preimage in G_1 of the stabilizer $G_{12}/K_1 \cong 2^5 : Sym_5$ of x_2 in $\overline{G}_1 \cong 3 \cdot \mathrm{Aut}\, M_{22}$. By (4.4.8) we know that (as a module for G_{12}/K_1) K_1/L_1 possesses a unique composition series $V^{(1)} < \ldots < V^{(5)} < K_1/L_1$. For $1 \leq i \leq 5$ let $K_1^{(i)}$ denote the full preimage of $V^{(i)}$ in K_1.

Lemma 11.5.2 *We have*

$$G_2 \sim 2^{2+1+4+8+2}.(Sym_5 \times Sym_3),$$

furthermore, if $\{x_1, y_1, z_1\}$ is the set of points incident to x_2, then

(i) *$K_1^{(5)} = K_1 \cap K_2$ has index 2 in K_1;*

(ii) $K^{(3)} = L_2$ *and* K_2/L_2 *is the tensor product of the extended natural module of* $K_2^-/K_2 \cong Sym_5$ *and the 2-dimensional module for* $K_2^+/K_2 \cong Sym_3$;

(iii) L_2 *is a maximal abelian subgroup in* K_1 *(of order* 2^7*)*;

(iv) $V := K^{(2)}$ *is elementary abelian of order* 2^3 *normal in* G_2;

(v) $K^{(1)} = \langle L(x_1), L(y_1), L(z_1)\rangle$ *is a normal subgroup of order* 4 *in* G_2 *and* $L_2/K^{(1)}$ *is the dual of the extended natural module of* K_2^-/K_2 *centralized by* $K_2^+/K_2 \cong Sym_3$.

Proof. Everything follows from (4.4.8). Notice that V is the largest subgroup in K_2 inside which all the chief factors of G_{12} are trivial. □

As an immediate consequence of (11.5.2 (v)) we obtain the following.

Lemma 11.5.3 *Let* φ *be the mapping of the point-set of* $\mathcal{G}$ *into* G *which sends* y *onto the unique involution in* $L(y)$. *Then* (G, φ) *is a* G*-admissible representation of* $\mathcal{G}$. □

The subgroup G_{12} is not maximal in G_1, since it is properly contained in $\widehat{G}_{12} = \langle G_{12}, X\rangle$, where X is a Sylow 3-subgroup of $O_{2,3}(G_1)$, so that $\widehat{G}_{12} = G_{12}O_{2,3}(G_1)$.

Lemma 11.5.4 *V is normal in* $\widehat{G}_{12}$.

Proof. The image $\overline{X}$ of X in $\overline{G}_1$ coincides with $O_3(\overline{G}_1)$. By (4.4.8), $\overline{X}$ normalizes $V^{(2)}$, which means that X normalizes V. □

By (11.5.2 (iv)) and (11.5.4), V is normal in both G_2 and $\widehat{G}_{12}$. Furthermore

$$G_{12} = G_2 \cap \widehat{G}_{12} \quad \text{and} \quad [G_2 : G_{12}] = [\widehat{G}_{12} : G_{12}] = 3.$$

Lemma 11.5.5 *Let* $C = C_{G_{12}}(V)$, $Q = O_2(C)$, $A = \mathrm{Aut}\, V \cong L_3(2)$, A_1 *and* A_2 *be the images in* A *of* $\widehat{G}_{12}$ *and* G_2, *respectively. Then*

(i) $\{A_1, A_2\}$ *is the amalgam of maximal parabolics in* A, *so that* A_1 *is the stabilizer of the* 1*-subspace* L_1 *and* A_2 *is the stabilizer of the* 2*-subspace* $K^{(1)}$ *in* V;

(ii) Q *is the normal closure of* $K^{(4)}$ *in* G_2 *of order* 2^{15} *and* $C/Q \cong Sym_5$;

(iii) C *is the largest subgroup in* G_{12} *normal in both* G_2 *and* $\widehat{G}_{12}$ *and*

$$C \sim 2^{1+1+1+4+4+4}.Sym_5.$$

Proof. Since K_1 is extraspecial by (11.5.1), it induces on V the group of all transvections with centre L_1. Since X acts on V non-trivially and X is fully normalized in $\widehat{G}_{12}$ it is clear that $\widehat{G}_{12}$ induces on V the full stabilizer of L_1 in A. Thus G_{12} induces the Borel subgroup D_8. Since G_2 induces Sym_3 on $K^{(1)}$, (iii) follows. By the above, K_2 induces on V an action of order 4, and hence (ii) follows from (11.5.2). The amalgam $\{A_1, A_2\}$ is simple and it is clear that

$$C_{G_2}(V) \leq G_{12} \text{ and } C_{\widehat{G}_{12}}(V) \leq G_{12},$$

hence (i) follows. □

By (11.5.5) we observe that

$$\widehat{G}_{12} \sim 2^{1+2+8+4}.(Sym_5 \times Sym_4), \quad G_2 \sim 2^{2+1+4+8}.(Sym_5 \times Sym_4).$$

Now we are going to make use of the T-subgeometries in $\mathcal{G}$. From Lemma 7.1.7 in [Iv99] and the paragraph before that lemma we can deduce the following.

Proposition 11.5.6 *The geometry $\mathcal{G}$ under consideration contains a family of T-subgeometries of rank 3, such that*

(i) *the element x_3 is contained in a unique subgeometry $\mathcal{S}$ from the family and* $\text{res}^-_{\mathcal{S}}(x_3) = \text{res}^-_{\mathcal{G}}(x_3)$*;*
(ii) *the stabilizer S of $\mathcal{S}$ in G acts on $\mathcal{S}$ flag-transitively;*
(iii) *the residue* $\text{res}_{\mathcal{S}}(x_1)$ *belongs to the family of $\mathcal{G}(3 \cdot S_4(2))$-subgeometries in* $\text{res}_{\mathcal{G}}(x_1) \cong \mathcal{G}(3 \cdot M_{22})$. □

By (11.5.6) $\{x_1, x_2, x_3\}$ is a maximal flag in $\mathcal{S}$ and $\{S_i = S(x_i) \mid 1 \leq i \leq 3\}$ is the amalgam of maximal parabolics associated with the action of S on $\mathcal{S}$ (we will see below that the action is not faithful).

Lemma 11.5.7 *The following assertions hold:*

(i) $S_3 = G_3 \sim [2^{18}].L_3(2)$;
(ii) $S_1 \sim 2^{1+6+6}.3.2^4.Sym_6$.

Proof. (i) follows from (11.5.6 (i)) while (ii) follows from (11.5.6 (iii)). □

Lemma 11.5.8 *Let K_S be the kernel of the action of S on $\mathcal{S}$ and $\overline{S} = S/K_S$. Then K_S is of order 2^{11} and $\mathcal{S} \cong \mathcal{G}(M_{24})$ or $\mathcal{S} \cong \mathcal{G}(He)$.*

Proof. By the classification of the rank 3 T-geometries $\mathcal{S}$ is isomorphic to

$$\mathcal{G}(M_{24}), \quad \mathcal{G}(He) \text{ or } \mathcal{G}(3^7 \cdot S_6(2)).$$

Suppose that $\mathcal{S}$ is isomorphic to the latter of the geometries and $\overline{S} \cong 3^7 \cdot S_6(2)$ (the only flag-transitive automorphism group of $\mathcal{G}(3^7 \cdot S_6(2))$). Then $S_1/K_S \cong 3 \cdot 2^4.Sym_6$ and it is easy to deduce from (11.5.7) that if X is a Sylow 3-subgroup of $O_{2,3}(S_1)$ then X acts faithfully on K_S. By considering the action of $S_6(2)$ on the set of hyperplanes of 3^7 it is easy to see that the smallest faithful $GF(2)$-representation of $\overline{S}$ has dimension 56. □

Thus K_S is of order 2^{11}, $S_1/K_S \cong 2^6 : 3 \cdot Sym_6$ and hence (compare (11.5.7 (ii))) $L_1 = L(x_1)$ is contained in K_S. Let φ_S be the restriction to $\mathcal{S}$ of the mapping as in (11.5.3). Then $\mathrm{Im}\,\varphi_S \leq K_S$ and $(\mathrm{Im}\,\varphi_S, \varphi_S)$ is an S-admissible presentation of $\mathcal{S}$. Clearly a quotient of $\mathrm{Im}\,\varphi_S$ over its commutator subgroup supports a non-trivial abelian representation of $\mathcal{S}$. By (4.6.1) every He-admissible representation of $\mathcal{G}(He)$ has dimension at least 51 and by (4.3.1) the only M_{24}-admissible representation of $\mathcal{G}(M_{24})$ is supported by the 11-dimensional Todd module, so we have the following.

Proposition 11.5.9 *The following assertions hold:*

(i) $\mathcal{S} \cong \mathcal{G}(M_{24})$;
(ii) $\overline{S} \cong M_{24}$;
(iii) $K_S \cong \overline{\mathcal{C}}_{11}$ *(the irreducible Todd module).* □

Now we are in a position to identify the subgroup $T = \langle \widehat{G}_{12}, G_2 \rangle$.

Lemma 11.5.10 *Let V be as in* (11.5.2 (iv)). *Then*

(i) *$N_S(V)$ contains K_S and $N_S(V)/K_S \cong 2^6 : (Sym_3 \times L_3(2))$ is the stabilizer of a trio in $\overline{S} \cong M_{24}$;*
(ii) *$\widehat{G}_{12} = (\widehat{G}_{12} \cap S)C$ and $G_2 = (G_2 \cap S)C$;*
(iii) *$T \cong 2^{3+12}.(Sym_5 \times L_3(2))$;*
(iv) *let $\psi : T \to \overline{T} = T/C \cong L_3(2)$ be the natural homomorphism and $\overline{\tau}$ be an involution from $\overline{T}$, then $\psi^{-1}(\overline{\tau})$ contains an involution.*

Proof. It is easy to notice that V is contained in K_S so that (i) follows from the basic properties of the irreducible Todd module $K_S \cong \overline{\mathcal{C}}_{11}$. Since $N_S(V)$ induces $L_3(2)$ on V, each of $\widehat{G}_{12} \cap S$ and $G_2 \cap S$ induces Sym_4, so (ii) follows from (11.5.5 (iii)). Finally (iii) is by (ii) and (11.5.5 (iii)).

In order to prove (iv), notice that $K_1 \cap C$ is the orthogonal complement to V with respect to the bilinear form induced by the commutator map on K_1. Hence $\psi(K_1)$ is an elementary abelian subgroup of order 2^2. Since all involutions in $\overline{T}$ are conjugate, we can assume that $\overline{\tau} \in \psi(K_1)$.

Since K_1 is extraspecial, it is easy to see (compare (4.4.7)) that there is an involution in $K_1 \setminus (K_1 \cap C)$. □

Let us take a closer look at the subgroup $S_1 = G_1 \cap S$ as in (11.5.7 (ii)). On the one hand, $K_1 \leq S_1$ and $S_1/K_1 \cong 2^4 : 3 \cdot Sym_6$ is the stabilizer in $\overline{G}_1 \cong 3 \cdot \mathrm{Aut}\, M_{22}$ of a $\mathcal{G}(3 \cdot S_4(2))$-subgeometry in $\mathrm{res}_{\mathcal{G}}(x_1) \cong \mathcal{G}(3 \cdot M_{22})$. On the other hand, $K_S \leq S_1$ and $S_1/K_S \cong 2^6 : 3 \cdot Sym_6$ is the stabilizer in $\overline{S} \cong M_{24}$ of x_1 considered as a point of $\mathcal{S}$.

Lemma 11.5.11 *The following assertions hold, where X is a Sylow 3-subgroup of $O_{2,3}(S_1)$:*

(i) *if $A = N_{S_1}(X) \sim [2^5].3 \cdot Sym_6$, then $O_2(A)$ is the indecomposable extension of a 1-dimensional module by the natural symplectic module of $A/O_{2,3}(A) \cong Sym_6 \cong S_4(2)$;*

(ii) *if $B = N_{G_1}(X) \sim 2.3 \cdot \mathrm{Aut}\, M_{22}$, then B' has index 2 in B, so B does not split over $L_1 = O_2(B)$;*

(iii) *$B' \cong 6 \cdot M_{22}$ is the unique covering group of M_{22} with centre of order 6;*

(iv) *G_1 splits over G_1';*

(v) *G_1 is isomorphic to the point-stabilizer of J_4 acting on $\mathcal{G}(J_4)$;*

(vi) *A splits over $O_2(A)$;*

(vii) *S splits over $K_S = O_2(S)$;*

(viii) *S is isomorphic to the stabilizer in J_4 of a $\mathcal{G}(M_{24})$-subgeometry in $\mathcal{G}(J_4)$.*

Proof. Since $O_2(A) \leq K_S$, (i) follows from Lemma 3.8.5 in [Iv99]. Since $O_2(A) \leq G_1'$ (ii) follows from (i). The Schur multiplier of M_{22} is cyclic of order 12 [Maz79], and since $\overline{G}_1$ does not split over its O_3, (iii) follows from (ii). In order to prove (iv) we need to show that $G_1 \setminus G_1'$ contains an involution. We follow notation as in (11.5.10 (iv)). By (11.5.5 (ii)) the images of $(G_1 \cap T)$ and $(G_1' \cap T)$ in $\overline{T}$ are isomorphic to Sym_4 and Alt_4, respectively. Hence the existence of the involution in $G_1 \setminus G_1'$ follows from (11.5.10 (iv)). Since $G_1/L_1 \cong 2^{12} : 3 \cdot \mathrm{Aut}\, M_{22}$ is the semidirect product of the universal representation module of the extended system of $\mathcal{G}(3 \cdot M_{22})$ and the automorphism group of this geometry, G_1/L_1 is uniquely determined up to isomorphism. Hence (v) follows from (iii) and (iv). Since A is contained in G_1, (v) implies (vi).

Let us prove (vii). Let $\mathcal{D}_1$ be the $\mathcal{G}(3 \cdot S_4(2))$-subgeometry in $\mathrm{res}_{\mathcal{G}}(x_1)$ such that S_1 is the stabilizer of $\mathcal{D}_1$ in G_1. Then $\mathcal{D}_1$ is the set of elements in $\mathrm{res}_{\mathcal{G}}(x_1)$ fixed by $O_2(S_1)/K_1 \cong 2^4$, in particular $\mathcal{D}_1$ is uniquely determined.

Let φ be the map from the point-set of $\mathrm{res}_{\mathcal{G}}(x_1)$ which turns K_1/L_1 into the representation module of the geometry. Then $K_S \cap K_1$ (of order 2^7) is the preimage in K_1 of $\varphi(\mathcal{D}_1)$. Furthermore, $K_S \cap K_1$ is the centralizer of $O_2(A)$ in K_1. Let $U_1 = [X, K_S \cap K_1]$. Then U_1 is a complement to L_1 in $K_S \cap K_1$ and it is a hexacode module for a complement $F \cong 3 \cdot Sym_6$ to $O_2(A)$ in A, which exists by (vi). Let $\mathcal{D}_2$ be another $\mathcal{G}(3 \cdot S_4(2))$-subgeometry in $\mathrm{res}_{\mathcal{G}}(x_1)$ such that the hexads in the Steiner system $S(3,6,22)$ (cf. Lemmas 3.4.4 and 3.5.8 in [Iv99]) corresponding to $\mathcal{D}_1$ and $\mathcal{D}_2$ are disjoint. Then the joint stabilizer $\overline{F}$ of $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\overline{G}_1$ is a complement to $O_2(S_1)/K_1$ in $S_1/K_1 \cong 2^4 : 3 \cdot Sym_6$. Without loss of generality we can assume that $\overline{F} = FK_1/K_1$ where F is the complement to $O_2(A)$ in A as above. Then F normalizes the subgroup U_2 in K_1 defined for $\mathcal{D}_2$ in the same way as U_1 was defined for $\mathcal{D}_1$. Since F acts irreducibly on U_1 and $U_1 \neq U_2$ (since $\mathcal{D}_1 \neq \mathcal{D}_2$) we have $U_1 \cap U_2 = 1$. Now $U_2F \cong 2^6 : 3 \cdot Sym_6$ is a complement to K_S in S_1. Since S_1 contains a Sylow 2-subgroup of S Gaschütz' theorem (8.2.8) gives (vii). Finally (viii) is immediate from (vii) and (11.5.9 (ii) and (iii)). □

By (11.5.11) and the paragraph before that lemma the type of the amalgam $\mathcal{E} = \{G_1, S\}$ is uniquely determined. Now we are going to identify it up to isomorphism.

Lemma 11.5.12

(i) $\mathrm{Out}\, S_1$ *is of order* 2;
(ii) $\mathcal{E} = \{G_1, S\}$ *is isomorphic to the analogous amalgam in* J_4.

Proof. We follow the notation introduced in (11.5.11), so that $F \cong 3 \cdot Sym_6$ is a complement to $O_2(S_1)$. Since $O_2(S_1)$ possesses the following chief series:

$$1 \leq L_1 \leq O_2(A) \leq O_2(A)U_1 \leq O_2(A)U_1U_2 = O_2(S_1),$$

the chief factors of F inside $O_2(S_1)$ are known. Since $H^1(F, U_i)$ is trivial for $i = 1,2$ while $H^1(F, O_2(A))$ is 1-dimensional (remember that $O_2(A)$ is indecomposable) we conclude that there are two classes of complements to $O_2(S_1)$ in S_1. Hence in order to prove (i) it is sufficient to show that every automorphism σ of S_1 which normalizes F is inner. Since $O_2(S_1)$ does not involve the module dual to U_1, σ induces an inner automorphism of F and hence we can assume that σ centralizes F. Notice that

$$K_S = C_{S_1}(O_2(A)), \quad \text{where} \quad A = N_{S_1}(O_3(F)),$$

and hence σ normalizes K_S and commutes with the action of F on K_S. Since $K_S = O_2(A) \oplus U_1$ (as a module for F), it is easy to see that σ must centralize K_S. Similarly σ must centralize the complement U_2 to K_S in $O_2(S_1)$. Thus (i) is proved. In order to prove (ii) we apply Goldschmidt's theorem (8.3.2). Since $H^1(M_{24}, \overline{\mathscr{C}}_{11})$ is non-trivial (cf. Table VI in Section 8.2), S possesses an outer automorphism. In fact it is easy to see that $\mathrm{Aut}\, S \cong \overline{\mathscr{C}}_{12} : M_{24}$ and the centralizer of S_1 in $\mathrm{Aut}\, S$ is trivial. Hence $\mathscr{E}$ is uniquely determined up to isomorphism and (ii) follows. □

Lemma 11.5.13 *The amalgam $\mathscr{F} = \{G_1, S, T\}$ is uniquely determined up to isomorphism.*

Proof. By (11.5.12), $\mathscr{E} = \{G_1, S\}$ is uniquely determined. Hence all we have to show is that the kernel K_T of the homomorphism onto T of the universal completion U_T of the amalgam $\{T_1, T_S\}$ is uniquely specified, where

$$T_1 = T \cap G_1 \cong 2^{3+12}.(Sym_5 \times Sym_4),$$
$$T_S = T \cap S \cong 2^{3+12}.(Sym_3 \times 2 \times L_3(2)).$$

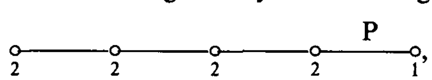
Clearly $Q = O_2(T)$ is contained and normal in both T_1 and T_S. Hence K_T is a complement to $V = Z(Q)$ in the centralizer of Q in U_T. In order to apply (8.4.3) all we have to show is that $2^3 : (Sym_5 \times L_3(2))$ is not a completion of the amalgam $\{T_1/Q, T_S/Q\} = \{Sym_5 \times Sym_4, Sym_3 \times 2 \times L_3(2)\}$, but this is quite obvious. □

Proposition 11.5.14 *All the amalgams of J_4-shape are isomorphic to $\mathscr{A}(J_4, \mathscr{G}(J_4))$ and the universal completion of such an amalgam is isomorphic to J_4.*

Proof. Since $G_2 \leq T$ and $G_3 \leq S$, the amalgam $\{G_1, G_2, G_3\}$ is contained in $\mathscr{F}$ and hence it is uniquely determined by (11.5.13). Hence the uniqueness of the amalgam follows by the standard remark that $\mathrm{res}_{\mathscr{G}}(x_4)$ is simply connected. The geometry $\mathscr{G}(J_4)$ is simply connected, as has been proved in [Iv92b], [ASeg91], [IMe99], which implies the conclusion about the universal completion. □

11.6 Truncated J_4-shape

In this section $\mathscr{G}$ is a rank 5 P-geometry with the diagram



such that $\mathrm{res}_{\mathscr{G}}(x_1) \cong \mathscr{G}(J_4)$, $G_1 \cong J_4$, and $G_5 \cong 2^{10}.L_5(2)$.

We will show that such a geometry does not exist by considering possible T-subgeometries. By Lemma 7.1.7 in [Iv99] (compare (11.5.6)) x_4 is contained in a unique subgeometry $\mathcal{S}$ which is a T-geometry of rank 4. Since $G_4 \sim [2^{16}].L_4(2)$ and the rank 3 T-subgeometry in $\mathrm{res}_{\mathcal{G}}(x_1) \cong \mathcal{G}(J_4)$ is $\mathcal{G}(M_{24})$, the classification of the flag-transitive T-geometries of rank 4 shows that $\mathcal{S} \cong \mathcal{G}(Co_1)$ and S (the stabilizer of $\mathcal{S}$ in G) is Co_1.

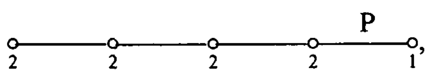
Now consider the stabilizer S_1 of x_1 in S. Since $S \cong Co_1$ we have $S_1 \cong 2^{11}.M_{24}$ and $O_2(S_1)$ is the irreducible Golay code module $\mathcal{C}_{11}$ (compare Section 12.6). On the other hand, S_1 is the stabilizer in $G_1 \cong J_4$ of a $\mathcal{G}(M_{24})$-subgeometry from $\mathcal{G}(J_4)$, so $S_1 \cong 2^{11}.M_{24}$, but from this point of view $O_2(S_1)$ must be the irreducible Todd module $\overline{\mathcal{C}}_{11}$ by (11.5.9). This is a contradiction and hence we have proved the following.

Proposition 11.6.1 *There is no P-geometry $\mathcal{G}$ of rank 5 possessing a flag-transitive automorphism group G such that $\mathcal{A}(G,\mathcal{G})$ is of truncated J_4-shape (that is with point stabilizer isomorphic to J_4).* □

Notice that $J = J_4$ itself contains a subgroup $L = 2^{10} : L_5(2)$. The action of J on the cosets of L preserves a graph Ξ of valency 31 which is locally projective. There is a family of Petersen subgraphs and a family of subgraphs isomorphic to the derived graph of $\mathcal{G}(M_{22})$, which are geometrical subgraphs of valency 3 and 7, respectively, but there is no family of geometrical subgraphs of valency 15. So this graph gives only a truncated version of P-geometry.

11.7 *BM*-shape

In this section $\mathcal{G}$ is a rank 5 P-geometry with the diagram



the residue $\mathrm{res}_{\mathcal{G}}(x_1)$ is isomorphic to $\mathcal{G}(Co_2)$ or $\mathcal{G}(3^{23}\cdot Co_2)$ and $\overline{G}_1 \cong Co_2$ or $3^{23}\cdot Co_2$, respectively; furthermore L_1 is of order 2 and K_1/L_1 is the 22-dimensional representation module of $\mathrm{res}_{\mathcal{G}}(x_1)$ isomorphic to the Co_2-section $\overline{\Lambda}^{(22)}$ of the Leech lattice taken modulo 2. Since the arguments for the two cases are basically identical, we assume that $\mathrm{res}_{\mathcal{G}}(x_1) \cong \mathcal{G}(Co_2)$ and $\overline{G}_1 \cong Co_2$. We start with the following

Lemma 11.7.1 *The group K_1 is extraspecial of plus type, so that $G_1 \sim 2_+^{1+22}.Co_2$.*

Proof. Arguing as in the proof of (11.5.1) it is easy to show that K_1 is non-abelian. Since Co_2 acts irreducibly on $K_1/L_1 \cong \overline{\Lambda}^{(22)}$ and $|L_1| = 2$, we have that K_1 is extraspecial. Since the action of Co_2 on $\overline{\Lambda}^{(22)}$ is absolutely irreducible (8.2.9), it preserves at most one non-zero quadratic form. The restriction to $\overline{\Lambda}^{(22)}$ of the form θ on $\overline{\Lambda}^{(24)}$ (the Leech lattice taken modulo 2) as in (5.1.1) is a Co_2-invariant quadratic form of plus type. Hence the proof follows. □

In this section θ will also denote the unique non-zero Co_2-invariant quadratic form on $K_1/L_1 \cong \overline{\Lambda}^{(22)}$. Put $\widetilde{G}_1 = G_1/L_1$ (so that $\widetilde{G}_1 \sim 2^{22}.Co_2$) and apply the tilde convention for subgroups in G_1. Then $\widetilde{K}_1 = O_2(\widetilde{G}_1)$ is isomorphic to $\overline{\Lambda}^{(22)}$.

Lemma 11.7.2 *$\widetilde{G}_1$ is determined uniquely up to isomorphism.*

Proof. Since L_1 is the centre of G_1, $\widetilde{G}_1$ is the image of G_1 in $A :=$ $\mathrm{Aut}\, K_1 \cong 2^{22}.O_{22}^+(2)$. Since Co_2 preserves a unique non-zero quadratic form on $\overline{\Lambda}^{(22)}$, $O_{22}^+(2)$ contains a unique conjugacy class of subgroups isomorphic to Co_2, such that the action on the natural module of $O_{22}^+(2)$ is isomorphic to that on $\overline{\Lambda}^{(22)}$. Hence $\widetilde{G}_1$ is specified as the full preimage of such a subgroup with respect to the homomorphism $A \to A/O_2(A)$. □

Since G_1 is a perfect central extension of $\widetilde{G}_1$ the next logical step is to calculate the Schur multiplier of $\widetilde{G}_1$.

Lemma 11.7.3 *The Schur multiplier of $\widetilde{G}_1$ is elementary abelian of order four.*

Proof. First we show that the Schur multiplier of $\widetilde{G}_1$ has order at least 4. Let $C_1 \cong 2_+^{1+24}.Co_1$ be the stabilizer in the Monster M of a point of $\mathcal{G}(M)$ and let $D \cong 2_+^{1+24}.Co_2$ be the preimage of a Co_2-subgroup in Co_1 with respect to the homomorphism $C_1 \to C_1/O_2(C_1) \cong Co_1$.

We know that $\overline{\Lambda}^{(24)}$, considered as a module for Co_2, is uniserial with the composition series

$$\langle \lambda \rangle < \overline{\Lambda}^{(23)} < \overline{\Lambda}^{(24)},$$

where λ is the unique non-zero vector in $\overline{\Lambda}^{(24)}$ stabilized by Co_2, $\overline{\Lambda}^{(23)}$ is the orthogonal complement of $\langle \lambda \rangle$ and $\overline{\Lambda}^{(22)} = \overline{\Lambda}^{(23)}/\langle \lambda \rangle$. This shows that the commutator subgroup D' of D has index 2 in D, it is perfect and the centre of D' is of order four.

Now we establish an upper bound on the Schur multiplier of $\widetilde{G}_1$. Let $\widehat{G}_1$ be the largest perfect central extension of $\widetilde{G}_1$ and $\widehat{Z}$ be the centre of

$\widehat{G}_1$. We apply the hat convention for subgroups in $\widetilde{G}_1$. The commutator map on $\widehat{K}_1$ defines a bilinear map

$$\chi : \widetilde{K}_1 \times \widetilde{K}_1 \to \widehat{Z}.$$

Since the Co_2-module $\widetilde{K}_1 \cong \overline{\Lambda}^{(22)}$ is absolutely irreducible (8.2.9), the image $\widehat{Z}_1 := [\widehat{K}_1, \widehat{K}_1]$ of the commutator map is of order at most two. On the other hand, $\widehat{K}_1/\widehat{Z}_1$ is abelian and it is rather easy to see that in fact it must be an elementary abelian 2-group, and since $\widehat{G}_1$ is perfect it must be indecomposable as a module for $\widehat{G}_1/\widehat{K}_1 \cong Co_2$. Since $H^1(Co_2, \overline{\Lambda}^{(22)})$ is 1-dimensional by (8.2.7 (ii)), the dimension of $\widehat{K}_1/\widehat{Z}_1$ is at most 23. Finally, $\widehat{G}_1/\widehat{K}_1$ is a perfect central extension of Co_2. Since the Schur multiplier of Co_2 is trivial by [Gri74], the proof follows. □

Lemma 11.7.4 *The isomorphism type of* G_1 *is uniquely determined.*

Proof. As a direct consequence of the proof of (11.7.3) we observe that the universal perfect central extension $\widehat{G}_1$ of G_1 is isomorphic to the subgroup $D' \cong 2^{1+1+22}.Co_2$ of $C_1 \cong 2_+^{1+24}.Co_1$. In terms of the proof of (11.7.3) let $\widehat{Z}_1$, $\widehat{Z}_2$ and $\widehat{Z}_3$ be the three subgroups of order two from $\widehat{Z} = Z(\widehat{G}_1)$. Then both $\widehat{G}_1/\widehat{Z}_2$ and $\widehat{G}_1/\widehat{Z}_3$ have extraspecial normal subgroups, and they are the only candidates for the isomorphism type of G_1. On the other hand, since $O_2(D)$ is extraspecial an element from $D \setminus D'$ conjugates $\widehat{Z}_2$ onto $\widehat{Z}_3$ and hence $\widehat{G}_1/\widehat{Z}_2 \cong \widehat{G}_1/\widehat{Z}_3$. □

The subgroup G_{12} is clearly the preimage in G_1 of the stabilizer $\overline{S}$ in $\overline{G}_1 \cong Co_2$ of the point x_2 of $\mathrm{res}_{\mathscr{G}}(x_1) \cong \mathscr{G}(Co_2)$, where $\overline{S} \cong 2^{10} : \mathrm{Aut}\, M_{22}$. We know that $O_2(\overline{S})$ is the irreducible Golay code module $\mathscr{C}_{10}$ for $\overline{S}/O_2(\overline{S}) \cong \mathrm{Aut}\, M_{22}$. By (5.2.3), $\overline{\Lambda}^{(22)}$, as a module for $\overline{S}$, is uniserial with the composition series

$$V^{(1)} < V^{(2)} < V^{(3)} < \overline{\Lambda}^{(22)},$$

where $V^{(1)}$ and $\overline{\Lambda}^{(22)}/V^{(3)}$ are 1-dimensional, $V^{(2)}$ is a maximal isotropic subspace with respect to the invariant quadratic form θ, $V^{(2)}/V^{(1)} \cong \overline{\mathscr{C}}_{10}$ and $V^{(3)}/V^{(2)} \cong \mathscr{C}_{10}$ (as modules for $\overline{S}/O_2(\overline{S}) \cong \mathrm{Aut}\, M_{22}$). So

$$G_{12} \sim 2^{1+1+10+10+1+10}.\mathrm{Aut}\, M_{22}.$$

Let $K_1^{(i)}$ be the full preimage of $V^{(i)}$ in K_1.

Lemma 11.7.5 *We have*

$$G_2 \sim 2^{2+10+20}.(\mathrm{Aut}\, M_{22} \times Sym_3).$$

Furthermore, if $\{x_1, y_1, z_1\}$ *is the set of points incident to* x_2*, then*

(i) $K_1^{(3)} = K_1 \cap K_2$ *has index* 2 *in* K_1*;*
(ii) $K_1^{(2)} = L_2$ *and* K_2/L_2 *is the tensor product of the* 10*-dimensional Golay code module* $\mathscr{C}_{10}$ *for* $K_2^-/K_2 \cong \mathrm{Aut}\, M_{22}$ *and the* 2*-dimensional module for* $K_2^+/K_2 \cong Sym_3$*;*
(iii) L_2 *is a maximal abelian subgroup of* K_1 *(of order* 2^{12}*);*
(iv) $Z_2 := K_1^{(1)} = \langle L(x_1), L(y_1), L(z_1)\rangle$ *is a normal subgroup of order* 4 *in* G_2*,* L_2/Z_2 *is the* 10*-dimensional Todd module* $\overline{\mathscr{C}}_{10}$ *and* $Z_2 = Z(K_2)$*.*

Proof. Since K_1 induces an action of order 2 on the point set $\{x_1, y_1, z_1\}$ of x_2 and $K_1^{(3)}$ is the only subgroup of index 2 in K_1 which is normal in G_{12}, we obtain (i). Now (ii) follows from (9.4.1). Finally (iii) and (iv) are immediate from the paragraph preceding the lemma. □

By (11.7.5 (iv)) there is a natural bijection between the point set $\{x_1, y_1, z_1\}$ of x_2 and the set of involutions in Z_2, which implies the equality

$$K_2^- = C_{G_{12}}(Z_2).$$

We follow the direct strategy, so our nearest goal is to calculate the automorphism group of K_2^-.

Lemma 11.7.6 K_2^- *splits over* $K_2 = O_2(K_2^-)$.

Proof. Let Y be a Sylow 3-subgroup of $O_{2,3}(G_2)$ (which is also a Sylow 3-subgroup of K_2^+) and $X = C_{G_2}(Y)$. Then by (11.7.5) and the Frattini argument

$$X \cong 3 \times 2^{10}.\mathrm{Aut}\, M_{22}$$

and (as a module for $X/O_{2,3}(X) \cong \mathrm{Aut}\, M_{22}$) $O_2(X)$ is isomorphic to $\overline{\mathscr{C}}_{10}$. By Table VI in Section 8.2 the group $H^2(\mathrm{Aut}\, M_{22}, \overline{\mathscr{C}}_{10})$ is trivial, which completes the proof. □

As a consequence to the proof of (11.7.6) we have the following.

Corollary 11.7.7 *The group* K_2^- *contains a subgroup* $D \cong \mathrm{Aut}\, M_{22}$ *which centralizes* Z_2. □

Lemma 11.7.8 $\mathrm{Out}\, K_2^- \cong Sym_4$.

Proof. By (11.7.5 (ii)) every subgroup of order 2^{10} in K_2/L_2 normalized by K_2^- coincides with $(K(\alpha) \cap K_2)/L_2$ for $\alpha = x_1$, y_1 or z_1 and the commutator subgroup of $K(\alpha) \cap K_2$ is exactly $L(\alpha)$. Since G_2 induces Sym_3 on $\{x_1, y_1, z_1\}$, we conclude that $\mathrm{Out}\, K_2^-$ induces Sym_3 on

$$\mathscr{T} = \{K(\alpha) \cap K_2 \mid \alpha = x_1, y_1, z_1\}.$$

Let T be the kernel of the action of $\mathrm{Out}\, K_2^-$ on $\mathscr{T}$. By the above discussion T centralizes Z_2 and normalizes every K_2^--invariant subgroup in K_2/L_2. Let τ be an automorphism of K_2^- which projects onto a non-trivial element of T. Since $K_2^-/K_2 \cong \mathrm{Aut}\, M_{22}$ is complete, we can assume (modulo inner automorphisms) that τ commutes with K_2^-/K_2. Every chief factor of K_2^- inside K_2/Z_2 is isomorphic either to $\mathscr{C}_{10}$ or to $\overline{\mathscr{C}}_{10}$, in particular, it is an absolutely irreducible $GF(2)$-module for K_2^-/K_2. Furthermore, τ stabilizes the series

$$Z_2 \lhd L_2 \lhd K_2 \lhd K_2^-$$

and $L_2/Z_2 \cong \overline{\mathscr{C}}_{10}$ is not involved in $K_2/L_2 \cong \mathscr{C}_{10} \oplus \mathscr{C}_{10}$. Hence τ centralizes K_2.

Let D be a complement to K_2 in K_2^- as in (11.7.7). Then $D^\tau \neq D$ (since otherwise τ would be trivial) and $\langle D, D^\tau\rangle \cap K_2 \leq Z_2 = Z(K_2)$ (since τ acts trivially on K_2 and so D^τ acts on K_2 exactly as D does). Hence

$$D^\tau \leq N_{K_2^-}(O^2(D)) = Z_2 \times D \cong 2^2 \times \mathrm{Aut}\, M_{22}.$$

Now it is easy to see that τ must coincide with the automorphism $\tau(\alpha)$, which centralizes $O^2(D)$ and multiplies every $d \in D \setminus O^2(D)$ by the only non-identity element of $L(p)$, where $\alpha \in \{x_1, y_1, z_1\}$ is a point on x_2. It is straightforward that $\tau(\alpha)$ is indeed an automorphism of K_2^- of order 2 and

$$\tau(x_1)\tau(y_1) = \tau(z_1).$$

Thus T is elementary abelian of order 2^2 and its non-identity elements are the images in $\mathrm{Out}\, K_2^-$ of the automorphisms $\tau(\alpha)$ for $\alpha \in \{x_1, y_1, z_1\}$. Now the proof is clear. □

Lemma 11.7.9 *The isomorphism type of G_2 is uniquely determined.*

Proof. Since K_1 is extraspecial and $K_1 \cap K_2 = C_{K_1}(Z_2)$, there is an involution $\sigma \in K_1 \setminus K_2$. Let Y be a Sylow 3-subgroup of $O_{2,3}(G_2)$, normalized by σ (such a subgroup exists by Sylow's theorem) and $E = \langle Y, \sigma\rangle$. Then $E \cong Sym_3$, E is a complement to K_2^- in G_2, which means that E maps isomorphically onto its image in $\mathrm{Out}\, K_2^-$ and this image coincides with the image of the whole of G_2. On the other hand, $\mathrm{Out}\, K_2^- \cong Sym_4$

by (11.7.8) and hence the action of E on K_2^- is determined uniquely up to conjugation in $\mathrm{Aut}\, K_2^-$, hence the proof. □

Let Y and σ as above denote also their images in $\mathrm{Aut}\, K_2^-$. Then it is easy to deduce from the proofs of (11.7.8) and (11.7.9) that

$$F = \langle \tau(x_1), \tau(y_1), \tau(z_1), Y, \sigma \rangle$$

is isomorphic to Sym_4 and maps isomorphically onto $\mathrm{Out}\, K_2^-$. Let $\widehat{G}_2$ denote the semidirect product of K_2^- and F with respect to the natural action. Then $\widehat{G}_2$ contains G_2 with index 4.

Now we consider the possibilities for the isomorphism type of $\mathscr{B} = \{G_1, G_2\}$.

Lemma 11.7.10 *The following assertions hold:*

(i) *each of the groups G_1 and G_2 is complete;*
(ii) $N_{G_j}(G_{12}) = G_{12}$ *for* $j = 1$ *and* 2*;*
(iii) $\mathrm{Aut}\, G_{12} \cong N_{\widehat{G}_2}(G_{12})/L_1$ *and* $|\mathrm{Out}\, G_{12}| = 2$*;*
(iv) *there are exactly two possibilities for the isomorphism type of* $\mathscr{B} = \{G_1, G_2\}$.

Proof. First of all by Goldschmidt's theorem (8.3.2), the assertion (iv) is immediate from (i), (ii) and (iii). The group G_1 is complete, since $G_1/K_1 \cong Co_2$ is complete (cf. [CCNPW]), the action of G_1/K_1 on $K_1/L_1 \cong \overline{\Lambda}^{(22)}$ is absolutely irreducible and $O^2(G_1) = G_1$. Since Sym_3 is self-normalized in Sym_4 and the centre of G_2 is trivial, it is easy to deduce from the proof of (11.7.8) that G_2 is complete, so (i) follows. Since $\overline{S} \cong 2^{10}.\mathrm{Aut}\, M_{22}$ is self-normalized (in fact maximal) in $\overline{G}_1$, we obtain (ii) for $j = 1$ ((ii) is completely obvious when $j = 2$). By (11.7.8)

$$N_{\widehat{G}_2}(G_{12}) = \langle G_{12}, \tau(x_1) \rangle$$

contains G_{12} with index 2. Thus in order to prove (iii) we only have to consider automorphisms of G_{12} which centralize K_2^-. We know that $G_{12} = K_2^- K_1$ and $K_1 \cap K_2^- = C_{K_1}(Z_2)$. Since K_1 is extraspecial, the only non-trivial automorphism of K_1 which centralizes $C_{K_1}(Z_2)$ is the inner automorphism induced by conjugation by an element from $Z_2 \setminus L_1$. □

Next we are going to show that at most one of the possible amalgams $\mathscr{B} = \{G_1, G_2\}$ extends to an amalgam $\mathscr{C} = \{G_1, G_2, G_3\}$ of correct shape. First let us look closer at the structure of G_3.

Lemma 11.7.11 *Let Z_3 denote the subgroup in G_3 generated by the subgroups $L(\alpha)$ taken for all the seven points α incident to x_3. Then*

(i) $G_3/K_3 \cong Sym_5 \times L_3(2)$;

(ii) *K_3/L_3 is the tensor product of the 5-dimensional module U_5 for $K_3^-/K_3 \cong Sym_5$ which contains a codimension 1 submodule U_4 and of the 3-dimensional natural module V_3 for $K_3^+/K_3 \cong L_3(2)$;*

(iii) *Z_3 is elementary abelian of order 2^3, which is the natural module for $K_3^+/K_3 \cong L_3(2)$, $Z_3 \leq L_3$ and $Z_3 = Z(K_3)$.*

Proof. The assertion (i) is immediate from the flag-transitivity of G_3 on $\text{res}_{\mathscr{G}}(x_3)$, while (ii) is by (9.4.1). It is easy to see that Z_3 is contained in K_1 and since the latter is extraspecial, (iii) follows. □

Lemma 11.7.12 *Let $\delta(x_1, x_2)$ denote the restriction to K_3^- of the automorphism $\tau(x_1)$ of K_2^-, defined in the proof of* (11.7.8). *Let Ψ be the set of the* 21 *similar automorphisms $\delta(\alpha, \beta)$ taken for all the maximal flags $\{\alpha, \beta\}$ in* $\text{res}^-_{\mathscr{G}}(x_3)$ *and A be the subgroup in* $\text{Aut}\,K_3^-$ *generated by the automorphisms from Ψ. Then A is elementary abelian of order 2^8 and A, as a module for $G_3/K_3^- \cong L_3(2)$, is isomorphic to the Steinberg module.*

Proof. Recall that $\tau(x_1)$ centralizes $O^2(K_2^-)$ and multiples every element from $K_2^- \setminus O^2(K_2^-)$ by the unique non-identity element of $L(x_1) = Z(G_1)$. Notice also that $O^2(K_2^-) = K_2 O^2(D)$, where $D \cong \text{Aut}\,M_{22}$ is the complement to K_2 in K_2^- as in (11.7.7). Let $F = (K_2^- \cap K_3^-)K_2/K_2$ and $H = (O^2(K_2^-) \cap K_3^-)K_2/K_2$. Then $F \cong 2^5 : Sym_5$, $H \cong 2^4 : Sym_5$, and $O_2(F)$ and $O_2(H)$ map onto the modules U_5 and U_4 as in (11.7.11 (ii)), respectively.

Let R_3 be the preimage in K_3 of the submodule $U_4 \otimes V_3$ in K_3/L_3 as in (11.7.11 (ii)). Then $K_3/P_3 \cong V_3 \cong Z_3$ (as modules for $G_3/K_3^- \cong L_3(2)$) and by the above R_3 is centralized by $\delta(x_1, x_2)$. Furthermore, if we put $I_3 = K_3^-/R_3$, then $I_3 \cong 2^3 \times Sym_5$ and $\delta(x_1, x_2)$ centralizes a complement to $O_2(I_3) = K_3/R_3$ in I_3. Put

$$J_3 = \bigcap_{\delta \in \Psi} C_{K_3^-}(\delta).$$

Since R_3 is normal in G_3 and centralized by $\delta(x_1, x_2)$, we conclude that R_3 is contained in J_3. We claim that $J_3/R_3 \cong Sym_5$, even more precisely, that the automorphisms in Ψ centralize a common complement to $O_2(I_3)$ in I_3. We have seen that $\delta(x_1, x_2)$ centralizes such a complement. In total there are exactly eight complements and 21 automorphisms in Ψ, transitively permuted by $G_3/K_3^- \cong L_3(2)$, hence the claim follows.

Thus we have shown that $K_3^-/J_3 \cong V_3 \cong Z_3$ (as modules for $G_3/K_3^- \cong L_3(2)$); an automorphism $\delta(\alpha, \beta)$ from Ψ centralizes the preimage of the hyper-plane in K_3^-/J_3 which corresponds to the line β and multiplies every element of K_3^- outside this preimage by the unique non-identity element of $L(\alpha)$. The latter elements taken for all points α incident to x_3 form the set of non-identity elements in the centre Z_3 of K_3^-as in (11.7.11 (ii)). Now the result is immediate from the definition of the Steinberg module. □

Lemma 11.7.13 *Let* $\{G_1^{(1)}, G_2^{(1)}\}$ *and* $\{G_1^{(2)}, G_2^{(2)}\}$ *be the possible amalgams* $\mathscr{B}$ *as in* (11.7.10 (iv)). *For* $j = 1$ *and* 2 *let* $P_1^{(j)}$ *and* $P_2^{(j)}$ *be the images in* $\operatorname{Out} K_3^-$ *of* $G_{13}^{(j)}$ *and* $G_{23}^{(j)}$, *respectively (so that* $P_1^{(j)} \cong P_2^{(j)} \cong Sym_4$). *Suppose that* $P_1^{(1)}$ *and* $P_2^{(1)}$ *generate* $L_3(2)$. *Then* $P_1^{(2)}$ *and* $P_2^{(2)}$ *generate* $2^8 : L_3(2)$ *(the semidirect product of* $L_3(2)$ *and the Steinberg module).*

Proof. By the proof of (11.7.10) we can assume that $G_{13}^{(1)}$ and $G_{13}^{(2)}$ have the same image in $\operatorname{Out} K_3^-$, while the images of $G_{23}^{(1)}$ and $G_{23}^{(2)}$ are conjugate by the image of the automorphism $\delta(x_1, x_2)$ as in (11.7.12). Now the proof follows from (11.7.12) and (8.5.4). □

By (11.7.13) at most one of the possible amalgams $\mathscr{B}$ extends to a rank 3 amalgam of the correct shape. Clearly the amalgam associated with the action of BM on $\mathscr{G}(BM)$ extends. Hence by (8.6.1) we obtain the final result of the section.

Proposition 11.7.14 *An amalgam* $\mathscr{A}$ *of BM-shape is isomorphic to either*

$$\mathscr{A}(BM, \mathscr{G}(BM)) \quad or \quad \mathscr{A}(3^{4371} \cdot BM, \mathscr{G}(3^{4371} \cdot BM))$$

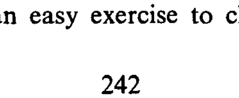
and the universal completion of $\mathscr{A}$ *is* BM *or* $3^{4371} \cdot BM$, *respectively.* □

12
Amalgams for T-geometries

In this chapter we consider the amalgams of maximal parabolics of flag-transitive actions on T-geometries with shapes given in Table VIIIb. It is an elementary exercise to show that up to isomorphism there is a unique amalgam of Alt_7-shape and we know (cf. Section 6.11 in [Iv99]) that it does not possess a faithful completion. In Section 12.2 we show that there is a unique isomorphism type of amalgams of $S_6(2)$-shape and in Section 12.3 that there are two types of M_{24}-shape. In Section 12.4 we show that there is a unique amalgam $\mathscr{A}_f$ of truncated M_{24}-shape and in Section 12.5 that the universal completion of $\mathscr{A}_f$ is isomorphic to M_{24} and it is not faithful. In Section 12.6 we show that there is a unique amalgam of Co_1-shape while in Section 12.7 we formulate the characterization of the Monster amalgam achieved in Section 5.13 of [Iv99]. In the final section of this chapter we classify the amalgams of symplectic shape with rank $n \geq 4$ (the classification was originally proved in [ShSt94]). Thus we have three amalgams for rank 3, two for ranks 4 and 5 and only one (of symplectic shape) for rank $n \geq 6$. These numbers coincide with the numbers of amalgams coming from the known examples in Table II, which proves Theorem 3 for T-geometries and by Proposition 4 and Theorem 2 completes the proof of Theorem 1 for T-geometries (see the Preface).

12.1 Alt_7-shape

Let $\mathscr{G}$ be a T-geometry of rank 3 with the diagram



G be a flag-transitive automorphism group of $\mathscr{G}$, such that $G_1 \cong 3 \cdot Alt_6$, $G_3 \cong L_3(2)$. It is an easy exercise to check that in this case G_2

must be isomorphic to $(Sym_3 \times Sym_4)^e$ (the stabilizer of a 3-element subsets in Alt_7). Then by Lemma 6.11.3 in [Iv99] the amalgam $\mathscr{A}_s = \{G_1, G_2, G_3\}$ is determined uniquely up to isomorphism. Let $(U(\mathscr{A}_s, \varphi)$ be the universal completion of $\mathscr{A}_s$. The computer calculations performed with the generators and relations for $U(\mathscr{A}_s)$ given in Section 6.11 in [Iv99] show the following lemma.

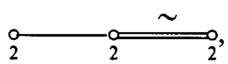
Proposition 12.1.1 *The following assertions hold:*

(i) $U(\mathscr{A}_s) \cong Alt_7$;
(ii) *the restriction of* φ *to* G_1 *has kernel of order* 3.

In particular there exists no pairs $(\mathscr{G}, G)$ *such that the amalgam* $\mathscr{A}(G, \mathscr{G})$ *is of* Alt_7*-shape (this means that* $\mathscr{G}$ *is a rank* 3 T*-geometry and* G *is a flag-transitive automorphism group of* $\mathscr{G}$ *with* $G_1 \cong 3 \cdot Alt_6$, $G_3 \cong L_3(2)$*).* □

12.2 $S_6(2)$-shape

In this section $\mathscr{G}$ is a T-geometry of rank 3 with the diagram



where $G_1 \sim 2^5.3 \cdot Sym_6$, $G_3 \sim 2^{3+3}.L_3(2)$, and

(a) $N_1 = 1$ and $L_1 = Z(G_1)$ is of order 2;
(b) $K_1 = O_2(G_1)$ and K_1/L_1 is the 4-dimensional symplectic module for $G_1/O_{2,3}(G_1) \cong S_4(2)$;
(c) L_3 is the natural module for $\overline{G}_3 = G_3/K_3 \cong L_3(2)$ and K_3/L_3 is the dual of the natural module.

Lemma 12.2.1 K_3 *is elementary abelian and as a module for* $\overline{G}_3 \cong L_3(2)$ *it is the even half of the* $GF(2)$*-permutation module for* $\overline{G}_3$ *on the set* $\mathscr{P}$ *of points in* $\mathrm{res}_{\mathscr{G}}(x_3)$.

Proof. For a point p incident to x_3 (a quint containing x_3) let z_p be the unique involution in $L(p) = Z(G(p))$ (compare (a)). If $p = x_1$, then z_p is centralized by $G_{13} \sim 2^5.(2 \times Sym_4)$, which shows that $z_p \in K_3$. On the other hand, L_3 is the dual natural module for $\overline{G}_3$ while z_p is centralized by a point stabilizer in G_3, hence $z_p \notin L_3$. If the involutions z_p taken for $p \in \mathscr{P}$ generate the whole K_3 then the proof follows, since $K_3 \leq G_{13}$ and $Z(G_1)$ is in the centre of G_{13}. Otherwise the involutions generate a G_3-invariant complement to L_3 in K_3 and K_3 is the direct sum of the natural module of $\overline{G}_3$ and the module dual to the natural one. We suggest that the reader rule out this possibility by looking at the structure of G_2 or otherwise. □

Lemma 12.2.2 *G_1 splits over K_1.*

Proof. Put $\overline{R} = O_2(G_{13}/K_3)$, which is elementary abelian of order 2^2. Then $\overline{R}$ coincides with the image of K_1 in $\overline{G}_3$. Since K_1 is elementary abelian, there is a subgroup R in G_{13} which maps isomorphically onto $\overline{R}$ and $K_1 \le C_{G_{13}}(R)$. In terms of (12.2.1), R has four orbits on $\mathscr{P}$ (one of length 1 and three of length 2), hence $\dim C_{K_3}(R) = 3$ and since $\overline{R}$ is self-centralized in $\overline{G}_3 \cong L_3(2)$, we conclude that

$$K_1 = C_{G_{13}}(R).$$

Let X be a Sylow 3-subgroup in G_{13}. Then

$$K_3 = C_{K_3}(X) \oplus [X, K_3],$$

where by (12.2.1) the centralizer and the commutator are 2- and 4-dimensional, respectively. Since all the involutions in $\overline{G}_3 \cong L_3(2)$ are conjugate and K_3R splits over K_3, there is an involution σ in G_{13} which inverts X. Since σ stabilizes every X-orbit on the point-set $\mathscr{P}$ of $\mathrm{res}_{\mathscr{G}}(x_3)$, it centralizes $C_{K_3}(X)$. Furthermore, since $C_{K_3}(X) \cap C_{K_1}(X)$ is 1-dimensional, there is 1-subspace W in $C_{K_3}(X)$ which is centralized by $\langle X, \sigma \rangle \cong Sym_3$. The commutator $[X, K_3]$ carries a 2-dimensional $GF(4)$-vector space structure and the set $\mathscr{T}$ of 2^2-subgroups in the commutator normalized by X is of size 5. Only one of these subgroups is in K_1 and σ induces on $\mathscr{T}$ a transposition. Hence there is a subgroup $T \in \mathscr{T}$ which is not in K_1 and which is normalized by $\langle X, \sigma \rangle$. Thus

$$\langle W, T, X, \sigma \rangle \cong 2 \times Sym_4$$

is a complement to K_1 in G_{13} and the proof is by Gaschütz' theorem (8.2.8). □

Lemma 12.2.3 *G_3 splits over K_3.*

Proof. By (12.2.2) G_{13} is the semidirect product of K_1 and a group $S \cong 2 \times Sym_4$. Furthermore, if $\Omega = \{1, 2, 3, 4, 5, 6\}$ is a set of size 6 then K_1 can be treated as the even half of the power space of Ω and S as the stabilizer in $Sym(\Omega) \cong Sym_6$ of a partition of Ω into three pairs, say

$$\Omega = \{1, 2\} \cup \{3, 4\} \cup \{5, 6\}.$$

Without loss of generality we assume that $K_1K_3 = K_1O_2(S)$, so that $K_3 = C_{G_{13}}(O_2(S))$ and $K_1 \cap K_3$ is 3-dimensional generated by the subsets $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$. Let $P \cong Sym_3$ be a complement to $O_2(S)$ in S (say $P = \langle \tau, \sigma \rangle$, where $\tau = (1,3,5)(2,4,6)$, $\sigma = (3,5)(4,6)$). Then the 2-subspace T in K_1 containing $\{1,3\}$, $\{3,5\}$, $\{1,5\}$ and the empty set generate together with P a complement to K_3 in G_{13}. As usual now the proof is by Gaschütz' theorem (8.2.8). □

Lemma 12.2.4 *The amalgam $\{G_2, G_3\}$ is determined uniquely up to isomorphism.*

Proof. By (12.2.1) and (12.2.3) G_3 is the semidirect product of K_3 and a group $L \cong L_3(2)$. Furthermore, K_3 is the even half of the $GF(2)$-permutation module of $\overline{G}_3$ on the set $\mathscr{P}$ of points incident to x_3. This means that G_{23} is the semidirect product of K_3 and the stabilizer $S \cong Sym_4$ of the line x_2 in L. The subgroup K_2^+ has index 2 in G_{23} and it is normal in G_2 with $G_2/K_2^+ \cong Sym_3$. So our strategy is to identify K_2^+ in G_{23} and to calculate its automorphism group.

We identify x_2 with the 3-element subset of $\mathscr{P}$ formed by the points incident to x_2. Then the subgroup $R := O_2(G_{23})$ is the semidirect product of K_3 and $O_2(S)$, so that $|R| = 2^8$ and $\widehat{G}_{23} := G_{23}/R \cong Sym_3$. If $R_0 = Z(R)$, then R_0 is elementary abelian of order 2^3 and as a module for $\widehat{G}_{23}$ we have

$$R_0 = R_0^{(1)} \oplus R_0^{(2)},$$

where $R_0^{(1)}$ is 1-dimensional generated by $\mathscr{P} \setminus x_2$ and $R_0^{(2)}$ is 2-dimensional irreducible generated by the 2-subsets of x_2. If it easy to see that there is a unique subgroup R_1 of index 2 in R which is normal in G_{23}, namely, the one generated by $O_2(S)$ and the subsets of $\mathscr{P}$ which intersect x_2 evenly. Furthermore, $R_0^{(1)} = [R_1, R_1]$, the quotient $\overline{R}_1 := R_1/R_0^{(1)}$ is elementary abelian and a Sylow 3-subgroup X of G_{23} acts fixed-point freely on that quotient. This shows that as a module for $\widehat{G}_{23}$ we have

$$\overline{R}_1 = R_0^{(2)} \oplus \overline{R}_1^{(3)}.$$

If $R_1^{(3)}$ is the preimage of $\overline{R}_1^{(3)}$ in R_1 then $R_1^{(3)}$ is extraspecial of plus type with centre $R_0^{(1)}$. Since $K_2 = O_2(G_2)$ and $G_2/K_2 \cong Sym_3 \times Sym_3$, we observe that $K_2 = R_1$. Let Y be a Sylow 3-subgroup of K_2^-. Then Y permutes transitively the points incident to x_2, normalizes R_1 and commutes with X modulo R_1, and Y is inverted by elements from $R \setminus R_1$. In view of the above described structure of R it is an elementary exercise to check that $\{G_2, G_3\}$ is indeed determined uniquely up to isomorphism. □

Now applying the standard strategy (compare the proof of (12.8.16) we prove uniqueness of $\mathscr{A} = \{G_1, G_2, G_3\}$. The universal completion of this amalgam was proved to be isomorphic to $3^7 \cdot S_6(2)$ independently in [Hei91] and in an unpublished work of the authors.

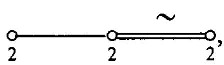
Proposition 12.2.5 *All the amalgams of $S_6(2)$-shape are isomorphic to*

$$\mathscr{A}(3^7 \cdot S_6(2), \mathscr{G}(3^7 \cdot S_6(2)))$$

and $3^7 \cdot S_6(2)$ is the universal completion of such an amalgam. □

12.3 M_{24}-shape

In this section $\mathscr{G}$ is a T-geometry of rank 3 with the diagram



G is a flag-transitive automorphism group of $\mathscr{G}$, such that $G_1 \sim 2^6.3{\cdot}Sym_6$, where $K_1 = O_2(G_1)$ is the hexacode module for $\overline{G}_1 \cong 3 \cdot Sym_6$ and $G_3 \sim 2.2^3.2^3.L_3(2)$. Our goal is to show that $\mathscr{A} = \{G_1, G_2, G_3\}$ is isomorphic either to the amalgam associated with the action of M_{24} on $\mathscr{G}(M_{24})$ or to the amalgam associated with the action of He on $\mathscr{G}(He)$.

Immediately by (8.2.4) we obtain

Lemma 12.3.1 *G_1 splits over K_1, in particular, G_1 is determined uniquely up to isomorphism.*

By (9.4.2) the subgroup G_{12} is specified in G_1 up to conjugation as the full preimage of a parabolic subgroup $Sym_4 \times 2$ in $\overline{G}_1$ which stabilizes a hyperplane in K_1. Thus by (12.3.1) G_{12} is determined uniquely up to isomorphism and hence it is isomorphic to the corresponding subgroup in M_{24} or He. Calculating in either of these groups or otherwise we obtain the following (we consider it easiest to calculate in M_{24} where G_{12} is contained in the stabilizer of a trio).

Lemma 12.3.2 *Let $D_0 = O^2(G_{12})$, $U = O_2(D_0)$ and let X be a Sylow 3-subgroup in G_{12}. Then*

(i) *U is elementary abelian of order 2^6;*
(ii) *X acts fixed-point freely on U;*
(iii) *G_{12} is the semidirect product of U and $N_{G_{12}}(X) \cong D_8 \times Sym_3$.* □

Observe that G_2 normalizes D_0. Indeed, G_2 normalizes the subgroup K_2^- which has index 2 in G_{12}, hence

$$D_0 = O^2(G_{12}) = O^2(K_2^-).$$

Lemma 12.3.3 *The subgroup D_0 has a trivial centralizer in G_2. In particular, G_2 is isomorphic to a subgroup of* $\mathrm{Aut}\, D_0$ *containing* $\mathrm{Inn}\, D_0$.

Proof. Suppose $R := C_{G_2}(D_0) \neq 1$. Since $C_{G_{12}}(D_0) = 1$, we must then have that $R \cong 3$ and $G_2 = RG_{12}$. On the other hand, since $K_1 \neq L_1$, we have that G_2 induces $Sym_3 \times Sym_3$ on the residue of the link x_2. Clearly, R, being normal in G_2, maps into one of the direct factors Sym_3. This means that either $R \leq G_3$, or $R \leq G_1$. The first option contradicts the fact that $R \not\leq G_{12}$. The second option also leads to a contradiction with the structure of G_3. □

We identify D_0 with the subgroup $\mathrm{Inn}\, D_0$ of $\mathrm{Aut}\, D_0$. By (12.3.2 (ii)) we conclude that $\mathrm{Aut}\, D_0$ is the semidirect product of $U \cong 2^6$ and $\Gamma L(3,4) = N_{GL(U)}(X)$. The latter group contains a normal subgroup $SL(3,4)$ and the corresponding factor-group is isomorphic to D_6. Since G_2 has a quotient $Sym_3 \times Sym_3$ and since G_2 contains the scalar subgroup X, the image of G_2 in $D_6 \cong \Gamma L_3(4)/SL_3(4)$ is of order two. Hence G_2 is a subgroup of $2^6 : \Sigma L(3,4)$.

Lemma 12.3.4 *The group G_2 is a semidirect product of U and $Sym_4 \times Sym_3$. It is uniquely determined up to isomorphism.*

Proof. By (12.3.2 (iii)), G_{12} is a semidirect product of U with $N_{G_{12}}(X) \cong D_8 \times Sym_3$ (and X is the group of scalars in $\Sigma L(3,4)$). If $G_2/U \not\cong Sym_4 \times Sym_3$ then the Sylow 3-subgroup of G_2/U is normal. This, however, contradicts the structure of G_3 (just check the number of 2-dimensional factors in G_{23}). Thus, $G_2/U \cong Sym_4 \times Sym_3$, and clearly, since X acts on U fixed-point freely, G_2 is the semidirect product as claimed.

To prove the second sentence, consider an involution $a \in N_{G_2}(X)$ in the direct factor Sym_3. Then a inverts X and hence it maps onto an outer involution (field automorphism) in $\Sigma L(3,4)$. We have that the centralizer in $\Sigma L(3,4)$ of the subgroup Sym_3 generated by the image of $\langle X, a\rangle$ is isomorphic to $L_3(2)$. Since in G_2 we already have a subgroup D_8 from this $L_3(2)$, there are exactly two ways to extend that D_8 to a Sym_4 (maximal parabolics in $L_3(2)$). We claim that only one of the resulting subgroups can be our G_2. Indeed, by our original assumption $Z(G_3)$ is of order 2, hence the unique involution t in $Z(G_3)$ is central in the subgroup G_{23} which has index 3 in G_2. Since $C_{G_2}(t)$ contains a Sylow 2-subgroup of G_2, it is clear that $t \in U$. Thus the subgroup Sym_4 which extends G_{12} to G_2 must centralize a vector in U, which uniquely specifies it. □

From (12.3.1) and (12.3.3) it is easy to deduce that the type of the amalgam $\mathscr{B} = \{G_1, G_2\}$ is uniquely determined. The next lemma shows that there are at most two possibilities for the isomorphism type of $\mathscr{B}$.

Lemma 12.3.5 *The order of* $\mathrm{Out}\, G_{12}$ *is at most* 2.

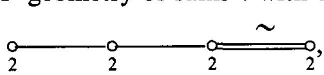
Proof. Let τ be an automorphism of G_{12}. Since $D_0 \cong 2^6 : 3$ is characteristic in G_{12} and X is a Sylow 3-subgroup of G_{12}, τ normalizes D_0 and without loss of generality we may assume that it normalizes X. Then τ normalizes $N := N_{G_{12}}(X) \cong Sym_3 \times D_8$ which is a complement to U in G_{12}. Let $S, D \leq N$, such that $S \cong Sym_3$, $D \cong D_8$ and $N = S \times D$. Then the centralizer of S in $\mathrm{Aut}\, D_0 \cong 2^6 : \Sigma L_3(4)$ is isomorphic to $L_3(2)$ in which D is self-normalized. Notice that S is generated by X and an involution a which is in the centre of a Sylow 2-subgroup of N and inverts x, while $D = C_N(S)$. This immediately shows that there are at most two direct product decompositions of N and the proof follows. □

Proposition 12.3.6 *An amalgam of M_{24}-shape is isomorphic to either $\mathscr{A}(M_{24}, \mathscr{G}(M_{24}))$ or $\mathscr{A}(He, \mathscr{G}(He))$ and its universal completion is isomorphic to M_{24} or He, respectively.*

Proof. Since $\mathscr{G}(M_{24})$ and $\mathscr{G}(He)$ are simply connected [Hei91], M_{24} and He are the universal completions of $\mathscr{A}^{(1)}$ and $\mathscr{A}^{(2)}$, respectively. In particular, the latter two amalgams are not isomorphic and it only remains to show that there are at most two possibilities for the isomorphism type of $\mathscr{A}$. By (12.3.5) and the remark before that lemma, there are at most two possibilities for the isomorphism type of $\mathscr{B}$. We claim that the isomorphism type of $\mathscr{B}$ uniquely determines that of $\mathscr{A}$. Indeed by the proof of (12.3.4) $Z_3 = Z(G_3)$ is determined in G_{12} up to conjugation. Hence $G_{i3} = C_{G_i}(Z_3)$ for $i = 1$ and 2. Thus the hypothesis of (8.5.2) holds and the claim follows. □

12.4 Truncated M_{24}-shape

In this section $\mathscr{G}$ is a T-geometry of rank 4 with the diagram



G is a flag-transitive automorphism group of $\mathcal{G}$ such that G_1 is isomorphic to M_{24} or He and $G_4 \sim 2^4.L_4(2)$. By (10.3.5 (i)) G_4 splits over K_4 (which is the natural module for $\overline{G}_4 \cong L_4(2)$.) In the present section we prove that the imposed conditions specify the amalgam $\mathscr{A}_f = \{G_i \mid 1 \leq i \leq 4\}$ up to isomorphism (the index f means 'fake') and in the next section we show that $\mathscr{A}_f$ has no faithful completions, which implies the non-existence of the geometry with the stated properties.

We apply the dual strategy and start with the following

Lemma 12.4.1 *The parabolic G_3 is the semidirect product of $\overline{G}_3 \cong L_3(2) \times Sym_3$ and K_3 which is the tensor product of the natural (2-dimensional) module of $K_3^-/K_3 \cong Sym_3$ and the dual of the natural module of $K_3^+/K_3 \cong L_3(2)$, so that $G_3 \cong 2^6 : (L_3(2) \times Sym_3)$.*

Proof. Clearly $G_{34} \sim 2^4 : 2^3 : L_3(2)$ is the preimage in G_4 of the stabilizer $2^3 : L_3(2)$ of the plane x_3 in the residual projective space $\text{res}_{\mathcal{G}}(x_4)$. Then K_3^+ is the kernel of the action of G_{34} on the vertex-set of the link x_3. Moreover K_3^+ is the only index 2 subgroup in G_{34}, in particular, K_3 is of order 2^6. Since G_4 acts faithfully on the set of vertices adjacent to x_4 in the derived graph, we conclude that $L_3 = 1$. Hence by (9.4.1), K_3 possesses the tensor product structure as stated in the lemma. Since a Sylow 3-subgroup of $O_{2,3}(G_3)$ acts fixed-point freely on K_3, it is easy to see that G_3 splits over K_3. □

Lemma 12.4.2 *Let $\mathscr{X} = \{G_4, G_3\}$. Then*

(i) $\text{Out}\, G_{34}$ *has order two;*
(ii) $\mathscr{X}$ *is isomorphic to one of two particular amalgams $\mathscr{X}^{(1)}$ and $\mathscr{X}^{(2)}$.*

Proof. Consider $K_3^+ \cong 2^{3+3} : L_3(2)$, which is the commutator subgroup of G_{34}. A complement $F \cong L_3(2)$ to $K_3 = O_2(K_3^+)$ in K_3^+ acts on K_3 as it does on the direct sum of two copies of the dual natural module. By the Three Subgroup Lemma, for an automorphism τ of G_{34} which centralizes K_3^+ we have

$$[G_{34}, \tau] \leq C_{G_{34}}(K_3^+) = 1,$$

and hence whenever an automorphism of G_{34} acts trivially on K_3^+, it is trivial. So $\text{Aut}\, G_{34}$ is a subgroup of $\text{Aut}\, K_3^+$, more precisely

$$\text{Aut}\, G_{34} = N_{\text{Aut}\, K_3^+}(\text{Inn}\, G_{34}).$$

By (8.2.5 (ii)), $H^1(L_3(2), 2^3)$ is 1-dimensional, hence K_3^+ contains exactly four classes of complements to K_3. Since K_3 is abelian, K_3^+ can be presented as a semidirect product of K_3 and any such complement F with respect to the same action. Hence $\mathrm{Out}\, K_3^+$ acts transitively on the set of classes of complements. To calculate the order of $\mathrm{Out}\, K_3^+$, suppose that $\tau \in \mathrm{Aut}\, K_3^+$ stabilizes the class of complements containing F. Then (adjusting τ by an inner automorphism) we may assume that τ normalizes F. Since K_3 involves only the dual natural module of F, τ induces an inner automorphism of F and again adjusting τ by an inner automorphism (induced by conjugation by an element of F), we can assume that τ centralizes F. In this case

$$\tau \in C_{GL(K_3)}(F) \cong L_2(2),$$

which shows that $\mathrm{Out}\, K_3^+$ has order at most 24. We claim that $\mathrm{Out}\, K_3^+$ acts faithfully on the classes of complements. Suppose $\tau \in \mathrm{Aut}\, K_3^+$ leaves invariant every class of complements. For each pair $\mathscr{C}_1$ and $\mathscr{C}_2$ of such classes, there is a unique 3-dimensional submodule U in K_3, such that $\mathscr{C}_1$ and $\mathscr{C}_2$ merge modulo U. Since τ stabilizes each of the four classes of complements, τ normalizes all the three submodules U. Now if we adjust τ by an inner automorphism, we can assume that it centralizes a particular complement F. Then τ centralizes each U and hence τ is the identity.

Thus, $\mathrm{Out}\, K_3^+ \cong Sym_4$ and the image of G_{34} in $\mathrm{Out}\, K_3^+$ is a subgroup T of order two. We claim that T is generated by a transposition. Indeed, since G_{34} contains a subgroup $2 \times L_3(2)$, some involution from $G_{34} \setminus K_3^+$ commutes with a complement $L_3(2)$ from K_3^+. Therefore the involution generating T fixes one of the four points. Since $|N_{\mathrm{Out}\, K_3^+}(T) : T| = 2$, (i) follows.

Since G_{34} is the normalizer in G_4 of a hyperplane from $K_4 = O_2(G_4)$ and G_{34} is the unique (up to conjugation) subgroup of index 3 in G_3, the type of $\mathscr{X}$ is uniquely specified. Since $H^1(\overline{G}_4, K_4)$ is trivial by (8.2.5) and $H^1(\overline{G}_3, K_3)$ is trivial because of the fixed-point free action of a subgroup of order 3, both $\mathrm{Out}\, G_3$ and $\mathrm{Out}\, G_4$ are trivial. Since G_{34} is self-normalized in G_3 and G_4, (ii) follows from (i) and Goldschmidt's theorem (8.3.2). □

Let $G^{(1)} \cong M_{24}$, $G_4^{(1)}$ be the stabilizer in $G^{(1)}$ of an octad B and $G_3^{(1)}$ be the stabilizer of a trio containing B. Let $G^{(2)} \cong L_5(2)$, $G_4^{(2)}$ be the stabilizer in $G^{(2)}$ of a 1-subspace U from the natural module and $G_3^{(2)}$ be the stabilizer of a 2-subspace containing U.

Lemma 12.4.3 *In the above terms (up to a reordering) we have* $\mathscr{X}^{(i)} = \{G_4^{(i)}, G_3^{(i)}\}$ *for* $i = 1$ *and* 2.

Proof. The fact that $\{G_4^{(i)}, G_3^{(i)}\}$ possesses the imposed conditions is an elementary exercise when $i = 2$ and it follows from the basic properties of the action of M_{24} on the Steiner system $S(5, 8, 24)$ when $i = 1$. Hence it only remains to show that $\mathscr{X}^{(1)}$ and $\mathscr{X}^{(2)}$ are not isomorphic.

For a faithful completion H of an amalgam $\mathscr{X}^{(i)} = \{G_4^{(i)}, G_3^{(i)}\}$, where $i = 1$ or 2 define a graph $\Delta(\mathscr{X}^{(i)}, H)$, whose vertices are the cosets of $G_4^{(i)}$ in H and two such cosets are adjacent if their intersection is a coset of $G_3^{(i)} \cap G_4^{(i)}$. If $\mathscr{X}^{(i)}$ is a subamalgam in the amalgam of maximal parabolics associated with a flag-transitive action of a T-geometry $\mathscr{G}$, then $\Delta(\mathscr{X}^{(i)}, G)$ is the derived graph of $\mathscr{G}$. Furthermore $\Delta(\mathscr{X}^{(1)}, G^{(1)})$ is the octad graph and $\Delta(\mathscr{X}^{(2)}, G^{(2)})$ is the complete graph on 31 vertices.

Let $\widetilde{G}^{(i)}$ be the universal completion of $\mathscr{X}^{(i)}$. Then $\widetilde{\Delta}^{(i)} = \Delta(\mathscr{X}^{(i)}, \widetilde{G}^{(i)})$ is of valency 30, every vertex is in 15 triangles and the vertices-triangles incidence graph is a tree. For a vertex $v \in \Delta^{(i)}$ there is a projective space structure Π on the set of triangles containing v. For every line l of Π there is a geometrical subgraph $\widetilde{\Sigma}^{(i)}$ of valency 6.

Let $\widetilde{G}_2^{(i)}$ be the stabilizer of $\widetilde{\Sigma}^{(i)}$ in $\widetilde{G}^{(i)}$, $K_2^{(i)}$ be the kernel of the action of $\widetilde{G}_2^{(i)}$ on $\widetilde{\Sigma}^{(i)}$ and $\widehat{G}_2^{(i)}$ be the image of $\widetilde{G}_2^{(i)}$ in $\operatorname{Out} K_2^{(i)}$, so that

$$\widehat{G}_2^{(i)} \cong \widetilde{G}_2^{(i)}/(K_2^{(i)} C_{\widetilde{G}_2^{(i)}}(K_2^{(i)}).$$

Then the structures of $K_2^{(i)}$ and $\widehat{G}_2^{(i)}$ are determined solely by that of the amalgam $\mathscr{X}^{(i)}$ but it is easier to calculate them in a finite completion of the amalgam.

Let $\Sigma^{(i)}$ be the image of $\widetilde{\Sigma}^{(i)}$ with respect to the covering

$$\widetilde{\Delta}^{(i)} \to \Delta(\mathscr{X}^{(i)}, G^{(i)}).$$

Then $\Sigma^{(1)}$ is the subgraph in the octad graph induced by the octads refined by a sextet (isomorphic to the collinearity graph of $\mathscr{G}(S_4(2))$) while $\Sigma^{(2)}$ is a complete subgraph on 7 vertices, induced by the 1-subspaces contained in a 3-space. This shows that

$$K_2^{(1)} \cong 2^6 : 3, \quad \widehat{G}_2^{(1)} \cong Sym_6,$$

$$K_2^{(2)} \cong 2^6, \quad \widehat{G}_2^{(2)} \cong L_3(2) \times Sym_3.$$

In particular, $\mathscr{X}^{(1)}$ and $\mathscr{X}^{(2)}$ are not isomorphic. □

Let us turn back to the amalgam $\mathscr{A}_f = \{G_i \mid 1 \leq i \leq 4\}$ of maximal parabolics associated with the action on a rank 4 T-geometry as in the beginning of the section.

Lemma 12.4.4 *The amalgam* $\mathscr{X} = \{G_4, G_3\}$ *is isomorphic to* $\mathscr{X}^{(1)}$.

Proof. Arguing as in the proof of (12.4.3) we produce a covering

$$\chi : \widetilde{\Delta}^{(i)} \to \Delta(\mathscr{G})$$

of the graph $\widetilde{\Delta}^{(i)}$ associated with the universal completion of $\mathscr{X}^{(i)}$ onto the derived graph $\Delta(\mathscr{G})$ of $\mathscr{G}$. If $\mathscr{X} \cong \mathscr{X}^{(2)}$ then we can easily deduce from the proof of (12.4.3) that G_2^+ possesses $L_3(2) \times Sym_3$ as a factor group, which is impossible. □

Notice that since $G_i = \langle G_{i3}, G_{i4}\rangle$ for $i = 1, 2$, the above lemma implies that the universal completion of $\mathscr{A}$ possesses a homomorphism onto M_{24}.

Lemma 12.4.5 *The amalgam* $\{G_4, G_3, G_2\}$ *is uniquely determined up to isomorphism.*

Proof. Let $\widetilde{G}_2$ be the universal completion of the amalgam $\{G_{23}, G_{24}\}$. Then K_2 (which is the largest subgroup normal in both G_{23} and G_{24}) is of the form $2^6 : 3$. We can check in M_{24} (which is a completion of $\{G_4, G_3\}$) that a 3-element from K_2 acts fixed-point freely on $O_2(K_2)$, which means that $Z(K_2) = 1$. In order to prove the lemma we have to show that the kernel of the homomorphism $\varphi : \widetilde{G}_2 \to G_2$ is uniquely determined. The kernel is contained in $C_{\widetilde{G}_2}(K_2)$ while by the proof of (12.4.3)

$$\widetilde{G}_2/(C_{\widetilde{G}_2}(K_2)K_2) \cong Sym_6.$$

Since $G_2/K_2 \cong 3 \cdot Sym_6$, the kernel is an index 3 subgroup in $C_{\widetilde{G}_2}(K_2)$. Suppose there are two such subgroups and let T be their intersection. Then $\overline{G}_2 = \widetilde{G}_2/TK_2 \cong 3^2.Sym_6$. Since the 3-part of the Schur multiplier of Alt_6 is of order 3, $\overline{G}_2$ has a factor-group isomorphic to Alt_3 or Sym_3. On the other hand, $\overline{G}_2$ is a completion of the amalgam $\{G_{23}/K_2, G_{24}/K_2\}$. It is an easy exercise to check that this is impossible (compare (8.5.3 (i))). □

Now we are in a position to establish the main result of the section.

Proposition 12.4.6 *All the amalgams* $\mathscr{A}_f$ *of truncated* M_{24}*-shape are isomorphic and*

$$G_1 \cong M_{24}, \quad G_2 \cong 2^6 : (3 \cdot Alt_6 \times 3).2,$$

$$G_3 \cong 2^6 : (L_3(2) \times Sym_3), \quad G_4 \cong 2^4 : L_4(2).$$

Proof. Since $\text{res}_{\mathscr{G}}(x_1)$ is simply connected the uniqueness of $\mathscr{A}_f$ follows directly from (12.4.5). We know that G_1 is either M_{24} or He and by the paragraph before (12.4.5) the universal completion of $\mathscr{A}_f$ possesses a homomorphism onto M_{24}. Since He is not a subgroup in M_{24} by the order consideration, $G_1 \cong M_{24}$. □

12.5 The completion of $\mathscr{A}_f$

In this section we show that the amalgam $\mathscr{A}_f$ as in (12.4.6) does not possess a faithful completion. More precisely we prove the following.

Proposition 12.5.1 *Let $\mathscr{A}_f$ be the unique amalgam of truncated M_{24}-shape as in* (12.4.6) *and $(U(\mathscr{A}_f), \varphi)$ be the universal completion of $\mathscr{A}_f$. Then*

(i) $U(\mathscr{A}_f) \cong M_{24}$;

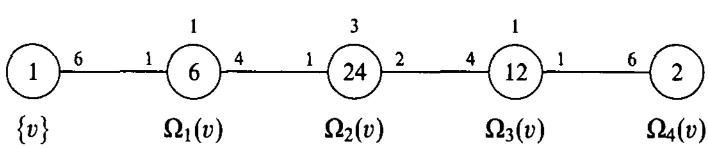
(ii) *the restriction of φ to G_2 has kernel of order* 3;

(iii) $\varphi(G_1) = U(\mathscr{A}_f)$.

We are going to show that starting with a tilde geometry $\mathscr{G}$ of rank at least 4 which possesses a flag-transitive automorphism group G and in which the residual rank 3 tilde geometries are isomorphic to $\mathscr{G}(M_{24})$, we can construct a geometry $\mathscr{H}$ with a locally truncated diagram. This construction generalizes the constructions of $\mathscr{H}(Co_1)$ and $\mathscr{H}(M)$ from $\mathscr{G}(Co_1)$ and $\mathscr{G}(M)$. The group G also acts flag-transitively on the geometry $\mathscr{H}$ and we achieve a contradiction in the case of the amalgam $\mathscr{A}_f$ when reconstructing one of the parabolics associated with the action on $\mathscr{H}$.

Thus let $\mathscr{G}$ be a T-geometry of rank n such that either $n = 3$ and $\mathscr{G} = \mathscr{G}(M_{24})$ or $n \geq 4$ and every rank 3 residual T-geometry in $\mathscr{G}$ is isomorphic to $\mathscr{G}(M_{24})$. Let G be a flag-transitive automorphism group of $\mathscr{G}$ (recall that M_{24} is the only flag-transitive automorphism group of $\mathscr{G}(M_{24})$).

Let $\Delta = \Delta(\mathscr{G})$ be the derived graph of $\mathscr{G}$ where as usual for an element y of $\mathscr{G}$ by $\Sigma[y]$ we denote the subgraph in Δ induced by the vertices (elements of type n in $\mathscr{G}$) incident to y. If y is of type $n-2$ then $\Sigma[y]$ is the collinearity graph Ω of $\text{res}^+_{\mathscr{G}}(y) \cong \mathscr{G}(3 \cdot S_4(2))$ which is an antipodal distance-transitive graph with the intersection diagram



There is an equivalence relation on Ω with classes of the form $\{v\} \cup \Omega_4(v)$ (the antipodal classes). These classes are exactly the fibers of the morphism from Ω onto the collinearity graph of $\mathcal{G}(S_4(2))$ which commutes with the action of the automorphism group.

Define a graph Ψ on the vertex set of Δ by the following rule: two distinct vertices are adjacent in Ψ if they are contained in a subgraph $\Sigma[y]$ for an element y of type $n-2$ and if they are antipodal in this subgraph. By the same letter Ψ we denote a connected component of Ψ containing x_n. We start by the following

Lemma 12.5.2 *If $\mathcal{G} = \mathcal{G}(M_{24})$ then Ψ is a complete graph on 15 vertices.*

Proof. Let φ be the morphism of Δ onto the octad graph which commutes with the action of M_{24}. The vertices of Δ are the central involutions in M_{24} and φ sends such an involution τ onto the octad formed by the elements of $S(5, 8, 24)$ fixed by τ. Then Ψ is a fiber of φ (compare Section 3.3 in [Iv99]) and the stabilizer of Ψ in M_{24} induces on Ψ the doubly transitive action of the octad stabilizer $A \cong 2^4 : L_4(2)$ on the cosets of $C_A(\tau) \cong 2^{1+3+3}.L_3(2)$ where τ is an involution from $O_2(A)$. □

Lemma 12.5.3 *Let H be the stabilizer of Ψ in G. Then*

(i) *H acts transitively on the vertex-set of Ψ;*
(ii) *the valency of Ψ is $2 \cdot \left[{n \atop 2}\right]_2$ and $H(x_n) = G(x_n)$ acts transitively on $\Psi_1(x_n)$.*

Proof. (i) follows from the flag-transitivity of G. Every element y of type $n-2$ incident to x_n corresponds to a pair $\{z^{(1)}(y), z^{(2)}(y)\}$ of vertices adjacent to x_n in Ψ (here $\{x_n, z^{(1)}(y), z^{(2)}(y)\}$ is the antipodal block of $\Sigma[y]$ containing x_n). By (9.2.3), G_n acts primitively on the set of such elements y and hence it is easy to deduce from (12.5.2) that $z^{(i)}(y) = z^{(j)}(y')$ if and only if $i = j$ and $y = y'$, which gives (ii). □

For $1 \leq i \leq n-2$ we associate a subgraph $\Psi[y_i]$ which is the connected component containing x_n of the subgraph in Ψ induced by the intersection $\Psi \cap \Sigma[y_i]$ with an element y_i of type i in $\mathcal{G}$ incident to x_n. We associate the subgraph $\Psi[y_{n-1}]$ induced by the union of the subgraphs $\Psi[z]$ taken for all the elements z of type $n-2$ (the quints) incident to y_{n-1} with an element y_{n-1} of type $n-1$ in $\mathcal{G}$ incident to x_n (a link containing x_n).

Lemma 12.5.4 *The following assertions hold:*

(i) *the valency of $\Psi[y_i]$ is $2 \cdot \left[{n-i \atop 2}\right]_2$ for $i \neq n-1$;*

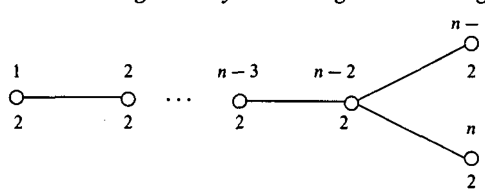
(ii) *for* $1 \leq i \leq j \leq n-2$ *we have* $\Psi[y_i] \subseteq \Psi[y_j]$ *if and only if* y_i *and* y_j *are incident in* $\mathscr{G}$*;*
(iii) $\Psi[y_{n-2}]$ *is a triangle in* Ψ*;*
(iv) $\Psi[y_{n-3}]$ *is a complete graph on* 15 *vertices;*
(v) $\Psi[y_{n-1}]$ *is a complete graph on* $(2^{n+1}+1)$ *vertices.* □

Proof. (i) follows from (12.5.3 (ii)) while (ii) and (iii) are by the definition. Since $\text{res}^+_{\mathscr{G}}(y_{n-3}) \cong \mathscr{G}(M_{24})$, (iv) follows from (12.5.2). By (iii), $\Psi[y_{n-1}]$ is the union of $2^n - 1$ triangles with x_n being the intersection of any two of them. Let z_1 and z_2 be elements of type $n-2$ incident to y_{n-1}. Then, since $\text{res}^-_{\mathscr{G}}(y_{n-1})$ is a projective space, there is an element of type $n-3$ incident to each of y_{n-1}, z_1 and z_2. Hence by (iv) the union $\Psi[z_1] \cup \Psi[z_2]$ induces a complete subgraph (on 5 vertices) and (v) follows. □

Let $\mathscr{D}$ be a subgeometry of rank n in $\mathscr{G}$ whose elements of type n are the vertices of Ψ and the elements of type i for $1 \leq i \leq n-1$ are subgraphs $\Psi[y_i]$ defined as above, where y_i is of type i in $\mathscr{G}$ incident to a vertex of Ψ. If z_i and z_j are elements of type i and j in $\mathscr{D}$ with $i \neq n-1 \neq j$, then z_i and z_j are incident in $\mathscr{D}$ if and only if $z_i \subset z_j$ or $z_j \subset z_i$. An element $\Psi[y_{n-1}]$ of type $n-1$ in $\mathscr{D}$ is incident to all the vertices it contains and to all the elements $\Psi[y_j]$ of type j for $1 \leq j \leq n-2$ defined with respect to elements y_j incident to y_{n-1} in $\mathscr{G}$. It is easy to check that $\Psi[y_{n-1}]$ and $\Psi[y_j]$, $1 \leq j \leq n-2$, are incident in $\mathscr{D}$ if and only if $\Psi[y_{n-1}] \cap \Psi[y_j]$ is of size $2^{n-j+1}+1$.

Proposition 12.5.5 *The geometry* $\mathscr{D}$ *belongs to the diagram*

$D_n(2)$:



and the stabilizer H *of* $\mathscr{D}$ *in* G *induces on* $\mathscr{D}$ *a flag-transitive action.*

Proof. We proceed by induction on n. If $n = 3$ then the result follows from (12.5.2) in view of the Klein correspondence. Thus we may assume that the residue in $\mathscr{D}$ of an element of type 1 belongs to the diagram $D_{n-1}(2)$. On the other hand, it is straightforward by the definition that the residues of x_n in $\mathscr{G}$ and $\mathscr{D}$ are isomorphic. Hence it only remains for us to show that the $\{1, n\}$-edge on the diagram is empty. But this is clear since the incidence in the residue of an element of type $n-1$ is via inclusion. □

In view of the classification of the spherical buildings [Ti74], [Ti82] and the description of their flag-transitive automorphism groups [Sei73], (12.5.5) implies the following.

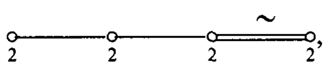
Lemma 12.5.6 *In terms of* (12.5.5) *we have the following:*

(i) *the action* $\overline{H}$ *of* H *on* $\mathscr{D}$ *is isomorphic to* $\Omega_{2n}^{+}(2)$*;*
(ii) *the image* $\overline{I}$ *of* $G(x_n)$ *in* $\overline{H}$ *is of the form* $2^{n(n-1)/2} : L_n(2)$*, where* $O_2(\overline{I})$ *is the exterior square of the natural module of* $\overline{I}/(O_2(\overline{I})) \cong L_n(2)$. □

Proof of Proposition (12.5.1). Since $G_4 \cong 2^4 : L_4(2)$ does not possess $2^6 : L_4(2)$ as a factor-group, (12.5.6) shows that $\mathscr{A}_f$ has no faithful completions. Since we already know that M_{24} is a completion of $\mathscr{A}_f$ the proof follows. □

12.6 Co_1-shape

In this section $\mathscr{G}$ is a rank 4 T-geometry with the diagram



$G_1 \sim 2^{11}.M_{24}$ with $K_1 = O_2(G_1)$ being the irreducible Golay code module $\mathscr{C}_{11}$ for $\overline{G}_1 = G_1/K_1 \cong M_{24}$. Since $H^2(M_{24}, \mathscr{C}_{11}) = 1$, G_1 splits over K_1 and we can choose a complement $N_1 \cong M_{24}$ to K_1 in G_1 so that G_1 is the semidirect product of K_1 and N_1 with respect to the natural action. Since $H^1(M_{24}, \mathscr{C}_{11}) = 1$ all such complements N_1 are conjugate in G_1. We follow the direct strategy, so our first goal is to determine the isomorphism type of the amalgam $\mathscr{B} = \{G_1, G_2\}$ (to be more precise we are going to show that $\mathscr{B}$ is isomorphic to the similar amalgam associated with the action of Co_1 on the T-geometry $\mathscr{G}(Co_1)$.)

The subgroup G_{12} is the preimage in G_1 of the stabilizer $\overline{S} \cong 2^6 : 3 \cdot Sym_6$ in $\overline{G}_1$ of a point of $\mathrm{res}_{\mathscr{G}}(x_1) \cong \mathscr{G}(M_{24})$. Since G_1 is a semidirect product, G_{12} is the semidirect product of K_1 and a subgroup S in N_1 which maps isomorphically onto $\overline{S}$.

By Lemma 3.8.3 in [Iv99] K_1, as a module for S, is uniserial with the composition series

$$1 < K_1^{(2)} < K_1^{(1)} < K_1,$$

where $K_1^{(2)} = C_{K_1}(O_2(S))$ is the natural 4-dimensional symplectic module for $S/O_{2,3}(S) \cong S_4(2)$, $K_1^{(1)} = [K_1, O_2(S)]$ has codimension 1 in K_1 and $K_1^{(1)}/K_1^{(2)}$ is the hexacode module for $S/O_2(S) \cong 3 \cdot Sym_6$. Hence

$$G_{12} \sim 2^4.2^6.2.2^6.3 \cdot Sym_6.$$

We need to identify the subgroup K_2^- which is the kernel of the action of G_2 on the point-set of the line x_2. Towards this end we classify the subgroups of index 2 in G_{12} (since K_2^- is one of them).

Lemma 12.6.1 *The group G_{12} contains exactly three subgroups $Y^{(1)}$, $Y^{(2)}$ and $Y^{(3)}$ of index two. If X is a Sylow 3-subgroup of $O_{2,3}(G_{12})$ and $N^{(i)} = N_{Y^{(i)}}(X)/X$ then up to reordering the following holds*

(i) *$Y^{(1)}$ is the semidirect product of K_1 and $S' \cong 2^6 : 3 \cdot Alt_6$ with $N^{(1)} \cong 2^5 : Alt_6$;*
(ii) *$Y^{(2)}$ is the semidirect product of $K_1^{(1)}$ and S with $N^{(2)} \cong 2^4 : Sym_6$;*
(iii) *$Y^{(3)}$ is the 'diagonal' subgroup with $N^{(3)} \cong 2^4 \cdot Sym_6$ (the non-split extension).*

Proof. A subgroup of index 2 in G_{12}, certainly contains the commutator subgroup G'_{12} of G_{12}. It is easy to see that G'_{12} is the semidirect product of $K_1^{(1)}$ and $S' \cong 2^6 : 3 \cdot Alt_6$. Thus $G_{12}/G'_{12} \cong 2^2$ and there are three subgroups of index 2 in G_{12}. The result is clear in view of the fact that $C_{K_1}(X)$ is an indecomposable extension of the natural symplectic module $K_1^{(2)}$ for $N_S(X)/X \cong S_4(2)$ by a trivial 1-dimensional module. □

Since K_1 induces a non-trivial action on the point-set of x_2, K_2^- does not contain the whole of K_1, so $K_2^- \neq Y^{(1)}$, but at this stage we are still left with two possibilities for K_2^-. In order to choose between the possibilities let us have a closer look at the possible structure of G_2. As usual let L_2 be the kernel of the action of G_2 on the set of elements y_2 of type 2 such that $\{x_1, y_2, x_3, x_4\}$ is a flag. Let $\mathscr{E}$ be the set of subgroups $K(u) \cap K_2$ taken for all the points incident to x_2 (so that $\mathscr{E}$ consists of three subgroups).

Lemma 12.6.2

$$G_2 \sim 2^{4+12}.(3 \cdot Sym_6 \times Sym_3),$$

and furthermore

(i) *$K_1 \cap K_2 = K_1^{(1)}$ has index 2 in K_1;*
(ii) *$K_2 = O_2(G_2)$ and K_2/L_2 is the tensor product of the hexacode module for $K_2^-/K_2 \cong 3 \cdot Sym_6$ and of the 2-dimensional module for $K_2^+/K_2 \cong Sym_3$;*

(iii) $L_2 = K_1^{(2)} = \cap_{E \in \mathcal{E}} E$ *and* $L_2 \cong 2^4$ *is the natural symplectic module for* $G_2/G_2^{\infty} \cong S_4(2)$*;*
(iv) *if* E *is an elementary abelian subgroup of order* 2^{10} *in* K_2 *which is normal in* K_2^- *then* $E \in \mathcal{E}$.

Proof. Since K_1 acts trivially on $\mathrm{res}^+_{\mathcal{G}}(x_2)$ and induces on $\mathrm{res}^-_{\mathcal{G}}(x_2)$ an action of order 2, (i) follows. Now (ii) follows from (9.4.1) and implies (iii). Since the action of the group $3 \cdot Sym_6$ on the hexacode module is absolutely irreducible by (8.2.9), (iii) implies (iv). □

Before identifying K_2^-, let us explain a minor difficulty we experience at this stage. What we know for sure, is that K_2^- contains $G'_{12} \sim 2^{4+6+6}.3 \cdot Alt_6$. The action of $3 \cdot Alt_6$ on the hexacode module H is not absolutely irreducible (it preserves a $GF(4)$-vector space structure). By (12.6.2 (ii)), $\widetilde{K}_2 = K_2/L_2$ is the direct sum of two copies of the hexacode module. Hence there are exactly five (the number of 1-subspaces in a 2-dimensional $GF(4)$-space) G'_{12}/K_2-submodules in $\widetilde{K}_2$, isomorphic to the hexacode module. Thus we cannot reconstruct $\mathcal{E}$ as in (12.6.2 (iv)) just looking at the action of G'_{12} on $\widetilde{K}_2$, since a priori the preimage in K_2 of any of the five hexacode submodules could be a subgroup from $\mathcal{E}$. But in fact at most three of the preimages are elementary abelian.

Lemma 12.6.3 *Let* E *be an elementary abelian subgroup of order* 2^{10} *in* K_2 *which is normal in* G'_{12}. *Then* $E \in \mathcal{E}$.

Proof. Since the second cohomology group of every chief factor of G'_{12} inside K_2 is trivial, G'_{12} splits over K_2. Let $T \cong 3 \cdot Alt_6$ be a complement so that $X = O_3(T)$. If $\mathcal{E} = \{E_1, E_2, E_3\}$ then, (treating E_i as a module for T) we have

$$E_i = L_2 \oplus V_h^{(i)},$$

$L_2 = C_{E_i}(X)$ and $V_h^{(i)} = [E_i, X]$ is the hexacode module for T.

Since G'_{12} is isomorphic to the corresponding subgroup associated with the action of Co_1 on $\mathcal{G}(Co_1)$, we know that K_2 must contain the subgroups E_i as above. Notice that the centralizer in T of a non-zero vector from $V_h^{(i)}$ for $i = 1$ or 2 centralizes a unique non-zero vector in L_2. Thus there is a unique surjective mapping

$$\lambda : V_h^{(1)} \to L_2,$$

which commutes with the action of T. Notice that we can treat the non-zero vectors in $V_h^{(1)}$ and L_2 as points of $\mathcal{G}(3 \cdot S_4(2))$ and $\mathcal{G}(S_4(2))$, respectively. Then λ is the morphism of the geometries, which commutes with the action of the automorphism group.

Since $K_2 = V_h^{(1)} V_h^{(2)} L_2$, it is easy to see that

$$V_h^{(3)} = \{h\varphi(h)l(h) \mid h \in V_h^{(1)}\},$$

where $l(h) \in L_2$ and $\varphi : V_h^{(1)} \to V_h^{(2)}$ is an isomorphism. Let $T(h) \cong Sym_4$ be the stabilizer of h in T. Since $V_h^{(3)}$ is the hexacode module for T, $h\varphi(h)l(h)$ must be centralized by $T(h)$, which means that

(a) either $l(h)$ is the identity for of all $h \in V_h^{(1)}$ or $l(h) = \lambda(h)$ for all $h \in V_h^{(1)}$;
(b) $\varphi(h)$ is contained in the 1-dimensional $GF(4)$-subspace in $V_h^{(2)}$ centralized by $T(h)$.

By reducing the product of $h\varphi(h)l(h)$ and $h'\varphi(h')l(h')$ to the canonical form $hh'\varphi(hh')l(hh')$, we deduce the following equality:

(c) $[h', \varphi(h)] = l(h)l(h')l(hh')$.

Since the mapping $(h_1, h_2) \mapsto [h_1, h_2]$ for $h_1 \in V_h^{(1)}$, $h_2 \in V_h^{(2)}$ is non-trivial, in view of (a) we conclude that $l(h) = \lambda(h)$ for all $h \in H$. This shows that $[h, \varphi(h)] = \lambda(h)^2 = 1$, which is consistent with the assumption that $V_h^{(3)}$ is an elementary abelian 2-group. We claim that the isomorphism φ is uniquely determined. Indeed, let $\{h_1 = h, h_2, h_3\}$ be the line in $V_h^{(1)}$ centralized by $T(h)$ and $\{k_1 = \varphi(h), k_2, k_3\}$ be the line in $V_h^{(2)}$ centralized by $T(h)$ (we may assume that $k_i = \varphi(h_i)$ for $i = 2$ and 3). Then

$$[h, k_2] = [h, \varphi(h_2)] = \lambda(h)\lambda(h_2)\lambda(hh_2) = \lambda(h)^3 \neq 1$$

and the proof follows. □

Lemma 12.6.4 $K_2^- = Y^{(2)}$.

Proof. By (12.6.1) and the paragraph after the proof of (12.6.1) it remains to show that $K_2^- \neq Y^{(3)}$. By (12.6.2 (iv)) and (12.6.3) K_2^- is the kernel of the action of G_{12} on the well-defined collection $\mathscr{E}$. Since $G_{12} = Y^{(2)}Y^{(3)}$ induces on $\mathscr{E}$ an action of order 2, K_2^- is characterized among $Y^{(2)}$ and $Y^{(3)}$ as that which normalizes at least two elementary abelian subgroups of order 2^{10} in K_2, normalized by G'_{12}. Clearly both $K_1^{(1)}$ and $L_2O_2(S)$ are contained in $\mathscr{E}$ and each of them is normalized by $Y^{(2)}$. Hence the proof. □

Lemma 12.6.5 $\mathrm{Out}\, K_2^- \cong Sym_3 \times 2$.

Proof. By (12.6.2 (iii)), $\mathrm{Out}\, K_2^-$ acts on $\mathscr{E}$ and since K_2^- is isomorphic to the corresponding subgroup associated with the action of Co_1 on $\mathscr{G}(Co_1)$, we know that $\mathrm{Out}\, K_2^-$ induces Sym_3 on $\mathscr{E}$. Let B be the subgroup in $\mathrm{Aut}\, K_2^-$ which acts trivially on $\mathscr{E}$ (notice that B contains all the inner automorphisms). We claim that $B/\mathrm{Inn}\, K_2^-$ has order 2. Let $\tau \in B$. Since X is a Sylow 3-subgroup in $O_{2,3}(K_2^-)$ we can adjust τ by an inner automorphism so that τ normalizes X. Then τ normalizes $N := N_{K_2^-}(X) \cong (3 \times 2^4) \cdot Sym_6$. We know by (12.6.1 (ii)) that N splits over $O_2(N)$. Since $H^1(N/O_{2,3}(N), O_2(N))$ is 1-dimensional (cf. Table VI in Section 8.2), there are two classes of complements to $O_2(N)$ in N. In order to complete the proof it is sufficient to show that whenever τ normalizes a complement $\widehat{T} \cong 3 \cdot Sym_6$ to $O_2(N)$ in N, τ is inner. Since $N/O_{2,3}(N) \cong Sym_6$ is self-normalized in $\mathrm{Out}\, O_2(N) \cong L_4(2)$, τ induces an inner automorphism of $\widehat{T}$ and hence we may assume that τ centralizes $\widehat{T}$. Recall that τ normalizes each $E_i \in \mathscr{E}$ and by the above the action of τ commutes with the action of $\widehat{T}$. As a module for $\widehat{T}$ the subgroup E_i possesses the direct sum decomposition

$$E_i = L_2 \oplus V_h^{(i)}$$

where L_2 and $V_h^{(i)}$ are non-isomorphic and absolutely irreducible by (8.2.9). This means that τ centralizes E_i and hence must be the identity automorphism. Now it remains to mention that since $O_2(N) = L_2 = Z(K_2)$, an automorphism of N which permutes the classes of complements to $O_2(N)$ can be extended to an automorphism of K_2^-. □

Since the centre of K_2^- is trivial, (12.6.5) implies that G_2 is the preimage of a Sym_3-subgroup in $\mathrm{Out}\, K_2^-$. By (12.6.5) there are exactly two Sym_3-subgroups in $\mathrm{Out}\, K_2^-$ and by the proof of (12.6.5) one of them, say D_1, is the kernel of the action on the classes of complements to K_2. We know that K_1 is contained in G_2 and that the image of K_1 in $\mathrm{Out}\, K_2^-$ has order 2. Furthermore, $C_{K_1}(X)$ is indecomposable and hence an element from K_1 permutes the classes of complements to K_2. Thus G_2 is the preimage in $\mathrm{Aut}\, K_2^-$ of the Sym_3-subgroup in $\mathrm{Out}\, K_2^-$ other than D_1.

By the above paragraph the type of $\mathscr{B} = \{G_1, G_2\}$ is uniquely determined. Also it is easy to deduce from the proof of (12.6.5) that every automorphism of G_{12} can be extended to an automorphism of G_2. In view of Goldschmidt's theorem (8.3.2) we obtain the following.

Lemma 12.6.6 *In the considered situation the amalgam $\mathscr{B} = \{G_1, G_2\}$ is isomorphic to the analogous amalgam associated with the action of Co_1 on $\mathscr{G}(Co_1)$.* □

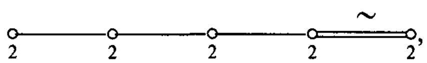
Now applying (8.6.1) we obtain the main result of the section.

Proposition 12.6.7 *All the amalgams of Co_1-shape are isomorphic to $\mathscr{A}(Co_1, \mathscr{G}(Co_1))$ and the universal completion of such an amalgam is isomorphic to Co_1.* □

In terms of generators and relations the amalgam of maximal parabolics associated with the action of Co_1 on $\mathscr{G}(Co_1)$ was characterized in [FS98].

12.7 M-shape

In this section $\mathscr{G}$ is a T-geometry of rank 5 with the diagram



the residue of a point is isomorphic to $\mathscr{G}(Co_1)$,

$$G_1 \sim 2.2^{24}.Co_1,$$

where L_1 is of order 2 and K_1/L_1 is the universal representation module of $\mathscr{G}(Co_1)$, isomorphic to the Leech lattice $\overline{\Lambda}^{(24)}$ taken modulo 2. Arguing as in the proof of (11.5.1) we obtain the following.

Lemma 12.7.1 $K_1 = O_2(G_1)$ *is extraspecial of plus type and* $G_1 \sim 2_+^{1+24}.Co_1$. □

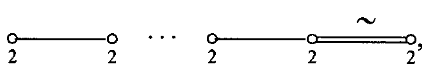
Since $\mathscr{A} = \{G_i \mid 1 \le i \le 5\}$ is the amalgam of maximal parabolics associated with an action on a T-geometry with $\mathrm{res}^+_{\mathscr{G}}(x_2) \cong \mathscr{G}(M_{24})$, it is immediate that the conditions in Definition 5.1.1 of [Iv99] are satisfied, which means that $\mathscr{C} = \{G_1, G_2, G_3\}$ is a *Monster amalgam* , in particular,

$$G_2 \sim 2^{2+11+22}.(Sym_3 \times M_{24}), \quad G_3 \sim 2^{3+6+12+18}.(L_3(2) \times 3 \cdot Sym_6).$$

By Proposition 5.13.5 in [Iv99] all the Monster amalgams are isomorphic, which means that $\mathscr{C}$ is isomorphic to the corresponding amalgam associated with the action of M on $\mathscr{G}(M)$.

12.8 $S_{2n}(2)$-shape, $n \ge 4$

In this section $\mathscr{G}$ is a T-geometry of rank $n \ge 4$ with the diagram



in which the residue of a point is isomorphic to $\mathcal{G}(3^{[{n-1 \atop 2}]_2} \cdot S_{2n-2}(2))$, G is a flag-transitive automorphism group of $\mathcal{G}$, such that

$$G_1 \sim 2.2^{2n-2}.3^{[{n-1 \atop 2}]_2} \cdot S_{2n-2}(2),$$

so that $Z_1 = Z(G_1)$ is of order 2 and K_1/Z_1 is the natural symplectic module for $\overline{G}_1/O_3(\overline{G}_1) \cong S_{2n-2}(2)$;

$$G_n \sim 2^{n(n-1)/2}.2^n.L_n(2),$$

so that L_n is the exterior square of the natural module of $\overline{G}_n \cong L_n(2)$ and $\widehat{K}_n := K_n/L_n$ is the natural module for $\overline{G}_n$. Our goal is to show that the amalgam $\mathcal{A} = \{G_i \mid 1 \le i \le n\}$ is isomorphic to the amalgam $\mathcal{A}^0 = \{G_i^0 \mid 1 \le i \le n\}$ associated with the action of

$$G^0 \cong 3^{[{n \atop 2}]_2} \cdot S_{2n}(2)$$

on its T-geometry $\mathcal{G}(G^0)$.

Let

$$\mu : G^0 \to \overline{G} = G^0/O_3(G^0) \cong S_{2n}(2)$$

be the natural homomorphism and let $\overline{G}_i = \mu(G_i^0)$ for $1 \le i \le n$. Then $\overline{G}_i \cong G_i^0/O_3(G_i^0)$ and

$$\overline{\mathcal{A}} := \{\overline{G}_i \mid 1 \le i \le n\}$$

is the amalgam of maximal parabolics associated with the action of $\overline{G} \cong S_{2n}(2)$ on its symplectic polar space $\mathcal{G}(S_{2n}(2))$ (where $\overline{G}_i$ is the stabilizer of the i-dimensional totally isotropic subspace from a fixed maximal flag). From this and the well-known properties of the parabolics in $S_{2n}(2)$ we make the following observation.

Lemma 12.8.1 *G_1^0 splits over $O_2(G_1^0)$ and G_n^0 splits over $O_2(G_n^0)$.*

In the next lemma we follow notation from (3.2.7). The proof is similar to that of (12.2.1) and therefore is not given here.

Lemma 12.8.2 *The subgroup K_n is an elementary abelian 2-group and as a module for $\overline{G}_n \cong L_n(2)$ it is isomorphic to the quotient $\mathcal{P}_e^1/\mathcal{X}(2)$ of the even half of the $GF(2)$-permutation module of $L_n(2)$ on the set of 1-subspaces in the natural module.* □

Let us consider K_n as a module for

$$\widehat{G}_{1n} := G_{1n}/K_n \cong 2^{n-1} : L_{n-1}(2).$$

The following result can be checked directly using the structure of K_n specified in (12.8.2).

Lemma 12.8.3 *The following assertions hold:*

(i) L_n, *as a module for* $\widehat{G}_{1n}$, *contains a unique submodule* $L_n^{(1)}$, *which is isomorphic to the natural module of* $\widehat{G}_{1n}/O_2(\widehat{G}_{1n}) \cong L_{n-1}(2)$ *and* $L_n/L_n^{(1)} \cong \bigwedge^2 L_n^{(1)}$;

(ii) $\widehat{K}_n$, *as a module for* $\widehat{G}_{1n}$, *contains a unique submodule* $\widehat{K}_n^{(1)}$ *which is* 1-*dimensional and* $\widehat{K}_n/\widehat{K}_n^{(1)}$ *is isomorphic to the dual of* $L_n^{(1)}$. □

Let us now allocate K_1 inside $O_2(G_{1n})$. Recall that in terms of the action of G on the derived graph the subgroup K_1 is the vertexwise stabilizer of the subgraph $\Sigma = \Sigma[x_1]$.

Lemma 12.8.4 *The following assertions hold:*

(i) $K_1 \cap L_n = L_n^{(1)}$;

(ii) $K_1 L_n/L_n = \widehat{K}_n^{(1)}$;

(iii) $K_1 K_n = O_2(G_{1n})$.

Proof. The elementwise stabilizer of $\Sigma_1(x_n)$ in G_1 induces on $\Sigma_2(x_n)$ an action of order $2^{(n-1)(n-2)/2}$, hence (12.8.3 (i)) gives (i). Since $K_1 \cap K_n$ fixes every vertex in $\Sigma_1(x_n)$, it induces on $\Delta_1(x_n)$ an action of order 2, which gives (ii). Finally (iii) is by the order consideration. □

Lemma 12.8.5 *The following assertions hold:*

(i) K_1 *is elementary abelian;*

(ii) K_1, *as a module for* $\overline{G}_1/O_3(\overline{G}_1) \cong S_{2n-2}(2) \cong \Omega_{2n-1}(2)$, *is isomorphic to the natural orthogonal module.*

Proof. Since G_1 acts irreducibly on K_1/Z_1 (isomorphic to the natural symplectic module), K_1 is either abelian or extraspecial and since G_1 does not preserve non-zero quadratic forms on the quotient, K_1 cannot be extraspecial and (i) follows. In view of (8.2.6), in order to prove (ii) it is sufficient to show that K_1 is indecomposable, which is easy to deduce from (12.8.4) and the structure of K_n as that in (12.8.3). □

Let us turn to the structure of G_2.

Lemma 12.8.6 *The following assertions hold:*

(i) $[K_1 : K_1 \cap K_2] = 2$;

(ii) G_2 *induces* Sym_3 *on the triple of points incident to* x_2;

(iii) *G_2 induces on* $\mathrm{res}^+_{\mathscr{G}}(x_2) \cong \mathscr{G}(3^{\left[{n-2 \atop 2}\right]_2} \cdot S_{2n-4}(2))$ *the full automorphism of the residue;*
(iv) $\overline{G}_2 \cong Sym_3 \times (3^{\left[{n-2 \atop 2}\right]_2} \cdot S_{2n-4}(2))$.

Proof. Since K_1 is contained in G_2 and K_1/Z_1 is non-trivial, K_1 induces an action of order 2 on $\mathrm{res}^-_{\mathscr{G}}(x_2)$ (clearly K_1 fixes $\mathrm{res}^+_{\mathscr{G}}(x_2) \subseteq \mathrm{res}_{\mathscr{G}}(x_1)$ elementwise). This gives (i) and (ii). The rest follows from the basic properties of the T-geometries of symplectic type (cf. Chapter 6 in [Iv99]). □

Lemma 12.8.7 *Put* $\widehat{K}_2^- = K_2^-/K_2 \cong 3^{\left[{n-2 \atop 2}\right]_2} \cdot S_{2n-4}(2)$. *Then*

(i) $|L_2| = 2$;
(ii) *K_2/L_2 is elementary abelian isomorphic to the tensor product of the natural $(2n-3)$-dimensional orthogonal module of* $\widehat{K}_2^-/O_3(\widehat{K}_2^-) \cong \Omega_{2n-3}(2)$ *and the 2-dimensional module of* $G_2/K_2^+ \cong Sym_3$;
(iii) *if X is a Sylow 3-subgroup of K_2^+ then $C_{G_2}(X) \cong X \times D$ where $L_2 \le D$ and $D/L_2 \cong \widehat{K}_2^-$.*

Proof. (ii) follows from (9.4.1) and implies (i) by the order reason. Finally (iii) is by (12.8.6 (iv)). □

Lemma 12.8.8 *In terms of* (12.8.7) *D splits over L_2 i.e., $D \cong L_2 \times D_0$, where $D_0 \cong \widehat{K}_2^-$.*

Proof. It is known (cf. [CCNPW]) that the Schur multiplier of $S_{2n-4}(2)$ is trivial unless $2n-4 \le 6$, thus we only have to handle the cases $n = 4$ and $n = 5$. Suppose first that $n = 5$ and that $D/O_3(D) \cong 2{\cdot}S_6(2)$ (the only non-split extension). It is known that the preimage in $2{\cdot}S_6(2)$ of a transvection of $S_6(2)$ has order 4, in particular, $O_2(D \cap G_n)$ is not elementary abelian, which is contradictory to (12.8.2), since $O_2(D \cap G_n) \le K_n$. Similarly, if $n = 4$, then, independent of whether D involves a non-split double cover of Alt_6 or is a semidirect product of $3 \cdot Alt_6$ with a cyclic group of order 4, $O_2(D \cap G_n)$ contains an element of order 4, which is not possible. □

Lemma 12.8.9 *G_1 splits over K_1.*

Proof. Let $D_0 \cong 3^{\left[{n-2 \atop 2}\right]_2} \cdot S_{2n-4}(2)$ be the direct factor as in (12.8.8). It follows from (12.8.6 (i)) that, considered as a D_0-module, $K_1 \cap K_2$ is an extension of two 1-dimensional modules by the natural symplectic module of the $S_{2n-4}(2)$-factor of D_0. By (8.2.6) this implies that $K_1 \cap K_2$ is a direct sum of a 1-dimensional module and a module Y of dimension $2n-3$. Since $(K_1 \cap K_2)/L_2$ is indecomposable, we have $K_1 \cap K_2 = Y L_0$ and hence $K_2 = (K_1 \cap K_2)Y^x$ where x is a generator of the Sylow 3-subgroup X of K_2^+. Finally $G_{12} = K_1(Y^x D_0)$ splits over K_1. Since G_{12} contains a Sylow 2-subgroup of G_1 the proof follows by (8.2.8). □

Lemma 12.8.10 *$G_1 \cong G_1^0$, in particular, G_1 splits over K_1.*

Proof. In view of (12.8.9) it only remains to establish the module structure of K_1. By our original assumption K_1 is an extension of the trivial 1-dimensional module by the natural symplectic module for $S_{2n-2}(2)$. It follows from (12.8.7 (ii)) that $[K_1, K_2] = L_2$, since $[K_1, K_2]$ clearly contains L_2 and $[x, K_2]$ covers the image of $K_1 \cap K_2$ in K_2/L_2. In particular, $[K_1, K_2]$ has dimension $2n-2$ which excludes the possibility that K_1 is a direct sum. Finally by (8.2.6), K_1 must be the only indecomposable extension, namely the natural orthogonal module of $S_{2n-2}(2) \cong \Omega_{2n-2}(2)$. □

Lemma 12.8.11 *$G_n \cong G_n^0$, in particular, G_n splits over K_n.*

Proof. By Gaschütz' theorem (8.2.8), G_n splits over K_n if and only if G_{1n} splits over K_n. Let $\psi : G_1^0 \to G_1$ be the isomorphism, whose existence is guaranteed by (12.8.10) and S_{12}^0 be a complement to $O_2(G_n)$ in $G_{1n}^0 \leq G_n$ (by (12.8.1) such a complement exists). Then $\psi(S_{12}^0)$ is a complement in $G_{12} = \psi(G_{12}^0)$ to $K_n = \psi(O_2(G_{12}^0))$ and the result follows. Notice that G_{12}^0 is uniquely determined in G_1^0 up to conjugation as the preimage of the stabilizer in $G_1^0/O_{2,3}(G_1^0) \cong S_{2n-2}(2)$ of a maximal totally isotropic subspace in the natural symplectic module. □

We follow the dual strategy and our nearest goal is to reconstruct up to isomorphism the amalgam $\mathscr{X} = \{G_n, G_{n-1}\}$. By (12.8.2) and (12.8.11) the structure of G_n is known precisely. Then $G_{n-1,n}$ is the full preimage of the stabilizer in $\overline{G}_n$ of the hyperplane x_{n-1} in the natural module of $\overline{G}_n \cong L_n(2)$. We denote x_{n-1} also by W and call it the natural module for $G_{n-1,n}/O_2(G_{n-1,n}) \cong L_{n-1}(2)$.

Lemma 12.8.12 *The following assertions hold:*

(i) *K_{n-1}^+ coincides with $O^2(G_{n-1,n})$ and is the unique subgroup of index* 2 *in $G_{n-1,n}$;*

(ii) *there is an elementary abelian subgroup T_0 in L_n, which is in the centre of $O_2(G_{n-1,n})$ and as a module for $G_{n-1,n}/O_2(G_{n-1,n})$ it is isomorphic to W;*

(iii) *$G_{n-1,n}$ contains within K_{n-1}^{+}/T_0 exactly three composition factors, each isomorphic to $\bigwedge^2 W$.*

Proof. Everything follows directly from the structure of G_n and the definition of $G_{n-1,n}$. In order to see (iii) we are using (9.2.4).

By (12.8.12) K_{n-1}^{+} has a trivial centralizer in G_{n-1} and therefore G_{n-1} can be identified with a suitable subgroup in $\mathrm{Aut}\, K_{n-1}^{+}$ such that

(P1) $G_{n-1,n}$ is a subgroup of index 2 in G_{n-1};
(P2) $G_{n-1}/K_{n-1}^{+} \cong Sym_3$.

Thus $\mathscr{X}$ is contained in the amalgam $\{G_n, \mathrm{Aut}\, K_{n-1}^{+}\}$, which is determined uniquely up to isomorphism.

Lemma 12.8.13 *Let $T = O_2(K_{n-1}^{+})$. Then $Z(T)$ involves exactly two chief factors of K_{n-1}^{+}, namely $T_0 \cong \bigwedge^2 W$ and $Z(T)/T_0 \cong W$. As a module for $K_{n-1}^{+}/T \cong L_{n-1}(2)$ the module $Z(T)$ is indecomposable.*

Proof. Clearly $Z(T)$ contains the centre Z of the Borel subgroup B. It is easy to deduce from (12.8.2) that Z is of order 4. Thus $Z(T)$ involves at least two chief factors. One of them is T_0 as in (12.8.12 (iii)). On the other hand, T covers the subgroup $O_2(G_{n-1,n})/K_n$ of G_n/K_n which acts non-trivially on L_n. Hence $Z(T) \leq K_n$ and $Z(T) \cap L_n = T_0$. Thus $Z(T)$ contains another chief factor, which is isomorphic to W.

It only remains to show that $Z(T)$ is indecomposable. Suppose on the contrary that

$$Z(T) = T_0 \oplus T_1 \quad \text{and} \quad T_1 \cong W.$$

For a point p of $\mathscr{G}$ incident to x_{n-1} let $z(p)$ be the unique non-zero element in the centre of $G(p)$. Since $G(p) \cap G_{n-1,n}$ contains a Sylow 2-subgroup of $G_{n-1,n}$, we conclude that $z(p) \in Z(T)$. Since $G(p) \cap G_{n-1,n}$ does not stabilize non-zero vectors in $T_0 \cong \bigwedge^2 W$, we must have $z(p) \in T_1$. Suppose now that l is a line incident to x_{n-1} and $\{p_1, p_2, p_3\}$ is the point set of l. Then, because of the isomorphism $T_1 \cong W$, we must have

$$z(p_1) + z(p_2) + z(p_3) = 0,$$

which shows that K_n splits over L_n contrary to (12.8.2). □

Now let us turn to the outer automorphism group of K_{n-1}^{+}. By (12.8.11) we have $K_{n-1}^{+} = TS$ for a subgroup $S \cong L_{n-1}(2)$. First let us consider the subgroup $\overline{K}_{n-1}^{+} = K_{n-1}^{+}/Z(T)$.

Lemma 12.8.14 $\operatorname{Out} K_{n-1}^{+} \cong Sym_3$ *if* $n \geq 5$ *and* $\operatorname{Out} K_{n-1}^{+} \cong Sym_4$ *if* $n = 4$.

Proof. The group $\overline{K}_{n-1}^{+}$ is a semidirect product of $\overline{T}$ and $\overline{S} \cong L_{n-1}(2)$. Since by (12.8.11), $\overline{K}_{n-1}^{+}$ is isomorphic to the corresponding subgroup in G_{n-1}^{0}, it possesses an outer automorphism group Sym_3. As a consequence we conclude that $\overline{K}_{n+1}^{+}$ must be the direct sum of two copies of the $\overline{S}$-module isomorphic to W. Since by (8.2.5), $H^1(\overline{S}, W)$ is trivial if $n \geq 5$ and 1-dimensional if $n = 4$, the proof follows (compare the proof of (12.4.1)). □

It remains for us to determine the image in $\operatorname{Out} K_{n-1}^{+}$ of the subgroup

$$A := C_{\operatorname{Aut} K_{n-1}^{+}}(\overline{K}_{n-1}^{+}).$$

Lemma 12.8.15 *The following assertions hold:*

(i) *if* $a \in A$ *then* a *acts trivially on* T*;*

(ii) *the image in* $\operatorname{Out} K_{n-1}^{+}$ *of the subgroup* A *is trivial if* $n \geq 5$ *and it is a normal subgroup of order* 2 *if* $n = 4$ *or* 5.

Proof. As above, let S be a subgroup in K_{n-1}^{+}, isomorphic to $L_{n-1}(2)$. Let $a \in A$. Notice first that if $s \in S$ then $s^a = s \cdot z_a$ for some $z_a \in Z(T)$. This means that a preserves the action of S on T. On the one hand, this implies that a acts trivially on $Z(T)$. On the other hand, the mapping

$$\lambda : t \mapsto [t, a]$$

from $T/Z(T)$ to $Z(T)$ must be linear, commuting with the action of S. By (12.8.13), $Z(T)$ contains no submodules isomorphic to W. Hence λ must be trivial, which gives (i).

Now as usual the question is reduced to the number of complements to $Z(T)$ in $Z(T)S$. By (12.8.13) we know that $Z(T)$ involves two factors isomorphic to W and $\bigwedge^2 W$, respectively. Hence we only have to consider the case when $n = 4$ (when both $H^1(S, W)$ and $H^1(S, \bigwedge^2 W)$ are non-trivial) and the case when $n = 5$ (when $H^1(S, W)$ is trivial, but $H^1(S, \bigwedge^2 W)$ is non-trivial). We do not present the relevant argument in full here (cf. Lemma (5.4) in [ShSt94]). □

Lemma 12.8.16 *The amalgam* $\mathscr{X} = \{G_n, G_{n-1}\}$ *is determined uniquely up to isomorphism.*

Proof. It was mentioned before (12.8.13) that $\mathscr{X}$ is a subamalgam in the uniquely determined amalgam $\{G_n, \mathrm{Aut}\,K^+_{n-1}\}$. Suppose that $n \geq 5$. Then by (12.8.14) and (12.8.15) we have

$$\mathrm{Aut}\,K^+_{n-1}/K^+_{n-1} \cong Sym_3 \text{ or } Sym_3 \times 2,$$

in particular, G_{n-1} is uniquely specified in $\mathrm{Aut}\,K^+_{n-1}$ by the conditions (P1) and (P2) stated before (12.8.13).

When $n = 4$ some further arguments are required, which we do not reproduce here (cf. Lemmas (5.6) to (5.8) in [ShSt94]). □

Lemma 12.8.17 *The amalgam $\{G_n, G_{n-1}, G_{n-2}\}$ is determined uniquely up to isomorphism.*

Proof. By (12.8.16) $\mathscr{X}$ is isomorphic to $\mathscr{X}^0 = \{G^0_n, G^0_{n-1}\}$ and since $O_3(G_n) = O_3(G_{n-1}) = 1$, also to $\overline{\mathscr{X}} = \{\overline{G}_n, \overline{G}_{n-1}\}$. Let $\widetilde{G}_{n-2}$ be the universal completion of the amalgam $\{G_n \cap G_{n-2}, G_{n-1} \cap G_{n-2}\}$ (as usual this amalgam is easily specified inside $\mathscr{X}$). Then in order to prove the lemma it is sufficient to show that the kernel N of the homomorphism of $\widetilde{G}_{n-2}$ onto G_{n-2} is uniquely determined.

Let $\overline{N}$ be the kernel of the homomorphism of $\widetilde{G}_{n-2}$ onto $\overline{G}_{n-2}$. Since $|O_3(G_{n-2})| = 3$ and in view of the existence of the homomorphism μ, we immediately conclude that N has index 3 in $\overline{N}$. Suppose there are two possible choices for N, say N_1 and N_2, and consider

$$\widehat{G}_{n-2} = \widetilde{G}_{n-2}/\langle K^+_{n-2}, N_1 \cap N_2\rangle \cong 3^2.Sym_6.$$

Since the 3-part of the Schur multiplier of Alt_6 is of order 3, $\widehat{G}_{n-2}$ possesses a factor group $\widehat{F}$ isomorphic to Sym_3 or Alt_3. On the other hand, $\widehat{G}_{n-2}$ (and hence $\widehat{F}$ as well) is a completion of the amalgam

$$\mathscr{J} = \{(G_n \cap G_{n-2})/K^+_{n-2}, (G_{n-1} \cap G_{n-2})/K^+_{n-2}\} \cong \{Sym_4 \times 2, Sym_4 \times 2\}$$

(notice that $\mathscr{J}$ is a subamalgam in Sym_6). Now it is easy to check that $\mathscr{J}$ could not possibly have $\widehat{F}$ as a completion. □

Since $\mathrm{res}^+_{\mathscr{G}}(x_i)$ is simply connected for $1 \leq i \leq n-3$ by the induction hypothesis, we obtain the following.

Proposition 12.8.18 *An amalgam of $S_{2n}(2)$-shape for $n \geq 4$ is isomorphic to the amalgam $\mathscr{A}^0 = \mathscr{A}(G^0, \mathscr{G}(G^0))$ and its universal completion is G^0.*

Concluding Remarks

Thus the exposition of the classification for the flag-transitive Petersen and tilde geometries is complete. The classification was announced in [ISh94b], while an outline of the history of the project along with the names of many people who contributed to it can be found in Section 1.12 in [Iv99].

Let us emphasize that we never assumed that the finiteness of the Borel subgroup and that our classification proof rely on results of computer calculations in the following instances:

(a) the non-existence of a faithful completion of the amalgam of Alt_7-shape (12.1.1);
(b) the simple connectedness of the rank 3 T-geometries $\mathcal{G}(M_{24})$, $\mathcal{G}(He)$ and $\mathcal{G}(3^7 \cdot S_6(2))$ established (computationally) independently in [Hei91] and in an unpublished work of the present authors;
(c) the universal representation module of $\mathcal{G}(He)$ (4.6.1);
(d) the universal representation group of the involution geometry of Alt_7 (6.2.1).

It would certainly be nice to achieve in due course a completely computer-free classification, but at the moment it seems rather complicated.

In our proof the construction, the simple connectedness proof and the classification via the amalgam method come separately and independently. One would like to see a uniform treatment, say of the Monster group M (starting with a 2-local structure and leading to the existence and uniqueness), based solely on the T-geometry $\mathcal{G}(M)$, as it was treated in [IMe99] for the fourth Janko group J_4 using its P-geometry $\mathcal{G}(J_4)$.

But, there is always a price to pay: we have to admit that some proofs in [IMe99] are quite complicated.

Another possibility for improving and refining the classification is to drop the flag-transitivity assumption. In Section 13.1 we report on the latest progress towards this.

13
Further developments

In this chapter we discuss two projects which lie beyond the classification of the flag-transitive P- and T-geometries. In Section 13.1 we report on the latest progress in the attempt to classify the P- and T-geometries when the flag-transitive assumption is dropped. In Section 13.2 we discuss Trofimov's theorem for locally projective graphs. Recall (cf. Chapter 9 in [Iv99]) that a 2-arc transitive action of G on Γ is locally projective if

$$L_n(q) \trianglelefteq G(x)^{\Gamma(x)} \leq P\Gamma L_n(q),$$

where $L_n(q)$ is considered as a doubly transitive permutation group on the set of 1-subspaces in the associated n-dimensional $GF(q)$-space. Trofimov's theorem shows in particular (cf. the following Table IX) that the exceptional cases of locally projective actions with $G_2(x) \neq 1$ are related to the actions of the automorphism groups of Petersen geometries on the corresponding derived graphs. We would like to classify all the amalgams $\mathscr{A} = \{G(x), G\{x,y\}\}$ of vertex- and edge stabilizers coming from locally projective actions. We believe that such a classification would demonstrate once again the very special rôle of P-geometries and their automorphism groups. Notice that the classification of the amalgams $\mathscr{A}$ as above is equivalent to the classification of the locally projective actions on trees.

13.1 Group-free characterizations

Our classification of the flag-transitive P- and T-geometry is essentially group-theoretical. So it is very far from being a purely geometrical theory. From this point of view, it is desirable to develop methods to study P-and T-geometries in a 'group-free' way. Ideally, the classification should be reproduced under purely geometrical assumptions. However, this goal seems to be too ambitious at present. The principal complications is that if the flag-transitivity assumption is dropped then the number of examples increases astronomically. To illustrate this point, let us consider the P-geometry $\mathcal{G}(3^{4371} \cdot BM)$. Factoring this geometry over the orbits of any subgroup of $O_3(3^{4371} \cdot BM)$, we again always get a P-geometry.

One possible solution to the above problem would be to classify only the 2-simply connected geometries. However, at present it is unclear how that condition of 2-simple connectedness can be utilized, and so new ideas are needed. Of course, even though a complete classification is beyond our reach, we can try and characterize the particular examples of P- and T geometries by some geometrical conditions.

The following result has been established in [HS00].

Proposition 13.1.1 *Suppose that $\mathcal{G}$ is a rank three P-geometry such that*

(i) *any two lines intersect in at most one point and*
(ii) *any three pairwise collinear points belong to a plane*

Then $\mathcal{G}$ is isomorphic either to $\mathcal{G}(M_{22})$ or to $\mathcal{G}(3 \cdot M_{22})$. □

If we drop the conditions (i) and (ii) in (13.1.1) then there is at least one further example: a 63-point geometry (discovered by D.V. Pasechnik and the second author) that is a quotient of $\mathcal{G}(3 \cdot M_{22})$ over the set of orbits of an element of order 11 from $3 \cdot M_{22}$ (which acts on $\mathcal{G}(3 \cdot M_{22})$ fixed-point freely).

In [CS01] the rank 4 case has been considered.

Proposition 13.1.2 *Suppose that $\mathcal{G}$ is a rank four P-geometry such that*

(i) *any two lines intersect in at most one point,*
(ii) *any three pairwise collinear points belong to a plane, and*
(iii) *the residue of every point is isomorphic to $\mathcal{G}(M_{22})$.*

Then $\mathcal{G}$ is isomorphic to $\mathcal{G}(Co_2)$. □

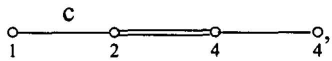
In the above theorem the condition (iii) eliminates the geometry $\mathcal{G}(3^{23} \cdot Co_2)$ and its numerous non-flag-transitive quotients and also the flag-transitive geometry $\mathcal{G}(J_4)$. The fourth (and last) example of flag-transitive P-geometry of rank four, namely $\mathcal{G}(M_{23})$, is eliminated by the condition (i).

In the final step of the proof of (13.1.2) the following result from [C94] has played a crucial role. Let Π denote the orbital graph of valency 891 (on 2 300 vertices) of the action of Co_2 on the cosets of $U_6(2).2$.

Proposition 13.1.3 *Let Σ be the collinearity graph of the dual polar space $\mathcal{D}_4(3)$ of $U_6(2)$. Let Δ be the distance 1–or–2 graph of Σ (i.e., Δ and Σ have the same set of vertices and two vertices are adjacent in Δ if and only if they are at distance 1 or 2 in Σ) then Π is the unique graph which is locally Δ.* □

The above proposition can be reformulated in geometrical terms as follows.

Proposition 13.1.4 *Let $\mathcal{E}$ be an extended dual polar space with the diagram*



such that

(i) *the residue of an element of type 1 is isomorphic to the dual polar space $\mathcal{D}_4(3)$ of $U_6(2)$;*
(ii) *two elements of type 1 are incident to at most one common element of type 2;*
(iii) *three elements of type 1 are pairwise incident to common elements of type 2 if and only if they are incident to a common element of type 4.*

Then $\mathcal{E}$ is isomorphic to the geometry $\mathcal{E}(Co_2)$ of the Conway group Co_2. □

We pose the following.

Conjecture 13.1.5 *Let $\mathcal{G}$ be a rank five P-geometry such that*

(i) *any two lines intersect in at most one point,*
(ii) *any three pairwise collinear points belong to a plane, and*
(iii) *the residue of every point is isomorphic to $\mathcal{G}(Co_2)$.*

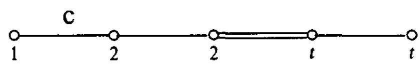
Then $\mathcal{G}$ is isomorphic to $\mathcal{G}(BM)$.

Recall that the *Baby Monster graph* is a graph Ω on the set $\{3,4\}$-transpositions in the Baby Monster group BM (the centralizer of such a transposition is $2\cdot{}^2E_6(2):2$), two vertices are adjacent if their product is a central involution in BM (with centralizer of the form $2^{1+22}_+.Co_2$). Locally Ω is the *commuting graph* of the central involutions (in other terms root involutions) in the group ${}^2E_6(2)$. (This means that two involutions are adjacent in the local graph if and only if they commute.) The suborbit diagram of Ω is given in Proposition 5.10.22 in [Iv99]. A crucial role in the simple connectedness proof for $\mathcal{G}(BM)$ was played by the fact that Ω is triangulable (cf. Proposition 5.11.5 in [Iv99]). In [IPS01] we have established the following group-free characterization of the Baby

Monster graph. We believe that this result can be used in a proof of Conjecture 13.1.5, in a similar way to the use of (13.1.3) in the proof of (13.1.2).

Proposition 13.1.6 *Let* Γ *be a graph which is locally the commuting graph of the central involutions in* ${}^2E_6(2)$. *Then* Γ *is isomorphic to the Baby Monster graph.* □

The maximal cliques in the Baby Monster graph Ω are of size 120. Let $\mathcal{E}(BM)$ be the geometry whose elements are the maximal cliques in Ω together with the non-empty intersections of two or more such cliques; the incidence is via inclusion. Then $\mathcal{E}(BM)$ is of rank 5, its elements of type 1, 2, 3, 4 and 5 are the complete subgraphs in Ω on 1, 2, 4, 8 and 120 vertices, respectively, and $\mathcal{E}(BM)$ belongs to the diagram.

$c.F_4(t)$: 

for $t = 4$, so that $\mathcal{E}(BM)$ is a c-extension of the F_4-building of the group ${}^2E_6(2)$. The geometry $\mathcal{E}(BM)$ was first mentioned in [B85]. In the geometrical terms (13.1.6) can be reformulated as follows.

Proposition 13.1.7 *Let* $\mathcal{E}$ *be a geometry with the diagram* $c.F_4(4)$, *such that*

(i) *any two elements of type* 1 *are incident to at most two elements of type* 2;
(ii) *three elements of type* 1 *are pairwise incident to common elements of type* 2 *if and only if they are incident to a common element of type* 5.

Then $\mathcal{E}$ *is isomorphic to* $\mathcal{E}(BM)$. □

The geometry $\mathcal{G}(BM)$ contains subgeometries $\mathcal{E}({}^2E_6(2))$ and $\mathcal{E}(Fi_{22})$ with diagrams $c.F_4(2)$ and $c.F_4(1)$. The stabilizers in BM of these subgeometries induce on them flag-transitive actions of ${}^2E_6(2) : 2$ and $Fi_{22} : 2$, respectively. Three further $c.F_4(2)$-geometries $\mathcal{E}(3{\cdot}{}^2E_6(2))$, $\mathcal{E}(E_6(2))$, $\mathcal{E}(2^{26} : F_4(2))$ and one $F_4(1)$-geometry $\mathcal{E}(3 \cdot Fi_{22})$ were constructed in [IPS01].

In [IW02] it was proved that every flag-transitive $c.F_4(1)$-geometry is isomorphic to either $\mathcal{E}(Fi_{22})$ or $\mathcal{E}(3 \cdot Fi_{22})$. The suborbit diagrams of the four known $c.F_4(2)$-geometries are calculated in [IP00]. The classification problem of the flag-transitive $c.F_4(2)$-geometries is currently under investigation by C. Wiedorn.

13.2 Locally projective graphs

In [Tr91a] V.I. Trofimov has announced that for a locally projective action of a group G on a graph Γ (which can always be taken to be a tree), the equality $G_6(x) = 1$ holds. The proof is given in the sequence of papers [Tr92], [Tr95a], [Tr95b], [Tr98], [Tr00], [Tr01], [TrXX] (the last one is still in preparation). The proof can be divided into the consideration of five cases (i) to (v); in addition the cases $p = 3$, $p = 2$, and $q = 2$ were considered separately. The case (v) for $q = 2$ seems to be the most complicated one (the papers [Tr00], [Tr01], [TrXX] deal solely with this situation). In some cases stronger bounds on the order of $G(x)$ were established, in fact it was claimed that $G_2(x) = 1$ except for the cases given in Table IX (in this table W_{n+1} denotes the direct product of two copies of $L_{n+1}(2)$ extended by a pair of commuting involutary automorphisms). In [Tr91b] some information on the structure of $G(x)$ in the case $G_2(x) = 1$ is given (although this information does not specify $G(x)$ up to isomorphism in all the cases).

Table IX

$(H/H_1)^\infty$	V_1	V_2	V_3	V_4	V_5	Examples
$L_2(2^n)$	2^{2n}	2^n				$\mathrm{Aut}\, S_4(2^n)$
$L_2(3^n)$	3^{2n}	3^{2n}	3^n			$\mathrm{Aut}\, G_2(3^n)$
$L_3(2^n)$	2^{6n}	2^{6n}	2^{3n}	2^{3n}	2^{2n}	$\mathrm{Aut}\, F_4(2^n)$
$L_3(3)$	3^3	3^3				$\mathrm{Aut}\, Fi_{22}$
$L_n(2)$	2^n	2				W_{n+1}
$L_3(2)$	2^3	2				$\mathrm{Aut}\, M_{22}$
$L_4(2)$	2^6	2^4	2			Co_2
$L_4(2)$	2^6	2^4	2^4			J_4
$L_5(2)$	2^{10}	2^{10}	2^5	2^5		BM

Thus Trofimov's theorem and its proof brings us very close to the description of all possible vertex stabilizers in locally projective action. Nevertheless (at least so far as the published results are concerned) a considerable amount of work is still to be done to get the complete list.

In fact, a final step in the classification of the locally projective actions would be the classification of all possible amalgams: $\mathscr{A} = \{G(x), G\{x,y\}\}$. Notice that the same $G(x)$ might appear in different amalgams. An example (not the smallest one) of such a case comes from the actions of $\Omega_{10}^{+}(2).2$ on the corresponding dual polar space graph and of J_4 on the derived graph of the corresponding locally truncated P-geometry. In both cases $G(x)$ is the semidirect product $Q : L$ where $L \cong L_5(2)$ and Q is the exterior square of the natural module of L.

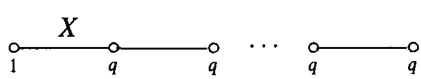
Thus it is very important to classify amalgams $\mathscr{A}$ of vertex and edge stabilizers that come from locally projective actions. This is of course equivalent to the classification of the locally projective actions on the trees. Let us mention some further motivation for this classification project.

In studying the locally projective actions, a very important role is played by so-called *geometrical subgraphs.* When the original graph Γ is a tree, a proper geometrical subgraph Σ is also a tree (of a smaller valency) and the setwise stabilizer $G\{\Sigma\}$ induces on Σ a locally projective action. Proceeding by induction, we can assume that the action of $G\{\Sigma\}$ on Σ is known, and in this case there is the possibility of simplifying the proof of Trofimov's theorem (of course, Trofimov also uses geometrical subgraphs, but only on the level of vertex stabilizers).

It is also useful to study the kernel K_Σ of the action of $G\{\Sigma\}$ on Σ. This is a finite normal subgroup in $G\{\Sigma\}$ and we can consider the natural homomorphism φ of $G\{\Sigma\}$ into the outer automorphism group of K_Σ. If O_Σ is the image of φ then the pair (O_Σ, K_Σ) is uniquely determined by the amalgam $\mathscr{A}$ and by the type (valency) of the geometrical subgraph Σ.

The pairs provide certain information on the possibilities of flag-transitive diagram geometries whose residues are projective spaces. We illustrate this statement in the case (v) (the collinearity case).

Let $\mathscr{G}$ be a geometry with the diagram



Then (ignoring some degenerated case) the collinearity graph Γ of $\mathscr{G}$ is locally projective with respect to the action of G and hence the amalgam

$\{G_1, G_2\}$ where G_1 is the stabilizer of a point and G_2 is the stabilizer of a line must be from the list. Furthermore we can deduce some restrictions on the leftmost edge of the diagram (the residue $\mathscr{H}$ of a flag of cotype $\{1,2\}$). Indeed, the residue $\mathscr{H}$ is the geometry of vertices and edges of the geometrical subgraph Σ of valency $q+1$. Let Σ_0 be the quotient of the corresponding tree (which is the universal cover of $\mathscr{H}$) over the orbits of $C_{G\{\Sigma\}}(K_\Sigma)K_\Sigma$. Then $\mathscr{H}$ is a covering of Σ_0.

As a continuation of the above example, we observe that when $G(x) \cong 2^{10} : L_5(2)$ the rank 2 residue $\mathscr{H}$ is either a covering of $K_{3,3}$ or a covering of the Petersen graph. We consider this as yet another justification of the importance of the classification of the flag-transitive Petersen geometries.

References

[A86] M. Aschbacher, *Finite Group Theory*, Cambridge Univ. Press, Cambridge, 1986.

[A94] M. Aschbacher, *Sporadic Groups*, Cambridge Univ. Press, Cambridge, 1994.

[A97] M. Aschbacher, *3-Transposition Groups*, Cambridge Univ. Press, Cambridge, 1997.

[ASeg91] M. Aschbacher and Y. Segev, The uniqueness of groups of type J_4, *Invent. Math.* **105** (1991), 589–607.

[ASeg92] M. Aschbacher and Y. Segev, Extending morphisms of groups and graphs, *Ann. Math.* **135** (1992), 297–324.

[Bel78] G. Bell, On the cohomology of finite special linear groups I and II., *J. Algebra* **54** (1978), 216–238 and 239–259.

[BI84] E. Bannai and T. Ito, *Algebraic Combinatorics I, Association Schemes*, Benjamin-Cummings Lect. Notes, Benjamin, Menlo Park, Calif., 1984.

[BI97] M.K. Bardoe and A.A. Ivanov, Natural representations of dual polar spaces, Unpublished report, 1997.

[BB00] A. Blokhuis and A.E. Brouwer, The universal embedding dimension of the binary symplectic polar spaces, Preprint 2000.

[BCN89] A.E. Brouwer, A.M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.

[B85] F. Buekenhout, Diagram geometries for sporadic groups. In: *Finite Groups – Coming of Age*, J. McKay ed., *Contemp. Math.* **45** (1985), 1–32.

[CCNPW] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[Coo78] B.N. Cooperstein, An enemies list of factorization theorems, *Comm. Algebra*, **6** (1978), 1239–1288.

[Coo97] B.N. Cooperstein, On the generation of some dual polar spaces of symplectic type over $GF(2)$, *Europ. J. Combin.* **18** (1997), 741–749.

[CS01] B. Cooperstein and S.V. Shpectorov, A group-free characterization of the P-geometry for Co_2, Preprint 2001.

[CS97] B. Cooperstein and E.E. Shult, Combinatorial construction of some near polygons, *J. Combin. Theory* (A) **78** (1997), 120–140.

[Cur70] C.W. Curtis, Modular representations of finite groups with split (B,N)-pairs, In: *Seminar on Algebraic Groups and Related Finite Groups*, Lecture Notes Math. **131**, pp. 57–95, 1970.

[C94] H. Cuypers, A graphical characterization of Co_2, Preprint 1994.

[DGS85] A. Delgado, D.M. Goldschmidt and B. Stellmacher, *Groups and Graphs: New Results and Methods*, Birkhäuser Verlag, Basel, 1985.

[Dem73] U. Dempwolff, On the second cohomology of $GL(n,2)$, *J. Austral. Math. Soc.* **16** (1973), 207–209.

[FLM88] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, Boston, 1988.

[FS98] A. Fukshansky and G. Stroth, Semiclassical parabolic systems related to M_{24}, *Geom. Dedic.* **70** (1998), 305–329.

[FW99] A. Fukshansky and C. Wiedorn, c-extensions of the Petersen geometry for M_{22}, *Europ. J. Combin.* **20** (1999), 233–238.

[GAP] GAP Group, *GAP: Groups, Algorithms and Programming*, Version 4.2, Aachen, St Andrews 1999.
(`http://www-gap.dcs.st-and.ac.uk/~gap`)

[Gol80] D. Goldschmidt, Automorphisms of trivalent graphs, *Annals Math.* **111** (1980), 377–406.

[Gri73] R.L. Griess, Automorphisms of extra special groups and nonvanishing degree 2 cohomology, *Pacific J. Math.* **48** (1973), 403–422.

[Gri74] R.L. Griess, Schur multipliers of some sporadic simple groups, *J. Algebra* **32** (1974), 445–466.

[Gri98] R.L. Griess, *Twelve Sporadic Groups*, Springer-Verlag, Berlin, 1998.

[HS00] J.I. Hall and S.V. Shpectorov, Rank 3 P-geometries, *Geom. Dedic.* **82** (2000), 139–169.

[H59] M.Jr. Hall, *The Theory of Groups*, Macmillan, New York, 1959.

[Hei91] St. Heiss, On a parabolic system of type M_{24}, *J. Algebra* **142** (1991), 188–200.

[Iv87] A.A. Ivanov, On 2-transitive graphs of girth 5, *Europ. J. Combin.* **8** (1987), 393–420.

[Iv92a] A.A. Ivanov, A geometric characterization of the Monster, in: *Groups, Combinatorics and Geometry*, Durham 1990, M. Liebeck and J. Saxl eds., London Math. Soc. Lect. Notes **165**, Cambridge Univ. Press, Cambridge, 1992, pp. 46–62.

[Iv92b] A.A. Ivanov, A presentation for J_4, *Proc. London Math. Soc.* **64** (1992), 369–396.

[Iv95] A.A. Ivanov, On geometries of the Fischer groups, *Europ. J. Combin.*, **16** (1995), 163–183.

[Iv97] A.A. Ivanov, Exceptional extended dual polar spaces, *Europ. J. Combin.* **18** (1997), 859–885.

[Iv98] A.A. Ivanov, Affine extended dual polar spaces, In: *Trends in Mathematics*, A. Pasini ed., Birkhäuser Verlag, Basel, 1998, pp. 107–121.

[Iv99] A.A. Ivanov, *Geometry of Sporadic Groups I. Petersen and Tilde Geometries*, Cambridge Univ. Press, Cambridge, 1999.

[Iv01] A.A. Ivanov, Non-abelian representations of geometries, In: *Groups and Combinatorics – in memory of Michio Suzuki*, ASPM 2001.

[ILLSS] A.A. Ivanov, S.A. Linton, K. Lux, J. Saxl and L.H. Soicher, Distance-transitive representations of the sporadic groups, *Comm. Algebra* **23(9)** (1995), 3379–3427.

[IMe99] A.A. Ivanov and U. Meierfrankenfeld, A computer free construction of J_4, *J. Algebra* **219** (1999), 113–172.

[IP00] A.A. Ivanov and D.V. Pasechnik, c-extensions of the $F_4(2)$-building, Preprint 00P/005, Imperial College, 2000.

[IPS96] A.A. Ivanov, D.V. Pasechnik and S.V. Shpectorov, Non-abelian representations of some sporadic geometries, *J. Algebra* **181** (1996), 523–557.

[IPS01] A.A. Ivanov, D.V. Pasechnik and S.V. Shpectorov, Extended F_4-buildings and the Baby Monster, *Invent. Math* **144** (2001), 399–433.

[ISh89] A.A. Ivanov and S.V. Shpectorov, Geometries for sporadic groups related to the Petersen graph. II, *Europ. J. Combin.* **10** (1989), 347–362.

[ISh90] A.A. Ivanov and S.V. Shpectorov, P-geometries of J_4-type have no natural representations, *Bull. Soc. Math. Belgique* (A) **42** (1990), 547–560.

[ISh93] A.A. Ivanov and S.V. Shpectorov, An infinite family of simply connected flag-transitive tilde geometries, *Geom. Dedic.* **45** (1993), 1–23.

[ISh94a] A.A. Ivanov and S.V. Shpectorov, Natural representations of the P-geometries of Co_2-type, *J. Algebra* **164** (1994), 718–749.

[ISh94b] A.A. Ivanov and S.V. Shpectorov, Flag-transitive tilde and Petersen type geometries are all known, *Bull. Amer. Math. Soc.* **31** (1994), 173–184.

[ISh94c] A.A. Ivanov and S.V. Shpectorov, Application of group amalgams to algebraic graph theory, in: *Investigations in Algebraic Theory of Combinatorial Objects*, Kluwer Acad. Publ., Dordrecht, NL, 1994, pp. 417–441.

[ISh97] A.A. Ivanov and S.V. Shpectorov, The universal non-abelian representation of the Petersen type geometry related to J_4, *J. Algebra*, **191** (1997), 541–567.

[IW02] A.A. Ivanov and C. Wiedorn, More on geometries of the Fischer group Fi_{22}, *J. Algebraic Combin.* (to appear).

[J76] Z. Janko, A new finite simple group of order 86,775,571,046,077,562,880 which possesses M_{24} and the full covering group of M_{22} as subgroups, *J. Algebra* **42** (1976), 564–596.

[JLPW] Ch. Jansen, K. Lux, R. Parker and R. Wilson, *An Atlas of Brauer Characters*, Clarendon Press, Oxford, 1995.

[JP76] W. Jones and B. Parshall, On the 1-cohomology of finite groups of Lie type, in: *Proc. of the Conference on Finite Groups*, W.R. Scott and R. Gross eds, Academic Press, New York, 1976, pp. 313–327.

[KL98] A. Kerber and R. Laue, Group actions, double cosets, and homomorphisms: unifying concepts for the constructive theory of discrete structures, *Acta Appl. Math.* **52** (1998), 63–90.

[Kur60] A.G. Kurosh, *The Theory of Groups*. II, Chelsea, New York 1960.

[Li00] P. Li, On the universal embedding of the $U_{2n}(2)$ dual polar space, Preprint 2000.

[Li01] P. Li, On the universal embedding of the $Sp_{2n}(2)$ dual polar space, *J. Combin. Theory* (A) **94** (2001), 100–117.

[MSm82] G. Mason and S.D. Smith, Minimal 2-local geometries for the Held and Rudvalis sporadic groups, *J. Algebra*, **79** (1982), 286–306.

[Mas67] W.S. Massey, *Algebraic Topology: An Introduction*, Harcourt Brace, New York, 1967.

[Maz79] P. Mazet, Sur le multiplicateur de Schur du groupe de Mathieu M_{22}, *C. R. Acad. Sci. Paris*, Serie A–B, **289** (1979), 659–661.

[McC00] P. McClurg, On the universal embedding of dual polar spaces of type $Sp_{2n}(2)$, *J. Combin. Th.* A **90** (2000), 104–122.

[MSh01] U. Meierfrankenfeld and S.V. Shpectorov, The maximal 2-local subgroups of the Monster, Preprint 2001.

[MSt90] U. Meierfrankenfeld and G. Stroth, On quadratic $GF(2)$-modules for sporadic groups and alternating groups, *Comm. Algabra.* **18** (1990), 2099–2139.

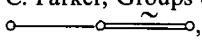
[MSt01] U. Meierfrankenfeld and G. Stroth, F-module for finite simple groups, Preprint, 2001.

[Mei91] T. Meixner, Some polar towers, *Europ. J. Combin.* **12** (1991), 397–417.

[Nor80] S.P. Norton, The construction of J_4, in: *Proc. Symp. Pure Math.* No. 37, B. Cooperstein and G. Mason eds., AMS, Providence, R.I., 1980, pp. 271–278.

[Nor85] S.P. Norton, The uniqueness of the Fischer–Griess Monster, in: *Finite Groups – Coming of Age*, Proc. 1982 Montreal Conf., J. McKay ed., Contemp. Math. **45**, AMS, Providence, R.I., 1985, pp. 271–285.

[Nor98] S.P. Norton, Anatomy of the Monster: I, in: *The Atlas of Finite Groups: Ten Years on*, R. Curtis and R. Wilson eds., London Math. Soc. Lect. Notes **249**, Cambringe Univ. Press, Cambridge 1998, pp. 198–214.

[Par92] C. Parker, Groups containing a subdiagram , *Proc. London. Math. Soc.* **65** (1992), 85–120.

[Pasi85] A. Pasini, Some remarks on covers and apartments, in: *Finite Geometries*, C.A. Baker and L.M. Batten eds., Marcel Dekker, New York, 1985, pp. 233–250.

[Pasi94] A. Pasini, *Diagram Geometries*, Clarendon Press, Oxford, 1994.

[Pol71] H. Pollatsek, First cohomology groups of some linear groups over fields of characteristic two, *Ill. J. Math.* **15** (1971), 393–417.

[Rich99] P.J. Richardson, *Sporadic Geometries and their Universal Representation Groups*, PhD Thesis, Imperial College London, 1999.

[Ron80] M.A. Ronan, Coverings and automorphisms of chamber systems, *Europ. J. Combin.* **1** (1980), 259–269.

[Ron81a] M.A. Ronan, Coverings of certain finite geometries, in: *Finite Geometries and Designs*, Cambridge Univ. Press, Cambridge, 1981, pp. 316–331.

[Ron81b] M.A. Ronan, On the second homotopy group of certain simplicial complexes and some combinatorial applications, *Quart. J. Math. (2)* **32** (1981) 225–233.

[Ron82] M.A. Ronan, Locally truncated buildings and M_{24}, *Math. Z.* **180** (1982), 489–501.

[Ron87] M.A. Ronan, Embeddings and hyperplanes of discrete geometries, *Europ. J. Combin.* **8** (1987), 179–185.

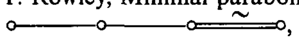
[RSm80] M.A. Ronan and S. Smith, 2-Local geometries for some sporadic groups, in: *Proc. Symp. Pure Math.* No. 37, B. Cooperstein and G. Mason eds., AMS, Providence, R.I., 1980, pp. 283–289.

[RSm86] M.A. Ronan and S.D. Smith, Universal presheaves on group geometries, and modular representations, *J. Algebra* **102** (1986), 135–154.

[RSm89] M.A. Ronan and S.D. Smith, Computation of 2-modular sheaves and representations for $L_4(2)$, A_7, $3S_6$ and M_{24}, *Comm. Algebra* **17** (1989), 1199–1237.

[RSt84] M.A. Ronan and G. Stroth, Minimal parabolic geometries for the sporadic groups, *Europ. J. Combin.* **5** (1984), 59–91.

[Row89] P. Rowley, On the minimal parabolic system related to M_{24}, *J. London Math. Soc.* **40** (1989), 40–56.

[Row91] P. Rowley, Minimal parabolic systems with diagram , *J. Algebra* **141** (1991), 204–251.

[Row92] P. Rowley, Pushing down minimal parabolic systems, in: *Groups, Combinatorics and Geometry*, Durham 1990, M. Liebeck and J. Saxl eds., London Math. Soc. Lect. Notes **165**, Cambridge Univ. Press, Cambridge, 1992, pp. 144–150.

[Row94] P. Rowley, Sporadic group geometries and the action of involutions, *J. Austral. Math. Soc.* (A) **57** (1994), 35–48.

[RowXX] P. Rowley, On the minimal parabolic system related to the Monster simple groups, *J. Combin. Theory* (A) (to appear).

[RW94a] P. Rowley and L. Walker, A characterization of the Co_2-minimal parabolic geometry, *Nova J. Algebra and Geom.* **3** (1994), 97–155.

[RW94b] P. Rowley and L. Walker, The maximal 2-local geometry for J_4, I, Preprint, UMIST, No *11*, *1994*.

[RW97] P. Rowley and L. Walker, A 11,707,448,673,375 vertex graph related to the Baby Monster, I and II, Preprint UMIST, No 4 and 14, 1997.

[Seg88] Y. Segev, On the uniqueness of the Co_1 2-local geometry, *Geom. Dedic.* **25** (1988), 159–212.

[Sei73] G. Seitz, Flag-transitive subgroups of Chevalley groups, *Ann. Math.* **97** (1973), 27–56.

[Ser77] J.-P. Serre, *Arbres, amalgams*, SL_2, Astérisque **46**, Soc. Math. de France, Paris, 1977.

[Sh85] S.V. Shpectorov, A geometric characterization of the group M_{22}, in: *Investigations in Algebraic Theory of Combinatorial Objects*, VNIISI, Moscow, 1985, pp. 112–123 [Russian, English translation by Kluwer Acad. Publ., Dordrecht, NL, 1994]

[Sh88] S.V. Shpectorov, On geometries with diagram P^n, preprint, 1988. [Russian]

[Sh92] S.V. Shpectorov, The universal 2-cover of the P-geometry $\mathcal{G}(Co_2)$, *Europ. J. Combin.* **13** (1992), 291–312.

[Sh95] S.V. Shpectorov, Natural representations of some tilde and Petersen type geometries, *Geom. Dedic.* **54** (1995), 87–102.

[ShSt94] S.V. Shpectorov and G. Stroth, Classification of certain types of tilde geometries, *Geom. Dedic.* **49** (1994), 155–172.

[Sm94] S.D. Smith, Universality of the 24-dimensional embedding of the .1 2-local geometry, *Comm. Algebra*, **22(13)** (1994), 5159–5166.

[Sp66] E.H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[Str84] G. Stroth, Parabolics in finite groups. in: *Proc. Rutgers Group Theory Year, 1983–1984*, Cambridge Univ. Press, Cambridge 1984, pp. 211–224.

[StW01] G. Stroth and C. Wiedorn, c-extensions of P- and T-geometries, *J. Combin. Theory* (A), **93** (2001), 261–280.

[Su86] M. Suzuki, *Group Theory II*. Springer-Verlag, 1986.

[Tay92] D.E. Taylor, *The Geometry of the Classical Groups*, Heldermann Verlag, Berlin, 1992.

[Th79] J.G. Thompson, Uniqueness of the Fischer–Griess Monster, *Bull. London Math. Soc.* **11** (1979), 340–346.

[Th81] J.G. Thompson, Finite-dimensional representations of free products with an amalgamated subgroup, *J. Algebra* **69** (1981), 146–149.

[Tim89] F.G. Timmesfeld, Classical locally finite Tits chamber systems of rank 3, *J. Algebra*, **124** (1989), 9–59.

[Ti74] J. Tits, *Buildings of Spherical Type and Finite BN-pairs*, Lect. Notes Math. **386**, Springer-Verlag, Berlin 1974.

[Ti82] J. Tits, A local approach to buildings, in: *The Geometric Vein (Coxeter-Festschrift)*, Springer-Verlag, Berlin, 1982, pp. 519–547.

[Ti86] J. Tits, Ensembles ordonnés, immeubles et sommes amalgameés, *Bull. Soc. Math. Belg.* **A38** (1986) 367–387.

[Tr91a] V.I. Trofimov, Stabilizers of the vertices of graphs with projective suborbits, *Soviet Math. Dokl.* **42** (1991), 825–828.

[Tr91b] V.I. Trofimov, More on vertex stabilizers of the symmetric graphs with projective subconstituents, In.: *Int. Conf. Algebraic Combin.*, Vladimir, USSR, 1991, pp. 36–37.

[Tr92] V.I. Trofimov, Graphs with projective suborbits, *Russian Acad. Sci. Izv. Math.* **39** (1992), 869–894.

[Tr95a] V.I. Trofimov, Graphs with projective suborbits. Cases of small characteristics. I, *Russian Acad. Sci. Izv. Math.* **45** (1995), 353–398.

[Tr95b] V.I. Trofimov, Graphs with projective suborbits. Cases of small characteristics. II, *Russian Acad. Sci. Izv. Math.* **45** (1995), 559–576.

[Tr98] V.I. Trofimov, Graphs with projective suborbits. Exceptional cases of characteristic 2. I, *Izvestiya RAN: Ser. Mat.* **62** (1998), 159–222.

[Tr00] V.I. Trofimov, Graphs with projective suborbits. Exceptional cases of characteristic 2. II, *Izvestiya RAN: Ser. Mat.* **64** (2000), 173–192 [Russian]

[Tr01] V.I. Trofimov, Graphs with projective suborbits. Exceptional cases of characteristic 2. III, *Izvestiya RAN: Ser. Mat.* **65** (2001) [Russian]

[TrXX] V.I. Trofimov, Graphs with projective suborbits. Exceptional cases of characteristics 2. IV, In preparation.

[Wil87] R.A. Wilson, Some subgroups of the baby monster, *Invent. Math.* **89** (1987), 197–218.

[Wil89] R.A. Wilson, Vector stabilizers and subgroups of Leech lattice groups, *J. Algebra* **127** (1989), 387–408.

[Yos92] S. Yoshiara, Embeddings of flag-transitive classical locally polar geometries of rank 3, *Geom. Dedic.* **43** (1992), 121–165.

[Yos94] S. Yoshiara, On some extended dual polar spaces I, *Europ. J. Combin.* **15** (1994), 73–86.

Index