METHODS OF BOSONIC AND FERMIONIC PATH INTEGRALS REPRESENTATIONS

Continuum Random Geometry in Quantum Field Theory

Luiz C. L. Botelho

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METHODS OF BOSONIC AND FERMIONIC PATH INTEGRALS REPRESENTATIONS: CONTINUUM RANDOM GEOMETRY IN QUANTUM FIELD THEORY

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For all those who have been struggling, resisting and finally prevailing in “Gulags” (even “Academic” ones.)

To Nelma, Rafael and Gabriel – always at my side.

Contents

About This Monograph (Foreword I)

“Therefore, conclusions based on the renormalization group arguments concerning the theory summed to all orders are dangerous and must be viewed with due caution. So is it with all conclusions from local relativistic field theories.”

J.D. Bjorken, S.O. Drell (1965)

“Because of improved divergence, Yang-Mills theories (without Higgs fields) can not be consistently interpreted by conventional perturbation theory.”

J.C. Taylor (1976)

“There are methods and formulae in science, which serve as master-key to many apparently different problems. The resource of such thing have to be refilled from time to time. In my opinion at the present time, we have to develop an art of handling sums over random surfaces.”

A.M. Polyakov (1981)

“QCD = String Theory´´ , A.M. Migdal (1981)

“Modern Physics = Quantum Geometry”, Luiz C.L. Botelho (2006)

About This Monograph (Foreword II)

When still a graduate student in 1980, I became acquainted with a set of CERN lectures on functional integrals written by V.N. Popov. Since that time I have been working steadly on the use of functional integrals methods in order to handle non-pertubative issues in Quantum Field Theory – specially about the problem of correct quantization of Yang-Mills Chromodynamics and Einstein Quantum Gravity (in terms of Astekar variables) through quantum geometric path integrals (Loop and Random Surfaces representations).

The general scheme to apply ideas of Quantum Geometry may be sketchy as follows.

1- By firstly, one should try to represent the formal path integrals of the theory under quantization, originally defined in terms of wave-field configurations, by means of purely geometrical objects (quantum loops and surfaces) and writing thus the relevant governing field motion equations in terms of these quantum geometrical variables.

2 - As a second step, one should try to solve the quantum geometric wave equations through string path integrals and finally one must use the whole formalism of continuum quantum geometric path integrals to make calculations of physical observable (loop space path integrals and string scattering amplitudes).

This monograph is writen on topics in the subject of Continuum Quantum Geometric Path Integrals applied to Yang-Mills theory and variants (QCD, Chern-Simons Theory, Ising Models, etc.) – the called Random Geometry in Quantum Field theory, which are hoped to be useful to graduate students of quantum physics and applied mathematics, with a focused weight towards to those interested in applying the concepts of continuum quantum geometry in other branches of modern physics, like superconductivity, nuclear physics, polymer theory, string theory, etc...

As a monograph, I have choose to present those topics which I subjectively in the path integral framework consider that are basic to give a sound understanding of quantum geometric path integrals representations. As a consequence of this choice our exposition is entirely based in our studies made in the subject in last 26 years (1980-2006).

The methodology used to write our monograph is the same exposed in our previous work in random classical physics: "Methods of Bosonic Path Integrals Representations – Random Systems in Classical Physics - Nova Science Publisher, (2006) U.S.A.": Expositions and formulas should be chewed, swallowed and digested. This process of analysis should not be abandoned until it yields a comprehension of the overall pattern of the proposed ideas and math, so after this step, one is ready to make improvements, corrections or criticisms on the path integrals representations of our book. Important material is frequently exposed in forum of appendixes to the main exposition with the unique objective of not divert our readers from the central discussions in his/her first lecture and to serve as "exercises" to our readers.

Another point I wish stress to our readers is that I have chosen to not give extensive references on this monograph, because I still consider the attitude to distribute scientific intellectual credits in an ongoing notoriously difficult subject like Quantum Geometry, a subjective, incomplete, sometimes "political oriented" and not less, a "dangerous" attitude: I am far away to claim to have some competent background to be a Science Historian. This monograph should be considered as another attempt to discuss theories and protocols which have never been completely understood and we wrote it with the sincere hope in mind that, although imperfect, it will stimulate deeper reflections in the subject of continuum quantum geometry by others – specially graduate students.

Cumbersome use of English and the certainly types and spelling mistakes existent in our monograph (reporting mostly our original results) naturally reflects the author's limitations and rather short time taken to write this book. The reader's criticism will be welcome.

Luiz Carlos Lobato Botelho
Full Professor - Universidade Federal Fluminense
Niterói/Rio de Janeiro/Brazil – 2006

Post Scriptum. As a basic set of text-books and lecture notes references on Path-integral methods for our monograph we point out the followings:

[1] A.M. Polyakov, *Gauge Field and Strings*, Harwood Academic Chur, Switzerland, (1987).

[2] A.M. Migdal, *Phys. Rep* (1983)102, 199.

[3] V.N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics*, Reidel Publishing Company (1983).

[4] A.A. Abrikosov, L.P. Gorkov, I.E. Dzyaloshinski, textitMethods of Quantum Field Theory in Statistical Physics, Dover Publications, Inc, New York, (1975).

[5] B. Durhuus, *Quantum Theory of Strings*, Nordita Lectures, (1982).

[6] I.M. Gelfand, N.Ya.Vilenkin, *Generalized Functions* (1964) vol. 4, N.Y. Academic Press,.

– Post scriptum II – Special thanks to Mr. Wilson Goes and Mr. Rogerio Trindade for the arduos work of typesetting my hand-written munuscripts.

Chapter 1

Loop Space Path Integrals Representations for Euclidean Quantum Fields Path Integrals and the Covariant Path Integral

1.1. Introduction

In this introductory chapter we present from an operational point of view the basic methodology of re-writing Euclidean quantum field path integrals in term of Loop Space path integrals, the important Feynman's idea of describing quantum phenomena by means of geometrical objects (Feynman trajectories made up of: paths; surfaces, metrics, etc...).

In section 2, we present the above Bosonic Loop space reformulation in the simplest example of a $O(N)$-scalar field theory with a $O(N)$-invariant quartic interaction.

In the section 3, we present similar Loop Space reformulation for Quantum Chromodynamics and finally in section 4, we present in details the theory of covariant path integration, the basic mathematical method to study the objects in the theory of Random Surfaces as exposed in the next chapters of this monograph.

Some Mathematical oriented studies on Euclidean Path Integrals are presented in chapter 19 "Domains of Bosonic Functional Integrals and Some Applications to the Mathematical Physics of Path Integrals and String Theory" and chapter 20 "Non-Linear diffusion in R^ν and in Hilbert Space, a Path Integral study".

1.2. The Bosonic Loop Space Formulation of the $O(N)$-Scalar Field Theory

Let us start our expositon in this section by considering the following $O(N)$-invariant path-integral in an Euclidean Space-time R^ν, the called Generating Functional of the Green function of the composite $O(N)$-invariant operator $(\sum_{h=1}^{N}(\phi^k\phi^k)(x))$.

$$Z[J(x)] = \frac{1}{Z(0)}\left\{\int\left(\prod_{n=1}^{N} D^F[\phi^k(x)]\right)\times\int D^F[\beta(x)]\right.$$
$$\exp\left(-\frac{1}{2}\left[\int d^\nu x(\phi^k(-\Delta+m^2+ig\beta+J)\phi^k)(x)\right]\right)$$
$$\left.\exp\left(-\frac{1}{2}\left[\int d^\nu x\beta^2(x)\right]\right)\right\} \tag{1.1}$$

Note that after the evaluation of the *Gaussian* $\beta(x)$-path integral, we obtain our quartic $O(N)$-invariant interaction term

$$\tilde{Z} = \int D^F[\beta(x)]\exp\left(-\frac{1}{2}\left[\int d^\nu x\beta^2(x)\right]\right)\exp\left(-\frac{1}{2}\left[ig\int d^\nu x\left(\sum_{h=1}^{N}\phi^k\phi^k\right)(x)\beta(x)\right]\right)$$
$$= \exp\left\{-\frac{g^2}{2}\int d^\nu x\left(\sum_{h=1}^{N}(\phi^k\phi^k)(x)\right)^2\right\} \tag{1.2}$$

In order to apply the Loop Space reformulation to the path-integral eq.(1.1), we realize the *Guaussian* functional integration related to the set of scalar (neutral) fields $\{\varphi^k(x)\}_{k=1,\ldots,N}$. Namely

$$\int\left(\prod_{k=1}^{N} D^F[\phi^k(x)]\exp\left(-\frac{1}{2}\left[\int d^\nu x(\phi^k(-\Delta+m^2+ig\beta+J)\phi^k)\right]\right)\right)$$
$$= \det{}^{-\frac{N}{2}}(-\Delta+m^2+ig\beta+J) = e^{-NW[J]} \tag{1.3}$$

which can be re-written as a trajectory path-integral for the effective action $W[J]$.

$$W[J] = \frac{+1}{2} lg\det(-\Delta+m^2+ig\beta+J)$$
$$= -\frac{1}{2}\lim_{\varepsilon\to 0^+}\left\{\int_\varepsilon^\infty \frac{dt}{t}\mathrm{Trace}_F(e^{-t[-\Delta+m^2+ig\beta+J]})\right\}$$
$$= -\frac{1}{2}\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{dt}{t}e^{-tm^2}\left[\int d^\nu x^\mu\int_{X^\mu(0)=X^\mu(t)=x^\mu}\left(\prod_{\mu=1}^{\nu} D^F[X^\mu(\sigma)]\right)\right.$$
$$\times\exp\left(-\frac{1}{2}\int_0^t d\sigma(\dot{X}^\mu(\sigma))^2\right)\times\exp\left(-\int_0^t d\sigma J(X^\alpha(\sigma))\right)$$
$$\times\exp\left(+ig\int_0^t d\sigma\beta(X^\mu(\sigma))\right) \tag{1.4}$$

where $\mathrm{Trace}_F \equiv Tr_F$ means the complete functional trace applied to operator in question.

Note that all the Feynman Wienner trajectories entering in the Bosonic Loop Space expression eq.(1.4) are very rough geometrical objects in R^ν, since they are non-differentiable paths possesing alone mathematical continuity. As a consequence one can not assign without a subtle analysis lenghts, topological properties, etc., to them as it is usually done for smooth geometrical objects in the field of the non-random geometry.

At this point we may consider the complete object in the Loop Space:

$$\begin{aligned}\tilde{Z}[J(x)] &= \int D^F[\beta(x)]e^{-\frac{1}{2}\int d^\nu x \beta^2(x)} e^{-NW[J]} \\ &= \int D^F[\beta(x)]e^{-\frac{1}{2}\int d^\nu x \beta^2(x)}(1 - NW[J] + \frac{N^2(W(J))^2}{2} + \dots) \\ &= 1 - \frac{N}{2}\lim_{\varepsilon\to 0^+}\left\{\int_\varepsilon^\infty \frac{dt}{t}e^{-tm^2}\left[\int d^\nu x^\alpha \int_{X^\mu(0)=X^\mu(t)=x}\left(\prod_{\mu=1}^{\nu} D^F[X^\mu(\sigma)]\right)\right.\right. \\ &\quad \times \exp\left(-\frac{1}{2}\int_0^t d\sigma(\dot{X}^\mu(\sigma))^2\right) \\ &\quad \times \exp\left(-\int_0^t d\sigma J(X^\alpha(\sigma)\right) \\ &\quad \times \exp\left(-\frac{g^2}{2}\int_0^t d\sigma\int_0^t d\sigma'\delta^{(\nu)}(X^\beta(\sigma)-X^\beta(\sigma'))\right) + O(N^2) \end{aligned} \tag{1.5}$$

We have, thus, reformulated all field dynamics in terms of random bosonic paths with a pure self-avoiding geometrical interaction with strenght g^2 as one can see from the last term in eq.(1.5) by considering a formal power series expansion on the N factor corresponding to the group order $O(N)$.

It is worth to see that the two-point Green function associate to the composite operators $\left(\sum_{h=1}^{N}(\phi^k\phi^k)(x)\right)$ is given entierely by a random loop geometrical intercept point object

$$\begin{aligned} &\frac{\delta^2}{\delta J(\overline{x})\delta J(\overline{y})}\tilde{Z}[J(x)]\Bigg|_{J\equiv 0} \\ &= -\frac{N}{2}\lim_{\varepsilon\to 0^+}\left\{\int_\varepsilon^\infty \frac{dt}{t}e^{-tm^2}\left[\int d^\nu x^\alpha\int_{X^\mu(0)=X^\mu(t)=x^\mu}\prod_{\mu=1}^{\nu}D^F[X^\mu(\sigma)]\exp\left(-\frac{1}{2}\int_0^t d\sigma(\dot{X}^\mu(\sigma))^2\right)\right.\right. \\ &\quad \times\exp\left(-\int_0^t d\sigma J(X^\alpha(\sigma))\right)\exp\left(-\frac{g^2}{2}\int_0^t d\sigma\int_0^t d\sigma'\delta^{(\nu)}(X^\beta(\sigma)-X^\beta(\sigma'))\right) \\ &\quad \times\left[\int_0^t d\sigma\int_0^t d\sigma'\delta^{(\nu)}(X^\alpha(\sigma)-\overline{x}^\alpha)\delta^{(\nu)}(X^\alpha(\sigma')-\overline{y}^\alpha)\right] + O(N^2). \end{aligned} \tag{1.6}$$

One can follow ref.[3], to see that the usual Feynman Diagramatic perturbative expansion can be easily obtained from the above written Bosonic Loop space path-integrals.

An important point to be called the reader attention for, is that the above Bosonic loop quantum field reformulation allows us traightforwardly to consider the field configurations to live in a compact space-time. For instance, the whole effect of considering our $O(N)$-invariant scalar fields living on spherical field configurations surface

$$\sum_{k=1}^{N}(\phi^k\phi^k)(x) = r \tag{1.7}$$

is to introduce a formal further path-integration on eq.(1.6) $\int D^F[\lambda(x)]e^{-iR\int d^\nu x\lambda(x)}\times$ same integrand of eq.(1.6) added with the factor $\exp\left[+i\int_0^t d\sigma\lambda(X^\beta(\sigma))\right]$ as a result of writing the (formal) classical constraint eq.(1.7) into the path-integral schene

$$\sum_{h=1}^{N}(\phi^k\phi^k)(x)=R\Leftrightarrow\lim_{n\to\infty}\left\{\left(\frac{1}{\sqrt{2\pi}}\right)^n\int_{-\infty}^{+\infty}d\lambda(x_1)\dots d\lambda(x_n)\right.$$
$$\left. e^{i\sum_{\ell=1}^{N}\left[\lambda(x_\ell)\left(\sum_{k=1}^{N}(\varphi^k\varphi^k)(x_\ell)-R\right)\right]}\right\}$$
$$\equiv\int D^F[\lambda(x)]\exp\left\{i\int d^\nu x\left[\lambda\left(\sum_{h=1}^{N}\varphi^k\varphi^k-R\right)\right](x)\right\} \tag{1.8}$$

At this point of our exposition, we refer our readers to the ref.[3], where it is attempted a rigorous mathematical analysis of the above written self-avoiding bosonic loop space theory eq.(1.6)-eq.(1.8).

Another important basic point to be called the reader attention for is that in the presence of charged $SU(N)$ scalar fields interacting with Yang-Mills fields, it appears as other object in the path-integral of the bosonic loop space representation eqs.(5)-(6), the famous Wilson Loop Phase Factor defined by the Yang-Mills field $A_\mu(x)=\sum_{i=1}^{N^2-1}\left(A_\mu^i(x)\lambda_i\right)$ in the $SU(N)$ fundamental representation ([1]-[3]).

$$W\left[A_\mu\,;\,X^\beta(\sigma)\right]=\frac{1}{N}Tr_{SU(N)}\left\{\mathbb{P}\left[\exp\, ig\left(\int_0^t d\sigma A_\mu(X^\beta(\sigma)\dot{X}^\beta(\sigma)\right)\right]\right\} \tag{1.9}$$

In the case of the presence of (formally) quantized Yang-Mills fields, one further considers the average on the Yang-Mills fields, with the (ill-defined) Yang-Mills Path-Integral (see appendix A)

$$\int D^F[A_\mu(x)]e^{-\frac{1}{4}\int d^\nu x(Tr_{SU(N)}[F_{\mu\nu}^2(A)])(x)} \tag{1.10}$$

where the Yang-Mills strenght is given by

$$F_{\mu\nu}(A)=\partial_\mu A_\nu-\partial_\nu A_\mu+ig[A_\mu,A_\nu] \tag{1.11}$$

Finally, we end this section to point out that all the Quantum Field analysis presented still remains in present days, a somewhat formal (high complex!) Mathematical Methods approach, even in the framework of the subject of Quantum Field Path Integrals. The aim of the further chapters of our monograph is to present studies in the problem of given a precise operational formulation of this Random Loop Space formalism in the context of Quantum Field Theory. Some rigorous mathematical analysis is however presented in the last chapters 19–20, of this monograph.

1.3. A Fermionic Loop Space for QCD

One of the most interesting problems in particle physics is that of understanding QCD in terms of colour single fields [1,2].

Our aim in this section 3 is to propose a generalisation of the usual bosonic loop space formulation for gauge theory in the case of fermionic interacting matter [3] (quantum chromodynamics).

Let us start our analysis by considering the QCD Euclidean partition functional with the fermionic quark degrees integrated out:

$$Z^{QCD} = \int DA_\mu \exp(-S[A_\mu]) \mathrm{Det}[\not{D}(A_\mu)] \tag{1.12}$$

where $S[A_\mu]$ denotes the Yang-Mills action and $\not{D}(A_\mu) = \gamma_\mu(i\partial_\mu + A_\mu)$ is the Euclidean Dirac operator in the presence of the external Yang-Mills field $A_\mu(X)$.

By using the proper-time definition for the above-mentioned functional determinant, we consider the formal relationship for its modulus [4]

$$\begin{aligned} &\log \mathrm{Det}|[\not{D}(A_\nu)]| \\ &= |\log[\mathrm{Det}(\frac{1}{2}[D(A_\mu)D^*(A_\mu) + D^*(A_\mu)D(A_\mu)])]^{1/2}| \\ &= -\int_0^\infty \frac{d\tau}{\tau} Tr[\exp(-\mathbb{H}(A_\mu)\tau)] \end{aligned} \tag{1.13}$$

where we have introduced the (Euclidean)Hamiltonian

$$\mathbb{H}(A_\mu) = [\frac{1}{2}(D(A_\mu)D^*(A_\mu) + D^*(A_\mu)D(A_\mu))]^{1/2}. \tag{1.14}$$

In the loop approach to QCD the next step is to write the propagator $Tr[\exp(-\mathbb{H}(A_\mu)\tau)]$ as a kind of "continuous" sum of closed trajectories [2]. At this point we introduce our suggestion: since the Hamiltonian $\mathbb{H}(A_\mu)$ corresponds to a particle possessing Lorentz and colour spin (interacting with the external Yang-Mills fields) the closed trajectories entering into the Feynman path-integral expression for $Tr[\exp(-\mathbb{H}(A_\mu)\tau)]$ should reveals in an explicit way their fermionic particle dynamical degrees of freedom.

A natural framework for analysing this case is pseudoclassical mechanics where the worldline of a spinning coloured particle is described by the usual vector position $X^\mu(\xi)$ added to a set of Grassmann complex variables $\{\theta_l(\xi), \theta_l^*(\xi)\}$ associated with the particle colour charges [5] and another set of real $\psi_\mu(\xi)$ Grassmann variables corresponding to the Lorentz spin [6].

In the simplest Abelian case, the above path integral expression was proposed by Rumpf [7] and given explicitly by the following expression:

$$\begin{aligned} &\mathrm{Tr}[\exp(-\mathbb{H}(A_\mu))] \\ &= \int d^D X \int_{X_\mu(0)=0=X_\mu(\tau)=X} D(X_\mu(\xi)) \underset{\mathrm{Dirac}}{\mathrm{Tr}} \\ &\quad \times \int_{\psi\mu(0)=\psi_\mu(\tau)} D(\psi_\mu(\xi)\exp[-L(X_\mu)(\xi),\ \Psi_\mu(\xi), A_\mu(\xi))] \end{aligned} \tag{1.15}$$

where the pseudoclassical Lagrangian $L[X_\mu(\xi), \psi_\mu(\xi), A_\mu(\xi)]$ is given by [6]:

$$L(X_\mu(\xi), \psi_\mu(\xi), A_\mu(\xi)) = \frac{1}{4}(\dot{X}_\mu)^2 + \frac{1}{4} i\dot{\psi}_\mu\psi_\nu + A_\mu(X)\dot{X}_\mu + \frac{1}{4} i[\psi_\mu, \psi_\nu]F_{\mu\nu}(X(\xi)). \tag{1.16}$$

In the non-Abelian case we propose to consider the analogues of (1.16) and (1.17), the coloured version of our previous pure coloured path integral

$$
\begin{aligned}
&Tr[\exp(-\mathbb{H}(A_\mu)\tau)] \\
&= \int dX^D \int_{X_\mu(0)=X_\mu(\tau)=X} D[X_\mu(\xi)] \underset{\text{Dirac}}{\mathrm{Tr}} \\
&\quad \times \int_{\psi\mu(0)=\psi_\mu(\xi)} D(\psi_\mu(\xi)) \int D(\theta(\xi)) D(\theta^*(\xi))\theta_i(0)\theta_i^*(\tau) \\
&\quad \times \exp[-L(X_\mu(\xi),\ \psi_\mu(\xi), \theta(\xi), \theta^*(\xi),\ A_\mu(X))]
\end{aligned} \tag{1.17}
$$

where our proposed pseudoclassical gauge-invariant Lagrangian for a spinning coloured particle is given by

$$
\begin{aligned}
&L(X_\mu(\xi), \theta(\xi), \theta^*(\xi), \psi_\mu(\xi), A_\mu(X)) \\
&\quad = \frac{1}{2}(\dot{X}_\mu)^2 + \frac{1}{4} i \dot{\psi}_\mu \psi_\nu + \frac{1}{2} i \left(\sum_{i=1}^{N} (\theta_i^* \dot{\theta}_i - \dot{\theta}_i^* \theta_i)(\xi) \right) - g(\theta_l^* (\lambda_i)_{lk} \theta_k)(\xi) \\
&\qquad \times A_\mu^i(X(\xi)) \dot{X}^\mu(\xi) + \frac{1}{4} i [\psi_\mu,\ \psi_\nu](\xi)(\theta_l^* (\lambda_i)_{lk} \theta_k)(\xi) F_{\mu\nu}^i (X(\xi)).
\end{aligned} \tag{1.18}
$$

By exactly integrating out the colour Grassmannian variables in (18), we can see the natural appearance of the fermionic Wilson loop factor considered in chapters 7–8 for quantum chromodynamics and quantum gravity

$$
W[X_\mu^{(F)}(S,\theta)] = \underset{\text{colour}}{\mathrm{Tr}}\, P \exp \left(\int_0^1 dS \int d\theta A_\mu (X_\mu^F(S,\theta)) D X_\mu^{(F)}(S,\theta) \right) \tag{1.19}
$$

where we have used a super-loop notation to write the Yang-Mills interacting term in (1.19) in a compact form (see chapters 7–8 for the super-loop notation).

1.4. Invariant Path Integration and the Covariant Functional Measure for Einstein Gravitation Theory

1.4.1. Introduction

The path integral for gravitational interactions has been discussed several times in the past ([9]–[12]) and the important problem of the gravitational path-integral measure has been reexamined.

In this section we intend to propose an approach for the quantization of Einstein's gravitational theory in the framework of path integrals suitable to the analysis of the above-mentioned problem of the path-covariant local measure.

The basic idea in our discussion [9], [13] is the introduction of a Riemann structure into the functional manifold of the metric field variables compatible with the invariance group of the theory and consider the associated partition functional as an infinite-dimensional version of an invariant integral in a Riemann manifold [13]. As a result we will not need to introduce the *add hoc* insertion of the Faddeev-Popov unity resolution into the path-integral measure in order to extract the gauge orbit volume [14], since we will be able to implement this calculation in a purely geometric way. So, in the proposed framework, it is not necessary to use a *posteriori* a constraint Halmiltonian path integral [15] to justify the Faddeev-Popov procedure; besides our approach leads to a natural and adequate local covariant pah measure.

1.4.2. Invariant Integration

We start our analysis by briefly reviewing the basic results of the theory of invariant integrals in Riemann manifolds.[13]

Let T be a homomorphism of a compact Lie group G in the isometry group of a given Riemann manifold M. Let us consider the integral

$$\int_M f(x)[d\mu](x), \tag{1.20}$$

where $f(x)$ is invariant under the action of $G[f(T(g)x) = f(x),\ \forall g \in G]$ and $[d\mu]$ is the measure in M induced by its Riemann metric. The orbit of a point $x \in M$ [the submanifold of M formed by all the points $\{T(g)x\}$, $g \in G$] will be denoted by $0(x)$. The orbit quotient space M/G can be realized as a submanifold of M which are not related by a group element. The measure induced by the M-Riemann metric in M/G is denoted by $[d\overline{\mu}]$ and that induced in $0(x)$ by $[d\nu]$. Now we can state the basic result of the theory [13]. We have the following relationship between the integral (1.1) and an integral defined only over the orbit quotient space M/G:

$$\int_M f(x)[d\mu](x) = \int_{M/G} f(x)[d\overline{\mu}](x)\nu(x) \tag{1.21}$$

with

$$\nu(X) = \int_{0(x)} [d\nu](X). \tag{1.22}$$

We remark that $[d\nu](x)$ is a G-invariant measure over the group G, since $O(x)$ can be realized as a "copy" manifold of G.

This result is fundamental for our analysis.

Another result of differential geometry which we will use is the coordinate expression for the induced metric in a given submanifold of M. Let $\{g_{hj}(x)\}$ denote the matrix of the metric tensor in M with $1 \leq h, j \leq N$ (N being the dimension of M). Here, x belongs to an M coordinate domain. Let H be a submanifold of M described by the parametric equations

$$X_j = R_j(z_l) \tag{1.23}$$

with $\{z_l\}$ $(1 \leq z_l \leq k; k \leq N)$ belonging to a domain D (coordinate domain for H). Assuming that the matrix $[A]_{jk}(z_l) = \partial R_j / \partial z_k(z_l)$ has maximal characteristic k in D the metric $\{g_{hj}(x)\}$ induces the following metric in H:

$$g^{(\mathrm{ind})}_{pq}(Z_k) = (g_{hj} A^{hp} A^{jq})(Z_k) \tag{1.24}$$

with the volume element given by

$$[dv](z_k) = [\det g^{(\mathrm{ind})}_{pq}(z_R)]^{1/2} dz^1 \ldots dz^k. \tag{1.25}$$

After having displayed the basic results of invariant integration we pass to the problem of the path-integral quantization for the Einstein theory.

1.4.3. A Quantum Path Measure for Einstein Theory

Let us start our analysis writing the Einstein-Hilbert action for the theory of gravitation defined in a d-dimensional Minkowinski space-time manifold E with fixed topology and without boundary (see Ref. [16] for the case of an open space-time):

$$S[\{g_{\alpha\beta}(x)\}] = \frac{1}{16\pi G}\int_E (\sqrt{-g}R)(x)d^Dx, \tag{1.26}$$

where the field variables are given by those metric tensors $\{g_{\alpha\beta}(x)\}$ that can be defined in E, i.e., compatible with its topological structure, $-g(x) = \det\{g_{\mu\nu}(x)\}$, $R(x)$ being the scalar of curvature induced by $g_{\mu\nu}$ in M and G the Newton graviational constant.

The starting point of the Feynman path-integral quantization for the Einstein theory is the formal continous sum over $\{g_{\mu\nu}(x)\}$ histories:

$$Z = \sum_{(\{g_{\mu\nu}(x)\}} \exp\left[\frac{i}{\hbar}S[g_{\mu\nu}(x)\}]\right]. \tag{1.27}$$

The precise meaning for the continous sum eq.(1.27) is achieved by introducting a path measure in the functional space of all possible field configurations (denoted by M); $[d\mu][g_{\alpha\beta}(x)]$, such that (1.27) can be written as

$$Z = \int_M [d\mu][g_{\alpha\beta}(x)]\exp\left[\frac{i}{\hbar}S[g_{\alpha\beta}(x)]\right]. \tag{1.28}$$

The fundamental problem in Eq.(1.28) is to define appropriately the path measure since the Einstein action possesses of the physical invariance under the action of the group of the coordinate transformations in M (the Einstein general-relativity principle) denoted by $G^{\text{diff}}(E)$:

$$x^\mu \to l^\mu(x^\alpha), \tag{1.29}$$

$$\begin{aligned} g_{\mu\nu}(x) \to & \frac{\partial l^\mu(x^\alpha)}{\partial x_\sigma} g_{\sigma\rho}(l^\mu(x^\alpha))\frac{\partial l^\nu(x^\alpha)}{\partial x_\rho} \\ & \equiv (Lg_{\sigma\rho})_{\mu\nu}(x^\alpha) \end{aligned} \tag{1.30}$$

and which in its infinitesimal version $G^{\text{diff}}(E)$ is given by

$$\delta x^\mu = \varepsilon^\mu(x^\alpha), \tag{1.31}$$

$$\delta g_{\mu\nu}(x^\alpha) = (\nabla_\mu\varepsilon_\nu + \nabla_\nu\varepsilon_\mu)(x^\alpha), \tag{1.32}$$

where ∇_α is the usual covariant derivative defined by the metric $\{g_{\alpha\beta}(x)\}$.

This invariance property leads us to treat the above path integral as an infinite-dimensional version $G^{\text{diff}}(E)$ -invariant integral in M [see Eq.(1.21)].

So, we intend to use the fundamental relation Eqs.(1.22) and (1.23) in its functional version in order to get its expression in the physical path manifold $M/G^{\text{diff}}(E)$. As a first step to implement the invariant integration theory we have to introduce a metric structure in $\mathbb{M}$ compatible with the group $G^{\text{diff}}(E)$. By folllwing DeWitt's analysis [9] we introduce a metric (functional) tensor $\gamma^{\mu\nu;\alpha\beta}[g_{\sigma\rho}](x,x')$ on the functional path space $\mathbb{M}$ for which the actions of $G^{\text{diff}}(E)$ are isometries.

The unique (ultralocal) functional metric satisfying the above condition is given by the following expression [9]-(17) (the well-known "DeWitt functional metric"):

$$ds^2 = \int_E d^D x \sqrt{-g(x)} \int_E d^D x' \sqrt{-g(x')} \delta g_{\mu\nu}(x) \times \gamma^{\mu\nu;\alpha\beta}[g_{\sigma\rho}](x,x') \delta g_{\alpha\beta}(x'), \tag{1.33}$$

where the ultralocal tensor density $\gamma^{(\mu\nu;\alpha\beta)}[g_{\sigma\rho}](x,x')$ is explicitly given by $(c \neq -2/D)$

$$\gamma^{\mu\nu;\alpha\beta}[g_{\sigma\rho}](x,x') = \frac{1}{\sqrt{2}} \frac{\delta^{(D)}(x-x')}{\sqrt{-g(x')}} \times (g^{\mu\alpha} g^{\nu\beta} + c g^{\mu\nu} g^{\alpha\beta})(x) \tag{1.34}$$

and $(\delta g_{\mu\sigma}(x)$ denotes the functional infinitesimal displacements on $\mathbb{M}$.

After introducing a Riemann structure on the path functional manifold $\mathbb{M}$ we can use the basic relationship, Eqs. (1.21) and (1.22), to give a precise meaning for the path integral:

$$Z = \int_M [d\mu][g_{\alpha\beta}](x) \exp\left[\frac{i}{\hbar} S[\{g_{\alpha\beta}(x)\}]\right]. \tag{1.35}$$

As a first step, we have to realize the abstract orbit quotiente space $M/G^{\text{diff}}(E)$ in $\mathbb{M}$. For this task we consider a set of D functionals $f^\mu(g_{\sigma\rho})(x))$ defined in $\mathbb{M}$ and in such a way that equations in $G^{\text{diff}}(E)$,

$$f^\mu(L g_{\alpha\beta}(x)) = 0,\ \mu = 1, \ldots, D\,, \tag{1.36}$$

have only the identity solution for a given $\{g_{\alpha\beta}(x)\}$; i.e., we have fixed our gauge. In order to simplify the discussion below we restrict our analysis to the class of the linear functionals $f^\mu(g_{\sigma\rho})(x))$ satisfying the following condition:
$\delta f^\mu(g_{\alpha\beta}(x))/\delta g_{\mu\nu}(x')$ is a functional independent
of the field variables

$$\{g_{\sigma\xi}(x)\}. \tag{1.37}$$

For instance, the well-known harmonic guage $\partial^\alpha g_{\mu\alpha}(x) = f^\mu(g_{\alpha\beta}(x))$ belongs to the above-cited class. Thus, we can realize the orbit quatient space $M/G^{\text{diff}}(E)$ in M as the path inequivalente manifold solution of Eq.(1.36) in M:

$$\overline{g}_{\alpha\beta}(x) \in M/G^{\text{diff}} \Leftrightarrow f^\mu(\overline{g}_{\alpha\beta}(x)) = 0. \tag{1.38}$$

With this implicit M/G^{diff} parametrization the induced path measure is, thus, given by the well-known DeWitt result [see Ref. 9, Eq.(14.52)]

$$[d\overline{\mu}][\overline{g}_{\alpha\beta}(x)] = \prod_{(x\in E)} [d g_{\alpha\beta}(x)] \det\{\gamma^{\mu\nu;\alpha\beta)}(x,x')\} \times \delta_F(f^\mu(g_{\sigma\rho}(x))), \tag{1.39}$$

where

$$\det\{\gamma^{\mu\nu;\alpha\beta)}(x,x')\} = (-1)^{D-1}\left[1 + \frac{cD}{2}\right] \times (\sqrt{-g})^{(D-4)(D+1)/4} \tag{1.40}$$

and the functional delta $\delta_F(f^\mu(g_{\sigma\rho}(x)))$ in the functional measure (1.39) restricts its support to the manifold of inequivalent metrics [Eq.(1.38)].

Now we have to evaluate the orbit (functional) volume defined by a given inequivalent configuration $\{\overline{g}_{\alpha\beta}(x) \in M/G^{\text{diff}}(E)\}$. For this purpose we need an explicit parametrization of the orbit submanifold $O(\overline{g}_{\alpha\beta}(x))$. Such an expression is given explicitly by the path integral:

$$Y_{\mu\nu}[L;\overline{g}_{\alpha\beta}] = \int_M \left[\prod_{x\in E} dg_{\rho\sigma}(x)\right] g_{\mu\nu}(x) \times \delta_F(f^\mu(g_{\rho\sigma}(x) - f^\mu((L\cdot\overline{g})_{\rho\sigma}(x))). \tag{1.41}$$

We remark that the $\{g_{\rho\sigma}(x)\}$ functional integration in Eq.(1.41) is defined over the whole functional manifold $\mathbb{M}$ and the $G^{\text{diff}}(E)$ is the parameter domain for the orbit manifold $O(\overline{g}_{\alpha\beta}(x))$.

The functional integration over $\mathbb{M}$ gives straightforwardly the result

$$Y_{\mu\nu}[L;\overline{g}_{\alpha\beta}(x)] = (L\overline{g})_{\mu\nu}(x) \times \left[\prod_{\mu=1}^{D} \det{}_F \left[\frac{\delta f^\mu(g_{\alpha\beta})}{\delta g_{\rho\sigma}}\right](x)\right]^{-1} \tag{1.42}$$

and since the functional determinants involved in Eq.(1.42) are $g_{\alpha\beta}(x)$ independent by the condition eq.(1.37) we find that $Y_{\mu\nu}[L;\overline{g}_{\alpha\beta}(x)]$ is an explicit parametrization of the orbit $O(\overline{g}_{\alpha\beta}(x))$; i.e., the image of $\mathbb{G}$ under $Y_{\mu\nu}[L;\overline{g}_{\alpha\beta}(x)]$ coincides with the orbit associated with the inequivalente metric $\{\overline{g}_{\alpha\beta}(x)\}$.

In order to evaluate the induced metric in $O(\overline{g}_{\alpha\beta}(x))$ by the DeWitt metric Eq.(33) we use the functional version of Eq.(1.25) with Eq.(1.42) playing the role of Eq.(1.23). So, the differential line element in $O(\overline{g}_{\alpha\beta}(x))$ is given by

$$\begin{aligned} ds^2_{\text{ind}} = \int d^Dx d^Dx' & \left(\frac{\delta}{\delta\varepsilon_\rho(x)} Y_{\mu\nu}[\varepsilon^\gamma,\overline{g}_{\alpha\beta}]\right)\delta\varepsilon_\rho(x) \\ & \times \sqrt{-\overline{g}(x)}\gamma^{(\mu\nu;\alpha\beta)}(\overline{g}_{\alpha\beta})(x,x')\sqrt{-\overline{g}(x')} \\ & \times \left[\frac{\delta}{\delta\varepsilon_\sigma(x')} Y_{\alpha\beta}[\varepsilon^\gamma,\overline{g}_{\alpha\beta}]\right]\delta\varepsilon_\sigma(x'), \end{aligned} \tag{1.43}$$

where we have considered the group transformation $\mathbb{L} \in G^{\text{diff}}(E)$ being infinitesimal and characterized by the infinitesimal generators $\{\varepsilon^\gamma(x)\}$ [see Eqs.(1.31) and (1.32)].

Evaluating the functional derivatives in Eq.(1.43),

$$\begin{aligned} \frac{\delta}{\delta\varepsilon_\rho} Y_{\mu\nu}[\varepsilon^\gamma,\overline{g}_{\alpha\beta}] &= \int_M \left[\prod_{\substack{x\in E \\ (\beta,\sigma)}} dg_{\beta\sigma}(x)\right] g_{\mu\nu}(x) \frac{\delta}{\delta\varepsilon_\rho(x)}[\delta_F(f^\mu(g_{\gamma\xi}) - f^\mu((L\cdot\overline{g})_{\gamma\xi}))] \\ &= \sum_{(\alpha',\beta')} \left[\int_M \left[\prod_{\substack{x\in E \\ (\beta,\sigma)}} dg_{\beta\sigma}(x)\right] g_{\mu\nu}(x) \left[-\frac{\delta}{\delta g_{\alpha'\beta'}(x)}[\delta_F(f^\mu(g_{\gamma\xi}) - f^\mu((L\cdot\overline{g})_{\gamma\xi}))\right] \right. \\ & \left. \times \prod_{\mu=1}^{D} \det{}_F \left[\frac{\delta f^\mu((L\cdot\overline{g})_{\gamma\xi})}{\delta\varepsilon_\rho(x)}\right]\right] \end{aligned} \tag{1.44}$$

and using the functional version of the usual relation

$$\int_{-\infty}^{+\infty} g(x)\frac{d}{dx}\delta(f(x)) = -\sum_{\{x_0\}\in\mathbb{S}} \frac{g'(x)}{f'(x)}\Big|_{x=x_0} \tag{1.45}$$

[where $\mathbb{S}$ denotes the set of zeros of $f(x)$] to evaluate the above functional integral; we get the (formal) result

$$\begin{aligned}\frac{\delta}{\delta\varepsilon_\rho}Y_{\mu\nu}[\varepsilon^\gamma;\bar{g}_{\alpha\beta}] &= \sum_{(\alpha'\beta')}\left\{\frac{\delta}{\delta g_{\alpha'\beta'}}\left[g_{\mu\nu}\prod_{\sigma=1}^{D}\det{}_F\left[\frac{\delta f^\sigma((L\cdot\bar{g})_{\gamma\xi})}{\delta\varepsilon_\rho}\right]\right]\right\}\\ &= \sum_{\alpha'\beta'}\delta_{\mu\alpha'}\delta_{\nu\beta'}\left[\prod_{\sigma=1}^{D}\det{}_F\left[\frac{\delta f^\sigma((L\cdot\bar{g})_{\gamma\xi})}{\delta\varepsilon_\rho}\right]\right],\end{aligned} \tag{1.46}$$

where we have used that $\delta f^\mu(g_{\alpha\beta})/\delta g_{\rho\sigma}(x)$ is a functional independent of the metric $\{g_{\gamma\xi}(x)\}$ and $\delta/\delta g_{\alpha\beta} f^\mu(L\cdot\bar{g}) \equiv 0$ since $\{\bar{g}_{\alpha\beta}(x)\}$ is a fixed metric.

By substituting Eq. (1.46) into Eq. (1.43) we thus obtain

$$\begin{aligned} ds^2_{\text{ind}} &= \int d^Dx d^Dx' \sqrt{-\bar{g}(x)}\det\left[\frac{\delta f^\mu((L\cdot\bar{g}))}{\delta\varepsilon_\rho(x)}\right][\delta\varepsilon_\rho(x)]\\ &\times Tr[\bar{\gamma}^{(\mu\nu;\alpha\beta)}(\bar{g})\sqrt{-\bar{g}(x')}\delta^{(D)}(x-x')\det\left[\frac{\delta f^\mu((L\cdot\bar{g})}{\delta\varepsilon_{\rho'}(x')}\right][\delta\varepsilon_{\rho'}(x')],\end{aligned} \tag{1.47}$$

where

$$Tr[\bar{\gamma}^{(\mu\nu;\alpha\beta)}(\bar{g}] = \sum_{(\sigma_1,\sigma_2,\sigma_3,\sigma_4)}\{[\delta^{\sigma_1}_\mu\delta^{\sigma_2}_\nu(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}+c\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta})\delta^{\sigma_3}_\alpha\delta^{\sigma_4}_\beta]\} \tag{1.48}$$

is the trace of the DeWitt metric defined by the fixed metric $\bar{g}_{\alpha\beta}(x)$.

The functional measure induced by Eq.(1.47) in $O(\bar{g}_{\mu\nu}(x))$ is then given by [see Eqs. (1.22)-(1.25)]

$$[dv][\bar{g}_{\alpha\beta}(x)] = \int\prod_{x\in E}(\sqrt{-\bar{g}}d\varepsilon^\rho)(x)\{Tr[\bar{\gamma}^{(\mu\nu;\alpha\beta)}(\bar{g})]\}^{1/2}\det[\delta f^\mu((L\cdot\bar{g})_{\alpha\beta})/\delta\varepsilon_\rho]. \tag{1.49}$$

Since we are considering the infinitesimal group transformations in Eq. (1.49) we can use the Taylor expansion for the functional $\delta f^\mu(L\cdot\bar{g})/\delta\varepsilon_\rho$ i.e.,

$$\frac{\delta f^\mu(L\cdot\bar{g})_{\alpha\beta})}{\delta\varepsilon_\rho} = \frac{\delta f^\mu(L\cdot\bar{g})_{\alpha\beta})}{\delta\varepsilon_\rho}\Big|_{\varepsilon_\rho\equiv 0}(x)+O(|\varepsilon|^2(x)) \tag{1.50}$$

and, as consequence of Eq. (1.50), we get the result where the invariant group volume is covariantly factorized from the path integral:

$$[dv][\bar{g}_{\alpha\beta}(x)] = \{Tr[\bar{\gamma}^{(\mu\nu;\alpha\beta)}(\bar{g})]\}^{1/2}\det\left[\frac{\delta f^\mu(L\cdot\bar{g})}{\delta\varepsilon_\rho(x)}\right]\left[\int_{G^{\text{diff}}}\prod_{x\in E}\sqrt{-\bar{g}(x)}(d\varepsilon^\rho)(x)\right]. \tag{1.51}$$

Finally by grouping together the obtained resuts Eqs.(1.39) and (1.51) [see Eqs.(1.22) and (1.36)] we obtain our proposed path measure for Einstein gravitation theory:

$$[d\mu](g_{\alpha\beta}) = \prod_{x\in E}[dg_{\alpha\beta}\det{}_{\gamma}^{(\mu\nu;\alpha\beta)}\delta_F(f^{\mu}(g))Tr(\gamma^{(\mu\nu;\alpha\beta)})^{1/2}](x)\det\left(\frac{\delta f^{\mu}(L\cdot\overline{g})}{\delta\varepsilon^{\rho}}\bigg|_{\varepsilon_{\rho}\equiv 0}\right). \tag{1.52}$$

At this point of our study it is instructive to point out that the above written measure differs from the original DeWitt measure by the factor $Tr(\gamma^{(\mu\nu;\alpha\beta)})$ [see Eq.(1.51)] which in our framework takes into account the contribution from the geometric intersection between the orbit submanifold $O(\overline{g}_{\alpha\beta}(x))$ [see Eq.(1.43) with the quotient space $M/G^{\text{diff}}(E)$ in M [see Eqs.(1.44)-(1.49)]. However, we can see that this factor is irrevelant in the physical space-time $D=4$, since the functional measure

$$\prod_{x\in E}(dg_{\alpha\beta}\det\gamma^{(\mu\nu;\alpha\beta)}(x))$$

becomes “flat” [see Eq. (1.40)]. So, we can now safely use the dimensional regularization scheme to vanish the “tadpole” contribution $Tr(\gamma^{(\mu\nu;\alpha\beta)})$. This result in turn coincides with that proposed in Ref. [17] by DeWitt in $D=4$.

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Appendix A.
A Grassmanian Loop Space Approach for Fermionic Bell Functional Integral

Analysis of quantum many-body systems by means of the so-called Bell functional integral [1] has proved to be an useful technique to understand phenomena such as superconductivity, superfluidity, etc., all features expected to be present in the systems' non-perturbative regime. [2]

Our aim in this appendix is to propose a generalization of the usual bosonic Bell functional integral for the case of existence of explicitly (two-body) spin interaction potential by using the Grassmanian Loop space formalism as proposed in chapter 1 and Ref. 3.

Let us start our analysis by considering the canonical partition functional associated with a system of N spin $\frac{1}{2}$ particles in a volume Ω and temperature $T=\frac{1}{k\beta}$

$$Z(N,\beta,\Omega)=\frac{1}{N!}\sum_{P}(\mathrm{sgn}P)\times\mathrm{Tr}(\exp(-\beta H)P). \tag{A1}$$

where P denotes the permutation operator in the Hilbert space of N spin particles and the N-body Hamiltonian is given explicitly by

$$H=\sum_{j=1}^{N}\left(-\frac{\hbar^2}{2M}\Delta_j+W(r_j)\right)+\frac{1}{2}\sum_{i<j}^{N}(V_{(0)}(r_i-r_j)+\mathbf{S}_i\cdot\mathbf{S}_jV_{(1)}(r_i-r_j)) \tag{A2}$$

We denote by $(V_{(0)},V_{(1)})$ the system's two-body (spin dependent) interaction. $W(r)$ is an external scalar field and $\mathbf{S}_i=(S_{ix},S_{iy},S_{iz})$ represents the spin operator associated with the spin degrees of freedom of the i-particle. We can thus write Eq.(A1) as a continuous sum of R^3 Grassmanian trajectories $(X_a(\sigma),\Psi^j_\alpha(\sigma))$ $(1\le j\le N)$ $(a=1,2,3)$, where, the j-particle trajectory is described by the usual R^3 vector position $X_a(\sigma)$ added to a set of R^3 Grassmanian real variables $\Psi_a(\sigma)$ corresponding to the spin variables:

$$\begin{aligned}Z(N,\beta,\Omega)=\frac{1}{N!}\prod_{j=1}^{N}\bigg[\int d^3r_j\int_{X_j(0)=r_i}^{X_j(\beta)=r_j}D^F(X_j(\sigma))\\ \times\int_{\Psi_j(0)=\Psi_j(\beta)}D^F(\Psi_j(\sigma))\exp\left(-\int_0^\beta d\sigma F(X_j(\sigma),\psi_j(\sigma))\right)\bigg].\end{aligned} \tag{A3}$$

The path integral weight in Eq.(A3) is given by the pseudo-classical Hamiltonian associated to the quantum Hamiltonian [3, 4] Eq.(A2)

$$F(X_j(\sigma),\psi_i(\sigma)) = +\frac{1}{2M}\sum_{j=1}^{N}\left\{\left(\frac{dX_j(\sigma)}{d\sigma}\right)^2 + \psi_j(\sigma)\left(i\frac{\partial}{\partial\sigma}\right)\psi_j(\sigma) + W(X_j(\sigma))\right\}$$
$$+\frac{1}{2}\sum_{i<j}^{N}\int_0^{\beta} d\sigma'\left(V_{(0)}(X_i(\sigma)-X_j(\sigma')\right)$$
$$+\psi_i(\sigma)\cdot\psi_j(\sigma')V_{(1)}\left(X_j(\sigma)-X_j(\sigma')\right). \tag{A4}$$

We can reduce the above lengthy expression by replacing the two-body non-local interactions by an independent local interaction of each particle with Gaussian fluctuating external fields followed by an average process over these stochastic felds.[5] We thus introduce a set of Gaussian random scalar and vector fields $(\Phi^{(0)}(r),\Phi^{(1)}(r))$ with two-point correlation functions as given by

$$\Phi^{(1)}(\mathbf{r}) = \left(\Phi_a^{(1)}(\mathbf{r})\right)_{a=1,2,3}$$
$$\left\langle \Phi^{(0)}(r)\Phi^{(0)}(r')\right\rangle = +V_{(0)}(r-r')$$
$$\left\langle \Phi_a^{(1)}(r)\Phi_b^{(1)}(r')\right\rangle = +V_{(1)}(r-r')\delta_{ab}\ . \tag{A5}$$

We, thus, rewrite the canonical partition functional in the following suitable form,

$$Z(N,\beta,\Omega) = \frac{1}{N!}\prod_{j=1}^{N}\left[\int d^3r_j\int_{X_j(0)=r_i}^{X_j(\beta)=r_j} D^F(X_j(\sigma))\right.$$
$$\left.\times\int_{\psi_j(0)=\Psi_j(\beta)} D^F(\psi_j(\sigma))\left\langle \exp\left(-\int_0^{\beta} d\sigma \hat{F}[X_j,\psi_j,W,\Phi^{(0)},\Phi^{(1)}](\sigma)\right)\right\rangle\right]. \tag{A6}$$

where the new path integral weight is now given by

$$\hat{F}[X_j,\psi_j,W,\Phi^{(0)},\Phi^{(1)},(\sigma)] = \sum_{j=1}^{N}\frac{1}{2M}\left(\frac{dX_j(\sigma)}{d\sigma}\right)^2 + \psi_j(\sigma)\left(i\frac{\partial}{\partial\sigma}\right)\psi_j(\sigma) + W(X_j(\sigma))$$
$$+\,i\left(\Phi^{(0)}(X_j(\sigma)\right) + \Psi_j^{(a)}(\sigma)\cdot\Phi_j^{(a)}(X_j(\sigma)) \tag{A7}$$

Analysis of thermodynamical properties of this quantum many-body system may be implemented by considering the associated grand canonical partition functional by introducing the system activity variable z,

$$Z(z,\beta,\Omega) = \sum_{N=0}^{\infty} Z(N,\beta,\Omega)z^N\ . \tag{A8}$$

In order to write (A8) as a functional integral over fields (the so-called Bell functional integral[6] we introduce the following Schrödinger-Pauli operator acting on complex spin$-\frac{1}{2}$ fields $(\psi^+(r,\sigma),\psi^-(r,\sigma))$.

$$\hat{S}[W,\Phi^{(0)},\blacksquare^{(1)}] = \frac{\partial}{\partial\sigma} - \frac{1}{2m}\Delta_r - W(r) - (\Phi^{(0)}(r) + \mathbf{S}\cdot\blacksquare^{(1)}(r)) + \mu\ .$$

The complex spin$-\frac{1}{2}$ fields defining the domain of the operator $\hat{S}[W,\Phi^{(0)},\blacksquare^{(1)}]$ are chosen as eigenfunctions of the z-component spin operator S_3 and, besides, these fields should satisfy a periodicity condition on the fictious time (temperature) σ, i.e.: $\Psi^{\pm}(r,\sigma+\beta) = \Psi^{\pm}(r,\sigma)$.[6]

Let us, thus, consider the functional determinant of the above-defined Schrödinger-Pauli operator $(\hbar = 1)$

$$\log\det(\hat{S}[W,\Phi^{(0)},\blacksquare^{(1)}]) = \sum_{m=-\infty}^{+\infty} e^{2\pi i m}\log\det\left[-\frac{2\pi i m}{\beta} - \frac{1}{2M}\Delta_r - W(r) - i(\blacksquare^{(0)}(r) + \mathbf{S}\cdot\blacksquare^{(1)}(r)) + \mu\right]. \quad \text{(A9)}$$

By using the proper-time definition for the above-written functional determinant,we can consider the following loop space representation for it.[3]

$$\log\det\left[\frac{1}{2M}\Delta_r - W(r) - i(\Phi^{(0)}(r) - \mathbf{S}\cdot\blacksquare^{(1)}(r)) + \left(\mu - \frac{2\pi i m}{\beta}\right)\right]$$
$$= -\sum_{m=-\infty}^{+\infty}\int_0^{\beta} e^{-\mu T}\exp\left(-2\pi i m\left(\frac{T-\beta}{\beta}\right)\right)$$
$$\int_{\Omega} d^3r \int_{X(0)=r}^{X(\beta)=r} D^F(X(\sigma))\int_{\Psi(0)=\Psi(\beta)} D^F(\psi(\sigma))$$
$$\exp\left(-\frac{1}{2M}\int_0^T\left(\dot{X}^2(\sigma) + \psi(\sigma) i\frac{\partial}{\partial\sigma}\psi(\sigma)\right)\right)$$
$$\exp\left(-\int_0^T d\sigma\left(W(x(\sigma)) + i\blacksquare^{(0)}(X(\sigma)) + i\blacksquare^{(1)}(X(\sigma))\cdot\psi(\sigma)\right)\right). \quad \text{(A10)}$$

By summing the series $\sum_{m=-\infty}^{+\infty} e^{i2\pi m(T/\beta-1)} = (\beta\delta(T-\beta))$ the integral over the proper-time T in Eq.(A10) is replaced by the Boltzman factor β. By identifying now the activity z with the parameter μ through the relationship $z = (e^{-\mu\beta})\beta$, we can see that the determinant of the Schrödinger-Pauli operator in Eq.(A9) averaged over the Gaussian fields $(\Phi^{(0)}(r),\blacksquare^{(1)}(r))$ coincides exactly with the grand canonical partition functional Eq.(A8). Explicitly we have obatined the following representation for the system's grand canonical partition functional Eq.(A8),

$$Z(z,\beta,\Omega) = \left\langle \det\left[\frac{\partial}{\partial\sigma} + \frac{1}{2M}\Delta_r - W(r) - i\Phi^{(0)}(r) - i\blacksquare^{(1)}(r)\cdot\mathbf{S} - \frac{1}{\beta}lg(\frac{z}{\beta})\right]\right\rangle \quad \text{(A11)}$$

Let us write the functional determinant of the Schrödinger-Pauli operator in Eq.(A11) as a Gaussian functional integral over spin$-\frac{1}{2}$ doublet complex field $\Psi(r,\sigma) = (\Psi^{+}(r,\sigma),\Psi^{-}(r,\sigma))$,

$$Z(z,\beta,\Omega) = \left\langle \int D^F[\Psi(r,\sigma)]D^F[\Psi^*(r,\sigma)]\right.$$
$$\times\exp\left\{-\int_{\Omega} d^3r\int_0^{\beta} d\sigma\Psi^*(r,\sigma)\left(\frac{\partial}{\partial\sigma} - \frac{1}{2M}\Delta_r - W(r) - i\blacksquare^{(0)}(r)\right.\right.$$
$$\left.\left.\left. -i\mathbf{S}\cdot\blacksquare^{(1)}(r) + \frac{1}{\beta}lg(\frac{z}{\beta})\right)\Psi(r,\sigma\right\}\right\rangle \quad \text{(A12)}$$

By evaluating straightforwardly the Guassian averages associated with the fields $(\Phi^{(0)}, \blacksquare^{(1)})$ we finally obtain our proposed Bell functional integral representation for the grand canonical partition functional associated with the Fermi many-body sustem described by the Hamiltonian Eq.(A2):

$$Z(z,\beta,\Omega) = \int D^F[\Psi(r,\sigma)] D^F[\Psi^*(r,\sigma)] \exp(-S[\Psi,\Psi^*]) \ . \tag{A13}$$

Here the functional integral weight $S[\Psi,\Psi^*]$ is given by the following action functional

$$\begin{aligned} S[\Psi,\Psi^*] = &\int_S d^3r \int_0^\beta d\sigma \Psi^*(r,\sigma)\left(\frac{\partial}{\partial\sigma} - \frac{1}{2} M\Delta_r - W(r) + \frac{1}{\beta} lg(\frac{z}{\beta})\right)\Psi(r,\sigma) \\ &+ \int_\Omega d^3r d^3r' \int_0^\beta d\sigma (|\Psi(r,\sigma)|^2 V_{(0)}(r-r')|\Psi(r',\sigma)|^2 \\ &+ \Psi^*(r',\sigma)\mathbf{S}\blacksquare((r',\sigma))V_{(1)}(r-r')\psi^*(r,\sigma)\mathbf{S}\blacksquare((r,\sigma))). \end{aligned} \tag{A14}$$

This expression is the main result of our appendix A. We remark that, for weak two-body interactions $\int_\Omega d^3r |V_{(0,1)}|^2 << 1$, it is possible to analyze perturbatively Eq.(A13). The free one-body Green function may be expressed in the following explicity form,[5, 6]

$$G(r,r',\sigma) = \sum_K \frac{\varphi_k(r)\varphi_k(r')e^{-i\sigma E_k}}{1-e^{E_k}} \times (\Phi(\sigma) + e^{-E_k}\Phi(-\sigma)) \ . \tag{A15}$$

where $\varphi_k(r)$ denotes the eigenfunctions of the one-body interaction Schrödinger-Pauli operator and E_k its associated eigenvalue:

$$\left(\frac{\partial}{\partial\sigma} - \frac{1}{2M}\Delta_r - W(r) + \frac{1}{\beta} lg(\frac{z}{\beta})\right)\varphi_k(r) = E_k\varphi_k(r) \ . \tag{A16}$$

It is worth observing the $i\beta$-periodicity of Eq.(A15) in the fictitious time variable σ. Finally we would like to point out that, by considering the Grassmanian variables associated with the $U(N)$ color degrees in the Grassmanian path integral representation, Eq.(A1), as in chapter 1, we can easily write the following Bell functional integral for a gas of spin$-\frac{1}{2}$ $U(N)$-charged particles interacting with and external $3D$ Yang-Mills field $A(r)$

$$Z(z,\beta,\Omega,\mathbf{A}(r)) = \int D^F[\Psi(r,\sigma)] D^F[\Psi^*(r,\sigma)] \exp(-S[\Psi,\Psi^*,\mathbf{A}]) \ . \tag{A17}$$

where the weight functional is now given by

$$\begin{aligned} S[\Psi,\Psi^*,\mathbf{A}] = &\int_S d^3r \int_0^\beta d\sigma \Psi^*(r,\sigma) \\ &\times \left(\frac{\partial}{\partial\sigma} - \frac{1}{2M}(\vec{\nabla} - \vec{A})^2 - W(r) + \frac{1}{\beta} lg(\frac{z}{\beta})\right)\Psi(r,\sigma) \\ &+ \int_0^\beta d\sigma \int_\Omega d^3r d^3r' (|\Psi(r,\sigma)|^2 V_{(0)}(r-r')|\Psi(r',\sigma)|^2 \\ &+ (\blacksquare^*(r,\sigma)\mathbf{S}\blacksquare((r,\sigma))V_{(1)}(r-r')(\blacksquare^*(r,\sigma)\mathbf{S^k}\blacksquare((r',\sigma)))) \\ &+ i\frac{g^2}{2M}\int_0^\beta d\sigma \int_\Omega d^3 \varepsilon^{ijk} F_{ij}(\mathbf{A})(\blacksquare^*(r,\sigma)S^k\Psi(r,\sigma)) \ . \end{aligned} \tag{A18}$$

The complex field $\blacksquare(r,\sigma)$ now belongs to the fundamental representation of the color group $U(N)$.

At this point our analysis it becomes worthwhile to remark that if we consider a further average in Eq.(A17) by considering the second quantized Yang Mills fields we are naturally led to considering it as a possible definition for the partition functional for a gas of strings in a volume Ω and temperature β, since the field excitations are now confined as a result of its interaction with the quantized $3D$-Yang Mills fields.[1, 4] Work on thermodynamic analysis of this string gas will be reported in next appendix.

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Appendix B. Bell Functional Integral for Gas of Strings

One of the most useful technique to analyze the statistical Physics of point-particle many-body systems with two-body interaction is to represent the system's grand canonical partition functional by means of a quantum field functional integral, the so-called Bell functional Integral. (see appendix A).

In the previous decades, the study of the statistical mechanics of strings (or random surfaces), has became a unifying concept Physics of collective phenomena.[1-4] (see section 3.7 – chapter 3).

Following our previous work (appendix A),in this appendix we propose a generalization of the usual point-particle Bell functional integral for the case of statistical systems of random surfaces with two-body interaction.[4, 5]

Let us start our analysis by considering the statistical system of N closed strings described by functions $r_j(\alpha)(0 \leq \alpha \leq L, r_j(0) = r_j(L))$, contained in a volume $\Omega \subset \mathbb{R}^D$ and at temperature $T = 1/k\beta$. The generalization of the point-particle Feynman path integral expression for the Canonical partition functional associated with this string ensemble will need functional integrals over random surfaces instead of the usual point-particle world line;[3] Let us, thus, introduce the associated string world sheet at temperature $T = 1/k\beta$.

It is described by a two-dimensional field $R_\mu(\alpha,\tau)$ in $\mathbb{R}^D$ with $0 \le \alpha \le L, 0 \le \tau \le \beta$ and satisfying the periodicity condition $R_\mu(\alpha,\tau+i\beta) = R_\mu(\alpha,\tau)$. The interactions will be given by the interaction of the random surface $R_\mu(\alpha,\tau)$ with external fields in $\mathbb{R}^D$, $V_0(r_\mu)$ and $V_1(r_\mu)$ represent the analogous one-body and two-body interaction for surfaces respectively.

In order to write the canonical partition functional for the above described ensemble of strings we follow the simplest generalization of the Boltzman weight of point-particles for strings.[5] It will be given by the path integrations below written

$$\begin{aligned} Z(N,\beta,\Omega) =& \frac{1}{N!}\prod_{j=1}^{N}\int_0^\infty dL \int_\Omega dr_\mu^{(j)} \\ &\times \sum_{\{r_\mu^{(j)}(\alpha)\}} \int_{R_\mu^{(j)}(\alpha,0)=r_\mu^{(j)}(\alpha)} D^F[R_\mu^{(j)}(\alpha,\tau)] \exp\left[-\frac{1}{2}\sum_{j=1}^{N}\int_0^\beta d\tau\int_0^L d\alpha \right. \\ &\left. \times \left(\frac{\partial R_\mu^{(j)}(\alpha,\tau)}{\partial\alpha}\right)^2 + \left(\frac{\partial R_\mu^{(j)}(\alpha,\tau)}{\partial\tau}\right)^2\right] \\ &\times \exp\left(-\sum_{j=1}^{N}\int_0^\beta d\tau\int_0^L d\alpha V_0(R_\mu^{(j)}(\alpha,\tau))\right) \\ &\times \exp\left(-\sum_{i,j=1}^{N}\int_0^\beta d\tau'\int_0^L d\alpha'\int_0^\beta d\tau\int_0^L d\alpha V_1(R_\mu^{(i)}(\alpha,\tau)-R_\mu^{(j)}(\alpha',\tau'))\right) \end{aligned} \quad \text{(B1)}$$

where the string "continuous sum" $\sum_{\{r_\mu^{(j)}(\alpha)\}}$, which replaces the usual integration dr_j over the particle position in the point-particle canonical partition functional, is defined by the following path integration over $r_\mu(\alpha)$, (see chapter 1).

$$\sum_{\{r_\mu(\alpha)\}} = \int_{r_\mu(0)=r_\mu(L)=r_\mu^{(j)}} D^F[r_\mu(\alpha)] \exp\left\{-\frac{1}{2}\int_0^L \dot{r}_\mu^2(\alpha)d\alpha\right\} . \quad \text{(B2)}$$

The weight $\exp\{-(1/2)\int_0^L \dot{r}_\mu(\alpha)^2\}$ is introduced in the string sum in order to make it formally convergent.[4]

Now, we can write the two-body non-local surface interaction by an independent local interaction of each random surface with a Gaussian fluctuating external Field followed by an average over this stochastic field [1]. We, thus introduce a Gaussian Random scalar field Φ with two-point correlation function given by $\langle \blacksquare(r_\mu), \blacksquare(r_\mu)' \rangle = V_1(r_\mu - r'_\mu)$ which by its turn enables us to rewrite Eq.(B1) in the following suitable form,

$$\begin{aligned} Z(N,B,\Omega) =& \frac{1}{N!}\prod_{j=1}^{N}\int_0^\infty dL \int_\Omega dr_\mu^{(j)} \sum_{\{r_\mu^{(j)}(\alpha)\}} \int_{R_\mu^{(j)}(\alpha,0)=r_\mu^{(j)}(\alpha)} D^F[R_\mu^{(j)}(\alpha,\tau)] \\ &\times \exp\left[-\frac{1}{2}\sum_{j=1}^{N}\int_0^\beta d\tau\int_0^L d\alpha \left(\frac{\partial R_\mu^{(j)}(\alpha,\tau)}{\partial\alpha}\right)^2 + \left(\frac{\partial R_\mu^{(j)}(\alpha,\tau)}{\partial\tau}\right)^2\right] \end{aligned}$$

$$\times \exp\left(-\sum_{j=1}^{N}\int_0^{\beta} d\tau \int_0^L d\alpha V_0(R_\mu^{(j)}(\alpha,\tau))\right)$$
$$\times \left\langle \exp\left(-\sum_{i,j=1}^{N}\int_0^{\beta} d\tau \int_0^L d\alpha \blacksquare(R_\mu^{(j)}(\alpha,\tau))\right)\right\rangle_{\Phi} \tag{B3}$$

Analysis of thermodynamical properties of this many-random-surface system may be done by considering the associated grand canonical partition functional by introducing the system activity variable $\mathfrak{z}$

$$Z(\mathfrak{z},\beta,\Omega) = \sum_{N=0}^{\infty} Z(N,\beta,\Omega)\mathfrak{z}^n\,. \tag{B4}$$

Now our aim is to write (B4) as a functional integral over complex disorder fields. For this task, we propose to consider a disorder field defined over the functional space of closed strigss $r_\mu(\alpha)$ and depending on a evolution parameter A in the range $0 \le A \le \beta$. This proposed field is denoted by $\psi[r_\mu(\alpha)),A]$ and is supposed to be periodic in A, i.e., $\psi[(r_\mu(\alpha)),A] = \psi[r_\mu(\alpha);A+i\beta]$.

Following closely the study of Appendix A, a natural candidate to be the Schrödinger operator to act on the proposed string disorder field $\psi[r_\mu(\alpha),A]$ is given by (see chapter 9)

$$\hat{S}([V_{(0)}],[\Phi]) = \frac{\partial}{\partial A} - \hat{\Delta}_{(r_\mu(\alpha))} + \int_0^L [V_{(0)}(r_\mu(\alpha)+\Phi(r_\mu(\alpha))]d\alpha + \mu \tag{B5}$$

where $\hat{\Delta}$ is the generalization of the point-particle Laplacean for strings (see section 3.7).

$$\hat{\Delta}_{(r_\mu(\alpha))} = \int_0^L d\alpha\left(\frac{1}{2}\frac{\delta^2}{\delta^2 r_\mu(\alpha)} - \frac{1}{2}|r'_\mu(\alpha)|^2\right)\,. \tag{B6}$$

Proceeding from our point-particle study,[2] we should consider the functional determinant of the Schrödinger operator Eq.(B6) after taking into account a Fourier expansion in the A-variable for the disorder field $\psi[r_\mu(\alpha),A]$.

$$\lg\left(\frac{\mathrm{DET}(\hat{S}([V_{(0)}],\blacksquare)}{\mathrm{DET}(\hat{S}([V_{(0)}=0,\blacksquare=0])}\right) = \sum_{m=-\infty}^{-\infty} e^{2\pi i m}\lg\,\mathrm{DET}\left(\frac{2\pi i m}{\beta} - \hat{\Delta}_{\{r(\alpha)\}}\right.$$
$$\left. + \int_0^L d\alpha\,(V_0,(r_\mu(\alpha)) + \blacksquare(r_\mu(\alpha))) + \mu\right). \tag{B7}$$

We use now the proper-time technique to define the functional determinant in Eq.(B7)

$$\lg\,\mathrm{DET}\left(+\frac{2\pi i m}{\beta} - \hat{\Delta}_{\{r(\alpha)\}} + \int_0^L d\alpha\left(V_{(0)},(r_\mu(\alpha)) + \blacksquare(r_\mu(\alpha))\right) + \mu\right)$$
$$= -\sum_{m=-\infty}^{-\infty}\int_0^{\infty}\frac{dT}{T}e^{\mu T}\exp\left(-2\pi i m\left(\frac{T}{\beta}-1\right)\right) \times \sum_{\{r_\mu(\alpha)\}}$$
$$\left\langle r_\mu(\alpha)\middle|\exp\left\{-T\left[-\hat{\Delta}_{\{r(\alpha)\}} + \int_0^L d\alpha\left(V_{(0)}(r_\mu(\alpha)) + \blacksquare(r_\mu(\alpha))\right)\right]\right\}\middle|r_\mu(\alpha)\right\rangle \tag{B8}$$

At this point of our exposition, it is instructive to survey the main result of chapter 11 which will be used in what follows. In this chapter it has been shown that the free string propagator $\langle r_\mu(\alpha)|\exp(-T\hat{\Delta})|r'_\mu(\alpha))\rangle = G(r_\mu(\alpha),T)$ satisfies the Schrödinger string equation

$$i\frac{\partial}{\partial A}G(r_\mu(\alpha),A) = \hat{\Delta}_{\{r(\alpha)\}}G(r_\mu(\alpha),r'_\mu(\alpha),A)\ , \tag{B9}$$

$$\lim_{A\to 0^+} G(r_\mu(\alpha),r'_\mu(\alpha),A) = \prod_{0\le\alpha\le L}\delta^{(d)}(r_\mu(\alpha)-r'_\mu(\alpha))\ . \tag{B10}$$

For Euclidean evolution parameter $A = iT$, we have the functional integral representation for this string propagator

$$\begin{aligned}\angle r'_\mu(\alpha)|\exp(-T\hat{\Delta})|r_\mu(\alpha)\rangle = \int_{\substack{R_\mu(\alpha,0)=r_\mu(\alpha)\\ R_\mu(\alpha,T)=r'_\mu(\alpha)}} D^F[R_\mu(\alpha,\tau)]\exp\left\{-\frac{1}{2}\int_0^T d\tau\int_0^L d\alpha\right.\\ \left.\times\left[\left(\frac{\partial R_\mu(\alpha,\tau)}{\partial\alpha}\right)^2+\left(\frac{\partial R_\mu(\alpha,\tau)}{\partial\tau}\right)^2\right]\right\}\ .\end{aligned} \tag{B11}$$

By taking into account the above results, we have the following random surface representation for the (Euclidean) string propagator in Eq.(B8) in the presence of external fields

$$\begin{aligned}&\left\langle r_\mu(\alpha)\left|\exp\left\{-T\left[-\hat{\Delta}_{\{r(\alpha)\}}+\int_0^L d\alpha\left(V_{(0)}(r_\mu(\alpha)\right)+(\blacksquare(r_\mu(\alpha)))\right]\right\}\right|r_\mu(\alpha)\right\rangle\\ &=\left\langle r_\mu(\alpha)\left|\int_{R_\mu^{(j)}(\alpha,0)=R_\mu^{(j)}(\alpha,\tau)=r_\mu(\alpha)} D^F[R_\mu(\alpha,\tau)]\exp\left\{-\frac{1}{2}\int_0^\beta d\tau\int_0^L d\alpha\right.\right.\\ &\left.\left.\times\left[\left(\frac{\partial R_\mu(\alpha,\tau)}{\partial\alpha}\right)^2+\left(\frac{\partial R_\mu(\alpha,\tau)}{\partial\tau}\right)^2+V_{(0)}(R_\mu(\alpha,\tau))+\blacksquare(R_\mu(\alpha,\tau))\right]\right\}\right|r_\mu(\alpha)\right\rangle\ .\end{aligned} \tag{B12}$$

By identifying the activity $\mathfrak{z}$ with the parameter μ through the relationship $\mathfrak{z} = e^{-\mu\beta}/\beta$, we obtain that the inverse of determinant Eq.(B7) averaged over the fluctuating field $\blacksquare(r_\mu)$ coincides exactly with the grand canonical partition functional (B4). Explicity, we have the result

$$Z(\mathfrak{z},\beta,\Omega) = \left\langle \mathrm{DET}^{-1}\left(\frac{\hat{S}[V_{(0)},\blacksquare]}{\hat{S}[V_{(0)}=0,\blacksquare=0]}\right)\right\rangle\ . \tag{B13}$$

We can rewrite (B13) in the form of a Gaussian functional integral over the proposed bosonic complex disorder field $\psi[r_(\alpha),A]$

$$\begin{aligned}Z(\mathfrak{z},\beta,\Omega) = \int D^F(\psi[(r_\mu(\alpha),A)])D^F(\psi^*[(r_\mu(\alpha),A])\\ \times\left\langle\exp\left\{-\frac{1}{2}\int_0^\beta dA\sum_{\{r_\mu(\alpha)\}}\left(\psi^*[(r_\mu(\alpha),A)]\left(\frac{\partial}{\partial A}-\hat{\Delta}\{r_\mu(\alpha)\}\right.\right.\right.\right.\\ \left.\left.\left.\left.+\int_0^L\left(V_{(0)}(r_\mu(\alpha)\right)+\blacksquare(r_\mu(\alpha))+\mu\right)\psi[(r_\mu(\alpha)),A]\right]\right)\right\}\right\rangle\ .\end{aligned} \tag{B14}$$

By evaluating the ■-average we obtain our proposed random surface Bell functional integral

$$
\begin{aligned}
Z(\mathfrak{z},\beta,\Omega) = \int & D^F(\psi[(r_\mu(\alpha),A)])D^F(\psi^*[(r_\mu(\alpha),A]) \\
& \times \exp\left\{-\frac{1}{2}\int_0^\beta dA \sum_{\{r_\mu(\alpha)\}} \left(\psi^*[(r_\mu(\alpha),A)]\left(\frac{\partial}{\partial A} - \hat{\Delta}\{r_\mu(\alpha)\}\right.\right.\right. \\
& + \left.\left.\left.\int_0^L \left(V_{(0)}(r_\mu(\alpha)) + \mu\right)\psi[(r_\mu(\alpha)),A]\right)\right)\right\} \\
& \times \exp\left\{-\int_0^\beta dA \int_0^\beta dA' \int_0^L d\alpha \int_0^L d\alpha' \sum_{\{r'_\mu(\alpha')\}}\sum_{\{r_\mu(\alpha)\}} |\psi[(r_\mu(\alpha)),A]|^2\right. \\
& \left.\times V_1(r_\mu(\alpha) - r'_\mu(\alpha'))|\psi[(r'_\mu(\alpha')),A']|^2\right\}
\end{aligned} \tag{B15}
$$

This expression is the main result of this Appendix B.

It is instructive to point out that in the situation of "collapsing strings" $r_\mu(\alpha)$ to a point r'_μ the above written Bell functional integral Eq.(B15) may be implemented by using the free propagator Eq.(B11) for the two-point propagator of the disorder string field $\psi[r_\mu(\alpha),A]$ (with $V_0(r) \equiv 0$). Results on thermodynamical properties of this quantum bosonic string gas will be left to our readers.

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Appendix C. Invariant Path Integral Quantization of Yang-Mills Theory

Let us start our study by considering the path-integral associated to a classical $SU(N)$ Yang-Mills theory defined in a finite valume $\Omega \subset R^4$.

$$
Z = \int D[A_\mu(x)]\exp\left\{-\frac{1}{4}\int_\Omega d^4x \mathrm{Tr}\left(F^2_{\mu\nu}(x)\right)\right\} \tag{C1}
$$

here the $SU(N)$ Gauge field has N^2-1 components

$$A_\mu(x) = \sum_{a=1}^{N^2-1} A_\mu^a \lambda_a \tag{C2}$$

with the hermitian traceless generators of $SU(N)$ satisfying the structure commutation relations below

$$[\lambda^a,\lambda^b] = i f_{abc}\lambda^c \tag{C3}$$

$$(\lambda^a)_{pq}(\lambda^a)_{p',q'} = (\delta^{pp'}\delta^{qp'} - \frac{1}{N}\delta^{pq}\delta^{p'q'}) \tag{C4}$$

Note the gauge covariant objects below defined

$$F_{\mu\nu}(x) = -\frac{1}{g}[D_\mu, D_\nu] = \sum_{a=1}^{N^2-1} F_\mu^a \lambda_a = \partial_\mu A_\mu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \tag{C5}$$

$$D_\mu = 1\partial_\mu - igA_\mu^a \lambda_a \tag{C6}$$

As noted in the bulk of this chapter, the path-integral infrared-divergent free (since $\text{vol}(\Omega) < \infty$, Ω being a compact set of R^4), has the gauge invariance under the local gauge group $G : C^\infty(\Omega, SU(N)) \subset \prod_{x\in R^4} (SU(N))_x$:

$$\Omega(x) \in \Pi(SU(N))_x$$

$$A_\mu \to A_\mu^\Omega = \Omega A_\mu \Omega^{-1} + \frac{i}{g}\Omega\partial_\mu\Omega^{-1} \tag{C7}$$

$$F_{\mu\nu} \to F_{\mu\nu}^\Omega = \Omega(x)F_{\mu\nu}(x)\Omega^{-1}(x) \tag{C8}$$

By introducing some ultra-violet cut-off in the free kinetic action associated to eq(C1), it can be showed that all “rough-distributional” aspects of the path-integrated gauge field configurations turns out to become point-functions and as a consequence the formal functional domain of eq(C1) becomes the space of $SU(N)$-valued functions $L^2(\Omega, SU(N)$ (see chapter 19), besides of producing a well-defined cylindrical measure in the infinite-dimensional Manifold Space

$$\mathcal{M} = \frac{L^2(\Omega, SU(N))}{C^\infty(\Omega, SU(N))} \otimes C^\infty(\Omega, SU(N))$$

In order to write exactly the path-integral measure in this functional Manifold $\mathcal{M}$, we follow our geometrical procedure described in section 1.4 by introducing a flat Riemmanian structure in the Functional Bundle $\mathcal{M}$

$$ds^2 = \int_\Omega d^4x' \int_\Omega d^4x \left[(\delta A_\mu^a)(x)(\delta^{ab}\delta_{\mu\nu})\delta^{(4)}(x-x')(\delta A_\nu^b)(x')\right] \tag{C9}$$

After this step has been taken we should choose a fixed-gauge (bundle section) base Manifold $N \subset M$, through a function (infinite-dimensional) hyper surface

$$F(A) = \sum_{a=1}^{N^2-1} f^a(A_\mu^c \lambda_c)\lambda_a = 0 \ ,$$

called the gauge-fixing functional.

Since we are in an infinite-dimensional setting, let us restrict our study to the class of those functionals which are linear in the gauge field variable,namely

$$\frac{\delta F(A_\nu)}{\delta A_\mu(x)} = \sum_{a=1}^{N^2-1}\left[\lambda^a \frac{\delta}{\delta A_\mu^a(x)} F(A)\right] = \text{field-independent} \tag{C10}$$

besides of satisfying the linear argument condition

$$F(A_\nu + B_\nu) = F(A_\nu) + F(B_\nu) \tag{C11}$$

Let us write an explict parametrization equation of a given orbit associated to an fixed-gauge field configuration $\bar{A}_\mu = \sum_{a=1}^{N^2=1} \bar{A}_\mu^a \lambda_a$

$$F(\bar{A}_\mu) = 0\ , \tag{C12}$$

namely

$$Y_\mu^a(\Omega,[\bar{A}]) = \det_F \left\{ \frac{\delta F(A)}{\delta A}\Big|_{A=\bar{A}^\Omega} \right\} \times \left(\int_{\mathcal{M}} D^F[A_\mu(x)](A_\mu^a(x) \times \delta^F(F(A_\mu - \bar{A}_\mu^\Omega)) \right) \tag{C13}$$

It is straightfoward to see that the path-integral eq.(C13) produces as a result of this explicit evaluation the gauge orbit Manifold passing through the gauge fixed configuration $\bar{A}_\mu$,namely

$$Y_\mu^a(\Omega(x),[\bar{A}_\mu]) = \frac{\det_F\left\{\left(\frac{\delta F(A)}{\delta A}\right)\Big|_{A=\bar{A}^\Omega}\right\}}{\det_F\left\{\left(\frac{\delta F(A)}{\delta A}\right)\Big|_{A=\bar{A}^\Omega}\right\}} \times ((\bar{A}_\mu^a)^\Omega) = (\bar{A}_\mu^a)^\Omega \tag{C14}$$

The induced functional Riemanian metric eq(C9) on this orbit Manifold is explicitly given by the operational formulae below

$$\begin{aligned} dS^2 = \int d^4x d^4x' \left\{ \left(\frac{\delta}{\delta\Omega} Y_\mu^a(\Omega,[\bar{A}])\right)(x) \right\} (\delta\Omega(x)) \\ \times \delta^{(4)}(x-x')\delta^{\mu\nu}\delta_{ab} \left\{ \left(\frac{\delta}{\delta\Omega} Y_\nu^b(\Omega,[\bar{A}])\right)(x) \right\} (\delta\Omega(x)) \end{aligned} \tag{C15}$$

where

$$\begin{aligned} \frac{\delta}{\delta\Omega} Y_\mu^a(\Omega,[\bar{A}]) = \int_{\mathcal{M}} D^F[A_\mu(x)] A_\mu^a(x) \left\{ -\left(\frac{\delta F(\bar{A}^\Omega)}{\delta\Omega}\right)(x)\right\} \\ \times (\delta')^{(F)}(F(A) - F(\bar{A}^\Omega)) \end{aligned} \tag{C16}$$

Here $(\delta')^{(F)}(\cdot)$ denotes the (functional) derivative of the delta functional.

The evaluation of eq(C15) by means of the result eq(C16) can be done by using the well known distributional formula expected to be correct in the infinite-dimensional setting $\mathcal{M}$.

$$\int_{-\infty}^{+\infty} dx h(x) \left(\frac{d}{dx}\delta(f(x))\right) = -\sum_{\{x_n\}} \left(\frac{h'(x)}{f'(x)}\right)\Big|_{x=x_n} \tag{C17}$$

where $\{x_n\}$ is the set of (single) zeroes of $f(x)$.

We have thus in $\mathcal{M}$

$$\begin{aligned}
&\int_{\mathcal{M}} D^F[A_\mu(x)]\left(A_\mu^a\left(-\frac{\delta F(\bar{A}^\Omega)}{\delta\Omega}\right)\right)(\delta')^{(F)}\left(F(A)-F(\bar{A}^\Omega)\right) \\
&=\left\{\frac{1}{\det^F\left\{\frac{\delta F(A)}{\delta A}\right\}}\left[-\sum_{(\nu,b)}\left(\frac{\delta}{\delta A_\nu^b}\left(A_\mu^a\frac{\delta F(\bar{A}^\Omega)}{\delta\Omega}\right)\right)\right]\right\}\Big|_{A=\bar{A}^\Omega} \\
&=-\sum_{(\mu,b)}\left[\left(\frac{\delta A_\mu^a}{\delta A_\nu^b}\right)\frac{\delta F(\bar{A}^\Omega)}{\delta\Omega}\right]=-(\delta_{\mu\nu}\delta_{ab})\frac{\delta F(\bar{A}^\Omega)}{\delta\Omega} \qquad \text{(C18)}
\end{aligned}$$

By substituting eq(C18), we obtain the explicit form of the functional Riemmanian metric of the orbit Manifold generated by a given fixed field configuration

$$\begin{aligned}
dS^2=&\int d^4x d^4x'\ \delta^{(4)}(x-x')\left(\frac{\delta F(\bar{A}^\Omega)}{\delta\Omega}(x)\right)(\delta\Omega(x)) \\
&\left(\sum_{\gamma,\gamma',e,e'}\left(\delta_{\mu\gamma}\delta_{ac}(\delta_{\mu\nu}\delta_{ab})\delta_{\nu\gamma'}\delta_{bc'})\right)\right)\left(\frac{\delta F(\bar{A}^\Omega)}{\delta\Omega}(x')\right)(\delta\Omega(x')) \qquad \text{(C19)}
\end{aligned}$$

which by its turn lead us to volume element of the orbit Manifold, as first deduced by Faddev-Popov as a "trick"

$$d\mu[\bar{A}^\Omega]=\det{}_F\left\{\frac{\delta F(\bar{A}^\Omega}{\delta\Omega}\right\}d^{Haar}\mu(\Omega)\ . \qquad \text{(C20)}$$

Here $d^{Haar}\mu(\Omega)=\prod_{x\in\Omega}(\Omega^{-1}d\Omega)(x)$ is the formal gauge invariant measure in the local gauge group $C^\infty(\Omega,SU(N))$.

As a consequence the path-integral measure takes a gauge fixed form in $\mathcal{M}$.

$$\begin{aligned}
Z=&\int D^F[\bar{A}^\mu]\exp\left\{-\frac{1}{4}\int_\Omega d^4x\mathrm{Tr}(F_{\mu\nu}^2(\bar{A}))\right\}\delta^{(F)}(F(\bar{A})) \\
&\times\left(\int_G \det{}_F\left\{\frac{\delta F(\bar{A}^\Omega)}{\delta\Omega}\right\}d^{Haar}\mu(\Omega)\right) \qquad \text{(C21)}
\end{aligned}$$

This is an important result of ours in the subject.

In the so called perturbative case, it is possible to evaluate the volume of the gauge orbit by using the "infinitesimal" local gauge group G^{inf}, which is formed by all infinitesimal gauge transformations in a neighborhood of the group element identit 1.

$$\begin{aligned}
&\det{}_F\left\{\frac{\delta}{\delta\Omega}F(\bar{A}^\Omega)\right\}\Big|_{\Omega=1+i\varepsilon\omega^a(x)\lambda_a} \\
&=\det{}_F\left\{\frac{\delta}{\delta\Omega}\left(F(\bar{A})+\frac{\delta F}{\delta\Omega}(\bar{A})\Big|_{\Omega=1}(\delta\Omega)+\frac{\delta^2F}{\delta\Omega\delta\Omega}(\delta\Omega)^2\Big|_{\Omega=1}+\cdots\right)\right\} \\
&=\det{}_F\left\{\frac{\delta F}{\delta\Omega}(\bar{A})\Big|_{\Omega=1}\right\}+O(\varepsilon^2) \qquad \text{(C22)}
\end{aligned}$$

which lead us to the famous weak field-perturbative result of the Faddev-Popov

$$\int_{G^{inf}} \det{}_F \left\{ \frac{\delta F}{\delta \Omega}(\bar{A}^{\Omega})\Big|_{\Omega=1} \right\} d^{Haar}\mu(\Omega) \tag{C23}$$

$$= \det{}_F \left\{ \frac{\delta F}{\delta \Omega}(\bar{A}^{\Omega})\Big|_{\Omega=1} \right\} \times \left(\int \prod_{a=1}^{N^2-1} D^F[\omega^a(x)] \right) , \tag{C24}$$

where the infinitesimal gauge group volume is absorbed in an over all factor, when evaluating observables the theory by means of path integrals, like the Wilson Loops, etc. (see Chapter 2).

It still to be an open problem to analyze in full, our result eq.(C21),including the important case of the existence non-trivial homotopical class of the local gauge group $G = C(\Omega, SU(N))$. We left this "infinite-dimensioanl homotopical" problem to the future inquiries of our readers with the very important mathematical remark that all gauge field configurations possessing differentiable structures supporting topological-differential structure assignments (Chern-Simon Classes, etc...) forms a set of functional zero measure in the functional domain of the path integral eq.(C21) (see chapter 19). As a consequence, these smooth C^∞-field configurations are relevant solely as objects to be used in saddle point evaluation of eq.(C22) (see A.M. Polyakov – Compact Gauge Fields and the infrared catastrophe – Phys. Lett. **59**B, 82 (1975)).

Appendix D. Polyakov Invariant Path Integral Quantization of Gravity in Two-Dimensional Manifolds

In this appendix, we intend to apply the method of invariant path integration to the "cosmological" action of a two-dimensional metric field path integral (Quantum Gravity in the two-dimensional domain $D \subset R^2$).

$$Z = \int d\mu[g_{ab}(x)] e^{\mu^2 \int_D d^2x (\sqrt{g}(x))} \tag{D1}$$

We note that the evaluation of eq.(D1) must preserve the invariance of the objects inside it under the action of the infnitesimal dipheomorfism-coordinates change group in D

$$\delta x^a = g^{ab}(x)\varepsilon_b(x) \tag{D2}$$

$$\delta g_{ab}(x) = (\nabla_a \varepsilon_b + \nabla_b \varepsilon_a)(x^b) , \tag{D3}$$

with the covariant objects

$$(\nabla_a \varepsilon_b)(x) = \left(\frac{\partial}{\partial x^a} \varepsilon_b - \Gamma^c_{ab} \varepsilon_c \right)(x) = (\partial_a \varepsilon_b - \Gamma^c_{ab})(x) \tag{D4}$$

$$\Gamma^c_{ab}(x) = \frac{1}{2} \left\{ g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \right\}(x) , \tag{D5}$$

It appears suitable to define eq.(D1) through the method of Functional Riemann metrics as exposed in section 1.4 of this chapter.

$$dS^2 = \int_D d^2x(\delta g_{ab})(x)\gamma^{ab,a'b'}[g_{cd}(x)](\delta g_{a'b'})(x) \tag{D6}$$

with$(c \neq 1/2)$

$$\gamma^{(ab,a'b')}[g_{cd}(x)] = \left(\sqrt{g}(g^{aa'}g^{bb'} + cg^{ab}g^{a'b'})\right)(x) \tag{D7}$$

with the conformal gauge $g_{ab}(x) = e^{\varphi(x)}\delta_{ab}$ as a gauge fixing functional.

It is worth call attention that due to the fact that there are only 3 independent components of the path-integrated metric field $g_{ab}(x)(g_{12}(x) = g_{21}(x))$, the functional metric takes the form below

$$dS^2 = \int_D d^2x \left\{ (\delta g_{11}, \delta g_{12}, \delta g_{22})[\overline{\gamma}^{ij}]_{\substack{1\le i\le 3\\ 1\le j\le 3}} (\delta g_{11}, \delta g_{12}, \delta g_{13})^T \right\}(x) \tag{D8}$$

with the coeficients

$$\begin{aligned}
\overline{\gamma}^{11} = & \quad \gamma^{(11,11)}[g],\ \gamma^{22} = \gamma^{(12,12)}[g],\ \overline{\gamma}^{(22,22)}[g] = \overline{\gamma}^{23} \\
\overline{\gamma}^{12} = & \quad \gamma^{(11,12)}[g],\ \overline{\gamma}^{13} = \gamma^{(11,22)}[g],\ \overline{\gamma}^{31} = \gamma^{(22,11)}[g] \\
\overline{\gamma}^{23} = & \quad \gamma^{(12,12)}[g],\ \overline{\gamma}^{32} = \gamma^{(22,12)}[g].
\end{aligned} \tag{D9}$$

We have thus for $g_{ab} = e^{\varphi}\delta_{ab}$

$$\det[\overline{\gamma}^{ij}] = \begin{vmatrix} \frac{1+c}{e^{\varphi}} & 0 & \frac{c}{e^{\varphi}} \\ 0 & \frac{1}{e^{\varphi}} & 0 \\ \frac{c}{e^{\varphi}} & 0 & \frac{1+c}{e^{\varphi}} \end{vmatrix} = \frac{1+2c}{e^{3\varphi}} \tag{D10}$$

We can see thus, that only for $c \neq -1/2$, eq.(D6) defines an infinite-dimensional Riemaniann structure on the space of the (distributional) metrical fields in D (see chapter 19).

A general dispalcement metric field $\delta g_{ab}(x)$, around the fixed-gauge configuration $\bar{g}_{ab} = e^{\varphi}\delta_{ab}$ is given by

$$(\delta g_{ab})(x) = \delta\varphi(x)\bar{g}_{ab} + (\nabla_a\varepsilon_b + \nabla_b\varepsilon_a)(x)$$

which by its turn leads to the more invariant form for the associated functional metric eq.(D6)

$$\begin{aligned}
dS^2 &= \int_D d^2x e^{\varphi(x)} \left\{ (\delta\varphi g^a_b + \nabla^a\varepsilon_b + \nabla_b\varepsilon^a)(\delta\varphi g^b_a + \nabla^b\varepsilon_a + \nabla_a\varepsilon^b) + 4c(\nabla_c\varepsilon^c + \delta\varphi)^2 \right\}(x) \\
&= \int_D d^2x e^{\varphi(x)} \left\{ (\nabla^a\varepsilon_b + \nabla_b\varepsilon^a - (\nabla_c\varepsilon^c)g^a_b)(\nabla^b\varepsilon_a + \nabla_a\varepsilon^b - (\nabla_c\varepsilon^c)g^b_a) \right. \\
&\qquad \left. + 2(1+2c)(\nabla_c\varepsilon^c + \delta\varphi)^2) \right\} \\
&= 2(1+2c)\left\{ \int_D d^2x e^{\varphi(x)}(\nabla_c\varepsilon^c + \delta\varphi)^2) \right\} \\
&\quad - 2\left\{ \int_D d^2x e^{\varphi(x)} \left\{ \varepsilon_a e^{-\varphi}(\nabla_c\nabla^c)\varepsilon_b + \varepsilon_a e^{-\varphi}[\nabla_a, \nabla_b]\varepsilon_b \right\}(x) \right\},
\end{aligned} \tag{D11}$$

where we have used the integration by parts formula with covariant derivatives, for general objects (tensors) T and S.

$$\int_D d^2x\sqrt{g}\,S(\nabla_d T) = -\int_D d^2x\sqrt{g}(\nabla_d S)T \tag{D12}$$

As a consequence the functional volume element takes the Faddev-Popov form

$$d\mu[G_{ab}] = \left(\prod_{x\in D} d(e^{\frac{\varphi}{2}(x)})\right)\times \det{}_F^{1/2}\left\{\nabla^c\nabla_c + [\nabla^b,\nabla_a]\right\}\times\left(\prod_{\substack{x\in D\\ a=1,2}} d\varepsilon_a(x)\right) \tag{D13}$$

As a result of the evaluation of the metric field path integral eq.(D1)) in the conformal gauge $g_{ab}(x) = \rho(x)\delta_{ab}$,it takes the form below after an explicitly evaluation of the Faddev-Popov functional determinant in eq.(D13) (A.M. Polyakov).

$$Z = \lim_{\delta\to 0}\left\{\int D^F[\sqrt{\rho}]\exp\left[-\frac{13}{12\pi}\int_D d^2x\left(\frac{\partial}{\partial x^a}\lg(\sqrt{\rho})\right)^2 + \left(\mu^2 - \frac{1}{2\pi\delta}\right)\int_D d^2x(\sqrt{\rho})^2(x)\right]\right\}$$
$$= \int \overbrace{\left(\prod_{x\in D} d\beta(x)\right)}^{D^F[\beta(x)]}\exp\left[-\frac{13}{6\pi}\left(\int_D d^2x\frac{1}{2}\left(\frac{\partial_a\beta}{\beta}\right)^2\right) + \mu_{ren}^2\int_D d^2x\beta^2(x)\right] \tag{D14}$$

Note that we have introduced the correct degree of freedom to describe induced (quantum) two-dimensional gravity in the region D (without boundary). It is worth call the reader attention that the appearance of a kind of “Goldstone Massive Bóson” $\beta(x)$ is due to the dynamical breaking of the conformal group (scaling) of the theory at the quantum level by means of the induction of a counter-term of the form of a cosmological constant $\lim_{\delta\to 0}\exp\{(\mu^2 - \frac{1}{2\delta})\int_D d^2x\sqrt{g}(x)\}$ in the induced σ-like model scalar action as given in eq.(D14).

A complete use of these formulae will appear in chapters 9–16.

At this point we comment that perturbative evaluation of the two-dimensional σ-model (scalar) field theory as expressed by eq.(D14) can be implemented through an natural flat background weak fluctuation metrical variations of small strength ε, namely $\beta = 1 + \varepsilon\beta^{(1)} + \varepsilon^2\beta^{(2)} + \cdots$. Perturbative analysis in this context will be left to our readers.

In the important string case of taking into account explictly the existence of non-trivial topology in D (a bounded / boundaryless open domain with holes inside), we must take into account the conformal gauge fixing with Teichmüller parameters (see M. Nakahara – Geometry, Topology and Physics – Graduate student Series in Physics – IOP Publishing Ltd. 1990 and Chapters 12, 14 and 17) $g_{ab}(x) = e^{\varphi(x)}\cdot h_{ab}(t^i,x)$, where t^i are the Teichmüller parameters in an open domain $L_{Teich(g)}$ in R^{6g-6} (if D possesses g holes) and R^2 (if D possesses a single hole).

The final answer is given by the scalar σ-model in the back-ground field $h_{ab}(t^i)$, as written below

$$Z = \sum_{\text{Topologies}} \left\{ \int d\mu[g_{ab}] e^{(\int_D d^2x(\sqrt{g}R)(x)} e^{-\mu^2 \int_D d^2x(\sqrt{g})(x)} \right\}$$
$$= \sum_{g=0}^{\infty} e^{-2\pi(2-2g)} \left\{ \int_{L_{\text{Teich(g)}}} \prod_{i=1}^{6g-6} \left[\int \left(\prod_{x\in D} d(\sqrt{h_{ab}(t_i)}\beta(x) \right) \right.\right.$$
$$\times \left\{ \exp - \left[\frac{13}{12\pi} \int_D d^2x \sqrt{h_{ab}(t_c)} \cdot h^{ab}(t_i) \left(\frac{\partial_a \beta}{\beta}\right) \left(\frac{\partial_b \beta}{\beta}\right)(x) \right] \right\}$$
$$\left.\left.\times \exp\left(-\frac{1}{2} \int_D d^2x (\sqrt{h_{ab}(t_c)}\beta^2(x)) \right) \right] \right\} \tag{D15}$$

In our opinion it remains an important problem in Quantum Field Theory to understand the exact meaning of the Liouville field theorie (in the correct form of "Scalar σ-model" – eq.(D14) in a Quantum Field two-dimensional Framework, with the very important remark to keep in mind that quantum geometric bosonic field configurations with C^∞-differentiable topological structure made up a set of zero functional measure in all non-trivial quantum geometric path integrals (A.Yu. Morozov, A.M. Porelomov, String theory and Complex Geometry, Phys. Reports, 1992).

Appendix E.
Functional Determinants Evaluations on the Seeley Approach

In this somewhat technical appendix, we intend to highlight the mathematical evaluation of the functional determinants involved in the theory of random surface path integral (Chap. 9, 10, 11, 12).

Let us start by considering a differential elliptic self-adjoint operator of second order acting on the space of infinitely differentiable functions of compact support in $R^2, C_c^\infty(R^2,\mathbb{C}^q)$ with values in $\mathbb{C}^q$

$$A = \sum_{|\alpha|\leq 2} a_\alpha(x) D_x^\alpha \tag{E-1}$$

with $\alpha = (\alpha_1, \alpha_2)$ multi-indexes and

$$D_x^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{1}{i}\frac{\partial}{\partial x_2}\right)^{\alpha_2} \tag{E-2}$$

and $A_\alpha(x) \in C_c^\infty(R^2,\mathbb{C}^q)$.

By introducing the usual square-integrable inner product in $C_c^\infty(R^2,\mathbb{C}^q)$ and making the hypothesis that A is a positive definite operator, one may consider the (contractive) semigroup generated by A and defined by the spectral calculus

$$e^{-tA} = \frac{1}{2\pi i} \left\{ \int_C d\lambda \frac{e^{-t\lambda}}{(\lambda \mathbf{1}_{q\times q} - A)} \right\} \tag{E-3}$$

with C being an (arbitrary) path containing the positive semi-axis $\lambda > 0$ (the spectrum of the operator A) with a counter-clock wise orientation.

According to Seeley, one must consider the symbol associated to the resolvent pseudo-differential operator $(\lambda \mathbf{1}_{q\times q} - A)^{-1}$ and defined by the relationship below

$$\sigma(A - \lambda\mathbf{1}) = e^{-ix\xi}(A - \lambda\mathbf{1})e^{ix\xi} = \sum_{|j|=0}^{2} A_j(x,\xi,\lambda) \tag{E-4}$$

with

$$A_j(x,\xi,\lambda) = \left(\sum_{|\alpha|=\alpha_1+\alpha_2=j} a_j(x)(\xi_1^{\alpha_1})(\xi_2^{\alpha_2}) \right) \quad 0 \le j < 2 \tag{E-5}$$

$$A_2(x,\xi,\lambda) = -\lambda\mathbf{1}_{q\times q} + \left(\sum_{|\alpha|=\alpha_1+\alpha_2=j} a_j(x)(\xi_1^{\alpha_1})(\xi_2^{\alpha_2}) \right) \quad j = 2 \tag{E-6}$$

It is basic for symbols calculations, the important scaling properties as written below

$$A_j(x, c\xi, c^2\lambda) = (c)^j A_j(x,\xi,\lambda) \tag{E-7}$$

It is too a fundamental result of the Seeley's theory of pseudo-differential operators that the resolvent operator $(A - \lambda\mathbf{1})^{-1}$ (the associated Green function of the operator A) has an expansion of the form below in a suitable functional space

$$\sigma(A - \lambda\mathbf{1})^{-1}) = \sum_{j=0}^{\infty} C_{-2-j}(x,\xi,\lambda) \tag{E-8}$$

and satisfies the relationship below

$$\sigma(A - \lambda\mathbf{1}) \cdot \sigma((A - \lambda\mathbf{1})^{-1}) = 1 \tag{E-9}$$

$$\frac{1}{a!} \left\{ \sum_{|\alpha|\le 2} \sum_{j=0}^{\infty} \left[\left(D_\xi^\alpha(\sigma(A - \lambda\mathbf{1})\right)(x,\xi) D_x^\alpha \left[C_{-2-j}(x,\xi,\lambda) \right] \right\} = 1 \tag{E-10}$$

Recurrence relationships for the explicitly determination of the Seeley coefficients of the resolvent operator eq. (E-8) can be obtained through the use of the scaling properties

$$\xi = p\xi'; \quad \lambda^{\frac{1}{2}} = p(\lambda')^{\frac{1}{2}} \tag{E-11}$$

$$C_{-2-j}(x, p\xi', (p(\lambda')^{\frac{1}{2}})^2) = p^{-(2+j)} C_{-2-j}(x,\xi',\lambda') \tag{E-12}$$

After using eq. (E-11) in eq. (E-10) and for each integer j, comparing the resultant power series in the variable $1/p$ $\left(1 = 1 + 0\left(\frac{1}{p}\right) + \cdots 0 \left(\frac{1}{p}\right)^n + \dots\right)$, one gets as a result

$$C_{-2}(x,\xi) = (A_2(x,\xi)^{-1}) \tag{E-13}$$

$$0 = a_2(x,\xi)C_{-2-j}(x,\xi) + \left\{ \frac{1}{\alpha!} \sum_{\substack{\ell<j \\ k-|\alpha|-2-\ell=-j}} D_\xi^\alpha a_k(x,\xi) \left((iD_x^\alpha C_{-2-\ell}(x,\xi,\lambda) \right) \right\} \tag{E-14}$$

For the explicit operator in $C_c^\infty(R^2,R^q)$ as given below

$$A = -\left(g_{11}\frac{\partial^2}{\partial x_1^2} + g_{22}\frac{ø^2}{\partial x_2^2}\right)\mathbf{1}_{q\times q} - (A_1)_{q\times q}\frac{\partial}{\partial x_1} - (A_2)_{q\times q}\frac{\partial}{\partial x_2} - (A_0)_{q\times q} \tag{E-15}$$

with all the coefficients in $C_c^\infty(R^2,R^q)$, one obtains the following results after calculations

$$\begin{aligned} A_2(x,\xi,\lambda) &= (g_{11}(x)\xi_1^2 + g_{22}\xi_2^2 - \lambda)\mathbf{1}_{q\times q} \\ A_2(x,\xi,\lambda) &= -iA_1(x)\xi_1 - iA_2(x)\xi_2 \\ A_0(x,\xi,\lambda) &= -A_0(x) \end{aligned} \tag{E-16}$$

and

$$\begin{aligned} &C_{-2}(x,\xi) = (g_{11}(x)\xi_1^2 + g_{22}(x)\xi_2^2 - \lambda)^{-1} \\ &C_{-3}(x,\xi) = i(A_1(x)\xi_1 + A_2(x)\xi_2)(C_{-2}(x,\xi))^2 \\ &-2ig_{11}(x)\xi_1\left[\left(\frac{\partial}{\partial x_1}g_{11}\right)(\xi_1)^2 + \left(\frac{\partial}{\partial x_2}g_{22}\right)(\xi_2)^2\right](C_{-2}(x,\xi))^3 \\ &-2ig_{22}(x)\xi_2\left[\left(\frac{\partial}{\partial x_2}g_{11}\right)(\xi_1)^2 + \left(\frac{\partial}{\partial x_2}g_{22}\right)(\xi_2)^2\right](C_{-2}(x,\xi))^3 \end{aligned} \tag{E-17}$$

By keeping in view evaluations of the heat kernel of our given differential operator, let us write its expansion in terms of the Seeley coefficients of (E-8):

$$\begin{aligned} Tr(e^{-tA}) &= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi}\right)^2\left\{\int_{R^2} d^2x\,\sigma(e^{-tA})(x,\xi)\right\} \\ &= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi}\right)^2\left(\frac{1}{2\pi i}\right)\left[\int_{+\infty}^{-\infty} d(-is)e^{ist}\left(\int_{R^2\times R^2} d^2x\,d^2\xi\, C_{-2-j}(x,\xi,-is)\right)\right] \\ &= \sum_{j=0}^{\infty}\left\{\frac{1}{t}\frac{1}{(2\pi)^3}\int_{R^2} d^2\xi\int_{R^2} d^2x\left[\int_{-\infty}^{+\infty} e^{is}C_{-2-j}\left(x,\xi,\frac{-is}{t}\right)ds\right]\right\} \\ &= \sum_{j=0}^{\infty}\left\{\frac{1}{t}\frac{1}{(2\pi)^3}\int_{R^2} d^2\xi\int_{R^2} d^2x\, e^{is}\, t^{\left(\frac{2+j}{2}\right)}C_{-2-j}(x,t^{\frac{1}{2}}\xi,-is)\right\} \\ &= \sum_{j=0}^{\infty}\left\{t^{\frac{(j-2)}{2}}(2\pi)^{-3}\int_{R^2} d^2\xi\int_{R^2} d^2x\int_{-\infty}^{+\infty} C_{-2-j}(x,\xi,-is)\right\} \end{aligned} \tag{E-18}$$

By applying eq. (E-17) to the differential operator as given by eq. (E-15), we obtain the short-time Seeley expansion as an asymptotic expansion in the variable t

$$
\begin{aligned}
Tr\left(e^{-tA}\right) \sim \frac{q}{4\pi t}\left(\int d^2x\sqrt{g_{11}\,g_{22}}\right) \\
+\frac{q}{4\pi}\left(\int d^2x\sqrt{g_{11}\,g_{22}}\left(-\frac{1}{6}R\right)\right) \\
+\int d^2x\left(-\frac{1}{2}\frac{1}{\sqrt{g_{11}\,g_{22}}}\operatorname{Tr}\left[\overbrace{\left(\frac{\partial}{\partial x_1}(\sqrt{g_{11}g_{22}}A_1\right)+\left(\frac{\partial}{\partial x_2}(\sqrt{g_{11}\,g_{22}}A_2)\right)}^{\left(-\frac{1}{2}\mathrm{div}_{\mathrm{cov}}\vec{A}\right)}\right]\right) \\
+\int d^2x\left(-\frac{1}{4}Tr\left[\frac{(A_1)^2}{g_{11}}+\frac{(A_2)^2}{g_{22}}+A_0\right]\right)+0(t)
\end{aligned}
\tag{E-19}
$$

For applying the above formulae for the Polyakov's covariant path integrals, let us by firstly introduce the R^2 complex structure

$$z = x_1 + x_2\,, \qquad \overline{z} = x_1 - ix_2 \tag{E-20}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x_1} - i\,\frac{\partial}{\partial x_2}\,, \quad \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2} \tag{E-21}$$

For each integer j, let us define two Hilbert spaces H_j and $\overline{H}_j$ as follows:

a) H_j is defined as the vector space of all complex functions $f(z,\overline{z}) = f_1(x,y) + i\,f_2(x,y)$ with the following tensorial behavior under the action of a conformal tranformation

$$z = z(w) \tag{E-22}$$

$$f(z,\overline{z}) = \left(\frac{\partial w}{`\partial z}\right)^{-j}\,\tilde{f}(w,\overline{w}) \tag{E-23}$$

Let us introduce into H_j the following inner product

$$(q,f)_{H_j} = \int_{R^2} dz\overline{z}(\rho(z,\overline{z})^{j+1}\,\overline{g}(z,\overline{z})\,f(z,\overline{z}) \tag{E-24}$$

with $\rho(\overline{z})$ a positive continuous real-valued function of compact support in R^2 and associated to a conformal metric

$$ds^2 = \rho(z,\overline{z})\,dz \wedge d\overline{z}$$

b) $\overline{H}_j$ is the same definition as above exposed with the following tensor low

$$f(z,\overline{z}) = \left(\left(\overline{\frac{\partial w}{\partial z}}\right)\right)^{-j}\,\tilde{f}(w,\overline{w}) \tag{E-25}$$

At this point, we can verify that the above written inner products are conformal invariant

$$(g,f)_{H_j} = \int_{R^2} dw d\overline{w} \left(\frac{\partial z}{\partial w}\right)\left(\frac{\partial \overline{z}}{\partial w}\right)\left(\left|\frac{\partial w}{\partial z}\right|^2 \tilde{\rho}(w,\overline{w})\right)^{j+1}$$
$$\left[\left(\frac{\partial}{\partial z} w\right)^{-j} \tilde{f}(w,\overline{w})\right]\left[\left(\overline{\frac{\partial w}{\partial z}}\right)^{-j} (\tilde{g}(w,\overline{w}))\right]$$
$$= \int_{R^2} dw d\overline{w} (\tilde{\rho}(w,\overline{w}))^{j+1} \tilde{f}(w,\overline{w})(\overline{\tilde{g}(w,\overline{w})}) \tag{E-26}$$

Let us now introduce the following weighted Cauchy-Riemann operators with a $U(1)$-real valued connection $A = (A_z, A_{\overline{z}})$ in $R^2 \equiv \mathbb{C}$.

$$\begin{aligned} &\text{a)} \quad L_j = H_j \longrightarrow \overline{H}_{-(j+1)} \\ &\qquad f \longrightarrow (\rho(z,\overline{z}))^j (\partial_{\overline{z}} + A_{\overline{z}}))f \\ &\text{b)} \quad \overline{L}_j = \overline{H}_j \longrightarrow H_{(j+1)} \\ &\qquad f \longrightarrow (\rho(z,\overline{z}))^j (\partial_z + A_z))f \end{aligned} \tag{E-27}$$

together with the adjoint operators $(L_j\varphi, f)_{\overline{H}_{-(j+1)}} = \langle \varphi, L_j^* f\rangle_{H_j}$, namely:

$$\begin{aligned} L_j^* = -\overline{L}_{-(j+1)} : \overline{H}_{-(j+1)} \longrightarrow H_j \quad (\text{and } \overline{L}_j^* = -L_{-(j+1)}) \\ f \longrightarrow -(\rho(z,\overline{z}))^{-(j+1)} (\partial_z + A_z))f \end{aligned} \tag{E-28}$$

For simplicity, we consider the case of $A_z = A_z \equiv 0$.

The second order positive definite operators below

$$\begin{aligned} \mathcal{L}_j &= L_j^* L_j = -\overline{L}_{-(j+1)} L_j : H_j \to H_j \\ \overline{\mathcal{L}}_j &= (\overline{L}_j)^* \overline{L}_j : \ -L_{-(j+1)} \overline{L}_j : \overline{H}_{-(j+1)} \to \overline{H}_{-(j+1)} \end{aligned} \tag{E-29}$$

possesses the explicitly expressions (for $\rho(z,\overline{z}) = e^{\varphi(z,z)}$)

$$\begin{aligned} \mathcal{L}_j &= -e^{-(j+1)\varphi(z,\overline{z})} \partial_z e^{j\varphi(z,\overline{z})} \partial_{\overline{z}} \\ \overline{\mathcal{L}}_j &= -e^{-(j+1)\varphi(z,\overline{z})} \partial_{\overline{z}} e^{j\varphi(z,\overline{z})} \partial_z \end{aligned} \tag{E-30}$$

They have the following Seeley expansion:

$$\lim_{t\to 0^+} Tr_{C_c^\infty(R^2)}\left(e^{-t\mathcal{L}_j}\right) = \int dz d\overline{z} \left(\frac{\rho(z,\overline{z})}{2\pi t} - \frac{(1+3j)}{j2\pi} \Delta \ell g \rho(z,\overline{z})\right) \tag{E-31}$$

$$\lim_{t\to 0^+} Tr_{C_c^\infty(R^2)}\left(e^{-t\overline{\mathcal{L}}_j}\right) = \int dz d\overline{z} \left(\frac{\rho(z,\overline{z})}{2\pi t} + \frac{(2+3j)}{j2\pi} \Delta \ell g \rho(z,\overline{z})\right) \tag{E-32}$$

The above written expressions come from the Seeley asymptotic expansion

$$\begin{aligned} \lim_{t\to 0^+} Tr_{C_c^\infty(R^2)}\left(e^{-tA}\right) \sim \int d^2x \Bigg\{ &\left(\frac{\sqrt{g}}{4\pi} Tr(\mathbf{1})_{2\times 2}\right)\left(\frac{1}{t}\right) \\ &- \left(\frac{1}{24\pi}\sqrt{g} R\right) Tr(\mathbf{1})_{2\times 2} \\ &+ \frac{1}{4\pi}\sqrt{g} B_0 \Bigg\} + O(t) \end{aligned} \tag{E-33}$$

where A is the elliptic second-order self-adjoint differential operator in the presence of a Riemann metric $ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$ (in a tensorial notation in the space $L^2(R^2, \sqrt{g}\,dx^1dx^2)$:

$$A = \left(-\frac{1}{\sqrt{g}}(\partial_\mu \mathbf{1}_{2\times 2} + B_\mu)\sqrt{g}\,g^{\mu\nu}(\partial_\nu \mathbf{1}_{2\times 2} + B_\nu)\right) - (B_0) \qquad \text{(E-34)}$$

with $B_\mu(x^\nu)$ denoting $C_c^\infty(R^2, R^2)$ functions.

After all these preliminaries discussions, we pass to the problem of evaluating functional determinant (without zero modes)

$$\ell g \det \mathcal{L}_j = \lim_{\varepsilon \to 0^+} -\left\{\int_\varepsilon^\infty \frac{dt}{t}\, Tr_{C_c^\infty(R^2)}(e^{-t\mathcal{L}_j})\right\} \qquad \text{(E-35)}$$

It is straightforward to verify that the following chain of equations related to the functional variations of the conformal structure hold true [Herewith $Tr \equiv Tr_{C_c^\infty(R^2)}$]

$$\begin{aligned}
\delta\,\ell g \det \mathcal{L}_j &= \lim_{\varepsilon \to 0^+}\left\{\int_\varepsilon^\infty dt\, Tr\left(\overbrace{\frac{\delta\mathcal{L}_j}{\delta\varphi}\delta\varphi}^{\delta\mathcal{L}_j}\, e^{-t\mathcal{L}_j}\right)\right\} \\
&= \lim_{\varepsilon \to 0^+}\left\{\int_\varepsilon^\infty dt\, Tr\left[\left(-(j+1)\delta\varphi\,\mathcal{L}_j - j\overline{L}_{(j+1)}\delta\varphi L_j\right)e^{-t\mathcal{L}_j}\right]\right\} \\
&= \lim_{\varepsilon \to 0^+}\left\{\int_\varepsilon^\infty dt\, Tr\left[-(j+1)\delta\varphi\mathcal{L}_j\, e^{-t\mathcal{L}_j} + j\,\delta\varphi\,\overline{\mathcal{L}}_{-(j+1)}e^{-t\overline{\mathcal{L}}_{-(j+1)}}\right]\right\}
\end{aligned} \qquad \text{(E-36)}$$

where we have used the functional identity

$$\begin{aligned}
&Tr\left(-j\overline{L}_{-(j+1)}\delta\varphi L_j\, e^{-t\mathcal{L}_j}\right) \\
&= Tr\left[-j\,\overline{L}_{-(j+1)}\left(\overline{L}_{-(j+1)}\right)^{-1}(\delta\varphi)L_j\,\overline{L}_{-(j+1)}\,e^{-t\overline{\mathcal{L}}_{-(j+1)}}\right] \\
&= Tr\left[-j\cdot\mathbf{1}\delta\varphi\left(-\overline{\mathcal{L}}_{-(j+1)}\right)e^{-t\overline{\mathcal{L}}_{-(j+1)}}\right]
\end{aligned} \qquad \text{(E-37)}$$

since

$$e^{-t\overline{\mathcal{L}}_{-(j+1)}} = \left(\overline{L}_{-(j+1)}\right)^{-1} e^{-t\mathcal{L}_j}\,\overline{L}_{-(j+1)} \qquad \text{(E-32)}$$

is a consequence of the operatorial relationship

$$\overline{\mathcal{L}}_{-(j+1)} = \left(\overline{L}_{-(j+1)}\right)^{-1}\mathcal{L}_j\,\overline{L}_{-(j+1)} \qquad \text{(E-33)}$$

As a consequence, we obtain the results below:

$$\begin{aligned}
\delta(\ell g \det \mathcal{L}_j) = &-(j+1)\lim_{\varepsilon \to 0^+} Tr\left(\delta\varphi\, e^{-\varepsilon\mathcal{L}_j}\right) \\
&+ j\lim_{\varepsilon \to 0^+} Tr\left(\delta\varphi\, e^{-\varepsilon\overline{\mathcal{L}}_{-(j+1)}}\right)
\end{aligned} \qquad \text{(E-34)}$$

$$\begin{aligned}
&= -(j+1)\left[\int_{R^2} d^2x\,\delta\varphi(x)\left\{\frac{1}{2\pi\varepsilon}e^{\varphi(x)} - \frac{(1+3j)}{12\pi}\Delta\varphi(x)\right\}\right]\Bigg|_{\varepsilon\to 0^+} \\
&+ j\left[\int_{R^2} d^2x\,\delta\varphi(x)\left\{\frac{1}{2\pi\varepsilon}e^{\varphi(x)} + \frac{(2+3j)}{12\pi}\Delta\varphi(x)\right\}\right]\Bigg|_{\varepsilon\to 0^+}
\end{aligned} \qquad \text{(E-35)}$$

Grouping together, we obtain our final "basic-brick" formulae of the quantum geometric path integrals for Random surfaces:

$$\begin{aligned}\delta \ell g \det \mathcal{L}_j = \lim_{\varepsilon\to 0^+}\left(-\frac{1}{2\pi\varepsilon}\right)\left[\int_{R^2} d^2x\,\delta\varphi(x)\,e^{\varphi(x)}\right] \\ + \frac{(1+6j(j+1))}{12\pi}\left[\int_{R^2} d^2x\,\delta\varphi(x)\,\Delta\varphi(x)\right]\end{aligned} \tag{E-36}$$

which produces the "brick" result (A.M. Polyakov)

$$\begin{aligned}\ell g \det \mathcal{L}_j = \left[\lim_{\varepsilon\to 0^+}\left(-\frac{1}{2\pi\varepsilon}\right)\int_{R^2} d^2x\,e^{\varphi(x)}\right] \\ \left[-\frac{1+6j(j+1))}{12\pi}\int_{R^2} d^2x\left\{\frac{1}{2}\,(\partial_a\varphi)^2(x)\right\}\right]\end{aligned} \tag{E-37}$$

The result in the presence of gauge fields can be obtained through bosonization techniques and only will lead to the following additional term to be added to eq. (E-37) in its right-hand side

$$\begin{aligned}\exp\left[-\frac{1}{2\pi}\int_{R^2} d^2x\,e^{\varphi(x)}\left\{A_1(x)A^1(x)+A_2(x)A^2(x)\right\}\right] \\ = \exp\left[-\frac{1}{2\pi}\int (dz\wedge d\overline{z})(A_z\cdot A_{\overline{z}})\right]\end{aligned} \tag{E-38}$$

References

[1] B. Durhuus, Quantum theory of string. Nordita Lectures, (1982).

Chapter 2

Path Integrals Evaluations in Bosonic Random Loop Geometry - Abelian Wilson Loops

2.1. Introduction

In this *somewhat long* chapter we present several basic elementary calculations on the use of Gaussian Euclidean Path Integrals in combination with the previously exposed Bosonic Loop Space for representing Euclidean Gauge Theories in the chapter 1. The main objective of these loop space-path integrals evaluations is to show the usefulness and the computational power of these non-perturbative mathematical techniques to obtain exactly results, otherwise extremely difficult to obtain by another mathematical methods like Feynman Diagrammatics; operatorial perturbative expansions, etc.

The content of this chapter is the following. In the section 2.2, we examine some features of longe-range interactions between electrically neutral systems represented by rectangular Wilson Loops in the presence of a heat reservoir. The temperature independence of the interaction is obtained. In the section 2.3, we present similar path-integrals analysis by evaluating explictly the quark-antiquark static potential in Quantum Chromodynamics $Q.C.D(SU(3))$ by using the Dimensional Regularization scheme in the context of the Mandelstam approximation for the Gluonic interaction. We obtain its charge confining behavior in oppostion to those non-confining of the section 2.2. In the section 2.4, we present path-integrals studies - based on the previous sections on the problem of confinement in the presence of fermiomic and scalar magnetic monopole fields.

2.2. Abelian Wilson Loop Interaction at Finite Temperature

a) Introduction

The analysis of the interaction of neutral colour states in non-abelian quantum gauge theories at zero temperature has revealed the existence of long-range forces, like van der Waals forces in atomic and molecular physics [1],[2],etc. On the other hand, it is well-known that the introduction of a heat reservoir can modify the zero-temperature physical phenomena.

In this section we analyse these long-range interactions in the simple case of a quantized electromagnetic field in contact with a heat reservoir by computing the interaction of electrically neutral systems represented by rectangular Wilson loops by means of elementary path integrals evaluations.

Our conclusion concerns the temperature independence of these long-range forces in these simple path examples in Quantum Field Theory, otherwise difficult result to be obtained in the operatorial framework.

b) Wilson Loop Evaluation at Zero-Temperature

We consider a neutral system simulated as an external current circulating around a rectangle $C_{(R,T)}$.

The interaction energy between two such neutral sources separated by a a space-like distance h is computed by evaluating the vacuum energy of the quantized electromagnetic field in the presence of these sources and then subtracting off their self-energies

$$E(h) = \lim_{T\to\infty} -\frac{1}{2T}\log\left[\frac{\langle\exp(ie\oint_{C^{(1)}_{(R,T)}} A_\mu dx_\mu)\exp(ie\oint_{C^{(2)}_{(R,T)}} A_\mu dx_\mu)\rangle}{\langle\exp(ie\oint_{C^{(1)}_{(R,T)}} A_\mu dx_\mu)\rangle\langle\exp(ie\oint_{C^{(2)}_{(R,T)}} A_\mu dx_\mu)\rangle}\right] \tag{2.1}$$

where the rectangle $c^{(2)}_{(R,T)}$ is translated through the distance h from the rectangle $C^{(1)}_{(R,T)}$ along its spatial direction. The factor 2 in eq. (2.1) prevents the double counting of the interaction energy.

The quantum average $,<\ \ >$ in eq. (2.1) is defined by the Euclidean generating fuctional of the quantized electromagnetic field, an Gaussian exactly soluble path integral

$$\langle 0(A_\mu) = \int d[A_\mu(x)]\exp\left(-\frac{1}{4}\int d^D x F^2_{\mu\nu}\right)0(A_\mu) \tag{2.2}$$

where $0(A_\mu)$ denotes an observable, and $D[A_\mu(x)]$ is the appropriately normalized functional measure $(<1>=1)$ including gauge fixing terms. We call attention to the usefulness of the representation of neutral objects by Wilson loops, since eq. (2.1) manifestly exhibits the gauge-invariance of the calcuation, a result impossible to be achieved in others quantum field theoretical calculations schemes.

In order to evaluate eq. (2.1) it is convenient to express the Wilson loops by means of external currents $J_\mu(x;C^{(i)}_{(R,T)})$ circulating around the contours $C^{(i)}_{(R,T)}$ parametrized by $x^{(i)}_\mu = x^{(i)}_\mu(s)$ with $i=1,2$ ([6],[7])

$$J_\mu(x,C^{(i)}_{(R,T)}) = ie\oint_{C^{(i)}_{(R,T)}} \delta^{(D)}(x_\mu - x^{(i)}_\mu(s))(\mu=0,1;\dots;D-1) \tag{2.3}$$

The interaction energy in eq.(2.1) can be exactly evaluated, as the euclidean functional integrals involved are of the Gaussian type, as it has been observed thus giving the following result:

$$E(h) = \lim_{T\to\infty} -\frac{1}{T}\log\left[\exp\{\frac{1}{2}\int d^D x d^D y J_\mu(x,C^{(1)}_{(R,T)})\Delta^{(E)}_{\mu\nu}(x-y)J_\nu(y;C^{(2)}_{(R,T)})\}\right] \tag{2.4}$$

$$\Delta^{(E)}_{\mu\nu}(x-y) = \delta_{\mu\nu} \int \frac{d^D k}{(2\pi)^D} e^{-ik\cdot(x-y)} \cdot \frac{1}{k^2}$$

$\Delta^{(E)}_{\mu\nu}(x-y)$ means the associated (Euclidean) Feynman propagator.

The evaluation of eq.(2.4) can be accomplished by writing it in momentum space

$$E(h) = \lim_{T\to\infty} -\frac{1}{2T}\left[\int \frac{d^D k}{(2\pi)^D} f_\mu(k;C^{(1)}_{(R,T)}) \frac{\delta_{\mu\nu}}{k^2} f_\nu(-k;C^{(2)}_{(R,T)})\right] \tag{2.5}$$

with

$$f_\mu(k;C^{(i)}_{(R,T)}) = ie \oint_{C^{(i)}_{(R,T)}} e^{-ik_\alpha \cdot x_\alpha(s)} dx_\mu(s), \ (\alpha,\mu = 0,1,\dots D-1) \tag{2.6}$$

As the rectangles $C^{(i)}_{(R,T)}$ are contained in a two-dimensional sub-space of the space-time R^D, we can decompose the vector $\vec{k}$ as $\vec{k} = k_0\vec{e}_0 + k_1\vec{e}_1 + \hat{k}$, where $\hat{k}$ is the projection of $\vec{k}$ over the sub-space perpendicular to the sub-space $\{\vec{e}_0;\vec{e}_1\}$ containing $C^{(i)}_{(R,T)}$. In addition, the space coordinate system is chosen so that the x -axis direction coincides with the one defined by the spatial sides of the rectangles $C^{(i)}_{(R;T)}$. This coordinate choice implies the validity of the following relations between the contour-functionals in eq. (2.6)

$$f_0(k;C^{(2)}_{(R,T)} = e^{-ik_1\cdot h} f_0(k;C^{(1)}_{(R,T)})$$

and

$$f_1(k;C^{(2)}_{(R,T)} = e^{-ik_1\cdot h} f_1(k;C^{(1)}_{(R,T)}) \tag{2.7}$$

A simple evaluation of eq.(2.6) provides the solutions

$$f_0(k;C^{(1)}_{(R,T)} = -\frac{4e}{k_0}\sin(\frac{k_0 T}{2})\sin(\frac{k_1 R}{2})$$

and

$$f_1(k;C^{(1)}_{(R,T)} = \frac{4e}{k_1}\sin((\frac{k_0 T}{2})\sin(\frac{k_1 R}{2}) \tag{2.8}$$

Inserting eqs.(2.7) and (2.8) into eq.(2.5), we obtain

$$E(h) = \lim_{T\to\infty} +\frac{8e^2}{T}\left\{\int_{-\infty}^{+\infty} \frac{dk_1}{(2\pi)} e^{-ik_1\cdot h} \frac{\sin^2(\frac{k_1 R}{2})}{k_1^2}\right.$$
$$\left.\left[\int \frac{d^{D-2}\hat{k}}{(2\pi)^{D-2}}\left(\int_{-\infty}^{+\infty} \frac{dk_0}{(2\pi)} \frac{(k_0^2+k_1^2)}{k_0^2} \frac{1}{(k_0^2+k_1^2+\hat{k}^2} \sin^2(\frac{k_0 T}{2})\right)\right]\right\} \tag{2.9}$$

The integration in k_0 -variable is easily performed by using the formulas 3.824-1 and 3.826-1 from Ref. 8. After taking the limit $T^{\to\infty}$, we get

$$E(h) = 2e^2\left[\int_{-\infty}^{+\infty} \frac{dk_1}{(2\pi)} e^{-ik_1\cdot h} \sin^2(\frac{k_1\cdot R}{2})\left(\int \frac{d^{D-2}\hat{k}}{(2\pi)^{D-2}} \frac{1}{(k_1^2+\hat{k}^2)}\right)\right] \tag{2.10}$$

In order to calculate eq.(2.10) we use the dimensional regularization scheme [9]. By making use of the relation (3.8) from Ref. 9 (analytically continued to Euclidean space-time) we can perform the integration in $\hat{k}$-variable

$$E(h) = -\frac{e^2\Gamma(2-\frac{D}{2})}{2^{D-1}\pi^{(D-2)/2}}\left[\int_{-\infty}^{+\infty}\frac{dk_1}{(2\pi)}|k_1|^{D-4}\left(e^{-ik_1(R+h)}+e^{-ik_1\cdot(R-h)}-e^{-ik_1h}\right)\right] \quad (2.11)$$

The Fourier transforms in eq. (2.11) are tabulated [10] in the form

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i\alpha\cdot x}|x|^{\beta}dx = -2\sin\frac{\beta\pi}{2}\cdot\Gamma(\beta+1)|\alpha|^{-\beta-1} \quad (2.12)$$

Finally, we obtain the expression for the interaction energy between Wilson loops at zero temperature

$$E(h) = \frac{e^2\Gamma(2-\frac{D}{2})\Gamma(D-3)}{2^{D-1}\cdot\pi^{D/2}}\sin((D-4)\frac{\pi}{2})\{(h+R)^{-D+3}+(h-R)^{-D+3}-2h^{-D+3}\} \quad (2.13)$$

In order to study eq. (2.13) for the physical limit $D=4$ we note that the pole of the gama-function $\Gamma(2-\frac{D}{2})$ cancels the zero of the sine function $\sin(D-4)\frac{\pi}{2}$, namely

$$\lim_{D\to 4}\Gamma(2-\frac{D}{2})\sin(D-4)\frac{\pi}{2} = -\pi \quad (2.14)$$

which provides the four-dimensional interaction energy as a multiple expansion

$$E(h) = \frac{e^2}{8\pi}\{(h+R)^{-1}+(h-R)^{-1}-2\cdot h^{-1}\} = -\frac{e^2}{4\pi}\left(\sum_{k=1}^{\infty}\frac{R^{2h}}{h^{2k+1}}\right) \quad (2.15)$$

From eq. 2.(15) we readily observe that the dominant term in the asymptotic limit $h\to\infty$ comes from the classical dipole-dipole interaction. Furthermore, the interaction is atractive since the dipolar moments of the neutral systems analysed are parallel.

For completeness we have evaluated the static potential of two sources by using Wilson loops and the dimensional regularization scheme, which is

$$V(R) = \lim_{T\to\infty} -\frac{1}{T}\log\langle\exp(ie\oint_{C_{(R,T)}}A_\mu dx_\mu)\rangle \quad (2.16)$$

The result yields the $(D-1)$ *Dimensional Coulomb law*

$$V(R) = \frac{e^2\Gamma(2-\frac{D}{2})}{\pi^{D/2}2^{D-2}}\sin\left(\frac{(D-4)\pi}{2}\right)\Gamma(D-3)|R|^{-D+3} \quad (2.17)$$

where we have used the dimensional regularization rule which assigns the value zero to the *tad pole R*-independent integral

$$\frac{e^2\Gamma(2-\frac{D}{2})}{2^{D-2}\pi^{(D-2)D/2}}\int_{-\infty}^{+\infty}\frac{dk_1}{(2\pi)}|k_1|^{D-4} = 0 \quad (2.18)$$

The usual Coulomb law in there dimensions is obtained by taking the physical limit $D\to 4$ in eq. 2.(17), resulting $V(R) = -e^2/4\pi R$.

c) Wilson Loop Evaluation at Non-zero-Temperature

We now examine the presence of a heat reservoir at temperature $T = 1/k_B\beta$ (k_B is the Boltzmann's constant) in the quantum gauge system.

We first evaluate the free energy of two static sources [3]

$$V(R;\beta) = -\frac{1}{\beta}\log\langle\exp\, ie\oint_{C_{(R;\beta)}} A^{\beta}_{\mu}(x)dx_{\mu}\rangle \tag{2.19}$$

where now the rectangle $C_{(R;\beta)}$ has its temporal sides extending from 0 to β. The quantum average $\langle \quad \rangle$ involved in eq.(2.19) is defined by the Euclidean partition functional of the quantized electromagnetic field at temperature T ([11])

$$\langle 0(A^{\beta}_{\mu}(x))\rangle = \int D[A^{\beta}_{\mu}(x)]\exp\,\{\frac{1}{4}\int_0^{\beta} dx^0 \int d^{D-1}\vec{x}(F^2_{\mu\nu})\}\cdot 0(A^{\beta}_{\mu}(x)) \tag{2.20}$$

Here $D[A^{\beta}_{\mu}(x)]$ means the normalized functional measure over all thermal gauge fields $A^{\beta}_{\mu}(x)$ satisfying the periodicity condition

$$A^{\beta}_{\mu}(\vec{x},0) = A^{\beta}_{\mu}(\vec{x},\beta) \tag{2.21}$$

A convenient interpretation for eqs.(2.19) and (2.21) consists in considering that at finite temperature the space-time possesses the topology of a cilinder $S^1\times R^{D-1}$ instead of the usual topology R^D.

The periodicity conditions in eq. (2.21) imply that the Wilson loop contour integration around $C_{(R;\beta)}$ is reduced to the contour integration along their temporal sides only, i.e.

$$\exp\,(ie\oint_{C_{(R;\beta)}} A^{\beta}_{\mu}(x)dx_{\mu}) = \exp\,(ie\int_0^{\beta} A^{\beta}_0(0,\tau)d\tau)\,\exp\,(-ie\int_0^{\beta} A^{\beta}_0(R,\tau)d\tau) \tag{2.22}$$

In order to evaluate eq.(2.19) we express the *Wilson Strings* in eq.(2.22) by means of external localized currents

$$\begin{aligned}\tilde{J}_0(\vec{x},\tau) &= ie[\delta^{(D-1)}(\vec{x}) - \delta^{(D-1)}(\vec{x}-\vec{R})]\\ \tilde{J}_i(\vec{x},\tau) &= 0 \qquad (i = 1;\ldots,D-1)\end{aligned} \tag{2.23}$$

and compute the Gaussian functional integration, yielding the result

$$V(R,\beta) = -\frac{1}{\beta}\left\{\frac{1}{2}\int_0^{\beta} d\tau \int_0^{\beta} d\tau' \int d^{D-1}\vec{x}\, d^{D-1}\vec{y}\, \tilde{J}_{\mu}(\vec{x},\tau)\Delta^{(E)}_{\mu\nu}(\vec{x}-\vec{y},\tau-\tau';\beta)\tilde{J}_{\nu}(\vec{y},\tau')\right\} \tag{2.24}$$

where $\Delta^{(E)}_{\mu\nu}(\vec{x}-\vec{y},\tau-\tau';\beta)$ denotes the thermal Euclidean Feynman propagator in the Feynman gauge [11], namely

$$\Delta^{(E)}_{\mu\nu}(x-y,\tau-\tau',\beta) = \frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\delta_{\mu\nu}\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}}\frac{e^{i\vec{k}\cdot(x-y)+i\omega_n(\tau-\tau')}}{(\vec{k}^2+\omega_n^2)} \qquad (\omega_n = \frac{2\pi n}{\beta}; n\in Z) \tag{2.25}$$

With the orthogonality relations

$$\int_0^\beta d\tau \int_0^\beta d\tau' \, e^{i\omega_n(\tau-\tau')} = \begin{cases} 0 & n \neq 0 \\ \beta^2 & n = 0 \end{cases} \tag{2.26}$$

which suppress the modes with $\omega_n \neq 0$ in eq.(2.25) we simplify eq.(2.24) to the form

$$V(R;\beta) = -e^2 \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \frac{(1-\cos\vec{k}\cdot\vec{R})}{\vec{k}^2} \tag{2.27}$$

We observe in eq.(2.27) the temperature independence of the free energy. Now it is convenient to choose the k_1-axis along the direction vector R. We thus obtain the result

$$V(R;\beta) = -e^2 \left[\int \frac{dk_1}{(2\pi)} \frac{1}{2}(e^{ik_1R} + e^{-ik_1R}) \int \frac{d^{D-2}\hat{k}}{(2\pi)^{D-2}} \frac{1}{(k_1^2+\hat{k}^2)} \right] \tag{2.28}$$

which is evaluated as before (see eqs.(2.11) and (2.12)), giving

$$V(R;\beta) = \frac{e^2}{2^{D-2}\pi^{D/2}} \Gamma(2-\frac{D}{2})\Gamma(D-3)\sin(\left(\frac{D-4}{2}\right)\pi) \cdot |R|^{-D+3} \tag{2.29}$$

From eq.(2.29) we notice its coincidence with the electrostatic potential at zero temperature. (see eq.(2.17)).

Finally, we evaluate the free energy of the previous Wilson loops simulating neutral objects in contact with a heat reservoir at temperature T.

The evaluation of eq.(2.1) is now performed by means of the quantum average furnished in eq.(2.20) and its result in coordinate space reads

$$E(h,\beta) = -\frac{1}{\beta}\Bigg\{ -\frac{e^2}{2}\int_0^\beta d\tau \int_0^\beta d\tau' \int d^{D-1}\vec{x} d^{D-1}\vec{y}$$
$$\left[\delta^{(D-1)}(\vec{x}) - \delta^{(D-1)}(\vec{x}-\vec{R})\right] \Delta_{\mu\nu}^{(E)}(\vec{x}-\vec{y},\tau-\tau',\beta) \left[\delta^{(D-1)}(\vec{y}-\vec{h}) - \delta^{(D-1)}(\vec{y}-\vec{R}-\vec{h})\right]\Bigg\} \tag{2.30}$$

Writing eq.(2.30) in momentum space we obtain the expression

$$E(h,\beta) =$$
$$= -\frac{1}{2}e^2\left\{ \int_{-\infty}^{+\infty} \frac{dk_1}{(2\pi)} \int \frac{d^{D-2}\vec{k}}{(2\pi)^{D-2}} \frac{1}{(\hat{k}^2+k_1^2)} (e^{-k_1\cdot(R+h)} + e^{-k_1\cdot(R-h)} - 2e^{-ik_1\cdot R}) \right.$$
$$= \frac{e^2\Gamma(2-\frac{D}{2})\Gamma(D-3)}{2^{D-1}\pi^{D/2}} \sin[(D-4)\frac{\pi}{2}] \left\{ (h+R)^{-D+3} + (h+R)^{-D+3} - 2h^{-D+3} \right\} \tag{2.31}$$

The above result result clearly shows that the free energy interaction of neutral systems repesented by rectangular Wilson loops is temperature independent and turns out to be of the same form as the corresponding quantity in the zero temperature regime (see eq.(2.13)).

We now make some concluding remark on the results in eqs.(2.29) and (2.31). We understand that these results imply that to detect the temperature effects in the interactions analysed above, one should consider the matter fields in the quantum system (quantum electrodynamics) since in this case the radiative corrections induced on the N-point photon propagator are temperature dependent, which results in an renormalized temperature dependent electronic charge $e(R;\beta)$ in the interactions in eqs.(2.29) and (2.31)

We left to our readers the introduction of the matter fields in the interactions analysed in this section 2.2.

2.3. The Static Confining Potential for Q.C.D. in the Mandelstam Model through Path Integrals

a) Introduction

One of the still unsolved problem in the Gauge theory for strong interactions as given by Quantum Chromodynamics with gauge group $SU(3)$ is to produce arguments for the color charge confinement of the related field excitations ([12]).

A long time ago ([13]), it was argued by S. Mandelstam through a somewhat intrincate non-perturbative analysis of the Q.C.D. Schwinger-Dyson equations that one should use as a first approximation for the small momenta (infrared regime) of the non-abelian quantum Yang-Mills path measure, including its non-perturbative aspects, an effective (somewhat phenomenological) purely abelian Gluonic action but with a free effective propagator already including the sum of a certain class of relevant Feynman diagrams for Gluons color-charge exchange. It was conjectured that the use of this scheme would be suitable if such an effective dynamics led directly to the color confinement.

It is the purpose of this section to evaluate the static potential between two statics charges with opposite signal on the above mentioned Mandelstam effective Gluon theory and show exactly its envisaged color-charge confining property; a basic physical requirement to use directly continuum Q.C.D. with improved Mandelstam-Feynman diagrammatics, at least on the level of Dyson-Schwinger equations as earlier proposed on ref [13] by S. Mandelstam.

b) The Wilson Loop in the Mandelstam Model

We start our analysis by considering the (Euclidean) Effective Mandelstam Gluonic action written in terms of a path-integral in a ν-dimensional space-time R^ν

$$Z = \int D^F[A_\mu(x)] \exp\left\{-\frac{1}{2}\int d^\nu x\, d^\nu y\, A_\mu(x)\, D_m(x-y) A_\mu(y)\right\} \tag{2.32}$$

where the Mandelstam (free) propagator with logarithmic term is given explicitly by the Fourier transform on the (Tempered) Schwartz distributional space

$$D_m(x-y) = \frac{1}{(2\pi)^\nu}\int d^\nu p\, e^{ip(x-y)} \frac{\ell g(|p|^{2\alpha})}{|p|^4} \tag{2.33}$$

with α a positive model resummation constant (including factor index groups, etc...). (see ref [13])

The static potential between a quark and an anti-quark in the Feynman picture for particle propagations in the space-time is given by the vacuum Gluonic energy as given by eq.(2.32), but in presence of the above spatial-static charges. This vacuum energy of such charges separated by a space-like distance R is computed by evaluating the temporal (ergodic) limit ([12,13,14]).

$$V(R) = \lim_{T\to\infty} -\frac{1}{T}\ell g \left[\left\langle \exp ie \oint_{C_{(R,T)}} A_\mu dx_\mu \right\rangle_A \right] \tag{2.34}$$

where the rectangle $C_{(R,T)}$ is the Feynman trajectory of the neutral pair in the space-time and the Mandelstam Gluonic normalized average as represented by the operation $\langle\ \rangle_A$ is given explicitly by the Gaussian path integral eq.(2.32).

In order to evaluate the static potential eq.(2.34) it is convenient to re-write the Wilson loop inside eq.(2.34) by means of an external current $J_\mu(x;C_{(R,T)})$ circulating around the pair finite-time propagation space-time trajectory $C_{(R,T)} = \{x_\mu(s)\}$, namely ([14])

$$J_\mu(x;C_{(R,T)}) = ie \oint_{C_{(R,T)}} \delta^{(\nu)}(x_\mu - x_\mu(s))dx_\mu(s) \tag{2.35}$$

The Gaussian path integral eq.(2.34) can be exactly evaluated and yielding the following result

$$V(R) = \lim_{T\to\infty} -\frac{1}{T}\ell g \left[\exp\left\{ +\frac{1}{2}\int d^\nu x d^\nu y J_\mu(x;C_{(R,T)})D_m(x-y)J^\mu(y,C_{(R,T)}) \right\} \right] \tag{2.36}$$

The evaluation of eq.(2.36) can be accomplished by writing it in momentum space

$$V(R) = \lim_{T\to\infty} -\frac{1}{T} \left[\int \frac{d^\nu p}{(2\pi)^\nu} f_\mu(p_\alpha, C_{(R,T)}) \times \frac{\alpha \ell g(p^2)}{p^4} \times f^\mu(-p_\alpha, C_{(R,T)}) \right] \tag{2.37}$$

with the contour form factors

$$f_\mu(p_\alpha, C_{(R,T)}) = ie \int_{C_{(R,T)}} e^{-ip_\mu x_\mu(s)} dx_\mu(s) \tag{2.38}$$

A simple evaluation of eq.(2.38) provides the solutions

$$f_0(p, C_{(R,T)}) = -\frac{4e}{p_0}\sin\left(\frac{p_0 T}{2}\right)\sin\left(\frac{p_1 R}{2}\right) \tag{2.39}$$

and

$$f_1(p, C_{(R,T)}) = +\frac{4e}{p_1}\sin\left(\frac{p_0 T}{2}\right)\sin\left(\frac{p_1 R}{2}\right) \tag{2.40}$$

After inserting the contour form factors eq.(2.39), eq.(2.40) into eq.(2.37), we obtain as a result

$$V(R)=\lim_{T\to\infty}\frac{1}{T}\left\{16e^2\alpha\int_{-\infty}^{+\infty}\frac{dp_1}{(2\pi)}\frac{\sin^2\left(\frac{p_1R}{2}\right)}{p_1^2}\right.$$
$$\left.\times\left[\int_{-\infty}^{+\infty}\frac{d^{\nu-2}\hat{p}}{(2\pi)^{\nu-2}}\left(\int_{-\infty}^{+\infty}\frac{dp_0}{(2\pi)}\frac{(p_0^2+p_1^2)}{p_0^2}\sin^2\left(\frac{p_0T}{2}\right)\times\frac{\ell g(p_0^2+p_1^2+\hat{p}^2)}{(p_0^2+p_1^2+\hat{p}^2)^2}\right)\right]\right\}. \tag{2.41}$$

Note that we have considered the pair spatial-static trajectory $C_{(R,T)}$ contained in a two-dimensional sub-space of the (Euclidean) space-time R^ν in a such way that we can decompose the vector $\vec{p}\in R^\nu$ as $\vec{p}=p_0\vec{e}_0+p_1\vec{e}_1+\hat{p}$, where $\hat{p}$ denotes the projection of $\hat{p}$ over the sub-space perpendicular to the sub-space $\{\vec{e}_0,\vec{e}_1\}$ containing the square $C_{(R,T)}=\{(x_0,x_1);-\frac{T}{2}\le x_0\le+\frac{T}{2};-\frac{R}{2}\le x_1\le+\frac{R}{2}\}$.

The ergodic limit of $T\to\infty$ and the p_0-integration is easily evaluated through the use of the Distributional limit

$$\lim_{T\to\infty}\frac{\sin^2\left(\frac{p_0T}{2}\right)}{p_0^2T}=2\pi\,\delta(p_0) \tag{2.42}$$

As a consequence we get the result

$$V(R)=16e^2\alpha\left[\int_{-\infty}^{+\infty}\frac{dp_1}{(2\pi)}\cdot\frac{\sin^2\left(\frac{p_1R}{2}\right)}{p_1^2}\times\int\frac{d^{\nu-2}\hat{p}}{(2\pi)^{\nu-2}}\cdot\frac{\ell g(p_1^2+\hat{p}^2)}{(p_1^2+\hat{p}^2)^2}\right] \tag{2.43}$$

Let us analyze the $(D-2)-\hat{P}$ dimensional integration. In order to evaluate such integral, we use the well-known formulae (from I.S. Gradshteyn & I.M. Ryzhik table of integrals – page 558 – eq.(14) – Academic Press – 1980.

$$\int\frac{d^{\nu-2}\hat{p}}{(2\pi)^{\nu-2}}\cdot\frac{\ell g(p_1^2+\hat{p}^2)}{(p_1^2+\hat{p}^2)^2}$$
$$=(\pi)^{\frac{\nu-2}{2}}\left\{\frac{\Gamma\left(\frac{6-\nu}{2}\right)}{\Gamma(2)}(|p_1|)^{\nu-6}\right\}$$
$$\times\left(\psi(2)-\psi(3-\frac{\nu}{2})+2\,\ell n(|p_1|)\right) \tag{2.44}$$

For the evaluation of the final p_1-integration we use the well-known Gelfand results of the Fourier Transform of Tempered (Finite-part) Distributions ([15]).

$$\sin^2\left(\frac{k_1R}{2}\right)=-\frac{1}{4}(e^{k_1R}+e^{-k_1R}-2) \tag{2.45}$$

and

$$\int_{-\infty}^{+\infty}e^{ip_1R}|p_1|^\beta\,dp_1=-2\sin\left(\frac{\beta\pi}{2}\right)\Gamma(\beta+1)|p|^{-\beta-1} \tag{2.45-a}$$

with

$$\int_{-\infty}^{+\infty} e^{ip_1R}|p_1|^{\beta}\ell n(|p_1|)dp_1$$
$$= i e^{i\beta\frac{\pi}{2}}\left\{\left[\Gamma'(\beta+1)+\frac{i\pi}{2}\Gamma(\beta+1)\right](|R|+i\varepsilon)^{-\beta-1}-\Gamma(\beta+1)(|R|+i\varepsilon)^{-\beta-1}\cdot\ell n(|R|+i\varepsilon)\right\}$$
$$- i e^{-i\beta\frac{\pi}{2}}\left\{\left[\Gamma'(\beta+1)-\frac{i\pi}{2}\Gamma(\beta+1)\right](|R|-i\varepsilon)^{-\beta-1}-\Gamma(\beta+1)(|R|-i\varepsilon)^{-\beta-1}\cdot\ell n(|R|-i\varepsilon)\right\} \tag{2.45-b}$$

By passing to the Physical limit of $\nu \to 4$ and noting that the pole of the Gamma function cancels out either with the sinus zero for $\nu \to 4$, namely.

$$\lim_{\nu\to 4}\sin\left(\frac{\pi}{2}(\nu-6)\right)\Gamma(\nu-4-1)$$
$$\sim -\frac{1}{(\nu-5)}\Gamma(\nu-4)\cdot\sin\left(\frac{\pi}{2}(\nu-4)\right)$$
$$= +\pi \tag{2.46}$$

We obtain, thus, the finite result for the static quark-antiquark potential in the Mandelstam Gluonic effective theory on the physical space-time R^4.

$$V(R) = (e^2\alpha)\cdot\bar{c}\cdot|R|(1+\ell n(|R|)) \tag{2.47}$$

Here $\bar{c}$ denotes a positive constant which depends on the Fourier Transform normalization factors, etc...

We see, thus, that the Effective Gluonic Mandelstam theory leads in a very natural way to a quark-antiquark confining potential and not to a dynamics of charge color screening as it would be expected in a first analysis ([12]). This is the main result of this section.

At this point, it is worth remark that if one has added to the logarithmic propagator eq.(2.33) a pure quartic term of the following form

$$\tilde{D}_m(x-y) = \frac{1}{(2\pi)^{\nu}}\int d^{\nu}p\, e^{ip(x-y)}\cdot\frac{1}{|p|^4} \tag{2.48}$$

one obtains the same result as given by eq.(2.47) without the logarithmic term.

Another important point to be called the reader's atention is that if one tries to evaluate the self-energy of the quark propagator with the effective Mandelstam propagator eq.(2.33), namely

$$\Sigma(p) \sim e^2\int\frac{d^{\nu}k}{(2\pi)^{\nu}}\left(\frac{\gamma_{\mu}(\not{p}-\not{k})\gamma_{\mu}}{(p-k)^2}\right)\frac{\ell g(k^2)}{k^4}$$
$$= 3\int_0^1 dx(1-x)\left\{\int\frac{d^{\nu}k}{(2\pi)^{\nu}}\left[\frac{((1-x)\ \not{p}-\not{k})\ell g((k+xp)^2)}{\{k^2+x(1-x)p^2\}^3}\right]\right\} \tag{2.49}$$

with the power series expansion for the logarithmic term in eq.(2.49) as given below

$$\ell g(k^2+x^2p^2+2(x\cdot p)x) = \ell g(k^2)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\frac{(2x(k\cdot p)+x^2p^2)}{(k)^2}\right]^n \tag{2.50}$$

one should arrives at the standard Mandelstam behavior after tedious calculations.

$$\Sigma(p) \sim \not{p} \left[\frac{A + B\ell g(p^2)}{p^4} \right] \tag{2.51}$$

with A and B constant p-independent, (including possible divergences at $\nu \to 4$!).

As a consequence one see that the quark-antiquark propagator should have a behavior of the form (in the Euclidean world)

$$\begin{aligned} G_{ij}^{\mu\nu}(x-y) &= \left\langle 0 \middle| T(\psi_i^\mu(x)\overline{\psi}_j^\nu(y)) \middle| 0 \right\rangle_{\text{Eucl.}} \\ &\sim \int d^4p \, \frac{(p^2) \, \not{p} e^{ip(x-y)}}{p^4 + B\ell g(p^2) + A} \end{aligned} \tag{2.52}$$

signaling again that at $p^2 \to 0^+$ (the L.S.Z´s asymptotic limit) we find branch-cuts instead of mass-physical poles. This indicates again that it is a completely ill-defined process to apply L.S.Z´s framework to Quarks and Gluons since the quark field excitations are not physically-quantum mechanical observable. This leads one to consider only composite operators from the very beginning, as Mandelstam did in ref. [13], in order to apply correctly the L.S.Z' Quantum Field Methods, even at the higher momenta region.

c) The Two-Dimensional Mandelstam-Schwinger Model: Its Chiral Path-Integral Bosonization

It is well-known that two-dimensional models has proved to be a useful theoretical laboratory to understand difficult dynamical features expected to be present in four-dimensional quantum chromodynamics. It is the purpose of this part c) to complement the analysis of confining of four-dimensional dynamical fermions in the infrared leading approximate Mandelstam model of part b) by means of a higher-derivative exactly soluble two-dimensional model.

Let us start this section by writing the (Euclidean) Hermitian Lagrangean of our proposed higher-derivative two-dimensional model

$$\begin{aligned} \mathcal{L}_\mu(\psi,\overline{\psi},A_\mu) &= (\psi,\overline{\psi}) \left\{ \begin{matrix} 0 & (\not{D}_A \not{D}_A^*)^\mu \not{D}_A \\ \not{D}_A^*(\not{D}_A^* \not{D}_A)^\mu & 0 \end{matrix} \right\} \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} \\ &\quad + \frac{1}{2} F_{\mu\nu}^2(A) + (\psi,\overline{\psi}) \begin{pmatrix} \eta \\ \overline{\eta} \end{pmatrix} \end{aligned} \tag{2.53}$$

where $(\psi,\overline{\psi})$ denotes the (independent Euclidean fermion fields two-dimensional) quarks; A_μ the usual (confining) two-dimensional eletromagnetic field with a quartic propagator on the Landau Gauge (see below) and $\not{D}_A$ is the (Euclidean) Dirac operator in the presence of this 2D quantum Gauge field. The Dirac γ matrices algebra we are using satisfy the relations

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu} \ , \ \gamma_\mu\gamma_5 = i\varepsilon_{\mu\nu}\gamma_\nu \ ; \ \gamma_5 = i\gamma_0\gamma_1 \tag{2.54}$$

Note that this γ-matrices algebra is choosen in a such way that the Dirac operator $\not{D}_A$ may by written in the chiral-phase form when one considers the general Hodge decomposition of the two-dimensional electromagnetic field.

$$A_\mu = \varepsilon_{\mu\nu} \, \partial_\nu \, \varphi + \partial_\mu \, \rho \tag{2.55}$$

$$\not{D}_A = e^{ig\rho}\, e^{ig\gamma_5\varphi} \cdot (\not{\partial}) e^{-ig\rho}\, e^{ig\gamma_5\varphi} \tag{2.56}$$

Here μ is a free-parameter ranging on the interval $[1,\infty)$ (eq.(2.53)).

Let us consider the associated path-integral expression for the 2D-quantum higher derivative model eq.(2.53) in the fermion sector.

$$Z[\eta,\overline{\eta}] = \frac{1}{Z(0,0)} \int D^F\psi D^F\overline{\psi}\, DA_\mu \times \exp\left(-\int d^2x \mathcal{L}_\mu(\psi,\overline{\psi},A_\mu)(x)\right) \tag{2.57}$$

In order to solve exactly the two-dimensional path-integral eq.(2.57) by means of the Gauge invariant Bosonization technique, we consider the change of variable on the field dynamics

$$A_\mu(x) = (\varepsilon_{\mu\nu}\,\partial_\nu)\varphi(x) \tag{2.58}$$

$$\psi(x) = e^{-ig\gamma_5\varphi(x)}(-\Delta)_x^{-\mu}\chi(x) \tag{2.59}$$

$$\overline{\psi}(x) = \overline{\chi}(x)\, e^{-ig\gamma_5\varphi(x)} \tag{2.60}$$

It is worth call the reader attention that in the Euclidean world $\overline{\psi}(x)$ is an independent field of $\psi(x)$, opposite in Minkowisky space where $\overline{\psi}(x) = (\psi^*(x))^T\gamma^0$. That is the reason about the difference between eq.(2.59) and eq.(2.60).

At the quantum level of the path measures we have the non-trivial jacobians (see refs. [16]) physically related to the dynamical breaking of the models axial (chiral) symmetry, namely

$$D^F[A_\mu(x)] = \det(-\Delta)\cdot D^F[\varphi(x)] \tag{2.61}$$

$$\begin{aligned}
D^F[\psi(x)]\,D^F[\overline{\psi}(x)] &= \frac{\det[(\not{D}_A\cdot \not{D}_A^*)^\mu\, \not{D}_A]}{\det[\not{\partial}]} D^F[\chi(x)]D^F[\overline{\chi}(x)] \\
&= \frac{\det[(\not{D}_A\, \not{D}_A^*)^\mu(\not{D}_A\, \not{D}_A^*)^{1/2}]}{\det(\not{\partial})} D^F[\chi(x)]D^F[\overline{\chi}(x)] \\
&= \left\{\frac{\det[(\not{D}_A\, \not{D}_A^*)^{\mu+\frac{1}{2}}]}{\det[\not{\partial}\,\not{\partial}^*]^{\mu+\frac{1}{2}}} \times \det(\not{\partial}\,\not{\partial}^*)^\mu\right\} D^F[\chi(x)]D^F[\overline{\chi}(x)] \\
&= \left\{\left(\det\left[\frac{\not{D}_A\,\not{D}_A^*}{\not{\partial}\,\not{\partial}^*}\right]\right)^{\mu+\frac{1}{2}} \times (\det(-\Delta))^\mu\right\} D^F[\chi(x)]D^F[\overline{\chi}(x)]
\end{aligned} \tag{2.62}$$

After implementing equations (2.58) - (2.62) on the fermionic generating functional eq.(2.57), we obtain the Bosonized associated model, where one can evaluate exactly all the models correlation field functions.

$$\begin{aligned}
Z[\eta,\overline{\eta}] = \frac{1}{Z(0,0)} &\int D[\varphi(x)]D^F[\chi(x)]D^F[\overline{\chi}(x)] \\
&\times \exp\left\{-\frac{g^2}{\pi}(\mu+\frac{1}{2})\int d^2x \left(\frac{1}{2}(\partial\varphi)^2\right)(x)\right\}
\end{aligned}$$

$$\times \exp\left\{-\frac{1}{2}\int d^2x((\partial^2\varphi)^2)(x)\right\}$$
$$\times \exp\left\{-\frac{1}{2}\int d^2x\left((\chi,\overline{\chi})\begin{bmatrix}0 & \vec{\partial}\\ \overleftarrow{\partial}^* & 0\end{bmatrix}\begin{pmatrix}\chi\\ \overline{\chi}\end{pmatrix}\right)(x)\right\}$$
$$\times \exp\left\{\int d^2x\left[\left(\overline{\eta}e^{-ig\gamma_5\varphi}(-\Delta)^{-\mu}\chi\right)(x)+(\overline{\chi}e^{-ig\gamma_5\varphi}\eta)(x)\right]\right\} \tag{2.63}$$

It is important to remark that we have used the basic identity below to arrive at eq.(2.63) with α a real positive parameter and used throughout on the formulae

$$\begin{aligned}(\not{D}_A\,\not{D}_A^*)^\alpha\,\not{D}_A &= e^{ig\gamma_5\varphi}[(\not{\partial}\not{\partial}^*)^\alpha\,\not{\partial}]e^{ig\gamma_5\varphi}\\ &= e^{ig\gamma_5\varphi}((-\Delta)^\alpha\,\not{\partial})e^{ig\gamma_5\varphi}\end{aligned} \tag{2.64}$$

It is important point out that the part of the Lagrangean with Fermions sources in the new field parametrization are not symmetric in its form as that of eq.(2.53) in the old field parametrization as a consequence of our asymmetric change of variable in the (independent in the Euclidean world!) two-dimensional quarks fields.

Finally we have the explicitly expression for our Fermion propagator in terms of the free-propagators of the Bosonized theory

$$\left\langle\psi(x)\overline{\psi}(y)\right\rangle=(-\Delta)_x^{-\mu}\left\{\left\langle\chi(x)\overline{\chi}(y)\right\rangle^{(0)}\times\right.$$
$$\left.\exp\left\{-\frac{1}{2}g^2\left[\frac{\pi}{g^2(\mu+\frac{1}{2})}\left((-\partial^2)^{-1}(x.y)-(-\partial^2+\frac{g^2}{\pi}(\mu+\frac{1}{2}))^{-1}(x,y)\right)\right]\right\}\right. \tag{2.65}$$

Here

$$\left\langle\chi_\alpha(x)\overline{\chi}_\beta(y)\right\rangle^{(0)}=\frac{1}{2\pi}(\gamma_\mu)_{\alpha\beta}\frac{(x_\mu-y_\mu)}{|x-y|^2} \tag{2.66}$$

and

$$(-\partial^2)^{-1}(x,y)=-\frac{1}{2\pi}\ell g|x-y| \tag{2.67}$$

$$(-\partial^2+\frac{g^2}{\pi}(\mu+\frac{1}{2}))^{-1}(x.y)=\frac{1}{2\pi}K_0\left(\sqrt{\frac{g^2}{\pi}(\mu+\frac{1}{2})}|x-y|\right) \tag{2.68}$$

Note that we have used the general decomposition in eq.(2.65)

$$(a(-\partial^2)^2+b(-\partial^2))^{-1}(x,y)=\frac{1}{b}\left\{-\frac{1}{2\pi}\ell g|x-y|-\frac{1}{2\pi}K_0\left(\sqrt{\frac{b}{a}}|x-y|\right)\right\} \tag{2.69}$$

The short-distance behavior of the fermion propagator is strong than the usual free case by a μ-power derivative (strong asymptotic freedom).

$$\lim_{|x-y|\to 0}\left\langle\psi(x)\overline{\psi}(y)\right\rangle\sim(-\Delta)_x^{-\mu}\left\langle\chi_\alpha(x)\overline{\chi}_\beta(y)\right\rangle \tag{2.70}$$

The long-distance behavior by its turn is exactly given by

$$\lim_{|x-y|\to\infty} \langle \psi(x)\overline{\psi}(y)\rangle \sim$$
$$\lim_{|x-y|\to\infty} \left\{ (-\Delta)_x^{-\mu} \left[\left\langle \chi(x)\,\overline{\chi}(y) \right\rangle^{(0)} \times |x-y|^{|4\mu+2|} \right] \right\} \qquad (2.71)$$

which shows an anomalous behavior in the infra-red limit and signaling the impossibility to use L.S.Z interpolating fields for the 2D fermion fields as similar phenomenon in the Mandelstam model analyzed on part b).

Anyway it is a straightforward procedure the exactly computation of all fermionic correlation function of the higher derivative model eq.(2.63) as in last references of ref. [16].

d) Color Charge Screening in the Mandelstam Model

Sometimes it is argued that it is important to realize that the absence of coulored states in the expected nuclear strong force theory of Quantum Chromodynamics may not be equivalent to the ethernal quark-gluon confinement as showed by us in the Effective Abelian Gluon Mandelstam model analyzed in part b) by an explicitly Wilson Loop evaluation.

The absence of color charged states can still be a result of these color quantum numbers just screened by the quark-antiquark pairs creation on the presence of the Gluon field and leading, thus, to the physical picture that the test charges (a static pair!) are surrounded by a cloud of quark-antiquark pairs playing the role of plasmons. It is, thus, expected that the resulting Wilson loop colorless object of part b) no longer leads to a rising linear confining potential as showed on that section, but rather to an exponentially falling potential characterizing the short range screened strong interactions like similiar screening phenomena in two-dimensional Q.E.D. (see part b)) for the case of $\mu = 0$).

In this section we intend to show such screening phenomena by an explicitly calculation in the above mentioned four-dimensional Effective Gluon Mandelstam model by considering the existence of totally reflecting walls on the point $z = 0$ and $z = a$ of the space-time which turns out to be of the cylindrical form $R^{\nu-1} \times [0,a]$. We further impose Dirichlet boundary conditions on the "effective" abelian Gluonic Mandelstam field at the walls $z = 0$ and $z = a$. Its propagator, thus, posseses the following analytical expression on momentum space by taking into account explicitly the above pointed out Boundary condition

$$G((\vec{r},z,t);(\vec{r}',z',t')) = \sum_{m=0}^{\infty} \left\{ \int_{-\infty}^{+\infty} \frac{d^{\nu-2}\vec{p}}{(2\pi)^{\nu-2}} \cdot \frac{dp_0}{(2\pi)} e^{-i\vec{p}(\vec{r}-\vec{r}')} e^{+ip_0(t-t')} \right.$$
$$\left. \times \sin\left(\frac{m\pi}{a}z\right) \sin\left(\frac{m\pi}{a}z'\right) \times [p_0^2 + \overline{p}^2 + \left(\frac{m\pi}{a}\right)^2]^2 \right\} \qquad (2.72)$$

The static-potential of such a screened pair separated by a space-like distance R on the sub-space perpendicular to the plane z (and with a coordinate $z = \overline{z}$) is given by the temporal (ergodic) limit result (see Wilson Loop´s discussions on section 1) namely

$$V(R) = e^2 \sum_{m=1}^{\infty} \left[\left(1 - \cos\left(\frac{2\pi m}{a}\overline{z}\right) \right) V_m(R) \right] \qquad (2.73)$$

with $\overline{p}=(\hat{p},p_1)\in R^{\nu-2}$

$$V_m(R)=\left\{\int_{-\infty}^{+\infty}\frac{d^{\nu-3}\hat{p}}{(2\pi)^{\nu-3}}\frac{dp_1}{(2\pi)}\frac{\sin^2\left(\frac{p_1R}{2}\right)}{p_1^2}\right.$$
$$\left.\times\lim_{T\to\infty}\left\{\int_{-\infty}^{+\infty}\frac{dk_0}{(2\pi)}\frac{\sin^2\left(\frac{p_0T}{2}\right)}{T}\left(1+\frac{p_1^2}{p_0^2}\right)\times\frac{1}{(\hat{p}^2+p_1^2+p_0^2+\left(\frac{m\pi}{a}\right)^2)^2}\right\}\right. \tag{2.74}$$

The evaluation of the ergodic limit on eq.(2.73) is similar to those analyzed in part a) and leading to the result

$$\begin{aligned}V_m(R)&=\int_{-\infty}^{+\infty}\frac{dp_1}{(2\pi)}\cdot\sin^2\left(\frac{p_1R}{2}\right)\left[\int\frac{d^{\nu-3}\hat{p}}{(2\pi)^{\nu-3}}\frac{1}{(\hat{p}^2+p_1^2+\left(\frac{m\pi}{a}\right)^2)^2}\right]\\&=\overline{c}(\nu)\int_{-\infty}^{+\infty}\frac{dp_1}{(2\pi)}\sin^2\left(\frac{p_1R}{2}\right)\left(p_1^2+\left(\frac{m\pi}{a}\right)^2\right)^{\frac{\nu-7}{2}}\end{aligned} \tag{2.75}$$

with $\overline{c}(\nu)$ a positive constant, finite for $\nu\to 4$ and depending on the Fourier integral definition normalization factors geometrical sizes of the loop $C_{(R,T)}$, etc... which exact value will not be of our interest here, since it is convergent for $\nu\to 4$ as a function of the space-time dimensionality ν. The evaluation of the integral on eq.(2.75) can be easily accomplished through the useful formula

$$\begin{aligned}&\int_{-\infty}^{+\infty}dx\frac{\sin^2(ax)}{(x^2+b^2)^{\mu}}=\int_0^{\infty}dx\frac{1}{(x^2+b^2)^{\mu}}-\int_0^{\infty}dx\frac{\cos(2ax)}{(x^2+b^2)^{\mu}}\\&=\left(\frac{b^{-2\mu+1}}{2}\right)\frac{\Gamma(\frac{1}{2})\Gamma(\mu-\frac{1}{2})}{\Gamma(\mu)}-\frac{1}{\sqrt{\pi}}\left(\frac{b}{a}\right)^{\mu+\frac{1}{2}}\cos\left(\pi(\mu+\frac{1}{2})\right)\Gamma(\mu+1)K_{-(\mu+\frac{1}{2})}(2ab)\end{aligned} \tag{2.76}$$

and leading to the envisaged result for the harmonic m-potential contributing to the Fourier expansion eq.(2.73)

$$\begin{aligned}V_m(R)=\overline{c}(\nu)&\left\{\left[\left(\frac{m\pi}{2a}\right)^{\nu-6}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{6-\nu}{2})}{\Gamma(\frac{D-\nu}{2})}\right]\right.\\&\left.-\left[\frac{1}{\sqrt{\pi}}\left(\frac{2m\pi}{aR}\right)^{\frac{8-\nu}{2}}\cos\left(\pi\left(\frac{8-\nu}{2}\right)\right)\Gamma\left(\frac{9-\nu}{2}\right)\times K_{(\frac{\nu-8}{2})}(\frac{m\pi}{a}R)\right]\right\}\end{aligned} \tag{2.77}$$

Now its straightforward to see directly from eq.(2.77) the Casimir vacuum-energy content of the Abelian Gluonic Mandelstam Field as given by the convergent Fourier series below

$$E_{\text{Casimir}}(\overline{z})=e^2\overline{c}(\nu)\sum_{m=1}^{\infty}\left[\left(1-\cos\left(\frac{2m\pi}{a}\overline{z}\right)\right)\right]\times\left[\left(\frac{m\pi}{2a}\right)^{\nu-6}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{6-\nu}{2})}{\Gamma(\frac{7-\nu}{2})}\right] \tag{2.78}$$

The expected exponential falling at large distance R of the static potential, signaling screening of color charges for our Mandelstam Gluonic Abelian field with pure quartic propagator, is given by the second term on eq.(2.77)

$$V(R) \underset{R\to\infty}{\sim} (-e^2)\sum_{m=1}^{\infty}\left[\left(1-\cos\left(\frac{2\pi m}{a}\bar{z}\right)\right)\right] \times \left[\frac{1}{\sqrt{\pi}}\left(\frac{2m\pi}{a}\right)^{\frac{8-\nu}{2}}\cos\left(\pi\left(\frac{8-\nu}{2}\right)\right)\Gamma\left(\frac{9-\nu}{2}\right)\, e^{-\frac{m\pi}{a}R}\right] \tag{2.79}$$

$$\sim e^{-\frac{\pi}{a}R}(-e^2)\left\{\sum_{m=1}^{\infty}\left[\left(1-\cos\left(\frac{2\pi m}{a}\bar{z}\right)\right)\right] \times \left[\frac{1}{\sqrt{\pi}}\left(\frac{2m\pi}{a}\right)^{\frac{8-\nu}{2}}\cos\left(\pi\left(\frac{8-\nu}{2}\right)\right)\Gamma\left(\frac{9-\nu}{2}\right)e^{-\frac{(m-1)\pi}{a}R}\right\} \sim (-e^2)(e^{-\frac{\pi}{a}R})\overline{W}(R) \tag{2.80}$$

where the harmonic sum on the integers m is convergent due to the Bessel function argument (see eq.(2.77)).

Finally, we call the reader attention that similiar result is obtained for a propagator with a logarithmic term as that one considered in part b).

Detailed calculations taking into account quantum corrections, finite temperature efects, etc... will left to our readers as an extensive calculation exercise.

Path-Integrals on Quantum Magnetic Monopoles

a) Introduction

The question of the existence of Magnetic Monopoles has been a fruitful research path on modern theoretical physics since the appearance of the seminal work of P.M. Dirac ([17]) in the subject. In the modern framework of Non-Abelian Gauge theories, most of the relevant dynamical questions about the physical modeling of particles interactions are transferred to the difficult and more subtle mathematical analysis of special gauge-field configurations (instantons, merons, strings, magnetic monopoles, etc...) which are expected to constitute the non-perturbative vacuum structure of the underlying Bosonic Yang-Mills Gauge theory. Among those special field configurations, the Magnetic Monopole has been considered as one of the basic hypothetical non-perturbative excitation expected to be connected to practically all non-trivial charge confining dynamical effects ocurring on non-abelian Gauge theories. This fact is due to the hope that Magnetic Monopoles are the best candidates for explain naturally the (electrical) charge confinement ([18]). However magnetic monopoles by themselves should not be observed in the particle spectrum as a physical excitation. Note that this last constraint on monopole confinement makes the use of the standard Quantum Field techniques to handle magnetic monopoles dynamics a very difficult task ([19], [20]).

In this section we address to these dynamical questions on Magnetic Monopole theory by path integrals analysis, specially the technique of four-dimensional chiral bosonization path-integral as earlier proposed by this author ([21]).

This secton is organized as follows

In part b), we show how to obtain by a direct evaluation, the area behavior for an abelian Wilson Loop phase Factor in the presence of an effective second quantized electromagnetic field generated by an (condensate) second quantized monopole fermion field, as much as envisaged as an dynamical mechanism in the famous Nambu-Mandelstam propose for the existence of a Meissner effect for magnetic manopoles vacuum condensation in Yang-Mills theory in order to explain the quark-gluon confinement. As a new result of our study, we claim, thus, to have produced a well-defined path integral procedure to prove the electric charge confining in the presence of a quantum dynamics of magnetic monopoles, with a Fermi-Dirac statistics.

In part c), we exactly analyze by path-integrals techniques the quantum field dynamics of (massless) fermions field interacting with Kalb-Ramond tensor fields, expected to represent dynamically quark fields interacting with rank-two tensor field, with the later field representing the disorder field of a vacuum structure formed by condensation of magnetic monopoles ([19]). We show, thus, that it is ill-defined to associated physical observables LSZ interpolating fields for the fermion fields in the theory as consequence of the explicitly Bosonized structure formulae obtained for the matter excitations interacting with rank-two tensor fields through a spin orbit coupling with the Kalb Ramond field strenght, which by its turn provides another support for electrical charge confining in the presence of magnetic monopoles.

b) The Abelian Confinement in Presence of Magnetic Monopoles, a Wilson Loop Gauge Invariant Path-Integral Evaluation

Let us start this section by considering the Euclidean path integral average associated to a $U(1)$-abelian field $A_\mu(x)$ whose dual strength field intensity has a second quantized magnetic monopole as a (chiral) electromagnetic source $(*F^{\mu\nu}(A) \equiv E^{\mu\nu\alpha\beta}F_{\alpha\beta}(A))$

$$\begin{aligned}\langle W[C_{(R,T)}]\rangle = \int D^F[A_\mu]D^F[\Omega]D^F[\overline{\Omega}]\delta^{(F)}[\partial^*_\mu F^{\mu\nu}(A)) - (g\overline{\Omega}\gamma^\nu\gamma^5\Omega)] \\ \times \exp\left(-\frac{1}{2}\int d^4x(\Omega,\overline{\Omega})\begin{bmatrix} 0 & i\,\partial\!\!\!/ + M \\ (i\,\partial\!\!\!/ + M)^* & 0\end{bmatrix}\begin{pmatrix}\Omega \\ \overline{\Omega}\end{pmatrix}\right) \\ \times \exp\left(ie\oint_{C_{(R,T)}} A_\mu(x^\alpha)dX_\mu\right)\end{aligned} \tag{2.81-a}$$

Here $(\Omega,\overline{\Omega})(x)$ are the Euclidean Fermion (second-quartized) point-like fundamental monopole fields with g denoting the magnetic charge which by its turn is supposed to be related to the $U(1)$-electric charge e by the Dirac quantization relation $eg = \frac{n}{4}$ (with $n \in \mathbb{Z}$). M denotes the magnetic monopole mass and $W[C] = \exp\{ie\oint_{C_{(R,T)}} A_\mu dX_\mu\}$ is the $U(1)$-Wilson Loop phase factor defined by the (Euclidean) space-time trajectory of two static eletric carrier external charges interacting with the fluctuating $A_\mu(x)$ field generated by the (fluctuating) second quantized magnetic monopole fermionic source (see the constraint on eq.(2.81-a)). Note that $C_{(R,T)}$ is the boundary of the square $S_{(R,T)}$ below

$$C_{(R,T)} = \partial S_{(R,T)}; \quad S_{(R,T)} = \left\{(x_0,x_1)\in R^2; -\frac{T}{2} \le x_0 \le +\frac{T}{2}; -\frac{R}{2} \le x_1 \le +\frac{R}{2}\right\} \subset R^4. \tag{2.81-b}$$

It is worth call the reader attention that the above written quantum Wilson Loop associated to static quarks charges can be physically replaced by the complete Generating functional of the second quantized Quark fields interacting with the Monopole Generated Electromagnetic field, namely

$$
\begin{aligned}
Z[\eta,\overline{\eta}] = \int & D^F[A_\mu]D^F[\Omega]D^F[\overline{\Omega}][\delta(\partial_\mu * F^{\mu\nu}(A) - (g\overline{\Omega}\gamma^\nu\gamma^5\Omega)] \\
& \times \exp\left(-\frac{1}{2}\int d^4x(\Omega,\overline{\Omega})\begin{bmatrix} 0 & i\,\partial\!\!\!/ + M \\ (i\,\partial\!\!\!/ + M)^* & 0 \end{bmatrix}\begin{pmatrix}\Omega \\ \overline{\Omega}\end{pmatrix}\right) \\
& \times \exp\left(-\frac{1}{2}\int d^4x(\psi,\overline{\psi})\begin{bmatrix} 0 & i\,\partial\!\!\!/ + A\!\!\!/ \\ (i\,\partial\!\!\!/ + A\!\!\!/)^* & 0 \end{bmatrix}\begin{pmatrix}\psi \\ \overline{\psi}\end{pmatrix}\right) \\
& \times \exp\left(i\int d^hx(\psi,\overline{\psi})\begin{pmatrix}\overline{\eta} \\ \eta\end{pmatrix}\right) \qquad \text{(2.81-c)}
\end{aligned}
$$

For static charges eq.(2.81-c) reduces to eq.(2.81-a) as it is showed in first ref. [22].

In order to evaluate the path-integral eq.(2.81-a) from the physical point of view of an effective field theory ([21]), we should consider firstly the magnetic monopole field as a London large mass excitation in the fermonic path-integral weight of the Wilson Loop path integral average eq.(2.81-a). The reason why we should evaluate our Wilson Loop average in this context can be related to the fact that very heavy monopoles (but with small quantum fluctuations) are expected to populating the non-perturbative vacuum phase of any non-abelian Gauge Theory (at least in its confining phase) ([18], [19]). Let us, thus, re-write the magnetic monopole axial current constraint in eq.(2.81-a) by means of an axial-vectorial Lagrange multiplier field $\lambda_\mu(x)$, namely:

$$
\begin{aligned}
\langle W[C_{(R,T)}]\rangle = \Big\{ \int & D^F[A_\mu]D^F[\Omega]D^F[\overline{\Omega}]D^F[\lambda_\mu] \\
& \times \exp\left(i\int d^4x[\lambda_\nu(\partial^*_\mu F^{\mu\nu}(A) - g\overline{\Omega}\gamma^\nu\gamma^5\Omega)](x)\right] \\
& \times \exp\left[-\frac{1}{2}\int d^4x(\Omega,\overline{\Omega})\begin{bmatrix} 0 & i\,\partial\!\!\!/ + M \\ (i\,\partial\!\!\!/ + M)^* & 0 \end{bmatrix}\begin{pmatrix}\Omega \\ \overline{\Omega}\end{pmatrix}\right) \times \exp\left(ie\int_{C_{(R,T)}} A_\mu dX_\mu\right)\Big\}
\end{aligned} \qquad \text{(2.82)}
$$

At this point we follow well known studies in the literature in order to give a correct meaning for the effective field theory associated to very heavier magnetic monopoles London large mass limit in the monopoles Fermionic determinants ([21]). It is a standard result in the subject that the (mathematical) leading limit of (renormalized) magnetic monopole large mass should be given by the auxiliary Gauge field mass term, (see refs. [21] for the calculational details at this London limit for Fermion determinants)

$$
\begin{aligned}
& \lim_{M_{\text{ren}}\to\infty} |\det(i\,\partial\!\!\!/ + M_{\text{ren}} + g\gamma^5\,\lambda\!\!\!/_\mu)|^2 \\
& \cong \exp\left\{-\frac{1}{2}(\Lambda_{QCD}\cdot g^2)\int d^4x(\lambda_\mu(x))^2\right\} + O(1/M_{\text{ren}}) \qquad \text{(2.83)}
\end{aligned}
$$

Note that the appearance [through the phenomenological QCD vacuum scale $\Lambda_{QCD} = (M_{\text{ren}})^{+2}$] of a mass term for the auxiliary vector field $\lambda_\mu(x)$ which by its turn, should signals the expected dynamical breaking of the $U(1)$-axial gauge invariance (with opposite parity ([20], [21]) of this (non-physical) vectorial field by the phenomenon of dimensional transmutation on the adimensional g-coupling constant. This result indicates strongly the dynamical breaking of the $U(1)$-axial symmetry of the fermionic magnetic monopole second quantized field $\{\Omega(x), \overline{\Omega}(x)\}$.

After inserting eq.(2.83) into eq.(2.82) and by realizing the Gaussian λ_μ-field path integral, we are led to consider the effective fourth-order Wilson Loop path integral average for eq.(2.81) as the leading London limit on the magnetic monopole mass M, namely:

$$\begin{aligned}\langle W[C_{(R,T)}]\rangle = \Big\{ \int & D^F[A_\mu(x)]\delta^{(F)}(\partial_\mu A_\mu) \\ & \times \exp\left(-\frac{1}{2(g^2\Lambda_{QCD})}\int d^4x (A_\mu[(-\partial^2)^2]A_\mu)(x)\right) \\ & \times \exp\left(ie\int_{C_{(R,T)}} A_\mu dX_\mu\right)\Big\} + O(M^{-1})\end{aligned} \tag{2.84}$$

The static inter-quark linear risen potential can be obtained from eq.(2.84) by using the dimensional regularization scheme of Bollini-Giambiagi for evaluating the Feynman-diagrams integrals as it is exposed in details on refs. ([22]). It yields the expected linear raising confining potential

$$\begin{aligned}V(R) &= (e^2 \cdot g^2)(\Lambda_{QCD})R \\ &= \overline{A}n^2(\Lambda_{QCD})\cdot R = \overline{A}\left(\frac{n^2}{2\pi\alpha'}\right)R = \alpha_{\text{eff}}(N^2)R\end{aligned} \tag{2.85}$$

Here $\overline{A}$ is a model-calculational positive adimensional constant, which details will not be needed for our study, and α' denotes the Regge Slope parameter associated to the non-perturbative vacuum scale $\Lambda_{QCD} \sim (\frac{1}{2\pi\alpha'})$. It is worth call the reader attention that we have obtained somewhat the infinite quantized number of parallel Regge trajectories from the Dirac topological quantization rule for electric and magnetic charges as it is suggested in the effective Regge slope parameter $\alpha_{\text{eff}}(n^2) = n^2/2\pi\alpha'$.

Thus we see that the effective path integral eq.(2.81) for the Wilson Loop in the presence of an electromagnetic field generated by a heavy quantum monopole leads naturally to a dynamics of Wilson Loop area behavior for the electrical charges in the theory, a result obtained by us explicitly through an exactly gauge invariant path-integral evaluation.

c) Monopoles Interacting with Kalb-Ramond Fields through Spin-Orbit Coupling

In the last years, Kalb-Ramond field theory has been widely studied as an alternative dynamical quantum field scheme to the Higgs mechanism, as well as in relation to the dynamics of strings in the problem of string representation for Q.C.D. at large number of colors as a dynamical disorder field representing the effects of existence of magnetic monopoles ([18], [19]). The basic formalism used to analyze such Kalb-Ramond non-perturbative quantum dynamics has been the path-integral formalism, which has shown itself to be a very powerful procedure to understand correctly the different phases of the associated Kalb-Ramond Quantum Field Theory [23].

One important problem in those Path-integral studies, still missing in the literature, is that one related to the presence of interacting dynamical fermions (simulating second quantized matter fields) in the Kalb-Ramond Gauge theory. In this Section 3 we shall describe the extension of previous path-integral dualization-bosonization studies [24] to the case of Fermionic matter coupling through a spin-orbit field quantum interaction as it is expected to be relevant to describe the interacting physics of quarks and magnetic monopoles.

Let us start by considering the Abelian Kalb-Ramond first order action but now in the presence of massless dynamical fermions in the four-dimensional Euclidean world.

$$S[H,B,\psi,\overline{\psi}] = \int_{R^4} d^4x \left\{ \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} - \frac{1}{6} H^{\lambda\mu\nu} \partial_{[\lambda} B_{\mu\nu]} + \overline{\psi}(i \not\partial + ig\gamma^\alpha\gamma^\beta\gamma^\mu H_{\alpha\beta\mu})\psi \right\}. \tag{2.86}$$

Here the dynamical fields are the independent three-form H, the KR gauge field B and the Dirac fermion fields $(\psi,\overline{\psi})$.

We shall apply the bosonization procedure in the path-integral framework through the following theory's generating functional (normalized to unity)

$$\begin{aligned} Z[J,\eta,\overline{\eta}] = \int & D^F[H] D^F[B] D^F[\psi] D^F[\overline{\psi}] \\ & \times \exp\{-S[H,B,\psi,\overline{\psi}]\} \\ & \times \exp\left\{ -i \int_{R^4} d^4x (\overline{\eta}\psi + \overline{\psi}\eta + J_{\mu\nu} B^{\mu\nu})(x) \right\}. \end{aligned} \tag{2.87}$$

It is worth call the reader attention that the Path-integral eq.(2.87) is invariant under the KR gauge symmetry, provide the external source corrent $J_{\mu\nu}$ is chosen to be divergence free and our proposed action term related to the direct interaction of the quantum fermionic matter with the Kalb-Ramond gauge field through its strenght three-form H – the spin orbit fermion interaction. (see eq.(2.86)).

The Path-Integral Bosonization analysis proceeds as usually by integrating exactly out the Kalb-Ramond gauge potential field which produces as a result the delta functional [24].

$$\begin{aligned} Z[J,\eta,\overline{\eta}] = \int & D^F[H] D^F]\psi] D^F[\overline{\psi}] \delta^{(F)}(\partial_\lambda H^{\lambda\mu\nu} - J^{\mu\nu}) \\ & \times \exp\left\{ - \int_{R^4} d^4x \left[\frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} + \overline{\psi}(i \not\partial + ig\gamma^\alpha\gamma^\beta\gamma^\mu H_{\alpha\beta\mu})\psi \right](x) \right\}. \end{aligned} \tag{2.88}$$

Let us note that the delta functional integrand inside of the path integral eq.(2.88) imposes the classical equations of motion on the three-form Kalb-Ramond strenght H which by its turn can be exactly solved by the Rham-Hodge theorem in terms of the effective dual scalar axion (zero-form) dynamical degree of freedom in the KR theory defined in a space-time topologically trivial as considered in our path integral eq.(2.88)

$$H_{\lambda\mu\nu} = g\varepsilon^{\lambda\mu\nu\rho} \partial_\rho \vartheta + \partial^{[\lambda} \frac{1}{\partial^2} J^{\mu\nu]}. \tag{2.89}$$

At this point we re-write the effective action eq.(288) in a four-dimensional bosonized chiral action [25]

$$
\begin{aligned}
Z[J,\eta,\overline{\eta}] = & \int D^F[\vartheta] \\
& \times \exp\left\{-\frac{1}{2}\int_{R^4} d^4x \left[g^2\partial_\mu\vartheta\partial^\mu\vartheta + \frac{1}{2}J^{\mu\nu}\left(-\frac{1}{\partial^2}\right)J_{\mu\nu}\right](x)\right\} \\
& \times \int D^F[\psi]D^F[\overline{\psi}]\exp\left\{-\frac{1}{2}\int_{R^4} d^4x(\overline{\psi}e^{ig\gamma_5\vartheta}\ \partial\!\!\!/ e^{ig\gamma_5\vartheta}\psi)(x)\right\} \\
& \times \exp\left\{-\frac{1}{2}\int_{R^4} d^4x\left(ig\overline{\psi}\left[\gamma^\alpha\gamma^\beta\gamma^\rho\partial^{[\alpha}\frac{1}{\partial^2}J^{\beta\rho]}\right]\right)\psi\right\}(x) \\
& \times \exp\left\{-i\int_{R^4} d^4x(\psi\overline{\eta}+\overline{\psi}\eta)(x)\right\}.
\end{aligned} \tag{2.90}
$$

After considering the chiral-fermion field variable change on the fermionic path-integral term of eq.(2.90)

$$\overline{\psi} = \overline{\chi}e^{-ig\gamma_5\vartheta} \tag{2.91-a}$$

$$\psi = e^{-ig\gamma_5\vartheta}\chi \tag{2.91-b}$$

$$
\begin{aligned}
D[\psi]D[\overline{\psi}] &= D[\chi]D[\overline{\chi}]\frac{\det[e^{ig\gamma_5\vartheta}\ \partial\!\!\!/ e^{ig\gamma_5\vartheta}]}{\det[\partial\!\!\!/]} \\
&= D[\chi]D[\overline{\chi}]J[\vartheta],
\end{aligned} \tag{2.91-c}
$$

we obtain the exactly bosonized path-integral representation for the KR first order theory as given by eq.(2.87), namely:

$$
\begin{aligned}
Z[J,\eta,\overline{\eta}] = & \int D^F[\vartheta]D[\chi]D[\overline{\chi}]J[\vartheta] \\
& \times \exp\left\{-\int_{R^4} d^4x\left[\frac{g^2}{2}\partial_\mu\vartheta\partial^\mu\vartheta - \frac{1}{2}J^{\mu\nu}(\partial^2)^{-1}J_{\mu\nu}\right](x)\right\} \\
& \times \exp\left\{-\frac{1}{2}\int_{R^4} d^4x(\overline{\chi}\ \partial\!\!\!/\chi)(x)\right\} \\
& \times \exp\left\{-\frac{1}{2}ig\int_{R^4} d^4x\left(\overline{\chi}\left(\gamma^\alpha\gamma^\mu\gamma^\nu\partial^{[\alpha}\frac{1}{\partial^2}J^{\mu\nu]}\right]\chi\right)(x)\right\} \\
& \times \exp\left\{-i\int_{R^4} d^4x\left(\overline{\chi}e^{-ig\gamma_5\vartheta}\eta + \overline{\eta}e^{-ig\gamma_5\vartheta}\chi\right)(x)\right\},
\end{aligned} \tag{2.92}
$$

here the functional Fermion Jacobian eq.(2.91-c) has been exactly evaluated in refs. [25] (see Appendix A e B - Chapter 18)

$$
\begin{aligned}
J_\varepsilon[\vartheta] = & \exp\left\{\frac{g^2}{4\pi^2\varepsilon}\int_{R^4} d^4x(\partial_\mu\vartheta)^2(x)\right\} \\
& \times \exp\left\{-\frac{g^2}{4\pi^2}\int_{R^4} d^4x(\partial^2\vartheta)(\partial^2\vartheta)(x)\right\} \\
& \times \exp\left\{\frac{g^4}{12\pi^2}\int_{R^4} d^4x[\vartheta(\partial_\mu\vartheta)^2(-\partial^2\vartheta)](x)\right\}.
\end{aligned} \tag{2.93}
$$

As a first remark to be made on the above written result we note that its first term has the effect of formally inducing a renormalisation of the g-charge after the cutt-off removing $\varepsilon \to 0$ on the complete result eq.(2.87), namely

$$g^2_{\text{bare}}(\varepsilon)\left(1+\frac{1}{4\pi^2\varepsilon}\right) = g^2_{\text{ren}}. \tag{2.94}$$

By secondly, we point out the appearance of the fourth-order kinetic term for the scalar effective KR field $\vartheta(x)$, a very important result for the model ultra-violet finiteness.

An another important physical result coming from the set eq.(2.92)–eq.(2.94) is the explicitly fermionic matter asymptotic freedom as can be see directly from the factorized – decoupled form of the full interacting matter fermionic propagator, namely

$$\frac{1}{(i)^3}\frac{\delta Z[\eta,\overline{\eta},J]}{\delta\eta_\alpha(x)\delta\eta_\beta(y)}\Big|_{\substack{J=0\\ \eta=\overline{\eta}=0}} = S_{\alpha\beta}(x-y)\times F(x,y) \tag{2.95}$$

with $S_{\alpha\beta}(x-y)$ denoting the free fermion propagator and the (decoupled) Kalb-Ramond form factor being given exactly by the (perturbative finite) fourth-order ϑ-path integral as remarked above.

$$\begin{aligned} F(x,y) = \int & D^F[\vartheta]e^{-\frac{1}{2}g^2_{\text{ren}}\int_{R^4}(\partial_\mu\vartheta)^2(x)d^4x} \\ & \times e^{-\frac{g^2_{\text{ren}}}{4\pi^2}\int_{R^4}(\partial^2_\mu\vartheta)^2(x)d^4x} \\ & \times e^{+\frac{g^2_{\text{ren}}}{4\pi^2}\int_{R^4}[\vartheta(\partial_\mu\vartheta)^2(-\partial^2_\mu\vartheta)](x)d^4x} \\ & \times \{(\exp -ig_{\text{ren}}\gamma_5\vartheta(x))(\exp -ie_{\text{ren}}\gamma_5\vartheta(y)\} \end{aligned} \tag{2.96}$$

which goes to 1 in the high energy limit of $|x-y| \to 0$ as a result of the path-integral super renormalizability associated to the effective axion scalar dual Kalb-Ramond theory eq.(2.86)) [the well-known phenomenon of asymptotic freedom in confining gauge theories]. A low energy study of the form-factor eq.(2.96) has been carried out in refs. [25] (Appendix). There, we have suggested that these bosonized fermionic fields do not possesses LSZ interpolating fields, since the associated two-point Euclidean correlation function eq.(2.95) defines Wightman functions which are ultra-distributions in Jaffe Distributional Spaces and not in the usual Schwartz Tempered Distributional Spaces naturally associated to the existence of LSZ interpolating fields (a well defined Scattering Matrix) in the quantum field theory eq.(2.87).

A calculational remark to be made at this point of this section is related to the straightforward exactly solubility for the Macroscopic radiative corrections evaluations of the Kalb-Ramond gauge potential propagator

$$\begin{aligned} & \frac{1}{i^2}\frac{\delta^2[J,\eta,\overline{\eta}]}{\delta J_{\mu\nu}(x)\delta J_{\alpha\beta}(y)}\Big|_{\substack{\eta-\overline{\eta}=0\\ J=0}} = \langle B_{\mu\nu}(x)B_{\alpha\beta}(y)\rangle \\ & = (-\partial^2)^{-1}(x,y) + e^2_{\text{ren}}\int d^4z d^4z'(-\partial^2)^{-1}(z-x)(-\partial^2)^{-1}(z-y) \\ & \quad \times \partial_z^{[\lambda}\partial_{z'}^{\lambda'}\langle(\overline{\chi}(z)(\gamma^\lambda\gamma^{[\mu}\gamma^{\nu]})\chi(z))(\overline{\chi}(z')(\gamma^{\lambda'}\gamma^{[\alpha}\gamma^{\beta]})\chi(z'))\rangle^{(0)}, \end{aligned} \tag{2.97}$$

here $\langle\ \rangle^{(0)}$ denotes the free fermion average path integral

$$\langle\ \rangle^{(0)} = \int D(\chi)D[\overline{\chi}]e^{-\frac{1}{2}\int_{R^4} d^4x(\overline{\chi}\partial\!\!\!/\chi)(x)}. \tag{2.98}$$

The exactly evaluation of the quantum correction eq.(2.97) is standard and can be easily obtained by just using the well-known Dirac matrixes relationship and will be left as an exercise to our readers

$$\gamma^\lambda\gamma^\mu\gamma^\nu = (S_{\lambda\mu\nu\sigma} + \varepsilon_{\lambda\mu\nu\sigma}\gamma_5)\gamma^\sigma \tag{2.99}$$

$$S_{\lambda\mu\nu\sigma} = (\delta_{\lambda\mu}\delta_{\nu\sigma} + \delta_{\mu\nu}\delta_{\lambda\sigma} - \delta_{\lambda\nu}\delta_{\mu\sigma}). \tag{2.100}$$

The above exposed results concludes our part c) on path-integral exactly studies on the four-dimensional path-integral Bosonization of our abelian interacting KR field.

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Chapter 3

The Triviality – Quantum Decoherence of Quantum Chromodynamics $SU(\infty)$ in the Presence of an External Strong White-Noise Eletromagnetic Field

3.1. Introduction

For a long time, a very interesting (and conceptually) important problem in Quantum Field theory has been the correct understanding of the triviality phenomena of interacting fields as a kind of "phase-transition"phenomena depending on external parameters including the famous space-time dimensionality. The basic formalism used to understand such an important phenomena is – until present time – the re-writing of the given interacting quantum field generating functional in terms of the famous Symanzik Loop Space (even at the Lattice) [1–3].

The purpose of this chapter is to point out quantum field triviality phenomena in another context, however in a more complicated Quantum Field theory than those analyzed on literature which is Quantum Chronodynamics at large number of colors but in the presence of an external random abelian field. The main idea is to show that exactly such a triviality result for Q.C.D. (SU(∞)) will be the systematic use of the Loop Space representation for Q.C.D. which, by its turn, allows us to exactly integrate out the external random abelian field when one is analyzing the Q.C.D (SU(∞)) on the physical sector (observable) of abelian quark currents (form factors).

In section II we present our ideas and a complete Loop Analysis of Q.C.D. (SU(∞)) triviality in the presence of randomness. In section III, we present a path-integral renormalization analysis of the resulting effective random surface theory. In section IV, we apply the previous Q.C.D. Loop analysis to the important case of non-relativistic (many-body) field theories. In section VI, we present a Tensor Model for improved QCD(SU∞)) and finally in section VII, we propose a string second-quantized field theroy for the random surface theory of section VI.

3.2. The Triviality – Quantum Decoherence Analysis

In order to show such a triviality – quantum decoherence on Bosonic Q.C.D(∞) let us consider the Euclidean generating functional of the abelian (for simplicity) quarks currents on the presence of an external white-noise electromagnetic field $B_\mu(x)$, simulating a kind of "dissipative" vaccum structure or quantum external resorvoir acting in the system (see second reference of refs.[2]).

$$Z\left[J_\mu(x), B_\mu(x)\right] = \left\langle \det_f^{N_c}\left(\not{D}(A_\mu, B_\mu, J_\mu)\, \not{D}^*(A_\mu, B_\mu, J_\mu)\right)\right\rangle_{A_\mu} \tag{3.1}$$

Here the Euclidean Dirac operator is explicitly given by

$$\not{D}(A_\mu, B_\mu, J_\mu) = i\gamma_\mu(\partial_\mu + eB_\mu + J_\mu + gA_\mu) \tag{3.2}$$

with gA_μ denoting the Yang-Mills non-abelian quantum field configurations averaged on eq.(3.1) by means the usual Yang-Mills Path integral, $J_\nu(x)$ is the auxiliary source field associated to the abelian quarks currents and $B_\mu(x)$ is a random external electromagnetic field with a strenght field $F_{\mu\gamma}(B)$ satisfying a Gaussian statistics with randomness of intensity $\lambda > 0$.

$$E_F\left\{F_{\mu\gamma}(B)F_{\alpha\beta}(B)(y)\right\} = \lambda\delta^{(D)}(x-y)\cdot(\delta_{\mu\alpha}\delta_{\gamma\beta} - \delta_{\mu\beta}\delta_{\gamma\alpha}) \tag{3.3}$$

Here E_F denotes the stochastic average on the ensemble of the external strenght abelian field $F(B)$.

In the Bosonic loop space framework [3] we can express the quark functional determinant eq.(3.1) – which was obtained as an effective generating functional for the color singlet quark current after integrating out the Euclidean quark action – , as a functional on the Bosonic loop space composed of all trajectories $C_{xx} = \left\{X_\mu(\sigma), X_\mu(0) = X_\mu(T) = x; 0 \le \sigma \le T\right\}$

$$\begin{aligned} &Z[J_\mu(x), B_\mu(x)] \\ &= \left\langle \exp - \left\{ N_c \sum_{C_{xx}} \left[\Phi[C_{xx}, B_\mu]\Phi[C_{xx}, J_\mu] Tr_c\left(W\left[C_{xx}, A_\mu\right]\right)\right]\right\}\right\rangle_{A_\mu} \end{aligned} \tag{3.4}$$

where $\Phi[C_{xx}, B_\mu]$ is the usual Wilson-Mandelstan loop variable defined by the random external electromagnetic field $B_\mu(x)$, $W[C_{xx}, A_\mu]$ is the same loop space object, however with a sum path order and defined by the non-abelian Yang-Mills quantum Euclidean field $A_\mu^a(x)\lambda_a$. Namely

$$\Phi[C_{xx}, B_\mu] = \exp\left(ie\oint_{C_{xx}} B_\mu(X_\beta(\sigma)dX_\mu(\sigma)\right) \tag{3.5}$$

$$W[C_{xx}, A_\mu] = P\left[\exp\left(i\oint_{C_{xx}} A_\mu(X_\beta(\sigma))dX_\mu(\sigma)\right)\right] \tag{3.6}$$

The sum over the closed loops C_{xx} with end-point x is given by the proper-time bosonic path integral below

$$\sum_{C_{xx}} = \int_0^\infty \frac{dT}{T} \int d^D x \int_{X(0)=x=X(T)} D^F[X(\sigma)] \exp\left\{ -\frac{1}{2} \int_0^T \dot{X}^2(\sigma) d\sigma \right\} \tag{3.7}$$

In refs.[3], the factorization of the color gauge invariant averages of the products of Wilson loops associated to the Yang-Mills fields A_μ at $SU(\infty)$ was presented on the basis of a diagrammatic analysis. As a consequence of this result the non trivial dynamical content of the generating functional of abelian quark currents is entirely given by the fermionic functional determinant written in the $SU(\infty)$ bosonic loop space functional with a factorized form in relation to the loop fields entering in its (loop space) structural form as given below

$$\begin{aligned} & -\ln Z[J_\mu(x), B_\mu(x)]_{SU(\infty)} \\ & = \left\{ \sum_{C_{xx}} \Phi[C_{xx}, B_\mu] \Phi[C_{xx}, J_\mu] \left\langle Tr_c W[C_{xx}, A_\mu] \right\rangle_{SU(\infty)} \right\} \end{aligned} \tag{3.8}$$

In order to show the triviality quantum decoherence of the bosonic loop space generating functional eq.(3.8) when averaging over the quark currents dependence on the external white-noise abelian field $B_\mu(x)$, we consider the stochastic average of the Wilson-Mandelstan phase factor defined by the abelian random field with the following result

$$\begin{aligned} E_F\left\{ \Phi[C_{xx}, B_\mu] \right\} &= E_F \left\{ \exp ie \int_{\Sigma(C_{xx})} F_{\mu\nu}(x) d\sigma^{\mu\nu}(x) \right\} \\ &= \left\{ -\frac{(e^2\lambda)}{2} \int_{\Sigma(C_{xx})} d\sigma^{\mu\nu}(x) \delta^{(D)}(x-y) d\sigma^{\mu\nu}(y) \right\} \end{aligned} \tag{3.9}$$

Let us analyze the behavior of the loop space functional eq.(3.9) in terms of the metric properties of the surface $\sum(C_{xx})$ bounded by the loop $C_{xx}(\sigma)$. In order to analyze such a geometrical behavior of eq.(3.9) we consider an explicitly parametrization of the (fixed) surface $\sum(C_{xx})$ possesing as boundary the loop C_{xx}:

$$\sum(C_{xx}) = \left\{ \varphi_\mu(s,\sigma), 0 \le s \le 2\pi; 0 \le \sigma \le T \right\} \tag{3.10}$$

In terms of this two-dimensional surface vector parametrization we re-write the loop functional eq.(3.9) in the coordinate invariant parametrization form, suitable to analyze its geometrical content

$$\begin{aligned} & \ln\left(E\left\{ \Phi[C_{xx}, B_\mu] \right\} \right) \\ & = -\frac{(e^2\lambda)}{2} \int ds d\sigma \int ds' d\sigma' \sqrt{h(s,\sigma)} \sqrt{h(s',\sigma')} \\ & \times \tau^{\mu\nu}\left(\varphi_\beta(s,\sigma) \right) \tau^{\mu\nu}(\varphi_\beta(s',\sigma')) \left(\delta^{(D)}\left(\varphi_\beta(s,\sigma) - \varphi_\beta(s',\sigma') \right) \right) \end{aligned} \tag{3.11}$$

Here the surface area tensor is given by

$$d\sigma^{\mu\nu}(x_\beta) \Big|_{x_\beta = \Phi_\beta(s,\sigma)} = \left(\sqrt{h(s,\sigma)} \tau^{\mu\nu}\left(\varphi_\beta(s,\sigma) \right) ds d\sigma \right) \tag{3.12}$$

with

$$\sqrt{h(s,\sigma)} = \left(\sqrt{\det(\partial_a\varphi_\beta\partial_b\varphi^\beta)(s,\sigma)}\right) \tag{3.13}$$

$$\tau^{\mu\nu}(\Phi_\beta(s,\sigma) = \left(\varepsilon^{ab}\partial_a\varphi^\mu\partial_b\varphi^\nu/\sqrt{h}(s,\sigma)\right) \tag{3.14}$$

By introducing a regularization form to the singular delta-function appearing on the surface function eq.(3.11)

$$\delta^{(D)}_{(\varepsilon)}\left(\varphi^\beta(s,\sigma)-\varphi^\beta(s',\sigma')\right) = \int_{|k|>1/\varepsilon} d^Dk \exp\left(ik_\alpha(\varphi_\alpha(s,\sigma)-\varphi_\alpha(s',\sigma'))\right), \tag{3.15}$$

one obtains as the leading geometrical functional associated to the trivial surface self-intersecting case $(\sigma,s)=(\sigma',s')$, the well-known Nambu-Goto area surface functional [4], and see section 3.3 of this chapter.

$$-\ln\left\{E\left(\Phi[C_{xx},B_\mu]\right)\right\} = \bar{c}(e^2\lambda)\int dsd\sigma\left(\sqrt{h}h^{ab}\partial_a\varphi^\mu\partial_b\varphi^\mu\right)(s,\sigma) \tag{3.16}$$

Here $\bar{c}$ is a positive R^D-dimensional constant related to the renormalization parameters ε used on the regularization form eq.(3.15) and somewhat related to the analogous expected phenomena of dimensional transmutation on Q.C.D$(SU(\infty))$. Note that we have used the normalization condition of the surface area tensor to obtain the area functional eq.(3.16):

$$\tau^{\mu\nu}(\varphi_\beta(s,\sigma))\tau^{\mu\nu}(\varphi_\beta(s,\sigma)) = 1 \tag{3.17}$$

At this point, it is straightforward to see that for a large white-noise external abelian field $\lambda\to\infty$ [2], the noise averaged Wilson loop on eq.(3.1) is vanishing small for any loop C_{xx}. It is worth call the reader attention that for a given fixed noise strenght $\lambda\neq 0$, all loops C_{xx} bounding large minimal areas surfaces $\sum[C_{xx}]$ are suppressed on the bosonic loop path integral eq.(3.8) and leading to a dynamics of Gluon condensates [3].

Note that the same loop C_{xx} appearing on eq.(3.9) enters in the definition of all loop space objects on eq.(3.8). This result in turns show us that at the very large noise strenght limit $\lambda\to+\infty$,we have the strong triviality of $SU(\infty)$-Quantum Chromodynamics in the sector of the abelian quark currents, since all closed loops $C_{xx}(\sigma)$ degenerate to the loop base point x, namely

$$\begin{aligned}&\lim_{\lambda\to\infty} E_B\left\{Z(J_\mu(x),B_\mu(x)\right\}\\ &\quad= \lim_{\lambda\to\infty}\exp\left\{-\sum_{(C_{xx}(\sigma)\to x)} e^{-\bar{c}\lambda e^2 Area(\Sigma[C_{xx}])}\Omega[C_{xx},J_\mu]\times\left\langle Tr_C(W[C_{xx},A_\mu])\right\rangle\right\}\\ &\quad= \exp(0) = 1\end{aligned} \tag{3.18}$$

This is the first main conclusion of this chapter about the Q.C.D$(SU(\infty))$ triviality – quantum decoherence.

A second result we wishe to present is related to the somewhat different situation of our abelian randon field is now originating from a source described by a manifold of random currents obeying a pure-white noise statistics in a physical our-dimensional space-time R^4

$$\Delta B_\mu(x) = j_\mu(x) \tag{3.19}$$

with the white-noise (spaghetti-vaccum [12]) current source correlation function (see chapter 2, page 65)

$$E_j\left\{j_\mu(x)j_\nu(y)\right\} = \lambda\delta^{(4)}(x-y)\delta_{\mu\nu} \tag{3.20}$$

In order to see the area behavior for the abelian phase factor $\Phi[C_{xx}, B_\mu]$ in eq.(3.4), we probe the system vacuum energy by considering a static pair of quark-antiquark interacting with the random electromagnetic field eq.(3.19)–eq.(3.20).

The binding electromagnetic energy between such static probing charges e, separated by a distance R is computed by evaluating the energy of the abelian white-noise field $B_\mu(x)$ in the presence of these static quark sources and given explicitly by the following Wilson loop average (see chapter 2).

$$V(R) = \lim_{T\to\infty} -\frac{1}{T} lg E_j\left\{\exp ie\oint_{C(R,T)} B_\mu(x,[j])dX_\mu\right\} \tag{3.21}$$

where the quark-antiquark static space-time trajectory is given by a rectangle $C_{(R,T)} = \left\{-\frac{T}{2} \le t \le +\frac{T}{2}; -\frac{R}{2} < \sigma < \frac{R}{2}\right\}$ and E_j denotes the stochastic average over the vacuum current sources eq.(3.20).

The evaluation of the binding energy $V(R)$ can be more invariantly accomplished by writing it in momentum space and using the dimensional regularization of Bollini and Giambiagi [5], after evaluating explicitly the source average on eq.(3.21)

$$V(R) = \lim_{T\to\infty} -\frac{1}{2T}\left[\int \frac{d^Dk}{(2\pi)^D} f_\mu(k;C_{(R,T)})\cdot\frac{\lambda\delta_{\mu\nu}}{(k^2)^2}\times f_\nu(-k,C_{(R,T)})\right] \tag{3.22}$$

with the rectangle form factor written as follows

$$f_\mu(k,C_{(R,T)}) = ie\oint_{C_{(R,T)}} e^{-ik_\alpha(\sigma)}\frac{dX_\alpha(\sigma)}{d\sigma} \tag{3.23}$$

As the rectangles $C_{(R,T)}$ is contained in a two-dimensional sub-space of the space-time R^D, we can decompose the vector $\vec{k}$ as $\vec{k} = k_0\vec{e}_0 + k_1\vec{e}_1 + \hat{k}$, where $\hat{k}$ is the projection of $\hat{k}$ over the sub-space perpendicular to the sub-space $\{\vec{e}_0, \vec{e}_1\}$ containing $C_{(R,T)}$. In addtion, the space coordinate system is chosen so that the x-axis direction coincides with the one defined by the spatial sides of the rectangles $C_{(R,T)}$, this coordinate choice leads us to the solutions

$$\begin{aligned} f_0(k,C_{(R,T)}) &= -\frac{4e}{k_0}\sin\left(\frac{k_0T}{2}\right)\sin\left(\frac{k_1R}{2}\right) \\ f_1(k,C_{(R,T)}) &= +\frac{4e}{k_1}\sin\left(\frac{k_0T}{2}\right)\sin\left(\frac{k_1R}{2}\right) \end{aligned} \tag{3.24}$$

After substituting eq.(3.24) into eq.(3.22), we face the problem of evaluating the following dimensionally regularized integral limit of $T\to\infty$. We get as a result:

$$\begin{aligned} V(R) = \lim_{T\to\infty}\frac{8(e^2\lambda)}{T}&\left\{\int_{-\infty}^{+\infty}\frac{dk_1}{(2\pi)}\frac{\sin^2(\frac{k_1R}{2})}{(k_1)^2}\right. \\ &\times\left.\left[\int\frac{d^{\nu-2}\hat{k}}{(2\pi)^{\nu-2}}\left(\int_{-\infty}^{+\infty}\frac{dk_0}{(2\pi)}\frac{(k_0^2+k_1^2)}{k_0^2}\frac{\sin^2(\frac{k_0T}{2})}{(k_0^2+k_1^2+\hat{k}^2)^2}\right)\right]\right\} \end{aligned} \tag{3.25}$$

By using the elementary improper integral formula for the evaluation of the k_0-integrand on eq.(3.25)

$$\lim_{b\to\infty}\frac{1}{b}\left\{\int_{-\infty}^{+\infty}\left(1+\frac{a^2}{x^2}\right)\frac{\sin^2(bx)}{(x^2+c^2)^2}\right\} = \frac{2\pi a^2}{c^4} \tag{3.26}$$

We arrive at the (partial) result

$$V(R) = +\frac{(e^2\lambda)}{(4\pi)^{\frac{D}{2}-1}}\left\{\int\frac{dk_1}{(2\pi)}\frac{\sin^2(\frac{k_1 R}{2})}{k_1^2} \times\Gamma\left(\frac{6-\nu}{2}\right)|k_1|^{\nu-4}\right\} \tag{3.27}$$

with the final result on the dimensional regularized form (a general space-time with a continuum dimension ν) and where we have introduced a Coulomb term (by hand) to eq.(3.27) associated to a $1/k^2$ propagator – just for completenness (see chapter 2).

$$V(R) = V_{Coul}(R)+V_{Conf}(R) \tag{3.28}$$

with

$$V_{Coul}(R) = +\frac{(e^2\lambda)}{(4\pi)^{\frac{\nu}{2}-1}}\times\left\{\Gamma(\nu-3)\frac{\sin((\nu-4)\frac{\pi}{2})}{2\pi}\Gamma\left(\frac{4-\nu}{2}\right)\right\}(R)^{-\nu+3} \tag{3.29}$$

and

$$V_{Conf}(R) = +\frac{(e^2\lambda)}{(4\pi)^{\frac{\nu}{2}-1}}\times\left\{\Gamma\left(\frac{6-\nu}{2}\right)\frac{\sin(\frac{\pi}{2}(\nu-6))}{2\pi}\Gamma(\nu-5)\right\}(R)^{-\nu+5} \tag{3.30}$$

At this point one can see that the potential energy term as given by eq.(3.30) at the physical four-dimensional space-time leads to the expected "confining" area behavior to the stochastic abelian phase factor

$$E_j\left\{\exp ie\oint_{C(R,T)}B_\mu(x,[j]dX_\mu\right\}\sim\exp\exp\left\{-\bar{c}T\cdot R(e^2\lambda)\right\} \tag{3.31}$$

whith $\bar{c}$ a positive adimensional constant.

It is worth remark that the term eq.(3.29) leads to the usual Coulom Law at $D=4$, namely

$$V_{Coul}(R) = -\frac{e^2\lambda}{4\pi R} \tag{3.32}$$

3.3. Random Surface Dynamical Factor in the Analytical Regularization Scheme

Sometimes, it is argumented on the literature [6], that one should consider a dynamical random surface path-integral sum to the surface functional as given by eq.(3.11) in the case of the existence of only trivial self-intersections $(\sigma, s) \equiv \xi = (\sigma', s') = \xi'$ on the domain functional

$$Z[\varphi](g_{bare}) = \frac{1}{Z(0)} \int D^F[\varphi(\xi)] \exp\left\{-\frac{1}{2}\int d^2\xi \left(\varphi^{\mu}(-\Delta)^{+\alpha}\varphi^{\mu}\right)(\xi)\right\}$$
$$\times \exp\left\{-g_{bare}\int d^2\xi \delta^{(D)}\left(\varphi_\mu(\xi) - \varphi_\mu(\xi')\right)\right\} \quad (3.33)$$

Here α is a regularizing theory's parameter $\alpha \geq 1$.

Let us address the problem of renormalization on this self-avoiding random surface functional eq.(3.33). Firstly, we point out that one can safely replace the surface self-avoidance on the path-integral interaction weight by an interaction with the tangent plane at the surface point $\varphi_\mu(\xi)$, namely;

$$\delta^{(D)}\left(\varphi_\mu(\xi) - \varphi_\mu(\xi')\right) = \delta^{(D)}\left(\varphi_\mu(\xi) - T_\mu(\xi)\right) \quad (3.34)$$

where the tangent plane equation is given by

$$T_\mu(\xi') = T_\mu(\xi) = t^0_\mu \cdot \xi_0 + t^{(1)}_\mu \xi_1 \quad (3.35)$$

with $\{t^{(0)}_\mu, t^{(1)}_\mu\}$ denoting the surface tangent vectors at $\varphi_\mu(\overline{\xi})$.

By a simple variable change

$$\varphi_\mu(\xi) \to \varphi_\mu(\xi) - T_\mu(\xi) \quad (3.36)$$

we obtain as an effective random surface path-integral to be analyzed from a renormalization point of view, the self-avoiding random surface interacting with the origin [6].

$$Z[\varphi](g_b) = \frac{1}{Z(0)} \int D^F[\varphi^{\mu}(\xi)] \exp\left\{-\frac{1}{2}\int d^2\xi \left(\varphi_{\mu}(-\Delta)^{+\alpha}\varphi_{\mu}\right)(\xi)\right\}$$
$$\times \exp\left\{-g_b\int d^2\xi \delta^{(D)}\left(\varphi_\mu(\xi)\right)\right\} \quad (3.37)$$

where g_b denotes the (positive) bare self-avoiding random surface coupling constant.

It is instructive to point out that the formal perturbation expansion around the massless $2D$ fluctuating surface vector position $\{\varphi_\mu(\xi)\}$ is ill defined in the case of $\alpha = 1$ on eq.(3.37) due to the severe infrared divergences of the associated Laplacean Green function on R^2. As a consequence of the above made remark, we start from the begining with the Riesz-Hadamard expression of the Seeley α-power of the Laplacean as written on the kinetic term of eq.(3.37).

$$\begin{aligned} G_\alpha(\xi_1, \xi_2) &= (-\Delta)^{-\alpha}(\xi_1, \xi_2) \\ &= \frac{e^{-i\pi\alpha}\Gamma(1-\alpha)}{4^\alpha(\pi)^{1/2}\Gamma(\alpha)} |\xi_1 - \xi_2|^{2(\alpha-1)} \\ &= \int d^2k e^{ik(\xi_1-\xi_2)} |k|^{-2\alpha} \end{aligned} \quad (3.38)$$

We, thus, renormalize eq.(3.33) from eq.(3.37) by means of the renormalization prescription at the physical case of $\alpha = 1$ (pure Laplacean).

$$g_b = \frac{g_{ren}}{(1-\alpha)^{D/2}} \tag{3.39}$$

$$Z_R[\varphi_\mu](g_{ren}) = \lim_{\substack{\alpha\to 1\\ \alpha>1}} Z[\varphi_\mu](g_b(\alpha)) \tag{3.40}$$

Let us show that eq.(3.40) is a well defined in a formal power expansion in the renormalized coupling constant g_{ren} as given by eq.(3.39)

In order to show this result, we make the power expansion of the α-regularized path-integral eq.(3.37)

$$Z_R[\varphi_\mu](g_{ren}) = \sum_{\ell=0}^{\infty} \frac{(-g_b)^N}{N!} \left\{ \prod_{j=1}^{N} \int d^2\xi_j \det{}^{-\frac{D}{2}} [G_\alpha(\xi_i,\xi_j)] \right\} \tag{3.41}$$

The finites of eq.(3.41) for each N under the renormalization prescription eq.(3.39) is a straightforward consequence of the following properties:

Firstly,

$$\lim_{\substack{\alpha\to 0\\ \alpha>1}} (\varphi_\alpha(\xi_1,\xi_2)) = \lim_{\substack{\alpha\to 1\\ \alpha>1}} \left\{ \frac{e^{-i\pi\alpha}\Gamma(1-\alpha)}{4^\alpha \pi^{\frac{1}{2}}\Gamma(\alpha)} (0)^{2\alpha-1)} \right\} = 0 \tag{3.42}$$

Secondly

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}} \det \begin{bmatrix} G_\alpha(\xi_1,\xi_2) & G_\alpha(\xi_1,\xi_2) \\ G_\alpha(\xi_2,\xi_1) & G_\alpha(\xi_2,\xi_1) \end{bmatrix} =$$

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}} \left[-\frac{e^{-2\pi i\alpha}}{4^{2\alpha}\pi} \left(\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)} \right)^2 \right] (|\xi_1-\xi_2|)^{4(\alpha-1)} = \frac{C_2}{(1-\alpha)^2} \tag{3.43}$$

Thirdly:

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}} \det \begin{bmatrix} 0 & G_\alpha(\xi_1,\xi_2) & G_\alpha(\xi_1,\xi_2) \\ G_\alpha(\xi_2,\xi_1) & 0 & G_\alpha(\xi_2,\xi_1) \\ G_\alpha(\xi_3,\xi_1) & G_\alpha(\xi_3,\xi_2) & 0 \end{bmatrix} =$$

$$= \frac{e^{-3\pi i\alpha}}{4^{3\alpha}\pi^{\frac{3}{2}}} \left(\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)} \right)^3 (1+1') = \frac{C_3(1)}{(1-\alpha)^3} \tag{3.44}$$

Finally;

$$\lim_{\substack{\alpha\to 1\\ \alpha>1}} \det_{NXN} [G_\alpha(\xi_1,\xi_2)] = \frac{e^{-\pi i\alpha N}}{4^{N\alpha}\pi^{\frac{N}{2}}} \cdot \frac{1}{(1-\alpha)^N} C_N \tag{3.45}$$

with

$$C_N = \det[A_{i,j}] = -(N-1)(-1)^N \tag{3.46}$$

where $[A_{i,j}]$ is the matrix whose entries are

$$[A_{i,j}] = \begin{cases} 0 & if \quad i = j \\ 1 & if \quad i \neq j \end{cases} \tag{3.47}$$

As a consequence of the analysis above exposed, we obtain our renormalization result for the eq.(3.41) at the limit $\alpha \to 1$.

$$Z_R[\varphi_\mu](g_{ren}) = \sum_{\ell=0}^{\infty} \frac{(-g_{ren})^\ell}{\ell!} C_\ell \cdot A^\ell < \infty \tag{3.48}$$

with $A = \int d^2\xi$ denoting the internal random surface area and $C_\ell = e^{-i\pi\ell}/4^\ell pi^{\ell/2} \times (-1)^\ell \times (1-\ell)$.

Finally, let us complement our studies on the area behavior of the surface functional as given as by eq.(3.9) in a more physical way. Let us see its area behavior by using distribution theory on sufaces [6]. Firstly, we introduce a R^D vector basis along the coordinate lines $\frac{\partial\phi^\mu}{\partial\sigma}$ and $\frac{\partial\phi^\mu}{\partial s}$. We have, thus, the surface-intrinsic distributional results

$$\delta^{(D)}\left(\phi_\mu(s,\sigma) - \phi_\mu(s',\sigma')\right) = \delta_\varepsilon^{(D-2)}(0) \times \left(\frac{1}{\sqrt{h((s,\sigma)}}\delta^{(1)}(s-s')\delta^{(1)}(\sigma-\sigma')\right) \tag{3.49}$$

and

$$d\sigma_{\mu\nu}(x)\Big|_{x^\alpha=\phi^\alpha(s,\sigma)} = \sqrt{h(s,\sigma)} \cdot dsd\sigma \cdot \tau_{\mu\nu}(\phi_\alpha(s,\sigma)) \tag{3.50}$$

Here $\delta_\varepsilon^{(D-2)}(0)$ means a regualrized form of the delta function singular value $\delta^{(D-2)}(0)$ and physically related to the non-trivial structure of the non-perturbative phenomenon of the coupling constant dimensional transmutation (see appendix of the first reference on ref.[4]).

After substituting eq.(3.49)–eq.(3.50) into the random surface term eq.(3.9) – section 3.2, we get our result

$$\begin{aligned} eq(11) &= \frac{e^2\lambda}{2}\int_0^T d\sigma \int_0^{2\pi} ds\sqrt{h(\phi^\alpha(s,\sigma))}\int_0^T d\sigma' \int_0^{2\pi} ds'\sqrt{h(\phi^\alpha(s',\sigma'))}\left\{\delta_\varepsilon^{(2)}(0)\frac{\delta(\sigma-\sigma')\delta(s-s')}{\sqrt{h(\phi^\alpha(s',\sigma'))}}\right\} \\ &= \frac{e^2\lambda}{2}\int_0^T d\sigma \int_0^{2\pi} d\sigma\sqrt{h(\phi^\alpha(s,\sigma))} = \text{Area}\left(\sum_{C_{xx}}\right) \times \frac{e^2\lambda}{2} \end{aligned} \tag{3.51}$$

3.4. The Non-relativistic Case

In this complementary section, we apply the analysis presented in section 3.2 for Quantum Chromodynamics at t'Hooft limit in a non-relativistic finite-temperature non-linear Schörindger theory (see appendix A, chapter 1).

Let us start our analysis by considering the partition functional of the following Schrödinger Bosonic many-body field theory with a quartic interaction at the temperature $T = (k\beta)^{-1}$ (k denotes the Boltzman constant in the physical space R^3 and the partition functional is written in the form of a Bell-Wiegel path integral [7]

$$\begin{aligned} Z[T,\vec{B}] &= \int_{\psi(r,0)=\psi(r,\beta)} D^F[\psi(r,t)] \int_{\bar{\psi}(r,0)=\bar{\psi}(r,\beta)} D^F[\bar{\psi}(r,t)] \\ &\times \exp\left\{-\frac{1}{2}\int_0^\beta dt \int_\Omega d^3r \psi^*(r,t)\left[-\frac{\hbar^2}{2m}\left(i\vec{\nabla} - \frac{e\hbar}{mc}\vec{B}\right)^2 + \frac{\partial}{\partial t}\right]\psi(r,t)\right\} \\ &\times \exp\left\{-\frac{1}{2}\int_0^\beta dt \int_\Omega d^3r d^3r \cdot |\psi(r,t)|^2 V(r-r')|\psi(r',t)|^2\right\} \end{aligned} \tag{3.52}$$

Note the presence of the external random magnetic vector potential supposed to satisfy the white-noise statistics with randomness strenght λ

$$E_B\left\{(\mathrm{rot}\vec{B})_i(\vec{r})(\mathrm{rot}\vec{B})_j(\vec{r}')\right\}=\lambda\delta^{(3)}(\vec{r}-\vec{r}')\delta_{ij} \tag{3.53}$$

and the non-relativistic field excitations interacting through a short-range pair potential $V(r-r')$.

At this point, we re-write the partition functional by means of the Siegert'st trick of reducing the non-local spatial pair interaction by an independent interaction of each Schörindger field excitation with a fluctuating external scalar field $\phi(\vec{r},t)$ with a Gaussian (non white) statistics:

$$E_\phi\left\{\phi(\vec{r},t)\phi(\vec{r}',t')\right\}=V(\vec{r}-\vec{r}')\delta(t-t') \tag{3.54}$$

One finds, thus, the following result for the partition functional written as statistics averages over ensembles of the physical random magnetic field rot $\vec{B}(r,t)$ and the auxiliary scalar field $\phi(\vec{r},t)$. Namely

$$\begin{aligned}&E_B\left\{Z(T,\vec{B})\right\}\\&=E_B\left\{E_\phi\left[\overset{-\frac{1}{2}}{\det}\left(\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\left(i\vec{\nabla}-\frac{e\hbar}{mc}\vec{B}\right)^2+i\phi(\vec{r},t)\right)\right]\right\}\end{aligned} \tag{3.55}$$

Let us go from the field path integrals on eq.(3.55) to the ensemble of spatial loops through a loop expansion for the functional determinant resulting from integrating out the Schrödinger Bosonic matter quantum fields. It yields as a result the following functional defined on the Bosonic three-dimensional loop space $\{\vec{x}(\sigma),0\le\sigma\le\beta,\vec{x}(0)=\vec{x}(\beta)=\vec{r}\}$

$$\begin{aligned}&lg\overset{-\frac{1}{2}}{\det}\left(\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\left(i\vec{\nabla}-\frac{e\hbar}{mc}\vec{B}\right)^2+i\phi(r,t)\right)\\&=+\frac{1}{2}\left\{N\int_\Omega d^3r\left[\int_{\vec{x}(0)=\vec{r}}^{\vec{x}(\beta)=\vec{r}}D^F[\vec{x}(\sigma)]\exp\left(-\frac{1}{2}m\int_0^\beta(\vec{x}(\sigma))^2d\sigma\right)\right.\right.\\&\left.\left.\exp\left(\frac{ie}{\hbar c}\int_0^\beta\vec{B}(\vec{x}(\sigma))\dot{\vec{x}}(\sigma)d\sigma\right)\times\exp\left(-\int_0^\beta\phi(\vec{x}(\sigma),\sigma)d\sigma\right)\right]\right\}\end{aligned} \tag{3.56}$$

where we have introduced explicitly the integer N, given by the number of different Bosonic matter species.

After substituting the purely Bosonic loop space eq.(3.56) into the statistics averages as given by eq.(3.55) and evaluating them by means of a cummulant expansion (in a generic from) and valid, at least for the limit $N\to 0$, [1].

$$E\{e^{Nf}\}=\exp\left\{N\langle f\rangle+\frac{1}{2}N^2\left(\langle f^2\rangle-\langle f\rangle^2\right)+O(N^3)\right\}, \tag{3.57}$$

one obtains explicitly that the dominant behavior of the random magnetic field average on eq.(3.56) is governed by the three-dimensional analogous of that area-surface functional eq.(3.9) of secton 1

$$E_{\vec{B}}\left\{\exp\left(\frac{ie}{\hbar c}\int_{\Sigma}(\text{rot}\vec{B})(\Sigma)d\vec{\sigma}\right)\right\}$$
$$= \exp\left\{-\frac{\lambda e^2}{\hbar^2 c^2}\int_{\Sigma_r}\int_{\Sigma_{r'}} d\vec{\sigma}(\vec{r})\delta^{(3)}(\vec{r}-\vec{r}')d\vec{\sigma}(\vec{r}')\right\} \tag{3.58}$$

where Σ is the "minimal" area surface bounded by the bosonic closed contour (loop) $\vec{x}(0)$ entering on the loop path integral eq.(3.56).

As a consequence of eq.(3.58), one can see that for a large white-noise magnetic field strenght $\lambda \to \infty$, this averaged phase-factor is only non-zero for a surface Σ of zero area, which is equivalent to the suppression of the quantum phenomena and reducing the quantum gas partition functional eq.(3.52) to a classical gas partition functional since all closed quantum trajectories reduce to the loop base point (see theorem 10.1 in second reference of ref.[7]).

Hence, one can see again that quantum phenomena in fluctuating magnetic field can be viewed as quantum phenomena in a dissipative media that destroys quantum phase coherence and leading to the theory's triviality.

3.5. The Static Confining Potential in a Tensor Axion Model

One of the still unsolved basic problem in the Gauge theory for strong interactions as given by Quantum Chromodynamics is to produce arguments for the colour charge confinement of the coloured Q.C.D.' field excitations, quarks and gluons [9].

Some time ago, through a somewhat intrincate path integral analysis, A.M.Polyakov [10] has proposed that – at least at the t'Hooft large number of colors limit – one should expect that the basic loop space dynamical variable as described by the averaged SU(∞) Wilson Loop (in the Euclidean world).

$$\langle W[C]\rangle^{(\infty)} = \left\langle Tr\mathcal{P}\left\{\exp\left[+i\int A_\mu dx^\mu\right]\right\}\right\rangle^{(\infty)} \tag{3.59}$$

should be equivalently represented from a calculational point of view by a string-like functional integral Ansatz based on a coupling of an abelian rank-two tensor field $B_{\mu\nu}(x)$ – the called by us of Polyakov's axion field – with the dynamics of random surfaces S_c living in a mathematical space [11], however possessing as boundary the previous loop C, understood as the quark-antiquark space-time physical (on-shell) Feynman trajectory, with the axion effective dynamics of $B^2 << (dB)^2$.

$$\langle W[C]\rangle_{Q.C.D.(\infty)} = \left\{\frac{\int D^F[B_{\mu\nu}]e^{-S[B_{\mu\nu}]}e^{(i\int_{S[C]}B_{\alpha\beta}(x)d\sigma^{\alpha\beta}(x)\delta(x-c))}}{\int D^F[B_{\mu\nu}]e^{-S[B_{\mu\nu}]}}\right\} = \langle\Phi[C]\rangle_{(B)}. \tag{3.60}$$

The effective local effective axion action is thus given explicitly by

$$S(B) = \frac{1}{4e^2}\int d^\nu x \left(B_{\mu\nu}^2 + dB \arcsin\left(\frac{dB}{m^2}\right) - \sqrt{m^4 - (dB)^2}\right)(x)$$
$$\stackrel{m^2\to\infty}{\sim} \frac{1}{4e^2_{\text{bare}}}\int d^\nu x \left(B_{\mu\nu}^2 + \frac{(dB)^2}{m^2} + m^2\sqrt{1-\left(\frac{dB}{m^2}\right)}\right)$$
$$\stackrel{m^2\to\infty}{\sim} \frac{1}{4e^2_{\text{bare}}}\int d^\nu x \left(B_{\mu\nu}^2 + \frac{(dB)^2}{m^2}\right)(x) \sim \frac{1}{4e^2_{\text{bare}}m^2}\int d^\nu x (dB)^2(x) \qquad (3.61)$$

where $m^2 = O(N)$ is a dimensional transmutation parameter on the $SU(\infty)$ gauge coupling constant $e^2_\infty = \lim_{N\to\infty}(e^2_{bare}N) < \infty$ and $\Phi(C) = \exp(i\int_{S[C]}(Bd\sigma)\delta(x-c))$ denotes the on-shell phase flux on the string C, boundary of the mathematical random surfaces $S[C]$. It is very important at this point of our exposition to call the reader attention that the phase flux factor on the left-hand side of eq.(3.60) is taken to be of a form of “string on-shell vertex” by considering the point x constrained to be on the physical pair trajectory C. It is the purpose of this section 3.6 – to evaluate the static potential between two static charges (with opposite charge signal) on the above mentioned random surface axion-rank-two tensor abelian field theory at $SU(\infty)$ eq.(3.61) by means of the dimensional regularization scheme [9] and show exactly its so much envisaged color-charge confining property; a first basic physical requirement to consider the axion-string propose eq.(3.60)–eq.(3.61) as an useful calculational scheme – at least as a leading effective quantum geometric field theory for Q.C.D (SU(∞)).

Finally in section 3.6, we address the problem of implementing a non-perturbative self-avoiding representation for a $\lambda\varphi^4$-closed string field theory by the same procedure used years ago by Symanzik in his non-perturbative self-avoiding contour representation for the usual $\lambda\phi^4$ (point like) (see chapter 3.1) field theory and underlying the Axion-String Polyakov framework for Q.C.D.$(SU(\infty))$.

3.6. The Confining Potential on the Axion-String Model in the Axion Higher-Energy Region

The static potential between two charges of opposite signal separated by a space-like distance R is computed in the path-integral framework by considering the vacuum energy of the rank-two tensor theory in the presence of the boundary flux of such colored charges, namely

$$V(R) = \lim_{T\to\infty} -\frac{1}{T}\langle\left[\left\langle W[C_{(R,T)}]\right\rangle^{(\infty)}\right] =$$
$$= \lim_{T\to\infty} -\frac{1}{T}\langle\left[\langle\Phi[C]\rangle_{(B)}\right] \qquad (62a)$$

where the rectangle $C_{(R,T)}$ denotes the space-time (euclidean) closed trajectory of the neutral pair and $<>_{(B)}$ average is defined explicitly by the Gaussian path integral eq.(7.2) with the rank-two tensor weight as given by the effective largee N free action eq.(3.61).

In order to evaluate the static potential at our proposed rank-two tensor theory at higher energy as given by eq.(3.62a) it appears convenient to re-write the on-shell abelian axion flux given by eq.(3.60) by means of an external current $J_\mu(x;C_{(R,T)})$ solely circulating around the pair finite-time propagation $(-\frac{T}{2} \leq t \leq +\frac{T}{2})$ space-time quark-antiquark trajectory $C_{(R,T)} = \{x_\mu(s), a \leq s \leq b\}$, namely

$$\begin{aligned} & i\int_{S[C_{(R,T)}]} B_{\alpha\rho}(x) d\sigma^{\alpha p}(x)\delta(x-c) = \\ & \equiv \int d^\nu x B_{\alpha\rho}(x)\left[i\oint_{C_{(R,T)}} \delta^{(\nu)}(x - x_\mu(s) x^\alpha(s) dx^\rho(s)\right] \end{aligned} \tag{3.62b}$$

Note that the point x on the planar surface $S[C_{(R,T)}] = \{\sigma x_\mu(s), 0 \leq \sigma \leq 1, a \leq s \leq b\}$ area tensor on the left-hand side of eq.(3.62b) is constrained to be on the physical planar loop $C_{(R,T)} = \{x_\mu(x), a \leq s \leq b\}$ as written on the right-hand side of this equation by mean of the delta function $\delta(x-c) \equiv \delta(\sigma - 1)$.

The Gaussian path integral eq.(3.61) can be exactly evaluated and yielding the following effective result where we note the appearance of the fourth-order Mandelstam effective propagator as the leading effective propagator in the analysis.

$$\begin{aligned} V(R) \sim \lim_{T\to\infty} -\frac{1}{T}\langle \Big[\exp\Big\{\frac{1}{2}\int d^\nu x d^\nu y J_\mu(x;C_{(R,T)}) \\ \times\ D_m(x-y) J^\nu(y, C_{(R,T)}\Big\}\Big]. \end{aligned} \tag{3.63a}$$

Here the purely fourth-order Mandelstam propagator [9] in momentum space is given by

$$D_m(x-y) = \frac{1}{(2\pi)^\nu}\int d^\nu p e^{ip(x-y)}\frac{1}{|p|^4} \tag{3.63b}$$

and the purely vectorial contour form factor is defined by the pair physical trajectory on the space-time and reads explicitly as (see eq.(3.62b)

$$J_\mu(x, C_{(R,T)}) = ie\oint_{C_{(R,T)}} \delta^{(\nu)}(x_\mu - x_\mu(s)) dx_\mu(s) \tag{3.63c}$$

The evaluation of eq.(3.63a) can be accomplished by writing it in momentum space

$$\begin{aligned} V(R) = \lim_{T\to\infty} -\frac{1}{T}\Big[\int \frac{d^\nu p}{(2\pi)^\nu} f_\mu(p_\alpha;C_{(R,T)}) \\ \times\ \frac{1}{p^4} f_\mu(-p_\alpha, C_{(R,T)})\Big] \end{aligned} \tag{3.64}$$

with the momentum-space contour form factors

$$f_\mu(p_\alpha, C_{(R,T)}) = ie\int_{C_{(R,T)}} e^{-ip_\mu x_\mu(s)} dx_\mu(s) \tag{3.65}$$

A simple evaluation of eq.(3.65) provides the solutions

$$f_0(p_\alpha, C_{(R,T)}) = -\frac{4e}{p_0}\sin\left(\frac{p_0 T}{2}\right)\sin\left(\frac{p_1 R}{2}\right) \tag{3.66}$$

and

$$f_1(p_\alpha, C_{(R,T)}) = +\frac{4e}{p_1}\sin\left(\frac{p_0T}{2}\right)\sin\left(\frac{p_1R}{2}\right) \tag{3.67}$$

After inserting the contour form factors eq.(3.66). eq.(3.67) into eq.(3.62), we obtain as a result

$$V(R) = \lim_{T\to\infty} +\frac{1}{T}\left\{+16e^2\int_{-\infty}^{+\infty}\frac{dp_1}{(2\pi)}\frac{\sin^2(\frac{p_1T}{2})}{p_1}\right.$$
$$\left.\times\left[\int\frac{d^{\nu-2}\hat{p}}{(2\pi)^{\nu-2}}\left(\int_{-\infty}^{+\infty}\frac{dp_0}{(2\pi)}\frac{(p_0^2+p_1^2)}{p_0^2}\sin^2\left(\frac{p_0T}{2}\right)\frac{1}{(p_0^2+p_1^2+\hat{p}^2)^2}\right)\right]\right\} \tag{3.68}$$

Note that we have considered the pair spatial-static trajectory $C_{(R,T)}$ contained in a two-dimensional sub-space of the (Euclidean) space-time R^ν in a such way that we can decompose the vector $\vec{p}\in R^\nu$ as $\vec{p} = p_0\vec{e}_0 + p_1\vec{e}_1 + \hat{p}$, where $\hat{p}$ is the projection of $\vec{p}$ over the sub-space perpendicular to the sub-space $\{\vec{e}_0, \vec{e}_1\}$ containing $C_{(R,T)}$.

The integration in the p_0-variable is easily perfomed by using the formulae given below.

$$\widetilde{F}_1(a,c) = \int_{-\infty}^{+\infty}dx\frac{\sin^2(ax)}{(x^2+c^2)} = \frac{\pi}{2c}(1-e^{-2ac}) \tag{3.69}$$

and

$$\int_{-\infty}^{+\infty}dx\frac{\sin^2(ax)}{(x^2+c^2)^2} = \widetilde{F}_2(a,c) = -\frac{d}{d(c^2)}(\widetilde{F}_1(a,c)) \tag{3.70}$$

As a consequence we have that

$$\lim_{a\to\infty}\frac{1}{a}\widetilde{F}_2(a,c) = -\frac{d}{d(c^2)}\left\{\lim_{a\to\infty}\frac{1}{a}\widetilde{F}_1(a,c)\right\} = 0 \tag{3.71}$$

Note either that we have the additional formulae:

$$\lim_{a\to\infty}\frac{1}{a}\int_{-\infty}^{+\infty}dx\sin^2(ax)\frac{1}{x^2(x^2+c^2)^2}$$
$$= \lim_{a\to\infty}\frac{1}{a}\left\{-\frac{d}{d(c^2)}\left[\int_{-\infty}^{+\infty}dx\sin^2(ax)\frac{1}{x^2(x^2+c^2)}\right]\right\}$$
$$= \lim_{a\to\infty}\frac{1}{a}\left\{-\frac{d}{d(c^2)}\left[\frac{\pi}{4c^2}\left(2a-\frac{1}{c}(1-e^{-2ac})\right)\right]\right\} = \frac{\pi}{2c^4} \tag{3.72}$$

As a consequence we get the following explicitly result for the p_0-integration and the associated ergodic limit, where only the result provenient from the analogous of eq(3.72) survive at the limit of $T\to\infty$.

$$\lim_{T\to\infty}\frac{1}{T}\left\{\int\frac{dp_0}{(2\pi))}\left(1+\frac{p_1^2}{p_0^2}\right)\sin^2\left(\frac{p_0T}{2}\right)\right.$$
$$\left.\frac{1}{(p_0^2+p_1^2+\hat{p}^2)^2}\right\} = \frac{4p_1^2}{\pi}\frac{1}{(p_1^2+\hat{p}^2)^2} \tag{3.73}$$

Inserting the result eq.(3.73) back the complete expression eq.(3.68), we have the partial result for the static potential between the neutral color charges in rank-two effective theory for Q.C.D. (∞)

$$V(R) = 16e^2 \left[\int_{-\infty}^{+\infty} \frac{dp_1}{(2\pi)} \frac{\sin^2(\frac{p_1 R}{2})}{p_1^2} \int \frac{d^{\nu-2}\hat{p}}{(2\pi)^{\nu-2}} \cdot \frac{4p_1^2}{(p_1^2+\hat{p}^2)^2} \right] \tag{3.74}$$

Let us evaluate the $(\nu-2)$-dimensional integration on eq.(3.74).

Firstly we note that in the dimensional regularization scheme

$$\int_{-\infty}^{+\infty} \frac{d^{\nu-2}\hat{p}}{(2\pi)^{\nu-2}} \cdot \frac{1}{(p_1^2+\hat{p}^2)^2} = \frac{\Gamma(2-\frac{(\nu-2)}{2})}{(2\pi)^{\frac{\nu-2}{2}}\Gamma(2)} |p|^{\nu-6} \tag{3.75}$$

where we have used the formula below to obtain explicity the above written result.

$$\int \frac{d^{\nu}p}{(2\pi)^{\nu}} \cdot \frac{1}{(p^2+a)^{\gamma}} = \frac{\Gamma(\gamma-\frac{\nu}{2})}{(4\pi)^{\frac{\nu}{2}}\Gamma(\gamma)} (a)^{\frac{\nu}{2}-\gamma} \tag{3.76}$$

We arrive, thus, at the final effective result for the $(\nu-2)$ integration on eq.(3.74), with a renormalized constnat $\bar{c}(\nu)$ finite at the physical limit of $\nu \to 4$.

$$\int \frac{d^{\nu-2}\hat{p}}{(2\pi)^{\nu-2}} \cdot \frac{1}{(p_1^2+\hat{p}^2)^2} = \bar{c}(\nu)|p_1|^{\nu-6} \tag{3.77}$$

We face, thus, the final (and last!) p_1-integration

$$V(R) = 16e^2 \times \frac{4}{\pi} \times \bar{c}(\nu) \int_{-\infty}^{+\infty} \frac{dp_1}{(2\pi)} \frac{\sin^2(\frac{p_1 R}{2})}{p_1^2} \times p_1^2 \times |p_1|^{\frac{\nu-6}{2}} \tag{3.78}$$

The p_1-integration is easily evaluated as a Fourier transform on the sense of Distribution theory [11] by means of the formula.

$$\int_{-\infty}^{+\infty} e^{ipx}|x|^{\beta}dx = -2\sin\left(\frac{\beta\pi}{2}\right)\Gamma(\beta+1)|p|^{-\beta-1} \tag{3.79}$$

and the trivial identity

$$\sin^2(x) = -\frac{1}{4}(e^{2ix}+e^{-2ix}-2) \tag{3.80}$$

Finally, we obtain the expression for the static inter-quark potential in the Axion Effective Gluon theory in space-time R^{ν}

$$\begin{aligned} V(R) = & -\frac{64e^2}{\pi} \times \bar{c}(\nu) \\ & \times \left\{ -\frac{2}{2\pi}\sin\left(\frac{\pi}{2}(\nu-6)\right)\Gamma(\nu-5)\left|\frac{R}{2}\right|^{-\nu+5} \right\} \end{aligned} \tag{3.81}$$

By passing to the Physical limit of $\nu \to 4$, and by taking into account that

$$\lim_{\nu\to 4} \sin\left(\frac{\pi}{2}(\nu-6)\right)\Gamma(\nu-4-1) \sim \frac{-1}{\nu-5} \times \left(\Gamma(\nu-4)\times\sin\left(\frac{\pi}{2}(\nu-4)\right)\right) \sim +\pi \tag{3.82}$$

We obtain the finite result for the static inter-quark potential in the Axion Gluonic effective theory in R^4

$$V(R) = +e^2\bar{A}|R| \tag{3.83}$$

Here $\bar{A}$ is a model-calculation postive constant, which detalils will be not nedeed on our study.

We see, thus, that the Effective Axion's path integral representation quark potential leads to the confining property and not to a dynamics of charge color screening as it would be expected in a first analysis [9]. This is the main result of this section.

Finally let us consider the generating functional of the color neutral quark vectorial abelian currents on Q.C.D(SU(∞)). Namely

$$\begin{aligned}
&\left\langle \exp\left\{ ie\int d^\nu x(\bar{\psi}\gamma^\mu\psi)(x)J^\mu(x)\right\}\right\rangle_{Q.C.D.(\infty)} \\
&= \left\langle \det\left[i\gamma^\mu(\partial_\mu + eA_\mu^{(\infty)} + J_\mu)\right]\right\rangle_{YM(\infty)} \\
&= Z[J_\mu(x)]
\end{aligned} \tag{3.84}$$

Here $<>_{YM}$ denotes the quantum average defined by the Yang-Mills theory $\{A_\mu^{(\infty)}\}$ at the topological t'Hooft limit of SU(∞)]9] (or chapter 4).

In the loop space "bosonization" framework of ref.[8]-ref-[10], we can re-write eq.(3.84) into the quantum geometrical (off-shell) form involving solely a dynamics of Loops, Random Surfaces with arbitrary topology and the general axion tensor field.

$$\begin{aligned}
&Z[J_\mu(x)]_{SU(\infty)} = \\
&\sum_{\{S(C_{xx})\}}\left\{\left\langle \exp - \left\{ \sum_{C_{xx}} \Phi[C_{xx}, J_\mu]\exp\left(ie\int_{S(C_{xx})} B_{\mu\nu}d\sigma^{\mu\nu}\right)\right\rangle_B\right\}\right\}
\end{aligned} \tag{3.85}$$

where $\Phi[C_{xx}, J_\mu]$ is the Wilson loop space variable associated to the quarks abelian current classical source $J_\mu(x)$.

$$\Phi[C_{xx}, J_\mu] = \exp\left(i\oint_{C_{xx}} J_\mu dx^\mu\right). \tag{3.86}$$

The sum over the closed loops C_{xx} with end-point x is given by the proper-time bosonic path integral below

$$\sum_{C_{xx}} = \int_0^\infty \frac{dT}{T}\int d^\nu x\int_{x(0)=x=x(T)} D^F[X(\sigma)]\exp\left\{ -\frac{1}{2}\int_0^T \dot{X}^2(\sigma)d\sigma\right\} \tag{3.87}$$

and the nedded sum over off shell random surfaces $S(C_{xx})$-bounding the "fractal" (Hausdorff dimension 2) on-shell physical countors C_{xx}-should be defined by the path-integrals of refs.[11] (chapter 2). It is worth remark that we have here withdraw from the flux-phase factor (which is a kind of string vertex) the on-shell condition used on our static-potential analysis as expressed by the constraint for the axion flux be restricted to the loop C_{xx} as imposed by eq.(3.62b).

As a consequence the generating functional of the abelian quark currents leads naturally to a purely dynamical quantum geometrical objects evaluations. For instance, the two-point Q.C.D. $(SU(\infty))$ abelian quark current – an physical observable – has the quantum geometrical closed expression in this phenomenological quantum geometric framework.

$$\left\langle (\bar{\psi}\gamma^{\alpha}\psi)(x)(\bar{\psi}\gamma^{\beta}\psi)(y)\right\rangle^{(\infty}) = F_{\alpha\beta}((x-y))_{Q.C.D.(\infty)}$$
$$\sim \sum_{\{S(C_{xx})\}} \left\{ \frac{\delta^2}{\delta J_{\alpha}(x)\delta J_{\beta}(y)} \left[\left\langle \exp - \left(\sum_{C_{xx}} \Phi[C_{xx}, J_{\mu}] \exp\left(ie\int_{S(C_{xx})} B_{\mu\nu} d\sigma^{\mu\nu} \right)\right)\right\rangle\right]\right\}_{J(x)\equiv 0} \tag{3.88}$$

After evaluating the functional derivatives, one obtains a quantum loop-surface space partitional functional for on-shell non-planar closed loops C_{xx} and off-shell random surfaces $S(C_{xx})$ bounding them. We get as a result, the exactly expression below for the vectorial current quark form factor at the t'Hooft limit of large number of colors

$$F_{\alpha\beta}((x-y))_{Q.C.D.(\infty))} = \sum_{\{S(C_{xx})\}} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \right.$$
$$\times \left[\sum_{\{C_{x_1x_1}\}} \cdots \sum_{\{C_{x_nx_n}\}} \left(\oint \delta^{(\nu)}(C_{x_1x_1}-x)\, dC^{\alpha}_{x_1x_1} \right) \left(\oint \delta(C_{x_1x_1}-y) dC^{\beta}_{x_1x_1} \right) \right.$$
$$\left. \times \cdots \left(\oint \delta^{(\nu)}(C_{x_nx_n-x})\, dC^{\alpha}_{x_nx_n} \right) \left(\oint \delta^{)\nu)}(C_{x_nx_n}-y)\, dC^{\beta}_{x_nx_n} \right) \right] \times$$
$$\left. \exp\left[-\frac{1}{2}e^2 \sum_{i,j=0}^{n} \left(\int_{S(C_{x_ix_i})} d\sigma^{\alpha\beta}(x_i) \int_{S(C_{x_jx_j})} d\sigma^{\alpha'\beta'}(x_j)(-\partial^2)^{-1}(x_i,x_j)\delta^{\alpha\alpha'}\delta^{\beta\beta'} \right)\right]\right\} \tag{3.89}$$

Studies on such dynamics of gas of loops and self-avoiding surfaces [11] will be presented in next section in a $\lambda\phi^4$-String Field theory closely related to the our proposed Q.C.D.$(SU(\infty))$-string representation eq.(3.60).

3.7. A $\lambda\phi^4$ String Field Theory as a Dynamics of Self Avoiding Random Surfaces

Let us start our analysis by considering the generating functional of the following mathematical $\lambda\phi^4$ Closed String Field Path Integral on the critical dimension $D = 26$ (see M. Kaku book of ref.[4] for the general covariant discussion in a closely related, but different string Q.F.T model).

$$Z[J(C)] = \int D^F[\Phi(C)] \times$$
$$\exp\left\{ -\sum_{[C]} \left(\Phi(C)\hat{\Delta}_C\Phi(C) + J(C) \right) \right.$$
$$\left. +\lambda^2 \left(\sum_{[C_0,C_1]} \delta^{(D)}(C_0-C_1)\Phi^2(C_0)\Phi^2(C_1) \right) \right\}. \tag{3.90}$$

The notation is as follows: i) the string field is given by a functional $\Phi(C)$ defined over the space of all closed string configurations $C = \{X_\mu(\sigma), -\pi \le \sigma \le \pi, X_\mu(-\pi) = X_\mu(\pi)\}$; ii) The sum over all closed string configurations is defined by the path integral

$$\sum_{(C)} = \int d^D x \left(\int_{X_\mu(-\pi)=X_\mu(\pi)=X_\mu} D^F[X_\mu(\sigma)] \right.$$
$$\left. \exp\left(-\frac{1}{2} \int_{-\pi}^{\pi} (\dot{X}_\mu(\sigma))^2 d\sigma \right) \right); \tag{3.91}$$

iii) The critical $D = 26$ string free kinetic term is associated to the string D'Alembertian (see chapters 9, 11, 12)

$$\hat{\Delta}_C = \frac{1}{2} \frac{\delta^2}{\delta^2 X_\mu(\sigma)} - \frac{1}{2\pi\alpha'} |X'_\mu(\sigma)|^2; \tag{3.92}$$

iv) The string functional measure in equation eq.(3.90) is given by the usual Feynman product measure

$$D^F[\Phi(C)] = \prod_{\{X_\mu(\sigma)\}} d\Phi(X_\mu(\sigma)); \tag{3.93}$$

and iv) The interaction action in equation eq.(3.90) is given by the following vertex with D-dimensional delta functions supported on the string configurations and involving a positive λ^2 coupling constant in the extrinsic space

$$\lambda^2 \sum_{\{C_0, C_1\}} \delta^{(D)}(C_0 - C_1). \tag{3.94}$$

The proposed interaction vertex was defined in such way that it allows the replacement of the four string field interaction in equation eq.(3.90) by an independent interaction of each string with an extrinsic Gaussian stochastic field $W(x)$ followed by an average over the fluctuating field $W(x)$. It is instructive to point out that similar procedure is well known in many-body path integral quantum field theory [15]. So, we can write equation eq.(3.59) in the following convenient form

$$Z[J(C)] = \left\langle \int D^F[\Phi(C)] \right.$$
$$\exp\left\{ -\sum_{\{C\}} \Phi(C)(\hat{\Delta}_C - i\lambda W(C))\Phi(C) \right.$$
$$\left. \left. + J(C)\Phi(C) \right\} \right\rangle_W. \tag{3.95}$$

Here, $W(C)$ means that the external stochastic field $W(x)$ is projected on the string configuration C

$$W(C) = \int_{-\pi}^{\pi} d\sigma W(X_\mu(\sigma)) \tag{3.96}$$

and satisfies the white noise stochastic correlation function with $x \in R^D$

$$\langle W(x) W(x') \rangle_W = \delta^{(D)}(x - x'). \tag{3.97}$$

In the free case, $\lambda = 0$, the String Path Integral Field Theory equation eq.(3.95) is exactly soluble with the following quantum string field generating functional

$$\frac{Z[J(C)]}{Z[J(C) \equiv 0]} = \exp\left\{ +\frac{1}{2} \sum_{[C,\bar{C}]} J(C)\hat{\Delta}^{-1}(C,\bar{C})J(\bar{C}) \right\}. \tag{3.98}$$

Here $\hat{\Delta}^{-1}(C,\bar{C})$ denotes the Green's Function for the string Laplacian and is given explicitly by the Random Surface Path Integral

$$\hat{\Delta}^{-1}(C,\bar{C}) = \int_0^\infty dA \left\langle C|e^{-A\hat{\Delta}_C}|\bar{C}\right\rangle, \tag{3.99}$$

with

$$\left\langle C|e^{-A\hat{\Delta}_c}|\bar{C}\right\rangle = \int_{\substack{X^\mu(\sigma,\bar{0})=C^\mu(\sigma)\\ X^\mu(\sigma,A)=\bar{C}^\mu(\sigma)}} D^F[X^\mu(\sigma,\tau)]\times$$
$$\exp\left(-\frac{1}{2}\int_0^A d\tau \int_{-\pi}^{\pi} d\sigma \left[(\partial_\sigma X^\mu)^2 + (\partial_\tau X^\mu)^2\right](\sigma,\tau)\right). \tag{3.100}$$

In order to reformulate the closed string field theory equation eq.(3.95) as a dynamics of self-Avoiding Random Surface, we evaluate formally the Gaussian Field Path Integral in equation eq.(3.95)

$$Z[J(C)] = \Big\langle [\det(\hat{\Delta}_C + i\lambda W(C)]^{-1/2}\times$$
$$\exp\left\{ +\frac{1}{2}\sum_{\{C,\bar{C}\}} J(C)(\hat{\Delta}_C + i\lambda(C))^{-1}J(\bar{C})) \right\}\Big\rangle \tag{3.101}$$

Let us define the string functional determinant in equation eq.(3.101) by the proper-time technique

$$\frac{1}{2}\log\det[\hat{\Delta}_C + i\lambda W(C)] =$$
$$= -\int_0^\infty \frac{dA}{A} \sum_{(C,\bar{C})} \delta^{(F)}(C-\bar{C})\times$$
$$\langle C|\exp(-A(\hat{\Delta}_C + i\lambda W(C))|\bar{C}\rangle \tag{3.102}$$

with

$$\langle C|\exp(-A(\hat{\Delta}_C + i\lambda W(C)))|\bar{C}\rangle =$$
$$\int_{\substack{X^\mu(\sigma,0)=C^\mu(\sigma)\\ X^\mu(\sigma,A)=\bar{C}^\mu(\sigma)}} D^F[X^\mu(\sigma,\tau)]\times$$
$$\exp\Big\{ -\frac{1}{2}\int_0^A d\tau \int_{-\pi}^{\pi} d\sigma (\partial^a X^\mu)^2(\sigma,\tau)$$
$$-i\lambda \int_0^A d\tau \int_{-\pi}^{\pi} d\sigma W(X^\mu(\sigma,\tau)) \Big\}. \tag{3.103}$$

By substituing equations eq.(3.102) and eq.(3.103) into equation eq.(3.101) and making a power expansion in the coupling constant λ, we obtain the String Field Theory equation eq.(3.90) as a Theory of Random Cylindrical Surfaces (with boundaries being closed string configurations) interacting with an external Gaussian Stochastic Field $W(x)$. The Gaussian average $< ... >_W$ may be straightforwardly evaluated at each order of the λ-power expansion and produces self-avoiding interaction among the cylindrical random surfaces similar to the usual self-avoiding Symanzik contour gas for the $\lambda\phi^4$ Field Theory. For instance, by neglecting the functional determinant on eq.(3.101), which physically means suppressing surfaces creation – annihilation (second-quantization) process, we have the following expression for free theory's propagator

$$\begin{aligned}
&\left\langle \Phi(C^{\text{in}})\Phi(C^{\text{out}})\right\rangle^{(0)} = \\
&\int_0^\infty dA \int_{\substack{X^\mu(\sigma,0)=C^{\text{in}} \\ X^\mu(\sigma,A)=C^{\text{out}}}} D^F[X^\mu(\sigma,\tau)]\times \\
&\exp\left\{ -\frac{1}{2}\int_0^A d\tau \int_{-\pi}^{\pi} d\sigma (\partial X^\mu)^2(\sigma,\tau) \right. \\
&\left. -i\lambda \int_0^A d\tau \int_{-\pi}^{\pi} d\sigma W(X^\mu(\sigma,\tau)) \right\}. \qquad (3.104)
\end{aligned}$$

Note that self-intersectińg lines are invariant under reparametrizations of the full string world sheet.

The next string quantum field correction for eq.(3.104) in our proposed framework will be given by

$$\begin{aligned}
&\left\langle \Phi(C^{\text{in}})\Phi(C^{\text{out}})\right\rangle^{(1)} = \\
&\int_0^\infty d\bar{A} \int_0^\infty \frac{dA}{A} \sum_{\{C,\bar{C}\}} \delta^{(F)}(C-\bar{C})\times \\
&\left\langle \left\langle C\right| \exp[-A(\hat{\Delta}_e + i\lambda W(c))] \left|\bar{C}\right\rangle \times \right. \\
&\left. \left\langle C_{\text{in}}\right| \exp[-\bar{A}(\hat{\Delta}_e + i\lambda W(c))] \left|C_{\text{out}}\right\rangle \right\rangle_W , \qquad (3.105)
\end{aligned}$$

where

$$\delta^{(F)}(C-\bar{C}) = \prod_{\pi\leq\sigma\leq\pi} \delta^{(D)}(C_\mu(\sigma) - \bar{C}_\mu(\sigma)). \qquad (3.106)$$

We may write eq.(3.106) in the form of a two body random surface path integral with self-avoiding interactions

$$\begin{aligned}
&\left\langle \Phi(C^{\text{in}})\Phi(C^{\text{out}})\right\rangle^{(1)} = \\
&\int_0^\infty d\bar{A} \int_0^\infty \frac{dA}{A} \sum_{\{C,\bar{C}\}} \delta^{(F)}(C-\bar{C}) \\
&\int_{\substack{X^\mu_{(1)}(\sigma,O)=C_\mu(\sigma) \\ X^\mu_{(1)}(\sigma,A)=\bar{C}_\mu(\sigma)}} D^F[X^\mu_{(1)}(\sigma,\tau)]
\end{aligned}$$

$$\int_{\substack{X^{\mu}_{(2)}(\sigma,O)=C^{\text{in}}_{\mu}(\sigma)\\ X^{\mu}_{(2)}(\sigma,A)=\bar{C}^{\text{out}}_{\mu}(\sigma)}} D^F[X^{\mu}_{(2)}(\sigma,\tau)]$$
$$\exp\left(-\frac{1}{2}\int_0^A d\tau\int_{-\pi}^{\pi} d\sigma(\partial X^{\mu}_{(1)})^2(\sigma,\tau)\right)\times$$
$$\exp\left(-\frac{1}{2}\int_0^{\bar{A}} d\tau'\int_{-\pi}^{\pi} d\sigma'(\partial X^{\mu}_{(2)})^2(\sigma',\tau')\right)\times$$

$$\exp\left(-\frac{\lambda^2}{2}\int_0^A d\tau\int_0^{\bar{A}} d\tau'\int_{-\pi}^{\pi}\right.$$
$$d\sigma\int_{-\pi}^{\pi} d\sigma'\delta^{(D)}\left(X^{\mu}_{(1)}(\sigma,\tau)-X^{\mu}_{(2)}(\sigma',\tau')\right)\Big)\times$$
$$\exp\left(-\frac{\lambda^2}{2}\int_0^A d\tau\int_0^{\bar{A}} d\tau'\int_{-\pi}^{\pi}\right.$$
$$d\sigma\int_{-\pi}^{\pi} d\sigma'\delta^{(D)}\left(X^{\mu}_{(1)}(\sigma,\tau)-X^{\mu}_{(1)}(\sigma',\tau')\right)\Big)\times$$
$$\exp\left(-\frac{\lambda^2}{2}\int_0^A d\tau\int_0^{\bar{A}} d\tau'\int_{-\pi}^{\pi}\right.$$
$$\left. d\sigma\int_{-\pi}^{\pi} d\sigma'\delta^{(D)}(X^{\mu}_{(2)}(\sigma,\tau)-X^{\mu}_{(2)}(\sigma',\tau'))\right). \tag{3.107}$$

Let us point out that the perturbative renormalizability of the interacting string propagator eq.(3.107), may be given by the renormalization group of the self avoiding random surfaces theories. An alternative regularization study for eq.(3.107) may be implemented in a pure geometrical framework as proposed in ref.[13] for the loop space formulation of point particle field theories). In order to implement this study for random surface, we start by extracting the trivial selfintersect points $X_{\mu}(\sigma,\tau)=X_{\mu}(\sigma',\tau')$ with $\sigma=\sigma',\tau=\tau'$ from the λ^2 interaction term of eq.(3.107). Thus, let us introduce a D-dimensional regularization parameter Λ on the self avoiding D-dimensional interaction in order to extract the (geometrical) infinities associated to the trivial self-intersect surface points

$$I[X_{\mu}(\sigma,\tau)]=\frac{\lambda^2}{2}\int_0^A d\tau\int_0^A d\tau'\int_{-\pi}^{\pi} d\sigma\int_{-\pi}^{\pi} d\sigma'$$
$$\left(\int_{|k|<\Lambda} d^DK\exp\left[iK_{\mu}(X_{\mu}(\sigma,\tau)-X_{\mu}(\sigma',\tau'))\right]\right) \tag{3.108}$$

The above equation may be written in the more suitable form after introducing the extrinsic λ coupling constant as a scaling of the $X_{\mu}(\sigma,\tau)$ – field, i.e.:

$$I[X_{\mu}(\sigma,\tau)]=\frac{1}{2}C(D)\int_0^A d\tau\int_0^{\bar{A}} d\tau'\int_{-\pi}^{\pi} d\sigma\int_{-\pi}^{\pi} d\sigma'$$
$$\left(\int_0^{\Lambda} d|K|\cdot|K|^{D/2}\frac{1}{\lambda^{D/2}}|X_{\mu}(\sigma,\zeta)-X_{\mu}(\sigma',\zeta')|^{1-D/2}\right.$$
$$\left.\mathcal{J}_{\frac{D}{2}-1}\left(\frac{|K|}{\lambda^{2/D}}|X_{\mu}(\sigma,\zeta)-X_{\mu}(\sigma',\zeta')|\right)\right) \tag{3.109}$$

where $C(D)$ is a constant depending only on the space time dimension and $\mathcal{J}_\nu(x)$ denotes the usual Bessel Function of order ν.

By power expanding the Bessel Function we reduce equation eq.(3.109) to a sum of the form

$$I[X_\mu(\sigma,\tau)] = \frac{1}{2}C(D) \sum_{K=0}^{\infty} \frac{(-1)^k(\lambda^2 2/D)^k}{k!2^{2k}\Gamma(\frac{D}{2}-k)} I^{(k)}(X_\mu(\sigma,\tau),\Lambda), \tag{3.110}$$

where the partial contribuitons in equation eq.(3.107) are of the form

$$I^{(K)}[X_\mu(\sigma,\tau),A] = \int^{\Lambda} d|K|\cdot|K|^{D+2K-1} \int_{-\pi}^{\pi} d\sigma \int_{-\pi}^{\pi} d\sigma' \int_0^A d\tau \int_0^A d\tau' |X_\mu(\sigma,\tau) - X_\mu(\sigma',\tau')|^{2K}. \tag{3.111}$$

To regularize the infinities in equation eq.(3.111), we propose to introduce the already used parameter Λ in eq.(3.109) on the two dimensional string space-time $\{(\sigma,\tau); -\pi \le \sigma \le \pi; 0 \le \tau \le A\}$ by using the following unity decomposition into the integrand of equation eq.(3.111)

$$1 = \delta^{(2)}_{(\Lambda)}((\sigma,\tau)-(\sigma',\tau')) + \left[1-\delta^{(2)}_{(\Lambda)}((\sigma,\tau)-(\sigma',\tau'))\right], \tag{3.112}$$

where the regularized two dimensional delta function is given explicity by

$$\delta^{(2)}_{(\Lambda)}((\sigma,\tau)-(\sigma',\tau')) \begin{cases} \Lambda & \sigma-\frac{1}{\Lambda} \le \sigma' \le \sigma+\frac{1}{\Lambda} \\ & \tau-\frac{1}{\Lambda} \le \tau' \le \tau+\frac{1}{\Lambda} \\ 0 & \text{otherwise} \end{cases} \tag{3.113}$$

By Taylor expanding the integrand of eq.(3.111) around the point $\xi = (\sigma',\tau')$, where $\xi = (\sigma,\tau)$,

$$|X_\mu(\xi) - X_\mu(\xi')|^{2k} = \left\{ \sum_{\ell=2}^{\infty} \left[\sum_{r_1\ge1,r_2\ge1}^{r_1+r_2=\ell} D^{r_1}_\xi X^\mu(\xi) D^{r_2}_\xi(\xi)|\xi-\xi'|^{r_1+r_2} \right] \right\}^k, \tag{3.114}$$

inserting the identity eq.(3.112) and eq.(3.113) into eq.(3.111) and making use of the result

$$\int_{-\pi}^{\pi} d\sigma \int_0^A d\tau \delta^{(2)}_\Lambda((\sigma,\tau)-(\sigma',\tau')) (\sigma-\sigma')^n(\tau-\tau')^m f(\sigma',\tau') = \begin{cases} \Lambda^{(n+m)/2} f(\sigma',\tau') & n,m = \text{even} \\ 0 & \text{otherwise,} \end{cases} \tag{3.115}$$

we are able to show that the most general extrinsic counter term arising from the non-trivial self-intersect limit $\Lambda \to \infty$ is an exponential of a four variable quadratic polinomial with a renormalized extrinsic λ^R coupling constant,

$$\int_{-\pi}^{\pi} d\sigma \int_0^A d\tau \exp\{\mathcal{P}[\partial_\sigma X^\mu, \partial_\tau X^\mu, \partial_\sigma^2 X^\mu, \partial_\tau^2 X^\mu]\}. \tag{3.116}$$

All other contributions on the derivative order greater than the *second derivative vanishes* on the trivial self intersect limit of $\Lambda \to \infty$.

The contribution of the non trivial self intersect points associated to the term $(1 - \delta_{\Lambda'}(\sigma,\tau) - (\sigma',\tau')))$ at $\Lambda \to \infty$ leads to a kind of surface self-avoiding topological index [6]

$$⨍_0^A d\tau d\tau' ⨍_{-\pi}^{\pi} d\sigma d\sigma' \delta^{(D)}(X_\mu(\sigma,\tau) - X_\mu(\sigma',\tau')). \tag{3.117}$$

The slash in the integration symbols f in eq.(3.117) means that the trivial self intersect points $\sigma = \sigma', \tau = \tau'$ are excluded from the integrand.

We remark that eq.(3.107), after being renormalized as described above, describes a two dimensional super-renormalizable field theory on the string space-time $\{(\sigma,\tau), -\pi \leq \sigma \leq \pi, 0 \leq \tau \leq A\}$ since the counter term, eq.(3.116), generates a term to be added to the "free kinetic extrinsic string action" with the form $\approx C(\Lambda^R)[(\partial_\sigma^2 X^\mu)^2 + (\partial_\tau^2 X^\mu)^2]$ where $C(\lambda^R)$ is a function of the extrinsic renormalized self-supressing coupling constant [6].

Finally we comment that our proposed string quantum field theory is, in principle, different from those already proposed by other authors since our interaction vertex, eq.(3.90), is a combination of D-dimensional delta functions and not as functional delta functions as in ref [12] and directly inspired on the pure self-avoiding trivial case of eq.(3.89) on the extrinsic ultra-violet regime, namely [7].

$$\begin{aligned} &\int_{S(C_{xx})} \int_{S(C_{xx})} d\sigma^{\alpha\beta}(x_i)(-\partial^2)^{-1}(x_i,x_j) d\sigma^{\alpha\beta}(x_j) \\ &\to \int_{S(C_{xx})} \int_{S(C_{xx})} d\sigma^{\alpha\beta}(x_i)\delta^{(D)}(x_i - x_j) d\sigma^{\alpha\beta}(x_j) \end{aligned} \tag{3.118}$$

Appendix A.
A Convariant Version of the Proposed $\lambda\phi^4$ String Field Theory

In this appendix we will make comments on the covariance of the theory under the action of the string diffeomorphism group.

In order to have from the beginning a covariant string field theory we must consider our theory for sub-critical strings $D \leq 26$. The main change in our study is that we have to take into account $2D$ induced pure quantum gravity which is needed by the dynamical status acquired by the intrinsic metric field $g_{ab}(\sigma,\tau)$. This step may be easily implemented on the random surface path integrals, eqs.(3.100)-(3.104). For instance, the theory's propagator, eq.(3.104), will take the *reparametrization invariant form*.

$$\begin{aligned}
&\left\langle \Phi(C^{\text{in}})\Phi(C^{\text{out}})\right\rangle^{(0)} = \\
&\int_0^\infty dA \int_{\substack{X^\mu(\sigma,O)=C^\mu_{\text{in}}(\sigma)\\ X^\mu(\sigma,A)=C^\mu_{\text{out}}(\sigma)}} D^C[X^\mu(\sigma,\tau)] \int D^C[g_{ab}(\sigma,\tau)]\times \\
&\exp\left[-\frac{1}{2}\int_0^A d\tau \int_{-\pi}^{\pi} d\sigma(\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu)(\sigma,\tau)\right] \\
&\times \exp\left[-\frac{\lambda^2}{2}\int_0^A d\tau \int_0^A d\tau' \int_{-\pi}^{\pi} d\tau \int_{-\pi}^{\pi}\right. \\
&d\sigma' \sqrt{g(\sigma,\tau)}\delta^{(D)}(X_\mu(\sigma,\tau)-X_\mu(\sigma',\tau')) \\
&\left.\times \sqrt{g(\sigma',\tau')}\right].
\end{aligned} \tag{A1}$$

Unfortunately, the theory of sub-critical strings was not exactly solved yet. However, at $D = 26$ we can show that the $g_{ab}(\sigma,\tau)$ field decouples, from the full string propagator, eq.(A1), at least for the weak perturbative coupling phase for the λ-constant (the result for $\lambda = 0$ was proved by Polyakov). This result afford us to choose and, thus, to fix the decoupling gauge $g_{ab}(\sigma,\tau) = \delta_{ab}$ in our proposed theory.

It is worth to point out that in a rigorous mathematical procedure one should consider, as in usual gauge theories, first Ward-Takahashi identities associated to the diffeomorphism (non-conformal) group at $D \leq 26$. Thus, take the limit $D = 26$ on the these identities. Anyway, the physical objects in string theories are not the string propagators but the scattering amplitudes which are physical observables and may be calculated directly from the eq. (A1) and tested to have the necessary invariances as shown by a perturbative analysis in λ coupling constant.

We remark that difficulties in considering *non gauge fixed* thoeries is shared by others string field theories considerd in the literature as the B.R.S.T. and light cone string field theories.

As a final comment we notice that the important problem of invariances in string field theory is waiting the solution of the theory of sub-critical string (see chapter 3.1 and supplementary appendixes A and B at the end of this book).

Appendix B.
Our Proposed $\lambda\phi^4$ String Field Theory as an Infinite Component Field Theory of String Excitations

Let us consider a harmonic oscilator expansion for the closed string configuration with $X_\mu(0) = x_\mu$; i.e.:

$$X_\mu(\sigma) = x_\mu + \sum_{\substack{n\neq 0\\ n=-\infty}}^{+\infty} \mathcal{A}^\mu_n e^{in\sigma}. \tag{B1}$$

In this base, the second quantized string field will be decompose in all possible string excitations

$$\Phi[X_\mu(\sigma)] = O(x) + A_\mu(x)\mathcal{A}^\mu_{(1)} + \cdots B_{\mu_1\ldots\mu_N}(x)\mathcal{A}^{\mu_1}_{(N)}\cdots\mathcal{A}^{\mu_N}_{(N)} + \cdots \tag{B2}$$

The sum over all closed string configurations are weighted by (see eq.(3.60)

$$\int_{-\infty}^{+\infty} d^D x \int \prod_{(N_\mu)} d\mathcal{A}^\mu_{(N)} e^{-|\mathcal{A}^\mu_{(N)}|^2}. \tag{B3}$$

The Feynman product measure, eq.(3.97), is factorized in the product of all Feynman measures associated to the point-like field string excitations, eq.(B2), and thus

$$D^F[\Phi(C)] = \prod_{N=1}^{\infty} D^F[B_{\mu_1\ldots\mu_N}(x)]D^F(O(x)], \tag{B4}$$

with

$$\hat{\Delta}_c = -\frac{\partial^2}{\partial x_\mu^2} + \sum_{\substack{(N\neq 0)\\ N=-\infty}}^{\infty} \frac{\partial^2}{\partial\mathcal{A}^\mu_N \partial\mathcal{A}^\mu_{-N}}. \tag{B5}$$

Finally our proposed vertex takes the form

$$\delta^{(D)}(C_0 - C_1) = \int d^D k \exp iK^\mu \left[\sum_{N=-\infty}^{+\infty} \mathcal{A}^{\mu,(0)}_N e^{iN\sigma} - \sum_{N=-\infty}^{+\infty} \mathcal{A}^{\mu,(1)}_N e^{iN\sigma}\right]. \tag{B6}$$

After substituting the above writen equations in our proposed action, eq.(3.59), we obtain an interacting infinite-component field theory associated to the string excitations.

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[17] It is worth either to consider that this abelian external (divergenceless) white-noise field cames mathematically from a standard Stratonovich – Hubbard parametrization of a non-local charged piece of quarks Lagrangean arising from a interaction with an external apparatus, namely:

$$\exp\left\{-\frac{(e^2\lambda)}{2}\int_{R^D} dx\left[\partial^\nu(\bar{\psi}\gamma^\mu\psi)(x)\cdot\Box^{-2}(x,y)\partial_\nu(\bar{\psi}\gamma^\mu\psi)(y)\right]\right\}$$

$$=\int D[F_{\mu\nu}]\exp\left\{-\frac{1}{2\lambda}\int_{R^D} dx(F_{\mu\nu})^2(x)\right\}\delta^{(F)}(\partial_\beta B_\beta=0)$$

$$\times\delta^{(F)}(\partial_\mu F_{\mu\nu})-\Box B_\nu)\exp\left\{ie\int_{R^D}(\bar{\psi}\gamma^\mu\psi)\cdot B_\mu)(x)\right\}$$

Chapter 4

The Confining Behaviour and Asymptotic Freedom for $QCD(SU(\infty))$ - A Constant Gauge Field Path Integral Analysis

4.1. Introduction

Since 1950, the quantum field theory of light and electrons (Q.E.D) has been a very consistent framework for the description of the interaction of light and charged matter. In 1967, this quantum field theory of particles has arrived at another success with the advent of the Weinberg-Salam quantum field theory which handled successfully the weak-electromagnetic component of the nuclear scattering processes.

These quantum field methods are based on a principle of minimal action with (local and global) symmetries and the existence of a mathematical Generating Functional (Schwinger) defined on the space of classical source fields (test functions in the language of Schwart Distribution Theory). This Generating Functional, by its turn, contains all the probabilities ocurrences associated to all physically possible quantum scatterings involving the elementary particle field excitations.

However, it remains until present time as a difficult challenge in the subject, the direct application of the above Scattering Quantum Field methods (L.S.Z methods) to describe the pure strong-nuclear interaction as a Particle Field theory based in the framework of the non-abelian Gauge theory of Quantum Chromodynamics - QCD. The basic and conceptual difficulty in applying the L.S.Z - quantum field method on Quantum Chromodynamics is rooted on the first QCD model assumption of the charge-color confinement to which must be subject all QCD particles which by its turn constrains particles only with a color-singlet compound structure to be subject to Physical L.S.Z. scattering process.

It is important to remark that strong mathematical clues for this charge-color confinement on QCD were obtained by K. Wilson (1974) in a discretized space-time by using as dynamical variables the well-known gauge-invariant discretized Mandelstam-Feynman phase factors instead of Gauge-variant discretized fields. Although there is a strong indication that it is possible to remove the difficulties of the direct use of a discrete space-time through a phase transition of second-order leading to zero lattice spacing limit, this step remains as an somewhat unsolved problem within the Wilson's program for QCD until present days.

The purpose of this chapter is to consider another Quantum Yang-Mills reduced model with an explicitly confining behavior at the limit of large number of charge-colours (t'Hooft limit), however defined on a continuum space-time. This quantum dynamical reduced model is defined by introducing directly on R^{ν}, a Functional Manifold of Constant Gauge Fields configurations ([1]), which by its turn are expected to generate an effective dynamics on the Manifold of the full Gauge Field configurations at the t'Hooft limit $SU(\infty)$ for the Yang-Mills path integral. We show the Wilson confining area-behavior for QCD$(SU(\infty))$ as described by our proposed $SU(\infty)$ effective reduced dynamics of constant gauge fields. We show exactly our $SU(\infty)$-model solubility when added with full dynamical quark fields and the related fermionic field asymptotic freedom. These studies are presented an Section 4.3 of this chapter.

Another interesting and conceptually important problem in Quantum Field Theory is to understand the triviality of quantum field theories as a "phase-transition" phenomena depending on external parameters, including the famous space-time dimensionality.

It is argued sometimes that there are no non-renormalizable quantum field theories. What is really happening is the appearance of the Quantum Field Theory Triviality phenomena. However, there is some analysis in literature pointing out that through resummations-specially by means of the large N expansions - one could be able to make such non-renormalizable Field theories (like the Thirring fermion quantum field model) turn out to be non-trivial renormalizable ones. We aim in section 3 to present an analysis, based on an approximate chiral path-integral bosonization and the E. Witten reduced constant gauge field dynamics of section 2, to show that such resummation renormalization phenomenon does not happen. In section 4 we complement our previous path integral analysis by presenting a triviality argument by means of a Loop space analysis for any N.

4.2. The Model and Its Confining Behavior

One of the basic quantum field variables used to probe in the nonperturbative phase of non-abelian Gauge field theories is the well-known (Euclidean) path integral average associated to the non-abelian Faraday flux defined by a space-time loop C-the so called Wilson-Mandelstam loop variable

$$W[C] = \frac{1}{W(0)} \left\{ \int_{S'(R^{\nu}\times SU(N))} D^F[A_\mu(x)] \times \exp\left(-\frac{1}{2} \int_{R^{\nu}} \mathrm{Tr}(F_{\mu\nu})^2(x) d^{\nu}x \right) \times \left(\frac{1}{N} \mathrm{Tr}\, \mathbb{P} \left[\exp\left(ig \oint_C A_\mu dx_\mu \right) \right] \right) \right\} \tag{4.4.1}$$

where the domain of the quantum average on equation (4.1) is composed of Schwartz-tempered $SU(N)$ valued connections associated to the bundle $R^{\nu} \times SU(N)$.

A long time ago ([1]), it was argued by E. Witten that at the limit of infinite-number of colors $N \to \infty$ with the diagrammatic restriction $\lim_{N\to\infty} (g^2 N) = g^2_{\infty} < \infty$, the full domain of the Yang-Mills functional integral eq.(4.1) would be expected to be reduced to a manifold of translation invariant constant gauge fields. Let us, thus, define our reduced Yang-Mills model by considering from the beginning only constant gauge fields configurations on the functional domain of equation (4.1) as our basic assumption.

We now show the usefulness of such effective dynamics by giving a proof of the colour-charge confining through an explicit evaluation of the Wilson-Mandelstam phase factor at $N \to \infty$, an important result supporting the possibility of the above reduction of degrees of freedom for Yang-Mills theory at $SU(\infty)$, as first conjectured in Refs. [1].

The main idea to make explicitly this path-integral evaluation for constant gauge-fields is to consider the [non gauge-invariant] Cartan decomposition of each constant gauge field A_μ entering in the path integral average equation (4.1).

$$A_\mu = B_\mu^a H_a + G_\mu^b E_b \tag{4.2}$$

where the Cartan basis $\{H_a, E_a\}$ of the $SU(N)$ Lie algebra have the following distinguished calculational properties ([2])

a) For $a, b = 1, 2, \ldots, N-1$

$$[H_a, H_b]_- = 0 \tag{4.3}$$

b) For $b = \pm 1, \ldots, \pm \frac{N(N-1)}{2}$

$$[H_a, E_b]_- = r_a(b) E_b \tag{4.4}$$

c) For $a = 1, 2, \ldots, \frac{N(N-1)}{2}$

$$[E_a, E_{-a}]_- = \sum_{\ell=1}^{N-1} r_c(a) H_a \tag{4.5}$$

d) For $a \neq -b$; $a, b = \pm 1, \ldots, \pm \frac{N(N-1)}{2}$

$$[E_a, E_b]_- = N_{ab} E_{a+b} \tag{4.6}$$

Since one has to fix the gauge on the path-integral equation (4.1) and at the same time one should preserve the non-abelian field variable character, which is expected to be dynamically significant for explain the charge confinement – we impose the vanishing of the abelian components as our gauge fixing condition (the Bollini-Giambiagi gauge - see last reference on ref. [1]).

$$B_\mu^a \equiv 0. \tag{4.7}$$

Note that the use of the Gauge fixing condition allows us to simplify considerably the objects to be path-integrated on our proposed $SU(\infty)$ constant gauge field model.

For instance, the constant gauge field Yang-Mills path integral weight is obtained by simple substituting eq.(4.2) in the Yang-Mills action and leading by its turn to a pure fourth-order pure polynomial action

$$
\begin{aligned}
S[G_\mu^b E_b] &= \frac{1}{2}\int_\Omega d^\nu x \left(\mathrm{Tr}(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu])^2\right) \\
&= -\frac{g^2}{2}\cdot V\ \mathrm{Tr}([G_\mu, G_\nu]^2) \\
&= -\frac{g^2}{2} V\, G_\mu^a G_\nu^b G_\mu^c G_\nu^d [\mathcal{L}_{abcd}]
\end{aligned}
\tag{4.8}
$$

Here we have introduced an appropriate finite-volume domain $\Omega \subset R^\nu$ such that $\mathrm{vol}(\Omega) = V$ and with the topology product form $\Omega = S \times [0,\ell_3] \times [0,\ell_4]$ in order to extract the area behavior of equation (4.1) at the limit of large area behavior $S \to \infty$ (infinite volume V). The colour indexes matrix $\mathcal{L}_{abcd}$ are given explicitly by (with $Tr(E_a E_b) = +2\delta_{ab}$)

$$
\begin{aligned}
\mathcal{L}_{abcd} &= \left(\sum_{i,\ell=1}^{N-1} r_i(a) r_\ell(c)\delta_{i\ell}\,\delta_{c,-d}\,\delta_{a,-b}\right) \\
&\quad + \left(N_{ab} N_{cd}(1-\delta_{a,-b})(1-\delta_{c,-d})\delta_{a+b,-(c+d)}\right).
\end{aligned}
\tag{4.9}
$$

We have the following exact result for the Mandelstam Phase factor as a straightforward consequence of the non-abelian Stokes theorem applied to the planar loop C, which is supposed to be entirely contained in the plane ($\mu = 0$, $\nu = 1$, (containing the Euclidean time axis) and S denotes the area of the minimal surface bounded by C with the disc topology (for a rigorous proof see section 3).

$$
\mathbb{P}\left\{e^{ig\oint_{C_{0,1}} A_\mu dx_\mu}\right\} = \exp\left(-g^2 S\ \mathrm{Tr}[A_0, A_1]\right), \tag{4.10}
$$

The leading limit of $N \to \infty$ in eq.(4.10) (similar to the deduction of the large number law in Statistics!) yields the closed result below

$$
\begin{aligned}
&\frac{1}{N}\,\mathrm{Tr}\,\mathbb{P}\left\{e^{iS\oint_{C_{0,1}} A_\mu dx_\mu}\right\} \\
&\quad = \exp\left\{+\frac{(g^2 S)^2}{2N}\left(\mathrm{Tr}[A_0, A_1]\right)^2\right\} + O\left(\frac{1}{N}\right) \\
&\quad = \exp\left\{+\frac{(g^2 S)^2}{2N} G_\mu^a G_\nu^b G_\mu^c G_\nu^d [\mathcal{L}_{abcd}]\delta_{\mu 0}\,\delta_{\nu 1}\right\}
\end{aligned}
\tag{4.11}
$$

At this point of our path-integral study, let us make a technical remark not used in what follows and related to the fact that the path-integral average equation (4.1) for constant gauge fields is fully $SU(N)$ gauge invariant and, as a consequence, one should in principle evaluate the Faddev-Papov Jacobian associated to our proposed gauge fixing equation (4.7). In order to implement this technical step, one considers the infinitesimal functional displacements through a gauge transformation with parameters $[\delta\omega^a, \delta\varepsilon^b\}$

$$
\begin{aligned}
\delta A_\mu = \Big\{&(\delta G_\mu^b)E_b + i(\delta\omega^a)(G_\mu^{b'} E_{b'})(-r_a(b')) \\
&+ i(\delta\varepsilon^b)\left[G_\mu^{b'}\,\delta_{b,-b'}\left(\sum_{\ell=1}^{N-1} r_\ell(b) H_\ell\right)\right] \\
&+ i(\delta\varepsilon^b)[G_\mu^{b'} N_{bb'} E_{b+b'}(1-\delta_{b,-b'})]\Big\},
\end{aligned}
\tag{4.12}
$$

which after substituting in the functional metric ([3]),

$$\begin{aligned} \delta s_A^2 &= \mathrm{Tr}\left(\int_\Omega (\delta A \cdot \delta A) d^\nu x\right) \\ &= [\delta\sigma, \delta\varepsilon, \delta\omega]^T M[\sigma, \varepsilon, \omega][\delta\sigma, \delta\varepsilon, \delta\omega] \end{aligned} \tag{4.13}$$

would lead us to the Faddev-Popov Jacobian as the functional metric determinant averaged over the Gauge group (with infinitesimal Gauge Group neighborhood implying the use of the Feynman measure!)

$$\Delta_{FP}[G_\mu] = \int_{SU(N)} D^F(\delta\varepsilon, \delta\omega) \det^{\frac{1}{2}} \{M[\bar{\sigma}, \delta\varepsilon, \delta\omega]\}. \tag{4.14}$$

However, it is expected that in the large N limit equation (4.14) does not affect the confining area behavior of the averaged Wilson loop equation (4.1). We thus neglect its contribution to the average equation (4.1).

$$\Delta_{FP}[G_\mu] = 1 + O\left(\frac{1}{N}\right) \tag{4.15}$$

By collecting equation (4.8) and equation (4.11), one finally obtains our proposed path integral representation for the Wilson loop for constant gauge fields at the large number of colours $N \to \infty$.

$$\begin{aligned} W[C_{01}] &= \lim_{N\to\infty} \left\{ \frac{1}{W(0)} \int \left(\prod_{a=1}^{N^2-N} \prod_{\mu=0}^{\nu-1} dG_\mu^a \right) \right\} \\ &\times \exp\left\{ +\frac{1}{2} G_\mu^a G_\nu^b G_\mu^c G_\nu^d \mathcal{L}_{abcd} \times \left[g^2 V + \delta_{\mu 0}\delta_{\nu 1} \frac{(g^2 S)^2}{N} \right] \right\}. \end{aligned} \tag{4.16}$$

Now the area behavior at the t'Hooft large number of colors $N \to \infty$ is exactly obtained after considering a simple rescaling on the G_μ^a-variables in both path integral factors in equation (4.16) (including the normalization factor $W(0)$!) namely $G_{(0,1)}^a \to G_{(0,1)}^a \left[g^2 V + \frac{(g^2 S)^2}{N} \right]^{-\frac{1}{4}}$ in the numerator and $G_\mu^a \to G_\mu^a [g^2 V]^{-\frac{1}{4}}$ in the denominator as well.

$$W[C] = \frac{\left[g^2 V \left(1 + \frac{g^2 S^2)}{NV} \right) \right]^{-\frac{(N^2-N)\nu}{4}}}{[g^2 V]^{-(N^2-N)\frac{\nu}{4}}} = \left(1 + \frac{g^2 S^2}{NV} \right)^{-\frac{N(N-1)\nu}{4}} \tag{4.17}$$

which in the large N limit gives us exactly the expected exponential area behavior in a four-dimensional space time of the cylindrical form $\Omega^{(\infty)} = R^2 \times [0, \ell_3] \times [0, \ell_4]$, with $S \to \infty$ (the area bounded by C).

$$W[C] \sim \exp_{S\to\infty}\left\{-\frac{\left(\lim\limits_{N\to\infty}(g^2(N-1))\right)}{(\ell_3\ell_4)S}\cdot S^2\right\} \sim \exp\left\{-\left(\frac{g_\infty^2}{(\ell_3\ell_4)}\right)S\right\} \qquad (4.18)$$

It is very important to point out the appearance of a kind of Dual Models-String slope parameter $\frac{g_\infty^2}{(\ell_3\ell_4)}$ as an over-all coefficient in the area behavior equation (4.18), which by its turn signals the existence of the phenomenon of dimensional transmutation on the adimensional $SU(\infty)$ gauge coupling constant in four-dimensional space-time, phenomena expected to be responsible for the existence of strings structures on $QCD(SU(\infty))$ besides of generating the expected scale of mass for Hadrons in the observed nuclear particle forces ([4]). Note that the string tension on eq.(4.18) depends solely of the "area vacuum cross section" $A = \ell_3\ell_4$ as expected ([4]). In the three-dimensional case one obtains a pure length behavior for the Wilson Loop on the basis of eq.(4.18).

Finally, in the two-dimensional case one obtains the area behavior, however without the phenomenon of dimensional transmutation for the $N = \infty$ coupling constant ([4]).

After producing arguments for the confining behavior in our reduced-constant Gauge Field Model through explicit evaluations, we now introduce full dynamical chiral Fermion fields in our proposed constant gauge field Yang-Mills $SU(\infty)$ theory.

The associated quark field generating functional in the presence of the background constant gauge fields can be explicitly evaluated.

Let us show briefly this result since we make a complete analysis in this problem in the next section 3. Firstly we have the following chiral quark field Euclidean path integral

$$\begin{aligned} Z[\eta,\bar\eta] &= \frac{1}{Z(0,0)}\int D^F[\psi(x)]D^F[\bar\psi(x)]\delta^{(F)}(\gamma_5\psi-\psi)\times\delta^{(F)}(\gamma_5\bar\psi-\bar\psi) \\ &\times \exp\left\{-\frac{1}{2}\int_\Omega d^\nu x(\psi,\bar\psi)\begin{bmatrix} \bigcirc & U(\phi)\,\partial\!\!\!/ U(\phi) \\ U(\phi)^*\,\partial\!\!\!/^* U^*(\phi) & \bigcirc\end{bmatrix}\begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}\right\} \\ &\times \exp\left\{-i\int_\Omega(\bar\psi\eta+\bar\eta\psi)d^\nu x\right\} \end{aligned} \qquad (4.19)$$

where the chiral $SU(N)$ phase $U(\phi)$ associated to the constant gauge fields configuration is given explicitly by the expression

$$U(\phi) = \{\exp[-ig\gamma_5(A_\alpha^a\cdot x^\alpha)\lambda_a]\} = \mathbb{P}\left\{e^{-ig\gamma_5\int_{-\infty}^x A_\alpha^a\cdot dx^\alpha}\right\} \qquad (4.20)$$

where $\phi = \phi^a\lambda_a = A_\alpha^a x^\alpha\lambda_a$ is the chiral phase.

We can proceed as in the chiral bosonization path integral framework in order to "Bosonize" (solve exactly) the quark field path integral equation (4.19) by means of the chiral change of variables ([5])

$$\begin{aligned} \psi(x) &= \exp\{-ig\gamma_5\,\phi(x)\}\chi(x) \\ \bar\psi(x) &= \bar\chi(x)\,\exp\{-ig\gamma_5\,\phi(x)\} \end{aligned} \qquad (4.21)$$

After the change equation (4.21), the generating functional takes the decoupled form

$$
\begin{aligned}
Z[\eta,\bar{\eta}] &= \frac{1}{Z(0,0)} \int D^F[\chi(x)] D^F[\bar{\chi}(x)] \\
&\exp\left\{ -\frac{1}{2} \int_\Omega d^\nu x (\chi,\bar{\chi})(x) \begin{bmatrix} \bigcirc & \partial\!\!\!/ \\ \partial\!\!\!/^{*} & \bigcirc \end{bmatrix} \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} (x) \right\} \\
&\exp\left\{ -\frac{i}{2} \int_\Omega d^\nu x \left(\bar{\chi} e^{-ig\gamma_5 \phi(x)} \eta + \bar{\eta} e^{-ig\gamma_5 \phi(x)} \chi \right)(x) \right\} \\
&\times \det{}_F^{+1}[U(\phi)\ \partial\!\!\!/ U(\phi)]
\end{aligned} \tag{4.22}
$$

At this point, we remark the validity of the free-field result for the Fermionic functional determinant in the path integrand equation (4.22) (see next section for detailed calculations)

$$
\det{}_F[U(\phi)\ \partial\!\!\!/ U(\phi)] = \det{}_F\,[\partial\!\!\!/] \tag{4.23}
$$

Here we have used the Alvarez-Romanov-Schwartz teorem ([5]), the condition $\int_\Omega d^\nu x \cdot x^\mu = 0$ and the non-existence of zero modes of the Dirac operator in presence of constant gauge field configurations in order to obtain equation (4.23).

As a consequence of the above displayed results, one gets the famous asymptotic freedom property of the quark fields in our $SU(\infty)$ constant gauge field model after writing explicitly the quark two-point function

$$
\begin{aligned}
\langle \psi(x)\bar{\psi}(y)\rangle &= \left. \frac{\delta^2 Z[\eta,\bar{\eta}]}{\delta\bar{\eta}(x)\delta\eta(y)} \right|_{\eta=\bar{\eta}=0} \\
&= \langle \chi(x)\bar{\chi}(y)\rangle^{(0)} \exp\left(-ig\gamma_5 \int_x^y A_\mu dx_\mu \right) \underset{|x-y|\to 0}{\sim} \langle \chi(x)\bar{\chi}(y)\rangle^{(0)}
\end{aligned} \tag{4.24}
$$

Here $\langle \chi(x)\bar{\chi}(y)\rangle$ denotes the free Fermion propagator coming from the “bosonized” action and the contour on the gauge field path-phase factor is a straight line connecting the points x^α and y^α, which reduces to unity at the higher-energy limit of $|x-y| \to 0$. (see eq.(4.20).

At this point, let us call the reader’s attention to the fact that phenomenon of asymptotic freedom should be analyzed for Gauge-invariant quark bilinear fields. For instance, we have the Gauge-invariant result:

$$
\begin{aligned}
\langle (\psi(x)\bar{\psi}(x))(\psi(y)\bar{\psi}(y))\rangle &\sim \langle \chi(x)\bar{\chi}(y)\rangle^{(0)} \langle \chi(y)\bar{\chi}(x)\rangle^{(0)} \\
&\times \left\{ Tr_{SU(\infty)} P(+ig \oint_{C_{xy}} A_\mu dx_\mu) \right\}
\end{aligned} \tag{4.24-b}
$$

Here C_{xy} denotes an arbitrary planar closed contour intercepting the ”marked” points x and y. We can see that for large $|x-y|$ separation, the above quark-bilinear field correlation function approximates to the free field fermion correlation functions as the family of planar loops C_{xy} entering in the gauge-invariant expression eq.(4.24-b) reduces to a point as the geometrical result of the superposition of the segments of straight-line connecting the points x and y (see eq.(4.24)), however with opposite orientation. Note that all those loops C_{xy} with a large area $|x-y|^2$ have a negligible contribution to eq.(4.24-b).

4.3. The Path-Integral Triviality Argument for the Thirring Model at $SU(\infty)$

We start our analysis by considering the chiral non-abelian $SU(N_c)$ Thirring model Lagrangean on the Euclidean space-time of finite volume $\Omega \subset R^4$ as done in Section 2

$$L(\psi,\overline{\psi}) = \frac{1}{2}\left[\overline{\psi}^a(i\overrightarrow{\gamma_\mu\partial_\mu}\psi^a) + (\overline{\psi}^a i\overleftarrow{\gamma_\mu\partial_\mu})\psi^a\right] + \left(\frac{g^2}{2}(\overline{\psi}_b\gamma^\mu\gamma^5(\lambda^A)_{bc}\psi_c)^2\right) \tag{4.25}$$

Here $(\psi^a,\overline{\psi}^a)$ are the Euclidean four-dimensional chiral fermion fields belonging to a fermionic fundamental representation of the $SU(N_c)$ non-abelian group with Dirichlet boundary condition imposed at the finite-volume region Ω. In the framework of path integrals, the generating functional of the Green's functions of the quantum field theory associated with the Lagrangean eq.(4.25) is given by $(\partial\!\!\!/ = i\gamma_\mu\partial_\mu)$

$$\begin{aligned}Z[\eta_a,\overline{\eta}_a] = {} & \frac{1}{Z(0,0)}\int\prod_{a=1}^{N^2-N} D[\psi_a]D[\overline{\psi}_a] \\ & \times \exp\left\{-\frac{1}{2}\int_\Omega d^4x(\psi_a,\overline{\psi}_a)\begin{bmatrix}0 & \vec{\partial\!\!\!/} \\ \overleftarrow{\partial\!\!\!/}^* & 0\end{bmatrix}\begin{pmatrix}\psi_a\\ \overline{\psi}_a\end{pmatrix}(x)\right\} \\ & \times \exp\left\{-\frac{g^2}{2}\int_\Omega d^4x\,(\overline{\psi}_b\gamma^5\gamma^\mu(\lambda^A)_{bc}\psi_c)^2(x)\right\} \\ & \times \exp\left\{-i\int_\Omega d^4x(\overline{\psi}_a\eta_a + \overline{\eta}_a\psi_a)(x)\right\}\end{aligned} \tag{4.26}$$

In order to proceed with a bosonization analysis of the fermion field theory described by the above path-integral, it appears to be convenient to write the interaction Lagrangian in a form closely parallel to the usual fermion-vector coupling in gauge theories by making use of an auxiliary non-abelian vector field $A^a_\mu(x)$, but with a purely imaginary coupling with the axial vectorial fermion current (at the Euclidean world).

$$\begin{aligned}Z[\eta_a,\overline{\eta}_a] = {} & \frac{1}{Z(0,0)}\int\prod_{a=1}^{N^2-N} D[\psi_a(x)]D[\overline{\psi}_a(x)]\int\prod_{a=1}^{N^2-N}\prod_{\mu=0}^{3} D[A^a_\mu(x)] \\ & \times \exp\left\{-\frac{1}{2}\int_\Omega d^4x(\psi_a,\overline{\psi}_a)\begin{bmatrix}0 & \partial\!\!\!/ + ig\gamma_5 A\!\!\!/ \\ (\partial\!\!\!/ + ig\gamma_5 A\!\!\!/)^* & 0\end{bmatrix}\begin{pmatrix}\psi_a\\ \overline{\psi}_a\end{pmatrix}(x)\right\} \\ & \times \exp\left\{-\frac{1}{2}\int_\Omega d^4x(A^a_\mu A^a_\mu)(x)\right\} \\ & \times \exp\left\{-i\int_\Omega d^4x(\overline{\psi}_a\eta_a + \overline{\eta}_a\psi_a)(x)\right\}\end{aligned} \tag{4.27}$$

In this point of our analysis we present our idea to bosonize (solve) exactly the above written fermion path integral. The main point is to use the old suggestion that at the strong coupling and at a large number of colors (the t'Hooft limit), one should expect a great reduction of the (continuum) vector dynamical degrees of freedom to a manifold of constant gauge fields living on the infinite dimensional Lie algebra of $SU(\infty)$ ([1], [6]). In t' Hooft limit of large number of colors, we can evaluate exactly the fermion path-integral by noting that the Dirac kinetic operator in the presence of the constant $SU(N)$ gauge fields can be written in the following suitable form

$$\exp\left\{-\frac{1}{2}\int_{\Omega} d^4x(\psi_a\overline{\psi}_a)\begin{bmatrix} 0 & U(\varphi)\,\partial\!\!\!/ U(\varphi) \\ U(\varphi)^* \,\partial\!\!\!/^* U^*(\varphi) & \end{bmatrix}\begin{pmatrix}\psi_a \\ \overline{\psi}_a\end{pmatrix}(x)\right\} \tag{4.28}$$

where the chiral hermitean phase-factor is given by

$$U(\varphi) = \exp[-g\gamma_5(A^a_\mu x^\mu)\lambda_a] \tag{4.29}$$

with the chiral $SU(N)$ valued phase defined by the constant gauge field configuration

$$\varphi(x^\mu) = \varphi^a\lambda_a = (A^a_\mu x^\mu)\lambda_a \tag{4.30}$$

Note that due to the attractive coupling of the axial current - axial current interaction of our Thirring model eq.(4.26), the axial vector coupling is made of an imaginary - complex coupling constant ig.

Now we can follow exactly as in the well-known chiral path-integral bosonization scheme ([5],[7]) in order to solve exactly the quark field path integral eq.(4.28) by means of the chiral change of variables

$$\psi(x) = \exp\{-g\gamma_5\,\varphi(x)\}\chi(x) \tag{4.31}$$

$$\overline{\psi}(x) = \overline{\chi}(x)\exp\{-g\gamma_5\,\varphi(x)\} \tag{4.32}$$

After implementing the variable change eq.(4.31)-eq.(4.32), the fermion sector of the Generating functional takes the form where the independent euclidean fermion fields are decoupled from the interacting - intermediating non-abelian constant vector field A^a_μ, namely

$$\begin{aligned} Z[\eta_a,\overline{\eta}_a] = {} & \frac{1}{Z(0,0)}\int\prod_{a=1}^{N^2-N} D[\chi_a(x)]D[\overline{\chi}_a(x)] \\ & \times\int_{-\infty}^{+\infty}\prod_{a=1}^{N^2-N} d[A^a_\mu]\times\exp\left\{+\frac{V}{2}Tr_{SU(N)}(A^2_\mu)\right\} \\ & \times\det\nolimits_F^{+1}[(\partial\!\!\!/ + ig\gamma_5\,A\!\!\!/)(\partial\!\!\!/ + ig\gamma_5\,A\!\!\!/)^*] \\ & \times\exp\left\{-\frac{1}{2}\int_{\Omega} d^4x(\chi_a,\overline{\chi}_a)\begin{bmatrix} 0 & \partial\!\!\!/ \\ \partial\!\!\!/^* & 0\end{bmatrix}\begin{pmatrix}\chi_a \\ \overline{\chi}_a\end{pmatrix}(x)\right\} \\ & \times\exp\left\{-i\int_{\Omega} d^4x(\overline{\chi}_a e^{-g\gamma_5\varphi(x)}\eta_a + \overline{\eta}_a e^{-g\gamma_5\varphi(x)}\chi_a)(x)\right\} \end{aligned} \tag{4.33}$$

Let us now evaluate exactly the fermionic functional determinant on eq.(4.33) which is given by the functional Jacobian associated to the chiral fermion field reparametrizations eq.(4.31)-eq.(4.32).

In order to compute this fermionic determinant, $\ell n \det_F^{+1}[(\not\partial + ig \not A)(\not\partial + ig \not A)^*]$, we use the well-known theorem of Schwarz-Romanov ([7]) by introducing a σ-parameter $(0 \leq \sigma \leq 1)$ dependent family of interpolating Dirac operators (see eq.(4.23) - section 4.2).

$$\not D^{(\sigma)} = (\not\partial + ig \not A^{(\sigma)}) = \exp\{-g\sigma\gamma_5\varphi(x)\}(\not\partial)\exp\{-g\sigma\gamma_5\varphi(x)\} \tag{4.34}$$

Since we have the relationship for the interpolating Dirac operators

$$\frac{d}{d\sigma}\not D^{(\sigma)} = (-g\gamma_5\varphi)\not D^{(\sigma)} + \not D^{(\sigma)}(-g\gamma_5\varphi) \tag{4.35}$$

and the usual proper-time definition for the functional determinants under analysis

$$\begin{aligned} &\log \det_F^{+1}(\not D^{(\sigma)} \not D^{(\sigma)^*}) \\ &= \lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{ds}{s} Tr_F(e^{-s(\not D^{(\sigma)}\not D^{(\sigma)^*}}), \end{aligned} \tag{4.36}$$

one obtains straightforwardly the following differential equation for the Fermionic functional determinant

$$\begin{aligned} &\frac{d}{d\sigma}\{\log \det_F^{+1}(\not D^{(\sigma)} \not D^{(\sigma)^*})\} \\ &= 4\lim_{\varepsilon\to 0}\left\{\int d^4x Tr_F\left[g\gamma_5\varphi \times \exp(-\varepsilon \not D^{(\sigma)} \not D^{(\sigma)^*})\right]\right\} \end{aligned} \tag{4.37}$$

where Tr_F denotes the complete trace over the color, Dirac and space-time indices. At this point we note that the diagonal part of $\exp(-\varepsilon \not D^{(\sigma)} \not D^{(\sigma)^*})$ has a well-known gauge - invariant asymptotic expansion in four-dimensions ([4]) (where $\sigma^{\mu\nu} = \frac{1}{2i}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$)

$$\begin{aligned} \exp(-\varepsilon \not D^{(\sigma)} \not D^{(\sigma)^*}) &= \frac{1}{4\pi^2}\left\{\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}(F^b_{\mu\nu}(\sigma A)\sigma^{\mu\nu}\lambda_b)\right. \\ &\left. +\frac{1}{4}\left(-\frac{1}{3}F^b_{\mu\nu}(\sigma A)F^{b'}_{\mu\nu}(\sigma A)\lambda_b\lambda_{b'} - \frac{1}{2}F^c_{\alpha\beta}(\sigma A)F^{c'}_{\alpha'\beta'}(\sigma A)\lambda_c\lambda_{c'}\gamma^\alpha\gamma^\beta\gamma^{\alpha'}\gamma^{\beta'}\right) + 0(\varepsilon)\right\} \end{aligned} \tag{4.38}$$

After substituting the Seeley-Hadamard expansion on eq.(4.38), by taking into account eq.(4.30), together with the fact that $Tr_{\text{Dirac}}(\gamma_5) = 0$ and $Tr_{\text{Dirac}}(\gamma_5\sigma^{\mu\nu}) = 0$, one obtains finally the only possible non-zero term in our evaluations

$$\begin{aligned} W[A^a_\mu] = 16&\left\{\left(\int_\Omega d^4x \frac{(-g)}{(4\pi)^2}\left(-\frac{1}{8}\right)x^\mu\right)\right. \\ &\left.\times(\sigma A^a_\mu)(F^c_{\alpha\beta}(\sigma A)^* F^{c'}_{\alpha\beta}(\sigma A) Tr_{SU(N)}(\lambda_a\lambda_c\lambda_{c'})\right) \end{aligned} \tag{4.39}$$

By supposing explicit space-time symmetry of the finite-volume region Ω, one has that the "symmetry integral" vanishes

$$\int_\Omega d^4x \cdot x^\mu \equiv 0 \tag{4.40}$$

As a consequence, we get the somewhat expected result that the Fermion functional determinant in the presence of constant gauge external fields coincides with the free one, (see eq.(4.23) namely:

$$\det_F\left[(\partial\!\!\!/ + ig A\!\!\!/)(\partial\!\!\!/ + ig A\!\!\!/)^*\right] \Big/ \det_F\left[(\partial\!\!\!/)(\partial\!\!\!/)^*\right] = 1 \tag{4.41}$$

Let us return to our "Bosonized" Generating functional (after substituting the above obtained results on its previous expression eq.(4.33).)

$$\begin{aligned} Z[\eta_a,\overline{\eta}_a] &= \frac{1}{Z(0,0)}\int \prod_{a=1}^{N^2-N} D[\chi_a(x)]D[\overline{\chi}_a(x)] \\ &\times \int_{-\infty}^{+\infty} \prod_{a=1}^{N^2-N} d[A_\mu^a] \exp\left\{+\frac{1}{2} V\, Tr_{SU(N)}(A_\mu)^2\right\} \\ &\times \exp\left\{-\frac{1}{2}\int_\Omega d^4x (\chi_a,\overline{\chi}_a)\begin{bmatrix} 0 & \partial\!\!\!/ \\ \partial\!\!\!/^* & 0\end{bmatrix}\begin{pmatrix}\chi_a \\ \overline{\chi}_a\end{pmatrix}(x)\right\} \\ &\times \exp\left\{-i\int_\Omega d^4x(\overline{\chi}_a e^{-g\gamma_5(A_\mu^a\lambda_a)x^\mu}\eta_a + \overline{\eta}_a e^{-g\gamma_5(A_\mu^a\lambda_a)x^\mu}\chi_a)(x)\right\} \end{aligned} \tag{4.42}$$

Let us argument in favor of the theory's triviality by analyzing the long-distance behavior associated to the $SU(N)$ gauge-invariant fermionic composite operator $B(x) = \psi_a(x)\overline{\psi}_a(x)$. It is straightforward to obtain its exact expression from the bosonized path-integral eq.(4.42)

$$\begin{aligned} &\left\langle B(x)B(y)\right\rangle \\ &= \left\langle (\chi_a(x)\overline{\chi}_a(x))(\chi_a(y)\overline{\chi}_a(y))\right\rangle^{(0)} \times G((x-y)) \end{aligned} \tag{4.43}$$

here the reduced model's Gluonic factor is given exactly in its structural-analytical form by the path-integral (without bothering us with the γ_5-Dirac indexes)

$$\begin{aligned} G((x-y)) \sim \frac{1}{G(0)}\int_{-\infty}^{+\infty}\prod_{a=1}^{N^2-N} d[A_\mu^a]\, \exp\left\{+\frac{1}{2}\text{ vol }(\Omega) Tr_{SU(N)}(A_\mu)^2\right\} \\ \times Tr_{SU(N_c)}\mathbb{P}\left\{\exp -g\oint_{C_{xy}} A_\alpha dx_\alpha\right\} \end{aligned} \tag{4.44}$$

with C_{xy} a planar closed contour containing the points x and y and possesing an area S given roughly by the factor $S = (x-y)^2$.

The notation $\langle\ \rangle^{(0)}$ means that the Fermionic average is defined solely by the fermion free action as given in the decoupled form eq (42).

Let us pass to the important step of evaluating the Wilson phase factor average eq (44) at the limit of t'Hooft of large number of colors $N \to \infty$. As the first step to implement such evaluation, let us consider our loop C_{xy} as a closed contour lying on the plane $\mu = 0$, $\nu = 1$ bounding the planar region S (see section 2).

We now observe that the ordered phase-factor for constant gauge fields can be exactly evaluated by means of a triangularization of the planar region S, i.e.,

$$S = \bigcup_{l=1}^{M} \Delta_{\mu\nu}^{(i)} \tag{4.45}$$

Here, each counter-clock oriented triangle $\triangle_{\mu\nu}^{(i)}$ is adjacent to next one $\triangle_{\mu\nu}^{(i)} \cap \triangle_{\mu\nu}^{(i+1)} =$ common side with the opposite orientations.

At this point we note that

$$\mathbb{P}\{e^{-g\int_{\Delta_{\mu\nu}^{(i)}} A_\alpha \cdot dx_\alpha}\} \cong e^{-gA_\alpha \cdot \ell_\alpha^{(1)}} \cdot e^{-gA_\alpha \cdot \ell_\alpha^{(2)}} \cdot e^{-gA_\alpha \cdot \ell_\alpha^{(3)}} \tag{4.46}$$

where $\{\ell_\alpha^{(i)}\}_{1=1,2,3}$ are the triangle sides satisfying the (vector) identity $\ell_\alpha^{(1)} + \ell_\alpha^{(2)} + \ell_\alpha^{(3)} \equiv 0$.

Since we have that

$$\mathbb{P}\{e^{-g\oint_{(x)} A_\alpha dx_\alpha}\} = \lim_{n\to\infty} \prod_{i=1}^{n} \mathbb{P}\{e^{-g\int_{\Delta_{\mu\nu}^{(i)}} A_\alpha dx_\alpha}\} \tag{4.47}$$

and by using the Campbel Hausdorff formulae to sum up the product limit eq.(4.47) with X and Y denoting general elements of the $SU(N)$ - Lie algebra:

$$e^X \cdot e^Y = e^{X+Y+\frac{1}{2}[X,Y]} + 0(g^2) \tag{4.48}$$

one arrives at the non-Abelian Stokes theorem for constant Gauge Fields (see second reference in refs. [1]).

$$\begin{aligned} \mathbb{P}\{e^{-g\int_{C_{xy}} A_\alpha dx_\alpha}\} &= \mathbb{P}\{e^{-g\iint_S F_{01} d\sigma^{01}}\} \\ &= \mathbb{P}\{e^{+(g)^2 [A_0,A_1]\cdot S}\} \end{aligned} \tag{4.49}$$

As a consequence, we have the following result (exact at $N \to \infty$) to be used in our analysis below

$$\begin{aligned} Tr_{SU(N)} \mathbb{P}\{e^{-g\int_{C_{xy}} A_\alpha dx_\alpha}\} \sim \exp\left\{+\frac{(g^2S)^2}{2}(Tr_{SU(N)}[A_0,A_1])^2\right\} \\ + O\left(\frac{1}{N}\right) \end{aligned} \tag{4.50}$$

Note that eq.(4.50) is a rigorous result and eq.(4.49) is a rigorous proof of the Non-Abelian Stokes theorem as used on section 2.

Let us now substitute eq.(4.50) into eq.(4.44) and taking into account the natural two-dimensional degrees of freedom reduction on the average eq.(4.44)

$$\begin{aligned} G((x-y)) = \frac{1}{\tilde{G}(0)} \int_{-\infty}^{+\infty} \prod_{a=1}^{N^2-N} d[A_1^a]\, d[A_0^a] \exp\left\{+\frac{1}{2}V\left[Tr_{SU(N)}(A_0^2 + A_1^2)\right]\right. \\ \left. \times \exp\left\{+\frac{(g^2S)^2}{2}(Tr_{SU(N)}[A_0,A_1])^2\right\}\right. \end{aligned} \tag{4.51}$$

where $\tilde{G}(0)$ is the normalization factor given explicitly by

$$\tilde{G}(0) = \int_{-\infty}^{+\infty} \prod_{a=1}^{N^2-N} d[A_1^a] d[A_0^a] \exp\left\{-\frac{1}{2} \text{ vol }(\Omega)[(A_0^a)^2 + (A_1^a)^2]\right\} \tag{4.52}$$

By looking closely at eq.(4.51)–eq.(4.52), one can see that the behavior of the Wilson phase factor average at large N is asymptotic to the value of the integral below

$$\begin{aligned} G((x-y))_{N>>1} \sim \Bigg\{ \int_{-\infty}^{+\infty} da \exp\left\{-\frac{1}{2} \text{ vol }(\Omega) a^2\right\} & \\ \times \exp\left\{-\frac{(g^2 S)^2}{2} a^4\right\} & \\ \times \left(\int_{-\infty}^{+\infty} da \exp\left\{-\frac{1}{2} \text{ vol }(\Omega) a^2\right\}\right)^{-1} \Bigg\}^{N^2-N} & \end{aligned} \tag{4.53}$$

By using the well-known result (see ref [9] - pag 307, eq(3). 323 -3)

$$\int_0^{\infty} \exp(-\beta^2 x^4 - 2\gamma^2 x^2) dx = 2^{-\frac{3}{2}} \left(\frac{\gamma}{\beta}\right) e^{\frac{\gamma^4}{2\beta^2}} K_{\frac{1}{4}} \left(\frac{\gamma^4}{2\beta^2}\right) \tag{4.54}$$

we obtain the closed result (at finite volume $V = \text{ vol }(\Omega) < \infty$).

$$\begin{aligned} G((x-y))_{N>>1} \sim \Bigg\{ \left(\frac{\sqrt{\text{vol }(\Omega)} N}{2 \cdot \left(\frac{g^2 S N}{\sqrt{2}}\right)}\right) & \\ \times e^{+\frac{(\text{ vol }(\Omega))^2}{\frac{32}{N^2}\left(\frac{g^2 S N}{\sqrt{2}}\right)^2}} K_{\frac{1}{4}} \left(\frac{(\text{ vol }(\Omega))^2 N^2}{16 g^4 N^2 S^2}\right) & \\ \times \left(\frac{\sqrt{\pi}}{2 \cdot \left(\frac{\text{vol }(\Omega)}{2}\right)^{\frac{1}{2}}}\right)^{-1} \Bigg\}^{N^2-N} & \end{aligned} \tag{4.55}$$

Let us now give a theoretical physicist's argument of the theory's triviality at infinite volume $\text{vol}(\Omega) \to \infty$ on the basis of the explicit representation. Let us firstly define the infinite-volume theory's limit by means of the following limit

$$\text{vol }(\Omega) = S^2 \tag{4.56}$$

and consider the asymptotic limit of the correlation function at $|x-y| \to \infty \quad (S \to \infty)$.

By using the standard asymptotic limit of the Bessel function

$$\lim_{z \to \infty} K_{\frac{1}{4}}(z) \sim e^{-z} \sqrt{\frac{\pi}{2z}} \tag{4.57}$$

one obtains the result $(\lim_{N\to\infty} g^2 N = g_\infty^2 < \infty)$ in four dimensions

$$G((x-y))_{\substack{N>>1\\|x-y|\to\infty}} \sim \lim_{S\to\infty}\left\{\frac{N}{S}\cdot e^{\frac{N^2S^4}{16S^2}}\sqrt{\frac{16\pi}{N^2S^2 2}}e^{-\frac{S^2N^2}{16}}\right\}^{N^2-N} \sim \frac{1}{|x-y|^{4(N^2-N)}} \tag{4.58}$$

So, we can see that for N a very large parameter, there is a fast decay of eq.(4.58) without any bound on the power decay law. However in the usual L.S.Z. framework for Quantum Fields, it would be expected the opposite behavior through a non decay of such factor as in the two-dimensional case (see eq.(4.58) for vol$(\Omega) = S$), meaning physically that one can observe fermionic scattering free states at large separation. However at $N\to\infty$, where we expect the full validity of our analysis, one obtains [on the basis of the formal behavior of eq.(4.58)] the vanishing of the above analyzed fermionic correlation function eq.(4.43), faster than any power of $|x-y|$ for large $|x-y|$. This result shows that g^2_{bare} may be zero from the very beginning and strongly signalling the fact that the chiral Thirring model - for large number of colors - may remain a trivial Quantum Field Theory, a result not fully expected at all in view of previous claims on the subject that large N resummations always turn non-renormalizable field theories in non-trivial renormalizable useful ones ([8]).- However, rigorous mathematical proofs are needed to establish such an important triviality result in full ([8]).

Finally and as a last remark on our formulae eq.(4.55)-eq.(4.58), let us point out that a mathematical rigorous sense to consider these results is by taking as our continuum space-time Ω, a set formed of n hyper-four-dimensional cubes of a side a - the expected size of the non-perturbative vacuum domain of our theory (see the first reference in [1]) - and the surface S being formed, for instance, by n squares on the Ω plane section contained on the plane $\mu = 0$, $\nu = 1$. As a consequence of the construction above exposed, we can see that the large behavior is given exactly by

$$\begin{aligned} G(na)_{N>>1} \overset{n\to\infty}{\sim} & \left\{\frac{N}{\overline{g}^2_\infty\cdot na^2}\, e^{\frac{(N^2n^2a^8}{32.\left(\frac{\overline{g}^2_\infty na^2}{\sqrt{2}}\right)^2}}\right. \\ & \left.\times K_{\frac{1}{4}}\left(\frac{N^2(n^2a^8)}{16(g^2_\infty)^2\, n^2a^4}\right)\right\}^{N^2-N} \\ & \sim \left(\frac{1}{na^4}\right)^{N^2-N} \sim e^{-N(N-1)\ell g(na^4)} \underset{N\to\infty}{\sim} 0 \end{aligned} \tag{4.59}$$

4.4. The Loop Space Argument for the Thirring Model Triviality

In order to argument one more time for the triviality phenomenon of the $SU(N)$ non-abelian thirring model of section 3 for finite N, let us consider the generating functional eq.(4.27) for vanishing fermionic sources $\eta_a = \overline{\eta}_a = 0$, the so-called vacuum energy theory's content or the theory's partition functional

$$Z(0,0) = \int \prod_{a=1}^{N^2-N} \prod_{\mu=0}^{3} D[A_\mu^a(x)]\, e^{-\frac{1}{2}\int_\Omega d^4x (A_\mu^a A_\mu^a)(x)} \times \det_F[(\partial\!\!\!/ + ig\gamma_5 A\!\!\!/)(\partial\!\!\!/ + ig\gamma_5 A\!\!\!/)^*] \quad (4.60)$$

At this point of our analysis, let us write the functional determinant on eq.(4.60) as a functional on the space of closed bosonic paths $\{X_\mu(\sigma),\ 0 \le \sigma \le T,\ X_\mu(0) = X_\mu(T) = x_\mu\}$, namely ([6] and first reference on [8]).

$$\ell g \det_F[(\partial\!\!\!/ + ig\gamma_5 A\!\!\!/)(\partial\!\!\!/ + ig\gamma_5 A\!\!\!/)^*] = \sum_{C_{xx}} \left\{ \mathbb{P}_{SU(N)} \cdot \mathbb{P}_{\text{Dirac}} \exp\left[-g \oint_{C_{xx}} A_\mu(X_\beta(\sigma) dX_\mu(\sigma) + \frac{i}{2}[\gamma^\alpha, \gamma^\beta] \oint_{C_{xx}} F_{\alpha\beta}(X_\beta(\sigma)) ds \right] \right\} \quad (4.61)$$

The sum over the closed loops C_{xy} with fixed end-point x_μ is given by the proper-time bosonic path integral below

$$\sum_{C_{xx}} = -\int_0^\infty \frac{dT}{T} \int d^4x_\mu \int_{\chi_\mu(0)=x_\mu=\chi_\mu(T)} D^F[X(\sigma)] \times \exp\left\{ -\frac{1}{2}\int_0^T \dot{X}^2(\sigma) d\sigma \right\} \quad (4.62)$$

Note the symbols of the path ordenation $\mathbb{P}$ of the both, Dirac and color indexes on the loop phase space factors in the expression eq.(4.61).

By using the Mandelstam area derivative operator $\delta / \delta\sigma_{\gamma\rho}(X(\sigma))$ ([4]), one can re-writes eq.(4.61) into the suitable form as an operation in the loop space-with Dirac matrices bordering the loop C_{xx}, namely:

$$\ell g \det_F[(\partial\!\!\!/ + ig\gamma_5 A\!\!\!/)(\partial\!\!\!/ + ig\gamma_5 A\!\!\!/)^*] = \sum_{C_{xx}} \mathbb{P}_{\text{Dirac}} \exp\left\{ \oint_{C_{xx}} d\sigma \frac{i}{2}[\gamma^\alpha, \gamma^\beta](\sigma) \frac{\delta}{\delta\sigma_{\alpha\beta}(X(\sigma))} \mathbb{P}_{SU(N)}\left[\exp(-g \oint_{C_{xx}} A_\mu(X_\beta(\sigma) dX_\mu(\sigma)) \right] \right\} \quad (4.63)$$

In order to show the triviality of functional fermionic determinant when averaging over the (white-noise!) auxiliary non-abelian fields as in eq.(4.60), we can use a cummulant expansion, which in a generic form reads as

$$\langle e^f \rangle_{A_\mu} = \exp\left\{ \langle f \rangle_{A_\mu} + \frac{1}{2}(\langle f^2 \rangle_{A_\mu} - \langle f \rangle_{A_\mu}^2) + \cdots \right\} \quad (4.64)$$

So let us evaluate explicitly the first order cummulant

$$\sum_{C_{xy}} \mathbb{P}_{\text{Dirac}} \left\{ \oint_{C_{xy}} ds \frac{i}{2} [\gamma^{\alpha}(\sigma), \gamma^{\beta}(\sigma)] \frac{\delta}{\delta\sigma_{\alpha\beta}(X(\sigma))} \right. \\ \times \left\langle \mathbb{P}_{SU(N)} [\exp(-g \oint_{C_{xx}} A_{\mu}(X_{\beta}(\sigma)) dX_{\mu}(\sigma)] \right\rangle_{A_{\mu}} \tag{4.65}$$

with the average $\langle \ \rangle_{A_\mu}$ defined by the path-integral eq.(4.60).

By using the Grassmanian zero-dimensional representation to write explicitly the $SU(N)$ path-order as a Grassmanian path integral ([10])

$$\mathbb{P}_{SU(N)} [\exp(-g \oint_{C_{xx}} A_{\mu}(X_{\beta}(\sigma)) dX_{\mu}(\sigma))] \\ = \int \prod_{a=1}^{N^2-N} D^F [\theta_a(\sigma)] D^F [\theta_a^*(\sigma)] (\sum_{a=1}^{N^2-N} \theta_a(0) \theta_a^*(T)) \\ \times \exp\left(\frac{i}{2} \int_0^T d\sigma \sum_{a=1}^{N^2-N} \left(\theta_a(\sigma) \frac{\vec{d}}{d\sigma} \theta_a^*(\sigma) + \theta_a^*(\sigma) \frac{d}{d\sigma} \theta_a(\sigma) \right) \right) \\ \times \exp\left(g \int_0^T d\sigma (A_{\mu}^a (X^{\beta}(\sigma)) (\theta_b (\lambda_a)_{bc} \theta_c^*)(\sigma) dX^{\mu}(\sigma)) \right) \tag{4.66}$$

one can easily see that the average over the $A_\mu(x)$ fields is straightforward and producing as a result the following self-avoiding loop action

$$\left\langle \mathbb{P}_{SU(N)} [\exp(-g \oint_{C_{xx}} A_{\mu}(X_{\beta}(\sigma)) dX_{\mu}(\sigma))] \right\rangle_{A_{\mu}} \\ = \int \prod_{a=1}^{N^2-N} D^F [\theta_a(\sigma)] D^F [\theta_a^*(\sigma)] (\sum_{a=1}^{N^2-N} \theta_a(0) \theta_a^*(T)) \\ \times \exp\left(\frac{i}{2} \int_0^T d\sigma \sum_{a=1}^{N^2-N} \left(\theta_a(\sigma) \frac{\vec{d}}{d\sigma} \theta_a^*(\sigma) + \theta_a^*(\sigma) \frac{\vec{d}}{d\sigma} \theta_a(\sigma) \right) \right) \\ \exp\left\{ \frac{g^2}{2} \int_0^T d\sigma \int_0^T d\sigma' \left[(\theta_b(\lambda_a)_{bc} \theta_c^*)(\sigma) (\theta_b(\lambda_a)_{bc} \theta_c^*)(\sigma') \right] \right. \\ \left. \times \delta^{(D)} (X_{\mu}(\sigma) - X_{\mu}(\sigma')) dX_{\mu}(\sigma) dX_{\mu}(\sigma') \right\} \tag{4.67}$$

At this point one can use the famous probabilistic - topological Parisi argument ([11]) to show the $\lambda\varphi^4$ triviality at the four-dimensional space-time [8]: due to the fact that Hausdorff dimension of our Brownian loops $\{X_\mu(\sigma)\}$ is two, and the topological rule for continuous manifold holds true in the present situation, one obtains that for ambient space greater than (or equal) to four, the Hausdorff dimension of the closed path intersection set of the argument of the delta function in eq.(4.67) is empty. So, we have as a consequence

$$\left\langle \mathbb{P}_{SU(N)} [\exp(-g \oint_{C_{xx}} A_{\mu}(X_{\beta}(\sigma) dX_{\mu}(\sigma))] \right\rangle_{A_{\mu}} = 1 \tag{4.68}$$

Proceeding in analogous way for higher-order cummulants, one uses again the aforementioned Parisi topological argument to arrive at the general results for a set of m Brownian paths $\{C_{xx}^{(\ell)}\}_{\ell=1,\dots,m}$

$$\left\langle \prod_{\ell=1}^{m} \left[P_{SU(N)} \exp\left(-g \oint_{C_{xx}^{(\ell)}} A_\mu(X_\beta^{(\ell)}(\sigma)) dX_\mu^{(\ell)}(\sigma) \right) \right] \right\rangle_{A_\mu} = 1. \qquad (4.69)$$

At this point we note that for finite N_c the following result holds true as a consequence of eq.(4.60) and eq.(4.69)

$$\begin{aligned} Z(0,0) = \Big\langle \exp\Big\{ \sum_{C_{xx}} P_{Dirac} \Big\{ \oint_{C_{xx}} d\sigma \frac{i}{2}[\gamma^\alpha,\gamma^\beta](\sigma) \frac{\delta}{\delta\sigma_{\alpha\beta}(X(\sigma))} \\ P_{SU(N)}[\exp(-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma))dX_\mu(\sigma)] \Big\} \Big\rangle_{A_\mu} \\ = \exp\Big\{ \sum_{C_{xx}} P_{Dirac} \Big\{ \oint_{C_{xx}} d\sigma \frac{i}{2}[\gamma^\alpha,\gamma^\beta](\sigma) \frac{\delta}{\delta\sigma_{\alpha\beta}(X(\sigma))} \\ \Big\langle P_{SU(N)}[\exp(-g\oint_{C_{xx}} A_\mu(X_\beta(\sigma))dX_\mu(\sigma)] \Big\rangle_{A_\mu} \\ + \frac{1}{2} \sum_{C_{xx}^{(1)}} \sum_{C_{xx}^{(2)}} \Big\{ \oint_{C_{xx}} d\sigma^1 \frac{i}{2}[\gamma^\alpha,\gamma^\beta](\sigma^1) \frac{\delta}{\delta\sigma_{\alpha\beta}(X^1(\sigma^1))} \\ \times \oint_{C_{xx}^{(2)}} \frac{i}{2}[\gamma^\rho,\gamma^J](\sigma^2) \frac{\delta}{\delta\sigma_{\rho J}(X^2(\sigma^2))} \\ \Big\langle P_{SU(N)}\Big[\exp(-g\oint_{C_{xx}^{(1)}} A_\mu(X_\beta^1(\sigma^1))dX_\mu^1(\sigma^1)\Big] P_{SU(N)}\Big[\exp(-g\oint_{C_{xx}^{(2)}} A_\mu(X_\beta^2(\sigma^2))dX_\mu^2(\sigma^2)\Big] \Big\rangle_{A_\mu} \\ + \cdots \Big\} = \exp(0) = 1 = \det{}_F(\partial\!\!\!/ \, \partial\!\!\!/^*], \end{aligned} \qquad (4.70)$$

which by its turn leads to the Thirring model's triviality for space-time R^D with $D \geq 4$.

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Chapter 5

Triviality - Quantum Decoherence of Fermionic Quantum Chromodynamics $SU(N_c)$ in the Presence of an External Strong $U(\infty)$ Flavored Constant noise Field

5.1. Introduction

In chapters 3 and 4 we have proposed a bosonic loop space formalism for understanding the important problem of triviality in interacting Gauge Field theories ([1], [2]). The basic idea used in our work above mentioned in order to analyze such kind of quantum triviality phenomena was the systematic use of the framework of the loop space to rewrite particle-field path integrals in terms of its ensembre of quantum trajectories and the introduction of a noisely electromagnetic field as an external quantized reservoir.

The purpose of this chapter - of complementary nature to the above mentioned chapters 3 and 4 is to point out quantum field triviality phenomena in the context of our previous loop space formalism for the case of Fermionic Quantum Chromodynamics with finite number of colors but in presence of an external non-abelian translation independent $U(\infty)$-flavor charged white noise simulating a quantum field reservoir ([1]).

In order to show exactly this triviality result for $Q.C.D(SU(N_c))$ in such a context of an external non-abelian reservoir, we use of Migdal-Makeenko loop space expression for the spin quark generating functional of abelian vectorial quarks currents ([3]) – associated to the physical abelian vectorial mesons, added with the explicitly evaluation of $U(M)$-flavor Wilson Loops at the t'Hooft $M \to \infty$ limit for translation invariant noise-flavor field configurations.

We finally arrive at our main result that the triviality of Quantum Chromodynamics at such a kind of flavor reservoir, is linked to the problem of quantum decoherence in Quantum Physics ([1]). In appendix A, we present an aplication of our study to the Physical Problem of Confining in Yang-Mills Theory. In appendix B, we present the detailed analysis of the problem of large N in Statistics, which mathematical ideas have underlyning our Path-Integral Analysis in the bulk of this chapter.

5.2. The Triviality - Quantum Decoherence Analysis for Quantum Chromodynamics

In order to show such a triviality - quantum decoherence on Fermionic $Q.C.D(SU(N_c))$ with finite number of colors in the presence of $U(\infty)$ flavored random reservoirs, let us consider the physical Euclidean generating functional of the Abelian quarks currents in the presence of an external translation invariant white-noise $U(M)$ non-abelian field $B_\mu^{(M)}$, considered here as a kind of "dissipative" non-abelian reservoir structure and corresponding to the interaction quarks flavor charges with a $U(M)$ vacuum-reservoir structure, namely

$$Z[J_\mu(x),B_\mu^{(M)}] = \left\langle \det{}_F \begin{bmatrix} 0 & \not{D}(A_\mu,B_\mu^{(M)},J_\mu) \\ \not{D}^*(A_\mu,B_\mu^{(M)},J_\mu) & 0 \end{bmatrix} \right\rangle_{A_\mu} \tag{5.1}$$

Here the Euclidean Dirac operator is explicitly given by

$$\not{D}(A_\mu,B_\mu^{(M)},J_\mu) = i\gamma_\mu(\partial_\mu + g^{(M)}B_\mu^{(M)} + eA_\mu + J_\mu) \tag{5.2}$$

with $eA_\mu(x)$ denoting the $SU(N_c)$ Yang-Mills non-Abelian quantum field (translation dependent) configurations averaged in eq.(5.1) by means of the usual Yang-Mills path integral denoted by $\langle \quad \rangle_{A_\mu}$, $J_\mu(x)$ is the auxiliary source field associated to the abelian quark currents and $g^{(M)}B_\mu^{(M)}$ is a random translation invariant external $U(M)$ flavor Yang-Mills field with a constant field strength

$$F_{\mu\nu}(B) = (ig_M)[B_\mu^{(M)},B_\nu^{(M)}]. \tag{5.3}$$

Here $E_F^{(M)}$ denotes the stochastic average on the ensemble of the external random $U(M)$ non-abelian strenght fields defined by the $U(M)$-invariant path-integral ([4])

$$\begin{aligned} E_F^{(M)}\{O(B_\mu)\} \equiv \frac{1}{E_F^{(M)}\{1\}} \Bigg(\int \Big(\prod_{\mu=1}^{D}\prod_{a=1}^{M^2} dB_\mu^a \Big) \\ \times \exp\left\{ -\frac{1}{2}\left[(ig_M)^2 \mathrm{Tr}_{U(M)}([B_\mu^{(M)},B_\nu^{(M)}]^2) \right] \right\} \\ \times O(B_\mu^{(M)}) \Bigg) \end{aligned} \tag{5.4}$$

with $O(B_\mu^{(M)})$ denoting an $U(M)$-flavor invariant observable on the presence of an external translation invariant random $U(M)$-valued non-abelian reservoir field $B_\mu^{(M)}$.

In the fermionic loop space framework ([1], [2], [3]), we can express the quark functional determinant, eq.(5.1) – which has been obtained as an effective generating functional for the color N_c-singlet quark current after integrating out the Euclidean quark action –

as a purely functional on the bosonic bordered loop space composed of all trajectories $C_{xx} = \{X_\mu(\sigma), X_\mu(0) = X_\mu(T) = x;\, 0 \leq \sigma \leq T\}$, namely

$$Z[J_\mu(x), B_\mu^{(M)}] = \Big\langle \exp\Big\{ -N_c \text{spur}\Big[\sum_{C_{xx}} \mathbb{P}_{\text{Dirac}}\Big[\exp\Big(\oint_{C_{xx}} d\sigma \frac{i}{2}[\gamma^\mu,\gamma^\nu](\sigma) \times \frac{\delta}{\delta\sigma_{\mu\nu}(X(\sigma))}\Big] \times \text{Tr}_{U(M)}(\Phi[C_{xx}, B_\mu^{(M)}])\Big\} \times \Phi[C_{xx}, J_\mu] \times \text{Tr}_{SU(N_c)}(W[C_{xx}, A_\mu]\Big)\Big]\Big\} \tag{5.5}$$

where $\Phi[C_{xx}, B_\mu^{(M)}]$ is the usual Wilson-Mandelstam path-ordered loop variable defined by the translation invariant random external (reservoir) $U(M)$ field $B_\mu^{(M)}$, and $W[C_{xx}, A_\mu]$ is the same loop space object for the dynamical quantum color gauge field $SU(N_c)$.

Note the appearance of the Migdal-Makeenko area – loop derivative operator with the Dirac index path ordenation in order to take into account explicitly the relevant spin-orbit interaction of the quarks Dirac spin with the set of interacting vectorial fields $\{A_\mu(x), B_\mu, J_\mu(x)\}$ in the theory described by eq.(5.1) ([3]) (the well-know bordered loops).

The sum over the closed bosonic loops C_{xx}, with end-point x is given by the proper-time bosonic path integral below ([1],[2])

$$\sum_{C_{xx}} = \int_0^\infty \frac{dT}{T} \int d^D x \int_{X(0)=x=X(T)} D^F[X(\sigma)] \times \exp\left\{-\frac{1}{2}\int_0^T \dot{X}^2(\sigma) d\sigma\right\}. \tag{5.6}$$

Following the idea of our previous work on Triviality-Quantum Decoherence of Gauge theories [1], we need to show in eq.(5.5) that at the t'Hooft topological limit of $M \to \infty$ in the ensemble of external white-noise reservoir fields $B_\mu^{(M)}$ as implemented in ref. [1], one obtains for the Wilson Loop $E_F(\Phi[C_{xx}, B_\mu^{(M)}])$ an area-power behavior on the (minimal) area $S[C_{xx}]$ bounded by the large area loops C_{xx} inside the loop space functional on eq.(5.5), after considering the average of the infinite-flavor limit on the external translation independent white-noise B_μ field eq.(5.3)–eq.(5.4).

In the context of a cummulant expansion for the loop space integrand in eq.(5.5) defined by the $U(M)$ path integral eq.(5.4), one should firstly evaluate the following Wilson Loop path integral (loop normalized to unity) on the $U(M)$-noise reservoir field $B_\mu^{(M)}$:

$$E_F^{(M)}\{\text{Tr}_{U(M)}(\phi[C_{xx}, B_\mu^{(M)}])\} = \frac{1}{E_F^{(M)}\{1\}} \int_{-\infty}^{+\infty} \left(\prod_{a=1}^{M^2}\prod_{\mu=1}^{D} dB_\mu^{a,(M)}\right) \times \exp\left\{+\frac{1}{2}(g_M)^2 \text{Tr}_{U(M)}([B_\mu^{(M)}, B_\nu^{(M)}]^2)\right\} \times \frac{1}{M}\text{Tr}_{U(M)}\mathbb{P}\left\{e^{ig_M \oint_{C_{xx}} B_\mu^{(M)} dX_\mu}\right\}. \tag{5.7}$$

By using the non-abelian Stokes theorem for constant gauge fields, one obtains the following result for large M ([4]):

$$\frac{1}{M}\left(\text{Tr}_{SU(M)}\mathbb{P}\left\{e^{ig_M \oint_{C_{xx}} B_\mu^{(M)} dx_\mu}\right\}\right) = \frac{1}{M}\left(\text{Tr}_{SU(M)}\mathbb{P}\left\{e^{ig_M \int_{S[C_{xx}]} F_{12}(B^{(M)}) S^{12}}\right\}\right) \tag{5.8-a}$$

or equivalently:

$$\frac{1}{M}\mathrm{Tr}_{SU(M)}\left(\mathbb{P}e^{-(g_M)^2[B_1,B_2]S[C_{xx}]}\right)=\exp\left\{+\frac{(g_M^2 S[C_{xx}])^2}{2M}(\mathrm{Tr}[B_1^{(M)},B_2^{(M)}])^2\right\}+O(\frac{1}{M}) \tag{5.8-b}$$

where we have choosen the large loop C_{xx} to be contained in the plane $\mu=1$, $\nu=2$ without loss of generality.

A simple field re-scaling on the path-integral eq.(5.7) as written below, after inserting the $M\to\infty$ leading exact result of the Wilson Loop noise factor eq.(5.8) on the cited equation (5.7):

$$B^a_{\mu=1}\to\tilde{B}^a_{\mu=1}\left[g_M^2+\frac{(g_M^2 S[C_{xx}])^2}{M}\right]^{-\frac{1}{4}} \tag{5.9}$$

$$B^a_{\mu=2}\to\tilde{B}^a_{\mu=2}\left[g_M^2+\frac{(g_M^2 S[C_{xx}])^2}{M}\right]^{-\frac{1}{4}} \tag{5.10}$$

$$B^a_{\mu\neq\{1,2\}}\to\tilde{B}^a_{\mu\neq\{1,2\}}[g_M^2]^{-\frac{1}{4}} \tag{5.11}$$

leads us to the exactly result at the t'Hooft limit of $U(\infty)$ flavor charge

$$\begin{aligned}\lim_{M\to\infty}(E_F^{(M)}\{\mathrm{Tr}_{U(M)}(\Phi[C_{xx},B_\mu^{(M)}])\})&=\lim_{M\to\infty}\left\{\frac{\left[g_M^2\left(1+\frac{g_M^2 S[C_{xx}]^2}{M}\right)\right]^{-\frac{1}{2}M^2}}{[g_M^2]^{-(\frac{(M^2D)}{4})}}[g_M^2]^{-\frac{M^2(D-2)}{4}}\right\}\\&=\exp\left\{-\frac{1}{4}(g_\infty)^2S^2[C_{xx}]\right\}+O(\frac{1}{M})\end{aligned} \tag{5.12}$$

where $g_\infty^2=\lim_{n\to\infty}((g_M)^2M)<\infty$ denotes the $U(\infty)$-flavor reservoir t'Hooft coupling constant. Note that we have used the leading $M\to\infty$ limit on the weight on the numerator of the reservoir field path integral eq.(5.7). For instance (here $B_\mu\equiv B_\mu^a\lambda_a$ with $[\lambda_a,\lambda_b]=f_{abc}\lambda_c$)

$$\begin{aligned}&\lim_{n\to\infty}\left\{\exp\left[\left(\frac{1}{2}(g_M)^2+\frac{(g_M)^4(S[C_{xx}])^2}{2M}\right)\times\left(\tilde{B}_1^a\tilde{B}_2^b\tilde{B}_1^{a'}\tilde{B}_2^{b'}f_{abc}f^{ca'b'}\right)\right]\right\}\\&\sim\exp\left\{\left[\left(\frac{1}{2}(g_M)^2\right)\times\tilde{B}_1^a\tilde{B}_2^b\tilde{B}_1^{a'}\tilde{B}_2^{b'}f_{abc}f^{ca'b'}\right]\right\}\\&+O\left(\frac{1}{M}\right)\end{aligned} \tag{5.13}$$

which produces as the only non-trivial result at $M\to+\infty$ in the average eq.(5.7), that one arising from the ratio of the Jacobians of the measure change associated to re-scalings eq.(5.9)–eq.(5.11) on the path integral numerator eq.(5.7) and the normalization path-integral denominator respectively.

As a result, we get an exponential behavior for our noise $U(\infty)$-averaged Wilson Loop with an power square area argument.

Finally, we can see that the loop space quark fermion determinant eq.(5.1) is entirely supported at those loops C_{xx} with vanishing small area $S[C_{xx}]$ for large values of the noise-field vacuum streinght $g_\infty^2\to+\infty$, since those of large area $S[C_{xx}]$ are suppressed on the loop space expression generating functional eq.(5.1) above mentioned, as much as similar K. Wilson mechanism for charge confining in Q.C.D.

Note that the same matter loop C_{xx} appearing in eq.(5.12) enters in the definition of all loop space objects in eq.(5.5). As a consequence, we have produced a loop space analysis supporting that at very large noise strenght $(g^{(\infty)} \to +\infty)$, one has exactly the strong triviality of the $SU(N_c)$ on the sector of the quark abelian currents, in the mathematical sense that the dominant loops on the loop path integral eq.(5.5) are degenerate to the loop base point x or to the straight line vector bilinear quark field excitations trajectories motion. It yield as a result, thus

$$\lim_{M\to\infty} E_F^{(M)}(Z[J_\mu(x), B_\mu^{(M)}]) = \exp(0) = 1. \tag{5.14}$$

This result leads us to the conclusion that the theory has on free field behavior ([5]) at very strong noise-reservoir of the type introduced in this work signaling a kind of quantum field phenomena in a flavored dissapative vacuum media that destroys quantum phase coherence and leading to the theory's triviality as much as similar mechanism underlying the phenomena which has been obtained in ref. [1] for white-noise abelian reservoirs.

Appendix A.
The Confining Property of the $U(\infty)$ - Charge Reservoir

We intend to show the own quantum decoherence/triviality of the $U(\infty)$-charged reservoir considered in the bulk of this work. Let us, thus, consider our translation invariant $U(M)$ non-abelian gauge field theory of the previous analysis. However defined in a finite volume domain Ω with $\mathrm{vol}(\Omega) = ma^4$, where m is an positive integer with a playing the rule of a fundamental lenght scale associated to the elementary cell of volume a^4 of our finite-volume space-times (euclidean). We introduce at this point of our argument a closed loop C contained in the plane (μ,ν) – section of the domain $\Omega \subset R^4$ and possesing area (planar) $S[C_{\mu\nu}] = na^2$. (See eq.(5.7)).

$$\langle W[C]\rangle^{(\infty)} = \lim_{M\to\infty}\left(\frac{I_M[C]}{I_M[0]}\right) \tag{5.A-1}$$

The explicitly expressions for the objects on eq.(5.A-1) are the following C. Bollini and J.J. Giambiagi translation invariant gauge field path integrals ([5])

$$\begin{aligned} I_M[C] = \int_{-\infty}^{+\infty}\left(\prod_{a=1}^{M^2-M}\prod_{\mu=1}^{4} dA_\mu^a\right) \times \exp\left\{\frac{g^2}{2}(ma^4)\mathrm{Tr}_{U(M)}([A_\mu,A_\nu]^2\right\} \\ \times\left(\frac{1}{M}\mathrm{Tr}_{U(M)}P\left[\exp\left(ig\int_C A_\mu dx_\mu\right)\right]\right) \end{aligned} \tag{5.A-2}$$

and

$$I_M(0) = \int_{-\infty}^{+\infty}\left(\prod_{a=1}^{M^2-M}\prod_{\mu=1}^{4} dA_\mu^a\right) \times \exp\left\{\frac{g^2}{2}(ma^4)\mathrm{Tr}_{U(M)}([A_\mu,A_\mu]^2)\right\} \tag{5.A-3}$$

The exactly evaluation of eq.(5.A-2) was presented in our previous analysis, with the result below, after considering the re-scaling integration variable

$$A_\mu^a \to A_\mu^a \left[\frac{g^2}{2}(ma^4) + \frac{g^4(na^2)^2}{M}\right]^{-\frac{1}{4}} \quad \mu = 1,2 \tag{5.A-4}$$

$$A_\mu^a \to A_\mu^a \left[\frac{g^2}{2}(ma^4)\right]^{-\frac{1}{4}} \quad \mu \neq 1,2 \tag{5.A-5}$$

with the result

$$I_M[C] = \left[\frac{g^2}{2}(ma^4) + \frac{g^4(na^2)^2}{M}\right]^{-(M^2)}. \tag{5.A-6}$$

The same procedure is applied too in eq.(5.A-3) with the associated re-scaling $A_\mu^a \to A_\mu^a \left[\frac{g^2}{2}(ma^4)\right]^{-\frac{1}{4}}$. It yields the exactly result for the path-integral normalization factor

$$I_M[0] = \left[\frac{g^2}{2}(ma^4)\right]^{-M^2}. \tag{5.A-7}$$

As consequence, we get the following result for the $U(\infty)$-Loop Wilson average $((g^\infty)^2 = \lim_{M\to\infty}(g^2 M) < \infty)$

$$\begin{aligned}\langle W[C]\rangle^{(\infty)} &= \lim_{M\to\infty}\left(\frac{I_M[C]}{I_M[0]}\right)\\ &= \exp\left\{-(g_\infty)^2\cdot\left(\frac{[na^2]^2}{ma^4}\right)\right\}\\ &= \exp\left\{-\left(\frac{(g_\infty)^2}{a^2}\right)\cdot\left[\left(\frac{n^2}{m}\cdot a^2\right)\right]\right\}.\end{aligned} \tag{5.A-8}$$

At this point of our study we call the reader attention that in the final result eq.(5.A-8), we have considered already the case $D = 4$, where one must taken into account the transmutation phenomena of the Gauge coupling constant $g^{(\infty)}$ by considering the existence of a vacuum area domain a^2 (the cell of our space-time) as much as the famous "Q.C.D. spaghetti vacuum" of Nielsen, Olesen et. al. ([1]).

The area behavior of eq.(5.A-6) is easily obtained for large area loops $n^2 >> m$ in the following situation: If one considers the relationship $n = \gamma m$, with γ an adimensional number $(\gamma < 1)$ which will be kept constant at the limit of infinite volume $m \to \infty$, one can see that eq.(5.A-8) gives area behavior for the Q.C.D. Wilson Loop for very large loop area

$$\begin{aligned}\langle W[C]\rangle^{(\infty)} &\underset{m\to\infty}{\sim} \exp\left\{-\left(\frac{\gamma(g_\infty)^2}{a^2}\right)\cdot na^2\right\}\\ &= \exp\left\{-\frac{(g_\infty)^2}{a^2_{\text{eff}}}\cdot \text{ Area } S[C]\right\}\end{aligned} \tag{5.A-9}$$

At this point, one should envisage to implement a formal Feynman diagrammatic field theoretic $\frac{1}{M}$ – expansion on the finite order group $U(M)$ – Gauge theory by considering next translation – dependent field corrections on our reservoir field configurations of the form $A_\mu^a(x) = A_\mu^{(\infty)} + \frac{1}{M} G_\mu^a(x)$ in the usual Path-Integral measure with the matter confining behavior eq.(5.A-9) already built in the formalism, namely:

$$\prod_{\mu=1}^{D} \prod_{a=1}^{M^2-M} dA_\mu^a(x) = \prod_{\mu=1}^{D} \prod_{a=1}^{M^2-M} dA_\mu^a \cdot dG_\mu^a(x) \tag{5.A-10}$$

$$\begin{aligned} &\exp\left\{ -\frac{1}{2} \int_\Omega \mathrm{Tr}_{SU(M)} (F_{\mu\nu})^2(x) d^D x \right\} \\ &\quad = \exp\Bigg\{ -\frac{1}{2} \int_\Omega \mathrm{Tr}_{SU(M)} \Big((\partial_\mu G_\nu - \partial_\nu G_\mu)(x) \\ &\qquad + \frac{ig^{(\infty)}}{\sqrt{M} a} \Big[A_\mu + \frac{1}{M} G_\mu(x), A_\nu + \frac{1}{M} G_\nu(x) \Big] \Big)^2 \Bigg\}. \end{aligned} \tag{5.A-11}$$

It is worth remarking that the Feynman's Diagrammatic associated to the Back-Ground field decomposition in eqs.(5.A-10)–(5.A.11) leads to an exchange of "massive" Gluons and leading, thus, to a infrared-free perturbation analysis of the theory's observables.

Appendix B.
On the Law of Large Number in Statistics

Let us present the usual mathematical methods procedure to define the large N limit in Statistics.

The large N problem in Statistics starts by considering a set of N-independent random variables $\{X_\ell(w)\}_{\ell=1,\ldots,N}$, with w belonging to a given fixed probability space $(\Omega, d\mu(w))$, besides of satisfying the following additionals constraints:

a) Theirs mean value posseses all the same value m:

$$\int_\Omega X_\ell(w) d\mu(w) = E\{X_\ell(w)\} = m \tag{5.B-1}$$

b) Theirs associated variance are all equals:

$$\sigma^2 \left[\left(\int_\Omega X_\ell^2(w) d\mu(w) \right)^2 - \left(\int_\Omega X_\ell(w) d\mu(w) \right)^2 \right] \tag{5.B-2}$$

The large N problem in Statistics can be stated now as the problem of defining mathematically the normalized limit of "large numbers" $N \to \infty$, of the sequence of random variables sum below

$$\lim_{N\to\infty} \hat{S}_N(w) = \lim_{N\to\infty} \left(\frac{1}{\sigma\sqrt{N}} \left(\sum_{\ell=1}^{N} X_\ell(w) - m \right) \right). \tag{5.B-3}$$

The path-integral solution for this problem contains all needed ideas and expose clearly the method which were implemented in our analysis in Gauge Field Theory.

Firstly, we define the associated Generating Functionals for each independent random variable $X_\ell(w)$, with $J \in R$. Namely:

$$Z_{\{X_\ell\}}((J)) = E\{e^{iJX_\ell(w)}\} = \int_\Omega e^{iJX_\ell(w)} d\mu(w)$$
$$= \sum_{k=0}^{\infty} \frac{i^k J^k}{k^1} \left(\int_\Omega (X_\ell(w))^k d\mu(w) \right) \tag{5.B-4}$$

It is straightforward to see that the Generating Functional associated to the finite N random variable sum eq.(5.3)

$$Z_N(J) = \prod_{\ell=1}^{N} \left[Z_{\{X_\ell\}} \left(J \left(\frac{X_\ell - m}{\sigma\sqrt{N}} \right) \right) \right]$$
$$= \left[\sum_{k=0}^{\infty} \frac{1}{k!} \frac{i^k M_k}{\sigma^k N^{k/2}} \cdot J^k \right]^N$$
$$= \left(1 - \frac{J^2}{2N} - \frac{iM_3 J^3}{6\sigma^3 N^{3/2}} + \frac{M_4 J^4}{24\sigma^4 N^2} + \dots \right)^N, \tag{5.B-5}$$

with the k-power averages given by the integral expressions below, which are supposed to be ℓ-independent

$$M_k = \int_\Omega (X_\ell(w)^k d\mu(w). \tag{5.B-6}$$

At this point, we define mathematically the large N limit by defining the effective statistics distribution parameters:

$$\lim_{N\to\infty} (\sigma\sqrt{N}) = \overline{\sigma}_{eff} < \infty \tag{5.B-7}$$

$$\lim_{N\to\infty} (mN) = \overline{m}_{eff} < \infty \tag{5.B-8}$$

and by taking the $N \to \infty$ limit of eq.(5.5) in the context of the definitions eq.(5.7)–eq.(5.8), by considering just for simplicity of our formulae writing $m = 0$ (see eq.(5.1)).

As a result, we have the simple expression below

$$\lim_{N\to\infty} [lg Z_N(J)] = N\, lg \left[1 - \frac{J^2}{2N} - \frac{iM_3 J^3}{6\sigma^2 N^{3/2}} + \frac{M_4 J^4}{24\sigma^4 N^2} + \dots \right]$$
$$= -\left(\frac{J^2}{2N} \right) N = -\frac{J^2}{2}, \tag{5.B-9}$$

or equivalently

$$\lim_{N\to\infty} Z_N(J) \equiv Z^{eff}_{N=\infty}(J) = e^{-\frac{J^2}{2}}, \tag{5.B-10}$$

which is nothing more than the Generating Functional associated to the Gaussian Statistics distribution:

$$Z^{eff}_{N=\infty}(J) = \frac{1}{\sqrt{2\pi} \cdot \overline{\sigma}} \times \int_{-\infty}^{+\infty} dx\, e^{-ixJ}\, e^{-\frac{x^2}{2\overline{\sigma}^2}} \tag{5.B-11}$$

which is formally the limit (with $m \neq 0$)

$$\lim_{N\to\infty} \left\{ \frac{1}{\sqrt{2\pi}(\sigma\sqrt{N})} e^{-\frac{(x-Nm)^2}{2N\sigma^2}} \right\} = \frac{1}{\sqrt{2\pi}\overline{\sigma}} e^{-\frac{(x-\overline{m})^2}{2\overline{\sigma}^2}} \tag{5.B-12}$$

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Chapter 6

Fermions on the Lattice by Means of Mandelstam-Wilson Phase Factors: A Bosonic Lattice Path-Integral Framework

6.1. Introduction

One of the long-standing unsolved problems in the lattice approach to QCD is how to handle discretized massless fermionic fields [1]. In this chapter we propose a solution for the above-mentioned problem by considering as the QCD natural field variable to be discretized on the lattice the Mandelstam-Wilson phase factor defined by the color-singlet quark currents, instead of the fermion field as proposed by previous studies. Additionally, we show the usefulness of this propose by obtaining, in an unambiguous way, the associated QCD Nambu-Jona-Lasinio fermionic model, which, upon being bosonized, leads to a low-energy theory of the mesons and baryons of QCD (see Chapter 18).

6.2. The Framework

Let us start our study by considering the Euclidean QCD [$\mathrm{SU}(N_c)$] generating functional for the color-singlet scalar and vectorial quark currents:

$$Z[\sigma+\gamma_5\beta, J_\mu+\gamma_5\widetilde{A}_\mu] = \int D^F[A_\mu(x)[\exp\left(-\frac{1}{4}\int d^4x\,\mathrm{Tr}[F^2_{\mu\nu}(A)](x)\right) \tag{6.1}$$

$$\times\left\{\int D^F[\overline{\psi}(x)]D^F[\psi(x)]\exp\left(-\int d^4x(\overline{\psi}[i\gamma^\mu\partial_\mu - ig\gamma^\mu A_\mu+\sigma+\gamma_5\beta+\gamma^\mu J_\mu+\gamma^\mu\gamma^5\widetilde{A}_\mu]\psi)(x)!\right)\right\}$$

where $\psi(x)$, $\overline{\psi}(x)$ are the independent Euclidean quark fields, $\sigma(x)+\gamma_5\beta(s)$ and $J_\mu(x)+\gamma^5\widetilde{A}_\mu(x)$ are the external sources for the scalar, scalar-axial, and axial-vectorial QCD quark currents. $A_\mu(x)$ denotes the $\mathrm{SU}(N_c)$ gluon field.

In order to obtain effective quark field theories from eq.(6.1) we propose to integrate out their gluon degrees of freedom in the lattice; i.e., let us first consider the pure gluonic functional integral

$$I[\psi,\tilde{\psi}] = \int D^F[A_\mu(x)]\exp\left(\left(-\frac{1}{4}\int d^4x\,\mathrm{Tr}[F^2_{\mu\nu}(A)](x)\right)\right.$$
$$\times\exp\left(ig\int d^4x(\bar{\psi}\gamma^\mu\psi)(x)A_\mu(x)\right). \tag{6.2}$$

Our procedure to evaluate eq.(6.2) is, first, to introduce a lattice space-time. At this point we put forward our idea to handle correctly fermonic fields on the lattice. As was shown in [1], it is impossible to have a well-defined procedure to define massless fermion fields on the usual lattice $\{x_\mu = [n_\mu], n_\mu \in \mathbb{Z}\}$ (with spacing a) [1]. We propose, thus, to consider directly the bosonic quark fermion current on the lattice by means of its associated Mandelstam-Wilson phase factor defined on each lattice link $([n_\mu],[n_\mu]+\alpha)$

$$\Phi_\alpha([n_\mu]) = \exp(iag(\bar{\psi}\gamma^\alpha\psi)([n_\mu])). \tag{6.3}$$

Note that the above-written phase factor has indices (i,j) on the group $\mathrm{SU}(N_c)$ and an index α related to the Lorentz group as it should be.

The associated gluon $\mathrm{U}(N)$ group-valued Mandelstam-Wilson phase factor is still given by the link lattice gluon variable

$$U_\mu([n_\alpha]) = \exp(iaA_\mu([n_\alpha])). \tag{6.4}$$

At this point of our study, we point out that the quark gluon coupling on the lattice may be written as a product of the Mandelstam-Wilson phase factor given by eqs.(6.3) and (6.4) since we have the formal continuum limit at the lattice space going to zero as one can see by expanding the exponentials

$$\lim_{a\to 0}\left[\sum_{\{[n_\alpha]\}} a^2\,\mathrm{Tr}(\{U_\mu([n_\alpha])-\mathbf{1}\}\times\{\phi_\mu([n_\alpha])-\mathbf{1}\}+hc)\right] = ig\int d^4x A_\mu(x)(\tilde{\psi}\gamma^\mu\psi)(x). \tag{6.5}$$

Our proposed gauge-invariant lattice version of the gluon functional integral, eq.(6.2), is, thus, given by

$$I[\psi,\tilde{\psi}] = \int D^H[U_\mu([n_\alpha])]\exp\left(\left(-\frac{2}{4g^2}\sum_{\{[n_\alpha]\}}\mathrm{Tr}\{U_\mu([n_\alpha])U_\nu([n_\alpha+\nu])U^\dagger_\mu([n_\alpha+\nu])U^\dagger_\nu([n_\alpha])\}\right)\right.$$
$$\times\exp\left(\sum_{\{[n_\alpha]\}} a^2\,\mathrm{Tr}\{U_\mu([n_\alpha])-\mathbf{1}\}\right)\times\left(\Phi_\mu([n_\alpha]-\mathbf{1})^\dagger\right). \tag{6.6}$$

The advantage of this lattice phase factor approach to analyze the gluonic path integral, eq.(6.2), is its allowance for an exact integration of the lattice gluon phase factor in both the perturbative and the nonperturbative regimes. Let us show its usefulness by evaluating in closed form eq.(6.6) in the leading limit of the number of colors and in the leading limit of strong coupling as in ([2] - Eq.(3.17)):

$$I[\psi,\tilde{\psi}]_{\substack{g^2 \to\infty \\ N_c \to\infty}} = \lim_{N_c\to\infty} \int D^H[U_\mu([n_\alpha])] \exp\left(-a^2 \sum_{\{[n_\alpha]\}} \mathrm{Tr}(U_\mu([n_\alpha]))\{\Phi^\mu([n_\alpha]) - \mathbf{1}\})^\dagger \right)$$

$$= \exp\left(\frac{\Lambda^{(a)}_{QCD/N_c}}{N_c} \sum_{\{[n_\alpha]\}} \mathrm{Tr}(\{\phi^\mu([n_\alpha]) - \mathbf{1}\}\{\phi_\mu([n_\alpha]) - \mathbf{1}\}^\dagger \right), \tag{6.7}$$

where $\Lambda^{(a)}_{QCD}$ is the QCD strong-coupling phenomenological scale with dimension of inverse area (the gluon nonperturbative condensate) which by its turn is lattice spacing dependent.

It is very important to remark that the Jacobian J of the variable change $U_\mu([n_\alpha]) \to U_\mu([n_\alpha]) + \mathbf{1}$ on the lattice functional integrals, Eqs. (6) and (7), is unity only at the continuum limit $a \to 0$ (or at large N_c) since it is explicitly given by the ratio

$$J_{(a)} = \prod_{([n_\mu])} \left\{ \frac{\det^{1/2}(M_{ij}\{U_\mu([n_\alpha]) + \mathbf{1}\})(a)}{\det^{1/2}(M_{ij}\{U_\mu([n_\alpha])\})(a)} \right\} \tag{6.8}$$

and for $a \to 0$ we have that $(\mathbf{1} + U^\dagger_\mu([n_\alpha])) \to \mathbf{1}$. Here the Haar measure $\prod_{[n_\alpha]} D^H\{U_\mu([n_\alpha])\}$ on the group $\prod_{[n_\alpha]} U(N_c)$ follows from the metric tensor group on each factor $U(N_c)$ [3]:

$$M_{ij} = \mathrm{Tr}\left(U^{-1}([n_\alpha]) \frac{\partial}{\partial t_i} U([n_\alpha]) \times U^{-1}([n_\alpha]) \frac{\partial}{\partial t_j} U([n_\alpha]) \right) \tag{6.9a}$$

$$D^H[U_\mu([n_\alpha])] = \prod_{(i)} (dt_i \det^{1/2}[M_{ij}(t)]), \tag{6.9b}$$

where the derivatives are with respect to the group parameters $\{t_i\}$; i.e.,

$$U_\mu([n_\alpha]_= \exp(it^l([n_\mu])\lambda_l). \tag{6.10}$$

The formal continuum limit $a \to 0$ of the result, eq.(6.7), after a Fierz transformation, leads to the following quartic fermionic action in the continuum:

$$I_{\mathrm{continuum}}[\psi,\tilde{\psi}]_{\substack{g^2 \to\infty \\ N_c \to\infty}} = \exp\left\{ \frac{g_F^2}{N_c} \int d^4x [(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^4\psi)^2 + \frac{1}{2}(\bar{\psi}\gamma^\mu\psi)^2 - \frac{1}{2}(\bar{\psi}\gamma^\mu\gamma^5\psi)^2](x) \right\}. \tag{6.11}$$

Here the fermion/ic effective coupling constant g_F^2 is defined in the continuum by thee formal limit $g_F^2 = \lim_{a\to 0} \Lambda^{(a)}_{QCD} \cdot g^2$ and signaling the usual QCD dimensional transmutation phenomenon.

After substituting eq.(6.11) into eq.(6.1) we get our proposed fermionization for quantum chromodynamics in the very low-energy region with the gluon field $U(N_c)$ integrated out for large N_c in the sense of Ref. [2]. We remark that by introducing the Hubbard-Stratonovich ansatz to linearize the quartic fermion interactions, we obtain the U(1) chiral scalar and vectorial bosonized QCD $[U(\infty)]$ meson theory which improves that considered in [4] which was deduced by using solely phenomenological guessing arguments:

$$Z[\sigma+\gamma_5\beta, J_\mu+\gamma_5\tilde{A}_\mu] = D^F[\hat{\sigma}]D^F[\hat{\beta}]D^F[\hat{J}_\mu]D^F[\hat{A}_\mu]\exp\left(-\frac{N_c}{g_F^2}\int d^4x[(\frac{1}{2}\hat{\sigma}^2+\frac{1}{2}\beta^2+\frac{1}{2}\hat{J}_\mu^2+\frac{1}{2}\hat{A}_\mu^2)](x)\right)$$
$$\times\{\det{}^{N_c}[i\gamma\partial+(\sigma+i\hat{\sigma})+\gamma_5(\beta+i\hat{\beta})+\gamma_\mu(J_\mu+i\hat{J}_\mu)+\gamma^5\gamma^\mu(i\hat{A}_\mu+\tilde{A}_\mu)]\}. \tag{6.12}$$

Note that in eq.(6.12), $(\hat{\sigma}+i\gamma_5\hat{\beta})$ and $(\hat{J}_\mu+i\gamma_5\hat{A}_\mu)$ should be identified with the U(1) chiral scalar and vectorial low-energy physical meson fields. Let us comment that the dynamics for the meson fields above comes from the evaluation of the quark function determinant [4]. In the limit of the heavy scalar meson mass $(\langle\hat{\sigma}\rangle\to\infty$, one can easily implement the technique of [5] to get the full effective hadronic action in terms of $1/\langle\hat{\sigma}\rangle$ power series (see Chapter 18).

In the case of baryonlike field excitations of the form $\Omega(x)=\varepsilon_{ijk}\psi_i(x)\psi_j(x)\overline{\psi}_k(x)$ it is still posible to analyze them in our proposed framework. For this task we consider a Hubbard-Stratonovich ansatz to write the generating functional for the baryonlike excitation $B(x)$: namely,

$$Z[B(x)] = \int D^F[\Delta]D^f[\lambda]D^F[A_\mu]D^F[\psi]D^F([\overline{\psi}]\exp\left(-\int d^4x\{\overline{\psi}_p[i\gamma^\mu\partial_\mu\delta_{pq}+i\lambda_{pq}+\gamma^\mu(A_\mu)_{qp}]\psi_q\}(x)\right)$$
$$\times\exp\left(-\int d^4x[B(x)\varepsilon_{ijk}\psi_i(x)\Delta_{jk}(x)]\right)\exp\left(-i\int d^4x[\lambda_{pq}(x)\Delta_{qp}(x)]\right). \tag{6.13}$$

where (p,q) ae $U(N_c)$ indices and the auxiliary fields (Δ,λ) belong to the adjoint $U(N_c)$ representation.

After integrating out the gluon field $A_\mu(x)$ folloowing the steps leading to eq.(6.7) and the quark field as in eq.(6.9), we get our proposed effective QCD-baryon field theory:

$$\begin{aligned}
Z[B(x)] &= \int D^F[\hat{\sigma}]D^F[\hat{\beta}]D^F[\hat{J}_\mu]D^F[\hat{A}_\mu]D^F[\Delta]D^F[\lambda] \\
&\times\exp\left(-\frac{N_c}{g_F^2}\int d^4x[\frac{1}{2}\hat{\sigma}^2+\frac{1}{2}\hat{\beta}^2](x)+[\frac{1}{2}\hat{J}_\mu^2+\frac{1}{2}\hat{A}_\mu^2](x)\right) \\
&\times\exp\left(\int d^4x\,\mathrm{Tr}(\lambda\Delta)(x)\right)\det\{[i\gamma\partial+(\hat{\sigma}+i\gamma_5\hat{\beta})+\gamma^\mu(\hat{J}_\mu+i\gamma_5\hat{A}_\mu)]_{pq}-i\lambda_{pq}\} \\
&\times\exp\left\{-\int d^4xd^4yB(x)\varepsilon_{ijk}(x)[(i\gamma\partial+(\hat{\sigma}+i\gamma_5\hat{\beta})+\gamma^\mu\hat{J}_\mu+i\gamma^\mu\gamma_5\hat{A}_\mu-\lambda]^{-1}_{ii'}(x,y)\varepsilon_{i'j'k'}\Delta_{j'k'}(y)B(y)\right\}.
\end{aligned} \tag{6.14}$$

It is instructive to remark that eq.(6.14) indicates the impossibility to consider baryon excitations in isolation from the meson excitations in our proposed bosonized effective QCD field theory.

It is worth pointing out that strong-coupling corrections from the neglected gluon field kinetic action in eq.(6.7) are straighforwardly implemented on the lattice by using the usual quantum field theory perturbation theory with the external lattice gluon source coupling [2]:

$\sum_{[n_\mu]} J_\mu([n_\alpha])U_\mu([n_\alpha])$;

$$I[\psi,\tilde{\psi}]_{N_c\to\infty} = \lim_{J_\nu([n_\alpha])\to 0}\Bigg\{\exp\Bigg[-\frac{1}{4g^2}\sum_{([n_\alpha])}\left(\frac{\delta}{\delta J_\mu([n_\alpha])}+\mathbf{1}\right)\left(\frac{\delta}{\delta J_\nu([n_\alpha+\mu])}+\mathbf{1}\right)$$
$$\times\left(\frac{\delta}{\delta J_\mu^{+}([n_\alpha]+\nu)}+\mathbf{1}\right)\left(\frac{\delta}{\delta J_\nu^{\dagger}([n_\alpha])}+\mathbf{1}\right)\Bigg]\widetilde{I}[\psi,\tilde{\psi}]_{\substack{g^2\to\infty\\ N_c\to\infty}}\Bigg\}, \quad (6.15)$$

where [see eq.(6.7)]

$$\widetilde{I}[\pi,\tilde{\psi}]_{\substack{g^2\to\infty\\ N_c\to\infty}} = \exp\left(\frac{\Lambda_{QCD}^{(a)}\cdot a^4}{N_c}\sum_{\{[n_\alpha]\}}\mathrm{Tr}\{(\Phi_\mu+J_\mu-\mathbf{1})([n_\alpha])(\Phi_\mu+J_\mu-\mathbf{1})^\dagger([n_\alpha])\}\right). \quad (6.16)$$

The associated $1/4g^2$ corrected fermionized QCD $[U(\infty)]$ effective theory will, thus, be given by nonlocal current-quark correlation functions averaged with the leading Namu-Jona-Lasinio quark field theory. eq.(6.11). Unfortunately, only at the limit of large mass (see Chapter 18), it is posible to implement reliable approximate calculations useful for nuclear physics at low evergy. Work on these applications for very low-energy nuclear hadron dynamics will be left to the future endeavors of our readers in this subject.

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Chapter 7

A Connection between Fermionic Strings and Quantum Gravity States – A Loop Space Approach

7.1. Introduction

The dynamical formulation of Einstein General Relativity in terms of a new set of complex $SU(2)$ coordinates has opened new perspectives in the general problem of quantization of the gravitation field by non-perturbative means. The new set of dynamical variables proposed by Ashtekar are the projection of the tetrads (the so called triads) on the three-dimensional base manifold M of our cylindrical space-time $M \times R$ added with the four-dimensional spin connection for the left-handed spinor again restricted to the embedded space-time base manifold M (the Ashtekar-Sen $SU(2)$ connections) [1] and paralleling successful procedure used to quantize canonically pure three-dimensional gravity [2].

The fundamental result obtained with this approach is related to the fact that it is possible to canonically quantizes the Einstein classical action in the same way one canonically quantizes others quantum fields [3]. As a consequence, the governing Schörindger-Wheeler-Dewitt dynamical equations which emerges in such gravity gauge field parametrization supports exactly highly non-trivial prospective explictly (regularized) functional solutions [4].

In this chapter we intend to present in Section 7.3 a Loop Space-Path integral supporting the fact that the formal continuum limit of a $3D$ Ising model, a Quantum Fermionic String on the space-time base manifold M, is a (formal operatorial) solution of the Wheeler-De Witt equation in the above mentioned Ashtekar-Sen parametrization of the Gravitation Einstein field. We present too a propose of ours on a Loop geometrodynamical representation for a kind of $\lambda\varphi^4$ third-quantized geometrodynamical field theory of Einstein Gravitation in terms of Ashtekar-Sen gauge fields.

7.2. The Loop Space Approach for Quantum Gravity

Let us start our analysis by writing the governing wave equations in the following operatorial ordered form [5].

$$\hat{C}[A]\psi[A] = \varepsilon^{ijk} \frac{\delta^2}{\delta A^i_\mu(x)\delta A^j_\nu(x)} \left\{ \left[F^k_{\mu\nu}(A(x))\psi[A] \right] \right\} = 0\ ; \tag{7.1}$$

$$\hat{C}_\mu[A]\psi[A] = \frac{\delta}{\delta A^k_\nu(x)} \left\{ F^k_{\mu\nu}(A(x))\psi[A] \right\} = 0\ ; \tag{7.2}$$

$$Q_\mu = D_i \left\{ \frac{\delta}{\delta A^i_\mu(x)} \psi[A] \right\} = 0\ ; \tag{7.3}$$

where we have considered in the usual operatorial-functional derivative form the Hamiltonian,diffeomorphism and Gauss law constraints respectively implemented in a functional space of quantum gravitational states formed by wave functions $\psi[A]$ [1].

At this point we come to the usefulness of possessing linear-functional field equations by considering explicitly functional solutions for the set eq.(7.1)-eq.(7.3).

Let us therefore, consider the space of bosonic loops with a marked point $x \in M$ and the associated Gauge invariant Wilson Loop defined by a given Ashtekar gauge field configuration $A^i_\mu(x)$.

$$W[C_{xx}] = Tr\left(P_{SU(2)} \left\{ \exp i \oint_{C_{xx}} A^i_\mu(X(\sigma) dX_\mu(\sigma) \right\} \right)\ ; \tag{7.4}$$

here the bosonic loop C_{xx} is explicitly parametrized by a continuous (in general everywhere non-differentiable) periodic function $X_\mu(\sigma) = X_\mu(\sigma + T)$ and such that $X_\mu(T) = X_\mu(0) = x_\mu$ (see refs.[6]–[7]).

Following refs [4], one shows that eq.(7.4) satisfies the diffeomorphism constraint, namely

$$\begin{aligned}
&\left(\frac{\delta}{\delta A^i_\nu(x)} \left[\partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \varepsilon^{irs} A^r_\mu A^s_\nu \right](x) \right) W[C_{xx}] \\
&\quad + F^i_{\mu\nu}(A(x)) \left(\frac{\delta}{\delta A^i_\nu(x)} W[C_{xx}] \right) \\
&= 2 \times \left[\left(\partial^x_\mu + \varepsilon^{irs} \delta^{si} A^r_\mu(x) \right) W[C_{xx}] \right. \\
&\quad \left. + \left(P \left\{ F^i_{\mu\nu}(A(X(0)) \cdot X'^\nu(0) W[C_{x(0)x(0)}] \right\} \right] \right. \\
&= 2 \times \left(\partial^x_\mu + \frac{\delta}{\delta X_\mu(0)} \right) W[C_{xx}] = 0\ ;
\end{aligned} \tag{7.5}$$

where we have the Migdal usual derivative relation for the marked Wilson Loop – note that the loop orientability is responsible for the minus signal on the Wilson Loop marked point derivative [6].

$$-\partial^x_\mu W[C_{xx}] = \lim_{\sigma \to 0} \left\{ \frac{\delta}{\delta X_\mu(\sigma)} W[C_{xx}] \right\}\ ; \tag{7.6}$$

If one had used the usual Smolin factor ordering as given below instead of that of Gambini-Pullin eq.(7.1)-eq.(7.3), one could not satisfy in a straightforward way the diffeomorphism constraint

$$\begin{aligned}
&F^i_{\mu\nu}(A(x))\frac{\delta}{\delta A^i_\nu(x)}W[C_{xx}] \\
&= P\left\{F_{\mu\nu}(A(X(0))\dot{X}^\nu(0)W[C_{xx}]\right\} \\
&= P\left\{\ddot{X}_\mu(0)W[C_{xx}]\right\} \neq 0\ ;
\end{aligned} \tag{7.7}$$

Note that we have assumed the validity of the Lorentz dynamical equation for the loops $X_\mu(\sigma)$ $(0 \leq \sigma \leq T)$ on the last line of eq.(7.7).

Also, the Schörindger-Wheeler-De Witt equation is solved by the marked point Wilson Loop within the same functional derivative procedure. Firstly, we note that the Smolin and Gambini-Pullin operator ordering coincides in the realm of the Wheeler-DeWitt equation. Namely:

$$\begin{aligned}
&\varepsilon^{ijk}\frac{\delta^2}{\delta A^i_\mu(x)\delta A^j_\nu(x)}\left\{F^k_{\mu\nu}(A(x))W[C_{xx}]\right\} \\
&= \left(\varepsilon^{ijk}\frac{\delta^2}{\delta A^i_\mu(x)\delta A^j_\nu(x)}F^k_{\mu\nu}(A(x))\right)W[C_{xx}] \\
&+ \varepsilon^{ijk}\left(\frac{\delta}{\delta A^i_\mu(x)}F^k_{\mu\nu}(A(x))\right)\left(\frac{\delta W[C_{xx}]}{\delta A^j_\nu(x)}\right) \\
&+ \varepsilon^{ijk}\left(\frac{\delta}{\delta A^j_\nu(x)}F^k_{\mu\nu}(A(x))\right)\left(\frac{\delta W[C_{xx}]}{\delta A^i_\mu(x)}\right) \\
&+ \varepsilon^{ijk}F^k_{\mu\nu}(A(x))\left(\frac{\delta^2}{\delta A^i_\mu(x)\delta A^j_\nu(x)}W[C_{xx}]\right) \\
&= 0+0+0+\varepsilon^{ijk}F^k_{\mu\nu}(A(x))\frac{\delta^2 W[C_{xx}]}{\delta A^i_\mu(x)\delta A^j_\nu(x)}\ ;
\end{aligned} \tag{7.8}$$

An important step should be implemented at this point of our analysis and related to a loop regularization process. We propose to consider a weak form of the Wheeler-DeWitt operatorial equation as expressed below

$$\hat{C}_{(\varepsilon)}[A] = \int_M dxdy\delta^{(\varepsilon)}(x-y)\varepsilon^{ijk}F^k_{\mu\nu}(A(x))\left(\frac{\delta^2}{\delta A^i_\mu(x)\delta A^j_\nu(y)}\right)\ ; \tag{7.9}$$

here $\delta^{(\varepsilon)}(x-y)$ is a $C^\infty(M)$ regularization of the delta function on the space-time base manifold M. Rigorously, one should consider eq.(7.9) in each local chart of M with the usual induced volume associated to the flat metric of R^4. Note that the validity of eq.(7.9) (at least locally) comes from the supposed cylindrical topology of our (Euclidean) space-time.

Proceeding as usual one gets the following result

$$
\begin{aligned}
&\hat{C}_{(\varepsilon)}[A]W[C_{xx}] \\
&= \int_M dx \int_M dy \delta_{(\varepsilon)}(x-y) \left\{ \oint_{C_{xx}} \delta(x - X(\sigma)) dX^\mu(\sigma) \oint_{C_{xx}} \delta(y - X(\sigma')) dX^\nu(\sigma') \right. \\
&\times \mathrm{Tr}_{SU(2)} P \left\{ F^k_{\mu\nu}(A(X(\sigma)))\varepsilon^{ijk} W[C_{X(0)X(\sigma')}] \right. \\
&\left.\left. \lambda^j W[C_{X(\sigma')X(\sigma)}]\lambda^i W[C_{X(\sigma)X(T)}] \right\}\right\} \\
&= \oint_{C_{xx}} \oint_{C_{xx}} \delta_{(\varepsilon)}(X(\sigma) - X(\sigma')) dX^\mu(\sigma) dX^\nu(\sigma') \\
&\times \mathrm{Tr}_{SU(2)} P \left\{ F^k_{\mu\nu}(A(X(\sigma)))\varepsilon^{ijk} W[C_{X(0)X(\sigma')}] \right. \\
&\left. \lambda^j W[C_{X(\sigma')X(\sigma)}]\lambda^i W[C_{X(\sigma)X(T)}] \right\} = 0 \, ; \qquad (7.10)
\end{aligned}
$$

As a consequence, one should expect that the cut-off removing $\varepsilon \to 0$ will not be a difficult technical problem in the case of everywhere self-intersecting Brownian loops C_{xx} [7]. Note that in the case of trivial self-intersections $\sigma = \sigma'$, the validity of eq.(7.10) comes directly from the fact that $dX^\mu(\sigma)dX^\nu(\sigma)$ is a symmetric tensor on the spatial indexes (μ,ν) and $F_{\mu\nu}(A(X(\sigma))$ is an antisymmetric tensor with respect to these same indexes. In the general case of smooth paths with non-trivial self-intersection [1], one should makes the loop restrictive hypothesis of the (μ,ν) symmetry of the complete bosonic loop space object $\delta_{(\varepsilon)}(X(\sigma) - X(\sigma'))dX^\mu(\sigma)dX^\nu(\sigma')$ [8,9], otherwise we can not obviously satisfy the Wheeler-DeWitt equation – a common non-trivial fact in the Literature of Wilson Loop as formal quantum states defined by smooth C^∞ – differentiable paths! [1].

At this point we remark that all the governing equations of the theory eq(7.1)-eq(7.3) are linear. As a consequence one can sum up over all closed Brownian loops C_{xx} (with a fixed back-ground metric) in the following way (see second ref. on [6]).

$$
\begin{aligned}
\Omega[A^i_\mu] &= -\int_M d^3x_\mu \int_0^\infty \frac{dT}{T} \int_{X_\mu(0)=X_\mu(T)=x_\mu} D^F[X_\mu(\sigma)] e^{-\frac{1}{2}\int_0^T (\dot{X}_\mu(\sigma))^2 d\sigma} W[C_{xx}] \\
&= \langle \det[\nabla_A \nabla^*_A] \, ; \qquad (7.11a)
\end{aligned}
$$

where one can see naturally the appearance of the functional determinant of Gauged-Klein-Gordon operator as a result of this loop sum.

At this point we introduce Fermionic Loops – an alternative procedure –, which do not have non-trivial spatial self-intersections on R^3 – and representing now closed path – trajectories of $SU(2)$ Fermionic particles on the Wilson Loop eq.(7.4) [8].

Here the Fermionic closed loop C^F_{xx} is described by a fermionic (Grassmanian) vector poistion $X^{(F)}_\mu(\sigma,\theta) = X_\mu(\sigma) + i\theta\psi_\mu(\sigma)$, with $X_\mu(\sigma)$ the ordinary periodic (bosonic) position coordinate and $\psi_\mu(\sigma)$ Grassman variables associated to intrinsic spin loop coordinates. The Fermionic Gauge-Invariant Wilson Loop is given as (see section 1.3, chapter 1).

$$
W[X^{(F)}_\mu(\sigma,\theta)] = \mathrm{Tr}_{SU(2)} \left\{ P \left[\exp\left(\int_0^T d\sigma \int d\theta A_\mu(X^{(F)}_\mu(\sigma,\theta)) \left(\frac{\partial}{\partial\sigma} + i\theta \frac{\partial}{\partial\sigma} \right) X^F_\mu(\sigma,\theta) \right) \right] \right\} ; \qquad (7.11b)
$$

we get as a result the following expression:

$$\begin{aligned}
&\hat{C}_{(\varepsilon)}[A]W[C^F_{xx}] \\
&= \oint_{C^F_{xx}} d\sigma d\theta \oint_{C^F_{xx}} d\sigma' d\theta' DX^{\mu}_{(F)}(\sigma',\theta') DX^{\nu}_{(F)}(\sigma,\theta)(\varepsilon^{ijk})\delta^{(3)}(X^{\mu}_{(F)}(\sigma,\theta) - X^{\mu}_{(F)}(\sigma',\theta'))\delta(\sigma-\sigma') \\
&\times \mathrm{Tr}_{SU(2)} P \left\{ F^k_{\mu\nu}\left(A(X^{(F)}(\sigma,\theta))\right) W\left[C^F_{X(0)X(\sigma')}\right]\right. \\
&\left.\lambda^j W\left[C^F_{X(\sigma')X(\sigma)}\right]\lambda^i W\left[C^F_{X(\sigma)X(T)}\right]\right\} ; \qquad (7.11c)
\end{aligned}$$

By proceeding analogously as in the bosonic loop case eq.(7.11a), we obtain as a (formal) operatorial quantum state of Gravity, the functional determinant of the Dirac Operator on M (with a fixed back-ground metric associated to the embedding of M on R^4! which is not relevant in our study!) as another formal Einstein gravitation quantum state to be used in the analysis which follows [11], an important result by itself.

$$\Omega[A^i_\mu(x)] = \langle \det[\not{D}(A)\ \not{D}^*(A)] \; ; \qquad (7.12)$$

We note that the others constraints eq.(7.2)-eq.(7.3) are satisfied in a straightforward manner in the same way one verifies them for the Bosonic Loop case [(eq.(7.5)-eq.(7.6)) and note the explicitly gauge invariance of the Fermionic Wilson Loop [8]].

Let us present our proposed Loop Space argument that one can obtain the continuum version of Ising models on M from the quantum gravity state $3D$ fermionic determinant eq.(7.12).

In order to see this formal connection let us consider an ensemble of continuous surfaces Σ on M and the restriction of the Ashtekar-Sen $SU(2)$ connection to each surface Σ. Since the Ashtekar-Sen connection is the M-restriction of the four-dimensional left-handed spin connection, one can see that the Σ-restricted quantum gravity state can be re-written as a fermionic path-integral of covariant two-dimensional fermions now defined on the surface Σ, namely (see section 7.2)

$$\begin{aligned}
&\left(Z^{(n)}[A^i_\mu]\right) = \exp\left(\widetilde{\Omega}_\Sigma[A^i_\mu(x)]\right) \\
&= \int d^{(\mathrm{cov})}[\Sigma^\mu(\xi,\sigma)] d^{\mathrm{cov}}[\psi^{(n)}(\xi,\sigma)] \\
&\times \exp\left(-\frac{1}{2\pi\alpha'}\int d\xi d\sigma\left(\sqrt{g}g^{ab}\partial_a\Sigma^\mu\partial_b\Sigma^\mu)(\xi,\sigma)\right)\right) \\
&\times \exp\left(-\frac{1}{2}\int d\xi d\sigma\left(\sqrt{g}\psi^{(n)}_\mu(\gamma\widetilde{\nabla}_a)\psi^{(n)}_\mu\right)(\xi,\sigma)\right) \; ; \qquad (7.13)
\end{aligned}$$

The main point of our argument on the connection of the string theory eq.(7.13) and the Ising model on M is basicaly related to the fact that the two-dimensional spin connection on the $2D$-fermionic action eq.(7.13) is exactly given by the restriction of the four-dimensional spin connection to the surface Σ or – in an equivalently geometrical way – the restriction of the three-dimensional Ashtekar-Sen connection to the surface Σ!.

Let us now give a Loop Space argument that the string theory eq.(7.13) represents a $3D$ Ising model at a formal replica limit on the geometrical fermionic degrees

$\{\psi_\mu^{(n)}, \overline{\psi}_\mu^{(n)}\}_{1\leq n\leq N}$. This can easily be seen by integrating out these geometrical fermion fields, writing the resulting surface two-dimensional determinant in terms of closed loops $\{C_L(t), L=1,2; C_L(t)\in\Sigma\}$ on the string world-sheet Σ by using the replica limit together with a surface proper-time representation for $2D$ fermion determinant [9]

$$\lim_{N\to 0}\left(\frac{Z^{(N)}[A_\mu^i]-1}{N}\right) = \langle \det[\widetilde{\nabla}_a] \\ = \sum_{\{C_L(t)\}}\left[\mathrm{Tr}_{SU(2)}\exp\left(i\int_{C_L}\omega_L(C^L)dC^L\right)\right] \; ; \tag{7.14}$$

At this point one verifies that the Wilson Loop on the string surface as given by eq.(7.14) and defined by the two-dimensional spin connection ω_L coincides with the Ising model sign factor of Sedrakyan and Kavalov [9] which is expected to underlying the continuum string representation of the partition functional of the three-dimensional Ising model on a regular lattice in R^3 at the critical point, namely

$$Z_{\text{ising}}[\beta\to\beta_{\text{crit}}] = \\ \lim_{\beta\to\beta_{\text{crit}}}\left\{(\cosh\beta)^N\sum_{\{\widetilde{\Sigma}\}CZ^3}\left\{\exp\left[-A(\widetilde{\Sigma})\ln\left(\frac{1}{\tanh\beta}\right)\right]\right\}\Phi[\widetilde{C}(\widetilde{\Sigma})]\right\} \; ; \tag{7.15}$$

where the sum in the above written equation is defined over the set of all closed two-dimensional lattice surfaces $\widetilde{\Sigma}CZ^3$ with a weight given by the (lattice) area of $\widetilde{\Sigma}$; N is the number of the plaquettes, $\beta = J/kT$ denotes the ratio of the Ising hope parameter and the temperature. The presence of the Ising wheight $\Phi[\widetilde{C}(\widetilde{\Sigma})]$ inside the partition functional expression eq.(7.15) is the well-known sign factor defined on the manifold of the lines of self-intersection $\widetilde{C}(\widetilde{\Sigma})$ appearing on the surface $\widetilde{\Sigma}$ with the explicitly Polyakov-Sedrakyan-Kavalov expression $\Phi[\widetilde{C}(\widetilde{\Sigma})] = \exp\{i\pi\text{length}[\widetilde{C}(\widetilde{\Sigma})]\}$.

As a consequence of the above made remarks, one can see that at the replica limit of $N\to 0$ eq.(7.14) should be expected to coincide at the critical point of the partition functional eq.(7.15), since the phase factor inside eq.(7.14) is the continuum version of the Ising model factor $\Phi[\widetilde{C}(\widetilde{\Sigma})]$ [9].

This completes the exposition of our Loop Space argument that critical Ising models on M may be relevant quantum states to understand the new physics of quantum gravity when parametrized by the Gauge field-like connections of Ashtekar-Sen.

All the above made analysis would be a mathematical rigorous proof if one had a mathematical result that Fermionic Loops (Grassmanian Wiener Trajectories) do not have non-trivial space-time self-intersections on eq.(7.11c) (see next chapter 8 and chapter 9).

On the other hand this formal mathematical fact about the nonexistence of non-trivial self-intersection fermionic paths that leads naturally to the triviality of the Thirring model (a "$\lambda\varphi^4$" – Fermion Field Theory!) in space-times with dimension greater than 2 (see chapter 4). Finally, let us comment that it is expected that the Ashtekar-Sen connections defining the above studied quantum gravity states are distributional objects with a functional measure given by a σ-model like path integral with a scalar intrinsic field $E(x,t)$ on $M\times R$, the geometrodynamical anologous of the σ-dimensional manifold particle covariant Brink-Howe-Polyakov path integral, namely

$$
\begin{aligned}
d\mu[A^i_\beta] = & \prod_{\substack{x\in M \\ t\in[0,\infty]}} \prod_{i=1}^{3} \left[d(A^i_\beta(x,t)d(E(x,t)) \right] \\
& \times \exp\left\{ -\frac{1}{16\pi G}\int_0^\infty dt \int_M d^3x (E(x,t))^{-1} \right. \\
& \times \left. \left[\left(\frac{\partial}{dt} A_{i,\mu} M^{\mu i,\nu j}[A] \frac{\partial}{\partial t} A_{j,\nu} \right)(x,t) \right] \right\} \\
& \times \exp\left\{ -\mu \int_0^\infty dt \int_M d^3x E(x,t) \right\} ;
\end{aligned}
\tag{7.16}
$$

where the invariant metric on the Wheeler-DeWitt super space of Ashtekar-Sen connections is given explicity by

$$M^{\mu i,\nu j}[A] = (b(A))^{-1} (J^{\mu i} J^{\nu j} - J^{\mu j} J^{\nu i})(A) ; \tag{7.17}$$

with

$$J^{\mu a}(A) = \frac{1}{2}\varepsilon^{\mu\alpha\rho} F^a_{\alpha\rho}(A) ; \tag{7.18}$$

and

$$b(A) = \det(J(A)^{\mu a}); \tag{7.19}$$

Work on the averaged, Wilson Loop eq.(7.4) with the functional path space measure eq.(7.16) – expected to be relevant to analyze the matter interaction with Quantum Gravity is presented in next section.

7.3. The Wheeler - De Witt Geometrodynamical Propagator

The starting point in Wheeler-De Witt Geometrodynamics is the Probability Amplitude for metrics propagation in a cylindrical Space Time $R^3 \times [0,T]$, the so called Wheeler Universe

$$G[{}^3g^{IN};{}^3g^{OUT}] = \int_{{}^3g^{IN}}^{{}^3g^{OUT}} d\mu[h_{\mu\nu}] \exp[-S(h_{\mu\nu})] \tag{7.20}$$

where the integration over the four metrics Functional Space on the cylinder $R^3 \times [0,T]$ is implemented with the Boundary conditions that the metric field $h_{\mu\nu}(x,t)$ induces on the Cylinder Boundaries the Classically Observed metrics ${}^3g^{IN}(x)$ and ${}^3g^{OUT}(x)$ respectivelly. The Covariant Functional measure averaged with the Einstein $S(h_{\mu\nu}) = \int_{R^3\times[0,T]} d^3x dt(\sqrt{g}R(g))$ is given explicitly in ref.4.

Unfortunatelly the use of eq.(7.20) in terms of metrics variables is difficulted by the "Conformal Factor Problem" in the Euclidean Framework. In order to overcome such difficulty I follow section 7.2 by using from the begining, the Astekar Variables to describe the Gometrodynamical Propagation.

Let me thus, consider Einstein Gravitation Theory Parametrized by the $SU(2)$ Three-Dimensional Astekar - Sen connection $A_\mu^a(x,t)$ associated to the Projected Spin Connection on the Space - Time Three - Dimensional Boundaries.

$$A_\mu^{a,IN}(x) = -i\omega_\mu^{0a}(x,0) + \frac{1}{2}\varepsilon_{bi}^a \omega_\mu^{bi}(x,0) \tag{7.21}$$

$$A_\mu^{a,OUT}(x) = -i\omega_\mu^{0a}(x,T) + \frac{1}{2}\varepsilon_{bi}^a \omega_\mu^{bi}(x,T) \tag{7.22}$$

An appropriate action on the Functional Space of Astekar-Sen connections is proposed by myself to be given explicitly by a slight modification of that proposed in chapter 1. My proposed action is given by a covariant σ-model like Path Integral with a scalar intrinsic field $E(x,t)$ on $R^3 \times [0,T]$. Here μ^2 denotes a scalar “mass” parameter which my be vanishing (massless Wheeler-Universes).

$$\begin{aligned} S_{\mu^2}[A_\mu^a(x,t),E(x,t)] &= \frac{1}{16\pi G}\int_0^T dt \int_{R^3} d^3x (E(x,t))^{-1} \\ &\left[\left(\frac{\partial}{\partial t}A_{a,\mu}\right) G^{\mu a,\nu b}[A] \left(\frac{\partial}{\partial t}\right) A_{b,\nu}\right] + \mu^2 \int_0^T dt \int_{R^3} d^3x E(x,t), \end{aligned} \tag{7.23}$$

where the invariant metric on the Wheeler-de Witt superspace of Astekar connections is given by

$$G^{\mu a,\nu b}[A] = (b(A))^{-1}(J^{\mu a}J^{\nu b} - J^{\mu b}J^{\nu a})(A), \tag{7.24}$$

with

$$J^{\mu a}(A) = \frac{1}{2} \in^{\mu\alpha\rho} F_{\alpha\rho}^a(A) \tag{7.25}$$

and

$$b(A) = \det(J(A))_{\mu;a}. \tag{7.26}$$

My proposed quantum geometrodynamical propagator will be given now by the following formal path integral:

$$\begin{aligned} G[A^{IN},A^{OUT}] &= \int_{A_\mu^a(x,0)=A_\mu^a(x,0)=A_\mu^{a,IN}(x)}^{A_\mu^a(x,T)=A_\mu^{a,OUT}(x)} d^{INV}(A_\mu^a(x,t)) \times \\ &\int \left(\prod_{(x,t)\in R^3\times[0,T]} (dE(x,t))\right) \exp(-S_{\mu^2}[A_\mu^a;E]) \end{aligned} \tag{7.27}$$

where the invariant functional measure over the Astekar-Sen connections is given by the invariant functional metric

$$dS_{INV}^2 = \int_{R^3\times[0,T]} d^3x dt [(\delta A_{\mu,a}) G^{\mu a,\nu b}[A](\delta A_{\nu,b})](x,t). \tag{7.28}$$

In order to show that the geometrodynamical propagator equation (7.27) satisfies the Wheeler-de Witt equation, I follow our procedure to deduce the functional wave equations from geometrical path integrals by exploiting the effective functional translation invariance on the functional space of the scalar intrinsic metrics $(E(x,t))$ at the boundary $t \to 0^+$

(chapter 9). As a consequence, we have that the propagator equation (7.27) satisfies the Wheeler-de Witt equation with the "mass" parameter μ^2.

$$\begin{aligned} &\in_{abc} F^{c}_{\mu\nu}(A^{IN})(x)\frac{\delta^2}{\delta A^{a,IN}_{\mu}(x)\delta A^{b,IN}_{\nu}(x)}G(A^{IN};A^{OUT}) = \\ &= -\mu^2 G(A^{IN};A^{OUT}) + \delta^{(F)}(A^{IN,a}_{\mu} - A^{OUT,a}_{\nu}), \end{aligned} \tag{7.29}$$

where we have used the Enclidean commutation relation

$$\left[\left(\frac{G^{\mu a,\nu b}[A]}{E}\times\left(\frac{\partial}{\partial t}A_{\nu,b}\right)\right)(x,t);A_{\mu,a}(x',t)\right] = \delta^{(3)}(x-x'). \tag{7.30}$$

It is instructive to remark that the classical canonical momentum written in eq. (7.30) is given by the Schrödinger functional representation in the euclidean quantum-mechanical equation (7.29)

$$\Pi^{\mu a}(x) = \frac{\delta}{\delta A^{a,IN}_{\mu}(x)} \tag{7.31}$$

It is worth pointing out that the usual covariant Polyakov path integral for Klein-Gordon particles may be considered as the 0-dimensional reduction of the geometrodynamical propagator equation (7.27).

At this point we remark that by fixing the gauge $E(x,t) = \frac{E}{\mu^2}$, with μ^2 the "mass" parameter, we arrive at the analogous proper-time Schwinger representation for this geometrodynamical quantum gravity propagator

$$G_{\tilde{E}}[A^{IN},A^{OUT}] = \int_0^\infty dt e^{-(\tilde{E}t)} \times \int d^{INV}(A^a_\mu)\exp(-S[A^a_\mu(x,t)]). \tag{7.32}$$

where $E = (E,\mu^2)\times vol(R^3)$ is the renormalized mass parameter in the Schwinger Proper-Time representation.

In the next we will use the proper-time-dependent propagator given below as usually is done in the Symanzik's loop space approach for quantum field theories (chapter 1) to write a third-quantized theory for gravitation Einstein theory in terms of Astekar-Sen variables.

$$\begin{aligned} &G[A^{IN},A^{OUT};T] = \int_{A^a_\mu(x,0)=A^{a,IN}_\mu(x)}^{A^a_\mu(x,T)=A^{a,OUT}_\mu} d^{INV}(A^a_\mu)\times \\ &\times \exp\left\{-\frac{1}{16\pi G}\int_0^T dt\int_{R^3}\left[\left(\frac{\partial}{\partial t}A_{a,\mu}\right)\times G^{\mu a,\nu b}[A]\times\left(\frac{\partial}{\partial t}A_{\nu,b}\right)\right](x,t)\right\}. \end{aligned} \tag{7.33}$$

Unfortunatelly exactly solutions for eq.(7.29) with $\mu^2 \neq 0$ or eq.(7.14) were not found yet. However its σ-like structure and $(SU(2)$ Gauge Invariance may afford to truncated aproximate solutions as usually done for the Wheeler-De Witt equations by means of the Mini-Super Space Ansatz. Finally let me comment on the introduction of a Quantized Matter Field represented by a massless field $\phi(x,t)$ on the Space Time.

By considering the effect of the introduction of this quantized field as a flucturation on the Geometrodynamical Propagator eq.(7.27) one should consider the following functional representing the interaction of this massless quantized matter and the Astekar-Sen connection as one can easily see by making $E(x,t)$ variations

$$S_{INT}[A_\mu^a;E,\varphi]=\int_0^T dt\int_{R^3}d^3x\left\{\left[\varphi\left(-\frac{\partial}{\partial t}\left(E\frac{\partial}{\partial t}\right)\right)\varphi\right](x,t)+\right.$$
$$\left.+\left(\varphi\left[\partial_\mu\left(\frac{1}{E}G^{a,\mu,b\rho}[A]\frac{\partial}{\partial t}A_{b,\rho}\times G^{\mu\sigma,a\nu}[A]\cdot\frac{\partial}{\partial t}A_{\sigma\mu}\right)\partial_\nu\right]\varphi\right)(x,t)\right\} \quad (7.34)$$

Now the effect on integrating at the scalar matter field in eq.(7.33) is the appearence of the further effective action to be added on the σ-like action of our Proposed Geometrodynamical Propagator.

$$S^{E_{FF}}[A_\mu^a,E,T]=-\frac{1}{2}\langle\det{}_F\left\{-\frac{\partial}{\partial t}\left(E\frac{\partial}{\partial t}\right)+\partial_\mu\left(\frac{1}{E}G^{a,\mu,bF}[A]\frac{\partial}{\partial t}A_{\rho b}\times G^{\mu\sigma}_{a\nu}[A]\frac{\partial}{\partial t}A_{\sigma\mu}\right)\partial_\nu\right\} \quad (7.35)$$

The coupling with (Weyl) Fermionic Matter is straight forward and leading to the Left-Handed Fermionic Functional determinant in the presence of the Astekar-Sen connection $A_\mu^a(x,t)$.

The joint probability for the massless field propagator in the presence of a fluctuating geometry parametrized by the Astekar-Sen connection is given by

$$G[A_\mu^{IN},A_\mu^{OUT};\langle\varphi(x_1,t_1)\varphi(x_2,t_2)\rangle]=\int_{A_\mu^a(x,-\infty)=A_\mu^{a,IN}}^{A_\mu^a(x,+\infty)=A_\mu^{a,OUT}} d^{INV}[A_\mu^a]\times$$
$$\exp\left\{-\frac{1}{16\pi G}\int_{-\infty}^{+\infty}dt\int_{R^3}d^3x\times\right.$$
$$\left.\times\left(\frac{1}{E(x,t)}\times\left(\frac{\partial}{\partial t}A_{a,\mu}\right)G^{\mu a,\nu b}[A]\left(\frac{\partial}{\partial t}A_{\mu,b}\right)(x,t)\right)+\mu^2\int_{-\infty}^{+\infty}dt\int_{R^3}d^3xE(x,t)\right\}\times$$
$$\times\det{}^{-\frac{1}{2}}\left[-\frac{\partial}{\partial t}\left(E\frac{\partial}{\partial t}\right)+\partial_\mu\left(\frac{1}{E}G^{a\mu,b\rho}[A]\frac{\partial}{\partial t}A_{\mu,b}\times G^{\mu\sigma}_{a\nu}[A]\times\frac{\partial}{\partial t}\frac{\partial}{\partial t}A_{\sigma,\mu}\right)\partial_\nu\right]\times$$
$$\times\lim_{J(x,t)\to 0}\frac{\delta}{\delta J(x_1,t_1)}\frac{\delta}{\delta J(x_2,t_2)}\times\exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty}dtdt'\int_{R^3}d^3xd^3y\times\right.$$
$$\left\{EJ\eta_{abc}\varepsilon^{\mu\nu\rho}\left(G^{a\mu,b'\rho'}\frac{\partial}{\partial t}A_{p',b'}\times G^{b\nu,b''\rho''}\frac{\partial}{\partial t}A_{p'',b''}\times G^{c\rho,b'''\rho'''}\frac{\partial}{\partial t}A_{p''',b'''}\right)\right\}(x,t)\times$$
$$\left[-\frac{\partial}{\partial t}\left(E\frac{\partial}{\partial t}\right)+\partial_\mu\left(\frac{1}{E}G^{a\mu,b\rho}[A]\frac{\partial}{\partial t}A_{\rho,b}G^{\mu\sigma}_{a\nu}\frac{\partial}{\partial t}A_{\sigma\mu}\right)\partial_\nu\right\}^{-1}((x,t),(y,t))\times$$
$$\left.\left\{EJ\eta_{abc}\varepsilon^{\mu\nu\rho}\left(G^{a\mu,b'\rho'}\frac{\partial}{\partial t}A_{p',b'}\times G^{b\nu,b''\rho''}\frac{\partial}{\partial t}A_{\rho'',b''}G^{c\rho;b'''\rho'''}\frac{\partial}{\partial t}A_{\rho''',b'''}\right)(y,t')\right\}\right. \quad (7.36)$$

7.4. A $\lambda\phi^4$ Geometrodynamical Field Theory for Quantum Gravity

Let me start the analysis by considering the generating functional of the following geometrodynamical field path integral as the simplest generalization for quantum gravity of a similar well-defined quantum field theory path integral of strings and particles [6]

$$\begin{aligned}
Z[J(Z)] = & \int D^F(\phi[A])\times \\
& \times \exp\left[-\int d\nu(A)\phi[A]\times\left(\int d^3x\left(\in_{abc} F^c_{\mu\nu}(A)\frac{\delta^2}{\delta A^a_\mu \delta A^b_\nu}\right)(x)\right)\phi[A]\right\}\times \\
& \times \exp\left\{-\lambda\int d^3x d^3y\int d\nu(A)d\nu(\bar{A})(\phi^2[A(x)](\phi^2[\bar{A}(y)])\times\delta^{(3)}(A_\mu(x)-\bar{A}_\mu(y)\right\} \\
& \times \exp\left\{-\int d\nu(A)J(A)\phi[A]\right\}. \qquad (7.37)
\end{aligned}$$

The notation is as follows: i) The quantum gravity third-quantized field is given by a functional $\phi[A]$ defined over the space of all Astekar-Sen connections configurations $M=\{A^a_\mu(x); x\in R^3\}$. The sum over the functional space M is defined by the gauge and diffeomorphism invariant and topological non-trivial path integral of a Chern-Simons field theory on the Astekar-Sen connections

$$d\nu(A)=\int\left(\prod_{x\in R^3} dA^a_\mu(x)\right)\times\exp\left\{-\int d^3x(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)(x)\right\}. \qquad (7.38)$$

ii) The third quantized functional measure in eq. (7.37) is given formally by the usual Feynman product measure

$$D^F(\phi[A])=\prod_{A\in M} d\phi[A]. \qquad (7.39)$$

iii) The $\lambda\phi^4$-like interaction vertex is given by a self-avoiding geometrodynamical interaction among the Astekar-Sen field configurations in the extrinsic space R^3

$$\lambda\sum_{a=1}^{3}\delta^{(3)}(A^a_\mu(x)-\bar{A}^a_\mu(y)). \qquad (7.40)$$

The proposed interaction vertex was defined in such a way that it allows the replacement of the Four Universe interaction in eq.(7.37) by an independent interaction of each Astekar-Sen connection with an extrinsic triplet of Gaussian stochastic field $W^a(x)$ followed by an average over W^a. A similar procedure is well known in the many-body and many-random surface path integral quantum field theory. So, we can write eq.(7.37) in the following form

$$\begin{aligned}
Z[J(A)] = & \left\langle\int D^F(\phi[A])\times\exp\left\{-\int d\nu(A)\left[\phi[A]\left(L(A)-\right.\right.\right.\right. \\
& \left.\left.\left.\left. -i\lambda\int d^3x\left(\sum_{a=1}^{3}W^a(A^a_\mu)\right)\right)\phi[A]+J(A)\phi[A]\right]\right]\right\rangle_W. \qquad (7.41)
\end{aligned}$$

Here, $W^a(A^a)$ means the external a-component of the triplet of the external stochastic field $\{W^a\}$ projected on the Astekar-Sen connection $\{A^a_\mu\}$, namely

$$W^a(A^a)=W^a(A^a_1(x),A^a_2(x),A^a_3(x)), \qquad (7.42)$$

and has the white-noise stochastic correlation function

$$\langle W^a(x^\mu)W^b(y^\nu)\rangle = \delta^{(3)}(x^\mu - y^\mu)\delta^{ab}. \tag{7.43}$$

The $L(A)$ operator on the functional space of the universe field is the Wheeler-de Witt operator defining the quadratic action in eq. (7.41).

In the free case $\lambda = 0$. The third-quantized gravitation path integral equation (7.37)is exactly soluble with the following generating functional:

$$\frac{Z[J(A)]}{Z[0]} = \exp\left\{+\frac{1}{2}\int d\nu(A)d\nu(\bar{A})J(A)\left(\int d^3x \in_{abc} F^c_{\mu\nu}(A)\frac{\delta^2}{\delta A^a_\mu \delta A^b_\nu}\right)^{-1}(A,\bar{A})J(\bar{A})\right\}. \tag{7.44}$$

Here the functional inverse of the Wheeler-de Witt operator is given explicitly by the geometrodynamical propagator equation (7.32) with $\tilde{E} = 0$

$$\left(\int d^3x \in_{abc} F^c_{\mu\nu}(A)\frac{\delta^2}{\delta A^a_\mu \delta A^b_\nu}\right)^{-1}(A,\bar{A}) = \int_0^\infty dT\, G[A,\bar{A},T]. \tag{7.45}$$

In order to reformulate the third-quantized gravitation field theory as a dynamics of self-avoiding geometrodynamical propagators, we evaluate formally the Gaussian $\phi[A]$ functional path integral in eq.(7.37) with the following result

$$\begin{aligned} Z[J(A)] = \Bigg\langle \det{}^{-\frac{1}{2}}\left[\int_{R^3} d^3x \in_{abc} F^c_{\mu\nu}(A)\frac{\delta^2}{\delta A^a_\mu \delta A^b_\nu} + i\lambda\left(\sum_{a=1}^{3} W^a(A^a(x))\right)\right] \times \\ \times \exp\left\{+\frac{1}{2}\int d\nu(A)d\nu(\bar{A}) \times J(A)\left[\int_{R^3} d^3x \in_{abc} F^c_{\mu\nu}(A)\frac{\delta^2}{\delta A^a_\mu \delta A^b_\nu} + \right.\right. \\ \left.\left. i\lambda\left(\sum_{a=1}^{3} W^a(A^a_\mu(x))\right)\right]^{-1}(A,\bar{A}) \times J(\bar{A})\right\}\Bigg\rangle \end{aligned} \tag{7.46}$$

Let us follow our previous studies implemented for particles and strings in previous chapters by defining the functional determinant of the Wheeler-de Witt operator by the proper-time technique

$$\begin{aligned} &-\frac{1}{2}\log\det\left[L(A) + i\lambda\int_{R^3} d^3x\left(\sum_{a=1}^{3} W^a(A^a_\mu(x))\right)\right] = \\ &= \int_0^{+\infty}\frac{dT}{T}\left\{\int d\nu(A)d\nu(\bar{A})\delta^{(F)}(A-\bar{A}) \times \right. \\ &\left. \times \left\langle A\right|\exp\left[-T\left(L(A) + i\lambda d^3x\left(\sum_{a=1}^{3} W^a(A^a_\mu(x))\right)\right)\right]\left|\bar{A}\right\rangle\right\}. \end{aligned} \tag{7.47}$$

with the geometrodynamical propagator (see eq.(7.33) in the presence of the extrinsic potential $\{W_\mu^a(x)\}$ which is given explicitly by the path integral below

$$\begin{aligned}
&\left\langle A\right| \exp\left[-T(L(A)) + i\lambda \int_{R^3} d^3x \left(\sum_{a=1}^{3} W^a(A_\mu^a(x))\right)\right] |\bar{A}\rangle = \\
&= \int d^{INV}[B_\mu^a(x,t)] \exp\left\{-\frac{1}{16\pi G}\int_0^T dt \int_{R^3} d^3x\right. \\
&\left.\left[\left(\frac{\partial}{\partial t}B_{a,\mu}\right)\times G^{\mu a,\nu b}[B]\left(\frac{\partial}{\partial t}B_{b,\nu}\right)\right](x,t)\right\}\times \\
&\times \exp\left[-i\lambda\int_0^T dt\int_{R^3} d^3x\left(\sum_{a=1}^{3} W^a(B_\mu^a(x,t))\right)\right]\Bigg]. \qquad (7.48)
\end{aligned}$$

By substituting eq.(7.48) and eq.(7.47) into eq.(7.46) and making a loop expansion of the functional determinant, we obtain eq.(7.37) as a theory of an ensemble of geometrodynamical propagators interacting with the extrinsic Gaussian stochastic field $\{W^a(x)\}$. The Gaussian average $\langle\rangle_w$ may be straightforwardly evaluated at each loop expansion producing the self-avoiding interaction among the geometrodynamical propagators (the Wheeler quantum universes) and leading to the picture of joining and splitting of these Wheeler Universes as necessary for the description of the Universe in its Space-Time Third Quantized form picture of Wheeler. For instance, by neglecting the functional determinant in eq.(7.46) we have the following expression for the geometrodynamical third quantized propagator:

$$\begin{aligned}
&\langle\Phi[A_\mu^{a,IN}]\Phi[A_\mu^{a,OUT}]\rangle^{(0)} = \\
&\int_0^\infty dT \times \int_{B_\mu^a(x,0)=A;B_\mu^a(x,T)=\bar{A}} d^{INV}[B_\mu^a(x,t)]\times \\
&\times \exp\left\{-\frac{1}{16\pi G}\int_0^T dt\int_{R^3} d^3x\left[\left(\frac{\partial}{\partial t}B_{a,\mu}\right)G^{\mu a,\nu b}[B]\left(\frac{\partial}{\partial t}B_{\nu,b}\right)\right](x,t)\right\}\times \\
&\times \exp\left\{-\frac{\lambda^2}{2}\int_0^T dt\int_0^T dt'\int_{R^3} d^3x d^3y\left(\sum_{a=1}^{3}\delta^{(3)}(B_\mu^a(x,t)-(B_\mu^a(y,t')))\right)\right\}. \qquad (7.49)
\end{aligned}$$

Next corrections will involve self-avoiding interactions among differents Wheeler Universes associated to different Astekar-Sen connections associated to different Geometrodynamical Propagators appearing from the functional determinant loop expansion equation (7.47).

Finally, I comment that calculations will be done sucessfully only if one is able to handle correctly the Gometrodynamical Propagator eq.(7.33) on eq.(7.36) and, thus, proceed to generalized for this Quantum gravity case the analogous framework used in the Theory of Random Lines and Surfaces.

Appendix A

In this short appendix we call the reader attention that there are (formal) states satisfying the Wheeler-DeWitt equation (7.1), the diffeomorphism constraint eq.(7.2), but not the gauge-invariant Gauss law eq.(7.3).

For instance, the non-gauge invariant “mass term” wave functional below

$$M[A] = \exp\left\{-\frac{1}{2}\int_M d^3x A^i_\mu (\delta^{i1}\delta^{j1}\delta^{\mu\nu}) A^j_\nu\right\}; \tag{A1}$$

satifies the Wheeler-DeWitt equation, since

$$\left(\varepsilon^{ijk} F^k_{\mu\nu}(A)(x) \frac{\delta^2}{\delta A^i_\mu(x)\delta A^j_\nu(y)} M[A]\right) \sim \varepsilon^{11k}(\cdots) = 0; \tag{A2}$$

and the diffeomorphism constraint

$$\begin{aligned} &\frac{\delta}{\delta A^i_\nu(x)}\left(F^i_{\mu\nu}(A) M[A]\right) = \partial^x_\mu M[A] \\ &+ F^i_{\mu\nu}[A] M[A]\left(-\frac{1}{2}A^1_\mu(x)\right)\delta^{i1}\delta^{\nu\mu} \\ &= 0 - \frac{1}{2}A^1_\mu(x)\cdot F_{\mu\mu}(A) = 0; \end{aligned} \tag{A3}$$

At this point and closely related to the above made remark it is worth call the reader attention that the 3D-fermionic functional determinant with a mass term still satisfies the Wheeler-DeWitt and the diffeomorhism constraint. However, at the limit of large mass $m \to \infty$, one can see the appearance of a complete cut-off dependent mass term like eq.(A1), added with Chern-Simon terms and higher order terms of the strenght field $F_{\alpha\beta}(A(x))$ in the full quantum state [12]. As a result one can argue that this “fermion classical limit” of large mass may be equivalent to the appearance of a dynamical cosmological constant, if one neglects the gauge-violating quantum induced mass term [1].

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Chapter 8

A Fermionic Loop Wave Equation for Quantum Chromodynamics at $N_c = +\infty$

8.1. Introduction

In last decades new quantization of Yang-Mills gauge fields has been pursued by several authors, which seems appropriate for handling its confining phase.It makes use of the so-called "quantum Wilson loop" as dynamical variable (see ref. [1] for an extensive review) which has the meaning of being the probability amplitude of a bosonic (Klein-Gordon) colored particle propagating along a closed world line $X_\mu(s)$ and in the presence of the vacuum of a pure gauge theory.

A closed wave equation for this dynamical variable at the 't Hooft topological limit was derived: the Migdal-Makeenko equation [1,2] which supports a string solution (see Chapter 9).

In this chapter, we consider the case that the above particle possesses Dirac spin degrees by making use of the pseudo-classical mechanics formalism as exposed in ref. [4].

8.2. The Fermionic Loop Wave Equation

The basic dynamical variable in the loop space formulation for euclidean $(\mathrm{QCD})_{N_c}$ at $N_c = +\infty$ is the amplitude for a quark loop propagating in the vacuum of a pure Yang-Mills. At this point our idea is implemented. Since the quark possesses Dirac spin degrees of freedom, its (euclidean) world line should reveals the existence of these fermionic degrees. A natural framework to implement this idea is pseudo-classical mechanics [4-6] where the world line of a spinning particle is described by a fermionic vector position $X_\mu^{(F)}(s,\theta) = X_\mu^{(B)}(s) + i\theta\psi_\mu(s)$ with s being the evolution parameter, $X_\mu^{(B)}(s)$ the ordinary (bosonic) position coordinate and $\psi_\mu(s)$ are Grassman variables associated to the spin coordinates.

In this framework, the quark loop amplitude associated to a given spinning closed world line $\{X_\mu^{(F)}(s,\theta);\ 0 \leq s \leq 1;\ X_\mu^{(B)}(0) = X_\mu^{(B)}(1) = X \in R^D\}$ in the presence of the vacuum of a pure $U(N)$ Yang-Mills gauge theory is proportional to the following dynamical factor (the fermionic version of the usual (bosonic) Wilson Loop) (see eq. (25) in ref. [4]):

$$W^{(F)}[X_\mu^{(F)}(s,\theta)] = \left\langle \mathrm{Tr}\left\{ P\left[\exp\left(\int_0^1 ds \int d\theta A_\mu(X_\mu^{(F)}(s,\theta))DX_\mu^{(F)}(s,\theta)\right)\right]\right\}\right\rangle \quad (8.1)$$

where $A_\mu(x)$ denotes the usual $U(N)$ Yang-Mills potential, P the path ordering of the $U(N)$ matrix indices of the exponent in (8.1) along the bosonic path $X_\mu^{(B)}(s)$ and $D = \partial/\partial\theta + i\theta\partial/\partial s$ the covariant derivative. The quantum average $\langle\ \rangle$ is defined by the partition functional of the pure Yang-Mills theory (see Chapter 1).

An important remark to be used below is that (8.1) possesses the fermionic mixing symmetry [4]

$$\delta X_\mu^{(B)}(s) = i\varepsilon\psi_\mu(s), \quad \delta\psi_\mu(s) = \varepsilon X_\mu^{(B)}(s), \quad (8.2)$$

with ε a grassmanian spinor parameter.

We note that by realizing the θ-integration in the phase in (8.1) we get in addition to the usual term $\int_0^1 ds A_\mu(X_\mu^{(B)}(s))dX^{(B)}(s))$, a term responsible for the interaction between the spin degrees and the field strength, namely: $\frac{1}{2}i[\psi_\mu,\psi_\nu]_+(s)F_{\mu\nu}(X_\mu^{(B)}(s))$.

In order to deduce a closed functional for the fermionic Wilson Loop (8.1) we shift the $A_\mu(x)$-variable and get the result [2]

$$\begin{aligned}
&\frac{1}{2g^2N}\left\langle \mathrm{Tr}\left\{ P\left[(D_\mu F_{\mu\nu})(x)\exp\left(\int_0^1 d\theta A_\mu(X_\mu^{(F)}(s,\theta))DX_\mu^{(F)}(s,\theta)\right)\right]\right\}\right\rangle \\
&= \int_0^1 d\sigma \int d\theta\, \delta^{(D)}(X_\mu^{(F)}(\sigma,\theta)-x)DX_\nu(\sigma,\theta)\left\langle \mathrm{Tr}\left\{ P\left[\exp\left(\int_0^\sigma ds \int d\theta A_\mu(X_\mu^{(F)}(s,\theta))DX_\mu^{(F)}(s,\theta)\right)\right]\right\}\right\rangle \\
&\times \left\langle \mathrm{Tr}\left\{ P\left[\exp\left(\int_\sigma^1 ds \int d\theta A_\mu(X_\mu^{(F)}(s,\theta))DX_\mu^{(F)}(s,\theta)\right)\right]\right\}\right\rangle. \quad (8.3)
\end{aligned}$$

Now we note the crucial fact that we have here a very irregular path $X_\mu^{(B)}(s)$ which intercepts itself at every point [7] and, further, ensures the gauge invariance of each fermionic Wu-Yang factor on the right-hand side eq.(8.3). As a consequence of this remark, the relation (8.3) takes a closed form at the 't Hooft limit $N_c \to \infty$ ($\lim_{N_c\to\infty} g^2N_c = \lambda^2$) (see Chapter 4).

$$\begin{aligned}
&\langle(\mathrm{Tr}\{P\{(D_\mu F_{\mu\nu})(x)\psi[X_\mu^{(F)}(s,\theta); 0\le s\le 1]\}\}\rangle \\
&= 2\lambda^2\int_0^1 d\sigma \int d\theta\, \delta^{(D)}(X_\mu^{(F)}(\sigma,\theta)-X)DX_\nu^{(F)}(\sigma,\theta)\langle\psi[X_\mu^{(F)}(s,\theta); 0\le s\le\sigma]\rangle\langle\psi[X_\mu^{(F)}(s,\theta);\sigma\le s\le 1]\rangle,
\end{aligned} \quad (8.4)$$

where we have introduced a more compact notation for the fermionic Wu-Yang factors in eq.(8.3)

$$\psi[X_\mu^{(F)}(s,\theta);\sigma_1\le s\le\sigma_2] = P\left[\exp\left(\int_{\sigma_1}^{\sigma_2} ds\int d\theta A_\mu(X_\mu^{(F)}(s,\theta))DX_\mu^{(F)}(s,\theta)\right)\right]. \quad (8.5)$$

At this point of the analysis it is convenient to multiply both sides of eq.(8.4) by the fermionic current density $j_\mu(x) = \delta^{(D)}(x - X_\mu^{(F)}(\tilde\sigma,\tilde\theta)DX_\mu(\tilde\sigma,\tilde\theta)$ and integrate out the result in relation to the space-time variable x. So we get

$$\begin{aligned}
&\langle \mathrm{Tr}\{P\{(D_\mu F_{\mu\nu})(X_\mu^{(F)}(\tilde\sigma,\tilde\theta)\psi[X_\mu^{(F)}(s,\theta;0\le s\le 1]\}\}\rangle \\
&= 2\lambda\int_0^1 d\sigma\int d\theta\,\delta^{(D)}(X_\mu^{(F)}(\sigma,\theta)-X_\mu^{(F)}(\tilde\sigma,\tilde\theta))DX_\mu^{(F)}(\sigma,\theta)DX_\nu^{(F)}(\tilde\sigma,\tilde\theta) \\
&\times \langle\psi[X_\mu^{(F)}(s,\theta);0\le s\le\sigma]\rangle\langle\psi[X_\mu^{(F)}(s,\theta);\sigma\le s\le 1]\rangle.
\end{aligned} \tag{8.6}$$

In order to write the left-hand side of relation (8.6) in a form similar to the usual string equations, we note the relations

$$\begin{aligned}
&\delta\,\mathrm{Tr}(\psi[X_\mu^{(F)}(s,\theta);0\le s\le\sigma])/\delta X_\mu^{(F)}(\tilde\sigma,\tilde\theta) \\
&= \mathrm{Tr}\{P\{F_{\mu\nu}(X_\mu^{(F)}(\tilde\sigma,\tilde\theta))DX_\nu^{(F)}(\tilde\sigma,\tilde\theta)\psi[X_\mu^{(F)}(s,\theta);\theta\le s\le\sigma]\}\},
\end{aligned} \tag{8.7}$$

and consequently (compare with the similar bosonic relation in Chapter 9:

$$\begin{aligned}
&\partial^2\,\mathrm{Tr}(\psi[X_\mu^{(F)}(s,\theta);0\le s\le 1])/\partial^2 X_\mu^{(F)}(\tilde\sigma,\tilde\theta) \\
&= \lim_{\zeta\to 0^+}\int_{-\zeta}^{+\zeta} d\zeta\frac{\delta^2}{\delta X_\mu^{(F)}(\tilde\sigma+\frac{1}{2}\zeta,\tilde\theta)\delta X_\mu^{(F)}(\tilde\sigma-\frac{1}{2}\zeta;\tilde\theta)}\ \mathrm{Tr}(\psi]X_\mu^{(F)}(s,\theta);0\le s\le 1]) \\
&= \mathrm{Tr}\{P\{(D_\mu F_{\mu\nu})(X_\mu^{(F)}(\tilde\sigma,\tilde\theta))DX_\nu^{(F)}(\tilde\sigma,\tilde\theta)\psi[X_\mu^{(F)}(s,\theta);0\le s\le 1]\}\}.
\end{aligned} \tag{8.8}$$

So we can rewrite eq.(8.6) in the form

$$\begin{aligned}
&\partial^2 W^{(F)}[X_\mu^{(F)}(s,\theta),0\le s\le 1]/\partial X_\mu^2(\tilde\sigma,\tilde\theta) \\
&2\lambda\int_0^1 d\sigma\int d\theta\,\delta^{(D)}(X_\mu^{(F)}(\sigma,\theta)-X_\mu^{(F)}(\tilde\sigma,\tilde\theta))DX_\mu^{(F)}(\sigma,\theta)DX_\mu^{(F)}DX_\mu^{(F)}(\tilde\sigma,\tilde\theta) \\
&\times W^{(F)}[X_\mu^{(F)}(s,\theta),0\le s\le\sigma]\cdot W^{(F)}[X_\mu^{(F)}(s,\theta);\sigma\le s\le 1].
\end{aligned} \tag{8.9}$$

This is the proposed fermionic loop wave equation for QCD at $N_c = +\infty$.

Note the initial condition imposed on the solutions of eq.(8.9) and related to the asymptotic freedom of QCD

$$W^{(F)}[X_\mu^{(F)}(s,\theta)\equiv 0] = 1. \tag{8.10}$$

Since our equation is deduced formally, the important problem of its regularization and renormalization shows up. At first, we note that in loop dynamics the paths $X_\mu^{(B)}(s)$ are very irregular geometric objects in euclidean space, so all Feynman diagrammatic perturbative analyses break down [8]. A probable useful scheme should be the introduction of its discrete version, as in ref. [9], and the continuous limit is taken together with other kinematical factors [4].

Another more interesting point of view is to solve formally eqs.(8.9), (8.10) in terms of the functional integral of a string theory (see Chapter 9). Due to the fermionic mixing symmetry (8.2) of the fermionic Wilson Loop (8.1), it appears naturally to consider as a string ansatz a fermionic string [10, 11] with all of its good spectral features (Chapter 16).

To summarize, we propose a fermionic loop wave function for QCD at $N_c = +\infty$ which supports hope for the existence of a QCD fermionic string ansatz (restricted to its bosonic sector as in Chapter 16).

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Chapter 9

String Wave Equations in Polyakov's Path Integral Framework

9.1. Introduction

In the Feynman path integral formulation for (first) quantization of a physical system,[1] the central object is the transition amplitude for the system evolution from a prescribed initial state to a prescribed final state. Its explicit expression is given by the continuous sum over all system trajectories connecting these states and weighted by the classical system action. This quantization procedure does not rely on the conventional operator Heisenberg-Schrödinger formulation of quantum mechanics. However, for most of the physical systems analyzed up to the present time, the formal equivalence between these two alternatives is implemented by showing that the above-mentioned Feynman transition amplitude satisfies the associated wave equation obtained from the operator approach.

The purpose of this chapter is to describe a simple procedure for writing string wave equations directly from the Feynman path integral for the covariant bosonic and fermionic string transition amplitude presented by Polyakov some years ago.[2] In Sec. 9.2 we present our ideas in the simple case of covariant particle dynamics. The reason for writing wave equations in the Polyakov path integral is that it may shed some light on the role of the Liouville conformal freedom degree in the string quantization below the critical dimension. This study is presented in Sec. 9.3. Another more important motivation is that the quantum chromodynamic $[SU(\infty)]$ (bosonic) contour average satisfies a closed stringlike evolution equation.[3] With a general procedure for writing string wave equations directly from the string path integral, the search for its (string) solutions becomes a simple and transparent task. This analysis is presented in Sec. 9.4. Finally in Sec. 9.5 we deduce a kind of Dirac-Ramond-Marshall string wave equation by extending the bosonic path integral formalism to the fermionic case.

9.2. The Wave Equation in Covariant Particle Dynamics

In the covariant description of a relativistic bosonic particle,[4] the particle trajectory is described by two degrees of freedom: the usual vector position $X_\mu(\zeta)$, with $0 < \zeta < 1$, and an additional one-dimensional metric $e(\zeta)$. Theparameter ζ describes the evolution of the system and the particle trajectory $X_\mu(\zeta)$ does not change its orientation in space-time $[X_\mu(\zeta) \neq X_\mu(\zeta'), \zeta \neq \zeta']$ (see Ref. 1).

The covariant classical action for this particle, moving under the influence of an external potential $V(x)$, is given by

$$S[X_\mu(\zeta), v(x)] = \int_0^1 \left(\frac{1}{2} \frac{X_\mu(\zeta)^2}{e(\zeta)} + \frac{1}{2} m^2 e(\zeta) + e(\zeta) V[X_\mu(\zeta)] \right). \tag{9.1}$$

where m^2 is the particle mass.

Following Feynman, the transition amplitude for which a particle initial state $(X_\mu^{\mathrm{in}}, e^{\mathrm{in}})$ propagates to a final state $(X_\mu^{\mathrm{out}}, e^{\mathrm{out}})$ is given explicitly by the path integral:

$$\begin{aligned} & G[(X_\mu^{\mathrm{out}}, e^{\mathrm{out}}); (X_\mu^{\mathrm{in}}, e^{\mathrm{in}})] \\ & \int_{\begin{pmatrix} X_\mu(0) = X_\mu^{\mathrm{in}} \\ X_\mu(1) = X_\mu^{\mathrm{out}} \end{pmatrix}} d\mu[X_\mu(\zeta)] \int_{\begin{pmatrix} e(0) = e^{\mathrm{in}} \\ e(1) = e^{\mathrm{out}} \end{pmatrix}} d\mu[e(\zeta)] \\ & \times \exp\{-S[X_\mu(\zeta), V(x)]\}. \end{aligned} \tag{9.2}$$

Here the covariant Feynman measures $d\mu[e(\zeta)]$ and $d\mu[X_\mu(\zeta)]$ are, respectively, defined as the volume element of the covariant functional metrics

$$||\delta e||^2 = \int_0^1 (\delta e \delta e)(\zeta) d\zeta$$

and

$$\int_0^1 e(\zeta)(\delta X_\mu \cdot \delta X_\mu) d\zeta.$$

It is possible to evaluate explicitly the above transition amplitude in the proper-time gauge $e(\zeta) = \mathrm{const}$, thus producing the (Euclidean) Green's function of the Klein-Gordon operator in the presence of the external potential $V(x)$.

An alternative way to obtain the above result is by closely following Feynman,[1] and by considering the identity that results by making varitions of the intrinsic metric at the end-point trajectory. Since a gauge exists where $e(\zeta)$ can be fixed as the trajectory proper-time parameter, we expect that this identiy should produce a covariant wave equation that (in the proper time gauge) reduces to the usual Klein-Gordon equation (see the Appendix of Ref. 1).

As a consequence of the invariance under functional translations of the functional measure $d\mu[e(\zeta)]$, we show that the following relation holds true:

$$\begin{aligned} 0 = & \int_{\begin{pmatrix} X_\mu(0) = X_\mu^{\mathrm{in}} \\ X_\mu(1) = X_\mu^{\mathrm{out}} \end{pmatrix}} d\mu[X_\mu(\zeta)] \int_{\begin{pmatrix} e(0) = e^{\mathrm{in}} \\ e(1) = e^{\mathrm{out}} \end{pmatrix}} d\mu[e(\tau)] \\ & \times \exp\{-S[X_\mu(\zeta), V(x)]\}. \end{aligned} \tag{9.3}$$

By considering the boundary $\overline{\zeta} \to 0$ in eq.(9.3) we show that transition amplitude, eq.(9.2), satisfies the identity

$$\int_{\begin{pmatrix} X_\mu(0)=X_\mu^{\text{in}} \\ X_\mu(1)=X_\mu^{\text{out}} \end{pmatrix}} d\mu[X_\mu(\zeta)] \int_{\begin{pmatrix} e(0)=e^{\text{in}} \\ e(1)=e^{\text{out}} \end{pmatrix}} d\mu[e(\zeta)] \exp\{S[X_\mu(\zeta),V(x)]\}$$
$$\lim_{\zeta\to 0^+} \left((\prod_\mu(\overline{\zeta}))^2 - \frac{1}{2}m^2 - V(x_\mu(\overline{\zeta})) \right) \tag{9.4}$$

where $\prod_\mu(\zeta) = X_\mu(\zeta)/e(\zeta)$ denotes the classical canonical momentum of the covariant particle.

In order to translate the path integral constraint equation (9.4) into an operator statement, we have to use the covariant Heisenberg commutation relation

$$[\prod_\mu(\zeta), x_\nu(\zeta')] = -[i/e(\zeta')]\delta(\zeta'-\zeta)\delta_{\mu\nu} \ (i=\sqrt{-1}),$$

which in the Schrödinger representation is given explicitly by

$$\prod_\mu(\zeta) = -\frac{i}{e(\zeta)} \frac{\delta}{\delta X_\mu(\zeta)}.$$

After fixing the particle proper-time gauge [since eq.(9.4) is invariant under the group of the trajectories reparametrization] and taking into account that the particle trajectory does not self-intersect in "time" $[X_\mu(\zeta) \neq X_\mu(\zeta')$ if $\zeta \neq \zeta']$, we finally, obtain that eq.(9.4) reduces to the Klein-Gordon wave equation in the presence of the external potential $V(x)$, namely

$$\left(-\Box_{X^{\text{in}}} + \frac{1}{2}m^2 - V(X^{\text{in}}) \right) G(X^{\text{out}}; X^{\text{in}}) = 0. \tag{9.5}$$

It is instructive to point out that by considering functional variations of the functional metric $dmu[X_\mu(\zeta)]$ we obtain constraints without dynamical content that are associated to the invariance of the theory under the action of the space-time translation Poincaré group.

9.3. The Wave Equation in the Covariant Bosonic String Dynamics

The basic object in the Polyakov approach[2,5] for the string covariant quantization (in the trivial topological sector) is that the following transition amplitude for an initial string state

$$C^{\text{in}} = \{(X_\mu^{\text{in}}(\sigma, e^{\text{in}}(\sigma)); \ 0 \leq \sigma \leq 1\}$$

propagates to a final string state $[C^{\text{out}} = \{(X_\mu^{\text{out}}(\sigma), e^{\text{out}}(\sigma))\}]$

$$G[e^{\text{out}}, e^{\text{in}}] = \int d\mu[g_{ab}] d\mu[\phi_\mu] e^{-I_0(g_{ab},\phi_\mu)}, \tag{9.6}$$

where the covariant string action is given by

$$I_0(g_{ab},\phi_\mu) = \int_D \left(\frac{1}{2}\sqrt{g}g^{ab}\,\partial_a\phi^\mu\partial_b\phi^\mu + \mu_0^2\sqrt{g}\right)(\sigma,\zeta)d\sigma\, d\xi. \tag{9.7}$$

The string surface parameter domain is taken to be the rectangle $D = \{(\sigma,\zeta), 0 \le \sigma \le 1, 0 \le \zeta \le T\}$. The functional measures $d\mu[g_{ab}]$ and $d\mu[\phi_\mu]$ are defined over all cylindrical quantum surfaces without holes and handles having as a boundary the string end configurations $\{C^{\text{in}};C^{\text{out}}\}$; i.e., $\phi_\mu(\sigma,0) = X_\mu^{\text{in}}(\sigma)$ and $\phi_\mu(\sigma,T) = X_\mu^{\text{out}}(\sigma)$. The intrinsic metric $\{g_{ab}(\sigma,\zeta)\}$ (which, roughly, plays the role of the covariant string proper-time parameter) can be chosen to satisfy the conformal gauge

$$g_{ab}(\sigma,\zeta) = \exp\,\beta(\sigma,\zeta)\delta_{ab}$$

and the initial end-point boundary condition $e^{\text{in}}(\sigma) = \exp(\beta(\sigma,0))$.

At this point a fundamental difference appears between the string and particle case (see 9.1). In the last case it is always possible to fix the proper-time gauge $e(\zeta) = \text{const} = 1$, where the intrinsic metric decouples from the dynamical description of the theory. This result reveals itself in the form of the associated wave equation [eq.(9.5), Sec. 9.2], where it does not have any functional dependence on the intrinsic metric. This decoupling phenomenon will not happen in the string case due to the conformal anomaly of the theory[2,5] unless it is canceled. Further, the associated string wave equation will depend on the intrinsic Liouville field at the boundary $\beta(\sigma,0) = \beta^{\text{in}}(\sigma)$, as we will show explicitly below.

Let us now proceed as in the particle case by considering the following identity related to the integrand invariance under translations in the conformal factor $\beta(\sigma,\zeta)$ functional space $[g_{ab}(\sigma,\zeta) = \exp(\beta(\sigma,\zeta)\delta_{ab}\}]$ in the string propagator eq.(9.6):

$$\int D[\beta(\sigma,\zeta)]\exp\left\{-\frac{26}{48\pi}\int_D\left(\frac{1}{2}(\partial_a\beta)^2 + \frac{1}{2}\mu_R^2 e^\beta\right)\right\}$$
$$\lim_{\zeta\to 0^+}\left(e^{-\beta(\bar\sigma,\bar\zeta)}\,\frac{\overleftrightarrow{\delta}}{\delta\beta(\bar\sigma,\zeta)}\,\delta_{ab}\right)F(\phi_\mu,g_{ab}), \tag{9.8}$$

where

$$F(\phi_\mu,g_{ab}) = \int d\mu[\phi_\mu]\exp(-I_0(\phi_\mu,g_{ab})) \tag{9.9}$$

denotes the pure string vector position term in eq.(9.6).

It is worthwhile to remark that this procedure for deducing a dynamical (wave) equation is the two-dimensional analog of that used to write the Wheeler-De Witt equation four-dimensional quantum gravity from the path integral expression for the universe propagator.[6]

The variation associated to the Faddeev-Popov term is given by

$$\int D[\beta(\sigma,\zeta)]\exp\left\{-\frac{26}{24\pi}\int_D\left(\frac{1}{2}(\partial_a\beta)^2 + \frac{1}{2}\mu^2 e^\beta\right)\right.$$
$$\left.\times(\sigma,\zeta)d\sigma\, d\zeta\right\}\frac{26}{24\pi}(R(e^\beta)+\mu^2)(\bar\sigma,\bar\zeta)F(\phi_\mu,g_{ab}), \tag{9.10}$$

where $R(e^{\beta}) = -(e^{-\beta}\Delta\beta)(\sigma,\zeta)$ denotes the scalar of curvature associated to the metric $g_{ab}(\sigma,\zeta) = \exp(\underline{\beta}(\sigma,\zeta))\delta_{ab}$.

The $\delta/\delta\beta(\bar{\sigma},\zeta)$ functional derivative of the term $F(\phi_\mu, g_{ab} = e^{\beta}\delta_{ab})$ is more subtle since the covariant functional measure $d\mu[\phi_\mu]$ [see Eq. (9) of Ref. 2] depends in a nontrivial way on the conformal factor $\beta(\sigma,\zeta)$ as a consequence of its definition as the functional volume element associated to the covariant functional metric

$$||\delta\psi^\mu|| = \int_D (e^{\beta}\delta\phi^\mu\delta\phi^\mu)(\sigma,\zeta)d\sigma\, d\zeta. \tag{9.11}$$

Its evaluation proceeds in the following way:

$$d\mu[\phi^\mu,(e^{\delta h+\beta})\delta_{ab}] - d\mu[\phi^\mu, e^{\beta}\delta_{ab}] \stackrel{\text{def}}{=} \frac{\delta}{\delta\beta} d\mu[\phi^\mu, e^{\beta}\delta_{ab}] + O(h^2). \tag{9.12}$$

Since, as a consequence of eq.(9.11), we have the result

$$d\mu[\phi^\mu, e^{\delta h+\beta}\,\delta_{ab}] = d\mu[e^{\delta h/2}\phi^\mu, e^{\beta}\delta_{ab}], \tag{9.13}$$

and effect of the functional string vector position measure under a conformal scale was evaluated exactly by Fujikawa [see Eqs. (3) and (39) in Ref. 5],

$$d\mu[e^{\delta h/2}\phi^\mu, e^{\beta}\delta_{ab}] = \exp\left\{\frac{D}{48\pi}\int_D (\partial_a\beta)^2 + \frac{1}{2}\mu^2 e^{\beta})\delta h\right\} d\mu[\phi^\mu, e^{\beta}\delta_{ab}], \tag{9.14}$$

we thus have the following esult by taking $h(\sigma,\zeta) = \varepsilon\delta(\sigma-\bar{\sigma})\delta(\zeta-\bar{\zeta})$ and considering the linear term in ε:

$$\begin{aligned}&\frac{\delta}{\delta\beta(\bar{\sigma},\zeta)} d\mu[\phi^\mu, e^{\beta}\delta_{ab}]\\ &\quad = \frac{1}{\varepsilon}\lim_{\varepsilon\to 0^+}(d\mu[\phi^\mu, e^{\delta h+\beta}\delta_{ab}] - d\mu[\phi^\mu, e^{\beta}\delta_{ab}]\\ &\quad = (D/24\pi)(R(e^{\beta(\bar{\sigma},\bar{\zeta})}) + \mu^2) \times d\mu[\phi^\mu, e^{\beta}\delta_{ab}].\end{aligned} \tag{9.15}$$

Finally the term $[\delta/\delta\beta(\bar{\sigma},\bar{\zeta})]I_0(\phi^\mu, g_{ab} = e^{\beta}\delta_{ab})$ is given by diagonal component of the string energy momentum tensor:

$$\left(e^{-\beta}\frac{\delta}{\delta\beta}\right)(I_0(\phi^\mu, g_{ab} = e^{\beta}\delta_{ab}))(\bar{\sigma},\bar{\zeta}). \tag{9.16}$$

By grouping together Eqs. (10), (15), and (16), we obtain that the string transition amplitude in the conformal gauge satisfies the dynamic constraint

$$\begin{aligned}0 = &\int d\mu[g_{ab}]\Big|_{g_{ab}=e^{\beta}\delta_{ab}} \int d\mu[\phi_\mu]\\ &\times \exp(-I_0(g_{ab},\phi_\mu))\left\{\frac{26-D}{48\pi}\lim_{\bar{\zeta}\to 0^+}(R((\bar{\sigma},\bar{\zeta})) + \mu^2)\right.\\ &\left.+\left(\frac{1}{2}\prod_{\mu}^{\text{in}}(\bar{\sigma})^2 - \frac{1}{2}|X'^{\mu}_{\text{in}}(\bar{\sigma})|^2\right)\right\},\end{aligned} \tag{9.17}$$

where $d\mu[g_{ab}]\big|_{g_{ab}=d^{\beta}\delta_{ab}}$ means that the functional measure over the intrinsic metric field $\{g_{ab}(\bar{\sigma},\bar{\zeta})\}$ is defined in the conformal gauge,

$$\prod_{\mu}^{\text{in}}(\bar{\sigma}) = \lim_{\bar{\zeta}\to 0^{+}} \partial_{\bar{\zeta}}\phi_{\mu}(\bar{\sigma},\bar{\zeta})$$

denotes the string canonical momentum and

$$X_{\text{in}}^{\prime\mu}(\bar{\sigma}) = \lim_{\bar{\zeta}\to 0^{+}} \partial_{\bar{\sigma}}\phi_{\mu}(\bar{\sigma},\bar{\zeta}).$$

In order to translate the above path integral relation into a wave equation form,[7] we introduce covariant string commutation relations[8]

$$\left[\prod_{\mu}^{\text{in}}(\bar{\sigma}), X^{\nu}(\bar{\sigma}')\right] = [i/\hbar^{(D)}][e^{\text{in}}(\bar{\sigma})]\delta(\bar{\sigma}-\bar{\sigma}'), \tag{9.18}$$

with $\hbar^{(D)}$ being the Planck constant in the physical space-time R^{D}. Using the Schrödinger representation for this commutation elation,

$$\prod_{\mu}^{\text{in}}(\sigma) = \frac{i}{\hbar^{(D)}e^{\text{in}}(\sigma)}\,\frac{\delta}{\delta X_{\mu}^{\text{in}}(\sigma)}, \tag{9.19}$$

we can express eq.(9.17) in the following form, which generalizes the usual $D=26$ Nambu-Virasoro wave equation[7]:

$$\begin{aligned}&\left\{-\frac{1}{2}\frac{e^{-2\beta_{\text{in}}(\sigma)}}{(\hbar^{D})^{2}\delta X_{\mu}^{\text{in}}(\sigma)\delta X_{\mu}^{\text{in}}(\sigma)} - \frac{1}{2}\,|X^{\beta_{\text{in}}}(\sigma)|^{2}\right.\\ &\left.+\frac{26-D}{24\pi}\left(-\frac{1}{2}\prod_{\beta}^{\text{in}}(\sigma)^{2} - \frac{1}{2}\beta_{\text{in}}'(\sigma)^{2} + \frac{1}{2}\mu^{2}e^{\beta_{\text{in}}(\sigma)}\right)\right\}\\ &\times G((X_{\mu}^{\text{in}}(\sigma,e^{\beta_{\text{in}}(\sigma)});(X_{\mu}^{\text{out}}(\sigma,e^{\beta_{\text{out}}(\sigma)}))=0,\end{aligned} \tag{9.20}$$

where we have written the conformal contribution in eq.(9.17) in the Polyakov proposed Liouville Hamiltonian,[2] with

$$\prod_{\beta}^{\text{in}}(\sigma) = \lim_{\zeta\to 0^{+}} \partial_{\zeta}\beta(\sigma,\zeta)$$

being the canonical momentum associated with the Liouville field $\beta(\sigma,\zeta)$ at the boundary. We note that it has the following representation:

$$\prod_{\beta}^{\text{in}}(\sigma) = \frac{i}{\hbar^{(2)}}\,\frac{\delta}{\delta\beta_{\text{in}}(\sigma)}. \tag{9.21}$$

Here $\hbar^{(2)}$ now denotes the Planck constant associated with two-dimensional string space-time D.

It is worth mentioning that the dynamical status acquired by the metric $g_{ab}(\sigma,\zeta) = \exp(\beta(\sigma,\zeta))\delta_{ab}$ in eq.(9.20) induced pure quantum gravity in D as a result of the dynamical breaking of the complete diffeomorphism ground of the action in eq.(9.7), denoted by $G\text{diff}(D)$, to the subgroup $G_{\text{diff}}(D)/G_{\text{W}eil}(D)_{\text{diff}}$, where $G_{\text{Weil,diff}}(D)$ is the subgroup of $G_{\text{diff}}(D)$ that acts on the metric field as a Weil scaling.

As a consequence of these remarks we can see that only at $D = 26$ can be choose the proper time string gauge $g_{ab}(\sigma,\zeta) = \delta_{ab}$ in an analogous way as in covariant particle dynamics (see Sec. 92), since now the invariance of the theory under $G_{\text{diff}}(D)$ is preserved by quantization.

9.4. A String Solution for the $QCD[SU(\infty)]$ Bosonic Contour Average Equation

There are several compelling arguments for the existence of a string representation for quantum chromodynamics (QCD) at the 't Hooft large number of colors. One of these arguments is that the $QCD[SU(\infty)]$ covariant loop average with an additional intrinsic global $SO(M)$ flavor group (see Appendix A),

$$\begin{aligned} &W_{ik}[C_{X(-\pi),X(\pi)}] \\ &= \frac{1}{N_c}\left\langle T_c^{\text{color}} \exp\left(i \oint_{C_{X(-\pi),X(\pi)}} A_\mu(X_\mu(\sigma)) \frac{dX_\mu(\sigma)}{e(\sigma)} \right) \right\rangle, \end{aligned} \tag{9.22}$$

satisfies the following (formal) stringlike contour equation[3] $[e(\sigma) = 1]$:

$$\begin{aligned} &\frac{\delta^{(2)}}{\delta X_\mu(\sigma)\delta X_\mu(\sigma)} W_{ik}[C_{X(-\pi),X(\pi)}] \\ &\lambda_0^2 \,⨏\!\!\oint_{C_{X(-\pi),X(\pi)}} d\bar{\sigma} X'_\mu(\sigma)\delta^{(D)}(X_\mu(\sigma) - X_\mu(\bar{\sigma}))X'_\mu(\bar{\sigma}) \\ &\times (W_{ij}[C_{X(-\pi),X(\sigma)}] W_{jk}[C_{X(\sigma),X(\pi)}]) \\ &-\gamma^2 |X'_\mu(\sigma)|^2 W_{ik}[C_{X(-\pi),X(\pi)}], \end{aligned} \tag{9.23}$$

where the contour integral $⨏\!\!\oint_{C_{X(-\pi),X(\pi)}}$ means that the coincident $\sigma = \bar{\sigma}$ does not contribute for the integrand (Cauchy principal value).

It is thus conjectured that some sort of string propagator should solve eq.(9.23) in some sense. Our aim in this section is to present an interacting string theory with an intrinsic fermionic structure that posesses as a string wave equation (in our proposed framework of Sec. 9.3) eq.(9.23) with a fixed flavor group $SO(22)$.

Let us start our analysis by describing the covariant string action of our proposed $QCD[SU(\infty)]$ string:

$$S[\phi_\mu(\sigma,\zeta),\psi_{(k)}(\sigma,\zeta),g_{ab}(\sigma,\zeta)] = S_0[\phi_\mu(\sigma,\zeta),g_{ab}(\sigma,\zeta)] + S_1[\psi_{(k)}(\sigma,\zeta),g_a(\sigma,\zeta)] + S_{\text{int}}[\phi_\mu(\sigma,\zeta),\psi_{(k)}(\sigma,\zeta),g_{ab}(\sigma,\zeta)], \tag{9.24}$$

where

$$S_0[\phi_\mu(\sigma,\zeta),g_{ab}(\sigma,\zeta)] = \frac{1}{2}\left(\int_D(\sqrt{g}g_{ab}\,\partial_a\phi^\mu\partial_b\psi^\mu)(\sigma,\zeta)d\sigma d\zeta\right), \tag{9.25a}$$

$$S_1[\psi_{(k)}(\sigma,\zeta),g_{ab}(\sigma,\zeta)] = \frac{1}{2}\int_D(\sqrt{g}\bar{\psi}_{(k)}\gamma_a(\sigma,\zeta)\partial_a\psi_{(k)})(\sigma,\zeta)d\sigma d\zeta, \tag{9.25b}$$

$$\begin{aligned} &S_{\text{int}}[\phi_\mu(\sigma,\zeta),\psi_{(k)}(\sigma,\zeta),g_{ab}(\sigma,\zeta)] \\ &\quad = \beta\bigg(\int_D d\sigma d\zeta\sqrt{g}(\bar{\psi}_{(k)}\psi_k)\widehat{T}^{\mu\nu}(\phi_\mu)(\sigma,\zeta) \\ &\quad \times\left(\int_D\sqrt{g(\bar{\sigma},\bar{\zeta})}\delta^{(D)}(\phi_\mu(\sigma,\zeta)-\phi_\mu(\bar{\sigma},\bar{\zeta}))\times\widehat{T}^{\mu\nu}(\phi_\mu(\bar{\sigma},\bar{\zeta}))\right)d\bar{\sigma}d\bar{\zeta}\bigg) \end{aligned} \tag{9.5c}$$

The notation is as follows: The bosonic degrees of freedom are $\{\phi_\mu(\sigma,\zeta),g_{ab}(\sigma,\zeta)\}$ as in Sec. 9.2. Additionally we introduce a set of intrinsic two-dimensional Weyl spinors in the string surface and belonging to the $SO(M)$ fundamental representation. They are denoted by $\{\psi_{(k)}(\sigma,\zeta),k=1,\dots,m\}$. We impose on them the Neumann boundary condition

$$\lim_{\zeta\to 0^+}\partial_\sigma\psi_{(k)}(\sigma,\zeta) = 0.$$

The bosonic $\{\psi_k(\sigma,\zeta),g_{ab}(\sigma,\zeta)\}$ string sector interacts with fermionic $\{\psi_{(k)}(\sigma,\zeta)\}$ sector through a self-avoiding interaction involving the surface orientation tensor

$$\begin{aligned} &\widehat{T}^{\mu\nu}(\phi_\mu(\sigma,\zeta)) = (\varepsilon^{ab}\partial_a\phi^\mu\partial_b\psi^\nu/\sqrt{h}(\sigma,\zeta), \\ &h = \det h_{ab}\,,\ h_{ab} = \partial_a\phi^\mu\partial_\mu\phi^\nu, \end{aligned}$$

and an attractive $(\beta < 0)$ delta function potential supported at the self-intersecting lines of the string surface. These non-trivial self-intersections are supposed to arise at those submanifolds where $X_\mu(\sigma,\zeta) = X_\mu(\sigma',\zeta')$ with $\sigma \neq \sigma'$ for every $\zeta \in [0,T]$. We notice that self-intersections of the form $X_\mu(\sigma,\zeta) = X_\mu(\sigma,\zeta')$ with $\zeta \neq \zeta'$ arise only in the case where the string surface possesses holes and handles, which is not the case here.

After having described our string theory, we consider the following $O(M)$ string transition amplitude[9]:

$$\begin{aligned} Z_{kl}[C_{X(-\pi),X(\sigma)}] = \int &d\mu[g_{ab}]d\mu]\phi_\mu] \\ &\times d\mu[\psi_{(k)}](\psi_{(k)}(-\pi,0)\bar{\psi}_l(\pi,0)) \\ &\times\exp\{-S[\phi_\mu,\psi_{(k)},g_{ab}]\}. \end{aligned} \tag{9.26}$$

In order to write the wave function equation associated with the above string Green's function, in the physical space-time R^4, we proceed as in Sec. 9.3 by considering the analogous identity of eq.(9.8), namely,

$$\begin{aligned}
&\int d\mu[g_{ab}]d\mu[\phi_\mu]d\mu[\psi_{(k)}](\psi_{(k)}(-\pi,0)\overline{\psi}_{(l)}(\pi,0))\exp\{-S[\phi_\mu,\psi_{(k)},g_{ab}\} \\
&\times\left(\frac{1}{2}\prod)\mu^{\text{in}}(\sigma)^2-\frac{1}{2}|X'_\mu(\sigma)|^2+\lim_{\zeta\to0^+}(\overline{\psi}_{(k)}\gamma_1\,\partial_a\psi_{(k)})(\sigma,\zeta)\right) \\
&=\frac{\beta}{2}\int_{-\pi}^{\pi}d\bar{\sigma}X'^{\mu}(\sigma)\delta^{(D)}(X_\mu(\sigma)-X_\mu(\bar{\sigma}))X'^{\mu}(\bar{\sigma})\int d\mu[g_{ab}d\mu[\phi_\mu]d\mu[\psi_{(k)}] \\
&\times\left(\sum_{(p)=1}^{22}\psi_{(p)}\overline{\psi}_{(p)}\right)(\sigma,0)(\psi_{(k)}(-\pi,0)\overline{\psi}_{(l)}(\pi,0))\exp\{-S[\phi_\mu,\psi_{(k)},g_{ab}]\}.
\end{aligned}\tag{9.27}$$

Our choice of the intrinsic "flavor" group to be $SO(22)$ is dictated by the fact that the $QCD[SU(\infty)]$ string should preserve the full invariance under the diffeomorphism group and this happens only in the case where the conformal anomaly of the theory vanishes (see Sec. 9.3). Since, in our proposed theory ($D=4$), the anomalous term is proportional to $[26-(D+M)]/24\pi$ we see that only for $M=22$ can we preserve the above-mentioned symmetry.

We thus can rewrite eq.(9.27) in the form

$$\begin{aligned}
&\left(-\frac{1}{2}\,\frac{\delta^{(2)}}{\delta X_\mu(\sigma)\delta X_\mu(\sigma)}-\frac{1}{2}\,|X'_\mu(\sigma)|^2\right)Z_{kl}[C_{X(-\pi),X(\sigma)}] \\
&=\frac{\beta}{2}\int_{-\pi}^{\pi}d\bar{\sigma}X'_\mu(\sigma)\delta^{(D)}(X_\mu(\sigma)-X_\mu(\bar{\sigma}))X'_\mu(\bar{\sigma})(Z_{kp}[C_{X(-\pi),X(\sigma)}]Z_{pl}[C_{X(\sigma),X(\pi)}]),
\end{aligned}\tag{9.28}$$

where we have used the string measure factorization properties

$$\begin{aligned}
&\int\prod_{\substack{-\pi<\beta<\pi\\0<\zeta<T\\1<k<22}}(d\psi_{(k)}(\beta,\zeta))(\psi_{(k)}(-\pi,0)\overline{\psi}_{(l)}(\pi,0))(\psi_{(p)}(\sigma,0)\overline{\psi}_{(p)}(\sigma,0))\exp\{-S[\phi_\mu,\psi_{(k)},g_{ab}]\} \\
&=\int\prod_{\substack{-\pi<\beta<\sigma\\0<\zeta<T\\1<k<22}}(d\psi_{(k)}(\beta,\zeta))(\psi_{(k)}(-\pi,0)\overline{\psi}_{(p)}(\sigma,0))\exp\{-S^{(1)}[\phi_\mu,\psi_{(k)},g_{ab}]\} \\
&\times\int\prod_{\substack{\sigma<\beta<\pi\\0<\zeta<T\\1<k<22}}(d\psi_{(k)}(\beta,\zeta))(\psi_{(p)}(\sigma,0)\overline{\psi}_{(l)}(\pi,0))\exp\{-S^{(2)}[\phi_\mu,\psi_{(k)},g_{ab}]\}
\end{aligned}\tag{9.29a}$$

and

$$\int\left(\prod_{\substack{-\pi<\beta<\pi\\0<\tau<T}} d\phi^{\mu}(\beta,\zeta)\big|_{C_{X(\pi),X(-\pi)}}\right)\exp\left\{-\frac{1}{2}\int_{D_{[-\pi,\pi]\times[0,T]}}(\partial_a\phi^{\mu})^2\right\}$$
$$=\int\left(\prod_{\substack{-\pi<\beta<\sigma\\0<\zeta<T}} d\phi^{\mu}(\beta,\zeta)\big|_{C_{X(-\pi)\times X(\sigma)}}\right)\exp\left\{-\frac{1}{2}\int_{D_{[-\pi,\sigma]\times[0,T]}}(\partial_a\phi^{\mu})^2\right\}$$
$$\times\int\left(\prod_{\substack{\sigma<\beta<\pi\\0<\zeta<T}} d\phi^{\mu}(\beta,\zeta)\big|_{C_{X(\sigma),X(+\pi)}}\right)\exp\left\{-\frac{1}{2}\int_{D_{[\sigma,\pi]\times[0,T]}}(\partial_a\phi^{\mu})^2\right\}. \qquad (9.29b)$$

Here

$$\left(\prod_{\substack{-\pi<\beta<\pi\\0<\zeta<T}} d\phi^{\mu}(\beta,\zeta)\big|_{C_{X(-\pi),X(\pi)}}\right)$$

means that the functional integration is done with the boundary condition $\phi^{\mu}(\beta,0)=C_{X(-\pi),X(\pi)}$.

We remark that these factorization properties hold true only in the case that the split string surfaces $\phi_{\mu}(D_{[-\pi,\sigma]\times[0,T]})$ and $\phi_{\mu}(D_{[\sigma,\pi]\times[0,T]})$ possess the same topology as in our case of trivial topology and are homotopical deformations of the loop boundary which by their turns are smooth and possessing only isolated double point at path self-intersections as a consequence of Pauli-Exclusion occupation number for fermions.

Let us now identify the string wave equation [eq.(9.28)] with the $QCD[SU(\infty)]$ contour average equation [eq.(9.23)]. The first step is to identify the $SU(\infty)$ gauge coupling constant λ_0^2 with the string interactoin coupling $-\beta$. Second, we make the identification of the constant $-\gamma^2$ (the Euclidean gluon condensate – see Appendix A) with the Regge slope parameter $1/\pi a'$, which was adjusted to unit in our study.

After these coupling constant identifications we see that the Euclidean self-suppressing string theory should represent Euclidean $QCD[SU(\infty)]$ in the gauge invariant observable algebra (color singlet currents, spectrum, etc.).

9.5. The Neveu-Schwarz String Wave Equation

Let us start by considering the open fermionic string action in a D-dimensional Euclidean space-time[10] $(\mu=1,\ldots,D,(A)=1,2,a=1,2)$:

$$S[\phi_{\mu}(\sigma,\zeta),\psi_{\mu}(\sigma,\zeta),e_a^{(A)}(\sigma,\zeta),\chi_a(\sigma,\zeta)]$$
$$=\int_D d\sigma\, d\zeta\, e(\sigma,\zeta)\left[\frac{1}{2}\partial_a\psi^{\mu}\partial_b\psi^{\mu}g^{ab}+\frac{1}{2}i\psi_{\mu}(\gamma\partial)\psi_{\mu}\right.$$
$$\left.-\frac{1}{2}F^2-\frac{1}{2}i(\chi_a\gamma^b\gamma^a\psi^{\mu})\left(\partial_b\psi^{\mu}-\frac{1}{4}i\chi_b\,\psi^{\mu}\right)\right](\sigma,\zeta)+\text{ boundary terms.} \qquad (9.30)$$

Here the fermionic string is characterized by two (external) fields: the usual bosonic vector position $\phi^\mu(\sigma,\zeta)$ and the Majorana spinor $\psi^\mu(\sigma,\zeta)$ describing the string Lorentz spin. The presence of the vierbein $e_a^{(A)}(\sigma,\zeta)$ and of the two-dimensional vector Majorana spinor $\chi_a(\sigma,\zeta)$ together with the auxiliary scalar field $F(\sigma,\zeta)$ ensures, respectively, the action's invariance under general Lorentz and coordinate transformations together with the world-sheet local supersymmetric transformations.

Following Polyakov the (formal) fermionic string propagator is given by the follwoing path integral connecting the initial C^{in} string state to a final string state C^{out}:

$$C[C^{\text{out}};C^{\text{in}}] = \int d\mu[\phi^\mu,\psi^\mu,e_a^{(A)},\chi_a]\exp\{-S[\phi^\mu,\psi^\mu,e_a^{(A)},\chi_a]\} \tag{9.31}$$

(here the boundary terms were absorbed in $G[C^{\text{out}};C^{\text{in}}]$).

In order to write dynamical wave equations we exploit the invariance under translations in the superconformal factor $(\varphi(\sigma,\zeta);\zeta(\sigma,\zeta))$ functional space of the fermionic string propagator [eq.(9.31)]

$$g_{ab}(\sigma,\zeta) = \exp(2\varphi(\sigma,\zeta)\delta_{ab}) \qquad \chi_a(\sigma,\zeta) = \gamma_a^{(B)}\zeta_{(B)}(\sigma,\zeta)],$$

which produces the following identities:

$$\int d\mu[\phi^\mu,\psi^\mu,e_a^{(A)},\chi_a]e^{-S[\phi^\mu,\psi^\mu.e_a^{(A)},\chi_a]}\left(-\frac{\delta S[\phi^\mu,\psi^\mu,e_a^{(A)},\chi_a]}{\delta\varphi(\bar{\sigma},\zeta)}\right) =$$

$$= \int\left(\frac{\delta}{\delta\varphi(\bar{\sigma},\zeta)}d\mu[\phi^\mu,\psi^\mu,e_a^{(A)},\chi_a]\right)e^{-S[\phi^\mu,\psi,\mu,e_a^{(A)},\chi_a]} \tag{9.32a}$$

and

$$\int d\mu[\phi^\mu,\psi^\mu,e_a^{(A)},\chi_a]e^{-S[\phi^\mu,\psi^\mu.e_a^{(A)},\chi_a]}\left(-\frac{\delta S[\phi^\mu,\psi^\mu,e_a^{(A)},\chi_a]}{\delta\zeta_{(B)}(\bar{\sigma},\zeta)}\right) =$$

$$= \int\left(\frac{\delta}{\delta\zeta_{(B)}(\bar{\sigma},\zeta)}d\mu[\phi^\mu,\psi^\mu,e_a^{(A)},\chi_a]e^{-S[\phi^\mu,\psi,\mu,e_a^{(A)},\chi_a]}\right) \tag{9.32b}$$

By noting that the fermionic string is defined at the quantum level only at $D=10$ (the so-called Neveu-Schwarz string) or at $D\to-\infty$,[11] we will consider $D=10$, which means that the functional measure variations in the right-hand side of Eqs. (32a) and(32b) vanish. In the superconformal gauge and using the Euclidean identity $\gamma_{(A)}\gamma_{(B)} = i\varepsilon_{(A)(B)}\gamma_s$, we rewrite Eqs. (32a) and (32b) as

$$\int d\mu[\phi^\mu,\psi^\mu]e^{-S[\phi^\mu,\psi^\mu]}\,\frac{1}{2}((\partial_\zeta\phi^\mu)^2-(\partial_\sigma\phi^\mu)^2+\psi^\mu\gamma_{(1)}\,\partial_\sigma\psi^\mu(\bar{\sigma},\bar{\zeta})=0, \tag{9.33a}$$

$$\int d\mu[\phi^\mu,\psi^\mu]e^{-S[\phi^\mu,\psi^\mu]}\,\frac{1}{2}(1+\gamma_5)\psi^\mu(\bar{\sigma},\bar{\zeta})(\partial_\zeta\phi^\mu-\partial_\sigma\phi^\mu)(\bar{\sigma},\bar{\zeta})=0 \tag{9.33b}$$

respectively.

In order to translate the above-written string path integral identities into a wave equation form we take its boundary limit $\zeta \to 0^+$ and translate the result into an operator equation by using the Schörindger quantum representation

$$\lim_{\zeta\to 0^+} \partial_\zeta \phi^\mu(\sigma,\zeta) \Leftrightarrow \frac{i}{\hbar^{(D)}} \frac{\delta}{\delta\phi_{\text{in}}(\sigma)}\,, \tag{9.34a}$$

$$\lim_{\zeta\to 0^+} \partial_\sigma \phi^\mu(\sigma,\zeta) \Leftrightarrow \phi'^\mu_{\text{in}}(\sigma), \tag{9.34b}$$

$$\lim_{\zeta\to 0^+} \psi^\mu(\sigma,\zeta) \Leftrightarrow \Gamma^\mu_{\text{in}}(\sigma). \tag{9.34c}$$

Here the quantum C^{in} string state in the operator framework is characterized by the coordinaes $(\Gamma^\mu_{\text{in}}(\sigma), \phi^\mu_{\text{in}}(\sigma))$ where the $\Gamma^\mu_{\text{in}}(\sigma)$ are string valued Dirac matrices obeying the space-time anticommuting relations[8]

$$\{\Gamma^\mu_{(A)\,\text{in}}(\sigma), \Gamma^\nu_{(B),\text{in}}(\sigma')\} = 2\delta(\sigma-\sigma')\delta_{\mu\nu}\delta_{(A),(B)}.$$

By noting that the Neveu-Schwarz string fermion field $\psi^\mu(\sigma,\tau)$ satisfies the Neumann condition

$$\lim_{\tau\to 0^+} \partial_\sigma \psi^\mu(\sigma,\tau) = 0,$$

we obtain a fermionic string wave equation

$$D^{(\pm)}_{C^{\text{in}}}\, G[C^{\text{in}}, C^{\text{out}}] = 0, \tag{9.35}$$

where

$$D^{(\pm)}_{C^{\text{in}}} = \frac{1}{2}(1+\gamma_5)\left(\frac{i}{\hbar^{(D)}}\Gamma^\mu_{\text{in}} \frac{\delta}{\delta\pi^\mu_{\text{in}}} - \Gamma^\mu_{\text{in}}\,\phi'^\mu_{\text{in}}\right)(\sigma). \tag{9.36}$$

It is instructive to remark that in eq.(9.35) the same $\Gamma^\mu_{\text{in}})\sigma)$ used in the momenta operator is also ued in the string length factor $\phi'^{\text{in}}_\mu(\sigma)$, opposite to the earlier proposed Ramond-Marshall fermionic string wave equation[8] where two different sets of $\Gamma^\mu(\sigma)$ matrices are used.

Finally we note that the formal anticommutator $\{D^{(\pm)}_{C^{\text{in}}}(\sigma); D^{(\pm)}_{C^{\text{in}}}(\sigma)\}$ is equal to the bosonic

$$-\Delta_{C^{in}} = -\frac{1}{2}\frac{\delta^{(2)}}{\delta\phi^{\text{in}}_\mu(\sigma)\delta\phi^{\text{in}}_\mu(\sigma)} - \frac{1}{2}\,|\phi'^{\text{in}}_\mu(\sigma)|^2$$

string wave D'Alembertian since we have preserved the superdiffeomorphism group of the theory, which, in turn, manifests itself in the following constraint imposed in the physical Hilbert space of Neveu-Schwarz string states:

$$\left(\phi'^\mu_{\text{in}}(\sigma)\frac{\delta}{\delta\phi^\mu_{\text{in}}(\sigma)}\right) G[C^{\text{in}}, C^{\text{out}}] = 0. \tag{9.37}$$

Appendix A.
The $QCD(S(\infty))$ Bosonic Contour Average

The basic dynamical variable in the loop space formulation for Euclidean $QCD[SU(\infty)]$ is the amplitude for a quark loop propagating in the quantum (confining) vacuum of a pure Yang-Mills field, since at the *t*' Hooft limit for a large number of colors the second-quantized quark matter effective action reduces to the quark first-quantized action, namely,[9]

$$\lim_{\substack{(g^2N_c)\text{ fixed} \\ N_c\to\infty}} (\det(i\gamma_\mu(\partial_\mu + A_\mu))) = \int d^D X \left(\sum_{\substack{C_{X(\pi),X(-\pi)} \\ X(\pi)=X(-\pi)=X}} \langle \operatorname{Tr} U[C_{X(\pi),X(-\pi)}] \rangle \right) \tag{9.A1}$$

where

$$U[C_{X(-\pi),X(\pi)}] = P\left\{ \exp \int_{-\pi}^{\pi} d\sigma A_\mu(X_\mu(\sigma)) \frac{dX_\mu(\sigma)}{e(\sigma)} \right\} \tag{9.A2}$$

denotes the covariant Wu-Yang phase factor defined by the closed (covariant) quark trajectory

$$C_{X(-\pi),X(\pi)} = \{(X_\mu(\sigma), e(\sigma));\ -\pi < \sigma < \pi\}$$

and representing the interaction of the pair with the Yang-Mills external field $A_\mu(x)$. The notation $\langle\ \rangle$ means the quantum average defined by the Yang-Mills functional integral at $N_c \to \infty$ (planar graphs).

In order to deduce a closed contour functional equation for the amplitude inside eq.(9.A2), we remark the validity of the classical second-order functional derivatives results[3] $[e(\sigma) = 1]$

$$\begin{aligned}
&\lim_{\sigma\to\sigma'} \frac{\delta^2}{\delta x_\mu(\sigma)\delta x_\mu(\sigma')} (\operatorname{Tr} U[C_{X(-\pi),X(\pi)}]) \\
&\lim_{\sigma\to\sigma'} \delta(\sigma-\sigma') \operatorname{Tr}((\nabla_\mu F_{\mu\nu})(x(\sigma))X'^{\nu}(\sigma)(U[C_{X(\sigma),X(\pi)}]U[C_{X(-\pi),X(\sigma)}])) \\
&+ \lim_{\sigma\to\sigma'} \theta(\sigma-\sigma') \operatorname{Tr}(U[C_{X(-\pi),X(\sigma')}]F_{\alpha\beta}(X(\sigma'))X'^{\beta}(\sigma')U[C_{X(\sigma'),X(\sigma)}]F_{\alpha\rho}(X(\sigma))X'^{\rho}(\sigma)U[C_{X(\sigma),X(\pi)}]) \\
&+ \lim \theta(\sigma'-\sigma) \operatorname{Tr}(\text{ above written expression with } \sigma \text{ exchanged by } \sigma').
\end{aligned} \tag{9.A3}$$

By using that $\theta(\sigma'-\sigma) = \frac{1}{2}$ if $\sigma = \sigma'$ and imposing the loop periodicity property

$$U[C_{X(a),X(a+2\pi)}] = U[C_{X(-\pi),X(\pi)}]\ (-\pi \le a \le \pi), \tag{9.A4}$$

we can finally rewrite eq.(9.A3) in the loop invariant form

$$\begin{aligned}
&\frac{\delta^2}{\delta X_\mu(\sigma)\delta X_\mu(\sigma')} \operatorname{Tr} U[C_{X(-\pi),X(\pi)}] \\
&= \operatorname{Tr}((\nabla_\mu F_{\mu\nu})(X(\sigma))X'^{\nu}(\sigma)(\operatorname{Tr} U[C_{X(-\pi),X(\pi)}])) \\
&+ \operatorname{Tr}(F_{\alpha\beta}(X(\sigma))X'^{\beta}(\sigma)F^{\alpha\rho}[X(\sigma))X'_\rho(\sigma)(\operatorname{Tr} U[C_{X(-\pi),X(pi)}])).
\end{aligned} \tag{9.A5}$$

In order to write the (unrenormalized) quantum analogous loop equation, we take the quantum ($N_c \to \infty$) average of both sides of eq.(9.4a) and observe the quantum results

$$\begin{aligned}&\langle \operatorname{Tr}(\nabla_\mu F_{\mu\nu})(X(\sigma))X'^{\nu}(\sigma)\operatorname{Tr}U[C_{X(-\pi),X(\pi)}]\rangle \\ &= \lambda_0^2 \oint_{X(-\pi),X(\pi)} X'_\mu(\sigma)\delta^{(D)}(X_\mu(\sigma)-X'_\mu(\bar{\sigma})\langle \operatorname{Tr}U[C_{X(-\pi),X(\sigma)}]\rangle\langle \operatorname{Tr}U[C_{X(\sigma),X(\pi)}]\rangle \quad (9.A6)\end{aligned}$$

and

$$\begin{aligned}&\langle \operatorname{Tr}(F_{\alpha\beta}(X(\sigma))X'^{\beta}(\sigma)F^{\alpha\beta}((\sigma))X'_\rho(\sigma)U[C_{X(-\pi),X(\pi)}])\rangle \\ &= \left(\int d^Dx\langle \operatorname{Tr}(F_{\alpha\beta}F^{\alpha\beta})(x)\rangle\right)|X'(\sigma)|^2\langle \operatorname{Tr}U[C_{X(-\pi),X(\pi)}]\rangle. \quad (9.A7)\end{aligned}$$

Equation (9.A7) was obtained by supposing the very existence of confining in $QCD\,SU(N)]$ for any value of the color parameter N signaled by the (formal) nonvanishing gauge invariant $SU(N)$ gluon condensate in R^D: (see chapter 4).

$$\int d^Dx\langle \operatorname{Tr}(F_{\alpha\beta}F^{\alpha\beta})(x)\rangle = -\gamma^2. \quad (9.A8)$$

By making the assumption that confining persists at $N_c \to \infty$ we obtain the $QCD\,[SU(\infty)]$ loop wave equaiton [eq.(9.23)] in the proper-time gauge $e(\sigma) = 1$.

Appendix B.
The β Term

In this Appendix we present the calculations leading to the β term in eq.(9.27).

Therefore let us consider the boundary value of the following quantity:

$$\lim_{\tau\to 0^+}\int_D d\bar{\sigma}\,d\bar{\zeta}\,\widehat{T}_{\mu\nu}(\phi_\mu(\sigma,\zeta))\delta^{(D)}(\phi_\mu(\sigma,\zeta)-\phi_\mu(\bar{\sigma},\bar{\zeta})\widehat{T}_{\mu\nu}(\bar{\sigma},\bar{\zeta})). \quad (9.B1)$$

We can evaluate eq.(9.B1) by taking into account the following results.

First, formally

$$\begin{aligned}&\lim_{\tau\to 0^+}\delta^{(D)}(\phi_\mu(\sigma,\zeta)-\phi_\mu(\bar{\sigma},\bar{\zeta})) \\ &\quad = \lim_{\tau\to 0^+}\delta^{(D)}(\phi_\mu(\sigma,\zeta)-\phi_\mu(\bar{\sigma},\bar{\zeta}))\delta(\zeta-\bar{\zeta}) \\ &\quad = \delta^{(D)}(X_\mu(\sigma)-X_\mu(\bar{\sigma}))\delta(\bar{\zeta}), \quad (9.B2)\end{aligned}$$

since our topologically trivial string surface does not possess self-intersections in the intrinsic string time variable ζ, which in turn, is related to the nonexistence of handles and holes in the string world sheet.

Second, in the asymptotic limit $\zeta \to 0^+$ the string surface has the behavior

$$\lim_{\tau\to 0^+}\phi_\mu(\sigma,\zeta) = \lim_{\tau\to 0^+}X_\mu(\sigma)(1+\zeta),$$

since the string surface is a homotopical (contractible) deformation of its boundary.

As a consequence of the above-mentioned remark, we obtain, in the string isothermal gauge $[X'_\mu(\sigma\cdot X_\mu(\sigma)=0]$, the value in eq.(9.B1):

$$\begin{aligned}&\lim_{\tau\to 0^+}\widehat{T}^{\mu\nu}(\phi_\mu(\sigma,\zeta))\widehat{T}(\phi_\mu(\bar\sigma,\bar\zeta))\\&=[X'_\mu(\sigma/\sqrt{X'_\mu(\sigma)^2}][X'_\mu(\bar\sigma)/\sqrt{X'_\mu(\bar\sigma)^2}],\end{aligned}\tag{9.B3}$$

where we have taken into account that $X_\mu(\sigma)=X_\mu(\bar\sigma)$ in eq.(9.B1). By making eq.(9.B3) covariant, i.e., $\sqrt{(X^1_\mu(\sigma))^2}\to e(\sigma)$, we obtain the β term in eq.(9.27), which for $M=22[e(\sigma)=\text{const}]$, is simply given by

$$\frac{\beta}{2}\int_{-\pi}^{\pi}d\bar\sigma X'_\mu(\sigma)(\delta^{(D)}(X_\mu(\sigma)-X_\mu(\bar\sigma)))X'_\mu(\bar\sigma).\tag{9.B4}$$

Appendix C.
The Migdal-Elfin String as a Particular Case

Our aim in this Appendix is to show how to obtain the proposed Migdal-Elfin string for $QCD\,[SU(\infty)]$ [12] as a particular case of our proposed self-suppressing fermionic string when the string world sheet does not possess nontrivial self-intersections, i.e., $\phi_\mu(\sigma,\zeta)=\phi_\mu(\bar\sigma,\bar\zeta)$ means that $\sigma=\bar\sigma$, $\zeta=\bar\zeta$.

In order to analyze this case let us introduce orthonormal coordinates on the string surface $\{\phi_\mu(\sigma,\zeta)\}$:

$$\begin{aligned}\partial_\sigma\phi_\mu\partial_\zeta\phi^\mu=0,\ (\partial_\sigma\phi^\mu)^2&=(\partial_\zeta\phi_\mu)^2,\\ h(\sigma,\zeta)=\det\{h_{ab}(\sigma,\zeta)\}&=\det\{\partial^a\phi^\mu\partial_b\phi^\mu\}\\&=(\partial_\sigma\phi^\mu)^2=(\partial_\zeta\phi^\mu)^2.\end{aligned}\tag{9.C1}$$

Not that this is possible since we have canceled the model's conformal anomaly by choosing $M=22$.

By introducing a tangent vector along coordinates lines $\partial\phi^\mu/\partial\zeta$ and $\partial\phi^\mu/\partial\sigma$, we have the relationship (see the Appendix of Ref. 13)

$$\begin{aligned}&\delta^{(D)}(\phi^\mu(\sigma,\zeta)-\phi^\mu(\bar\sigma,\bar\zeta))\\&=\delta_\varepsilon^{(D-2)}(0)([1/h(\sigma,\zeta)^{1/2}]\delta^{(2)}((\sigma-\bar\sigma),(\zeta-\bar\zeta))),\end{aligned}\tag{9.C2}$$

where $\delta_\varepsilon^{(D-2)}(0)$ means a regularized form of the delta function singular value $\delta^{(D-2)}(0)$ (see Ref. 13).

Substituting eq.(9.C2) into the string self-interaction term [eq.(9.25)] we obtain the more invariant expression for the fermion action:

$$\begin{aligned}&\beta^{(R)}\int_D(\bar\psi_{(k)}\psi_{(k)})(\delta,\zeta)\\&\times\left(\sum_{\{\phi_\mu(\sigma,\zeta)=\phi_\mu(\bar\sigma,\bar\zeta)\}}\widehat{T}^{\mu\nu}(\phi_\mu(\sigma,\zeta))\widehat{T}^{\mu\nu}(\phi_\mu(\bar\sigma,\bar\zeta))\right),\end{aligned}\tag{9.C3}$$

where $\beta^{(R)} = \beta\delta_\varepsilon^{(D-2)}(0)$ is the regularized string constant.

At this point we can see that eq.(9.C3) reduces to a mass term for the intrinsic $SO(22)$ fermion field $\psi_\zeta(\sigma,\zeta)$, which, in the case of the string world sheet has only the trivial self-intersection

$$\phi_\mu(\sigma,\zeta) = \phi_\mu(\bar{\sigma},\bar{\zeta}) \Rightarrow \sigma = \bar{\sigma},\ \zeta = \bar{\zeta},$$

wince

$$\widehat{T}^{\mu\nu}(\phi_\mu(\sigma,\zeta))\widehat{T}_{\mu\nu}(\phi_\mu(\sigma,\zeta)) = 1.$$

We thus get

$$\sum_{k=1}^{22} \beta^{(R)} \int_D (\bar{\psi}_{(k)}\psi_{(k)})\sqrt{h}(\sigma,\zeta)d\sigma\, d\zeta. \tag{9.C4}$$

For the non-trivial self-intersecting case [σ multivalued $\phi^\mu(\sigma, zeta)$ functions] we have to add to eq.(9.C4) the term responsible for the theory's interaction, which is supported at the nontrivial string's surface self-intersection lines $\phi_\mu(\sigma,\zeta) = \phi_\mu(\bar{\sigma},\bar{\zeta})$ with $\sigma \neq \bar{\sigma}$ as given by our interaction action [eq.(9.25C)] and previously conjectured in Ref. 14.

Appendix D.
On Polyakov's Bosonic String Path Integral - Revisited on the Light of Correct Measures Definition

In opinion of A.M. Polyakov "there are methods and formulae in science, which serve as master-key to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned sum over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics" (A. M. Polyakov).

The general picture has been envisaged as follows: one should try to solve loop-space or generalized Schrödinger functional wave equations by the appropriate flux lines functionals represented by transition amplitudes given by the sums over all possible surfaces with fixed boundary.

$$G(C) = \sum_{(S_C)} \exp\left\{-\frac{1}{2\pi\alpha'}A(S_c)\right\} \tag{9.1-d}$$

here C is some loop (smooth or a random closed path), S_C is a surface bounded by the loop C and $A(S_C)$ is the area of this surface and α' an extrinsinc (lenght square) constant (the Regge slope parameter).

The main point on Polyakov's propose is to introduce besides the surface parametrization $X_\mu(\xi_1,\xi_2)$, an intrinsic metric tensor $g_{ab}(\xi_1,\xi_2)$ and a quadratic functional on the random surface $X_\mu(\xi_1,\xi_2)$ field substituting the area functional in eq.(9.1-d) (with $2\pi\alpha' = 1$)

$$A(S_C) = \frac{1}{2}\int_D d^2\xi(\sqrt{g}g^{ab}\partial_a X_\mu \partial_b X^\mu)(\xi) \tag{9.2-d}$$

It is very important to remark that the above 2D-gravity induced surface functional has the geometrical meaning of the area spanned by the surface $X_\mu(\xi_1,\xi_2)$ *only at the classical level* $\alpha' \to 0$.

In order to proceed to the quantum theory, A.M. Polyakov has proposed that the quantum surface average of any extended reparametrization invariant functional $\Phi[X_\mu(\xi_1,\xi_2);g_{ab}(\xi_1,\xi_2)]$ should be given by the following expression

$$\int d\mu[S]\phi(S_C) \stackrel{\text{def}}{\equiv} \int [Dg_{ab}(\xi)] \exp(-\mu_{bare}\int \sqrt{g}d^2\xi) \int [D[X_\mu(\xi)] \left[\exp\left(-\frac{1}{2}\int_D (\sqrt{g}g^{ab}\partial_a X_\mu \partial_b X_\mu)(\xi)d^2\xi\right)\right] \Phi[X_\mu(\xi),g_{ab}(\xi)] \tag{9.3-d}$$

The reparametrization invariant functional measures on eq.(9.3-d) are associated to the following functional measures

$$\|\delta X^\mu\|^2 = \int d^2\xi (g(\xi))^{1/2}\, \delta X_\mu(\xi)\delta X_\mu(\xi) \tag{9.4-d}$$

and

$$\|\delta g_{ab}\|^2 = \int d^2\xi [g(\xi)]^{1/2}\, (g^{aa'}g^{bb'} + Cg^{ab}g^{a'b'})\delta g_{ab}\delta g_{a'b'} \tag{9.5-d}$$

where $C \neq -\frac{1}{2}$ is an arbitrary constant.

The reparametrization invariant gaussian functional integral $X_\mu(\xi_1,\xi_2)$ is easily evaluated with the result in the conformal gauge $g_{ab} = \rho^2\delta_{ab}$ (for closed boundary-less 2D-compact Riemannian manifolds)

$$\det{}^{-D/2}(-\Delta_{g_{ab}=\rho^2\delta_{ab}}) = \exp\left\{\frac{D}{48\pi}\int d^2\xi \left[\frac{(\partial_a\rho)^2)}{\rho^2} + \left(\lim_{\varepsilon\to 0}\frac{D}{4\pi\varepsilon}\right)\rho^2\right]\right\} \tag{9.6-d}$$

The functional integration on the intrinsic metric field is well-known with infinitesimal coordinate transformation $\{\in_a(\xi_1,\xi_2)\}$ around the conformal orbit (i.e., $\nabla^c_{g_{ab}=\rho^2\delta_{ab}} \cdot \in_c = 0$)

$$\|\delta g_{ab}\|^2 = (1+2c)\int d^2\xi\, \delta\rho(\xi)\delta\rho(\xi) + \int d^2\xi \sqrt{g}\phi_a^b\phi_b^a \tag{9.7-d}$$

Here

$$\phi_{ab} = (\nabla_a \in_b + \nabla_b \in_a)_{g_{ab}=\rho^2\delta_{ab}} \tag{9.8-d}$$

From eq.(9.7-d) we derive the correct integration measure in terms of the Feynman measures, denoted by the symbol $D^F(\cdot) = \prod_\xi d(\cdot)$

$$[Dg_{ab}(\xi)] = D^F[\rho(\xi)]D^F[\in_a(\xi)](det^{1/2}\mathcal{L}) \tag{9.9-d}$$

Here the Polyakov's operator $\mathcal{L}$ is obtained from eq.(9.7-d) and given by

$$(\mathcal{L} \in)_a = \nabla^b(\nabla_a \in_b + \nabla_b \in_a)|_{g_{ab}=\rho^2\delta_{ab}} \tag{9.10-d}$$

and its functional determinant was exactly evaluated (acting on smooth C^∞ compact support vector-sections on S)

$$-\frac{1}{2}\log\det\mathcal{L} = \frac{13}{6\pi}\int_\xi \left(\left[\frac{1}{2}\frac{(\partial_a\rho)^2}{\rho^2}\right] + \int_\xi \left(\lim_{\varepsilon\to 0}\frac{2}{4\pi\varepsilon}\right)\rho^2(\xi)\right)d^2\xi \tag{9.11-d}$$

By combining eq.(9.6-d) with eq.(9.11-d) and eq.(9.3-d), we obtain the partition function for the closed surfaces *defined in terms of the natural conformal quantum degrees of freedom* $\rho(\xi_1,\xi_2)$

$$Z=\int D^F[\rho(\xi)]\exp\left(-\frac{(26-D)}{12\pi}\int_\xi\left[\frac{1}{2}\frac{(\partial_a\rho)^2}{\rho^2}\right]+\int_\xi\mu_R^2\rho^2\right) \tag{9.12-d}$$

This expression shows the origin of the commonly known critical dimension 26 in the string theory: at this value of the dimension one does not have dynamics for the metric field $g_{ab}(\xi)=\rho^2(\xi)\delta_{ab}$. However for $D<26$ one must examine the "σ-model like" in eq.(9.12-d) which is not the Liouville field theory as originally stated by A.M. Polyakov because the natural theory's dynamical variable in this framework is the scalar field $\rho(\xi)$ instead of that proposed initially by Polyakov $2lg\,\rho(\xi)=\varphi(\xi)$. These above cited 2D-theories coincides only for very weak fluctuations around the 2D-flat metric $\rho(\xi)=1+\varepsilon\overline{\rho}\,(\varepsilon\to 0)$ in our opinion.

Note that the quantum field equation associated to the obtained effective partition functional is given by (the *the two-dimensional* effective Einstein equations for this induced 2D-gravitation!)

$$(\partial^a\partial_a)\rho(\xi)=\frac{12\pi\mu_R^2}{(26-D)}(\rho(\xi))^3+\frac{12\pi}{(26-D}\frac{(\partial_a\rho)^2}{\rho^2}(\xi) \tag{9.13-d}$$

Note that our σ-model like (Euclidean) lagrangian (with $\mu_R^2=\mu_{bare}^2+\lim_{\in\to 0^+}\frac{(2-D)}{4\pi\varepsilon}$) describing the closed random surface sum

$$\mathcal{L}(\rho,\partial_a\rho)=\frac{26-D}{12\pi}\int_\xi\left[\frac{1}{2}\partial_a\left(\frac{1}{\rho}\right)\partial_a(\rho)\right](\xi)d^2\xi+\mu_R^2\int_\xi\rho^2(\xi)d^2\xi \tag{9.14-d}$$

does not possesses in principle a full conformal symmetry as a consequence of the correct variable to be quantized. It is worth remark that even in the original Polyakov's work the symmetry which remains after specification of the conformal gauge are the conformal transformation of the ξ-domain $|\frac{dw}{dz}|^2=1$ for $\phi(\overline{z})$ defined as a scalar field. We *conjecture* that the only phase in which the 2D-quantum field theory makes sense is its perturbative phase around the "flat" configuration $\rho^2(\xi)=1+\frac{1}{D}\rho_q^{-2}(\xi)$ in a $\frac{1}{D}$-expansion of other suitable classical $\rho_{cl}(\xi)$ solution of eq.(9.13-d) $\rho^2(\xi)=\rho_{cl}^2(\xi)+\frac{1}{D}\rho_q^{-2}(\xi)$.

The intercept point probabilities (the scalar N-scattering amplitude) in this random surface theory is straightforwardly reduced to the average

$$\begin{aligned}A^{(\delta)}(p^1,\cdots,p^N)&=(\delta)^{(\sum_{i=1}^N p_i^2)}\int_\xi\Pi_{i=1}^N d^2\xi_j\,(\Pi_{i<j}^N|\xi_i-\xi_j|^{p^i\cdot p^j})\\&\times\int D^F[\rho]e^{-\mathcal{L}(\rho,\partial_a\rho)}\,(\Pi_{i=1}^N[\rho(\xi_j)]^{+2(1-p_i^2)})\end{aligned} \tag{9.15-d}$$

It is possible to show that only for (Euclidean) values of external momenta $1-p_i^2=-1,-2,\cdots$ or $p_i^2=0,-1,-2,\cdots$, and suggesting, thus, to a spectrum without the usual lowest state being a tachyon.

So, our main conclusion is that the summation of Bosonic random surface understood as 2D-induced quantum gravitation as originally proposed by A.M. Polyakov is reduced to a massive σ-model scalar field lagrangean obtained in eq.(9.12-d), and not to the Liouville somewhat ill-defined $2D$-quantum model as originaly put forward by A.M. Polyakov. Note that the simplest supersymmetric version of the Bosonic Quantum Field eq.(9.12-d) describes the sum of fermionic random surfaces with critical dimension $D = 10$ and will be analyzed in the next section.

Let us finally point out that there is a formal propose to describe the closed random surface partitional functional eq.(9.12-d) by means of Liouville-Polyakov degree of freedom $\phi(\xi) = 2lg\rho(\xi)$ which has the advantages of taking into account directly in the path integral *the positivity* of the quantum field $\rho(\xi)$. The important formal step in this study is the variable functional change

$$D^F[\rho(\xi)] = \Pi_\xi\, d[e^{\frac{\phi}{2}(\xi)}] = \Pi_\xi(\det(e^{\frac{\phi}{2}})(\xi))\, d(\phi(\xi)) \tag{9.16-d}$$

Unfortunately the *functional Jacobian* $\det(e^{\frac{\phi}{2}})$ does not makes sense as a functional change of functional measures. However, one can propose a definition for the above cited Jacobian as in the original Fujikawa's "hand-wave" prescription to handle the axial anomaly as follows:

$$\begin{aligned}
&\det{}_F[(e^{\frac{\phi}{2}})(\xi)] = \lim_{\varepsilon\to 0^+} \exp Tr_{(\xi)}\, [lg(e^{\frac{\phi}{2}})(\xi) e^{-\varepsilon\Delta_{g_{ab}} = e^{\phi}\delta_{ab}}] = \\
&\lim_{\varepsilon\to 0^+} \exp\left\{\int d^2\xi\, e^{\phi(\xi)} \frac{\phi}{2}(\xi) \left[\frac{1}{4\pi\varepsilon} - \frac{1}{12\pi}(e^{-\phi}\Delta\phi)\right](\xi)\right\} = \\
&\exp\left\{\frac{1}{48\pi}\int_\xi \left[\frac{1}{2}(\partial_a\phi)^2\right]\right\} \exp\left\{\frac{1}{8\pi\varepsilon}\int_\xi e^{\phi(\xi)}\phi(\xi)\right\}
\end{aligned} \tag{9.17-d}$$

By analyzing eq.(9.17-d) we feel that is not sound as it stands since 1) one could use other regularizing operator as that one of eq.(9.17-d); 2) the term in front of kinetic term for the Liouville weight *decreases* and leading to a new (incorrect) critical dimension for string theory, etc... Anyway eq.(9.17-d) deserves further studies and will be left to our readers.

Appendix E.
On Polyakov's Fermionic String Path Integral - Revisited

In the last section of our chapter we review the original paper by A.M. Polyakov (Quantum Geometry of Fermionic Strings (Phys. Lett. 103B, 211, 1981) with *corrections and improvements* on the concepts exposed there.

In this previous Appendix D, we have clarified and improved the Polyakov's procedure for quantizing Bosonic strings as 2D quantum gravity models by a carefull analysis of the involved path-integrals.

Let us begin from the supersymmetric extension of the Bose string (quantum gravity!)

lagrangean.

$$S = \frac{1}{2\pi\alpha'}\left\{\int d^2\xi\left[\frac{1}{2}\sqrt{g}g^{\alpha\beta}\partial_\alpha X^A\partial_\beta X_A + \frac{1}{2}\overline{\psi}^A(i\gamma^\alpha\partial_\alpha)\psi_A\right.\right.$$
$$\left.\left.\overline{\chi}_\alpha\gamma^\beta\gamma^\alpha(\partial_\beta\chi^A + \frac{1}{2}\chi_\beta\psi^A)\psi_A\right]\right\} + \mu\int_D d^2\xi\sqrt{g}(\xi) \quad (9.1\text{-e})$$

Here, the surface is parametrized by $X_A = X_A(\xi)$, $(A = 1\cdots D)$; ψ^A is a ξ-two component Majorana spinor, $g_{\alpha\beta}(\xi)$ is a metric tensor and χ_α is a spinor gravitino field. The Polyakov's strategy as exposed in the previous paper, was to integrate out the χ^A and ψ^A fields firstly and, then, he has examined the resulting theory of "induced ξ-supergravity". By choosing the "super-conformal" gauge

$$g_{\alpha\beta}(\xi) = \rho^2(\xi)\delta_{\alpha\beta}\,;\,\chi_\alpha(\xi) = (\gamma_\alpha\chi)(\xi) \quad (9.2\text{-e})$$

Polyakov has showed that the only expression which satisfies all ξ-supersymmetries not destroyed by the super-conformed gauge eq.(9.2-e) is the direct supersymmetric extension of the Bosonic action given in Appendix D, namely

$$e^{-W} = \int D\psi^A DX_a e^{-S} \quad (9.3\text{-e})$$

In terms of the original fields $\rho(\xi)$ and $\chi(\xi)$, the component form of eq.(9.3-e) can be (correctly) rewritten as (with $2\pi\alpha' = 1$)

$$W[\rho,\chi] = \frac{10-D}{8\pi}\int[\frac{1}{2}\left(\frac{\partial_\xi\rho}{\rho}\right) + \left[\frac{1}{2}i\overline{\chi}(\gamma\partial)\chi + \frac{1}{2}\mu(\overline{\chi}\gamma_5\chi)\rho + \frac{1}{2}\mu^2\rho^2\right](\xi)d^2\xi \quad (9.4\text{-e})$$

Note that in the usual Liouville field parametrization the induced 2D-supergravity is written as $(\rho = e^{\varphi/2})$

$$W[\varphi,\chi] = \frac{10-D}{8\pi}\int_\xi\left[\frac{1}{2}(\partial W)^2 + \frac{1}{2}i\overline{\chi}(\gamma\partial)\chi + \frac{1}{2}\mu(\overline{\chi}\gamma_5\chi)e^\varphi + \frac{1}{2}\mu^2 e^{2\varphi}\right](\xi) \quad (9.5\text{-e})$$

At this point it is worth remark that the intrinsic fermionic degrees of freedom in eq.(9.5-e) may be easily integrated out with the following result: [if one considers $\chi(\xi)$ as an usual 2D-Dirac fermion field]

$$\int D^F[\chi(\xi)D^F[\overline{\chi}(\xi)]\exp\left\{-\left(\frac{10-D}{8\pi}\right)\int_\xi d^2\xi\left[\frac{1}{2}i\overline{\chi}(\gamma\partial)\chi + \frac{1}{2}\mu(\overline{\chi}\gamma_5\chi)\rho\right]\right\}$$
$$= \det\left[i\gamma\partial + \frac{1}{2}\mu\gamma_5\rho\right] = I(\rho) \quad (9.6\text{-e})$$

At this point, we note that (after introducing the notation $\sigma_+ = \beta\left(\frac{1+\gamma_5}{2}\right)\overline{\beta}$ and $\sigma_- = \beta\left(\frac{1-\gamma_5}{2}\right)\overline{\beta}$, we have the μ-expansion $(\mu << 1)$

$$I(\rho) = \sum_{n=0}^{\infty}\frac{(-\frac{1}{2}\mu)^n}{n!}\int d^2\xi_1\cdots d^2\xi_n$$
$$\int D\beta D\overline{\beta}\exp\left[-\frac{1}{2}\int_\xi(\overline{\beta}(i\gamma\partial)\beta)\right]$$
$$(\sigma_+\rho - \sigma_-\rho)(\xi_1)\cdots(\sigma_+\rho - \sigma_-\rho)(\xi_n) \quad (9.7\text{-e})$$

and it is a result of a well-known theorem on 2D-Fermionic model's that the only non zero terms of eq.(9.7-e) are those with equal number of $\sigma_{+}^{`}s$ and $\sigma_{-}^{`}s$. We get, thus, that eq.(9.6-e) becomes the bosonized path-integral below written

$$I(\rho) = \int D^F[a(\xi)] \exp\left(-\frac{1}{2}\int d^2\xi (\partial a)^2(\xi)\right) \exp\left(-\int d^2\xi \left[\frac{1}{2}\mu e^{\Delta(0)} \frac{1}{4\pi} \operatorname{sen}(\sqrt{4\pi a} + \rho)\right](\xi)\right) \quad (9.8\text{-e})$$

where the (bare) ξ-cosmological constant μ (gets a multiplicative ultraviolet) renormalization $\mu_R = \frac{1}{2}\mu(\varepsilon)^{-\frac{1}{2\pi}}$.

As a final comment let us use as dynamical degrees of freedom the Polyakov's original conformal factor $\varphi(\xi) = lg\,\rho(\xi)$. In terms of this variable the bosonized theory's path integral is written as

$$\begin{aligned} Z = &\int D^F[e^{\varphi(\xi)}] \exp\left\{-\frac{1}{2}\int_\xi \left[(\partial\varphi)^2(\xi) + \mu^2\left(\frac{10-D}{8\pi}\right) e^{2\sqrt{\frac{8\pi}{10-D}}\varphi}\right](\xi)\right\} \\ &\times \left\{\int D^F[a(\xi)] \exp\left(-\frac{1}{2}\int d^2\xi (\partial a)^2(\xi)\right)\right. \\ &\left.\exp\left(-\int d^2\xi \left[\frac{1}{8\pi}\mu_R \sin\,(\sqrt{4\pi a}) e^{\sqrt{\frac{8\pi}{10-D}}\varphi}\right](\xi)\right)\right\} \end{aligned} \quad (9.9\text{-e})$$

It is worth to note that the one must use as the Feynman product measure that written in eq.(9.9-e) $\Pi_\xi(e^{\varphi(\xi)}d\varphi(\xi))$ since the associated functional (ξ-covariant) functional metric is given by

$$\|\delta g_{ab}\|^2 = \int_\xi (e^{2\varphi(\xi)}\delta\varphi\cdot\delta\varphi)(\xi)d^2\xi = \int_\xi [\delta(e^\varphi)\delta(e^\varphi)](\xi)d^2\xi \quad (9.10\text{-e})$$

Note that only for weak intrinsic metric fluctuations (or for $D = 10-\varepsilon$) $e^{\varphi(\xi)}$ may be replaced directly by $\varphi(\xi)$ inside the Feynman product measure as it was supposed in Polyakov's original propose

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Chapter 10

A Random Surface Membrane Wave Equation for Bosonic Q.C.D. $(SU(\infty))$

10.1. Introduction

In last decades, representations for Quantum Chromodynamics as extended objects have been pursued by several authors ([1], [2], [3], [4]). Among these, the representation of the meson wave functional by the quantum amplitude of a closed trajectory of a colored particle in the vacuum of a pure Yang-Mills field has strongly suggested the equivalence between Bosonic – non supersymmetric QCD $(SU(\infty))$ and a dynamic of strings ([1], [2]).

In this chapter, we propose to replace the one-dimensional closed trajectory in the above quantum amplitude by a two-dimensional random surface possessing color degrees as another collective non-perturbative variable for probing non-pertubative structures on $Q.C.D(SU(\infty))$. Thus, we deduce (formally) its associated surface wave equation in the t'Hooft topological limit of large number of colors $N_c = +\infty$. This study is presented in section 2. On the section 3 we suggest a path-integral argument on the connection of our proposed Random Surface Wave functional and the Path-Integral Partition Functional of the usual (Bosonic) Yang-Mills Gauge theory. Finally on Section 4 and Appendix B, we make some comments on previous work on the subject and on the regularization program.

10.2. The Random Surface Wave Functional

Let us start our analysis by considering the problem of associating a wave functional for a random surface Σ possessing $SU(N)$ color degrees of freedom interacting with an external quantized Yang-Mills field $A_\mu(X)$, the most simple geometrical gauge-invariant generalization of the usual Wilson Loop variable for Q.C.D.

The colored random surface is characterized by two fields: first, by the usual (bosonic) vector position $X_\mu(\xi)$, $\xi \in D$ ($\mu = 1,\ldots,D$, where D is the space-time dimension), and second, by the random surface color variable $g(\xi)$ which is an element in the fundamental representation of the $SU(N)$ group. Here, we have fixed the two-dimensional flat domain D to be the rectangle

$$D_{|0,2\pi|\times|0,T|} = \{(\xi_0,\xi_1); \quad 0 \leq \xi_0 \leq 2\pi \quad \text{and} \quad 0 \leq \xi_1 \leq T\}.$$

The classical action for this membrane is naturally given by ([5], [6], [7]).

$$S = S_0 + S_1^{(B)} \tag{10.1}$$

with

$$S_0 = \frac{1}{2}\int_D d^2\xi(\partial_a X^\mu \partial_a X^\mu)(\xi) \tag{10.1a}$$

$$S_1^{(B)} = \frac{1}{4\pi m}\int_D T_R^{(c)}(g^{-1}\partial_a g)^2(\xi)d^2\xi + 4\pi i\Gamma_{WZ[g]}, \tag{10.1b}$$

where $\Gamma_{WZ[g]}$ denotes the two-dimensional Wess-Zumino functional. Its existence, together with the integer m in the above written σ-model on the action of $g(\xi)$'s afford us to consider the bosonized fermionic equivalent action

$$S_1^{(F)} = \int_D \psi(\xi)(i\gamma_a\partial_a)\overline{\psi}(\xi)d^2\xi, \tag{10.2}$$

where the two-dimensional Dirac field $\psi(\xi)$ belongs to the fermionic fundamental $SU(N)$ representation.

At this point, the simplest action taking into account the interaction with the external non-Abelian field is given by

$$S^{\text{int}}[\psi(\xi);A_\mu(X)] = \int_D \overline{\psi}(\xi)(\gamma_a\partial^a X^\mu(\xi)A_\mu(X(\xi))\psi(\xi)d^2\xi \tag{10.3}$$

The complete classical interacting action (eqs. (10.1a), (10.2) and (10.3)) is invariant under the gauge transformations

$$\begin{aligned} A_\mu(X_\mu(\xi)) &\to (h^{-1}A_\mu h + h^{-1}\partial_\mu h)(X_\mu(\xi)) \\ \psi(\xi) &\to h(X_\mu(\xi))\psi(\xi) \\ \overline{\psi}(\xi) &\to \overline{\psi}(\xi)h^{-1}(X_\mu(\xi)) \end{aligned} \tag{10.4}$$

Before turning to the construction of a quantum wave functional for the above system, it is instructive to remark that eqs. (10.1a), (10.2) and (10.3) are the random surface generalizations of the analogous formulae in the one-dimensional string case, where the colored string is described by the position vector $X_\mu(\sigma)$ and the one-dimensional complex fermion (Grassmanian) field $\{\theta(\sigma),\theta^\star(\sigma)\}$ in the $SU(N)$ fundamental representation. The associated action is

$$\begin{aligned} &S[X_\mu(\sigma),\theta(\sigma),\theta^\star(\sigma),A_\mu(X_\mu(\sigma))] \\ &= \int_0^T \frac{1}{2}\dot{X}_\mu(\sigma)^2 d\sigma + \int_0^T \theta^\star(\sigma)\dot{\theta}(\sigma) + \int_0^T \dot{X}^\mu(\sigma)A_\mu^I(X(\sigma))(\theta(\sigma)\lambda_I\theta^\star(\sigma))d\sigma \end{aligned} \tag{10.5}$$

where $\{\lambda_I\}$ denotes the Hermitian generators of the $SU(N)$ Lie algebra.

In this string case, a quantum wave functional is given by the following path integral[(1)]

$$\begin{aligned} &W[X_\mu(\sigma),A_\mu(X)] \\ &= \int d[\theta(\sigma)]d[\theta^\star(\sigma)]\sum_{\alpha=1}^{N^2-1}\theta_\alpha(0)\theta_\alpha^*(T)\exp\{-S[X_\mu(\sigma),\theta(\sigma),\theta^\star(\sigma),A_\mu(X_\mu(\sigma))]\} \end{aligned} \tag{10.6}$$

which leads to the well-known Wilson Loop factor defined by the closed string $\{X_\mu(\sigma)\}$. The complete quantum wave functional is defined by the average $\langle W[X_\mu(\sigma), A_\mu(X)]\rangle$ where $\langle\ \rangle$ denotes the partition functional of the pure Yang-Mills theory ([1]).

We shall now use eq.(10.6) to propose the following functional integral as a quantum wave functional for a $SU(N)$ colored random surface Σ interacting with the quantum vacuum of a $SU(N)$ Yang-Mills theory.

$$\begin{aligned}\mathrm{Tr}^{\mathrm{color}}(\psi[\Sigma]) \overset{\mathrm{def}}{=} \sum_{R=1}^{N^2-1} \int d[\psi(\xi)]d[\overline{\psi}(\xi)](\overline{\psi}(0,0)\frac{\lambda^R}{N_c}\psi(2\pi,0)) \\ \times \exp\{-S[X_\mu(\xi), A_\mu(X(\xi),\psi(\xi))]\}.\end{aligned} \tag{10.7}$$

Notice that our above proposed random surface phase factor $\mathrm{Tr}^{\mathrm{color}}(\psi[\Sigma])$ is a 2×2 matrix in the flat domain $D(a=1,2)$.

In order to deduce a closed wave functional for the quantum average $\langle \mathrm{Tr}^{\mathrm{color}}(\psi[\Sigma])\rangle$ in the limit $N_c = +\infty$, we proceed as in the string case[1,2] by shifting the $A_\mu(X)$ field variable, which by its turn, produces the following result ($\lambda_0^2 = \lim_{N^c\to\infty}(g_0^2 N_c) < \infty$)

$$\begin{aligned}&\frac{1}{4\lambda_0^2}\langle \mathrm{Tr}^{\mathrm{color}}\{(D_\mu F_{\mu\nu})(X)\psi[\Sigma]\}\rangle \\ &= \int_D \delta^{(D)}(X - X_\mu(\sigma,\xi))\partial_c X^\mu(\sigma,\tau)\langle \mathrm{Tr}^{\mathrm{color}}\psi[\Sigma_1]\rangle\langle \mathrm{Tr}^{\mathrm{color}}\gamma^{(c)}\psi[\Sigma_2]\rangle,\end{aligned} \tag{10.8}$$

where the split membranes $\Sigma_{(1)}$ and $\Sigma_{(2)}$ are respectively defined by the restriction of the mapping $X_\mu(\xi_1,\xi_2)$ for the (split) domains

$$D_{(1)} = \{(\xi_0,\xi_1);\ 0\le\xi_0\le\sigma;\ 0\le\xi_1\le T\}$$

and

$$D_{(2)} = \{(\xi_0,\xi_1)\mid \sigma\le\xi_0\le 2\pi;\ 0\le\xi_1\le T\}.$$

It is now convenient to multiply both sides of eq.(10.8) by the membrane current density

$$J_{(a)}(X) = \delta^{(D)}(X - X_\mu(\overline{\sigma},\overline{\tau})\partial_a X^\mu(\overline{\sigma},\overline{\tau})$$

and integrate out the result relative to the space-time variable X. So, we get the result

$$\begin{aligned}&\langle \mathrm{Tr}^{\mathrm{color}}\{(D_\mu F_{\mu\nu})(X^\mu(\overline{\sigma},\overline{\tau}))\partial_a X^\mu(\overline{\sigma},\overline{\tau})\psi[\Sigma]\}\rangle \\ &= 4\lambda_0^2\int_D \delta^{(D)}(X_\mu(\overline{\sigma},\overline{\tau}) - X_\mu(\sigma,\tau))\partial_c X^\mu(\sigma,\tau)\partial_a X^\mu(\overline{\sigma},\tau) \\ &\quad \times \langle \mathrm{Tr}^{\mathrm{color}}\psi[\Sigma_{(1)}]\rangle\gamma^{(c)}\langle \mathrm{Tr}^{\mathrm{color}}\psi[\Sigma_{(2)}]\rangle.\end{aligned} \tag{10.9}$$

In order to write the left-hand side of the above result in a form similar to the random surface wave equation of ref. [11] we use the relations

$$\left\{\frac{\delta}{\delta X_\mu(\sigma,\tau)}\right\}\mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma)) = \mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma_1)F_{\mu\nu}(X(\sigma,\tau))\partial_c X^\nu(\sigma,\tau)\gamma^{(c)}\psi(\Sigma_2)), \tag{10.10a}$$

$$P_F\left\{\frac{\delta^2}{\delta X_\mu(\overline{\sigma},\overline{\tau})\delta X^\mu(\sigma,\tau)}\right\}\mathrm{Tr}^{\text{color}}(\psi(\Sigma))$$
$$= \mathrm{Tr}^{\text{color}}(D_\mu F_{\mu\nu}(X_\mu(\overline{\sigma},\overline{\tau}))\partial_c X^\nu(\sigma,\tau)\gamma^{(c)}\psi(\Sigma)), \tag{10.10b}$$

where the derivative-finite part operations is given by ([1]).

$$P_F\left\{\frac{\delta^2}{\delta X_\mu(\overline{\sigma},\overline{\tau})\delta X^\mu(\overline{\sigma},\overline{\tau})}\right\}$$
$$\equiv \lim_{\varepsilon\to 0^+}\int_{-\varepsilon}^{\varepsilon} d\beta \frac{\delta^2}{\delta X_\mu(\overline{\sigma}+\beta,\overline{\tau}+\beta)\delta X^\mu(\overline{\sigma}-\beta,\overline{\tau}-\beta)}. \tag{10.10c}$$

By substituting eq.(10.10b) into eq.(10.9), we obtain our proposed random surface version of the string Migdal-Makkenko wave equation (compare with eq.(10.9), ref. 2, and eq.(10.7), ref, 4).

$$P_F\left\{\frac{\delta^2}{\delta X_\mu(\overline{\sigma},\overline{\tau})\delta X^\mu(\overline{\sigma},\overline{\tau})}\right\}\langle \mathrm{Tr}^{\text{color}}(\psi(\Sigma))\rangle$$
$$= 4\lambda_0^2\int_D \delta^{(D)}(X_\mu(\overline{\sigma},\overline{\tau})-X_\mu(\sigma,\tau))\partial_b X^\mu(\sigma,\tau)\partial_c X^\mu(\overline{\sigma},\overline{\tau})$$
$$\langle \mathrm{Tr}^{\text{color}}\gamma^{(b)}\psi[\Sigma_{(1)}]\rangle\langle \mathrm{Tr}^{\text{color}}\gamma^{(c)}\psi[\Sigma_{(2)}]\rangle. \tag{10.11a}$$

To summarize, we propose a continuum random surface version of the string Migdal-Makkenko loop wave equation in $SU(\infty)$, which we hope to open a new path to understand the non-pertubative structure of Quantum Chromodynamics as a dynamics of random surfaces as much sucessuful studies implemented in Loop Space approach for Quantum Gravity ([11]).

10.3. A Connection with $Q.C.D(SU(\infty))$

In this section we present an path-integral argument connecting our proposed random surface wave functional eq.(7) to the $Q.C.D(SU(\infty))$, thus, showing the usefulness of our propose on Section 10.2.

In order to achieve such goal, let us consider the quantum vaccum of the Yang-Mills theory as an ensemble of random $SU(N)$ connections with *an uniform distribution* interacting with the random surface Σ constraint to remains on the sphere S^{D+1} on R^D. Formally one is considering the strong bare coupling $g^2_{\text{bare}}\to\infty$ vaccum limit on the Yang-Mills quantum average and the random surface rigid limit $X_\mu(\xi)=\overline{X}_\mu+\sqrt{\alpha'}Y_\mu(\xi)$, with $\alpha'\to 0$ denoting the physical observable Regge slope constant, namely:

$$\langle \mathrm{Tr}^{\text{color}}(\psi(\Sigma))\rangle_{g^2\to\infty} = \int (\Pi_{(\overline{X},\mu,a)} dA^a_\mu(\overline{X}))^{\text{Haar}}\int D^F[X^\mu(\xi)]$$
$$\times\exp\left[-\frac{1}{2}\int_{R^2} d^2\xi(\partial_A X^\mu\partial^A X_\mu)(\xi)\right]\delta^{(F)}((X_\mu X^\mu(\xi)-1)$$
$$\times\int D^F[\psi_a,\overline{\psi}_a]\exp\left[-\frac{1}{2}\int_{R^2} d^2\xi(\overline{\psi}(i\gamma^A\partial_A)\psi)(\xi)\right]$$
$$\times\exp\left[ie\int_{H^2} d^2\xi\left[A^i_\mu(X^\beta(\xi))(\overline{\psi}_a\gamma^A(\lambda_i)_{ab}\psi_b)(\xi)(\partial_A X^\mu)(\xi)\right]\right]. \tag{10.11b}$$

In order to connect eq.(10.11b) with $Q.C.D(SU(\infty))$, we consider the "Harmonic gauge" fixing in the Haar-Yang-Mills path integral in eq.(10.11b), namely $(X_\mu(\xi)-\overline{X}_\mu)\cdot A_\mu(X^\beta(\xi))=0$, which allow us in its turn to rewrite the interaction term in eq.(10.11b) in terms of the Yang-Mills strength field in the chart $V(\overline{X})$ at large random surface scale $(\alpha'\to 0)$, since in this harmonic gauge we have the expansion $A_\mu(X^\beta(\xi)) = -\frac{1}{2}F_{\mu\nu}(\overline{X}^\alpha)\sqrt{\alpha'}Y^\nu(\xi)+O(\sqrt{\alpha'})$

$$\begin{aligned} I_{V(\overline{X})}[A^\mu_a(\overline{X})] &= \int D^F[Y^\mu(\xi)\exp\left[-\frac{1}{2}\int_{R^2} d^2\xi(\partial_A Y^\mu\partial^A Y_\mu)(\xi)\right] \\ &\times\left(\lim_{\lambda\to\infty}\exp\left[-\langle\lambda\rangle\int_{R^2} d^2\xi[(Y^\mu Y_\mu)(\xi)-1]\right]\right) \\ &\times\int D^F[\psi_a,\overline{\psi}_a]\exp\left[-\frac{1}{2}\int_{R^2} d^2\xi(\overline{\psi}(i\gamma^A\partial_A)\psi)(\xi)\right] \\ &\times\exp\left[-ie\alpha'\int_{R^2} d^2\,\xi\,\frac{1}{2}Y^P(\xi)F^i_{\rho\mu}(\overline{X}^\alpha)(\overline{\psi}_a\gamma^A(\lambda_i)_{ab}\psi)(\xi)(\partial_A Y^\mu)(\xi)\right]. \end{aligned} \tag{10.12}$$

Note that we have used the condensate Polyakov approximation ([1]) for the functional delta inside eq.(10.11), expected to hold true in the limit of $\alpha'\to 0$ and effectively generating a mass term for the random surface vector position field

$$\begin{aligned} &\delta^{(F)}((X^\mu X_\mu)(\xi)-1) \\ &= \int D^F[\lambda(\xi)]e^{+i\int_{R^2}d^2\xi\lambda(\xi)[(X^\mu X_\mu)(\xi)-1]} \\ &\sim \lim_{\langle\lambda\rangle\to\infty}\left\{e^{+i\int_{R^2}i\langle\lambda\rangle_{\text{conden}}[(X^\mu X_\mu)(\xi)-1]d^2\xi}\right\} \\ &\sim \lim_{\langle\lambda\rangle\to\infty} e^{-\langle\lambda\rangle_{\text{conden}}\int_{R^2}d^2\xi[(X^\mu X_\mu)(\xi)]} \end{aligned} \tag{10.13-a}$$

At this point we evaluate the $Y_\mu(\xi)$-Gaussian functional integral with the exact result

$$\begin{aligned} &I_{V(\overline{X})}[A^\mu_a(\overline{X})] \\ &= \lim_{\langle\lambda\rangle\to\infty}\left\langle \det{}^{-\frac{1}{2}}\left[(-\partial^2)_\xi\eta^{\mu\nu}(\overline{X})+\frac{1}{2}(F^{\mu\nu}_i(\overline{X})j^i_a(\xi))\partial^a_\xi+\langle\lambda\rangle\right]\right\rangle_{\psi,\overline{\psi}} \end{aligned} \tag{10.13-b}$$

where $\langle,\rangle_{\psi,\overline{\psi}}$ denotes the functional integral over the $SU(N)$ string intrinsic Dirac fields and $j^i_a(\xi)$ is the conserved fermion $SU(N)$ current on the random surface sheet.

At the condensate value $\langle\lambda\rangle\to\infty$, we obtain the following result for eq.(10.13)

$$I_{V(\overline{X})}[A^a_\mu(\overline{X})]\sim\left\langle\exp\left[-\frac{1}{16\pi}F^i_{\mu\nu}(\overline{X})F^\rho_{\mu\nu}(\overline{X})(j^a_i(\xi)j^a_\rho(\xi))\right]\right\rangle_{\psi,\overline{\psi}} \tag{10.14}$$

which at large N, give us the final result depending only the "infinite-tensioned random surface macroscopic space-time fixed vector position $\overline{X}$"

$$I_{V(\overline{X})}[A^a_\mu(\overline{X})]_{(N\to\infty)} = \exp\left[-\left(\frac{\langle\int d^2\xi j^a_i(\xi)j^a_i(\xi)\rangle^{(N\to\infty)}_{\psi,\overline{\psi}}}{16\pi}\right)F^i_{\mu\nu}(\overline{X})F^i_{\mu\nu}(\overline{X})\right]. \tag{10.15}$$

The complete path integral equation (10.15) is, thus, exactly the $SU(\infty)$ Yang-Mills quantum field path integral for the space-time at large random surface scale (after integrating out the space-time macroscopic surface space-time point $\overline{X}$)

$$\int D^F[A^a_\mu(\overline{X})]\exp\left[-\frac{1}{16g^2_{QCD}}\int d^D\overline{X}(F^2_{\mu\nu}(\overline{X}))\right]=\sum_{\text{(membranes)}}\left\{<\mathrm{Tr}^{\text{color}}(\psi(\Sigma))>^{N_c\to\infty}_{g^2_{\text{bare}}\to 0}\right\} \quad (10.16)$$

Note that the $QCD_{N_c=+\infty}$ coupling constant is expressed in terms of the intrinsic Fermion fields in an explicitly form

$$(g_{QCD})^2=\frac{1}{\pi}\left\langle\int d^2\xi\, j^a_i(\xi)\, j^a_i(\xi)\right\rangle^{(N\to\infty)}_{\psi,\overline{\psi}} \quad (10.17)$$

Appendix A. Rank Two Antisymmetric Path-Integrals the Q.C.D String: Some Comments

The most important problem in the present days of theoretical and mathematical physics is how to quantize correctly Non-Abelian Gauge Field Theories defined on the physical continuum space-time. The only result in this direction still remains a somewhat formal Ansatz from the experimental and theoretical point of view of the use of the Higgs mechanism. Probably, this Ansatz is formal from a strict quantum field theoretic point of view since its makes heavier use of a trivial $\lambda\phi^4$-field theory in four dimensions and of the associated gauges of t'Hooft for the Yang-Mills Fields (see the comments on pag. 38 the J.C. Taylor book "Gauge theories of weak interactions - Cambridge Monographs on Mathematical Physics). However, it was realized by K. Wilson that in the Ising like euclidean path integral crude approximation framework (Lattice Gauge Theory) theses non-abelian gauge field theories in the lattice at a bare strong coupling regime are naturally expressed in terms of the *Euclidean* Wilson Loops defined by the matter content trajectories $C=\{X_\mu(\sigma);\ 0\le\sigma\le 1$ $\sigma=$proper-time parameters$\}$

$$W[C]=Tr\mathbb{P}\left\{\exp\left[+i\oint_C A_\mu dX^\mu\right]\right\}. \quad (10.1\text{-A})$$

Note that typical interaction energy densities, such as $\overline{\psi}\psi,\overline{\psi}\gamma^5\psi,\overline{\psi}\gamma^\mu\psi A_\mu$ which are real function (distributions) in the Minkowski space-time are complex on the Euclidean world.

It was argued on ref. [10] by A.M. Polyakov, an euclidean string functional integral Ansatz for eq.(10.1-A) based on a coupling of an abelian rank-two antisymmetric tensor field $B_{\mu\nu}(x)$ (the Polyakov's axion field) with the string orientation area tensor previously proposed by this author but with an important difference: This rank-two antisymmetric tensor field B has a non trivial dynamic content. Namely (see eq.((10.12)-(10.15)) - ref. [10]).

$$W[C]=\frac{\int D^F[B_{\mu\nu}]\,e^{-S[B_{\mu\nu}]}\,e^{(i\int_{\Sigma_C}Bd\sigma)}}{\int D^F[B_{\mu\nu}]\,e^{-S[B_{\mu\nu}]}} \quad (10.2\text{-A})$$

where the axion action is given by

$$S(B) = \frac{1}{4e^2}\int d^\nu x(B^2_{\mu\nu} + dB \cdot \text{arc sen}\,\frac{dB}{m^2} - \sqrt{m^4 - (dB)^2}) \tag{10.3-A}$$

At this point we point out that the functional integral weight eq.(10.3-A) makes sense only for those field configurations which makes eq.(10.3-A) a *real number*, namely: $\sup_{x\in R^\nu} |dB(x)| \leq m^2$.

Unfortunately this bound on the kinetic energy of the axion field is impossible for those distributional fields configurations making the domain of the axion functional integral eq.(10.2-A), unless $m^2 \to \infty$ and comments below eq.(10.40) of ref. [10]. (A quantum field may be bounded but not its kinetic energy!).

So, in the deep infrared regime of $Q.C.D(SU(\infty))$ eq.(10.3-A) should turns into a pure White-Gaussian action for the axion field B dominated by almost constant gauge field configurations

$$S[B] \sim \frac{1}{4e^2}\int B^2(x)d^\nu x \tag{10.4-A}$$

One has, thus, the following effective result for the Wilson loop surface dependence in the very low momenta regime

$$W[C] \sim \exp\left[-F(C;\sum_C)\right] \tag{10.5-A}$$

where the surface functional weight is given by the self-avoiding extrinsic action firstly proposed in a minimal area context solution for the Q.C.D-Loop wave equation in ref. [1] with β a (positive) coupling constant

$$F(C,\sum_C) = \beta\int_\Sigma d\sigma_{\mu\alpha}(x)(\delta^{\mu\lambda}\delta^{\alpha\rho}\delta^\nu(x-y))d\sigma_{\lambda\rho}(y). \tag{10.6-A}$$

It is straightforward to see that for fixed constant e^2, the limit $m^2 \to \infty$ leads to a pure Nambu-Goto action strongly coupled ([10])

$$\begin{aligned} F(C,\sum_C) &\sim \lim_{m^2\to\infty} c_1(e^2 m)\int d^2\xi\sqrt{g}(\xi) + \lim_{m^2\to\infty} c_2(e^2/m)\int d^2\xi(\nabla t_{\mu\nu})^2\sqrt{g} \\ &+ O\left(\frac{1}{m}\right) \sim \frac{1}{2\pi\alpha'}\int d^2\xi\sqrt{g}(\xi) \end{aligned} \tag{10.7-A}$$

and by its turn suggesting a random surface wave functional behavior like eq.(10.7) for the quantum averaged $Q.C.D(SU(\infty))$ Wilson loop eq.(10.6)-eq.(10.1-A). in the $Q.C.D(SU(\infty))$ deep infrared regime.

Appendix B.
On the Self-avoiding Membrane Wave Functional

In this appendix we present some comments on the renormalization program to the random surface wave functional associated to the self-avoiding extrinsic reparametrization functional for $QCD(SU(\infty))$ in R^D as given by eq.(10.5-A) of the previous Appendix A

$$F[X^\alpha(\xi)] = \beta \int_\xi \sqrt{h}(X(\xi)) \int_\xi \sqrt{h(X(\xi'))}(\Gamma^{\mu\nu}(X(\xi))\Gamma_{\mu\nu}(X(\xi')) - 1) \times \delta^{(D)}(X^\alpha(\xi) - X^\alpha(\xi')). \tag{10.1-B}$$

With $X^\alpha(\xi)$ denoting the parametrization of the Σ-surface on the surface wave function ansatz eq.(10.5-A).

Here the surface area tensor responsable by the extrinsic properties of $QCD(SU(\infty))$ quantum geometry is explicitly given by

$$\Gamma^{\mu\nu}(X(\xi)) = \frac{(\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu)_{(\xi)}}{\sqrt{h(X(\xi))}} \tag{10.2-B}$$

and the random surface scalar area is written as

$$\sqrt{h(X(\xi))} = \sqrt{\det\{\partial_a X^\mu \partial_b X_\mu\}(\xi)}. \tag{10.3-B}$$

As a first step to analyze eq.(10.1-B), one should extract the pure string world sheet U.V divergence associated to the trivial self-avoiding surface case $X_\mu(\xi) = X_\mu(\xi')$ with $\xi = \xi'$.

Let is follow our study.

Firstly we note that a regularized form for eq.(10.1-B) in the U.V case $\xi = \xi'$ is explicitly given by

$$W_{(\Lambda)}[X(\xi)] = \beta \sum_{p=0}^{\infty} \frac{(-1)^p}{p! 2^{2p} \cdot \Gamma(\frac{D}{2} + p)} \left(\frac{\Lambda^{D+2p}}{D+2p}\right) \cdot \delta_\Lambda(\xi,\xi') \times X \left\{ \int_\xi d_\xi^2 d_{\xi'}^2 \sqrt{h(\xi)}\sqrt{h(\xi')}(\Gamma^{\mu\nu}(X(\xi)\Gamma_{\mu\nu}(X(\xi')) - 1)|X(\xi) - X(\xi')|^{2p} \right\} \tag{10.4-B}$$

with

$$\delta_\Lambda(\xi,\xi') = \begin{cases} \Lambda & \text{if } \begin{cases} \xi_1 - \frac{1}{\Lambda} \le \xi_1' \le \xi_1 + \frac{1}{\Lambda} \\ \xi_2 - \frac{1}{\Lambda} \le \xi_2' \le \xi_2 + \frac{1}{\Lambda} \end{cases} \\ 0 & \text{otherwise} \end{cases} \tag{10.5-B}$$

By considering the taylor expansion around $\xi = \xi'$

$$\Gamma^{\mu\nu}(X(\xi))\Gamma_{\mu\nu}(X(\xi')) - 1 = -(\partial_a \Gamma^{\mu\nu})(\partial_b \Gamma_{\mu\nu}(X(\xi))(\xi - \xi')_a(\xi - \xi')_b + \text{higher terms} \tag{10.6-B}$$

one can see that all reparametrization invariant counter-terms are of the second order derivative on the surface vector position and on the area tensor object namely, at one-loop case $(p \le 1)$; one has the following explicit counter-terms involving the extrinsic geometry (note the subtraction of the pure self-avoiding term in eq.(10.1) which at the level of loop equations means that a non-vanishing Gluon condensate was already taken into account by considering a non-zero Regge slope parameter, i.e., $(2\pi\alpha')^{-1} = \langle 0|F^2|0\rangle \ne 0$:

$$W_1[X(\xi)] \sim \beta(\Lambda)^4 \int_\xi \sqrt{h(\xi)}(\partial_a \Gamma^{\mu\nu} \partial^a \Gamma_{\mu\nu})(\xi)$$

$$W_2[X(\xi)] \sim \beta(\Lambda)^4 \int_\xi \sqrt{h(\xi)}\{(\partial_a \Gamma^{\alpha\beta})(\partial^a \Gamma^{\mu\beta})(\partial_b X_\alpha)(\partial^b X_\mu) + \cdots\}$$

$$W_3[X(\xi)] \sim \beta \int_\xi \sqrt{h(\xi)}\{(\partial^2 \Gamma^{\alpha\beta}(\partial^2 \Gamma_{\alpha\beta})\} \tag{7-B}$$

At this point we consider the extrinsic ultraviolet divergences $X_\mu(\xi) = X_\mu(\xi')$ *but with* $\xi \neq \xi'$.

In the physical situation of line self-intersections, where the equation $X_\mu(\xi) = X_\mu(\xi')$ defines a sub-manifold of dimension 1 (the Σ-surface is generically discribed by the union of vertical surfaces cylinders locally in contact along self-intersecting vertical lines passing through the points $\sigma_j = \{\xi_j^1, \tau\}$ with $X_\mu(\xi_j', \tau) = X_\mu(\xi_{j+1}^1, \tau) \quad 1 \leq j \leq m)$. The resulting random surface wave functional path integral still remains formally renormalizable. In order to show the correctness of this claim, one can see that $\Gamma_{\mu\nu}(X(\sigma_j))\Gamma^{\mu\nu}(X(\sigma_{j+1})) = \cos X(\sigma_j; \sigma_{j+1})$, the constant angle between the extrinsic surface tangent planes possesing the comon self-intersecting non-trivial line $X_\mu(\sigma_j)(\text{or} X_\mu(\sigma_{j+1})!)$. Now it is straightforward to see that the action eq.(10.1-B) reduces to a pure (intrinsic) self-avoiding action of the cylinder surfaces branches with the associated tangent plane above cited. In this simple case one can follow our previous exposed results in ([3]) to show its formal renormalizability as a two-dimensional Quantum Field Theory.

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Chapter 11

Covariant Functional Diffusion Equation for Polyakov's Bosonic String

11.1. Introduction

The attempt to formulate a convariant quantum theory of strings in terms of the line functional has a basic object the string transition amplitude.([1]−[3]) The main idea in this framework is to consider the string world-sheet area playing the role of a proper time. The string propagator, thus, should satisfy a kind of functional diffusion equation in the area space variable.[2]

In this chapter we analyze the associated functional diffusion equation in Polyakov's quantum bosonic string theory by taking into account in an explicit way the theory's conformal anomaly (see Chapters 1 and 19).

11.2. The Covariant Equation

The transition amplitude for an initial (Euclidean) string state

$$\{(x_\mu^{\text{in}}(\sigma), e^{\text{in}}(\sigma)),\ 0 \leq \sigma \leq 1\}$$

propagating to a final string

$$\{(x_\mu^{\text{out}}(\sigma), e^{\text{out}}(\sigma)),\ 0 \leq \sigma \leq 1\}$$

in Polyakov's theory is given by ([1] and Chapters 1 and 19)

$$G[c^{\text{out}}, c^{\text{in}}] = \int d\mu[g_{ab}]d\mu[\phi_\mu]\exp[-I_0(g_{ab}, \phi_\mu, \mu^2, \lambda)], \tag{11.1}$$

where the covariant string action with a cosmological term μ^2 and a "quark-mass" parameter λ is the Brink-Di Vecchia-Howe action [4]

$$I_0(g_{ab}, \phi_\mu, \mu^2, \lambda) = \frac{1}{2}\int_D d\sigma\, d\zeta(\sqrt{g}\, g^{ab}\partial_a\phi^\mu\partial_b\phi^\mu + \mu_0^2) + \lambda_0\int_{\partial D} ds. \tag{11.2}$$

The string surface parameter domain is taken to be the rectangle $D = \{(\sigma,\zeta), 0 \leq \sigma \leq 1, 0 \leq \zeta < T\}$. The covariant functional measures $d\mu[g_{ab}]d\mu[\phi_\mu]$ are defined over all cylindrical (random) surfaces without holes and handles with the string configurations as nontrivial boundaries: i.e., $\phi_\mu(\sigma,) = x_\mu^{\text{in}}(\sigma)$. $\phi_\mu(\sigma,T) = x_\mu^{\text{out}}(\sigma)$.

In order to write an area functional diffusion equation for the string propagator, Eq.(11.1), we rewrite it in a form where the string's world-sheet area plays a role as a string proper time:

$$G[C^{\text{out}},C^{\text{in}}] = \exp\left[-\lambda_0 \int_{C^{\text{in}}} ds - \lambda_0 \int_{C^{\text{out}}} ds\right] \int_0^\infty dA\, e^{-\mu^2 A} \overline{G}[C^{\text{out}},C^{\text{in}},A], \tag{11.3}$$

where $\overline{G}[C^{\text{out}},C^{\text{in}},A]$ is the fixed-area string propagator

$$\overline{G}[C^{\text{in}},C^{\text{out}},A] = \int d\mu[g_{ab}]d\mu[\phi_\mu]\delta\left(\left(\int_D d\sigma\, d\zeta \sqrt{g(\sigma,\zeta} - A\right)\right) \exp[-I_0(g_{ab},\phi_\mu,\mu^2 \equiv 0)]. \tag{11.4}$$

The δ-function constraint in Eq.(11.4) ensures that only the random surfaces with fixed area A contribute.

Let us evaluate the area partial derivative of the area-fixed propagator: namely,

$$\frac{\partial}{\partial A}\overline{G}[C^{\text{in}},C^{\text{out}},A] = -\int d\mu[g_{ab}]d\mu[\phi_\mu]\delta'\left[\int_D d\sigma\, d\zeta\sqrt{g(\sigma,\zeta)} - A\right] \tag{11.5}$$

with $\delta'(x)$ being the first derivative of the δ distribution.

At this point we consider the identity

$$= \delta'\left[\int_D d\sigma\, d\zeta\sqrt{g(\sigma,\zeta)} - A\right] = \lim_{\zeta\to 0^+}\left[\frac{1}{2\sqrt{gg^{00}}}\frac{\delta}{\delta g_{00}}\right](\overline{\sigma},\zeta)\delta\left[\int_D d\sigma\, d\zeta\sqrt{g(\sigma,\zeta)} - A\right] \tag{11.6}$$

which can be easily verified by using the Fourier integral representation for the δ functional and the relationship $\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{00}\delta g_{00}$.

By substituting Eq.(11.6) into Eq.(11.5) we obtain the result (see Chapter 9)

$$\frac{\partial}{\partial A}\overline{G}[C^{\text{out}},C^{\text{in}},A] = \lim_{\zeta\to 0^+}\int d\mu[g_{ab}]\left[-\frac{1}{2\sqrt{gg^{00}}}\frac{\overleftrightarrow{\delta}}{\delta g_{00}}\right](\overline{\sigma},\zeta)F(\phi_\mu,g_{ab}), \tag{11.7}$$

where $\delta/\delta g_{00}(\overline{\sigma},\zeta)$ acts on the measure $d\mu[g_{ab}]$ and on the string-field term

$$F(\phi_\mu,g_{ab}) = \int d\mu[\phi_\mu]\exp[-I_0(\phi_\mu,g_{ab},\mu^2 \equiv 0)]. \tag{11.8}$$

The $\delta/\delta g_{00}(\overline{\sigma},\zeta)$ functional derivative of the term $F(\phi_\mu,g_{ab})$ is subtle since the covariant functional measure $d\mu[\phi_\mu]$ depends in a nontrivial way on the metric $g_{ab}(\sigma,\zeta)$ as a consequence of its definition as the functional volume element associated with the covariant functional metric[5]

$$||\delta\phi_\mu||^2 = \int_D (\sqrt{g}\,\delta\phi_\mu\delta\phi_\mu)(\sigma,\zeta)d\sigma\, d\zeta. \tag{11.9}$$

Its evaluation proceeds in the following way. The $g_{00}(\bar{\sigma},\zeta)$ functional derivative of the Brink-Di Vecchia-Howe action without the boundary term is trivially given by the $(0,0)$ component of the stress-energy tensor:[3]

$$\frac{\delta}{\delta g_{00}(\bar{\sigma},\zeta)} I_0(g_{ab},\phi_\mu,\mu^2 \equiv 0) = (\partial_0\phi^\mu\partial_0\phi^\mu - \frac{1}{2} g_{00}g^{cd}\partial_c\phi^\mu\partial_d\phi^\mu)(\bar{\sigma},\tau). \tag{11.10}$$

In the conformal gauge $g_{ab} = e^\rho\delta_{ab}$ Eq.(11.10) takes the simple form below at the boundary limit $\zeta \to 0^+$ with $\pi_\mu^{\text{in}}(\bar{\sigma}) = \lim_{\zeta\to 0^+} \delta_0\phi^\mu(\bar{\sigma},\zeta)$ being the string canonical momentum and $x_\mu^{\prime\text{in}}(\bar{\sigma}) = \lim_{\zeta\to 0^+} \partial_1\phi_\mu(\bar{\sigma},\zeta)$:

$$\frac{1}{2}[\pi_\mu^{\text{in}}(\bar{\sigma})^2 - x_\mu^{\prime\text{in}}(\sigma)^2]. \tag{11.11}$$

Let us evaluate the $\delta/\delta g_{00}(\bar{\sigma},\zeta)$ functional derivative of the functional measure $d\mu[\phi_\mu]$ in the conformal gauge where the results are given by local expressions.

The Frechet derivative of the functional measure is (by its definition) given by the relationship (see Chapters 1 and 9)

$$e^{\rho(\bar{\sigma},\bar{\zeta})}\frac{\delta}{\delta\rho(\bar{\sigma},\zeta)}(d\mu[\phi_\mu;e^\rho\delta_{ab}]) = \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}(d\mu[\phi_\mu;e^{\rho+\delta h}\delta_{ab}] - d\mu[\phi_\mu,e^\rho\delta_{ab}]) \tag{11.12}$$

with $\delta h = \varepsilon\delta(\sigma-\bar{\sigma})\delta(\zeta-\bar{\zeta})$.

Since we have, as a straightforward consequence of the theory's covariance [see Eq.(11.9)],

$$d\mu[\phi_\mu,e^{\rho+\delta h}\delta_{ab}] = d\mu[e^{\delta h/2}\phi_\mu,e^\rho\delta_{ab}] \tag{11.13}$$

and the effect of the functional measure $d\mu[\phi_\mu]$ under a conformal rescaling can be exactly evaluated, [6] (Chapters 1 and 9)

$$d\mu[\phi^\mu,e^{\rho+\delta h}\delta_{ab}] = d\mu[\phi^\mu,e^\rho\delta_{ab}]\exp\left[\frac{D}{24\pi}\left[\int_D \frac{1}{2}(\partial_a\rho)(\partial_a\delta h) + \mu^2(\varepsilon)e^\rho\delta h + \lambda_0(\varepsilon)\int_{\partial D} e^\rho\delta h\right]\right], \tag{11.14}$$

we thus have the result

$$e^{-\rho(\bar{\sigma},\zeta)}\frac{\delta}{\delta\rho(\bar{\sigma},\zeta)} d\mu[\phi^\mu,e^\rho\delta_{ab}] = \frac{D}{24\pi}[R(\rho(\bar{\sigma},\zeta)) + \mu_0^2(\varepsilon) + \lambda_0(\varepsilon)]d\mu[\phi^\mu,e^\rho\delta_{ab}], \tag{11.15}$$

where $R(\rho(\bar{\sigma},\zeta)) = e^{-\rho(\hat{\sigma},\zeta)}\Delta\rho(\bar{\sigma},\zeta)$ is the scalar of curvature associated with the intrinsic metric $e^{-\rho}\delta_{ab}$ and $\mu_0(\varepsilon)$, $\lambda_0(\varepsilon)$ are infinite constants which depend on the regularization scheme used to evaluate the functional determinants of two-dimensional Beltrami-Laplace operators in Polyakov's effective action (Chapter 1).

It is instructive to remark that one can implement the above calculation without choosing the conformal gauge since the measure functional derivative may be alternatively defined by the ratio

$$\frac{\delta}{\delta g_{00}(\bar{\sigma},\zeta)} d\mu[\phi_\mu,g_{ab}] = \frac{\det^{-D/2}[\Delta g_{ab} + \delta g_{00}(\bar{\sigma},\bar{\zeta})]}{\det^{-D/2}(\Delta g_{ab})} \tag{11.16}$$

and we have the general covariant result

$$\text{In } \det(\Delta g_{ab}) = \frac{1}{4\delta\pi}\int_0 d\sigma d\zeta \int_D d\sigma' d\zeta' (\sqrt{g}R)(\sigma,\zeta)\Delta^{-1}_{g_{ab}}(\sigma-\sigma,\zeta-\zeta')(\sqrt{g}R)(\sigma',\zeta'), \tag{11.17}$$

where $\Delta^{-1}_{g_{ab}}(\sigma-\sigma',\zeta-\zeta')$ denotes the Green's function of the Laplace Beltrami operator $\Delta_{g_{ab}} = (1/\sqrt{g})\partial_a(g^{ab}\partial_b)$ in the presence of the intrinsic metric $\{g_{ab}\}$.

However, it is important to note that only in the conformal gauge do our calculations take a local form as a functional of the intrinsic metric tensor. This is the technical reason that we use the conformal gauge at the end of our calculations.

Finally the $g_{00}(\bar{\sigma},\zeta)$ derivative of $d\mu[g_{ab}]$ in the conformal gauge is easily evaluated: [3,5]

$$e^{-\rho(\bar{\sigma},\zeta)}\frac{\delta}{\delta\rho(\bar{\sigma},\zeta)}d\mu[g_{ab}=e^{\rho}\delta_{ab}] = -\frac{26}{24\pi}[R(\rho(\bar{\sigma},\zeta))+\mu_0^2(\varepsilon)+\lambda_0(\varepsilon)]d\mu[g_{ab}=e^{\rho}\delta_{ab}], \tag{11.18}$$

since we have explicitly

$$d\mu[g_{ab}=e^{\rho}\delta_{ab}] = D^{\text{cov}}[\rho]\exp\left[-\frac{26}{48\pi}\int_D[\frac{1}{2}(\partial_a\rho)^2+\mu^2(\varepsilon)e^{\rho}]+\lambda(\varepsilon)\int_{\partial D}e^{\rho}ds\right]$$
$$\left[D^{\text{cov}}[\rho] = \prod_{(\sigma,\zeta)\in D} e^{\rho(\sigma,\zeta)}\,d\rho(\sigma,\zeta)\right]. \tag{11.19}$$

By grouping together Eqs.(11.11), (11.15), (11.18), and introducing the covariant string commutation relation[1]

$$[\pi^{\mu}_{\text{in}}(\sigma),x^{\nu}(\sigma')] = \frac{i\delta(\sigma-\sigma')}{\hbar e_{\text{in}}(\sigma)}\{e_{\text{in}}(\sigma)=\lim_{\zeta\to 0^+}\exp[+\rho(\sigma,\zeta)]\}$$

which produces the Schrödinger representation $\pi^{\mu}_{\text{in}}(\sigma) = -\hbar e^{-1}_{\text{in}}(\sigma)\delta/\delta x^{\text{in}}_{\mu}(\sigma)$, we can finally write Eq.(11.7) as a covariant diffusion equation for Polyakov's bosonic string which takes into account in an explicitly and local way the presence of the world sheet intrinsic metric

$$\exp[\rho(\sigma,\zeta)]\left[-\frac{1}{2}\frac{\delta^2}{e_{\text{in}}(\bar{\sigma})^2\delta x^{\text{in}}_{\mu}(\bar{\sigma})\delta x^{\text{in}}_{\mu}(\bar{\sigma})}-\frac{1}{2}|x'^{\text{in}}_{\mu}(\bar{\sigma})|^2+\frac{26-D}{24\pi}\lim_{\zeta\to 0^+}[R(\rho(\bar{\sigma},\zeta))+C_\infty]\right]\overline{G}[C^{\text{out}},C^{\text{in}},A]$$
$$=\frac{\partial}{\partial A}\overline{G}[C^{\text{out}},c^{\text{in}},A]. \tag{11.20}$$

The above -written string wave equation is the main result of this chapter.

Let us comment that at $D=26$, where the invariance of Polyakov's string theory under the world-sheet diffeomorphism group is restored (otherwise it is partially broken to the quotient group of the complete diffeomorphism group by the Weyl diffeomorphism subgroup) we can fix $e_{\text{in}}(\sigma)=1$ and the above area diffusion equation takes the simple form

$$\frac{\partial}{\partial A}\overline{G}[C^{\text{out}},C^{\text{in}},A] = \left[-\frac{1}{2}\frac{\delta^2}{\delta x^{\text{in}}_{\mu}(\bar{\sigma})\delta x^{\text{in}}_{\mu}(\bar{\sigma})}-\frac{1}{2}|x'^{\text{in}}_{\mu}(\bar{\sigma})|^2\right]\overline{G}[C^{\text{out}},C^{\text{in}},A]. \tag{11.21}$$

A simple functional solution of Eq.(11.21) is

$$\overline{G}[C^{\text{out}}, C^{\text{in}}, A] = e^{-EA}\Phi[C^{\text{in}}]\Phi[C^{\text{out}}], \tag{11.22}$$

where the string functional $\Phi[C^{\text{in}}]$ satisfies the string wave equation

$$\left[-\frac{1}{2}\frac{\delta^2}{\delta x_\mu^{\text{in}}(\overline{\sigma})\delta x_\mu^{\text{in}}(\overline{\sigma})} - |x_\mu'^{\text{in}}(\overline{\sigma})|^2\right]\Phi_E[C^{\text{in}}] = -E\Phi_E[C^{\text{in}}] \tag{11.23}$$

Here we can see that the possible values of E are exactly the eigenvalues of the "functional Klein-Gordon" operator on the left-hand side of Eq.(11.23) which can be identified with the $-L_0$ Virasoro constraint written in the Schrödinger representation (see Chapter 20, Appendix D) – Supplements.

11.3. The Wheeler - De Witt Equation as a Functional Diffusion Equation

We aim in this section to present a path integral framework where the three-metric quantum gravity propagator Ref. ([10]-[22]) in Einstein theory satisfies a kind of functional diffusion equation with the Space-Time four volume playing the role of a proper time for Space-Time quantum evolution as much as similar analysis presented in 11.2.

We, thus, recover the Wheeler - De Witt equatIon in the situation of vanishing Space-Time four volume.

Let us start our analysis by considering a Space-Time M which has topology of a cylinder. This means that M can be considered as a homotopical deformation of a three-dimensional manifold S.

In Four-Dimensional Einstein Gravitation Theory (Chapter 1), the dynamical fields are rank two symmetric tensor $h_{\mu\nu}(x)$ and defining metric strucures in M compatible with its cylindrical topology. The basic object in the (formal) Feynman path integral approach for quantization is the number of quantum gravitational field states with a fixed four volume V and satisfying the boundary condition that the metric field $h_{\mu\nu}(x)$ induces in the three-dimensional manifold S a given field (classical observable) metric $\hat{g}_{ij}(\vec{x})$

$$\begin{aligned} N(V) = \int D^c[h_{\mu\nu}] \exp\left\{-\frac{1}{8G_N^2}\int_M d^4x(\sqrt{h}R)(x)\right\} \\ \times\,\delta\left(\int_N d^4x(\sqrt{h}(x) - V)\right) \end{aligned} \tag{11.24}$$

The Delta function in Eq.(11.24) ensures that only the gravitational states $h_{\mu\nu}(x)$ with a fixed four-volume V contribute. The covariant functional measure $D^c[h_{\mu\nu}]$is given explicitly in Chapter 1. The metric boundary condition and the topology of M is taken into account by using the Lapse-Shift form of the metric field ([14])

$$h_{\mu\nu}(x) = -(N(\vec{x},\zeta)^2(h)^2 + g_{ij}(\vec{x},\zeta)(dx^i + N^i(\vec{x},\zeta)d\zeta) \times (dx^j + N^j(\vec{x},\zeta)d\zeta) \tag{11.25}$$

where $\vec{x} \in S$, $\zeta \in [0,T]$; $N(\vec{x},0) = N^i(\vec{x},0) = 0$ and $\boxed{g_{ij}(\vec{x},0) = \hat{g}_{ij}(\vec{x})}$.

In order to write a four-volume functional diffusion equation for the Quantum Gravity propagator Eq.(11.24), we re-write it is a form where the new field variables are given by the lapse-shift (scalar and vectorial) fields (N, N^i) and the three-dimensional metric field $g_{ij}(\vec{x}, \zeta)$

$$N(V; \hat{g}_{ij}) = \int D^c[N]D^c[N^i]D^c[g_{ij}] \\ \times \exp\left\{-\frac{1}{8\pi G_N^2}\int_0^T d\zeta \int_S d^3x S^{ADM}[N, N^i, g_{ij}]\right\} \\ \delta\left(\int_0^T d\zeta \int_S d^3x(\sqrt{g}N)(\vec{x}, \zeta) - V\right) \tag{11.26}$$

where $\sqrt{g} = \det^{(3)}(g_{ij})$ and $S^{ADM}[N, N^i, g_{ij}]$ denotes the Arnowitt, Deser and Misner expression for the Einstein-Hilbert action is term of the three dimensional geometric intrinsic objects (N, N^i, g_{ij}) and the extrinsic curvatue K_{ij} ([14])

$$S^{ADM}[N, N^i, g_{ij}] = \int_M (\sqrt{h}R)(x)d^4x \\ = \int_M (N\sqrt{g}(K_{ij}K^{ij} - K^2 + {}^{(3)}R))(x)d^4x \tag{11.27}$$

It is important to point out that the (formal) Jacobian of the field transformation $(h_{\mu\nu} \to (N, N^i, g_{ij}))$ is the tad-pole term $\exp\left(-\delta^{(4)}(0)\int_S dx^3 \int_0^T d\zeta N(x, \zeta)\right)$ which may be assigned the value 1 by using the *Dimensional Regularization Scheme* since the general covariant functional measure $D^c[h_{\mu\nu}]$ reduces to the usual Feynman Measure (see Chapter 1).

Another remark is related to the fact that the object $N(V)$ does not depends on the (homotopical) parameter T since it is integrated out in the formal definition of the product Feynman measure $D[N(x, \zeta)] = \prod_{\zeta \in [0,T]} (dN(x, \zeta))$.

Let us now evaluate the V-derivative of the $N(V)$

$$\frac{\partial}{\partial V} N(V, \hat{g}_{ij}) = \int D^F[N]D^F[N^i]D^F(g_{ij}] \\ \times \exp\left(-\frac{1}{8\pi G_n^2}\int S^{ADM}[N, N^{\cdot} g_{ij}]\right) \\ \times -\delta'\left(\int_0^T dx \int_S d^3x \sqrt{g}N - V\right) \tag{11.28}$$

with $\delta'(x)$ being the usual first-derivative of the $\delta(x)$ distribution.

At this point we consider the identity

$$-\delta'\left(\int_S d^3x \int_0^T d\zeta(N\sqrt{g} - V)\right) \\ = \lim_{\zeta \to 0^+} \frac{1}{\sqrt{g_{ij}(\vec{x}, \zeta)}} \frac{\delta}{\delta N(\vec{x}, \zeta)} \delta\left(\int_S d^3x \int_0^T d\zeta N\sqrt{g} - V\right) \tag{11.29}$$

which can be verified by using the usual Fourier Integral representation for the δ-function.

By substituting Eq.(11.29) into Eq.(11.18) and using the fact that $D^c[N(x,\zeta)]$ is the usual Feynman Measure we can re-write Eq.(11.28) in more invariant form after doing a partial functional N-integration:

$$\begin{aligned}&\frac{\partial}{\partial V}N(V,\hat{g}_{ij}) = \int D^c[N,N^i,g_{ij}]\times\\&\lim_{\zeta\to 0^+}\frac{1}{\sqrt{g(\vec{x},\zeta)}}\frac{\delta}{\delta N(\vec{x},\zeta)}\left(-\frac{1}{8\pi G_N^2}S^{ADM}[N,N^i,g_{ij}]\right)\\&\quad\exp\left(-\frac{1}{8\pi G_n^2}S^{ADM}[N,N^i,g_{ij}]\right)\times\\&\quad\times\left(\int_0^T d\zeta\int_M d^3x\sqrt{g}N - V\right)\end{aligned} \tag{11.30}$$

Now we have the result

$$\frac{\delta}{\delta N(\vec{x},\zeta)}S^{ADM}[N,N^i,g_{ij}] = \sqrt{g_{ij}}(K^{ij}K_{ij}-K^2+{}^{(3)}R)(\vec{x},\zeta) \tag{11.31}$$

In the functional integral fremework we have in a formal way the usual Schrödinger representation inside Eq.(11.30)

$$\lim_{\zeta\to 0^+}K_{ij}(\vec{x},\zeta) = (\sqrt{\hat{g}})^{-\frac{1}{2}}\left(\frac{\delta}{\delta\hat{g}_{ij}}-\frac{1}{2}\hat{g}_{ij}\hat{g}^{k\ell}\frac{\delta}{\delta\hat{g}^{k\ell}}\right)(\vec{x},0) \tag{11.32}$$

$$\lim_{\zeta\to 0^+}K_{ij}(\vec{x},\zeta) = -\frac{1}{2}\left((\sqrt{\hat{g}})^{-1}\hat{g}^{ij}\frac{\delta}{\delta\hat{g}^{ij}}\right)(\vec{x}) \tag{11.33}$$

After substituting Eq.(11.32), Eq.(11.33) into Eq.(11.31)-Eq.(11.30) and taking the limit of $\zeta\to 0^+$ (see Eq.(11.25)) we obtain our proposed Four-volume gravitational diffusion equation (in the *Euclidean section* of the space-time M ([17])

$$\begin{aligned}\frac{\partial}{\partial V}N(V,\hat{g}_{ij}) = &-\left(\frac{1}{\sqrt{\hat{g}}}G_{(i,j),(k\ell)}(\hat{g})\frac{\delta^2}{\delta\hat{g}_{ij}\delta\hat{g}_{k\ell}}\right.\\&\left.+\frac{{}^{(3)}R}{\sqrt{\hat{g}}}\right)(\vec{x})N(V,\hat{g}_{ij})\end{aligned} \tag{11.34}$$

where $G_{(i,j),(k\ell)}(\hat{g})$ denotes the Wheeler - De Witt metric over metrics (see Chapter 1)

$$G_{(i,j),(k\ell)}(\hat{g}) = \frac{1}{2\sqrt{\hat{g}}}(\hat{g}_{ik}\hat{g}_{j\ell}-\hat{g}_{ij}\hat{g}_{k\ell}) \tag{11.35}$$

It is worth point out the similar equation for two-dimensional quantum gravity obtained by the author in section 11.2 ([18]).

A simple solution of Eq. (33) is given by

$$N(V,\hat{g}_{ij}) = e^{-EV}\,\Psi_{(E)}(\hat{g}_{ij}) \tag{11.36}$$

where $\Psi_{(E)}(\hat{g}_{ij})$ are the formal eingenfunctions of the functional Wheeler - De Witt "Laplacian" $\mathbb{L}$

$$\begin{aligned}\left(\mathbb{L}\,\psi_{(E)}\right)(\hat{g}_{ij}) &= \frac{1}{\sqrt{\hat{g}}}\left[G_{(i,j),(k\ell)}(\hat{g})\,\frac{\delta^2}{\delta g_{ij}\,\delta g_{k\ell}}\right. \\ &\quad \left. - {}^{(3)}R(\hat{g})\right]\psi_{(E)}(\hat{g}_{ij}) \\ &= E\psi_{(E)}(\hat{g}_{ij}) \end{aligned} \tag{11.37}$$

Now we can see that for zero eigenvalue $E = 0$:

$$\psi_{[E=0]}(\hat{g}_{ij}) = \lim_{V\to 0^+} N(V,\hat{g}_{ij})$$

satisfies the "Universe Wheeler - De Witt wave equation".

The most general solution should be given by a superposition of eigenfunctions

$$N(V,[\hat{g}_{ij}]) = \int_{\mathrm{Spec}(\mathbb{L})} \psi_E[\hat{g}_{ij}]e^{-EV}\,\rho(E)dE \tag{11.38}$$

where the spectral weight $\rho(E)$ is determined by some unkown boundary - initial condition on $N(V,[\hat{g}_{ij})$ ([17]). These further enquiries on universe initial conditions are left to our readers. (See Appendix D in the supplementary appendixes in the end of this book).

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Chapter 12

Covariant Path Integral for Nambu-Goto String Theory

12.1. Introduction

To attempt to understand collective phenomena in field theories has become the central problem of quantum field theory [1,2]. One of the most promising frameworks to solve this fundamental prolem in quantum field theories is to write the associated field theory path integral in loop space and, thus, search for string solutions for the loop space field equations of motion (see Chapter 9). This effort, in turn, has recently led to intensive research into the problem of the correct meaning for the string path integral. Most of these studies were based on Polyakov's analysis of the conformal anomaly of two-dimensional massless fields interacting with induced DeWitt quantum gravity in two dimensions (see Chapter 1 and Chapter 19).

Unfortunately the Polyakov proposal of DeWitt two-dimensional quantum gravity as the correct meaning for the string path integral may be considered only as a guessed effective action study for the full Nambu-Goto area functional, since it involves the full use of a mean field approximation [1].

It is purpose of this chapter to solve the above mentioned problem by quantizing directly the Geometrical non effective Nambu-Goto string path integral and thus solving this long-standing unsolved problem in Quantum Geometry.

12.2. The Nambu-Goto Full Path Integral

Let us start our analysis by considering the original Polyakov path integral for the Nambu-Goto string propagator in a form useful for non-Abelian gauge theories, (Eq. (9.76) of Ref. [1]) and Ref. [3]:

$$G(C) = \sum_{[g_{ab}]} \sum_{[X_\mu]} \exp\left[-\frac{1}{2\pi\alpha'} \int_D (\sqrt{g})(\xi) d^2\xi\right] \delta^{(F)}_{\text{cov}}[g_{ab}(\xi) - h_{ab}(X^\mu(\xi))]. \tag{12.1}$$

The continuous sum over the string world sheet vector position $X_\mu(\xi)$ and the intrinsic two-dimensional $(2D)$ metric $g_{ab}(\xi)$ in Eq. (1) are defined by DeWitt functional metrics on hemispherical manifolds possessing as non-trivial boundaries the string configuration $\{C\}$ (see Chapter 1)

$$||\delta X^{\mu}||^2 = \int_D (\sqrt{g}\,\delta X^{\mu}\delta X^{\mu})(\xi)d^2\xi, \tag{12.2a}$$

$$||\delta g_{ab}|| = \int_D [\sqrt{g}\,\delta g_{ab}(g^{aa'}g^{bb'})\delta g_{a'b'}](\xi)d^2\xi. \tag{12.2b}$$

The δ functional inside Eq.(12.1) restricts the nonphysical variable (intrinsic metric) $g_{ab}(\xi)$ to be the world sheet induced metric (see Chapter 1)

$$h_{ab}(X^{\mu}(\xi)) = (\partial_a X^{\mu})(\partial_b X^{\mu})(\xi).$$

Let us biefly recall Polyakov's covariant analysis. In his explicitly convariant scheme one writes the delta functional by means of a covariant Fourier path integral:

$$G(C) = \sum_{[g_{ab}]} \exp\left[-\frac{1}{2\pi\alpha'}\int_D (\sqrt{g})(\xi)d^2\xi\right]\left[\sum_{[X^{\mu}]}\sum_{[\lambda_{ab}]} \exp\left[i\int_D d^2\xi[\sqrt{g}\lambda_{ab}(\partial^a X^{\mu}\partial^b X^{\mu} - g^{ab}]\right]\right]. \tag{12.3}$$

By making the guess of the exact validity of the covariant mean field average for the Lagrange multiplier (see Eq. (9.88a) of Ref. [1]),

$$\lambda_{ab}(\xi) = i\langle\lambda\rangle g_{ab}(\xi), \tag{12.4}$$

one obtains Polyakov's result of $2D$ massless scalar fields interacting with DeWitt two-dimensional quantum gravity as a definition for the string path integral Eq.(12.1) after substituting Eq.(12.4) into Eq.(12.3) and defining an effective cosmological constant:

$$\mu_0 = 1/2\pi\alpha' + \langle\lambda\rangle.$$

Unfortunately, in string theory the conditions for the full validity of Eq.(12.4) on the string energy phase space is still an open question. This, in turn, makes Polyakov's approach [1] at most a path integral effective theory for string quantization.

We, thus, make a departure from the above Polyakov approximate analysis and try to consider exactly the original expression Eq.(12.1) with the δ function without making any mean field approximation of the sort of Eq.(12.4).

The invariant measure associated with the DeWitt supermetric Eq.(12.2b) on the functional space of the fields $g_{ab}(\xi)$ in the path integral formalism was shown in chapter 1 to be correctly defined by the DeWitt measure

$$\sum_{[g_{ab}]}\int\prod_{(\xi,a,b)}[[dg_{ab}(\xi)](\sqrt{g(\xi)})^{-6/4}]\delta^{(F)}[M(g_{ab})]\left[\left[\prod_{(\xi,c)} d\varepsilon_c(\xi)\sqrt{\det(\sqrt{g}g^{ab})}\right]\det[\delta M(g^{\varepsilon}_{ab})/\delta\varepsilon]\right], \tag{12.5}$$

where $M(g_{ab})$ is a gauge-fixing functional and $[\varepsilon_c(\xi)]$ denotes the infinitesimal vector field generators of a general coordinate transformation in D. The powers of $\sqrt{g(\xi)}$ in the above written equation come from the root square of the DeWitt super metric determinant in the invariant measure (Eq.(12.2) of Ref. [5] for R^2).

We point out that direct use of Eq.(12.5) for calculations is very subtle since it contains the usual Feynman product measure on the variables $dg_{ab}(\xi)$ and $d\varepsilon_c(\xi)$ weighted with factors of the form $(\sqrt{g(\xi)})^m$ which, in turn, lead to the use of a new field reparametrization in the path integral in order to reduce the functional measure to the usual Feynman measure. For instance, if one wants to evaluate formally a path integral of the form

$$I = \sum_{g_{ab}]} \exp\left[-\int_D d^2\xi L(g_{ab}(\xi))\right], \tag{12.6}$$

where $L(g_{ab})$ denotes an invariant coordinate transformation action functional for the $g_{ab}(\xi)$ field, we must consider first the variable change

$$\frac{\partial\varphi_{ab}(\xi)}{\partial\xi_l} = \left[\frac{\partial}{\partial\xi_l} g_{ab}(\xi)\right](\sqrt{g(\xi)})^{-3/2}, \tag{12.7}$$

which will reduce the weighted measure Eq.(12.5) to the usual Feynman product measure

$$I = \int \left[\prod_{(\xi,a,b)} d\varphi_{ab}(\xi)\right] \exp\left[-\int_D d^2\xi \widetilde{L}(\varphi_{ab}(\xi))\right], \tag{12.8}$$

where $\widetilde{L}(\varphi_{ab})$ is the new expression of the action in terms of the new variable Eq.(12.7) added with the Faddeev-Popov ghost action. It is worth remarking that in the functional integral form Eq.(12.8), practical calculations are very cumbersome and not explicitly covariant under the action of the diffeomorphism group.

Fortunately, in two dimensions it is possible to obtain a closed expression for Eq.(12.6) in the conformal gauge $g_{ab}(\xi) = e^{\varphi(\xi)}\delta_{ab}$ as has been shown by Polyakov by directly using the DeWitt super metric Eq. (2b) to rewrite the covariant measure Eq.(12.5) in terms of the conformal factor (see Chapter 1 and Chapters 9/10)

$$\begin{aligned}\sum_{[g_{ab}=e^{\varphi(\xi)}\delta_{ab}]} &= \int \prod_{\xi} [d(e^{\varphi(\xi)}\delta_{11}) d(e^{\varphi(\xi)}\delta_{22}) e^{-3\varphi(\xi)/2}] \exp\left[-\frac{26}{48\pi}\int_D d^2\xi \left[\frac{1}{2}(\partial\varphi)^2 + \mu^2 e^{\varphi}\right](\xi)\right] \\ &= \prod_{\xi} d(e^{\varphi(\xi)/2}) \exp\left[-\frac{26}{48\pi}\int_D d^2\xi \left[\frac{1}{2}(\partial\varphi)^2 + \mu^2 e^{\varphi}\right](\xi)\right]. \end{aligned} \tag{12.9a}$$

By making the choice $e^{\varphi(\xi)/2} = \gamma(\xi)$ as the correct dynamical degree of freedom, we get the final expression for the g_{ab} invariant measure to be used in our study:

$$\sum_{g_{ab}} \int \prod_{\xi} d[\gamma(\xi)] \exp\left[-\frac{26}{48\pi}\int_D d^2\xi \frac{1}{2}\left[\frac{\partial_a(\gamma^2)}{\gamma^2}\right]^2\right] \exp\left[\lim_{\delta\to 0^+} \frac{1}{4\pi\delta}\int_D d^2\xi \gamma^2(\xi)\right]. \tag{12.9b}$$

Next, we consider the $X_\mu(\xi)$ functional integral [16]. In order to reduce the covariant path integral over the world sheet string vector position to a Feynman functional measure as in Eq.(12.9b) we first consider the following covariant Gaussian functional integral which may be used to define the covariant sum in Eq.(12.1) [see Eq. (2a)]:

$$\widehat{I}[g_{ab}] = \prod_{(\xi,\mu)} [dX^\mu(\xi) \sqrt[4]{g(\xi)}] \exp\left[-\frac{1}{2}\int_D d^2\xi [\sqrt{g} X^\mu(-\Delta_g) X^\mu](\xi)\right], \tag{12.10}$$

where Δ_g is the Laplace Beltrami operator associated with the metric $g_{ab}(\xi)$. Now we note that Eq.(12.10) is a Gaussian path integral:

$$\widehat{I}[g_{ab}] = \det{}^{-D/2}(-\Delta_g). \tag{12.11}$$

It is possible to write the above functional determinant as a local field action for the conformal factor $\gamma(\xi)$ [1]: namely

$$\widehat{I}[g_{ab} = \gamma^2\delta_{ab}] = \exp\left[\frac{D}{4u\pi}\int_D d^2\xi \frac{1}{2}\left[\frac{\partial_a(\gamma^2)}{\gamma^2}\right]^2(\xi)\right]\exp\left[+\lim_{\delta\to 0^+}\frac{D}{\delta}\int_D d^2\xi\gamma^2(\xi)\right]. \tag{12.12}$$

Let us now consider a metric conformal scaling in Eq.(12.10) [6]:

$$g_{ab}(\xi) = e^{\lambda(\xi)}\hat{g}_{ab}(\xi). \tag{12.13}$$

We, thus, write (12.10) as well as

$$\widehat{I}[g_{ab}] = \int \prod_{(\xi,\mu)}[dX^\mu(\xi)\hat{g}(\xi)]^{1/4}e^{\lambda(\xi)/2}]\exp\left[-\frac{1}{2}\int_D d^2\xi[\sqrt{\hat{g}}X^\mu(-\Delta_g)X^\mu](\xi)\right]. \tag{12.14}$$

We remark that the classical action of massless scalar fields on a compact manifold without boundary (the domain D) is conformally scale invariant, so it does not depend on the conformal factor. The effects of the conformal scaling are nontrivial only at the quantum level or, equivalently, at the level of the functional measures as may be seen from Eq.(12.14).

Now we note tht change on the functional measure Eq.(12.13) is taken into account entirely by a Jacobian $J[\lambda(\xi)]$ which is a functional of the conformal scale factor (the well-known Fujikawa conformal anomaly factor [6,9]:

$$\left[\prod_{(\xi,\mu)} dX^\mu(\xi)e^{\lambda(\xi)/2}[\hat{g}(\xi)]^{1/4]}\right] = J[\lambda(\xi)]\left[\prod_{(\xi,\mu)} dX^\mu(\xi)[\hat{g}(\xi)]^{1/4]}\right]. \tag{12.15}$$

After substituting Eq.(12.15) into Eq.(12.14) and evaluating the resulting Gaussian covariant functional integral, we get the explicit expression for the above-mentioned Jacobian:

$$J[\lambda(\xi)] = \det{}^{-D/2}(-\Delta_{e^\lambda\hat{g}})/\det{}^{-D/2}(-\Delta_{\hat{g}}). \tag{12.16}$$

Let us make use of Eqs.(12.14)-(12.16) for $\hat{g}_{ab} = \delta_{ab}$ and $\lambda(\xi) = 21n\gamma(\xi)$, since we can always consider the conformal gauge in Eq.(12.10)

$$g_{ab}(\xi) = \gamma^2(\xi)\delta_{ab}.$$

As a result we obtain the following relation between the covariant measure and the Feynman product measure parametrization:

$$\left[\prod_{(\xi,\mu)}[dX^\mu(\xi)\gamma(\xi)]\right] = \exp\left\{\frac{D}{48\pi}\int_D d^2\xi\left(\frac{1}{2}\left[\frac{\partial_u(\gamma^2)}{\gamma^2}\right]^2 + \mu^2\gamma^2\right](\xi)\right\}\prod_{(\xi,\mu)}[dX^\mu(xi)]. \tag{12.17}$$

At this point, we return to the original Eq.(12.3) and rewrite it in the conformal gauge by using the Feynman functional measure parametrization Eqs.(12.9) and (12.17):

$$G[C] = \int \left[\prod_{\xi} d\gamma(\xi)\right] \left[\prod_{(\xi,\mu)} [dX^{\mu}(\xi)]\right] \exp\left\{-\frac{26-D}{48\pi}\int_D d^2\xi \left[\frac{1}{2}\left[\frac{\partial_a \gamma^2}{\gamma^2}\right]^2 + \mu^2\gamma^2\right](\xi)\right\}$$
$$\times \exp\left[-\frac{1}{2\pi x'}\int_D d^2\xi (\partial_a X^{\mu})^2(\xi)\right] \delta^{(F)}_{\text{cov}}[\gamma^2(\xi)\delta_{ab} - h_{ab}(X^{\mu}(\xi))]. \qquad (12.18)$$

It is instructive to remark that we must rewrite the covariant delta functional inside Eq.(12.18) in a Feynman parametrization form. In order to implement this step of our study we consider the covariant Fourier path integral representation written directly in the conformal gauge $g_{ab}(\xi) = \gamma^2(\xi)\delta_{ab}$ [see Eq. (3)]:

$$\delta^{(F)}_{\text{cov}}[\gamma^2(\xi)\delta_{ab} - h_{ab}(X^{\mu}(\xi))] = \int \left[\prod_{\xi}[d\lambda_{11}(\xi)\gamma^{-1}(\xi)]\right] \left[\prod_{\xi}[d\lambda_{22}(\xi)\gamma^{-1}(\xi)]\right]$$
$$\times \exp\left\{i\int_D \frac{\lambda_{11}(\xi)}{\gamma(\xi)} \frac{[\partial^1 X^{\mu}\partial_1 X^{\mu})(\xi) - \gamma^2(\xi)]}{\gamma(\xi)}\right\}$$
$$\times \exp\left\{i\int_D \frac{\lambda_{22}(\xi)}{\gamma(\xi)} \frac{[\partial^2 X^{\mu}\partial^2 X^{\mu})(\xi) - \gamma^2(\xi)]}{\gamma(\xi)}\right\}. \qquad (12.19)$$

The covariant functional measure $\sum_{[\lambda_{ab}]}$ for the Fourier tensor field variable $\lambda_{ab}(\xi)$ in the conformal gauge $g_{ab}(\xi) = \gamma^2(\xi)\delta_{ab}$ used in Eq.(12.9) is still defined by us with the DeWitt covariant measure Eq.(12.2b) for two-dimensional tensors $\lambda_{ab}(\xi)$:

$$||\delta\lambda_{ab}||^2 = \int_D d^2\xi \left[\gamma^2(\xi)(\delta\lambda_{ab})(\xi)\frac{\delta^{aa'}}{\gamma^2(\xi)}\frac{\delta^{bb'}}{\gamma^2(\xi)}(\delta\lambda_{a'b'})(\xi)\right]. \qquad (12.20)$$

Following the discussion after Eq.(12.6) about the correct meaning of a convariant path integral, we note that by making the variable change

$$\tilde{\lambda}_{11}(\xi) = \lambda_{11}(\xi)/\gamma(\xi),\ \tilde{\lambda}_{22}(\xi) = \lambda_{22}(\xi)/\gamma(\xi), \qquad (12.21)$$

the covariant delta functional Eq.(12.19) in the conformal gauge has the same form of the delta functional defined from the usual Feynman product measure definition:

$$\delta^{(F)}_{\text{cov}}[\gamma^2(\xi)\delta_{ab} - h_{ab}(X^{\mu}(\xi))] = \int \left[\prod_{\xi} d\tilde{\lambda}_{11}(\xi)\right] \left[\prod_{\xi} d\tilde{\lambda}_{22}(\xi)\right] \exp\left[i\int_D d^2\xi \left[\tilde{\lambda}_{11}\frac{(\partial^1 X^{\mu}\partial^1 X^{\mu} - \gamma^2)}{\gamma}\right](\xi)\right]$$
$$\times \exp\left[i\int_D d^2\xi \left[\tilde{\lambda}_{22}\frac{(\partial^2 X^{\mu}\partial^2 X^{\mu} - \gamma^2)}{\gamma}\right](\xi)\right]$$
$$= \delta^{(F)}[\gamma^2(\xi)\delta_{ab} - h_{ab}(X^{\mu}(\xi))]. \qquad (12.22)$$

Next, we can evaluate exactly the root-square conformal factor $\gamma(\xi)$ auxiliary functional integral due to the usual delta functional Eq.(12.21) which produces the result [4]

$$G[C] = \int \left[\prod_{(\xi,\mu)} dX^\mu(\xi)\right] \exp\left[-\frac{1}{2\pi\alpha'}\int_D d\xi^+ d\xi^-](\partial_+ X^\mu)(\partial_- X^\mu)](\xi^+,\xi^-)\right]$$
$$\times \exp\left[-\frac{26-D}{48\pi}\int_D d\xi^+ d\xi^- \left[\frac{(\partial_+^2 X^\mu)(\partial_- X^\mu)(\partial_-^2 X^\mu)(\partial_+ X^\mu)}{[(\partial_+ X^\mu)(\partial_- X^\mu)]^2}\right](\xi^+,\xi^-)\right]. \quad (12.23)$$

Note that the use of the conformal gauge in Eq.(12.1) implicitly constrains the use of the orthonormal coordinates for the string world sheet vector position (Ref. [3], Appendix C):

$$(\partial_+ X^\mu)(\partial_+ X^\mu) = (\partial_- X^\mu)(\partial_- X^\mu) \equiv 0, \ (\partial_+ X^\mu)^2 = (\partial_- X^\mu)^2; \quad (12.24)$$

Equation (12.23) is, thus, the exact path integral meaning to the sum over surfaces Eq.(12.1) in the string world sheet orthonormal gauge as originally conjectured in Ref. [4].

At this point of our chapter we remark that scalar scattering amplitudes as random surfaces which intercept point probabilities at the critical dimension $D = 26$ [1] are given exactly by the usual nontachyonic dilaton scattering amplitudes which solve the problem of tachyonic excitation on string theory.

If we now consider a further term, taking into account the surface rigidity extrinsic functional in Eq.(12.1), namely,

$$\exp\left[-\frac{k}{2}\int_D d^2\xi[\sqrt{g}(-\Delta_g X^\mu)^2](\xi)\right], \quad (12.25)$$

we obtain straightforwardly a well-defined path integral quantization of the extrinsic string on the conformal gauge, a result which was used in Ref. [7] on an suggestion basis:

$$\overline{G}[C] = \int \left[\prod_{(\xi,\mu)} dX^\mu(\xi)\right] \exp\left[-\frac{1}{2\pi\alpha'}\int_D d\xi^+ d\xi^-[(\partial_+ X^\mu)(\partial_- X^\mu)](\xi^+,\xi^-)\right]$$
$$\exp\left\{-k\int_D d\xi^+ d\xi^- \left[(\partial_+\partial_- X^\mu)(\partial_+\partial_- X^\mu)\frac{1}{(\partial_+ X^\mu)(\partial_- X^\mu)}\right](\xi^+,\xi^-)\right\}$$
$$\exp\left[-\frac{26-D}{48\pi}\int_D d\xi^+ d\xi^- \left[\frac{\partial_+^2 X^\mu)(\partial^- X^\mu)(\partial_-^2 X^\beta)(\partial_+ X^\beta)}{(\partial_+ X^\mu \partial_- X^\mu)^2}\right](\xi^+,\xi^-)\right]. \quad (12.26)$$

Let us recall that it is a subtle problem if the Liouville terms Eqs.(12.23) and (12.26) do not disturb the ultraviolet theory renormalizability. In addition, by considering complex fermionic degress of freedom belonging to the fundamental representation of an intrinsic group such as $SU(22)$ we can cancel this nonpolynomial Liouville piece of the action [3].

Finally we call attention to the fact that if we had followed Polyakov [1] by using the complete conformal factor $\rho(\xi) = e^{\varphi(\xi)}$ instead of its square root $e^{\varphi(\xi)/2}$ as the scalar dynamical degree of freedom to be quantized in the g_{ab}-functional integral,

$$\sum_{[g_{ab}]} = \int \prod_\xi [d\rho(\xi)] \exp\left\{-\frac{26}{48\pi}\int_D d^2\xi \left[\frac{1}{2}\left[\frac{\partial_a \rho}{\rho}\right]^2\right](\xi)\right\} \exp\left[\lim_{\delta\to 0_+} \frac{1}{4\pi\delta}\int_D \rho(\xi) d^2\xi\right]. \quad (12.27)$$

we would have obtained the following delta functional for Eq.(12.22):

$$\delta^{(F)}_{\mathrm{cov}}[\rho(\xi)\delta_{ab} - h_{ab}(X^\mu(\xi))] = \delta^{(F)}\left[\frac{\partial^1 X^\mu \partial^1 X^\mu - \rho}{\sqrt{\rho}}\right]\delta^{(F)}\left[\frac{\partial^2 X^\mu \partial^2 X^\mu - \rho}{\sqrt{\rho}}\right]$$
$$= \sqrt{(\partial^1 X^\mu \partial^1 X^\mu)(\partial^2 X^\mu \partial^2 X^\mu)}\,\delta^{(F)}(\partial^1 X^\mu \partial^1 X^\mu - \rho)\delta^{(F)}(\partial^2 X^\mu \partial^2 X^\mu - \rho) \qquad (12.28)$$

as a simple result of the usual identity

$$\delta[(y-a)/\sqrt{a}] = \sqrt{a}\,\delta(y-a)$$

used in its functional integral version.

The result implied by Eq.(12.28) will lead us to consider a further weight of the form $\sqrt{h(X^\mu(\xi))}$ on the Feynman differentials $dX^\mu(\xi)$ in our final Eqs.(12.23) and (12.26) for a sum over surfaces in the orthonormal coordinates [see Eq, (12.24)]; and it is worth pointing out that a similar weighted path integral result was put forward some decades ago in Ref. [8] without proof from first principles. (See Appendixes A), B) and C) of supplementary appendixes at the end of this book for studies on string moving on manifolds).

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Chapter 13

Topological Fermionic String Representation for Chern-Simons Non-Abelian Gauge Theories

13.1. Introduction

It was suggested in Ref. 1 that topological non-Abelian quantum field theories in three dimensions $(3D)$ may be solved exactly by means of a noncritical fermionic string theory. The correctness of this string representation holds great potential for high-T_c superconductivity since it produces evidence in favor of a fermionic string picture for the fermionic magnons advocated in Ref. 2.

In this short chapter we address the problem of solving exactly the Chern-Simons loop wave equation in the formalism proposed in Chapter 1 and Chapters 9-16.

13.2. The Fermionic String Representation

Let us start our analysis by considering a set of multiplet scalar field $\beta(x)$ interacting with an $SU(N)$ non-Abelian Chern-Simons gauge theory (in the Euclidean sector) in $3D$ with a nongauged "flavor" group $SO(M)$:

$$\mathcal{L}(\beta,\beta^{\dagger},A_i^{(a)}) = \frac{1}{4}|(\partial_i - gA_i)\beta|^2(x) + \varepsilon^{ijk}\,\mathrm{Tr}[A_i(\partial_j A_k - \partial_k A_j + \frac{2}{3}[A_j, A_k](x),\ [i=1,2,3;(a)=1,\ldots,M]. \tag{13.1}$$

Physically the Lagrangian in Eq.(13.1) may be thought of as the effective Lagrangian obtained by integrating out the quark sector of the Weinberg-Salam electroweak theory at finite temperature and in the very-low-energy regime.[5] After integrating out the Gaussian action of the scalar field $\beta(x)$ and expressing the resulting funcitonal determinant as a functional in the bosonic loop space (Chapter 1 and [5], [6]) we get the following expression for the theory's Euclidean vacuum energy (Ref. 7):

$$Z = \left\langle \exp\left[-\sum_{C_{xx}} \mathrm{Tr}^{(c)}\, \Phi^{CS}[C_{xx}] \right] \right\rangle, \tag{13.2}$$

where $\Phi^{CS}[C_{xx}]$ is the usual (normalized) Mandelstam loop defined by the loop C_{xx} and the Chern-Simons gauge field $A_i^{(a)}(x)$.The quantum average in Eq.(13.2) is defined by the pure Chern-Simons action of Eq.(13.1) and the sum over the loops $C_{xx} = \{X_i(\sigma), 0 < \sigma < T\}$ is given by the bosonic loop path integral

$$\sum_{C_{xx}} = -\int_0^\infty \frac{dT}{T} \int d^3x \int_{\substack{X(0)=x\\X(T)=x}} D^F[X(\sigma)] \exp\left(-\frac{1}{2} \int_0^T \dot{X}^2(\sigma) d\sigma \right). \tag{13.3}$$

In Ref. 1 the factorization (Ref. 6) of the averages of the products of Wilson loops on the basis of a diagrammatic analysis was presented. As a consequence of this result the nontrivial dynamical content of Eq.(13.2) is entirely given by the quantum Wilson loop which is turn is a matrix in the "flavor" space $SO(M)$:

$$W_{(a)(b)}[C_{xx}] = \frac{1}{N} \left\langle \mathrm{Tr}^{(c)}\, P\left[\exp\left(i \oint_{C_{xx}} A_i(X(\sigma)) dX_i(\sigma) \right) \right] \right\rangle. \tag{13.4}$$

In order to deduce a loop wave equation for $W_{(a)(b)}[C_{xx}]$, as in chapter 9, we at first consider the covariant version of the loop C_{xx} by introducing an intrinsic metric $e(\sigma)$ on it,

$$C_{xx} = \{(X_i(\sigma), e(\sigma)); 0 \le \sigma \le 2\pi; X_i(0) = X_i(2\pi) = x\},$$

and by replacing in Eq.(13.4) the tangent loop vector $dX_\mu(\sigma)$ by its covariant version $dX_i(\sigma)/e(\sigma)$. By shifting the $A_i(x)$ variable and introducing the Mandelstam scalar area derivative $\delta|\sigma|(X(\sigma'))$ at an arbitrary point $X(\sigma') \in C_{xx}$, we get the following unrenormalized covariant loop equation $(\lambda = g^2 N)$: [1,4]

$$\begin{aligned} \frac{\delta}{\delta|\sigma|(X(\sigma'))} W_{(a)(b)}[C_{X(0),x(2\pi)}] = \lambda \not\!\!\int_0^{2\pi} \frac{dX_i(\sigma)}{e(\sigma)} \frac{dX_j(\sigma')}{e(\sigma')} X_k(\sigma') \varepsilon_{ijk} \delta^{(3)}(X(\sigma) - X(\sigma')) \\ \times W_{(a)(c)}[C_{X(0),X(\sigma)}] W_{(c)(b)}[C_{X(\sigma),X(2\pi)}] \end{aligned} \tag{13.5}$$

where the line integral $\not\!\!\int_0^{2\pi}$ means that only the nontrivial self-intersection loop points $X_i(\sigma) = X_i(\sigma')$ with $\sigma \neq \sigma'$ contribute to the integrand in Eq.(13.5) since the condensate term $\langle F^2(x) \rangle$ vanishes identically in Chern-Simons gauge theories (see Appendix A of Ref. 4).

In order to solve Eq.(13.5) by means of a string theory as exposed in Chapters 9 and 10 let us consider an arbitrary (but fixed) $3D$ surface

$$\sum = \{\phi_i(\sigma, \zeta); 0 \le \sigma \le 2\pi; 0 \le \zeta \le T; i = 1,2,3\}$$

possessing as a boundary the loop C_{xx} [this will always be possible if $\sum$ is a homology three-sphere (Ref. 1)].

Let us introduce in $\sum$ an $O(M)$ (neutral) spinor structure $\psi_{(a)}(\sigma,\zeta)$ together with a metric structure $\{g_{\mu\nu}(\sigma,\zeta);\mu,\nu=1,2]$. We, thus, consider the following $O(M)$ fixed-area string propagator (the reader should compare this with the $QCD[SU(\infty)]$ string propagator of Chapter 9):

$$\begin{aligned} G_{(a)(b)}(C_{xx};A) = \int & D^c[g_{\mu\nu}]D^c[\psi_a]\{[\psi_{(a)}(0,0)\bar{\psi}_{(b)}(2\pi,0)\} \\ & \times\delta\left(\int_0^{2\pi}d\sigma\int_0^T d\zeta\sqrt{g(\sigma,\zeta)}-A\right)\exp\left(-\int_0^{2\pi}d\sigma\int_0^T d\zeta(\bar{\psi}\,\not{D}_g\psi)(\sigma,\zeta)\right) \\ & \times\left(-\lambda\,{\not}\!\!\int_0^{2\pi}d\sigma\frac{\dot{X}_l(\sigma)}{e(\sigma)}\,{\not}\!\!\int_0^T d\zeta'[\sqrt{g(\sigma',\zeta')}(\bar{\psi}\psi)(\delta',\zeta') \right. \\ & \left.\times\,\delta^{(3)}(\phi_l(\sigma',\zeta')-X_l(\sigma))\varepsilon^{ijk}T_{jk}(\phi_l(\sigma',\zeta'))]\right) \end{aligned} \tag{13.6}$$

where $\not{D}_g$ denotes the covariant Dirac operator associated with the intrinsic metric $g_{\mu\nu}$,

$$T_{jk}(\phi(\sigma',\zeta'))=[(1/\sqrt{h})e^{\mu\nu}\partial_\mu\phi^j\partial_\nu\phi^k](\sigma',\zeta')$$

is the (normalized) orientation tensor of the surface $\sum$ at the point $\phi_l(\sigma',\zeta')$ and ${\not}\!\!\int$ means that only the nontrivial self-intersection points of the surface $\sum$ with its boundary C_{xx} contribute. The intrinsic metric $g_{\mu\nu}$ satisfies the boundary condtion $\lim_{\zeta\to 0^+}\sqrt{g(\sigma,\zeta)}=e(\sigma)$ and the intrinsic fermions $\psi(\sigma,\zeta)$ satisfy the Neumann condition $\lim_{\zeta\to 0^+}\partial_\sigma\psi(\sigma,\zeta)\equiv 0$.

Let us remark that the λ-interaction term in Eq.(13.6) for nondynamical fermions, $(\bar{\psi}_{(a)}\psi_{(a)})(\sigma,\zeta)=\mu=$ const, is topologically invariant, being an entanglement index of the loop C_{xx} with respect to the surface $\sum$. As a result our string propagator depends functionally only on the topological class of the $\sum$ surface. This is one of the reasons that we do not consider surface fluctuations in the above-written string propagator.

It is important to point out that it is inconsistent to consider string solutions for Eq.(13.5) which have surface fluctuations since these flucutuations will lead one to consider second-order loop wave equations for $W_{(a)(b)}[C_{xx}]$ as in $QCD[SU(\infty)]$ which is not the case in Chern-Simons gauge theory since it has a nondynamical content $\langle\nabla_i F_{ik}(x)W[C_{xx}]\rangle\equiv 0$. However, the area A induced by the intrinsic fluctuating metric $g_{\mu\nu}$ still is a variable quantity since the metric structure on $\sum$ is fluctuating in Eq.(13.6). So, our string representation differs from that suggested in Ref. 1. Eq.(13.22). Another important remark to be pointed out is related to the conformal invariance of the $O(M)$ string propagator in Eq.(13.6). This propagator has its conformal anomaly canceled if $M=26$, producing, thus, a noncritical string.

Let us show that $G_{(a)(b)}(C_{xx},A)$ satisfies the same loop equation, Eq.(13.5). In order to write the area equation for $G_{(a)(b)}(C_{xx},A)$ we evaluate its area partial derivative as it is exposed in Chapter 11:

$$\frac{\partial}{\partial A}G_{(a)(b)}(C_{xx},A) = -\lim_{\zeta\to 0^+}\left[\int D^c[g_{\mu\nu}]\delta\left(\int_0^{2\pi} d\sigma\int_0^T d\zeta\sqrt{g(\sigma,J)}-A\right)\right.$$
$$\left.\times\left(-\frac{1}{2\sqrt{g}g^{00}}\frac{\overset{\leftrightarrow}{\delta}}{\delta g_{00}(\bar\sigma,\zeta)}\right)I_{(a)(b)}[\psi_{(a)},g_{\mu\nu}]\right], \tag{13.7}$$

where the pure fermionic string propagator is

$$I_{(a)(b)}[\psi,g_{\mu\nu}] = \int D^c[\psi]\psi_{(a)}(0,0)\bar\psi_{(b)}(2\pi,0)\exp\left(-\int_0^{2\pi} d\sigma\int_0^T d\zeta(\bar\psi\, \not{D}_g\psi)(\sigma,\zeta)\right)$$
$$\exp\left(-\lambda\oint_0^{2\pi} d\sigma\frac{\dot X_l}{e(\sigma)}\oint_0^{2\pi} d\sigma'\int_0^T d\zeta\sqrt{g(\sigma',\zeta')}(\bar\psi\psi)(\sigma',\zeta')\delta^{(3)}\right.$$
$$\left.\times(\phi_l(\sigma',\zeta')-X_l(\sigma))e^{ijk}T_{jk}(\phi_l(\sigma',\zeta'))\right). \tag{13.8}$$

By canceling the conformal anomaly by choosing $M=26$ and evaluating the boundary limit of Eq.(13.8) as in Chapter 9 we get the following result for the right-hand side of Eq.(13.7):

$$\frac{\partial}{\partial A}G_{(a)(b)}C_{X(0);X(2\pi)};A) = \lambda\oint_0^{2\pi} dX_i(\sigma)dX_j(\sigma')X_k(\sigma')\varepsilon_{ijk}\delta^{(3)}(X_l(\sigma)-X_l(\sigma'))$$
$$\times G_{(a)(c)}(C_{X(0)X)\sigma)};A)G_{(c)(b)}(C_{X(\sigma)X(2\pi)};A). \tag{9}$$

The above-written equation coincides with the Chern-Simons loop wave equation in the loop proper-time gauge.

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Chapter 14

Fermionic String Representation for the Three-Dimensional Ising Model

14.1. Introduction

It is well known that the partition functional of the three-dimensional $(3D)$ Ising gauge model can be rigorously described on a regular lattice in R^3 by a sum over self-intersecting surfaces [1] on this lattice manifold (here after denoted by Z^3):

$$Z[\beta] = (\cosh\beta)^N \sum_{\{S\}\subset Z^3} \left\{ \exp\left[-A(S)\left(\ln\frac{1}{\tanh\beta} \right) \right] \Phi[\widetilde{C}(S)] \right\}, \tag{14.1}$$

where the sum in the above written equation is defined over the set of all closed two-dimensional surfaces $S \subset Z^3$ with a weight given by the (lattice) area of S; N is the number of the plaquettes, $\beta = J/kT$ denotes the ratio of the Ising hope parameter and the temperature. The presence of the Ising model functional $\Phi[\widetilde{C}(S)]$ inside Eq.(14.1) is a further weight given by the famous sign factor defined on the manifold of the lines of self-intersection $\widetilde{C}(S)$ of a given surface S on the sum Eq.(14.1). Its explicit expression is given by

$$\Phi[\widetilde{C}(S)] = (-1)^{l[\widetilde{C}(S)]} = \exp\{i\pi l[\widetilde{C}(S)]\}, \tag{14.2}$$

where $l]\widetilde{C}(S)]$ denotes the total length of $\widetilde{C}(S) \subset S$.

It has been argued elsewhere [2] that the dependence of the $3D$ Ising model partition functional Eq.(14.1) on the area of the lattice closed surfaces S is a strong indication that, near its critical point, some formal continuum string theory representation should be possible.

In this chapter we address the problem of writing a geometric string path integral involving only the string world-sheet geometry as in our previous work [3], which upon fermionization possesses formally on the lattice the same partition functional given by Eq.(14.1) after a "replica" limit. This study is presented in Sec. I. In the same section we show the usefulness of our proposed string framework for the $3D$ Ising model by writing in the lattice the associated partitional functional in the presence of an external magnetic field.

14.2. The Proposed String Theory

In our previous study, we proposed on formal mathematical grounds the following geometrical path-integral as a continuum limit of the sum Eq.(14.1) without the sign factor [3]:

$$Z(\alpha') = \int d_\mu^{\text{cov}}[g_{ab}, X^i]\delta_{\text{cov}}^{(F)}(g_{ab} - \partial_a X^i \partial_b X_i) \times \exp\left[-\frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty} d^2\xi \left(\frac{1}{2}\sqrt{g}g^{ab}\partial_a X^i \partial_b X_i\right)(\xi)\right]. \quad (14.3)$$

The above written string path integral is the same as that considered by Polyakov [2], but with a fundamental difference: we have used a covariant functional restricting the intrinsic metric field $g_{ab}(\xi)$ to be the string world-sheet-induced metric. As a result, the physical quantum theory obtained after integrating the $g_{ab}(\xi)$ field depends only on the string vector position [after considering $((2\pi\alpha')^{-1} = 1)$] and the metric piece $h_{ab}^{(J)}(\xi)$ related to the metric module space associated with the nontrivial topology of S (see Chapter 12):

$$\begin{aligned} Z = &\int d_\mu^{(\text{Weyl})}[h_{ab}^{(J)}(\xi)]\int D^{\text{cov}}_{\sqrt{h^{(J)}}}[X^i(\xi)] \\ &\times \exp\left[\left(-\frac{1}{2} + \mu_0^2\frac{(26-3)}{48\pi}\right)\right. \\ &\times \left.\int_{-\infty}^{+\infty} d^2\xi(\sqrt{h^{(J)}}h_{ab}^{(J)}\partial^a X^i \partial^b X_i)(\xi)\right] \\ &\times \exp\left[-\left(\frac{26-3}{48\pi}\right)\int_{-\infty}^{+\infty} d^2\xi\{\sqrt{h^{(J)}}h_{mn}^{(J)}\partial_m\right. \\ &\times \left.[\ln(h_{ab}^{(J)}\partial^a X^i \partial^b X_i)]\partial_n[\ln(h_{a'b'}^{(J)}\partial^{a'} X^i \partial^{b'} X_i)]\}(\xi)\right]. \end{aligned} \quad (14.4)$$

At this point we proceed by analogy by searching a continuum functional defined on the physical geometrical string degrees of freedom leading formally on the lattice to the sign factor $\Phi[\widetilde{C}(S)]$. Our purpose is to consider a new intrinsic field $\Omega(\xi)$ taking values on the $SO(3)$ group with a similar role of the intrinsic metric field in Eqs.(14.1)-(14.3). We have, thus, to consider in Eq.(14.1) besides the terms already written there, a further path integral over the $\Omega(\xi)$ field with a weight given by a σ model action added with a Wess-Zumino functional $\Gamma_{WZ}(\Omega)$ and the following $SO(3)$-invariant δ functional:

$$\delta_{\text{Haar}}^{(F)}(\Omega_{ij}(\xi) - \widehat{C}_{ij}(\xi, [X^i], [g_{ab}])). \quad (14.5)$$

Here $\widehat{C}_{ij}$ denotes the (covariant) Cartan matrix relating the orthonormal basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$ to the orthonormal basis defined by the tangent vectors $\{v_1(\xi), v_2(\xi)\}$ and the normal vector $\{v_3(\xi)\}$ on the string surface at the point $\{X^i(\xi)\}$ [4]:

$$e_i = \widehat{C}_{ij}(\xi,]X^i], [g_{ab}])v_j(\xi), \quad (14.6)$$

where

$$v_1^{(i)}(\xi) = \partial_1 X^{(i)}(\xi)/(\partial_1 X^a g^{11}\partial_1 X_a)^{1/2}, \quad (14.7)$$

$$v_2^{(i)}(\xi) = \partial_2 X^{(i)}(\xi)/(\partial_2 X^a g^{22}\partial_2 X_a))^{1/2}, \quad (14.8)$$

$$v_2^{(i)}(\xi) = \left(\frac{v_1(\xi) \wedge v_2(\xi)}{|v_1(\xi) \wedge v_2(\xi)|} \right)^{(i)} \tag{14.9}$$

The geometrical string path integral to be considered now is given by (see Chapter 10)

$$\begin{aligned} \overline{Z}(\alpha') = \int d^{\rm cov}\mu[g_{ab};X^i] D^{\rm cov}_{\rm Haar}[\Omega] \delta^{(F)}_{\rm cov}(g_{ab} - \partial_a X^i \partial_b X_i) \delta^{(F)}_{\rm Haar}(\Omega(\xi) - \widehat{C}(\xi,[X^i],[g_{ab}])) \\ \times \exp\left\{ -\frac{1}{2} \int_{-\infty}^{+\infty} d^2\xi (\sqrt{g} g^{ab} \partial_a X^i \partial_b X_i)(\xi) \right\} \\ \times \exp\left\{ -\frac{1}{2} \int_{-\infty}^{+\infty} d^2\xi (\sqrt{g}\, {\rm Tr}(\Omega^{-1}\partial_a(\Omega)^2)(\xi) \right\} \\ \times \exp\{4\pi i \Gamma_{WZ}[\Omega]\}. \end{aligned} \tag{14.10}$$

where the quantum meaure defining the σ-quantum model is the invariant $SO(3)$ measure associated with the invariant metric

$$dS^2 = \int_{-\infty}^{+\infty} d^2\,\xi [\sqrt{g}\,{\rm Tr}(\Omega^{-1}\delta(\Omega)^2](\xi). \tag{14.11}$$

It is an important step in our study to consider the fermionic version of the above displayed σ-model path integral as a result of the presence of the Wess-Zumino functional in Eq.(14.10): (see Appendix 22-E).

$$\begin{aligned} \overline{Z}(\alpha') = \int d^{\rm cov}\mu[g_{ab};X^i] d^{\rm cov}\mu[g_{ab};X^i] d^{\rm cov}[\psi_A;\overline{\psi}_A] \delta^{(F)}_{\rm cov}(g_{ab} - \partial_a X^i \partial_b X_i) \\ \times \exp\left(-\frac{1}{2\pi a'} \int_{-\infty}^{+\infty} d^2\xi (\sqrt{g}\ {}^{ab} \partial_a X^i \partial_b X_i)(\xi) \right) \\ \times \exp\left\{ -\frac{1}{2} \int_{-\infty}^{+\infty} d^2\xi \left[\sum_{A=1}^{3} (\sqrt{g}\overline{\psi}_A(\gamma^a \nabla_a)\psi_A)(\xi) \right] \right\} \end{aligned} \tag{14.12}$$

Here, the Dirac curved space-time matrices satisfy the usual (Euclidean) anticommuting relationship $\{\gamma^a(\xi),\gamma^b(\xi)\}_+ = g^{ab}(\xi) = e^a_{b'}(\xi) e^{bb'}(\xi)$ and the spin connection is given by the following expression involving the surface Cartan matrix:

$$\omega_a(\xi) = e^{a'}_a \gamma_{a'}(\xi)(\widehat{C}^{-1}\partial_a \widehat{C})(\xi). \tag{14.13}$$

Let us now give a formal argument that the string theory Eq.(14.12) represents the $3D$ Ising model at a replica limit on the geometrical fermionic degrees of freedom. In order to implement such an argument, we introduce N copies of the fermionic field $\{(\psi_A^{(m)}.\overline{\psi}_A^{(m)})_i\, 1 \le m \le N\}$ in the fermionic action Eq.(14.12). After integrating out these fermion fields, writing the fermionic functional determinant by the Grassmanian proper-time technique implemented on the surface loop space (see [5], Appendix B) and using the well-known replica limit on the fermion species, we have the following loop space path integral for the fermionic effective action in Eq.(14.12) (see Chapter 18 for details):

$$\lim_{N\to 0}(\det{}^N(\gamma^a\nabla_a)-1)/N$$
$$=\frac{1}{2}\int_0^\infty dT\ \exp(-l(C_a)T)\int_{-\infty}^{+\infty} d^2\xi\sqrt{g(\xi)}$$
$$\times \mathrm{Tr}_{\mathrm{Dirac}}\left\{\int_{C_a(0)=C_a(T)=\xi_a} D[C_a(t)]D[\pi_a(t)]\right.$$
$$\times \exp\left(i\int_0^T dt\,\pi_a(t)dC_a(t)\right)$$
$$\times P_{\mathrm{Dirac}}\left\{\exp\left(i\int_0^T dt(\gamma^a\pi_a)(t)\right)\right\}$$
$$\left.\times \mathrm{Tr}_{SO(3)}\left\{\exp\left[i\int_0^T dt\left((\widehat{C}^{-1}\partial_a\widehat{C})(t)\,\frac{dC^a(t)}{dt}\right)\right]\right\}\right\} \qquad (14.14)$$

where $\{l_a(t)\}$ belongs to the manifold of closed bosonic trajectories on the string surface and $\{\pi_a(t)\}$ the Grassmanian degrees of freedom associated to the $2D$ Dirac indexes. If one considers formally the above replica limit on the lattice, one can see that the Wilson loop defined by the Cartan matrix in Eq.(14.14) coincidess exactly with the sign factor as Sedrakyan and Kavalov showed by using topological-homotopical techniques.

As a consequence, we have the following string representtion at the critial point for the $3D$-Ising model with $\beta=\ \mathrm{arctanh}(e^{-1/2\pi\alpha'})$,

$$Z_{\text{critical point}}[B]=\int d^{\mathrm{cov}}\mu[g_{ab};X^i]\delta^{(F)}_{\mathrm{cov}}(g_{ab}-\partial_aX^i\partial_bX_i)$$
$$\times\exp\left(-\frac{1}{2\pi\alpha'}\int_{-\infty}^{+\infty}d^2\xi(\sqrt{g}g^{ab}\partial_aX^i\partial_bX_i)(\xi)\right)$$
$$\times\lim_{N\to 0}\left\{\frac{\det^N(\gamma^\mu\nabla_a)-1}{N}\right\} \qquad (14.15)$$

This is our main result in this chapter.

It is worth remarking that all of the above results are of a formal mathematical nature and real checks will be to compute (at least numerically) physical quantities. However, one can use Eq.(14.15) to suggest some new formulas on the lattice. Let us show the usefulness of Eqs.(14.12)-(14.15) by coupling the proposed Ising string theory to an external magnetic field $\vec{H}(\xi)$ by means of the well-known string electromagnetic flux action (see Chapter 10):

$$\exp\left\{-\frac{1}{2}e\int_{-\infty}^{+\infty}d^2\xi\sqrt{g(\xi)}H^i[X^j(\xi)]\partial_aX_i(\xi)\right\}\left(\sum_{A=1}^{3}\bar{\psi}_A(\gamma^a)\psi_A\right)(\xi). \qquad (14.16)$$

By considering the replica limit of the resulting string path integral as in Eq.(14.14), we obtain as a candidate for the partition Ising model in the presence of the external magnetic field the following sum over closed surfaces on the lattice:

$$Z[\beta,e\vec{H}]=(\cosh\beta)^N\sum_{\{S\}\subset Z^3}\left\{\exp\left[-\left(\ln\frac{1}{\tanh\beta}\right)A(S)\right]\right.$$
$$\left.\times\Phi[\widetilde{C}(S)]\times W[C(S)]\right\}, \qquad (14.17)$$

where we note the appearance of the usual Wilson loop defined by the $2D$-closed loops $\{l^a(t)\}$ on the surface S and the external magnetic field:

$$W[C(S)] = \prod_{\{C(S)\subset S\}} \left\{ \exp\left(ie \int_{C(S)} \widetilde{H}^a[C^b(t)] \frac{dC_a(t)}{dt} \right) \right\}, \tag{14.18}$$

where $\widetilde{H}^a[C^b(t)]$ is the restriction of the surface magnetic flux $H^i(X^j(\xi))\partial_a X_i(\xi)$ to the $2D$ loop $\{l^a(t), a = 1,2\}$ which are obtained from the string surface parametrization by supposing an implicit relation of the form $[\xi = (\xi_1, \xi_2)]$

$$\xi_2 = \beta(\xi_1) \Rightarrow X^i(\xi_1, \beta(\xi_1)) = X^i(C_1(\xi), C_2(\xi)). \tag{14.19}$$

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Chapter 15

A Polyakov Fermionic String as a Quantum State of Einstein Theory of Gravitation

15.1. Introduction

In recent years a new quantization of Einstein gravitation theory has been pursued by several authors, which seems appropriate for writing explicit solutions of the Wheeler-De Witt equation. It makes use of the so called $SU(2)$-Ashtekar-Sen connection as dynamical variable (see Refs. [1,2]) which has the geometrical meaning of being the projected spin connection on the space-time (three-dimensional) boundary [3].

A linear wave equation for this new quantum gravity dynamical variable was derived which supports a Wilson Loop solution (see Chapter 7).

In this chapter, following our previous studies in this subject (Chapter 9-Chapter 10; [4]), we consider a new solution for the above mentioned equation defined by a Polyakov fermionic string functional integral (Chapter 9 and [5]).

15.2. The Quantum Gravity String

Let us start our analysis by considering the following Polyakov string functional integral in the presence of a $SU(2)$ connection $A_\mu(x)$

$$G_{AS}[A_\mu(x);l_\mu(\sigma)] = \int a^{\text{cov}}\mu[g_{AB}]d^{\text{cov}}\mu[\psi,\overline{\psi}]d^{\text{cov}}\mu[X^a]\left(\sum_{i=1}^{3}[\psi^i_A(0,0)\overline{\psi}^i_B(0,2\pi)]\right)$$

$$\times \exp\left\{-\mu_0^2\int_{-\infty}^{+\infty}d\xi\int_0^{2\pi}d\sigma(\sqrt{g}(\xi,\sigma))\right\}\exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty}d\xi\int_0^{2\pi}d\sigma(\sqrt{g}g^{AB}\partial_A X^\mu\partial_B X_\mu)(\xi,\sigma)\right\}$$
$$\times \exp[\left\{-\frac{1}{2}\int_{-\infty}^{+\infty}d\xi\int_0^{2\pi}d\sigma\right.$$
$$\left.\times\left[(\psi,\bar{\psi})\begin{pmatrix} 0 & (\overrightarrow{\partial}_g+e(\gamma^A\partial_A X^\mu)(A_\mu^\ell(X^p)\lambda_l) \\ (\overleftarrow{\partial}_g+e(\gamma^A\partial_A X^\mu)(A_\mu^\ell(X^p)\lambda_l) & 0\end{pmatrix}\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix}\right](\xi,\sigma)\right\} \quad (15.1)$$

The open string surface $\{X^\mu(\xi,\sigma),\mu=1,2,3\}$ is immersed in the space-time (three-dimensional) boundary and does not possess holes and handles. The string surface parameter domain is taken to be the half-strip $R_{2\pi}^2=\{(\xi,\sigma,-\infty\le\xi\le+\infty;0\le\sigma\le 2\pi\}$ without loss of generality. The Polyakov two-dimensional quantum gravity (string) metric is denoted by $(g_{AB}(\xi,\sigma))]$ and satisfies the trivial topological condition $\int_{-\infty}^{+\infty}d\xi\int_0^\pi d\sigma(\sqrt{g}R(g))(\xi,\sigma)=2\pi$. The two-dimensional intrinsic fermions Dirac fields belong to a complex $SU(2)$ fundamental representation and are denoted by $\{\psi_A^i(\xi,\sigma);\bar{\psi}_A^i(\xi,\sigma)\}$ with the subscript A associated to the two-dimensional string (Euclidean) Lorentz Group $SO(2)$ and the superscript i associated to the $SU(2)$ group index. The interaction of the Polyakov string and the $SU(2)$ three-dimensional Ashtekar-Sen connection is given by the explicit interaction of the $SU(2)$ connection flux and the intrinsic fermion current as in the $SU(2)$. *QCD* gauge theory (see Chapter 10).

The functional measures in the Polyakov string funcitonal integral are the well-known De-Witt covariant functional measures with boundary terms (we take the string boundary $X_\mu(0,\sigma)=l_\mu(\sigma)$ to have zero geodesic induced curvature [6,7]).

Let us show that Eq.(15.1); which may be considered as the Polyakov string propagator with a $SU(2)$ *QCD* action and describing the “creation” of a string $\{l_\mu(\sigma)\}$ from the vacuum $\{0\}$ in the (Euclidean) space-time boundary; satisfies the Wheeler-De Witt equation in terms of Ashtekar variables [4]

$$\left(\varepsilon^{ijk}F_{\mu\nu}^i(A)(x)\times\frac{\delta^2}{\delta A_\mu^j(x)\delta A_\mu^k(s)}\right)G'_{AB}[A_\mu(x),l_\mu(\sigma)]=0. \quad (15.2)$$

A straightforward calculation shows that [4]

$$\int_M d^3x\left(\varepsilon^{ijk}F_{\mu\nu}^i(A)\frac{\delta^2}{\delta A_\mu^j\delta A_\nu^k}\right)(x)G_{AB}[A_\mu(x),l_\mu(\sigma)]$$
$$=\left\langle\int_{R^2_{(2\pi}}d\xi d\sigma\int_{R^2_{(2\pi)}}d\xi' d\sigma'(\sqrt{g})(\xi,\sigma)(\sqrt{g})(\xi',\sigma')(\partial^C X^\mu)(\xi,\sigma)(\partial^D X^\mu)(\xi',\sigma')\right.$$
$$\times(e^2(\mu)_{AB}(\delta^{(3)}(X^\mu(\xi,\sigma)-X^\mu(\xi',\sigma'))(\bar{\psi}\gamma_C\lambda^j\psi)(\xi,\sigma)(\bar{\psi}\gamma_D\lambda^k\psi)(\xi',\sigma')$$
$$\left.\times(\varepsilon^{ijk}F_{\mu\nu}^i(A))(X^\rho(\xi,\sigma))\right\rangle \quad (15.3)$$

where $\langle\rangle$ denotes the string average defined by the covariant string path integral equation (Eq.(15.1)). $e^2(u)_{AB} = \sum_{i=1}^{3} \psi_A^i(0,0)\overline{\psi}_B^i(0,2\pi))$ is the constant matrix fermion number density projected on the string boundary $\ell_\mu(\sigma)$

In order to evaluate Eq.(15.3), we note that the condition that the string surface $\{X^\mu(\xi,\sigma), \mu = 1,2,3\}$ does not posseses self-intersections leads to the following regularized expression for the delta-function string surface term in Eq.(15.3)

$$\delta^{(3)}(X^\rho(\xi,\sigma) - X^\rho(\xi',\sigma')) = \frac{1}{\sqrt{h(X^\rho(\xi,\sigma))}}\delta(\xi-\xi')\delta(\sigma-\sigma')\cdot\delta_0^{(1)}(\varepsilon) \tag{15.4a}$$

with

$$\begin{aligned} h &= \det(h_{AB}) \\ h_{AB} &= (\partial_A X^\mu B_B X_\mu)(\xi,\sigma) \end{aligned} \tag{15.4b}$$

where $\delta_\varepsilon^{(1)}(0)$ is a regularized form of the singular term $\delta^{(1)}(0)$ (see Ref. [5] for details).

The evaluation of the fermionic functional integral average in Eq.(15.3) is straightforward, since in two-dimensional *QCD* $(SU(2))$ one can use the Roskies gauge decoupling fermion gauge [9.9] and thus, the ultra-violet limit implied by Eq.(15.4) leads that the average of the fermion currents in Eq.(15.3) is effectively defined by Fermion free fields (asymptotic freedom). It yields terms of the form

$$\begin{aligned} &\frac{a_{\bar{A}}}{a^2}\frac{a_{\bar{B}}}{a^2}(\varepsilon^{\bar{A}A}\varepsilon^{\bar{B}B} \pm \delta^{\bar{A}A}\delta^{\bar{B}B} \pm \delta\bar{A}\varepsilon^{\bar{B}B} \pm \delta^{\bar{B}B}\varepsilon^{\bar{A}A})[e^2\delta_\varepsilon^{(1)}(0)]\frac{(\partial_A X^\mu)(\xi,\sigma)(\partial_B X^\nu)(\xi,\sigma)}{\sqrt{h(X^p(\xi,\sigma))}} \\ &\times F_{\mu\nu}(A(X^\alpha(\xi,\sigma)) \end{aligned} \tag{15.5}$$

where the *UV* regularized form of the fermion propagator used to obtain Eq.(15.5) is given by

$$S_{AB}^{ij}((\xi,\sigma);(\xi',\sigma'))^{(a)} = \frac{i(\gamma^1)(\xi-\xi'+a)+i(\gamma^2)(\sigma-\sigma'+a)}{(\xi-\xi'+a)^2+(\sigma-\sigma'+a)^2}\delta^{ij}. \tag{15.6}$$

By absorbing the two-dimensional UV infinity $a \to 0$ in the bare model coupling constant $e^2\delta_\varepsilon^{(1)}(0)$, we can follow the argument of Refs. [2,4] to conclude that Eq.(15.5) vanishes identically as a consequence of being a contraction of the antisymmetric (μ,ν) tensor $F_{\mu\nu}(A(X^\alpha(\xi,\sigma)))$ and the (μ,ν) symmetric tensor in front of the above mentioned tensor in Eq.(15.5). It is worth pointing out that we have used the Polyakov conformal gauge

$$g_{AB}(\xi,\sigma) = e^{\varphi(\xi,\sigma)}\,\delta_{AB} \tag{15.7}$$

in the above calculations in order to factorize the metric field dependence of the fermionic propagator under analysis.

Another important observation to be made is that proposed Polyakov string quantum gravity state Eq.(15.11) contains the usual Wilson loop quantum gravity state celebrated in the literature [1-3] as a simple overall factor. In order to show this claim it is enough to integrate the fermions fields in the string path integral to obtain the result ([2])

$$\begin{aligned} G_{AB}[A_\mu(x);l_\mu(\sigma)] &= \mathrm{Tr}_{SU(2)}\, P\left\{\exp ie\oint_{l_\mu} A_\mu\, dX^\mu\right\}(\mu^{-1})_{AB} \\ &\times \det\big((\not\partial_g + \not B^\ell[X^\alpha(\xi,\sigma)]\lambda_\ell)(\not\partial_g + \not B^\ell[X^\alpha(\xi,\sigma)]\lambda_i)^*\big) \end{aligned} \tag{15.8}$$

where the two-dimensional QCD external $SU(2)$ gauge field entering in the fermion funcitonal determinant in Eq.(15.8) is given explicitly by the $2D$ surface induced $SU(2)$ gauge field

$$B_A^\ell[X^\alpha(\xi,\sigma)] = (A_\mu^\ell(X^\alpha(\xi,\sigma))\lambda_\ell)(\partial_A X^\mu(\xi,\sigma)) \tag{15.9}$$

Note that the appearance of the Wilson Loop functional in Eq.(15.8) is nothing more than the (boundary) fermion propagator associated to our dermion boundary current in Eq.(15.1) projected on the spatial loop $l_\mu(\sigma) = X^\mu(0,\sigma)(X^\mu(0,0) = x^\mu)$.

Let us comment the results presented in this chapter differ some what from those of Ref. [4] since here we have not considered the theory of self-avoiding string neither the restrictive Ashtekar-Sen connection boundary condition $(\partial_\mu^x F^{\mu\nu}(A)(x) \equiv 0)$; both conditions necessary to obtain the validity of the results presented in this reference.

At this point of our chapter, the question of physical observable suitable to our string quantum gravity state Eq.(15.1) should be considered. We start our discussion on this very important question by calling attention that it remains an open problem to understand canonical quantum gravity in light of the Copenhagen school interpretation of quantum mechanics. In the Wheeler-De Witt (canonical) frame work, there is no time parameter in the associated quantum gravity Schrödinger equation (the well-known Wheeler-De Witt equation (Eq.(15.2))) (see Chapter 11).

As a consequence, the operation of taking quantum system averages makes no sense physically for the observer. Not that there are no bound-states, currents, energy observables, etc. in the canonical Wheeler-De Witt quantum gravity framework. There is only, in principle, the "vacuum" state of the $3D$ geometry satisfying the homogeneous Wheeler-De Witt equation and that was the main reason for the search of new field parametrization in Einstein quantum gravitation theory. We remark that among these frameworks for Quantum Gravity the Ashtekar-Sen parametrization is the most promising scheme devised until now, since it leads to a mapping of the $3D$ metric field to the well studied $SU(2)$ gauge theory (the old Faraday line interpretation for fields) and making, thus, the original nonlinear Wheeler-De Witt equation a linear wave quation in terms of these new variables (Chapter 7).

However, some geometrical (non-physical) objects have been studied [9] and leading to the result that the Ashtekar-Sen-Smolin Wilson Loop associated to smooth loops are eigenstates of these geometrically operators with eigenvalues given by the entanglement index of these infinitely differentiable loops with the smooth surface and smooth volume which are fixed by an (somewhat unphysical) observer measuring area and volume in the $3D$ geometry.

Following these attempts to evaluate formal observables in order to get a better insight in this very difficult problem, we remark that our string quantum stte may be useful to evaluate a kind of spatial gravitation propagator given by the dollowing quantum state average (see appendix of Ref. [5])

$$\left\langle G[A_\mu;\ell_\mu]|(\widehat{\sqrt{g}g^{ij}})(x)(\widehat{\sqrt{g}g^{kl}})(x)|0\right\rangle. \tag{15.10}$$

This object has a formal meaning of describing the process of a "spatial graviton" propagation from the pure vacuum state (nothing) to our proposed string state equation (Eq.(15.1)) defined by the Ashtekar-Sen connection $A_\mu(x)$ and loop $l_\mu(\sigma)$.

Following the Copenhagen School interpretation, we substitute the metric operations below [2]

$$(\widehat{\sqrt{g}g^{ij}})(x) = \delta^2/\delta A^i_\mu(x)\delta A^j_\mu(x) \tag{15.11}$$

$$(\widehat{\sqrt{g}g^{kl}})(y) = \delta^2/\delta A_\nu^k(y)\delta A_\nu^l(y) \tag{15.12}$$

inside the Polyakov string path integral representing the non-trivial quantum state in Eq.(15.10). As a consequence, we can easily write the "3D graviton propagator" as a two-point Polyakov string scattering amplitude associatred to our proposed string theory Eq.(15.1). Studies of the possible relevance of these scattering amplitudes for quantum gravity will be intentionally left to our readers.

Finally the argument that another surface solution with a topology of a cylinder may be obtained by simply taking the Wilson loop of the Ashtekar-Sen connection along a one parameter family of closed loops in the spatialmanifold, and integrate the resulting one-parameter family of numbers over the parameter is not correct since this object is not defined as a functional over the surface vector position $\{X_\mu(\xi,\sigma)\}$ and, thus, losing all meaning of a functional of the cylinder surface. The above cited construction is nothing more than a superposition of the Wilson Loop solutions which still satisfies the Wheeler-De Witt equation written in terms of Ashtekar-Sen variables, since this Schrödinger quantum gravity equation is linear in this $SU(2)$ gauge field parametrization. As a consequence of these remarks, this kind of superposition loop solutions do not bring new features besides those already studied in Ref. [9]. Note that our proposed solution being a string theory *opens* the possibility of using all machinery of 2D-quantum field models (see Chapters 16 and 17 and [10]-[11]) to understand four-dimensional Einstein quantum gravity.

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[12] For the case of vanishing small string lenght scale $\alpha' \to 0$, which means that $X^{\alpha}(\xi,\sigma) = \overline{X}^{\alpha} + \sqrt{\alpha^{a}}Y^{\alpha}(\xi,\sigma)$ with $Y^{\alpha}(\xi,\sigma)$ denoting the minimal surface bounded by the loop $\ell_{\mu}(\sigma)$, we have the following leading approximate expression for the Fermionic Functional Determinant under analysis (see chapter 21)

$$\det[(\not\partial_g + \not B)] \underset{\alpha'\to 0}{\sim} \exp\left\{-\frac{1}{\pi}\int_D d\xi d\sigma\sqrt{g}Tr_{SU(2)}(\not B_A(\xi))^2\right\}$$

$$\overset{\alpha'\to 0}{\sim} \left[\exp\left\{-\frac{\alpha'}{\pi}\int_D d\xi d\sigma\sqrt{g}(\partial_a Y^{\alpha})^2\right\}\exp\{-\frac{\alpha'}{\pi}[Tr_{SU(2)}F^2(A)(\overline{X}^{\alpha}]\}\right]$$

Chapter 16

A Scattering Amplitude in the Quantum Geometry of Fermionic Strings

16.1. Introduction

Polyakov [1,2] has developed a formalism for closed strings quantization, later further generalized by including the case of open strings [3-5].

An important problem in the formalism concerns the definition of a scattering amplitude for these strings, whose knowledge affords (in principle) the determination of the associated spectrum. A natural definition for these scattering amplitudes remains, however, the main problem. Probably its complete solution will require the determination of the exact *QCD* string (Chapter 9).

In the lack of a *QCD* scattering definition, a suggestion for the closed bosonic string was put forward by Polyakov [1] and generalized for the bosonic open string case in ref. [3]. A remarkable feature of these scattering amplitudes is that the standard dual (Veneziano) model can be easily obtained in a saddle point approximation [3].

Our aim in this chapter is to propose a scattering amplitude for the open fermionic string [2,5] with the property that the spectrum does not possess the usual tachionic excitation in the saddle point approximation $D \to -\infty$, and leading thus to the solution of a long-standing problem in Quantum Geometry of strings as the correct Dual Model theory for Strong Interactions.

16.2. The Scattering Amplitude

Let us start our analysis by considering the fermionic string action in a D-dimensional euclidean space-time [2,5,8,9]; namely

$$S[\phi^{(A)}(\xi),\psi^{(A)}(\xi),e^{a}_{\mu}(\xi,\chi_{\nu}(\xi)] = \int_{\mathcal{D}} d^2\xi e(\xi)[\frac{1}{2}d_{\mu}\phi^{(A)}\partial_{\nu}\phi^{(A)}g^{\mu\nu} + \frac{1}{2}i\psi^{(A)}\gamma_{\mu}D_{\mu}\psi^{(A)}$$
$$-\frac{1}{2}F^2 - \frac{1}{2}i(\chi_{\mu}\gamma^{\nu}\gamma^{\mu}\psi^{(A)}(\phi_{\nu}\phi^{(A)} - \frac{1}{4}i\chi_{\nu}\psi^{(A)})](\xi). \quad (16.1)$$

Here the fermionic string is charactereized by two fields; firstly, the vector-position $\phi^{(A)}(\xi)$ $(A=1,\dots,D)$ and secondly by $\psi^{(A)}(\xi)=(\psi_1^{(A)}(\xi),\psi_2^{(A)}(\xi))$, a two-dimensional Majorana spinor describing the string fermionic degrees of freedom. $\mathcal{D}$ denotes a two-dimensional parameter domain (embedded in the euclidean space) with the boundary denoted by $\partial\mathcal{D}$. The presence of the vierbein $e_\mu^a(\xi)$ and of the two-dimensional vector-Majorana spinor $\chi_\mu(\xi)$ together with the auxiliary scalar field $F(\xi)$ insure respectively that the action (16.1) is invariant under general Lorentz and coordinate trnsformations, and local supersymmetry transformation [5,8,9].

The average of a functional $W(\phi^{(A)}(\xi),\psi^{(A)}(\xi))$ defined on the fermionic string random surface is given by the following prescription:

$$\langle W[\phi^{(A)}(\xi),\psi^{(A)}(\xi)]\rangle_F=\frac{1}{Z}\Bigg(\int D[\phi^{(A)}(\xi)]D[\psi^{(A)}_{(\xi)}]$$
$$\times D[\varepsilon_\mu^a(\xi)]\cdot D[\chi_\mu(\xi)\exp\{-S[\phi^{(A)}(\xi),\psi^{(A)}(\xi),e_\mu^a(\xi),\chi_\nu(\xi)]\}W[\phi^{(A)}(\xi),\psi^{(A)}(\xi)]\Bigg),\quad(16.2)$$

where Z denotes the usual measure normalization factor.

The functional measures in (16.2) are invariant under local supersymmetry, and general Lorentz and coordinate transformations. They are obtained as the functional element of volume associated to the following functional Riemann metrics (Chapter 1):

$$||\delta\phi^{(A)}||^2=\left(\int_{\mathcal{D}}d^2\xi\, e(\xi)[\delta\phi^{(A)}(\xi)\cdot\delta\phi^{(A)}(\xi)]\right)+\Gamma_1(\chi_\mu(\xi),\phi^{(A)}(\xi),\psi^{(A)}(\xi),e_\mu^a(\xi)),\quad(16.3a)$$

$$||\delta\psi^{(A)}||^2=\left(\int_{\mathcal{D}}d^2\xi\, e(\xi)[\delta\psi^{(A)}(\xi)\cdot\delta\psi^{(A)}(\xi)]\right)+\Gamma_2(\chi_\mu(\xi),\phi^{(A)}(\xi),\psi^{(A)}(\xi),e_\mu^a(\xi)),\quad(16.3b)$$

$$||\delta e_\mu^a||^2=\left(\int_{\mathcal{D}}d^2\xi\, e(\xi)[e_a^{\mu'}e_{a'}^{\mu}(\delta e_\mu^a)(\delta e_{\mu'}^{a'})+ce_a^\mu e_{a'}^{\mu'}(\delta e_\mu^a)(\delta e^{a'}\mu')+c'e_a\mu e^{a\mu'}(\delta e_a^\mu e^{a\mu'}(\delta e_\mu^a)(\delta e_{a\mu'})]\right)$$
$$\Gamma_3(\chi_\mu(\xi),\phi^{(A)}(\xi),\psi^{(A)}(\xi),e_\mu^a(\xi)),\quad(16.3c)$$

$$||\delta\chi_\mu(\xi)||^2=\left(\int_{\mathcal{D}}d^2\xi\, e(\xi)[g^{\mu\nu}\delta\chi_\mu\cdot\delta\chi_\nu(\xi)]\right)+\Gamma_4(\chi_\mu(\xi),\phi^{(A)}(\xi),\psi^{(A)}(\xi),e_\mu^a(\xi)),\quad(16.3d)$$

where c and $c'>1$ are arbitrary constants and $\Gamma_i(\chi_\mu(\xi),\phi^{(A)}(\xi),\psi^{(A)}(\xi),e_\mu^a(\xi))$ $(i=1,\dots,4)$ represents term of these functional metrics which vanish for $\chi_\mu(\xi)\equiv 0$ and insure invariance of the associated element of volume by local supersymmetry transformations. As we will explain below, its explicit expression is not necessary.

For the evaluation of the average (16.2), one has to fix the gauge associated to the local symmetries of the action (16.1), quoted above. As proposed by Polyakov [2], a natural gauge is the super-conformal gauge specified by the relations

$$e^a_\mu(\xi) = \exp[\delta(\xi)]\delta^a_\mu\,,\; e(\xi) = \exp[2\delta(\xi)] = \rho(\xi))\,,\; \chi_\mu(\xi) = \frac{1}{2}\gamma_\mu\chi(\xi) = \exp[-\frac{1}{2}\delta(\xi)]\gamma_\mu\zeta(\xi). \tag{16.4}$$

Thus, the integrand becomes an effective functional of the fields $\delta(\xi)$, $\zeta(\xi)$ and an auxiliary field $f(\xi)$ necessary to insure the remnants of the local analytic supersymmetry, which are not destroyed by the gauge (16.4). Because of this residual symmetry, we can evaluate (16.2) for $\chi_\mu(\xi) \equiv 0$ and use this residual super symmetry to determine the dependence of the effective integrand in terms of the fields $\zeta(\xi)$ and $f(\xi)$. We notice that, as a consequence of this fact, we need not know the exressions $\Gamma_i(\chi_\mu(\xi), \phi^{(A)}(\xi), \psi^{(A)}(\xi), e^a_\mu(\xi))$ in (16.3a)-(16.3d).

After having described above the formalism to compute averages in the theory, we now pass on to the problem of defining an off-shell scattering amplitude. For this task, we follow our basic idea: the proposed N-point of-shell scattering amplitude is given by the sum over all fermionic random surfaces which contains a given set of fixed points $\{X_j\}$ $(j = 1,\ldots,N)$, i.e.: (see Chapter 8)

$$A(X_1,\ldots,X_N) = \left\langle \prod_{j=1}^{N} d^2\xi_j^{(H)} e(\xi_j) d\theta_1^{(j)} d\theta_2^{(j)} \delta^{(D)}(\phi^{(A)}(\xi_j) + i\theta_1^{(j)}\psi_1^{(A)}(\xi_j) + i\theta_2^{(j)}\psi_2^{(A)}(\xi_j) - X_j) \right\rangle, \tag{16.5}$$

where $\phi^{(A)}(\xi_j) + i\theta_1^{(j)}\psi_1^{(A)}(\xi_j) + i\theta_2^{(j)}\psi_2^{(A)}(\xi_j)$ denotes the "fermionic-position" of the fermionic string random surface with $(\theta_1^{(f)}, \theta_2^{(f)})$ grassmanian parameters, and $\prod_{j=1}^{N} d^2\xi_j^{(H)}$ is the Möbius invariant Haar measure, which takes into account the (physical) residual symmetry of the projective group not fixed by the conformal gauge $e^a_\mu(\xi) = \exp[\delta(\xi)]\delta^a_\mu$. Their explicit expression is given by

$$\prod_{j=1}^{N} d^2\,\xi_j^{(H)} = \prod_{\substack{j=1\\ j\neq a,b,c}}^{N} d^2\xi_j |\xi_b - \xi_a|^2\, |\xi_c - \xi_a|^2. \tag{16.6}$$

The indices a, b, c are fixed but choosen arbitrarily. We observe that the effective number of integrated variables in (16.6) is $N-3$ and is related to the maximum number of mutually non-overlapping channels of the scattering process.

The physical spectrum is determined by considering the poles in the $\{X_j\}$-Fourier transformed expression for such amplitude, whose associated residues are identified with the on-shell scattering amplitudes.

In order to evaluate (16.5) is convenient to write (16.5) in momentum-space:

$$\widehat{A}(P_1,\ldots,P_N) = \left\langle \int_{\mathcal{D}} \prod_{j=1}^{N} d^2\xi_j^{(H)} e(\xi_j) \exp[i(P_j^{(A)};\phi^{(A)}(\xi_j))](P_j^{(A)};\psi_1^{(A)}(\xi_j))(P_j^{(A)};\psi_2^{(A)}(\xi_j)) \right\rangle_F, \tag{16.7}$$

where $(\,;\,)$ means the euclidean scalar product over the Lorentz indices.

On the super-conformal gauge (16.4), the interaction lagrangian involving the vector-spinor $X_\mu(\xi)$ vanishes and the functional integration over the "matter" fields $(\phi^{(A)}(\xi,\psi^{(A)}(\xi))$ becomes of the gaussian type. In order to evaluate these functional integrations we have to choose appropriate boundary conditions since we are in the presence of a quantum theory defined in a two-dimensional space-time $\mathcal{D}$ with a non-trivial boundary. At this point we fix the domain $\mathcal{D}$ as the upper-half plane R_2^+ with the real axis being the boundary. Then, we assume as in ref. [5] that the "matter fields" satisfy the supersymmetric boundary conditions corresponding to the Neveu-Schwarz model (see eqs.(16.3)-(16.7) in ref. [1], we also ref. [5]) and the Faddeev-Popov determinants associated to (16.4), the boundary condtions as discussed in ref. [4].

By introducing the family of self-adjoint operators acting on an appropriate space of two-component real functions on R_2^+ with boundaary conditions indicated by N (Neumann) or D (Dirichlet) [4],

$$\mathcal{L}_j = \big(-\rho^{-(j+1)}\,\partial_{\bar{z}}\rho^j\,\partial_z\big), \tag{16.8}$$

we can thus perform the gaussian functional integration over the scalar field $\phi^{(A)}(\xi)$ with the result

$$\mathrm{Det}^{-D/4}(\mathcal{L}_0^{NN})\exp\left[-\left(\sum_{(i,j)=1}^{N}(P_i^{(A)},P_j^{(A)})K^{(e)}(z_j,z_j,2\delta(z_i,z_i^*))\right)\right], \tag{16.9}$$

where $K^{(e)}(z,z',2\delta(z,z^*))$ is the conformally regularized Green function for the laplacian in the metric $g_{\mu\nu}(z,z^*)=\exp[2\delta(z,z^*)]\delta_{\mu\nu}$ with the Neumann boundary conditions along the real axis [3]. Its expression reads:

$$\begin{aligned} K^{(\varepsilon)}(z,z',2\delta(z,z^i)) &= -(1/2\pi)(\ell n|z-z'||z-z'^*|) \;\; z\neq z' \\ &= \delta(z,z^*)/2\pi-(1/4\pi)\ell n\,\varepsilon-(1/2\pi)\ell n|z-z| \;\; z=z' \end{aligned} \tag{16.10}$$

The integration over the Majorana fields $\psi^{(A)}(\xi)$ is carried out by using the fact that the Green function $(i\gamma_\mu D_\mu)^{-1}_{(N)}(z_i,z_j)$, with the Neumann boundary conditions along the real axis, is related to the corresponding flat propagator $(i\gamma_a\partial_a)^{-1}_{(N)}(z_i,z_j)$ by (see eq. (6.11) in ref. [10])

$$(i\gamma_\mu D_\mu)^{-1}(z_i,z_j) = \exp[-\delta(z_i,z_i^*)](i\gamma_a\partial_a)^{-1}_{(N)}(z_i,z_j)\exp[-\delta(z_j,z_j^*)], \tag{16.11}$$

where

$$(i\gamma_a\partial_a)^{-1}_{(N)}(z_i,z_j) = (i\gamma_a\partial_a)[-(1/2\pi)\ell n(|z_i-z_j|\,|z_i-z_j^*|)]. \tag{16.12}$$

As again the functional integration over the Majorana fields are gaussian, we get the result:

$$\mathrm{Det}^{D/4}(\mathcal{L}^{ND}_{-1/2})\left\{ \exp\left[-\left(\sum_{i=1}^{N}\delta(z_i,z_j^*)\right)\right] \right.$$
$$\left. \times \sum\left(\prod_{(i,j)}^{N}(P_i^{(A)};P_j^{(A)})\prod_{(\alpha_1,\alpha_2)}((i\gamma_a\partial_a)^{-1}_{(N)}(z_i,z_j))_{\alpha_1\alpha_2}\right)\right\}, \tag{16.13}$$

where the Σ in (16.13) means that we have to sum over all ways of pairing the fermion fields in (16.7) and the subscripts (α_1,α_2) denotes the matrix indices of the propagator (16.12).

We note that N should be an even number. This implies that the Polyakov fermionic string model possesses a quantum number which is subject to conservation and can be related to the $NS-G$ parity [11]. By evaluating the Faddeev-Popov determinants associated to the gauge (16.4), we get the effective action and hence the final expression conformally regularized for the n-point off-shell scattering amplitude.

$$\widehat{A}^{(E)}(P_1,\dots,P_N)=\frac{1}{2}\left\{\int D[\delta]D[\zeta]D[f]\exp(-S_{\mathrm{eff}}[\delta,\zeta,f])\right.$$
$$\times\left\{\int_{R^2_+}\prod_{j=1}^{N}d^2\xi_j^{(H)}\exp\left(\sum_{j=1}^{N}2\delta(z_j,z_j^*)\right)\exp\left[-\left(\sum_{(i,j)}^{N}(P_i^{(A)};P_j^{(A)})K^{(\varepsilon)}(z_i,z_j,2\delta)\right)\right]\right.$$
$$\left.\left.\times\left[-\left(\sum_{j=1}^{N}\delta(z_i,z_j^*)\right)\right]\left(\sum\prod_{(i,j)}^{N}(P_i^{(A)};P_j^{(A)})\prod_{(\alpha_1,\alpha_2)}((i\gamma_a\partial_a)^{-1}_{(N)}(z_i,z_j))_{\alpha_1\alpha_2}\right)\right\}\right\}, \tag{16.14}$$

where the effective action is given by the expression [5]:

$$S_{\mathrm{eff}}[\delta,\zeta,f]=\frac{10-D}{8\pi}\left[\int_{R_2^+}d^2\xi[\frac{1}{2}(\partial\delta)^2-\frac{1}{2}i\zeta^T(\gamma\cdot\partial\cdot)\zeta-\frac{1}{2}f^2]\right.$$
$$+\frac{1}{4}i\left(\int_{-\omega}^{+\infty}d\xi_0(\zeta\gamma_5\zeta)|_{\xi_1=0}\right)+\frac{D}{8\pi}\left(\mu\cdot\int_{R_2^+}d^2\xi\exp[\delta(\xi)](f-\frac{1}{2}i\zeta\gamma_5\zeta)(\xi)-\mu\int_{-\infty}^{+\infty}d\xi_0(e^{\delta})|_{\xi_1=0}\right.$$
$$\left.\left.-\int_{-\infty}^{+\infty}d\xi_0[f+(\partial/\partial\xi_1)\delta]|_{\xi_1=0}\right)\right]. \tag{16.15}$$

It was pointed out in ref. [5] that the term $f(\xi)\exp[\delta(\xi)]$ in (16.15) produces a Liouville term after being formally integrated over f, a very important remark on the analysis.

Since the complete solution of the supersymmetric Liouville field theory in R_2^+ was not found yet, which would provide the complete solution of (16.14), we implement a saddle-point approximation to evaluate (16.14) as introduced in refs.[2,5]: we take the Majorana field $\zeta\equiv 0$ and consider the classical motion equaiton for the resulting action [5]:

$$\Delta\delta=[D^2/(10-D)^2]\mu^2e^{2\delta}-\delta'(\xi_1)([D/(10-D)]-\xi_1\{[D^2/(10-D)^2]\mu e^{\delta}+[D/(10-D)]\mu e^{\delta}+\partial_{\xi_1}\delta\}). \tag{16.16}$$

A solution of (16.16) having the property of vanisnhing automatically at the boundary conditions is the Poincaré metric in R^2_+, namely:

$$\delta(\xi_1,\xi_2) = \ell n\{[D/(10-D)]/\mu\xi_1\} = \ell n\{[D/(10-D)]/\mu|z-z^*|\}. \tag{16.17}$$

By substituting this expression in eq. (16.14) and taking into account that the action evaluated in (16.17) cancels out with the same term arising from the normalization factor, we finally get:

$$\begin{aligned}\widehat{A}^{(E)}(P_1,\dots,P_N) &= \int_{R_2^+}\prod_{j=1}^{N} d^2 z_j^{(H)} (\varepsilon^{\Sigma_{i=1}^N[(P_i^2)/2\pi]})\left(\frac{D}{(10-D)\mu}\right)\sum_{i=1}^{N}[(1-P_i^2)/2\pi] \\ &\times \left(\prod_{i<j}^{N}(|z_i-z'_j|\,|z_i-z'^*_j|)\right)(P_i^{(A)};P_j^{(A)})/\pi\left(\prod_{j=1}^{N}|z_i-z_i^*|^{P_i^2/\pi-1}\right) \\ &\times \left(\sum\prod_{\substack{(i,j)\\ i\neq j}}^{N}(P_i^{(A)};P_j^{(A)}(\prod_{(\alpha_1,\alpha_2)}((i\gamma_a\partial_a)^{-1}_{(N)}(z_i,z_j))_{\alpha_1\alpha_2}\right).\end{aligned} \tag{16.18}$$

In order to isolate the on-shell scattering amplitudes, we first have to find the poles in the external momentum variables $(P_i)^2 = (P_i^{(A)};P_i^{(A)})$. Such poles occur when z_i and z_i^* come close together, i.e. the only contribution for the associated residues comes only from the region $\ell m(z_i)\to 0$ in the integrand in (16.18). This phenomenon reduces the integration over R_2^* to the integration along the real axis. As a result, there exist (euclidean) poles when

$$(P_i)^2/\pi - 1 = -1,-2,\dots \quad \text{or} \quad (P_i)^2/\pi = 0,-1.-2,\dots. \tag{16.19}$$

This fact implies that the proposed scattering amplitude (16.5) leads to a spectrum without the usual lowest state being a tachyon [compare with the bosonic case, eq. (4.21) in ref. [3]].

For the lowest massless excitation, we obtain an expression similar to the S-matrix elements encountered in the Neveu-Schwarz model [11]

$$\begin{aligned}S(P_1,\dots,P_N) &= \left(\frac{D}{(10-D)\mu}\right)^N\left[\int_{-\infty}^{+\infty}\prod_{j=1}^{N} d^1 z_j^{(H)}\left(\prod_{i<j}^{N}|z_i-z_j|\,\frac{2(P_i^{(A)};P_j^{(A)}}{\pi}\right)\right. \\ &\left.\times\left(\sum\prod_{\substack{(i,j)\\ i\neq j}}^{N}2(P_i^{(A)},P_J^{(A)})\prod_{(\alpha_1,\alpha_2)}((i\gamma_a\partial_a)^{-1}(z_i,z_j))_{\alpha_1\alpha_1}\right)\right],\end{aligned} \tag{16.20}$$

where now

$$((i\gamma_a\partial_a)^{-1}(z_i,z_j))_{\alpha_1\alpha_1} = ((i\gamma_a\partial_a)[-(1/2\pi)\ell n|z_i-z_j|])_{\alpha_1\alpha_2}, \tag{16.21}$$

and the Möbius invariant Haar measure $\Pi_{j=1}^N d^1 z_j^{(H)}$ is taken over the real axis.

The next $1/D$-corrections to the saddle-point analysis presented in this chapter are left to our readers (see Chapter 7).

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Chapter 17

Path-Integral Bosonization for the Thirring Model on a Riemann Surface

17.1. Introduction

Analysis of quantum field models defined on Riemann surface as two-dimensional space-time is a fundamental issue for strings field theory in Polyakov's approach [1,2].

It is the purpose of this chapter to solve exactly the Abelian-Thirring model defined on a Riemann surface in the framework of chiral path integrals, an useful calculational path-integral result for our *QCD* string representation presented in Chapter 9–Chapter 16, for the case of non-trivial string world sheet topology (next $\frac{1}{N_c}$-corrections).

17.2. The Path-Integral Bosonization on a Riemann Surface

We start our analysis by considering the Abelian-Thirring model associated to a complex spin field associated to a spin structure (θ^i, ϕ^i) of a genus g Riemann surface $D^{(g)}$

$$\mathcal{L}(\psi, \bar{\psi})_{(\theta^i, \phi^i)} = \bar{\psi} i \gamma^\mu D_\mu \psi + \frac{g^2}{2} (\bar{\psi} \gamma^\mu \psi)^2. \tag{17.1}$$

Here the Dirac operator is given by

$$i g^\mu D_\mu = i \gamma^a \hat{e}_a^\mu \left(\partial_\mu + \frac{1}{8} \omega_{\mu ab}(\hat{e}) \varepsilon^{ab} \gamma_5 \right), \tag{17.2}$$

where $\hat{e}_a^\mu$ are fixed background two-beins satisfying the topological genus constraint

$$\int_{D^{(g)}} \sqrt{\hat{g}} R(\hat{g}) = 2\pi(2-2g). \tag{17.3}$$

$R(\hat{g})$ is the scalar of curvature associated to $\hat{g}_{\mu\nu}$ and $\omega_{\mu ab}(\hat{g})$ is the spin connection defined by the relation $\nabla_\mu \hat{e}_\nu = 0$.

The $\gamma^\mu = \hat{e}^\mu_a \gamma_a$ Euclidean (curved) Dirac matrices are defined by the relationship below ($\xi \in D^{(g)}$):

$$\begin{cases} \{\gamma_\mu, \gamma_\nu\}_+(\xi) = 2\hat{g}_{\mu\nu}(\xi), \\ \gamma^\mu(\xi)\gamma_5 = i\left(\dfrac{\varepsilon^{\mu\nu}\gamma_\nu}{\sqrt{\hat{g}}}\right)(\xi), \end{cases} \tag{17.4}$$

where γ_a are the usual flat-space Dirac matrices.

In the framework of path integrals, the generating functional of the Green's function of the (mathematical) quantum field theory associated with the Lagrangian eq. (17.1) is defined by the following covariant functional integration (Chapter 1):

$$Z[\rho,\bar{\rho}] = \frac{1}{Z(0,0)} \int d^c[\psi] d^c[\bar{\psi}] \times \exp\left[-\int_{D^{(g)}} d^2\xi (\sqrt{\hat{g}} \mathcal{L}(\psi,\bar{\psi}))(\xi)\right] \\ \times \exp\left[-\int_{D^{(g)}} d^2\xi(\sqrt{\hat{g}}(\bar{\rho}\psi + \bar{\psi}\rho))(\xi)\right]. \tag{17.5}$$

It is worth pointing out that the classical action in eq. (17.5) is invariant under the local diffeomorphism group and the global Abelian-chiral groups acting on the spin field restrict to any local region R of $D^{(g)}$. These symmetries have the associated Noether covariant conserved currents

$$\nabla_\mu(\bar{\psi}\gamma^5\gamma^\mu\psi) = 0; \quad \nabla_\mu(\bar{\psi}\gamma^\mu\psi) = 0. \tag{17.6}$$

In order to implement the path-integral gauge and local diffeomorphism invariant bosonization, we rewrite the fermion interaction term in the Hubbard-Stratonovitch form by using an auxiliary vector field $A_\mu(\xi)$

$$Z[\rho,\bar{\rho}] = \frac{1}{Z(0,0)} \int d^c[\psi] d^c[\bar{\psi}] d^c[A_\mu] \times \exp\left[-\int_{D^{(g)}} d^2\xi \sqrt{\hat{g}} \left[\bar{\psi} i\gamma^\mu (D_\mu + gA_\mu)\psi + \frac{1}{2} A_\mu A^\nu\right](\xi)\right] \times \\ \times \exp\left[-\int_{D^{(g)}} d^2\xi \sqrt{\hat{g}}(\bar{\rho}\psi + \bar{\psi}\rho)(\xi)\right]. \tag{17.7}$$

Let us now proceed as in [4-6] by making the local field change in eq. (17.7)

$$A_\mu(\xi) = -\left(\frac{\varepsilon^{\mu\nu}\partial_\nu\eta}{\sqrt{\hat{g}}}\right)(\xi) + A_\mu^H(\xi), \tag{17.8}$$

$$\psi(\xi) = (\exp[i\gamma_5\eta(\xi)])\cdot\chi(\xi), \tag{17.9}$$

$$\bar{\psi}(\xi) = \bar{\chi}(\xi)\cdot\exp[i\gamma_5\eta(\xi)], \tag{17.10}$$

where $\nabla^\mu(A_\mu - A_\mu^H) \equiv 0$ and $A_\mu^H(\xi)$ is the Hodge topological vector field which is explicitly given in terms of canonical Abelian differentials ω_i and their complex conjugates $\bar{\omega}_i$ [7]:

$$A_\mu^H(\xi) = 2\pi\sum_{l=1}^{g}(p_i\cdot\alpha_\mu^i(\xi) + r_i\beta_\mu^i(\xi)), \tag{17.11}$$

$$\alpha_\mu^i(\xi) = -\bar{\Omega}_{ik}(\Omega - \bar{\Omega})_{kj}^{-1}\,\omega_\mu^i(\xi) + c;c;. \tag{17.12}$$

$$\beta_\mu^i(\xi) = (\Omega - \bar{\Omega})_{ij}^{-1}\,\omega_\mu^j(\xi) + c.c. \tag{17.13}$$

The period matrix Ω is defined by

$$\int_{a^j} a^i = \delta_{ij}\,, \quad \int_{b^i} a^j = \Omega_{ij} \tag{17.14}$$

where a^i and b^i are (canonical) homology cycles on $D^{(g)}$.

As it has been shown by Fujikawa [5], the transformation of eqs. (17.9)-(17.10) are not free of cost, since the functional measures $d^c[\psi]d^c(\bar{\psi}]$ are defined in terms of the normalized eigenvectors of the covariant and $U(1)$ gauge invariant Dirac operator eq. (2) in the presence of the auxiliary vector field A_μ.

The associated Jacobian of eqs. (17.9), (17.10) is given by [6]

$$d^c[\psi]d^c[\psi] = d^c[\chi]d^c[\bar{\chi}] \times \frac{\det[i\gamma^\mu(D_\mu + cA_\mu)]}{\det[i\gamma^\mu(D_\mu + cA_\mu^H)]}. \tag{17.15}$$

At this point we note that after the chiral change takes place the new quantum fermionic vacuum is defined by the fermionic field $\chi(\xi)$ (with the same spin structure of $\psi(\xi)$) in the presence solely of the Hodge topological field A_μ^H (eq. (17.11)).

The Jacobian associated to eq. (17.8) is [7]

$$d^c[A_\mu] = d^c[\eta]\left((2\pi)^{2g}\prod_{l=1}^{g} dp_i\, dr_i\right) \times \det{}^{1/2}\begin{pmatrix} \langle\alpha^i_\mu,\alpha^i_\mu\rangle & \langle\alpha^i_\mu,\beta^j_\mu\rangle \\ \langle\beta^i_\mu,\alpha^j_\mu\rangle & \langle\beta^i_\mu,\beta^i_\mu\rangle \end{pmatrix}, \tag{17.16}$$

where the covariant scalar product in the space of vector fields in $D^{(g)}$ is defined by

$$\langle\Sigma_\mu,\theta_\mu\rangle = \int_{D^{(g)}} d^2\xi\left(\sqrt{\bar{g}}\bar{g}^{\alpha\beta}\,\Sigma_\alpha\theta_\beta\right)(\xi). \tag{17.17}$$

Let us remark that with this definition we have

$$\langle\omega^i_\mu,\omega^j_\mu\rangle = 2\ell m\Omega_{ij}\,. \tag{17.18}$$

So, we face the problem of the evaluaiton of the ratio of two Dirac determinants related themselves by a chiral rotation:

$$J[A_\mu] = \frac{\det[\exp[ic\gamma_5\eta]iga^\mu(D_\mu + cA_\mu)\exp[ic\gamma_5\eta]]}{\det[i\gamma^\mu(D_\mu + cA^H_\mu)]}. \tag{17.19}$$

By following the procedure of ref. [6] we, at first, introduce a one-parameter family of Dirac operators interpolating the Dirac opeator $i\gamma^\mu(D_\mu + cA^H_\mu) = \mathbb{D}(A^H_\mu)$ and the chirally rotated $\exp[ic\gamma_5\eta)\mathbb{D}(A^H_\mu)\cdot\exp[i\gamma_5\eta]$:

$$\mathbb{D}^{(\zeta)}(A_\mu) = \exp[i\gamma_5\zeta\eta]\mathbb{D}(A^H_\mu)\cdot\exp[ic\gamma_5\zeta\eta], \quad (0 \le \zeta \le 1). \tag{17.20}$$

By using a proper-time prescription to define the funcitonal determinant of $\mathbb{D}^{(\zeta)}$ (after making the analytic extension $c = -i\bar{c}$), we have the following differential equation for log det $\mathbb{D}^{(\zeta)}$:

$$\begin{aligned}\frac{d}{d\zeta}\log\det\mathbb{D}^{(\zeta)} = &-2\lim_{\varepsilon\to 0^+}\mathrm{Tr}[\bar{c}\gamma_5\eta\exp[-\sigma\mathbb{D}^{(\zeta)^2}]\times(1-\mathbb{P}^{(\zeta)}]^{1/\varepsilon}_\varepsilon + \\ &+\lim_{\varepsilon\to 0^+}\int_\varepsilon^{1/\varepsilon}\frac{d\sigma}{\sigma}\,\mathrm{Tr}\left(\exp[-\sigma\mathbb{D}^{(\zeta)^2}\frac{d}{d\zeta}\mathbb{P}^{(\zeta)}\right) = I^{(\zeta)}_{(1)}[A_\mu] + I^{(\zeta)}_{(0)}[A_\mu],\end{aligned} \tag{17.21}$$

where $\mathbb{P}^{(\zeta)} = \sum_n \langle\, , \phi_n^{(0),(\zeta)} \rangle \phi_n^{(0),(\zeta)}$ denotes the projection over the zero modes $\phi_n^{(0),(\zeta)}$ of the Dirac interpolating operator $\mathbb{D}^{(\zeta)}$. These zero modes are related by an analytically continued chiral rotation to those of $\mathbb{D}(A_\mu^H)$:

$$\phi_n^{(0),(\zeta)} = \exp[-\hat{c}\gamma_5 \zeta \eta] \cdot \tilde{\phi}_n^{(0)} \tag{17.22}$$

and

$$\mathbb{D}(A_\mu^H) \cdot \tilde{\phi}_n^{(0)} = 0. \tag{17.23}$$

Since $\mathbb{D}^{(\sigma)^2}(A_\mu)$ is a self-adjoint invertible operator in the manifold orthogonal to the subspace generated by the zero modes, we can use the Seeley-De Witt technique to evaluate the first term in eq. (17.21) which yields

$$\begin{aligned} I_{(1)}^{(\zeta)}[A_\mu] &= \lim_{\varepsilon \to 0^+} \mathrm{Tr}[\hat{c}\gamma_5 \eta \exp[-\sigma \mathbb{D}^{(\eta)^2}](\mathbf{1} - \mathbb{P}^{(\zeta)}]\big|_{\varepsilon}^{1/\varepsilon} = \\ &= -\frac{2}{\pi}\zeta\, \mathrm{Tr}\left[-i\bar{c}\left(\eta \frac{1}{\sqrt{\hat{g}}}\partial_\alpha(\hat{g}^{\alpha\beta}\partial_\beta)\eta\right) + \frac{\varepsilon_{\mu\nu}}{2} F^{\mu\nu}(A^H)\right]. \end{aligned} \tag{17.24}$$

The second term on the left side of eq. (21) is easily evaluated giving the result

$$\begin{aligned} I_{(0)}^{(\zeta)}[A_\mu] &= \lim_{\varepsilon \to 0^+} \int_\varepsilon^{1/\varepsilon} \frac{d\sigma}{\sigma}\, \mathrm{Tr}\left(\exp[-\sigma \mathbb{D}^{(\zeta)^2}]\, \frac{d}{d\zeta}\mathbb{P}^{(\zeta)}\right) = \\ &= \lim_{\varepsilon \to 0^+} (4 \cdot \log \varepsilon \cdot \bar{c}) \sum_n \int_{D^{(g)}} d^2\xi (\sqrt{\hat{g}}\, \tilde{\phi}_n^{(0)} \cdot \eta \bar{\tilde{\psi}}_n^{(0)})(\xi). \end{aligned} \tag{17.25}$$

The final result for the functional determinants ratio eq. (17.19) is thus given by

$$\begin{aligned} J[A_\mu] &= \frac{(c^{(R)})^2}{\pi} \int_{D^{(g)}} d^2\xi \frac{1}{2} (\sqrt{\hat{g}} \partial_\alpha \eta \hat{g}^{\alpha\beta} \partial_\beta \eta)(\xi) + \\ &\quad + \frac{c^{(R)}}{\pi} \int_{D^{(g)}} d^2\xi \big[(\varepsilon_{\mu\nu} F^{\mu\nu}(A_\mu^H) \cdot \eta)\sqrt{\hat{g}}\big](\xi) - e^{(R)} \sum_n \int_{D^{(g)}} d^2\xi (\sqrt{\hat{g}} \tilde{\phi}_n^{(0)} \cdot \eta \bar{\tilde{\phi}}_n^{(0)})(\xi), \end{aligned} \tag{17.26}$$

here $c^{(R)}$ is the usual multiplicative infrared coupling constant renormalization due to zero-mode terms.

The generating funcitonal thus takes the more invariant form:

$$Z[\rho,\bar{\rho}] = \frac{1}{Z(0,0)} \int dm(p_i,r_i) Z^{(0)}[\rho,\bar{\rho},(p_{i,r_i}], \quad (17.27)$$

where the measure over the (p_i,r_i) parameters is given by [7]

$$dm(p_i,r_i) = (2\pi)^{2g} \cdot \prod_{l=1}^{g} dp_l \cdot dr_l \times \det \begin{pmatrix} \langle \alpha^i_\mu, \alpha^i_\mu \rangle & \langle \alpha^i_\mu, \beta^j_\mu \rangle \\ \langle \beta^j_\mu, \beta^i_\mu \rangle & \langle \beta^j_\mu, \beta^j_\mu \rangle \end{pmatrix} \times$$
$$\times \exp\left[-2\pi^2 \int_{D^{(g)}} d^2\xi \{ \sqrt{\hat{g}} [(p_k \bar{\Omega}_{ki} - r_i)(\ell m \Omega)^{-1}_{ij} (\Omega_{ji} p_l - r_l)] \} (\xi) \right]. \quad (17.28)$$

The (bosonized) generating functional is explicitly given by

$$Z|^{(0)}[\rho,\bar{\rho}] = \frac{1}{Z(0)} \int d^c[\eta] \exp[iW[\hat{\phi}^{(0)}_n, \bar{\hat{\phi}}^{(0)}_n, A^H_\mu]] \times$$
$$\times \int d^c[\chi] d^c[\bar{\chi}] \exp\left[-\frac{1}{2}\left(1 - \frac{c^{(R)^2}}{\pi}\right) \int_{D^{(g)}} d^2\xi \sqrt{\hat{g}} [\hat{g}^{\alpha\beta} \partial_\alpha \eta \partial_\beta \eta)(\xi) + \right.$$
$$\left. + (\bar{\chi} i \gamma^\mu (D_\mu + e^{(R)} A^H_\mu)\chi)(\xi) + (\bar{\chi} \exp[ic^{(R)}\gamma_5 \eta]\rho + \bar{\rho} \exp[ic^{(R)}\gamma_5 \eta]\chi)(\xi) \right], \quad (17.29)$$

where the functional $W[\hat{\phi}^{(0)}_n, \bar{\hat{\phi}}^{(0)}_n, A^H_\mu]$ is defined by the interaction with the (external) zero-mode fermion fields $\hat{\phi}^{(0)}_n, \bar{\hat{\phi}}^{(0)}_n$:

$$W[\hat{\phi}^{(0)}_n, \bar{\hat{\phi}}^{(0)}_n, A^H_\mu] = \int_{D^{(g)}} d^2\xi \sqrt{\hat{g}} \left[\left(-i \frac{c^{(R)}}{\pi} \varepsilon_{\mu\nu} F^{\mu\nu}(A^H)\eta \right) + (-c^{(R)} \hat{\phi}^{(0)}_n \eta \bar{\hat{\phi}}^{(0)}_n) \right] (\xi). \quad (17.30)$$

We remark that the fermions $\chi(\xi)$ still interact with the Hodge topological field A^H_μ by the minimal gauge invariant interaction $\mathbb{D}(A^H_\mu)$ and with the $\eta(\xi)$ field by the coupling with the source term.

Let us exemplify our main result, eq. (17.29), by displaying the general structure of the two-point fermion correlation function

$$\langle\psi_\alpha(\xi_1)\overline{\psi}_\beta(\xi_2)\rangle = \frac{1}{Z(0)}\int_{-\infty}^{+\infty} dm(p_i,r_i)\det[i\gamma^\mu(D_\mu + c^{(R)}A_\mu^H)]\cdot$$
$$\cdot\exp\left[-\frac{1}{2}\frac{c^{(R)^2}}{1-(c^{(R)^2/\pi)}}\Delta^{-1}(\xi_1,\xi_2)\right]\mathbb{D}^{-1}(A_\mu^H), \qquad (17.31)$$

where $\Delta^{-1}(\xi_1,\xi_2)$ is the Green's function of the Laplace operator on the Riemann surface $D^{(g)}$ and $\mathbb{D}^{-1}(A_\mu^H) = (i\gamma^\mu(D_\mu + c^{(R)}A_\mu^H))^{-1}(\xi_1,\xi_2)$ is the Green's function of the Dirac operator with spin structure (θ_i,ϕ_i) in the presence of the topological Hodge vector field $A_\mu^H(\xi)$ [1].

The determinant in eq. (17.31) was exactly evaluated ref. [1] and expressed in terms of ϑ-functions

$$\det i\gamma^\mu(D_\mu + c^{(R)}A_\mu^H) = |l(\Omega)|^2\cdot\left|\vartheta\begin{bmatrix}\frac{1}{2}+\theta^i\\ \frac{1}{2}-\phi^i\end{bmatrix}(0|\Omega)\right|. \qquad (17.32)$$

The Green's function of the laplace operator may be expressed in terms of the theta-functions

$$\Delta^{-1}(\xi_1,\xi_2) = -\frac{1}{4\pi}\log|\vartheta[(\xi_1|\Omega)] - \vartheta[(\xi_2|\Omega)]| + \frac{\text{lm}(\xi_1-\xi_2)^2}{\delta(\text{Im}\Omega)}\cdot \qquad (17.33)$$

Finally a formal expression for the Green's function of the Dirach operator is given by [3]

$$\exp\left[-i\frac{c^{(R)}}{2}\int_{C_{\xi_1,\xi_2}}(A_\mu^J + \gamma_5\varepsilon_{\mu\nu}A^{\nu,H})d\xi^\mu\right]\times(i\gamma^\mu D_\mu(A^H)^{-1}_{(\phi^i,\theta^i)}(\xi_1,\xi_2)\times$$
$$\times\exp\left[+i\frac{c^{(R)}}{2}\int_{C_{\xi_1,\xi_2}}(A_\mu^J + \gamma_5\varepsilon_{\mu\nu}A^{\nu,H})d\xi^\mu\right], \qquad (17.34)$$

where C_{ξ_1,ξ_2} is an arbitrary contour on the Riemann surface $D^{(g)}$ which has a nonempty intersection with each canonical homology cycles on $D^{(g)}$ and connecting the points ξ_1 and ξ_2 .

As we have shown, chiral changes in path integrals even for fermion model on a Riemann surface provide a quick, mathematically and conceptually simple way to analyse these models, with potential for exactly evaluations on Quantum Geometry of Riemann surfaces.

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Chapter 18

A Path-Integral Approach for Bosonic Effective Theories for Fermion Fields in Four and Three Dimensions

18.1. Introduction

Analysis of fermionic quantum models in four-dimensional space-time always have been a very difficult mathematical problem [1]. Fortunately, nonperturbative effective actions have shown its usefulness to analyzing new phenomena in these theories, It is the purpose of this chapter to propose a new technique to arrive at an effective bosonic action, suitably adapted from similar exactly obtained results on two dimensions. This main result of our study is the content of Secs. 2 and 3. In Sec. 4 we present our study of Polyakov's Fermi-Bose transmutation in the Abelian Thirring model in detail [3].

Finally in Sec. 5 we comment on some papers in the literature related to the topic of higher-dimensional bosonization and in Sec. 6 we present a loop space proof of the model triviality as a quantum field theory (Chapter 4).

18.2. The Bosonic High-Energy Effective Theory

We start this section by considering the generating functional for the correlation functions generated by vectorial and axial currents in a theory of Euclidean Abelian massive fermions in a Euclidean four-dimensional space-time R^4

$$\begin{aligned} Z[V_\mu,A_\mu](m) = \frac{1}{Z(0,0)} \int D^F[\psi(x)]D^F[\bar{\psi}(x)] \\ \times \delta^{(F)}(\partial_\mu(\bar{\psi}^\mu\psi)(x))\delta^{(F)}([\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) - 2im\bar{\psi}\psi](x)) \\ \times \exp\left(-\int d^4[\bar{\psi}(i\gamma^\mu\partial + m + \gamma_\mu\gamma_5 A_\mu + \gamma_\mu V_\mu)\psi](x) \right), \end{aligned} \tag{18.1}$$

where we have taken into account in an explicit way, in the functional domain of integration of Eq. (18.1), the current-charge law for the theory, in response to *phase* local variable field change

$$\begin{aligned} \psi(x) \to e^{ig_V\theta(x)}\, e^{ig_A\gamma_5\omega(x)}\psi(x), \\ \bar{\psi}(x) \to \bar{\psi}(x)e^{-ig_V\theta(x)}\, e^{ig_A\gamma_5\omega(x)}. \end{aligned} \tag{18.2}$$

It is worth pointing out that our fermionic functional measues are defined in terms of the spectral set (eigenfunctions and eigenvalues) associated with the free massless Dirac operator $\partial\!\!\!/ \equiv i\gamma^\mu\partial_\mu$ instead of the full massive Dirac operator $\partial(A,V) - m \equiv i\gamma^\mu(\partial_\mu + V_\mu + \gamma_5 A_V) + m$, since the external sources (A_μ, V_μ) are not dynamical and thus leading to the absence of the axial-anomaly piece in the chiral current law associated with these fields. Besides, the mass term is defined as a perturbation of the massless case as in 2D models [4]. We now write the generating functional eq. (18.1) in a local way by expressing the functional Delta constraints in Fourier functional domain:

$$\begin{aligned} Z[V_\mu,A_\mu](m) = \frac{1}{Z(0,0)} \int D^F]\psi(x)]D^F]\bar{\psi}(x)] \int D^F[\theta(x)]D^F[\omega(x)] \\ \times \exp\left[-\int d^4x ig_A(\bar{\psi}\gamma^\mu\gamma^5\psi)(x)\partial_\mu\omega(x) - 2m\int d^4x(\bar{\psi}\psi)(x)\omega(x) \right] \\ \times \exp\left[-i\int d^4x g_V(\bar{\psi}\gamma^\mu\psi)\partial_\mu\theta(x) \right] \\ \times \exp\left[-\int d^4x\bar{\psi}[i\ \partial\!\!\!/(A,V) + m]\psi \right. \end{aligned} \tag{18.3}$$

At this point of our study, we implement the phase variable change Eq. (18.2) into Eq. (18.3) by taking into consideration the nonunity Jacobian associated with the chiral rotation [Ref. 5 - Eq. (9)] and Appendix 22-E].

$$\begin{aligned} D^F[\bar{\psi}(x)]D^F[\psi(x)] = D^F[(\bar{\psi}(e^{+ig_A\gamma_5\omega}\, e^{-ig_V\theta}))(x)] \\ \times D^F[((e^{ig_A\psi_5\omega}\, e^{ig_V\theta})\psi)(x)]\, \frac{\det_F[e^{ig_A\gamma_5\omega}(i\ \partial\!\!\!/)e^{ig_A\gamma_5\omega}]}{\det_F[i\ \partial\!\!\!/]}. \end{aligned} \tag{18.4}$$

The ratio of the functional Dirac determinants was evaluated in ref. 5 [Eqs. (18.17) and (18.18)] and yielded the following functional weight for the chiral dynamical phase $\omega(x)$ (with a UV cutoff Λ):

$$\begin{aligned} &\det{}_F[e^{ig_A\gamma_5\omega}(i\not\partial)e^{ig_A\gamma_5\omega}]/[\det{}_F(i\not\partial)] \\ &\times\exp\left[\left(\frac{g_A}{\Lambda}\right)^2\int d^4x\omega(-\partial^2)\omega\right] \\ &\times\exp\left[-\frac{(g_A)^2}{4\pi^2}\int d^4x(-\partial^2\omega)(-\partial^2\omega)(x)\right] \\ &\times\exp\left[\frac{(g_A)^4}{12\pi^2}\int d^4(\omega)\partial_\mu\omega)^2(-\partial^2\omega))(x)\right]. \end{aligned} \tag{18.5}$$

By substituting Eq. (18.4) into Eq. (18.3) and by noting the validity of the equation

$$\begin{aligned} &\int D^F[\overline{\psi}(x)e^{(+ig_A\gamma_5\omega-ig_V\omega)(x)}]D^f[e^{(+ig_A\gamma_5\omega+ig_V\theta)(x)}\psi(x)] \\ &\times\exp\left[-\int d^4x\{(\overline{\psi}e^{ig_A\gamma_5\omega-ig_V\theta})[i\not\partial(A,V)\right. \\ &\left.+me^{-2(ig_A\gamma_5\omega)}(1+2\omega)](e^{ig_A\gamma_5\omega+ig_V\theta}\psi)\}(x)\right] \\ &=\det{}_F[i\not\partial(A,V)+m(1+2\omega)\exp(-2ig_A\gamma_5\omega)]. \end{aligned} \tag{18.6}$$

we finally obtain the result sought in the leading limit of high ultraviolet region $m\to 0$, which improves those models studied in the second reference of Ref. 1.

$$\begin{aligned} &\widetilde{Z}[V_\mu,A_\mu](m) \\ &\quad=\frac{1}{\widetilde{Z}(0,0)}\int D^F[\theta(x)]D^F[\omega(x)] \\ &\quad\exp\left(\int d^4x\omega(x)\left\{-\frac{\Lambda_F^2}{(g_A)^{-1}\pi^2}\left[\frac{1}{-(\partial^2)+(\frac{2\pi}{\Lambda_F})^2}-\frac{1}{-\partial^2}\right]^{-1}(x,y)\right\}\omega(y)\right) \\ &\quad\times\exp\left(\frac{(g_A)^4}{12\pi^2}\int d^4(\omega(\partial_\mu\omega)^2)(-\partial^2\omega))(x)\right) \\ &\quad\times\exp\left\{-2\int d^4xd^4y[(m(1+2\omega)e^{2i(g_A\gamma_5\omega)})(x)(i\not\partial)^{-1}(x,y)\right. \\ &\quad\left.\times(V_\mu+\gamma_5A_\mu)(y)(i\not\partial)^{-1}(y,x)]+O(m^2)\right\}. \end{aligned} \tag{18.7}$$

Comments related to this effective high-energy bosonic field theory for the current algebra of observables are made in Sec. 4 of this chapter.

18.3. The Bosonic Low-Energy Effective Theory

Let us start our analysis in this section by writing the generating functional for the correlations functions generated by vectorial and axial currents in a theory of free massive Euclidean fermion fields in R^4

$$\widetilde{Z}[V_\mu, A_\mu] = \frac{1}{Z(0,0)} \int D^F[\psi(x)] D^F[\overline{\psi}(x)] \times \exp\left[-\int d^4x \overline{\psi}(i \not\partial(A,V,m)\psi)(x)\right]. \tag{18.8}$$

The main point of our approximate bosonization procedure for Eq. (18.8) is to introduce a massive feermion field theory invariant under the field rotation Eq. (18.2) by elevating the involved local $(\omega(x), \theta(x))$ to being dynamical degrees of freedom and functionally integrating them out. As a consequence, we propose to approximate Eq. (18.8) in the infrared region by means of the chiral-invariant functional integral with a mass parameter term,

$$\widetilde{Z}[V_\mu, A_\mu]_{\mathbb{R}} = \lim_{m\to\infty} \int D^F[\omega(x)] D^F[\theta(x)] \int D^F[\overline{\psi}^{\theta,\omega}(x)] D^F[\psi^{(\theta,\omega)}(x)] \times \exp\left\{-\int d^4x \overline{\psi}^{(\theta,\omega)}(i \not\partial(A,V) + m)\psi^{(\theta,\omega)}(x)\right\}, \tag{18.9}$$

where the fields rotated in Eq. (18.9) are given by Eq. (18.2):

$$\begin{aligned} \psi^{(\theta,\omega)}(x) &= e^{ig_V\theta(x)} e^{ig_A\gamma_5\omega(x)}\psi(x), \\ \overline{\psi}^{(\theta,\omega)}(x) &= \overline{\psi}(x) e^{-ig_V\theta(x)} e^{ig_A\gamma_5\omega(x)}. \end{aligned} \tag{18.10}$$

We thus proceed in the inverse path of that followed in Sec. 18.2 by using the inverse field variable change Eq. (18.4):

$$\widetilde{Z}[V_\mu,A_\mu]_{\mathbb{R}} = \lim_{m\to\infty}\int D^F[\omega(x)]D^F[\theta(x)]$$
$$\times \exp\left(\frac{-(g_{1_\Lambda}}{12\pi^2}\right)\int(\omega(\partial_\mu(\omega)^2\times(-\partial^2\omega)(x)))d^4x$$
$$\times \exp\left(\int d^4x\omega(x)\left\{\frac{\Lambda_F}{(g_A)^4 4\pi^2}\left[\frac{1}{(-\partial^2)+(\frac{2\pi}{\Lambda_F})^2}-\frac{1}{(-\partial^2)}\right]^{-1}\right\}(x,y)\omega(y)\right)$$
$$\times \det{}_F[i\not\partial(V_\mu+ig_V\partial_\mu\theta, A_\mu+ig_A\partial_\mu\omega)+m\exp(2ig_A\gamma_5\omega)], \tag{18.11}$$

where Λ_F denotes the intrinsic cutoff from the original fermion field theory (see Chapter 6), which, by its turn, determines the effective energy scale where our effective bosonic theory is expected to be working.

Let us now analyze the fermion functional determinant involving the sources in this low-energy limit $m\to\infty$. At this limit, we can easily improve the asymptotic expansion in terms of the inverse power of the bare mass parameter m of Ref. 6 by approximating the term $m\exp(2g_A\gamma_5\omega)$ by the simple mass term m (this procedure being correct only at this limite of $m\to\infty$).

We thus consider the following differential equation for this functional determinant, where the parameter s ranges in the interpolating $0\le s\le 1$:

$$\lim_{m\to\infty}\frac{d}{ds}\{\det{}_F[i\not\partial(s(V_\mu+ig_V\partial_\mu\theta);s(A_\mu+ig_A\partial_\mu\omega))+m]\}$$
$$\sim \int_0^\infty dt\, e^{-tm^2}\times \mathrm{Tr}_F[(\gamma^\mu V_\mu+\gamma_5\gamma_\mu A_\mu)]$$
$$\times[i\not\partial(s(V_\mu+ig_V\partial_\mu\theta);s(A_\mu+g_A\partial_\mu\omega))+m]$$
$$\exp\{-t[i\not\partial(s(V_\mu+ig_V\partial_\mu\theta);s(A_\mu+ig_A\partial_\mu\omega))]^2\}. \tag{18.12}$$

By applying the saddle oint technique to evaluate the Laplace transform (see Ref. 6), we obtain the leading effective infrared effective source-dependent action

$$S_{\mathrm{eff}}(A_\mu,V_\mu)_{\mathbb{R}} = \exp\left\{+\frac{(m|\Lambda_f)^2}{4\pi}\int d^4x([V_\mu+ig_V\partial_\mu\theta)^2\right.$$
$$+(A_\mu+ig_A\partial_\mu\omega)^2(x)]+[c_1F^2_{\mu V}(A_\mu)-c_2F^2_{\mu V}(A_\mu)$$
$$\left.+c_3F_{\mu\nu}(V_\mu)F^{\mu\nu}(A_\mu)(x)+0((m|\Lambda_F)^{-2})\right\} \tag{18.13}$$

Hence c_1 and c_2 are positive constants whose values depend on the regularization scheme used and the Dirac matrices representation. By substituting the massive Abelin gauge field (source) action above into the functional integral Eq. (18.11), we get our propose IR effective bosonic theory for the algebra generated by vectorial and axial currents of a massive free fermion field theory. At this point the reader should compare the UV-effective action Eq. (18.13) with IR-effective action given by Eq. (18.7).

It is instructive to point out that in the important use of $D \equiv 2$, all functional integrals are of Gaussian type and leading to the following result in the IR-region:

$$\begin{aligned}
\widetilde{Z}[A_\mu, V_\mu] &= \int D^F[\omega(x)] D^F[\theta(x)] \\
&\quad \times \exp\left[-\frac{1}{2\pi}\int d^2x[(\partial\theta)^2 + (\partial\omega)^2(x)]\right] \\
&\quad \times \exp\left[-\frac{m^2}{2\pi}\int d^2x[(V_\mu + ig_V\partial_\mu\theta)^2 + (A_\mu + ig_A\partial_\mu\omega)^2](x)\right] \\
&= \exp\left\{\int d^2x d^2y V_\mu(x)\left[m^2\delta_{\mu\nu} - 4\frac{g_V^2}{\left(\frac{1+m^2g_V^2}{\pi}\right)}\frac{\partial_m u\partial_V}{(-\partial^2)}\right]V_\nu(y)\right\} \\
&\quad \times \exp\left\{\int d^2x d^2y A_\mu(x)\left[m^2\delta_{\mu\nu} - \frac{4g_A^2}{\left(1+\frac{m^2g_A^2}{\pi}\right)}\frac{\partial_\mu\partial_\nu}{(-\partial^2)}\right](x,y)A_\nu(y)\right\}. \qquad (18.14)
\end{aligned}$$

By analyzing the two-dimensional effective bosonic theory we conclude that the result is clearly not gauge invariant on the source gauge fields as the gauge symmetry is dynamically broken in two-dimensional space-time.

In the important case of the presence of a quantized electromagnetic field $G_\mu(x)$, we can follow our previous procedure of the section. The main difference is the introduction of the "topological charge" of the electromagnetic field in the delta function of Eq. (18.1):

$$\delta^{(F)}([\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) - 2im\bar\psi\psi]) \to \delta^{(F)}([\partial_\mu(\bar\psi\gamma^\mu\gamma^5\psi) - 2im\bar\psi\psi - \frac{1}{32\pi^2}\int d^4x(*F_{\mu\nu}F^{\mu\nu})(G_\mu)]) \qquad (18.15)$$

and the replacing of the full Dirac operator below in Eq. (18.12),

$$\partial\!\!\!/(A,V) + m \to \partial\!\!\!/(A, V+G) + m.$$

It is worth pointing out the natural appearance of an "axion like" interaction between the chiral phase neutral field $\omega(x)$ and the electromagnetic field $G_\mu(x)$, namely

$$S_{\text{axion}}[\omega, G_\mu] = \exp\left\{ i \int d^4x \omega(x) (*F_{\mu\nu}F^{\mu\nu})(G)(x) \right\}. \tag{18.16-a}$$

The generalization of our study for the non-Abelian case is straightforward and left to our readers and leading to the non-Abelian generalization of our previous study (see Ref. 12) where the non-Abelain evaluation of the chirality-rotated Jacobian Eq. (18.4) is presented in full details.

Finally, it is instructive to point out that one should show explicitly the "Euclideanicity" of our approach by considering the *nonunitary* (Euclidean) variable change below

$$\begin{aligned} \psi(x) &\to e^{g_V \theta(x)} e^{g_V \gamma_5 \omega(x)} \psi(x), \\ \bar{\psi}(x) &\to \bar{\psi}(x) e^{g_A \gamma_5 \omega(x)} e^{-g_V \theta(x)}. \end{aligned} \tag{18.16-b}$$

Instead of the classical unitary Eq. (2), the Jacobain will now be a functional involving the nonunitary phases $(\theta(x), \omega(x))$. Note that Eq. (18.16') is allowed in Euclidean space-time since the energy densities $\bar{\psi}\psi$, $\bar{\psi}\gamma^5 A_\mu$ are not real as $\bar{\psi}$ and ψ are independent anticommuting Euclidean fields and, thus, living in different functional spaces.

18.4. Polyakov's Fermi-Bose Transmutation in 3D Abelian-Thirring Model

Polyakov's Fermi-Bose transmutation in the infrared regime of the CP^1 model has became a basic phenomenon for understanding approximate bosonization in fermion field theory in three-dimensional space-time. In this section we present in detail the above-cited phenomenon in the Thirring model. This study is based on our unpublished research (7), prior to all the results that appeared on the subject since then.

Let us start our study in this section by considering the massive three-dimensional Thirring Lagrangian in the Euclidean space-time with a repulsive interaction

$$\mathcal{L}(\psi, \bar{\psi}) = \bar{\psi}(i\gamma\partial)\psi + m\bar{\psi}\psi - \frac{g^2}{2}(\bar{\psi}\gamma^\mu\psi)^2. \tag{18.17}$$

The 3D Euclidean Hermitian γ^μ matrices which we are using obey the relationship

$$\{\gamma^\mu, \gamma^\nu\} = \delta^{\mu\nu}, \quad [\gamma^\mu, \gamma^\nu] = \frac{1}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho. \tag{18.18}$$

The independent Euclidean fields $\psi^{(\alpha)}(x)$ and $\bar{\psi}^{(\beta)}(x)$ satisfy the Euclidean anticommuting relation $(\alpha, \beta = 1, 2, 3)$

$$\{\psi^{(\alpha)}(x), \overline{\psi}^{(\beta)}(y)\} = \delta^{\alpha\beta}\delta^{(3)}(x-y). \tag{18.19}$$

The Lagrangian (17) is invariant under the *global Abelian group* $\psi \to \exp(i\Omega)\psi_1$, $\overline{\psi} \to \exp(-i\Omega)\overline{\psi}$ with the Noetherian conserved current

$$\partial_\mu(\psi\gamma_\mu\overline{\psi}) \equiv 0. \tag{18.20}$$

In order to analyze Polyakov's boson-fermion transmutation, we consider the generating function

$$Z[\eta,\overline{\eta}] = \frac{1}{Z(0,0)} \times \left\{ \int D^F[\psi(x)]D^F[\overline{\psi}(x)] \times \exp\left[\int d^3x(\mathcal{L}(\psi,\overline{\psi}) + \eta\overline{\psi} + \psi\overline{\eta})(x) \right] \right\}. \tag{18.21}$$

By making use of the Hubbard-Stratonovich field reparametrization, we rewrite Eq. (18.21) in a form useful for our bosonization purpose:

$$\begin{aligned} Z[\eta,\overline{\eta}] = \frac{1}{Z(0,0)} \times \Bigg\{ \int & D^F[\psi(x)]D^F[\overline{\psi}(x)]D^F[A_\mu(x)] \\ & \times \exp\left(-\frac{1}{2}\int d^3x A_\mu^2(x)\right)\delta^{(F)}[(\partial_\mu A_\mu)] \\ & \times \exp\left(-\int d^3x[\overline{\psi}(i\gamma\partial + g\gamma A + m)\psi + \eta\overline{\psi} + \psi\overline{\eta}](x)\right)\Bigg\}, \end{aligned} \tag{18.22}$$

where $A_\mu(x)$ is an auxiliary Euclidean Abelian real vector field satisfying the Landau gauge as a consequence of Eq. (18.20), since it should coincides with the vectorial current at the operator level.

At this point, it becomes important to remark that the fermionic measures $D^F[\overline{\psi}_1(x)]D^F[\psi(x)]$ in Eq. (18.22) are defined in terms of the normalized eigenvectors of the self-adjoint Euclidean Dirac operator $i\gamma_\mu(\partial - igA_\mu)$ since we want to keep the model's physical local gauge invariance in the pure fermion sector of the theory:

$$\begin{aligned} &\psi(x) \to \psi(x)\exp(ig\Omega(x)), \\ &\overline{\psi}(x) \to \overline{\psi}(x)\exp(-ig\Omega(x)), \\ &A_\mu(x) \to A_\mu(x). \end{aligned} \tag{18.23}$$

Note that this local Abelian gauge invariance in the fermionic parametrization Eq. (18.17) is a consequence of the current conservation Eq. (18.20) at the quantum level of the generating functional Eq. (18.21), and differs from the usual local gauge invariance of the gauge models involving the shift $A_\mu \to A_\mu + g\partial_\mu \Omega$. The local invariance Eq. (18.23) is a consequence of the following path integral identity:

$$\begin{aligned}
&\int D^F[\psi(x)e^{ig\Omega(x)}]D^F[\overline{\psi}(x)e^{-ig\Omega(x)}] \\
&\quad \times \exp\left\{-\int d^3\mathcal{L}(\psi(x)e^{ig\Omega}(x), \overline{\psi}(x)e^{-ig\Omega}(x))\right\} \\
&= \int D^F[\psi(x)]D^F[\overline{\psi}(x)] \exp\left\{-\int d^3x\mathcal{L}(\psi(x), \overline{\psi}(x))\right\} \\
&\quad \times \exp\left[-i\int d^3\Omega(x)(\partial_\mu(\overline{\psi}\gamma_\mu\psi))(x)\right]. \qquad (18.24)
\end{aligned}$$

In this quantum field path-integral framework, the infrared Polyakov's Fermi-Bose transmutation (3) may be understood as the large fermion mass limit of the otherwise trivial 3D Abelian quantum field Thirring model (Chapter 4).

Explicitly, we first introduce an ultra-violet cutoff in Eq. (18.22) and integrate out the Euclidean Fermi fields. Let us, thus consider the effective path integral

$$\begin{aligned}
Z[\eta, \overline{\eta}] &= \frac{1}{Z(0,0)} \times \int D^F[A_\mu(x)] \\
&\times \exp\left(-\frac{1}{2}\int d^3x A_\mu^2(x)\right) \times \delta^{(F)}[(\partial_\mu A_\mu)] \\
&\times \det[i\gamma\partial + g\gamma A + m] \\
&\times \exp\left\{+\frac{1}{2}\int d^3x d^3y (\overline{\eta}(x)(i\gamma\partial + g\gamma A + m)^{-1}(x,y)\eta(y)]\right\}. \qquad (18.25)
\end{aligned}$$

The fermion vacuum loops associated with the fermion functional determinant may be easily evaluated in the limit of large mass by using the proper-time definition for this functional determinant (see App. A):

$$\log\det(i\gamma\partial + g\gamma A + m) = +\frac{1}{2} \times \lim_{\varepsilon\to 0^+} \int_\varepsilon^\infty \frac{dt}{t} \times \mathrm{Tr}_{(F)}[\exp(-t[i\gamma\partial + g\eta A + m]^2)]. \qquad (18.26)$$

where $\mathrm{Tr}_{(F)}$ denote the functional trace.

We thus have the following result for the family of interpolating Dirac operator $i\gamma\partial + sg\gamma A + m (0 \le s \le 1)$:

$$
\begin{aligned}
&\frac{d}{ds}(\log\det[i\gamma\partial + sg\gamma A + m]) \\
&\quad \times \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{\infty} dt\, e^{-tm^2}\, \mathrm{Tr}_{(F)}\{(g\gamma A)(i\gamma\partial + sg\gamma A - M) \\
&\quad \times \exp(-t[i\gamma\partial + sg\gamma A + m]^2)\}
\end{aligned} \tag{18.27}
$$

By taking the limit of large fermion mass as in Ref. 6 and App. A, we get the result below, after integrating the interpolating parameter in the range $0 \le s \le 1$,

$$
\begin{aligned}
&\log[\det(i\gamma\partial + g\gamma A + m)/\det(i\gamma\partial + m)]_{(\varepsilon)} \\
&\quad = \frac{g^2 m}{(4\pi)^{\frac{3}{2}}} \cdot \left(\frac{1}{\varepsilon}\right) \cdot \int d^3x \left(\frac{1}{2} A_\mu^2(x)\right) \\
&\quad - g^2 \frac{\sqrt{\pi}}{2} \frac{m}{|m|} \int d^3x (A_\mu \varepsilon^{\mu\nu\rho} F_{\nu\rho}(A))(x) + 0\left(\frac{1}{m}\right).
\end{aligned} \tag{18.28}
$$

It is worth pointing out the existence (in principle) of an induced (cutoff dependent) mass term for the auxiliary ector field (this auxiliary vector at the quantum level coincides with the Noetherian $U(1)$ global current: $A_\mu(x) = (\bar{\psi}\gamma_\mu\psi)(x)$.

Note that this mass term signals the dynamic breaking of the usual gauge invariance in the pure fermionic sector of Eq. (18.25) which involves the gauge change $A_\mu(x) \to A_\mu(x) + g\partial_\mu\Omega(x)$ as in $2D$ models (see Eq. (18.23)).

The physical consequence of this term is a formal renormalization of the bare gauge field mass m_R at one loop, as similar phenomenon happened in the Jacobian evaluation of Eq. (18.4):

$$
m_R = \frac{m}{\varepsilon}\cdot \tag{18.29}
$$

The second term in the right-hand side of Eq. (18.28) is the Chern-Simons Lagrangian. By substituting Eqs. (18.28) and (18.29) in Eq. (18.25) we get the result with fermion loops integrated out at large mass,

$$Z[\eta,\bar{\eta}] \sim \frac{1}{Z(0,0)} \times \int D^F[A_\mu(x)] \times \exp\left\{-\frac{1}{2}\left(1-\frac{g^2 m_R}{(4\pi)^{\frac{3}{2}}}\right)\cdot\int d^3x A_\mu^2(x)\right\}$$
$$\times \exp\left\{-\frac{g^2\sqrt{\pi}}{2}\int d^3x (A_\mu \varepsilon^{\mu\nu\rho} F_{\gamma\rho}(A))(x)\right\} \times \delta^{(F)}[(\partial_\mu A_\mu)]$$
$$\times \exp\left\{+\frac{1}{2}\int d^3x d^3y \bar{\eta}(x)(i\gamma\partial + g\gamma A + m)^{-1}(x,y)\eta(y)\right\}. \qquad (18.30)$$

Following closely Ref. 3 now we analyze the large bare m limit of the external fermion sources by considering the Feynman path integral representation for the Feynman-Green function of the Dirac operator in the presence of $A_\mu(x)$:

$$(i\gamma\partial + g\gamma A + m)^{-1}_{\alpha\beta}(x,g) = \int_0^\infty dt\, e^{-mt} \times \left\{ \int_{\substack{X^\mu(0)=x^\mu \\ X^\mu(t)=y^\mu}} D^F[X^\mu(\sigma)]\right.$$
$$\left. \times \Phi_{\alpha\beta}(x,y)\cdot \exp\left(ig\int_0^t d\sigma A_\mu(X(\sigma))\dot{X}^\mu(\sigma)\right)\right\}, \qquad (18.31)$$

where the sping-factor is explicitly given by

$$\Phi_{\alpha\beta}(x,y) = \int D^F[\pi^\mu(\sigma)]\exp\left(i\int_0^t d\sigma(\pi^\mu(\sigma)\cdot\dot{X}^\mu(\sigma))\right)$$
$$\times \mathbb{P}\left\{\exp i\int_0^t d\sigma(\gamma^\mu\cdot\pi_\mu(\sigma))\right\}. \qquad (18.32)$$

Here $\mathbb{P}$ means the path order of the $3D$ γ^μ matrices along Feynman trajectory $X_\mu(\sigma)\cdot(0\leq \sigma\leq t)$.

In the limit of large m, only the classical straight-line trajectory entering the path integral leads to Eqs. (18.31) ande (18.32), producing the result

$$(i\gamma\partial + g\gamma R + m)^{-1}_{\alpha\beta}(x,y) \sim (U_\alpha^{(1)}U_\beta^{(2)})\exp\left(iy\int_x^y A_\mu(x)dX^\mu\right), \qquad (18.33)$$

where $U_\alpha^{(1),(2)}$ are the usual Euclidean spinorial bases associated with the free massive fermion fields $\{\bar{\psi}(x),\psi_\alpha(x)\}$.

By grouping Eqs. (18.33) and (18.34), we finally obtain our Polyakov's infrared bosonic theory for the $3D$ Thirring model:

$$
\begin{aligned}
Z[\eta,\bar{\eta},(m\to\infty)] = \int D^F[A_\mu(x)]\exp\left\{-\frac{1}{2}\left(1-\frac{g^2m_R}{(4\pi)^{\frac{3}{2}}}\cdot\int d^3xA_\mu^2(x)\right)\right\} \\
\times\exp\left\{-\frac{g^2\sqrt{\pi}}{2}\frac{m}{|m|}\int d^3x(A_\mu\varepsilon^{\mu\nu\rho}F_{\nu\rho}(A)(x)\right\} \\
\times\delta^{(F)}[(\partial_\mu A_\mu)]\exp\left\{+\frac{1}{2}\int d^3xd^3y(\bar{\eta}_\alpha(x)\eta_\beta(y))\cdot(U_1^\alpha\overline{U}_2^\beta)\right. \\
\left.\times\exp\left(ig\int_x^y A_\mu(X)\dot{X}^\mu\right)\right\}. \qquad (18.34)
\end{aligned}
$$

Now it is a straightforward consequence of Eq. (18.34) *the infrared (large mass) bosonization formulae of* 3*D Abelian Thirring model* analogous to those associated to 2*D* Thirring model

$$
\begin{aligned}
\psi_\alpha^1(x) \overset{m\to\infty}{\sim} \psi_{\alpha,\,\text{free}}^1(x)\times\exp\left(ig\int_{-\infty}^x A_\mu(X)\cdot X^\mu\right), \\
\psi_\alpha^2(x) \overset{m\to\infty}{\sim} \psi_{\alpha,\,\text{free}}^2(x)\times\exp\left(ig\int_{-\infty}^x A_\mu(X)\cdot X^\mu\right),
\end{aligned} \qquad (18.35)
$$

Here $A_\mu(x)$ is the quantum field associated with the "massive" Chern-Simon theory

$$
\begin{aligned}
\overline{\mathcal{L}}(A_\mu) = \frac{1}{2}\left(1-\frac{g^2m_R}{(4\pi)^{\frac{3}{2}}}\right)\cdot\int d^3dxA_\mu^2(x) \\
-\frac{g^2\sqrt{\pi}}{2}\frac{m}{|m|}\cdot\int d^3x(A_\mu\varepsilon^{\mu\nu\rho}F_{\nu\rho}(A))(x). \qquad (18.36)
\end{aligned}
$$

Equations (28) and (36) are our main result in this section about approximate bosonization for the Thirring model in the large mass limit.

In the important case for high-T_c superconductivity, modeled by the Thirring model coupled to an external divergence free current

$$
W(\bar{\psi},\psi,J_\mu) = \mathcal{L}(\psi,\psi)+\int d^3xJ_\mu(x)(\bar{\psi}\gamma_\mu\psi)(x), \qquad (18.37)
$$

we can proceed as exposed above and obtain the associated Polyakov's full bosonized generating functional for correlation function involving vectorial currents from the 3*D*-Thirring model Eq. (18.17):

$$
\begin{aligned}
W_{\text{eff}}(J_\mu) = & \int D^F[A_\mu(x)] \cdot \delta^{(F)}[(\partial_\mu A_\mu)] \\
& \times \exp\left\{ -\frac{1}{2} \int d^3x (A_\mu - J_\mu)^2(x) \right\} \\
& \times \exp\left\{ -\frac{1}{2}\left(1 - \frac{g^2 m_R}{(4\pi)^{\frac{3}{2}}}\right) \int d^3x (A_\mu^2(x)) \right. \\
& \left. - \frac{g^2\sqrt{\pi}}{2} \frac{m}{|m|} \cdot \int d^2x (A_\mu \varepsilon^{\mu\nu\rho} F_{\nu\rho}(A))(x) \right\}. \qquad (18.38)
\end{aligned}
$$

Finally, we point out that we have neglected in Eq. (18.25) the zero modes of the 3D Dirac operator which will be left to our readers.

18.5. Effective Four-Dimensional Bosonic Actions – Some Comments

The effective bosonic action obtained in Secs. 2 and 3 are higher-order four-dimensional bosonic field theories, and this should be considered only as an approximate and effective action as it shares all the drawbacks and usefulness of all effective action proposed in the literature) ([9], [12]). However, there are some hints that theories of the kind obtained in this chapter may be given a meaning by nonperturbative procedures and this point may be advantageous for implementing realizable approximate calculations useful for realistic 4D field theories.

In three dimensions, we disagree with similar studies presented in Ref. 11, since in this reference it used the Deser-Jackiw interpolating field to rewrite the effective action in terms of Maxwell-Chern-Simon field theory, which does not hold true when one is analyzing observables and leads to a cumbersome theory in the non-Abelian case (a theory in the strong limit $g^2_{\text{phy}} \to \infty$). Finally, the use of Wilson loops of Ref. 11 is unclear since the non-Abelian Stokes theorem was proved only in R^2, namely for R^n $(n > 2)$ it was not proved rigorously that

$$
\text{Tr}, P\left(\exp \oint_e \cdot A_\mu \cdot \dot{X}^\mu\right) = \text{Tr}_S\left[\left(\exp \int_\Sigma d\Sigma^{\mu\nu}\, \text{Tr}_t\left(\frac{\delta}{\delta\Sigma_{\mu\nu}} \cdot W[\widetilde{C}^S W_t]\right)\right)\right], \qquad (18.39)
$$

where $\widetilde{C}_t^S$ *are closed trajectories in the surface* Σ (*see Ref. 13 for the notation*) *in* R^3. (Unless for those planar surfaces homotopical to its boundary).

As an alternative for the study of Ref. 11 one should writes the loop wave equation for the Wilson Loops. Eq. (18.39), and solve them by means of effective theory of Chern-Simons string as exposed in previous Chapter 13.

We start this final part of our chapter by considering the fermionic determinant of the self-adjoint Dirac operator in $L^2(R^3)$:

$$\log\det(D(A)+m) = S_{\text{eff}}(S,s) = -\frac{1}{2}\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{dt}{t}\,\text{Tr}_{(F)}(e^{[-t(\not{D}_S(A)+m)]^2}), \tag{18.40}$$

where we have introduced a one-parameter family of Dirac operators interpolating the free operator and that in the presence of an external gauge field

$$\not{D}_s(A)+m = i\gamma^\mu(\partial_\mu - isgA_\mu)+m. \tag{18.41}$$

We have regulated the fermion determinant by the proper-time method. At this point we remark that $S_{\text{eff}}(A;s)$ satisfies the differential

$$\begin{aligned}\frac{d}{ds}S_{\text{self}}(A;s) = \lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty dt\,\text{Tr}_{(F)}[(g(\gamma^\mu A_\mu)\cdot(\not{D}(A)+m)\\ \times\exp(-t(\not{D}_S^2(A)+m^2+2m\ \not{D}_S(A)))].\end{aligned} \tag{18.42}$$

Since we are interested in the large fermion mass limit $m\to\infty$, we neglect the term $\exp(-2m\not{D}(A))\sim 1$ inside the trace operation of Eq. (18.42). We have thus, at large m,

$$\begin{aligned}\lim_{m\to\infty}\frac{d}{ds}S_{\text{eff}}(A,s) &\sim \lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty dt\,e^{-tm^2}\,\text{Tr}_{(F)}[+g(A)(\not{D}(A)+m)\cdot\exp(-\ \not{D}^2(A))]\\ &\sim -\frac{g}{(4\pi)^{\frac{3}{2}}}\sum_{\ell=0}^{\infty}\left(\int_0^\infty dt\,e^{-tm^2}\cdot t^{\ell-\frac{3}{2}}\right)\\ &\times\int d^3x\,\text{Tr}_{(F)}((\not{\gamma}_A)[\not{D}(A)+m]\times b_\ell(x,r,A,s)),\end{aligned} \tag{18.43}$$

where $b_\ell(x,x,A,s)$ are the Seeley-De Witt coefficients associated with the asymptotic short-time $t\to 0^+$ of Eq. (18.43) since we are considering the asymptotic limit of $m\to\infty$ by means of the Laplace method for handling saddle-point of integral [6]. Explicit expressions for these coefficients are easily calculated [5]. In the large fermion mass limit, only the first 2 Seeley-De Witt coefficients will be needed in R^3, namely

$$b_0(x,x,A,s) = \mathbf{1}_{\text{iden}} \tag{18.44}$$

and

$$b_1(x,x,A,s) = -\frac{g_s}{2}[\gamma^\mu,\gamma^\nu]mF_{\mu\nu}(A) + g^2A_\mu^2 + ig(\partial_\mu, A_\mu). \qquad (18.45)$$

After substituting Eqs. (18.44) and (18.45) into Eq. (18.43) and solving the s-differential equation we get Eq. (18.28) as displayed in the text.

We point out that a similar procedure may be used to evaluate the fermion propagator in the large mass limit. However, this evaluation is of no help in deducing infrared bosonization formulae of the kind of Eq. (18.25).

Finally we remark that the same procedure, now involving the Seeley-De Witt coefficient $b_2(x,x,A,s)$, was used to deduce Eq. (18.13).

18.6. The Triviality of the Abelian-Thirring Quantum Field Model

One of the most interesting problems in D-dimensional Euclidean field theories is the appearance of a critical dimensionality above which the associated field theory becomes trivial ([15], [16]).

Our aim in this section is to present the Parisi geometrical analysis [17] generalized to the fermionic case by analyzing the critical space-time dimension for the vectorial four-fermion interaction (the Abelain-Thirring model).

Let us start our analysis by considering the Thirring model Euclidean partition functional in $\mathbb{R}^D$ with the fermionic fields integrated out

$$Z(g) = \int DA_\mu \exp\left[-\frac{1}{2}\int dx^D A_\mu^2(x)\right] \det\, \not{D}(A_\mu), \qquad (18.46)$$

where $\not{D}(A_\mu \equiv \gamma_\mu(\partial_\mu + gA_\mu)$ is the Euclidean Dirac operator in the presence of the external auxiliary vectorial field and g is the bare theory's coupling constant.

We aim to show that $Z[g] = Z[g=0]$ when $D > 2$ since this result will lead, formally at least, to triviality of Eq. (18.46).

By using the fermionic loop representation for det $\not{D}(A_\mu)$, as displayed in Chapter 8, we can write this funcitonal determinant as a Grassmannian path integral:

$$\det \not{D}(A_\mu) = \sum_{[\chi_\mu^F(\xi,\theta)]} \exp\left(\int_0^1 d\xi \int_0^1 d\theta A_\mu[\chi_\mu^F(\xi,\theta)][D\chi_\mu^F(\xi,\theta)]\right)$$
$$= \sum_{[\chi_\mu^F(\xi,\theta)]} \int d^D x A_\mu(x) J_\mu^F[\chi_\mu^F(\xi,\theta)], \tag{18.47}$$

where the $\sum_{[\chi_\mu^F(\xi,\theta)]}$ is defined in Ref. 18 and $J_\mu^F[\chi_\mu^F(\xi,\theta)]$ is the current associated with the Grassmannian loop $\chi_\mu^F(\xi,\theta) = \chi_\mu(\xi) + i\theta\psi_\mu(\xi)$ $(\theta^2 = 0; 0 \le \xi \le 1)$. Through a g-power series expansion and integrating the Gaussian $A_\mu(x)$ functional integral we get, for instance, for its first coefficient $\left.\frac{dZ[h]}{dg}\right|_{g=0} = Z_1$ the following expression:

$$Z_1 = \sum_{[\chi_\mu^F(\xi,\theta)]} \exp \frac{1}{2} \int_0^1 d\xi d\theta \int_0^1 d\xi' d\theta' D\chi_\mu^F(\xi,\theta)\delta^{(D)}$$
$$\times (\chi_\mu^F(\xi,\theta) - \chi_\mu^F(\xi',\theta'))(D\chi^F(\xi',\theta'). \tag{18.48}$$

We can understand Eq. (18.48) as the partition functional associated with a gas of closed polymers $[\chi_\mu^F(\xi,\theta)]$ possessing a Grassmannian structure and interacting among themselves by a self-avoiding interaction $\delta^{(D)}[\chi_\mu^F(\xi,\theta) - \chi_\mu(\xi',\theta')]$ (see Chapter 9).

In order to argue for the triviality of the fermionic polymer gas we follow Parisi [17] by assigning a Hausdorff dimension d_H for the "set" $[\chi_\mu^F(\xi,\theta).\theta^2 = 0; 0 \le \xi \le 1]$. A natural Hausdorff dimension for this set is given by the exponent of the fermion free-field propagator in the momentum space which is 1, so $d_H[\chi_\mu^F(\xi,\theta)] = 1$.

By using now the geometrical intersection rule $d_H(A \cap B) = d_H(A) + d_H(B) - D$ [17] with D being the space-time dimensionality, we obtain that the support set of the self-avoiding interaction $[\delta^{(D)}(\chi_\mu(\xi,\theta) - \chi_\mu(\xi',\theta'))]$ has a negative Hausdorff dimension for $D > 2$, which means that this set is empty.

As a consequence we have the analytical relation

$$\int_0^1 d\xi d\theta \int_0^1 d\xi' d\theta' D^F \chi_\mu(\xi,\theta)\delta^{(D)}(\chi_\mu^F(\xi,\theta) - \chi_\mu^F(\xi',\theta'))D\chi_\mu^F(\xi',\theta') = 0, \tag{18.49}$$

which indicates, in turn, the triviality of the theory, since this argument can be straightforwardly applied for any arbitrary coeficient Z_n, leading to the result $Z_n = 0$.

Finally we remark that by reformulating the Thirring theory in the loop space, we can in principle define the theory for any general manifold m as space-time by including the constraint $[X_\mu^F(\xi,\theta)] \subset m$ in the path integral Eq. (18.46). Note that m may be fluctuating [17].

Work in this direction is left to our readers.

Appendix A

Let us write a formal path integral for Dirac particles by using only bosonic trajectories $X^{\mu}(\sigma)$, instead of the supersymmetric trajectories of Refs. 8-18.

By using the usual plane wave Euclidean spinor basis

$$|x,\alpha\rangle = e^{ipx}U_{\alpha}^{(1)}(p), \quad \langle y,\beta| = U_{\beta}^{(2)}(p)e^{ipy}, \tag{18.A.1}$$

where the spinors $\{U_{\alpha}^{(1)}(p), U_{\beta}^{(2)}(p)\}$ satisfy the free Dirac equation and the completeness relation

$$U_{\alpha}^{(1)}(p)\cdot\overline{U_{\beta}^{(2)}(p)} = \delta_{\alpha\beta}\,, \tag{18.A.2}$$

one can write the fermion propagator in the presence of an external field in the following form (see Ref. 3):

$$\begin{aligned} S_{\alpha\beta}(x-y) &= \int_0^{\infty} dt\ \langle x,\alpha|\exp(-T(-i\gamma\partial + g\gamma A + m))|\,y,\beta\rangle \\ &\times \int_0^{\infty} dT\, e^{-mT} \int_{\substack{X_{\mu}(0)=x \\ X_{\mu}(T)=y}} D^F[X_{\mu}(\sigma)] \int [p_{\mu}(\sigma)] \\ &\times \exp\left(i\int_0^T d\sigma\, p_{\mu}(\sigma)\cdot\dot{X}^{\mu}(\sigma)\right) \\ &\times \mathbb{P}_{\text{Dirac}}\left\{\exp\left(i\int_0^T \gamma^{\mu}(p_{\mu}(\sigma) + gA_{\mu}(x(\sigma))d\sigma\right)\right\}. \end{aligned} \tag{18.A.3}$$

where $\mathbb{P}_{\text{Dirac}}$ means the order along the bosonic trajectory of the Dirac indexes coming from the γ^{μ}-exponential involving the external gauge fields $A_{\mu}(X)$. Note that the $p_{\mu}(\sigma)$ path integral is free at the end points.

Let us now consider the formal variable change in the path integral in Eq. (18.A.3)

$$p_{\mu}(\sigma) + gA_{\mu}(X(\sigma)) = \pi_{\mu}(\sigma). \tag{18.A.4}$$

As a consequence of Eq. (18.A.5), we get the path integral fermion propagator formal expression used in Eq. (18.33) of the text.

$$S_{\alpha\beta}(x-y) = \int_0^\infty dT\, e^{-T} \int_{\substack{X_\mu(0)=x \\ X_\mu(T)=y}} D^F[X_\mu(\sigma)]$$
$$\times \int D^F[pi_\mu(\sigma)] \times \exp\left(i\int_0^T d\sigma \pi_\mu(\sigma)\cdot \dot{X}^\mu(\sigma)\right)$$
$$\times \exp\left(-ig\int_0^T d\sigma A_\mu(X(\sigma))\dot{X}^\mu(\sigma)\right)$$
$$\times \mathbb{P}_{\text{Dirac}}\left\{\exp i\int_0^T d\sigma(\gamma^\mu \pi_\mu)(\sigma)\right\}. \qquad (18.A.5)$$

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Chapter 19

Domains of Bosonic Functional Integrals and Some Applications to the Mathematical Physics of Path Integrals and String Theory

19.1. Introduction

Since the result of R.P. Feynman on representing the initial value solution of Schrodinger Equation by means of an analytically time continued integration on a infinite - dimensional space of functions, the subject of Euclidean Functional Integrals representations for Quantum Systems has became the mathematical - operational framework to analyze Quantum Phenomena and stochastic systems as showed in the previous decades of research on Theoretical Physics ([1]–[3]).

One of the most important open problem in the mathematical theory of Euclidean Functional Integrals is that related to implementation of sound mathematical approximations to these Infinite-Dimensional Integrals by means of Finite-Dimensional approximations outside of the always used [computer oriented] Space-Time Lattice approximations (see [2], [3] - chap. 9). As a first step to tackle upon the above cited problem it will be needed to characterize mathematically the Functional Domain where these Functional Integrals are defined.

The purpose of this chapter is to present in section 19.2, the formulation of Euclidean Quantum Field theories as Functional Fourier Transforms by means of the Bochner-Martin-Kolmogorov theorem for Topological Vector Spaces ([4], [5] - theorem 4.35) and suitable to define and analyze rigorously Functional Integrals by means of the well-known Minlos theorem ([5] - theorem 4.312 and [6] - part 2) and presented in full details in section 3.

In section 4, we present news results on the difficult problem of defining rigorously infinite-dimensional quantum field path integrals in general space times $\Omega \subset R^\nu \quad (\nu = 2,4,\dots)$ by means of the analytical regularization scheme.

19.2. The Euclidean Schwinger Generating Functional as a Functional Fourier Transform

The basic object in a scalar Euclidean Quantum Field Theory in R^D is the Schwinger Generating Functional (see refs. [1], [3]).

$$Z[j(x)] = \langle \Omega_{VAC}| \exp\left(i \int d^D x\, j(\vec{x}, it) \phi^{(m)}(\vec{x}, it) \right) |\Omega_{VAC}\rangle \tag{19.1}$$

where $\phi^{(m)}(\vec{x}, it)$ is the supposed Self-Adjoint Minkowski Quantum Field analytically continued to imaginary time and $j(x) = j(\vec{x}, it)$ is a set of functions belonging to a given Topological Vector Space of functions denoted by E which topology is not specified yet and will be called the Schwinger Classical field source space. It is important to remark that $\{\phi^m(\vec{x}, it)\}$ is a commuting Algebra of Self-Adjoints operators as Symanzik has pointed out ([7]).

In order to write eq.(19.1) as an Integral over the space E^{alg} of all linear functionals on the Schwinger Source Space E (the called Algebraic Dual os E), we take the following procedure, different from the usual abstract approach (as given - for instance - in the proof of th IV - 11 - [2]), by making the hypothesis that the restriction of the Schwinger Generating Functional eq.(19.1) to any finite-dimensional R^N of E is the Fourier Transform of a positive continuous function, namely.

$$Z\left(\sum_{\alpha=1}^{N} C_\alpha \bar{j}_\alpha(x) \right) = \int_{R^N} \exp\left(i \sum_{\alpha=1}^{N} C_\alpha P_\alpha \right) \tilde{g}(P_1, \ldots, P_N) dP_1, \ldots, dP_N \tag{19.2}$$

Here $\{\vec{j}_\alpha(x)\}_{\alpha=1,\cdots,N}$ is a fixed vectorial base of the given finite-dimensional sub-space (isomorphic to R^N) of E.

As a consequence of the above made hypothesis (based physically on the Renormalizability and Unitary of the associated Quantum Field Theory), one can apply the Bochner - Martin - Kolmogorov Theorem ([5] - theorem 4.35) to write eq.(19.1) as a Functional Fourier Transform on the Space E^{alg} (see appendix A)

$$Z[j(x)] = \int_{E^{alg}} \exp(ih(j(x)) d\mu(h) \tag{19.3}$$

where $d\mu(h)$ is the Kolmogorov cylindrical measure on $E^{alg} = \Pi_{\lambda\in A}(R^\lambda)$ with A denoting the index set of the fixed Hamel Vectorial Basis used in eq.(19.2) and $h(j(x))$ is the action of the given Linear (algebric) Functional (belonging to E^{alg}) on the element $j(x) \in E$.

At this point, we relate the mathematically non-rigorous physicist point of view to the Kolmogorov measure $d\mu(h)$ eq.(19.3) over the Algebraic Linear Functions on the Schwinger Source Space. It is formally given by the famous Feynman formulae when one identifies the action of h on E by means of an "integral" average

$$h(j) = \int_{R^D} dx^D j(x) h(x) \tag{19.4}$$

Formally we have the equation

$$d\mu(h) = \left(\prod_{x \in R^D} dh(x) \right) \exp\{-S(h(x))\} \tag{19.5}$$

where S is the classical action of the Classical Field Theory under quantization, but with the necessary coupling constant renormalizations need to make the associated Quantum Field Theory well-defined.

Let us outline these proposed steps on a $\lambda\phi^4$ - Field Theory on R^4.

At first we will introduce the massive free field theory generating functional directly in the infinite volume space R^4.

$$Z[j(x)] = \exp\left\{ -\frac{1}{2} \int d^4x d^4x' j(x)((-\Delta)^\alpha + m^2)^{-1}(x,x') j(x') \right\} \tag{19.6}$$

where the Free Field Propagator is given by

$$((-\Delta)^\alpha + m^2)^{-1}(x,x') = \int d^4k \frac{e^{ik(x-x')}}{k^{2\alpha} + m^2} \tag{19.7}$$

with α a regularizing parameter with $\alpha > 1$.

As the source space, we will consider the vector space of all real sequences on $\Pi_{\lambda \in (-\infty,\infty)}(R)^\lambda$, but with only a finite number of non-zero components. Let us define the following family of finite-dimensional Positive Linear Functionals $\{L_{\Lambda_f}\}$ on the Functional Space $C(\prod_{\lambda \in (-\infty,\infty)} R^\lambda; R)$

$$L_{\Lambda_f}\left(e(P_{\lambda_{s_1}}, \cdots P_{\lambda_{s_N}}) \right) = \int_{(\prod_{\lambda \in \Lambda_f} R^\lambda)} g(P_{\lambda_{s_1}}, \ldots, P_{\lambda_{s_N}}) \exp\left\{ -\frac{1}{2} \sum_{\lambda \in \Lambda_f} (\lambda^{2\alpha} + m^2)(P_\lambda)^2 \right\}$$
$$\left(\prod_{\lambda \in \Lambda_f} d(P_\lambda \sqrt{\pi(\lambda^{2\alpha} + m^2)}) \right) \tag{19.8}$$

Here $\Lambda_f = \{\lambda_{s_1}, \cdots, \lambda_{s_N}\}$ is an ordered sequence of real number of the real line which is the index set of the Hamel Basis of the Algebraic Dual of the proposed source space.

Note that we have the generalized eingenproblem expansion

$$((-\Delta)^\alpha + m^2)e^{i\lambda x} = (\lambda^{2\alpha} + m^2)e^{i\lambda x} \tag{19.9}$$

By the Stone-Weirstrass Theorem or the Kolmogoroff Theorem applied to the family of finite dimensional measure in eq.(19.8), there is a unique extension measure $d\mu(\{P_\lambda\}_{\lambda \in (-\infty,\infty)})$ to the space $\Pi_{\lambda \in (-\infty,\infty)} R^\lambda = E^{alg}$ and representing the Infinite-volume Generating Functional on our chosen source space (the usual Riesz-Markov theorem applied to the linear functional $L = \lim_{\{\Lambda_f\}} \sup L_{\Lambda_f}$, on $C(\Pi_{\lambda \in (-\infty,\infty)} R^\lambda, R)$ leads to this extension measure) ([10]).

$$Z[j(x)] = Z[\{j_\lambda\}_{\lambda\in\Lambda_f}] = \int_{\prod_{\lambda\in(-\infty,\infty)} R^\lambda} d\mu^{(0)\cdot(\alpha)}(\{P_\lambda\}_{\lambda\in(-\infty,\infty)})$$

$$\times \exp\left(i \sum_{\lambda\in(-\infty,\infty)} j_\lambda P_\lambda\right) = \exp\left\{-\frac{1}{2}\sum_{\lambda\in\Lambda_f} \frac{(j_\lambda)^2}{\lambda^{2\alpha}+m^2}\right\} \quad (19.10)$$

At this point it is very important remark that the generating functional eq.(19.10) has continuous natural extension to any test space ($S(R^N), D(R^N)$, etc) which contains the continuous functions of compact support as a dense sub-space.

At this point we consider the following Quantum Field interaction functional which is a measurable functional in relation to the above constructed Kolmogoroff measure $d\mu^{(0)\cdot(\alpha)}(\{P_\lambda\}_{\lambda\in(-\infty,\infty)})$ for α non integer in the original field variable $\phi(x)$

$$V^{(\alpha)}(\phi) = \lambda_R\phi^4 + \frac{1}{2}(Z_\phi^{(\alpha)}(\lambda_R, M) - 1)\phi((-\Delta)^\alpha)\phi - \frac{1}{2}[(m^2 Z_\phi^{(\alpha)}(\lambda_R, m) - 1$$
$$-(\delta m^2)^{(\alpha)}(\lambda_R)]\phi^2 - [Z_\phi^{(\alpha)}(\lambda_R, m)(\delta^{(\alpha)}\lambda)(\lambda_R, m)]\phi^4 \quad (19.11)$$

Here the renormalization constants are given in the usual analytical finite-part regularization form for a $\lambda\phi^4$ - Field Theory. It still a open problem in the mathematical-physics of quantum fields to prove the integrability in some Distributional space of the cut-off removing $\alpha \to 1$ limit of the interaction lagrangean $\exp(-V^{(\alpha)}(\phi))$ (see section 19.4 for a analysis of this cut off removing on space of functions).

19.3. The Support of Functional Measures - The Minlos Theorem

Let us now analyze the measure support of Quantum Field Theories generating functional eq.(19.3).

For higher dimensional space-time, the only available result in this direction is the case that we have a Hilbert structure on $E([4],[5],[6])$.

At this point of our paper, we introduce some definitions. Let $\varphi : \mathbb{Z}^+ \to R$ be an increasing fixed function (including the case $\varphi(\infty) = \infty$). Let E be denoted by H and H^Z be the sub-space of $H^{alg} = (\Pi_{\lambda\in A=[0,1]} R^\lambda)$ (with A being the index set of a Hamel basis of H), formed by all sequences $\{x_\lambda\}_{\lambda\in A} \in H^{alg}$ with coordinates different from zero at most a countable number

$$H^Z = \{(x_\lambda)_{\lambda\in A|x_\lambda} \neq 0 \text{ for } \lambda \in \{\lambda_\mu\}_{\mu\in\mathbb{Z}}\} \quad (19.12)$$

Consider the following weighted sub-set of H^{alg}

$$H^Z_{(e)} = \{\{x_\lambda\}_{\lambda\in A} \in H^Z\}$$

and

$$\lim_{N\to\infty}\left\{\frac{1}{\varphi(N)}\sum_{n=1}^N (x_{\lambda\sigma(\mu)})^2\right\} < \infty\}$$

for any $\sigma : N \to N$, a permutation of the natural numbers.

We now state our generalization of the Minlos Theorem.

Theorem 3. Let T be an operator, with Domain $D(T) \subset H$, and $T; D(T) \to H$ such that for any finite-dimensional space $H^N \subset H$, the sum is bounded by the function $\varphi(N)$

$$\left(\sum_{(i,j)=1}^{N} \langle Te_i, Te_j \rangle^{(0)} \right) \leq \varphi(N) \tag{19.13}$$

Here $\langle , \rangle^{(0)}$ is the inner product of H and $\{e_p\}_{1 \leq p \leq N}$ is a vectorial basis of the sub-space H^N with dimension N.

Suppose that $Z[j(x)]$ is a continuous function an $D(T) = \overline{(D(T), \langle , \rangle^{(1)}})$ where $\langle , \rangle^{(1)}$ is a new inner product defined by the operator $T(\langle j, \overline{j} \rangle^{(1)} = \langle T j, T \overline{j} \rangle^{(0)})$ we have, thus, that the support of the cylindrical measure eq.(19.3) is the measurable set H_e^Z.

Proof: Following closely references ([1]) - Theorem 2.2., [4]) let us consider the following representation for the characteristic function of the measurable set $H_e^Z \subset H^{alg}$

$$\begin{aligned} &\cdot X_{H_e^Z}(\{x_\lambda\}_{\lambda \in A}) = \\ &\lim_{\alpha \to 0} \lim_{N \to \infty} \exp \left\{ -\frac{1\alpha}{2\varphi(N)} \sum_{\ell=1}^{N} x_{\lambda_l}^2 \right\} \\ &= 1 \qquad \text{if} \qquad \lim_{N \to \infty} \frac{1}{\varphi(N)} \sum_{\ell=1}^{N} x_{\lambda_l}^2 < \infty \\ &0 \qquad \text{otherwise} \end{aligned} \tag{19.14}$$

Now its measure satisfies the following inequality

$$\int_{H^{alg}} d\mu(h) = \mu(H^{alg}) = 1 > \mu(H_e^Z) \tag{19.15}$$

But

$$\begin{aligned} &\mu(H_e^Z) = \lim_{\alpha \to 0} \lim_{N \to \infty} \int_{H^{alg}} d\mu(h) \exp - \left\{ \frac{\alpha}{2\varphi(N)} \sum_{\ell=1}^{N} x_{\lambda_\ell}^2 \right\} = \\ &\lim_{\alpha \to 0} \lim_{N \to \infty} \left\{ \frac{1}{(\frac{2\pi\alpha}{\varphi(N)})^{N/2}} \right\} \int_{R^N} d j_1, \cdots d j_N \\ &\exp \left(-\frac{1}{2} \left(\frac{\varphi(N)}{\alpha} \right) \sum_{\ell=1}^{N} j_\ell^2 \right) \tilde{Z}(j_1, \cdots, J_N) \end{aligned} \tag{19.16}$$

where

$$\tilde{Z}(j_1, \cdots, j_N) = \int_{\substack{\pi R^\lambda \\ \lambda \in A}} d\mu(\{x_\lambda\}) \exp \left(i \sum_{\ell=1}^{N} x_{\lambda_\ell} j_\ell \right) = \int_{H^{alg}} d\mu(h) \exp \left(i \sum_{\ell=1}^{N} x_{\lambda_\ell} j_\ell \right) \tag{19.17}$$

Now due to the continuity and positivity of $Z[j]$ in $D(T)$; we have that for any $\varepsilon > 0 \to \exists\delta$ such that the inequality below is true since we have that: $Z(j_1,\cdots j_N) \geq 1-\varepsilon-\frac{2}{\delta^2}(j,j)^{(1)}$

$$\begin{aligned}
&\frac{1}{\left(\frac{2\pi\alpha}{\varphi(N)}\right)^{N/2}}\int_{R^N} dj_1\cdots dj_N \exp\left(-\frac{1}{2}\frac{\varphi(N)}{\alpha}\sum_{\ell=1}^{N} j_\ell^2\right)\tilde{Z}(j_1\cdots,j_N)\\
&\geq 1-\varepsilon-\frac{2}{\delta^2}\left\{\sum_{(m,n)=1}^{N}\frac{1}{\left(\frac{2\pi\alpha}{\varphi(N)}\right)^{N/2}}\int_{R^N} dj_1\cdots dj_N \exp\left(-\frac{1}{2}\left(\frac{\varphi(N)}{\alpha}\right)\sum_{\ell=1}^{N} j_\ell^2\right) j_m j_n < e_m, e_n >^{(1)}\right\} \qquad (19.18)\\
&= 1-\varepsilon-\frac{2}{\delta^2}\left\{\left(\frac{\alpha}{\varphi(N)}\right)\sum_{(m,n)=1}^{N}\delta_{mn} < Te_n, Te_m >^{(0)}\right\}\\
&\geq 1-\varepsilon-\frac{2}{\delta^2}\left(\frac{\alpha}{\varphi(N)}\right)\varphi(N) \geq 1-\varepsilon-\frac{2}{\delta^2}\alpha
\end{aligned}$$

By substituting eq.(19.18) into eq.(19.15), we get the result

$$1 \geq \mu(H_e^Z) \geq 1-\varepsilon-\frac{2}{\delta^2}\left(\lim_{\alpha\to 0}\alpha\right) = 1-\varepsilon \qquad (19.19)$$

Since ε was arbitrary we have the validity of our theorem.

As a consequence of this Theorem in the case of $\varphi(N)$ being bounded (so TT^* is an operator of Trace Class), we have that $H_e^Z = H$ which is the usual Topological Dual of H.

At this point, a simple proof may be given to the usual Minlos Theorem on Schwartz Spaces ([5], [6],).

Let us consider $S(R^D)$ represented as the countable normed spaces of sequences ([8])

$$S(R^D) = \bigcap_{m=0}^{\infty} \ell_m^2 \qquad (19.20)$$

where

$$\ell_m^2 = \{(x_n)_{n\in\mathbb{Z}}, x_n \in R \Big| \sum_{n=0}^{N}(x_n)^2 n^m < \infty\} \qquad (19.21)$$

The Topological Dual is given by the nuclear structure sum ([8])

$$S'(R^D) = \bigcup_{n=0}^{\infty} \ell_{-n}^2 = \bigcup_{n=0}^{\infty} (\ell_n^2)^* \qquad (19.22)$$

We, thus, consider $E = S(R^D)$ in eq.(19.3) and $Z[j(x)] = Z[\{j_n\}_{n\in\mathbb{Z}}]$ as a continuous on $\bigcap_{n}^{\infty}\ell_n$. Since $Z[\{j_n\}_{n\in\mathbb{Z}}] \in C(\bigcap_{n=0}^{\infty}\ell_n^2, R)$ we have that for any fixed integer $p, Z[\{j_n\}_{n\in\mathbb{Z}}]$ is continuous on the Hilbert Space ℓ_p^2 which, by its turn, may be considered as the Domain of the following operator.

$$\begin{aligned}
T_p : \ell_p^2 c\ell_0 &\to \ell_0\\
\{j_n\} &\to \{n^{p/2} j_n\} \qquad (19.23)
\end{aligned}$$

It is straightforward to have the estimate

$$\Big|\sum_{(m,n)=1}^{N}\langle T_p e_m\,,\,T_p e_n\rangle^{(0)}\Big| \le N^{(B_p)} \tag{19.24}$$

for some positive integer B and $\{e_i\}$ being the canonical orthonormal basis of l_o^2. By an application of our theorem for each fixed p; we get that the support of measure is given by the union of weighted spaces

$$\operatorname{supp} d\mu(h) = \bigcup_{p=0}^{\infty}(\ell_p^2)^* = \bigcup_{p=0}^{\infty}\ell_{-p}^2 = S'(R^D) \tag{19.25}$$

At this point we can suggest, without a proof a straightforward (non topological) generalization of the Minlos Theorem.

Theorem 4. Let $\{T_\beta\}_{\beta\in C}$ be a family of operators satisfying the hypothesis of Theorem 3. Let us consider the Locally Convex space $\bigcup_{\beta\in C}\overline{Dom(T_\beta)}$ (supposed non-empty) with the family of norms $\|\,\psi\,\|_\beta = <T_\beta\psi, T_\beta\psi>^{1/2}$

If the Functional Fourier Transform is continuous on this Locally Convex Space, the support of the Kolmogoroff measure eq.(19.3) is given by the following sub-set of $[\bigcup_{\beta\in C}\overline{Dom(T_\beta)}]^{alg}$, namely

$$\operatorname{supp} d\mu(h) = \bigcup_{\beta\in C} H^2_{\varphi_\beta} \tag{19.26}$$

where φ_β are the functions given by Theorem 3. This general theorem will not be applied in what follows.

Let us now proceed to apply the above displayed results by considering the Schwinger Generating Functionals for two-dimensional Euclidean Quantum Eletrodynamics in Bosonized Parametrization ([9])

$$Z[j(x)] = \exp\left\{-\frac{1}{2}\int_{R^2} d^2x \int_{R^2} d^2y\, j(x)((-\Delta)^2 + \frac{e^2}{\pi}(-\Delta))^{-1}(x,y)\, j(y)\right\} \tag{19.27}$$

where in eq.(19.27), the electromagnetic field has the decomposition in Landau Gauge

$$A_\mu(x) = (\varepsilon_{\mu\nu}\partial_\nu\phi)(x) \tag{19.28}$$

and $j(x)$ is, thus, the Schwinger Source for the $\phi(x)$ field taken as a basic dynamical variable ([9]).

Since eq.(19.27) is continuous in $L^2(R^2)$ with the inner product defined by the trace class operator $((-\Delta)^2 + \frac{e^2}{\pi}(-\Delta))^{-1}$, we conclude on basis of theorem 3 that the associated Kolmogoroff measure in eq.(19.3) has its support in $L^2(R^2)$ with the usual inner product. As a consequence, the Quantum Observable Algebra will be given by the Functional Space $L^1(L^2(R^2), d\mu(h))$ and usual orthonormal Finite - Dimensional approximations in Hilbert

Spaces may be used safely i.e if one considers the basis expansion $h(x) = \sum_{n=1}^{\infty} h_n e_n(x)$ with $e_n(x)$ denoting the eigenfunctions of the operator in eq.(19.27) we get the result

$$\bigcup_{n=1}^{\infty} L^1(R^N, d\mu(h_1, \cdots, h_N)) = L^1(L^2(R^2), d\mu(h)) \tag{19.29}$$

It is worth mentioning that if one uses the Gauge Vectorial Field parametrization for the $(Q.E.D)_2$ - Schwinger Functional

$$Z[j_1(x), j_2(x)] = \exp\left\{ -\frac{1}{2} \int_{R^2} d^2x \int_{R^2} d^2y\, j_i(x) \left(-\Delta + \frac{e^2}{\pi} \right)^{-1} (x,y) \delta_{il}\, j_l(x) \right\} \tag{19.30}$$

the associated measure support will now be the Schwartz Space $S'(R^2)$ since the operator $(-\Delta + \frac{e^2}{\pi})^{-1}$ is an application of $S(R^2)$ to $S'(R^2)$. As a consequence it will be very cumbersome to use Hilbert Finite Dimensional approximations ([8]) as in eq.(19.29).

An alternative to approximate tempered distributions is the use of its Hermite expansion in $S'(R)$ distributional space associated to the eigenfunctions of the Harmonic-oscillator $V(x) \in L^\infty(R) \bigcup L^2(R)$ potential pertubation (see ref. [3] for details with $V(x) \equiv 0$).

$$(-\frac{d^2}{dx^2} + x^2 + V(x)) H_n(x) = \lambda_n H_n(x) \tag{19.31}$$

Another important class of Bosonic Functionals Integrals are those associated with an Elliptic Positive Self-Adjoint Operator A^{-1} on $L^2(\Omega)$ with suitable Boundary conditions. Here Ω denotes a D-dimensional compact manifold of R^D with volume element $d\nu(x)$.

$$Z[j(x)] = \exp\left\{ -\frac{1}{2} \int_{\Omega} d\nu(x) \int_{\Omega} d\nu(y)\, j(x) A^{+1}(x,y)\, j(y) \right\} \tag{19.32}$$

If A is an operator of trace class on $(L^2(\Omega), d\nu)$ we have, thus, the validity of the usual eigenvalue Functional Representation

$$Z[\{j_n\}_{\mu \in \mathbb{Z}}] = \int \left(\prod_{\ell=1}^{\infty} d(c_\ell \sqrt{\lambda_\ell}) \right) \exp\left(-\frac{1}{2} \sum_{\ell=1}^{\infty} \lambda_\ell c_\ell^2 \right) X_{\ell^2}(\{c_n\}_{n \in \mathbb{Z}}) \exp(i \sum_{\ell=1}^{\infty} c_\ell j_\ell)$$

with the spectral set

$$\begin{aligned} A^{-1}\sigma_\ell &= \lambda_\ell \sigma_\ell \\ j_\ell &= \langle j, \sigma_\ell \rangle \end{aligned} \tag{19.33}$$

and the characteristic function set

$$X_{\ell^2}(\{c_n\}_{n \in \mathbb{Z}}) = \begin{cases} 1 \text{ if} & \sum_{n=0}^{\infty} c_n^2 < \infty \\ 0 & \text{otherwise} \end{cases} \tag{19.34}$$

It is instructive point out the usual Hermite functional basis (see 5.4 - [5]) are a complete set in $L^2(E^{alg}, d\mu(h))$, only if the Gaussian Kolmogoroff measure $d\mu(h)$ is of the class above studied

A criticism to the usual framework to construct Euclidean Field Theories is that is very cumbersome to analyze the infinite volume limit from the Schwinger Generating Functional defined originally on Compact Space Times. In two dimensions the use of the result that the massive Scalar Field Theory Generating Functional

$$\exp\left\{-\frac{1}{2}\int_{R^2} d^2x \int_{R^2} d^2y\, j(x)(-\Delta+m^2)^{-1}(x,y)\, j(y)\right\} \tag{19.35}$$

with $j(x) \in S(R^2)$; is given by the limit of Finite Volume Dirichlet Field Theories

$$\lim_{\substack{L\to\infty \\ T\to\infty}} \exp\left\{-\frac{1}{2}\int_{-L}^{L} dx^0 \int_{-T}^{T} dx^1 \int_{-L}^{L} dy^0 \right.$$
$$\left.\int_{-T}^{T} dy^1\, j(x^0,x^1)(-\Delta_D+m^2)^{-1}(x^1,y^1,x^0,y^0)\, j(y^0,y^1)\right. \tag{19.36}$$

may be considered, in our opinion, as the similar claim made that is possible from a mathematical point of view to deduce the Fourier Transforms from Fourier Series, a very, difficult mathematical task (see appendix B).

Let us comment on the functional integral associated to Feynman propagation of fields configurations used in geometrodynamical theories in the scalar case

$$G[\beta^{in}(x);\beta^{out}(x),T](j) = \int_{\substack{\phi(x,0)=\beta^{in}(x) \\ \phi(x,T)=\beta^{out}(x)}}$$
$$\exp\left\{-\frac{1}{2}\int_0^T dt \int_{-\infty}^{+\infty} d^\nu x \left(\phi\left(-\frac{d^2}{dt^2}+A\right)\phi\right)(x,t)\right\}$$
$$\exp\left(i\int_0^T dt \int_{-\infty}^{+\infty} d^\nu x\, j(x)t)\phi(x,t)\right) \tag{19.37}$$

If we define the formal functional integral by means of the eigenfunctions of the self-adjoint Elliptic operator A, namely:

$$\phi(x,t) = \sum_{\{k\}} \phi_k(t)\psi_k(x) \tag{19.38}$$

where

$$A\psi_k(x) = (\lambda_k)^2\psi_k(x) \tag{19.39}$$

it is straightforward to see that eq.(19.36) is formally exactly evaluated in terms of an infinite product of usual Feynman Wiener - path measures

$$G[\beta^{in}(x);\beta^{out}(x),T](j) =$$
$$= \prod_{\{k\}} \int_{\substack{c_k(0)=\phi_k(0)\\ c_k(T)=\phi_k(T)}} D^F[c_k(t)] \exp\left\{-\frac{1}{2}\int_0^T \left(c_k\left(-\frac{d^2}{dt^2}+\lambda_k^2\right)c_k\right)(t)dt \exp\left(i\int_0^T dt\, j_k(t)c_k(t)\right)\right\}$$
$$= \prod_{\{k\}}\left\{\sqrt{+\frac{\lambda_k}{\sin(\lambda_k T)}}\exp\left\{-\frac{\lambda_k}{2\sin(\lambda_k T)}\left[(\phi_k^2(T)+\phi_k^2(0))\cos(\lambda_k T)-2\phi_k(0)\phi_k(T)\right\}\right\}$$
$$-\frac{2\phi_k(T)}{\lambda_k}\int_0^T dt\, j_k(t)\sin(\lambda_k t)-\frac{2\phi_k(0)}{\lambda_k}\int_0^T dt\, j_k(t)\sin(\lambda_k(T-t))$$
$$\left\{-\frac{2}{(\lambda_k)^2}\int_0^T dt\int_0^t ds\, j_k(t)j_k(s)\sin(\lambda_k(T-t))\sin(\lambda_k s)\right]\right\} \quad (19.40)$$

Unfortunately, our theorems do not aply in a straightforward way to infinite (continuum) measure product of Wiener measures in eq.(19.40) to produce a sensible measure theory on the functional space of the infinite product of Wiener trajectories $\{c_k(t)\}$ (Note that for each x fixed, a sample field configuration $\phi(t,0)$ in eq.(19.36) is a Hölder continuous function, result opposite to the usual functional integral representation for the Schwinger generating functional eq.(19.1)- eq.(19.5)) where it does not make a mathematical sense to consider a fixed point distribution $\phi(t,0)$ - see section 19.4 - eq.(19.74).

Let us call attention that still there is a formal definition of the above Feynman Path propagator for fields eq.(19.37) which at large time $T \to +\infty$ gives formally the Quantum Field Functional integral eq.(19.5) associated to the Schwinger Generating Functional.

We thus consider the functional domain for eq.(19.37) as composed of field configurations which has a classical piece added with another fluctuating component to be functionally integrated out, namely

$$\sigma(x,t) = \sigma_{CL}(x,t) + \sigma_q(x,t) \quad (19.41)$$

Here the classical field configuration problem (added with all zero modes of the free theory) defined by the kinetic term £

$$\left(-\frac{d^2}{dt^2} + \text{£}\right)\sigma^{CL}(x,t) = j(x,t) \quad (19.42)$$

with

$$\sigma^{CL}(x,-T) = \beta_1(x); \sigma^{CL}(x,T) = \beta_2(x) \quad (19.43)$$

namely

$$\sigma_{CL}(x,t) = \left(-\frac{d^2}{dt^2} + \text{£}\right)^{-1} j(x,t) + (\text{ all projection on zero modes of £}) \quad (19.44)$$

As a consequence of the decomposition eq.(19.41), the formal geometrical propagator with an external source below

$$G[\beta_1(x),\beta_2(x),T,[j]]$$
$$=\int_{\substack{\sigma(x,-T)=\beta_1(x)\\ \sigma(x,+T)=\beta_2(x)}} D[\sigma(x,t)]\exp\left(-\frac{1}{2}\int_{-T}^{T}dt d^{\nu}x\sigma(x,t)\left(-\frac{d^2}{dt^2}+\pounds\right)\sigma(x,t)\right)$$
$$\exp(i\int_{-T}^{T}dt\int d^{\nu}xj(x,t)\sigma(x,t)) \quad (19.45)$$

may be defined the following mathematically well defined Gaussian functional measure

$$\exp\left\{-\frac{1}{2}\int_{-T}^{T}dt\int d^{\nu}xj(x,t)\sigma^{CL}(x,t)\right\}\times$$
$$\int_{\substack{\sigma_q(x,-T)=0\\ \sigma_q(x,+T)=0}} d\sigma_q(x,t)\exp\left\{-\frac{1}{2}\int_{-T}^{T}dt\int d^{\nu}x\sigma_q(x,t)\left(-\frac{d^2}{dt^2}+\pounds\right)\sigma_q(x,t)\right\} \quad (19.46)$$

The above claim is a consequence of the result below

$$\int_{\substack{\sigma_q(x,-T)=0\\ \sigma_q(x,T)=0}} D[\sigma_q(x,t)]\exp\left\{-\frac{1}{2}\int_{-T}^{T}dt\int d^{D}x\sigma_q(x,t)\left(-\frac{d^2}{dt^2}+\pounds\right)\sigma_q(x,t)\right\}$$
$$=det_{Dir}^{-\frac{1}{2}}\left[-\frac{d^2}{dt^2}+\pounds\right] \quad (19.47)$$

where the sub-script Dirichlet on the functional determinant means that one must impose formally the Dirichlet condition on the domain of the operator $\left(-\frac{d^2}{dt^2}+\pounds\right)$ on $D'(R^D\times[-T,T])$ (or $L^2(R^D\times[-T,T]$ if $\pounds^{-1}$ belongs to trace class). Note that the operator $\pounds$ in eq.(19.46) does not have zero modes by the construction of eq.(19.41).

At this point, we remark that at the limit $T\to+\infty$ eq.(19.45) is exactly the Quantum Field functional eq.(19.5) if one takes $\beta_1(x)=\beta_2(x)=0$ (Note that the classical vacuum limit $T\to\infty$ of Wiener measures is mathematically ill-defined (see theorem 5.1. of ref [1]).

It is a important point to remark that $\sigma_{CL}(x,t)$ is a regular $C^{\infty}([-T,T]\times\Omega)$ solution of the Elliptic problem eq.(19.42) and the fluctuating component $\sigma_q(x,t)$ is a Schwartz distribution in view of the Minlos - Dao Xing theorem 3, since the Elliptic operator $-\frac{d^2}{dt^2}+\pounds$ in eq.(19.47) acts now on $D'([-T,T]\times\Omega)$ with range $D([-T,T]\times\Omega)$, which by its turn shows the difference between this framework and the previous one related to the infinite product of Wiener measures since these objects are functional measures in different Functional Spaces

Finally we comment that Functional Schrodinger equation, may be mathematically defined for the above displayed field propagators eq.(19.37) only in the situation of eq.(19.40). For instance, with $\pounds=-\Delta$ (the Laplacean), we have the validity of the Euclidean field wave equation for the Geometrodynamical path-integral eq (37)

$$\frac{\partial}{\partial T}G[\beta_1(x),\beta_2(x),T,[j]]=$$
$$=\int_{\Omega}d^{\nu}x\Big[+\frac{\delta^2}{\delta^2\beta_2(x)}-|\nabla\beta_2(x)|^2+j(x,T)\Big]G[\beta_1(x),\beta_2(x),T,[j]] \quad (19.48)$$

with the functional initial - condition

$$\lim_{T\to 0^+}G[\beta_1(x),\beta_2(x),T]=\delta^{(F)}(\beta_1(x)-\beta_2(x)) \quad (19.49)$$

19.4. Some Rigorous Quantum Field Path Integral in the Analytical Regularization Scheme

In this core section of our paper we address the important problem of producing concrete non-trivial examples of mathematically well - defined (in the ultra - violet region!) path integrals in the context of the exposed theorems on the previously sections of this paper, specially section 19.2 - eq.(19.11).

Let us thus start our analysis by considering the Gaussian measure associated to the (infrared regularized) α-power $(\alpha > 1)$ of the Laplacean acting on $L^2(R^2)$ as an operational quadratic form (the Stone spectral theorem)

$$(-\Delta)^{\alpha}_{\varepsilon} = \int_{\varepsilon_{IR} \leq \lambda} (\lambda)^{\alpha}\, dE(\lambda) \tag{19.50-a}$$

$$\begin{aligned} Z^{(0)}_{\alpha,\varepsilon_{IR}}[j] &= \exp\left\{ -\frac{1}{2} \left\langle j, (-\Delta)^{-\alpha}_{\varepsilon}\, j \right\rangle_{L^2(R^2)} \right\} \\ &= \int d^{(0)}_{\alpha,\varepsilon}\mu[\varphi]\, \exp\left(i \left\langle j, \varphi \right\rangle_{L^2(R^2)} \right) \end{aligned} \tag{19.50-b}$$

Here $\varepsilon_{IR} > 0$ denotes the infrared cut off.

It is worth call the reader attention that due to the infrared regularization introduced on eq (50-a), the domain of the Gaussian measure is given by the space of square integrable functions on R^2 by the Minlos theorem of section 19.3, since for $\alpha > 1$, the operator $(-\Delta)^{-\alpha}_{\varepsilon_{IR}}$ defines a classe trace operator on $L^2(R^2)$, namely

$$Tr_{\oint_1}((-\Delta)^{-\alpha}_{\varepsilon_{IR}}) = \int d^2k \frac{1}{(|K|^{2\alpha} + \varepsilon_{IR})} < \infty \tag{19.50-c}$$

This is the only point of our analysis where it is needed to consider the infra-red cut off considered on the spectral resolution eq (50-a). As a consequence of the above remarks, one can analize the ultra-violet renormalization program in the following interacting model proposed by us and defined by an interaction $g_{\text{bare}} V(\varphi(x))$, with $V(x)$ denoting a compact support function on R such, that it posseses an essentially bounded Fourier transform and g_{bare} denoting the positive bare coupling constant.

Let us show that by defining a renormalized coupling constant as (with $g_{ren} < 1$)

$$g_{\text{bare}} = \frac{g_{\text{ren}}}{(1-\alpha)^{1/2}} \tag{19.51}$$

one can show that the interaction function

$$\exp\left\{ -g_{\text{bare}}(\alpha) \int d^2x V(\varphi(x)) \right\} \tag{19.52}$$

is an integrable function on $L^1(L^2(R^2), d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi])$ and leads to a well-defined ultra-violet path integral in the limit of $\alpha \to 1$.

The proof is based on the following estimates.

Since almost everywhere we have the pointwise limit

$$\exp\left\{-g_{\text{bare}}(\alpha)\int d^2x V(\varphi(x))\right\}$$
$$\lim_{N\to\infty}\left\{\sum_{n=0}^{N}\frac{(-1)^n(g_{\text{bare}}(\alpha))^n}{n!}\int_R dk_1\cdots dk_n\tilde{V}(k_1)\cdots\tilde{V}(k_n)\int_{R^2}dx_1\cdots dx_n e^{ik_1\varphi(x_1)}\cdots e^{ik_n\varphi(x_n)}\right\} \tag{19.53}$$

we have that the upper-bound estimate below holds true

$$\left|Z^{\alpha}_{\varepsilon_{IR}}[g_{\text{bare}}]\right| \leq \Big|\sum_{n=0}^{\infty}\frac{(-1)^n(g_{\text{bare}}(\alpha))^n}{n!}\int_R dk_1\cdots dk_n\tilde{V}(k_1)\cdots\tilde{V}(k_n)$$
$$\int_{R^2}dx_1\cdots dx_n\int d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi](e^{i\sum_{\ell=1}^{N}k_\ell\varphi(x_\ell)})\Big| \tag{19.54-a}$$

with

$$Z^{\alpha}_{\varepsilon_{IR}}[g_{\text{bare}}]=\int d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi]\exp\left\{-g_{\text{bare}}(\alpha)\int d^2x V(\varphi(x))\right\} \tag{19.54-b}$$

we have, thus, the more suitable form after realizing the d^2k_i and $d^{(0)}_{\alpha,\varepsilon_{IR}}\mu[\varphi]$ integrals respectivelly

$$\left|Z^{\alpha}_{\varepsilon_{IR}=0}[g_{bare}]\right| \leq \sum_{n=0}^{\infty}\frac{(g_{bare}(\alpha))^n}{n!}\left(||\tilde{V}||_{L^\infty(R)}\right)^n$$
$$\Big|\int dx_1\cdots dx_n \det{}^{-\frac{1}{2}}\left[G^{(N)}_{\alpha}(x_i,x_j)\right]_{\substack{1\leq i\leq N\\1\leq j\leq N}}\Big| \tag{19.55}$$

Here $[G^{(N)}_{\alpha}(x_i,x_j)]_{\substack{1\leq i\leq N\\1\leq j\leq N}}$ denotes the $N\times N$ symmetric matrix with the (i,j) entry given by the Green-function of the α-Laplacean (without the infra-red cut off here! and the needed normalization factors !).

$$G_{\alpha}(x_i,x_j)=|x_i-x_j|^{2(1-\alpha)}\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)} \tag{19.56}$$

At this point, we call the reader attention that we have the formulae on the asymptotic behavior for $\alpha\to 1$.

$$\left\{\lim_{\substack{\alpha\to 1\\ \alpha>1}}\det{}^{-\frac{1}{2}}[G^{(N)}_{\alpha}(x_i,x_j)]\right\}\sim(1-\alpha)^{N/2}\times\left(\left|\frac{(N-1)(-1)^N}{\pi^{N/2}}\right|\right)^{-\frac{1}{2}} \tag{19.57}$$

After substituting eq.(19.57) into eq.(19.55) and taking into account the hypothesis of the compact support of the non-linearily $V(x)$ (for instance: $suppV(x)\subset[0,1]$), one obtains the finite bound for any value $g_{rem}>0$, and producing a proof for the convergence of the perturbative expansion in terms of the renormalized coupling constant.

$$\lim_{\alpha\to 1}\left|Z^{\alpha}_{\varepsilon_{IR}=0}[g_{bare}(\alpha)]\right| \leq \sum_{n=0}^{\infty}\frac{(\|\tilde{V}\|_{L^\infty(R)})^n}{n!}\left(\frac{g_{ren}}{(1-\alpha)^{\frac{1}{2}}}\right)^n\times\frac{(1^n)}{\sqrt{n}}(1-\alpha)^{n/2}$$
$$\leq e^{g_{ren}\|\tilde{V}\|_{L^\infty(R)}}<\infty \tag{19.58}$$

Another important rigorously defined functional integral is to consider the following α-power Klein Gordon operator on Euclidean space-time

$$\mathcal{L} = (-\Delta)^{\alpha} + m^2 \tag{19.59}$$

with m^2 a positive "mass" parameters.

Let us note that $\mathcal{L}^{-1}$ is an operator of class trace on $L^2(R^\nu)$ if and only if the result below holds true

$$Tr_{L^2(R^\nu)}(\mathcal{L}^{-1}) = \int d^\nu k \frac{1}{k^{2\alpha}+m^2} = \bar{C}(\nu)\, m^{(\frac{\nu}{\alpha}-2)} \times \left\{\frac{\pi}{2\alpha}\text{cosec}\frac{\nu\pi}{2\alpha}\right\} < \infty \tag{19.60}$$

namely if

$$\alpha > \frac{\nu}{2} \tag{19.61}$$

In this case, let us consider the double functional integral with functional domain $L^2(R^\nu)$

$$\begin{aligned} Z[j,k] = &\int d_G^{(0)}\beta[v(x)] \\ &\times \int d^{(0)}_{(-\Delta)^\alpha+v+m^2}\mu[\varphi] \\ &\times \exp\left\{i\int d^\nu x\,(j(x)\,\varphi(x)+k(x)\,v(x))\right\} \end{aligned} \tag{19.62}$$

where the Gaussian functional integral on the fields $V(x)$ has a Gaussian generating functional defined by a $\oint_1$-integral operator with a positive defined kernel $g(|x-y|)$, namely

$$\begin{aligned} Z^{(0)}[k] &= \int d_G^{(0)}\beta[v(x)]\, \exp\left\{i\int d^\nu x\, k(x)v(x)\right\} \\ &= \exp\left\{-\frac{1}{2}\int d^\nu x \int d^\nu y\,(k(x)\,g(|x-y|)\,k(x))\right\} \end{aligned} \tag{19.63}$$

By a simple direct application of the Fubbini-Tonelli theorem on the exchange of the integration order on eq.(19.62), lead us to the effective $\lambda\varphi^4$ - like well-defined functional integral representation

$$\begin{aligned} Z_{\text{eff}}[j] = &\int d^{(0)}_{((-\Delta)^\alpha+m^2)}\mu[\varphi(x)] \\ &\exp\left\{-\frac{1}{2}\int d^\nu x d^\nu y\,|\varphi(x)|^2\, g(|x-y|)\,|\varphi(y)|^2\right\} \\ &\times \exp\left\{i\int d^\nu x\, j(x)\varphi(x)\right\} \end{aligned} \tag{19.64}$$

Note that if one introduces from the begining a bare mass parameters m^2_{bare} depending on the parameters α, but such that it always satisfies eq.(19.60) one should obtains again eq.(19.64) as a well-defined measure on $L^2(R^\nu)$. Of course that the usual pure Laplacean limit of $\alpha \to 1$ on eq.(19.59), will needed a renormalization of this mass parameters ($\lim_{\alpha\to 1} m^2_{bare}(\alpha) = +\infty$!) as much as done in the previous example.

Let us continue our examples by showing again the usefulness of the precise determination of the functional - distributional structure of the domain of the functional integrals in order to construct rigorously these path integrals without complicated limit procedures.

Let us consider a general R^ν Gaussian measure defined by the Generating functional on $S(R^\nu)$ defined by the α-power of the Laplacean operator $-\Delta$ acting on $S(R^\nu)$ with a of small infrared regularization mass parameter μ^2

$$\begin{aligned} Z_{(0)}[j] &= \exp\left\{-\frac{1}{2}\left\langle j, ((-\Delta)^\alpha+\mu_0^2)^{-1} j\right\rangle_{L^2(R^\nu)}\right\} \\ &= \int_{E^{alg}(S(R^\nu))} d_\alpha^{(0)}\mu[\varphi] \exp(i\varphi(j)) \end{aligned} \tag{19.65}$$

An explicitly expression in momentum space for the Green function of the α-power of $(-\Delta)^\alpha+\mu_0^2$ given by

$$((-\Delta)^{+\alpha}+\mu_0^2)^{-1}(x-y) = \int \frac{d^\nu k}{(2\pi)^\nu} e^{ik(x-y)} \left(\frac{1}{k^{2\alpha}+\mu_0^2}\right) \tag{19.66}$$

Here $\bar{C}(\nu)$ is a ν-dependent (finite for ν-values !) normalization factor.

Let us suppose that there is a range of α-power values that can be choosen in such way that one satisfies the constraint below

$$\int_{E^{alg}(S(R^\nu))} d_\alpha^{(0)}\mu[\varphi](\|\varphi\|_{L^{2j}(R^\nu)})^{2j} < \infty \tag{19.67}$$

with $j = 1,2,\cdots,N$ and for a given fixed integer N, the highest power of our polinomial field interaction. Or equivalently, after realizing the φ-Gaussian functional integration, with a space-time cutt off volume Ω on the interaction to be analyzed on eq.(19.70)

$$\begin{aligned} \int_\Omega d^\nu x[(-\Delta)^\alpha+\mu_0^2]^{-j}(x,x) &= \mathrm{vol}(\Omega)\times\left(\int\frac{d^\nu k}{k^{2\alpha}+\mu_0^2}\right)^j \\ &= C_\nu(\mu_0)^{(\frac{\nu}{\alpha}-2)}\times\left(\frac{\pi}{2\alpha}\mathrm{cosec}\frac{\nu\pi}{2\alpha}\right) < \infty \end{aligned} \tag{19.68}$$

For $\alpha > \frac{\nu-1}{2}$, one can see by the Minlos theorem that the measure support of the Gaussian measure eq.(19.65) will be given by the intersection Banach space of measurable Lebesgue functions on R^ν instead of the previous one $E^{alg}(S(R^\nu))$

$$\mathcal{L}_{2N}(R^\nu) = \bigcap_{j=1}^{N}(L^{2j}(R^\nu)) \tag{19.69}$$

In this case, one obtains that the finite - volume $p(\varphi)_2$ interactions

$$\exp\left\{-\sum_{j=1}^{N}\lambda_{2j}\int_\Omega(\varphi^2(x))^j\,dx\right\} \le 1 \tag{19.70}$$

is mathematically well-defined as the usual pointwise product of measurable functions and for positive coupling constant values $\lambda_{2j} \ge 0$. As a consequence, we have a measurable functional on $L^1(\mathcal{L}_{2N}(R^\nu); d_\alpha^{(0)}\mu[\varphi])$ (since it is bounded by the function 1). So, it would make sense to consider mathematically the well-defined path - integral on the full space R^ν with those values of the power α satisfying the contraint eq.(19.67).

$$Z[j] = \int_{\mathcal{L}_{2N}(R^\nu)} d_\alpha^{(0)}\mu[\varphi] \exp\left\{ -\sum_{j=1}^{N} \lambda_{2j} \int_\Omega \varphi^{2j}(x)dx \right\} \times \exp(i\int_{R^\nu} j(x)\varphi(x)) \tag{19.71}$$

Finally, let us consider a interacting field theory in a compact space-time $\Omega \subset R^\nu$ defined by an iteger even power $2n$ of the Laplacean operator with Dirichlet Boundary conditions as the free Gaussian kinetic action, namely

$$\begin{aligned} Z^{(0)}[j] &= \exp\left\{ -\frac{1}{2}\left\langle j, (-\Delta)^{-2n} j \right\rangle_{L^2(\Omega)} \right\} \\ &= \int_{W_2^n(\Omega)} d_{(2n)}^{(0)}\mu[\varphi] \exp(i\langle j,\varphi\rangle_{L^2(\Omega)}) \end{aligned} \tag{19.72}$$

here $\varphi \in W_2^n(\Omega)$ - the Sobolev space of order n which is the functional domain of the cylindrical Fourier Transform measure of the Generating functional $Z^{(0)}[j]$, a continuous bilinear positive form on $W_2^{-n}(\Omega)$ (the topological dual of $W_2^n(\Omega)$).

By a straightforward application of the well-known Sobolev immersion theorem, we have that for the case of

$$n - k > \frac{\nu}{2} \tag{19.73}$$

including k a real number the functional Sobolev space $W_2^n(\Omega)$ is contained in the continuously fractional differentiable space of functions $C^k(\Omega)$. As a consequence, the domain of the Bosonic functional integral can be further reduced to $C^k(\Omega)$ in the situation of eq.(19.73)

$$Z^{(0)}[j] = \int_{C^k(\Omega)} d_{(2n)}^{(0)}\mu[\varphi] \exp(i\langle j,\varphi\rangle_{L^2(\Omega)}) \tag{19.74}$$

That is our new result generalizing the Wiener theorem on Brownian paths in the case of $n = 1$, $k = \frac{1}{2}$ and $\nu = 1$

Since the bosonic functional domain on eq.(19.74) is formed by real functions and not distributions, we can see straightforwardly that any interaction of the form

$$\exp\left\{ -g\int_\Omega F(\varphi(x))d^\nu x \right\} \tag{19.75}$$

with the non-linearity $F(x)$ denoting a lower bounded real function $(\gamma > 0)$

$$F(x) \geq -\gamma \tag{19.76}$$

is well-defined and is integrable function on the functional space $(C^k(\Omega), d_{(2n)}^{(0)}\mu[\varphi])$ by a direct application of the Lebesque theorem

$$\left| \exp\left\{ -g\int_\Omega F(\varphi(x))\,d^\nu x \right\} \right| \leq \exp\{+g\gamma\} \tag{19.77}$$

At this point we make a subtle mathematical remark that the infinite volume limit of eq.(19.74) - eq.(19.75) is very difficult, since one looses the Garding - Poincaré inequalite at this limit for those elliptic operators and, thus, the very important Sobolev theorem. The probable correct procedure to consider the thermodynamic limit in our Bosonic path integrals is to consider solely a volume cut off on the interaction term Gaussian action as in eq.(19.71) and there search for $\mathrm{vol}(\Omega) \to \infty$.

As a last remark related to eq.(19.73) one can see that a kind of "fishnet" exponential generating functional

$$Z^{(0)}[j] = \exp\left\{ -\frac{1}{2} \left\langle j, \exp\{-\alpha\Delta\}\, j \right\rangle_{L^2(\Omega)} \right\} \tag{19.78}$$

has a Fourier transformed functional integral representation defined on the space of the infinitelly differentiable functions $C^{\infty}(\Omega)$, which physically means that all field configurations making the domain of such path integral has a strong behavior like purely nice smooth classical field configurations.

As a general conclusion of this central section of our work, we can see that the technical knowledge of the support of measures on infinite dimensional spaces-specially the powerfull Minlos theorem of section 19.3 is very important for a deep mathematical physical understanding into one of the most important problem in Quantum Field theory and Turbulence which is the problem related to the appearance of ultra-violet (short-distance) divergences on perturbative path integral calculations.

19.5. Remarks on the Theory of Integration of Functionals on Distributional Spaces and Hilbert-Banach Spaces

Let us first consider a given vector space E with a Hilbertian structure $\langle\, , \rangle$, namely $\mathcal{H} = (E, \langle\rangle)$, where $\langle\ \rangle$ means a inner product and $\mathcal{H}$ a complete topological space with the metrical structure induced by the given $\langle\, , \rangle$. We have, thus, the famous Minlos theorem on the support of the cylindrical measure associated to a given quadratic form defined by a positive definite class trace operator $A \in \oint_1(\mathcal{H}, \mathcal{H})$ (see apendix for a discussion on Fourier Transforms in Vector Spaces of Infinite-Dimension)

$$\begin{aligned} \exp\left\{ -\frac{1}{2} \langle b, Ab \rangle \right\} &= \exp\left\{ -\frac{1}{2} \langle |A|^{\frac{1}{2}} b, |A|^{\frac{1}{2}} b \rangle \right\} \\ &= \int_{\mathcal{H}} d_A\mu(v) \cdot \exp(i\langle v, b \rangle) \end{aligned} \tag{19.79}$$

since any given class trace operator can be always be considered as the composition of two Hilbert-Schmidt, each one defined by a function on $L^2(M \times M, d\nu \otimes d\nu)$.
Here the cylindrical measure $d_A\mu(v)$, firstly defined on the vector space of the linear forms of E, with the topology of pontual convergence – the so called algebraic dual of E –, has its support concentrated on the Hilbert spaces $\mathcal{H}$, through the isomorphism of $\mathcal{H}$ and its dual $\mathcal{H}'$ by means of the Riesz theorem.

This result can be understood more easily, if one represents the given Hilbert space $\mathcal{H}$ as a square-integrable space of measurable functions on a complete measure space

$(M,d\nu)\,L^2(M,d\nu)$. In this case, the class trace positive definite operator is represented by an integral operator with a positive-definite Kenel $K(x,y)$. Note that it is worth to re-write eq.(19.79) in the Feynman path integral notation as written below

$$\exp\left\{-\frac{1}{2}\int_{M\times M} d\nu(x)d\nu(y)\, f(x)\,K(x,s)\,\overline{f(s)}\right\}$$
$$= \frac{1}{Z(0)}\int_{L^2(M,d\nu)}\left(\prod_{x\in M} d\varphi(x)\right)\exp\left[-\frac{1}{2}\int_{M\times M} d\nu(x)d\nu(y)\,\overline{\varphi(x)}K^{-1}(x,y)\varphi(y)\right]$$
$$\times \exp\left\{\int_M d\nu(x)\,f(x)\overline{\varphi(x)}\right\} \tag{19.80}$$

Here the "inverse Kenel" of the operator A is given by the relationship below

$$\int_M d\nu(y)K(x,y)K^{-1}(y,x') = \text{ identity operator} \tag{19.81}$$

and the path-inegral normalization factor is given by the functional determinant $Z(0) = \det^{-\frac{1}{2}}(K) = \det^{\frac{1}{2}}(K^{-1})$.

A more invariant and rigorous representation for the Gaussian path-integral eq.(19.79)-eq.(19.80) can be exposed through an eigenfunction-eigenvalue harmonic expansion associated to our given class trace operator A, namely

$$A\beta_n = \lambda_n\beta_\mu \tag{19.82-a}$$
$$v = \sum_{n=0}^{\infty} v_n\beta_n\,;\ \varphi = \sum_{n=0}^{\infty}\varphi_n\beta_n \tag{19.83-b}$$

$$\exp\left\{-\frac{1}{2}\sum_{n=0}^{\infty}\lambda_n|v_n|^2\right\}$$
$$= \lim_N \sup\left\{\int_{R^N} d\langle\varphi|\varphi_1\rangle\ldots d\langle\varphi|\varphi_N\rangle e^{-\frac{1}{2}\left(\sum_{n=0}^{N}\frac{|\langle\varphi|\varphi_n\rangle|^2}{\lambda_n}\right)}\left(\prod_{n=0}^{N}\frac{1}{2\pi\lambda_n}\right)^{\frac{1}{2}}\exp\left(i\sum_{n=0}^{N}\varphi_n\,\bar{v}_n\right)\right\} \tag{19.84}$$

The above cited theorem for the support characterization of Gaussian path integrals can be generalized to the highly non-trivial case of a non-linear functional $Z(v)$ on E, satisfying the following conditions:

a) $Z(0) = 1$

b) $\sum_{j,k}^{N} Z(v_j - v_k)z_j\bar{z}_k \geq 0$, for any $\{z_i\}_{1\leq i\leq N} : z_i \in \mathbb{N}$ (19.85)

and $\{v_i\}_{1\leq i\leq N} : v_i \in E$.

c) there is a a H-subspace of E with a inner product $\langle\,,\rangle$, such that $Z(v)$ is continuous in relation to a given inner product $\langle\,,\rangle_A$ coming from a quadratic form defined by a positive definite class trace operator A on $\mathcal{H}$, in others words, we have the sequential continuity criterium (if $\mathcal{H}$ is separable):

$$\lim_{n\to\infty} Z(v_n) = 0 \tag{19.86}$$

if

$$\lim_{n\to\infty} \langle v_n, Av_n\rangle = 0 \qquad (A \in \oint_1(\mathcal{H})) \tag{19.87}$$

We have thus, the following path integral representation

$$Z(v) = \int_H d\mu(\varphi)\exp(i\langle v,\varphi\rangle) \tag{19.88}$$

where

$$\int_H d\mu(\varphi) = 1. \tag{19.89}$$

Another less mathematically rigorous result is that one related to an inversible self-adjoint positive-definite operator A in a given Hilbert space $(H,\langle,\rangle)$ – not necessarily a bounded operator in the class trace operator as considered previously. In order to write somewhat formal path-integrals representations for the Gaussian functional

$$Z(j) = \exp\left\{-\frac{1}{2}\langle j, A^{-1}j\rangle\right\} \tag{19.90}$$

with $f \in \text{Dom}(A^{-1}) \subset \mathcal{H}$,we start by considering the usual spectral expression for the following quadratic form

$$\langle\varphi, A\varphi\rangle = \int_{\sigma(A)} \lambda\,\langle\varphi, dE(\lambda)\varphi\rangle \tag{19.91}$$

with $\sigma(A)$ denoting the spectrum of A (a subset of R^+!) and $dE(\lambda)$ are the spectral projections associated to the spectral representation of A.

In this case one has the result for the path-integral weight

$$\begin{aligned}\exp\left\{-\frac{1}{2}\int_{\sigma(A)}\lambda\,\langle\varphi, dE(\lambda)\varphi\rangle\right\} &= \exp\left\{-\frac{1}{2}\langle\varphi, A\varphi\rangle\right\}\\ = \lim\sup\left\{\prod_{\lambda\in\sigma_{\text{Fin}}(A)}\exp\left(-\frac{1}{2}\langle\varphi,\lambda dE(\lambda)\varphi\rangle\right)\right\},\end{aligned} \tag{19.92}$$

here $\sigma_{\text{Fin}}(A)$ denotes all sub-sets with a finite number of elements of $\sigma(A)$.

As a consequence one should define formally the generating functional as

$$\begin{aligned}Z(j) &= \exp\left\{-\frac{1}{2}\langle j, A^{-1}j\rangle\right\}\\ &= \lim\sup\left\{\prod_{\lambda\in\sigma_{\text{Fin}}(A)}\int_{-\infty}^{+\infty} dx_\lambda\cdot e^{-\frac{1}{2}\lambda(x_\lambda)^2}\,e^{ij_\lambda x_\lambda}\sqrt{\frac{\lambda}{2\pi}}\right\}\end{aligned} \tag{19.93}$$

associated to the self-adjoint operator A acting on a Hilbert space $\mathcal{H}$

$$Z(j) = \int_{\mathcal{H}} d_A\mu(\varphi)\,e^{i\langle j,\varphi\rangle}. \tag{19.94}$$

Otherwise, one should introduces formal redefinitions of parameters entering in the definition of our action operator A, in such a way to render finite the functional determinant in eq.(19.80). Let us exemplify such calculational point with the operator $(-\Delta+m^2)$, acting on $L^2(R^N)$ (with domain being given precisely by the Sobolev space $H^2(R^N)$). We note that

$$\begin{aligned} & Tr_{L^2(R^N)}(\exp(-t(-\Delta+m^2))) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \left[\int_{-\infty}^{+\infty} d^N k\, e^{-tk^2}\, e^{-tm^2}\right] \\ &= e^{-tm^2} C(N) \times \begin{cases} \dfrac{(N-2)!!}{2(2t)^{\frac{N-1}{2}}}\sqrt{\frac{\pi}{t}} & \text{it} \quad N-1 \text{ is even} \\ \\ \dfrac{((N-2/2))!}{2}\,\frac{1}{(t)^{\frac{N}{2}}} & \text{if} \quad N-1 \text{ is odd} \end{cases} \end{aligned} \tag{19.95}$$

with $C(N)$ denoting a N-dependent constant.

It is worth to note that one must introduce in the path-integral eq.(19.93), some formal definition for the functional determinant of the self-adjoint operator A, which by its term leads to the formal process of the "Infinite Renormalization" in Quantum Field Path Integrals

$$\begin{aligned} \int_{\mathcal{H}} d_A\mu(\varphi) &= \lim_{(R\text{-valued net})} \sup \left\{ \prod_{\lambda\in\sigma_{\text{Fin}}(A)} \int_{-\infty}^{+\infty} \frac{dx_\lambda}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\lambda(x_\lambda)^2} \right\} \\ &= \lim\sup\left(\prod_{\lambda\in\sigma_{\text{Fin}}(A)} \left\{\frac{1}{\sqrt{\lambda'}}\right\}\right) = \det{}_F^{-\frac{1}{2}}[A] \\ &= \lim_{\varepsilon\to 0^+} \left\{ \prod_{\lambda\in\sigma_{\text{Fin}}(A)} \exp\left[+\frac{1}{2}\int_{\varepsilon}^{1/\varepsilon} \frac{dt}{t}\, e^{-t\lambda}\right]\right\} \\ &= \lim_{\varepsilon\to 0^+} \left\{ \exp\left(+\frac{1}{2}\int_{\varepsilon}^{1/\varepsilon} \frac{dt}{t}\, Tr(e^{-tA})\right)\right\} \end{aligned} \tag{19.96}$$

In the case of the finitude of the right hand side of eq.(19.96) (which means that e^{-tA} is a trace class operator and its finitude up the proper-time parameter t), one can proceed as in the appendix to define mathematically the Gaussian path-integral.

Let us now consider the proper-time (Cauchy principal value sense) integration process as indicated by eq.(19.95), for the case of N be an even space-time dimensionality

$$I(m^2,\varepsilon) = \int_{\varepsilon}^{1/\varepsilon} \frac{dt}{t}\,\frac{e^{-tm^2}}{t^{N/2}} = \int_{\varepsilon}^{1/\varepsilon} dt\,\frac{e^{-tm^2}}{t^{\frac{N+2}{2}}} \tag{19.97}$$

The whole idea of the renormalization/regularization program means a (non-unique) choice of the mass parameter as a function of the proper-time cut-off ε in such way that the otherwise infinite limit of $\varepsilon \to 0^+$ turns out to be finite, namely

$$\lim_{\varepsilon\to 0^+} I(m^2(\varepsilon),\varepsilon) < \infty. \tag{19.98}$$

A slightly generalization of the above exposed Minlos-Bochner Theorem in Hilbert spaces is the following theorem

Theorem 1. Let $(\mathcal{H},\langle\,,\rangle_1)$ be a separable Hilbert space with a inner product $\langle\,,\rangle_1$. Let $\mathcal{H}_0,\langle\,,\rangle_0)$ be a sub-space of $\mathcal{H}$, so there is a trace class operator $T:\mathcal{H}\to\mathcal{H}$, such that the inner product $\langle\,,\rangle_0$ is given explicitly by $\langle g,h\rangle_1 = \langle g,Th\rangle_0 = \langle T^{1/2}g,T^{1/2}h\rangle_0$. Let us, thus, consider a positive definite functional $Z(j)\in C((\mathcal{H},\langle\,,\rangle_1),R)$ [if $j_n \xrightarrow{\|\,\|_1} j$, then $Z(j_n)\to Z(j)$ on R^+]. We obtain that the Bochner path integral representation of $Z(j)$ is given by a measure supported at these linear functionals, such that their restrictions in the sub-space H_0 are continuous by the norm induced by the "trace-class" inner product $\langle\,,\rangle_1$.

With this result in our hands, it became more or less straightforwardly to analyze the cylindrical measure supports in Distributional Spaces. For instance, the basic Euclidean Quantum Field Distributional Spaces of Tempered Distributions in R^N: $S'(R^N)$, can always be seen as the strong topological dual of the inductive limit of Hilbert spaces below considered

$$s_p = \left\{ (x_n)\in\mathbb{C} \mid \lim_{n\to\infty} n^p x_n = 0,\ \text{with the inner product}\ \langle (x_n),(y_n)\rangle_{s_p} = \sum_{n=1}^{\infty} n^{2p} x_n \bar{y}_n \right\} \tag{19.99}$$

We note now that

$$S(R^N) = \bigcup_{p\geq 1}^{\infty} s_p \tag{19.100}$$

and

$$S'(R^N) = \bigcup_{p\geq 1}^{\infty} s_{-p}\,. \tag{19.101}$$

An important property is that $s_p \supset s_{p+1}$ and they satisfy the hypothesis of the Theorem 1, since

$$\sum_{n=1}^{\infty} n^{2p}|x_n|^2 = \sum_{n=1}^{\infty} n^{(2p+2)}\,\frac{1}{n^2}|x_n^2| \tag{19.102}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}. \tag{19.103}$$

As a consequence

$$||(x_n)||_{s_p} \leq \frac{\pi^2}{6}\,||(x_n)||_{s_{p+1}} \tag{19.104}$$

If one have an arbitrary continuous positive-definite functional in $S(R^N)$, necessarily its measure support will always be on the topological dual of s_{p+1}, for each p. As a consequence its support will be on the union set $\bigcup_{p=1}^{\infty} s_{-p}$ (since $s_{-p}\subset s_{-(p+1)}$) and thus it will be the whole Distribution Space $S'(R^N)$.

Similar results hold true in others Distributional Spaces.

The application of the above cited result in Gaussian Path-Integrals is always made with the use of the famous result of the kernel theorem of Schwartz-Gelfand.

Theorem 2 (Gelfand): Any continuous bilinear form $B(j,j)$ defined in the teste space of the tempered distribuiton $S'(R^N)$ has the following explicitly representation:

$$B(j,j) = \int d^N x d^N y\, j(x)(D_x^m D_y^n F)(x,y)\, \bar{j}(y) \tag{19.105}$$

with $j \in S(R^N)$, $F(x,y)$ a continuous function of polynomial growth and D_x^m, D_y^n are distributional derivatives of order m ande n respectively.

In all cases of application of this result to our study presented in the previous chapters were made in the context that $(D_x^m D_y^n F)(x,y)$ is a fundamental solution of a given differential operator representing the kinetic term of a given Quantum Field Lagrangean.

As a consequence we have the basic result in the Gaussian Path Integral in Euclidean Quantum field theory

$$e^{-\frac{1}{2}B(j,j)} = \int_{T \in S'(R^N)} d\mu(T) \exp\{i(T(j))\} \tag{19.106}$$

where $T(j)$ denotes the action of the distribution T on the test function $j \in S(R^N)$.

At this point of our exposition let us show how to produce a fundamental solution for a given differential operator $P(D)$ with constant coefficients, namely

$$P(D) = \sum_{|\rho| \leq m} a_p D^p \tag{19.107}$$

A fundamental solution for eq.(19.107) is given by a (numerique) distribution $E \in S'(R^N)$ such that for any $\varphi \in S(R^N)$, we have:

$$(P(D)E)(\varphi) = \delta(\varphi) = \varphi(0) \tag{19.108}$$

or equivalently

$$E({}^t P(D)\varphi) = \delta(\varphi) \tag{19.109}$$

where ${}^t P(D)$ is the transposte operator through the duality of $S(R^N)$ and $S'(R^N)$.

By means of the use of a Harmonic-Hermite expansion for the searched fundamental solution

$$E \overset{S'(R^N)}{=} \sum_{p=1}^{\infty} (E, H_p) H_p = \sum_{p=1}^{\infty} E_p H_p \tag{19.110}$$

with H_p denoting the appropriate Hermite Polinomials in R^N, together with the test function harmonic expansion

$$\varphi \overset{S(R^N)}{=} \sum_{p=1}^{\infty} (\varphi, H_p) H_p = \sum_{p=1}^{\infty} \varphi_p H_p \tag{19.111}$$

and the use of the relationship between Hermite polinomials

$$ {}^{t}P(D)H_p = \sum_{|q|\leq \ell(p)} M_{pq} H_q \tag{19.112} $$

with $\ell(p)$ depending on the order of H_p and the order of the differential operator ${}^{t}P(D)$. (For instance in $S(R)$: $\frac{d}{dx} H_n(x) = 2n H_{n-1}(x)$;

$$ \frac{d^2}{d^2x} H_n(x) = 2n \frac{d}{dx} H_{n-1}(x) = 4n(n-1)H_{n-2}(x), \text{ etc...}), $$

one obtains the recurrence equations for the searched coefficients E_p in eq.(19.110)

$$ \sum_n \varphi_n \left[\sum_{|q|\leq \ell(n)} M_{nq} E_q \right] = \sum_n \varphi_n H_n(0) \tag{19.113} $$

for any $(\varphi_n) \in \ell^2$
or equivalently:

$$ \sum_{|q|\leq \ell(n)} M_{nq} E_q = H_n(0) \tag{19.114} $$

the solution of the above written infinite-dimensional system produces a set of coefficients $\{E_p\}_{p=1,\dots,\infty}$ satisfying a condition that it belongs to some space s_{-r} , where r is the order of the fundamental solution being searched (the rigorous proof of the above assertions is left as an exercise to our mathematically oriented reader!).

Finally let us sketech the connection between path integrals and the operator framework in Euclidean Quantum Field Theory, both still mathematically non rigorous from a strict mathematical point of view. In ther former approach, one has a self-adjoint operator $\overline{H}(j)$ indexed by a set of functions (the field classical sources of the Quantum Field Theory under analysis) belonging to the Distributional Space $S(R^N)$. This self-adjoint operator is formally given by the space-time integrated Lagrangean Field Theory and the basic object is the Generating functional as defined by the vacuum-vacuum transition amplitude

$$ Z(j) = \langle e^{iH(j)} \Omega_{VAC}, \Omega_{VAC} \rangle \tag{19.115} $$

with Ω_{VAC} denoting the theory vacuum state. It is assumed that $Z(j)$ is a continuous positive definite functional on $S(R^N)$. As a consequence of the above exposed theorems of Minlos and Bochner, there is a cylindrical measure $d\mu(T)$ on $S'(R^N)$ such that the Generating Functional $Z(j)$ is represented by the Quantum Field Path integral defined by the above mentioned measure:

$$ Z(j) = \int_{S'(R^N)} d\mu(T) \exp\{i(T(j)\} \tag{19.116} $$

which in the Feynman symbolic notation express itself in the following symbolic-operational Feynman notation

$$ Z[j(x,t)] = \frac{1}{Z(0)} \int_{S'(R^N)} D^F[T(x,t)] e^{-\frac{1}{2}\int_{-\infty}^{+\infty} d^{n-1}x\, dt\, \mathcal{L}(T,\partial_t T,\partial_x T)}\, e^{i\int_{-\infty}^{+\infty} d^{n-1}x\, dt\, j(x,t)T(x,t)} \tag{19.117} $$

with $\mathcal{L}(T,\partial_t T,\partial_x T)$ means generically the Lagrangean density of our Field Theory under quantization.

Let us exemplify the Feynman symbolic Euclidean Path Integral as given by eq.(19.117) in the Gaussian case (free Euclidean Field Massless Theory in R^N, $N \geq 2$)

$$\begin{aligned}
&\exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^N x \int_{-\infty}^{+\infty} d^N y\, j(x)\left(\frac{(-1)}{(n-2)\Gamma(n)|x-y|^{n-2}}\right) j(y)\right\} \\
&= \int_{S'(R^N)} D^F[T(x)] \exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^N x \int_{-\infty}^{+\infty} d^N y\, T(x)\big((-\Delta)_x \delta^{(N)}(x-y)\big) T(y)\right\} \\
&\quad \det{}^{+\frac{1}{2}}\big((-\Delta)x\delta^{(N)}(x-y)\big) \\
&\quad \times \exp\left\{i\int_{-\infty}^{+\infty} d^N x\, j(x) T(x)\right\} \qquad (19.118)
\end{aligned}$$

Or for the heat differential operator in $S'(R^N \times R^+)$

$$\begin{aligned}
&\exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^N x \int_0^{\infty} dt \int_{-\infty}^{+\infty} d^N y \int_0^{\infty} dt'\, J(x,t)\left(\frac{1}{\big(\sqrt{2\pi(t-t')}\big)^N}\exp\left(-\frac{|x-y|^2}{4(t-t')}\right)\right)\theta(t-t')J(y,t')\right\} \\
&= \int_{S'(R^N\times R^+)} D^F[T(x,t)] \exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^N x \int_0^{\infty} dt\, T(x,t)\left[\left(\Delta_x - \frac{\partial}{\partial t}\right) T(x,t)\right]\right) \\
&\quad \times \cdot \det{}^{1/2}\left[\Delta_x - \frac{\partial}{\partial t}\right] \times \exp\left\{i\int_{-\infty}^{+\infty} d^N x \int_0^{\infty} dt\, J(x,t)\, T(x,t)\right\} \qquad (19.119a)
\end{aligned}$$

It is worth to point out that the fourth-order path integral in $S'(R^4)$:

$$\begin{aligned}
&\exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty} d^4 x \int_{-\infty}^{+\infty} d^4(x)\, j(x)\left(\frac{1}{8\pi}|x-y|^2\,\ell n(x-y)\right) j(y)\right\} \\
&= \int_{S'(R^4)} D^F[\varphi(x)] e^{-\frac{1}{2}\int_{-\infty}^{+\infty} d^4 x(\varphi(x)(\Delta^2)\varphi(x))} \\
&= \int_{R'(R^4)} d_{\Delta^2}\mu(\varphi) e^{i\varphi(j)}\, e^{i\int_{-\infty}^{+\infty} d^4 x\, j(x)\varphi(x)} \qquad (19.119\text{-b})
\end{aligned}$$

holds mathematically true since the locally integrable function $\frac{1}{8\pi}|x|^2\,\ell n|x|$ is a fundamental solution of the differential operator Δ^2: $S'(R^4) \to S(R^4)$ when acting on Distributional Spaces.

As a last important point of this section, we present an important result on the geometrical characterization of massive free field on an Euclidean Space-Time.

Firstly we announcing a slightly improved version of the usual Minlos Theorem.

Theorem 3. Let E be a nuclear space of tests functions and $d\mu$ a given σ-measure on its topologic dual with the strong topology. Let $\langle\,,\,\rangle_0$ be an inner product in E, inducing a Hilbertian structure on $\mathcal{H}_0 = \overline{(E,\langle\,,\,\rangle_0)}$, after its topological completation.

We suppose the following:

a) There is a continuous positive definite functional in $\mathcal{H}_0$, $Z(j)$, with an associated cylindrical measure $d\mu$.

b) There is a Hilbert-Schmidt operator $T\colon \mathcal{H}_0 \to \mathcal{H}_0$; invertible, such that $E \subset$ Range (T), $T^{-1}(E)$ is dense in $\mathcal{H}_0$ and $T^{-1}\colon \mathcal{H}_0 \to \mathcal{H}_0$ is continuous.

We have thus, that the support of the measure satisfies the relationship

$$\text{support } d\mu \subseteq (T^{-1})^*(\mathcal{H}_0) \subset E^* \tag{19.120}$$

At this point we give a non-trivial application of ours of the above cited Theorem 3.

Let us consider an differential inversible operator $\mathcal{L}\colon S'(R^N) \to S(R)$, together with an positive inversible self-adjoint elliptic operator $P\colon D(P) \subset L^2(R^N) \to L^2(R^N)$. Let H_α be the following Hilbert space

$$H_\alpha = \left\{\overline{S(R^N), \langle P^\alpha\varphi, P^\alpha\varphi\rangle_{L^2(R^N)} = \langle\,,\rangle_\alpha}, \text{ for } \alpha \text{ a real number}\right\}. \tag{19.121}$$

We can see that for $\alpha > 0$, the operators below

$$\begin{aligned} P^{-\alpha}\colon L^2(R^N) &\to \mathcal{H}_{+\alpha} \\ \varphi &\to (P^{-\alpha}\varphi) \end{aligned} \tag{19.122}$$

$$\begin{aligned} P^{\alpha}\colon \mathcal{H}_{+\alpha} &\to L^2(R^N) \\ \varphi &\to (P^{\alpha}\varphi) \end{aligned} \tag{19.123}$$

are isometries among the following sub-spaces

$$\overline{D(P^{-\alpha}), \langle\,,\rangle_{L^2})} \text{ and } H_{+\alpha}$$

since

$$\langle P^{-\alpha}\varphi, P^{-\alpha}\varphi\rangle_{\mathcal{H}_{+\alpha}} = \langle P^\alpha P^{-\alpha}\varphi, P^\alpha P^{-\alpha}\varphi\rangle_{L^2(R^N)} = \langle\varphi,\varphi\rangle_{L^2(R^N)} \tag{19.124}$$

and

$$\langle P^\alpha f, P^\alpha f\rangle_{L^2(R^N)} = \langle f, f\rangle_{H_{+\alpha}} \tag{19.125}$$

If one considers T a given Hilbert-Schmidt operator on H_α, the composite operator $T_0 = P^\alpha T P^{-\alpha}$ is an operator with domain being $D(P^{-\alpha})$ and its image being the Range (P^α). T_0 is clearly an invertible operator and $S(R^N) \subset$ Range (T) means that the equation $(TP^{-\alpha})(\varphi) = f$ has always a non-zero solution in $D(P^{-\alpha})$ for any given $f \in S(R^N)$. Note that the condition that $T^{-1}(f)$ be a dense subset on Range $(P^{-\alpha})$ means that

$$\langle T^{-1} f, P^{-\alpha}\varphi\rangle_{L^2(R^N)} = 0 \tag{19.126}$$

has as unique solution the trivial solution $f \equiv 0$.

Let us suppose too that $T^{-1}: S(R^N) \to H_\alpha$ be a continuous application and the bilinear term $(\mathcal{L}^{-1}(j))(j)$ be a continuous application in the Hilbert spaces $H_{+\alpha} \supset S(R^N)$, namely: if $j_n \xrightarrow{L^2} j$, then $\mathcal{L}^{-1}: P^{-\alpha} j_n \xrightarrow{L^2} \mathcal{L}^{-1} P^{-\alpha} j$, for $\{j_n\}_{n\in\mathbb{Z}}$ and $j_n \in S(R^N)$.

By a direct application of the Theorem 3, we have the result

$$Z(j) = \exp\left\{-\frac{1}{2}[\mathcal{L}^{-1}(j)(j)]\right\} = \int_{(T^{-1})^* H_\alpha} d\mu(T) \exp(iT(j)) \tag{19.127}$$

Here the topological space support is given by

$$\begin{aligned}(T^{-1})^* \mathcal{H}_\alpha &= \left[\left(P^{-\alpha} T_0 P^{\alpha}\right)^{-1}\right]^* \left(\overline{(P^{\alpha}(S(R^N)))}\right) \\ &= \left[(P^{\alpha})^* (T_0^{-1})^* (P^{-\alpha})^*\right] P^{\alpha}(S(R^N)) \\ &= P^{\alpha} T_0^{-1} (L^2(R^N))\end{aligned} \tag{19.128}$$

In the important case of $\mathcal{L} = (-\Delta + m^2): S'(R^N) \to S(R^N)$ and $T_0 T_0^* = (-\Delta + m^2)^{-2\beta} \in \oint_1(L^2(R^N))$ since $Tr(T_0 T_0^*) = \frac{1}{2(m^2)^\beta} \left(\frac{m^2}{1}\right)^{\frac{N}{2}} \frac{\Gamma(\frac{N}{2})\Gamma(2\beta - \frac{N}{2})}{\Gamma(\beta)} < \infty$ for $\beta > \frac{N}{4}$ with the choice $P = (-\Delta + m^2)$, we can see that the support of the measure in the path-integral representation of the Euclidean measure field in R^N may be taken as the measurable sub-set below

$$\text{supp}\ \{d_{(-\Delta+m^2)}\, u(\varphi)\} = (-\Delta + m^2)^{-\alpha} \cdot (-\Delta + m^2)^{+\beta} (L^2(R^N)) \tag{19.129}$$

since $\mathcal{L}^{-1} P^{-\alpha} = (-\Delta + m^2)^{-1-\alpha}$ is always a bounded operator in $L^2(R^N)$ for $\alpha > -1$.

As a consequence each field configuration can be considered as a kind of "fractional distributional" derivative of a square integrable function as written below

$$\varphi(x) = \left[\left(-\Delta + m^2\right)^{\frac{N}{4}+\varepsilon-1} f\right](x) \tag{19.130}$$

with a function $f(x) \in L^2(R^N)$ and any given $\varepsilon > 0$, even if originally all fields configurations entering into the path-integral were elements of the Schwartz Tempered Distribution Spaces $S'(R^N)$ certainly very "rough" mathematical objects to characterize from a rigorous geometrical point of view.

We have, thus, make a further reduction of the functional domain of the free massive Euclidean scalar field of $S'(R^N)$ to the measurable sub-set as given by eq.(19.130) denoted by $W(R^N)$

$$\begin{aligned}\exp\left\{-\frac{1}{2}\left[(-\Delta + m^2)^{-1} j\right](j)\right\} &= \int_{S'(R^N)} d_{(-\Delta+m^2)}\mu(\varphi)\, e^{i\varphi(j)} = \\ &= \int_{W(R^N)\subset S'(R^N)} d_{(-\Delta+m^2)}\tilde{\mu}(f)\, e^{i\langle f, (-\Delta+m^2)^{\frac{N}{4}+\varepsilon-1} f\rangle_{L^2(R^N)}}\end{aligned} \tag{19.131}$$

Appendix A

In this appendix we give new functional analytic proofs of the Bochner-Martin-Kolmogorov Theorem of section II.

Theorem of Bochner-Martin-Kolmogorov (Version I) let $f : E \to R$ be a given real function with domain being a vector space E and satisfying the following properties

1) $f(0) = 1$

2) The restriction of f to any finite-dimensional vector sub-space of E is the Fourier Transform of a real continuous function of compact support.

Then there is a measure $d\mu(h)$ on a σ-algebra containing the Borelians if the Space of Linear Functionals of E with the topology of pontual convergence denoted by E^{alg} such that for any $y \in E$

$$f(g) = \int_{E^{alg}} \exp(ih(g))d\mu(h) \tag{19.A.1}$$

Proof: Let $\{\hat{e}_{\lambda \in A}\}$ be a Hamel (Vectorial) basis of E and $E^{(N)}$ a given sub-space of E of finite-dimensional. By the hypothesis of the Theorem, we have that the restriction of the functions to $E^{(N)}$ (generated by the elements of the Hamel basis $\{\hat{e}_{\lambda_1}, \dots, \hat{e}_{\lambda_N}\} = \{e_\lambda\}_{\lambda \in \Lambda_F}$ is given by the Fourier Transform

$$f\left(\sum_{\ell=1}^{N} \sigma\lambda_\ell \hat{e}\lambda_\ell\right) = \int_{\prod_{\lambda \in \Lambda_F} R^\lambda} (dP_{\lambda_1} \cdots dP_{\lambda_N}) \exp\left[\sum_{\ell=1}^{N} a_{\lambda_\ell} P_{\lambda_\ell}\right] \hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N}) \tag{19.A.2}$$

with $\hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N}) \in C_c\left(\prod_{\lambda \in \Lambda_F} R^\lambda\right)$

As a consequence of the above written result we consider the following well-defined family of linear positive functionals on the space of continuous function on the product space of the Alexandrov Compactifications of R denoted by R^w :

$$L_{\lambda_F} \in \left[C\left(\prod_{\lambda \in \Lambda_F} (R^w)^\lambda ; R\right)\right]^{Dual} \tag{19.A.3}$$

with

$$L_{\Lambda_F}[\hat{g}(P_{\lambda_1}, \dots, P_{\lambda_N})] = \int_{\prod_{\lambda \in \Lambda_F} (R^w)^\lambda} \int \hat{g}(P_{\lambda_1}, \dots, P_{\lambda_N})(dP_{\lambda_1} \dots dP_{\lambda_N}) \tag{19.A.4}$$

Here $\hat{g}(P_{\lambda_1}, \cdots, P_{\lambda_N})$ still denotes the unique extension of eq.(19.A-2) to the Alexandrov Compactification R^w.

We remark noe that the above family of linear continuous functionals have the following properties:

1) The norm of L_{λ_F} is always the unity since

$$||L_{\lambda_1}|| = \int_{\prod_{\lambda \in \Lambda_F} (R^w)^\lambda} \hat{g}(P_{\lambda_1}, \dots, P_{\lambda_N}) dP_{\lambda_1} \dots dP_{\lambda_N} = 1 \tag{19.A.5}$$

2) If the index set Λ_F, contains Λ_F the restriction of the associated linear functional Λ_F, to the space $C\left(\prod_{\lambda\in\Lambda_f}(R^w)^\lambda, R\right)$ coincides with L_{Λ_F}.

Now a simple application of the Stone-Weisrtress Theorem show us that the topological closure of the union of the sub-space of functions of finite variable is the space $C\left(\prod_{\lambda\in A}(R^w)^\lambda, R\right)$, namely

$$\overline{\bigcup_{\Lambda_F\subset A} C\left(\prod_{\lambda\in\Lambda_F}(R^w)^\lambda, R\right)} = C(\prod_{\lambda\in A}(R^w)^\lambda, R) \tag{19.A.6}$$

where the union is taken over all family of sub-sets of finite elements of the index set A.

As a consequence of the remark 2 and eq.(19.A-6) there is a unique extension of the family of linear functionals $\{L_{\Lambda_F}\}$ to the whole space $C\left(\prod_{\lambda\in A}(R^w)^\lambda, R\right)$ and denoted by L_∞. The RieszMarkov Theorem give us a unique measure $d\overline{\mu}(h)$ on $\prod_{\lambda\in A}(R^w)^\lambda$ representing the action of this functional on $C\left(\prod_{\lambda\in A}(R^w)^\lambda, R\right)$.

We have, thus, the following functional integral representation for the function $f(g)$:

$$f(g) = \int_{(\prod_{\lambda\in A}(R^w)^\lambda)} \exp(i\overline{h}(g))d\mu(\overline{h}) \tag{19.A.7}$$

Or equivalently (since $\overline{h}(g) = \sum_{i=1}^{N} p_i a_i$ for some $\{p_i\}_{i\in N} < \infty$), we have the result

$$f(g) = \int_{(\prod_{\lambda\in A} R^\lambda)} (\exp ih(g))d\mu(h) \tag{19.A.8}$$

which is the proposed theorem with $h \in (\prod_{\lambda\in\lambda_F} R^\lambda)$ being the element which has a the image of $\overline{h}$ on the Alexandrov Compactification $\prod_{\lambda\in\lambda_F}(R^w)^\lambda$.

The practical use of the Bochner-Martin Kolmogorov Theorem is difficulted by the present day non existence of an algorithm generating explicitly a Hamel (Vectorial) Basis on Function of Spaces. However, if one is able to apply the theorems of section III one can construct explicitly the functional measure by only considering Topological Basis as in the Gaussian Functional integral eq.(19.32).

Theorem of Bochner-Martin-Kolmogorov (Version 2)

We have now the same hypothesis and results of theorem version 1 but with the more general condition.

3) The restriction of f to any finite-dimensional vector sub-space of E is the Fourier Transform of a real continuous function vanishing at “infinite”.

For the proof of the theorem under this more general mathematical condition, we will need two lemmas and some definitions.

Definitions 1. Let X be a normal Space, locally compact and satisfying the following σ-compacity condition

$$X = \bigcup_{n=0}^{\infty} K_k \tag{19.A.9}$$

with

$$K_n \subset \text{int}(K_{n+1}) \subset K_{n+1} \tag{19.A.10}$$

we define the following space of continuous function “vanishing” at infinite

$$\tilde{C}_0(X,R) = \left\{ f(x) \in C(X,R) | \lim_{n\to\infty} \sup_{x\in(K_n)^c} |f(x)| = 0 \right\} \tag{19.A.11}$$

We have, thus, the following lemma.

Lemma 1. The Topological closure of the functions of compact support contains $\tilde{C}_0(X,R)$ in the topology of uniform convergence.

Proof: Let $f(x) \in \tilde{C}_0(X,R)$ and $g_\mu \in C(X,R)$, the (Uryhson) functions associated to the closed disjoints sets $\overline{K}_n$ and (K^c_{n+1}). Now it is straightforwardly to see that $(f \cdot g_n)(x) \in C_i(X,R)$ and converges uniformly to $f(x)$ due to the definition (19.A-11).

At this point, we consider a linear positive continuous functional L on $\tilde{C}_0(X,R)$. Since the restriction of L to each sub-space $C(K_n,R)$ satisfy the conditions of the Riesz-Markov Theorem, there is a unique measure $\mu^{(n)}$ on K_n containing the Borelians on K_n and representing this linear functional restriction. We now use the hypothesis eq.(19.A-10) to have a well defined measure on a σ-algebra containing the Borelians of X

$$\overline{\mu}(A) = \lim\sup \mu^{(n)}(A \bigcap K_n) \tag{19.A.12}$$

for A in this σ-algebra and representing the functional L on $\tilde{C}_0(X,R)$

$$L(f) = \int_X f(x) d\overline{\mu}(x) \tag{19.A.13}$$

Note that the normally of the Topological Space X is a fundamental hypothesis used in this proof by means of the Uryhson lemma.

Unfortunatelly, the non-countable product space $\prod_{\lambda\in A} R^\lambda$ is not a Normal Topological Space (the famous Stone counter example) and we can not, thus, apply the above lemma to our Vectorial case eq.(19.A-8). However, we can overcame the use of the Stone Weirstrass Theorem in the Proof of the Bochner-Martin-Kolmogorov Theorem by considering directly a certain Functional Space instead of that given by eq.(19.A-6).

We define, thus, the following Space of Infinite-Dimensional functions vanishing at finite

$$C_0(R^\infty,R) \equiv C_0\left(\prod_{\lambda\in A} R^\lambda, R\right) \stackrel{\text{def}}{=} \overline{\bigcup_{\Lambda_F\subset A} \tilde{C}_0\left(\prod_{\lambda\in\lambda_F} R^\lambda, R\right)} \tag{19.A.14}$$

where the closure is taken in the topology of uniform convergence.

If we consider a given continuous linear functional L on $C_0\left(\prod_{\lambda\in A} R^\lambda, R\right)$ there is a unique measure μ^∞ on the union of the Borelians $\prod_{\lambda\in\Lambda_F} R^\lambda$ representing the action of L on $C_0(R^\infty,R)$.

Conversely, given a family of consistent measures $\{\mu_{\Lambda_F}\}$ on the finite-dimensional spaces $(\prod_{\lambda\in\Lambda_F} R^\lambda)$ satisfying the property of $\mu_{\Lambda_F}\left(\prod_{\lambda\in\Lambda_F} R^\lambda\right) = 1$, there is a unique measure on the cylinders $\prod_{\lambda\in A} R^\lambda$ associated to the functional L on $C_0\left(\prod_{\lambda\in A} R^\lambda, R\right)$.

Collecting the results of the above written lemmas we get the Proof of eq.(19.A-8) in this more general case.

Appendix B
On the Support Evaluations of Gaussian Measures

Let us show explicitly by one example of ours of the quite complex behavior of cylindrical measures on infinite dimensional spaces R^∞.

Firstly we consider the family of Gaussian measures on $R^\infty = \{(x_n)_{1\le n\le\infty}, x_n \in R\}$ with $\sigma_n \in \ell^2$.

$$d^{(\infty)}\mu(\{x_n\}) = \lim_N \sup \left\{ \prod_{n=1}^N (dx_n \frac{1}{\sqrt{\sigma_n\pi}}) e^{-\frac{x_n^2}{2\sigma_n^2}} \right\} \tag{19.B-1}$$

Let us introduce the measurable sets on R^∞

$$E_{(\alpha_n)} = \left\{ (x_n) \in R^\infty \,;\, \|x\|^2_{(x_n)} = \sum_{n=1}^\infty \alpha_n^2 x_n^2 < \infty \right\}$$
$$\text{and } \sum_{\ell=1}^\infty \alpha_n^2\sigma_n^2 < \infty \tag{19.B-2}$$

Here $\{\alpha_n\}$ is a given sequence suppose to belonging to ℓ^2 either.

Now it is straightforward to evaluate the "mass" of the infinite-dimensional set $E_{(x_n)}$, namely

$$
\begin{aligned}
{}^{(\infty)}\mu(E_{(\alpha_n)}) &= \int_{R^\infty} d^\infty \mu(\{x_n\}) \left[\lim_{\varepsilon\to 0^+} e^{-\varepsilon(\sum_{n=1}^{\infty} \alpha_n^2 x_n^2)} \right] \\
&= \lim_{\varepsilon\to 0^+} \left\{ \lim \sup_{0\le \ell \le n} \left[\prod_{\ell=1}^{n} (1+2\varepsilon\,\alpha_n^2\sigma_n^2)^{-\frac{1}{2}} \right] \right\} \qquad (19.B\text{-}3)
\end{aligned}
$$

Note that

$$
\left(\prod_{\ell=1}^{n} (1+2\varepsilon\,\alpha_n^2\sigma_n^2)^{-\frac{1}{2}} \right) \le \frac{1}{1+\sum_{\ell=1}^{n} \alpha_n^2\sigma_n^2} \qquad (19.B\text{-}4)
$$

As a consequence one can exchange the order of the limits on eq.(19.B-3) and arriving at the result

$$
\begin{aligned}
{}^{(\infty)}\mu(E_{(\alpha_n)}) &= \lim \sup_{0\le \ell \le n} \left\{ \lim_{\varepsilon\to 0^+} \left[\prod_{\ell=1}^{n} (1+2\varepsilon\,\alpha_n^2\sigma_n^2)^{-\frac{1}{2}} \right] \right\} \\
&= \lim \sup_{0\le \ell \le n} \{1\} = 1 \qquad (19.B\text{-}5)
\end{aligned}
$$

So we conclude on basis if eq.(19.B-5) that the support of the measure eq.(19.B-1) is the set $E_{(\alpha_n)}$ for any possible sequence $\{\alpha_n\} \in \ell^2$. Let us show that $(E_{(\alpha_n)})^C \cap E_{(\beta_n)} \neq \{\phi\}$, so these sets are not coincident.

Let be the sequences

$$
\begin{aligned}
\sigma_n &= n^{-\sigma} \\
\alpha_n &= n^{\sigma-1} \\
\beta_n &= n^{\sigma-\lambda} \qquad (19.B\text{-}6)
\end{aligned}
$$

with $\gamma > 1$ and $\sigma > 0$.

We have that

$$
\sum \alpha_n^2\sigma_n^2 = \sum \frac{1}{n^2} = \frac{\pi^2}{6} \qquad (19.B\text{-}7)
$$

$$
\sum \beta_n^2\sigma_n^2 = \sum n^{-2\lambda} < \infty \qquad (19.B\text{-}8)
$$

So $E_{\{\alpha_n\}}$ and $E_{\{\beta_n\}}$ are non-empty sets on R^∞.

Let us consider the point $\{\bar{x}_n\} \in R^\infty$ and defined by the relationship

$$
\bar{x}_n^2 = n^{-2(\sigma-1)-\varepsilon} \qquad (19.B\text{-}9)
$$

We have that

$$
\begin{aligned}
\sum (\bar{x}_n)^2 \alpha_n^2 &= \sum n^{-2(\sigma-1)-\varepsilon} \cdot n^{n^2(\sigma-1)} \\
&= \sum n^{-\varepsilon}
\end{aligned}
$$

and

$$\sum(\bar{x}_n)^2\beta_n^2 = \sum n^{-2(\sigma-1)-\varepsilon}\cdot n^{2(\sigma-\lambda)}$$
$$= \sum n^{2-\varepsilon-2\lambda}$$

If we choose $\varepsilon = 1$; $\gamma > 1$ ($\gamma = \frac{3}{2}!$), we obtain that the point $\{\bar{x}_n\}$ belongs to the set $E_{\{\beta_n\}}$ (since $\sum n^2 = \frac{\pi^2}{6}$), however it does not belongs to $E_{\{x_n\}}$ (since $\sum_{n=0}^{\infty} n^{-1} = +\infty$), although the support of the measure eq.(19.B-1) is any set of the form $E_{\{\gamma_n\}}$ with $\{\gamma_n\} \in \ell^2$.

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Chapter 20

Non-linear Diffusion in R^D and in Hilbert Spaces, a Path Integral Study

20.1. Introduction

The deterministic non-linear diffusion equation is one of the most important topics in the Mathematical-Physics of the non-linear evolution equation theory [1-3]. An important class of initial-value problems in turbulence has been modeled by non-linear diffusion stirred by random sources [4].

The purpose of this chapter 20 in Mathematical methods for Physics is to provide a model of non-linear diffusion were one can use and understand the compacity functional analytic arguments to produce theorems of existence and uniqueness on weak solutions for deterministic sitirring in $L^\infty([0,T]\times L^2(\Omega))$. We use these results to give a first step "proof" for the famous Rosen path integral representation for the Hopf charactheristic functional associated to the white-noise stirred non-linear quantum field diffusion model. These studies are presented on section II.

In section III we present a study of a Linear diffusion equation in a Hilbert Space, which is the basis of the famous Loop Wave Equations in String and Polymer surface theory of the previous presented studies.

20.2. The Non-linear Diffusion

Let us start our chapter by considering the following non-linear diffusion equation in some strip $\Omega\times[0,T]$ with $\overline{\Omega}$ denoting a C^∞-compact domain of R^D.

$$\frac{\partial U(x,t)}{\partial t} = (+\Delta U)(x,t) + \Delta^{(\wedge)}(F(U(x,t)) + f(x,t)) \tag{20.1}$$

with initial and Dirichlet boundary conditions as given below.

$$U(x,0) = g(x) \in L^2(\Omega) \tag{20.2}$$

$$U(x,t)\mid_{\partial\Omega}\equiv 0 \quad (\text{for } t>0) \tag{20.3}$$

We note that the non-linearitity of the diffusion-spatial term of the parabolic problem eq(1) takes into account the physical properties of non-linear porous medium's diffusion saturation physical situation where this model is supposed to be applied [1] - by means of the hypothesis that the regularized Laplacean operator $\Delta^{(\wedge)}$ in the non-linear term of the governing diffusion eq.(20.1) has a cut-off in its spectral range. Additionaly we make the hypothesis that the non-linear function $F(x)$ is a bounded real continuously differentiable function on the extended interval $(-\infty,\infty)$ with its derivative $F'(x)$ strictly positive there. The external source $f(x,t)$ is supposed to belong to the space $L^\infty([0,T]\times L^2(\Omega))$ or to be a white-noise external stirring of the form ([2] - pp. 61) when in the random case

$$F(\cdot,t)=\frac{d}{dt}\left\{\sum_{n\in Z}\sqrt{\lambda_n}\beta_n(t)\varphi_n(\cdot)\right\}=\frac{d}{dt}w(t). \tag{20.4}$$

Here $\{\varphi_n\}$ denotes a complete orthonormal set on $L^2(\Omega)$ and $\beta_n(t)$, $n\in Z$ are independent Wiener processes.

Let us show the existence and uniqueness of weak solutions for the diffusion problem above stated by means of Galerking Method for the case of deterministic $f(x,t)\in L^\infty([0,T]\times L^2(\Omega))$.

Let $\{\varphi_n(x)\}$ be spectral eigen-functions associated to the Laplacean Δ. Note that each $\varphi_n(x)\in H^2(\Omega)\cap H_0^1(\Omega)$ [3]. We introduce now the (finite-dimensional) Galerkin approximants

$$\begin{aligned} U^{(n)}(x,t)&=\sum_{i=1}^{n}U_i^{(n)}(t)\varphi_i(x)\\ f^{(n)}(x,t)&=\sum_{i=1}^{n}(f(x,t),\varphi_i(x))_{L^2(\Omega)}\varphi_i(x) \end{aligned} \tag{20.5}$$

subject to the initial-conditions

$$U^{(n)}(x,0)=\sum_{i=1}^{n}(g(x),\varphi_i)_{L^2(\Omega)}\varphi_i(x) \tag{20.6}$$

here $(\ ,\ )_{L^2(\Omega)}$ denotes the usual inner product on $L^2(\Omega)$.

After substituting eqs.(20.5), (20.6) in eq.(20.1), one gets the weak form of the non-linear diffusion equation in the finite-dimension approximation as a mathematical well-defined systems of ordinary non-linear differential equations, as a result of an application of the Peano existence-solution theorem.

$$\begin{aligned} &\left(\frac{\partial U^{(n)}(x,t)}{\partial t},\varphi_j(x)\right)_{L^2(\Omega)}+\Big(-\Delta U^{(n)}(x,t),\varphi_j(x)\Big)_{L^2(\Omega)}\\ &=\Big(\nabla^{(\wedge)}\cdot[(F'(U^{(n)}(x,t))\nabla^{(\wedge)}U^{(n)}(x,t)],\varphi_j(x)\Big)_{L^2(\Omega)}+(f^{(n)}(x,t),\varphi_j(x))_{L^2(\Omega)} \end{aligned} \tag{20.7}$$

By multiplying the associated system eq.(20.7) by $U^{(n)}$ we get the diffusion equation in the finite dimensional Galerking sub-space in the integral form:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|U^{(n)}\|^2_{L^2(\Omega)}\\&+(-\Delta U^{(n)},U^{(n)})_{L^2(\Omega)}\\&+\int_\Omega d^3x(F'(U^{(n)})(\nabla^{(\wedge)}U^{(n)}\cdot\overline{\nabla^{(\wedge)}U^{(n)}})(x,t)=(f,U^{(n)})_{L^2(\Omega)}\end{aligned} \tag{20.8}$$

This result, by its turn, yields a prior estimate for any positive integer p:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|U^{(n)}\|^2_{L^2(\Omega)}\right)+\gamma(\Omega)\|U^{(n)}\|^2_{L^2(\Omega)}+\|(F'(U^{(n)}))^{\frac{1}{2}}(\nabla U^{(n)})\|^2_{L^2(\Omega)}\\&\leq\frac{1}{2}\left\{p\|f(x,t)\|^2_{L^2(\Omega)}+\frac{1}{p}\|U^{(n)}\|^2_{L^2(\Omega)}\right\}\end{aligned} \tag{20.9}$$

Here $\gamma(\Omega)$ is the Garding-Poincaré constant on the inequalite of the quadratic form associated to the Laplacean operator defined on the domain $H^2(\Omega)\cap H^1_0(\Omega)$.

$$\|U^{(n)}\|^2_{H^1(\Omega)}=\left(-\Delta U^{(n)},U^{(n)}\right)_{L^2(\Omega)}\geq\gamma(\Omega)\|U^{(n)}||^2_{L^2(\Omega)}. \tag{20.10}$$

By chosing the integer p big enough and applying the Gronwall lema, we obtain that the set of function $\{U^{(n)}(x,t)\}$ forms a bounded set in $L^\infty([0,T],L^2(\Omega))\cap L^\infty([0,T],H^1_0(\Omega))$ and in $L^2([0,T],L^2(\Omega))$. As a consequence of this boundeness property of the function set $\{U^{(n)}\}$, there is a sub-sequence weak-star convergent to a function $\overline{U}(t,x)\in L^\infty([0,T],L^2(\Omega))$, which is the candidate for our "weak" solution of eq.(20.1).

Another important estimate is to consider again eq.(20.9), but now considering the Sobolev space $H^1_0(\Omega)$ on this estimate eq.(20.9), namely:

$$\begin{aligned}&\frac{1}{2}\left(\|U^{(n)}(T)\|^2_{L^2(\Omega)}-\|U^{(n)}(0)\|^2\right)+\overline{C}_0\int_0^T dt||U^{(n)}||^2_{H^1_0(\Omega)}\\&\leq\frac{1}{2}p\left(\int_0^T||f||^2_{L^2(\Omega)}dt\right)+\frac{1}{2p}\left(\int_0^T\|U^{(n)}\|^2_{L^2(\Omega)}dt\right)<\overline{M}<\infty\end{aligned} \tag{20.11}$$

since we have the coerciviness condition for the Laplacean operator

$$(-\Delta U^{(n)},U^{(n)})_{L^2(\Omega)}\geq\overline{C}_0(U^{(n)},U^{(n)})_{H^1_0(\Omega)}. \tag{20.12}$$

Note that $\|U^{(n)}(0)\|^2\leq 2\|g(x)\|^2_{L^2(\Omega)}$ (see eq.(20.8)) and $\{\|U^{(n)}(T)\|^2_{L^2(\Omega)}\}$ is a bounded set of real positive numbers.

As a consequence of a prior estimate of eq.(20.11), one obtains that the previous sequence of functions $\{U^{(n)}\}\in L^\infty([0,T],H^1_0(\Omega)\cap H^2(\Omega))$ forms a bounded set on the vector valued Hilbert space $L^2([0,T],H^1_0(\Omega))$ either.

Finally, one still has another a prior estimate after multiplying the Galerkin system eq.(20.7) by the time-derivatives $\dot{U}^{(n)}$, namely

$$\begin{aligned}
&\int_0^T dt \left\| \frac{dU_n(t)}{dt} \right\|^2_{L^2(\Omega)} \\
&\leq \ \mathrm{Real}(AU_n(T),U_n(T)) - (AU_n(0),U_n(0)) \\
&\quad + \int_0^T dt \left\| \Delta^{(\wedge)} F(U_n(t)) \frac{dU_n}{dt} \right\|_{L^2(\Omega)} \\
&\leq \frac{1}{2}p \left(\int_0^T \left\| \Delta^{(\wedge)} F(U_n(t)) \right\|^2_{L^2(\Omega)} dt \right) \\
&\quad + \frac{1}{2p} \left(\int_0^T dt \left\| \frac{dU_n}{dt} \right\|^2_{L^2(\Omega)} \right)
\end{aligned} \tag{20.13}$$

By noting that

$$\begin{aligned}
&\int_0^T \|\Delta^{(\wedge)} F(U_n(t))\|^2_{L^2(\Omega)} dt \\
&\leq \|\Delta^{(\wedge)}\|^2_{op} \times \left(\sup_{x \in [-\infty,\infty]} \{F(x)\} \right)^2 \\
&\times \int_0^T dt \|U_n(t)\|^2_{L^2(\Omega)} < \infty
\end{aligned} \tag{20.14}$$

one obtains as a further result that the set of the derivatives $\{\frac{dU_n}{dt}\}$ is bounded in $L^2([0,T],L^2(\Omega))$ (so in $L^2([0,T],H^{-1}(\Omega))$.

At this point we apply the famous Aubin-Lion theorem [3] to obtain the strong convergence on $L^2(\Omega)$ of the set of the Galerkin approximants $\{U_n(x,t)\}$ to our candidate $\overline{U}(x,t)$, since this set is a compact set in $L^2([0,T],L^2(\Omega))$ (see appendix A).

By collecting all the above results we are lead to the strong convergence of the $L^2(\Omega)$-sequence of functions $F(U_n(x,t))$ to the $L^2(\Omega)$ function $F(\overline{U}(x,t))$.

We now assemble the above obtained rigorous mathematical results to obtain $\overline{U}(x,t)$ as a weak solution of eq.(20.1) for any test function $v(x,t) \in C_0^\infty([0,T]),H^2(\Omega) \cap H_0^1(\Omega))$

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T dt \Big[\Big(U^{(n)}, -\frac{dv}{dt} \Big)_{L^2(\Omega)} + (-\Delta U^{(n)}, v)_{L^2(\Omega)} \\
&\big(F(U^{(n)}), -\Delta^{(\wedge)} v)_{L^2(\Omega)} \Big] \\
&= \lim_{n\to\infty} \int_0^T dt (f^{(n)}, v)
\end{aligned} \tag{20.15}$$

or in the weak-generalized sense above mentioned

$$\begin{aligned}
&\int_0^T dt \Big(\overline{U}(x,t), -\frac{dv(x,t)}{dt} \Big)_{L^2(\Omega)} \\
&+ \big(\overline{U}(x,t), (-\Delta v)(x,t) \big)_{L^2(\Omega)} \\
&+ \big(F(\overline{U}(x,t), -(\Delta^{(\wedge)} v(x,t) \big)_{L^2(\Omega)} \\
&= \int_0^T dt (f(x,t), v(x,t))_{L^2(\Omega)},
\end{aligned} \tag{20.16}$$

since $v(0,x) = v(T,x) \equiv 0$ by our proposed space of time-dependent test functions as $C_0^\infty([0,t]$,
$H^2(\Omega) \cap H_0^1(\Omega))$, suitable to be used on the Rosens path integrals representations for stochastic systems (see equations (22a)-(22b) in what follows).

The uniqueness of our solution $\overline{U}(x,t)$, comes from the following lemma [4].

Lemma 1. If $\overline{U}_{(1)}$ and $\overline{U}_{(2)}$ in $L^\infty([0,T] \times L^2(\Omega))$ are two functions satisfying the weak relationship below

$$\int_0^T dt \left\{ \left(\overline{U}_{(1)} - \overline{U}_{(2)}, -\frac{\partial v}{\partial t} \right)_{L^2(\Omega)} + (\overline{U}_{(1)} - \overline{U}_{(2)}, +\Delta v)_{L^2(\Omega)} \left(F(\overline{U}_{(1)} - F(\overline{U}_{(2)}); +\Delta v \right)_{L^2(\Omega)} \right\} \equiv 0 \tag{20.17}$$

then $\overline{U}_{(1)} = \overline{U}_{(2)}$ a.e in $L^\infty([0,T] \times L^2(\Omega))$. The proof of eq.(20.17) is easily obtained by considering the family of test functions on eq.(20.16) of the following form $v_n(x,t) = g_{(\varepsilon)}(t)e^{+\alpha_n t}\varphi_n(x)$ with $-\Delta\varphi_n(x) = \alpha_n\varphi_n(x)$ and $g(t) = 1$ for $(\varepsilon, T-\varepsilon)$ with $\varepsilon > 0$ arbitrary. We can see that it reduces to the obvious identity $(\alpha_n > 0)$.

$$\int_\varepsilon^{T-\varepsilon} dt \exp(\alpha_n t) \left(F(\overline{U}_{(1)}) - F(\overline{U}_{(2)}), \varphi_n \right)_{L^2(\Omega)} \equiv 0, \tag{20.18}$$

which means that $F(\overline{U}_{(1)}) = F(\overline{U}_{(2)})$ a.e on $(0,T) \times \Omega$ since ε is an arbitrary number. We have thus $\overline{U}_{(1)} = \overline{U}_{(2)}$ a.e, as $F(x)$ satisfies the lower bound estimate by our hypothesis on the kind of non-linearity considered in our non-linear diffusion eq.(20.1).

$$|F(x) - F(y)| \geq \begin{pmatrix} \inf(F'(x)) \\ -\infty < x < +\infty \end{pmatrix} |x-y| \tag{20.19}$$

Let us now consider a path-integral solution of eq.(20.1) (with $g(x) = 0$) for $f(x,t)$ denoting the white-noise stirring [4].

$$E(f(x,t)f(x',t')) = \lambda\delta^{(D)}(x-x')\delta(t-t') \tag{20.20}$$

where λ is the noise-strenght.

The first step is to write the generating process stochastic functional (the Euclidean Quantum Field Diffusion) through the Rosen-Feynman path integral identities [4] (see chapter 1)

$$Z[J(x,t)] = E_f \left[\exp\left\{ i \int_0^T dt \int_\Omega d^D x U(x,t,[f]) J(x,t) \right\} \right] \tag{20.21-a}$$

$$= E_f \left[\int D^F[U] \delta^{(F)}(\partial_t U - \Delta U - \Delta^{(\wedge)}(F(U) - f)) \right] \times \exp\left\{ i \int_0^T dt \int_\Omega d^D x U(x,t) J(x,t) \right\} \tag{20.21-b}$$

$$= E_f\left[\int D^F[U]D^F[\lambda]\exp\left\{i\int_0^T dt\int_\Omega d^Dx\lambda(x,t)\right.\right.$$
$$\left.\left.\times\left(\partial_t U-\Delta U-\Delta^{(\wedge)}(F(U))-f\right)\right\}\right]$$
$$\times\exp\left\{i\int_0^T dt\int_\Omega d^DxU(x,t)J(x,t)\right\} \tag{20.21-c}$$

$$=\int D^F[U]\exp\left\{-\frac{1}{2\lambda}\int_0^T dt\int_\Omega d^Dx\right.$$
$$\left.\times\left[(\partial_t U-\Delta U-\Delta^{(\wedge)}(F(U))\right]^2(x,t))\right\}$$
$$\times\exp\left\{i\int_0^T dt\int_\Omega d^DxU(x,t)J(x,t)\right\} \tag{20.21-d}$$

The important step made rigorous mathematically possible on the above written (still formal) Rosen's path integral representation by our previous rigorous mathematical analysis is the use of the delta functional identity on eq.(20.21-b) which is true only in the case of the existence and uniqueness of the solution of the diffusion equation in the weak sense at least for multiplier Lagrange fields $\lambda(x,t)\in C_0^\infty([0,T],H^2(\Omega)\cap H_0^1(\Omega))$.

As an important mathematical result to be pointed out is that in general case of a non-porous medium [4] in R^3, where one should model the diffusion non linearity by a complete Laplacean $\Delta F(U(x,t))$, one should observes that the set of (cut-off) solutions $\{\overline{U}^{(\wedge)}(x,t)\}$ of eq.(20.1) still remains a bounded set on $L^\infty([0,T],L^2(\Omega))$. Since we have the a priori estimate uniform bound for the $U^{(n)}$-derivatives below in $D=3$ (with $G'(x)=F(x)$). Namely:

$$\left|\int_0^T dt\left\|\frac{dU^{(n)}}{dt}\right\|^2_{L^2(\Omega)}\right|\leq\left|\int_0^T dt\left(\int_\Omega d^3x(\Delta F(U^{(n)}(t)))\cdot\left(\overline{\frac{dU^{(n)}(t)}{dt}}\right)\right)\right|$$
$$+\left|\int_\Omega d^3xf(x,t)\frac{d(U^{(n)}(x,t))}{dt}\right|$$
$$\leq\left|\int_0^T dt\ \text{Real}\left\{\frac{d}{dt}\int_\Omega d^3x\Delta G(U^{(n)}(t)\right\}\right|+\frac{1}{2}\left\{\sup_{0\leq t\leq T}p\|f(x,t)\|^2_{L^2(\Omega)}+\frac{1}{p}\|\dot{U}^{(n)}\|^2_{L^2(\Omega)}\right\}$$
$$\leq\left|\text{Real}\left(\int_\Omega d^3x(\Delta G(U^{(n)}(T,x))-\Delta G(U^{(n)}(0,x))\right)\right|$$
$$+\frac{1}{2}\sup_{0\leq t\leq T}\left\{p\|f(x,t)\|^2_{L^2(\Omega)}+\frac{1}{p}\|\dot{U}^{(n)}\|_{L^2(\Omega)}\right\}$$
$$\leq\frac{1}{2}p\|f\|^2_{L^\infty((0,t),L^2(\Omega))}+\frac{1}{2p}\|\dot{U}^{(n)}(t)\|_{L^\infty((0,t),L^2(\Omega))}<\infty \tag{20.22}$$

Where $U^{(n)}(T,x)\Big|_{\partial\Omega}=U^{(n)}(0,x)\Big|_{\partial\Omega}=0$ (see eq.(20.3). The uniform bound for the derivatives is achieved by choosing $\frac{1}{2p}<1$.

As another point worth to call the attention for we note that the above considered function space is the dual of the Banach space $L^1([0,T],L^2(\Omega))$. So, one can extract from the above set of cut-off solutions a candidate $\overline{U}^{(\infty)}(x,t)$, in the weak-star topology of $L^\infty([0,T],L^2(\Omega))$ for the above cited case of cut-off removing $\wedge = +\infty$ [6]. However, we will not proceed throughly in this straightforward technical question of cut-off removing in our model of non-linear diffusion in this chapter for general spaces R^D.

Finally, we remark that in the one-dimensional case $\Omega \in R^1$, one can further show by using the same compacity methods the existence and uniqueness of the diffusion equation added with the hydrodynamic advective term $\frac{1}{2}\frac{\partial}{dx}(U(x,t))^2$, which turns the diffusion eq.(20.1) as a kind of non-linear Burger equation on a porous medium.

It appears very important to remark that Galerking methods applied directly to the finite-dimensional stochastic eq.(20.7) (see eq.(20.4)) may be saving-time computer simulation candidates for the "turbulent" path-integral eq.(22a)-eq.(22d) evaluations by approximate numerical methods ([2]-second reference).

20.3. The Linear Diffusion in the Space $L^2(\Omega)$

Let us now present some mathematical results for the diffusion problem in Hilbert Spaces formed by square-integrable functions $L^2(\Omega)$ [5], with the domain Ω denoting a compact set of R^D.

The diffusion equation in the infinite-dimensional space $L^2(\Omega)$ is given by the following functional differential equation (see first reference of [5] for the mathematical notation).

$$\frac{\partial\psi[f(x);t]}{\partial t} = \frac{1}{2} Tr_{L^2(\Omega)}\left([QD_f^2\psi[f(x,t)]\right)$$
$$\psi[f(x),t \to 0^+] = \Omega[f(x)], \tag{20.23}$$

Here $\psi[f(x),\cdot]$ is a time-dependent functional to be determined through the governing eq.(20.23) and belonging to the space $L^2(L^2(\Omega),d_Q\mu(f))$ with $d_Q\mu(f)$ denoting the Gaussian measure on $L^2(\Omega)$ associated to Q – a fixed positive self-adjoint trace class operator $\oint_1(L^2(\Omega))$ – and D_f^2 is the second – Frechet derivative of the functional$\psi[f(x),t]$ which is given by a $f(x)$-dependent linear operator on $L^2(\Omega)$ with associated quadratic form $(D_f^2\psi[f(x),t]\cdot g(x),h(x))_{L^2(\Omega)}$.

By considering explicitly the spectral base of the operator Q on $L^2(\Omega)$

$$Q\varphi_n = \lambda_n\varphi_n, \tag{20.24}$$

The $L^2(\Omega)$-infinite – dimensional diffusion equation takes the usual form:

$$\Psi[\sum_n f_n\varphi_n,t] = \psi^{(\infty)}[(f_n),t] \tag{20.25a}$$

$$\Omega[\sum_n f_n\varphi_n] = \Omega^{(\infty)}[(f_n)] \tag{20.25b}$$

$$\frac{\partial\psi^{(\infty)}[(f_n),t]}{\partial t} = \sum_n[(\lambda_n\Delta_{f_n})\psi^{(\infty)}[(f_n),t]] \tag{20.25c}$$

$$\psi^{(\infty)}[(f_n),0] = \Omega^{(\infty)}[(f_n)] \tag{20.25d}$$

or in the Physicist's functional derivative form (see ref. [5]).

$$\frac{\partial}{\partial t}\psi[f(x),t] = \int_\Omega d^D x \int_\Omega d^D x' Q(x,x') \frac{\delta^2}{\delta f(x')\delta f(x)} \psi[f(x),t] \tag{20.26a}$$

$$\psi[f(x),0] = \Omega[f(x)] \tag{20.26b}$$

Here the integral operator Kernel of the trace class operator is explicitly given by

$$Q(x,x') = \sum_n (\lambda_n \varphi_n(x)\varphi_n(x')) \tag{20.26c}$$

A solution of eq.(26a) is easily written in terms of Gaussian path-integrals [5] which reads on the physicist's notations

$$\psi[f(x),t] = \int_{L^2(\Omega)} D^F[g(x)]\Omega[f(x)+g(x)] \times \det{}^{+\frac{1}{2}}\left[\frac{1}{2t}Q^{-1}\right] \times \exp\left\{-\frac{1}{2t}\int_\Omega d^D x \int_\Omega d^D x' g(x)\cdot Q^{-1}(x,x') g(x')\right\} \tag{20.27}$$

Rigorously, the correct functional measure on eq.(20.27) is the normalized Gaussian measure with the following Generating functional

$$Z[j(x)] = \int_{L^2(\Omega)} d_{tQ}\mu[g(x)] \exp\left\{i\int_\Omega j(x)g(x)d^D x\right\} = \exp\left\{-\frac{t}{2}\int_\Omega d^D x \int_\Omega d^D x' j(x) Q^{+1}(x,x') j(x')\right\} \tag{20.28}$$

At this point, it becomes important remark that when writting the solution as a Gaussian-path integral average as done in eq.(20.27), all the $L^2(\Omega)$ functions in the functional domain of our diffusion functional field $\psi[f(x),t]$ belongs to the functional domain of the quadratic form associated to the classe trace operator Q the so-called reproducing kernel of the operator Q which is not the whole Hilbert Space $L^2(\Omega)$ as naively indicated on eq.(20.27), but the following subset of it:

$$\mathrm{Dom}(\psi[\cdot,t]) = \{f(x)\in L^2(\Omega) | Q^{-\frac{1}{2}} f \in L^2(\Omega)\} \overset{\subset}{\neq} L^2(\Omega) \tag{20.29}$$

The above written result gives a new generalization of the famous Cameron-Martin theorem that the usual Wienner measure (defined by the one-dimensional Laplacean with Dirichlet conditions on the interval end-points) is translation invariant, i.e $d^{\mathrm{Wien}}\mu[f+g] = d^{\mathrm{Wien}}\mu[f] \times \left(\frac{d^{\mathrm{Wien}}\mu[f+g]}{d^{\mathrm{Wien}}\mu[f]}\right)$, if and only if the shift function $g(x)$ is absolutely continuous with derivative on $L^2([a,b])$. In other words $g \in H_0^1([a,b]) = \mathrm{Dom}\left\{\sqrt{-\frac{d^2}{d^2x}}\right\}$.

Another point important to call the reader attention is that one can writte eq.(20.27) in the usual form of Diffusion in finite dimensional case (see appendix B)

$$\psi[f(x),t] = \int_{L^2(\Omega)} D^F[g(x)]\Omega[f(x)+\sqrt{t}g(x)] \times \exp\left\{-\frac{1}{2t}\int_\Omega d^D x \int_\Omega d^D x' g(x) Q^{-1}(x,x') g(x)\right\}, \tag{20.30}$$

At this point is worth call the reader attention that $d_{tQ}\mu$ and $d_Q\mu$ Gaussian measures are singular to each other by a direct application of Kakutani theorem for Gaussian infinite dimensional measures for any time $t > 0$.

$$d_{tQ}\mu[g(x)]\big/ d_Q\mu[g(x)] = +\infty \tag{20.31}$$

Let us apply the above results for the Physical diffusion of Polymer Rings (closed strings) described by Periodic Loops $\vec{X}(\sigma) \in R^D, 0 \leq \sigma \leq T, \vec{X}(\sigma + T) = \vec{X}(\sigma)$ with a non-local diffusion coeficient $Q(\sigma, \sigma')$ (such that $\int_0^T d\sigma \int_0^T d\sigma' Q(\sigma, \sigma') = Tr[Q] < \infty$). The functional governing equation in Loop Space (formed by Polymer rings) is given by

$$\frac{\partial \psi^{(\varepsilon)}[\vec{X}(\sigma); A]}{\partial A} = \int_0^T d\sigma \int_0^T d\sigma' Q_{ij}^{(\varepsilon)}(\sigma, \sigma') \frac{\delta^2}{\delta \vec{X}_i(\sigma) \delta \vec{X}_j(\sigma)} \psi^{(\varepsilon)}[\vec{X}(\sigma), A] \tag{20.32a}$$

$$\psi^{(\varepsilon)}[\vec{X}(\sigma); 0] = \exp\left\{ -\frac{\lambda}{2} \int_0^T d\sigma \int_0^T d\sigma' \vec{X}_i(\sigma) M_{ij}(\sigma, \sigma') \vec{X}_j(\sigma') \right\}. \tag{20.32b}$$

Here the ring polymer surface probability distribuition $\psi^{(\varepsilon)}[\vec{X}(\sigma), A]$ depends on the area parameter A, the area of the cylindrical polymer surface of our surface-polymer chain. Note the presence of a parameter ε on the above written objects takes into account the local (the integral operator kernel) case $Q(\sigma, \sigma') = \delta(\sigma - \sigma')$ as a limiting case of the rigorously mathematical well-defined (class trace) situation on the end of the observable evaluations

$$Q_{ij}^{(\varepsilon)}(\sigma, \sigma') = \frac{1}{\sqrt{\pi \varepsilon}} \left[\exp\left(-\frac{(\sigma - \sigma')^2}{\varepsilon^2} \right) \right] \tag{20.33}$$

The solution of eq.(32a) is straightforwardly written in the case of a self-adjoint kernel M on $L^2(\Omega \times \Omega)$.

$$\begin{aligned} &\exp\left\{ -\frac{1}{2} \int_0^T d\sigma \int_0^T d\sigma' \vec{X}_i(\sigma) M_{ij}(\sigma, \sigma') \vec{X}_j(\sigma') \right\} \\ &\times \det{}^{-\frac{1}{2}}[1 + A\lambda M(Q^{(\varepsilon)})^{-1}] \\ &\exp\left\{ +\frac{1}{2} \int_0^T d\sigma \int_0^T d\sigma' (\vec{M}X)_i(\sigma) \left((\lambda M + (Q^{(\varepsilon)})^{-1} \cdot \frac{1}{A} \right) (\vec{M}X)_j(\sigma') \right\} \end{aligned} \tag{20.34}$$

The functional determinant can be reduced to the evaluation of an integral equation

$$\begin{aligned} &\det{}^{\frac{1}{2}}[1 + A\lambda M(Q^{(\varepsilon)})^{-1}] \\ &= \exp\left\{ -\frac{1}{2} Tr_{L^2(\Omega)} \lg(1 + \lambda A M(Q^{(\varepsilon)})^{-1} \right\} \\ &= \exp\left\{ -\frac{1}{2} Tr_{L^2(\Omega)} \int_0^\lambda d\lambda' [(Q^{(\varepsilon)})^{-1} M)(1 + \lambda' A (Q^{(\varepsilon)})^{-1} M)^{-1}] \right\} \\ &= \exp\left\{ -\frac{1}{2} Tr_{L^2(\Omega)} \int_0^\lambda d\lambda' R(\lambda') \right\} \end{aligned} \tag{20.35}$$

Here the kernel operator $R(\lambda')$ satisfies the integral equation (accesible for numerical analysis)

$$R(\lambda')(1 + \lambda' A (Q^{(\varepsilon)})^{-1} M = (Q^{(\varepsilon)})^{-1} M \tag{20.36}$$

Which in the local case of $\varepsilon \to 0^+$, when considered in the final result eq.(20.34) - eq.(20.35), produces the explicitly candidate solutions for our Polymer-surface probalility distribuition with M a class trace operator on the Loop space: $L^2([0,T])$.

$$
\begin{aligned}
\psi[\vec{X}(\sigma),A] &= \exp\left\{-\frac{1}{2}Tr_{L^2(\Omega)}\int_0^\lambda d\lambda'[M(Q^{(\varepsilon)}+\lambda' AM)^{-1}]\right\} \\
&\times \exp\left\{-\frac{\lambda}{2}\int_0^T d\sigma\int_0^T d\sigma' X_i(\sigma)\cdot M_{ij}(\sigma,\sigma')\vec{X}_j(\sigma')\right\} \\
&\times \exp\left\{+\frac{1}{2}\int_0^T d\sigma\int_0^T d\sigma' (\vec{MX})_i(\sigma)\left(\lambda M+(Q^{(\varepsilon)})^{-1}\cdot\frac{1}{A}\right)(\sigma,\sigma')(\vec{MX})_j(\sigma')\right\} \qquad (20.37)
\end{aligned}
$$

It is worth call the reader attention that if $A \in \oint_1$ and B is a bounded operator - so $A\cdot B$ is a class trace operator-, the functional determinant $\det[1+AB]$ is a well-defined object as a direct result of the obvious estimate, result which was used to arrive at eq.(20.37).

$$
\lim_{N\to\infty}\prod_{n=0}^{N}(1+\lambda_n) \leq \exp\left(\sum_{n=0}^{N}\lambda_n\right) = \exp(TrAB)
$$

As a last comment on the linear infinite-dimensional diffusion problem eq.(20.23), let us sketchy a (rigorous) proof that eq.(20.27) is the unique solution of eq.(20.23). Firstly, let us consider the initial condition on eq.(20.23) as belonging to the space of all mappings $G\colon L^2(\Omega)\to R$ that are twice Fréchet differentiable on $L^2(\Omega)$ with uniformly continuous and bounded second derivative $D_f^2 G$ (a bounded operator of $\mathcal{L}(L^2(\Omega))$ with norm $\overline{C}$). This set of mappings will be denoted by $uC^2[L^2(\Omega),R]$. It is, thus, straightforward to see through an application of the mean value theorem that the following estimate holds true

$$
\begin{aligned}
&\sup_{f(x)\in Q^{\frac{1}{2}}L^2(\Omega)} |\psi[f(x),t]-G[f(x)]| \\
&\leq \int_{L^2(\Omega)} |G(f(x)+g(x))-G(g(x))|d_{tQ}\mu[g(x)] \\
&\leq \int_{L^2(\Omega)} \left[|DG(f(x),g(x))_{L^2(\Omega)}+\int_0^1 d\sigma(1-\sigma)(D^2G[f(x)+\sigma g(x)]g(x),g(x)_{L^2(\Omega)}|\right] \\
&\times \quad d_{tQ}\mu[g(x)] \\
&\leq 0+\overline{C}\int_0^1 d\sigma(1-\sigma)\int_{L^2(\Omega)} \|g(x)\|^2_{L^2(\Omega)} d_{tQ}\mu[g(x)] \\
&\leq \overline{C}\left(\int_{L^2(\Omega)} \|g(x)\|^2_{L^2(\Omega)} d_{tQ}\mu[g(x)]\right) \\
&\leq \overline{C}Tr(tQ) = (\overline{C}Tr(Q)t \to 0 \quad as \quad t\to 0^+. \qquad (20.38)
\end{aligned}
$$

We have thus defined a strongly continuous semi-group on the Banach Space $UC^2[L^2(\Omega),R]$ with infinitesimal generator given by the infinite-dimensional Laplacean $Tr[QD^2]$ acting on the space $L^2(Q^{\frac{1}{2}}(L^2(\Omega)),R)$. By the general theory of semi-groups on Banach spaces we obtain that eq.(20.27) satisfies the infinite-dimensional diffusion initial value problem eq.(20.23), at least for initial conditions on the space $uC^2[L^2(\Omega),R]$.

Since purely Gaussian functionals belong to $uC^2[L^2(\Omega),R]$ and they form a dense set on the space $L^2(L^2(\Omega),d_Q\mu)$, we get the proof of our result for general initial condition on $L^2(L^2(\Omega),d_Q\mu)$.

Finally, we point out that the general solution of the diffusion problem on Hilbert Space with sources and sinks, namely

$$\frac{\partial}{dt}\psi[f(x),t] = \frac{1}{2}Tr_{L^2(\Omega)}[QD_f^2\psi[f(x),t]] - V[f(x)]\psi[f(x),t] \tag{20.39}$$

with

$$\psi[f(x),t\to 0^+] = \Omega[f(x)], \tag{20.40}$$

posseses a generalized Feynman-Wiener-Kac Hilbert $L^2(\Omega)$ space valued path integral representation, which in the Feynman Physicist formal notation reads as

$$\begin{aligned}\psi[h(x),T] &= \int_{C([0,T],L^2(\Omega))} D^F[X(\sigma)]\\ &\times \exp\left\{-\frac{1}{2}\int_0^T d\sigma\left(\frac{dX}{d\sigma},Q^{-1}\frac{dX}{d\sigma}\right)_{L^2(\Omega)}(\sigma)\right\}\\ &\times \Omega\left[\left(\int_0^T X(\sigma)d\sigma\right)+X(0)\right]\\ &\times \exp\left\{-\int_0^T d\sigma V\left[\left(\int_0^T X(\sigma')d\sigma'\right)+X(0)\right]\right\}\end{aligned} \tag{20.41}$$

Where the paths satisfy the end-point constraint $X(T) = h(x) \in L^2(\Omega); X(0) = f(x) \in L^2(\Omega)$.

Appendix A. The Aubin-Lion Theorem

Just for completenesse in this mathematical appendix for our mathematical oriented readers, we intend to give a detailed proof of the basic result on compacity of sets in function spaces of the form $L^2(\Omega)$ and througout used on section 2. We have, thus, the Aubin-Lion Theorem[3] in the Gelfand triplet $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) = (H_0^1(\Omega))^*$

"**Aubin-Lion** - If $\{U_n(x,t)\}$ is a sequence of time-differentiable functions in a bounded set of $L^2([0,T],H_0^1(\Omega))$ such that its time derivatives forms a bounded set of $L^2([0,T],H_0^{-1}(\Omega))$, we have that $\{U_n(x,t)\}$ is a compact set on $L^2([0,T],L^2(\Omega))$".

Proof: the basic fact we are going to use to give a mathematical proof of this theorem is the following identity (Ehrling's lemma): For any given $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that

$$||U_n||_{L^2(\Omega)} \leq \varepsilon||U_n||_{H_0^1(\Omega)} + C(\varepsilon)||U_n||^2_{H^{-1}(\Omega)} \tag{20.A-1}$$

As a consequence, we have the following estimate

$$
\begin{aligned}
&\int_0^T \|U_n - U_m\|^2_{L^2([0,T]),L^2(\Omega)} \\
&\leq \int_0^T dt(\varepsilon\|U_n - U_m\|_{H^1_0(\Omega)} + C(\varepsilon)\|U_n - U_m\|_{H^{-1}(\Omega)})^2 \\
&\leq \varepsilon^2\left(\int_0^T dt\|U_n - U_m\|^2_{H^1_0(\Omega)}\right) + (C(\varepsilon)^2\left(\int_0^T dt\|U_n - U_m\|^2_{H^{-1}(\Omega)}\right) \\
&\quad + 2\varepsilon C(\varepsilon)\left(\int_0^T dt(\|U_n - U_m\|_{H^1_0(\Omega)} \times \|U_n - U_m\|_{H_0^{-1}(\Omega)})\right) \\
&\leq \varepsilon^2\left(\int_0^T dt\|U_n - U_m\|^2_{H^1_0(\Omega)}\right) + C(\varepsilon)^2\left(\int_0^T dt\|U_n - U_m\|^2_{H^{-1}(\Omega)}\right) \\
&\quad + 2\varepsilon C(\varepsilon)\left(\int_0^T dt\|U_n - U_m\|^2_{H^1_0(\Omega)}\right)^{\frac{1}{2}} + \left(\int_0^T dt\|U_n - U_m\|^2_{H^{-1}(\Omega)}\right)^{\frac{1}{2}} \\
&\leq 2\varepsilon^2 M + 2\varepsilon C(\varepsilon)M^{\frac{1}{2}}\left(\int_0^T dt\|U_n - U_m\|^2_{H^{-1}(\Omega)}\right)^{\frac{1}{2}} \\
&+ (C(\varepsilon)^2)\left(\int_0^T dt\|U_n - U_m\|^2_{H^{-1}(\Omega)}\right) \qquad (20.A2)
\end{aligned}
$$

At this point, we use the Arzela-Ascoli theorem to see that $\{U_n(x,t)\}$ is a compact set on the space $C([0,T],H^{-1}(\Omega))$ since we have the set equicontinuity:

$$
\begin{aligned}
\|U_n(t) - U_m(s)\|_{H^{-1}(\Omega)} &\leq \int_s^t \|U_n'(\tau)\|_{H_0^{-1}(\Omega)} d\tau \\
&\leq |t-s|^{(1-\frac{1}{2})} \times \left(\int_0^T \|U_n'(\tau)\|^2_{H^{-1}(\Omega)} d\tau\right)^{\frac{1}{2}} \\
&\leq \overline{M}|t-s|^{\frac{1}{2}} \qquad (20.A3)
\end{aligned}
$$

It is a crucial step now by remarking that $H^1_0(\Omega)$ is compactly immerse in $L^2(\Omega)$ (Rellich Theorem). Let us not that for each t (almost everywhere in $[0,T]$), $U_n(x,t)$ is a bounded set on $H^1_0(\Omega)$ since $U_n(x,t)$ belongs to a bounded set $L^2([0,T], H^1_0(\Omega))$ by hypothesis. As a consequence, $\{U_{n_k},(x,t)\}$ is a compact set on $L^2(\Omega)$ (Rellich Theorem) and so in $H^{-1}(\Omega)$ almost everywhere in $[0,T]$ since $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. By an application of the Arzela-Ascoli theorem, there is a sub-sequence$\{U_{nk}(x,t)\}$ of $\{U_n(x,t)\}$ (and still denoted by $\{U_n(x,t)\}$ such that it converges uniformly to a given function $\overline{U}(x,t) \in C([0,T],H^{-1}(\Omega))$. As a direct result of this fact we, have that (for $T < \infty$!) for $(n,m) \to \infty$.

$$
\left(\int_0^T \|U_n - U_m\|^2_{H^{-1}(\Omega)}\right)^{\frac{1}{2}} \leq (\sup|U_n - U_m|_{C([0,T],H^{-1}(\Omega))}) \times \left(\int_0^T 1 \cdot dt\right)^{\frac{1}{2}} \to 0 \quad (20.A4)
$$

Returning to our estimate eq.(20.A2), we see that this sub-sequence is a Cauchy sequence in $L^2([0,T],L^2(\Omega))$. As a consequence, for each fixed $t \in [0,T]$ (almost everywhere), $U_n(x,t)$ converges to $\overline{U}(x,t)$ in $L^2(\Omega)$.

Appendix B.
The Linear Diffusion Equation in Hilbert Spaces

Let us show mathematically the basic functional integral representation eq.(20.30) for the $L^2(\Omega)$-Space Diffusion Equation eq.(20.23) .

As a first step for such proof, let us call the reader attention that one should consider the second order (Laplacean) $D^2U(x,t)$ as a bounded operator in $L^2(\Omega)$ in order to the operatorial composition with the positive definete class trace operator Q still be a class trace operator as it is explicitly supposed in the right-hand side of eq.(20.23).

We thus impose as the sub-space of initial condition the Diffusion Equation eq.(20.23) for the (dense) vector sub-space of $C(L^2(\Omega),R)$ composed of all functionals of the form.

$$f(x)=\int_{L^2(\Omega)} d_Q\mu(p)F(p)\exp\left(i\langle p,x\rangle_{L^2(\Omega)}\right) \tag{20.B1}$$

with $F(p)\in L^2(L^2(\Omega),d_Q\mu)$.

By substituting the inital condition eq.(20.B1) into the integral representation eq.(20.30) and by using the Fubbini-Toneli Theorem to exchange the needed integrations order in the estimate below, we get:

$$\begin{aligned} U(x,t) &= \int_{L^2(\Omega)} f(x+\sqrt{t}\xi)d_Q\mu(\xi) \\ &= \int_{L^2} d_Q\mu(\xi)\left\{\int_{L^2} d_Q\mu(p)F(p)e^{i\langle p,x+\sqrt{t}\xi\rangle_{L^2}}\right\} \\ &= \int_{L^2} d_Q\mu(p)F(p)\cdot e^{i\langle p,x\rangle_{L^2}}e^{-\frac{1}{2}t\langle p,Qp\rangle_{L^2}}. \end{aligned} \tag{20.B2}$$

Note that we have already proved that $U(x,t)$ is a bounded functional of $C\left(L^2(\Omega)\times[0,\infty];R\right)$ on the basis of our hypothesis on the initial functional date eq.(20.B1).

At this point we observe that the second order Frechet derivatives of the Functional $\exp i\langle p,x\rangle_{L^2}$ are easily (explicitly) evaluated as [(7)]

$$QD^2\left(e^{i\langle p,x\rangle_{L^2}}\right)=\left(\sum_{\ell=1}^{\infty}\lambda_\ell\frac{\partial^2}{\partial^2 x_\ell}\right)\left[e^{i(\Sigma_{n=1}^{\infty}p_nx_n)}\right]=-\left(\langle p,Qp\rangle_{L^2}\right)e^{i\langle p,x\rangle_{L^2}} \tag{20.B3}$$

We have thus a straightforward proof of our claim above cited on the basis again of the chosen initial date sub-space

$$\begin{aligned} &Tr[QD^2U(x,t)] \\ &\leq \int_{L^2(\Omega)} d_Q\mu(p)|F(p)|\langle p,Qp\rangle_{L^2} \\ &\leq \left(\int_{L^2(\Omega)} d_Q\mu(p)|F(p)|^2\right)^{\frac{1}{2}}\left(\int_{L^2(\Omega)} d_Q\mu(p)|\langle p,Qp\rangle_{L^2(\Omega)}|^2\right)^{\frac{1}{2}} \\ &\leq (TrQ)^2\|F\|^2_{L^2(L^2(\Omega),d_Q\mu)}<\infty. \end{aligned} \tag{20.B4}$$

Now, it is a simply application to verify that eq.(20.B2) satisfies the Diffusion Equation in $L^2(\Omega)$ (or in any other Separable Hilbert Space). Namely:

$$\frac{\partial U(x,t)}{\partial t} = \int_{L^2(\Omega)} d_Q\mu(p) F(p) e^{i\langle p,x\rangle_{L^2}} \left\{ -\frac{1}{2}\langle p, Qp\rangle_{L^2(\Omega)} \right\} \times e^{-\frac{t}{2}\langle p,Qp\rangle_{L^2(\Omega)}} \tag{20.B5}$$

$$\begin{aligned} Tr_{L^2(\Omega)}[QD^2U(x,t)] &= \int_{L^2(\Omega)} d_Q\mu(p) F(p) Tr_{L^2(\Omega)} \left\{ QD^2 e^{i\langle p,x\rangle} \right\} e^{-\frac{t}{2}\langle p,Qp\rangle_{L^2(\Omega)}} \\ &= \int_{L^2(\Omega)} d_Q\mu(p) F(p) \{-\langle p, Qp\rangle_{L^2}\} e^{i\langle p,x\rangle} e^{-\frac{t}{2}\langle p,Qp\rangle_{L^2(\Omega)}} \end{aligned} \tag{20.B6}$$

with

$$U(x,0) = \int_{L^2(\Omega)} d_Q\mu(p) F(p) e^{i\langle p,x\rangle} \left\{ \lim_{t\to 0^+} e^{-\frac{t}{2}\langle p,Qp\rangle_{L^2(\Omega)}} \right\} = f(x). \tag{20.B7}$$

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Chapter 21

Basics Integrals Representations in Mathematical Analysis of Euclidean Functional Integrals

In this complementar chapter, we expose additional rigorous mathematical concepts and theorems behind Euclidean Functional Integrals as proposed by us in Chapters 19-20 of this book and used throughly in another chapters.

In Section 1.1, we present a pure topological proof of the basic measure theory Riesz-Markov theorem, mathematical concept basic to construct rigorously functional integrals. In Section 1.2, we present analogous results on the mathematics structure of the L. Schwartz Distributions.

In Section 1.3, we present the important Kakutani theorem on the Equivalence of Gaussian Measures in Hilbert Spaces, the mathematical basis for the rigorous framework for Jacobian Transformations in Euclidean Path Integrals.

21.1. On the Riesz-Markov Theorem

"The words set and function are not as simple as they may seem. They are potent words. They are like seeds, which are primitive in appearance but have the capacity for vast and intrincate developments - G.F. Simmons."

The Riesz Representation Theorem

Theorem 1. Let X be a Topological Compact Hausdorff Space. Let L be a positive linear functional on $C(X)$. There exists a unique positive measure $d_L\mu$ and an associated σ-algebra on X which represents L in the sense that

$$L(f) = \int_X f(x) d_L u(x) \tag{1}$$

Let us begin our proof by introducing the ring of compact subsets of X.

Another mathematical structure we needed is the following Banach space. Let ${}_pC(X)$ be the vector space formed by all linear combinations of the elements of $C_0(X)$ and the characteristic functions of the compact sets of X. We introduce the sup norm on this vector space and take its completation still denoted by ${}_pC(X)$ (which is a Banach Space). It is a straightforward consequence of the Hahn-Banach Theorem that the given positive linear functional L has an unique extension to ${}_pC(X)$ still denoted by L in what follows.

We define now an equivalence relation on the Algebra of sets above introduced through the relationship

$$\forall \beta, \alpha \in A \quad \text{and} \quad \alpha \sim \beta \Leftrightarrow L(\chi_{\overline{\alpha\Delta\beta}}) = 0, \tag{2}$$

here $\overline{\alpha\Delta\beta}$ denotes the topological closure of the difference set $\alpha\Delta\beta = (\alpha - \beta) \cup (\beta - \alpha) = (\alpha \cap \beta') \cup (\beta \cap \alpha')$. On this Coset Algebra of sets $A/\sim$, denoted by $E_{\text{Baire}}(X)$, we introduce a metrical structure by means of the metric set function

$$d_L(A,B) = L\left(\left(\chi_{\overline{A\Delta B}}\right)\right). \tag{3}$$

By considering the topological completation of the metric space $(E_{\text{Baire}}(X), d_L)$ we obtain our proposed σ-algebra on X and a measure defined by the simple metrical relation

$$\mu_L(\alpha) = d_L(\alpha, \phi). \tag{4}$$

At this point it is evident that the Extension Theorem of Caratheodory is a simple rephrazing of eq(4), since for a given μ_L-measurable set $\Omega \in (\overline{E}_{\text{Baire}}(X), \mu_L)$ and $\varepsilon > 0$, there is a finite family of disjoint compact sets on X: $\{K_e\}_{e=1,\dots,N(c)}$ such that

$$d\left(\Omega, \bigcup_{\ell=1}^{N(\varepsilon)} K_\ell\right) \leq \varepsilon \Leftrightarrow \left(\sum_{\ell=1}^{N(\varepsilon)} \mu_L(K_e)\right) - \varepsilon \leq \mu_L(\Omega) \leq \varepsilon + \left(\sum_{\ell=1}^{N(\varepsilon)} \mu_L(K_e)\right) \tag{5}$$

Let us introduce the large Banach Space $C_{\text{bounded}}(\overline{E}_{\text{Baire}}(X), R)$ with the usual sup norm. It is straightforward to see that the given functional $L \in ({}_pC(X))^*$ has an unique extension $\widetilde{L}$ to this new space of continuous function on $\overline{E}_{\text{Baire}}(X)$ [which is straightforwardly identified with the measurable functions on $(\overline{E}_{\text{Baire}}(X), u_L)$!] Since the characteristic functions of compact sets are elements of $C_{\text{bounded}}(\overline{E}_{\text{Baire}}(X), R)$ any $f \in C(x)$ is the limit on the topology of $C(\overline{E}_{\text{Baire}}(X), R)$ of the simple functions (monotone non-decreasing) sequence below written

$$f(x) = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n2^n} \frac{j-1}{2^n} \chi_{E_{n,j}}(x) + n\chi_{F_n}(x) \right\} \equiv \lim_{n\to\infty} S_n(f)(x). \tag{6}$$

Here the (compact!) sets $E_{n,j}$ and F_n in X are defined by

$$E_{n,j} = f^{-1}\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]\right) \tag{7-a}$$

$$F_n = f^{-1}\left([n, ||f||_{C(x)}]\right) \tag{7-b}$$

Now the assertive expressed by eq(1) is a simple result of the definition of integration

$$\begin{aligned} L(f) &= \widetilde{L}(f) = \widetilde{L}(\lim_{n\to\infty} S_n(f)) = \lim_{n\to\infty} L(S_n(f)) \\ &= \lim_{n\to\infty}\left(\sum_{j=1}^{n2^n} \frac{j-1}{2^n}\mu_L(E_{n,j}) + n\mu_L(F_\mu)\right) \\ &= \lim_{n\to\infty}\left(\int_X f_n(x) d_L\mu(x)\right) \stackrel{\text{def}}{\equiv} \int_X f(x) d_L\mu(x) \end{aligned} \tag{8}$$

which proves the Riesz-Markov theorem.

As a last point of this section let us give a criterion for the existence of invariant sets in relation to a given (measurable) transformation

$$T:\ (\overline{E}_{\text{Baire}}(X), d_L) \to (\overline{E}_{\text{Baire}}(X), d_L). \tag{9}$$

If the measurable transformation is a contraction (or some of its power!) between the above Complete Metric Spaces Namelly: if there is $c < 1$ and an integer $P \in \mathbb{Z}^+$ such that

$$d_L(T^\rho A, T^\rho B) = \int_X d_L\mu(x)\,\chi_{(T^\rho A \Delta T^\rho B)}(x) \leq e\left(\int_X d_L\mu(x)\,\chi_{A\Delta B)}(x)\right), \tag{10}$$

then there is a point-fixed set $\overline{A}$, such that

$$T(\overline{A}) = \overline{A} \tag{11}$$

in the sense that

$$d_L(T(\overline{A}), \overline{A}) = \int_X d_L\mu(x)\, X_{(T(\overline{A})\Delta\overline{A})}(x) = 0. \tag{12}$$

We now show a concrete version of the Riesz representation theorem

Theorem 2. Let L be a continuous linear functional on the $C(\overline{\Omega})$, the space of the continuous function defined in a compact set $\overline{\Omega} \subset R^4$ and satisfying the following property

$$L\left(e^{i\left(\sum_{j=1}^{N} p_j x_j\right)} \chi_{\overline{\Omega}}(x)\right) = f_\Omega(p_1,\dots,p_n) = f_\Omega(p) \in L^1(R^N) \tag{13}$$

Then there is a (unique) function $\phi_L(x) \in C(\overline{\Omega})$ representing the action of the functional eq(13) on $C(\overline{\Omega})$ by the integral representation

$$L(g(x)) = \int_{\overline{\Omega}} d^N x \, \phi_L(x) \, g(x). \tag{14}$$

Proof: Let us firstly consider the Fourier Transform of the function $f_\Omega(P)$. Namely

$$\phi_L(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{R^N} d^N p \cdot e^{iP\cdot x} f_\Omega(p). \tag{15}$$

Obviously $\phi_L(x) \in C_0(R^N)$.

Due to supposed continuity of the functional L, one can show that $f_\Omega(p) \in C^\infty(R^N)$ and we have the differentiality relation below written $(M = (\ell_1,\dots,\ell_N))$ (exercise)

$$\frac{\partial^{|M|}}{\partial p_1^{\ell_1}\dots\partial p_n^{\ell_N}} (f_\Omega(p)) = L\left\{(ix_1)^{\ell_1}\dots(ix_n)^{\ell_N}\left(\exp i\left(\sum_{j=1}^{N} p_j x_j\right)\right)\chi_{\overline{\Omega}}(x)\right\} \tag{16}$$

which means that the inversion Fourier transform theorem holds true

$$\begin{aligned} L\big((x_1)^{\ell_1}\dots(x_n)^{\ell_n}\big) &= \left(\frac{1}{\sqrt{2\pi}}\right)^N \int_{R^N} d^N x e^{-ipx}\left\{(x_1)^{\ell_1}\dots(x_n)^{\ell_n}\phi_L(x)\right\}\Big|_{p=0} \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^N \left\{\int_{R^N} d^N x (x_1)^{\ell_1}\dots(x_n)^{\ell}\phi_L(x)\right\} \end{aligned} \tag{17}$$

By the Weistrass Theorem, we have finally our envisaged result on $C(\overline{\Omega})$

$$L(g(x)) = \int_{R^N} d^N x g(x)\phi_L(x)\chi_{\overline{\Omega}}(x) = \int_{\overline{\Omega}} d^N x g(x)\phi_L(x). \tag{18}$$

21.2. The L. Schwartz Representation Theorem on $C^\infty(\Omega)$ (Distribution Theory)

"– The Quantum and Random World is an application of Cantor Set Theory in its developments – Luiz Botelho."

After have exposed the fundamental abstract result of Riesz-Markov on the structure of the elements of the dual space of continuous linear functionals in $C(X)$, with X denoting a general compact topological space, we pass on to the problem of describing continuous functionals on the vector space $C^\infty(\Omega)$, with Ω denoting a open set of R^N.

Let us thus start by considering a sequence of compact sets K_n, with the property interior $(K_{n+1}) \supset K_n$ and such that $\Omega = \bigcup_{n=1}^{\infty} K_n$, together with the Complete Metrical Space $C^\infty(K_n)$, defined by the vector space of infinitely differentiable functions in Ω, with support in K_n with the Frechet metric

$$d(f,g) = \sum_{m=0}^{\infty} \frac{2^{-m}||f-g||_m}{1+||f-g||_m}, \text{ where } ||f||_m\colon \sup_{x\in\Omega}\sup_{|p|\le m} |D^p f(x)|$$

The basic contribution of L. Schwartz is to consider the Topology of the inductive limit on $C^\infty(\Omega)$ as writing formally as topological spaces $C^\infty_{\text{ind}}(\Omega) = \bigcup_{n=0}^{\infty} C^\infty(K_n)$ rigorously meaning that the topology in $C^\infty(\Omega)$ is the weakest topology which makes all the canonical injections

$$D_N\colon C^\infty(K_n) \to C^\infty(\Omega) \tag{19}$$

continuous applications.

A Topological Basis for the origin of $C^\infty(\Omega)$ is formed by all those convex and barreleds sets $U \subset C^\infty(\Omega)$ such that $U \cap C^\infty(K_n)$ is always a neighborhood of the origin in $C^\infty(K_n)$. The main result and reason for introducing such Inductive Topology in $C^\infty(\Omega)$ is that it leads to the fundamental result that $C^\infty_{\text{ind}}(\Omega)$ is a Sequentially Complete Toplogical Vector Space.

At this point is worth call the reader attention that the usual non-distributional topological definition of $C^\infty(\Omega)$ as $\bigcup_{m=0}^{\infty} C^m(\Omega)$ [always used in others approach of Generalized Functions] is stronger than the L. Schwartz inductive topology above introduced.

We always re-write $C^\infty_{\text{ind}}(\Omega)$ in the well-known L. Schwartz motation as $D(\Omega)$: the Schwartz Test function space. The description of the notion of convergence in $D(\Omega)$ is straightforward since we have the sequential completeness topological property. Namely: a sequence $\varphi_n(x) \in D(\Omega)$ converges in $D(\Omega)$ if there is a set K_n such that $\varphi_n \to \overline{\varphi}$ in $C^\infty(K_n)$.

Another basic result as consequence of the introduction of the Inductive limit topology in $C^\infty(\Omega)$ is the straightforward description of the Dual Space of $D(\Omega)$, denoted by $D'(\Omega)$ and named as the L. Schwartz Distribution Space in Ω

$$D'(\Omega) = \left(\bigcup_n C^\infty(K_n)\right)^* = \bigcup_n (C^\infty(K_n))^*. \tag{20}$$

Note that the structural description of $(C^\infty(K_n))^*$ is expected to be closely related to that one of $C(K_n)^*$ (the Riesz-Markov Theorems). In fact, we have the L. Schwartz generalization of the Functional Integral representation of Riesz-Markov theorem.

Theorem 2 (Laurent Schwartz). Any given continuous linear functional $L \in D'(\Omega)$ may be represented by a sequence of complex Borel measures $d_n u(x)$ in K_n, a sequence of multi-indexes $\{P_n\} = \{p_n^1, \dots, p_n^N\}$ through the integral representation

$$L(\varphi) = \sum_{n=0}^{\infty} \left(\int_{K_n} d\mu_n(x)(D^{P_n}\varphi)(x) \right), \tag{21}$$

there

$$(D^p\varphi)(x) = \frac{\partial^{(P_n^1+\cdots+P_n^N)}}{\partial x_1^{P_n^1} \dots \partial x_N^{P_n^N}} \varphi(x_1, \dots, x_N). \tag{22}$$

Proof: Let $L \in D^q(\Omega)$, but with compact support $K_s \subset \Omega$. By the inductive limit topology (exercise 1, there is a constant $c_s > 0$, and an integer $m_s > 0$, such that for any $\varphi \in D(\Omega)$, we have the estimate

$$|L(\varphi)| \leq c_s \sup_{x\in K_s} \big(\sup_{|p|\leq m_s} |D^p\varphi(x)| \big) \tag{23}$$

Note that the triple (C, K_s, m_s) is not unique. We now consider the following Elliptic operator $\mathcal{L}_p = \left(\frac{\partial}{\partial x_1}\right)^{p_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{p_n}$ $(p = p_1 + \cdots + p_n)$. Since $\mathcal{L}_p$ is an injetive application of $C^\infty(\Omega)$ into $C^\infty(\Omega)$ and this T restricts to the dense subspace $\mathcal{L}_p[D(\Omega)]$ (range of $\mathcal{L}^p$ in $D(\Omega)$, satisfies the obvious estimate below

$$f \in C(K_s) \colon L\big(\mathcal{L}_p^{-1} f\big) \leq c_s \cdot \sup_{x\in K_s} |f(x)| \tag{24}$$

we can apply the Riesz-Markov Theorem 1 to the composed functional $L \circ \mathcal{L}_p^{-1}$ in $C(K_s)$.

$$L(\mathcal{L}_p^{-1} f) = \int_{K_s} d_s\mu(x) \cdot f(x) \tag{25}$$

or equivalently (exercise) for any $\varphi \in D(\Omega)$, we have the functional integral representation

$$L(\varphi) = \int_{K_s} d_s\mu(x)(D^{p_s}\varphi)(x). \tag{26}$$

In the general case, we just consider an unity partition subbordinate to a given open cover of Ω $(1 = \sum_n h_n(x), K_n \subset \operatorname{supp} h_n \subset K_{n+1}, K_n$ compact set of Ω and $UK_u = \Omega$ and $h_n(x) = 1$, for $x \in K_n)$

$$L = \sum_{n=1}^{\infty} h_n(x) L = \sum_{n=1}^{\infty} L_n\,. \tag{27}$$

At this point we introduce the weak-* topology in $D'(\Omega)$ through a sequential criterion: A sequence of Distributions $L_n \in D'(\Omega)$ converges weak-star if the sequence of measures in eq(25) converges in the weak-star topology of $C(K_s)^*$.

After the proof of L. Schwartz representation theorem, let us introduce the operation of derivation in the Distributional sense. Firstly, let us recall some definitons in Functional Analysis of Vector Topological Spaces. Let E and F be two vector spaces with topologies compatible with its vectorial structure and $U\colon E \to F$ a linear continuous application between them. For any $y' \in F'$ (dual of F), we can associate the element x' of E' through the definition $({}^tU\colon F' \to E')$

$$(x') = x'(x) = y'(U(x)) = ({}^tU(y)). \tag{28}$$

It can be showed that if U is continuous, the ${}^tU$ remains continuous if E and F are Frechet Spaces like $C^\infty(K_s))$.

As a consequence of the above remarks, the usual derivative operator is a linear continuous application between $D(\Omega)$. Namelly

$$D\colon D(\Omega) \to D(\Omega).$$

By the duality eq(28) above mentioned, one has a natural derivative application in $D^q(\Omega)$

$$(-DL) \overset{\text{def}}{\equiv} ({}^tD)(L)(f) \overset{\text{def}}{\equiv} L(Df), \tag{29}$$

besides of being always a continuous operation in $D'(\Omega)$ if $L_n \overset{D'(\Omega)}{\longrightarrow} \Leftrightarrow {}^tDL_n \overset{D'(\Omega)}{\longrightarrow} {}^tDL$.

At this point we call our reader to how that the sequence of functions $f_n(x) = \dfrac{1}{n}\,\text{sen}(nx)$ as seen as kernels of distributions in $D'(R)$ obviously converges to the zero distribution in $D'(R)$:

$$\lim_{n\to\infty} \int_R dx \left(\frac{1}{n}\,\text{sen}\,nx\right) \varphi(x) = 0. \tag{30}$$

As a consequence of the above made remark, we have the validity of the result called the Riemann-Lebesgue Lemma

$$\lim_{n\to\infty} \int_R dx \cos(nx) \varphi(x) = 0. \tag{31}$$

Namelly

$$\frac{d}{dx}\left(\frac{\operatorname{sen} nx)}{n}\right)^{D'(R)} = \cos(nx) \xrightarrow{D'(R)} 0 \tag{32}$$

A further finner structural analysis can be implemented to eq(21) by means of an application of the Radon-Nikodym theorem to the pair of complex Borel measures $(du_s(x), d^N x)$ on the Borelians of Ω.

$$\begin{aligned} L(\varphi) = \sum_{s=0}^{\infty}\left(\int_{\Omega}(D^{p_s}\varphi)(x)\left(\overbrace{\frac{du_s(x)}{d^n x}}^{h_s(x)}\right)d^n x\right) \\ + \sum_{s=0}^{\infty}\left(\int_{\Omega}(D^{p_s}\varphi)(x)d\nu_s^{\text{sing}}(x)\right), \end{aligned} \tag{33}$$

where $h_s(x) \in L^1(\Omega, d^N x)$ and $d\nu_s^{\text{sing}}(x)$ is a singular measure (in relation to the Lebesgue measure $d^N x$ in Ω) with support at points (Dirac delta functions) and on sets of Lebesgue zero measure

$$d\nu_s^{\text{sing}}(x) = \sum_{\ell=0}^{\infty} a_{\ell,s}\,\delta(x - x_{\ell,s}) + dV_s^{\text{(continuous singular)}}(x) \tag{34}$$

Another important Distributional Space in the (Topological) dual of the Space of test functions with polynomial decreasing $S(R^N)$, a very basic object in Wave Fields Quantum Path Integral (see Chapter 19)

$$S(R^N) = \left\{u \in C^{\infty}(R^N) \mid ||\varphi||_{n,m} = \sup_{x \in R^N} |x^n D^m \varphi(x)| < (\infty)\right\}. \tag{35}$$

We have the following structural theorem, analogous to the Theorem 1 of L. Schwartz.

Theorem 3. Given a functional L in $(S(R^n))'$, we can always represent L by a Borel complex measure $du(x)$ in R^N by means of $(x^p = x_1^{p_1} \dots x_N^{p_n}, \text{etc..})$

$$L(\varphi) = \int_{R^N} d\mu(x)(x^p D^q \varphi)(x) \tag{36}$$

The proof of the above written integral representation for distributions in $S'(R^N)$ is based on the fact that for a given $L \in S'(R^N)$, these are multi indexes (p,q), such that there is a positive constant c with

$$|L(\varphi)| \leq c||\varphi||_{p,q} \tag{37}$$

Note that the Elliptic operator $\mathcal{L}_p = x^p D^q = x_1^{p_1} \dots x_n^{p_n} \left(\frac{\partial}{\partial x_1}\right)^{q_1} \dots \left(\frac{\partial}{\partial x_N}\right)^{q_N}$ is an surjective application of $S(R^N)$ into $S(R^N)$ and $\mathcal{L}_p(S(R^N))$ is dense in $C_0(R^N)$ (continuous functions vanishing at ∞), the great usefulness of $S^1(R^N)$ in Quantum Field Theoretic Path Integrals is related by the fact the usual Fourier Transform is a vectorial/topological isomorphism in $S(R^N)$. By duality, one straightforwardly define the Fourier Transforms in $S'(R^N)$ which remains a topological isomorphism in the Distributional Space $S'(R^N)$

$$\mathcal{F} : S(R^N) \to S(R^N) \tag{38}$$

$$^t\mathcal{F} : S'(R^N) \to S'(R^N) \tag{39}$$

$$^t\mathcal{F}(L)(\varphi) = L(\mathcal{F}(\varphi)) \tag{40}$$

the above written equations are important results in Applications of Distribution Theory of L. Schwartz is given by the following result: Let A be a continuous linear application between a locally convex topological vector space E with values in the topological dual of another locally convex topological vector space F'. Then the bilinear form in $E \times F$ defined by the relation $B(f,g) = (Af)(g)$ is continuous in $E \times F'$, when one introduces the weak topology on F'. As a consequence, every continuous bilinear form on $S(R^N)$ is of the linear supperposition of forms below written for a pair of multi-indexes (m,n)

$$B(f,g) = \int_{R^N \times R^N} F(x,y)(D^m f(x))(D^n g(x))d^n x\, d^n y. \tag{41}$$

Here $F(x,y)$ is a continuous function of polynomial grow in R^N

$$\left(\exists\, p \in \mathbb{Z}^+ \mid \underset{\substack{|x|\to\infty \\ |y|\to\infty}}{\ell g}\; F(x,y)(|x|^2 + |y|^2)^{-p} = 0 \right)$$

21.3. Equivalence of Gaussian Measures in Hilbert Spaces and Functional Jacobians

In this somewhat long section, we present the mathematical analysis of the Jacobian Change of Variable in Gaussian Functional Integrals in Hilbert Spaces through the formalism of the Kakutani Theorem.

Let A^{-1} and B^{-1} be positive definite trace class operators in a given Hilbert Space $(H, \langle,\rangle)$ and operators inverse of the operators A and B.

The spectral representations for theses operators

$$A\varphi_n = \lambda_n \varphi_n \tag{42-a}$$

$$B\sigma_n = \alpha_n \sigma_n \tag{42-b}$$

define Gaussian measures $d_{A^{-1}}u(\varphi)$ and $d_{B^{-1}}u(\varphi)$ in the Borelian Algebra of the cylinders sets in H and are defined by

$$d_{A^{-1}}\mu(\varphi) = \limsup_{N} \left\{ \prod_{n=1}^{N} d\langle\varphi,\varphi_n\rangle exp\left\{ -\frac{\lambda_n}{2}\langle\varphi,\varphi_n\rangle^2 \right\} \times \left(\sqrt{\frac{\lambda_n}{2\pi}} \right) \right\} \tag{43-a}$$

$$d_{B^{-1}}\nu(\varphi) = \limsup_{N} \left\{ \prod_{n=1}^{N} d\langle\varphi,\sigma_n\rangle exp\left\{ -\frac{\alpha_n}{2}\langle\varphi,\sigma_n\rangle^2 \right\} \times \left(\sqrt{\frac{\alpha_n}{2\pi}} \right) \right\} \tag{43-b}$$

We have thus the Kanutani theorem and the measure equivalence of the above written measues.

Kanutani Theorem 4. The two measures eqs(43-a), eq(43-b) are mutually equivalent or singular. In the first case we have the criterium that

$$\sum_n \left(\frac{\lambda_n^2 - \alpha_n^2}{\lambda_n \alpha_n} \right) < \infty \tag{44}$$

and the Radon-Nykodin derivative of the above measures is given by

$$\frac{d_{A^{-1}}\mu(\varphi)}{d_{B^{-1}}\nu(\varphi)} = \lim_{N\to\infty} \left\{ \prod^{N} \left[\frac{\lambda_n}{\alpha_n} \exp\left(-\frac{1}{2}(\lambda_n - \alpha_n) \left(\sum_n (\varphi,\sigma_n)^2 \right) \right) \right] \right\} \tag{45}$$

Note that in the case for the Radon-Nykodim derivative

$$\frac{d_{A^{-1}}\mu(\varphi)}{d_{B^{-1}}\nu(\varphi)} = \det(AB^{-1}) \exp\left\{ -\frac{1}{2}\langle\varphi,(A-B)\varphi\rangle \right\} \tag{46}$$

On basis of eq(45)-(46), one can show the Wiener result about translation invariant of Gaussian Measures: Let $T_h \colon H \to H$ be the translation operator in H. Let us consider the translated Gaussian measure

$$d_{A^{-1}}\mu(T_h\varphi) = d_{A^{-1}}\mu(\varphi+h) = \left(\prod_{n=1}^{\infty} d\langle\varphi|\varphi_n\rangle e^{-\frac{1}{2}\lambda_n[\langle\varphi,\varphi_n\rangle + \langle h,\varphi_m\rangle]^2} \right\} \tag{47}$$

By the Kakutani theorem, the translated measure $d_{A^{-1}}\mu(T_h\varphi)$ is equivalent to the measure $d_{A^{-1}}\mu(\varphi)$ if and only if

$$\sum_n \frac{|\langle h,\varphi_n\rangle|^2}{\frac{1}{\lambda_n}} = \langle Ah,h\rangle < \infty, \tag{48}$$

or equivalently: the translational-invariance of the measure is insured if h belongs to the domain of the operator A.

At this point we remark that $Dom(A)$ is a set of zero measure for $(H, d_{A^{-1}}\mu(\varphi))$. [Finite action smooth field configurations, makes a set of zero functional measure – see Chapter

1]. Let us give a simple proof of such important result in practical calculations with Path Integrals.

Firstly, let us re-write the Finite Action set of path integrated configurations ($=Dom(A)$) in the following form (for $\varepsilon > 0$)

$$\chi_{Dm(A)}(\varphi) = \lim_{\alpha\to 0^+}\lim_{N\to\infty} \exp\{-\alpha\langle p_N\varphi, AP_N\varphi\rangle\} = \begin{cases} 1 \text{ if } \langle\varphi, A\varphi\rangle < \infty \\ 0 \text{ otherwise} \end{cases} \tag{49}$$

where the orthogonal projections $P_n \to \mathbf{1}$ in the strong sense.

Let as evaluate formally in the “Physical way” its functional measure content

$$\begin{aligned} M_{A^{-1}}(\chi_{Dm(A)}(\varphi)) &= \lim_{\alpha\to 0}\left\{\lim_{N\to\infty}\left[\int_H d_{A^{-1}}\mu(\varphi)e^{-\alpha\langle P_N\varphi, AP_N\varphi\rangle}\right]\right\} \\ &\lim_{\alpha\to 0^+}\left(\lim_{N\to\infty}\exp\left(-\frac{N}{2}\ell g(1+\alpha)\right)\right) = \lim_{\alpha\to 0}\lim_{N\to\infty}\left(e^{-\frac{N}{2}\alpha}\right) \\ &= e^{-\infty} = 0 \end{aligned} \tag{50}$$

At this point one can see that the usual Schwinger procedure to deduce functional equations for the Quantum Field Generating functional of Chaper 19 does not make sense in the Euclidean framework of Path Integrals since the measure is not translational invariant. Namely in the usual Feynman notation

$$\int_H D^F[\varphi]\,\frac{\delta}{\delta\varphi}\left\{e^{-\frac{1}{2}\langle\varphi, A\varphi\rangle}\,e^{-V(\varphi)}\right\} \neq 0. \tag{51}$$

As one can see from the above exposed result, the Minlos theorem is a power “tool” in the Functional Integration Theory in Infinite Dimension Vectorial Spaces.

Chapter 22

Supplementary Appendixes

Appendix 22.A. String Theory in Embeddings Manifolds

In modern quantum field theory, the framework of strings moving in manifolds has been successfully used to shed light in the basic problem of quantizing the Gravitation field ([1]). Moreover, until now the severe problems of the infrared divergencies of the string theory path integral when viewed as a σ-model two-dimensional field theory in the parameter string domain R^2 has been an issue not completely understood ([2]). Although there is a strong indication that it is possible to remove such quantum field theoretic difficulties of the use of a mathematically ill-defined 2D-massless quantum scalar-field [represented by the string vector position] by means of a string third quantization (the so called String Field Theory), this step remains an unsolved problem in the present framework of String Theory.

The purpose of this long appendix to Chapter 12 is to consider another fremework for the problem of the infrared divergencies in String Theory by applying the Nash theorem of Riemann metrics parametrized by immersions in order to show the appearance of a string mass effective matrix as a result of the dynamical interaction with the positive curvature of the given string ambient space-time M, considered as a smooth C^∞-differentiable manifold.

2 – The String Mass from the smooth Space-Time manifold Shape-Bending in the Extrinsic Space.

Let us start our analysis by considering the following convenient euclidean Polyakov's string functional integral in the presence of a given back-ground fixed Riemannian metric in the manifold M where the string dynamics takes place.

$$\begin{aligned} Z = \Bigg\{ \int d^{\text{cov}}\mu[g_{ab}(\xi)]d^{\text{cov}}\mu[X^\mu(\xi)] \\ \times \exp\Bigg\{ -\frac{1}{2\pi\alpha'}\int_{R^2} d^2\xi\{\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X(\xi))\} \\ \times \Bigg\{ \prod_{\ell=1}^{(s(d)-d)} \delta^{(F)}\big(H^\ell(f^A(X^B(\xi)))\big)\Bigg\}\Bigg\} \end{aligned} \tag{1}$$

the (closed) string surface $\{X^\mu(\xi), \mu = 1,\ldots,d\}$ is immersed in the space-time M given by a manifold possessing a $C^\infty(M)$-smooth Riemannian structure (metric) $\{G_{\mu\nu}(x^\gamma)\}_{\substack{\mu=1,\dots,d\\ \nu=1,\dots,d}}$ the manifold parametric explicit set of equations is denoted here by $H^\ell(f^A(x^B)) \equiv 0$, $A = 1,\dots,s(d))$ and $f^A\colon M \to R^{s(d)}$ is the set of real-valued immersions such that we have for them the Nash theorem for our smooth given space-time manifold metric $\{G_{\mu\nu}(x^\gamma)\}$. Namely ([3])

$$G_{\mu\nu}(x^\gamma) = \sum_{A=1}^{s(d)} \left[\frac{\partial f_A}{\partial x^\mu}\frac{\partial f_A}{\partial x^\nu}\right](x^\gamma). \tag{2}$$

Here $s(d)$ is the minimal Whitney immersion dimension of the manifold M in $R^d(S(d) > 2d)$.

The covariant functional measures in the Polyakov path integral eq(1) are the well-known De-Witt covariant functional metrics without boundary terms. Namely: ([4])

$$dS^2[g_{ab}] = \int_{R^2} d^2\xi \left[\sqrt{g}(\delta g_{ab})[g^{aa'}g^{bb'} + c\,g^{ab}g^{a'b'}](\delta g_{a'b'})\right](\xi) \tag{3}$$

$$dS^2[X^\mu] = \int_{R^2} d^2\xi \left[\sqrt{g}\,\delta X^\mu(\xi)\,G_{\mu\nu}(X^\gamma(\xi))\,\delta X^\nu(\xi)\right](\xi) \tag{4}$$

Let us show the announced phenomenon of geometrical mass generation for the 2D-scalar string vector-position fields $\{X^\mu(\xi), \xi = 1,\dots,d\}$, in the situation of a weakly space-time manifold of positive curvature.

Our main propose is to consider the following variable change in the string vector position dynamical degree of freedom (see eq(2)) in the full String Partition Functional Path Integral eq(1).

$$Y^A(\xi) = f^A(X^\mu(\xi)), \quad A = 1,\dots,s(d) \tag{5-a}$$

$$S[Y^A(\xi)] = \frac{1}{2\pi\alpha'}\int_{R^2} d^2\xi\,\sqrt{g}\,g^{ab}(\partial_a Y^A \partial_b Y_{A'})(\xi) \tag{5-b}$$

$$dS^2(Y^A(\xi)] = \int_{R^2} d^2\xi \left[\sqrt{g}\,\delta Y^A\,\delta Y_A\right](\xi) \tag{5-c}$$

At this point of our study we point at the usefulness on the explicitly use of the geometrical constraint that the string world-sheet Σ is in M through the writing of the supposed known set of the Space-Time Manifold parametric equations $\{H^\ell(Y^A) = 0, \ell = 1,\dots,s(d) - d, \{Y^A\} \in M\}$ defining M as an embedding geometrical-positional sub-manifold of the (Absolute-Extrinsic) Euclidean Whitney Space $R^{s(d)}$. This last step is the basic mechanism for our proposal of generating mass for the mean effective string vector position $\{Y^A(\xi), A = 1,\dots,s(d), \xi \in R^2\}$.

In order to show these string mass generation mechanism by geometric means, let us suppose that we have a manifold with very low positive curvature.

In this case we can replace the delta functional geometrical constraint in eq(5) by the effective string mass term as written below

$$\prod_{\ell=1}^{(s(d)-a)} \delta_{\text{cov}}^{(F)}\big(H^{\ell}(Y^{A}(\xi))\big) \underset{\substack{\text{(low extrinsic}\\ \text{curvature)}}}{\cong} \prod_{\ell=1}^{(s(d)-d)} \delta_{\text{cov}}^{(F)}\left[\left(\frac{\alpha'}{2}\frac{\partial^2 H^{\ell}}{\partial Y^A \partial Y^B}(\overline{Y}^c)\widetilde{Y}^A\widetilde{Y}^B\right)(\xi)\right] \quad (6)$$

where we have used the zero mode of mean string vector position variable in terms of the constant mode $\overline{Y}$ and its α'-vanishing small fluctuation

$$Y^A(\xi) = \overline{Y}^A + \sqrt{\pi\alpha'}\,\widetilde{Y}^A(\xi). \quad (7)$$

By making the usual hypothesis of the exact validity of the covariant mean field average for the Lagrange multiplier in the Path-Integral representation for the effective functional delta eq(6), we get the following explicitly results

$$\prod_{\ell=1}^{s(d)-d} \delta^{(F)}\left[\left(\alpha'\frac{\partial^2 H^{\ell}}{\partial Y^A \partial Y^B}(\overline{Y}^c)\widetilde{Y}^A\widetilde{Y}^B\right)(\xi)\right] \cong$$

$$\prod_{\ell=1}^{s(d)-d}\left[\int d^{\text{cov}}\mu[\lambda^{\ell}(\xi)]\exp\left\{i\alpha'\int_D d^2\xi\sqrt{g}\left[\lambda^{\ell}\left(\frac{1\delta^2 H^{\ell}(\overline{Y}^c)}{2\partial Y^A \partial Y^B}\widetilde{Y}^A\widetilde{Y}^B\right)\right](\xi)\right]\right\}$$

$$\sim \exp\left\{-\mu_{AB}(\overline{Y}^c)\int_D d^2\xi(\sqrt{g}\,\widetilde{Y}^A\widetilde{Y}^B)(\xi)\right\} \quad (8)$$

Here the string mass matrix is given explicitly by the combination of the curvature position Hessian Space-Time manifold matrix at the point $\{\overline{Y}^c\} \in M$ and the (positive) condensate value of Lagrange multiplier field $\lambda_{ab}^{\ell}(\xi) \cong \langle\lambda\rangle$, producing thus the result

$$\mu_{AB}(\overline{Y}^c) = \frac{1}{2}\langle\lambda\rangle\left(\frac{\partial^2 H^{\ell}}{\partial Y^A \partial Y^B}(\overline{Y}^c)\right) \quad (9)$$

At this point appears worthing mentioning that the non-linearity of the original theory appears fully as a consequence of the highly non-trivial re-writing of the string vertexs in terms of the somewhat decoupling-ambient geometry eq(5-A).

Now we proceed to the Nambu-Goto string path integral which depends functionally solely on the string world sheet imbedding $X^{\mu}(\xi)\colon R^2 \to R^D$, namely

$$Z = \int d_h\mu[X^{\mu}(\xi)]\exp\left\{-\frac{1}{2\pi\alpha'}\int_{R^2} d^2\xi\big(\sqrt{h(X^{\alpha}(\pi))}\big)\right\} \quad (10)$$

here the string world sheet metric tensor is always given by the imbedding variable $X^{\mu}(\xi)$

$$h_{ab}(X^{\alpha}(\xi)) = \partial_a X^{\mu}(\xi)G_{\mu\nu}(X^{\beta}(\xi))\partial_b X^{\nu}(\xi). \quad (11)$$

In this string theory, the main difficulty comes from the diffeomorphism invariant measure $D^{\text{cov}}[X^{\mu}(\xi)]$ which is strongly non-linear when written as a Feynman product measure as given below

$$d_h\mu[X^{x}(\xi)] = \prod_{\xi\in R^2}\left[\big(h(X^{\mu}(\xi))\big)^{1/4}\big(G(X^{\mu}(\xi))\big)^{1/2}dX^{\mu}(\xi)\right] \quad \text{(12-a)}$$

$$d^2S[X^{\mu}(\xi)] = \int_{R^2} d^2\xi\sqrt{h(X^{\alpha}(\xi))}\big(\delta X^{\mu}G_{\mu\nu}(X^{\gamma})\delta X^{\nu}\big)(\xi) \quad \text{(12-b)}$$

In order to overcome such problem, we proceed as in the previous chapter by considering the 2D-fluctuating metric tensor fields $g_{ab}(\xi)$ as a purely auxiliary Lagrange multiplier field without any singled out geometrical-physical role and whose dynamics must be suppressed at the end of the path integrals evaluations

$$\begin{aligned} Z = & \int d\mu[g_{ab}(\xi)] \int d\mu[X(\xi)] \exp\left\{ -\frac{1}{2\pi\alpha'} \int_{R^2} d^2\xi (\sqrt{g})(\xi) \right\} \\ & \times \exp\left\{ -\frac{1}{2} \int_{R^2} d^2\xi (\sqrt{g} g^{ab} \partial_a X^\mu G_{\mu\nu}(X) \partial_b X^\nu)(\xi) \right\} \\ & \times \delta^{(F)}_{\text{cov}}\big([g_{ab} - (\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)](\xi)\big). \end{aligned} \tag{13}$$

It is worth call the reader attention that the original Polyakov's propose eq(1) must be considered as an effective (analytical) path integral proceedure in the light of the Nash Theorem when applied to the string world sheet as a two-dimensional manifold immersed (not fully embedded) in R^D $(0 \geq 4)$ since there is a clear over counting of the degrees of freedom in eq(1) parametrizing the string dynamics: For each two-dimensional metric field $g_{ab}(\xi)$ in the string world sheet tangent bundle there is an immersion $X^\mu(\xi,[g])\colon \Sigma \to R^{\bar{J}}$, in some Whitney ambient space $R^{\bar{d}}$ $(\bar{d} > 3)$ and satisfying the metrical constraint

$$g_{ab}(\xi) = \frac{\partial X^\mu(\xi,[g])}{\partial \xi^a} \frac{\partial X_\mu(\xi,[g])}{\partial \xi^b}. \tag{14}$$

As a consequence of the above remark, one can see that our propose eq(13) already takes into account this deep geometrical-topological constraint between the string world-sheet metrical fields and the immersion/string vector position in the extrinsic space in a correct mathematical may by means of the (covariant) delta functional inside eq(13).

By proceeding as in the bulk of this chapter we can evoluate the covariant path integrals in terms of the usual Feynman product measures in the light-cone gauge

$$\begin{aligned} Z = & \int D^F[Y^A(\xi)] \exp\left\{ -\frac{1}{2\pi\alpha'} \int_{R^2} d\xi^+ d\xi^- \left[(\partial_+ Y^A \partial_- Y_A)(\xi^+,\xi^-) \right] \right\} \\ & \times \exp\left\{ -\frac{(26 - s(D))}{48\pi} \int_{R^2} d\xi^+ \xi^- \left[\frac{(\partial_+^2 Y^A)(\partial^- Y_A)(\partial_-^2 Y^B)(\partial_+ Y_B)}{(\partial_+ Y^A \partial_- Y_A)^2} \right] (\xi^+,\xi^-) \right\} \\ & \times \exp\left\{ -\frac{1}{2} \int_{R^2} d\xi^+ d\xi^- (\mu_{AB}(\gamma)(Y^A Y_B)(\xi)) \right\}. \end{aligned} \tag{15}$$

The introduction of non-trivial topology in the string world sheet is now straightforward in our Path-Integral analysis and the suppression of the Liouville dynamics for the unphysical field $g_{ab}(\xi)$ can be made by introducing N fermion species in order to change the conformal anomaly coefficient to the new factor $\frac{26-(S(D)+N)}{48}$, which can vanishes if one choose $N = S(D)$.

As a last remark in this Appendix, let us point out that in the case of a compact string parameter domain $D \subset R^2$ (not the fully R^2), one should introduces in the path integral eq(8)/eq(10) a further sum over these domains, in order to obtain full covariance. For instance, if one choose the rectangle $D_A = \{(\xi_1,\xi_2), 0 \le \xi_1 \le A;\ 0 \le \xi_2 \le 2\pi\}$, one should introduce a further integration in relation to the "moduli" A, namely

$$Z = \int_0^\infty dA \left\{ \int d_h\mu[X^\mu(\xi)] \exp\left[-\frac{1}{2\pi\alpha'} \int_{D_A} d^2\xi (\sqrt{h}(X^\mu(\xi))) \right] \right\}. \tag{16}$$

Note that the Green function associated to the compact domain D_A does not posseses infrared divergencies as in R^2, as one can see for its explicitly expression below (see chapter 18).

$$\langle X^\mu(z,\bar{z}) X^\nu(\zeta,\bar{\zeta})\rangle_{D_A} = \left(-\frac{1}{2\pi} \operatorname{Re}\left\{ \log\left[\frac{\sigma(z-\zeta,w_1,w_2)\sigma(z+\bar{\zeta},w_1,w_2)}{\sigma(z-\bar{\zeta},w_1,w_2)\sigma(z+\zeta,w_1,w_2)} \right] \right\} \right) \delta^{\mu\nu}. \tag{17}$$

Here

$$z = x+iy, \quad \zeta = \xi + i\pi$$
$$w_1 = A, \quad w_2 = 2\pi$$

and the Weirstrass-Elliptic σ-function has the expression

$$\sigma(z) = z \prod_w \left[\left(1 - \frac{z}{2w}\right) e^{\left(\frac{3}{2w} + \frac{z^2}{8w^2}\right)} \right],$$

$w = kA + \ell k\pi i, \quad (k = 0, \pm 1, \dots); \quad (\ell = 0, \pm 1, \dots).$

The reader should compare with the String Green function in R^2

$$\langle X^\mu(z,\bar{z}) X^\nu(\zeta,\bar{\zeta})\rangle = \delta^{\mu\nu} \left(-\frac{1}{4\pi} \ell g |z-\zeta| \right). \tag{18}$$

References for This Appendix A/Chapter 12

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Appendix 22.B. The Einstein-Hilbert Action as an effective theory for Random (Stringy) Fluctuations of the Space-Time

In this somewhat appendix, we intend to show how the Einstein-Hilbert action for Einstein Gravitation Theory appears in a rather natural way from a Bosonic Polyakov's String interacting with the ambient (extrinsic) manifold fluctuating metrical structure.

Let us thus firstly write the Polyakov's string path integral in the presence of the metric tensor $G_{\mu\nu}(X^\alpha)$:

$$\begin{aligned} Z[G_{\mu\nu}(X^\alpha(\xi)] &= \int \left[\prod_{\xi\in R^2} \left(\sqrt{G}\, G_{\mu\nu}(X^\mu(\xi))\right)^{1/2} dY^\mu(\xi) \right] \\ &\times \exp\left\{ -\frac{1}{2\pi\alpha'} \int_{R^2} d^2\xi \left[(\sqrt{G}\, G_{\mu\nu})(Y^\beta)\partial_a Y^\mu \partial^a Y^\nu \right](\xi) \right\} \end{aligned} \tag{1}$$

In order to see how (Higher order) Einstein-Hilbert actions emerges as an effective theory from eq(1), let us consider the geodesic expansion for the metrical objects in eq(1) through a power series expansion in the string lenght extrinsic scale α'. (Here $\sigma^{\alpha\beta}(\xi) = (X^\alpha X^\beta)(\xi)$):

$$Y^\mu(\xi) = \overline{Y}^z\mu + \sqrt{\alpha'}\, X^\mu(\xi) \tag{2}$$

$$\begin{aligned} \sqrt{G(Y^\mu(\xi))} &= 1 - \frac{\alpha'}{6} R_{\mu\nu}(\overline{Y}^B)\cdot(\sigma^{\mu\nu})(\xi) - \frac{(\alpha')^{3/2}}{12}(\nabla_\alpha R_{\mu\nu})(\overline{Y}^\beta)(X^\alpha\sigma^{\mu\nu})(\xi) \\ &+ \frac{(\alpha')^2}{24}\left\{ \left[-\frac{3}{5}(\nabla_\mu\nabla_\nu R_{\alpha\beta})(\overline{Y}^\beta) + \frac{1}{3}(R_{\mu\nu}R_{\alpha\beta})(\overline{Y}^\beta) \right.\right. \\ &\left.\left. - \frac{2}{15} R_{\mu\sigma\nu w}(\overline{Y}^\beta) R_{\alpha\sigma\beta w}(\overline{Y}^\beta) \right] (\sigma^{\mu\nu}\sigma^{\alpha\beta})(\xi) \right\} + O((\alpha')^{2+n}) \end{aligned} \tag{3}$$

$$\begin{aligned} G_{\mu\nu}(Y^\mu(\xi)) &= \delta_{\mu\nu} - \frac{(\alpha')}{3} R_{\alpha\mu\beta\nu}(\overline{Y})(X^\alpha X^\beta)(\xi) \\ &- \frac{1}{6}(\alpha')^{3/2}(\nabla_\alpha R_{\beta\mu\sigma\nu}(\overline{Y}^\beta)(X^\alpha X^\beta X^\sigma)(\xi) \\ &+ \frac{(\alpha')^2}{36}\left\{ \left[-18\nabla_\alpha\nabla_\beta R_{w\mu\sigma\nu} + 16 R_{\alpha\mu\beta\gamma} R_{w\nu\sigma\gamma} \right] (\overline{Y}^\beta) \right. \\ &\left. \times (X^\alpha X^\beta X^w X^\sigma)(\xi) \right\} + O((\alpha')^{2+n}). \end{aligned} \tag{4}$$

At this point let us re-write eq(1) in terms of the composite operator $\sigma^{\alpha\beta}(\xi) =$

$(X^{\alpha}X^{\beta})(\xi)$ by considering the identity insertion

$$\begin{aligned}
&\delta^{(F)}\big(\sigma^{\alpha\beta}(\xi)-(X^{\alpha}X^{\beta})(\xi)\big)\\
&=\int\left(\prod_{\xi\in R^2}d\lambda(\xi)\right)\exp\left\{i\int_{R^2}d^2\xi\sqrt{G(\overline{Y})}G_{x\beta}(\overline{Y})X\,\lambda(\xi)\right.\\
&\qquad\left.[\sigma^{x\beta}(\xi)-(X^{x}X^{\beta})(\xi)]\right\}\\
&\cong\exp\left\{-\langle\lambda\rangle\int_{R^2}d^2\xi\sqrt{G(\overline{Y})}\,G_{x\beta}(\overline{Y})[\sigma^{x\beta}(\xi)-(X^{x}X^{\beta})(\xi)\right\};
\end{aligned}\tag{5}$$

As a consequence we have the result

$$Z[G_{\mu\nu}(X^{\alpha})]=\prod_{\overline{Y}\in M}\widetilde{Z}[G_{\mu\nu}(\overline{Y})]\tag{6-a}$$

with

$$\begin{aligned}
&\widetilde{Z}[G_{\mu\nu}(\overline{Y})]=\int\prod_{\xi\in R^2}(\sqrt{G}\,G_{\mu\nu}(\overline{Y}))^{1/2}\,dX(\xi)\\
&\exp\left\{-\frac{1}{2\pi\alpha'}\int_{R^2}d^2\xi(\text{eq(3)})(\text{eq(4)})\partial_a X^{\mu}\partial^a X^{\nu}\right\}\\
&=\det{}^{-\frac{1}{2}}\left[\left(\delta_{\mu\nu}+\frac{\alpha'\langle\sigma\rangle}{3}R_{\alpha\mu\alpha\nu}(\overline{Y})+\frac{4}{9}(\alpha')^2\langle\sigma\rangle^2\big(R_{\alpha\mu\alpha\gamma}R^{\gamma}_{\beta\nu\beta}\big)(\overline{Y})\right.\right.\\
&\qquad\left.+\dots\big)(-\partial_a\partial^a)_{\xi}+\langle\lambda\rangle\delta_{\mu\nu}\right]
\end{aligned}\tag{6-b}$$

$$\begin{aligned}
&\times\exp\left\{-\frac{1}{2}\int_{R^2\times R^2}d^2\xi\,d^2\xi'\left[-\frac{(\alpha')^{3/2}}{12}(\nabla_{\alpha}R_{\mu\nu}(\overline{Y})+\dots\right]_{\mu\zeta}\right.\\
&\quad\left[\left(\delta_{\mu\nu}+\frac{\alpha'\langle\sigma\rangle}{3}R_{\alpha\mu\alpha\nu}+\dots\right)(-\partial_a\partial^a)_{\xi}+\langle\lambda\rangle\delta_{\mu\nu})\right]^{-1}_{\zeta\sigma'}(\xi,\xi)\\
&\quad\left.\times\left[-\frac{(\alpha')^{3/2}}{12}(\nabla_{\alpha}R_{\mu\mu})(\overline{Y})+\dots\right]_{\sigma'\mu}\right\}
\end{aligned}\tag{7}$$

where we have supposed another time the condensate formation for the bilinear field $\sigma^{\alpha\beta}(\xi)=\langle\sigma\rangle G^{\alpha\beta}(\overline{Y})$ and the implicity use of the saddle-point limit of $\langle\lambda\rangle\to\infty$ for the Lagrange multiplier.

At this point and for pedagogical purpose let us evaluate the following sample calculations of eq(7).

$$\begin{aligned}
&\lim_{\left\{\substack{\alpha'\to 0\\ \langle\lambda\rangle\to\infty}\right\}}\left\{\det{}^{-\frac{1}{2}}\left[\left(\delta_{\mu\nu}+\frac{\langle\sigma\rangle}{3}(\alpha')R_{\mu\alpha\nu\alpha}(\overline{Y})\right)(-\partial^2)_{\xi}+\langle\lambda\rangle\delta_{\mu\nu}\right]\right\}\\
&=\lim_{\left\{\substack{\alpha'\to 0\\ \langle\lambda\rangle\to\infty}\right\}}\det{}^{-\frac{1}{2}}\left[(-\partial^2)_{\xi}\,\delta_{\mu\nu}+\langle\lambda\rangle\left(\delta_{\mu\nu}-\frac{\langle\sigma\rangle}{3}\alpha' R_{\mu\alpha\nu\alpha}(\overline{Y})\right)\right]
\end{aligned}\tag{8}$$

Now one can see (details as exercise for our readers)

$$\begin{aligned}
&\log \det{}^{-\frac{1}{2}}\left[(-\partial^2)_\xi\,\delta_{\mu\nu}+\langle\lambda\rangle\left(\delta_{\mu\nu}-\frac{\langle\sigma\rangle}{3}\,\alpha'\,R_{\mu\alpha\nu\alpha}(\overline{Y})\right)\right]\\
&=\lim_{\substack{\alpha'\to 0\\ \langle\lambda\rangle\to\infty}}\int_\varepsilon^\infty \frac{dt}{t}\,e^{-t\langle\lambda\rangle}\cdot \mathrm{Tr}\exp\left\{-t\left[(-\partial^2)_\xi\,\delta_{\mu\nu}+\langle\lambda\rangle\left(\delta_{\mu\nu}-\frac{\langle\sigma\rangle}{3}\,\alpha'\,R_{\mu\alpha\nu\alpha}(\overline{Y})\right)\right]\right\}\\
&=\int_\varepsilon^\infty \frac{dt}{t}\,e^{-t\langle\lambda\rangle}\lim_{t\to 0^+}\mathrm{Tr}\exp\{-t\,[\text{above written operator}]\\
&\qquad\sim\sqrt{G(\overline{Y})}\left\{c_0(\varepsilon)\int_\varepsilon^\infty \frac{dt}{t}\,e^{-t\langle\lambda\rangle}\right\}\\
&-\sqrt{G(\overline{Y})}\,R(\overline{Y})\left\{\frac{\langle\lambda\rangle\langle\sigma\rangle\alpha'}{3}\,c_1(\varepsilon)\int_\varepsilon^\infty \frac{dt}{t}\,e^{-1\langle\lambda\rangle}\right\}+O((\alpha')^2). \qquad (9)
\end{aligned}$$

After inserting eq(9) into eq(6-a), we get as the leading limit of $\alpha' \to 0$ of the String Theory eq(1), the Einstein-Hilbert action with an effective cosmological constant and Newton Gravitation constant

$$\tilde{Z}[G_{\mu\nu}(\overline{Y})]=\exp\left\{-\mu^{\mathrm{eFF}}\int_M d\overline{Y}\,\sqrt{G(\overline{Y})}-\frac{1}{8\pi G_N^{\mathrm{eFF}}}\int_N d\overline{Y}\,\sqrt{G(\overline{Y})}\,R(\overline{Y})\right\} \qquad (10)$$

where $\left(A=\int_{R^2} d^2\xi\right)$

$$\mu^{\mathrm{eFF}}(\xi)\sim A\int_\varepsilon^\infty \frac{dt}{t^2}\,e^{-t\langle\lambda\rangle} \qquad (11)$$

$$\frac{1}{8\pi G_N^{\mathrm{eFF}}}\underset{(\varepsilon)}{\sim}A\left(\int_\varepsilon^\infty \frac{dt}{t}\,e^{-t\langle\lambda\rangle}\right)\left(\frac{\langle\lambda\rangle\langle\sigma\rangle\alpha'}{3}\right) \qquad (12)$$

If one consider the fluctuations of our metrical tensor $G_{\mu\nu}(\overline{Y})$ on M, one should consider a further path-integral on eq(1) as in Chapter 1.

At this point we leave as on exercise to our readers to evaluate next higher-order derivatives terms and to consider the Supersymmetric case in order to obtain Supergravity Theories.

Appendix 22.C. Nash Bosonization in Path Integral for Quantum Riemannian Geometry

Introduction

One of the most challenge mathematical problems in modern field theory is certainly the problem of choice of the correct dynamical variable to be quantized (or path integrated) in the theory of Random Geometry of metric fields in a given (fixed) manifold M. Several frameworks on the last decades have been proposed (see Chapters 1, 7), however without producing yet a consistent quantum field theoretic framework, useful to implement evaluations outside the usual (non-renormalizable) coupling constant perturbation Feynmann-Dhyson scheme.

In this Appendix C we intend to contribute for such a difficult problem of quantizing Quantum Gravitly by proposing as suitable variables to be quantized on phenomenological grounds, the field of the immersions applications of a given manifold of dimension n in a convenient ambient extrinsic Euclidean. Space R^d (with $d > n$): The famous Whitney & Nash imbeddings/immersion-embeddings theorems applied to our C^∞ space-time manifold M where the dynamics takes place. These ideas are proposed in thiss complementary appendix and can be considered as an approximate Bosonization of the usual metric variable theory in terms of "stress-strain" degrees of freedom associated to the Nash parametrization of the metric tensor.

We show the usefulness of this phenomenological path integral scheme for Quantum Riemannian Geometry, by evaluating straightforwardly the Classical Newton Potential by means of a Wilson Loop evaluation associated to a static trajectory of a pair of massive particle and quantum averaged in an effective induced quantum gravity dynamics of fermionic matter at the leading semi-classical limit of $c \to \infty$ (here c denotes the light-velocity parameter).

1 – Quantum Riemannian Geometry as a dynamics of bosonic quantum immersions and the Newton Gravitation law.

Let us start this section by recalling the Nash Theorem that asserts that every Riemannian metric in a C^∞-manifold M $\{g_{\mu\nu}(x)\}$ (a $C^2(M)$-tensor field) can be always obtained from an immersion $f^A\colon M \to R^{s(d)}$ ($f^A \in C^1(M)$ and rank $D_x f = d$) in a suitable Euclidean space $R^{s(d)}$, here the dimension of the Euclidean ambient space is strictly greater than d (a better lower bound is given by the inequalite $s(D) \geq 2d-1$) ([1])

$$g_{\mu\nu}(x) = \sum_{A=1}^{s(d)} \frac{\partial f_A}{\partial x^\mu}\frac{\partial f_A}{\partial x^\nu} = \frac{\partial f^A}{\partial x^\mu}\frac{\partial f_A}{\partial x^\nu} \tag{1}$$

We would thus expect that in this vectorial like bosonization all equations and path-integrals in Riemannian Geometry should acquires a more invariant and suitabe expressions for analysis. Let us thus set up some formulae related to this new metrical variable parametrization as pointed out by eq(1).

Let us consider the context of an effective scheme, where one should consider "lenght" scales appropriated for the governing quantum dynamics under analysis. In this context it appears important to consider already built in the formulae, the important non-relationship limit represented by the hypothesis of the analicity of the geometrical objects in relation to the inverse of light velocitye. As a consequence one should envisages an expansion in powers of $\frac{1}{c}$ for the Nash scalar immersion fields

$$f_A(x^\gamma) = x^\alpha \delta_A^\alpha + \sum_{\ell=1}^{\infty} \left(\frac{1}{c}\right)^\ell \varphi_A^\ell(x^\gamma). \tag{2}$$

The metrical vaiable takes the simple form at the leading $c \to \infty$ limit:

$$g_{\mu\nu}(x^\gamma) = \left(\delta_\mu^A + \frac{1}{c}\frac{\partial}{\partial x^\mu}\varphi_A^{(1)}\right)\left(\delta_A^\nu + \frac{1}{c}\frac{\partial}{\partial x^\nu}\varphi_A^{(1)}\right)$$
$$= \delta_{\mu\nu} + \frac{1}{c}\left(\frac{\partial}{\partial x^\mu}\varphi_\nu^{(1)} + \frac{\partial}{\partial x^\nu}\varphi_\mu^{(1)}\right)(x^\gamma) \tag{3}$$

$$g^{\mu\nu}(x^\gamma) = \delta_{\mu\nu} - \frac{1}{c}\left(\frac{\partial\varphi_\nu^{(1)}}{\partial x^\mu} + \frac{\partial}{\partial x^\nu}\varphi_\mu^{(1)}\right)(x^\gamma) \tag{2-b}$$

The Christofell connections are straightforwardly computed at this leading limit and take the very simple form

$$\Gamma^\mu_{\alpha\beta}(x) = \frac{1}{2}g^{\mu\gamma}\left(\frac{\partial}{\partial x^\alpha}g_{\beta\gamma} + \frac{\partial}{\partial x^\beta}g_{\alpha\gamma} - \frac{\partial}{\partial x^\gamma}g_{\alpha\beta}\right)$$
$$= \frac{1}{c}\frac{\partial^2\varphi_\mu^{(1)}(x^\gamma)}{\partial x^\alpha\partial x^\beta} + O\left(\frac{1}{c^2}\right) \tag{3}$$

The Riemann four-tensor is simply given by

$$R^\mu_{\gamma,\alpha\beta}(x) = \frac{1}{c^2}\left\{\frac{\partial^2\varphi_\mu^{(1)}}{\partial x^\alpha\partial x^{\gamma'}}\frac{\partial^2\varphi_{\gamma'}^{(1)}}{\partial x^\gamma\partial x^\beta} - \frac{\partial^2\varphi_\mu^{(1)}}{\partial x^\beta\partial x^{\gamma'}}\frac{\partial^2\varphi_{\gamma'}^{(1)}}{\partial x^\gamma\partial x^\alpha}\right\} + O\left(\frac{1}{c^4}\right), \tag{4-a}$$

which produces the following expression for the Ricci tensor

$$R_{\alpha\beta}(x) = R^\mu_{\alpha,\mu\beta} = \frac{1}{c^2}\left\{\frac{\partial^2\varphi_\mu^{(1)}}{\partial x^\mu\partial x^{\gamma'}}\frac{\partial^2\varphi_{\gamma'}^{(1)}}{\partial x^\alpha\partial x^\beta} - \frac{\partial^2\varphi_\mu^{(1)}}{\partial x^\beta\partial x^{\gamma'}}\frac{\partial^2\varphi_{\gamma'}^{(1)}}{\partial x^\alpha\partial x^\mu}\right\} + O\left(\frac{1}{c^4}\right), \tag{4-b}$$

and the associated scalar of curvature

$$R(x) = \left(g^{\beta\alpha}R_{\alpha\beta}\right)(\xi) = \frac{1}{c^2}\left[\frac{\partial^2\varphi_\mu^{(1)}}{\partial x^\mu\partial x^\gamma}\frac{\partial^2\varphi_\gamma^{(1)}}{\partial x^\beta\partial x^\beta} - \frac{\partial^2\varphi_\mu^{(1)}}{\partial x^\beta\partial x^\gamma}\frac{\partial^2\varphi_\gamma^{(1)}}{\partial x^\beta\partial x^\mu}\right] \tag{4-c}$$

At the quantum geometrical level the functional-path integral measure leads to the usual Feynman path integral measure as defined by the $c \to \infty$ leading Nash immersion fields $\{\varphi_\mu^{(1)}\}_{\mu=1,\dots,d}$ as one can see from the simple variable change written below

$$ds^2 = \int_M d^Dx\left\{\sqrt{g}\,\delta g_{ab}\left(g^{aa'}g^{bb'} + g^{ab}g^{a'b'}\right)\delta g_{a'b'}\right\}(x)$$
$$= \frac{1}{c^2}\int_M d^Dx\left[(\delta\varphi_\mu^{(1)})\left(-\frac{\partial^2}{\partial x^\mu\partial x^\nu}\right)(\delta\varphi_\nu^{(1)}\right](x) \tag{5}$$

and thus

$$d\mu[g_{\alpha\beta}] \cong D^F[\varphi_\mu^{(1)}] = \left\{\prod_{\mu=1}^{D}\left(\prod_{x\in M}d\varphi_\mu^{(1)}(x)\right)\right.$$
$$\times \det{}^{-\frac{1}{2}}\left[-\frac{2}{c^2}\frac{\partial^2}{\partial x^\alpha\partial x^\beta}\right]; \tag{6}$$

At this point is worthing call the reader attention that next $\frac{1}{c}$-corrections can be easily taken into account in the formulae above written generating now a fixed degree polinomial non-linearity on then and a non-trivial Faddev-Popov determinant in the new product Feymman measure eq(6) (Chapter 1).

We take as the weight for our Wilson Loop averages in our leading Nash fields a higher order Einstein-Hilbert action as given by the effective action obtained after integrating out a massive femionic matter field at the limit of large mass (Chapter 18)

$$\begin{aligned}
&\det\left[\not\partial + \Gamma^{\mu}_{\alpha\beta}\sigma^{\alpha\beta} + m\right]_{\substack{c\to\infty \\ m\to\infty}} \\
&\sim \lim_{c\to\infty}\left\{\exp\left\{-\frac{1}{2G^{\text{eff}}}\int_M d^{\nu}x\left[\sqrt{g}\,\Gamma^{\mu}_{\alpha\beta}\,g^{\alpha\alpha'}(-\Delta)g^{\beta\beta'}\,\Gamma^{\mu}_{\alpha'\beta'}\right](x)\right\}\right\} \\
&= \exp\left\{-\frac{1}{8\pi G^N}\int_M d^{\nu}x\left[\varphi^{(1)}_{\alpha}(-\Delta)^3\varphi^{(1)}_{\alpha}\right]\right\}.
\end{aligned} \tag{7}$$

Here G^N is the (somewhat effective) Newton Gravitation constant.

Let us deduce the Newton Gravitation Law from the above written formulae in terms of the Nash field.

In the Riemannian quantum geometry, the above written Holonomy factor defined by the $SO(d)$-valued vector field $\Gamma^{\mu}_{\alpha\beta}(x)\,\sigma^{\alpha\beta}$, here $\sigma^{\alpha\beta}$ are the generators of the $SO(D)$ Group (the Euclidean Lorentz Group) is expected to lead to the Newton law in the non-relativistic and dimension mean-field limits $D\to\infty$ evaluation of its quantum average for a static (non-fluctuating) trajectory

$$\begin{aligned}
\langle W[C_{(R,T)}]\rangle \sim \frac{1}{Z}\int D^F[\varphi^{(1)}_{\mu}(x)]\exp\left\{-\frac{1}{8\pi G_N}\int d^D x\left[\varphi^{(1)}_{\alpha}(-\Delta)^3\varphi^{(1)}_{\alpha}\right]\right\}(x) \\
\times\frac{1}{D}\,\mathrm{Tr}_{SO(D)}\left\{\mathbb{P}\left[\exp i\left(\oint_{C_{R,T)}}\Gamma^{\mu}(C(\sigma))\dot{C}_{\mu}(\sigma)d\sigma\right)\right]\right\}
\end{aligned} \tag{8}$$

here $\mathbb{P}$ is path $SO(D)$-indexes ordenation operator along the static trajectory $C_{(R,T)} = \{C_{\mu}(\sigma), 0\le\sigma\le T\}$ and given by the boundary of a rectange $\left\{-\frac{T}{2}\le x^0\le\frac{T}{2}, -\frac{R}{2}\le x^1\le\frac{R}{2}\right\}$.

The Newton gravitation potential should be given by the lowest quantum energy state associated to the quantum propagation of the gravitation interacting pair and it is given explicitly by the ergodic-temporal (non-relativistic) limit of eq(8)

$$V(R) = \lim_{T\to\infty} -\frac{1}{T}\,\ell g\left\{\langle W[C_{(R,T)}]\rangle\right\}. \tag{9}$$

In order to evaluate the quantum non-abelian Holonomy factor eq(8) at the Gravitaton mean field limit $D\to\infty$, as much as similar calculations done in Yang-Mills Theory ([4]), we write the Holonomy path ordered object as the one-dimensional fermion (Grasmanian variables) living on the contour $C_{(R,T)}$ (Chapters 1 and 4)

$$\frac{1}{D}\,\mathrm{Tr}_{SO(D)}\left\{\mathbb{P}\left[\exp i\oint_{C_{(R,T)}}\Gamma^{\mu}_{\alpha\beta}(C(\sigma))(\sigma^{\alpha\beta})\dot{C}_{\mu}(\sigma)\right]\right\}$$
$$\int_{\substack{\theta_\alpha(0)=\theta_\alpha(T)\\ \theta^*_\alpha(0)=\theta^*_\alpha(T)}}\prod_{\sigma\in[0,T]}(d\theta_\alpha(\sigma))(d\theta^*_\alpha(\sigma))\exp\left(\frac{i}{2}\int_0^T d\sigma\left(\theta^*_\alpha\frac{d}{d\sigma}\theta_\alpha+\theta_\alpha\frac{d}{d\sigma}\theta^*_\alpha)(\sigma)\right)\right.$$
$$\times\frac{1}{D}\left(\sum_{\alpha=1}^{D}(\theta_\alpha(\sigma)\theta^*_\alpha(T))\right)\times\exp\left[i\int_0^T d\sigma\big(\theta^*_\alpha(\sigma)\sigma^{\alpha\beta}\theta_\beta(\sigma)\big)\Gamma^{\mu}_{\alpha\beta}(C(\sigma))\dot{C}_{\mu}(\sigma)\right]$$

$$=\frac{1}{D}\,\mathrm{Tr}_{SO(D)}\left\{\exp\left(iM_{\mathrm{eff}}\left[\int_0^T d\sigma\,\Gamma^{\mu}(C(\sigma)\dot{C}_{\mu}(\sigma))\right]\right)_{\alpha\beta}\right\}$$
$$=\frac{1}{D}\,\mathrm{Tr}_{SO(D)}\left\{\exp i\,\frac{M_{\mathrm{eff}}}{C}\left[\int_0^T d\sigma\,\frac{\partial^2\varphi^{(1)}_{\mu}}{\partial x^{\alpha}\partial x^{\beta}}(C(\sigma))\dot{C}_{\mu}(\sigma)\right]\right\} \tag{9}$$

Here we have used the gravitational charge (mass) of our static pairs circulating around the loop $C_{(R,T)}$ throught a cumulant (leading order) evaluation of the Grassmanian variables. Namelly

$$M_{\mathrm{eff}}=\int_{\substack{\theta_\alpha(0)=\theta_\alpha(T)\\ \theta^*_\alpha(0)=\theta^*_\alpha(T)}}D^F[\theta_\alpha(\sigma)]D^F[\theta^*_\alpha(\sigma)]\left(\sum_{\alpha=1}^{D}\big(\theta_\alpha(\sigma)\theta^*_\alpha(T)\big)\right)\times\big(\theta^*_\alpha(\sigma)\sigma^{\alpha\beta}\theta_\beta(\sigma)\big). \tag{10}$$

As a consequence, one should expect that (at least for large dimensionality $D\to\infty$), the effective Holonomy Factor can be written as follow in the Fourier Space

$$W[C_{(R,T)}]=\exp\left\{iM_{\mathrm{eff}}\left[\int d^Dk\,\widetilde{\varphi}_\alpha(-k)k^2\,j_\alpha(k,C_{(R,T)})\right]\right\}+O\left(\frac{1}{D}\right). \tag{11}$$

Here the Fourier Transformed scalar immersion Nash field $\varphi^{(1)}_{\mu}$ is explicitly given by

$$\widetilde{\varphi}_\alpha(-k)=\frac{1}{(2\pi)^{D/2}}\int d^0k\,e^{ik_\beta x_\beta}\,\varphi^{(1)}_\alpha(x); \tag{12}$$

We have used the dimensional regularization rule of Bollini-Giambiagi for handling the $SO(D)$ indexes inside the ordinary integrals $k_\alpha k_\beta=\dfrac{k^2}{D}\delta_{\alpha\beta}$ and the contour form factor inside eq(11) is given explicitly by

$$j_\alpha(k,C_{(R,T)})=\frac{1}{D}\left[\oint_{C_{(R,T)}}e^{-ik_\mu C_\mu(\sigma)}\dot{C}_\mu(\sigma)d\sigma\right]. \tag{13}$$

After inserting all the above results into our effective sixth-order Gaussian path-integral eq(8), one obtains the following expression for the Newton potential in our Bosonized-metric framework of Nash immersions for Quantum Phenomenological Gravity

$$V(R)=\lim_{T\to\infty}\left\{-\frac{M_{\mathrm{eff}^2}}{T}\left[\int\frac{d^Dk}{(2\pi)^D}\,\frac{|j_\alpha(k,C_{(R,T)})|^2}{k^2}\right]\right\}. \tag{14}$$

This potential can be explicitly evaluated (Chapter 2) and leading to the Newton Law of Gravitation in this phenomenological scheme for quantizing Riemann metric fields

$$V(R) = -\left(4\pi |M_{\text{eff}}|^2 G_N \cdot \frac{1}{R}\right). \tag{15}$$

References

[1] Masahisa Adache - *Embeddings and Immersions*, vol. 24, Translations of Mathematics Monographs, American Mathematical Society, Providence, Rhode Island, 1993.

Appendix 22.D. The Eigenvalue Problem for Diffusion Equation in Loop Spaces: Elementary Comments

Let us consider the following eigenvalue problem for the Diffusion Equation in a given Hilbert (separable) space H to be solved in $L^2(H, d_Q u)$:

$$\text{Tr}_H \left[Q D^2 U_\lambda(x) \right] = -\lambda U_\lambda(x)\,. \tag{2a}$$

It is straightforward to see that all eigenvalues of the positive definite trace class operator Q, satisfies eq(1) with the Hilbert Space (Infinite-Dimensional) Plane Waves, of the special form given below. Namely

$$\text{Tr}_H \left[Q D^2 \left(e^{i\langle P_n, x\rangle_H} \right) \right] = -\lambda_n e^{i\langle P_n, x\rangle_H}. \tag{2b}$$

At this point one can add perturbation terms of the following forms:

a) $$V(x) = \int_H d_Q\mu[q]\, F(q)\, e^{i\langle q,x\rangle_H} \tag{3}$$

b) As in the explictly case of $H = L^2_{\text{periodic}}([0,2\pi])$ (Loop Space) one may considere the self-avoiding intersection useful in Polymer Theory of Chapter 20

$$V(x) = \int_0^{2\pi} d\sigma \int_0^{2\pi} d\sigma' V_0\left(|x(\sigma) - x(\sigma')|^2\right) \tag{4}$$

with $V_0(x) \in C_c(R)$, a positive funciton of compact support in R; and now trying to evaluate by the usual Rayleigh-Schörindger perturbation series framework the eigenvalues and eigenfunctionals of the perturbed Diffusion Equation bellow

$$\text{Tr}_H (Q D^2 U_\lambda(x)] + V(x) U_\lambda(x) = -\lambda U_\lambda(x). \tag{5}$$

Finally let us point out that the usual Gaussian Functional in $L^2(\Omega)$ defined by a symmetric kernel $K(y,y') = K(y',y) \in L^\infty(\Omega \times \Omega)$ and associated to a positive definite trace class operator

$$\overline{\Psi}[f] = \exp\left\{ -\frac{1}{2} \int_\Omega d^\nu y \int_\Omega d^\nu y'\, f(y)\, K_\Omega(y,y')\, f(y') \right\} \tag{6}$$

satisfies the following Poisson like functional equation in a space of finite volume Ω

$$\int_{\Omega\times\Omega} d^\nu x d^\nu z K_\Omega^{-1}(x,z)\,\frac{\delta^2}{\delta f(x)\,\delta f(z)}\,\overline{\Psi}[f] = (-\operatorname{vol}(\Omega))\overline{\Psi}[f] + \int_{\Omega\times\Omega} d^\nu y d^\nu y'\, f(y) K_\Omega(y,y')\, f(y')\,\overline{\Psi}[f] \tag{7}$$

Let us now pass to the problem of solving the functional Schörindger wave equation in $L^2(L^2(\Omega), d_K\mu(f))$ below written

$$i\frac{\partial}{\partial t}\psi[f,t] = \left(\int_{\Omega\times\Omega} d^\nu x d^\nu x' K^{-1}(x,x')\,\frac{\delta^2}{\delta f(x)\,\delta f(x')}\,\psi[f,t]\right) - \left(\int d^\nu y d^\nu y'\, f(y) K(y,y')\, f(y')\psi[f,t]\right) \tag{8-a}$$

$$\psi[f,0] = \Omega[f]. \tag{8-b}$$

By applying perturbation methods, we have the following result at the first perturbative order

$$E_n = -\big(\lambda_n^{(0)} + \varepsilon\lambda_n^{(1)}\big) + O(\varepsilon^2) \tag{9-a}$$

$$\psi_n[f] = \psi_n^{(0)}[f] + \varepsilon\,\psi_n^{(1)}[f] + O(\varepsilon^2) \tag{9-b}$$

Here

$$\lambda_n^{(1)} = -\left\{\int d^\nu y d^\nu y' K(y,y')\left[\int d_K\mu[f]\, f(y)\, f(y')\right]\right\} = -1 \tag{9-c}$$

$$\psi_n^{(0)}[f] = \exp\left(i\int_\Omega f(x)\, g_n^{(0)}(x)\, dx\right) \tag{9-d}$$

$$\psi_n^{(1)}[f] = \sum_m C_{nm}^{(1)}\psi_m^{(0)}[f] \tag{9-e}$$

$$C_{nm}^{(1)} = \frac{1}{\lambda_n^{(0)} - \lambda_n^{(1)}}\left\{-\int d^\nu y d^\nu y'\left[\int d_k u(f)\, f(y) f(y') \times \exp i\left(\int d^\nu x (g_n^{(0)} - g_m^{(1)})(x)\, f(x)\right)\right]\right\} \tag{9-f}$$

As usual, one should consider the ansatz for the full wave functional

$$\psi[f,t] = \sum_{\{n\}} C_n e^{iE_n t}\,\psi_n[f] \tag{10}$$

with the coefficients C_n adjusted from the initial date. Calculations are left as exercise to our readers.

On the basis of the mathematical (rigorous) results presented in the Chapter 20, one can see that the correct framework to solve functional Schörindger equations in Quantum Field Theory as exposed in Chapters 9-11 is to consider the "regularized" form below with cut-offs $\Lambda^2 > 0$ and $\alpha = 1+\varepsilon$ takes as an example of a $\lambda\varphi^4$-scalar field theory

$$\begin{aligned}\frac{\partial\psi[\varphi,t]}{\partial t} &= \mathrm{Tr}_H\left\{-\int_{\Omega\times\Omega} d^\nu y\, d^\nu y' (-\Delta+\Lambda^2)^{-\alpha}(y,y')\,\frac{\delta^2}{\delta\varphi(y)\delta\varphi(y')}\,\psi_{(\Lambda,\varepsilon)}[\varphi,t]\right\}\\ &+\mathrm{vol}(\Omega)\psi_{(\Lambda,\varepsilon)}[\varphi(y)] - \left\{\int_{\Omega\times\Omega} d^\nu y |\nabla\varphi|^2(y)\right\}\psi_{(\Lambda,\varepsilon)}[\varphi,t]\\ &-\left(m^2_{\mathrm{bare}}+\Lambda^2\right)\left(\int_{\Omega\times\Omega} d^\nu y\,\varphi^2(y)\right)\psi_{(\Lambda,\varepsilon)}[\varphi,t]\\ &+\lambda_{\frac{\mathrm{bare}}{4!}}\left(\int_{\Omega\times\Omega} d^\nu y\,\varphi^4(y)\right)\psi_{(\Lambda,t)}[\varphi,t] \qquad (11)\end{aligned}$$

Again, extensive calculations of solutions for eq(11) in the space of Euclidean $\lambda\varphi^4$-quantum field functionals $L^2(L^2(\Omega), d_{(-\Delta+m^2)^{1/2}}u(\varphi)) = \mathcal{V}$ will be left to the inquires of our mathematically oriented readers.

Appendix 22.E.
Some Calculations of the Q.C.D. Fermion Functional Determinant in Two-Dimensions and (Q.E.D.)$_2$ solubility

Let us firstly define the functional determinant of a self-adjoint, positive definite operator A (without zero modes) by the proper-time method

$$\log \det_F(A) = -\lim_{\varepsilon\to 0^+}\left\{\int_\varepsilon^\infty \frac{dt}{t}\,\mathrm{Tr}_f(e^{-tA})\right\} \qquad (1)$$

where the subscript f remainds us of the functional nature of the objects under study and so its trace.

It is thus expected that the definition eq(1) has divergents counter terms as $\varepsilon \to 0^+$, since $\exp(-tA)$ is a class trace operator only for $t \geq \varepsilon$. Asymptotic expressions at the short-time limit $t \to 0^+$ are well-known in mathematical literature (see Appendix E of Chapter 1). However this information is not useful in a first sight of eq(1) since one should know $\mathrm{Tr}_F(e^{-tA})$ for all t-values in $[\varepsilon,\infty)$.

An useful remark on the exactly evaluation is in the case where the operator A is of the form $A = B + m^2\mathbf{1}$ and one is mainly interested in the effective asymptotic limit of large mass $m^2 \to \infty$. In this particular case, one can use a Saddle-Point analysis of the expression in eq(1)

$$\lim_{m^2\to\infty}[\log \det_F(B+m^2\mathbf{1})] = -\lim_{\varepsilon\to 0}\left\{\int_\varepsilon^\infty \frac{dt}{t}\,e^{-m^2 t}\left[\lim_{t\in 0^+}\mathrm{Tr}_F(e^{-tB})\right]\right\} \qquad (2)$$

The above effective evaluation has been used extensively in Chapters 10, 18 and in the previous supplementary appendixes A, B.

Another very important case is covered by the (formal) Schwarz-Romanov Theorem announced below (see Chapter 17).

Theorem 1. Let $A(\sigma)$ be an one-parameter family of positive-definite self-adjoints operators and satisfying the parameter derivative condition $(0 \leq \sigma \leq 1)$

$$\frac{d}{d\sigma}A(\sigma) = f\,A(\sigma) + A(\sigma)\,g \tag{3-a}$$

where f and are σ-independents objects (may be operators).

Then we have the explicitly result

$$\log\left(\frac{\det_F(A(1))}{\det_F(A(0))}\right) = \\ = \left\{\int_0^1 d\sigma \lim_{\varepsilon\to 0^+} \mathrm{Tr}_F\left[f\,e^{-\varepsilon(A^{(\sigma)})^2}\right] + \int_0^1 d\sigma \lim_{\varepsilon\to 0^+} \mathrm{Tr}_F\left[q\,e^{-\varepsilon(A^{(\sigma)})^2}\right]\right\} \tag{3-b}$$

The proof of the equation (3) is based on the validity of the differential equation in relation to the σ-parameter

$$\frac{d}{d\sigma}\left[\log \det_F(A(\sigma))^2\right] = \lim_{\varepsilon\to 0^+} 2\left\{\mathrm{Tr}_F\left(f\,e^{-\varepsilon(A(\sigma))^2}\right) + \mathrm{Tr}_\varepsilon\left(\mathcal{G}\,e^{-\varepsilon(A(\sigma))^2)}\right)\right\} \tag{4}$$

which can be seen from the obvious calculations written down in the above equation

$$\left[\log \det_F(A(\sigma))^2\right] = \\ = -\lim_{\varepsilon\to 0^+}\left\{\int_\varepsilon^\infty dt\ \mathrm{Tr}_F\left[(fA(\sigma)+A(\sigma)g)A(\sigma)+A(\sigma)(fA(\sigma)+A(\sigma)g)\right.\right. \\ \left.\left.\left(-(A(\sigma))^{-2}\right)\frac{d}{dt}\left(\exp(-t(A(\sigma))^2)\right)\right]\right\} \tag{5}$$

$=$ eq(4).

Let us apply the above formulae in order to evaluate the functional determinant of the "Chrially transformed" self-adjoint Dirac operator in a two-dimensional space-time

$$D(\sigma) = \exp(\sigma\gamma^5\varphi^2(x)\lambda_a)|\partial|(\exp(\sigma\gamma^5\varphi^a(x)\lambda_a). \tag{6}$$

Here the Chiral Phase $W[\psi]$ in eq(6) takes value in $SU(N)$ for instance.

One can see that

$$(\not{D}(\sigma))^2 = -(\partial_\mu + iG_\mu(\sigma))^2\mathbf{1} - \frac{1}{4}[\gamma_\mu,\gamma_\nu]F_{\mu\nu}(-i\,G_\mu(\sigma)) \tag{7-a}$$

with the Gauge Field

$$\gamma_\mu G_\mu = \gamma_\mu(W^{-1}\partial_\mu W) \tag{7-b}$$

The asymptotics of the operator eq(7-a) are easily evaluated (see Chapter 1 - Appendix E)

$$\lim_{\varepsilon\to 0^+} \mathrm{Tr}_F\left(\exp(-\varepsilon(D(\sigma))^2)\right) \\ = \lim_{\varepsilon\to 0^+} \mathrm{Tr}\left\{\frac{1}{4\pi\varepsilon}\left\{\mathbf{1} + \frac{g(i\varepsilon_{\mu\nu}\gamma_5)}{2}F_{\mu\nu}(-i\,G_\mu(\sigma))\right\}\right\} \tag{8-a}$$

where we have used the Seeley expansions below for the square of the Dirac operator in the presence of a non-abelian connection

$$A\colon C_0^\infty(R^2) \to C_0^\infty(R^2) \tag{8-b}$$

$$A\varphi = \left((-\Delta) - V_1(\xi_1,\xi_2)\frac{\partial}{\partial\xi_1} - V_2(\xi_1,\xi_2)\frac{\partial}{\partial\xi_2} - V_0((\xi_1,\xi_2)\right)\varphi \tag{8-c}$$

$$\begin{aligned}\lim_{t\to 0^+}\left\{\mathrm{Tr}_{C_c^\infty(R^2)}(e^{-tA})\right\} = \frac{\mathbf{1}}{4\pi t} + \Bigg\{\left(-\frac{1}{8\pi}\left(\frac{\partial}{\partial\xi_1}V_1 + \frac{\partial}{\partial\xi_2}V_0\right)\right) \\ -\frac{1}{16\pi}(V_1^2+V_2^2) - \frac{1}{4\pi}V_0\Bigg\}(\xi_1,\xi_2) + O(t)\end{aligned} \tag{8-d}$$

$$\begin{aligned}(\not\partial - ig\not G_\mu)^2 = (-\partial^2)_\xi + (2ig\,G_\mu\partial_m u)_\xi \\ + \left[ig(\partial_\mu G_\mu) + \frac{ig\sigma^{\mu\nu}}{2}F_{\mu\nu}(G) + g^2 G_\mu^2\right]_\xi\end{aligned} \tag{8-e}$$

Which leads to the following exactly integral (non-local) representation for the non-Abelian Dirac Determinant

$$\log\left\{\frac{\det_F(D(1))}{\det_F(\partial)}\right\} = \frac{i}{2\pi}\int d^2x\,\mathrm{Tr}_{SU(N)}\left\{\varphi_a(x)\lambda^a \times \left[\int_0^1 d\sigma\,\mathcal{E}_{\mu\nu}F^{\mu\nu}(-iG_\mu(\sigma))\right]\right\} \tag{9}$$

A more invariant expression for eq(9) can be seen by considering the decomposition of the "$SU(N)$ gauge Field" $G_\mu(\sigma)$ in terms of its vectorial and axial components:

$$(W^{-1}(\sigma)\,\partial_\mu W(\sigma)) = V_\mu(\sigma) + \gamma^5 A_\mu(\sigma) \tag{10}$$

or equivalently $(\varepsilon_{\mu\nu}\gamma_n u, \gamma_5 = it_{\mu\nu}Y_\nu, \gamma_s = i\gamma_0\gamma_1, [\gamma_\mu,\gamma_\nu] = -2i\mathcal{E}_{\mu\nu}\gamma_5)$:

$$G_\mu(\sigma) = V_\mu(\sigma) + i\mathcal{E}_{\mu\nu}A_\nu(\sigma). \tag{11}$$

At this point we point out the formulae

$$F_{\mu\nu}(-iG_\mu(\sigma)) = \left\{(iD_\alpha^V(\sigma)A_\alpha(\sigma))\mathcal{E}^{\mu\nu} - [A_\mu(\sigma),A_\nu(\sigma)] + F_{\mu\nu}(V_\mu(\sigma)).\right\} \tag{12}$$

Here

$$D_\alpha^V A_\beta = \partial_\alpha A_\beta + [V_\alpha, A_\beta]. \tag{13}$$

Note that $A_\mu(\sigma)$ and $V_\mu(\sigma)$ are not independents fields since the Chiral Phase $W(\sigma)$ satisfies the integrability condition

$$F_{\mu\nu}(W\,\partial_\mu W) \equiv 0 \tag{14}$$

or equivalently

$$F_{\mu\nu}(V_\beta(\sigma)) = -[A_\mu(\sigma),A_\nu(\sigma)] \tag{15}$$

$$D_\mu^V A_\nu(\sigma) = D_\nu^V A_\mu(\sigma). \tag{16}$$

After substituting eqs(12)-(16) in eq(9), one obtains the result

$$\log\left\{\frac{\det_F(D(1))}{\det_F(\partial)}\right\}$$
$$=\frac{i}{2\pi}\left\{\int d^2x\,\mathrm{Tr}_{SU(N)}\left(\lambda^a\phi_a(\alpha)\times\int_0^1 d\sigma(2iD_\mu(\sigma)A_\mu(\sigma)\right.\right.$$
$$\left.+\,\mathcal{E}_{\mu\nu}\big(\overbrace{F_{\mu\nu}(V_\alpha(\sigma))}^{-[A_\mu(\sigma),A_\nu(\sigma)]}\big)-[A_\mu(\sigma),A_\nu(\sigma)]\right) \tag{17}$$
$$=\frac{i}{2\pi}\left\{\int d^2x\,\mathrm{Tr}_{SU(N)}\left(\lambda^a\phi_a(x)\int_0^1 d\sigma(2iD_\mu(\sigma)A_\mu(\sigma))\right)\right\}$$
$$+\frac{i}{2\pi}\left\{\int d^2x\,\mathrm{Tr}_{SU(N)}\left(\lambda^a\phi_a(x)\Big(\int_0^1 d\sigma(-2[A_\mu(\sigma),A_\nu(\sigma)])\Big)\right)\right\}$$
$$=I_1(\phi)+I_2(\phi). \tag{18}$$

Let us show now that the term $I_1(\phi)$ is a mass term for the physical Gauge Field $A_\mu(\sigma=1)=A_\mu$.

Firstly we observe the result

$$\mathrm{Tr}_{\mathrm{Dirac}}\otimes\mathrm{Tr}_{SU(N)}\left\{\gamma^5A_\mu(\sigma)\overbrace{\frac{d}{d\sigma}(W\partial_\mu W)(\sigma)}^{L_\mu(\sigma)}\right\}$$
$$=\mathrm{Tr}_{\mathrm{Dirac}}\otimes\mathrm{Tr}_{SU(N)}\left\{\gamma^5\phi_a(x)\lambda^a\left(\gamma^5(\partial_\mu A_\mu(\sigma))-[\gamma^5A_\mu(\sigma),L_\mu(\sigma)]\right)(x)\right\}$$
$$=2\,\mathrm{Tr}_{SU(N)}\left\{\lambda_a\phi^a(x)\left(\partial_\mu A_\mu(\sigma)+[V_\mu(\sigma),A_\mu(\sigma)](x)\right)\right\}$$
$$=2\;\mathrm{Tr}_{SU(N)}\left\{\lambda_a\phi_a(x)D_\mu(\sigma)A_\mu(\sigma)\right\}=I_1(\phi). \tag{19}$$

By the other side

$$\mathrm{Tr}_{\mathrm{Dirac}}\otimes\mathrm{Tr}_{SU(N)}\left\{\gamma^5A_\mu(\sigma)L_\mu(\sigma)\right\}$$
$$=\mathrm{Tr}_{\mathrm{Dirac}}\otimes\mathrm{Tr}_{SU(N)}\left\{\gamma^5A_\mu(\sigma)\left(\frac{d}{d\sigma}V_\mu(\sigma)+\gamma_5\frac{d}{d\sigma}A_\mu(\sigma)\right)\right\}$$
$$=\mathrm{Tr}_{SU(N)}\left(\frac{d}{d\sigma}(A_\mu(\sigma)A_\mu(\sigma))\right). \tag{20}$$

At this point it is worth to see the appearance of a dynamical Higgs mechanism for Q.C.D. in two-dimensions.

Let us now analyse the second term $I_2(\phi)$ in eq(18)

$$\exp\left\{-\frac{i}{2\pi}\int d^2x\,\mathrm{Tr}_{SU(N)}\left(\int_0^1 d\sigma\,\mathcal{E}_{\mu\nu}(2\phi_a(x)\lambda^a[A_\mu(\sigma),A_\nu(\sigma)](x))\right)\right\}$$
$$=\exp\left\{-\frac{i}{2\pi}\int d^2x\,\mathrm{Tr}_{\mathrm{Dirac}\otimes SU(N)}\left(\int_0^1 d\sigma(\overbrace{\gamma_5\gamma_5}^{1}\mathcal{E}_{\mu\nu}\phi_a(x)\lambda^a\right.\right.$$
$$\left.\left.\times\left[\gamma_5W^{-1}(\sigma)\partial_\mu W(\sigma)-\gamma_5V_\mu(\sigma),\gamma_5W^{-1}(\sigma)\partial_\nu W(\sigma)-\gamma_5V_\nu(\sigma)\right]\right)\right\} \tag{21}$$

Since we have the identity as a consequence of the fact that $[\sigma\gamma_5\,\phi_a(x)\lambda^a, \gamma_5\,\phi_b(x)\lambda^b] = \sigma(\phi_a(x)\phi_b(x)[\lambda^a,\lambda^b] \equiv 0$;

$$W^{-1}(\sigma)\,\frac{\partial}{\partial\sigma}W(\sigma) = \gamma_5\,\phi_a(x)\lambda^a, \tag{22}$$

we can see the appearance of a term of the form of a Wess-Zumino-Novikov topological functional for the Chiral Group $SU(N)$, namelly

$$\begin{aligned} &I_2(\phi) = \exp\left\{ -\frac{i}{2\pi}\int d^2x\,\mathrm{Tr}_{\mathrm{Color}\otimes\mathrm{Dirac}} \right. \\ &\left.\left(\mathcal{E}_{\mu\nu}\gamma_5\,W^{-1}(\sigma)\,\frac{\partial}{\partial\sigma}W(\sigma)[\gamma_5\,W^{-1}(\sigma)\partial_\mu W(\sigma),\gamma_5\,W^{-1}(\sigma)\partial_\nu W(\sigma)]\right)\right\} \\ &+\ \text{terms}\ (\phi,V_\mu), \end{aligned} \tag{23}$$

which after the one-point compactification of the space-time to S^3 and considering only smooths phases $(\phi(x) \in C^\infty(S^3))$ one can see that the Wess-Zumino-Novikov functional is a homotopical class invariant. For the Closed Ball $S^3 \times [0,1] = (\{\bar{x} \equiv (x^1,x^2,\sigma)\})$

$$\begin{aligned} &\int_{S^3\times[0,1]} d^3\bar{x}\,\mathrm{Tr}_{SU(N)_{\mathrm{axial}}}\Big\{(\gamma_5(W^{-1}\,\partial_\alpha W)(\bar{x})) \\ &\qquad (\gamma_5(W^{-1}\,\partial_\mu W)(\bar{x})(\gamma_5\,W^{-1}\partial_\nu W))(\bar{x})\Big\} = \alpha\pi n \end{aligned} \tag{24}$$

with α an over all factor and $n \in \mathbb{Z}^+$.

Finally let us call our readers attention that the Dirac operator in the presence of a Non-Abelian $SU(N)$ Gauge Field $A_\mu(x) = A_\mu^a(x)\lambda_a$, can always be re-written in the "Chiral Phase" in the so called Roskies Gauge Fixing

$$i\gamma^\mu(\partial_\mu - gG_\mu) = e^{i\gamma_5\widetilde{\phi}^a(x)}(i\gamma^\mu\partial_\mu)e^{i\gamma_5\widetilde{\phi}^a(x)} = \widetilde{W}[\phi](i\gamma^\mu\partial_\mu)\widetilde{W}[\phi]. \tag{25}$$

Here

$$\widetilde{W}[\phi] = e^{i\gamma_5\widetilde{\phi}^a(x)} \equiv \mathbb{P}_{\mathrm{Dirac}}\left\{\mathbb{P}_{SU(N)}\ e^{i\gamma_5\int_{-\infty}^{x} d\xi^\mu(\varepsilon_{\mu\nu}G\nu)(x)}\right\} \tag{26}$$

Since

$$(\gamma_\mu\,G_\mu)(x) = +\gamma_\mu(\widetilde{W})\partial_\mu(\widetilde{W})^{-1}(x) \tag{27}$$

It is worth now to use the formualism of Invariant Functional Integration – Appendix Chapter 1 to change the quantization variables of the Gauge Field $A_\mu(x)$ to the $SU(N)$-axial phases $\widetilde{W}(\phi)$. This task is easily accomplished through the use of Riemannian functional metric on the manifold of the Gauge connections

$$\begin{aligned} dS^2 &= \int d^2x\,\mathrm{Tr}_{SU(N)}(\delta\,G_\mu\,\delta\,G^\mu)(x) \\ &= \frac{1}{4}\int d^2x\,\mathrm{Tr}_{SU(N)\otimes\mathrm{Dirac}}[(\gamma_\mu\,\delta G_\mu)(\gamma^\mu\,\delta G^\mu)](x) \\ &= \det{}_F[\not{D}\not{D}^*]_{\mathrm{adg}} \times \left\{\int d^2x\,\mathrm{Tr}_{SU(N)_{\mathrm{Axial}}}[(\delta\widetilde{W}W^{-1})(\delta\widetilde{W}W^{-1})]\right\}, \end{aligned} \tag{28}$$

since

$$\gamma_\mu(\delta G_\mu) = \gamma_\mu \left\{ \partial_\mu(\delta\widetilde{W})\widetilde{W}^{-1} + \partial_\mu \widetilde{W}(-\widetilde{W}^{-1}(\delta\widetilde{W})\widetilde{W}^{-1}) \right\}$$
$$= \gamma_\mu(\partial_\mu - [G_\mu,])(\delta\widetilde{W}\widetilde{W}^{-1})$$

and

$$[\gamma_\mu, \delta\widetilde{W}\widetilde{W}^{-1}] = \delta\widetilde{W}\{\gamma_\mu, \widetilde{W}^{-1}\} + \{\gamma_\mu, \delta\widetilde{W}\}\widetilde{W}^{-1} \equiv 0 \tag{29}$$

and leading to new parametrization for the Gauge Field measure

$$D^F[G_\mu(x)] = \left[\det_{F,\mathrm{adj}}(\not{D}\not{D}^*) \right]^{\frac{1}{2}} D^{\mathrm{Haar}}[\widetilde{W}(x)]. \tag{30}$$

Here $\det_{F,\mathrm{adj}}(\not{D}\,\not{D}^*)^{\frac{1}{2}}$ is the functional Dirac Operator in the presence of the Gauge Field and in the adjoint $SU(N)$-representation. Its explicitly evaluations is left to our readers.

The full Gauge-Invariant Expression for the Fermion Determinant is conjectured to be given on explicitly integration of the Gauge parameters considered now as dynamical variables in the Gauge-fixed result. For instance in the Abelian Case and in the Gauge Fixed Roskies Gauge eq(6) result, we have the Schwinger result,

$$\begin{aligned}
\det_F[i\gamma^\mu(\partial_\mu - ieA_\mu)] &= \frac{1}{2}\int D^F[W(x)] \\
&\times \exp\left\{ -\frac{e^2}{\pi}\int d^2x \, \frac{1}{2}\,(A_\mu - \partial_\mu W(x))^2 \right\} \\
&= \int D^F[W(x)] \exp\left\{ -\frac{e^2}{2\pi}\int d^2x \big[A_\mu^2 + (\partial_\mu W)^2 + 2A_\mu \partial_\mu W\big] \right\} \\
&= \exp\left\{ -\frac{e^2}{\pi}\int d^2x \left[A_\mu \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{(-\partial^2)} \right) A_\nu \right](x) \right\}.
\end{aligned} \tag{31}$$

This result generalized to the $SU(2)$ in an approximate form case has been used in Chapter 15 - footnote [12].

Note that the Haar measure on the Abelian Group $U(1)$ is

$$\begin{aligned}
\delta S_W^2 &= \int d^2x \big[\delta\big(e^{iW}\partial_\mu e^{-iW}\big)\delta\big(e^{iW}\partial_\mu e^{-iW}\big)\big](x) \\
&= \int d^2x \big(\delta W(-\partial^2)\delta W\big)(x)
\end{aligned} \tag{32}$$

Let us solve exactly the two-dimensional Quantum Electrodynamics.

Firstly, the equation (10) takes the simple form in term of the chiral phase in the Roskies Gauge

$$G_\mu(x) = \big(\varepsilon_{\mu\nu}\,\partial_\nu\,\phi\big)(x). \tag{33}$$

Now the somewhat cumbersome non-abelian eq(3) has a straightforward form in the Abelian case

$$D^F[G_\mu] = \det_F(-\Delta)\,D^F[\phi] \tag{34}$$

and we have thus the exactly soluble expression for the (Q.E.D.)$_2$-Generating Functional (a non-gauge invariant object!)

$$
\begin{aligned}
Z[J_\mu,\eta,\overline{\eta}] = \frac{1}{2}\Bigg\{ \int D^F[\phi]\, D^F[\chi]\, D^F(\overline{\chi}] \\
\exp\left\{ -\frac{1}{2}\int d^2x \left[\phi\left(-\partial^4 + \frac{e^2}{\pi}\partial^2\right)\phi + \mathcal{E}_{\mu\nu}(\partial_\nu J_\mu)\phi\right](x)\right\} \\
\times \exp\left\{ -\frac{1}{2}\int d^2x(\chi,\overline{\chi}) \begin{bmatrix} 0 & i\not\partial \\ i\not\partial & 0 \end{bmatrix} \begin{pmatrix} \chi \\ \overline{\chi} \end{pmatrix} \right\} \\
\exp\left\{ -\int d^2x(\chi,\overline{\chi}) \begin{bmatrix} e^{-ig\gamma_5\phi} & 0 \\ 0 & e^{-ig\gamma_5\phi} \end{bmatrix} \begin{pmatrix} \eta \\ \overline{\eta} \end{pmatrix} \right\} \Bigg\}
\end{aligned}
$$

For instance, correlations functions are exactly solved and possessing an "coherent state" factor given by the ϕ-average below (after "normal ordenation" at the coincident points) and explicitling given a proof of the confinement of the fermionic fields since one can not assign LSZ-scattering fields configuration for them by the Coleman Theorem since then grown as the factor $|x-y|^{\frac{1}{2g^2}}$ at large separation distance

$$
\begin{aligned}
&\left\langle e^{-i(\gamma_5)_x\phi(x)}\, e^{-i(\gamma_5)_y\phi(y)} \right\rangle_\phi = \\
&= \langle\cos(\varphi(x))\,\cos(\varphi(y))_\phi\, \mathbf{1}_x \otimes \mathbf{1}_x \\
&+ \gamma_s \otimes \mathbf{1}\, \langle \text{sen}(\varphi(x))\, \cos(\varphi(y))\rangle_\phi \\
&+ \gamma_s \otimes \mathbf{1}\, \langle \cos(\varphi(x))\, \text{sen}(\varphi(y))\rangle_\phi \\
&- \gamma_s \otimes \gamma_s\, \langle \text{sen}(\varphi(x))\, \text{sen}(\varphi(s))\rangle_\phi \\
&= e^{-g^2\left[\left(-\partial^2+\frac{g^2}{\pi}\right)^{-1} - (-\partial^2)^{-1}\right](x,y)} \\
&= \exp\left\{ +\frac{\pi}{g^2}\left(\frac{1}{2\pi}K_0\left(\frac{g}{\sqrt{\pi}}|x-y|\right) + \frac{1}{2\pi}\ell g|x-y|\right\}(\mathbf{1}_x \otimes \mathbf{1}_y). \qquad (35)
\end{aligned}
$$

The 2-point function for the 2D-Electromgnetic field shows clearly the presence of a massive excitation (Fotons have acquired a mass term by dynamical means)

$$
\begin{aligned}
\left\langle G_\mu(x)G_\nu(g)\right\rangle_\phi &= \int \frac{d^2k}{(2\pi)}(\mathcal{E}_{\mu\alpha}\mathcal{E}_{\nu\beta})(k^\alpha k^\beta)\frac{1\, e^{ik(x-s)}}{k^2(k^2+\frac{e^2}{\pi})} \\
&= \frac{1}{2\pi}\int d^2k \overbrace{[\mathcal{E}_{\mu\alpha}\mathcal{E}_{\nu\beta}]}^{\delta^{\mu\nu}} \frac{\delta^{\alpha\beta}k^2 e^{ik|x-y|}}{k^2(k^2+\frac{e^2}{\pi})} \qquad (36)
\end{aligned}
$$

As an important point of this supplementary appendix, we wish to point out that the chirially transformed Dirac operator eq(6) in four-dimensions, still have formally an exactly integrability as expressed by the integral representation eq(3) (see the asymptotic expansion eq4.38) - Chapter 4). It reads as of

$$\log\left[\frac{\det(\not{D}(\sigma)^2)}{\det(\partial)^2}\right] = -\frac{i}{2\pi^2}\int d^4x\,\mathrm{Tr}_{SU(N)}\left\{\phi_a(x)\lambda^a \times\left[\int_0^1 d\sigma F^c_{\alpha\beta}(-iG_\mu(\sigma))F^{c'}_{\mu\nu}(-iG_\mu(\sigma))\varepsilon^{\alpha\beta\mu\nu}\lambda_c\lambda_{c'}\right.\right\} \quad (37)$$

where

$$-i\gamma_\mu G_\mu(\sigma) = \exp\left(\sigma\gamma^5\phi^a(x)\lambda_a\right)(i\partial)\exp(\sigma\gamma^5\phi^a(x)\lambda_a) \quad (38)$$

and we have the formulae [C.G. Collor, Jr., S. Coleman, J. Wess and B. Zumino] – "Structure of Phenomenological Lagrangians" - II, Phys. Rev., 177, 2247 (1969).

$$-i\gamma_\mu G_\mu(\sigma) = V_\mu(\sigma) + \gamma^5 A_\mu(\sigma) \quad (39\text{-a})$$

$$A_\mu(\sigma) = \Delta^{-1}_{\gamma_s\phi^a\lambda_a}\left\{\mathrm{sen}\,h(\Delta_{\gamma_s\phi_a\lambda^a})\circ\partial_\mu(\gamma_5\phi^a\lambda_a)\right\} \quad (39.\text{b})$$

$$V_\mu(\sigma) = \Delta^{-1}_{\gamma_s\phi^a\lambda_a}\left\{(1-\cos h(\Delta_{\gamma_s\phi^a\lambda_a}))\circ\partial_\mu(\gamma_5\phi^a\lambda_a)\right\} \quad (39\text{-c})$$

with the matrix operation

$$\Delta_X \circ Y = [X,Y] \quad (39\text{-d})$$

and $\Delta_X^{(n)}$ denoting its n-power.

For a complete quantum field theoretic analysis of the above formulae in on Abelian (theoretical) axial model we point out our work Luiz C.L. Botelho: Path-integral bosonization for a non renormalizable axial four-dimensional Fermion Model; Phys. Rev. D39, 10, 3051-3054, (1989) and Chapters 6 and 18.

Finally and just for completeness and pedagogical purposes, let us deduce the formal short-time expansion associated to the second-order positive differential elliptic operator in R^ν used in the previous cited reference

$$\mathcal{L} = -(\partial^2)_x + a_\mu(x)(\partial_\mu)_x + V(x). \quad (40)$$

Its evolution kernel $k(x,y,t) = \langle x| - exp(^{-t}\,\mathcal{L})\mid y\rangle$ satisfies the heat-kernel equation

$$\frac{\partial}{\partial t}K(x,y,t) = -\mathcal{L}_x K(x,y,t) \quad (41)$$

$$K(x,y,0) = \delta^{(\nu)}(x-y). \quad (42)$$

After substituting the asymptotic expansion below into eq(41) [with $K_0(x,y,t)$ denoting the Free Kernel ($a_\mu \equiv 0$ and $V \equiv 0$)]

$$K(x,y,t) \overset{t\to 0^+}{\simeq} K_0(x,y,t)\left[\sum_{n=0}^{\infty} t^n H_n(x,y)\right] \quad (43)$$

and by taking into account the obvious relationship for $t > 0$

$$\left(\partial_\mu K_0(x,y,t)\right)|_{x=y} = 0 \tag{44-a}$$

we obtain the following recurrence relation for the coefficients $H_n(x,x)$:

$$(n+1)H_{n+1}(x,x) = -\left\{(-\partial_x^2)H_n(x,x) + a_\mu(x)\cdot\partial_\mu H_n(x,x) + V(x)\right\}. \tag{45}$$

For the Axial Abelian Case in R^4, we have the result:

$$\begin{aligned}\left(e^{g\gamma_s\phi}(i\partial)e^{g\gamma_s\phi}\right)^2 = (-\partial^2)\mathbf{1}_{4\times 4} + \left(\left(\frac{1}{2}g\gamma^s[\gamma^\mu,\gamma^\nu]\right)\partial_\mu\phi(x)\right)\partial_\nu \\ + \left[-g\gamma_s\partial^2\phi + (g)^2(\partial_\mu\phi)^2\right]\end{aligned} \tag{46}$$

Note that in R^4

$$\begin{aligned} H_0(x,x) &= \mathbf{1}_{4\times 4} \\ H_1(x,x) &= -V(x) \\ H_2(x,x) &= \frac{1}{2}[-\partial^2 V + a_\mu\partial_\mu V + V^2](x) \end{aligned} \tag{47}$$

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