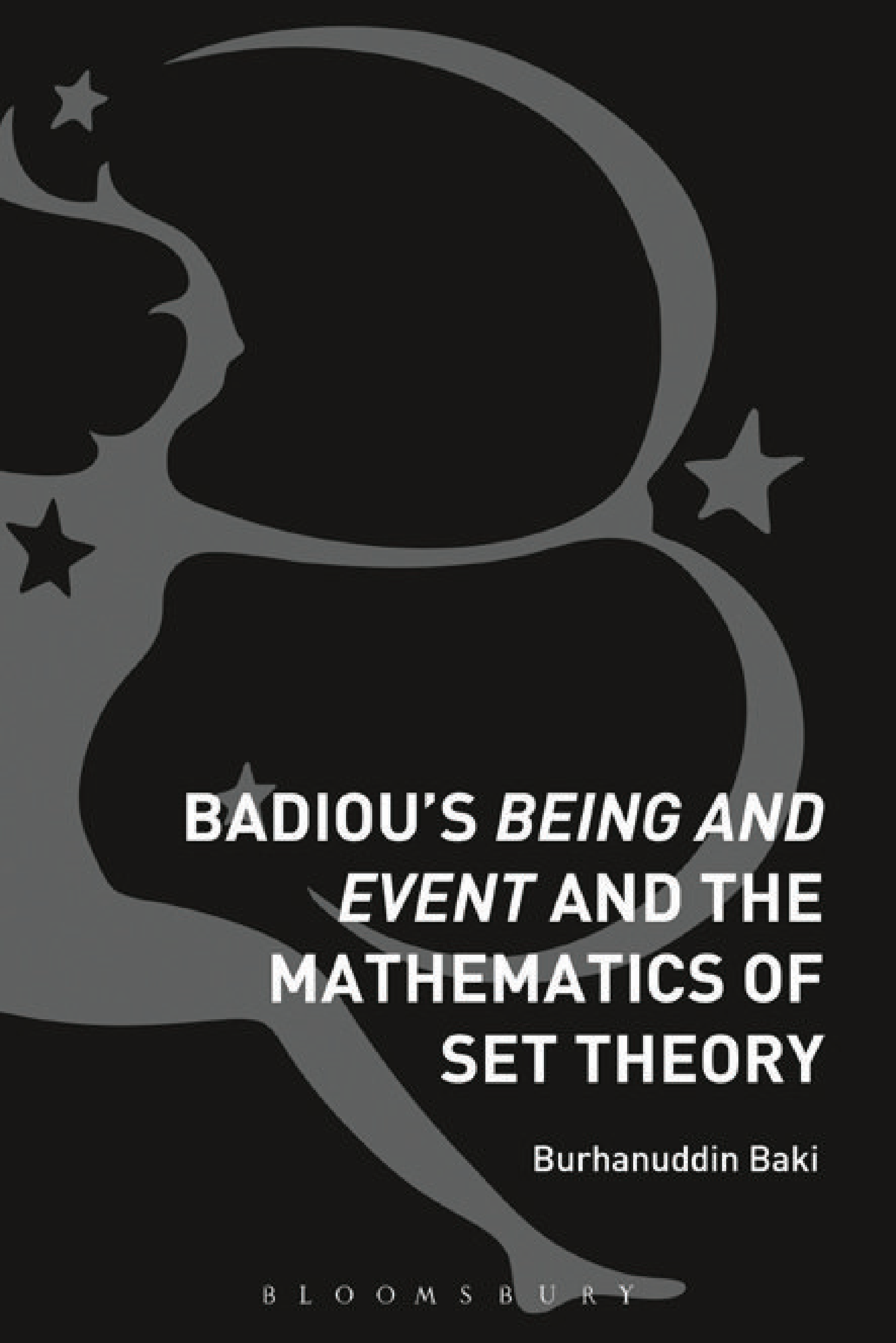

BADIOU'S *BEING AND EVENT* AND THE MATHEMATICS OF SET THEORY

Burhanuddin Baki

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By
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I dedicate this work to my maternal grandmother Hajjah Chik bt. Jol who passed away during the period of my writing this book.

Note on Abbreviations, Citations and Translations

I rely mostly on Oliver Feltham's text *Being and Event* for the English translation of my citations from Alain Badiou's *L'Être et L'Événement*. Abbreviations for three frequently cited texts by Badiou, where also I rely on the current English translations, are as follows:

BE	*Being and Event* (trans. Oliver Feltham, 2005)
MP	*Manifesto for Philosophy* (trans. Norman Madarasz, 1999)
TW	*Theoretical Writings* (trans. Ray Brassier and Alberto Toscano, 2004)

Introduction

Anyone sufficiently informed about the current scene in contemporary philosophical thought will no doubt bear witness to the explosive surge of global interest in the work of Alain Badiou. We need not go further than observing the enormous popularity of his written works; the extremely active publishing industry dedicated to translating his texts; the hundreds of listeners attending his public lectures or viewing online recordings of them; and the exponential growth in the amount of books, journal articles, conferences, courses, editorials, manifestos, art works, blog posts, podcasts and tweets devoted to explaining, appropriating, praising, criticizing and critiquing his philosophy. Many introductions to this philosophy often begin by repeating the proposal that Alain Badiou is, since the death of Jacques Derrida in 2004, probably the most important living French philosopher and, along with Slavoj Žižek and Jürgen Habermas, possibly the most important living European philosopher. All of this provides adequate proof that anyone attempting to engage with the contemporaneity of intellectual thought must find some way to reckon, either positively or negatively, though never indifferently, with the imposing forcefulness of Alain Badiou's oeuvre.

Anyone who knows enough about the reception of this oeuvre will no doubt be aware of the well-accepted proposition among scholars that *Being and Event* [*L'Être et L'Événement*], Alain Badiou's major treatise published in 1988, is his masterpiece. Peter Hallward even went as far as to proclaim it as 'the most ambitious and most compelling single philosophical work written in France since Sartre's *Critique de la raison dialectique*' (2003, xxi). And anyone who knows about *Being and Event* will no doubt be informed about its innovative, even revolutionary, deployment of mathematical thinking to reconfigure the entire landscape of our present investigations into the most central philosophical issues, from various discipline-specific questions in science, art, history, politics, economics, sociology, psychoanalysis and theology to problems concerning the philosophy of language, truth, subjectivity, knowledge, ethics and, above all, ontology. *Being and Event* seizes results from modern mathematics, specifically from the field of nineteenth- to twentieth-century set theory, to construct an extraordinarily original and robust metaphysical schema for understanding what it means to be. The treatise begins specifically with the counter-intuitive militant decision to equate ontology with mathematics. For our part, we also engage in hyperbole by proclaiming that Badiou's introduction of mathematical thinking into the philosophy of ontology is, perhaps, the most profound and most provocative development in the history of discursive investigations into Being since Martin Heidegger's *Being and Time* (1927). Moreover, we proclaim that Badiou's re-conceptualization of ontology as mathematics is the most radical, far-reaching and surprising philosophical equation since Emmanuel Levinas's redefinition of first philosophy as ethics in *Totality and Infinity* (1961) and perhaps since Baruch Spinoza's redefinition of God as substance in *Ethics* (1677).

So anyone who wishes to engage on a sufficiently serious level with *Being and Event* must attempt to come to terms with Badiou's unique deployment of mathematical thinking. Despite the significant growth in the field of Badiouian scholarship, the basic details of his proposed metaphysical framework are still not well understood. We read the secondary literature[1] and concluded that much work still needs to be conducted on the rudimentary level of trying to grasp the book's proposed metaphysics of Being and how it is affected by certain discoveries from mathematical set theory. These gaps in the literature can be attributed among other things to the fact that the field of Badiou studies is still relatively young and that the discipline of mathematical philosophy has been, with only a few exceptions, relatively latent for the last few decades, particularly within what many of us continue to call continental philosophy. We expect that many of the straightforward implications of Badiou's new conceptual framework have yet to emerge – the fresh intellectual terrain not only contains regions that will take time to traverse and map but also topographically displays itself akin to a fractal where any effort at honing onto a restricted nanoscopic region reveals yet another continent of possibilities. We also recognize that *Being and Event* is an extraordinarily difficult book. Moreover, acquiring the relevant formal background demands a markedly high level of erudition and specialized mathematical knowledge that has not been demanded since perhaps the work of Albert Lautman from before the Second World War.

To repeat everything more programmatically, we observe, in the secondary literature on Badiou's *Being and Event*, a specific gap that needs to be filled. This gap involves understanding the fundamentals of the book's proposed metaphysics of Being, and one distinguishing feature of this metaphysics is its indispensable relationship to certain discoveries in the modern science of sets. In our attempt to fill this gap and explicate Badiou's philosophy of ontology, we will have to tackle the issue of how this philosophy appropriates the mathematics. In this regard, my specific strategy is to reconstruct Badiou's metaphysics in relation to its specific 'conditioning' by the standard mathematics of sets from the inaugural work of Georg Cantor up to Paul Cohen's discovery of forcing [*forçage*]. In other words, I study, first, the relevant areas in the science of sets and, second, the relevant facets of Badiou's philosophy of ontology, and then try to rebuild, as far as I can, the details of the latter based on how they are informed by the former.

The question of the event

Being and Event is a massive book, massive in its relative size as well as in the radical novelty and refined complexity of its ideas. It appears even more complex in the deceptive clarity of its opaque language and in the ascetic succinctness of its all-too-schematic explanations.[2] As previously mentioned, despite the currently observed high level of activity involving Badiouian scholarship, the field itself is relatively recent. One is never able to select any topic that is specifically concentrated at this point in time. Nevertheless, for our purpose, we limit our purview to Badiou's construction of his philosophy in relation to the specific question of the event [*l'événement*]. Badiou fashions his metaphysical framework by connecting the question of Being to several philosophical thematics, but I concentrate on those clustered around one concept: the event, the ontological rupture, the transformative novel break.

This is an obvious choice because it is already suggested in the book's title. We are informed not only that the book will think the questions of Being and the event but also that the question of Being will be thought through the question of the event, which will be conceived, more or less, as its dialectical other, as That-Which-Is-Not-Being-qua-Being. Already we can expect that the idea of the event will play a role in Badiou's metaphysics that is analogous or at least comparable to the conceptions of time and nothingness in Martin Heidegger and Jean-Paul Sartre's early philosophies, respectively. Badiou's 'metaontological' investigations characterize and dialectically perpetuate themselves in tandem with his 'metaevental' investigations.

So the question is how to think the event and how to think its non-ontology – or, rather, how to think through the relation between Being and event from the viewpoint of Being. The event, is there even such a thing? That is, would it make sense? And, if so, in what, or what sort of, sense? Or at what limit of what makes sense, what position with respect to what limit point of comprehensibility? It would be an easy exercise to illustrate that this topic has been, since some obscure point in recent history, one of the central concerns of philosophical thought, particularly within the continental school. A straightforward overview of the literature would show that many major thinkers from the last few decades have offered, in one way or another, one or several philosophies of the event. We abandon ourselves for a moment to the zeal of listing and illustrating:

1. The later Martin Heidegger understands *ereignis*, his name for the event, as the emergence of a new world within which all entities appear, a non-ontic and epochal disclosure of a new configuration of Being.
2. Gilles Deleuze insists, however, on the event's absolute immanence to the order of Being. Ever the vitalist, Deleuze understands events as singular points of folding within the system, the turning points and points of inflection, the connected series of prehensions where the desubstantialized pure becomes sense, and the virtual becomes actual.
3. Jacques Derrida's later work concerns the various 'impossible possibilities' of the event, each of which is the impossibility of some aporetic passage – the impossibility of the event of forgiveness, the gift, creative invention, hospitality, mourning and so on.
4. As is well known, some of Derrida's ideas originated from his reading of the work of Emmanuel Levinas, whose well-known notion of the ethical and asymmetrical face-to-face encounter with the Other might be said to name or at least take the exterior form of an event.
5. Jean-François Lyotard's libidinal philosophy associates the event with the critical collision sites between the various energies that flow through a complex system.
6. Cognitivists and complexity theorists themselves – the cyberneticians; the chaos and systems theorists; and philosophers who read them, such as Gilles Deleuze, Michel Serres and Manuel DeLanda – relate the event to a variety of processual phenomena: the splitting of a bifurcation point; the threshold transition towards or from turbulent nonlinearity; the spontaneous breaking of symmetry; and the mysteries of emergence, those magically emerging complexities that are irreducible to any isolated collection of parts in the system.

7. And last, but not least, the various theorizations of the event in epistemology and in the philosophy of science (Thomas Kuhn's paradigm shifts, Gaston Bachelard's epistemological breaks, the various examples of materialist metaphysics invoking the Lucretian clinamen, etc.), in Marxist philosophy (revolution, utopia, violence, the breakdown of capitalism, Rosa Luxembourg's spontaneism, Louis Althusser's philosophy of the encounter, etc.), and in psychoanalytic theory (the primal scene, trauma, Oedipalization, encounter with the real, etc.).

In various philosophical writings, the event has been conceptualized as unexpected and undetermined; real or phantastic; constitutive and destructive; concrete or metaphorical; horrific, disgusting and violent; and un-assimilatable, though possibly premeditated and pursued. It has also been described as the disruption of the normal progressive flow of space-time, a space-time which it also makes and regulates; and as the birth, death or persistence of the subject. Employing the almost trivial dialectical gesture, the event can even be predicated as the non-event, as eventlessness itself. The end of the Cold War; Global Warming; September 11; the Severe Acute Respiratory Syndrome (SARS) and avian influenza pandemics; the Indian Ocean tsunami of 2004; the birth of the internet; the election of Barack Obama; the ongoing financial crisis; the introduction by Apple Inc. of the Mackintosh, the Ipod and the Iphone; the self-immolation of Mohamed Bouazizi – these recent historical incidents have at one time or another been called events.

On a more topical note, the event has permeated everyday consciousness as the topic of bestselling books in management theory, economics, sociology and popular science. We see the problematic of the rupture in Nassim Nicholas Taleb's concept of the 'black swan'[3] (2007); Malcolm Gladwell's expositions on the 'tipping point' (2000) and 'outliers' (2008); Philip Ball's 'critical mass' (2004); and the coming of the 'technological singularity' that various science fiction writers, such as Vernor Vinge (1993), and future studies scholars, such as Ray Kurzweil (2005), have foretold. Ever since the pioneering work by Joseph Schumpeter (1942) and then by Peter Drucker (1985), the all-too-present contemporary discourse in management science and the theory of entrepreneurship have their own unique positive name for the event, innovation,[4] a name that is haunted by a negative, economic collapse. All of this indicates that the topic comprises an area that is of tremendous interest not only to the intellectual class in particular and the public in general but also to the policy and decision makers in administration, business and industry. However, it remains to be seen if management theorists would consider *Being and Event* itself as an event, or whether Alain Badiou's philosophy could inform, contribute or be made consistent with the mainstream fields within management theory, some of which are partly implicated in the hegemony of the practico-utilitarian capitalist paradigm and in the repression of political action and scientific revolutions in favour of management thinking and technological pragmatism. Nevertheless, the main point of my lists and illustrations is to demonstrate the significance of the question of the event for contemporary philosophical thought. Since *Being and Event* is an important book, and since the problematic of the rupture is an important question in contemporary thought, any attempt at dealing with both of them together is bound to be useful.

Statement of purpose and delimitation of purview

In researching Badiou's metaphysics, my objective is a creative reconstruction, not an interpretation. The principal aim is not to closely read or produce an exhaustive *explique* of *Being and Event* but to *think* and *meditate* on the philosophy, and to do so through my process of reconstructing it. This process requires me to have a hermeneutic but undetermined relation to the textual body of the book, which serves only as a guide and not as a scriptural body of truth in its immanent textuality and authorial intentions. I shall not operate under the assumption that 'there is nothing outside the text'. What I offer is a report of my relationship with the book during the limited period of three years – an indirect personal chronicle, in the format of a philosophical book, of my tentative endeavours at grappling and coming to terms with what is offered by Badiou's treatise. In a sense, Badiou should not have the complete and final word on his own philosophy. In the same way as it continues to be viable for an ordinary physicist to resume and to add, albeit incrementally, to the trajectory of truth that is Relativity Theory – a truth that is not completely owned by Albert Einstein in all his greatness – it is possible to think and expand on *Being and Event* beyond the personal imprimatur of Alain Badiou himself. It is therefore necessary to secularize our relationship to the body and to the body of work that falls under Alain Badiou's authorship. This book can be understood as a finite fragment in the aleatory trajectory of my own specific 'following-through' with the event of Badiou's philosophy, with the Badiou-event that is *Being and Event*.

I must, however, qualify my intentions because what I attempt here is far from a thorough re-enactment of all of Badiou's ideas. The methodological principle is that it is often practical to sacrifice breadth in favour of attaining, hopefully, some possible depth of analysis. I am not intending some feat in meticulous erudition or bibliographic thoroughness in relation to Badiou scholarship. Despite my restrictive purview, I hope it leaves room to make some small and humble contribution. First, I focus on a particular yet fundamental aspect of Badiou's work: his philosophy of ontology. In particular, I focus on rebuilding the architecture of his metaphysical system and how it relates to his analytic of the event. I cannot stress enough that my research is not a treatment in political philosophy, historical analysis, ethics or aesthetics, even though those disciplines are perfectly legitimate and serious matters of investigation and are also implicated in Badiou's ontological conceptions. I will be reading *Being and Event* primarily as a text in the philosophy of ontology.

I limit myself by examining how Badiou's metaphysical framework constructs itself within one particular text: *Being and Event*. The principal chapters for me are mostly what Badiou has called the 'conceptual' and 'metaontological' meditations from that book (*BE*, 18–19). Badiou's framework is later supplemented by what he gives in *Logics of Worlds* (2009), a treatise that is not the subject of my research. Sometimes, but not always, I examine, when appropriate, some of Badiou's essays and short books that expand on the system set forth by *Being and Event*. But I will not engage too much with comparative studies in relation to other texts by Badiou or by other writers. I emphasize that I am concentrating not on the historical development of Badiou's philosophy throughout his career, but on how a certain systematic framework fits together within one specific treatise. It is always possible, and often tempting, to analyse this system in relation to ideas and propositions by other philosophers, some of whose works are discussed in the 'textual' meditations in *Being and Event*. But that is not my main focus. This book is not primarily a work in comparative analysis or in the history of ideas – although I will mention some possible points of contrast and possible sites for extending Badiou's investigations, but without necessarily checking, expanding or following through on their viability or feasibility.

In regard to the set theory, I look only at the mathematics itself, not its history and certainly not its philosophical foundations (although we admit that the distinction between the mathematics of set theory and the history and philosophy of set theory is often obscure). Moreover, I talk only about the standard formalization of set theory: first-order set theory under the well-established system known as the Zermelo-Fraenkel Axioms plus the Axiom of Choice (*ZFC*). Set theory and mathematics have been formally systematized in different ways and under different formal orders of language, but my focus is first-order *ZFC*.

My strategy is as follows. As explicitly given in the book's *Introduction*, Badiou philosophically seizes the mathematical truths originating from five individual 'bulwarks [*massifs*]' (*BE*, 20):

1. the Zermelo-Fraenkel Axioms of Set Theory, including the Axiom of Choice
2. the theory of ordinal numbers
3. the theory of cardinal numbers
4. Kurt Gödel's work on the constructible universe, which led to his proof of the consistency of the Axiom of Choice and the Continuum Hypothesis with respect to the Zermelo-Fraenkel Axioms
5. Paul Cohen's work on generic models and his technique of forcing, which proved the independence of the Axiom of Choice and the Continuum Hypothesis with respect to Zermelo-Fraenkel

After a discussion in Chapter 1 about Badiou's equation, I do two things for each of these bulwarks:

i. I gather together and clarify the relevant details of the technical mathematics. I believe this is important because some of these details remain obscure, not all of them being explicitly mentioned in Badiou's book.
ii. I try to reconstruct, as far as possible, Badiou's conceptual and metaontological meditations from the book in relation to their conditioning by the mathematics. I pay particular attention towards rebuilding the basic framework of Badiou's metaphysics of Being and towards his philosophy of the event – the latter being informed mostly by the last mathematical bulwark involving Cohen's concept of forcing. My reconstruction is based on my investigations into how the mathematics 'conditions' the philosophy.

The mathematical expositions and philosophical reconstructions are clearly divided into separate chapters or separate portions of chapters so as to differentiate clearly between the ontology and the metaontology. Badiou's genius must not be confused with the genius of the mathematicians or with the magisterial gravitas of mathematics itself. The reader, I hope, might then be in a better position to judge Badiou's philosophical inventiveness independently of the innovation inherent in the mathematical results.

I have decided not to exclude the mathematical expositions for several reasons. With a few exceptions, such expositions are missing from much of the secondary literature on Badiou's work, and I believe that providing a direct instruction in the specific mathematical technicalities would in itself comprise a contribution to the current scholarship. It is not immediately obvious which exact results from the five bulwarks are pertinent to Badiou's philosophy and how they fit together within some narrative under Badiou's rather distinctive treatment following from his militant proposal that mathematics is ontology. Yet it would not be realistic for me to expand on each and every single mathematical idea referenced in *Being and Event*. Even when the technical demonstrations are reproduced in detail, not all of them play an immediate role in the philosophy. My mathematical expositions are a product of my own negotiations with such difficulties. By orienting my narrative towards Badiou's idiosyncratic and often opaque appropriation of the technical material, I hope that I do not expend a disproportionate amount of this book on a mere repetition of what could be acquired from referring to a more exhaustive treatment of set theory in some textbook. I wish to provide as inclusive a discussion as possible in regard to mathematically preparing ourselves for the task of reconstructing Badiou's metaphysics of Being, under the assumption that direct exposure to the formal technicalities could offer a more nuanced understanding of Badiou's masterpiece. My objective is to explain set theory 'as a mathematician would', while simultaneously but gently orienting my narrative towards Badiou's unique treatment. The reader will then, I hope, be uniquely equipped to read and approach *Being and Event* from the viewpoint of someone who has already familiarized himself, to some sufficient degree of rigor, with much of the mathematics involved and not just as someone whose initial exposure to them is only through Badiou's work and his rather distinctive though revolutionary interpretation.

1

Mathematics = Ontology

How does one make sense of this militant equation by which the entire conceptual framework and philosophical topos of *Being and Event* is delimited? Before we pursue the philosophical implications of Badiou's wager, it would be appropriate to instruct ourselves as to what it says and what it means. Our goal is not the closed security of an etymologically exhaustive or analytically precise translation, which could only be realized and verified as an infinite truth procedure. But it would be pedagogically beneficial, by way of a preliminary instruction, to initiate ourselves into the meaning of each side of this equation, sketch their respective relations to philosophy, and try to understand what is being proposed in the commitment to treat the two terms as identical. Granted, the act of equating revises, reconfigures and redistributes the meanings of mathematics and ontology. But a first point of departure must be provided and we shall be open towards correcting ourselves as we go along.

Mathematics, ontology and philosophy

Any answer to the question of defining the first side of the equation is guaranteed to be complicated and controversial. The question has constituted the most dominant and contentious topic for almost the entirety of what has been called the 'philosophy of mathematics' for the past one hundred years and more, particularly involving the project of defining mathematics by proposing a foundational philosophical definition of its 'objects', of 'entities' deemed 'mathematical'. In the process of formulating such a foundation, a huge menagerie of philosophical orientations, oftentimes competing, cooperating and cross-breeding with each other, have been proposed, some dating back to the very beginning of mathematics as an investigative discourse:

antifoundationalism	idealism	predicativism
atomism	inflationism	psychologism
computationalism	instrumentalism	Pythagoreanism
conceptualism	intuitionism	realism
constructivism	logicism	reductionism
fictionalism	Meinongism	social constructivism
finitism	neutralism	social realism
formalism	nominalism	structuralism
foundationalism	phenomenalism	verificationism
holism	Platonism	and so on.

Badiou's equation purports to be not only a solution but also a dissolution of the matter, for a problem can be resolved by either providing a correct answer or showing the speciousness of its aims. He proposes not just another definition – which equates mathematics with ontology – but also the beginning of a demonstration that the questions 'How can mathematics be founded?' and 'What is a mathematical object?' are actually pseudo-problems at best.

However, most of us would easily recognize what is meant by mathematics: it is simply the investigative discourse comprising arithmetic, geometry, algebra and calculus. Depending on our degree of erudition, we might add various other advanced subfields to the list: topology; combinatorics; probability theory; the theory of computation; differential equations; numerical analysis; dynamical systems; category theory; statistics; mechanics; set theory; and so on. Mathematics is what mathematicians study, practise and do. This is a sufficiently instructive way to begin and for the moment we will not expand further.

In the historicity of its respective discursive threads, mathematical thought has often penetrated, informed and inspired philosophical thought. To use the Badiouian vocabulary, we say that mathematical thought has often constituted a 'condition' that 'forces' philosophical thought. A cursory inspection of the history shows that this has been the case ever since the latter's inception going back at least to the time of Thales. We examine the history of what has been called continental philosophy for the last one hundred years and observe such a mathematical conditioning at work, to various degrees of success, controversy and thoroughness, within the oeuvres of Albert Lautman, Jules Vuillemin, Jean Cavaillès, Gilles Deleuze, Jacques Derrida, Friedrich Kittler, Gaston Bachelard, the 'Speculative Realists', Michel Serres, Jean-Toussaint Desanti, François Laruelle and, in particular, Jacques Lacan, whose employment of logical and topological mathemes motivated Badiou's philosophy and whose identification of mathematics with the science of the real – the real that is the impasse of formalization – can be compared to Badiou's militant equation. But I would say that, without doubt, the most fruitful, methodical and influential 'forcing' took place in the early decades of what we call the analytic school of philosophy through the paradigmatic roles played by mathematical rationality following from certain developments by Gottlob Frege, Georg Cantor, members of the Vienna Circle, and Cambridge logicians such as Bertrand Russell and Ludwig Wittgenstein. The mathematical revolutions from the late nineteenth to the early twentieth century provided not just new spaces for the circulation of philosophical thought but also a new epoch of science where the very 'axiomatic' basis of mathematical rationality revealed itself.

As for the task of formulating a sufficiently instructive definition of the right-hand side of the equation, a more straightforward answer based on the simple etymology of the term is available: ontology is the study of Being or, more precisely, of Being-qua-Being. The matter in question involves examining not the particularity of beings, not specific entities or the specificities of their being-present, but investigating, at the most general level and with the most direct attention, into what is and only insofar as it is.[1]

While subtracting itself from any particular presentation, Being forms, by definition, the substantivation of the presentative basis and immanent presentativity for every being. So ontology is linked with the study of first causes, the *primum movens* and the *fundamentum absolutum*. However, it is necessary for Badiou that this study be scrupulously laicized and its 'captivating aura' be thoroughly exorcized. Theology is one thing, ontology another. When it comes to ontology per se, the halting point of investigations must be Being itself. Jean-Toussaint Desanti is correct to call Badiou's ontology 'intrinsic' (2004) because the focus here is pure and immanent Being, without any recourse towards unifying or undermining it in favour of some originary sovereignty, primordial exteriority, grounding monism, or distinct *theos*, be it God, Geist, language, the Other, the Idea, the Self-Same Subject, the Real, the Big Bang, the Unconscious, Capital, Vital Energy, or the ever-shifting vicissitudes of some substance. Badiou takes this rejection seriously and at the most uncompromising point, so much so that he recruits the propositions 'The one is not' and 'Being is pure multiplicity' as a second wager to supplement his militant equation. This rejection can be understood as a more radical continuation of Heidegger's deconstruction of metaphysics, particularly in its rejection of any onto-theological approach towards understanding Being.

In lieu of everything that is implied by Heidegger's name, we can provide yet another description of ontology by repeating what was often understood, since at least the time of Aristotle, to be its fundamental relationship to philosophy. Ontology is the 'first philosophy', and Badiou agrees with Heidegger's return to the question of the meaning of Being as the inaugural question of philosophy. 'Along with Heidegger,' as Badiou writes at the beginning of *Being and Event,* 'it will be maintained that philosophy as such can only be reassigned on the basis of the ontological question' (*BE*, 2). Ontology is one field within philosophy – in fact, the most central and most essential field, the kernel of philosophical discourse that frames, conditions and secures the site for all the others such as epistemology, philosophical logic, philosophy of mind, political philosophy, philosophy of morality, aesthetics, philosophy of science, philosophy of law, and so on. The study of Being-qua-Being has often been understood as the field most native to philosophy, and any other field could be recomposed to take the form of 'philosophy of X', a philosophy conditioned by the investigation into something that is, strictly speaking, outside of ontological considerations per se, be it knowledge, reason, the subject, politics, ethics, art, science or law. At the very least, ontology is the core branch within metaphysics, another field that is often understood to be most intrinsic to philosophy. But the difference is a matter of semantics: sometimes the word 'metaphysics' is meant to be synonymous with ontology; sometimes ontology is called 'General Metaphysics'; and sometimes metaphysics, following the well-known Heideggerian line of thought, is said to be the vulgarized version of a study that has forgotten the inaugural question of Being by seeking some causal, theological or quasi-theological explanation to undermine Being-qua-Being. Nevertheless, much of what is given in *Being and Event* has direct implications, as we shall see, for many of the central issues in traditional metaphysics – questions involving identity, predication, modality, universals, reality and so on. But our two main points remain:

1. Ontology is a branch of philosophy
2. The question of Being-qua-Being forms the very basis of philosophy as such.

Badiou preserves only the second of these two points. Since what mathematicians do is different from what philosophers do, and since ontology equals mathematics, then ontology cannot be philosophy. But ontology remains the kernel for philosophy. It is not the first philosophy, but the first condition for philosophy, albeit one condition among other non-primary conditions. When recognized as being equivalent to ontology, mathematics is the literal inscription of Being into discourse. Philosophy still has Being as its central question, except that it cannot study it directly at the first-order level – that particular task is reserved for ontology, that is, for mathematics. Badiou appropriately names 'metaontology' [*métaontologie*] as the part within philosophy that immediately concerns and is directly forced by ontology. So we correct ourselves by saying that the first philosophy is not ontology but metaontology. The role of what, following the Heideggerian return, used to be conceived as ontology is now played by a field involving a second-order thinking that is conditioned by ontology. Philosophy is at most the study of Being at the second-order level. Philosophy no longer deals with first-order questions as the letter of Being enters into discourse directly as mathematics.

The other conditions of philosophy and the compossibilization of truths

One upshot of Badiou's wager is that the possibility and possible meaningfulness of metaphysics and metaphysical knowledge (or at least the part of metaphysical philosophy that now belongs to the separate discourse of ontology) can be understood in terms of the possibility and possible meaningfulness of mathematics itself. Badiou's equation delegates and merges such questions with those involving the epistemology of mathematics itself. For example, if one accepts Badiou's equation and agrees that mathematical knowledge is partly *a priori*, then one can be led to conclude that some metaphysical knowledge is partly gained outside the realm of experience and sense-perception. The well-known problem posed by Rudolf Carnap (1950) regarding the possible futility of metaphysics has been partly converted to a problem of how to make sense of the strange meaningfulness of mathematical knowledge that has no concrete referent.

We must nevertheless remember that mathematics has not been the sole condition for philosophy, for it is trivial that philosophers are informed by other forms of activities, investigations and experiences. First, there is the internal condition: the history and the textual archive of philosophy itself, an archive to which every philosopher must relate and respond. Second, every branch within philosophy concerns itself with extra-philosophical realms. Epistemology, philosophy of mind, aesthetics and political philosophy have as their principal objects something that, strictly speaking, lies at least partly outside of philosophy proper. For example, epistemology studies knowledge and is informed by recent discoveries in psychology, neuroscience, cognitive science and even the socio-political science of knowledge. As implied by their own names, the philosophical fields of aesthetics, political philosophy and the philosophy of science study art, politics and science, respectively. So every philosophy is always a philosophy of something outside philosophy. 'Almost all our "philosophers",' writes Badiou, 'are in search of a diverted writing, indirect supports, oblique referents, so that the evasive transition of a site's occupation may befall to philosophy's presumably uninhabitable place' (*MP*, 28).

Badiou's wager posits that the study concerning the most intrinsic focus of philosophy, namely Being-qua-Being, belongs wholly to the mathematicians, who are not philosophers when they are doing mathematics. '[A]ffirming that mathematics accomplishes ontology unsettles philosophers because this thesis absolutely discharges them of what remained the centre of gravity of their discourse, the ultimate refuge of their identity. Indeed, mathematics today has no need of philosophy, and thus one can say that the discourse on [B]eing continues "all by itself"' (*BE*, 10). Absented from within itself, hollowed from its own essence, philosophy is fundamentally a cross-, inter- and trans-disciplinary investigative discourse that simultaneously invents new disciplinary classifications outside of the university and the encyclopaedia. Philosophy resides in the neutral in-between spaces of disciplines and lives in the heterogeneous times of truths. Every philosophy takes the form of a program for the 'compossibility' of truths. Badiou writes:

> I have assigned philosophy the task of constructing thought's embrace of its own time, of refracting newborn truths through the prism of concepts. Philosophy must intensify and gather together, under the aegis of systematic thinking, not just what its time imagines itself to be, but what its time is – albeit unknowingly – capable of. (*TW*, 15)

One of the most controversial and well-known propositions in Badiou's philosophy – although he does not sufficiently elaborate it in *Being and Event* – is that every external philosophical condition must belong to four very specific but general domains: science, politics, art and love. Badiou places mathematics under the domain of science (which is a controversial move in itself) and takes the former as the rational and paradigmatic basis for the latter. If physics is the scientific study of physical matter and chemistry is the scientific study of chemical reactions, then mathematics is the scientific study of Being. Mathematics is what is left of science when it is without any object.

We read *Being and Event* and find nothing in the philosophical commentary on the mathematics that justifies Badiou's decision to group the external philosophical conditions into these four domains. But we observe that, despite constituting a finite number, the domains are not as restrictive as they appear, particularly when we note that the word 'science' could be understood to cover a huge territory that includes not just the natural sciences (physics, chemistry, astronomy, and so on), the social sciences (economic science, political science, linguistics, and so on), but any systematic field of investigation that conserves the figure of mathematical rationality as a paradigm for thought.[2] Nevertheless, scientific thinking rejects certain modes of knowing and certain forms of knowledge such as mythology, alchemy and religion – unless, of course, it was possible to render such modes under a different condition, such as art or love.

One fundamental common denominator that defines each of Badiou's domains is that it must allow for the possibility for events, for ruptures that completely reconfigure the individual situations corresponding to each domain on an ontological level. For example, amorous encounters occur in the domain of love, and revolutions erupt in the domains of science, art and politics. The eruption of the event allows for the possible emergence of new truths, with each truth realized as an infinite truth procedure that is contemporary with the weaving of a new subjectivity that is militantly committed to it. Philosophy, for Badiou, proposes a conceptual framework in which the contemporary compossibilization of its conditions can be grasped in the rupture of an evental truth. Philosophy conceptually seizes and houses the site of heterogeneous truths by circulating between the procedures that arise from science, art, politics and love.

So philosophy can be forced only by scientific knowledge, political activity, artistic practices and amorous experiences. Despite its essential relation to these four domains of truth, philosophy has its own discursive sovereignty that is fundamentally independent of its conditions. A philosopher, when he is doing philosophy, is not a scientist, a political activist, an artist or a lover. Philosophy must not be sutured to its conditions. The domains themselves do not wholly predetermine the individual manner in which a condition is philosophically seized. There is nothing 'necessary' about the way Badiou understands and makes use of mathematical set theory to construct a new metaphysics of Being. The task of the philosopher is not to verify or rationalize, under some scientific framework, the current state of affairs and current intuitions regarding the state of science, politics, art and love. Ray Brassier and Alberto Toscano write that Badiou 'refuses the premise of a fundamental transitivity between the philosophical and pre-philosophical; the idea that philosophical insight is already latent in pre-philosophical experience and that the philosopher's task consists in extracting the former from the latter in order to purify it' (2004, 264–5). The subject of truth, which defines itself prior to its infinite verification, emerges partly through this lack of necessity within the relational space of forcing. The essential job of philosophy is not to map the totality of its domains or to secure some universal summation, some fixed ground, upon which they reside. Badiou writes:

> [T]he philosophical operators must not be understood as summations, totalizations. The eventful and heterogeneous nature of the four types of truth procedures entirely exclude their encyclopaedic alignment. . . . Philosophical concepts weave a general space in which thought accedes to time, to *its* time, so long as the truth procedures of this time find shelter for their compossibility within it. The appropriate metaphor is thus not of the register of addition, not even of systematic reflection. It is rather the liberty of movement, of a moving-itself of thought within the articulated element of the state of its own conditions. . . . Philosophy does not pronounce truths but its *conjuncture*, that is, the thinkable conjunction of truths. (*MP*, 38)

Moreover, any philosophical encounter with contemporary mathematics need not result in just another development concerning philosophical foundations or even simply a reading, interpretation, critique or *explique*. An evental encounter could also be the occasion for a new thinking, a philosophy conditioned by mathematical truths, but without it being simply a commentary of what is only inherent in the mathematical text. Every novel development in contemporary mathematics presents a truth that has the potential of being housed by philosophy. Every mathematical revolution provides an occasion for a new meta-ontological thinking via the liberated and aleatory movement of truth.

Algebra, geometry, arithmetic, calculus, topology, combinatorics, differential equations, and so on – all these fields now have a direct role to play within philosophy and outside of their contribution to the natural and human sciences. For *Being and Event* in particular, Badiou chooses to examine the mathematics of set theory up to Paul Cohen's work on forcing and generic sets. With Badiou's gamble, a new philosophical project is set forth. We can thus say and hope that

Badiou's militant wager could serve as a prolegomena that might orientate us towards what could be a new truth procedure in philosophy, a 'mathematical turn' that follows after the various other 'turns' – linguistic, ontological, political, cultural, religious, and so on – during the last one hundred years. In order to participate in the actual development of ontology, one must be a scholar in the actual development of mathematics. In view of everything that philosophy has ignored about the great innovations in mathematics from the last one hundred years, the vista of possibilities is almost endless.

The grand style versus the little style of philosophical inquiry

'Metaontology' in Badiou's philosophy should not be confused with meta-mathematics, which studies mathematics itself using mathematical means. It is also worth repeating that 'metaontology' is not the same as analytic meta-ontology or meta-metaphysics, which are active fields in analytic philosophy involving the epistemology and philosophical foundations of ontology and metaphysics themselves. *Being and Event* is a treatise that recruits set theory as one of the conditions for housing a new site for philosophical investigations. Badiou does not examine the various schools that try to provide some philosophical basis for mathematics. He does not deal with questions such as 'What is mathematical knowledge?' or 'How is mathematical knowledge acquired?', even though his militant wager could be used to help rethink some of the previous answers to those inquiries.

On Badiou's side, we have his metaontology and his conception-prescription of philosophy as being the liberty of a thinking that is subject to conditions originating from the domain of, among other things, mathematics. On the other side, we have the fields of analytic meta-ontology and meta-metaphysics, as well as philosophical inquiries into the foundations of mathematics and other investigative discourses or regions of experience. The distinction between the two sides implicates two separate styles of philosophical inquiry, what Badiou calls the 'grand style' versus the 'little style'. Badiou addresses this methodological difference in his highly-charged polemical essay 'Mathematics and Philosophy' (*TW*, 3–21). This polemic specifically contrasts the heroic methodology of mathematical philosophy with the timid investigations in the philosophy of mathematics. But we can generalize its discussions so as to concern the relation between philosophy and other domains of truths.

The little style compulsively constructs mathematics – or art, politics, love, science, knowledge, ethics, and so on – as a subservient and neutered 'object' for philosophical inspection. The function of this 'object' is to help perpetuate a ready-made and already well-defined philosophical specialization, with 'its own specialized bureaucracy in those academic committees and bodies whose role is to manage a personnel comprising teachers and researchers' (*TW*, 3). Thought asserts its grip upon its sterilized 'object' through scrupulous historicizations and technical classifications. We observe the endless typological markings and re-markings of different orientations in not only the philosophy of mathematics but also the philosophy of science, art, knowledge, morality, language and so on. In the little style, any philosophical identification of mathematics must be conducted via the route of linguistic reduction and logicization. By proceeding through a consideration of language and logic, this meek style is made entirely consistent with stale categories and the cataloguing of thought in terms of pre-existing proper names. Language is intended to take the form of perfectly systematic, measurable and constantly fine-tuned grammar of epistemic and positivistic inscriptions that leave nothing for any rupture or any eruption from the outside. Any possibility for an evental eruption – which constantly haunts the little style from without – is either deemed irrelevant or matched with controllable facts, verifiable testimonies, physical marks or pedantic thought-experiments. Badiou links the operations of the little style to the operations of neoclassicism and what he calls, in *Being and Event*, a constructivist orientation of thought [*pensée constructiviste*]. In its philosophical manifestation, this orientation is only a continuation of the arcane scholasticism of the medieval ages, akin to the extensive studies trying to figure out how many angels can dance on the head of a pin.

The grand style of mathematical philosophy 'stipulates that mathematics provides a direct illumination of philosophy' and that this illumination should be 'carried out through a forced or even violent intervention at the core of these issues' (*MP*, 8). Mathematics is a condition and not a mere 'object' for thought. Rather than simply being an object for the specific instance of an existing philosophical question, mathematics is able to challenge and even undermine existing philosophical questions, and can provide striking or paradoxical solutions to them. Here we have an alternative understanding of the philosophical action implemented by this prefix 'meta': between, on Badiou's side, the grand movement of the philosophical subject whose inquiries are conditioned by evental truths from certain domains and, on the other side, the little movement of 'going meta' with theoretical investigations carried out in higher orders of abstraction and with the aim of establishing some firm and unified ground for the theory in question. Philosophy, for Badiou, does not dissolve but seizes the sovereignty and self-sufficiency of truths in their unrivalled aristocratic positions. Moreover, 'philosophy must enter into logic via mathematics, not into mathematics via logic' (*TW*, 16). The practice of this grand style has a long history dating back at least to Plato, whose theory of forms is a procedure whose conditioning truth is the matheme. This history, according to Badiou, continues in the works of, among others, Descartes, Spinoza, Kant and Hegel, and even in Lautréamont's *Le Chants de Maldoror* (*TW*, 8–15).

One outcome of Badiou's wager is that, in this grand style, the conditioning relationship between mathematics and philosophy does not need to be that of mere analogical reasoning. One does not inspect mathematical definitions and theorems[3] and then take them as similes for justifying some unrelated philosophical conceit. Mathematics, whose corresponding propositions are not mere formal representations for metaphysical conclusions, is precisely ontology itself and its conditional relationship with metaontology is not simply that of whimsical metaphorical induction or provocation. Since mathematics presents nothing and means nothing, it does not make sense to distinguish between a 'direct' or 'metaphorical' understanding of its results and its so-called objects, for mathematics is the sole discourse where one knows what one is talking about. Badiou's wager offers another way to understand how mathematics can inform philosophy other than the route that has often been denigrated by scientists such as Alan Sokal and Jean Bricmont (1998). By equating mathematics with ontology, Badiou was able to cut through the problematic status of propositions gained with the direct aid of metaphor and image.[4] Metaontology is not necessarily a philosophical 'interpretation' or 'reading' of mathematics, and *Being and Event* is not merely a 'translation' of set theory into philosophical language. Badiou's wager dissolves this search for mathematical foundations because the absolute and necessary truth of mathematics is pronounced directly by Being itself. There is no need to ask what lies underneath mathematical objectivities because Being, following Badiou laicization of ontology, should be the halting point for every investigation. The wager dissolves the problem concerning the nature of mathematical objects, a question whose answers have been aligned to the various typology of orientations that we have listed. If mathematics is ontology, then there are no mathematical objects because ontology presents nothing but pure presentation itself, and without simply constituting the empty game of formalism. Ontology is what is left of thought when everything else has flowed back into the void. There is no need to dig in search of some grounding for mathematics because that would presume a deeper unity behind Being, which would make Being one and not multiple.

Nevertheless, set theory, which is the focus of his book, is a field that has been proposed as a viable meta-mathematical foundation for the formal discourse of mathematics itself. But what is relevant to *Being and Event* is not the actual validity of this proposal but the historical fact of its having been proposed, the fact that it has been suggested by mathematicians that the formal language of mathematics is entirely reducible to the formal language of set theory. Badiou takes the proposal only as a symptom that conditions his construction of a new possibility for philosophy, a possibility that is explored in his following through with his truth procedure initiated by his wager that mathematics equals ontology. In the 'Introduction' to *Being and Event*, Badiou writes:

> [I am not] saying that these domains [i.e. in set theory] are in a foundational position for mathematical discursivity, even if they generally occur at the beginning of every systematic treatise. To begin is not to found. My problem is not, as I have said, that of [philosophical] foundations, for that would be to advance within the internal architecture of ontology whereas my task is solely to indicate its site. However, what I do affirm is that historically these domains are symptoms, whose interpretation validates the thesis that mathematics is only assured of its truth insofar as it organizes what, of [B]eing-qua-[B]eing, allows itself to be inscribed. (*BE*, 14)

In particular, Badiou focuses only on the best known and most widely accepted axiomatic systemization of set theory, the first-order formulation based on the Zermelo-Fraenkel Axioms plus Choice. He finds this formulation to be the one most appropriate and most useful for the following through with his truth procedure, and either rejects or ignores other set-theoretic alternatives.[5]

In fact, I would say that if there is one unnecessary constraint to his metaontology, it is the fact that most of his selections of mathematical symptoms have only concerned topics involving the meta-mathematical formulation of definitions to mathematical concepts. For example, in another work in mathematical philosophy, *Number and Numbers* (2008), Badiou studies the various meta-mathematical attempts at founding the concept of number from the Greeks up to the invention of surreal numbers by John Horton Conway. This narrow purview does not, however, constitute what I would consider to be a serious objection to the legitimacy and general robustness of his philosophical project. His general methodology is not so constrained and not so specific as to prevent its being extended by examining other topics that do not involve meta-mathematical foundations. One can conduct a metaontological study of, for example, enumerative combinatorics, harmonic analysis, analytic number theory or extremal graph theory. For instance, one can take the notion of graph (i.e. network) as a possible mathematical figure for thinking the metaphysics of social relationality, but without needing any preparatory inquiry into some grounding definition of the idea of graph. We must realize that mathematics is a huge discipline, and one studies what one can.

Badiou is not a structuralist

Moreover, Badiou's equation is not a reassertion of the formalist or structuralist ontological paradigm. Even though *Being and Event* can be understood as a return to a more systematic and architectonic treatment involving the philosophy of ontology, Badiou's thesis does not say that Being is essentially structure. The structurality of any structure is linked to the consistency of presentation, while Being, for Badiou, forms an inconsistency prior and exterior to any consistent presentation. Algebra, the particular science that studies mathematical structures, does not necessarily cover the entirety of mathematics, an entirety that is not wholly reducible to the formal relational play of empty signs. In his rejection of the question of foundations, Badiou rejects the possibility of accepting the particular orientation in the philosophy of mathematics known as structuralism.

We know that the term 'structuralism' has a different but related meaning in the twentieth-century history of the human science, philosophy and the philosophy of science. But Badiou's thesis should not be misconstrued as a simple regress to the old philosophical project identified under Ferdinand de Saussure's name. Structuralism is one thing, mathematics another. Following Badiou's equating of ontology with mathematics and his proclaiming that Being is essentially pure multiplicity, our investigations into ontology do not need to be supplemented with entirely separate investigations into relationality. The hypothesis 'Entities are one thing, their interrelations another' no longer holds.

The play of interrelations does not reside in a dimension separate from Being because a relation is just another species of presentation. Badiou writes:

> Even if I have accumulated relational or functional abbreviations, even if I have continually spoken of 'objects', even if I have ceaselessly propagated the structuralist illusion, I am guaranteed that I can immediately return, by means of a regulated interpretation of my technical haste, to original definitions, to the Ideas of the multiple: I can dissolve anew the pretension to separateness on the part of functions and relations, and re-establish the reign of the pure multiple. (*BE*, 447)

It can be shown that any relation between mathematical objectivities is itself a mathematical objectivity, just as any relation between sets is just another set.

Badiou's philosophy nevertheless announces the need to laicize, formalize and de-hystericize our understanding regarding certain figures for alterity, excess, chaos, and the anti-structural, which includes, among others, the infinite, the new, the Other, the Real, and Being itself. *Being and Event* takes into account many of the defining thematics and contributions conferred by the radical movements within poststructuralist and postmodernist thought, while simultaneously declaring the necessity to go beyond them and recognizing the need for a return to systematic thinking. We also understand the intention behind assertions that try to align Badiou's philosophy with 'neoclassicism', although we should add that this term has a specific meaning in *Being and Event*, referring to a constructivist orientation of thinking that is in contradistinction to Badiou's own.

Being is not essentially mathematical

Badiou's wager should not be construed to imply that Being equals mathematical objectivities, or that concrete entities such as marble tables, mobile phones, humpback whales, Uzbekistan, the asteroid belt or nucleic acids are composed essentially of mathematical objectivities. The equivalence between ontology and mathematics conserves the trivial ontological difference between parrots and polynomial equations. Badiou does, however, make a second wager that Being is essentially multiple, which is not the same as claiming that Being is essentially the mathematical set. He distinguishes between the inconsistent multiplicity of Being and the consistent multiplicity of sets. It is the discourse of Being, not

Being itself, and it is the discourse of mathematics, not mathematical objectivities themselves, that Badiou hypothesizes to be identical. His equation 'affirms that mathematics, through the entirety of its historical becoming, pronounces what is expressible of [B]eing-qua-[B]eing' (*BE*, 8). He only proclaims that, insofar as a general and direct investigation of Being-qua-Being is possible, that investigation must form the discourse of what, all this while, we have been calling mathematics.

The assumption that 'mathematics = ontology' implies 'Being = mathematical objectivities' is a common misunderstanding, and it is worth delineating our reply more thoroughly. Insofar as we can directly speak and study Being-qua-Being, the ensuing discourse is effectively mathematics itself. The agreement between the two stratified realms of Being and mathematics begins at the first level of discourse and not the zeroth levels of the subject matters themselves. Badiou employs mathematics as a crucial condition for philosophy, but he never goes as far as to think that Being is essentially mathematical, that Being and beings are equivalent to mathematical objectivities. The equivalence of mathematics and ontology does not entail the equivalence of their defined matters of inquiry. Mathematics, by definition, studies mathematical objectivities while ontology, by definition, studies Being. But the equivalence between mathematics and ontology does not imply the equivalence between the 'occupants' within the realm of mathematical objectivities and the beings inside the realm of Being-qua-Being. Mathematics involves a domain of discourse whose objectivities are definitionally characterized by the non-predicate of being-mathematical, while ontology involves a domain of discourse whose objectivities are defined by the non-predicate of being-Being. But the equation between the two discourses does not mean that the non-predicates are identical.

But how is this possible? How can equivalent discourses involve themselves with different matters? How can mathematics be equal to ontology when one implicates mathematical objectivities and the other implicates Being-qua-Being? We repeat that ontology implicates Being without actually being philosophical, for that level of discourse only begins at the upper level of metaontology. So the equation of mathematics with ontology does not make mathematics itself a philosophical discourse. Ontology forms the basis of philosophy while assuming a separate disciplinary sovereignty. Philosophers should recognize that ontology is self-sufficient and does not need philosophy.

The so-called 'objects' of mathematics and ontology are not really objects per se. In their subtraction from any specific concrete objectivity, they present themselves without presentation. The shared paradox of mathematics and ontology is that they speak of what lies behind every presentation without actually involving a presentation in the usual sense. Mathematics and ontology are both discourses without entities. This is the paradox of abstract and subtractive presentation. The so-called abstract and 'abstractive' dimension of mathematics and of its so-called 'objects' – a dimension often non-rigorously understood to inhabit some transcendental domain above and beyond the material, the concrete and the particular – is explained away and supplanted by the subtractive and laicized dimension of ontological discourse. Mathematics is abstract because, in the generality of ontological discourse, it withdraws from the particularity of presentation. Mathematics is involved only with the Being of entities and with Being in general. It no longer makes sense to ask about the specific Being of the abstract as it leads to the specious question about the 'Being of Being' and not the valid question about Being-qua-Being that is immanent to itself.

One deploys ontological and mathematical investigations without actually possessing precise answers to the questions of 'What is Being?' and 'What is a mathematical object?'. Such investigations are deployed in the reflexive foreclosure of their identity. Even though mathematical objectivities provide the ontological schemas and form-multiples for certain facets of presentation (e.g. number for quantity and geometric figure for certain spatialities), such schemas are formalized without constituting a presented singular object. The form-multiples are objectivities without objects. So the discourse of mathematics is not defined by the singularity of its objectivities, by the singular object-ness of numbers, geometric shapes, algebraic structures and so on. Not only is there no meaning to being a mathematical 'object' but it also no longer makes sense to derive some foundational basis for mathematics by inquiring into the basis of its objects. This false assumption that mathematics has objects is, in fact, a symptom of the onto-theological mode of the forgetting of Being as diagnosed by Heidegger.[6]

Moreover, according to Badiou, a mathematician conducts research in mathematics without knowing its equivalence to ontology. Even if the mathematician knows of Badiou's equation and has read *Being and Event*, this knowledge is at no point represented in his mathematical work. Otherwise, this would place Being in the general unified position of a presented object, which would corrupt the exigency of ontology itself as a subtractive discourse. The working mathematician, who has no ultimate need for philosophy when he is doing mathematics, works only with what directly concerns the latest contemporary innovation. In a sense, every research mathematician is a pragmatist when it comes to his research.

> Even if practical mathematics is necessarily carried out within the forgetting of itself – for this is the price of its victorious advance – the option of de-stratification is always available . . . it restitutes the multiple alone as what is presented, there being no object everything being woven from the proper name of the void [*le vide*]. This availability means quite clearly that if the forgetting of being is the law of mathematical effectivity, what is just as forbidden for mathematics, at least since Cantor, is the forgetting of the forgetting. (*BE*, 447)

When Badiou proclaims that mathematics is ontology, there is nothing directly 'necessary' about this equivalence. The fact that mathematics and ontology, in their own ways, formalize our knowledge of That-Which-Subtracts-Itself-From-Presentation is not enough reason to equate the two discourses together. Still, *nothing* is stopping us from doing so. The verification that proceeds after the equation only shows that it is not contradictory for us to take mathematics as being 'effectively' equal to ontology. The gap of modality, the abyss of reason, introduced by this 'effectively' is precisely a demand for a committed decision one way or another. The pure decision cannot verify itself prior to its formulation. Perhaps mathematics is ontology, and the audacity of pursuing this 'perhaps' is the mad gamble of a militant who follows through with the trajectory of a truth procedure. We will see later that every subject is sustained by such a militant madness.

Metaontology versus the mathematical sciences

Badiou's equation partially clarifies the powerful and, to use Eugene Wigner's phrase, the 'unreasonable effectiveness' of mathematics for the natural sciences (1960). This does not mean that the empirical investigations and mathematical modellings of the scientists are completely reducible to the formal and non-empirical activity of the pure mathematicians. 'Physics, itself, enters into presentation. It requires more, or rather, something else, but its compatibility with mathematics is a matter of principle' (*BE*, 7). The physical realm involves physical entities that present themselves in some situation. Presentation must obey the laws of Being-qua-Being and must be consistent with the totality of what we currently know of ontology via mathematical knowledge and mathematical investigations.

However, it is worth reminding ourselves that *Being and Event* is not meant to be a treatise in either pure or applied mathematics. We are not presented with a chronicle of ontology comparable to what one finds in mathematics textbooks or in the various volumes labelled under the heading of popular mathematics. Badiou is not a mathematics journalist and his book is not meant to be a popular expository treatment of set theory for the layman. Even though it is conditioned by mathematical set theory, *Being and Event* is meant to be a philosophical work, albeit under a newly proposed definition-prescription of philosophy. None of the mathematics in the book is original and Badiou does not prove any new theorems. The decision 'mathematics = ontology' is a metaontological thesis, not a mathematical or ontological one. The book's formal and extensive demonstrations are only meant to explicate and frame the relevant details for its reader.[7]

Since *Being and Event* is not a mathematics textbook, it cannot be expected to explain in complete and rigorous detail all the related concepts and theorems in set theory and forcing. Instead of the full proofs, Badiou often just presents sketches or what he calls 'accounts of demonstrations' (*BE*, 271) to the theorems. The level of attention to the formal details depends entirely on whether they are pertinent to the metaontological issue at hand – or, rather, on the extent to which Badiou is able to extract some metaontological truth from them. This methodology is comparable to what occurs when a philosopher reads a literary work closely with the purpose of extracting some philosophical truth from it. One stops reading once one is unable to do anything more with the text in question. The essential tasks of the critic and the scholar are to interpret and to learn, while the essential task of the philosopher is to think.

Moreover, Badiou's metaontology is not meant to be a scientific modelling of the relevant philosophical concepts. *Being and Event* does not fall under the Galilean paradigm of mathematizing what we know of the world via scientific practice and experiment. Science is one thing, philosophy another. Badiou's mathematicization falls under the different registers of conditioning and compossibilization. Even though his metaphysical framework presents itself as a return to architectonic thinking, this systematicity does not necessarily make it a science, even though it is conditioned by a particular non-empirical science, the science of sets. Furthermore, Badiou's book begins its ontological analysis by beginning with the question of pure Being and not from the direction of concrete and empirical beings.

A different misunderstanding disapproves Badiou's choice of using comparatively 'old' results from mathematical logic and set theory as the main conditions for his philosophy. It was claimed that those fields have ceased to be the centre for current mathematical research. This is incorrect because much of the current research in logic and set theory continues to play a huge role in the active fields of discrete mathematics, combinatorics, the theory of computation and computer science, although we note that mathematicians in these fields often have to battle for the recognition and respectability of their work when dealing with the monastic 'patricians of the abstract' among researchers in algebraic geometry and topology. In fact, Badiou does not pretend that his mathematics is informed from the most up-to-date or most dynamic fields in contemporary mathematics. Set theory is only a symptom of what is the true focus of his inquiry, the truth organized by his equating of mathematics with ontology. Badiou encourages his readers to look into more contemporary topics, although he appears to have an ambiguous attitude towards any possible 'knighting' of his work by the mathematicians.

The question of this 'knighting' is a difficult one and partly concerns the complex socio-politics within academia concerning inter- and cross-disciplinary pursuits, particularly between the humanities and the sciences. We will not be able to expand thoroughly on this issue. Among others, it relates, in my experience, to the strangely elitist attitude and the perverse 'superiority complex' of mathematicians towards the work of their comrades in philosophy and in the humanities. 'Empirically, the mathematician always suspects the philosopher of not knowing enough about mathematics to have earned the right to speak' (*BE*, 11). For various reasons, it is true that the typical philosopher is usually extremely unlearned when it comes to basic mathematics, and even more with contemporary mathematics. The standard grumble is that the descriptive remarks about mathematics by philosophers, even by philosophers in analytic philosophy, are devoid of contemporary validity or relevance. Moreover, mathematicians, when they are doing mathematics, have no necessary use for philosophical or metaphysical considerations of their field.

Badiou accuses his critics among the mathematicians of being overly severe in their requirements of how much and at what depth a philosopher needs to know about the technicalities in contemporary mathematics. He also accuses these mathematicians of being overly lenient in judging the immanent value of philosophical pronouncements qua philosophy. 'We thus find ourselves, for our part, compelled to suspect mathematicians of being as demanding concerning mathematical knowledge as they are lax when it comes to the philosophical designation of the essence of that knowledge' (*BE*, 13). When encountering a philosophical work that involves mathematics, the mathematician often seems to be concerned only with the immaterial question of whether it is 'allowed', through a sort of aristocratic procedure, to knight the author as belonging to the austere club of genuine scholars in contemporary mathematics – hence, the bravura in the mathematicians' response of displaying their own erudition on their subject when encountering, reading and commenting on the work of philosophers.

But being a mathematics scholar is different from being a mathematical philosopher, although it is understandable that mathematicians 'only trust whomever works hand in hand with them grinding away at the latest mathematical problem' (*BE*, 11). A philosopher's quest for mathematical knighthood is, in principle, unproductive in relation to the immanent tasks of philosophy. Nevertheless, the ethics of philosophical appropriation with respect to its conditions remains, as with any ethics and any practice of appropriation, essentially obscure. But an event must take place in the ethical encounter between philosophy and mathematics.

Metaontology versus humanistic philosophy

The resistance on the side of the 'humanistic' philosophers concerns a different matter. Whereas mathematicians are impatient with the philosophy, non-mathematicians are impatient with the mathematics. 'In the end,' Badiou writes, 'it turned out that due to my having company with literature, the representatives of analytic philosophy . . . attempted to denigrate my use of mathematical formalism. However, due to that very use, the pure continentals found me opaque and expected a literary translation of the mathemes' (*BE*, xiv). This was one of the reasons why *Being and Event* did not lend itself to immediate comprehension when it first came out in 1988. 'Mathematics,' Badiou writes, 'has a particular power to both fascinate and horrify which I hold to be a social construction: there is no intrinsic reason for it' (*BE*, 19). When reading Badiou's work, it is therefore necessary to be courageous and try to dispel as much as possible any anxiety, timidity, impatience and phobia when confronting the majestic splendour of the concepts and theorems.[8]

The highlight in the latter half of *Being and Event* involves a piece of mathematical technology called forcing, which was discovered in the 1960s.

We acknowledge with humility that attaining some workable understanding of the technical machinery behind forcing is a difficult task, even for the average mathematics graduate. The technique is almost never taught in the undergraduate mathematics curriculum, and is usually only formally communicated to postgraduates majoring in set theory or in a small handful of closely-connected fields. The difficulty in understanding forcing presents itself, among other ways, first, when one is trying to conceptually grasp the abstract ideas involved and, second, when one is meticulously learning the intricate details of the mathematical technology. Our focus will be mainly on overcoming the former difficulty at the expense of the latter, although details of the proofs of the most important theorems will be indicated.

Badiou thinks that, for the reader lacking a mathematical background, the main psychological obstacle 'probably resides in the assumption that mathematical competence requires years of initiation. Whence the temptation for the philosophical demagogue, either to ignore mathematics altogether or act as if the most primitive rudiments are enough in order to understand what is going on there' (*TW*, 18). We supplement this diagnosis with the remark that the unnecessary division of intellectual life into the Two Cultures has led to a prejudice that bars any form of sophisticated mathematical thinking from entering philosophical discourse. This discrimination relates to the fact that, for some time now, and with only a handful of exceptions, the main domains for conditioning philosophy in the continental school have been primarily politics and, in particular, art. When attempting to shed light on philosophical issues, philosophers have no qualms about closely reading and quoting Franz Kafka, Marcel Proust, Arthur Rimbaud, Fyodor Dostoevsky, Wallace Stevens, William Shakespeare or T. S. Eliot. But some of these philosophers actively seek either to minimize or to avoid any contact with advanced mathematics unless it is absolutely necessary and unavoidable. Moreover, the average philosophy graduate today is expected to have extensive knowledge about the arts, the humanities and politics, but little or no acquaintance with elementary mathematics. Badiou writes:

> It is striking that . . . no justification whatsoever seems to be required for quoting poetry, but no-one would ever dream of quoting a piece of mathematical reasoning. No-one seems to consider it acceptable to dispense with Hölderlin or Rimbaud or Pessoa in favour of Humpty Dumpty, or to ditch Wagner for Julio Iglesias. . . . Philosophers are able to understand a fragment by Anaximander, an elegy by Rilke, a seminar on the real by Lacan, but not the 2,500-year-old

> proof that there are an infinity of prime numbers. This is an unacceptable, anti-philosophical state of affairs. . . . (*TW*, 18–19)

We are not dictating that philosophers must be professional mathematicians or mathematics scholars, any more than we are proposing that philosophers must be literary critics and political scholars. Art, the humanities and the sciences are all capable of contributing to the current philosophical discourse. Conducting any form of serious and high-level research is always challenging, and one cannot be expected to be a total polymath. But there should be room in philosophical thought for the participation of mathematical thought. Badiou claims 'the right to quote instances of mathematical reasoning, provided they are appropriate to the philosophical theses in the context of which they are being inscribed' (*TW*, 18).

Part of the reasons why Badiou's wager is so surprising is due to the long and widespread prominence of the post-Heideggerian doctrine that identifies the question of Being with a mytho-poetic framework of thought. Badiou would disagree with Heidegger's linking of mathematics to the principal manifestation of a nihilism that forgets the inaugural question of Being. Ideologically enslaved to a doctrinal conception of ontology, Heidegger links the withdrawal and unveiling of Being with endowment, with the gift of *aletheia*. Haunted by the mythological loss of origins and the dissipation of presence, such an ontological framework is poetic. Badiou calls it an 'ontology of presence' (*BE*, 27), instead of an ontology based on the rigorous subtraction from presentation. Badiou opposes this poetic offering of an ontological dimension that radically withdraws from any representation and presentation. When it withdraws, and as it withdraws, what is left of Being can, according to Badiou, be subject to systematic investigation, an investigation whose conditions are met by mathematics in its axiomatic and deductive form. 'I will say that [B]eing-qua-[B]eing does not in any manner let itself be approached, but solely allows itself to be sutured in its void to the brutality of a deductive consistency without aura. Being does not diffuse itself in rhythm and image, it does not reign over metaphor, it is the null sovereign of inference' (*BE*, 10). Here we find ourselves returning to the original Platonic decision to interrupt the poem and supplant it with mathematics. 'Let no one ignorant of geometry enter.' Badiou replaces poetic ontology with a mathematical ontology that is not supported by the nihilism of technology. He writes:

> Mathematics regulates in and by itself the possibility of deconstructing the apparent order of the object and the liaison, and of retrieving the original

> 'disorder' in which it pronounces the Ideas of the pure multiple and their suture to [B]eing-qua-[B]eing by the proper name of the void. It is both the forgetting of itself and the critique of that forgetting. It is the turn towards the object, but also the return towards the presentation of presentation. (*BE*, 447)

When faced with the almost universal prominence of the post-Heideggerian doctrine, which is then supplemented, first, with the centuries-old theologization of ontology and, second, with the hegemony of the Two Cultures that seek to separate humanistic and scientific discourse, it is no wonder that it should seem so counter-intuitive that ontology is none other than mathematics. I would say that the surprise in encountering Badiou's wager is comparable, perhaps, to that of Levinas's equation 'first philosophy = ethics' several decades ago and Spinoza's equation 'God = substance' several centuries ago. What the three equations share, among other things, is a participation in the constant theologization and desacralization within the history of thought with respect to the dialectical suture between what is central and what is other than philosophy. Through the boldness of a total identity (instead of a mere predication), these propositions link ideas and concepts that have long been held to be non-related and whose respective discourses had seemed to involve absolutely separate sovereignties of experiences. Theology is directed towards substance; first philosophy is directed towards the other; and mathematics is directed towards Being.

2

Ontology of Axiomatic Set Theory

The first half of this chapter is devoted to explaining the relevant essentials of the first mathematical bulwark, which Badiou simply lists as 'the axioms of set theory' (*BE*, 20). Much is at stake there. Before dealing with those axioms, first we elucidate the rudiments of 'naive' set theory, a topic that is indispensable for appreciating those axioms in the first place and understanding that they are meant to describe the formal framework for a mathematics of multiplicity. We then expound on two advanced but necessary topics: the Completeness and Incompleteness Theorems of Kurt Gödel. Some of the basics in the mathematics of model theory are supplied along the way.

Let us begin in the most obvious way, by providing preliminary instruction on the concept of the set. This concept can be introduced via either the route of intuition or the route of rigor – that is, either by appealing to some intuitive understanding of the set and its related notions, or by describing some rigorous system where a list of precise statements, called axioms, is written down and its structural implications followed through. The first method leads to naive set theory, while the second method gives axiomatic set theory, the focus of *Being and Event*. The first route can also be described as more semantically and informally motivated, and the second as more syntactic and formal. In the former, the 'concept' of the set takes precedence over the 'science', while the reverse is true for the latter. In a certain sense, the difference between the two methods can be compared, at least in part, not only to the difference between the intuitionist and formalist schools in the philosophical conception of mathematics – as typified in the eminent figures of Luitzen Brouwer and David Hilbert – but also to the difference between mathematics understood in terms of 'models' or in terms of 'formal axiomatic systems'. In the former, we explore, like a tourist, some situation or 'world', immersing ourselves in the universe, familiarizing ourselves with the entities that they present. In the latter, we receive and obey a certain list of rules, descriptions and formalisms, but without appropriating or internalizing them, without necessarily knowing what they mean, and then derive their consequences mechanically like a computer without a soul – or, if I may say so, a zombie, a dead biological machine, without a subjectivity.

Mathematical concept of the set

This might be a mere accident in the etymological contingencies of Anglo-American language and culture, but perhaps there is a deep reason, deep within the hidden recesses of human language, history and civilization, for the fact that the word 'set' was, for a long time, the one with most meanings in the English language:

1. a group of things of the same kind that belong together and are so used
2. a permanent inclination to react in a particular way
3. the descent of a heavenly body below the horizon
4. the process of becoming hard or solid
5. an unofficial association of people
6. to put in a certain place
7. to fix in a border
8. to determine
9. to establish
10. to specify
11. to locate
12. to plant; etc.

Whatever these various definitions have in common, it might imply that the very general notion of the set, of sethood, is something very primitive not only to Anglo-American culture and civilization but also to human thinking, that it points to some fundamentally intuitive arche-concept within human culture – or maybe even to the ever-shifting profundities of Being itself, to our conception of Being-qua-Being. This arche-concept corresponds, in part, and in ever-shifting degrees of correlation, to the notions of unity and multiplicity, of the singular and the plural – in short, of the multiple.

For our purpose, let me 'set' the stage by stating that, in informal terms, a set is a simple multiple of objects taken as a single entity, or a single entity understood as a simple multiplicity of objects. We immediately observe that every existent entity is a set, and that every entity exists in the form-multiple of a set. Everything is a collection and everything is collected into something else. To be is to be a set.

Besides 'set' or 'multiple', other more or less synonymous English words or collective nouns can be used here: accumulation, aggregate, aggregation, anthology, arrangement, array, assemblage, assembly, catalogue, category, choice, class, compilation, crowd, cumulation, diversity, flock, gang, gathering, group, horde, host, list, mass, mixture, multitude, network, pack, posse, range, school, selection, series, span, squad, summation, swarm, team, total, totality, type, universe, variation, variety, etc.[1] To give a few examples, we might have:

1. a set of rusty razor blades and bicycle chains
2. a set consisting of the number 92 and the letters 'n', 'm' and 'r'
3. a set of all even numbers larger than 51
4. a set of phrases in Latin, Esperanto and Yiddish
5. a set of important events in a Brazilian footballer's career
6. a set of discoveries in the fields of genetics and molecular biology before the twentieth century
7. the set of all philosophers
8. the set of all Marxist philosophers
9. the set of all living Marxist philosophers
10. and the specific set consisting of the five living Marxist philosophers: Alain Badiou, David Harvey, Slavoj Žižek, Fredric Jameson and Antonio Negri

The entities that a set collects together are called its members or elements. The elements belong to the set and the set contains its elements. A set is a multiple of its members that consist together as one. The elements are what the set 'counts' in, what counts as members in the set. To use Badiou's terminology, the set 'counts-as-one' its elements, and the elements on the whole are counted as one by the set. Alain Badiou is a member of each of the four examples of sets (7)–(10) that we have listed earlier. He is also a member of the set of all former students of Louis Althusser, the set of all Morocco-born *normaliens,* and the set of all people who wear spectacles and whose surname starts with the letter 'B'.

Without losing any internal consistency, the link between sethood and the count-as-one [*le compte-pour-un*] allows itself to become complicated in various ways:

1. We can have sets whose members seem to have very little in common, like the following set *T* that gathers these five entities: the philosopher Alain

Badiou, the poet Friedrich Hölderlin, the word 'situation', the number 35, and the month of May 1968. There is not one single discernible unifying property that all these elements share, other than the fact that they belong to *T*. A set is nothing more than its own relation of belonging, nothing more than the 'property' of belonging to it. This is crucial because, in set theory, all predicates return in the end to the 'property' of belonging.

2. A set can also be a singleton that has only one member, for 'collection' can also mean collecting just one element. For instance: the set that includes only the country Cuba, or only the authors of *Being and Event*, i.e. the set with Alain Badiou as its sole member. Moreover, set theory understands the singleton as nominally distinct from the object it contains. The set consisting of Badiou is different from Badiou himself.
3. In fact, nothing prevents a set from having no elements at all, while still being a set. A collection can collect nothing, while still remaining a multiple. This is the empty set, which is unique, for there cannot be two or several different empty sets. Even though it is not exactly identical to the void, the void is what it gathers into itself. Badiou would say that the empty set lies 'at the edge of the void'.
4. A set can also have an infinite amount of elements, like the set containing all the numbers 0,1,2,3,4 and so forth; or the set of points within some single continuous spatial interval; or the set of all possible sentences in Spanish. As we shall soon discover in our discussion of cardinal numbers, the possibility for infinite sets can be made to become extremely complex, even infinitely complex: infinitely, obscurely and undecidably complex.
5. What is more, nothing prevents a set from collecting other sets within itself. Those sets may themselves contain sets, and so on. We can have a set consisting of all the examples of set that we have mentioned so far. We can have a set consisting of three elements: Alain Badiou; the singleton set consisting only of Alain Badiou; and the empty set.
6. And not only can we have an empty set but we can also have sets that contain, not just nothing, but the empty set itself: the singleton that has the empty set as its sole member.
7. And we can even have sets containing sets containing sets containing the empty set. It is possible for a set to contain nothing but sets that contain, way down, just the empty set: nothing at all but the empty set at the very bottom. The name we give to such sets, pure sets, is quite fitting, for not only do such sets contain nothing but sets, but they are also, in themselves,

nothing but sets: pure multiplicity in and of itself. The possibility of pure sets is crucial since, for Badiou, ontology is precisely the science of pure multiplicity.

A unique formal language is needed in order to talk and write about sets to some sufficient degree of precision. Mathematicians have developed the standard notation, which is built upon the grammar of first-order logic. A set can be specified by inscribing an unordered list of its elements, surrounded usually by curly brackets. Ellipses can be used to indicate missing elements that are understood in context. The previously mentioned set T of five disparate elements can be written as:

T = {Alain Badiou, Friedrich Hölderlin, 'situation', 35, May 1968}.

Since the order in which we present the members does not matter, this set is identical to the alternative specification:

T = {'situation', Friedrich Hölderlin, 35, Alain Badiou, May 1968}.

The set of all whole numbers from zero up to infinity, the natural numbers, usually denoted as $\mathbb{N}$, can be written as:

$$\mathbb{N} = \{0,1,2,3,4, \ldots\},$$

listing all the numbers in ascending order. Since order does not matter, we can list all the odd numbers first, up to infinity, and then the even numbers, up to infinity:

$$\mathbb{N} = \{1,3,5,7, \ldots, 0,2,4,6, \ldots\}.$$

When we say that a set fundamentally disregards order, this does not mean that we cannot somehow codify this order into the set, that we cannot make that order 'count' somehow in the language of specifying sets. An ordered set[2] can be constructed in principle, but we omit here how it is done. For the moment, if we want to include order into the set, we simply replace the curly brackets with round brackets (some conventions use angular brackets instead). So one ordered analogue to the aforementioned set T would be:

(Alain Badiou, Friedrich Hölderlin, 'situation', 35, May 1968).

This is the ordered set where Alain Badiou is the first element, Friedrich Hölderlin the second, and so on. We hope the reader can trust our claim that there is a way to write ordered sets as unordered sets using curly brackets while allowing the ordered procession to be coded directly into the more fundamental notation.

We write $x \in S$ to say that x is a member of some set S, and $x \notin S$ to say that it is otherwise. The latter is simply a short form for writing '$x \in S$ is false'. So *Alain Badiou* $\in T$ and *battery* $\notin T$ are true propositions. What is more, an entity must always either belong or not belong to a set, but never both at the same time. In other words, if x is some possible element and S is some set, then both of these propositions must be simultaneously true:

1. either $x \in S$ or $x \notin S$, i.e. one of the propositions $x \in S$ or $x \notin S$ must be true,
2. not both $x \in S$ and $x \notin S$

Formally, a set defines itself fundamentally by its belonging relation, by its membership relation.

The empty set is denoted with a pair of empty curly brackets {} or using the Scandinavian letter ∅. The set containing the empty set would then be denoted as either {{}} or {∅}. The set containing that set would then be {{{}}} or {{∅}}. Examples of pure sets – sets that contain nothing but sets all the way down to the empty set – would include these onion-like structures: {{{{}}}}, {{}, {{}}, {{{}}}}, {∅, {∅}, {{∅}}, {{{∅}}}} and {∅, {{∅}, {{{∅}}, {{{∅}}}}}}. The empty set is, of course, the first example of a pure set as all pure sets are built from it using curly brackets. In other words, a pure set is either the empty set or a set that contains other pure sets.

Badiou gives much of the details of first-order logic in his 'Technical Note' (*BE*, 49–51) that follows from Meditation 3 in *Being and Event*. The result of grounding set theory in the language of first-order logic creates a limited but sufficiently powerful formalism that consists fundamentally of only one relation, namely that of belonging, of membership. Any other additional relations can be defined in terms of this arche-relation of belonging.

When used, this formal language will always be situated with respect to some 'world', some domain of discourse. (This is due to the Axiom of Separation, which we will explain soon.) Every domain of discourse specifies the individual entities that exist within that 'world'. In standard set theory, the domain is sometimes denoted with the letter U, for universe.

Lowercase letters will be used in the language to introduce variables within the universe for *single* indeterminate individuals. I stress the word 'single' here. If the language and the domain involve, for example, a particular room of people, then those individual entities might be the individual persons in that room. If we are talking about the universe specified by set theory, then those entities might be all the sets that exist – whatever 'exists' means, and whatever constitutes the ontological commitment within set theory. The simplest possible propositions in the language, the atomic propositions, are those that assert the relation of belonging, using the symbol '$\in$' that we mentioned earlier. In addition to the use of semicolons and parenthesis to ensure unique readability, the additional symbols used are the following:

1. '$\sim$' abbreviates 'not'
2. '$\vee$' abbreviates 'or'
3. '$\wedge$' abbreviates 'and'
4. '$\rightarrow$' abbreviates 'implies'
5. '$\leftrightarrow$' abbreviates 'if and only if'
6. '$=$' abbreviates 'is equal to'
7. '$\neq$' abbreviates 'is not equal to'
8. '$\forall$' abbreviates 'for all'
9. '$\exists$' abbreviates 'there exists'

The symbols '$\vee$', '$\wedge$', '$\rightarrow$', '$\leftrightarrow$' and '$\sim$' are the typical logical connectives used to concatenate different propositions. To give the details, let P be any proposition, which can be either true or false, atomic or complex. The more complex propositions are built using these logical connectives. Let P be the proposition '$x \in a$':

1. Then $\sim P$ is true if P is not, i.e. if x is not a member of a.

Let Q denote another proposition, say '$y \in a$'. Then:

2. $P \vee Q$ is true if either one, or both, of P and Q are true, i.e. if at least one of x or y belong to a.
3. $P \wedge Q$ is true if both P and Q are true, i.e. if both x and y belong to a.
4. $P \rightarrow Q$ is true if, when P is true, then Q is true. If P is already false, then this concatenated expression is always true, i.e. it is only false when x belongs to a but y does not.
5. $P \leftrightarrow Q$ is true if P and Q are mutually true or false, i.e. if either both x and y belong to a or both x and y do not belong to a.

Special care must be taken in explaining the symbols '$\forall$' and '$\exists$', which correspond to the universal and existential quantifiers, respectively. Given some proposition $S(x)$ depending on the free variable x, the statement $\forall x$: $S(x)$ says that the proposition is always true for all possibilities of x from the individuals that make the domain of discourse, while $\exists x$: $S(x)$ says that there exists at least one possibility for x, one existing individual from the domain, where it is true. Say $S(x)$ denotes '$x \in S$', then $\forall x$: $x \in S$ means that each individual in the domain is a member of S, while $\exists x$: $x \in S$ means that at least one entity belongs to S.

Since variables within first-order logic are meant to indicate only single individuals within the domain of discourse, some limitation is placed on the range of quantification, which must cover only single individuals from U, and never multiples of them (or their properties – and we will learn later that every property is ultimately a set, the property of belonging to some set). Quantification in first-order logic can only directly talk about single individuals, not multiples of them. Statements can be made only about individual entities, not sets of them. 'The fundamental principle,' writes Badiou, 'is that the formulations "for all" and "there exists" only affect terms (individuals) and never properties. In short, the stricture is that properties are not capable, in turn, of possessing properties (this would carry us into a second-order logic)' (*BE*, 49). This limit of quantification is what makes first-order logic.

We must also explain some basic facts about equality and negation in standard set theory. Now the symbols '=' and '≠' correspond to the usual relations of equality and inequality. As noted previously, the relation of belonging is the most basic and primitive relation in set theory, even more basic and primitive than the relation of equality. The following statement:

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

called the Axiom of Extensionality, says that if every element z of x is also an element of y, and vice-versa, then x is identical to y. It says that 'two sets are equal (identical) if the multiples of which they are the multiple, the multiples whose set-theoretical count as one they ensure, are "the same"' (*BE*, 60). In other words, two sets are equal when all of their elements are equal. So the relation of equality is definable in terms of the relation of belonging.

Negation in standard first-order logic obeys two basic laws:

1. The Law of Excluded Middle states that, for every proposition, either the proposition is true or its negation is. In other words, $P \vee \neg P$ holds for any proposition P.
2. The Law of Non-contradiction disallows paradoxes. So $\neg(P \wedge \neg P)$. Every proposition cannot be both true and not true. For every proposition, it

> cannot be that both its affirmation and its negation hold. The main reason for not allowing paradoxes is that they can be made to imply anything, from the notorious Principle of Explosion.

We mention in passing that some alternative logics differ from standard first-order logic in their treatment of these two laws. Intuitionistic logic does not accept the excluded middle, and instead allows for alternatives to true and false. Paraconsistent logic allows for self-contradictions by reducing the damage done by the Principle of Explosion. There is also the related proposition of dialetheism, which claims that a proposition can be simultaneously true and false.

Intensional specification of multiples

Having explained the basics of the language of first-order set theory, we can now provide the alternative way to specify multiples. The previous method, by the listing of the elements, by the naked presentation of the members, is called the extensional specification. There, a set is identified solely by what belongs to it, by its belonging relation. Although it is not always the case, a set can sometimes define itself intensionally by specifying a concept or predicate, a single and finitely expressed predicate shared by each of its members and only its members. For the extensional set $\{0,2,4,6, \ldots\}$, this necessary and sufficient concept is the predicate of 'being an even number', or 'being a whole number and divisible by 2'. The ways of expressing the predicate are potentially endless. Written in the standard notation, the corresponding intensional specifications make use of what is called a dummy free variable. It is written as{x: *x is an even number*} or $\{x: x \in \mathbb{N} \wedge x/2 \in \mathbb{N}\}$. The last example can also be written as $\{x \in \mathbb{N}: x/2 \in \mathbb{N}\}$, meaning that the value of the free variable x is pre-quantified to the whole numbers as the domain of discourse.

The notion of 'predicate' can be made mathematically rigorous by identifying it with an expression or formula, written in the language of first-order set theory, that is satisfied by the elements having it. As Badiou writes, '"set" is what counts-as-one a formula's multiple of validation' (*BE*, 39). The concept is reduced to a predicate expressed in the language of first-order set theory. In the previous example of the set of even numbers, the formulas are $\lambda(x)$ = '*x is an even number*' and $\lambda(x)$ = '$x \in \mathbb{N} \wedge x/2 \in \mathbb{N}$'. Being 'finitely-expressed' means the formula only requires a finite sequence of symbols in order to be written. Being 'single' means there is only one formula for each intensional specification, although a finite collection of formulas can be merged into one by stringing them together with the conjunction '$\wedge$'. Moreover, any extensional specification of a finite set can be converted into an intensional specification by concatenating, using the disjunction '$\vee$', the corresponding relations of equality. For example, the extensional set $\{a,b,c,d,e\}$ translates into the intensional set $\{x: x = a \vee x = b \vee x = c \vee x = d \vee x = e\}$. The conversion does not work automatically for infinite sets.

The extensional versus intensional distinction in the specification of multiples is crucial because the former dominates over the latter in set theory, for a set is identified fundamentally by its belonging relation and not by some predicate or unified principle governing it. This follows immediately from the Axiom of Extensionality. It also refutes Leibniz's well-known law in Section 9 of his *Discourse on Metaphysics* (1686) on the identity of indiscernibles. The identity of two sets is not governed ultimately by the fact that they share the same concept, that they are predicatively indiscernible. Suppose that Leibniz's Law holds. Since every predicate corresponds to a formula expressed in the formalized language, this implies, as Badiou writes, that the 'control of language (of writing) equals control of the multiple' (*BE*, 39), that 'the master of words is also the master of the multiple' (*BE*, 40). As a result, Being can never be in excess of language, or at least any sufficiently well-constructed logical language.

Russell's Paradox and the Axiom of Separation

Unfortunately, not every formula gives us a set of terms validating it (although many do). The multiple created by the identification of a predicate is informally called a class in mathematics. Every intensionally specified multiple is, strictly speaking, a class, but not automatically a set. Not all sets are classes and not all classes are sets. The following proposition, called the Axiom of Comprehension is false:

$$\varphi \text{ is a predicate} \rightarrow \text{the class}\{x: x \text{ is } \varphi\} \text{ is also a set.}$$

The most famous counter-example to the proposition that every class is a set was given by Bertrand Russell when he had φ correspond to the self-referential predicate of 'not being a member of oneself'. The class X satisfies this predicate φ if $X \notin X$. In the terminology given in *Being and Event*, such classes are called 'ordinary' [*ordinaires*], and 'evental' [*événementielles*] if otherwise.[3] An example of an evental class would be one consisting of all entities that are not Alain

Badiou, i.e. $\{x: x \neq \textit{Alain Badiou}\}$. Such a class includes everything, except for Alain Badiou. Since the class itself is obviously not Alain Badiou, it is therefore a member of itself.

Now consider something a little bit more complex, the class of all ordinary sets:

$$C = \{X : X \notin X\}.$$

This class cannot possibly be a set since that would be self-contradictory, for C having the predicate implies its not having it, and vice-versa. If $C \notin C$ then $C \in C$, and vice-versa, a paradox. To put it more explicitly:

1. Suppose $C \notin C$, that the class is not a member of itself. But C contains, by definition, everything that is not a member of itself. So $C \in C$, because C has to be a member of itself.
2. Now suppose that $C \in C$, that the class is a member of itself. But C contains everything that is a member of itself, contradicting our original supposition.

Russell's Paradox does not necessarily imply that the peculiar predicate 'not being a member of oneself' should be banned altogether from set theory. It is possible to 'fix' or at least minimize the damage incurred by requiring that all predicates used in the specification of sets concern themselves only with elements that are already members of other sets – pre-existing elements, in other words. A set can only collect, can only count as one, elements that are members of other existing sets. In other words:

$$\varphi \text{ is a predicate} \wedge Y \text{ is set} \rightarrow \text{the class } \{x \in x: x \text{ is } \varphi\} \text{ is a set.}$$

We call this statement the Axiom of Separation. Given some set, it is possible to 'separate' all its elements that satisfy some predicate and create another set. Predication works in set theory only for pre-existing elements, for 'a predicate only determines a multiple under the supposition that there is already a presented multiple' (*BE*, 45). In this case, it is the multiple, the logic of belonging, of some pre-belonging, that precedes language.

By accepting the Axiom of Separation and by considering Russell's Paradox, we must also require that the class of all pure sets, usually denoted as V, is not itself a set, otherwise the Axiom of Separation would be equivalent to the false Axiom of Comprehension. Suppose V, the class of all sets, was also a set. The schema of separation says that $\{x: x \textit{ is } \varphi \textit{ and } x \in Y\}$ is a set for every predicate φ and every pre-existing set Y. Replace Y with V and φ the predicate not being a member of oneself and we have Russell's Paradox all over again. This class of all multiples is the first multiple that is in excess of the concept of set.

The Axiom of Separation also explains why language must be evaluated with respect to a domain of discourse. The axiom says that language can also 'create' sets whose elements are extracted from a previously existing set. That previously existing set is precisely the domain of discourse to which the language evaluates itself. Language must always be situated with respect to a domain, a previously existing set. So an intensionally-specified multiple relates to the set to which it is equal and the set over which its elements range. We must also mention that the *expression* of a formula also involves parameters that can also be sets. Expressing a particular predicate involves a particular vocabulary that mentions other sets as well.

Relation of inclusion and the notions of subset and power set

We continue to the set-theoretic notion of the subset. In addition to the relations of belonging and equality, there is also the relation of set inclusion. One set is included in another, is the subset of another, is part of another, if all of its members are also members of that other set. The smaller set {*Alain Badiou*, *May* 1968}, for example, is a subset of

T = {Alain Badiou, Friedrich Hölderlin, 'situation', 35, May 1968}.

Moreover:

1. Every set is also, by definition, a subset of itself.
2. The empty set $\varnothing$ is the subset of every set, even the subset of itself.

The difference between the relation of element membership and the relation of set inclusion will prove to be crucial in Badiou's philosophy. Formally, we write $Q \subseteq S$ if Q is a part of S. So $Q \subseteq S$ if and only if for every element $x \in Q$, it is also true that $x \in S$. So, by definition:

$$(Q \subseteq S) \leftrightarrow \forall x\,(x \in Q \to x \in S).$$

Given a set, there is a number of possible parts it can have, including itself and $\varnothing$. This number depends on how many members it has. All the existing subsets of a set S form what is called its power set, written as $P(S)$:

$$P(S) = \{X : X \subseteq S\}.$$

For example, the three-element set {*a*,*b*,*c*} has, in addition to itself and ∅, six other possible subsets: {*a*},{*b*},{*c*},{*a*,*b*},{*a*,*c*} and {*b*,*c*}. So the power set of this set has exactly eight members:

$$P(\{a,b,c\})=\{\varnothing,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$$

The relation of inclusion is subordinate to the arche-relation of belonging as the former can be defined wholly in terms of the latter. To be included in a set is the same as to belong to the set's power set. So the relation of inclusion is extensionally equivalent to the power set's belonging relation.

When I wrote that a power set contains all the *existing* parts, the word 'existing' is crucial, for some of the possible subsets could be 'missing'. The size of *P*(*S*) could be 'smaller' than its maximally possible size. For the moment let us keep in mind that certain sets, though definable, do not necessarily exist. They have to pre-exist before they are definable.

It is easy to prove that if a finite set has *n* elements, then its total number of possible parts is exactly the 'definite and calculable' value of 2^n, i.e. 2 times itself *n* times (*BE*, 277). This explains why the power set of *S* can also be written as 2^S. So a set that has, say, five elements, would have $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ parts. A set that has zero elements, the empty set, has only itself as a member: $2^0 = 1$. So $P(\varnothing) = \{\varnothing\}$. The proof of the formula 2^n can be intuited if we understand a part as simply one of the possible ways to either 'allow' or 'disallow' elements from the original set. For example, the subset {*a*,*c*} of {*a*,*b*,*c*} corresponds to the case where we admit *a*, bar *b* and admit *c*. Since {*a*,*b*,*c*} has three elements, and since every element has two possible 'positions' in the same subset, either 'inside' or 'outside', then it is easy to see why the number of parts must be $2 \times 2 \times 2 = 8$. This explanation helps us understand why the empty set and the whole set must also be subsets; they constitute the cases where all the elements are in the 'outside' and 'inside' states, respectively. We can also understand the proof this way: every time we add one more element to a set, the number of possible parts must always double. This is because all of the subsets of the new sets can be divided into two groups of equal size. The first group contains all the subsets of the previous set. The second group has those same subsets, but with the added extra element inserted in each.

There is another crucial theorem about power sets in Badiou's philosophical enterprise, although it is not usually given in the standard syllabus of basic set theory. The proof itself is fairly easy and is intimately connected to Russell's proof of ordinary and evental multiples.

Theorem of the Point of Excess [*théorèm du point d'excès*]: Every set fails to contain at least one subset as a member. As a result, a set is never equivalent to its power set, by virtue of the Axiom of Extensionality.

Proof: Let X be a set, and let C be the subset consisting of all the elements of X that are ordinary, i.e. non-self-belonging. Then C is a subset of X that does not belong to X. Suppose C was an element. Then it is either ordinary or evental. But each case leads to a contradiction:

1. If C was ordinary, then C does not belong to itself. But not to belong to C means that C is evental, which means that C belongs to itself, a contradiction.
2. If C was evental, then C belongs to itself. But to belong to C means that C is ordinary, which means that C does not belong to itself, a contradiction.

So we have here an example of a non-belonging subset of every set, proving the theorem.

Basic set operations

We are now able to express, to some degree of rigour, some basic operations on sets, most of which are not only straightforward but also the ones that we would most naturally employ when it comes to manipulating multiples. To demonstrate the use of these operations, let us take the examples $F = \{a,b,c\}$ and $G = \{c,d\}$. We can take the union of two sets, using the symbol '∪', by 'merging' them together. The result would be the set that contains all the elements from F and G:

$$F \cup G = \{x\colon x \in F \vee x \in G\} = \{a,b,c,d\}.$$

Note here that, since both sets have the element c in common, it is only counted once in the merged set. This follows from the Axiom of Extensionality. The common element forms the intersection of both sets, which is created using the symbol '∩'. So:

$$F \cap G = \{x\colon x \in F \wedge x \in G\} = \{c\}.$$

Next we have the Cartesian product of two sets, created using the '×' symbol. This is simply the set of ordered pairs where the first element comes from the first set and the second element from the second set. So:

$$\begin{aligned} F \times G &= \{(f,g)\colon f \in F \wedge g \in G\} \\ &= \{(a,c), (a,d), (b,c), (b,d), (c,c), (c,d)\}. \end{aligned}$$

The operations of union, intersection and Cartesian product can, of course, be generalized when we have more than two sets.

When the universal set *U*, the domain of discourse, is understood, we are able to define the complement of a particular set, using the symbol '∼', which is simply all the elements it misses from the domain. So ∼*F*, the complement of *F*, would be all the letters which are not *a*, *b* or *c*:

$$\sim F = \{f\colon f \notin F \wedge f \in U\} = \{d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z\}.$$

The complement of the complement is obviously the set itself.

Supplementary properties, relations and functions

Next we must speak of properties, relations and functions in their general construction via the belonging relation. Three relations have been defined so far: belonging, equality and inclusion – with the last two being reducible to the first. But we also need the freedom to add and invent more relations, perhaps even an infinite number of them, depending on the context, need and situation. Those extra relations must be definable in terms of belonging. Since the ontology of set theory only admits sets, then those relations should ultimately be sets as well.

Let us reconstruct everything by first noting that, on a fundamental level, every property is a set, an existing set. In set theory, a property is ultimately the count-as-one of all the elements that satisfy it. The property φ is extensionally the set of all existing sets that satisfy φ. So if *x*, *y* and *z* are all the only entities that satisfy this property, then $\varphi = \{x,y,z\}$.

(Now there is a gap between the extensional versus the intensional understanding of properties, between understanding the word 'property' in terms of its extensional equivalent versus in terms of a predicate. In the former, a property corresponds fundamentally to the count-as-one operation of some set. In the latter, a property is a linguistic description, a formula with free variables. The Axiom of Extensionally ensures that multiples are defined fundamentally as count-as-one operations, not linguistic constructs. However, confusion can often take place, and this is why Badiou renames extensional properties as 'representations'. Since a set's belonging relation can be understood as the property of being any one of its members, a set can be said to represent what belongs to it.)

But what is relevant for the moment is that a property, in the extensional sense, contains all its instantiations. Relations are simply the 'higher-dimensional' analogue of this insight. A relation is the count-as-one of all the *ordered sets of elements* for which it holds, all the 'instances' where it is effectuated. Relations have a 'degree' or 'adicity' or 'arity' corresponding to the number of objects among which they can relate. For example, if the binary relation R holds only:

between a and x
between b and x
between x and b
and between b and y

then:

$$R = \{(a,x),(b,x),(x,b),(b,y)\}.$$

Relations can also correspond to infinite sets. For example, the usual relation '$<$' between two numbers, the relation of being smaller than, would correspond to the infinite set of ordered pairs where the first number is smaller than the second. If the relation is of order more than two, if it relates more than two entities, then longer ordered sets can be used. For example, consider the relation SUM between three numbers x, y and z, defined as '$x + y = z$'. Then the ordered triples (1,2,3), (10,0,10) and (212,3131,334) are all elements of the set SUM corresponding to this relation. SUM is an infinite set that contains all possible triples of numbers where the last is the sum of the first two. Likewise, a property can be understood as a relation of order one, a unary or monadic relation. We use the term 'property' to cover both monadic properties and polyadic relations of degree more than one.

A function is a specific type of binary relation. Every function associates an 'input' with an 'output'. The input is called the argument of the function and its output is called its value. For every argument taken from a specific set, called the domain, the function churns out a value from another specific set, called the codomain. 'A function f causes the elements of one multiple to correspond to the elements of another' (*BE*, 268). Formally, we write $f: X \to Y$ to say that f is the function, X is the domain and Y the codomain. Every element of X is assigned to some element of Y. Maybe several elements of X might be assigned to the same element in Y and maybe there are elements in Y that are not the output of any element of X. A function is thus a relation between X and Y that satisfies several criteria. First, every element from *X* must appear within the list of all first elements in the set of ordered pairs associated with the function. And they must appear exactly once.

Notion of the formal axiomatic system (*FAS*)

The turn has come for us to continue to axiomatic set theory. The intention is to reduce the totality of what we know about sets into a few fundamental atomic statements, called the axioms. In a sense, we want to describe and replicate everything that is true about sets to some axioms, but without sacrificing any logical coherence. The result is called a formal axiomatic system (*FAS*) for set theory. An *FAS* is technically specified by two formal systems built on top of one another. The first is a formal language, consisting of a set of symbols together with a grammar, which is a syntactic system for combining the symbols into propositions whose truth values can be either true or false. This syntax is built on that of first-order logic, but by supplementing it with some additional names, properties and relations peculiar to the *FAS* in question. The formal language stipulates the rules to generate all the possible grammatically correct propositions. To determine whether these propositions are true or false, we need the second system, the axioms. We must also include the rules of derivation, a deductive system to capture, codify and record correct inferences. The objective is to devise this second system such that every true proposition can be derived from the axioms, and vice versa. Otherwise, that proposition would be false, and so its negation can be derived from those propositions. The sequence of derivations from the axioms to the true proposition is called a proof. A proven proposition is called a theorem. The language generates all the grammatically correct propositions, while the axioms help sieve the true ones out. In a sense, the *FAS* is a kind of machine or algorithm that takes the axioms as its input and produces theorems as its output.

Isaac Newton's famous Three Laws of Motion can be considered as an *FAS* for mechanics, a system of axioms whose purpose is to describe the 'mechanical universe', the mechanical aspects of the physical world. Other examples of *FAS* in physics include the Four Laws of Thermodynamics, James Maxwell's Four Equations of Electromagnetism and the Navier-Stokes Equations for Fluid Motion. One of the earliest *FAS*s that appear in mathematical literature are Euclid's Five Postulates. This *FAS* axiomatizes planar geometry. Everything that is geometrically true about the planar world, about the geometric world of flat surfaces, should be derivable from these five postulates. Other examples of mathematical *FAS*s include Eilenberg-Steenrod Axioms of Homology Theory, the Tarski Axioms for the Real Numbers and, most importantly, the five Peano Axioms for the Natural Numbers.

Introduction to model theory

We need to describe the last example in detail as it will prove useful in introducing the mathematical notion of model and the mathematical field of model theory. Now the intention behind the Peano Axioms is simply to 'model' all the natural numbers, all the numbers 0,1,2,3,4 and so on. We have mentioned earlier that the set of all natural numbers is conventionally denoted with the symbol $\mathbb{N}$. Here are the five axioms. Central to them are the notions of number, zero and successor:

1. Zero is a number.
2. Every successor of a number is also a number.
3. Zero is not the successor of any number.
4. If two numbers have the same successor, then those two numbers are equal.
5. If a set of numbers contains zero and the successor of every number, then that set contains the set of all numbers.

The Peano Axioms are meant to describe the arithmetic universe of the natural numbers $\mathbb{N}$. The set $\mathbb{N} = \{0,1,2,3, \ldots\}$ is said to be a model for the *FAS*, and the Peano Axioms are meant to axiomatize this set. They describe what is true within the closed horizon of the natural numbers. Within the immanence of the 'world' constituted by the set $\mathbb{N}$, each of the axioms is true. For example, it is true that 'zero' is a number, an existing entity within the ontological commitment given by the world of $\mathbb{N}$. Likewise, it is true that the 'successor' of every element in the world is also an element in the world. And so on for all the other Peano Axioms.

A model for an *FAS* prescribes a world within which the axioms are satisfied. The *FAS* is said to be satisfied by the model. A model also specifies a particular interpretation of the axioms or, more precisely, a particular interpretation of the language upon which the axioms are expressed. For example, with respect to the standard model $\mathbb{N}$, the word 'number' that appears in the Peano Axioms means natural number; the word 'zero' means the first natural number 0; and the 'successor' of a number n means the natural number after it, $n + 1$.

The details behind the notion of interpretation must be described. Every language should have a vocabulary, a semantic system dictating some regime of representation. Appearing in the language are the names, properties and relations, each demanding to be decoded. The model, which provides the semantic correspondences, specifies a domain of discourse within which the language evaluates itself, and one where the axioms are satisfied within the corresponding universe. In addition to stipulating an ontological commitment, a complete set of existing individual entities, the model also identifies a decoding system for the vocabulary. First-order logic provides the 'syntax' while the model provides the 'semantics'.

The mathematics behind this decoding system is complicated, and becomes even more complicated with Badiou's metaontological intervention. We know that each of the names listed in the vocabulary is linked to the individual entities presented within the model. Every individual name refers to an individual element of the world. For example, in the case of the Peano Axioms and its standard model ℕ, the name 'zero' refers to the element 0, while the name '1' refers to the successor of 'zero'. The mathematical semantics for properties and relations is more complicated, and we will get to it later.

We add that every model prescribes an interpretation for what might be called the 'unifying concept' of the *FAS*, the concept corresponding to the property of belonging to the model as an existing element. In the case of the Peano Axioms, that concept is 'number' and for the standard model ℕ, that concept is interpreted as 'natural number', which is identified with belonging relation of the set ℕ. But the lesson of the moment is that every model specifies, with respect to an *FAS* and its language, two things: a domain of discourse and a regime of representation. In *Being and Event*, Badiou will rename the former as 'situation' and the latter as 'state'.

We must expand on the contingency of the relation between an *FAS* and its model. An interpretation need not be the only interpretation of the language. Given any world, there could be more than one way to axiomatize it into an *FAS*. Likewise, given a set of axioms, there are various worlds that might satisfy them. For example, we can also have one world and extend it by adding more objects to it while still fulfilling the axioms. Here is an alternative model for the Peano Axioms. 'Number' here now means a particular type of pure set, called a finite ordinal. In this case, we interpret 'zero' as the empty set ∅. The 'successor' of a finite ordinal n is $n \cup \{n\}$, the finite ordinal created by appending the set n to its own list of elements. The unifying concept is now no longer the concept of natural number but the concept of finite ordinal, corresponding to the set denoted by the symbol ω. We can check that the interpreted five axioms still hold true in the domain of finite ordinals. This set serves as an alternative model for the Peano Axioms.

This new interpretation can seem strange because we are used to interpreting number as the elements of {0,1,2, …}. Moreover, we became familiarized with the natural numbers before their axiomatization into the Peano Axioms. But the axioms can be subjected to different interpretations, and not just the one corresponding to our intuitive understanding of the numbers 0,1,2,3,4 and so on. Instead of beginning with the natural numbers as we know them, and constructing some *FAS* from them, the reverse could have happened. We could start by encountering numbers axiomatically instead of naively. It could be that we begin by knowing absolutely nothing about $\mathbb{N}$ before encountering the axioms and then devise a model of them, which need not be the natural numbers that we love and hate. This can be compared to learning, from a distance, the doctrines of some foreign tribal religion and then gradually finding oneself converted and becoming one of the faithful.

Deduction, consistency, completeness and undecidability

So an *FAS* can have different models, different interpretations, different situations where the axioms hold true. Being different, one model may have properties or features that are absent in other models of the same *FAS*. There can be statements that are true in one model but false in another model of the same *FAS*. The truth of such statements is contingent on the model used. However, what all the models must have in common is that, in their respective universes, all of the *FAS* axioms must hold. And, by implication, so must all the propositions that are directly deducible from those axioms, and only from those axioms. For all the models of some single *FAS*, what must be true are the axioms and any proposition that is directly and wholly derivable from those axioms. To expand on this in detail, we need to provide a more rigorous explanation of the idea of proof and deduction in mathematics. This will also help provide the background for us to explain the mathematical notions of consistency, completeness and undecidability, as well as, for our purposes, the first important result in model theory: Gödel's Completeness Theorem.

As we mentioned previously, a grammatically correct proposition, written in the language of some *FAS*, is a theorem of that *FAS* if a deductive thread can be produced linking the proposition from the axioms. As Badiou writes, a deduction is 'a chain of explicit propositions which, starting from axioms [. . .] results in the deduced proposition via intermediaries such that the passage from those which precede to those which follow conforms to defined rules' (*BE*, 242). We will not explain in detail the basic rules of deduction, but it suffices to say that they are relatively simple.

Given a collection of statements, it can happen that they either directly contradict each other or that a self-contradiction is deducible from them. In both cases, the collection is said to be inconsistent. It is likewise said to be consistent if it can never lead to a contradiction. A consistency proof is one that establishes consistency or inconsistency. It is obviously desirable for a collection of axioms to be consistent – and not just for the sake of logical coherence. Via a result known as the Principle of Explosion, it can be shown that *any* proposition in the language can be deduced from a self-contradiction. An inconsistent *FAS* would thus be useless in winnowing out truth from falsehood.

Another desirable state of affairs is for the axioms to be able to either prove or disprove every statement from its language. The truth and falsehood of every grammatically correct proposition is decidable with respect to the *FAS* if the axioms prove either the proposition or its negation – but never both, otherwise this would be the very definition of a paradox. An *FAS* that is able to prove or disprove every proposition from its language is called complete. A proposition that can be neither proven nor disproven is called undecidable or independent.

In sum, the two most important of the various desirable criteria for an *FAS* are completeness and consistency:

1. A complete *FAS* is one that is able to determine every grammatically correct proposition to be either true or false. Given any grammatically correct proposition, either the complete *FAS* proves it or its negation.
2. But not both, otherwise that could create a paradox and the *FAS* would be inconsistent. If a proposition is derivable from the *FAS*, then its negation is not derivable, and vice versa. The axioms must not contradict each other if they need to be consistent.

There is also the notion of relative consistency. A proposition is consistent with respect to an *FAS* if appending the proposition as an extra axiom does not lead to an inconsistent collection, provided the *FAS* was already consistent in the first place. In other words, if the *FAS* was consistent, then it remains consistent if the proposition is inserted as an additional axiom. The extension would not be responsible for a contradiction. Instead of directly proving a proposition from the axioms, a consistency proof simply shows that the consistency of the axioms implies the consistency of the proposition relative to the axioms. So there is a crucial distinction between a direct proof and a consistency proof.

This notion of relative consistency enables us to provide an alternative definition for deduction and undecidability. Let Γ be any consistent collection of axioms and let φ be a proposition in the language of Γ, with $\sim\varphi$ being its negation:

1. It can be shown that every proposition that is deducible from Γ would also be consistent with respect to Γ. Moreover, the negation of every provable proposition must be inconsistent relative to Γ.
2. In fact, the deducibility of a proposition from Γ is equivalent to its consistency with respect to those axioms and the inconsistency of its negation with respect to Γ. The axioms Γ prove the proposition φ if and only if φ, but not $\sim\varphi$, is consistent with Γ. So we do not need to provide an explicit deductive chain in order to determine that φ is a theorem. We can just provide two consistency proofs.
3. It is also possible that neither φ nor $\sim\varphi$ can be proved from the axioms, in which case both are deemed undecidable. The proposition φ is undecidable if and only if both φ and $\sim\varphi$ are consistent with respect to Γ.

We can see that supplementing a consistent collection of axioms with a proved proposition does not add any extra 'information' to them. This is not the case when we add an undecidable proposition. Since the truth of the proposition is independent with respect to the axioms, then something novel is added and we can say that an 'event' has occurred to the system.

Gödel's Completeness Theorem and the two types of consistency

Remember that a model refers to a *set* of entities that satisfy some *FAS*. In our usage of the term, the domain of a model is, by default, a set, although we can make informal reference to a class model whose domains are proper classes. An *FAS* that can be satisfied by some model, some set model, is called satisfiable. The notion of satisfaction is linked with the notion of consistency through the

Completeness Theorem of Kurt Gödel, a result that, as we shall see in Chapters 3 and 4, constitutes the most crucial but unacknowledged mathematical condition used in the early parts of *Being and Event*. Mathematicians attach two separate meanings to the idea of consistency:

1. Syntactic Consistency: This is the meaning that we have already introduced. An *FAS* can be consistent if there is no way for it to be used to prove both some statement φ and its negation $\sim\varphi$. The system must be structurally secure and syntactically coherent. Syntactically inconsistent systems are self-contradictory if some self-contradiction $\varphi \wedge \sim\varphi$ is derivable from them.
2. Semantic Consistency: An *FAS* can also be consistent if it is satisfiable. This means that the system is 'semantically' fulfilled, that some set model interprets it.

The Completeness Theorem simply says that, subject to certain conditions, these two consistencies coincide. Syntactic and semantic consistency imply each other. An *FAS* can never produce a contradiction if it is satisfiable by a set model. Likewise, a set model for an *FAS* ensures that the axioms are logically coherent. To put it more formally:

Gödel's Completeness Theorem: Let Γ be any *FAS* built on the language of first-order logic. Then Γ is syntactically consistent if and only if Γ has a set model.

In other words, the axioms of any first-order *FAS* are guaranteed to be non-contradictory if a set model can be found, if the domain of discourse constitutes a set. The consistency of the axioms coincide with the consisting together of the model's domain. A set is thus 'consistent' for two reasons: first, because its elements consist together as one and, second, because any satifiable axiomatic system brooks no paradoxes.

We must also note that there is another better known but equivalent formulation of Gödel's Completeness Theorem, linking truth with deducibility:

Gödel's Completeness Theorem: Let Γ be any *FAS* built on the language of first-order logic. Let φ be a sentence that is expressed in the language of Γ. Then Γ proves φ if and only if φ is true for every model of Γ.

In other words, if a proposition is true then that truth can be proved. If a proposition has been proved, then it is true.

The crucial proviso of the Completeness Theorem is that it concerns only first-order *FAS*s. In first-order languages, quantification only ranges over the individual elements of the domain. For example, the proposition

$$\exists x \in S: x = 1$$

says that 1 is an element of the set *S* or, more precisely, that there exists some element *x* belonging to set *S* that is equal to 1. In this case, the variable *x* is quantified to range over all the *elements of S*. Thus, the proposition is expressed in the language of first-order logic. However, the following proposition

$$\exists x \subseteq S: x = 1$$

states that 1 is a subset of *S*, and that there exists some subset *x* of *S* that is equal to 1. The variable *x* is quantified to range over the *subsets*, which is not allowed in first-order logic. This particular expression uses *second-order logic*, which permits quantification over subsets. Likewise, *third-order logic* allows quantification over subsets of subsets, and for *fourth-order logic*, subsets of subsets of subsets. And so on for other *higher-order logics*. The higher the logic, the greater the level of abstraction.

It turns out that correspondence between syntactic and semantic truths does not occur for any *FAS* built on second-order logic and above. Subject to certain interpretations, first-order logic is the highest level of abstraction that we can reach before truth and provability do not imply each other. This is the reason for the word 'Completeness' in the theorem. First-order logic is complete in the sense that *all* truths can be reached from the axioms via a finite sequence of derivations. True statements are not always provable in second-order logic or above.

In sum, we are given another way to understand deducibility, undecidability and relative consistency. Let Γ be any *FAS* built on the language of first-order logic. Let φ be a sentence that is expressed in the language:

1. φ is consistent with respect to the axioms of Γ if and only if φ is true within *some* model satisfying Γ
2. φ is a theorem of Γ if and only if φ is true within *any* model satisfying Γ
3. φ is undecidable with respect to Γ if and only if φ is true within some model satisfying Γ and false within a different model satisfying Γ

The Zermelo-Fraenkel Axioms of Set Theory plus the Axiom of Choice

The set theory axioms that are now commonly abbreviated as *ZFC* were the result of the joint effort during the first quarter of the twentieth century by several mathematicians, including Ernst Zermelo, Abraham Fraenkel, Thoralf Skolem and John von Neumann. Each of the axioms can be expressed under various formulations and under various names. Stronger or less restrictive versions are sometimes given. For the sake of minimalism, axioms are sometimes dropped if they can be proved from the others in *ZFC*, if they are theorems. Here, we adopt a configuration that is very nearly equivalent the one used by Badiou in *Being and Event*. The only difference is that we choose to add the Axiom of Pairing, which is not controversial because, first, it is derivable from the other axioms and, second, because it is explicitly mentioned in *Being and Event*.

1. *Axiom of the Void, the Empty Set*: 'There exists a set which does not have any element' (*BE*, 501), a set of which no set is a member. What is being referred to here is, of course, the empty set $\varnothing$.
2. *Axiom of Extensionality*: 'Two sets are equal if they have the same elements' (*BE*, 500). In other words, two sets are equal if their elements are equal. Given any set α and any set β, if for every set γ, γ is a member of α if and only if γ is a member of β, then α is equal to β. The converse of this axiom also holds: the elements of two sets are equal if the sets are themselves equal. This already follows from first-order logic, which forms part of the language of set theory.
3. *Axiom (Schema) of Separation*: Given a set and a property, there exists a second set consisting of all those elements from the first set that satisfy the property. 'If α is given, the set of elements of α which possess an explicit property (of the type $\lambda(\beta)$) also exists. It is a part of α, from which it is said to be separated by the formula λ' (*BE*, 501). The version given here is slightly more general than the one described before. This version is really several axioms in one, a scheme that consists of an infinite sequence of axioms, one axiom for each predicate. This is why it is more correct to call it the Axiom Schema of Separation. Some versions of *ZFC* omit this axiom because it is a consequence of the Axiom of Replacement and the Axiom of the Empty set.

4. *Axiom of Infinity*: There exists a set containing infinitely many elements. 'There exists a limit ordinal' (*BE*, 500). We postpone discussing this axiom until later, where ordinal numbers will be introduced. For the moment, let me just comment that it guarantees the existence of one particular infinite set, the set of all natural numbers $\mathbb{N}$. It guarantees that collecting all the natural numbers together creates a set. Set theory permits us to collect an infinite amount of elements together and have a set.
5. *Axiom of Union*: 'There exists a set whose elements are elements of the elements of a given set. If α is given, the union of α is written $\cup\, \alpha$' (*BE*, 501). Given a set, there exists a second set containing all the elements that are elements of that set. In other words, the union of all the elements of x exists – at least all the elements that are themselves sets. Given α, we can take all the sets that are members of α and then create their union. The union of all the elements of a set itself exists as a set.
6. *Axiom of Pairing*: Given two sets, there exists a third set consisting of those two sets as its members. Given sets α and β, the set $\{\alpha,\beta\}$ exists. So we are allowed to collect a finite number of elements together into a set.
7. *Axiom of the Power Set, of Parts or of Subsets*: Given a set, there exists a second set whose members are all the existing subsets of that first set. 'There exists a set whose elements are subsets or parts of a given set. This set, if α is given, is written $P(\alpha)$. What belongs to $P(\alpha)$ is included in α' (*BE*, 501). Given any set α, there is a set $P(\alpha)$ such that, given any set β, β is a member of $P(\alpha)$ if it is a subset of α. This axiom simply says that the power set exists. Or, more precisely, given α, there exists a set of which the power set of α is one subset. All the subsets of any set are counted as one.
8. *Axiom (Schema) of Replacement*: Given a set, we can create other sets by replacing its members with other existing elements. 'If a set α exists, the set also exists which is obtained by replacing all the elements of α by other existing multiples' (*BE*, 500). This is another axiom that is really an infinite sequence of axioms in one. Given a set, we can create other sets by replacing its members with other existing elements. The image of any set under any definable mapping is also a set.
9. *Axiom of Foundation, or Regularity*: 'Any non-void set possesses at least one element whose intersection with the initial set is void; that is, an element whose elements are not elements of the initial set. One has $\beta \in \alpha$ but $\beta \cap \alpha = \varnothing$. Therefore, if $\gamma \in \beta$, we are sure that $\sim(\gamma \notin \alpha)$' (*BE*, 500). Every non-empty set contains a 'foundational' element that shares no members

with the set itself. One important consequence of this axiom is the banning of self-membership. No set α is allowed to be a member of itself, otherwise the singleton $\{\alpha\}$ would have no foundational element since both $\{\alpha\}$ and α share one member, namely α itself. The axiom also prevents the possibility of an infinitely long chain of membership that does not terminate with the empty set.

10. *Axiom of Choice* (*AC*): Given a set of non-empty sets that have no elements in common, there exists another set that has exactly one element in common with each of those sets. In other words, for every non-empty set, there exists another set that contains exactly one element from each element of the first set. Moreover, every non-empty set has a choice function that takes it as its domain. The choice function selects one element out of every element from its domain. The Axiom of Choice guarantees, for every non-empty set, the existence of a choice function. This axiom used to be quite controversial and this is the reason why it is often considered apart from the other *ZFC* axioms. *ZFC* minus this axiom is often called as simply *ZF*.

Almost everything that we have said in the previous section can be developed from these ten statements. For example, to find the union of two sets, all we have to do is collect them together into one set using the Axiom of Pairing, then merge them together using the Axiom of Union. In fact, the axioms ensure that every possible finite set exists and it is always legal to create a set that is the union, intersection or Cartesian product of a finite collection of other sets.

So these ten axioms are what makes set theory work. We must, however, admit that *ZFC* is not the only *FAS* available for set theory. Even though there has been no compelling reason for a majority of mathematicians to use a different system, there is good reason to be aware of the alternatives. Instead of Zermelo-Fraenkel, there is also the von Neumann–Bernays–Gödel axiomatic set theory, which extends *ZFC* slightly by allowing for classes in addition to sets, but with some extra restrictions. We also have the Tarski-Grothendieck axiomatization, as well as Quine's New Foundations. Still, *ZFC* constitutes the standard and most commonly used axiomatization of set theory. It is the axiomatization used by Badiou in *Being and Event*, although it goes without saying that a substantially different metaontology would result from choosing a different axiomatic.

Gödel's Incompleteness Theorems and the foundation of mathematics

The *ZFC* axioms also provide the necessary and sufficient conditions for something to be a set. *ZFC* defines what a set means. By being an *FAS* for set theory, *ZFC* axiomatizes not only the theory, but also the idea of set. It describes both the universe and the theory. At the most minimal level, it defines what a *pure set* means. *ZFC* is consistent with the existence of sets consisting of concrete objects. But that is not necessarily the case. At the most minimal level, it only describes the domain of pure sets *V*. In *ZFC*, only pure sets are explicitly given. In addition to providing the fundamental machinery for set theory, they also sufficiently describe the universe of pure sets *V*. For *ZFC*, not only is each thing a set, but each thing is a set of sets, a multiple of multiples.

In principle, everything in mathematics, every mathematical entity, is a pure set. The language of mathematics can be reduced to the language of pure sets.[4] This would mean that the axioms of set theory are exactly the axioms for all of mathematics. By systematically axiomatizing set theory, mathematicians have also axiomatized all of mathematics. So if mathematics is fundamentally set theory, then everything about mathematics originates from *ZFC*. If set theory is the meta-mathematical foundation of mathematics, then *ZFC* is all of mathematics encapsulated in ten statements. If all mathematical theorems lead back to *ZFC*, this would culminate in the realization of the great mathematician David Hilbert's dream during the 1920s of systematically formalizing all of mathematics, of clarifying the foundations of mathematics.

The dream of mathematical formalism was crushed by another famous work of Kurt Gödel in 1931: his two Incompleteness Theorems. They prove the fundamental limitations behind the prospect of having an *FAS* for all mathematics. Earlier we have noted that two of the most important criteria for an *FAS* are consistency and completeness. In order to count as an *FAS* for mathematics, the purported axioms must satisfy certain precise features peculiar to mathematics itself. In particular, they should be sufficiently effective in a certain sense. For example, since numbers, particularly the natural numbers, are essential in mathematics, an *FAS* should be able to express basic ideas about arithmetic in the set $\mathbb{N}$. But if they satisfy those features, it turns out that they cannot be both consistent and complete at the same time. Consistency and completeness are not simultaneously achievable for an *FAS* that purports to axiomatize all of mathdom. This is the statement of Gödel's First Incompleteness Theorem.

The implication is that if *ZFC* is complete, then it is not consistent and, if *ZFC* is consistent, then it is not complete. Moreover, such an *FAS* is not capable of proving that it is consistent, that its consistency, if true, cannot be derived from the axioms. This is the statement of Gödel's Second Incompleteness Theorem. So the proposition '*ZFC* is consistent' is never a consequence of *ZFC*. Even if *ZFC* were consistent, we would not be able to prove it from the framework of *ZFC*. Let us put it more formally:

1. First Incompleteness Theorem: Any *FAS* that is capable of expressing and deducing elementary arithmetic cannot be both consistent and complete. Any candidate *FAS* for the whole of mathematics is, thus, bound to fail. If such an *FAS* was consistent, then there must exist undecidable propositions with respect to it.
2. Second Incompleteness Theorem: One undecidable proposition is the statement that the *FAS* is itself consistent. Any *FAS* capable of doing elementary arithmetic cannot be both consistent and capable of proving its own consistency. A consistent *FAS* of this type can neither prove nor disprove the statement 'I am consistent'.

What are the consequence of these two theorems for *ZFC*? Since they are meant to be axioms for mathematics, since they can express arithmetic statements, they cannot be both consistent and complete. That is, either the axioms disagree with each other, or there exist mathematical statements whose truth or non-truth is not derivable from them. Gödel's Second Incompleteness Theorem implies that *ZFC*, if consistent, cannot prove its own consistency by means limited to itself. We will need something outside of *ZFC* to prove that *ZFC* is consistent. Moreover, we would be in bigger trouble if *ZFC* was actually inconsistent. Inconsistency means, among other things, that the axioms are self-contradictory. Self-contradictions are highly undesirable because they can be made to prove anything via the Principle of Explosion. This is why most statements about *ZFC* often assume from the outset that the axioms are consistent.

The Second Incompleteness Theorem means that the consistency or inconsistency of *ZFC* does not contradict *ZFC* itself. In other words, appending either the statement '*ZFC* is consistent' or '*ZFC* is inconsistent' to the list of *ZFC* axioms does not create an inconsistent list of axioms. Since every model is an interpretation of the words in those statements, every consistent list of statements corresponds to an interpretation. The facts that '*ZFC* is consistent' and '*ZFC* is inconsistent' are both consistent with respect to *ZFC* itself means that *ZFC* can have different models and different interpretations. The words in the *ZFC* statements can be interpreted in different ways: the interpretations are wholly contingent and are not explicit in the statements themselves. The axioms do not provide a direct interpretation of sethood, a single finitely-expressed predicate concerning what it means to be a set. *ZFC* is an *FAS* without a concept of set. The axioms talk about the set without explicitly defining it.

3

Metaontology of Situations and Presentation

We begin with a summary of the mathematics that we have presented so far:

1. Informally, a set is a multiple of its members that consist together as one. These members can be absolutely disparate from each other and can themselves be sets. Their number can be zero, one, more than one, and even infinite. The empty set ∅ is the unique set lacking members. Pure sets contain nothing but other pure sets, all the way down to ∅ – which is the first pure set.
2. Formally, a set defines itself fundamentally by its belonging relation, by what Badiou metaontologically names as its count-as-one operation. Belonging constitutes the arche-relation in set theory and any other additional relation or property must be definable in terms of it by being extensionally equivalent to some set. A property is ontologically the complete set of entities satisfying it. A relation is ultimately the complete set of ordered multiples consisting of entities between which it holds. A function is a relation that assigns each element (input) from one specific domain set with another element (output) from another specific codomain set.
3. The two other basic set-theoretic relations, equality and inclusion, are already reducible to the belonging relation. The Axiom of Extensionality ensures that two sets are equal when their elements are equal, when their belonging relations are equal. One set is included in another if the former is a subset of the latter, if belonging to the former implies belonging to the latter. Every set includes itself and ∅. A set's inclusion relation is extensionally equivalent to the power set that collects all its subsets. The Theorem of the Point of Excess states that a set is always different from its power set. This is because a set cannot contain the subset consisting of all its ordinary (i.e. non-self-belonging) elements.

4. The set-theoretic language, which is built on the grammar of first-order logic, is always evaluated with respect to some presented domain of discourse (This follows from the Axiom of Separation). In first-order logic, negation obeys the Laws of Excluded Middle and Non-contradiction, and quantification within the domain ranges over the individual elements, never multiples of them or their properties. Given several sets, we can create new ones via the operations of union, intersection, complementation and taking Cartesian products.
5. Sets might also specify themselves via formulas with free variables, via single predicates finitely expressed in the set-theoretic language. Multiples associated with these predicates are informally called classes. Not all predicates create sets and the set-theoretic ontology disqualifies proper classes such as the Russell Class of all ordinary sets and the universe of all pure sets *V*. The Axiom of Separation ensures that predication only creates sets that are already subsets of existing sets.
6. A formal axiomatic system (*FAS*) is a list of axioms written using a formal language, plus the derivation rules. Two important examples are the Peano Axioms of Arithmetic and the Zermelo-Fraenkel Axioms of Set Theory plus the Axiom of Choice (*ZFC*).
7. A model specifies some set that forms some domain of discourse within which the *FAS* is satisfied, any specific 'world' within which all the axioms hold. An *FAS* might have several different models, each corresponding to a particular interpretation of its terms, a particular meaning of the names, properties and relations that appear in the language upon which the *FAS* is built. A model must always specify an interpretation in addition to its domain of discourse. On its own, and especially when it has several different models, an *FAS* does not explicitly state the meaning of its terms. Surrounding an *FAS* is a semantic abyss that can only be crossed using a model, an interpretation of the vocabulary used by the language.
8. *ZFC* provides both the formal axiomatic machinery for set theory and the set of conditions to determine, without explicitly defining, what constitutes a set. At the most minimal level, it specifies the conditions for what constitutes a pure set. In the version we use, *ZFC* consists of the Axioms of Extensionality, the Empty Set, Infinity, Union, Pairing, Power Set, Replacement, Separation, Foundation and Choice. The universe *V* of all pure sets is 'almost' a model for *ZFC*, albeit one that is a class and not a set model. But there could be set models of *ZFC* within *V*, consisting of a limited selection of pure sets.

9. It is desirable for an *FAS* such as *ZFC* to be complete and consistent. Complete means 'encounters no undecidable statements from the language', and a statement is undecidable when the axioms can neither prove nor disprove it. Consistency can mean semantic consistency (i.e. 'satisfied by some set model') or syntactic consistency (i.e. 'not self-contradictory'). Gödel's Completeness Theorem states that these two consistencies imply each other, provided that the *FAS* is built on first-order logic, like *ZFC* itself. The axioms of any first-order *FAS* are guaranteed to be non-contradictory if a set model can be found, if the domain of discourse constitutes a set. A set is thus a consistent multiple for two reasons: first, because its elements consist together as one; and, second, because the domain to which it specifies brooks no logical paradoxes. So the *ZFC* axioms are non-contradictory if a set model exists.
10. We can append an undecidable statement or its negation to a consistent *FAS* and still get a consistent *FAS*. The original consistent *FAS* can have models where the undecidable statement is true and others models where it is false. A statement is consistent relative to a consistent *FAS* if it is either undecidable or provable by that *FAS*. If both a statement and its negation are individually consistent with respect to an *FAS*, then that statement is obviously undecidable.
11. *ZFC* could serve as a foundation for mathematics because every mathematical entity is in principle constructible as a pure set. Unfortunately, Gödel's First Incompleteness Theorem implies that *ZFC* cannot be simultaneously complete and consistent, whereas Gödel's Second Incompleteness Theorem implies that the statement '*ZFC* is consistent' is undecidable with respect to *ZFC* itself. Zermelo-Fraenkel set theory cannot prove its own consistency – a consistency that has to be either axiomatically assumed or rejected from the onset. Following from Gödel's Completeness Theorem, this means that Zermelo-Fraenkel set theory can neither construct its own set model nor define the concept of set. There is no 'world' within *ZFC*, no world of sets. Non-standard interpretations of *ZFC* could exist where the word 'set' need not refer to multiples at all. So it can only be implicit within *ZFC* that it is consistent, that it has a set model, that a concept of set exists.

Whereas the previous chapter began and was centred in part on explaining the mathematical concept of the set, much will be gained if we structure our current exposition on describing the Badiouian metaontological concept of the situation. The first ten meditations of *Being and Event* can, in fact, be read as Badiou's explication of the formal machinery and constitutive features of situations in general and ontological situations in particular. So what is a situation and how is this concept conditioned by set theory?

Precedents to the concept of situation

We are not focused here on supplying an extensive comparative analysis or an excursion into the history of ideas, but I think it might be useful to draw our attention briefly towards a few conceptual and etymological precedents to the Badiouian concept, just so that we are aware of them. We can trust to a certain extent the everyday semantics of the word 'situation', which is roughly the same in French as in English: a position; an environment; a state of affairs; a combination of circumstances; a frame of reference. Within some world, certain relevant entities exist, and certain discourses, interactions, languages, properties, relations, rules, structures and truths take hold. Examples include the economic situation in Greece following the recent financial crisis; the situation in twentieth century Pop Art; the situation in English Premier League football; the situation in the science of string theory from year 1990 onwards; the tragic romantic situation between Abelard and Héloïse; the uncertain and fraught political situation in contemporary North Korea; the situation of arithmetic under the Peano Axioms within the domain of natural numbers $\mathbb{N}$; and the set theory situation under *ZFC* within some domain of pure sets. Several features in the quotidian meaning already orient us towards Badiou's specific metaontological conceptualization:

1. A situation is not necessarily a universe, a complete world consisting of every presentation, every single existing entity. Situations can just be smaller environments, sometimes even 'world-fragments' within the backdrop of a supposedly existing complete universe. The political situation in contemporary North Korea can be thought as a world-fragment of the global political setting of today. The arithmetical situation only concerns natural numbers, and any other mathematical entity, such as quadratic equations, differential manifolds or computational algorithms, is pertinent only if it relates to the question of what is in $\mathbb{N}$.
2. That everything and every verified truth presents itself within and contingent to a situation, and that different situations exist alongside each other, often competing with each other – this is what we call relativism.

The relative frames of reference can be, among others, anthropological, biological, class-related, cultural, constitutional, economic, educational, ethnic, geographical, ideological, legislative, linguistic, neurological, physical, psychological, racial and socio-political. Nevertheless, the multiplicity of different situations is a fact that Badiou denigrates today as trivial. Moreover, 'it is pointless to search amongst differences for anything that might play a normative role. If truths exist, they are certainly indifferent to differences' (*BE*, xii). Badiou's philosophical concept of the event provides one way to break out of relativism by allowing the possibility for a situation to shatter apart and create new ones.

3. Since a situation collects together all the relevant entities constituting it, a situation is a multiple. The situation in Pop Art, for example, might be the multiple consisting of all the relevant artists, art works, museums, galleries, critics, critical discourses, histories and artistic inscriptions related to that movement. Moreover, this quotidian concept of situation is flexible enough to cover each and every thing as well. Not only can every 'environment' but also every single existing entity can be called a situation because every entity delimits a closed environment of its own, be it a country, a city, a house, a room, a person, an organ, a cell, a molecule or a particle. So every entity is both a multiple and a situation. So every situation is a presented set and every presented set is a situation.

As with sets, this adaptable omnipresence of situations could explain the wide range of near-synonyms for the word, many of which Badiou could have easily used in his philosophy instead: area, atmosphere, circle, circumstances, condition, context, dimension, domain, ecosystem, enclosure, environment, frame, habitat, locale, milieu, multitude, neighbourhood, network, occasion, orbit, place, planet, range, realm, region, scene, scope, setting, setup, site, sphere, state, state of affairs, status, strata, structure, surrounding, system, vicinity, world, zone, etc.[1] We can see Badiou's analysis in *Being and Event* as taking place within a certain thematic thread running throughout his philosophical oeuvre, a thematic we can roughly describe, among other ways, as a series of attempts at rethinking various ontological and epistemological issues via some concept of worldhood. In his first book, *The Concept of Model* (1969), this thematization – albeit one that is well known to be incomplete – was centred on Badiou's reading of the mathematical concept of model, while in *Logics of Worlds*, his sequel to *Being and Event*, situation was renamed 'world' [*le monde*], a concept conditioned not just by set theory but also by the mathematical concepts of category, sheaf, scheme and topos. We can also argue that Badiou's choosing of the specific term 'situation' in *Being and Event* was so as to invoke, continue, merge and disavow two previous philosophical lineages concerning its usage, one from European continental philosophy and the other from Anglo-American analytic philosophy:

1. The continental thread is associated with the philosophical history in France involving the Sartrean thematic of situation, starting from its appearance in *Being and Nothingness* (1943), and continuing in Sartre's philosophical series *Situations* (1947–76), up to its resurgence in some of the leftist philosophical movements of the late 1960s revolving around Lettrism, Situationism and the Situationist International.
2. The analytic thread concerns developments in the positivistic philosophy of situation semantics and situation theory that was pioneered in the 1980s by Jon Barwise and John Perry (1983), a field that intersects with applied linguistics, theoretical semantics, computer science and natural language processing. This field, which was originally an investigation into alternatives for possible-world semantics and extensional model theory, can be seen as a continuation of more general explorations, dating back to Bertrand Russell, concerning the metaphysics of states of affairs as a new ontological category that accounts for how and why propositions become true or false. Parts of Badiou's basic ontological framework, particularly in the way it makes use of the language of set theory to understand the question of Being, inherits a lot from most of the classic early works in analytic philosophy – compare the framework to, for example, W. V. Quine's study 'On What There Is' (1948), which is probably the most famous paper in analytic philosophy.

Since we are not offering a comparative analysis, we will not go further than simply mentioning these two lineages, although it is worth revealing that Badiou refers to them in the endnotes to *Being and Event*, specifically in the Note to Page 24 (*BE*, 484). We must remark though that Badiou considers the opposition between Anglo-American and Continental philosophy to be null and artificial (*BE*, xiv–xv).

Two confusions about situations

Having taken into account some notable aspects of the everyday semantics and philosophical etymology of the word 'situation', we are now a little bit more equipped to approach Badiou's handling of the term. For expository purposes, and for the sake of introducing some instructive schematic into our descriptions, I believe that extra care should be taken to clarify two distinctions that are often the cause of misunderstanding among readers of *Being and Event* when it comes to understanding the Badiouian concept:

1. *Indeterminate Situations in General versus Ontological Situations in Particular*

The main difference between non-ontological and ontological situations concerns how they 'usually' decide between whether the one is or is not. Ontological situations, which will be discussed later, are a specific but essential type of situation. Our exposition should reasonably converge from the general to the specific, although we must admit that an unavoidable circularity presents itself because much of the basis to Badiou's thinking about general situations has its original justification in his commentary on ontology. On one hand, it is hard to describe ontological situations without first describing the general concept of situation. On the other hand, providing a metaontology of situations requires some presupposed ontological explanations, validations and commitments, which can only be given by talking about Badiou's analysis of ontological situations (provided, of course, that we commit ontology to be a situation in the first place). This inescapable circularity is not the result of some paradox in the foundational self-consistency of Badiou's philosophical enterprise because his metaontology is by nature self-founding by an event. This will be comprehensible when we discuss Badiou's philosophy of the event and the way it negates the Axiom of Foundation. For our purposes, we choose to introduce the concept of situation first.

2. *Situations as Entities, as 'Consistent Presented Multiples', versus Situations as Worlds, as the Domain of Models*

Badiou invokes this distinction in his first mention of the word 'situation' in *Being and Event*:

> I term situation any presented multiplicity [*toute multiplicité présentée*]. Granted the effectiveness of the presentation, a situation is the place of taking-place [*le lieu de l'avoir-lieu*], whatever the terms of the multiplicity in question. (*BE*, 24)

A situation is both 'any presented multiplicity' and 'the place of taking-place'. Two descriptions are given here and we discuss each in turn.

Situation as set

Badiou more or less repeats this first definition of situation in the glossary at the end of his book: 'Any consistent presented multiplicity, thus: a multiple, and a regime of the count-as-one' (*BE*, 522). Four distinct concepts – consistency, presentation, set and situation – are equated together and sutured to a fifth concept – the multiple.[2] We schematize these relations as follows:

1. Any presentation presents itself as a multiple and by virtue of being a multiple. '[W]hat *presents* itself is essentially multiple' and 'the multiple is the regime of presentation' (*BE*, 25), the 'general form of presentation' (*BE*, 514). Presentation is multiple because entities exist in various forms: this leaf, this book, this person, this place, this action, this quality, this atom, this number, this relation, this concept and so on. Presentation is also multiple because every existing being presents itself as a multiple of its constituents. So anything that is not a multiple is certainly absent from the ontology.
2. So multiples are all that are presented – and these are what Badiou defines as situations. To be a situation is to be a multiple that is presented.
3. Every presented object, that is, every situation, is *consist*-ent in the sense that its elements 'consist' together as one. Every presented multiple composes 'the terms of the presentation as units of a multiple' (*BE*, 25). This multiplicity of composition is the forming-into-consistency of its result. To be presented is to be consistent, to cohere together.
4. A presentation is not just any multiple; it is a consistent multiple – in other words, a set, for a set is really none other than a multiple that consists together as a compositional collective of its terms. The concept of the set defines what it is to be presented and consistent.

In sum:

situation = presentation = presented multiple = consistent multiple = set

So sets are what exist and the concept of set is an articulation of what-exists in their being-existing. Perhaps the most controversial part of this equation is the suture between presentation and consistency. We had casually taken the concept of the presented multiple as equivalent to the concept of the set, or at least the concept of the collection. When delimited to the question of presentation, the question of Being is the question of being consistent. But it is not clear why becoming-present and becoming-a-set constitutes the same operation. The essential multiplicity of presentation seems to be less provocative than its essential consistency. In his employment of set theory, Badiou recruits this mathematical concept of the set as the main primeval figure for understanding multiplicity. In hindsight, it seems almost obvious that set theory should have something informative to say because a set is not just literally a multiple of elements but also the simplest and most general entity as well as the least complex of the various closed structures that can be used to inscribe multiplicity. Anything bigger, denser or more complicated might prove to contain redundancies. Besides being the simplest structure, the figure of the set is also the basic unit of structure because, in principle, the structurality of any other structure – geometric figures, algebraic equations, probabilistic distributions and so on – can be defined on the basis of the structurality of sets alone. The set therefore provides the most fundamental articulation of structure and of multiplicity at the most general level.

This synonymy between the set and multiplicity should be questioned, although the link will become clearer and more justified once we understand Badiou's demonstration that set theory under *ZFC* forms a suitable rubric to approach what he calls the 'presentation of presentation'. Enlisting a different figure as a condition for a philosophy of multiplicity would result in a substantially different metaontology from what Badiou has given us. Other philosophers have recruited various alternatives from mathematics and the mathematical sciences. Before continuing to the second definition of situation, let us briefly list these alternatives, just so that we are aware of them:

1. One obvious choice, particularly in the current philosophical environment, is the graph, which is the mathematical name for the network consisting of nodes linked to each other by lines of relation. This is a figure that we see employed, either explicitly or subconsciously, in Bruno Latour and Michel Callon's Actor-Network Theory; Antonio Negri and Michael Hardt's analyses of empire and multitudes; Gilles Deleuze and Felix Guattari's commentaries on the rhizome; and the many current philosophical studies of relational ontology, internet culture, social grids, epidemiology and the topological dynamics of globalization.
2. Yet another paradigm would be the various names that engineers, scientists and applied mathematicians have used to name chaos or think chaos: entropy, noise, randomness, stochasticity, ergodicity, nonlinearity, non-commutativity, and so on. Some physicists might be tempted to refer back

to the Galilean concept of a physical frame of reference, a concept that founds modern physics. A biologist or ecologist might be inclined to use the concepts of ecosystem, biosphere, organism, or evolutionary system.

3. Systems theorists, or philosophers influenced by them, might look at the general concept of a dynamical, complex or adaptive system, or at particular dynamical systems, like, for example, when Michel Serres investigated the metaphysical implications of fluid and thermodynamics in *Genesis* (1995).
4. Besides the rhizome, it could be argued that much of Deleuze's philosophical enterprise was to examine, critique and philosophize a whole series of scientific and cultural figures for the multiple. The most well-known of these is the mathematical concept of the differential manifold, sometimes restricted to the Riemannian differential manifold.[3]
5. As previously mentioned, Badiou himself examined the alternative mathematical concepts of category, scheme, sheaf and topos in *Logics of Worlds*.

Situation as model

The second definition of situation is 'the place of taking-place' – a world. This is closer to the everyday semantic meaning that we discussed. The obvious mathematical figure here would be the concept of model, of set model. Situations are thus Badiou's translated metaontological name for models or, more precisely, for the domain of set models.

We say that the elements of the situation are presented within it and by it. The situation's unique belonging relation, its operator of the count-as-one, is called its 'structure'. The one of the count determines, with respect to the world, what is and what is not. The structure corresponds to a 'unifying concept', the central property, defining what is and what is not in the domain. The structure 'prescribes, for a presented multiple, the regime of its count-as-one' (*BE*, 24). The domain of presentation is precisely the end result of the count-as-one, of belonging to a situation. 'When anything is counted as one in a situation, all means is that it belongs to the situation in the mode particular to the effects of the situation's structure' (*BE*, 24). Whatever satisfies the structure is. The belonging relation of the situation constitutes its law of Being within its domestic immanence. The oneness of the belonging relation, the structurality of the count-as-one, is the unity of Being. Without this oneness, Being would be multiple therein.

So the structures of situations constitute ontological commitments and prescriptions within some designated environment. Moreover, the very concept of structurality – structurality as we usually understand the word – is inscribed by the prescriptive structurality of situations. In the same way that the structurality of any mathematical figure is constructible in terms of the language of sets, the structurality of any structure, any complex formalism, is reducible to the structuration of some or several situations. In the ontological closure specified by a situation, the only presented multiples are its elements, those entities that satisfy the belonging relation and undergo the count-as-one operation. A situation's list of members is *precisely* the list of its presented multiples, nothing more, nothing less. This leads to two noteworthy corollaries:

1. The world is not generally presented within its own immanent closure – unless, of course, it belongs to itself, an impossible state of affairs since *ZFC*, under the Axiom of Foundation, bans self-belonging. Within any world, the world itself is absent, so long as *ZFC*, the background rules of ontology, hold true, at least those rules that prevent self-belonging. 'Since self-belonging is prohibited, [the set] a does not belong to a,' writes Badiou. 'Consequently, an inhabitant of a does not know a. The universe of an inhabitant does not exist for that inhabitant' (*BE*, 513). If a situation does not belong to itself, then that situation is absented from itself. To inhabit a world does not necessarily mean that the world is presented as a single totality. Within a situation, that situation generally does not exist. Later, we will learn that the event – the evental multiple that presents itself as an 'ultra-one' – is an exception to this.
2. Every presented multiple, every element, in a situation is also a set. So if an entity belongs to an element within the situation, but that entity does not itself belong to the situation, then that entity itself is not presented, even though the element to which it belongs does. If $x \in y \in z$ but $x \notin z$ (which is possible because membership is not a 'transitive' relation), then from the viewpoint within the larger situation z, the element x is not presented, even though it is present within situation y. Moreover, if x is the only element within situation y, then from the viewpoint of z, the presented element y contains nothing and constitutes an 'abyss of the void'. From the viewpoint within situation z, the presented multiple y is at the edge of the void since what it contains is absent from z. We will expand more of this later.

The flat plane of presentation

So a situation is both an entity and a world. Situations are presented, first, by virtue of their own belonging relation and, second, by virtue of satisfying the belonging relation of another larger super-situation. Briefly we can say that a situation is:

1. a set when encountered from the 'outside', from 'beyond its curly brackets'
2. a model when encountered from the 'inside', when seized in its domestic immanence.

To avoid some easy misinterpretations, the link between situation and presentation must be qualified further with three points:

1. The concept of situation fully captures the ontological aspects of any concrete object and any world. The phrase 'ontological aspects' is crucial because only the presentative features of objects and worlds are encapsulated by the Badiouian term, not the ontic, concrete, contingent or specific characteristics of their presentation – unless those characteristics only concern whether the concrete specifics are or are not, in which case those specifics constitute other separate presentations, the presentations of the concrete specifics of the earlier presented multiple. When we speak of some empirical situation *S*, we are really referring to the 'situationality' of *S* as a situation, its ontological and presentative structure, its situation-qua-situation and its Being-qua-Being. In any interaction with situations, one never steps outside of intrinsic ontology. Remember that the situation is not the 'place' itself but 'the place of taking-place'.
2. The general realm of ontology, as the discourse of Being-qua-Being, admits a sort of 'Meinong jungle' where every multiple exists – although even this paradoxically contains undecidable 'gaps' as ontology is doomed to be incomplete. As noted by Oliver Feltham, 'Badiou clearly plumps for the opposite of Ockham's razor and admits as many multiples into existence as possible' (2008, 94). Moreover, presentation – that is, situation – is not the same as phenomena, for not every presentation is operative as an appearance. To be present is not the same as to appear. (In *Logics of World*, Badiou will later supplement his philosophy of ontology with a phenomenological account on appearances.)
3. The metaontological concept of situation has obvious comparisons to the modal concept of possible worlds. But presentation is not the same as the

actualization of a possibility because the multiple transcends the distinction between possible versus actual. The possible and the actual are just different species of presented multiples. The general realm of ontology contains every actuality and possibility.

Now the fact that entities, worlds, appearances, possibilities and actualities merge into one univocal ontological category – in fact, ultimately the only ontological category – is testament to the flatness, uniformity and homogeneity of Badiou's metaontological framework. It is not that there are entities and there are worlds; there are only entity–worlds with equal ontological dignity, only situations comprising the single and flat plane of presentation. As far as ontology is concerned, at least on the most basic level, Being admits no intrinsic categorization. At the most basic level, set theory does not differentiate different types of sets. This is because, in set theory, there is only

1. one ultimate concept: the set
2. one ultimate predicate: to be a set
3. one ultimate relation: to belong to another set.

Likewise, in ontology, there is also only

1. one ultimate concept: Being
2. one ultimate predicate: to be
3. one ultimate relation: to be within another being.

The temptation confronting us is to cite the all-too-famous opening from Hamlet's soliloquy. We must however remember that the mathematical concepts of set and model, though related, are not exactly equivalent in the *de dicto* sense. Yes, the domain of every set model is, by definition, a set. But not all sets are models, even though every set could potentially be taken as the domain of some world. What is missing is that every model must also be linked to a structured discourse of truth expressed in some interpretative system. A model is linked not only to a domain but also to a representational system, plus the list of statements that hold true within its world. Those statements could be reduced to some *FAS* modeled by the domain.

What is true and what is false within the situation depends on what is presented in the situation by its structure, its unifying concept. To distinguish truth within a situation from his own unique concept of truth (we will define it later once we discuss the event), Badiou renames the former 'veridicity' [*véridicité*].

A statement is veridical if it is true within the specified model, otherwise it is 'erroneous'. Every situation specifies a horizon of veracity [*vérité*], a state of affairs verifying what is veridical and invalidating what is erroneous. When taken as a world, a situation is specified by this triad of domain-representation-veridicity. We will expand further on Badiou's analysis of representation when we get to his metaontology of subsets and power sets. But let us continue towards Badiou's investigation into ontological situations.

Universes and quasi-complete situations

To be purely and properly ontological, a situation has to be some immediate discourse about Being-qua-Being. We know that philosophy is automatically disqualified because, being at most metaontological, it never reaches directly to the first-order level of ontology. To be a situation, ontology has to be a set and a model: a consistent presented multiple whose intrinsic horizon of veracity is specified by a structured discourse corresponding to a list of statements expressed in some language. All of this leads to a series of closely interrelated questions:

1. What is presented as ontology's situation-set? What can be said about its particular presentation, its multiplicity, its consisting-together? What can be said of the particular belonging relation, the count-as-one, of that set?
2. What is the discourse about? How does it compare to the discourse of set theory? What is the formal language of that discourse?
3. Which statements are veridical within the ontology model? Can they all be reduced to a specific collection of axioms? How do those axioms compare with those of *ZFC*? What can be said about the consistency of those axioms?

We expect that the answers to these questions should connect to the question of Being-qua-Being. The discourse should provide some *logos* to Being-qua-Being, and the unifying concept corresponding to the ontology set should shed some light on what it means to be. In fact, the unifying concept should be precisely the concept of Being, corresponding to the essential structure of Being as such, a concept that, as we shall learn, is essentially sutured to the concept of multiplicity.

We recall Badiou's equation 'mathematics = ontology' and infer that an ontological situation, a situation deemed ontological, would be any structured discourse that is wholly and solely mathematical. This requirement is satisfied by any branch of pure mathematics, and even by any subfield, however small, provided that it forms a consistent and sufficiently self-contained conversation. But the phrase 'ontological situation' can also refer to a situation *as* ontology – that is, a situation that refers to a specific subcategory of what we have just described: a universal ontological super-situation that purports to found ontology itself. Such an ontological super-situation should be able to describe all the statements that are veridical of and within any ontological situation. Those statements describe the minimal background rules governing Being-qua-Being, the '*a priori* conditions of any possible ontology' (*BE*, 23). Every ontological situation should be subsumable under such a super-situation.

Following Badiou's equation of mathematics with ontology, we are referring therefore to a meta-mathematics, a mathematical discourse that might serve as a formal foundation to mathematics, a mathematics of mathematics and of mathematicity as such. In Badiou's *Being and Event*, this meta-mathematical discourse is chosen to be set theory under the formal axiomatic of *ZFC*. In a moment, we will rehearse Badiou's justification for why this is the case. In a sense, the ontological super-situation is roughly intended to describe our current universe, or at least the ontological aspects of the current state of affairs. The totality that is our universe is thus almost a model – 'almost' because it is a class and not a set model – for the *FAS* corresponding to the ontological super-situation.

Any model satisfying the rules of ontology – which Badiou proclaims are exactly the *ZFC* axioms – must, in light of Gödel's First Incompleteness Theorem, be incomplete. To construct, exhibit or even justify the existence of such a set model implies that *ZFC* is consistent (because of Gödel's Completeness Theorem), which can happen only if that model is incomplete, if some statements, deemed undecidable, are only contingently veridical therein. To present the argument more systematically:

1. Say a set model for *ZFC* exists
2. The Completeness Theorem implies that having a set model means being consistent. So *ZFC* is assumed to be consistent.
3. The First Incompleteness Theorem implies that *ZFC* cannot be simultaneously consistent and complete. So *ZFC* is incomplete.

Hence Badiou's own name for ontological super-situations, for models in which the *ZFC* rules of ontology are veridical: 'quasi-complete situations [*situations* *quasi complète*]'. So the conditions for any ontology correspond to the *FAS* of a quasi-complete situation.[4]

As a set, the domain of the quasi-complete situation can only be *decided* to exist within *ZFC*. This follows from the Gödel's Second Incompleteness Theorem, which states that the statement '*ZFC* is consistent' is undecidable with respect to *ZFC*. So *ZFC* cannot present its own set model within itself. Badiou was certainly aware of this issue when he meditated on the 'profound' question of whether a quasi-complete situation exists:

> Such a situation 'reflects' a large part of ontology in one of its terms alone. . . . We know that a total reflection is impossible, because it would amount to saying that we can fix within the theory a 'model' of all of its axioms, and consequently, after Gödel's Completeness Theorem, that we can demonstrate within the theory the very coherency of the theory. The Theorem of Incompleteness by the very same Gödel assures us that if that were the case then the theory would in fact be incoherent: any theory which is such that the statement 'the theory is coherent' may be inferred from its axioms is incoherent. The coherency of ontology – the virtue of its deductive fidelity – is in excess of what can be demonstrated by ontology. (*BE*, 360)

To present the argument more systematically:

1. According to the Second Incompleteness Theorem, the statement '*ZFC* is consistent' is independent with respect to *ZFC*. The axioms, on their own, are incapable of proving their own consistency or inconsistency.
2. The Completeness Theorem implies that the statement '*ZFC* is consistent' is equivalent to the statement '*ZFC* has a set model'.
3. So the existence of a set model for *ZFC* is undecidable with respect to the axioms.

ZFC cannot establish or prescribe its own interpretation of sethood. The unifying concept of sethood, which corresponds to the belonging relation of the undecidable set model, cannot be provided by the axioms themselves. The ontological super-situation is not just quasi-complete; it can never be explicit to the axioms themselves. *ZFC* is, first and foremost, an FAS before it presents anything. The Incompleteness Theorems make it unique compared to most other FASs.

Having explained two crucial aspects concerning ontological super-situations, let us return to the distinction between indeterminate versus ontological situations. The difference is crucial and, I believe, a common source of confusion among Badiou's readers. A situation is any indeterminate set, any consistent presented multiple whatsoever. As such, the *ZFC* rules of ontology are not necessarily veridical there. For example, it is not necessarily true that the Axiom of the Power Set holds. An element within the indeterminate set need not have a power set also inhabiting the situation. A situation is just the simple presentation of some consistent collection of elements. The veridical statements within its world can be anything, so long as they are consistent.

Now the specific situations that satisfy *ZFC* are called quasi-complete. By definition, all the *ZFC* axioms are veridical there. The axioms are that of ontology, the discourse of Being-qua-Being. However, the specifics of the quasi-complete situation's model are obscure, first, because it cannot be exhibited and, second, because exhibition would also make it incomplete. Our current universe can be thought as almost a model, although it is not a set. The difference here is between world as any regime of presentation versus world as a quasi-complete regime where all the rules of ontology apply. To distinguish the latter from the former, let us refer to it using the term 'universe'. A world is any situation, while a universe is a situation that can be said to simulate all the fundamental rules of ontology.

In sum:

1. Situation = any consistent presented multiplicity and world
2. Quasi-Complete Situation = Ontological Super-Situation = Universe = any situation where the fundamental rules of ontology are satisfied and where the world could constitute not just a domain concerning ontology but also a maximal and inclusive totality where presentation is quasi-complete

Whenever we see Badiou mentioning the term 'situation' in *Being and Event*, it is important to examine its appearance in context to determine whether it is not a shorthand for 'quasi-complete situation'.

The ontological decision that the one is not

Even though the term 'quasi-complete situation' appears only later in *Being and Event*, much has already been described, explained and justified about it in Meditation 1. We do not intend to add more to the close exegetical discussions, save for a short recitation concerning the general thread of its argument while expounding its mathematical conditioning. The key divergence between ontological and non-ontological situations is their general immanent position towards the relation between Being and multiplicity, particularly in regard to the Being of the one. The dilemma is that we are confronted with two mutually exclusive choices: Being is essentially one versus Being is essentially multiple. This translates into choosing whether the one is or is not. If Being is one, then the one *is* and is fundamental. If Being is not one, then the multiple precedes the one, which therefore is not, at least on a fundamental level.

As is well known and oft-discussed among Badiou commentators, this dilemma harks back to the ancient metaphysical issue of whether unity ontologically precedes multiplicity, or vice versa. It could be argued that the question of one-versus-many is really *the* central impasse of ontology because it connects intimately to so many of the other well-known metaphysical dilemmas: Being-versus-becoming, identity-versus-difference, necessity-versus-contingency, objects-versus-properties, universals-versus-particulars, idealism-versus-materialism, whole-versus-parts, and so on. We can also see that the notion of the set is possible because of the fact that several entities can be considered as one entity and a single entity can be taken as consisting of several: a multiple-one or a one-multiple. Sethood therefore lies on the neutral border, some obscure point of negotiation and compromise, between our understanding of one versus many, unity versus multiplicity.

In that case, if every presentation is a set, then what comes first, its unity or its multiplicity?

We explicate what is at stake. Being – that is, Being-qua-Being, the immediate focus of any ontology – is it one or is it multiple? 'We find ourselves on the brink of a decision,' writes Badiou (*BE*, 23):

Hypothesis A: Being is essentially one because '*what* presents itself is essentially one' (*BE*, 23). Badiou cites Leibniz's formulation: 'What is not *a* being is not a *being*'. A presentation can only be itself when 'what it presents can be counted as one' (*BE*, 23), when it is single and unified. 'Presentation is only this multiple inasmuch as what it presents can be counted as one' (*BE*, 23). Moreover, Being must be one because Being is *the* Being. If it presents itself, that presentation must constitute *a* presentation. So the one should be.

Hypothesis B: Being is essentially multiple because 'what *presents* itself is essentially multiple' (*BE*, 23). Since every presentation is a consistent presented multiple and since we 'cannot see how there could be an access to [B]eing outside all presentation' (*BE*, 23), it does not make sense for Being to present itself if presentation is not. So the one should not be.

The first hypothesis historically constitutes 'the inaugural axiom of philosophy' (*BE*, 23) or, at the very least, of investigations into ontology throughout western history since its inaugural organization by Parmenides. Badiou even goes as far as to proclaim that the reciprocity of one and Being continued to be the conscious and unconscious premise of ontology up to his *Being and Event* – even when we take into account the history of flux-oriented Heraclitean thinking, and even when we deliberate on the work of Gilles Deleuze who, as Badiou himself admits, was the first thinker in recent times 'to properly grasp that a contemporary metaphysics must consist in a theory of multiplicities' (*TW*, 68). Up to the advent of axiomatic set theory, philosophy has not had the conditional means to force the thesis in Hypothesis B. *Being and Event* is meant to be the first work that is militantly dedicated to that alternative as a radically disconnected supplement to the current philosophical situation.

The philosophical situation has always taken Being to be one because it is also generally possible, within the immanence of any situation, that Being is one in lieu of the unifying effect of its structure. As Badiou writes:

> In general, a situation is not such that the thesis 'the one is not' can be presented therein. On the contrary, because the law is the count-as-one, nothing is presented in a situation which is not counted: the situation envelops existence with the one. Nothing is presentable in a situation otherwise than under the effect of structure, that is, under the form of the one and its composition in consistent multiplicities. . . . In a non-ontological (thus non-mathematical) situation, the multiple is possible only insofar as it is explicitly ordered by the law according to the one of the count. . . . [A]n indeterminate situation . . . necessarily identifies [B]eing with what is presentable, thus with the possibility of the one. (*BE*, 52)

Being is possibly one within the domestic closure of a situation under the particular count-as-one that forms its unifying concept. Sets are one as a result of the oneness of their belonging relations. The structure of the situation corresponds to a unifying concept that counts all the elements as one. Every presented set, every situation, corresponds to the concept of belonging to that set. The prescription of the count-as-one enables Being to be one with respect to the closure of the situation. So long as the situation's presentations are united under the one of a single concept, Being is one therein. So the previous philosophical situation has always taken Being to be one, and explicit within any general situation is the Being of the one.

Standing on the precipice of the decision, Badiou nevertheless firmly, axiomatically and militantly commits to the alternative. Being is multiple as far as ontology is concerned. Hypothesis B forms the other main axiomatic proclamation in *Being and Event* after the decision to equate mathematics with ontology.

In fact, as the discourse of Being-qua-Being, *ontology is a situation*. Being is multiple and ontology is a consistent presented multiple. This is yet another axiomatic commitment on Badiou's side. 'I will maintain, and it is the wager of this book, that ontology is a situation' (*BE*, 27). Consistent situations about Being-qua-Being can be constructed where certain entities are presented and certain statements hold. Ontological super-situations *are* and, in such situations, the one is not. As a result, for any ontological situation, it is generally the case that the one is not. Or, to put it more precisely, it is not necessarily the case that Being is one for an ontological situation.

Consistency and militant commitment

Badiou's methodology of deciding on the essential multiplicity of Being relates back to the distinction we gave about direct proof versus consistency proof. We recall the relationship between consistency, provability and undecidability. Let Γ denote a consistent collection of axioms and let φ be some statement, with $\sim\varphi$ being its negation.

1. We say that φ is consistent with Γ if appending the statement to the axioms does not lead to a system where a logical contradiction could be derived.
2. We say that Γ directly proves φ if the latter could be wholly derived from the former using the rules of deduction. This is equivalent to saying that φ, but not $\sim\varphi$, is consistent with Γ.
3. We say that φ is undecidable with respect to Γ if neither φ nor $\sim\varphi$ can be proven from Γ. This is equivalent to saying that both φ and $\sim\varphi$ are consistent with respect to Γ.

In other words, if φ, but not $\sim\varphi$, is consistent with Γ then φ is directly provable from Γ; but if φ and $\sim\varphi$ are both consistent with Γ, then both φ and $\sim\varphi$ are undecidable with respect to Γ. We prove a proposition directly by deriving it, using the rules of deduction, from some assumed collection of premises. If the premises are true, so are any propositions that are directly proven from them. The mathematical notion of the direct proof serves as the mathematical condition for what we usually understand as logical argument.

In the case of direct proof, there is a linear and necessary deductive thread from the premise to the propositional end result. But in the case of consistency proofs, the proposition is only *decided* to be true, and that is already enough. No such deductive linear thread is provided, save from the consistency of the premises to the relative consistency of the proposition with respect to the premises. Instead of deriving the proposition from the premise, one derives the statement that the conclusion is consistent with respect to the premises, if the premises are themselves consistent. If the premises are true, then the end result need not be true, unless it can also be proven that its negation is inconsistent with respect to the premises. But the point is that it *can* be true. A consistent proof simply establishes that nothing prevents one from deciding the proposition to be true if the premises are true as well. It does not provide a direct logical or necessary link from the premise to the end result – it simply says that the end calls for a decision, that is leads to the precipice of a decision. And that, for most cases, is enough. It is a legitimate form of argumentation.

Badiou's genius – which must be acknowledged – was to recognize the notion of consistency proof as another reasonable mathematical figure from which an alternative form of argumentation can be conditioned. We do not need to argue the conclusion from our basic assumptions; we can just demonstrate that it is not illogical for us to decide the conclusion to be true. Between the axioms and the end result lies a decision, a militant commitment. This gives us a great flexibility for pursuing any line of reasoning. Whenever a detailed and straightforward argument cannot be given, one can decide for the proposition to be true, with this deciding not being a simple recourse to subjective prejudices, but to the event of decision itself, which Badiou will later link to an event of subjectivity itself, the emergence of a new subject.

How does all this relate to Badiou's deciding that the one is not, and that mathematics equals ontology? We read *Being and Event* and we do not find any direct line of reasoning establishing those propositions. Badiou only claims that there is nothing wrong with taking Being to be multiple and mathematics to be equivalent to ontology. The validation of the propositions does not precede but comes after the decision. Badiou commits to the 'conclusion', and then follows through with 'justifying' and explicating them. This justification-to-come is precisely what he means by fidelity [*fidélité*] to the decision via the faithful sequence of truth procedures. Badiou begins *Being and Event* with the proposition 'the one is not' and spends the rest of the book justifying and following through with it. A consistency proof is a license for decidability. For example, Gödel's proof that the statement '*ZFC* is consistent' relative to *ZFC* is license for us to take *ZFC* to be consistent.

However, the reality of Badiou's relation to the notion of consistency proof, at least as a condition for his metaontology, is more complicated. This will become clear once we understand the details behind forcing, which is a well-known technique invented by Paul Cohen for proving consistency. At the risk of getting ahead of myself, let me just end here with a few preparatory remarks. It is not that a collection of basic assumptions are given; then a proposition is proven to be consistent relative to them; then a militant commitment to the proposition is made; then the deductive process of fidelity, as a sequence of truth procedures, is put into motion. In fact, when it comes to forcing, the consistency proof and the exposition of the truth procedures are tied together. The truth procedures are meant, at their end horizon, to establish consistency. In reality, Badiou never gives a complete consistency proof to the statements 'the one is not' and 'mathematics is ontology'. The following through of the truth procedures, which is an infinite process, is itself the consistency proof. Consistency and the decision are tied together. Instead of demonstrating that the decided proposition does not lead to a contradiction, logical consistency can also be established with the existence of a world, a set model, where the axioms and the decided proposition all hold true. The construction of the truth procedure is precisely the construction of such a world through the process called forcing, a process that is essentially different from the relations of causality, influence or logical implication.

Following through the multiplicity of being

We postpone further explication on this and return to Badiou's axiomatic commitment that Being is multiple – what we called Hypothesis B. The task then is to proceed from its aftermath via the deductive construction of a truth procedure, one that merges with the previous one instantiated by the first axiomatic decision that mathematics equals ontology. The following is a schematic summary of Badiou's findings:

1. What remains of the one when Being is multiple? The answer takes the form of a declaration: the one, though never presented, exists solely as the operation that is the count-as-one. '[T]here is no one,' writes Badiou, 'only the count-as-one' (*BE*, 24). This declaration implies that the operation of the

count is precisely that of presentation as such, the operation of set-making, of situation-creating, of creating consistent multiples. Multiples are present by virtue of being the result of a count-as-one operation. Everything is a set in mathematics, save for the general set-making operation. The one is only the operational result because the count-as-one only operates and it is never presented. What is at stake in the count-as-one is the very operation of Being as such.

2. As an operation, the count-as-one is a function,[5] albeit a very strange one. Like a function, the count-as-one operates from a domain where it receives its input and to a codomain where it takes its output. We have established that the output is a consistent presented multiple. What about the input? The response is complex and will only become fully explained in when we turn to Badiou's commentaries on the empty set and the power set. For the moment, here is Badiou's preparatory description in Meditation 1:

   > The fact that the one is an operation allows us to say that the domain of the operation is not one (for the one is not), and that therefore this domain is multiple; since, within presentation, what is not one is necessarily multiple. . . . What will have been counted as one, on the basis of not having been one, turns out to be multiple. (*BE*, 24)

   So the domain is a multiple, though of a very different kind from what we have discussed so far. It is the inertia of the operation that is retroactively discerned as a 'haunting' within the consistent presented multiple.

3. Never presenting itself, the domain is what the codomain once was, and is only recognized as such retrospectively from within the codomain. The domain is not consistent but what Badiou calls an inconsistent multiple:

   > A situation (which means a structured presentation) is, relative to the same terms, their double multiplicity; inconsistent and consistent. This duality is established in the distribution of the count-as-one; inconsistency before and consistency afterwards. Structure is both what obliges us to consider, via retroaction, that presentation is a multiple (inconsistent) and what authorizes us, via anticipation, to compose the terms of the presentation as units of a multiple (consistent). (*BE*, 25)

   So the count-as-one operation takes an inconsistent multiple as its 'input' and produces a consistent multiple as its 'output'. Both the domain and codomain are multiples. The count-as-one makes consistent an inconsistency and makes present what was subtractively absent.

So Badiou axiomatically decides that the one is not and that Being is multiple. This multiplicity has two separate components:

1. the multiplicity of beings, of particular existents when they are presented (the consistent codomain of the count wherein lies each existent)
2. the multiplicity of Being, of pre-presentation, of what precedes every presentation before it is presented (the inconsistent domain of the count wherein lies the Being of each entity)

Between these two components is the count-as-one, which is what remains of the one when it is only an operation (Figure 3.1). And since ontology studies Being-qua-Being, it is no wonder that it should also be the study of inconsistent multiplicity in general.

How do we account for the multiplicity and consistency of beings? Because every presentation is a set, and because the world of presentation presents a multiplicity of beings. The apparent oneness of beings, their consisting together, is only due to being the result of the count-as-one.

How do we account for the multiplicity and the inconsistency of Being? Since every presentation is consistent, and since Being is never directly presented as such, Being is inconsistent. This inconsistency is due to being prior to the result of the count-as-one. Since the one is not, Being as such must be a multiple. *Qua* Being, it only 'presents' itself as a presentation without presentation, as a haunting within the domain of presentation, the ghostly inertia that remains after any presentation flows forth. However, we will not go as far as to proclaim that Being-qua-Being is not, that inconsistency is not. That would imply that all is consistent multiplicity, which implies that everything is structure, which would be an affirmation of a radical version of structuralism and formalism, a radically structuralist ontology. This, in turn, would lead us to the false ontological thesis of mathematical structuralism, that mathematics is none other than the science of structure, that the entirety of mathematics reduces to algebra, and that mathematical objects are ultimately nothing but structures. Inconsistency *is* – even though it is never presented. This means that inconsistency is multiple. The basis behind pre-presentation is not a unity, not some monism, but an inconsistent multiplicity. This philosophical liaison between ontology and multiplicity can be seen as consistent with the rejection of any onto-theological direction towards the investigation of Being-qua-Being.

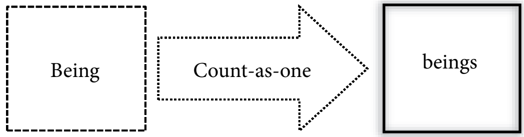

Figure 3.1 From Being to beings via the count-as-one.

ZFC as the a priori conditions for ontology

Further details are given about inconsistent multiplicity, particularly its relation to what Badiou understands as 'pure multiplicity', when he justifies his assertion that set theory under *ZFC* serves as an appropriate *FAS* for ontology. We confront this assertion by returning to the question on how ontology could reasonably comprise a situation. It could very well be that ontology never falls under a situation and that it can never constitute a presentation and a structured discourse. Badiou sees two main hurdles, both of which *ZFC* resolves. We describe these hurdles, then two specific features of *ZFC*, and then how those two features overcome the hurdles.

Hurdle (I): If ontology is a situation, then it must be a presented multiple, moreover one where Being *immediately* presents itself into discussion. This might prove impossible because Being only seems to be indirectly included in what presents itself, and we 'cannot see how it could be presented *qua [B]eing*' (*BE*, 25). One cannot speak of Being in itself because every spoken entity refers to a specific being, which differs from Being-qua-Being in the latter's subtraction from any specificity. Providing an ontological situation is tantamount to providing a presentation within and about which Being can be spoken in an immediate sense. But is it possible to present directly what lies behind any presentation when any presenting of Being can only seem to be indirect, when, to use Heidegger's formulation, 'The Being of entities "is" not an entity' (1978, 26)?

Hurdle (II): As some mode of the count-as-one, an ontological situation admits some plane of presentation, some unifying concept corresponding to the structure's belonging relation. Would not this therefore mean that Being is one therein? This goes against our inaugural decision that the one is not when it comes to ontology. One must present a situation that, despite corresponding to a count-as-one, still respects Badiou's second wager. Such an ontological situation must annul the general veridicity of any ontological unity.

When encountering Hurdles (I) and (II) mentioned earlier, the temptation is to retreat from the challenge and declare that ontology can never attain the status of a situation. The difficulties are thus 'made to vanish in the promise of an exception' (*BE*, 27). This had led to what Badiou denigrates as 'ontologies of presence', which he rejects for their 'captivating grandeur' (*BE*, 27). Among others and in various forms, Badiou lists the culprits as:

1. first, beginning with the Platonic analysis of the Good as both being-supremely-being and 'beyond substance' (*BE*, 26)
2. negative theologies, 'for which the exteriority-to-situation of [B]eing is revealed in its heterogeneity to any presentation and to any predication' (*BE*, 26)
3. experiences of mystical obliteration 'in which, on the basis of an interruption of all presentative situations, and at the end of a negative spiritual exercise, a Presence is gained, a presence which is exactly that of the [B]eing of the One as non-[B]eing, thus the annulment of all functions of the count of One' (*BE*, 26)
4. ontology conditioned solely by the poeticity of language, 'through its sabotage of the law of nominations, which is capable of forming an exception – within the limits of the possible – to the current regime of situations' (*BE*, 26–7).

Badiou's rejection of the last culprit is significant as it signifies his break from the specific mytho-poetic and quasi-theological approach that came about following from the later oeuvre of Martin Heidegger and continued with some of the adherents to the post-Heideggerian movement that it engendered.

Resolving the two Hurdles

Here are the two features of *ZFC*. They concern the standard interpretation of those axioms:

Feature (I): The axioms can be interpreted to be only about pure sets. We have discussed this before. It is enough simply to interpret the word 'set' as those multiples that contain nothing but other pure sets all the way down to $\varnothing$. 'The multiple from which ontology makes up its situation is composed solely of multiplicities. There is no one. In other words, every multiple is a multiple of multiples' (*BE*, 29). The axioms provide the essential scientific framework for a theory of multiplicities while also listing the conditions for something to count as a set. And it is enough, at the most minimal level, for those sets simply to be sets that ultimately contain nothing but ∅.

Feature (II): The main characteristic of the axiomatic route is the separation between the axioms and their interpretation. The axioms never directly prescribe a semantics, an interpretation of the terms that appear therein. 'The count-as-one is no more than the system of conditions through which the multiple can be recognized as multiple' (*BE*, 29). The meaning of 'set' is never explicit in the statements of *ZFC*. An *FAS* can have many different models. Its link to consistent multiples is separate from the statements themselves.

Here is how Feature (I) dissolves Hurdle (I). Three crucial notions – 'presentation of presentation'; 'pure multiplicity'; and 'inconsistent multiplicity' – are linked together by Badiou's line of argument:

1. A pure discourse of ontology must be about the presentation of presentation. 'If there cannot be a presentation of [B]eing because [B]eing occurs in every presentation – and this is why it does not present itself – then there is one solution left for us: that the ontological situation be the presentation of presentation' (*BE*, 27).
2. Since every presentation is essentially a multiple, and since the multiple is reciprocal with presentation, any ontology worthy of the name must concern multiplicity as such, the 'there is' of being-multiple, pure multiplicity, freed from the particularity and concreteness of a specific presentation. Presented therein are multiples with no other property than their multiplicity: multiplicities of multiplicity. What is required is a doctrine of the unadulterated multiple.
3. Remember that presentation splits the multiple into inconsistency and consistency. The consistent component is the specific multiple comprising some specific law of structure. So it cannot directly and purely concern presentation as such because the multiplicity in question was already contaminated by its particularity. This follows from the Axiom of Replacement, which implies that what is essentially at stake in the concept of set does not rely on which specific elements are presented. One can replace the specific elements with other elements, and still get a set. The Axiom 'thinks multiple-[B]eing (consistency) as transcendent to the particularity of elements' (*BE*, 500).

4. The inconsistent component is more latent to the side of presentation in general, of presentation as such. So the Being of the count is precisely on the side of inconsistent multiplicity. The inconsistency 'allows, within the retroaction of the count, a kind of inert irreducibility of the presented-multiple to appear, an irreducibility of the domain of the presented-multiple for which the operation of the count occurs' (*BE*, 28). So ontology investigates inconsistent multiplicity.

So what is at stake in the axioms is the pure operation of the count-as-one, the 'there is' of set-making – an operation with which *ZFC* is directly involved. Being about the presentation of presentation, ontology is thus a theory of pure multiplicity that is simultaneously a theory of inconsistent multiplicity. Ontology must concern presentation of presentation, which must concern pure multiplicity, which must concern inconsistent multiplicity.

Here is how Feature (II) dissolves Hurdle (II):

1. So what is required is an ontology where Being is not unified. This amounts to demanding a discourse without an explicit set model, without an explicit specification of its 'object', its general province of investigation. An FAS is ideally suited for this because its terms are never defined while, at the same time, the system is still able to impose the rules for their manipulation. As an *FAS*, *ZFC* is able to bracket out the meaning of Being from discourse and yet is able to talk about it using the axioms. What is more, in the case of pure sets, 'an explicit definition of what an axiom system counts as one, or counts as its object-ones, is never encountered' (*BE*, 30) since what is presented is precisely nothing but multiplicity per se. All there is in pure sets is ultimately the void, the empty set.
2. We add the following, which is not explicitly mentioned in *Being and Event*: *ZFC* is doubly special by virtue of Gödel's Completeness and Incompleteness Theorems being applicable therein. The limit imposed by the Incompleteness Theorems (on the capacity of *ZFC* to found mathematics) turns out to be a blessing. Pure multiplicity serves as a model for *ZFC* and an interpretation for the word 'set'. But this model, being only part of the universe *V* (since *V* itself, being a class, cannot serve as a consistent model), can never be constructed or exhibited within set theory – a corollary from the Completeness and the Second Incompleteness Theorems that we have repeated many times here.
3. As a result, the interpretability and the meaningfulness of the word 'set' – that is, the very notion of the set – is independent with respect to *ZFC*.

> We can only add this meaningfulness as an extra axiom in *ZFC*. We can only axiomatically commit to the notion of the set. Not only is the notion not explicit in the *ZFC* axioms, but this notion can also not be directly derived from them in any way. Being is multiple in *ZFC* because the structure prescribed by *ZFC* is not unified as a count, at least within the immanence of *ZFC* itself.

The non-existence of the complete universe – whose form-multiple is V – means that Being admits no structure and is never unified under the singularity of a universal and all-encompassing count-as-one. Being is multiple because V is not a set, and vice versa.

ZFC as a laicized and consistent science of inconsistent multiplicity

For Being to be one, *ZFC* has to be consistent. But this consistency is not provable from *ZFC*. So the situational and structural harmony of *ZFC* is not explicitly true within *ZFC*. It can only be true if we add the extra axiom '*ZFC* is consistent' to it. *ZFC* is an axiomatic before it is a situation.

The discourse on Being, when previously linked to the arcana of the one, had been filled with paradoxes. As a result, ontology was described by Badiou as akin to a 'ruined temple', a 'phoenix of its own sophistic consumption' (*BE*, 23). Badiou then writes that even a book of the size of *Being and Event* 'is not excessive for resolving such paradoxes, far from it' (*BE*, 27), although he thinks that his work has moved in the right direction by severing the association between the one and Being.

So has Badiou resolved all those classical paradoxes of ontology? Is *Being and Event* free of self-contradictions? We doubt that we would be able to answer this question without undertaking a tortuous close reading and arduous assessment of every line of argument that appears in the text, fine-combing, like the scholastics of yore, for any unsubstantiated leaps in logic, hidden assumptions or unexpected inferences. We can however repeat what we learnt from Gödel's Theorems. To say that ontology is free from paradoxes is to say that it is logically consistent. In lieu of the Completeness Theorem, this amounts to saying that *ZFC* has a set model. But the existence of a set model can only be axiomatically decided and not established within ontology, because of the Second Incompleteness Theorem.

So has Badiou resolved all the paradoxes of ontology? We read *Being and Event* and fail to find any official claim to this effect. No detailed resolution is described by him. Even if there was such a claim, it would be more accurate to say that Badiou has *axiomatically decided* the paradoxes as resolved, and nothing in ontology can argue against this because *ZFC* cannot prove anything about its own consistency or inconsistency. To simply commit that his ontology is paradox-free is enough for it to be paradox-free, so long as one follows through with the aftermath of the commitment without necessarily justifying or proving it. There is no direct claim or argument in *Being and Event* that this new ontology is without self-contradiction. But it does suggest that it is consistent to say that the ontology is consistent. And that is enough. It is enough to provide, not a direct proof, but a consistency proof through the construction of a consistent world.

The theorems of Gödel establish that there is no fact of the matter regarding the consistency and inconsistency of *ZFC*. Ontology cannot determine the truth-value of every proposition regarding Being. The compositional consistency of ontology lies outside of ontology itself. The realm of Being, paradoxically, contains gaps. These gaps must be filled with pure decisions, with militant subjectivities. The emergence of subjectivity makes up for the incompleteness of ontology. When coupled with Badiou's decision that *ZFC* forms the a priori conditions for ontology, the Incompleteness Theorems of Gödel establish the necessity for there to be subjects.

Conceived now as a situation, ontology is conferred certain new capabilities regarding its investigation of Being, all of which were missing from the so-called 'ontologies of presence', with their prophetic, mytho-poetic, aporetic and quasi-theological invocations. The discourse of inconsistent multiplicity becomes, under *ZFC*, laicized. This places the discourse apart from the previous reading by the mathematician Georg Cantor who, despite being the father of modern set theory, chose to understand inconsistent multiplicity under some holy doctrine of the absolute. For Cantor, the untotalizable inconsistency of certain multiples, such as the Russell Class and the universe of pure sets *V*, orients itself towards the absolute infinity of some Supreme-Being. Divinity stands at the failure of the count-as-one. For Cantor, God is 'the transcendence through which a divine infinity in-consists' (*BE*, 42). However, for Badiou, set theory under *ZFC* (whose complete axiomatization came after Cantor's death) provides the condition to decide that, as far as ontology is concerned, there is no oneness. Cantor's religious reading of inconsistent multiplicity was just another recourse to the captivating grandeur of yet another ontology of presence. An onto-theology is not needed to understand inconsistent multiplicity because inconsistency is other than what post-Cantorian set theory assigns to the side of pure Being in the most immanent and non-theological sense.

When linked to *ZFC*, ontology can be treated as a radically laicized science of inconsistent multiplicity, as a deductive, demonstrative, formal, symbolic, technical, inventive, rigorous and systematic discourse, a science that can be communicated and whose statements can be positively predicated within the regime of knowledge. Moreover, unlike the other systematic sciences such as physics or linguistics, the axiomatization of ontology into the *ZFC* axioms is:

> not an artifice of exposition, but an intrinsic necessity. [B]eing-multiple, if trusted to natural language and to intuition, produces an undivided pseudo-presentation of consistency and inconsistency, thus of [B]eing and non-[B]eing, because it does not clearly separate itself from the presumption of the [B]eing of the one. . . . Axiomatization is required such that the multiple, left to the implicitness of its counting rule, be delivered *without concept*, that is, *without implying the [B]eing-of-the-one.* (*BE*, 43)

Reducing a discourse to an FAS enables many things: a standard of rigour; a common systematic framework for expression; the possibility for a further illumination of ideas and properties; the possible prediction of new connections, causes and outcomes; and the determination of veracity versus error through the derivations of proof and disproof. But ontology is a unique case because it *has* to be an *FAS* in order to be what it is. The Newtonian Laws of Motion would not make sense if we did not explicitly understand that 'body' meant any physical body in our real world and 'velocity' meant the speed at which that physical body moves. But it is imperative in ontology that the concept of Being and the word 'set' remain without any explicit semantic correspondences. And *ZFC* is even more suited to this because the Theorems of Gödel imply that this particular *FAS* cannot predicate anything about Being and about sethood as such.

For axiomatic set theory, the multiple is not a formal concept but a real whose internal impasse is deployed by the theory itself. As the 'unconscious' of mathematics, the set provides the impossible limits of theorization while at the same time constituting its core matter of concern. As Oliver Feltham writes:

> These moments of impossibility in mathematics are 'real' in so far as they provide negative indexes: *indexes* in that they signal an unavoidable obstacle, *negative* in that the obstacle has no substantial positivity; it is not an external

> referent of mathematical discourse. . . . The 'real' is to be understood here not as substantial reality, as 'the world', but rather in the Freudian sense of reality testing: whatever proves an obstacle to our mechanisms of wish fulfilment, or our fantasy of reality, is real. . . . There is a strange conversion of impossibility into necessity at work here in this use of the term 'real'. (2008, 89–90)

This is consistent with what we have established in Chapter 1 that there are no mathematical 'objects' per se as there is no definition of the set, no meaning to pure multiplicity, to the *il y a* of the count, within *ZFC*. Not only does a mathematician not know the equivalence between mathematics and ontology, but he never really explicitly knows what 'objects' are in question in his work and what it means to be a 'mathematical' entity, just as ontology, in order to become itself, can never really represent Being or inconsistent multiplicity. As quoted by Gary Zukav, the great mathematician John von Neumann once said, 'In mathematics you don't understand things. You get used to them' (1990, 226).

We end here with one note regarding Badiou's conclusion that ontology is an axiomatic science. Even when we take into account how *ZFC* overcomes the two 'hurdles' for any ontology, the crucial extra step lies in Badiou's decision that ontology must be a situation. As a result, ontology must form a domain of discourse corresponding to a set, a consistent multiple containing members. This domain is a consistent fragment from the universe *V*. As a situation, ontology must also form a model, which corresponds to an axiomatic system that Badiou argues is none other than *ZFC*. Following from the Completeness Theorem, this axiomatic system must be consistent and admit no paradoxes. So deciding that ontology is the situation of *ZFC* is tantamount to deciding '*ZFC* is consistent', a statement that we know is independent of *ZFC* itself because of the Second Incompleteness Theorem. The First Incompleteness Theorem states that *ZFC* can never be simultaneously consistent and complete. Badiou, as noted by Paul Livingston in *The Politics of Logic* (2011), effectively chooses on the side of consistency. By choosing ontology to be a situation, by rejecting the all of completeness in favour of the one of consistency, Badiou has to accept that this situation must accept undecidability, which eventually allows for the possibility of the subject, of the pure subjectivity that decides the undecidable.

I suggest that it is still possible to weaken Badiou's metaontology framework by admitting only half of his militant decision 'mathematics = ontology'. We might accept that mathematics is ontology, but not that ontology is mathematics, that every first-order investigation into Being must be mathematical. Mathematics could only be part of ontology, the structured, scientific and situational part that allows itself to be axiomatized. Even though *ZFC* is able to overcome the two hurdles, it could still be that there is some portion, still wholly within the domain of direct investigations into Being-qua-Being, that does not fall under mathematical discourse or under any consistent situation or presented multiple whatsoever. There we might find some room to pursue an 'ontology of presence' or, at the very least, a 'metaontology' that is not conditioned solely by pure mathematics – for it is not clear within Badiou's framework, at least from my reading, whether *every* philosophy of ontology, whether the ontological basis of philosophy itself, must be derived solely from mathematical examinations. If ontology is mathematics, then should every direct philosophical inquiry into Being concern only what is given by mathematical texts? Has every previous philosopher been a conscious or unconscious consumer or theorizer of mathematical truths? From reading Badiou's work, it is not clear whether every philosophy of Being – particularly those ideas that Badiou accepts from previous non-mathematical philosophers and writers – is conditioned by some unacknowledged mathematical framework. There appears to be a conflict between taking ontology to be mathematics and accepting certain propositions about Being from Heidegger, Rousseau, Hegel and Hölderlin. In order to accept other conditions, the metaontological question of Being cannot derive itself solely from mathematical thought.

Metaontology of the Axiom of the Void

More can be said about ontology as a general science of inconsistent multiplicity if we continue with Badiou's metaontological commentary of the Axiom of the Void. We motivate our discussion by pointing out that it still seems peculiar that ontology should concern itself with pure sets, with these onion-like structures surrounding the set that has nothing between its curly brackets. The discourse of Being is a very serious and, perhaps, in its own way, the most serious of discourses, the seriousness behind every serious discourse and, following from Heidegger, the kernel behind philosophy as such. So what does ontology have to do with empty curiosities such as $\emptyset$, $\{\emptyset\{\emptyset\}\}$ or $\{\{\emptyset\}, \{\emptyset,\{\emptyset\}\}, \{\{\{\emptyset\}\}\}\}$? Perhaps it is true that ontology should be about the presentation of presentation, even about pure multiplicity and the multiplicity of multiplicity. And perhaps pure sets are the most appropriate mathematical figures for thinking multiplicity per se, for thinking the 'there is' of set-making. But it still seems strange that pure sets such as $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\{\{\emptyset\}\}\}\}$ should be connected with the question of the Being of the coffee mug or even the existence of one's own *cogito*. The connection should be pursued, commented on and clarified further.

Axiom of Existence

We confess that the Axiom of the Void is not indispensable to the axiomatic formulation of the horizon of veracity corresponding to *ZFC*. Various expressions of the FAS, all of which correspond to the same horizon, omit mentioning this axiom altogether as it is deducible from the others. The thesis 'The empty set exists' still holds true, although as a theorem and not an axiom at the zeroth level of a decision that precedes deduction. Kenneth Kunen's *Set Theory: An Introduction to Independence Proofs* (1992), which is the standard textbook in the mathematics community for explaining set theory up to forcing, replaces the Axiom of the Void with the alternative Axiom of Existence (xv). Instead of declaring that there exists an empty set, one can proclaim that there exists a set that is equal to itself:

Axiom of Existence: $\exists x(x = x)$.

It would be an interesting and noteworthy exercise to reconfigure Badiou's entire metaontological framework by focusing on and beginning with this alternative axiom where the emphasis is not on the concept of the void but on self-identity, self-similarity without any negative or dialectical excess. We can nevertheless defend Badiou's usage of the Axiom of the Void because of the foundational role it plays within the structure of the universe of pure sets V. Every pure set is built ultimately from the empty set and the structure of V can be partitioned into hierarchies that correspond to a set's 'distance' from $\emptyset$. Moreover, the empty set is the only unique set whose existence is posited within set theory. There are a lot of sets that are equal to themselves, but there is only one empty set. The uniqueness of $\emptyset$ will prove to be crucial to Badiou's enterprise. However, the point is that it is possible to begin, not with the axiomatic decision 'the void is', but with 'identity is'. But we will not pursue this possibility further.

Foundational edge-of-the-void elements

Badiou's metaontological commentary of the void provides a crucial link to the Axiom of Foundation. Recall that this axiom states that every non-empty set contains at least one element that is wholly disjoint from it. Every situation presents an element whose members are not presented in that set. This foundational element serves as a 'halting point' to belonging. From the point of view within that set, this elements lies at the precipice just before the void or, to be precise, before the *local* void of the situation.

This is a complicated idea, so an example is needed. Consider the following situation-set: $\{a, \{a\}, \{c\}, \varnothing\}$, with the elements a and c meant to be non-equal. This set constitutes a world where only four entities ultimately exist and are presented. Within this situation, only a, $\{a\}$, $\{c\}$ and the empty set exist. Consider the second element $\{a\}$. It can be discerned *within this situation* that $\{a\}$ is a singleton whose sole element is a. One encounters the presented element $\{a\}$ and can confirm that the belonging relation holds for the presented element a. This can be done because the element a itself exists within the situation and one can check that $a \in \{a\}$. But now consider the third element $\{c\}$. From the viewpoint within the situation, this set is a 'blackbox', an edge-of-the-void element. Its constituents cannot be identified. One cannot check that c is a member of this set because c itself does not exist within the situation. From the viewpoint of the set, c is not presented and is nothing. There is no way to know that c exists in $\{c\}$. In other words, $\{c\}$ is 'indistinguishable' from the empty set itself. From the viewpoint, within this situation $\{c\}$ and $\varnothing$ are both foundational since, peering into each of them, one encounters nothing but the abyss and the nothing as abyss. In fact, since none of the elements of a can be discerned, it is also equivalent to the empty set. So $a = \{c\} = \varnothing$ with respect to the limits of the situation's horizon of veracity. Within the situation, within the horizon of veracity presented therein, there exist only two entities: $\varnothing$, which is indiscernible from a and $\{c\}$; and $\{\varnothing\}$, which is indiscernible from $\{a\}$. There is a crucial difference between seeing a situation from the inside and seeing it from the outside.

Since every set must have an element that is disjoint from itself, every situation presents an element whose elements are not presented in the situation. In other words, every situation has a blackbox element, an empty-set-equivalent that, for all purposes, is at the edge of the void since no presentation belongs to it. Every situation has foundational members.

Our main remark is that anything that is not presented by the situation is an abyss that is indistinguishable from the void itself. The Axiom of Extensionality states that equivalence is solely a matter of being able to equate the individual elements between two sets. What lies inside the 'empty' element, whose existence is guaranteed by Foundation, is indistinguishable from the void because the situation has no means by which to differentiate it from the empty set. The 'empty' element might very well be ∅, but there is no way to know this from the intrinsic viewpoint of the situation. For all practical purposes, they both mark the event horizon surrounding a black hole. The foundational element will be the situation's own name for the void. Every void must be examined with respect to a situation, and every set admits the otherness counted as the edge of the void. The foundational element is minimal with respect to the situation's structure, the effect of the count, and so it is never the composition of other elements within the situation. As Badiou writes, the foundational element:

> cannot itself result from an internal combination of the situation. One could call it a primal-one of the situation; a multiple 'admitted' into the count without having to result from 'previous' counts. It is in this sense that one can say that in regard to structure, it is an undecomposable term. It follows that evental sites block the infinite regression of combinations of multiples. Since they are on the edge of the void, one cannot think the underside of their presented-Being. (*BE*, 175)

We will see later that a foundational element has the potential of being what Badiou calls an 'evental site' [*site événemential*]. This potential is realized only if the element is non-empty, although the fact that the site contains elements cannot be discerned from within the situation itself.

Inconsistency and the void

So anything not presented by the situation, anything that is not listed between the curly brackets, is a non-entity with respect to the situation's ontological closure. The void would thus refer to anything that is exterior to the outcome of the situation's compositional consistency, anything that fails to satisfy its structure. Situating itself as anterior to the composition, the inconsistent multiplicity of pure Being is consequently void as well.

The operationality of compositional consistency points to an inert antecedent wherein lies inconsistency. The inconsistent mass of the count-as-one, the Being of composition, is retroactively detected within the immanence of the situation as a haunting, as something of the multiple that does not coincide completely with the product of the count. This product marks out some spectrality that must be counted. But this must-be-counted causes the situation to waver towards the phantom of its inconsistency that is the Being of the situation.

However, it would be wrong to simply accept that all is consistent, that inconsistency is not. While Badiou is committed to the decision 'The one is not', he rejects the possibility that 'Inconsistency is not', for that would lead us to the false thesis of structuralism and legalism, the false thesis that all is structure, that there is nothing external to the realm of structured multiplicity and the sphere of situational structure. Inconsistency is not presented as the consequence of a count. But it still is. Inconsistency, therefore, forms a multiple, an inconsistent multiple. The Being of the count is an inconsistent, spectral and non-presented multiplicity. This inconsistency is nothing, but with this being-nothing forming the basis for compositional consistency. '[E]very situation implies the nothing of its all' (*BE*, 54). By definition, this abyss is not a term in the situation. It haunts the entirety of the situation by being retrospectively discerned as an abyssal nothing.

And this is why inconsistent multiplicity, pure multiplicity, is intimately sutured to the empty set. Both inconsistency and ∅ present a lack, and yet both must remain multiplicities. With respect to an ontological situation, the empty set is the only suitable figure to present the unpresentable, which is the inconsistent Being of the count-as-one. The empty set is the void-multiple of inconsistency and points to presentation of presentation.

To our previous summarized identity between situation, presentation, presented multiple, consistent multiple and set, we can now add the opposite equivalence:

Being = unpresented multiple = inconsistent multiplicity = void

We must clarify the semantic suture between the empty set and the nothing. The set ∅ is not precisely the nothing, but rather the presentative suture to the nothing. The void is the localization, with respect to the situation, of nothingness as a multiple. It designates the fissure between the result-one of multiplicity and the basis on which there is multiplicity. It is 'the one-term of any totality', 'the non-one of any count-as-one', 'the nothing particular to the situation' (*BE*, 55). The void is properly named as a compositionally consistent multiple: the set ∅, even though it composes nothing. Within ontology, unpresentation occurs under the presentative forcing of the Axiom of the Void. In order for the nothing to be presented, the only solution left is to name it as a consistent multiple.

The void is and is not the nothing. It is the nothing because it is the only proper way for the nothing to be a consistent presented multiple, for the nothing to become a thing. It is not nothing, but only contains the nothing in between its curly brackets. As it presents itself within ontology, the void is the proper name of the nothing, the unique nominal suture of the nothing to the ontological situation.

We repeat ourselves:

1. In relation to the immanence of the situation, anything not presented in the domain of the count-as-one is nothing.
2. Even though it can be discerned as the spectral inertia of the count-as-one, the inconsistency of pure multiplicity lies on the side of the domain of the count-as-one. Since this domain is not presented in the situation, inconsistent multiplicity is nothing.
3. The void is the consistency of this nothing in relation to the situation. It is not precisely the nothing but the situation's suture to it, the presentative access to the void. The empty set is, consequently, the suture designating the gap between the result and the Being of the count-as-one. The void is the situation's proper nomination and exclusive structuration of the nothing.

'If ontology is the particular situation which presents presentation, it must also present the law of all presentation – the errancy of the void, the unpresentable as non-encounter' (*BE*, 57). Since the void is the situation's 'rabbit hole' towards its own Being, it is no wonder that ontology, as the theory of inconsistent pure multiplicity, should be a theory of the void. Pure multiplicity can never lie on the side of the result of the count, of specific multiples. As implied by the Axiom of Replacement, a multiple remains a multiple even when its elements are replaced with others. Multiplicity-qua-multiplicity remains invariant with respect to the result of the count. This is why inconsistent multiplicity must lie on the side of pure multiplicity, the multiplicity of multiplicity. Moreover, all the terms within the ontology-set should, thus, be 'empty', and this is why they are composed from the void alone. 'The void is thus distributed everywhere, and everything that is distinguished by the implicit count of pure multiplicities is a modality-according-to-the-one of the void itself' (*BE*, 57).

The void as the ontological atom

So the void should be the first presentation, the originary multiple. Hence, the Axiom of the Void, which posits, via the excendary choice of the most proper and unique of names, the existence of the void:

> Given that ontology is the theory of the pure multiple, what exactly could be composed by means of its presentative axiom system?. . . . What is the absolutely original existential position, the first count, if it cannot be a first one? There is

> no question about it: the 'first' presented multiplicity without concept has to be a multiple of nothing, because if it was a multiple of something, that something would then be in the position of the one. And it is necessary, thereafter, that the axiomatic rule solely authorize compositions on the basis of this multiple-of-nothing, which is to say on the basis of the void. (*BE*, 57–8)

The void is the sole 'atom', the fundamental monad, making up the situation of ontology. Badiou's metaontology of *ZFC* rules out mixing existential physical claims with purely ontological questions. This mistake had already been made when the physical sciences were still in their infancy during the times of ancient Greek materialist philosophy. 'If there are "atoms", they are not, as the materialists of antiquity believed, a second principle of [B]eing, the one after the void, but compositions of the void itself, ruled by the ideal laws of the multiple whose axiom system is laid out by ontology' (*BE*, 58). Note that Badiou never claims that physical matter is not made ultimately from elementary physical particles such as quarks, leptons or bosons. He is only claiming that, as far as the discourse of Being-qua-Being is concerned, everything can only return to the void. Ontology, as the investigation of Being-qua-Being, can say nothing about the fundamental but contingent particularities of our physical world. It can make no positive claim about, for example, the existence of the Higgs boson because that would be the job of experimental high-energy physics. As far as ontology is concerned, nothing can be posited to exist, except for nothing itself as a presentation, which is named as the void. When nothing is presented, the only name, the only existential claim possible, concerns the empty set. Mathematics is similarly a discourse where no specific existential claims are made at the onset, except for the nothingness of presentation. Badiou's equation of mathematics with ontology redefines and refocuses the scope of ontology itself. Perhaps ontology, as the general science of Being-qua-Being, can make no direct claims about this or that coffee mug, unless the mug has some direct and specific suture to the general question of what it means to be. Ontology can only talk about Being in the general sense, subtracted from any particularity. Might ontology shed light on the existence of the *cogito* or, more generally, of the subject? To answer that question, we will have to wait until we get to the mathematics of forcing, which, according to Badiou's metaontology, provides what he calls a post-Cartesian 'Law of the Subject'.

4

Metaontology of the State and Representation

We recall the remarks made in Chapter 3. Metaontologically speaking, a situation specifies a set and a model, with the former constituting the domain of the latter. But a model must also make reference to a horizon of veracity expressed using a semantics. Veracity within a situation is determined solely by what is presented therein. The formal language takes first-order set theory as its general syntax. But a general syntax is not enough as the language must have its own semantics by being linked to an interpretative system, a regime of representation, which is Badiou's version of the Lacanian Symbolic. This is the particular vocabulary of names, properties and relations that make the interpretative system what it is as a representational capacity and a hegemonic rule. Each name refers to an element within the situation and it is sometimes the case that not every element is named by the semantic system. Elements can also have properties and relations between them, and it is often the case that not all possible properties and relations are presented. The vocabulary, together with the rules of grammar, enables the construction of statements. Some of those statements are veridical while the others are erroneous or undecidable with respect to the situation. Some of those veridical statements can form the collection of axioms. Moreover, every model provides a structure: the unifying concept corresponding to the particular count-as-one of the set forming the domain. We can think of this structure as the property corresponding to the belonging relation of the situation.

In order to illustrate everything we have just said, let us expand on the example of arithmetic in more detail. The standard domain is the set $\mathbb{N} = \{0,1,2,3, \ldots\}$ of the natural numbers as we usually understand them, while the standard axiomatic is the *FAS* of Peano Axioms. The arithmetic discourse is linked to its particular vocabulary: the element 0, for example, is named using the symbol '0', while the element 92 is represented by the symbol '92'. There are also the relevant arithmetic properties, such as 'being an even number', 'being smaller than 92' or 'being either one of the numbers 13, 26 or 52'. An example of a relation concerning two elements would be 'adding up to 10', which is satisfied, for example, by the pairs 5 and 5 and also 0 and 10, but not 6 and 9. It so happens that all the numbers in the arithmetic discourse of natural numbers are named, but not necessarily all the properties and relations.

So the question remains of how an ontology based on *ZFC* can condition a thinking for a regime of representation such as this.

Properties, subsets and representations

In order to describe this thinking concerning the semantic system, we will have to examine what set theory says about the ontology of names, properties and relations. Remember that a set is defined ultimately and ontologically by its belonging relation. The count-as-one is such a primitive relation that it takes precedence even over the naive understanding of the set as a 'collection', a gathering-together, or even a multiple. From the naive treatment of set theory, we might initially understand a set as a kind of 'space' with some things 'inside' of it and some things 'outside' of it, a 'club' which admits certain members and excludes others. This image is instructive at the beginning as it helps to hone certain intuitions we already have about multiplicity. But it must take a backseat to the purely inscriptive and axiomatic understanding of the set as a specific relation of '$\in$', a literal foundational inscription of the count-as-one operation. A set and its inscribed membership relation are equivalent. Badiou writes:

> The multiple is implicitly designated here in the form of a logic of belonging, that is, in a mode in which the 'something = α' in general is presented according to a multiplicity β. . . . What is counted as *one* is not the concept of the multiple; there is not inscribable thought of what *one*-multiple is. There one is assigned to the sign $\in$ alone; that is, to the operator of denotation for the relation between the 'something' in general and the multiple. (*BE*, 44)

A set is identified ultimately as a radically neutral relation, the count-as-one operation that is exclusive to itself, which it self-indexes. Instead of understanding a set β primarily as a gathering-together of its members, we should grasp it as $\in_\beta$, as the belonging relation to which it is indexed.

So every presentation is the result of a count-as-one. This includes, as we have established in Chapter 2, every property and relation, at least on the fundamental level of ontology. We established that properties and relations, when they present themselves, must be extensionally equivalent to some set. A property is ultimately the set of entities satisfying it while a relation, which is really a higher-order property, is ultimately the set consisting of all the groups of elements among which it holds. The linguistic, idealist and transcendental character of the properties and relations must take a backseat to its corresponding set, since every presentation is ultimately a set, because Being is essentially multiple. Extension precedes intension, as prescribed by the Axiom of Extensionality. Existence precedes language, at least formal language as prescribed by the Axiom of Separation. Just because a set is definable does not mean it exists. Every property is ontologically the operational count-as-one of some existing set, and every set corresponds to the property of belonging to it. The predication is not some linguistic transcendental; it is nothing but the set itself since, for set theory, the only things that exist are sets. To exist is to be a consistent multiple. This is coherent with our strictly ontological purview, and with the Badiouian equation of mathematics with the science of Being-qua-Being. Concrete predicates are subtracted, leaving the pure multiple, pure and simple presentation. Ontology involves the question of pure multiplicity, subtracted from any particularity, concreteness or predication. What is in question is not an object's particularities, not its properties or the relations to which it inheres, but the simple brute fact that it is, that it is presented.

This does not mean that any mention of properties is completely absent from set theory, but that they are fundamentally reduced to the question of their presentation, not their predication and not their partaking in specific objects. What is ontologically at issue with respect to any property or relation is the brute fact that it is, that it presents itself, that it constitutes a set; the question of a property partaking in an entity is reduced to the question of the entity belonging to the property's corresponding extensional specification. So everything returns to the count-as-one, to the membership relation. This is why set theory is so suited for investigating ontology, specifically when the latter is restricted to the question of Being-qua-Being and pure multiplicity. This does not negate the fact that properties can be specified intensionally using the resource of formal language. Even though they reduce to belonging relations, the fact remains that they can be specified as a predicate, as formulas with free variables. Moreover, an extensionally-specified property can have different intensional equivalents. We shall use the term 'predicate' or 'formula' to specify intensionally specified properties and the Badiouian term 'representation' for extensionally specified properties. An intensional set can be understood as a *re*-presentation, a second presentation, of the terms that it counts as one. The term *x* in the set {*x*,*y*,*z*}, for example, is re-presented in the corresponding extensional property 'being either *x* or *y* or *z*'.[1]

Names and singletons

So properties and relations are ultimately sets, and representation is just another presentation because there is ultimately only one category of presentation, one flat plane of existence, as far as ontology in concerned. The count-as-one subsumes the predicative and relational feature of formal language under the realm of pure presentation.

Nomination is a particular type of representation. When we have a representation that extensionally matches with a singleton, with a set containing one member – this is what we call a name, or nameability, the 'nominal seal' (*BE*, 90) of a proper name. Ontology understands the name as a singularly instantiated property. For example, if the structure of {*x*,*y*,*z*} corresponds to the property of 'belonging to {*x*,*y*,*z*}', of 'being either *x, y* or *z*', then the structure of {*x*} corresponds to the property of 'being *x*', to the proper name of *x*. The singleton of *x* presents the nominal seal of *x*. If the set {*x*} does not exist, then *x* is not nameable or, to use Badiou's term, is not represented in the relevant situation, is absent from the Symbolic, from the regime of representation. Badiou writes:

> [I]n set theory, what I count as one under the name of a set *a*, is multiple-of-multiples. It is thus necessary to distinguish *the count-as-one, or structure*, which produces the one as a nominal seal of the multiple, and the *one as effect*, whose fictive [B]eing is maintained solely by the structural retroaction in which it is considered. (*BE*, 90)

Take, for example, the person Alain Badiou as a consistent multiple. The name of Alain Badiou is a separate set altogether, i.e. the singleton set {*Alain Badiou*}. By 'name' we do not mean the linguistic inscription of the word 'Alain Badiou' with its eleven letters and space in between, but its representational quiddity, its nameability. Alain Badiou exists, is presented, if Alain Badiou is a set, a consistent presented multiple. Alain Badiou is named if {*Alain Badiou*} is an existing set, if the property 'being Alain Badiou' is a consistent multiple. Badiou calls this nominational count as the 'forming-into-one':

> The operation by which the law indefinitely submits to itself the one which it produces, counting it as one-multiple, I term forming-into-one. Forming-into-one is not really distinct from the count-as-one; it is rather a modality of the latter which one can use to describe the count-as-one applying itself to a result-one. (*BE*, 90–1)

So the property of 'being x' is allowed only if this being x constitutes some relation of belonging, belonging to $\{x\}$. So it has to be collected into the set containing itself. So, from the viewpoint from within the situation, an element really 'exists' only if its singleton does, if the referencing of x by means of names and formal language is itself allowed. So a set is really nameable only if there is a set that contains it. (However, things become a little more complicated when it comes to the event being considered as a set).

What this also means is the following. Within the larger situation, an element can only be spoken of if there is another smaller situation within the larger situation that finds the element solely presented. The proper name of x is the situation where only x exists. Likewise, a representation is the situation where only its exemplifications are presented.

In fact, the axioms of *ZFC* ensure that every element presented by a quasi-complete situation is also named, that the presentation of any element ensures the presentation of its singleton, at least from the general viewpoint of ontology as a situation, as a universe. When *ZFC* holds, then the presentation of element x in the situation implies the existence of $\{x\}$. As Badiou writes:

> [I]t is always possible to count as one an already counted one-multiple; that is, to apply the count to the one-result of the count. This amounts, in fact, to submitting to the law, in turn, the names that it produces as seal-of-the-one for the presented multiple. In other words: any name, which marks that the one results from an operation, can be taken in the situation as a multiple to be counted as one. (*BE*, 90–1)

The proof of this theorem – that $\{x\}$ is a set if x is a set – is given in Meditation 7 (*BE*, 91), and we shall not reproduce it. We will however add that *ZFC* also implies that the collecting together of any *finite* collection of elements always creates a set. Any property is allowed if the number of presented elements satisfying it is finite. So every representation is presented so long as the number of its represented elements is finite. Ontology ensures that every individual presentation is named and every property is allowed as long as only a finite number of entities satisfy it.

The power set and the regime of representation

Say we have any situation. As a structured discourse, it must be linked to some semantics, which will be linked to various relevant representations, which are all the extensionally specified names, properties and relations used by its discourse. This includes the property corresponding to the situation's structure, which is simply the belonging relation of the situation's corresponding domain. If all those representations are collected together and counted as one into a single and separate set, then the ensuing multiple must, by definition, prescribe its own structure of presentation. This structure is none other than the Symbolic of the original situation. The representations are said to obey a regime prescribed by the structure.

Of course, there will be a regime only if there exists that multiple of all the representations, if all those representations are counted as one. Ontology would provide for a general Being of representation only if the existence of every situation implies the existence of its semantic structure. Badiou's remarkable metaontological insight was to understand that the regime of representation is ontologically schematized by the situation's power set.

Since the power set collects together all the existing subsets of a particular situation, it specifies all the possible properties and proper names that can be made only from the members of that situation, all the possible ways of representing and 'talking' about its elements. The power set is simply the tabulation of all the proper names and properties that exist, that are allowed, of that set. These names and properties are representations made from that set. The power set is thus ontologically none other than the regime of representation, or what Badiou calls the state [*l'état*] of the situation, the 'metastructure' of the structure. To put it more schematically:

1. Proper names are singletons. Properties are the set of entities satisfying it.
2. Given any set, any situation, the set of all the proper names and properties that can be made out of its elements corresponds to the set of all its subsets.
3. So the power set contains precisely the set of all the proper names and properties that can be made from the situation. It corresponds to the state of the situation, at least when we are talking about names and properties.

The power set of a set S counts as one all the existing parts of S. It constitutes a second distinct count of S, presenting everything a second time as a re-presentation. As a count, the set $P(S)$ has a structure of its own, one that Badiou calls the 'metastructure' of the first (Figure 4.1). The Axiom of the Power Set ensures that $P(S)$ exists if S exists. Moreover, the non-equality of S and $P(S)$ ensures a gap exists between set versus power set, element versus subset, belonging versus inclusion and structure versus metastructure.

If the power set contains *all* the possible subsets, this means the regime of representation is maximally complete, that all possible representations exist, including the proper names for each and every one of the presentations. The power set must obviously contain the set itself – since the set exists and is a subset of itself. So the power set ontologically prescribes the unifying concept of the set, the concept corresponding to the property of belonging to the set.

The link between parts and representations can be made clearer if we return to the Axiom of Separation, which states that predication only creates sets that are already subsets of other sets. In other words, the use of formalized language is allowed only if it concerns subsets of other existing sets. Those existing sets are precisely the situations in question here. When counted as one, the subsets of a set S, therefore, present a limit of how much predication can be done with respect to S as a situation. Predication must always be quantified with respect to the subsets of an existing situation, and the power set is precisely the presentation of all those subsets as a count-as-one. This is why the notion of property is tightly knit to that of the power set. This is also why language, when it comes to the question of presentation, cannot operate solely on its own but must always evaluate itself with respect to a domain of discourse.

The fact that the state must be a separate multiple from the situation follows from the Theorem of the Point of Excess that we proved in Chapter 2. The set and its power set are separate because there exists a subset, consisting of all the ordinary and non-evental elements, that can never belong to the set. The theorem also explains why the situation, on its own, is incapable of providing for representation, as there will always be representations that are not presented within it. In order to represent the elements of a situation, in order for presentation to be fully named, predicated and related together, the demand is to step outside the situation into some larger situation and enter into the separate situation that is the regime of representation. The situation can be properly represented and its consistency can be verified only by stepping into the situation's power set. A structure can be recognized as such only via a separate metastructure.

Now every set is located within a larger universe satisfying the fundamental rules of ontology, i.e. *ZFC*. In order for the Axiom of the Power Set in *ZFC* to be completely veridical, the universe must either be empty or infinite, since taking the power set successively creates an infinite sequence of new elements: the power set, the power set of the power set, the power set of the power set of the power set, *ad infinitum*. Moreover, we cannot provide an example of a quasi-complete super-situation because it cannot be exhibited, owing to Gödel's Second Incompleteness Theorem.

In order to avoid what is bound to be a common confusion, we must note that representation is not the same as appearance. Within a state, not all the subsets are 'operative' as appearances (although Badiou will only provide a phenomenological framework in *Logic of Worlds*). The parts collected into the state are only an ontological inventory of every presented representation, regardless of whether they are operative as phenomena. Still, various other ontological factors might determine which representations are presented. If both the situation and the state are installed within a larger super-situation (as in the case of *U* earlier), then the presented subsets depend on which ones are counted within that larger super-situation. If that larger super-situation is the quasi-complete universe of ontology, then we know, as a corollary to the *ZFC* axioms, not only that every situation has a state but also that every finite subset must be presented. As a result, the state of every finite situation must always be complete in ontology and count every possible subset. If the situation is finite and contains n elements, then the regime of representation must contain 2^n parts within ontology.

One conceptual benefit of taking the state to correspond to the power set is that a new point of compromise is offered between the relation of transcendence and immanence with respect to language and the Symbolic. On one hand, the state is not completely unrelated to the situation as it contains members that share elements with the situation. On the other hand, the state is not absolutely sutured to the situation because, owing to the Theorem of the Point of Excess, the set and power set are not equal or members of each other. 'As such, the state of a situation can either be said to be separate (or transcendent) or to be attached (or immanent) with regard to the situation and its native structure. This connection between the separated and the attached characterizes the state as metastructure, count of the count, or one of the one' (*BE*, 98). In fact, as we shall see in the case of infinite sets, the absolute size of its power set is undetermined and undecidable with respect to ontology.

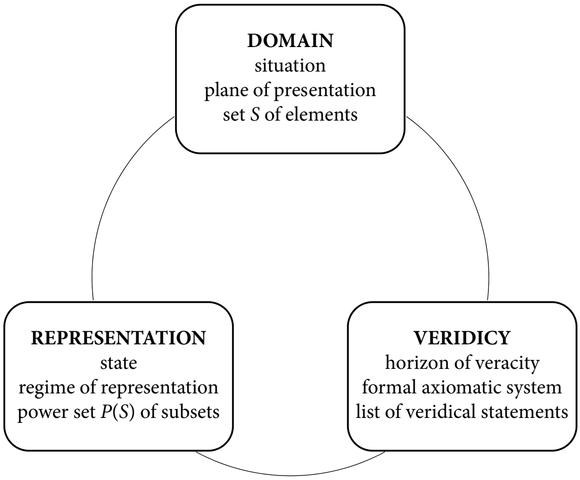

Figure 4.1 The triad of domain-representation-veridicy.

Power set of Cartesian products and the regimes of relation

Just as a property is present only if its corresponding set exists, a relation is present only if its corresponding set of ordered sets also exists. This is where the operation of taking Cartesian products becomes important. Now the operation of taking Cartesian products is a legal operation of *ZFC*. It can be proved that if two sets are presented in *ZFC*, then so is their Cartesian product. Just as the power set is the list of all possible properties that can be made from some set, the power set of a Cartesian product specifies all the possible relations that can be made among the elements of several sets. It determines what we would like to name now as the regimes of relation. The Cartesian product counts as one all the presented ordered pairs, while the power set of the Cartesian product counts as one all the presented sets containing any of those ordered pairs.

For example, say we have two situations $S = \{a,b\}$ and $T = \{x,y\}$. The Cartesian product collects all the existing ordered pairs where the first element is from S and the second from T. Say the Cartesian product $S \times T$ contains the three existing pairs (a,x), (b,x) and (b,y), with all the other possible pairs being nonexistent. The power set of this Cartesian product collects all its existing subsets, all the subsets of $S \times T = \{(a,x), (b,x), (b,y)\}$. So any existing relation between elements of S and elements of T must take its elements from what is in the power set of $S \times T$. Say $P(S \times T)$ has only the three relations $\{(b,x)\}$, $\{(b,y)\}$ and $\{(b,x), (b,y)\}$ – that is, the only relations are those that contain ordered pairs where the first element is always b. This means that, within set S, only the element b can have a relation with an element of T.

This notion of the regime of relation can be extended for relations between more than two elements, that is, for Cartesian products of three or more situations. For example, the regime of ternary relations among three situations S, T and M would be the power set $P(S \times T \times M)$. We can also speak of relations within the immanence of a single situation *among* its members. In this case, we must refer to the Cartesian powers. The regime of binary relations within situation S would then be the power set of the Cartesian product $S \times S = S^2$. This power set determines the limits of relation, specifically relations between two elements, within S itself. Presentations within S can relate to one another only by obeying what is allowed by $P(S^2)$. To sum up:

plane of presentation within S, i.e. domain of $S = S$
regime of properties within S, i.e. regime of representation within $S = P(S)$
regime of binary relations within $S = P(S^2)$
regime of ternary relations within $S = P(S^3)$
⋮
regime of k-ary relations within $S = P(S^k)$
⋮

We can understand the regimes of relation beyond the level of unary relation, of properties, as regimes of representation correspond to different situations, the Cartesian powers of situations. This explains why Badiou does not provide any thought for a regime of relation in *Being and Event* as the question of relation is partly subsumable under the question of representation. The regime of binary relation within some situation S can be understood as the regime of representation within the separate situation S^2. We know that, within the quasi-complete super-situation, a presented situation always has a regime of representation. This follows from the Axiom of the Power Set: the existence of set S implies the existence of set $P(S)$. But is the existence of the regimes of relation also ensured? Yes, because the existence of S also implies the existence of $P(S^2)$, $P(S^3)$, $P(S^4)$, and so on. The proof of this hinges on the fact that the operation of taking finite Cartesian products is a legal operation with respect to *ZFC*.

Representation versus predication

We repeat that the count-as-one is fundamentally an operation, not of language, but of Being in all its presentative quiddity without oneness. Beings are presented qua consistent multiples and not by virtue of being discernible by predicates. The structuration performed by the count is not on the higher order of formal language but on the zeroth order of presentation. The primordial relation of structuration is simply that of belonging. All the other relations – inclusion, predication, language and knowledge – are fundamentally derivable from that of belonging, from that of primordial structuration. For example, the set {2,4,6} is simply the consistent presented multiple whose belonging relation is satisfied only by 2, 4 and 6. It is the multiple fundamentally discerned by the property 'being a member of {2,4,6}', or 'being either 2, 4 or 6'. Ontology differentiates this multiple from those discerned by more complicated predicates that do not simply involve the fundamental belonging relation, such as 'being an even number between 1 and 8'. As a belonging relation, a consistent multiple must be separated from its predicative discernibility.

So the count-as-one as such does not require a linguistic transcendental domain. Ontology does not assume a divine 'counter' behind the count as such. Since the one is not, then the count-as-one 'exists' only as a neutral and anonymous operation. This is linked to Badiou's laicization of ontology. In its laicization, ontology defers the question of finding some *fundamentum absolutum*, some divine God, implementing the count. The discourse of Being stops at Being and digs no further. Now it is true that the set theory axioms provide a system of conditions through which a multiple can be realized qua multiple. But the unique feature is that these axioms, as we explained in previous chapters, do not unify Being, which still remains multiple therein.

Owing to the Axiom of Extensionality, a predicate is only semantically meaningful if it is liaised with a presented multiple. Just because a predicate can be formulated does not mean that it is semantically fulfilled, that an extensional equivalent of it exists. We cannot accept the Axiom of Comprehension. But the Axiom of Separation gives language some restricted ontological powers. Language can 'create' a set by collecting together elements under some discernible predicate, but only if those elements are already presented within some already existing set. In other words, language can only create subsets of other existing sets. A predicate can create a set only by separating elements out of another set.

So language links predicates not with sets themselves but with subsets. Now all the existing subsets of a given set are collected together as the set's power set. Remember that the power set need not collect all the possible sets, just the ones that already exist. And remember that the power set need not exist at all, unless the situation is installed within the larger quasi-complete situation of ontology where the Axiom of the Power Set holds. We know that each finite subset exists within the larger quasi-complete situation of ontology, owing to the Axiom of Pairing. We also know, because of the Axiom of Separation, that each subset separated by a predicate exists. Still, the power set is a complete inventory of all the existing subsets. It is the end result of, among other things, the possible predications with respect to the set, the regime of representation.

Since every finite subset exists in ontology, we can be sure that if a subset is missing then that subset must be infinite and ineffable (although not all ineffable subsets are missing). The subset must contain an infinite number of elements and its belonging relation cannot be liaised with any predicate constructed by the formal language. We can thus say that the power set specifies the limit to the predicative liaisons of formal language with respect to the particular set. Any missing subset is un-representable, indiscernible and ineffable. Likewise, the existence of a subset in the power set is none other than the possible predicative suture of that subset to formal language. This is why Badiou links the power set with the regime of representation. And this is why subsets provide the ontological schema for representations. A subset is simply a representation. The presentation of a subset within the power set allows the possibility for a predicative suture with the Symbolic.

The distinction between representation and predication is subtle and could be the cause of confusion. So it is worth schematizing everything:

1. A representation is simply a subset of a situation-set. Predication involves the formulation of a formula in first-order language with one or more free variables. The formulation of that formula often involves making use of names, relations and properties, which are also representations and, thus, make use of the state operations of the situation-set in question, or maybe other situations-sets. A predicate can only semantically function if it is quantified over the elements of some existing situation-set. This is dictated by the Axiom of Separation. Predication creates a multiple consisting of all the elements of the situation-set that satisfy the formula in question.
2. A representation is a subset of a situation-set and, thus, a member of the situation's state. The representation's structure corresponds to the property of being any one of the representation's elements. A representation with only one element is called a name. The name's structure corresponds to the property of being that single element. A representation can be part of a situation that forms a Cartesian product, in which case the representation is a relation, a higher-dimensional property.
3. A multiple can be intensionally defined by collecting together all the entities discerned by a predicate. Such a multiple is called a class. Because of the

> Axiom of Extensionality, a defined class exists and is presented only through its extensionally equivalent set. Multiples are presented qua sets and not qua predicates. The Axiom of Separation guarantees that certain predicates have extensionally equivalent sets. On its own, a complex predicate cannot create a set *ex nihilo*. It can do so only by collecting together elements that are separated out of an existent situation. So the operation of a complex predicate must be quantified over the elements of some existing situation. Predicates can only create subsets of other situations. We say that a representation is semantically liaised with a predicate when it is extensionally equivalent to the multiple that collects together all the elements that have been separated from its parent situation by that predicate. A predicate is semantically fulfilled and logically consistent if it has been liaised with some existing representation. A subset could be liaised with a variety of predicates, most of all with the unique atomic predicate corresponding to the property of satisfying its belonging relation.

When operative, an intensionally defined multiple therefore involves three types of sets:

i. the situation over whose elements it quantifies
ii. the subset-representation with which it is liased and to which it is extensionally equivalent
iii. the representations comprising the parameters in the formulation of the predicate.

To give a concrete example, say we have a situation-set comprising all living French philosophers. Consider the intensionally defined multiple consisting of all the French philosophers who are shorter than Alain Badiou, corresponding to the formula '*x* is shorter than Alain Badiou'. This multiple is quantified over all the living French philosophers. It is extensionally equivalent to the set of all living French philosophers shorter than Alain Badiou. In the explicit formulation of the formula '*x* is shorter than Alain Badiou', one of the parameters is Alain Badiou himself, or rather the name of Alain Badiou, the representation that is the singleton {*Alain Badiou*}.

We emphasize that a representation is only the possibility of a predicative suture with some formula. A subset is characterized as a linguistic 'transcendental' only via its interaction with predicates whose corresponding multiple is extensionally equivalent to it. For example, ontology recognizes the subset $\{3,5,7\} \subset \mathbb{N}$ as simply the belonging relation that admits the three numbers 3, 5, and 7. In the primordiality of its presentation, this representation is a consistent multiple like any other. It becomes predicated only when it is liaised with the relevant formulas, such as 'being odd number smaller not exceeding 7' or 'being an odd prime number smaller than 11'. The representation designates only the ontological and extensional possibilities for language. A representation can be liaised with several predicates. A predicate might not be liaised with any set at all and, thus, does not correspond to any presentation. The 'radicalism of ontology . . . suppresses liaisons in favour of the pure multiple' (*BE*, 293). '[W]e have at our disposal a whole arsenal of properties, or liaison terms, which unequivocally designate that such a named thing maintains with another such a relationship, or possesses such a qualification' (*BE*, 287).

Badiou will later link knowledge with the capacity to inscribe controllable predications in legitimate liaisons. Even though Being promotes the extensional over the intensional character of presentations, it is possible to have an ontological orientation where the predicates have a say in which possible multiples get presented. This orientation, called constructivism, will be discussed in Chapter 6. For the moment, we will say that the language of a situation is its capacity to liaise representation-subsets with predicates. An element in the representation-subset is discerned by the liaised predicate. If an element from the situation-set does not belong to any of the existing subsets in the power set, then that element is not only unrepresented but also ineffable with respect to the language. If a representation-subset cannot be liaised with any complex predicate, then that representation-subset is indiscernible. An indiscernible-subset is either absent from the power set or can be defined only in terms of the property of satisfying its belonging relation. With reference to a situation-set, language provides the means for predicatively discerning and separating subsets from the terrain of its power set. These predications are also linked to the operation of knowledge because they establish a variety of epistemic mappings with respect to the situation. An encyclopaedic determinant is simply a collection of such mappings that are operative at a given time. A situation might be subjected to different encyclopaedic determinants and a subset falls under a determinant if it is liaised with one of its predicates.

The state prevents the situation from encountering its own inconsistency

Badiou provides a curious commentary on the relationship between, on one hand, the state of the situation and, on the other hand, the situation's possible exposure to its own inconsistency, to the 'anxiety of the void'. This relationship is complicated, but we can shed light on it by recognizing its connection to the mathematical equivalence between syntactic and semantic consistency in lieu of Gödel's Completeness Theorem. We reconstruct Badiou's narrative:

1. Remember that the count-as-one splits the multiple into inconsistency and consistency. Inconsistency never presents itself directly within the consistent multiple, within the result of the count-as-one, but is only retrospectively discerned as a haunting and as the anxiety of the void. We have already discussed in the previous chapter the intimate ontological connections between inconsistent multiplicity, the void, and the Being of presentation. Within ontological situations, the Being of presentation is fixed under the name of the void, the monad around which every presented multiple is constructed.
2. But what about non-ontological situations, situations where the Axiom of the Void does not necessarily hold and where the void neither appears as a term nor constitutes the sole atom of every presentation? The spectre of Being, how is it exorcized? Every presentation, every situation, is haunted by the danger and the anxiety of the void that is its Being. In itself, the count is insufficient because it does not count itself. The result of the count-as-one is consistent because of its avoidance of the catastrophe of its inconsistency, of the unpresented multiple that is the inertial precursor of the count. The apparent firmness of presentation is due to some structure, some kind of presentation, preventing any encounter with the situation's own void, an encounter that would have been the ruin of the One.
3. But the structure of the non-ontological situation, its count-as-one, is unable to prohibit the errancy of the void from fixing itself. This is because something within the structure escapes the count, namely the count itself. The structure of the situation is unable to hinder the presentational occurrence of inconsistency as such because inconsistency is its very Being. The structure cannot 'structure off' its own structurality. Since it is always possible for the structure itself to be the point where the void is given, structure itself has to be structured. The 'there is Oneness' has to undergo a second count-as-one, a re-count. All situations are to be structured twice: they are always both presented and re-presented. The consistency of presentation requires structure to be doubled by a metastructure. The metastructure would attest that, in the situation, the one is.
4. In order to be free of the spectre of the void and thereby guarantee the structural integrity of the situation, there appears the exigency for this

second count. But what exactly is counted? The second count cannot simply re-count the terms from the first count, otherwise the former would lose its function and its separate identity. Moreover, the second count cannot directly count the first count-as-one, which exists only as an operation and never as a term. The must-be-counted of the void, it is neither local to the situation (since it is not a term) nor is it global (since it is the nothing of the whole). The remaining option is for the metastructure to count the parts, which are neither points nor whole, and neither local nor global. So the state counts the existing subsets of the situation. The Being of the state is the situation's power set.

5. Moreover, this state counts the first count because it contains as an element the situation itself, the 'total part', as a subset of itself. As Badiou writes:

> The completeness of the initial one-effect is thus definitely, in turn, counted as one by the state in the form of its effective whole. . . . The state of a situation is the riposte to the void obtained by the count-as-one of its parts. This riposte is apparently complete, since it both numbers what the first structure allows to in-exist (supernumerary parts, the excess of inclusion over belonging) and, finally, it generates the One-One by numbering structural completeness itself. (*BE*, 98)

We examine Badiou's thinking on the connection between the state and the inconsistency of the situation, and we might assume that this thinking is conditioned, at least in part, by Badiou's previous investigations of his philosophical masters. For example, the fact that he uses the words 'metastructure' and 'state' indicates the political genesis of these concepts in relation to Marxist, Leninist and Althusserian conceptualizations of superstructure and of the Ideological State Apparatus. (The relation is explored somewhat in Meditation 9 when Badiou talks about historico-social situations.) Badiou goes as far as to find precedents to this link between the state and inconsistency in Thomas Hobbes's work on political philosophy (*BE*, 109). I have already indicated that it is also possible that Badiou's thinking of the state might relate to Lacan's psychoanalytic study on the Symbolic Order, the Master Signifier and the Name-of-the-Father.

But a mathematical condition is also at work. Gödel's Completeness Theorem becomes relevant again as it implies that the horizon of veracity must be free of paradoxes when it is linked to the composition of the situation. We quote Badiou at length:

> The structure of structure is responsible for establishing, in danger of the void, that it is universally attested that, in the situation, the one is. Its necessity resides entirely in the point that, given that the one is not, it is only on the basis of its operational character, exhibited by its double, that the one-effect can deploy the guarantee of its own veracity. This veracity is literally the fictionalizing of the count via the imaginary being conferred upon it by it undergoing, in turn, the operation of a count. . . . Any ordinary situation thus contains a structure, both secondary and supreme, by means of which the count-as-one that structures the situation is in turn counted-as-one. (*BE*, 94–5)

Being consistent can also mean being without self-contradiction. The horizon of veracity within a situation is guaranteed to be free of self-contradictions so long as the situation counts as one. A presented multiple is consistent not just because it consists together but also because its structure is logically and internally coherent, provided that its horizon of veracity is articulated by some interpretative formalism that is the 'fictionalizing' of the first count-as-one. The veridical statements within a situation do not contradict each other because the domain of the situation constitutes a set.

This implies that a presentation can never contain paradoxes. Existence implies logical coherence, and vice versa. The Law of Non-contradiction becomes an inherent law of presentation. The world, which might include our world, is logically consistent if it is, on the whole, a set. Let us call the first consistency, where elements consist together as one, as 'compositional consistency' and let us call the second consistency, the logical coherence of the axiomatic system, as 'veridical consistency'. The two are clearly the metaontological names for semantic and syntactic consistency, respectively.

The semantic system allows a regime of representation to be presented and to be counted as one. Since the system allows for the expression of the situation's horizon of veracity, we can say, by virtue of Gödel's Completeness Theorem, that representation prevents the situation from encountering its own inconsistency. The situation becomes veridically consistent because of the expressive and interpretative capacity. Veridical consistency is conditioned by the compositional consistency of the semantic system, for veracity can be articulated as such only from the presentation of representation, from the consistency of representation, from the fictionalization of the first count.

Badiou writes that the state verifies the consistency of the situation by preventing an encounter with the haunting inconsistency that is its Being. This can be understood in relation to the Completeness Theorem, which links veridical with compositional consistency. The situation is consistent, but it cannot be verified as such from within its own immanence. The exigency, then, is to step outside of the situation itself into a separate situation, the state. The metastructure corresponding to the state allows the situation to be represented and statements to be posited. The logical consistency of all the veridical statements is equivalent to the compositional consistency of the situation. The equivalence is established through Gödel's Completeness Theorem. So the metastructure wards off the inconsistency of the situation by enabling it to be represented by the state. The consistency of the state, the count-as-one of the regime of re-presentation, verifies the consistency of the situation. So Badiou's claims about the relation between inconsistency and the state is not purely philosophical, or even political, but is conditioned by mathematics.

The state of the empty set and of a quasi-complete situation

We end by examining the unique states of the most minimal and the most maximal situations: the empty set and the quasi-complete situation of ontology.

Remember that there are two unique properties of ∅ when it comes to the inclusion relation: it is the subset of every set, and it is the sole subset of itself. 'The first property,' writes Badiou, 'testifies to the omnipresence of the void' (*BE*, 86). Nothing belongs to the void, and yet the void is included in everything. 'On the basis of everything which is not presentable it is inferred that the void is presented everywhere in its lack: not, however. as the one-of-its-unicity, as immediate multiple counted by the one-multiple, but as inclusion, because subsets are the very place in which a multiple of nothing can err, just as the nothing itself errs within the all' (*BE*, 86). Even though it satisfies the property of being the universal subset, it could be that ∅ is not presented in its power set, which collects only the subsets that exist in the larger super-situation. But we must also remember that every set includes the empty set. The void is always named. The void is always represented.

What about the case when the situation is empty, when it is equal to the empty set itself? The second property says that the power set of ∅ contains only ∅, i.e. $P(\varnothing) = \{\varnothing\}$. The empty set is named even when nothing is presented. The name of the void is consistent with the non-existence of anything. This is why it should be the first set of ontology, which never posits any specific presentation. The void is the only name that is compatible with the empty situation. When nothing is presented, all that remains is for this nothing to be named.

The maximal situation of ontology must be stateless, for there can be no metastructure to a quasi-complete situation. This is compatible with what we have said in the previous chapters. Ontology does not need a metastructure to ward off the inconsistency of its Being because it already presents this Being via the presentation of the empty set and via the fact that the domain contains only pure sets that are woven from the empty set. The need for a separate multiple to structure representation arises only for non-ontological situations. A separate regime of representation would contradict the axiomatic exigency of ontology by prescribing a definition of its terms and a definition of Being, which would go against the Badiouian decision that the one is not.

If ontology had a state, then that state would place its presented subsets outside of all consistent presentation. A quasi-complete situation should by definition contain every presentation. Since the state must lie external to it, then the state must contain and count an absolutely distinct species of Being-multiple, corresponding to the classes. Badiou writes:

> If indeed there existed a state of the ontological situation, not only would pure multiples be presented therein, but also represented; consequently there would be a rupture, or an order, between a first 'species' of multiples, those presented by the theory, and a second 'species', the submultiples of the first species, whose axiomatic count would be ensured by the state of the ontological situation alone, its theoretical metastructure. . . . [T]there would be two axiom systems, one for elements and one for parts. . . . This would certainly be inadequate since the very stake of the theory is the axiomatic presentation of the multiple of multiples as the unique general form of presentation. (*BE*, 100)

To accept a state of the situation of ontology (a situation that, by definition, contains every other situation and its corresponding state) would mean supplementing our ontological commitment with an additional category of Being. But we are already committed to the flatness and uniformity of presentation. The commitment to the existence of classes would give us a second-order and not first-order theory of sets because quantification would range over not just the individual elements but also multiples of them. One would have to dedicate oneself to a different axiomatic foundation from first-order *ZFC*, like the various alternative second-order theories, which include the von Neumann-Bernays-Gödel axioms (*NBG*) and the Morse-Kelley axioms (*MK*). Badiou has not chosen that route.

5

Ontology and Metaontology of the Cardinal and Ordinal Numbers

The current chapter and the next will provide the mathematical exposition and metaontological commentary of the second to fourth bulwarks referenced in *Being and Event*: the ordinal numbers, the cardinal numbers, and Gödel's the constructible. The relevant meditations are contained in Parts III and VI of the book. Much is given there, but our aim here is to be as succinct as possible without losing too much in terms of being sufficiently comprehensive in regard to the main focus of our book, which is understanding Badiou's metaontology of the event by concentrating particularly on how it is conditioned by the mathematics of set theory and forcing. The mathematics of the second to fourth bulwarks are essential for supplying the basic background knowledge to Badiou's philosophy, but they are not as central to understanding the event as the mathematics of *ZFC* and forcing.[1] So we do not intend to spend a disproportionately large part of our exposition on explaining the mathematics. At the same time, a lot of material must be presented, and the relevant technical details are notoriously abstract.

Basic extensions to the notion of number

In addition to being an extension of the usual natural numbers $\mathbb{N}$, the ordinal and cardinal numbers provide different conceptions of the idea of number and numericity. The cardinals understand numbers as indications of 'size' or magnitude, while the ordinals link number with 'order' or positional progression. What is more, each of these two conceptions provides different ways to mathematize the idea of infinity by extending the idea of number to include infinite numbers.

We review some of the other well-known extensions of the natural numbers that we might already know:

1. Most of us were started off by being introduced to the whole numbers 1,2,3,4 and so on. At some point, we were then informed of the strange whole number 0. Along the way, we were taught about the operations of addition, subtraction, multiplication and division. From the operation of division came the fractions, which are the result of dividing one whole number by another. Fractions take the form n/m, with the extra requirement that m is not 0. Every whole number is numerically equivalent to some fraction. For example, the number 4 is equivalent to 4/1 or 8/2 or 44/11. Then we learnt about negative numbers. This naturally led to the negative fractions such as −1/2 or −22/7. Numbers and fractions that are not negative were called positive. Only 0 is neither positive nor negative.
2. Then we were told that all these numbers can be imagined to be located on a continuous line, the continuum that continues left and right infinitely. The number 0 is located somewhere at the centre of the line, with 1 being usually one unit to its right and −1 located one unit to its left. After 1 comes 2, and before −1 is −2. And so on. The number 1/2 would lie right on the half-point between 0 and 1. And so on. All the whole numbers, fractions, negative numbers and negative fractions are just points on this number line.
3. But is every point on the number line either a whole number, fraction, negative number or negative fraction? No. Consider one counter-example: the number denoted by $\sqrt{2}$, the square root of 2. This is one of the possible numbers that produces 2 when multiplied by itself. So $\sqrt{2}\times\sqrt{2}=2$. It was proved as far back as the time of the ancient Greeks that $\sqrt{2}$ is not any of the kinds of numbers we have mentioned. This means that it cannot be expressed as a fraction of any kind; $\sqrt{2}$ can never be the result of one whole number divided by another. So the number line contains at least one number that is not a fraction.
4. However, all numbers on the continuum can be given in the decimal notation. The fraction 1/2 is the same as 0.5, while 3/4 is 0.75. Some fractions have an infinitely long trail of decimal digits. For example:

$$1/3 = 0.3333333\ldots$$
$$250/99 = 2.52525252\ldots$$
$$22/7 = 3.142857142857142857\ldots$$

In fact, the decimal expression of the square root of 2 also requires an infinite trail of digits that appear to be randomly distributed:

$$\sqrt{2} = 1.41421356\ldots$$

To provide the standard terminologies, here are the names of some of the numbers we have described so far.

1. The integers $\mathbb{Z}$ are all the natural numbers and negative natural numbers, plus 0. So $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
2. The rational numbers $\mathbb{Q}$ are all the numbers that can be represented as fractions, positive or negative. So $\mathbb{Q} = \{n/m: n \in \mathbb{Z}, m \in \mathbb{Z}, m \neq 0\}$. The integers are a subset of the rational numbers.
3. The real numbers $\mathbb{R}$ are those on the continuous number line, the continuum. Real numbers that are not rational, such as $\sqrt{2}$ are called irrational. The rational numbers are a subset of the real numbers.

Cardinal numbers, set sizes and one-to-one correspondences

Nevertheless, all of these numbers return in the end to $\mathbb{N}$ as their starting point. Our first acquaintance with the natural numbers might have begun when the need arose to understand and compare the size of sets. The natural numbers denote the sizes of sets or, to use the technical terminology, the cardinalities of sets. In *Being and Event*, and in French mathematical parlance, the cardinality of a set is also called its power [*la puissance*], although we shall avoid using this term as it could be confused with the notion of the power set. We will, however, use the notation $|S|$ to denote the cardinality of the set S.

Sets of the same size correspond to the same cardinal numbers. The number corresponding to the cardinality of a set is information about how 'big' it is. The relation 'smaller than' is used to compare two numbers when the corresponding set of one has less members than another. The notion of set cardinality can be understood more generally in terms of one-to-one correspondences. Two sets have the same cardinalities when their elements can be mutually paired. For example, if there is a way to pair each male in a room with each female, then the set of males has the same size as the set of females. The notion of one-to-one correspondence corresponds mathematically to what is called a bijection: a function that pairs every element from its domain to a unique element from its codomain. Two sets have equal cardinalities if they are the domain and codomain of some bijective function. We thus have this criterion for cardinal identity:

> $(|X| = |Y|) \leftrightarrow$ Every element of X can be paired with an element of Y, and vice-versa.

The criterion also works for the case where *X* or *Y* is each infinite. This basic fact about set sizes allows us to extend the concept of cardinality to include the size of infinite sets. It permits thinking an infinite cardinal number, extending the idea of number beyond finite numbers. To demonstrate this, consider the set *E* of all even numbers and the set *O* of all odd numbers:

$$E = \{0,2,4,6,\dots\}$$
$$O = \{1,3,5,7,\dots\}.$$

Both sets are obviously infinite. But they have the same cardinality because every even number can be uniquely paired with the unique odd number after it: 0 is paired with 1, 2 with 3, 4 with 5, and so forth. Since a one-to-one correspondence between the two sets exists, they have the same corresponding cardinal number, the first infinite cardinal number called aleph null, which is written using the first Hebrew letter: $\aleph_0$.

It is easy to see that an infinite set always has cardinality $\aleph_0$ if its elements can be arranged individually and sequentially as a list. Thus, a set is of size aleph null if it is infinite and if it is countable, if we can count all the elements systematically one by one. The set itself must be discrete in the sense that each of its elements is distinct and separate from all the others. This is why the set of all natural numbers $\{0,1,2,3,4,\dots\}$ also has cardinality $\aleph_0$, which might seem counter-intuitive at first because the sets of odd numbers and even numbers are both subsets of the natural numbers. A part of an infinite set might have the same cardinality as itself. In fact, the set $\mathbb{Z}$ of all integers also has size $\aleph_0$. It can also be shown – although the proof is omitted here – that the rational numbers $\mathbb{Q}$ are also countable. The cardinal number of the set of all fractions is also $\aleph_0$ as there is a sophisticated way to list all the rational numbers as a sequence.

Cantor's Theorem and the uncountability of the continuum

Is the continuum, the set of all real numbers $\mathbb{R}$, also countable? Is there a way to sequentially list all the points on the continuum of numbers, all numbers represented decimally? Unlike $\mathbb{N}$ and $\mathbb{Z}$, the continuum is not discrete but continuous. In fact, it is uncountable as there is no way to create a one-to-one correspondence between $\mathbb{R}$ and $\mathbb{N}$. To prove this, we will have to consider the cardinality of another kind of set, the power set. We prove the following theorem by Georg Cantor:

Cantor's Theorem: Given a set *X*, its cardinality is always strictly smaller than that of its power set *P*(*X*). There is no way to create a one-to-one correspondence between the elements of a set and its subsets.

Proof: Suppose that a correspondence exists. So every element of *X* is matched with a unique subset of *X* and vice versa. We divide all the elements of *X* into two types:

Type I: elements matched with subsets that contain them
Type II: elements matched with subsets that do not contain them.

We collect the elements of Type II as the set *B*, which is obviously a subset of *X*. Being a subset, this means that some element must be matched to it. Is that element of Type I or II?

1. If it is of Type I, it means it is an element of *B*, the set to which it is matched. But *B* contains, by definition, all the elements of X that are members of their corresponding set.
2. If it is of Type II, it means it is not an element of *B*. But this means that it should be an element of *B*, because *B* contains all the elements that are not members of the subsets to which they are matched.

So a contradiction arises from our assumption that a one-to-one correspondence exists, proving Cantor's Theorem.

Cantor's Theorem is closely linked to the Theorem of the Point of Excess that we proved in Chapter 2: the power set contains an element that is not presented in the initial set. Even though their proofs are similar, their results are not. The Theorem of the Point of Excess implies that the power set is not identical to the set, while Cantor's Theorem implies that the cardinality of the power set is larger than the cardinality of the set itself.

Now consider the power set of $\mathbb{N}$ itself, which corresponds to the set of all possible sets of natural numbers. Cantor's Theorem implies that the size of this power set exceeds the size of $\mathbb{N}$. The cardinality of $P(\mathbb{N})$ is larger than $\aleph_0$, so it is the first example we have of an uncountable set. So $\aleph_0 < 2^{\aleph_0}$. And it can be shown – although we omit the proof here – that the cardinality of the continuum is, in fact, equal to the cardinality of $P(\mathbb{N})$. There is a way to associate every set of natural numbers (finite or infinite) with every real number. This means that $\mathbb{R}$ is also uncountable. So, $|\mathbb{N}| < |\mathbb{R}|$.

(In fact, Cantor also gives a related but specific proof of the uncountability of the continuum using a well-known method called 'diagonalization'. We postpone providing the details until Chapter 8 as they play an important role in Badiou's metaontology of forcing and the 'trajectory' of subjectivation.)

The Continuum Hypothesis

So we have a cardinal number, $2^{\aleph_0}$, that we know is larger than $\aleph_0$. But just how much larger? Is there some other infinite cardinal number lying between $\aleph_0$ and $2^{\aleph_0}$? Can we find an infinite set whose cardinality exceeds $\aleph_0$ but not $2^{\aleph_0}$? Or is $2^{\aleph_0}$ the 'smallest infinity' after $\aleph_0$?

To put it in more technical terms, we define the other aleph numbers. We have said that $\aleph_0$ is the first aleph number, the first infinite cardinal number. We define the next aleph number, aleph one, or $\aleph_1$, as the smallest cardinal number larger than $\aleph_0$. For example, 5,6,7,8,9 and so on, are all cardinal numbers larger than 4. But 5 is the smallest of them, the smallest cardinal exceeding 4. Likewise, there are many other cardinal numbers larger than $\aleph_0$, but $\aleph_1$ is defined as the smallest of them. It is the smallest uncountable cardinal number, the smallest possible set size larger than $\aleph_0$. We can define the other aleph numbers similarly: $\aleph_2$ is the smallest cardinal number larger than $\aleph_1$; $\aleph_3$ is the smallest cardinal number larger than $\aleph_2$; and so on. So we have an infinite hierarchy of aleph numbers: $\aleph_0,\aleph_1,\aleph_2,\aleph_3,\aleph_4,\aleph_5$, etc. By definition, no other cardinal numbers can exist in between them. $\aleph_1$ is the set size that comes immediately after $\aleph_0$, and so on. This presumes, of course, that all the possible cardinal numbers can be arranged as a single list, sequentially arranged according to size.

Both the fact that the cardinality of an infinite set is exceeded by the cardinality of its power set and the fact that there exists a strictly increasing sequence of infinite aleph numbers provide a way to think the infinite scientifically and not theologically. The infinite, which is now no longer an absolute alterity, a vertical orientation towards divinity, has now been laicized with these infinite proliferations of different infinite quantities.

If $2^{\aleph_0}$ is much larger than $\aleph_0$, then it could be some other aleph number. But which one? If there is no cardinal number between $\aleph_0$ and $2^{\aleph_0}$, it means that $2^{\aleph_0}$

is precisely $\aleph_1$. The Continuum Hypothesis (*CH*) is precisely the hypothesis that $2^{\aleph_0}$ is $\aleph_1$:

$$\text{The Continuum Hypothesis } (CH)\text{: } \aleph_1 = 2^{\aleph_0}.$$

If this hypothesis was false, if ~*CH*, then $2^{\aleph_0}$ is bigger than $\aleph_1$. Since the discrete set $\mathbb{N}$ and the continuum $\mathbb{R}$ are primary examples of sets with cardinalities $\aleph_0$ and $2^{\aleph_0}$, the Continuum Hypothesis can be understood as the statement that there exists no 'middle ground' between the discrete and the continuous, the digital and the analogue, the striated and the smooth. If *CH* was true, then we cannot find an infinite set lying between a discrete and continuous space. The dialectical opposition between discrete and continuous cannot be synthesized, at least when it comes to set sizes.

The Continuum Hypothesis is the first specific case of the Generalized Continuum Hypothesis (*GCH*), which states that there is no cardinal number between any infinite set and the power set of that infinite set. In other words, *GCH* says that the next aleph number is the cardinality of the power set of the current aleph number. The sequence of denumerably infinite aleph numbers, $\aleph_1, \aleph_2, \aleph_3, \ldots$, is precisely the sequence of the result of successively taking the size of the power sets of aleph null: $2^{\aleph_0}, 2^{2^{\aleph_0}}, 2^{2^{2^{\aleph_0}}}, \ldots$:

$$\text{The Generalized Continuum Hypothesis } (GCH)\text{: } \aleph_\alpha = 2^{\aleph_{\alpha+1}}$$

How do we determine, then, the truth or falsity of *CH*, and, more generally, *GCH*? By proving or disproving it. Perhaps, by proving or disproving it from the axioms of set theory, from *ZFC*, which are meant to be a quasi-complete meta-mathematical foundation for mathematics. The task is to derive either *CH* or its negation from *ZFC*.

Here is where Paul Cohen's work takes centre stage. Using a mathematical technique he created, called forcing, he showed in 1963 that *CH* cannot be derived from *ZFC*. Kurt Gödel, the author of the Completeness and Incompleteness Theorems, had shown earlier in 1939 that ~*CH* itself cannot be derived from *ZFC*. This means that *CH* constitutes an undecidable statement whose truth cannot be decided within the framework of *ZFC*. So there is no way to create a sequence of deductive derivations from *ZFC* to either *CH* or ~*CH*. It cannot be proved to be true or false within the context of *ZFC*.

Moreover, a generalization of Cohen's work by William Easton in 1970, named the 'Easton Theorem' or 'Cohen-Easton Theorem' in *Being and Event*, implies that $2^{\aleph_0}$ could be equal to any one of $\aleph_1, \aleph_2, \aleph_3, \aleph_4$ and so on, and nothing in *ZFC* would prevent this. The cardinal $2^{\aleph_0}$ could take any infinite value above $\aleph_0$. As Badiou writes: '[I]t is coherent with the axioms to posit that $|P(\omega_0)| = \omega_1$ (this is the continuum hypothesis), but also to posit $|P(\omega_0)| = \omega_{18}$ or that $|P(\omega_0)| = \omega_{S(\omega_0)}$, etc' (*BE*, 502).

The first few ordinal numbers

We continue to the ordinal numbers. The many different, but equivalent, descriptions of them pose a dilemma for the would-be expositor who wishes to be concise but sufficiently comprehensive and clear. Our objective here is to supply enough information about these different descriptions and then to show that they are all equivalent, that they ultimately describe the same class of mathematical entities. Whereas the cardinals conceive the numbers as indications of size, the ordinals take them as indications of position. They indicate some position located in some progressive order. Here the numbers 1,2,3 and so on, are understood in terms of 'first', 'second', 'third' and so on. The relations in question are not 'smaller' or 'larger', but 'before' and 'after'.

This division between the cardinal and ordinal conceptions is so subtle that it appears at first to be specious. The differences, in fact, do not really matter when the numbers in question are finite. It is only when we have infinite sets that the distinction becomes prominent and practical.

We instruct ourselves by describing a procedure for generating a particular infinite sequence of symbols, a sequence that consists in itself of an infinite sequence of infinite sequences. The first ordinals are just the usual finite numbers, with an extra element at the end limit:

$$0,1,2,3, \ldots, \omega$$

This sequence ends with another ordinal number, the first infinite ordinal number ω_0 (called omega zero), or simply ω (omega). Mathematicians, for all practical purposes, often just take ω as equivalent to the first aleph number $\aleph_0$ (even though they connote different things). So the cardinals and ordinals coincide for the finite numbers and this first infinite number.

But what is interesting about the ordinal conception of number is that we can have a number that *directly* succeeds this limit ordinal. This successor, $\omega + 1$, exists and is an ordinal number as well. So do the numbers that come after it: $\omega + 2$, $\omega + 3$, $\omega + 4$ and so on. The ordinal conception of number allows us to attach a meaning to the idea of something coming directly after infinity, of attaching a coherent meaning to the idea of post-infinity. So now we have the following enlarged sequence, which consists of one infinite sequence appended to another, with another element at the end:

$$0,1,2,3, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\ \omega \cdot 2.$$

At the end of that second infinite sequence is $\omega + \omega$, or simply $\omega \cdot 2$. The numbers that proceed after that are also defined, $\omega \cdot 2 + 1$, $\omega \cdot 2 + 1$, $\omega \cdot 2 + 2$, and so on.

$$\begin{gathered} 0,1,2,3, \ldots, \omega, \\ \omega + 1, \omega + 2, \omega + 3, \ldots, \omega \cdot 2, \\ \omega \cdot 2 + 1, \omega \cdot 2 + 1, \omega \cdot 2 + 2 \ldots \end{gathered}$$

The last number of that sequence is $\omega \cdot 3$, which leads to sequence $\omega \cdot 3$, $\omega \cdot 3 + 1$, $\omega \cdot 3 + 2$, and so on. We can generate the ensuing infinite sequences similarly, producing sequences of ordinals, each of the form $\omega \cdot n + m$, where n and m are natural numbers.

$$\begin{gathered} 0,1,2,3, \ldots \\ \ldots\omega, \omega + 1, \omega + 2, \omega + 3, \ldots \\ \ldots\omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \omega \cdot 2 + 3, \ldots \\ \ldots\omega \cdot 3, \omega \cdot 3 + 1, \omega \cdot 3 + 2, \omega \cdot 3 + 3, \ldots \\ \vdots \\ \ldots\omega \cdot n, \omega \cdot n + 1, \omega \cdot n + 2, \omega \cdot n + 3, \ldots \\ \vdots \\ \omega^2 \end{gathered}$$

Now the limit of *this* sequence, this infinite sequence of infinite sequences, is also defined, which we write as $\omega \cdot \omega = \omega^2$. The generating process continues in the same fashion as with ω, creating $\omega^2 + 2$, $\omega^2 + 3$, $\omega^2 + 4$, … $\omega^2 \cdot 2$, $\omega^2 \cdot 2 + 1$, … and so on, with ordinals of the form $\omega^2 \cdot n + m$. The limit of that infinite sequence is $\omega \cdot \omega \cdot \omega = \omega^3$, and so on – so forth, creating ordinals that generalize to the form $\omega^k \cdot n + m$. And the end of *that* sequence, that infinite sequence of infinite sequences of infinite sequences, is ω^ω. The resulting infinite sequence continues to ordinals of the form $\omega^\omega \cdot n + m$, then of the form $\omega^{\omega^\omega} \cdot n + m$, then $\omega^{\omega^{\omega^\omega}} \cdot n + m$.

The limit of that infinite exponential tower of ω is called epsilon zero, ε_0. And the sequence continues in the same way with this new number:

$$
\begin{gathered}
0,1,2,3,\ldots \\
\ldots\omega,\ \omega+1,\ \omega+2,\ \omega+3,\ \ldots \\
\ldots\omega\cdot 2,\ \omega\cdot 2+1,\ \omega\cdot 2+2,\ \omega\cdot 2+3,\ \ldots \\
\ldots\omega\cdot 3,\ \omega\cdot 3+1,\ \omega\cdot 3+2,\ \omega\cdot 3+3,\ \ldots \\
\vdots \\
\ldots\ \omega\cdot n,\ \omega\cdot n+1,\ \omega\cdot n+2,\ \omega\cdot n+3,\ \ldots \\
\vdots \\
\ldots\ \omega^2,\ \omega^2+1,\ \omega^2+2,\ \omega^2+3,\ \ldots \\
\vdots \\
\ldots\ \varepsilon_0,\ \varepsilon_0+1\ \ldots
\end{gathered}
$$

Ordinal numbers and well-orderings

We expand our understanding of this generating process by interpreting the ordinals in terms of what are called well-ordered sets. The ordinals classify the well-ordered sets according to their progressive structure, according to their type of progressive ordering. In other words, the well-ordered sets can be catalogued according to their order type, with each ordinal number indexing each type.

A well-ordered set is simply an ordered set satisfying a certain condition, the condition of well-foundedness. We use the standard general notation '$\leq$' for an order relation. So, for example, '$i \leq j$' would say that element i in the set comes before element j under this relation. Now the relation must arrange all the elements of the set in some linear succession. The ordered set must consist of a single, though possibly infinitely long, queue. Whereas two sets have the same cardinality if there is a bijective function from one to the other, two well-ordered sets have the same well-ordering if there is a specific kind of bijective function between them, called an isomorphism, a bijection that preserves their well-ordered structure.

What is the condition of well-foundedness that makes an ordered set into a well-ordered set? If we impose the same relation of order on any of the set's non-empty subsets, one of the subset's elements must be the first element, the element that comes first in the subset. For an ordered set to be a well-ordered set, each of its subset that is not the empty set must contain within itself a first element under the relation.

It is easy to see that any ordering of a finite set will always create a well-ordered set. An ordered finite set will always be automatically well-ordered. Moreover, the usual way of ordering the natural numbers ℕ, where smaller numbers precede larger numbers (0,1,2,3, …) also gives us a well-ordered set. The natural numbers ordered according to cardinality creates a well-ordered set. But the set of natural numbers can be well-ordered in many other ways – in fact, uncountably many ways. We can, for instance, list 7,0,8,2,5,3 first before the rest of the numbers: (7,0,8,2,5,3,1,4,6,9,10,11,12,13, …). We can also list the even numbers before the odd numbers, juxtaposing two infinite sequences next to each other and still get a well-ordered set: (0,2,4,6, …, 1,3,5,7, …). The following ordering of ℕ is not, however, a well-ordered set: (… 3,2,1,0). Here, all the natural numbers are given in reverse cardinal order, with 0 the last element, 1 the second last, and so on. Because this ordering does not provide a first element for ℕ itself, it does not constitute a well-ordered relation. The condition of well-foundedness does not allow for an ellipsis that points backwards instead of forwards.

What about the uncountable sets, such as the continuum of real numbers ℝ? So far, no explicit well-ordering has ever been explicitly exhibited. We can, of course, order the real numbers in the usual way according to their location on the continuum, but this unfortunately does not satisfy the condition of well-foundedness. For example, the ordered subset of ℝ that consists of all real numbers larger than 10: ($x \in \mathbb{R}$: $x > 10$) has no first element. The number 10, which is the lower limit for this set, is not itself a member of this set.

There is also the more general question whether every set can be well-ordered. Is the statement 'There exists a well-ordering for every set' true or false? In fact, it can be shown that this statement is exactly equivalent to the Axiom of Choice in *ZFC*. In other words, the Axiom of Choice – that every set has a choice function – implies and is implied by the statement that every set can be arranged into a well-ordered set. The Axiom of Choice is, thus, sometimes called the Well-Ordering Theorem or the Principle of Maximal Order. The existence of a well-ordering for any set constitutes an axiom of set theory. (We will not establish here the equivalence between the Axiom of Choice and the Well-Ordering Theorem, although Badiou supplies the proof in Meditation 26 (*BE*, 277). It can, in fact, be shown that the Well-Ordering Theorem is equivalent to the special case that the set of real numbers ℝ can be well-ordered.)

When well-ordered, a set assumes a certain structure, a certain order type, a certain 'shape'. Well-ordered sets of equal shape can be grouped together. If two ordered finite sets have the same number of elements, then they have the same order type. To be more precise, they have the same ordinal structure because there is a way to substitute every element from one set with every element from another set in such a way that the sequential ordering remains unchanged.

Let us move to the case of infinite sets. Any ordered set that is an infinitely long single sequence of elements, such as the following,

$$(0,1,2,3,\ldots)$$
$$(0,2,4,6,\ldots)$$
$$(1,3,5,7,\ldots)$$

belongs to the same single ordinal structure. Substitutions can be done between these sets that would preserve their structure, their 'shape'. The following infinite sets, however, are not of the same order type:

$$(0,1,2,3,\ldots,a)$$
$$(0,1,2,3,\ldots,a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z)$$
$$(0,2,4,6,\ldots,1,3,5,7,\ldots)$$

The first example is an infinite sequence of elements, followed by one element at the end. The second example is an infinite sequence of elements, followed by a sequence of 26 elements, the 26 letters of the alphabet. The third example is a sequence of two infinite sequences, the even numbers and the odd numbers. Even though these three ordered sets are infinite, they do not have the same ordinal structure. Substitutions cannot be done that would leave the orderings of each set intact.

Each order type corresponds to a unique ordinal number and vice versa. Well-ordered sets of the same structure are matched to the same ordinal. We re-examine the generating sequence of ordinals that we gave a few sections earlier:

$$0,1,2,3,\ldots$$
$$\ldots\ \omega,\ \omega+1,\ \omega+2,\ \omega+3,\ldots$$
$$\ldots\ \omega\cdot 2,\ \omega\cdot 2+1,\ \omega\cdot 2+2,\ \omega\cdot 2+3,\ldots$$
$$\ldots\ \omega\cdot 3,\ \omega\cdot 3+1,\ \omega\cdot 3+2,\ \omega\cdot 3+3,\ldots$$
$$\vdots$$

$$\ldots\ \omega \cdot n,\ \omega \cdot n+1,\ \omega \cdot n+2,\ \omega \cdot n+3,\ \ldots$$
$$\vdots$$
$$\ldots\ \omega^2,\ \omega^2+1,\ \omega^2+2,\ \omega^2+3,\ \ldots$$
$$\vdots$$
$$\ldots\ \varepsilon_0\ \ldots$$

The matching between ordinal and order type is obvious for the case of well-ordered sets that are finite. Sets of size *n* are matched to ordinal number *n*. The order type of sets with 52 elements, for example, is matched to ordinal number 52. As for the infinite sets, the matching is conducted as follows. The first observation we can make is that any order type corresponds to an initial segment in the sequence of ordinals given previously. Every well-ordered set has the same structure as some well-ordered set consisting of all the ordinals from 0 up to a certain point.

For example, the infinite sequence (0,1,2,3 …) is matched with all the ordinals before ω. The set (0,1,2,3, …, *a*) is matched with all the ordinals *up to* ω. The set (0,1,2,3, …, *a*,*b*,*c*, … *z*), which is an infinite sequence followed by 26 extra elements, is matched with (0,1,2,3, …, ω, $\omega + 1$, $\omega + 2$, …, $\omega + 25$). Sets consisting of two consecutive infinite sequences such as (0,2,4,6, …, 1,3,5,7, …) are matched with all the ordinals before $\omega \cdot 2$, i.e. (0,1,2,3, …, ω, $\omega + 1$, $\omega + 2$, $\omega + 3$, …).

An ordinal is the set of the ordinals preceding it

Now there is another property of the ordinals that we have postponed mentioning. Except for 0, every ordinal number is defined as equivalent to the set of ordinals preceding it. In other words, the ordinals are exactly equal to the initial segments we mentioned. The ordinal 6, for example, is the set of ordinals from 0 to 5:

$$6 = \{0,1,2,3,4,5\}.$$

So, in general, every natural number *n*, when conceived as an ordinal, is precisely the set of all natural numbers preceding *n*. So $n = \{0,1,2, \ldots, n\}$ for every $n \in \mathbb{N}$, and likewise for the infinite ordinals $\lambda = \{x\colon x < \lambda\}$ for every ordinal λ except for 0. Thus, every ordinal is identical to the sequence of ordinals preceding it. Since every order type can be matched to an initial segment of the ordinals and every initial segment is, in fact, an ordinal itself, we have therefore provided a match between the order types and the ordinals.

Now the principle behind set theory is that every mathematical object is a set. As given previously, an ordinal is simply the set of all ordinals preceding it:

$$\begin{aligned}
&1 = \{0\}\\
&2 = \{0,1\}\\
&3 = \{0,1,2\}\\
&4 = \{0,1,2,3\}\\
&\vdots\\
&\omega = \{0,1,2,3, \ldots\}\\
&\omega + 1 = \{0,1,2,3, \ldots \omega\}\\
&\vdots
\end{aligned}$$

Since every ordinal refers back to all the ordinals before it, every ordinal ultimately comes back to the first ordinal, namely 0. Hence, each ordinal is built from zero. The set-theoretic construction is completed with the decision to equate the ordinal 0 with the empty set ∅ itself. This makes sense since zero and the empty set are simply different proper names for the void itself.

$$\begin{gathered}
0 = \varnothing\\
1 = \{\varnothing\}\\
2 = \{\varnothing, \{\varnothing\}\}\\
3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\\
4 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}\\
\vdots\\
\omega = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots \}\\
\omega + 1 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\},\\
\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots \}\}\\
\vdots
\end{gathered}$$

Moreover, every ordinal is a pure set, since there is nothing inside them other than the empty set and the onion layers of brackets.

The link between ordinals and well-foundedness implies what is called the Principle of Unique ∈-Minimality for Ordinals. Badiou expounds this principle in the first appendix to *Being and Event*. 'If there exists an ordinal which possesses a given property, there exists a smallest ordinal which has that property: it possesses the property, but the smaller ordinals, those that belong to it, do not' (*BE*, 519). Since every ordinal is well-founded when ordered according to belonging, every set of ordinals has a first element, the smallest ordinal of that set. This means that any collection of ordinals defined under some predicate has a unique first element.

Ordinals and homogeneous transitivity

One important property of the ordinals is that they are transitive sets. Transitivity means that every element of the set is also a subset. A set *S* is transitive when belonging implies inclusion:

$$x \in S \rightarrow x \subseteq S.$$

The arrow of implication is irreversible here. The converse of the transitive set is impossible in set theory because a set cannot contain all of its subsets as elements. This, of course, is a corollary to the Theorem of the Point of Excess.

It is easy to see that transitivity means that the set contains the elements of all its elements. So a set *S* is transitive when:

$$(y \in x \wedge x \in S) \rightarrow y \in S.$$

That is, if y is an element of x, and x is an element of the transitive set S, then y is also an element of S. This condition is exactly equivalent to the condition of transitivity described earlier. If a set includes all of its elements as subsets, then it contains the elements of all of its elements. Not only that, the transitive set contains all the elements in all the elements of all its elements, and so on.

Since every ordinal is transitive, and since every ordinal contains only other ordinals, it is easy to see that every ordinal contains only transitive sets, which themselves contain only transitive sets, and so on. A transitive set that contains only transitive elements all the way down is called homogeneously transitive (sometimes 'hereditarily' transitive). The property of transitivity is homogeneously dispersed within the set.

In fact, this property of homogeneous transitivity sufficiently defines the ordinal numbers as pure sets. Any homogeneously transitive pure set is an ordinal and every ordinal is a homogeneously transitive pure set. This alternative definition of the ordinals will prove crucial to Badiou's metaontology, even more crucial than the conventional understanding that ordinals are well-ordered pure sets, which Badiou calls the 'classic' definition (*BE*, 487–8).

Cardinals as specific ordinals

We have defined the ordinal numbers as a specific kind of pure set. We have yet to do so for the cardinals. The cardinals are, in fact, a certain type of ordinal number. They constitute a subclass to the class of ordinals.

Now the ordinals, being sets, also have the property of cardinality, which is simply the number of its elements. The ordinal number 26, for example, has cardinality 26. The ordinal ω, being infinite and countable, has size $\aleph_0$.

At some point in the linear progression of ordinality, the infinite ordinals will start to become uncountable sets. There exists a point in the series where all the ordinals thereafter are of a size larger than $\aleph_0$. So there exists the first uncountable ordinal, which we denote with ω_1. Every ordinal after it must also be uncountable and every ordinal before it must either be finite or countably infinite. The size of ω_1 is by definition the smallest uncountable cardinal, namely $\aleph_1$.

In fact, if we assume the Axiom of Choice, then each cardinal is simply the first ordinal of the corresponding size. The cardinality of a set is simply the first ordinal that has a one-to-one correspondence with it. So, if we take *AC* to be true, then $\aleph_1$ is precisely ω_1. We can also define the next omega number ω_2, which is the first ordinal of cardinality larger than ω_1, as $\aleph_2$. And so on, and so forth.

What is interesting is that since every ordinal is simply the set of all ordinals preceding it, and since the set of all ordinals preceding ω_1 is, by definition, the set of all countable ordinals (i.e. ordinals that are either finite or countably infinite), it means ω_1 equals the set of all finite and countable ordinals. The first uncountable ordinal is precisely the set of all ordinals that are not uncountable. This implies that the set of all countable ordinals has cardinality $\aleph_1$. The number of countable ordinals equals the first uncountable cardinal $\aleph_1$. This relates us back to the Continuum Hypothesis, which states that the power set of a countably infinity set has the same cardinality $\aleph_1$. This is equivalent to the statement that there exists a one-to-one correspondence between the set of all countable ordinals and the subsets of ω.

We end our mathematical exposition by stating two advanced topics related to the ordinals and cardinals, which we summarize:

1. The answer to whether some set satisfies some predicate can be either a relative or absolute matter. A property is relative if the answer depends on the model in which the set and predicate are situated. It is absolute if the answer is indifferent to the specifics of the situation. Being an ordinal

number is an absolute property, and so a set remains an ordinal regardless of which situation it inhabits. Other examples of absolute properties include the following: being a finite ordinal; being ∅; and being the first infinite ordinal ω. An important example of a relative property is the property of being the cardinal number $P(\omega)$ because the size of the continuum can vary depending on situations.

2. Like the class V, the multiple of all the ordinal numbers and the multiple of all cardinals are not sets. The proof that the ordinals constitute a proper class is given by Badiou at the end of Meditation 12, with the main point being that the class of all ordinals must itself be an ordinal, which would mean that it belongs to itself, thereby contravening the prohibition against self-belonging. As for the proof that the cardinals are not a set, this can be replicated by realizing that the existence of such a set would make its cardinality greater than any cardinality – another contradiction.

Summary of the mathematics

A lot of material has been presented so far. We pause to provide a reconstructive summary:

1. The cardinal numbers classify all the sets – finite or infinite – according to size. Sets of the same cardinality have a one-to-one correspondence, a bijection, between their elements. The cardinal numbers for the finite sets correspond to the natural numbers. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ all have cardinality $\aleph_0$, and so do all infinite sets whose elements can be enumerated one by one, and whose elements are individually discrete as opposed to continuous. Such sets are called countably infinite or denumerable. Infinite sets that are not countable are called uncountable or nondenumerable.
2. Cantor's Theorem states that the cardinality of a set is strictly always smaller than the cardinality of its power set. So the size of the power set of any infinite countable set is a different cardinal number $2^{\aleph_0}$, which is larger than $\aleph_0$. The power set $P(\mathbb{N})$ is uncountable, and has a bijection with the set of real numbers $\mathbb{R}$. So the continuum is also an uncountable set.
3. The ordinals are the pure sets that are homogeneously transitive. A set is transitive if it includes all of its own elements as subsets, which is equivalent to the set containing all the elements of its elements all the way down. A transitive set is homogeneous if each of its elements is also homogeneously transitive.

4. With the exception of 0, which is the empty set $\emptyset$, each ordinal number is the set of ordinals preceding it. Ordinals are categorized into the empty set, the successor ordinals and the limit ordinals. The finite ordinals are the natural numbers. Ordinals also correspond to the order type of well-ordered sets, which are well-founded ordered sets where every subset contains a first element. The Axiom of Choice is equivalent to the statement that every set can be well-ordered.
5. The first infinite ordinal is $\omega = \omega_0$, which is equivalent to the cardinal $\aleph_0$. The first uncountable ordinal ω_1 is equal to the set of all countable ordinals. The Axiom of Choice implies that the first ordinals of a particular cardinality are each exactly equivalent to the cardinal number corresponding to that size. The cardinals therefore obey a linear ordering according to their size.
6. A predicate is absolute if it obtains or does not obtain throughout any model for which it is verifiable. The following predicates are absolute: being an ordinal number; being a finite ordinal; being $\emptyset$; and being the first infinite ordinal ω. So a set remains an ordinal regardless of the situation. The multiples containing all the ordinals and all the cardinals are proper classes. A unifying concept of ordinality and cardinality is not present in any model of *ZFC*, in any universe.
7. The Continuum Hypothesis (*CH*) proposes that $P(\omega)$ and ω_1 have the same cardinality, that $\aleph_1 = 2^{\aleph_0}$, that there is no set size between the denumerable and the continuum. The Generalized Continuum Hypothesis (*GCH*) proposes that $\aleph_\alpha = 2^{\aleph_{\alpha+1}}$ for every ordinal α. Kurt Gödel proved that *CH* (and *GCH*) is consistent with *ZFC*. Paul Cohen proved that ~*CH* (and ~*GCH*) is consistent with *ZFC*, thereby establishing its undecidability. A later result, called the Cohen-Easton Theorem, proved that it is consistent to fix the size of $P(\omega)$ to be equal to any aleph number after $\aleph_0$.

We now provide a general recitation concerning Badiou's thinking on the metaontology of the ordinal numbers, a thinking that proceeds from his amazing observation that the ordinals, conceived as homogeneously transitive pure sets, constitute the form-multiple of what he calls natural multiples. This, in turn, connects to a remarkable rethinking about the ontology of nature. As for the cardinal numbers, Badiou does not seem to go further than taking them as the obvious and non-controversial mathematical figure for thinking quantity. That said, the various mathematical concepts, theorems and results that we have just restated about cardinality – Cantor's Theorem, the aleph numbers, the

Continuum Hypothesis, and the Cohen-Easton Theorem – will play a crucial role in Badiou's mathematical and metaontological explications on the constructible and on forcing.

Ontology versus onticology of nature

As our first point of departure, we meditate on the question posed by the title of Badiou's Meditation 11 (*BE*, 123), which asks whether nature should be thought via the resources of the poem or the matheme – or, to be precise, whether the *Being* and the multiplicity of nature should be thought via the resources of the poem or the matheme. My insertion of the word 'Being' is crucial, for it limits our purview to the strictly ontological, to what can be said of nature not in terms of the concrete and contingent specificities of its current presentation (as atoms, cells, plants, animals, landscapes, climates, planets, galaxies, the cosmos, and so on) but in terms of its very Being and its being-multiple.

To avoid any misconceptions, we admit that there is nothing wrong with investigating nature via the resources of poetic language, as all the great writers of the earth such as Ovid, William Wordsworth, Robert Frost and even Martin Heidegger have demonstrated to us with great depth and insight. But such enquiries, at least for Badiou, are never directly ontological or even metaontological. That the scope of Badiou's question lies within ontology, and not the ontic, is relevant, as this would place it outside investigations associated with the natural sciences such as physics, chemistry, biology, geology or astronomy. Badiou writes:

> What is the fate and the scope of this concept [nature] within the framework of mathematical ontology? It should be understood that this is an ontological question and has nothing to do with physics, which establishes the laws for particular domains of presentation ('matter'). The question can also be formulated as follows: is there a pertinent concept of nature in the doctrine of the multiple? Is there any cause to speak of 'natural' multiplicities? (*BE*, 127)

We reiterate the distinction between the scope of ontology and the scope of physics or, more generally, the scope of the natural sciences, that area of the natural sciences that is sufficiently formal, systematic, deductive and empirical. The natural sciences deal with the laws governing a very specific sphere of presented multiples, the laws for the physical part of our real world, the physical aspect of our contingently existing state of affairs. What presents itself therein are physical multiples: atoms, ecosystems, planets and so on. Nevertheless, the science of Being-qua-Being – at least when it begins on the essential side of Being and not on the empirical side of beings – makes no commitment to the particularity of a concrete multiple, save for the void within the presentation of presentation. There are no elementary entities in ontology, save for the non-entity that is the empty set. The 'object' of ontology is not a particular reality, and certainly not our own contingently presented physical reality, but the deep presentative structure of reality itself, the anonymous 'there is' of reality in its Being. It is not necessary that our world be the way it is because our physical laws need not be true. But the veracity of mathematics and ontology holds in any universe.

This is what sets Badiou's approach apart from some empiricist, ontic or object-oriented approach towards understanding Being-qua-Being. Badiou's equating of ontology with mathematics, thus, delimits the scope of ontology so that it concerns only pure multiplicity at the general level and most direct purview, and not inferences from some particular concrete presentations. Perhaps something philosophically illuminating can be gained by trying to understand Being via examinations of, for example, string theory; the contingent Laws of Thermodynamics; the gravitational structure of black holes; the dynamics of evolutionary or biological processes; the science of data and information; or the statistics of human populations. Badiou takes the natural sciences as a possible condition for philosophy. However, when it comes to speaking about Being as such, about Being-qua-Being at the zeroth and most direct level, the focus should be the deep structure behind presentation as such. And Badiou axiomatically equates this discourse with mathematics.

The typology of relations between structure and metastructure

Badiou's mathematical ontology of nature and metaontology of the ordinals relates to a particular mode of relation between the local structure of a situation and its statist metastructure. We know, following from the Cohen-Easton Theorem, that this relation is variable, at least in the general sense for infinite situations and their respective cardinalities. An undecidable gap separates an infinite set and its power set, and therein 'is the key to the analysis of [B]eing, of the typology of multiples-in-situation' (*BE*, 99). The typology of Being is determined by the errant meta- and extra-ontological suture between the situation and its state, the structure and its metastructure.

We need to expand on this complicated idea. The Cohen-Easton Theorem implies that the values of various infinite cardinalities such as $2^{\aleph_0}$ can, depending on our choice and compatiblilty with the rules of ontology, take a variety of values within the sequence of aleph numbers. '[I]t is impossible to determine where on the scale of alephs the set of parts of an infinite set is situated' (*BE*, 278). In itself, the situation is unable to determine how much it is exceeded by its state. In the end it necessary to decide on the power of the state over the situation, and this wager introduced randomness at the heart of what can be said of Being-qua-Being. 'Action . . . endeavours in vain when it attempts to precisely calculate the state of the situation in which its resources are disposed. Action must make a wager in this matter, rather than a calculation' (*BE*, 278). The tolerance towards the complete arbitrariness of a decision leads to pure subjectivity. 'Ontology unveils in its impasse a point at which thought – unconscious that it is [b]eing itself which convokes it therein – has always had to divide itself' (*BE*, 280).

The plurality of the possible relations between the situation and the state is schematized by Badiou as follows. Given a non-empty consistent multiple, there are three possibilities for its relation with respect to a situation and that situation's state:

1. The multiple belongs to both the situation and the state. It is a presentation and a representation, an element and a subset. The multiple, as well as each of its elements, belongs to the situation. Such multiples are called 'normal' by Badiou.[2] 'Normality consists in the re-securing of the originary one by the state of the situation in which that one is presented' (*BE*, 99).
2. The multiple belongs to the situation, but not the state. It is a presentation, but not a representation. The multiple belongs to the situation, but it contains some elements that do not belong to the situation. Such multiples are called 'singular'.[3] '[T]his term exists – it is presented – but its existence is not directly verified by the state' (*BE*, 99).
3. The multiple belongs to the state, but not the situation. It is a representation, but not a presentation. The multiple does not belong to the situation, but each of its elements do. The Theorem of the Point of Excess implies that there will always be such multiples, which are called 'excrescent' by Badiou. '[A]n excrescence is a one of the state that is not a one of the native

structure, an existent of the state which in-exists in the situation of which the state is the state' (*BE*, 100).

In lieu of the normal-singular-excrescent triad, Badiou also provides a typology of situations:

1. Natural situations: All their elements are normal, and all the elements of those elements are also normal with respect to them, and so on. In other words, all the elements contained within a natural situation are representations as well. The property of normality extends all the way down to the void.
2. Historical situations: At least one of their elements is singular. That is, at least one non-empty element is not a representation, does not belong to the state.
3. Neutral situations: The situations are neither natural nor historical. Because they are non-historical, all their elements are normal, are representations. But not all of them are normal all the way down, since the situation is non-natural.

Nature and homogeneous transitivity

It is easy to see that that a set is transitive if all its elements are normal because the concept of normality refers, by definition, to those elements whose elements also belong to the set. So natural and neutral situations are transitive. And it is also easy to see that a set is homogeneously transitive if and only if it is a natural situation. Since the ordinals are, by definition, homogeneously transitive *pure* sets, they constitute the mathematical figures for natural situations, the form-multiple for the concept of nature.

We repeat the network of relations between the concepts of transitivity, homogeneous transitivity, ordinality, normality and nature:

1. To be transitive is to contain only normal multiples.
2. To be homogeneously transitive is to contain only normal and only homogeneously transitive multiples. Being homogeneously transitive is the same as being a natural multiple, being a totally homogeneous normality.
3. The ordinal numbers are, by definition, homogeneously transitive pure sets. They are, thus, the only natural multiples in ontology. The concept of ordinality, thus, coincides ontologically with the Being of nature.

The ordinals, which include all the natural numbers, 1, 2, 3 and so on, provide the ontological scheme for natural multiples, for natural situations. The ontological link between ordinal numbers and natural situations implies the interchangeability between the concepts of number and nature. This does not mean that nature is number and that number is nature, but that the ontological scheme for nature is thinkable precisely as ordinal.

The foundational element of historical situations

Following from the Axiom of Foundation, we know that every situation has a foundational edge-of-the-void element whose intersection with the situation is empty. If this foundational element is not actually the empty set and if it contains no elements from the situation, then we call that element totally singular. An element of a situation is singular if it is not a representation – that is, if it contains *some* members that are not members of the situation itself. An element is totally singular if *none* of its members are members of the situation. Since only a historical situation can contain singular elements, it follows that only a historical situation can have foundational elements that are not empty. However, this happens only if the singular elements are totally singular, otherwise the foundational element is empty. But this can be discerned only from the outside of the situation as there is nothing within the historical situation that can be used to examine the constituents of the singular element. Within the situation, the singular element appears as an abyss. For natural and neutral situations, the foundational element can only be the empty set. The intersection of the empty set with the situation is clearly void in this case. We shall see that totally singular elements provide the site for the possible eruption of events.

To summarize:

1. The Axiom of Foundation ensures that every situation contains at least one foundational edge-of-the-void element.
2. For natural and neutral situations, the foundational element is the empty set.
3. For historical situations, the foundational element can be non-empty. Such elements correspond to totally singular elements. If the singular elements of the historical situation are not totally singular, then the foundational element is the empty set.

The ontological stability and homogeneity of nature

Let us, however, return to Badiou's curious decision to suture the being-multiple of nature to homogeneously transitive pure multiplicity. This decision provides his most distinctive metaontological contribution to the philosophical thinking of the ordinals. We do not wish to expand too much on it here and will only recount the basic narrative given in *Being and Event*. Badiou develops his analysis of nature by borrowing and re-conceptualizing Martin Heidegger's analysis of the Greek φύσις as the stability of what has opened forth and maintains itself. This is the remaining-there of the stable, the equilibrium of that which remains standing of itself within the opening forth of its limit. The notion of normality is the most appropriate predicate for φύσις in its stability because of its balancing between presentation and representation. For '[w]hat could be more stable than what is, as multiple, counted twice in its place, by the situation and by its state? Normality, the maximum bond between belonging and inclusion, is well suited to thinking the natural stasis of a multiple. Nature is what is normal, the multiple re-secured by the state' (*BE*, 127–8).

A homogeneously transitive situation, by symmetrising its structure with its metastructure, and by symmetrising the structure and metastructure of all its presentations, provides the perfect ontological figure for the equilibrium and the remaining-there-in-itself of nature. Every element is a subset, every presentation is a representation; there are no gaps, otherwise, '[t]he stable remaining-there of a multiple could be internally contradicted by singularities, which are presented by the multiple in question but not re-presented. To thoroughly think through the stable consistency of natural multiples, no doubt one must prohibit these internal singularities, and posit that a normal multiple is composed, in turn, of normal multiples alone' (*BE*, 128). Nature, in the remaining-there-of-itself, is self-homogeneous self-presentation. Ordinals, particularly infinite ordinals, should no longer be thought as an abstract curiosity but as a mathematical figure with direct metaphysical consequences. Badiou complains:

> It is remarkable that despite Cantor's creative enthusiasm for ordinals, since his time they have not been considered by mathematicians as much more than a curiosity without major consequence. This is because modern ontology, unlike that of the Ancients, does not attempt to lay out the architecture of [B]eing-in-totality in all its detail. The few who devote themselves to this labyrinth are specialists whose presuppositions concerning onto-logy, the link between language and the sayable of [B]eing, are particularly restrictive. (*BE*, 133)

We can understand this further by repeating the alternative definition of homogeneous transitivity: a situation that presents everything contained within its elements. Nature is, likewise, internally complete. It contains not only humans, trees and dogs, but also everything within humans, trees and dogs: fingers, lungs, hairs, eyeballs, leaves, roots, flowers, branches, paws, tails, blood cells, hydrocarbon atoms, chlorophyll, and so on. Nature presents a homogeneous equilibrium of homogeneously presented being. Badiou clarifies this with his metaphor of a biological cell:

> [B]elonging 'transmits itself' from an ordinal to any ordinal which presents it in the one-multiple that it is: the element of the element is also an element. If one 'descends' within natural presentation, one remains within such presentation. Metaphorically, a cell of a complex organism and the constituents of that cell are constituents of that organism just as naturally as its visible functional parts are. every natural multiple is connected to every other natural multiple by presentation. There are no holes in nature. (*BE*, 136)

He continues:

> The idea that we have now come to is much stronger. It designates the universal intrication, or co-presentation, of ordinals. Because every ordinal is 'bound' to every other ordinal by belonging, it is necessary to think that multiple-Being presents nothing separable within natural situations. Everything that is presented, by way of the multiple, in such a situation, is either contained within the presentation of other multiples, or contains them within its own presentation. This major ontological principle can be stated as follows: Nature does not know any independence. . . . Nature is thus universally connected; it is an assemblage of multiples intricated within each other, without a separating void . . . (*BE*, 136)

Moreover, the fact that the multiple of all ordinals forms a proper class implies that nature, as a totality (the nature that Galileo declared to be written in the language of mathematics), does not exist in Badiou's ontological framework. If the ordinals provide the being-multiple of natural multiples, and if the set of all ordinals cannot exist, then ontology does not provide a being of nature in totality. Lacking any ontological structure, nature therefore does not exist as a structure. 'Nature has no sayable [B]eing,' writes Badiou. 'There are only some natural beings' (*BE*, 140). The natural is only a predicate, only an intensional multiple, and not an actualized entity, not an extensional multiple.

The non-existence of nature links with the non-existence of various totalities that we have explained:

1. Within a situation, that situation itself does not exist because a set cannot contain itself.
2. The multiple of all pure sets cannot exist because *V* forms a proper class. As a result, ontology can never provide for a Being in totality.
3. The existence of a quasi-complete situation of ontology, a situation of pure sets where *ZFC* is veridical therein, is undecidable with respect to *ZFC*. As a result, the rules of ontology cannot say whether ontology itself can or cannot form a situation.
4. As a totality, nature cannot exist because the multiple of ordinals forms a proper class. Ontology says that there is no being-multiple to nature.

Badiou's ontological rethinking of nature removes the poetic and theological aura from the stability of *aletheia*. While taking into account its internal equilibrium, the ontology of nature can now be thought formally and scientifically. With the invention and scientific investigation of the ordinal numbers, we now have the means to bracket out the 'captivating grandeur' from the ontological equilibrium of nature and treat it using the laicized tools of mathematical understanding.

6

Ontology and Metaontology of the Constructible

The Löwenheim-Skolem Theorem and Skolem's Paradox

Before continuing to Gödel's Work on constructibility, we first present the Löwenheim-Skolem Theorem. We begin by explaining the cardinality of a formal axiomatic system and the cardinality of a set model. The former is simply the number of axioms in the *FAS*, while the latter is the number of entities contained in the domain corresponding to the set model. For example, there are five Peano Axioms in the standard formulation and the standard Peano model $\mathbb{N}$ has countably many entities. What about the *ZFC* axioms? The model *V* of all pure sets is a class, not a set, so the idea of cardinality makes no sense here. How many axioms are there in *ZFC*? It depends on how the system is formulated. The one we provided has ten, although in reality, there are countably many axioms. The Axioms of Replacement and Separation are each really schemas of axioms, each being a countably infinite sequence of statements. So the number of axioms in *ZFC* is countably infinite. The cardinality is thus $\aleph_0$.

The Löwenheim-Skolem Theorem has many formulations but we will deal with only one of them:

Löwenheim-Skolem Theorem: Every consistent formal axiomatic system built on first-order logic must have a countable model. Moreover, if it has an infinite model, then that infinite model can be reduced to a countable submodel.

The relevant implication of the Löwenheim-Skolem Theorem is that there exists a situation, consisting of a countable number of entities, where the *ZFC* axioms are veridical. Moreover, the theorem says that if we can exhibit an uncountable model of *ZFC*, then that model can be reduced to a countable submodel of *ZFC*.

This might lead to some confusion. The Axiom of Infinity states that the first countably infinite ordinal $\omega = (0,1,2,\ldots)$ must exist. So every model of *ZFC*, including the countable ones, must have ω in its universe. But the Axiom of the Power Set states that every power set of every set must exist. So the power set $P(\omega)$ must exist in every model of *ZF* as well, including every countable model. But Cantor's Theorem proves that $P(\omega)$ must be uncountable. So countable models of *ZFC* have uncountable sets within them. Is there something wrong here? Let us repeat ourselves:

1. The Löwenheim-Skolem Theorem implies that *ZF* has a countable model. Let us call it model *C*.
2. The Axiom of Infinity in *ZF* implies says that *C* contains the set $\omega = (0,1,2, \ldots)$, while the Axiom of the Power Set in *ZF* implies that *C* also contains the power set $P(\omega)$.
3. Cantor's Theorem implies that $P(\omega)$ is uncountable.
4. But how can *C*, which is countable, contain an uncountable element?

This confusion, known as Skolem's Paradox, is resolved as follows. Now, being uncountable is not an absolute but a relative property. The power set $P(\omega)$ is uncountable only relative to the horizon of veracity that is immanent within the model. It is uncountable only within the countable universe of *C*, only from the viewpoint of model *C*. If we examine $P(\omega)$ from outside of the model *C*, then this power set might not remain uncountable. Likewise, the model *C* is countable only from a viewpoint outside of itself. Being uncountable means not having a one-to-one correspondence with the countable set ω. Within model *C*, the power set $P(\omega)$ fulfils this criteria. In *C*, the size of $P(\omega)$ is in excess of ω. Outside of model *C*, the power set ceases to remain uncountable. Since it is inside the countable model *C*, the only number of subsets of ω it can have is really countable. So Skolem's Paradox is not really a paradox as the *ZF* axioms have a countable model. That countable model has its own version of the power set $P(\omega)$. This power set is the set of all subsets of ω, but subsets that are already contained in the countable model, that already exist according to the model.

Remember that the statement of the Axiom of Power Set does not say that the set of all subsets must exist, but that the set of all *existing* subsets must exist, with existence being limited to what is provided by ontology. We cannot collect all the possible subsets into a set, just all the subsets that already pre-exist, that are already located in the countable model. From the viewpoint inside that model, this power set is uncountable. From the viewpoint of that model, there is no one-to-one correspondence between its elements and ω. But from the viewpoint outside of that countable model, this power set is really countable. Since the number of sets that it contains cannot be more than countable, the power set does not remain uncountable outside of the countable set. The notion of absolute is important because, as it turns out, many properties hold relative only to the specifics of the situation while others are true regardless of the circumstances.

Introduction to Gödel's result on constructibility

Gödel showed that the Axiom of Choice and the Continuum Hypothesis are each consistent with *ZFC*, provided that those axioms are consistent in the first place. This result is the same as saying that if we take the *ZF* and *ZFC* axioms and append *AC* or *CH* to them, respectively, then the new set of axioms is consistent if the original axioms were already consistent in the first place. Yet another equivalent formulation: the *ZF* and *ZFC* axioms, if consistent, can be made to prove *AC* and *CH*. The truth of *AC* and *CH* would not contradict *ZF* and *ZFC*.

In reality, Gödel's result is stronger as it concerns *GCH* and not just *CH*. But only the specific case will concern us here. Note that when we say that *ZFC* can be made to prove *AC* and *CH*, it does not mean that *AC* and *CH* are wholly provable with respect to *ZFC*. We are not saying that *ZFC*, on its own, is enough to establish *AC* and *CH*, that *ZFC* logically implies *AC* and *CH*. We are saying that *ZF* can be used to prove *AC* and *CH*, that there is a way to interpret *ZF* so that *AC* and *CH* are true. A model of *ZFC* exists such that *AC* and *CH* are both true in the model. In this model, the negation of both *AC* and *CH* is unprovable.

Because of the length and technical complexity of Gödel's proof, we will not be able to provide all its details. Badiou's demonstration in *Being and Event* is considerably more detailed and we wish to provide only a general grasp of how the proof works, just enough to continue to the main focus of this book, which is Badiou's metaontology of the event. Here is a sketch of Gödel's proof:

1. First, Gödel provided a constructible hierarchy of a particular type of set. This hierarchy gives a class model to *ZFC*, a class called the constructible universe, which is conventionally denoted as *L*. The hierarchy also constitutes, by definition, a smaller submodel within the larger class model *V*. All the sets belonging to *L* are said to be constructible. If we accept

the Axiom of Constructibility, if we axiomatically assume that all sets are constructible, then the class of all sets equals the class of constructible sets: $V = L$.

2. When interpreted with the class L, the *ZF* axioms imply *AC*. In other words, the Axiom of Constructibility reduces *AC* to a theorem and not an axiom. Via its proving of *AC*, the Axiom of Constructibility also proves *CH*. Thus both *AC* and *CH* can be made to follow from the *ZF* axioms. So *AC* and *CH* are consistent with *ZF* plus the Axiom of Constructibility.

The proof makes direct use of the syntactic conception of consistency, not the semantic conception. Instead of directly providing a set model where the *ZF* axioms plus *AC* plus *CH* are true, Gödel demonstrated that *AC* and *CH* follow from *ZF*.

Transfinite induction and defining the sequence of ordinals

To describe the constructible hierarchy, we first describe how it is formulated, which is via transfinite induction. From that, we give the simpler cumulative hierarchy of sets. Transfinite induction also allows us to define the sequence of ordinal numbers by the fact that every ordinal is either the empty set, a successor ordinal, or a limit ordinal.

First, the general notion of induction must be introduced. Instead of defining certain classes of elements directly, mathematics allows us to do so by induction where more complex elements are defined in terms of simpler ones. If the elements of the class constitute a sequence, then later elements are defined in terms of earlier ones. For example, consider the following sequence of the *factorials*. The zeroth term of the sequence, $0!$, is set to 1, while the nth element, $n!$ is the result of multiplying all the first n natural numbers: $n! = 1 \times 2 \times 3 \times \ldots \times n$. So, for example, $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$. The sequence of factorials can be defined this way:

$$0! = 1$$
$$n! = 1 \times 2 \times 3 \times \ldots \times n \text{ when } n > 0$$

Instead of completing the long sequence of multiplications, we can replace the last line by defining the factorial in terms of the value before it. Every $(n + 1)$th element of the sequence is simply the product of the preceding element and $n + 1$. So the sequence of factorials can be defined alternatively as follows:

$$0! = 1$$
$$(n + 1)! = n! \times (n + 1) \text{ when } n > 0$$

Each term is thus defined by the smaller term before it. This countable sequence of factorials is defined by induction. The order type of this sequence corresponds to the infinite ordinal $\omega = (0,1,2,3,4, \ldots)$. The first line in the definition concerns the zeroth term, a term indexed by the first ordinal 0. The second line concerns the values indexed by the other subsequent finite ordinals.

Defining the ordinals via transfinite induction

Now, in reality, induction can also work for sequences of a larger order type. Instead of being indexed only by the sequence of natural numbers, we can use larger ordinals. Ordinals come in three types:

1. The first ordinal 0.
2. Successor ordinals: ordinals that are successors of other existing ordinals. For example, 10 is a successor ordinal because it succeeds the ordinal 9. $\omega + 1$ is another successor ordinal because it succeeds the ordinal ω.
3. Limit ordinals: ordinals that are the limit to the union of ordinals before them. These ordinals cannot be understood as successors at all. For example, there is no ordinal before ω, $\omega \cdot 2$ or ω^ω. Each is simply the sequence of ordinals preceding it.

Infinite sequences defined by transfinite induction contain an extra third line for elements indexed by limit ordinals. For example, the ordinals, when understood as pure sets, can be defined by transfinite induction as follows:

$$0 = \varnothing$$
$$\alpha + 1 = \alpha \cup \{\alpha\} \text{ if } \alpha + 1 \text{ is a successor ordinal}$$
$$\lambda = \cup_{\beta<\lambda} \beta \text{ if } \lambda \text{ is a limit ordinal, i.e. } \lambda \text{ is the union of all the ordinals before it}$$

Badiou describes this very construction of ordinals in Meditation 14. We can thus see that every transfinite sequence can be defined by transfinite induction by being indexed to the class or ordinals. Every transfinite sequence can be defined in three lines for elements indexed by the first ordinal, by successor ordinals and by limit ordinals. In fact, the class of pure sets can be broken up into a transfinite sequence of mutually inclusive parts or hierarchies.

The cumulative hierarchy of pure sets V

Now the class of all sets V can be partitioned into a transfinite hierarchical sequence of subclasses that together creates a hierarchy within V. In this hierarchy, a set's rank is defined based on its 'distance' from the empty set ∅. This distance is the minimum number of steps it takes to get from the void of the empty set to that set via the operation of creating power sets.

To be more precise, the empty set ∅ constitutes the zeroth hierarchy V_0. The next hierarchy, the first hierarchy V_1, is the power set of V_0. So $V_1 = \{\varnothing\}$. The empty set is the only element in this hierarchy. This means that the distance between the void and the empty set is 1.

The second hierarchy V_2 comprises all the sets in the power set of the previous hierarchy V_1. So $V_2 = \{\varnothing,\{\varnothing\}\}$. In addition to the empty set from V_1, this second hierarchy has the new set $\{\varnothing\}$. So the distance between $\{\varnothing\}$ and the void is 2.

The third hierarchy V_3 is the power set of V_2. So $V_3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$. The new elements here are $\{\{\varnothing\}\}$ and $\{\varnothing,\{\varnothing\}\}$.

And so on and so forth. Every hierarchy includes all the subsets of the previous hierarchy, plus some additional elements of its own, the subsets of the previous hierarchy that do not belong to that previous hierarchy. This operation of taking the power set of the preceding hierarchy does not end with V_ω because we can continue this process for any ordinal index. If the index is non-zero ordinal *i*, then V_i is simply the power set of the preceding hierarchy: $P(V_{i-1})$. But if the ordinal $i-1$ does not exist because *i* is not a successor of any ordinal, then *i* must be some limit ordinal. In that case V_i is simply the union of all the hierarchies before it. This *cumulative hierarchy* of sets is defined more formally via transfinite induction as follows:

$$V_0 = \varnothing$$

$$V_{\alpha+1} = P(V_\alpha) \text{ if } \alpha + 1 \text{ is a successor ordinal}$$

$$V_\lambda = \cup_{\beta<\lambda} V_\beta \text{ if } \lambda \text{ is a limit ordinal, i.e. } V_\lambda \text{ is the union of all the } V_\beta\text{'s before it.}$$

And the class V of all sets is simply the union of all these hierarchies:

$$V = \cup_{\alpha \text{ is an ordinal}} V_\alpha$$

This hierarchy provides some structure to the class V. We can visually imagine this class as an inverted triangle, with the empty set at the base point and all the other sets emerging from it. The higher the index, the larger the width of the triangle.

The constructible hierarchy of pure sets L

In the cumulative hierarchy, each new level admits every subset belonging to the levels below it. In Gödel's constructible hierarchy, this entry process is more restrictive. Subsets are admitted only if they are 'constructible' from the sets contained in the lower level. To be more precise, a subset is admitted if it is:

1. definable in the first-order language of set theory
2. using a predicate whose parameters come from the lower level sets and
3. quantified over the elements of lower level sets. So the new sets are those that are definable in terms of those already constructed.

Let the set *Def*(X) denote all those sets that are definable from the elements of set X. *Def*(X) contains all subsets separated from X, which are defined by predicates formulated using parameters that are elements from X itself. *Def*(X) includes the set X within itself, since every existing set can define itself. So *Def*(X) is X plus some newly defined sets, which we denote by X' (Be*ing and Event* uses the notation $D(X)$ instead). So:

$$Def(X) = X \cup X'.$$

The constructible hierarchy can be given as follows:

$$L_0 = \varnothing$$
$$L_{\alpha+1} = Def(L_\alpha) \text{ if } \alpha + 1 \text{ is a successor ordinal}$$
$$L_\lambda = \cup_{\beta<\lambda} L_\beta \text{ if } \lambda \text{ is a limit ordinal}$$
$$L = \cup_{\alpha \text{ is an ordinal}} L_\alpha.$$

The constructible hierarchy L is, by definition, contained in the cumulative hierarchy V. The sets contained in L are called constructible and L itself is often just called the constructible universe. Each constructible set is definable in terms of other constructible sets of lower rank. Since all of the levels return in the end to the first level $L_0 = \varnothing$, this means that every constructible set is a sequence of constructions out of the empty set, a sequence of sentences onto the set $\varnothing$.

Since the levels are indexed by every ordinal, this sequence of sentences can be transfinite. The index ordinals denote the minimum order type of the sequence. If L_α is the lowest level to which some constructible set belongs, then the number of steps it takes from the empty to the set is the cardinality of α.

Now *L* is a proper class, like *V*. And, like *V*, the universe *L* is also a class model of the *ZF* axioms. We can examine each of the *ZF* axioms and prove that they also hold in the constructible universe. In fact, it can also be proved that the construction of any constructible set is also, itself, constructible. Given any constructible set, its construction can be wholly relativized to *L*. In this model *L*, 'set' is interpreted as a constructible set. Every set is assumed to be constructible here. In addition to the *ZF* axioms, the statement 'Every set is constructible' is also true here. We can add this statement as an extra axiom in *ZF*, the Axiom of Constructibility. Another way to state this axiom is that the universe of sets is exactly equal to the universe of constructible sets, that is $V = L$:

Axiom of Constructibility: $V = L$.

Proving the Axiom of Choice

The remaining task is to show that both *AC* and *CH* are theorems in *L*. We do this by deriving them from the *ZF* axioms plus the Axiom of Constructibility. The formulation of *AC* that we shall use is the equivalent Well-Ordering Theorem: there exists a well-ordering for every set.

All we need to show is that every constructible set *S* can be well-ordered. This can be done by providing just one well-ordering of the class *L* itself. The elements in a constructible set *S* are ordered according to when they appear in the well-ordering of *L*. If *x* and *y* belong to *S* and *x* precedes *y* in the well-ordering of *L*, then *x* also precedes *y* in the well-ordering of *S*.

The well-ordering of *L* is achieved by well-ordering the set of all the new sets formed in each hierarchy level. All the new sets formed in every level L_α are collected together and given a well-ordering. The class *L* is thus partitioned into a transfinite sequence of well-ordered sets. The creation of a well-ordering on the whole *L* is achieved by simply juxtaposing these well-ordered sets next to each other.

So what remains is to provide a well-ordering of all the new sets formed in some level L_α. These are all the new sets definable from sets in the lower levels.

We can assume that all those lower levels are already well-ordered. Being definable, each new set corresponds to a sentence in the language of set theory with parameters from lower level sets. A well-ordering can be easily imposed on the set of all such possible sentences. We have already assumed that the set of all such parameters, the set of all lower level sets, is already well-ordered. And the set of all sentences already has an obvious 'alphabetical' well-ordering. Since the set of all sentences can be well-ordered, the set of all the new sets in L_α can also be well-ordered. This completes the proof.

To summarize:

1. We need to prove that the Axiom of Choice is true in the constructible universe.
2. Consider the set of all the new sets formed in some level L_α within the constructible universe. We assume that all the sets in the levels before L_α have already been well-ordered. Since every new set in L_α corresponds to a sentence that takes parameters from sets located at the lower levels, and since the set of all such sentences can be well-ordered, this implies that the set of all those new sets can be well-ordered as well.
3. A well-ordering on L can now be induced by appending together the well-orderings of all the new sets formed at each level. Every constructible set can now be well-ordered as well. Its elements are ordered based on how they are already well-ordered in L.
4. So every constructible set can be well-ordered. This conclusion is precisely the statement of the Axiom of Choice.

Besides the Axiom of Choice, the Axiom of Constructibility also directly prohibits self-belonging sets, but without the need of going through the Axiom of Foundation. Badiou demonstrates this in Meditation 29 (*BE*, 304–5):

1. Suppose the constructible set a belongs to itself.
2. Being constructible, the set a first appears at some level in the constructible universe. Let the set a first appear in $L_{\beta+1}$ but not in L_β.
3. This means that a, though not a member of L_β, is a definable subset of L_β. So $a \subset L_\beta$ but $a \notin L_\beta$.
4. However, the fact that a belongs to itself implies that $a \in L_\beta$, a contradiction.

In other words, self-belonging is prohibited as it contradicts the hierarchic linearity of the constructible hierarchy. 'We can see here how hierarchical generation bars the possibility of self-belonging. Between cumulative construction by levels and the event, a choice has to be made. If, therefore, every multiple is constructible, no multiple is evental' (*BE*, 304–5).

Proving the Continuum Hypothesis

That *CH* is true in *L* requires a more difficult demonstration. We present it as the result of three lemmas or 'mini-theorems' that we state without proof. Recall the standard statement of *CH*:

$$|P(\omega)| = \aleph_1.$$

The power set $P(\omega)$ collects all possible sets of natural numbers. Since $\aleph_1$ is, by definition, the first cardinal number after $\aleph_0$, and since Cantor's Theorems implies that the cardinality of $P(\omega)$ exceeds $\aleph_0$, the task of proving the Continuum Hypothesis is reduced to proving

$$|P(\omega)| \leq \aleph_1.$$

Here are the three lemmas:

Lemma 1: *cardinality of X = cardinality of Def(X),* if *X* is infinite.

This lemma means that adding all the sets definable from those in infinite set *X* does not increase its cardinality. This follows from our proof to *AC*. Since the number of sentences do not exceed the size of *X*, those new sets do not increase the cardinality. This implies that consecutive levels always have the same cardinality if that cardinality is infinite. The cardinalities of L_α and $L_{\alpha+1}$ are equal when $\alpha > 0$.

Lemma 2: *cardinality of* α = *cardinality of* L_α if α is infinite.

This lemma says that the cardinality of a level equals the cardinality of its index ordinal when that index is infinite. To arrive at levels of uncountable cardinality, we need an uncountable index.

Lemma 3: If $x \in L_\alpha$ and α is infinite, then every subset y of x belongs to some level L_β, where the cardinality of the index ordinal β is the same as the cardinality of α.

If we set $x = \alpha = \omega$, the lemma translates into the statement that every set of natural numbers is created at a level that is indexed by some countable ordinal.

In other words, every element of $P(\omega)$ is constructed after countably many steps up the constructible hierarchy.

Since every subset of natural numbers belongs to some L_β, where β is countable, this means that $P(\omega)$ is a subset of the union to all levels indexed by countable ordinals:

$$P(\omega) \subseteq \cup_{\alpha \text{ is a countable ordinal}} L_\alpha.$$

But the set of all countable ordinals is precisely ω_1. So the result above becomes:

$$P(\omega) \subseteq \cup_{\alpha \in \omega_1} L_\alpha.$$

This implies:

$$|P(\omega)| \leq |\cup_{\alpha \in \omega_1} L_\alpha|.$$

Lemma 1 implies that the cardinality of $\cup_{\alpha \in \omega_1} L_\alpha$ equals the cardinality of $Def(\cup_{\alpha \in \omega_1} L_\alpha)$. And $Def(\cup_{\alpha \in \omega_1} L_\alpha)$ is precisely the definition of L_{ω_1}. Lemma 2 implies that the cardinality of L_{ω_1} equals the cardinality of ω_1, which is $\aleph_1$. This implies that the cardinality of $P(\omega)$ does not exceed $\aleph_1$, which is what we wanted to prove.

Summary of the Mathematics of the Constructible

1. The Löwenheim-Skolem Theorem implies that *ZFC* has a countable model and that every uncountable model of *ZFC* can be reduced to a countable submodel. The 'paradox' is that such a countable model contains an uncountable set $P(\omega)$. This 'paradox' is resolved when we realize that the cardinality is uncountable only within the countable model and countable from the viewpoint outside the model.
2. The universe of pure sets V can be partitioned into a transfinite cumulative hierarchy. The first level is the empty set $\emptyset$. A successor level is the power set of its preceding level, while a level indexed by a limit ordinal is the union of all its preceding levels. All the possible subsets of a previous level become elements of the next level.
3. The universe of constructible sets L has the same transfinite hierarchical architecture as V, with the difference being that, in the movement from the lower to higher levels, only the constructible subsets are taken in. A subset is constructible if it is definable in terms of the elements from the previous levels. The Axiom of Constructibility states that $V = L$.

4. The Axiom of Constructibility proves *AC* by showing that every set can be well-ordered. In the constructible hierarchy, *AC* is reduced to a theorem. The hierarchy also directly prohibits evental sets.
5. The Axiom of Constructibility also implies *CH* (as well as *GCH*). This means that *ZFC*, when appended with the Axiom of Constructibility, can be made to prove *CH*. This also means that *CH* is consistent with *ZFC*. A model of $ZFC + CH$ exists if there exists a model of *ZFC*.

In his commentary on Gödel's work on the consistency of *CH*, Badiou's concentrates on the inherent metaontological orientation behind the commitment towards the Axiom of Constructibility and towards the specific structural edifice of the universe V when it becomes equated with L. We recall that the universe of ontology must be stateless because every presentation must be presented within ontology itself. This includes every representation as well since a power set is a set like any other. But we already proved, via the Theorem of the Point of Excess, that a situation cannot contain its own state. Even though there is no state to ontology, the quasi-complete situation contains the states of every other situation. On its own, ontology does not prescribe its own regime of representation. But the specific structure of its universe prescribes the metastructure of all the other situations within itself.

We repeat ourselves. The Axiom of the Power Set, which is veridical in the quasi-complete situation, dictates that, for every existing situation, there exists another situation within ontology that prescribes the regime of representation. Every situation and every state is an element within the universe. In other words, the structure of the universe determines what becomes represented out of any situation. In order to understand the determinations governing representation, we need to examine the determinations dictating which subsets are collected within each power set. Crucial information about these determinations is given in the cardinalities of each individual power set.

Metaontological orientations of thought

One of the most important of these power sets is the continuum $P(\omega)$. We know that it is the first power set whose cardinality is not completely determined by *ZFC*. Since *ZFC* prescribes the existence of every finite set (a corollary from the Axiom of Pairing), every subset of every finite set exists. As a result, given a finite set of cardinality n, its power set will contain all its possible subsets, which are 2^n in number. So the errancy involving the cardinality of power sets begins only with the first infinite set ω, an errancy demonstrated by the Cohen-Easton Theorem.

Since the structure and size of $P(\omega)$ is undecidable and variable with respect to *ZFC*, this means that ontology, on its own, cannot say everything about which sets of natural numbers are presented in a universe. And since the structure and the size of the continuum are independent with respect to the basic conditions of ontology that constitute *ZFC*, it can be thought that what is at stake there is none other than Being itself, the anonymous Real of ontology itself. This anonymity of cardinality detains the impossible real of being-Being, which is oriented precisely towards the ontological abyss between presentation and representation. Since thinking this abyss is beyond the realm of ontology, what is demanded now is an extra-ontological thinking that would counter the errancy established by ontology itself.

Badiou goes as far as to say that every orientation within thought is framed by such a metaontological thinking since thought is precisely the desire to fill the gap of excess created by the state, by the regime of representation whose ontological figure is the power set. He writes that:

> [t]hought occurs for there to be a cessation – even if it only lasts long enough to indicate that it has not actually been obtained – of the quantitative unmooring of [B]eing. It is always a question of a measure being taken of how much the state exceeds the immediate. Thought, strictly speaking, is what un-measure, ontologically proven, cannot satisfy. (*BE*, 282)

Any determination concerning representation, therefore, must be independent of ontology. It is possible to have different universes with different determinations of what is and what is not represented, at least when it comes to infinite universes. The general laws of the Symbolic therefore must constitute extra axioms, extra commitments, that lie outside of ontology itself.

Different laws would correspond to different extra-ontological commitments and different metaontological orientations of thought. In Meditation 27, Badiou schematized four ontological destinies with respect to these orientations of thought, with each destiny promoting, among others, a distinct metaontological attitude towards thinking the specific errancy separating ω from $P(\omega)$ and the general abyss of un-relation between presentation and representation. In lieu of Gödel's work on *CH*, we shall only be concerned in this chapter with the first orientation, the constructivist orientation. Explaining the details of Badiou's philosophical commentary on this orientation will be the focus of the rest of this chapter. We omit explicating Badiou's discussion on what he calls the 'theological' third orientation, corresponding to the specific mathematical doctrine of 'large cardinals' whose existence exceeds ontology. We also postpone Badiou's discussion of the 'generic [*générique*]' second orientation, corresponding to Cohen's work, and Badiou's unique thinking of the fourth orientation, corresponding to the Law of the Subject, an orientation that seeks to go beyond the standard philosophical account of forcing.

The constructivist orientation of thought

The commitment to the Axiom of Constructibility reduces the universe of sets to those within the constructible hierarchy. Here, presentation equals construction, at least in a very specific sense. Only definable subsets are admitted in the upward movement from one level in the hierarchy to the next, from the set to the power set, from the situation to the state, from ω to $P(\omega)$. As shown by Gödel, this limitation establishes CH: $|P(\omega)| = \aleph_1$. Here the fault at the origin of the un-measure of the continuum lies in what is definable, that is in language itself. The state, which differentiates between what can be considered a part, has to do so explicitly and with the force of legislation that is legitimized and reasoned by a well-made language. An unrecognized part is held as unconstructible and ineffable. Whatever is not definable by a well-made language is not. Language is tied to existence, and vice versa. This metaontological allegiance corresponds to an ontological orientation of thought described as constructivist, and sometimes 'grammarian' or 'programmatic'.

We reiterate the two different relations between the state and language in constructivist thought:

1. Every presented multiple has its power set, which specifies the properties that can be separated out of the terms contained within the presented multiple in lieu of the Axiom of Separation. Here, the state corresponds to the situation's regime of representation that prevents the situation from encountering its inconsistency. This regime provides the proper names and properties for the language limited to the situation.
2. Presentation within the quasi-complete situation – the universal situation that contains all the other situations and presented multiples – is limited to multiples contained within the constructible hierarchy. In this hierarchy, the only multiples that exist are constructible ones or, more precisely, multiples

> that are constructible from those lower down in the hierarchy. This is done by limiting the promotion of subsets extracted from lower levels to those that are definable in terms of lower-level presentations. The regime of statehood within the universal situation determines its entire plane of presentation. Here presentation is linked to the definability limited by language, or any other comparable apparatus of recognition determined by well-made grammatical and legislative rules.

We can understand the constructible hierarchy as the legislation of a closed feedback loop between existence and language. Remember that the Axiom of Separation, which is a restriction of the false Axiom of Comprehension, dictates that existence must precede language. A formula, constructed by language, can create a set only if that set was already a subset of an existing set. The laws of ontology demand that a multiple should already have existed as a subset before it can be defined by language. The paradoxes that emerge from the Axiom of Comprehension imply that language cannot fully dictate ontology.

The most that can be allowed is by instituting the weaker Axiom of Constructibility. This extra axiom, which is compatible with the laws of ontology, enables language to precede existence, but with certain restrictions in order to avoid circularity and the various paradoxes that would arise with a complete acceptance of the Axiom of Comprehension. A formula invents a new set if and only if it uses terms from simpler sets inhabiting a lower level in the constructible hierarchy. The division of L into strata helps avoid the self-contradictions of unrestricted Comprehension.

Nominalist metaontology

Grammarian thinking is a consistent metaontological doctrine that is often associated with the radical ontology connected with the so-called 'linguistic turn' in formalist philosophy: a nominalist orientation where presentation equals nomination, where what exists is nameable and what is nameable exists. The existential quantifier 'there exists' translates into 'there exists a named term' and the universal quantifier 'for all' translates into 'for all named terms'.

There is always a temptation to supplant existence with language, to equate the state with the situation, which is impossible because of the Theorem of the Point of Excess. Without its absolute separation from the situation, the state would have no meaning, no efficacy. But a longing for this impossible solution subsists unconsciously in programmatic thought. The most it can do is attempt to valorise and imitate the maximum equilibrium of natural situations where every presentation is a representation. Moreover, the inherent power and attraction of nominalism, of the constructivist orientation of thought, is the impossibility of refuting it. In order to refute nominalism, the requirement is to produce a counter-example, to provide an irrefutable part that is, at the same time, undifferentiated, anonymous, indiscernible and indeterminate with respect to constructed language. But to indicate such a part would require the construction of its very indication, turning the counter-example into an example. This is why the refutation of any nominalist ontology must be decided in an act of pure subjectivity, an act that would not conflict with the basic rules of ontology itself. This is also why nominalism constitutes a radically robust and unavoidable ontological commitment. Constructivist thought is, in fact, 'the latent philosophy of all human sedimentation, the cumulative state into which the forgetting of [B]eing is poured to the profit of language and the consensus of recognition it supports' (*BE*, 294). 'Knowledge calms the passion of [B]eing: measure taken of excess, it tames the state, and unfolds the infinity of the situation within the horizon of a constructive procedure shored up on the already-known' (*BE*, 294). The constructivist orientation of thought naturally prevails within situations that have long sedimented.

To demonstrate the empirical manifestations of the nominalist ontology within the three other conditions of art, science and politics, Badiou examines its philosophical underpinning within neoclassical art, positivist epistemologies and programmatic politics:

1. In neoclassical art, the language takes the form of a perfect and perfectly systematic repetition of aesthetic inscriptions, so perfect that any modification or rupture would make the artwork unrecognizable as art. From the viewpoint of neoclassicism, its so-called rivals from within modern art, which it deems barbaric, promote chaos and valorise the empty complexities of disorder.
2. In positivist epistemologies, the language takes the form of a well-made, measurable and constantly fine-tuned syntax where all presented multiples are systematically marked and all procedures are named and constructed to the utmost degree. The indistinct is vigorously hunted out and everything is confined to controllable facts by matching up clues, testimonies, experiments and statistics.

3. In programmatic politics, the endeavour, which is shared among the political parties, is to render language compatible with the reformist moderation of the political procedures, which aim to follow the form of some program that is realized by the state of the situation.

In the constructible universe, a state is not simply an ontological regime of representation in its presentative primacy, but also a linguistic discourse that legislates the existence of the subsets. The multiplicity of different situations becomes the multiplicity of different languages, of different 'language-games' as Wittgenstein proposed in his *Philosophical Investigations* (1953). We must pass over in silence that of which we cannot speak – which, as prescribed by Wittgenstein at the end of his early *Tractatus* (1921), is the outside of language, an outside whose existence is irrelevant and unprovable.

This leads to Leibniz's Law on the Identity of Indiscernibles there cannot exist two entities whose difference cannot be linguistically marked by the well-made language. Language thereby assumes the role of ontological law. Any constructivist orientation of thought is equal to the ontological principle of a logical grammar. It ensures that a logically well-made language prevails as the norm for what exists, that is for what is one-multiple among presentations. The state legislates on existence by stamping out any concept-less relations and unnameable parts. The link between existence and language implies the subsuming of the relation to being within the dimension of knowledge. Leibniz's Law comes down to rejecting the existence of that which is unclassifiable within knowledge.

Knowledge and encyclopaedic determinants

The constructible universe sacrifices the wealth of Being with a measurable and calculable order. But the knowledge prescribed by the constructible is, rather, knowledge without knowledge, inauthentic knowledge, knowledge lacking the necessary rupture of knowing.

We end here by expanding on what is at stake in this knowing without knowing. Now ontology dictates the representational and inventive capacities of language. But constructibility is able to dictate this language back into ontology. By the partitioning of the universe of sets into an ordered hierarchy, every non-empty set is the subset of some other simpler set. And since language restricts the actual presentation of every subset, this means that language has a say in what exists and what does not exist.

The subsets in the constructible universe are those that have already been legitimized by language. The power set contains only constructible subsets and the operations of the state are merged with the operations of language, of a regime of knowledge that is linguistically realized. Such a regime employs two constitutive operations:

1. The operation of discernment connects language with presentation by dictating how multiples are recognized via the means of predicates constructed by the relevant metastructures. Discernment pronounces the capacity to judge and to speak of predicates.
2. The operation of classification connects language with parts by dictating how multiplies are collected together into presentations via the means of the subsets that are counted by the relevant metastructures. Classification dictates the capacity to link judgments and to speak of parts.

What Badiou calls an encyclopaedia is any realization of knowledge as a summation of judgements under a closed and consistent determinant and systemic compartmentalization. The details are technical and we will only be able to formulate them after explaining the mathematical concept of the filter, which will be given in the next chapter. For the moment, we can only say that a regime of knowledge can be consummated under numerous encyclopaedic determinants, some of which might contradict and compete with each other. With respect to a situation, an encyclopaedic determinant is any constructible subset of the power set.

7

Ontology of Forcing and Generic Sets

Between late 1963 and early 1964, the mathematical world was stunned by the appearance of a series of two relatively short but dense research papers in the *Proceedings of the National Academy of Sciences of the United States of America* (1963, 1964), the same journal where Kurt Gödel had published his consistency proof nearly 25 years earlier (1939). The author of the two papers was the 29-year-old Paul Joseph Cohen, a mathematician who was already well known at the time for his brilliant work in mathematical analysis. His papers contained an outline of his proof that the negation of the Continuum Hypothesis is consistent with *ZFC*. In 1966 Cohen would publish a monograph containing the full details of his proof. In addition to the result of the Continuum Hypothesis, the monograph contained Cohen's proof that the negation of the Axiom of Choice is consistent with the *ZF* axioms. Cohen's discovery led him to his being awarded the Fields Medal, the highest honour in mathematics.

Cohen showed that if *ZFC* was consistent, so is *ZFC* + ~*CH*. Since Kurt Gödel had already established, using the Axiom of Constructibility, that *CH* is consistent with *ZFC*, Cohen's result implied that *CH* is independent and undecidable with respect to *ZFC*. In other words, there exist different models of Zermelo-Fraenkel set theory where the Continuum Hypothesis can or cannot hold.

We remind ourselves that *CH* is by no means the first interesting statement proven to be undecidable in set theory. We already encountered independence in the Second Incompleteness Theorem where Gödel proved that the consistency of a formal axiomatic system, of a certain general kind, is undecidable with respect to the system itself. What is significant in Cohen's case is the way he established his result, which relied on a new technique he devised, now simply called forcing. This technique made essential use of generic sets, a particular mathematical structure that Cohen borrowed from his background in mathematical analysis. Forcing and generic sets supplied the machinery and the mathematical infrastructure for Cohen to demonstrate his result. Moreover, this infrastructure later turned out to be very profound, powerful and flexible. Cohen's technique could be extended to prove other consistency and independence results, to prove that certain statements are consistent or independent with respect to certain sets of axioms that include *ZFC*. The technique of forcing can be generalized so that it could work with other statements besides *CH* and *AC*. We stress that it does not always work. Even when forcing does work, its technology often has to be designed and modified extensively to fit with the particular circumstances. Nevertheless, hundreds of other consistency results have been discovered using forcing and the area now forms its own special and active sub-branch within mathematics, finding applications not only in set theory but in many areas of contemporary mathematical research. A large majority of the consistency results discovered after Cohen's work are more or less adaptations of it and, to date, a fundamentally different mathematical infrastructure has yet to be constructed.

The specific employment of forcing in *Being and Event*

We must bear in mind the distinctiveness of Badiou's idiosyncratic explication of forcing in *Being and Event*:

1. He never shies away from reproducing the advanced details and formal demonstrations, all of which are, if not similar, at least generally compatible with the typical presentation of the topic in the standard mathematical texts such as Cohen's monograph (1966), Kenneth Kunen's *Set Theory* (1983), and Thomas Jech's *Set Theory* (2006). Any Sokalist-type argument would not be applicable here, for there is no question that Badiou displays a sufficiently exhaustive and deep understanding of the topic, certainly deeper than anything acquired by a typical mathematical dilettante reading a more introductory exposition.
2. Badiou also does not avoid employing, at times, symbolic and formal notations, although he appears to keep this at a minimum, perhaps in order to make the text more reader-friendly for those not mathematically trained. The downside of this is that the language loses in terms of being precise, concise and clear from the viewpoint of a more mathematically trained reader, who would benefit from referring to a more standard and rigorous elucidation of the material from the standard mathematical texts

beforehand. This dilemma, of choosing between being formal and being friendly, is also unavoidable for any work (such as this book) that wishes to discuss Badiou's mathematics and metaontology.

3. Since they are often presented in tandem, the ontological explanations and metaontological commentaries are often indistinguishable in the text. Badiou frequently renames many of the standard mathematical concepts and theorems, mainly – although we are speculating here – to fit with his metaontological framework and to accentuate certain features that were not obvious in the original mathematics. For example, with only a playful hint of explanation, Badiou famously renames the standard textual mark for the generic filter *G* as the Venus symbol '♀', probably in order to insinuate some Lacanian connection that he leaves open to speculation. Another by-product of these renominations is that the metaontological concept and its empirical realization become separated from their respective ontological figures, often with no explicit guide to the correspondences in the book. The mathematically trained reader, who is sufficiently well-versed in the mathematics, faces the task of finding the ontological figure for each metaontological concept.

We should also bear in mind that there are three different facets of forcing at issue in Badiou's book:

1. There is Paul Cohen's *ad hoc* employment of forcing from the early 1960s. Not everything there is relevant for Badiou. As a theorem, the undecidability of the Continuum Hypothesis plays some role in *Being and Event*, although Badiou takes only certain aspects of the proof as conditions for his philosophy. For example, the specific construction of the generic filter does not appear to be important for Badiou, save for the fact that the conditions involve 'finite partial functions'. Moreover, Badiou's metaontological commentary does not appear to have much to say about the specific undecidability of the Axiom of Choice, although he makes use of Gödel's work involving its provability from the Axiom of Constructibility.
2. There is the *general* technology of forcing, the general mathematical infrastructure (invented by Cohen and developed by others) that enables the demonstration of various consistency results. The technology has no standard version and no standard narrative to describe its process. This has to do partly with the sociological and plural nature of mathematical

discourse. It is also due to the fact that the field is still being actively developed, with only a few periods of seeming to congeal into some permanent and universal doctrine. Another reason is the inherent technicality of the mathematics itself, the technological quiddity of mathematics in its dynamic ontology and ever-shifting details.

3. There is the *space of possibility* circulating within the general technology of forcing in its dynamic ontology. Various features within the technology are flexible and adaptable, subject to certain conditions. And there is a lot of room within the technology for it to prove other consistency results. The technology can involve different ground models of *ZFC* and the generic filter can be constructed in various ways. As we shall see, this space of possibility allows for the emergence of what Badiou conceives to be the subject. Whenever the Law of Forcing is enforced within the ontology of a situation, we find the weaving of a subject.

We also note that the mathematical field of forcing experienced three well-known advancements following Cohen's work. Badiou does not write about any of them in any detail in *Being and Event*, although it is useful to mention them briefly, just so that we are aware of them:

1. The first occurred through the efforts of the mathematicians Dana Scott, Robert Solovay and Petr Vopěnka to understand and simplify Cohen's work. They showed that Cohen's technique of generic sets is effectively equivalent to the technique of Boolean-valued models. This result, which places less emphasis on the 'action' of forcing, provides an alternative and more extended reading of Cohen's technique.
2. The concept of forcing is also related to Luitzen Brouwer, Arend Heyting and Saul Kripke's work of trying to interpret and provide a semantics for modal and intuitionistic logic. This is often simply referred as 'Kripkean forcing'. The task of trying to construct alternative effective models of modal and intuitionistic logic has deep conceptual connections to Cohen's work. Kripke also employed certain features within the mathematics of forcing to construct a metaphysical framework for understanding possible worlds. Before Badiou's *Being and Event*, this is perhaps the most famous and most influential employment of Cohen's work as a condition for philosophical thinking.
3. Developments in the various attempts at proposing category theory as an alternative foundation for mathematics and logic have led Cohen's work to

be interpreted in the context of topos and sheaf theory. William Lawvere and Myles Tierney showed that forcing is generally the construction of a certain topos of sheaves.[1]

We will not expand further as none of these three advancements are discussed in any detail by Badiou in *Being and Event*. But he might have indicated some awareness of them:

1. In regard to Boolean-valued models, Badiou mentions in the end notes that this version of forcing was presented in two particular set theory textbooks: J. -L. Krivine's *Théorie Axiomatique des ensembles* (1969) and K. J. Devlin's *The Joy of Sets: Fundamentals of Contemporary Set Theory* (1979). Badiou calls such presented treatments more 'realist' and less 'conceptual' than his own (*BE*, 496).
2. In regard to Kripkean forcing, Badiou never mentions it in any of his philosophical works, although his concept of situation could be easily linked to the modal concept of possible world.[2] Badiou does, however, express disagreement with certain aspects of the general understanding of intuitionism as a meta-mathematical position, particularly in regard to the topic of double negation (*BE*, 249–50).
3. In regard to the sheaf-theoretic re-conceptualization of forcing, Badiou also never mentions it in *Being and Event*. But his ensuing work, *Logics of Worlds*, which is meant to supplement the metaontology of *Being and Event*, contains a thorough discussion involving the mathematics of topoi and sheaves. But sheaf-theoretic forcing is never explicitly mentioned there.

The semantic versus syntactic approach to establishing consistency

We return to Cohen's proof of the following proposition:

$$\textit{Consistency of ZFC} \rightarrow \textit{Consistency of ZFC} + {\sim}\textit{CH}.$$

Consistency here can mean 'does not prove a contradiction' or 'is satisfied by a model'. Since we are working within the language of first-order logic, this means that Gödel's Completeness Theorem can be used and so syntactic and semantic consistencies are effectively interchangeable. To prove that *ZFC* + ~*CH* is consistent, we can show that it does not give rise to any paradox or that there exists a model for it. In the former, we show that any valid proposition derivable from $\sim CH$ and *ZFC* cannot be paradoxical. In the latter, we prove the existence of some model whereby *ZFC* is veridical but *CH* is not.

Cohen chose the latter option. This decision is based on his careful articulation of a series of insightful strategic intuitions as to what methodology must be used to establish the consistency of $\sim CH$. He also relied on some lessons learnt from a mathematical result by the logician John Shepherdson and from an examination of Gödel's previous work on the constructible universe *L*. Gödel showed that *ZFC* can be made to prove *CH* when we assume the Axiom of Constructibility '$V = L$', when we limit our ontological commitment to the constructible universe *L*. Gödel's work was more syntactic because it relied more on an explicitly formulated property, the property of constructibility, rather than on the direct ontological presentation of *L*. Gödel took the *ZFC* axioms, added the Axiom of Constructibility to them, and derived *CH*. By concentrating less on the 'material quiddities' of *L* itself, Gödel's work was oriented more towards proof theory than model theory. Cohen realized that one must abandon the Axiom of Constructibility and leave the constructible universe if one wants to establish the consistency of $\sim CH$. Since *CH* is veridical in *L*, this means it can only be erroneous in models outside of *L* where the Axiom of Constructibility does not hold. We must stop assuming that every set is constructible, that each set is built out of the empty set via a sequence of explicit constructions, via the application of a series of predicates. We must construct, without actually 'constructing', a non-constructible universe outside of *L* where *CH* is erroneous but *ZFC* obtains, a universe that admits non-constructible sets.

This requirement of 'constructing without constructing' explains the difficulties inherent in using the syntactic interpretation of consistency. To show that $\sim CH$ is syntactically consistent means that we have to show that it derives no paradox. We might thus be led to examine the essential structure of any proof that is derivable from $ZFC + \sim CH$. But it is difficult to do this outside of *L* because the idea of a 'proof' and of 'proving' is very close to the idea of 'construction' and 'constructing'. Demonstrating $\sim CH$ by looking at the structure of proofs might end up being a double-bind because it is difficult to move away from the Axiom of Constructibility.

Moreover, it is not enough to simply replace the Axiom of Constructibility with some other statement and then use it to prove $\sim CH$. We cannot, again, just limit the universe of sets *V* to some other class, restricting the sets to some property other than constructibility, and then show that *CH* does not hold in the delimited model. This technique of limiting *V* to some smaller model, some 'inner model', is called the 'method of inner models'. About a decade before Cohen discovered forcing, John Shepherdson, in a series of three papers in *The Journal of Symbolic Logic* (1951–3), proved that this method cannot be used to prove ~*CH*. All of this provides strong evidence for the desirability of working with semantic as opposed to syntactic consistency.

Universe-building

So the task at hand is now translated into proving the following statement:

ZFC has a model → ZFC + ~CH has a model.

We assume beforehand that *ZFC* has a set model *M*, which we call the ground model. Because of Gödel's Second Incompleteness Theorem, this model cannot actually be exhibited, at least within the confines of *ZFC* itself, which is supposed to be the axiomatic foundation of all mathematics (and of ontology itself). Assuming the existence of *M*, the statement above says that another model must also exist, for *ZFC* + ~*CH*. In keeping with the standard mathematical notation, we call this second model *N*, or the generic extension. The statement says that the existence of *M* implies the existence of *N*, that the existence of a ground model satisfying *ZFC* implies the existence of a generic extension where *ZFC* is veridical and *CH* is erroneous.

The proof strategy used to establish the statement corresponds to the one that would probably immediately come to mind. We simply use the model *M* to get to the model *N*. We manipulate the first model in such a way that we get the second model, or at least prove that it exists. We find a universe where *ZFC* is veridical and extend it to get another universe where all the axioms of the *ZFC* are veridical, but the Continuum Hypothesis is guaranteed to be erroneous. Without loss of generality, we can already assume that the Continuum Hypothesis was veridical in *M* or not yet erroneous. We want to modify this ground model and end up with a generic extension where the Continuum Hypothesis becomes erroneous, but the *ZFC* axioms still hold. We want to preserve the truth of *ZFC* but change *CH* from veridical to erroneous.

How is this change performed? Remember that the Continuum Hypothesis and its negation say the following:

$$CH\colon |P(\omega)| = |\omega_1|$$
$$\sim CH\colon |P(\omega)| > |\omega_1|$$

In the ground model where *CH* is veridical, the sets contained in its version of $P(\omega)$ – the set of all its sets of natural numbers, of finite ordinals – have a one-to-one correspondence with the sets contained in its version of ω_1, the set of all its countable ordinals. In generic extension where ~*CH* is veridical, the number of its sets of natural numbers is larger and exceeds the size of its ω_1. Cohen realized that we can make *CH* erroneous simply by adding more 'new' sets into $P(\omega)$ to make it larger than ω_1. So the question of converting *M* into *N* is a question of 'changing' the size of the power set of ω by adding those 'missing' sets of natural numbers while still maintaining the truth of *ZFC*.

The highlight of Cohen's proof was a general and powerful technique for 'universe-building', for constructing new quasi-complete situations from old ones. And the technology can be used for a whole host of other statements besides *CH*. In Cohen's case, we take a consistent set of statements, namely *ZFC*, and append ~*CH* to it while still retaining consistency. This result, together with Gödel's result on the consistency of *CH*, implies that *CH* and *ZFC* are independent with respect to each other, that they bear no direct relation, either of implication or negation. *ZFC* neither implies nor negates *CH*. In the case of Cohen's result on the Continuum Hypothesis, we take a universe, modify its structure, add some more elements to it – which include the extra sets of natural numbers – and get a new universe, but now one where the Continuum Hypothesis is erroneous. Cohen found a way to get from one consistent universe to another, from one coherent system of statements to another. When understood in terms of 'universe-building', Cohen's proof leads to a general but profound way of thinking the dynamic development of new entities and notions over space and time, a way of comprehending how different universes are linked together through the addition, subtraction or modification of the entities or statements that differ between them. Given two different ontologies that contain different entities and satisfy different statements, we could turn one world into another using this relation of emergence. The name of that relation is forcing.

The general technology of forcing

So we have two different models, *M* and *N*, for *ZFC*, each specifying a particular universe of entities and a particular horizon of veracity – a particular commitment to what is and what is not the case. We want to build one model from another via some 'construction without construction'. Forcing is the name of this relation of emergence between the two universes. We want to 'force' the emergence of the generic extension to take place – the modification and extension of the elements in the ground model. We want to 'force' everything that needs to be veridical in *N*. In Cohen's case, we want to force *N* to admit the extra subsets needed in order, first, for the Continuum Hypothesis to be erroneous and, second, for the *ZFC* axioms to continue being veridical.

Later, we will formally describe the construction of the generic extension. So we end here with a rough sketch. The new universe of *N* will be an expansion of the universe of *M*. So the domain of *M* is a 'subset' of *N* in the sense that every element in *M* has a corresponding version in *N*. The generic extension includes *M*, plus some extra sets. The most crucial of these extra sets is the one commonly denoted as the set *G*, which is an infinite subset of *M* that is not itself an element of *M*. The nature of *G* depends on which extra statements we wish to force – in Cohen's case ~*CH*. The domain of *N* contains every element from *M*, plus the infinite set *G*, plus a minimal number of the other extra elements needed for *N* to be a model of *ZFC*. In fact, *N* is the 'smallest' and simplest possible model that includes *M*, contains *G*, and satisfies *ZFC*. Since *N* depends on *M* and *G*, it is also often written as the model $M[G]$.

We will schematize the process of constructing the generic extension into three more or less distinct steps:

1. The preparation of *M*
2. The codification of *G* in preparation for its being 'absorbed'
3. The creation of the new model $N = M[G]$ and the forcing of every statement that needs to be veridical in it.

As a precaution, we note that there is nothing metaphysically or metaontologically important about the distinction between these three steps. I make this division only to make my exposition more systematic. The three steps are executed more or less in tandem with each other, so their order does not really matter.

Cohen discovered that, in order to allow this process to work, the ground model *M* has to be reduced to one of its smaller submodels that, while satisfying *ZFC*, also fulfils three other requirements. Cohen also discovered that the added infinite set *G* has to take a particular structure: being a generic filter. Such a generic filter is constructed through what is known as a notion of forcing, which codifies all the possible 'information' that *could* be in *G*. The generic filter *G* will be an 'oracle' that contains all the information to determine everything that is veridical and erroneous within the generic extension. It will prescribe a complete horizon of veracity for $M[G]$. Moreover, there is enough in M to prepare itself to read only the necessary partial information about G and thereby determine the horizon of veracity corresponding to N.

In the process of creating N, several theorems will prove important:

1. The Rasiowa-Sikorski Lemma ensures that a generic filter G will always exist.
2. The Indiscernibility Theorem ensures that G exists outside of M.
3. The Forcing Theorem ensures that veracity within N is completely determined by the information in G.
4. The Local Definability of Forcing Theorem ensures that the process of determining truth within N can be conducted wholly within M upon the local approximation of the generic filter G and the information contained therein.
5. The Generic Model Theorem ensures, among other things, that N is a model of *ZFC* that includes M and contains G.

Reducing the ground model to a countable transitive model of *ZFC*

So we assume the existence of a set model M for *ZFC* (the ground model) and the mission is to prove the existence of a second set model $N = M[G]$ for *ZFC* (the generic extension). In particular, we assume that *CH* is veridical within the ground model and we want to make it erroneous within the generic extension. In order for everything to work, the ground model must be transformed first into one of its smaller submodels of *ZFC*, but one which is countable and transitive:

1. A countable model is one whose domain contains a countable number of elements. The ground model is reduced to the countable submodel for reasons of convenience because one important result, the Rasiowa-Sikorski Lemma, will become applicable if we work with it. If the ground model was not already countable, then we might think that a reduction is possible because of the Löwenheim-Skolem Theorem. However, this creates various other complications. We can get around them by working not with *ZFC* itself, but repeatedly with every finite portion of those axioms, which works up to the totally of *ZFC* in the end with the help of what are known as the Reflection Theorems (*BE*, 494–5).

2. A set model is transitive if its corresponding domain contains the elements of all its elements. This is important so that we can impose a certain desirable architecture to the ground model within the cumulative and constructible hierarchy. (This definition is slightly different from Badiou's metaontological concept of nature and neutrality as the ground model has no state.) Transitivity also implies that the model will ultimately contain all the ordinals before a certain point, some upper limit ordinal α. All the ordinals preceding this upper limit are contained in the reduced model. Since the model is also countable this means that this upper limit cannot be uncountable – or, more precisely, uncountable from the viewpoint outside of the model. But can a transitive submodel be found? Yes, because of a result known as the Mostowski Collapse Theorem, which says that if a model of *ZFC* is not transitive, we can convert it into one of exactly the same inner structure.

For simplicity, hereafter when we mention the ground model M, we will be referring to its reduced version as a countable transitive model.

The poset P of forcing conditions

We now need to construct the appropriate infinite set G that will be added to make the generic extension. The nature of G depends partly on what new statements – in this case $\sim CH$ – that we want to be veridical in the new model $N = M[G]$. We present here the *general* mathematical machinery for the construction, which is fairly complicated and technical. Later we will offer an indication of how Cohen codified the *specific* set G that is needed to make CH false.

The machinery is as follows. We design, within the ground model, the appropriate poset P. The elements of G will be a selection of the elements in P. In addition to being a subset of P, the set G must also satisfy the two properties with respect to the ground model (1) being a filter (2) being generic. Even though G is a subset of P, and P belongs to M, G will be constructed so that it does not belong in M.

Before moving to the more intuitive understanding, first we describe the technical definition of the poset. The word is short for 'partially-ordered set'. The poset P is a set of elements from the ground model that are structured by a binary relation of partial order, usually denoted as $\leq$. If $p \leq q$ but the elements p and q are not equal, then we write that $p < q$. One unique element, usually denoted as 1_P, is singled out as being the 'last' or 'largest' element of the poset. The relation corresponding to the partial order $\leq$ must also be reflexive, transitive and antisymmetric:

1. The order is reflexive when every element is related to itself, so $p \leq p$ for every element $p \in P$.
2. The order is transitive when for any three elements $p, q, r \in P$, if $p \leq q$ and $q \leq r$, then $p \leq r$.
3. The order is antisymmetric when $p \leq q$ and $q \leq p$ imply that elements p and q are equal.

A poset is comparable to an ordered set, except that we cannot necessarily arrange the elements in a single line of order. Given two elements q and r, either $q \leq r$, $r \leq q$, or r and q are incomparable. Only the largest element 1_P is comparable to every element in the poset in that $p \leq 1_P$ for every $p \in P$. We can, in a sense, understand a poset as a 'network', with the element 1_P as the 'mother root' and all the other elements smaller than 1_P emanating as 'branches' from it. One easy example of a poset is the power set $P(S)$ of any set S, taking the relation of reverse set inclusion $\supseteq$ as the partial order $\leq$ and 1_P as the empty subset $\varnothing$.

In relation to forcing, the elements of the poset constitute the elements that *could* end up belonging to the generic filter G. The set G transmits complete 'information' concerning the horizon of veracity corresponding to the generic extension. Each element from the poset that ends up in G provides partial details about the information transmitted by G. An element in the generic set approximates the complete information in G. In relation to forcing, the elements of the poset P are called forcing conditions, while the poset P is called a notion of forcing. Each notion of forcing provides a structured set of all the possible pieces of information that could be in the generic set G.

The partial order $\leq$ structures these pieces of information in an appropriate and convenient way. If p and q are conditions, and $p < q$, then we say the information in p exceeds the information in q. In other words, the condition p contains the information in q, and more. We also say that the condition p is stronger than condition q and dominates it. (Note the reverse order here: '$<$' translates into 'stronger', not 'weaker'). Being the 'last' element, 1_P contains the least amount of information and is, thus, the weakest condition. This is often why 1_P is often fixed as the empty set $\varnothing$.

Incomparability of information implies incompatibility and inconsistency. Suppose condition p is comparable and, thus, consistent with the information in q. Being compatible, this means that the information can exist together in some merged third condition that contains all the information in p and q (The information in any third condition, which is a set, must be consistent, because of the Completeness Theorem). In other words, conditions are compatible when there exists a third condition that is at least as strong as each of them. Such a third condition dominates and mutually extends them.

In order for forcing to work, the poset of forcing conditions must also satisfy two further requirements:

1. The first requirement is called the Countable Antichain Condition (or, sometimes, confusingly, the Countable Chain Condition, or c.c.c.). An antichain for the poset denotes a set consisting of mutually incomparable conditions. The Countable Antichain Condition is the requirement that any possible antichain contain a countable number of conditions. This requirement does not appear to play an explicit metaontological role in Badiou's philosophy, although he discusses the relevant mathematics in Appendix 12 of *Being and Event*.
2. The second requirement states that every condition in P can be dominated by at least two stronger conditions that are incompatible with each other. In other words, for every $p \in P$, there exists $q \in P$ and $r \in P$, such that $q \leq p$, $r \leq p$, and q and r are incompatible. What this implies is that there is a real choice when it comes to extending a condition. Any condition can be extended in at least two ways. Let us call this the Requirement of Real Choice.

Concept of a filter

We return to the generic filter G. We can intuitively understand the concept of a filter by thinking of it as a closed and consistent body of information. As a non-empty subset of the poset P, the set G must contain a non-empty selection of conditions while also inheriting the same relation of partial order $\leq$. To be a filter, G must satisfy three requirements in particular:

1. G contains the weakest condition 1_P. So $1_P \in G$.
2. G contains all the conditions weaker than at least one condition in it. So if $q < r$ and $q \in G$, then $r \in G$.

3. Given any two conditions in G, there exists another element in G that is stronger than both of them. In other words, any pair of conditions in G are mutually compatible in G. So if $p, q \in G$, then there exists $r \in G$ such that $r \leq q$ and $r \leq p$.

Requirement (1) means that the filter contains the most minimal condition that transmits no information. Requirement (2), which is called 'Rule Rd_1' throughout *Being and Event*, means that the information in G is 'deductively closed'. Information in weaker conditions is deducible from those in stronger ones, and this requirement implies that any information deducible from information already in G must also be in G. Requirement (3), which is called 'Rule Rd_2', means that the information in G is coherent and 'conjunctively closed'. Given any pair of conditions within G, there is a third condition within G containing their conjuncted information. So a filter contains enough pieces of information to be closed and consistent.

G is a generic filter

We will see in the last chapter that, for Badiou, a generic filter provides the being-multiple of truth, while a non-generic filter provides the being-multiple for a 'correct subset', which, in the constructible universe, is supported by encyclopaedic determinants. To explain the meaning of genericity, first we must describe the concept of being a dense subset with respect to the poset and the ground model.

We can think a dense subset as a set of conditions that contains *every* possible piece of information that could be given by the poset. A subset D is dense if, given any condition in P, there will always be one element in D that is at least as strong. In other words, any forcing condition is always weaker than some condition contained in the dense subset. More formally, if D is a dense subset of P, then $p \in P \rightarrow \exists d \in D$: $d \leq p$. Since every condition contains pieces of information, we can say that every possible information is implied by some information given in the dense subset, and without the dense subset needing to contain only compatible conditions. Since it contains items of information that dominate any other possible information, we can say that D contains *every* possible information.

Within the ground model M, the poset P can lead to several possible non-empty dense subsets. The subset G is generic over M when it always has at least one element in common with every of those possible dense subsets. That is, the generic filter intersects with every non-empty dense subset of P. The poset P can have one, several or no dense subsets. But, if it does have a dense subset, then the generic filter G shares at least one element with it.

The existence of a generic filter and its exteriority with respect to the ground model

The two main theorems are as follows:

1. *Indiscernibility Theorem:* no generic filter belongs to M
2. however, one will always exist outside of M

The specific proofs of these two theorems form the most crucial mathematical result in Badiou's metaontology of forcing, and it is worth explaining them in great detail. The demonstration of the Indiscernibility Theorem relies on the following proposition:

Proposition: For any filter F in M that is a subset of P, the complement of that filter with respect to P must be dense. That is, the set $P - F$, consisting of conditions not in that filter, must be dense.

Proof: This proof makes use of the transitivity of M and also the fact that the poset P satisfies the Requirement of Real Choice: every condition p from P is dominated by two mutually incompatible conditions q and r. The set $P - F$ must be dense because the condition q and r cannot both be in F, which is a filter. So one of them must be the condition in $P - F$ that dominates p. This proves the proposition.

If the complement of every filter is dense, and if a generic filter intersects every dense set in M, then that generic filter cannot be a filter in M. The generic filter will always have one element outside of every filter in M. So the generic set exists outside of M, completing the proof of the Indiscernibility Theorem.

The second theorem makes use of the countability of M, which also implies that the number of dense subsets in P must be countable. We can then apply the following result:

Rasiowa-Sikorski Lemma: If the number of dense subsets in P is countable, then a generic filter exists and can be constructed. Moreover, for any condition $\pi \in P$, there exists a generic filter containing it.

Proof: Let the complete countable sequence of dense subsets be denoted as D_1, D_2, …, and so on. With respect to the chosen condition $\pi \in P$, we construct a sequence of conditions π_0, π_1, …, and so on. There is no standard name for them in the mathematical literature and so, in keeping with the Badiouian terminology, we shall call them 'enquiries'. The first enquiry π_0 is simply π itself, and all the other enquiries are defined recursively. The $(n + 1)$th enquiry π_{n+1} is simply any condition in the $(n + 1)$th dense subset D_{n+1} that is either equal to the previous nth enquiry π_n or dominates it. The generic filter is the set of all conditions whose strength does not exceed at least one of those enquiries constructed in this sequence. So $G = \{g \in P\text{: } \exists n = \mathbb{N}, g \leq \pi_n\}$. The genericity of this set is obvious, and a short demonstration easily confirms it to be a filter.

The generic filter from the viewpoint of the ground model

Even though the generic filter lies outside of the ground model, its elements and finite subsets are all inside M. As a generic filter, G is still definable within M because the terms used to define its predicate – dense subset, intersection, conditions – are all definable with respect to the immanence of the ground model itself. The generic filter is definable, even though it does not exist. In M, the set G exists only 'intensionally', as a proper class, and not 'extensionally', as a set.

We re-examine the proof of the Rasiowa-Sikorski Lemma. The construction of G involves collecting all the conditions whose strength does not exceed any enquiry π_i in the countable sequence π_0, π_1, …, etc. Each enquiry in this sequence is, in turn, taken from the elements of some D_j in the countable sequence of dense subsets D_1, D_2, …, etc. In other words, the information contained in the conditions in G are already contained in the enquiries. Every condition in G is a finite subset of some enquiry π_i. Since the enquiries are arranged according to increasing strength, we can think of it as a constantly expanding set of information. Each new enquiry appends new information to this expanding set that sums up to the generic filter.

From the viewpoint of the ground model, we can thus understand the generic filter as being determined by the limit of some sequence of enquiries of increasing strength, as the infinite end result of an expanding set of information. The first enquiry of this sequence, $\pi_o = \pi$, is randomly selected, while the nth enquiry is randomly selected from the nth dense subset D_n, with the requirement that it be at least as strong as its preceding enquiries. Each new element of the sequence adds new information to the generic filter, a set that, in the end, is meant to be a closed and consistent body of information.

Since the sequence of dense subsets contains the complement of every non-generic filter in the ground model, we can say that the construction of the sequence of enquiries also involves 'avoiding', in turn, each filter, each closed and consistent body of information in *M*. In other words, the generic filter is approximated within the ground model as follows: the first piece of information $\pi = \pi_o$ is chosen randomly while all the rest are chosen by inspecting, in turn, each dense subset and randomly selecting a stronger condition than the previous ones. Some of these dense subsets – but not necessarily all of them – are conditions outside of some filter, of some closed and consistent body of information. This means that, with respect to *P*, the generic filter is a consistent and closed body of information outside of *M* that eludes being subsumed by any consistent and closed body of information inside of *M*.

The countability of the ground model also implies the countability of the number of filters. So the partial construction of the generic filter involves inspecting, in turn, each filter, and selecting a condition that avoids it. To put it more procedurally, the generic filter *G* is determined through the infinite weaving of a sequence of enquiries, a sequence of initial information fragments:

1. The first enquiry $\pi = \pi_o$ is randomly chosen.
2. All the other enquiries are constructed by inspecting, one by one, each filter and each dense subset in the ground model. When encountering a new dense subset, a new enquiry π_{n+1} is randomly chosen from among the conditions of that subset that is at least as strong as the previous enquiry π_n. When encountering a new filter, a new enquiry π_{n+1} is randomly chosen from among the conditions outside of that filter that is at least as strong as the previous enquiry π_n.

The sequential realization of the enquiries determines which conditions are collected into *G*. In Badiou's metaontology, we will learn that each enquiry will correspond ontologically to a generic enquiry, while the generic filter itself will correspond to a truth. We will also learn that the subject, or rather its trace within the ground model, corresponds ontologically to the trajectory of the enquiries.

The forcing language and the *P*-names

Having explained the notion of a poset *P* and the construction of a generic filter *G*, we can now continue to the construction of the generic extension $M[G]$ from within *M*. The idea is that we can prepare, within the ground model, the language of the generic extension. This language enables the construction of all the possible propositions that can be either veridical or erroneous within $M[G]$. We call this the forcing language, which is also grammatically based on first-order logic.

The main feature of the forcing language is that we can construct, within the ground model, the names for all the possible sets that will or will not be in the extension. The names are elements in M, but the referential value of those names, the signified sets, will be elements of $M[G]$. All names will be constructed in relation to the forcing poset P and they are thus called the P-names. Their translated referential values depend on the identity of the generic filter G and they are called the corresponding G-referents for the P-names.

(Note that the idea of a P-name here is different from what we have described in previous chapters about the singleton sets. Every element in the extension will have a proper name in the extension, which is its singleton set. The P-name is the name for that element from the viewpoint of the ground model M.)

The domain of the generic extension will be, obviously, the set of all the corresponding G-referents for all the P-names. Every sentence constructed by the forcing language can be translated into a proposition about what might be veridical in $M[G]$ after translating the individual P-names in the sentence.

The formal definition of a P-name contains a self-referential structure that might be confusing. We can clear this misunderstanding by taking the definition as analogous to the idea of a pure set. We know that a pure set can be defined as either (1) the empty set $\varnothing$ or (2) any multiple of other pure sets. The second part of this definition appears self-referential, but we can clear this up by recognizing the hierarchical architecture of every pure set: every non-empty pure set is made up of other simpler pure sets, which are themselves made of simpler pure sets, all the way down to the empty set.

The analogy is that a non-empty P-name is also made up of other simpler P-names all the way down to the empty set, which is also a P-name. The main difference with pure sets is that each element of a P-name is 'tagged' with a condition from the poset P. A P-name has a similar architectural stratification as a pure set, only that the elements are really pairings of other P-names with conditions. So we can understand a P-name as simply a relation between simpler P-names and conditions.

One last point: not all possible P-names can exist in the ground model because the multiple of all the P-names is really a proper class. With 'P-names', we are limiting ourselves to those that are constructible only by the ontological means available in M. We denote M^P as the set of all P-names in M.

The values of the P-names and the construction of the generic extension $M[G]$

The value of the P-name, its translated referent in the generic extension, depends on which conditions end up in the generic filter G. The evaluation can be said to 'sieve' the elements of a P-name from those not tagged with conditions from the generic filter. To find the G-referents of a P-name, one simply constructs the set consisting of the G-referents for only the elements tagged with conditions from G. Since those elements are also P-names, the process of evaluation continues again: one finds the G-referents for the P-names inside those elements that are tagged with conditions from G. And so on until we encounter the empty set ∅, whose G-referent is itself. The tagging of the elements of the P-name enables a sieving process that depends on which conditions are 'taken in' by the generic filter G. To translate a P-name, one has to translate the simpler P-names, which are the elements conditioned by G. In the standard notation, the G-referent of a P-name τ is usually written as τ_G or $val(\tau,G)$.

We can now define the domain of the generic extension as simply the set of the G-referents for all the P-names in the ground model: $M[G] = \{val(\tau,G), \tau \in M^P\}$.

The relation of forcing and the Forcing Theorems

Forcing is a possible relation between:

1. the conditions in P and
2. the sentences of the forcing language in M.

It is more specifically an actualized relation between:

i. the conditions from P that end up in G and
ii. the 'veridical' sentences of the forcing language, that is the sentences in the forcing language in M that, when translated, are veridical in the generic extension $M[G]$.

When this relation holds between condition p and sentence λ, then we say that p forces λ, or that λ is conditioned by p, or that $p \Vdash \lambda$. The criterion for the satisfaction of this forcing relation is as follows:

> $p \Vdash \lambda \leftrightarrow p$ being a condition in G implies that the translated version of λ is veridical in $M[G]$.

In other words, the statement 'p forces λ' means that the veracity, in $M[G]$, of the translated sentence λ is 'caused' by G containing the information corresponding to p.

Paul Cohen proved this remarkable result:

Forcing Theorem: For every veridical proposition in the generic extension, its pre-translated version is forced by some condition in the generic filter.

The implication here is that the horizon of veracity for $M[G]$ is *completely* prescribed by G. The generic filter is an 'oracle of truth' that contains all the information necessary to know what is veridical and erroneous in the generic extension. Every question asked within $M[G]$ can be correctly answered by inspecting the elements of the generic filter, by asking the 'oracle'. A proposition becomes veridical if and only if the corresponding condition is in the generic filter.

We note that each side of the forcing relation – the conditions from P and the sentences from the forcing language – are all defined in the ground model. We might think that the forcing relation is not itself definable in the ground model as it makes reference to sets G and $M[G]$, both of which exceed M itself. Nevertheless, both of these sets can be approximated by elements in M.

The remarkable fact is that forcing is equal to a relation that is *internal* to M itself, that makes no mention of generic sets at all. We can define, wholly within M, a relation that does the same job as forcing.

Local Definability of Forcing Theorem: The forcing relation can be verified wholly within the ground model.

The external version of the forcing relation is often called its 'semantic' or 'strong' version, while the inner version of this relation is called its 'syntactic' or 'weak' version. The Local Definability of Forcing Theorem states that these two relations are equal.

The details behind the weak forcing relation are complex, and so is the proof of the theorem. Badiou provides some indication in Meditation 36 and Appendices 6 and 7 from *Being and Event*. The basic idea is to provide an inductive definition for the relation of weak forcing for every possible sentence from the forcing language. The definition is given, first, for atomic sentences that state belonging or equivalence. It is also given for more complex sentences, involving the creation of new sentences using the logical connectives and logical quantifiers.

The Generic Model Theorem

The last major theorem involving the technology of forcing, the Generic Model Theorem, is more difficult to formulate, mainly because it is really several propositions in one, each playing various functions within the formal machinery of the technology. We will state only a few of them here. The main implication here is that, like the ground model M, the generic extension $M[G]$ also satisfies the property of being a universe, a consistent multiple where the *ZFC* axioms are veridical. Moreover, there is flexibility for different statements to be veridical in the extension.

The Generic Model Theorem:

1. $M[G]$ is a model of *ZFC*. Moreover, like M, $M[G]$ is also standard, countable and transitive. This means that the generic extension is also quasi-complete and its domain constitutes a quasi-complete universe. The proof to this involves justifying that each of the *ZFC* axioms are veridical in $M[G]$.
2. $M \subset M[G]$. Every element in the ground model has a 'version' in the generic extension. Moreover, $M[G]$ is the 'smallest' possible superset of M that is also a transitive model of *ZFC*.
3. $G \in M[G]$. So the generic set is one of the extra elements 'added' to M to make $M[G]$.
4. M and $M[G]$ have the same ordinals. So forcing preserves the ordinals. One implication involves the change in the structure of the cumulative hierarchy of M. In the movement to $M[G]$, this hierarchy does not get higher and the new elements are added on the 'sides' of the inverted cone, making it wider. The proof to this involves using the Countable Antichain Condition.

Say some condition forces some formula from the forcing language. We can understand this to mean the following: this condition encodes the relevant information that 'causes' the formula to be veridical in the generic extension. The generic filter encodes all the necessary information for determining anything that is veridical or false in the specific generic extension.

Here are three relevant features about the forcing relation. Let p be a condition in the poset and let φ be a formula in the forcing language:

1. p cannot force both φ and its negation $\sim\varphi$
2. p can be extended to a stronger condition q that forces either φ and $\sim\varphi$. So the information in p can be extended so as to determine the veracity of any formula

3. If p forces φ, then any condition stronger than p, any condition that contains the same information in p, must also force φ.

Even though the exact identity of G is undetermined, it still has certain features common to all generic filters. For example, we know that, being a filter, G must contain at least the empty condition $\varnothing$. So if a formula is forced by this empty condition, then we know it will be veridical in the extension regardless of the individual specifics of G. Universally forced formulas are thus provable from *ZFC*. Moreover, if $\varnothing$ forces the negation of a formula, then that formula contradicts *ZFC*. The interesting possibility is when the veracity of the formula depends on the nature of G. Some conditions force the formula, while others force its negation. If this is the case, then the formula is undecidable with respect to *ZFC*. Its veracity is contingent on the specifics of the model in question.

Two *ad hoc* features in Cohen's proof

The general technology of forcing has now been fully described. If we desire to prove that some statement is consistent with *ZFC*, then we need to design the appropriate notion of forcing P so that the statement is veridical in the corresponding generic extension. We shall see later that the very ontological process of designing a particular poset and weaving a particular generic filter links very intimately to the emergence of what Badiou conceives as the subject. Cohen can be seen as the first mathematician to apply this technology, which he discovered. The technology had not yet been generalized in his version, which was restricted to proving the consistency of ~*CH* with *ZFC* and, later, ~*AC* with *ZF*. Badiou's metaontology retains, with a slight modification, two *ad hoc* features from Cohen's proof:

1. The first feature is that ground model M is not just a model of *ZFC* but also a submodel of the constructible universe L. The Axiom of Constructibility must be veridical in M and every element must be constructible therein. This includes all the non-generic filters in M. Any new sets outside of M that are added to create $M[G]$ must be non-constructible. The generic filter G must itself be non-constructible with respect to M and remain non-constructible in $M[G]$.
2. The second feature is that, in Cohen's version of forcing, the conditions take the form of what are called 'finite partial functions'.

Finite partial functions

The details behind finite partial functions are highly technical. We begin with a simplified description and intuitive motivation for them. Now it is often the case that, for some new statement to be veridical, some specific new subset of the ground model needs to be added into the generic extension as a new element. For various reasons, this intended new subset cannot actually be the generic filter G itself, even though it is indirectly added with the help of G being adjoined into the extension. It would be too inflexible to require that the intended new subset only contain conditions from P and satisfy the restrictive structure of being generic and being a filter. Rather than being the intended subset, the generic filter G will encode complete information that says, for each 'possible' element, whether it belongs or does not belong to the intended subset. Let us call the intended subset $\overline{G}$.

In particular, the relevant conditions contain 'specifications' that are defined with respect to some infinite set S in the ground model. Each specification takes one element from S and 'tags' it with either the number 0 or 1. The result of this tagging is that each specification is a simple commitment for the element to belong or not belong to the intended new subset $\overline{G}$. A condition is simply a finite sequence of compatible specifications that tags, one by one, only a *finite* number of elements from the infinite set S. The generic filter will contain an infinite sequence of such conditions. The information contained therein will provide a complete and infinite sequence of compatible specifications where *each and every* element from S will be tagged with either 0 or 1. As an element of the generic extension, the generic filter will also add another new element to $M[G]$, namely the set of all elements from S that are tagged with 1. This will be the new subset $\overline{G}$ that we intend to add into the generic extension.

The formal details are as follows:

1. Let S be any infinite set in the ground model. The cardinality of S can be any infinite aleph number from the viewpoint of within M. However, M is countable from the viewpoint of outside itself, and so is S.
2. Each forcing condition is a schema that takes a *finite* number of elements from S and tags each element with either the number 0 or 1. (The choice for the tag names is arbitrary and can be replaced with any two elements from M). In other words, the poset P consists of all the possible functions whose domain is a *finite* subset of S and whose codomain is $\{0, 1\}$. Such functions

are what we call finite partial functions. The tagging specifications within each condition must be compatible because no element is tagged twice with both 0 and 1.

3. The partial order for the poset is reverse inclusion ⊇, while the unity element 1_p is simply the empty function ∅ with zero specifications. A condition is stronger if it implements the same tagging schema to the same elements, and more. Two conditions are compatible if the union of their specifications is also internally compatible, that is if the union is also a condition. It can be proved that this poset satisfies the Countable Antichain Condition in the ground model. And it is easy to see that this poset satisfies the Requirement of Real Choice because any condition can be extended by tagging a new element with either 0 or 1. As a finite partial function, a condition specifies two non-intersecting finite subsets of *S*, the subset of all elements tagged with 0 and the subset of all elements tagged with 1. So every condition, in a sense, selects two disjoint elements from the power set of *S*.
4. Every filter must contain the empty condition ∅. It can be proved that, given any finite or infinite collection of compatible specifications, the set of all *finite* subsets to this collection must form a filter. In fact, once *S* has been set, *all* filters in *M* take this form. So a set of conditions forms a filter when each condition is a finite subset of the same collection of compatible specifications. Moreover, every finite set of compatible conditions is the subset of some filter. Such a finite set is, more specifically, a subset of the following finite filter: the set of finite subsets to the strongest condition, that is any set of specifications given in the strongest condition. Being finite, this filter must exist in *M*. In other words, given any finite chain of conditions of increasing strength, we know that this chain is included in some existing filter in *M*, a filter that collects all the specifications within the chain, that is all the specifications in the strongest condition.
5. From the Rasiowa-Sikorski Lemma, we know that there exists a generic filter *G* with respect to the poset forcing conditions. This generic filter belongs not to the ground model but to the corresponding generic extension. The generic filter accumulates a collection of consistent specifications that, as a whole, identifies some partial function from *S* to {0, 1}. Let the function be denoted as *f*. It is precisely the union ∪ *G* of all the conditions in the filter. In fact, the domain of *f* covers the entirety of *S* and so *f* is a total function. The generic filter contains specifications that tag each and every element from

S to either 0 or 1. The proof involves the fact that genericity implies that G intersects with every dense subset $D_s = \{p \in P: s \in \textit{domain of } p\}$ for every $s \in H$.

6. It can be proved that, like the generic filter, the function f exists in the generic extension, but not in the ground model. It can also be proved that the set of elements mapped to 1 also exists in the extension but not in the ground model. This is the intended subset, which we denoted as $\bar{G}$.

So the addition of the generic filter G adds another element $\bar{G}$, which is a subset of S and, thus, of the ground model. Each condition in the generic filter is simply a finite sequence of specifications, with each specification saying whether some element from the ground model belongs or does not belong to the new subset S. The generic filter gives complete information about the structure, the belonging relation of $\bar{G}$. This information is specified at the extensional and not intensional level. Moreover, $\bar{G}$ must be infinite. Since $\bar{G}$ is not in M, and since M should contain every finite subset of S, $\bar{G}$ cannot be finite.

Note that, in this case, three new sets are added to the generic extension. They are really mirror images of one another:

1. The set $\bar{G}$, which is the intended subset that we want to adjoin to the generic extension in order to make some statement veridical.
2. The infinite total function $f = \cup G$ that specifies, for each element of S, whether it belongs or does not belong to $\bar{G}$.
3. The generic filter G, which contains an infinite sequence of compatible conditions, with each condition being a finite sequence of specifications, and with each specification saying whether some elements of S belong or do not belong to $\bar{G}$.

Now the finite partial functions involved in Cohen's consistency proof are extremely complicated and we shall not describe them here. We will note, though, that the addition of the new set $\bar{G}$ in Cohen's case also simultaneously adds an infinite number of new and individually distinct sets to the generic extension. Each new set is a distinct infinite subset of ω that was 'missing' from the ground model. The number of those new subsets can be fixed to be any cardinal number κ exceeding $\aleph_1$. As a result, the size of $P(\omega)$ is no longer $\aleph_1$ in the extension, which is a violation of the Continuum Hypothesis. The Cohen-Easton Theorem is established by the fact that the number κ of the new subsets can be almost any cardinal number from $\aleph_2$ and above. Badiou provides details in *Being and Event* (*BE*, 420–6). They do not appear to be directly relevant to his metaontology, so we will not expand on them here.

The generic set G completely specifies, for each element of infinite set S, whether it belongs or does not belong to the newly adjoined subset $\overline{G}$. The function f can be seen as a procedure that inspects, one by one, all the elements in S and specifies whether or not they belong to $\overline{G}$. In an abuse of terminology, sometimes the set $\overline{G}$ is also called the generic set. But its distinction from G and f must be noted. The set $\overline{G}$ can contain any element from S, which can be any infinite set in S. The function f contains specifications about each element from S. The generic filter contains conditions that, in turn, contain a finite sequence of specifications.

Now M is a countable and transitive ground model of *ZFC*, while S is an infinite member of M. I believe that the distinction between M and S is relevant to Badiou's ontology, although he appears to equivocate on the distinction. The set S is called the 'situation' in question, while M is also sometimes called the 'situation', and at other times, the 'fundamental situation S' or fundamental 'quasi-complete situation S' in *Being and Event*.

The technology of forcing allows us to equate M with S. The mathematics would still work if they are equal. However, in Badiou's metaontology, the set S, unlike the ground model, cannot be transitive, otherwise it would not allow for the existence of a 'site' for the eruption of what Badiou understands to be the event. This is why S cannot be equal to the ground model. We shall see that Badiou's metaontology requires that the infinite set S be historical, that it contain a totally singular element that will serve as the site for the event. In other words, the set S cannot be transitive. We shall also learn that f corresponds precisely to the operator of connection and non-connection to the event. We can say that elements mapped to 1 are connected to the event, while elements mapped to 0 are not connected to the event. The construction of the specifications in f is in tandem with the infinite weaving of the generic filter G, which is determined by the weaving of the sequence of enquiries $\pi_0, \pi_1, \ldots$, etc.

8

Metaontology of the Subject, Truth, the Event and Intervention

We begin with a summary of what we have said about forcing:

1. Cohen proved that *~CH* is consistent and undecidable with respect to *ZFC*. He accomplished this by proving that the existence of a model for *ZFC*, the ground model *M*, implies the existence of a model for *ZFC* + *~CH*, the generic extension *N*. In Cohen's proof, the Axiom of Constructibility is veridical in *M*, but not *N*.
2. In general, given any ground model of *ZFC* where certain statements hold and certain elements are presented, forcing involves enlarging it into a generic extension of *ZFC* where different statements hold and additional elements are presented. The most important of these additional elements is a particular infinite subset *G* of *M*. This is why the generic extension *N* is also written as *M*[*G*].
3. For forcing to work, the ground model must be reduced to a smaller countable transitive model of *ZFC*. Such a reduction is performed using the Mostowski Collapse Theorem and the Reflection Theorems.
4. The set *G* is constructed by taking an infinite number of elements from a poset *P* within the ground model. The poset contains forcing conditions, and each condition in *G* approximates it by revealing some partial information about it. Conditions are partially ordered according to reverse strength, according to the information contained. A pair of conditions can reveal compatible or incompatible information. For forcing to work, one of those conditions must be the weakest element with the least information, usually denoted as the condition 1_P. The poset must also satisfy the Countable Antichain Condition and the Requirement of Real Choice.
5. The set *G* must satisfy a particular structure: being a generic filter with respect to the poset and the ground model. To be a filter, *G* must be a closed

and consistent body of information. It must be a non-empty set of mutually compatible conditions such that it contains every condition weaker than any of its members. To be generic, *G* must intersect with every dense set of conditions in the ground model, and a set of conditions is dense if every condition from the poset is never stronger than some condition from the dense set. The Indiscernibility Theorem ensures that a generic filter can exist only outside of the ground model. The Rasiowa-Sikorski Lemma ensures that an external generic filter can always be constructed.

6. The generic extension has a domain specifying an ontological commitment, a language specifying the construction of propositions, and a horizon of veracity specifying which propositions are veridical and erroneous. Each of the three is definable from within the ground model. The domain of the generic extension contains all the elements from the ground model, plus the additional generic filter *G*, plus a minimal number of additional elements to ensure that the extension is consistent and satisfies *ZFC*.
7. The language of the extension is constructed using a forcing language in the ground model, which is grammatically based on first-order logic. Every sentence from this forcing language translates into a proposition in the extension. The names used by the forcing language, which are constructed with respect to the poset of forcing conditions *P*, are called the *P*-names.
8. If a set contains other sets, which themselves contain other sets, all the way down to the void, then a *P*-name similarly contains other *P*-names all the way down to the void, with the only difference being that every element of a *P*-name is tagged with a forcing condition. Each *P*-name in the ground model points to an element in the generic extension, an element whose full identity depends on the identity of the generic filter and the conditions it takes. The translated *G*-referent of a *P*-name is the set containing the *G*-referents for all its *P*-names that are tagged only with conditions from the generic filter *G*. The generic extension is the set of translated referents for all the *P*-names, evaluated with respect to the generic filter *G*.
9. Every sentence constructed using the forcing language translates, with respect to the generic filter, into a proposition that is either veridical or erroneous in the generic extension. Forcing is a possible relation between the forcing conditions and the sentences from the forcing language. This relation holds when the following is satisfied: if the condition is a member of *G,* then the sentence, when translated, is veridical within the extension.

10. The Forcing Theorem ensures that, for every veridical statement in extension, its pre-translated version is forced by some condition in *G*. The generic filter *G* is thus an 'oracle' containing all the necessary information about the horizon of veracity corresponding to the generic extension. Even though *G* lies outside *M*, it can be proved that the forcing relation itself is definable in *M* as it is equivalent to a relation that is internal to the immanence of *M*. This is known as the Local Definability of Forcing Theorem.
11. The Generic Model Theorem ensures, among other things, that the extension is a model of *ZFC*, that it includes the reduced ground model, and that it contains *G*. The theorem also ensures that the extension and the reduced ground model have the same ordinal numbers.
12. By designing the appropriate conditions, the method of forcing can be used to prove the consistency of certain statements with respect to *ZFC*. Badiou's metaontology will make use of Cohen's employment of finite partial functions as the forcing conditions, which are defined in relation to an infinite set *S* in the ground model *M*. In addition to the generic filter *G*, forcing in this case will add a new set $\overline{G}$, which is an infinite subset of *S* that was missing from *M* and from *P*(*S*) in *M*.
13. A finite partial function is a finite sequence of compatible specifications, with each specification saying whether some element of *S* will belong or not belong to $\overline{G}$. The ensuing generic filter *G* will contain conditions that collect together a complete set of specifications saying, for every element of *S*, whether it will belong or will not belong to $\overline{G}$. Those complete specifications will form a function *f*, which will also be a new set added to the generic extension.

Forcing from within *M* and *S*

Now the process of forcing with finite partial functions can be understood entirely from the viewpoint of its domestic deployment within the ground model *M* in relation to the infinite set *S*. The mathematics behind this domestic deployment is central to Badiou's conditioning of his metaontology, so we need to present it explicitly. There are two stages: (1) The 'Ontological' Phase, involving the fixing of the ground model *M* and the infinite set *S* and (2) The 'Militant' Phase, involving the weaving of the specific set $\overline{G}$ and its 'twins' *G* and *f*.

The Ontological Phase (Figure 8.1):

Everything begins when an 'inhabitant' has been thrown into some infinite situation S within some countable and transitive model M of ZFC.

Remember that the Axiom of Constructibility is veridical in the ground model and the domain of M consists only of constructible sets. So the set S is also constructible, contains only constructible sets, and satisfies the Axiom of Constructibility. Since M is countable, so is S. We can thus recast S as an infinite sequence of its elements. But the specific ordering of S depends on the realization of the generic filter that will be constructed in the Militant Phase. We will see that G will be constructed within S by encountering, one by one, the elements of S. The sequence through which these elements are encountered is not pre-determined and totally aleatory. Nevertheless, without loss of generality, let us simply denote the ordering as $S = (s_0, s_1, s_2, \ldots)$. Since the power set axiom is veridical in M, S must have a power set and a regime of representation. Since the Axiom of Constructibility is veridical in M, then, by the result of Gödel, so is the Continuum Hypothesis. So the cardinality of $P(S)$ must be $\aleph_1$ in the ground model.

Three new sets, namely the generic set G, the function f and the generic filter $\overline{G}$, will be added to M to determine the generic extension N. The three sets are really different facets of one another and so, for simplicity, and without loss of generality, we can centre our discussion on $\overline{G}$, which is an infinite subset of S that is not already an element of M.

The existence of the generic set $\overline{G}$ can only be imagined from within the ground model. An 'inhabitant' of M can only have faith that $\overline{G}$ exists, without having knowledge of its exact identity. The necessity of this faith is relevant because it supports the imagined genericity of G. With the fixing of the ground model and the set S, and with this faith in the existence of $\overline{G}$, the relevant poset P of forcing conditions is automatically defined. These conditions will take the form of finite partial functions that contain a finite sequence of compatible specifications. With the construction of the forcing conditions, the relevant filters and dense subsets in M will also be automatically defined, as well as the P-names and the forcing language. The generic filter G will also be defined, though not constructed. We repeat the details schematically:

1. A specification is a minimal or atomic report that says whether some element x of S is tagged with either 1 or 0. Such a specification is a commitment that the element will belong or not belong to the imagined set $\overline{G}$. So the minimal reports take the form $x \in \overline{G}$ or $x \notin \overline{G}$ for some $x \in S$.

2. A condition is simply any finite sequence of specifications. The sequence must also be internally consistent in the sense that it cannot contain incompatible specifications – that is, it cannot have a pair of specifications where one says $x \in \overline{G}$ and another says $x \notin \overline{G}$ for the same element $x \in S$. One condition is stronger than another if it has the same specifications and more. Two conditions are compatible if the union of their specifications is internally consistent, if there are no incompatible specifications – which is equivalent to saying that there is a third condition stronger than both of them.
3. The *P*-names and their corresponding forcing language are defined in relation to the set of conditions *P*. The relevant filters and dense sets in *M* are also defined. Remember that a filter is simply the set of all finite subsets to some finite or infinite set of compatible specification. If *C* is a finite sequence of compatible conditions, that is, if *C* is a finite chain containing conditions of increasing strength, then *C* belongs, among others, to the finite filter containing the finite subsets of the specifications in the strongest condition in the chain.
4. Since *M* is countable, the set of all filters and all dense sets is also countable and forms two infinite sequences, F_0, F_1, …, etc. and D_0, D_1, …, etc., respectively. The two sequences are connected because the complement of every filter must be a dense set.
5. A generic filter would be any set of conditions that intersects every dense set and avoids every filter in *M*. We know that no generic filter can exist within *M* itself, because of the Indiscernibility Theorem. However, it can be proved to exist outside of *M*, because of the Rasiowa-Sikorski Lemma.

The Militant Phase (Figure 8.2):

The second phase involves the finite constructions, within the ground model *M*, of the subset $\overline{G}$, as well as its twins, the function *f* and generic filter *G*. Since $\overline{G}$ does not belong to *P*(*S*) within *M*, its construction is never completed but is only partially realized as an infinite chain of conditions of increasing strength. These conditions correspond to finite initial segments that are finite subsets of $\overline{G}$.

Remember that, even though $\overline{G}$ is absent from *M*, these finite subsets still exist in *M* because the ground model, due to the Axiom of Pairing, contains every possible finite set. Since $\overline{G}$ is a subset of *S*, each of its finite initial segments exists in the power set of the situation-set *S*. Moreover, every segment, since it is a finite chain of compatible conditions, is included in a non-generic filter in *M*.

So the partial construction of $\bar{G}$ can be seen as the construction of an infinite sequence of finite initial segments of this missing subset. Each segment exists in M as an element of $P(S)$. The formation of each new segment adds new elements from S to $\bar{G}$. The complete construction of this infinite sequence is never fully realized within M, but every finite moment of its construction is actualized within M. And each finite moment can be subsumed under some non-generic filter in M. The intention of this sequence is to provide a complete determination to the structure, the belonging relation, of the imaginary set $\bar{G}$.

For example, suppose the first few elements of $\bar{G}$ are committed to be s_0, s_2, s_3, s_4, s_9, s_{13}, s_{14} and s_{28}. The trajectory of finite initial segments might take the following form:

$$\bar{G}\text{: } \{s_0\}, \{s_0, s_2, s_3\}, \{s_0, s_2, s_3, s_4, s_9, s_{13}\}, \{s_0, s_2, s_3, s_4, s_9, s_{13}, s_{14}, s_{28}\}, \ldots$$

Now, in reality, something more complex is at work in the background. The trajectory of the segments is defined by the sequence of enquiries π_0, π_1, …, etc. constructed via the algorithm given by the Rasiowa-Sikorski Lemma. Each enquiry π_i is some forcing condition from P, some finite sequence of specifications involving a finite initial segment of the set $S = (s_0, s_1, s_2, \ldots)$. Each enquiry is a partial examination of sequence of elements in S and commits whether each of them belongs or does not belong to $\bar{G}$. The elements that are committed to belong to $\bar{G}$ form the corresponding finite initial segment of $\bar{G}$. We can think of each enquiry as a finite instantaneous moment in the complete inspection of every element in the set S, specifying whether or not the element belongs or does not belong to $\bar{G}$. The trajectory of enquiries forms an expanding set of information, with each new enquiry adding new specifications on later elements of S. The difference between the enquiry and the finite initial segment is that the former also specifies elements from S that will not belong to $\bar{G}$. To illustrate this, let us consider $\{s_0, s_2, s_3, s_4, s_9, s_{13}\}$, the third finite initial segment of $\bar{G}$ given in the earlier example. This segment corresponds to the enquiry constituting the following sequence of specifications:

$$\begin{gathered} f\text{: } (s_0 \in \bar{G}, s_1 \notin \bar{G}, s_2 \in \bar{G}, s_3 \in \bar{G}, \\ s_4 \in \bar{G}, s_5 \notin \bar{G}, s_6 \notin \bar{G}, s_7 \notin \bar{G}, s_8 \notin \bar{G}, \\ s_9 \in \bar{G}, s_{10} \notin \bar{G}, s_{11} \notin \bar{G}, s_{12} \notin \bar{G}, s_{13} \in \bar{G}). \end{gathered}$$

This condition inspects the first fourteen elements of S and says that only s_0, s_2, s_3, s_4, s_9 and s_{13} belong to $\bar{G}$.

At the imaginary completion of the infinite construction of these enquiries, we will have a function f that will specify, for every element in S, whether it will belong or not belong to $\overline{G}$. Like $\overline{G}$, the existence of this function can only be imagined from within M. The inhabitant, who has faith in this function, can only commit to the completion of its construction. The generic filter G is constructed by collecting all enquiries plus the conditions weaker than them. At every instantaneous moment, new conditions are added to G by taking in all the conditions weaker than the currently realized enquiry, all the finite subsets of information from the most current π_i. The generic filter G is, at any finite moment, the collection of finite information from the most current enquiry.

Now the construction of the sequence of enquiries π_0, π_1, …, etc. obeys the aleatory algorithm dictated by the Rasiowa-Sikorski Lemma. We use the qualifier 'aleatory' because some randomness is still allowed in the algorithm, and so ontology does not completely prescribe how G will turn out. The algorithm involves inspecting and randomly taking stronger conditions from the infinite sequence of dense subsets D_0, D_1, …, etc. and avoiding conditions from the infinite sequence of non-generic filters F_0, F_1, …, etc. in M.

It is worth repeating the details from the previous chapter. The first enquiry π_0 is randomly chosen from the forcing conditions in P. The ensuing enquiries are chosen by encountering, one by one, every dense subset and randomly selecting a new condition from them, provided that it is not weaker than the previous enquiries, that is it contains the same specifications, and possibly more. Now the complement of every filter in M is also a dense subset. So the construction of the sequence of enquiries involves encountering, one by one, every filter in M and selecting any new condition that does not belong to it. The enquiries avoid every closed and consistent collection of conditions from M.

The generic extension and its horizon of truth are also partially constructed in tandem with the partial collection of the enquiries and the conditions into G. The construction of the extension involves the evaluation of all the P-names. Each P-name is evaluated by evaluating, in turn, all its elements that are paired with conditions from G. In order to know whether some sentence in the forcing language will be veridical in the generic extension, an inhabitant of M only needs to check which particular conditions would force its truth and whether this condition belongs to the generic filter G. This condition is identified using the forcing relation, an identification that we know can be done wholly from within the ground model. In order to know if some proposition is veridical in $M[G]$, we identify, using the internal forcing relation, the relevant information, the relevant finite set of specifications encoded in a condition within G. We then inspect G to see how this information is realized or, rather, we commit to the truth or falsehood of the proposition as we are committing to a particular realization of the relevant condition in G. Both the horizon of truth and the structure of G are constructed in tandem with each other through a single trajectory of commitment.

At every finite moment of this trajectory, certain elements from the situation-set are made to belong and not belong to G, certain sets are inserted into $M[G]$, and certain truths hold for the extension. The generic filter has a metaphorical affinity to the 'input tape' of a computational machine. This tape is 'read' in order to determine the distinguishing features of the new universe and the construction. The construction of the new universe, in turn, determines the information given in the input tape. It would also be tentatively helpful to understand the generic filter as analogous to a 'virus' that penetrates through the 'cellular' wall of M and then manipulates its impending actions and propagations.[1]

However, these images break down because both the 'input tape' and the 'virus' do not enter into the 'machine' or 'cell' from a transcendental outside but are constructed, part by part, within them. Moreover, this is 'enough' for them to operate. It is enough to know only a finite initial segment of G, only partial information about $\overline{G}$ and f, in order to determine whether a specific condition belongs to it. To know if the condition is in G, the inhabitant only needs to proceed with the construction of the sequence of enquiries up to a certain finite point. The inhabitant does not need to construct all of $\overline{G}$ in order to determine the horizon of truth. To deploy the Law of Forcing, the inhabitant only needs to check and commit to whether the relevant information has already been given by the enquiries that have been constructed so far. This checking is executed in tandem with determining the truth of a proposition from the forcing language.

Forcing from within *M* and *S* (summary)

Our explication of the domestic deployment of forcing cannot avoid being technical. But here is a summary:

1. The inhabitant is thrown into the infinite situations of S within the quasi-complete situation M.
2. The specifications, conditions, filters and dense subsets are defined in M, and so is the forcing language.

3. Something 'happens' and the inhabitant assumes a faith in some new and imaginary $\overline{G} \subseteq S$.
4. $\overline{G}$ is constructed part by part within M and S. The inhabitant inspects each member of S and decides whether or not it will belong to $\overline{G}$. These elements are inspected through an infinite and never-to-be-completed trajectory that is entirely aleatory.
5. At each moment in the trajectory, the inhabitant has a finite and constantly expanding sequence of specifications of belonging and not-belonging with respect to the imaginary $\overline{G}$. Each finite sequence, each finite moment in the trajectory, is called an enquiry. Each enquiry corresponds to a finite set of elements decided to belong to $\overline{G}$ and another finite set of elements decided not to belong to $\overline{G}$.
6. The trajectory of $\overline{G}$ is constructed in tandem with the trajectory of constructing $M[G]$ and deciding the veridicy or erroneousness of every sentence in the forcing language in $M[G]$ using the forcing relation.
7. Every enquiry collects a finite and compatible set of specifications. The set of all finite subsets to those specifications forms a filter in M. So every enquiry is included in some filter in M. But $\overline{G}$ is constructed so as to contain at least one specification outside of every filter in M and at least one specification inside every dense subset in M.

Initial hints of Badiou's metaontology of forcing

Many of the further structural details behind Badiou's metaphysics can already be gleaned by first understanding the general mathematical framework behind forcing in its domestic deployment, and then by tracing its contingent line of conditioning towards the metaontological framework. We can already begin with the following almost trivial observations:

1. the ground model M and the infinite set S specify the ontological form-multiples relevant to some default state-of-affairs
2. the generic extension $M[G]$ specifies the ontological form-multiple to some new emergent universe
3. the sets $\overline{G}$, f and G provide the ontological means to think and encode the new information about the new universe
4. the forcing relation, the forcing language and P-names provide the ontological means for the old universe to relate to the domain, the language and the horizon of veracity corresponding to the new universe

To a certain extent, these observations are correct and the rest of this chapter will be concerned with fleshing everything out.

First we bear in mind that there is one obvious way to make use of forcing as a condition for philosophical thought: by treating the ground model and generic extension as environments linked by modal relations of 'accessibility'. In particular, M and $M[G]$ can be thought as mathematical figures for ontological state of affairs or for fragments within some ontological state of affairs. $M[G]$ is understood to be modally 'accessible' from M. The ground model can be thought of as our current state-of-affairs and the extension can be thought as a possible future state of affairs proceeding from our current universe.

For example, M could be the ontological state of affairs corresponding to our current universe at this moment in time, and $M[G]$ could be the ontological state of affairs for a universe where Michelle Obama has been elected as the president of the United States of America in 2024. The ontology of N is an expansion of the ontology of M since it contains new entities and new facts through the passage of time because of the election of the second President Obama. With this connection in mind, we can employ various technical details behind the mathematical relation of forcing to help achieve a new philosophical conception about the connection between accessible environments. To a certain extent, such a conception would be related to Saul Kripke's employment of forcing in his celebrated formulation of possible world semantics.[2] The modal environments would be possible worlds, and forcing would be connected to the simple actualization of possibility in a modal sense. The movement of the forcing relation would be contemporaneous to the motion of simple temporality, the temporal conversion of the possible into the actual.

Still, my main point is that, in general, it is easy to see how forcing would help us form a metaphysical system for understanding the developments of new notions and new truths over time. I think that it would have been nearly impossible for Badiou not to be informed of such a prospect, first, because it is so palpable for anyone who understands the mathematics of forcing and, second, because the relation between forcing and Kripkean semantics is so well known. The mathematics of forcing shows that, if they exist at all, then there can never be just one unique model of *ZFC*. As suggested by the mathematician Joel David Hamkins (2011), this would imply a 'multiverse' view of set theory and, thus, a 'multiverse' view of ontology.

Badiou's genius – which should be recognized and celebrated – was his discovery of a substantially different, though related, way of employing the mathematics of forcing for philosophical thought. For him, the ground model M is still the form-multiple for a current state of affairs, though one localized to some world-fragment, the situation S. The generic extension $M[G]$ will be an imaginary and 'utopian' universe, a situation-to-come, that is constructed using the ontological resources available in the ground model. Such a construction is never fully completed but is realized through an infinite trajectory that is the sequential production of finite fragments to the utopian universe of $M[G]$. This sequential production is accomplished in tandem with the construction of the sets G, f and $\overline{G}$, which are also built, finite fragment by finite fragment, through a trajectory of their finite initial segments.

In a brilliant move, Badiou recognized that the construction of this infinite trajectory is supported by none other than a pure subject understood as a faithful militant, by an 'inhabitant' of S who has become faithful to the infinite procedure of realizing G, f and $\overline{G}$ in their imaginary existence. Moving beyond the simple understanding of forcing and genericity, Badiou formulates a different orientation of thought, the generic orientation supported by a subject and truth.[3] The ontological trace of the pure subject equates to the infinite aleatory trajectory of constructing G, f and $\overline{G}$.

Badiou will more audaciously equate the subject with its faithful militancy. A subject, in Badiou's philosophy, is re-defined as what supports the infinite construction of some generic filter with respect to some situation S, as what constructs an utopian universe. We thus have the possibility of conceiving a new metaphysics of subjectivity and truth. The sets G and $\overline{G}$, in their infinitely complete forms, will correspond to a truth, a new regime of truth. The horizon of veracity for the to-be-completed $M[G]$ is supported by the to-be-completed G and $\overline{G}$. Truth supplants veridicy. What holds for the new universe is not based on the verification of judgements, but on the faith sustaining the following through of the trajectory, which is the subject of truth. Moreover, the relation of emergence from M to $M[G]$ is not simply the actualization of a simple possibility through the accessible passage of time. The construction of the generic extension properly constitutes a rupture and is instigated by a rupture. This is what Badiou calls an event. In particular, Badiou will link the faithfulness of a militant subject to the fidelity towards the name of an event that has erupted within the situation S. We thus have the possibility for a new metaphysics of time and of modality.

Badiou's schema of metaontological translation

A great many of the notions and terms that appear in the standard terminology of forcing are renamed, re-conceptualized and reconfigured by Badiou in *Being and Event* so as to fit with his philosophical framework and to distinguish metaontological from ontological concepts. Much can be gained if we begin, first, by attempting to rebuild Badiou's 'translation schema' from the ontology to the metaontology and, second, by examining how everything is then reconstructed into a suitable and sufficiently robust philosophical narrative. Our assumption is that any reader sufficiently informed about the mathematics of forcing can quickly learn a great deal about the metaontology, about acclimatizing himself or herself more easily in the new conceptual landscape, if he or she is supplied with some of the details involved in the translation schema. The conditioning relation between, on one hand, the mathematics of forcing and, on the other hand, Badiou's metaontological commentary of it implies some structural isomorphism between the two sides. We are entitled to complain about the fact that, save in the official 'Dictionary' at the end of *Being and Event* and in a few scattered offhand mentions in the text, Badiou often omits to provide most of the details behind his schema. We can, however, say the following:

1. The infinite set S corresponds to the initial situation in question. Since the situation is essentially linked to its state, it is really the matrix of S plus its state, its semantics. Badiou's metaontology requires that this situation be historical – that is, it must contain a non-empty, foundational and totally singular element. So S cannot be a transitive set.
2. The countable and transitive ground model M of *ZFC* corresponds to what Badiou calls the 'fundamental situation' of S or the 'fundamental quasi-complete situation' of S. This fundamental situation contains the smaller situation S. The Axiom of Constructibility is veridical for M and so all the situations in M, which includes S and all the sets in S, must be constructible. Badiou often equivocates on the distinction between the situation S and the fundamental situation S, but the distinction is important because M, unlike S, is transitive.
3. Badiou also often equivocates on the generic filter G and its corresponding generic set $\overline{G}$. Both are called the 'generic set' and marked with the Venus symbol ♀. All of these correspond to the form-multiple of truth. The

function *f*, or its infinite weaving, corresponds to a 'faithful procedure', a 'procedure of fidelity' or a 'generic procedure'.

4. The symbol $S(♀)$, depending on the context, can be understood to mean either the generic extension $M[G]$ or to what happens, in that extension, to the matrix of situation S and its corresponding state $P(S)$. Now the domain of $M[G]$ includes the domain of M, plus the generic filter and the additional elements produced in the interaction between G and the elements of M. The domain of S does not change in the extension. What changes is its regime of representation where new subsets, corresponding to $\overline{G}$, are added.
5. A minimal specification, an atomic commitment of the form $x \in \overline{G}$ or $x \notin \overline{G}$ for some $x \in S$, corresponds to a minimal or atomic enquiry. If $x \in \overline{G}$, then it is said that x has been positively investigated, and if $x \notin \overline{G}$ then it is said that x has been negatively investigated. These are also written as $x(+)$ and $x(-)$, respectively.
6. A forcing condition, that is, a finite partial function, corresponds to a finite series of compatible minimal enquiries. The notion of forcing, which is the poset of conditions P within the ground model, corresponds to what Badiou marks as ©. A filter of conditions corresponds to a correct subset of conditions, while a dense subset D of conditions corresponds to what Badiou calls a domination.
7. The enquiries π_0, π_1, ..., etc. are also called enquiries and, sometimes, generic enquiries when they have been recruited to construct a generic filter. Each enquiry is a finite instantaneous Being of a trace to the fidelity.
8. A P-name τ corresponds to a ©-Name μ. In the mathematical notation, the evaluation of this P-name, with respect to the generic filter G, is written as $val(\tau, G)$ or τ_G. In Badiou's metaontological notation, the referential value of the ©-Name μ with respect to the truth ♀, the value of the ♀-Referent, is written as $R_♀(\mu)$.
9. The forcing language corresponds to the subject-language and the relation of forcing $\Vdash$ corresponds to what Badiou writes as the relation $\equiv$.

In this forcing schema, Badiou equivocates between:

i. the 'situation *S*' and the 'fundamental quasi-complete situation *S*' (i.e. the ground model *M*) within which it is embedded
ii. using the symbol $S(♀)$ to mean the entire generic extension $M[G]$ and using it to mean the version of S within $M[G]$ and its new state

Both the M and $M[G]$ are countable and transitive universes that satisfy *ZFC*. M contains S, while $M[G]$ contains the version of S after forcing has been completely implemented. Badiou appears synecdochally to mix, first, M with S and, second, $M[G]$ with the version of S in $M[G]$.

Our resolution of this ambiguity relies on various hypotheses that we have already made while reading *Being and Event*:

1. By the concept of situation, Badiou refers not only to some domain but also to its semantics, its state with respect to the quasi-complete situation within which it locates itself. So the situation is the matrix of the situation and the state.
2. The state refers not only to the power set of the situation (the semantics given by formulas in one variable) but also to any higher-order power set involving the situation itself. If S is the situation, then the state refers to all the formulas with one or more variables whose parameters and variables range only over S itself. So the state contains not just $P(S)$ but also, for example, $P(P(S))$, $P(S^2)$ and $P(P(P(S^6))$.
3. What Badiou calls the 'situation S' refers to or is conditioned by the infinite domain of the finite partial functions involved in forcing.
4. The operations that function within S reduce to the belonging relation of set theory.

Forcing is always locally implemented with reference to the elements that range over situation S. The process operates only on those sets directly connected to S within the ground model and within the generic extension. These sets include S itself and its corresponding matrix of states. The forcing procedure never goes outside of the domestic immanence of S itself or the various states connected to S. Forcing only concerns itself with those sets within M and $M[G]$ that are directly connected to S. This explains why Badiou sometimes equivocates on the situation and the quasi-complete situation containing it. When supporting the operation of forcing, the subject never leaves the immanence of the situation S and does not enter into the general universe of M and $M[G]$ outside of the situation. Forcing is never implemented in a language outside of the situation. This is why, for all intents, S is 'effectively' equal to M and S in $M[G]$ is effectively equal to $M[G]$ itself. Anything else that requires the subject to move outside of S, like the construction of the generic filter G, is supported by faith.

So the basic translation schema, from the mathematics to the metaontology of forcing, has been provided. In the same way that a perfect memorization of a dictionary will prove insufficient in gaining proficiency of a new language, even a complete translation table of Badiou's vocabulary cannot serve as the sole source of instruction in his philosophy of the event. The following sections will be devoted towards filling this gap. Even after everything from the mathematics has been translated, the task of reconstructing everything into a robust philosophical framework remains.

Moreover, Badiou investigates concepts that have no complete correspondence in the ontology. The two most crucial of these concepts are the event and the subject, both of which Badiou understands to be in excess of ontology itself by connecting to what is not Being-qua-Being. However, both the event and the subject figure into the metaontology by partaking in the presentative form of rigorously defined mathematical figures: there is a matheme of the event and the subject appears as that which supports the trajectory of forcing conditions and generic enquiries that are woven into the generic filter. Moreover, what is outstanding and unique about Badiou's metaontology involves what happens *before* forcing enters into the picture and what his theory says about the aleatory construction of the generic filter via a truth procedure of fidelity that is woven in the aftermath of an event.

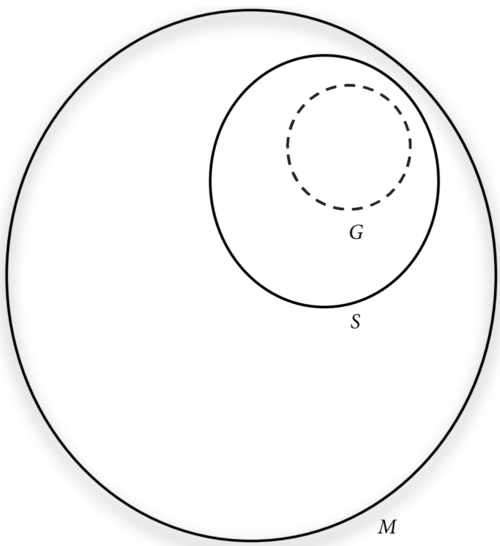

Figure 8.1 The ground model M (before forcing): The infinite situation S is a presented subset of the ground model M, while the set G is an unpresented subset of S and M.

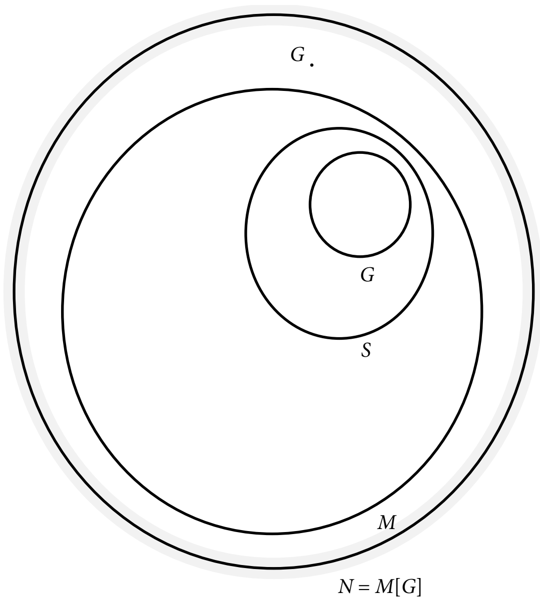

Figure 8.2 The generic extension N (after forcing): The ground model M has been expanded to become a part of the generic extension N and the subset G has become a presented element in the newly added region.

Pure versus empirical philosophy of ontology

Before continuing to our analysis of the event and the subject, let us first provide one crucial remark. Enlisting set theory as a condition for thinking the event is an extraordinarily peculiar, novel and non-trivial choice on Badiou's part. This is a mathematical field that is usually identified as part of what has traditionally been called 'pure' or 'theoretical' mathematics, which stands apart from 'applied' mathematics and from the mathematical methods used in the empirical extrapolations in the quantitative sciences. The task of describing the precise distinction between pure and applied mathematics is difficult, contentious and, perhaps, misguided. However, *prima facie*, we expect that many mathematicians and mathematical scientists who would be interested in theorizing about ruptures, breaks or revolutions would more likely attempt a more empirical and quantitative methodology. The concept of the event would have no meaning without making explicit or implicit reference to its empirical, though undecidable, manifestations in the concrete world. A philosopher might be justified in examining carefully the data relevant to one hundred concrete examples and case studies of events, searching for some patterns or commonalities, and then constructing an appropriate model or formulating a suitable hypothesis. Along the way, the philosopher might use mathematical and scientific tools – for instance, from statistical data analysis, partial differential equations, actuarial science, differential geometry or mathematical physics. A mathematically minded philosopher with a taste for broad abstractions might also be tempted to explore the general mathematics involved in the manifestation of an event. If a rupture occurs within some spatio-temporal system, and if a general equation can be formulated that describes some of the algebraic structure, dynamic or geometry of that rupture, then the philosopher might investigate whatever general forms are involved in the solution to that equation. For example, the mathematical philosopher can go further by investigating the general mathematics behind the study of temporal systems and differential equations.[4]

With the employment of applied mathematics, we see an alternative conditioning at work when it comes to the question of thinking the event. I would say that, compared to using the abstract mathematics of set theory, the application of empirical and quantitative methodologies from the natural and social sciences has, by their very strategic intention, a more effective, predictive and even descriptive power of understanding the event on the concrete level of local and particular manifestations. Despite the essential alterity, undecidability and subtractive character of the event, I would say that Badiou's metaontological formulations in *Being and Event* are not directly meant to compare with what is known by a mathematical economist about financial meltdowns, by a management scientist about innovation, by a political scientist about people's uprisings, by a physicist about the spontaneous breaking of physical symmetry, or by a specialist in nonlinear dynamics about the transition between chaos and order. Badiou's philosophy was never meant to be a theoretical abstraction of the event's various specific manifestations in particular concrete situations. His aim is a general metaontology, not a science, of the event. Even though it is conditioned by science, the activity of the philosopher must be held separate from it. It is worth repeating that ontology does not deal with specific physical and contingent presentations, but is subtracted from them – unless, of course, developments in the study of physics or chemistry have ramifications for what happens in pure mathematics itself.

As Badiou writes in his *Manifesto for Philosophy*, 'Science, *qua* science, grasped in its truth procedure is, moreover, profoundly *useless*, save that it avers thought as such in an unconditioned way' (*MP*, 54). This subtraction from the empirical, perhaps, indicates a weakness to Badiou's metaontological framework. Or perhaps it indicates what Badiou has already pointed out to be an essential limit of ontology itself as a discourse of Being-qua-Being and not qua-beings. Metaontology should not pretend to do more than what falls within its purview. Perhaps one can trace this strangeness of ontology-qua-ontology to the abstractive or, more precisely, subtractive dimension of mathematical thought. Researchers in pure mathematics, like those in the humanities, often struggle to justify the claim that their work is intrinsically meaningful outside of its possible practical connections to technology and the applied sciences. There would be no future for Badiou's mathematical philosophy of ontology, no future for the truth procedure following from the equation 'mathematics = ontology', without some further affirmation of the intrinsic meaningfulness of pure mathematics and of intrinsic ontology as such, beyond their presentative manifestations as concrete entities or their possible technological employment.

The 'trigger' and the 'dynamic'

Say we have an infinite situation S, linked to its state $P(S)$ and embedded within some fundamental quasi-complete situation M that is countable and transitive. The technology of forcing provides the platform, the ontological infrastructure, for a new generic set ♀ – that is, G and $\overline{G}$ – to be added to S(♀) and adjoined into the power set $P(S)$. The mathematics behind the technology only proves the consistency between ontology and the infinite construction of the generic extension. Being-qua-Being allows the new universe to transpire, but without necessitating its emergence. Veridical statements, though never contradictory with respect to ontology, are either provable or undecidable in the generic extension.

For this technology to be realized under some particular implementation, something extra-ontological must set it off and then sustain the aleatory dynamic development, which is the weaving of the generic filter, the construction of the generic extension, and the following through with the new horizon of truth. In particular, two new entities must impose themselves on M and S at the onset:

1. a 'trigger' instigating the enforcement of the technology
2. a 'dynamic' steering and compelling the continued trajectory of its operation.

The trigger and the dynamic, which are connected to each other, will define the specifics regarding the generic filter, the generic extension and the new horizon of truth. Both of them operate within the space of implementation allowed by the technology, but without being part of the technology itself. So the trigger and the dynamic must remain, to a certain extent, extra-ontological. To use an image (which, I admit, will aid us only to a certain extent), the technology can be thought as an automobile, with the trigger being the intention to reach some destination and the dynamic being the driver who manoeuvres the vehicle throughout its journey. In Badiou's metaontology, the role of this trigger is played by what he calls the event, while the role of the dynamic is played by what he calls the subject.

We choose to discuss the subject first because the relevant details are closer to what is given by the mathematics of forcing. The event, on the other hand, involves a different philosophical framework altogether. We must, however, stress that the event chronologically precedes the subject, which is born, as we shall see, through the nomination and recognition of the existence of the event-qua-event.

Much of what Badiou says about the event and the subject can be perplexing, partly because of the fact that his theorizations necessarily occupy the obscure intersection between ontology and philosophical thinking. We read the relevant later meditations in *Being and Event* and detect a certain hesitation and tentativeness in Badiou's usually confident and almost stoic language. The event and the subject are properly extra-ontological entities and so Badiou's conceptual framework cannot be entirely metaontological, cannot be entirely conditioned by mathematics. His aim in the later meditations is to think the other of ontology from the viewpoint of ontology itself. We are in a sense attempting to reconstruct a metaontology of the non-ontological, to ontologize That-Which-Is-Not-Being-Qua-Being.

In that case, Badiou's philosophy of the subject and the event cannot find its complete conditional basis in the technology of forcing. In order to understand his philosophy of non-ontology, we should also trace its line from the internal condition – the history and archive of philosophy – and from the other conditions – politics, art, love and science. However, in keeping with the restricted purview of this book, we only discuss how this non-ontology connects to the mathematics. But it is worth mentioning, although only in passing, that Badiou's philosophy of the subject and the event is also strongly conditioned by what he considers to be a new and specific paradigm of thought. In the Introduction to *Being and Event*, he observes and accepts a contemporary post-Cartesian understanding that has unfolded through the various schools following from certain philosophical, scientific, political and clinical developments under the names of, among others, Heidegger, Marx, Lenin, Freud, Lacan and various analytic philosophers (*BE*, 1–4). Maintaining itself in the pure void of its subtraction, and ex-centered from the place of transparency, the subject is now taken to be ir-reflexive, cleaved, a-substantial and dehumanized, and any analysis of it must be conducted in the context of certain rigorous and formal processes. A new doctrine of truth follows from the dissolution of its organic relation to adequation, correspondence, verified knowledge, the object, and subjective pronouncements. According to Badiou, every new conceptual apparatus must be consistent with this new epoch in our understanding of the subject and truth.

Correct subsets, knowledge and the encyclopaedia

Let us continue to Badiou's metaontology of the subject by returning to the infinite situation S and its state $P(S)$. We repeat and expand on Badiou's remarks about predication, knowledge and the encyclopaedia. The situation S corresponds to a count, a flat plane of presentation, while the state corresponds to a re-count, a regime of re-presentation. The members of S are presentations, while the members of $P(S)$ are representations, which are semantically liaised with predicates quantified over the members of S. Let $\varphi(x)$ be any formula, written in the first-order language of set theory, with its only free variable being x and all its other terms being elements of the situation. Then the collection of all the values of x from the situation satisfying this formula forms a representation. The function φ separates all the elements of the situation and forms an element of the state. The power set, in other words, is the inventory of all the subsets that could result from the operation of language, from the collecting together of all elements from the situation that satisfy a predicate whose other terms also range over the elements of the situation. To be precise, language predicatively involves two operations:

1. the operation of discernment, which discerns whether or not an element from the situation satisfies a predicate
2. the operation of classification, which collects together all the elements from the situation that satisfy a predicate.

The result of classification is semantically liaised with a representation. All the existing representations of a given situation are collected into the power set. This power set need not collect 'all' the possible subsets, just the ones that already exist in the larger fundamental situation. We are guaranteed the existence of each finite subset in the state, because of the Axiom of Pairing. We also know, because of the Axiom of Separation, that each subset separated by a predicate also exists in the state. The power set specifies the limit to the predicative liaisons of language with respect to the particular set. Any missing subset must be un-presentable, indiscernible and ineffable. The presentation of a subset in the power set provides the possible predicative suture of that subset of language to various possible formulas. This is why Badiou links the power set to the situation's regime of representation, and why the subsets form the ontological schema for representation.

Remember that the Axiom of Constructibility is veridical in the fundamental situation, which means that the power set collects *only* constructible representations, only subsets separated by predicates. If a subset is 'missing' from the state then that subset must be infinite and unconstructible. This absent subset must contain an infinite number of elements and its structure cannot be liaised with any predicate constructed by the first-order language of set theory whose variables and terms range over the elements of the situation. With reference to a situation in the constructible universe, language provides the means for predicatively discerning and separating subsets from the terrain of its power set. These predications are also linked to the operation of knowledge because they establish a variety of epistemic mappings with respect to the situation. An encyclopaedic determinant is a collection of such epistemic mappings that are operative at a given time. A situation might be subject to different encyclopaedic determinants and a subset might fall under a determinant if it is liaised with one of its predicates. Every subset, every representation in the state, is a subject of knowledge and falls under the encyclopaedic dimension of knowledge.

Under the nominalist ontology, knowledge subsumed Being: that which is not the object of encyclopaedic classifications is not. Unlike in the general ontology of *ZFC*, where multiplicity precedes language, the nominalist ontology draws the guarantee of Being for those multiples whose presentation is consented by the state. The predicative liaisons of representations are subjected to the operations of language, knowledge and various encyclopaedic determinants. Any 'missing' representation, any subset that is absent from $P(S)$, is infinite, unconstructible and subtracted from the language, knowledge and the various encyclopaedic determinants linked to the situation.

The generic set and knowledge

We recall that the generic set ♀ is one of those missing subsets. An inhabitant, who has faith in ♀, inspects the elements of S one by one and commits to whether that element belongs or does not belong to ♀. Given an encountered $x \in S$, an atomic enquiry (a specification) takes the form of either $x \in$ ♀ or $x \notin$ ♀, respectively – or, to use Badiou's notation, $x(+)$ or $x(-)$, respectively. A condition is a finite set of compatible atomic enquiries.

We will discuss this 'faith' when we talk about the event, so let us continue by recalling two points that we previously made about non-generic correct subsets (non-generic filters) in M:

1. A correct subset collects all conditions that are finite correct subsets to a finite or infinite set of compatible atomic enquiries. If F is a correct subset, then there exists a collection Q of compatible commitments such that $x \in F$ if and only if x is finite and $x \subseteq Q$.
2. Given any finite set of compatible atomic enquiries, we can be sure that those enquiries are collected into some correct subset in M. In particular, they are collected into the finite correct subset containing all subsets containing those commitments. So if π is a finite collection of compatible commitments, then π is a subset of the correct subset containing all the subsets of specification in π.

The atomic enquiries within a correct subset specify two sets, those specified to belong to ♀ and those specified not to belong to ♀. Each of these two sets is a subset of S. These two subsets belong to M and $P(S)$ if the correct subset belongs to M. They all certainly belong to M if they are finite. Any non-generic correct subset, any correct subset in M, identifies two existing subsets of S in $P(S)$. This means that any correct subset is subject to the operations of knowledge and the encyclopaedia. It is always possible to place any correct subset in M under some epistemic mapping and encyclopaedic determinant. Being outside of M, and specifying two subsets of S outside of M, the generic correct subset ♀ is guaranteed to be outside the operations of knowledge and cannot be subsumed by any encyclopaedia. A compatible set of possibly infinite specifications discerns two epistemically and linguistically controlled multiples. The corollary is that any set of consistent conditions, any collection of compatible specifications, about the belonging relation of ♀ can always be said to be subject to the operations of knowledge. A set of conditions avoids a correct subset if there are specifications in those conditions that are either compatible with those in the correct subset or involve elements not specified by the correct subset.

Supporting the trajectory of constructing ♀

Whenever the Law of Forcing is locally enforced within a situation, there we have the subject. The question of the subject is the question of circulating within the ontological architecture that the mathematics has prepared. The trace of the subject is precisely the empirical implementation and adaptation of the technology behind forcing at the ontological level through the eternal construction of enquiries. The Being of truth ♀ is constructed, part by part, within M and S through this infinite trajectory of enquiries. This constructed trajectory is supported by something outside of M, a pure subject, which is the result of the inhabitant militating on behalf of its faith in the truth of ♀. We will see that this faith is born out of the interventive nomination of some event. For the moment, we sketch the metaontological justification for the necessity of such a militant subjectivity, for the existence of a fidelity. The word 'faith' has already been mentioned several times, and the corresponding mathematics must be presented more explicitly.

Ontology, whose formal axiomatic is *ZFC*, cannot paradoxically account for everything that is the case about what is. The arrival of the subject is needed to fill these gaps left by ontology. The proven existence of statements that are independent of mathematics – such as the Continuum Hypothesis, the statement '*ZFC* is consistent', and so on – shows that not everything can be established by the a priori conditions of ontology. Moreover, as long as we accept 'mathematics = ontology', every consistent formal axiomatic system for ontology must admit the existence of independent statements – this was the lesson of Gödel's Incompleteness Theorems. Since undecidable statements are unavoidable with respect to Being, the only way to continue would be for their truth or falsehood to be waged as a militant gesture, for the undecidable to be decided with the support, not of Being, but of the commitment by a pure subjectivity with only the chasm of faith compelling its dynamic. A subject is only possible with the purity of a fidelity without any foundation from Being, not even from language or from knowledge. The subject only has the fidelity that defines it.

We mention in passing that the more erudite among us might detect remnants of this 'deciding the undecidable' in the work by some of Badiou's philosophical forebears such as Jean-Paul Sartre, Jacques Lacan and Jacques Derrida, whose later philosophy of the pure decision could be read as being conditioned by Gödel's mathematics of undecidability. Badiou goes further by examining forcing, which is a later mathematical development that provides an elaboration of what could be at work behind the demonstrable undecidability of statements with respect to *ZFC*. Under a new metaontological register, Badiou investigates the technicalities involved in Paul Cohen's forcing, particularly in the construction of new generic extensions where new statements hold true via the act of pure decision.

We have to return once more to the construction of the generic set ♀ that is immanently deployed within the ground model, the infinite situation *S* and its relevant regimes of representation. The aleatory algorithm given by the Rasiowa-Sikorski Lemma ensures that the domestic weaving of the generic sets follows an arbitrary trajectory. The selection of the first enquiry π_0 involves pure chance – it can be any condition containing a finite set of atomic enquiries about the belonging relation of the generic set. The ensuing enquiries are also randomly selected, so long as they belong to a domination (dense set), avoid a non-generic correct subset (non-generic filter), and add to the atomic enquiries from their preceding enquiries. The subject encounters, one by one, each element from *S* and decides to insert or not insert it into ♀ without obeying any rule determined from the *M* or, more specifically, from *S* and its relevant regimes of representation and relation. The fact that the complete structure of ♀ is not dictated by anything in *M* ensures that we can have undecidable statements in *S*(♀).

We need to illustrate more clearly the construction of ♀ because the details are quite intricate. Suppose that the elements in *S* are encountered and examined as the sequence s_1, s_2, s_3, …, and so on. Remember the order is not pre-determined and depends entirely on the random and eternal traversing of the inhabitant within *S*. Table 8.1 might specify the details to the early parts for one generic trajectory of ♀.

Each enquiry is simply a string of belonging and non-belonging statements. For simplicity, we can write each enquiry as a finite string of 0s and 1s, with 0 meaning that a particular newly encountered element is committed not to belong to the generic set, and 1 if otherwise. For example, using the enquiries in the Table 8.1: $\pi_0 = 01$, $\pi_1 = 011$, $\pi_2 = 011011$, and $\pi_3 = 01101100$. The generic set ♀

Table 8.1 Example of the first four generic enquiries

Generic enquiry	Elements of enquiry	Current partial value of $\bar{G}$	Newly added elements into $\bar{G}$	Current conditions in G
π_0	$s_1 \notin \bar{G}, s_2 \in \bar{G}$	$\{s_2\}$	s_2	any of the 4 subsets of the specifications in π_0, i.e. $\varnothing$, $\{s_1 \notin \bar{G}\}$, $\{s_2 \in \bar{G}\}$ and $\{s_1 \notin \bar{G}, s_2 \in \bar{G}\}$
π_1	$s_1 \notin \bar{G}$, $s_2 \in \bar{G}$, $s_3 \in \bar{G}$	$\{s_2, s_3\}$	s_3	any of the 8 subsets of the specifications in π_1, i.e. $\varnothing$, $\{s_1 \notin \bar{G}\}$, $\{s_2 \in \bar{G}\}$, $\{s_1 \notin \bar{G}, s_2 \in \bar{G}\}$, $\{s_3 \in \bar{G}\}$, $\{s_1 \notin \bar{G}, s_3 \in \bar{G}\}$, $\{s_2 \in \bar{G}, s_3 \in \bar{G}\}$, and $\{s_1 \notin \bar{G}, s_2 \in \bar{G}, s_3 \in \bar{G}\}$
π_2	$s_1 \notin \bar{G}, s_2 \in \bar{G}, s_3 \in \bar{G}, s_4 \notin \bar{G}, s_5 \in \bar{G}, s_6 \in \bar{G}$	$\{s_2, s_3, s_5, s_6\}$	s_5 and s_6	any of the 64 subsets of the specifications in π_2
π_3	$s_1 \notin \bar{G}, s_2 \in \bar{G}, s_3 \in \bar{G}, s_4 \notin \bar{G}, s_5 \in \bar{G}, s_6 \in \bar{G}, s_7 \notin \bar{G}, s_8 \notin \bar{G}$	$\{s_2, s_3, s_5, s_6\}$	none	any of the 64 subsets of the specifications in π_3
⋮	⋮	⋮	⋮	⋮

can be given as an infinite string of 0s and 1s. At each moment in the infinite construction of ♀, the subject decides whether to add a new 0 and 1, whether to append or not append a newly encountered element from *S*. The trajectory of ♀ is a path down the infinite binary Cantor decision tree whose first five levels are given in Figure 8.3.

Cantor's technique of diagonalization

The local construction of ♀ must be executed so that the completed correct subset satisfies the requirement of genericity by intersecting every domination and avoiding every other correct subset related to the situation *S*. We say ♀ diagonalizes through the correct subsets. The weaving of the generic filter can be understood as a more sophisticated version of what was given in Georg Cantor's

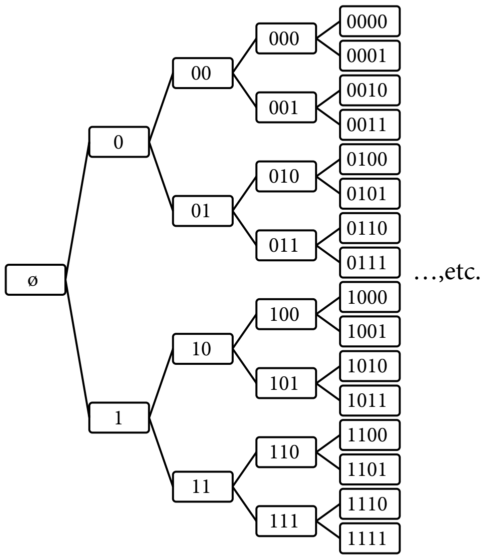

Figure 8.3 The Cantor decision tree.

proof involving his celebrated method of diagonalization[5] (a method whose essential principle does not really involve spatial 'diagonals' at all).

Cantor's proof demonstrates the uncountability of the power set $P(\omega)$, corresponding to all sets of natural numbers. We have already mentioned this result in an earlier chapter, but have postponed demonstrating it until now, although only the specific method of diagonalization shall concern us here.

For the task at hand, it would helpful to understand the proof as demonstrating the uncountability of a different set: the set of all infinite strings of 0s and 1s. It can be shown that every infinite string has a one-to-one correspondence with a unique set of natural numbers. If the digit at the nth position of the string is 0, then the natural number n does not belong to the corresponding set, while if the digit is 1, then n belongs to the corresponding set. So, for example, the string 10101010101010… etc. corresponds to the set of all even natural numbers {0,2,4,6,8, …}.

Theorem: Let c be the set of all infinite strings consisting of 0s and 1s. Then c is uncountable.

Proof: We prove by contradiction. Suppose c is countable. Then the infinite strings can be arranged as a single infinite sequence. We denote this sequence as the following matrix:

1st infinite string:	$c(1,1)\ c(1,2)\ c(1,3)\ c(1,4) \ldots$
2nd infinite string:	$c(2,1)\ c(2,2)\ c(2,3)\ c(2,4) \ldots$
3rd infinite string:	$c(3,1)\ c(3,2)\ c(3,3)\ c(3,4) \ldots$
4th infinite string:	$c(4,1)\ c(4,2)\ c(4,3)\ c(4,4) \ldots$
⋮	⋮

where $c(n, m)$ denotes the value – either 1 or 0 – of the *m*th digit for the nth infinite string.

We define the following new infinite string *g* using the following rule for determining its digits. We examine the infinite sequence of digits from the diagonal in the matrix above: $c(1,1)$, $c(2,2)$, c(3,3), c(4,4), etc. The *k*th digit of *g* is defined by taking the opposite value to the *k*th element $c(k,k)$ in this diagonal. If $c(k,k)$ is 0, then the *k*th digit of g is 1, and vice versa. So the infinite string *g* will be different from every infinite string in *c*. In particular, *g* differs from the *k*th infinite string in the matrix at the *k*th digit. But *c* is defined to be the set of *all* infinite strings of 0s and 1s. This is a contradiction of the supposition that *c* is countable, which proves the theorem.

Aleatory diagonalization by the subject

We can generalize the technology given by this proof by removing two unnecessary requirements:

1. We do not have to fix the cardinality of *c* as countably infinite. We can simply fix *c* to be any size ω_0, ω_1, ω_2, and so on (in fact, so long as it does not exceed ω_{ω_0}, i.e. the limit of ω_0, ω_1, ω_2, … etc.). As a result, the size of *c* can always, subject to certain conditions, be made to exceed whatever infinite cardinal. We thus have an intuitive justification for the Cohen-Easton Theorem on the errant excess of the size of $P(\omega)$.
2. We do not have to require that *g* be defined by differing specifically at the diagonal digits from the matrix. We can simply define *g* as the string that differs from every infinite string from *c* at some undetermined position. We can defer choosing the position of that differing digit up to a later indefinite point as we traverse down the matrix. So the weaving of the digits of *g* follows an aleatory trajectory. It is always possible at some possible future time to stop and choose the digit from which to differ. We have 'all the time

in the world' to choose the position. As Zachary Luke Fraser, the Badiou scholar, writes in his paper 'The Law of the Subject':

> It is therefore impossible to decide, based on empirical evidence, whether any procedure is or is not generic. Strictly speaking, the truthfulness of a procedure does not disclose itself in extensionally determinate evidence; it can be testified to only in the interiority of the sequence, with respect to its projected intension. Any declaration concerning the existence of a truth must, therefore, always remain hypothetical and anticipatory, without the hope of sufficient evidence ever arriving. (2006, 125)

When encountering a new infinite string, g can always be set to differ from it at some unspecified position. So the exteriority of g remains potential at every point of its construction. The following, for example, could occur: the first one hundred digits of g consists only of 1, meaning that it could end up containing only 1s, meaning that it could be equal to the identifiable string 11111111111… etc. But it is always possible at some point, which can be postponed indefinitely, for one of its digits to be set to 0. The string g is defined so that, for every infinite string, g will differ from it at some point. That is, g will avoid being subsumed by every infinite string in the sequence c. The exteriority of g is based on the projection towards its completion and exteriority, in the dynamic of the subjective 'faith' driving its trajectory. The existence of g is always anticipatory and the essence of this subjectivity is its freedom to choose differing digits.

The weaving of the generic set ♀ is simply a more sophisticated version of this idea, distilled to its technical essence. We replace every infinite string in c with the non-generic correct subsets in M. We replace the infinite string g with the expanding sequence of atomic enquiries pursuing the generic set ♀. The generic set itself is defined as distinct from every non-generic correct subset at some unspecified point.

Suppose the subject has executed all the enquiries up to π_n. The specifications collected in π_n can always be subsumed under the non-generic correct subset containing all finite subsets of those specifications. A non-generic correct subset is the product of the operations of language, knowledge and encyclopaedic determinants. So it can always be said that π_n was constructed, not by the subject, but by existing knowledge. Moreover, it can always be said that the next enquiry π_{n+1}, whatever it may be, was also constructed by existing knowledge, since it can also be subsumed under some non-generic correct subset in M. In fact, however ♀ is realized, its infinite string of specifications, if known explicitly, can always be subsumed by an infinite correct subset in *M*.

The construction of ♀ can always choose to avoid such correct subsets by differing from each of them at some indefinite point. This is the very definition of genericity. Even if ♀ matches with some correct subset at the first 1,000,000 enquiries, the point is that the subject, whose trajectory is independent of the situation, can always stop and choose differing specifications at the 1,000,001th enquiry. Badiou writes:

> [T]he faithful procedure is random, and in no way predetermined by knowledge. The multiples encountered by the procedure do not depend upon any knowledge. . . . There is no reason, in any case, for an enquiry not to exist which is such that the multiples positively evaluated therein by the operator of faithful connection form a finite part which avoids a determinant; the reason being that an enquiry, in itself, has nothing to do with any determinant whatsoever. (*BE*, 337)

The genericity of ♀ is always potential. The freedom, exteriority and non-constructibility of its genericity is supported by the essential freedom of the subject that is necessarily outside of *M* and *S* since it must be independent of the linguistic, epistemic and encyclopaedic operations therein. For ♀ to be generic, it is necessary that there be a pure subject that decides without the support of knowledge. In the construction of ♀ within the *M* and *S*, it is always possible to say that this construction follows or will follow some rule of knowledge. However, the defining feature of the subject is this freedom to avoid being subsumed by knowledge.

The subject versus the state

Let us examine more carefully the opposition between the aleatory fidelity of the truth procedure and the state's regime of representation. The Axiom of Pairing ensures that every finite representation is presented by the state, but not every infinite representation. Knowledge discerns and classifies every finite part, every encyclopaedic determinant realized as a finite multiple. The only way for truth to be distinguished from the veridical is for it to be infinite. For truth to be indiscernible and unclassifiable by the encyclopaedia, the one-multiple of truth must be infinite. So the atomic enquiries in ♀ must contain an infinite number of belonging and non-belonging relations.

Even though fidelity does not group together terms according to the rule of knowledge, knowledge itself knows nothing of this fact. It can always be said that any finite grouping by fidelity was actually a grouping executed by knowledge. At every finite path down the Cantor decision tree, it can always be said that the trajectory had followed some operation of knowledge. The results of faithful enquiry necessarily coincide with the result of the encyclopaedic classification. As Badiou writes:

> Here we have the paradox of a multiple – the finite result of an enquiry – which is random, subtracted from all knowledge, and which weaves a diagonal to the situation, yet which is already part of the encyclopaedia's repertory. It is as though knowledge has the power to efface the event in its supposed effects, counted as one by a fidelity; it trumps the fidelity with a peremptory 'already-counted!'. (*BE*, 333)

At each moment of its realization, each enquiry corresponds to a finite subset, an element of the state of the situation. Moreover, each enquiry is subsumed by a non-generic correct subset in *M*, particularly the correct subset collecting all the finite subsets to the atomic enquiries in the enquiry. Being a finite multiple, the enquiry must be a part of the situation, since the state always counts all the finite parts. So the result of the enquiry coincides with the encyclopaedic determinant. It can always be said that the result of enquiries cannot be radically subtracted from the nominations provided by the state. As a result, 'fidelity operates in a certain sense on the terrain of the state of the situation. A fidelity can appear, according to the nature of its operations, like a counter-state, or a sub-state' (*BE*, 233). Even though fidelity is not presented in the situation, each of the finite moments of its vector are definable by the fundamental situation. At any moment, the finite initial fragments allow themselves to be ontologically projected as a finite part. But the non-ontological operation of subjective fidelity is, at most, parenthetical to the situation.

Every local finite instance of the generic procedure, being finite, is thus an object of knowledge because it is an identifiable part of the encyclopaedia. But knowledge cannot anticipate how they will turn out. Knowledge cannot know anything of the subject that lies in between the terms in the infinite thread of truth, whose entire being takes the form of a random trajectory. In the constructivist and encyclopaedic ontological orientation, knowledge never encounters anything without pre-encountering it beforehand. In the generic orientation, the subject encounters the terms of the situation without anything prescribing such a term. Its incalculable trajectory does not fall under any determinant of the encyclopaedia. Veridical statements are controlled by knowledge, but statements of truth are controlled by the procedure of fidelity. Just like the operation of discernment in knowledge, an enquiry discerns whether a multiple satisfies the predicate of being connected or not connected to the name of the event. The difference is that a subject is always a dissident subject, and thought is always a dissident act. To be more precise, the subject lies properly at the intersection of knowledge and truth. It is 'a knowledge suspended by a truth whose finite moment it is' (*BE*, 406). The subject produces a truth whose infinity transcends it.

The being of truth, the subject-language and the generic extension *S*(♀)

Following the event, the process of forcing begins with the inscription of singular nominations that bring into play the additional signifier of the truth ♀. The subject intervenes in the gap between what is sayable of Being (the definable concept of genericity) and the non-Being from which it originates (the generic set itself).

The process of truth, the construction of the generic set, is never fully realized in its becoming. Even though the subject seeks to attain the status of a complete oracle, it always has the un-nameability of the infinite ♀ as a limit. Being a subset of *S* that was missing from the state, the generic set ♀ is non-constructible from the viewpoint of both the situation and the generic extension. Eluding every predicate with respect to *S* and *M*, truth is indiscernible. Moreover, the generic set is positively defined via the positively defined concept of genericity ('intersects every dense set') and not the indirect negative concept of indiscernibility. Genericity implies indiscernibility – the latter characterized only as an exception.

The aleatory weaving of ♀ is simultaneous with the commitment to a horizon of truth and the erection of the generic extension *S*(♀) as a situation whose consistency is to come. The subject specifies the formalization of the subject-language (forcing language), which is syntactically based on the language of first-order set theory and contains the ©-names (*P*-names), whose referents lie outside the situation and do not repeat the established language. The marks of these names are assembled wholly within the situation by combining, reworking and redirecting existing elements within the situation. The identities of the hypothetical referents of the names depend on the identity of ♀. The ♀-referents will have been presented in the future anterior within the generic extension. Containing signifiers without signification, the statements in the subject-language are, from the viewpoint of the situation, devoid of meaning. The complete correspondences of the ♀-referents are suspended from the unfinished construction of truth.

On its own, ontology cannot formalize the subject, which begins, as we shall see, at the borders of an event outside of Being and perpetuates itself via chance encounters. Ontology can think the Law of the Subject only via the formulation of the internal relation of forcing, without the birth and dynamic of the subject contradicting ontology itself. The subject supports the displacement of the horizon of veracity with the horizon of truth encoded by the generic set. The generic procedure is an infinite work of truth and the subject supports the finite approximations of truth. Remember that the relation of forcing, which is internal and verifiable from within the fundamental situation M, is thus determined by the encyclopaedic determinants of knowledge. The Law of the Subject says that it can be known, in a situation where a truth is being woven, whether a statement of the subject-language has a chance of being true in the generic extension.

The domain of the generic extension forms a situation not just for the basic rules of ontology but also for new statements, such as the negation of the Continuum Hypothesis, that are undecidable with respect to ontology. As a model, the generic extension specifies a new interpretation of the undecidable statements, a new 'world' of entities where the compossibility of new truths is collectively seized in a novel space of veridicy and a new regime of representation, a new semantics. To construct an extension is to invent a new consistent interpretation of ontology along with new truths. Moreover, the subject pursues the possibility of truth not on a representational but on a directly ontological level. The generic set lies exterior to the situation while simultaneously inscribing the truth and the very Being of the situation's totality. The construction of the generic set is attached directly to its belonging relation, to its specific count, without recourse to a predicate, to the operations of the state, language and knowledge. The generic set exhibits 'as one-multiple the very Being of what belongs insofar as it belongs' by referring to 'to being-in-situation as such' and not 'what language carves out therein as recognizable particularities' (*BE*, 339). Truth therefore touches upon the very Being of the situation and is closest to the initial state of things. Badiou writes:

> An indiscernible *inclusion* – and such, in short, is a truth – has no other 'property' than that of referring to *belonging*. This part is anonymously that which has no

> other mark apart from arising from presentation, apart from being composed of terms which have nothing in common that could be remarked, save belonging to *this* situation which, strictly speaking, is its [B]eing, qua [B]eing. But as for this 'property' – [*B*]*eing*, quite simply – it is clear that it is shared by *all* the terms of the situation, and that it is coexistent with every part which groups together terms. Consequently, the indiscernible part, by definition, solely possesses the 'properties' of any part whatsoever. (*BE*, 338–9)

The generic set has no predicate other than its respective structure, the predicate of belonging to itself. The only thing shared by its elements is the fact that they belong to ♀, which can only be qualified by saying that its elements are. The structure of ♀ therefore touches on the very structure of the *S* itself 'since in a situation "[B]eing" and "being-counted-as-one-in-the-situation" are one and the same thing' (*BE*, 340). A constructed part, which is the object of knowledge, refers not to the structure of the situation as such, but to what has already been prepared by language. The faithful procedure, however, re-links back to the 'truth' of the entire situation.

And yet the generic, by being exterior to the situation, is essentially independent of Being and of ontology. Unlike in Heideggerian philosophy, truth is no longer intimately equated with Being. 'The sayable of [B]eing is disjunct from the sayable of truth' (*BE*, 355), even though both are compatible with each other. Moreover, truth remains indiscernible to the constructing subject, which only inhabits a local and finite configuration of the generic set in the situation. Truth is infinite, but the subject is finite. The subject can only encounter terms within the situation and only has, at its disposal, the language of the situation and the name of the event.

The evental site and the matheme of the event

The subject supports the local construction of a truth procedure. In order to escape being subsumed by language and by knowledge, it must also properly lie outside of ontology. The question now is how to think this exteriority.

We begin with a summary. The birth of the subject constitutes an intervention with respect to the situation *S* and fundamental situation *M*. This intervention begins with the nomination and recognition of an event as evental. It perpetuates its subsequent trajectory by being faithful to the event's name. The founding of the subject is co-extensive with its decision that such and such exists as an event, while the aleatory and exterior dynamic of the subject is co-extensive with the faith in following through the implications of the event's nomination. The eruption of the event unleashes the discipline of time that is the trajectory of the faithful procedure. In particular, an element of the situation is decided to belong to the generic set ♀ when it is connected to the name of the event. The event itself, which must have actually occurred independently and prior to its nomination, takes the ontological structure of a self-belonging multiple. This form-multiple links to a matheme, which is defined with respect to a pre-existing and totally singular element in the situation called the evental site. To schematize the chronology:

1. First, the preparatory existence of the evental site within the situation
2. Then, the eruption of the evental multiple, taking the ontological form of a self-belonging matheme with respect to the evental site
3. Then, the nomination and recognition of the existence of the event as event
4. Then, the following through with the faith in the name of the event and the eternal construction of the truth ♀.

An event and a subject must be independent with respect to Being. In keeping with Badiou's philosophical orientation towards immanent materialism, the event and the subject cannot, however, be absolutely transcendental with respect to the situation. They must be both external and internal to *S*. The event will be localized with respect to some evental site. This site satisfies the requirement of interiority by being an element in *S*, and fulfils the requirement of exteriority by being foundational with respect to *S*. In other words, the site must be a totally singular element of the situation, and so the situation itself must be historical, not natural or neutral. Moreover, even though the event is properly outside of the situation and outside of ontology, it will leave a trace in the form of a multiple. This evental multiple will take the form of a matheme that contains the same elements as the evental site as well as 'interposing itself within itself'.

We examine Badiou's relevant remarks in Parts IV and V of *Being and Event*. As his first point of departure, he meditates directly on the question of understanding what lies external to the grasp of ontology. How can we think That-Which-is-Not-Being-qua-Being? He rejects the possibility of hastily equating it with the simple negation that is non-Being and, instead, offers the event as the appropriate candidate for 'the first concept external to the field of mathematical ontology' (*BE*, 184). Badiou names the evental multiple as the trace of the event in the relevant situation *S*. This multiple is what, from the viewpoint of the situation, is offered as the event. Such a multiple can erupt only if it is localized within the situation by a site. With respect to the situation, the site must take the form of a totally singular multiple, a non-empty foundational element. Hence, the situation must be historical. Using Badiou's notation, we denote this evental site as X (note the capitalization of this letter).

As with his investigation of nature and natural multiples, Badiou begins his analysis by borrowing and disavowing some themes and ideas from Martin Heidegger, who distinguished That-Which-is-Not-Being-qua-Being as that whose opening forth is set to work solely by art and whose appearance is confirmed and made accessible as a nothing. Badiou only retains Heidegger's proposition that the site of That-Which-is-Not-Being-qua-Being, corresponding to the site X of the event, lies external to nature and can never be a normal multiple. So the place of the otherwise-than-Being must be the anti-natural and the abnormal. It cannot submit to the stability of nature and the transitivity of ordinals. We know that the opposites of nature and normality are history and singularity, respectively. The form-multiple of history lies within the instability of singular multiples. This being-presented-but-not-represented is the limit that is beyond the grasp of the metastructure, the point of subtraction from the state's re-securing of the count. The site is this totally singular multiple that is the localization of the event within the situation S. None of the elements of X are presented in the situation S. A site must, thus, be foundational and lie on the edge of the void with respect to the situation, without actually being the empty set. Badiou writes:

> One could call it a primal-one of the situation; a multiple 'admitted' into the count without having to result from 'previous' counts. It is in this sense that one can say that in regard to structure, it is an undecomposable term. It follows that evental sites block the infinite regression of combinations of multiples. Since they are on the edge of the void, one cannot think the underside of their presented-[B]eing. It is therefore correct to say that sites found the situation because they are the absolutely primary terms therein; they interrupt questioning according to combinatory origin. (*BE*, 175)

In order to have a totally singular element, the situation S must be historical and not neutral or natural. The evental site is the point where the historicity of S is concentrated.

Note that this requirement of historicity sets the situation apart from the general requirements imposed by the technology of forcing with finite partial functions. In the mathematics of forcing, the infinite set S need not contain non-empty edge-of-the-void elements. Forcing can still operate if S is natural, contains only normal elements and is founded by the empty set. In fact, the simplest implementation of forcing in mathematics involves setting S to be the most minimal infinite set possible, the ordinal $\omega = \{0,1,2, \ldots\}$. And we know that this ordinal is not historical as all its elements, the natural numbers or finite ordinals, are also subsets. In the case of $S = \omega = \mathbb{N} = \{0,1,2, \ldots\}$, forcing adds a new infinite subset $\overline{G}$ of natural numbers. Only in Badiou's case is S required to be historical as he takes forcing as being instigated by an event. The situation must present a totally singular element as the evental site.

To avoid any confusion, we note that the evental site X, which is singular, must also be distinguished from the set $\overline{G}$ linked to the generic filter G. The site is an existing element of the historical situation S. None of the elements of X are elements of S, and so X is certainly not a member of the state $P(S)$. The set $\overline{G}$ also does not belong to the state. It also does not belong to the situation S or to the fundamental situation M. But it is added to the version of the state in the generic extension.

We also remark that singularity is not an absolute but a relative concept. The property of being-evental is always made with respect to the immanence of the historical situation. On a global level, history does not exist. A multiple can be singular in one situation but normal in another if all its terms are presented. Moreover, many non-transitive situations, as proved by the Mostowski Collapse Theorem, can be converted into transitive situations. A singularity can be normalized and an evental site can always undergo a state normalization. So every historical situation can be converted into a non-historical situation.

Despite their relationship, the evental multiple and the evental site are separate, with the latter being the localization of the former.[6] The event is localized to what is already there in the historical situation. However, the presentation of a totally singular multiple is not a sufficient condition for there to be an event in S. A site is evental only anterior to the irruption of the event. '[T]he existence of a multiple on the edge of the void merely opens up the possibility of an event. It is always possible that no event actually occur. Strictly speaking, a site is only "evental" insofar as it is retroactively qualified as such by the occurrence of an event' (*BE*, 179). If the eventual site is a presented multiple containing non-presented elements, what then, if the event occurs, is the evental multiple? Say we have an evental site X. This site is an element of the historical situation S, but none of its elements belong to that situation. Localized by this site, what form-multiple would be taken by the trace of the event?

'The event is not actually internal to the analytic of the multiple. Even though it can always be *localized* within presentation, it is not, as such, presented, nor is it presentable. It is – not being – supernumerary' (*BE*, 178). Badiou offers a conceptual construction corresponding to a formula, the mathematical structure for the trace of the rupture. He proposes a curious, though oftentimes confusing, matheme for this event, which he denotes as e_x. The letter '*e*' in this symbol obviously stands for the word 'event' and for the French '*événement*', and we must observe that the subscript in this symbol is the small letter x, not the capital letter X referring to the evental site. We will explain later that x refers to one of the unpresented elements of X, an element that becomes sutured to the name of the event e_x. The small x appears in Badiou's matheme for e_x, written as:

$$e_x = \{x \in X, e_x\}.$$

The notation used here is quite unlike the standard way that sets are intensionally defined. From the way this matheme is written, the formula could be confused to mean, among others: $e_x = \{x \in X: e_x\}$, $e_x = \{x \in X \vee x \in e_x\}$ or $e_x = \{x \in X \wedge x \in e_x\}$. Nevertheless, Badiou assists us by writing that the event of the site X is 'a multiple such that it is composed of on the one hand, elements of the site, and on the other hand, itself' (*BE*, 179). He elucidates this with his statement that the event is 'a one-multiple made up of, on the one hand, *all*[7] the multiples which belong to its site, and on the other hand, the event itself' (*BE*, 77).

This means that the event of site X is, in Badiou's definition, simply the set X and with the added element of the event itself. In other words, the event contains itself as an extra element, self-appended to the set X. We can rewrite this as:

$$e_x = X \cup \{e_x\}.$$

This leads to a cascading nest of self-belonging relations:

$$\begin{gathered} e_x = X \cup \{X \cup \{e_x\}\} \\ e_x = X \cup \{X \cup \{X \cup \{e_x\}\}\} \\ e_x = X \cup \{X \cup \{X \cup \{X \cup \{e_x\}\}\}\} \\ \vdots \\ e_x = X \cup \{X \cup \{X \cup \{X \cup \{X \cup \{X \cup \{X \cup \{X \cup \{X \cup \{X \ldots .\}\}\}\}\} \end{gathered}$$

The event contains all the elements of the site, plus an extra element that also contains all the elements of the site, plus an extra element that also contains all the elements of the site, and so forth. Badiou illustrates one intuitive justification for this matheme with the empirical example of the French Revolution, with its corresponding evental site being France between 1789 and 1794. The event must obviously contain everything already contained by the site. 'The historian ends up including in the event "the French Revolution" everything delivered by the epoch as traces and facts' (*BE*, 180). As a result, the site constitutes an essential subset of the event as a multiple. But it cannot be equal to the event itself, otherwise the latter would constitute a catalogue of the site's elements to the point of being only a collection of the gestures, inscriptions and traces relevant to itself. The halting point for this series of cataloguing would be the point when the event becomes a term within itself, where, in its irruption, it self-recognizes and becomes self-conscious of itself as a multiple. In his specific exposition of the French Revolution, Badiou writes that:

> [t]he halting point for this dissemination is the mode in which the Revolution is a central term of the Revolution itself; that is, the manner in which the conscience of the times-and the retroactive intervention of our own-filters the entire site through the one of its evental qualification. . . . Of the French Revolution as event it must be said that it both presents the infinite multiple of the sequence of facts situated between 1789 and 1794, and, moreover, that it presents itself as an immanent resume and one-mark of its own multiple. (*BE*, 180)

One noteworthy feature of this matheme of the event is that it does not make reference to the language of spatial or temporal discontinuity. For Badiou, the eventality of an event lies not in its attachment to some inflection, jump, gap or tear in the continuity of space-time. The site admits an event because of an essential disorder in the structure of its multiplicity, to a torsion within the configuration of its Being and not its spatio-temporality. Badiou's matheme describes the ontology of the event solely in set-theoretic and not geometric terms.

Exceeding the count of the site, the event is said to be 'supernumerary' with respect to the elements of the site. The event, in its presentation as a multiple, erupts when the multiple of the site becomes 'self-aware' and incorporates its own count into itself. In the supernumerary gesture, the event, collecting itself into itself, self-presents itself. This immanent self-presentation of the evental multiple severs any necessary originary relation to the count-as-one corresponding to the structures within the historical situation. The event is effectively self-caused or, to be precise, its causation cannot be wholly linked to the environment into which it erupts. The event's count is called ultra-one because it is not linked to the oneness that is the count of the situation. The evental multiples takes the presentative structure of a Droste-effect whose foundation lies nowhere within the site or the situation.

Non-ontology of the event

Badiou's formulation of the matheme $e_x = X \cup \{e_x\}$ describes two essential structural features of the event:

1. the event contains non-presented elements that together constitute a single presented element X
2. the event contains itself.

The first feature ensures that the event is sutured to the interiority of the situation via its relation to the presented X, while also being sutured to the exteriority by collecting the unpresented elements from X. Since e_x contains unpresented elements, it cannot be a subset and, thus, its existence is not guaranteed by the rule of the state. The second feature ensures that the event lies properly outside of ontology, as That-Which-Is-Not-Being-qua-Being, by contradicting the Axiom of Foundation. As Badiou writes:

> [O]ntology has nothing to say about the event. Or, to be more precise, ontology demonstrates that the event is not, in the sense in which it is a theorem of ontology that all self belonging contradicts a fundamental Idea of the multiple, the Idea which prescribes the foundational finitude of origin for all presentation. The axiom of foundation de-limits [B]eing by the prohibition of the event. It thus brings forth That-Which-Is-Not-[B]eing-qua-[B]eing as a point of impossibility of the discourse on [B]eing-qua-[B]eing . . . (*BE*, 190)

Since it belongs to itself, the matheme of the event contradicts Foundation. The singleton $\{e_x\}$ lacks a foundational element because its sole member e_x intersects itself. So the existence of the event is inconsistent with ontology as the *ZFC* axioms directly disprove the statement that e_x exists. The multiplicity of the event lies not in the realm of consistent presentation but in the realm of Being itself, in the void that is the inconsistency of the situation. Without existing, without presenting itself, the event *is*. It exists in its own disappearance, in the evanescence of its eventality.

Lacking any consistent multiplicity, the event thereby functions in the situation not in the specific presentation of a non-presented matheme, but as its name, a proper and empty name without referent. This name is imposed via the act of intervention that detains the event and decides on its existence. The situation is forced to confess its own void, its own inconsistency and 'thereby let forth, from inconsistent [B]eing and the interrupted count, the incandescent non-being of an existence' (*BE*, 183). The event is named through the moment of intervention. The event exists in the situation only through its name.

Undecidability of the event

The inconsistency of the event's existence with respect to ontology must not be confused with another statement made by Badiou, that the existence of an event is *undecidable qua event*. The extensional basis of the evental multiple cannot be determined using the resources available to the situation. The requirement that the event belong to itself leads any answer about its existence to be unverifiable with respect to the domestic standpoint within the situation itself. As Badiou explains:

> If one wishes to verify that the event is presented, there remains the other element of the event, which is the signifier of the event itself, e_x. The basis of this undecidability is thus evident: it is due to the circularity of the question. In order to verify whether an event is presented in a situation, it is first necessary to verify whether it is presented as an element of itself. (*BE*, 181)

The proposition in question is whether such and such event has occurred, whether a specific e_x exists, or had existed, qua taking a self-belonging multiplicity. To check whether e_x is evental (if it exists), it is necessary that one check whether $e_x \in e_x$, which leads to an infinite regress, because this checking can only be implemented if the first e_x, at the left-hand side of the belonging relation, was already presented. One can only determine whether a supposed element belongs to a set if the element already exists in the first place.

So the existence of any particular event, qua event, cannot be determined by the situation. The brink of a decision unfolds, and we examine the two possibilities:

Possibility (I) The event does not exist. This makes the evental multiple equal to its site, at least from the immanent viewpoint of the historical situation itself. Nothing has taken place within the situation, save for the place itself. When the event adds itself to the situation that does not already present it, the constituents of what appears are unrecognizable since it contains only unpresented elements - itself and the elements of the site. As a result, the event is void with respect to the situation. Its specific address is towards the void itself.

Possibility (II) The event exists. We know the evental multiple is singular since it contains the unpresented multiples of its site *X*. So the event is not counted by the state. But the evental multiple is not totally singular since the remaining element, namely itself, belongs to the situation. The event blocks its own total singularization. The event cannot be equal to its site because of the excess which is itself, which we know cannot be an element of the site because it is presented in the situation, unlike all the other elements of the site. The event is separated from the foundational multiple which is its site. It is separated from the void by itself. Being separated from the void by itself is what Badiou calls being 'ultra-one' because the sole and unique term that guarantees it is not at the edge of the void is the-one-that-it-is. So belonging to the situation comes down to saying that the event is conceptually distinguished from its site by the interposition of itself between the void and itself. This interposition is called the ultra-one as it is counted as one twice, first as a presented multiple and second as a multiple within itself.

In the first possibility, the event, being equal to its site, directly evokes the void that it names. In the second possibility, the event interposes itself between the void and itself. In both cases, the event, with its relationship to the void, deploys the Being of the non-Being of the situation, the unpresented elements of site *X*.

In many ways, the undecidability of the event's existence is the first question about the event. Is there such a thing as a rupture? Is any event not unlike any other or is one permitted – and under what authority, by whose consent, and verifiable by what knowledge? – to use the operation of singularity to speak of *the Event*? And, thus, to speak of the whole thematic that it implies, the thematic of the rupture, the turn, the break, the new, the revolution, the encounter and so on? We are confronted with a particular dichotomy, an opposition between the possibilities that we mentioned earlier. Is the unfolding as such – of time, space, Being, language, relation, hermeneutics, phenomena, logic and so on – always out of a pre-prepared continuation, the simple analytic development of an internal momentum, or could there be these 'leaps', these discontinuous jumps that are a response to what is otherwise than itself? Does the event happen from or to the unfolding? Can all events be completely attributed to an inflected involution, a fold within the unfolding, or is it a tear that can be traced to an exposure to a separate exteriority that comes from without – the e-vent, the ex-venir – instead of in-vented from within? Does the event happen all the time? Can an extension be conducted such that the event becomes a mere coordinate in some renormalized space-time and becomes simply what happens, what is happening everywhere all the time in that space-time (Possibility I)? Or it is possible for there to be a becoming where becoming itself is at stake (Possibility II)? Badiou's analysis proposes that the answer to this question can only take the form of a pure decision of pure subjectivity. Ontology – the question of what is, at least when formalized under the *ZFC* axiomatic – cannot determine anything about the Being of the event. At the brink of a decision, when Being and knowledge cannot provide any support, the task is to choose and become militant about that choice.

Nomination and recognition of the event as event

In its evanescence, in its appearance as disappearance, the undecidable occurrence of an event is not sufficient to affect the situation of ontology, for what must proceed from it is an intervention. The event must be confirmed by the birth of a subject that recognizes and nominates it. The subject begins with the decision that such and such event has occurred and that it takes the form of the evental matheme. The subject thereby nominates the event. This is a procedure, also independent from ontology, that recognizes the existence of the event qua event and gives it a name. It is impossible to separate recognition from nomination as each pre-supposes the other. One can only recognize what has been named, and vice versa. Now recognition involves two simultaneous but contradictory steps:

1. designating the form of the supposed multiple as evental and self-belonging – and, thus, as one whose belonging to the situation is undecidable
2. nullifying this undecidability by deciding the evental multiple as belonging to the situation.

The self-nullification of intervention constitutes the various aporias of recognizing the event as event. The event can only be recognized as such if its belonging to the situation is undecidable. To recognize is to possibilize the impossible.

In the naming of the event as event, what is the ontological basis for the event's nominal suture to some signifier? It cannot come from what is presented in the situation: all the names have already been used up and the effect of a homonymy would efface the unpresentation within the evental multiple. So the situation cannot structure the intervention. Moreover, the evental site itself cannot name the event because it is already a term in the situation, despite being at the edge of the void. As a result, the only possibility left would be the unpresented elements within the evental site. The basis for the nomination of the event is what the situation unpresents, not what it presents. Intervention makes a nominal suture out of some element in the site, a naming power that qualifies the event. This unpresented element is precisely the small x that indexes the evental multiple $e_x = \{x \in X, e_x\}$. This explains Badiou's idiosyncratic quasi-mathematical formulation of this matheme. 'The name of the event is drawn from the void at the edge of which stands the intra-situational presentation of its site' (*BE*, 204).

The unpresented element x is thus both an element of the event and the nominal suture, which, in the act of intervention, indexes the event to a name. The fact that x is void with respect to the situation makes its indexing to the event radically arbitrary and anonymous. It cannot even be differentiated from the other unpresented elements from the site. All that can be properly said about x is that it belongs to the site. It is, strictly speaking, not really the name of the event, but its namelessness. 'The event has the nameless as its name: it is with regard to everything that happens that one can only say what it is by referring it to its unknown Soldier' (*BE*, 205). The illegality of nomination relates back to its nonconformity to the situation's regime of representation. Intervention extracts the supernumerary signifier from the void bordering the site, thereby interrupting the law of the state. The nomination of the event arises from the void and touches upon the exterior inconsistency of the situation. With this nominal suture, it becomes possible to speak of the event's proper name, which is the singleton $\{e_x\}$. Even though it is not exactly identical to the event, is represents the event with respect to the situation.

The event of intervention proceeds after the eruption of the evental multiple. Intervention cannot erupt on its own as a total beginning or an absolute commencement. The subject does not authorize itself on the basis of itself alone or some negative will with respect to the situation. This is why the intervention constitutes a second event proceeding from the first, which it authorizes. By 'event', we are referring to the matrix of the evental site and the evental multiple. From the viewpoint of the situation itself, the evental multiple is recognized by the name, which is the singleton $\{e_x\}$. So intervention creates the Two of the event, which is the site and the name: $\{X, \{e_x\}\}$. The only legal parts or subsets of the evental multiple, the only subsets that are present in the situation, would be the elements of this pair, the site and the singleton. However, in the Two between these two parts, there is no relation. This is because the nominal suture *x*, being unpresented with respect to the situation, has no discernible relation to the site to which it belongs. 'From the standpoint of the state, the name has no discernible relation to the site. Between the two there is nothing but the void' (*BE*, 208). The power set of the event is a heteroclite pairing, a lawless and incoherent one-multiple. The state is incapable of rationalizing the link between the site and the event as a name.

Unconstructibility of the event

Since it belongs to itself, the event is undecidable and lies external to ontology. The issue become even more complicated when the situations *M* and *S* are located within the constructible universe and satisfy the Axiom of Constructibility. Badiou writes:

> [W]ithin the constructivist vision of [B]eing . . . there is place for an event to take place. . . . Constructivism has no need to decide upon the non-being of the event, because it does not have to know anything about the latter's undecidability. Nothing requires a decision with respect to a paradoxical multiple here. It is actually of the very essence of contructivism – this is its total immanence to the situation – to conceive neither of self-belonging, nor of the supernumerary; thus it maintains the entire dialectic of the event and intervention outside thought. (*BE*, 289)

Recall that the Axiom of Constructibility directly disproves, without going through the Axiom of Foundation, any possibility of self-belonging sets. So Constructibility bars the possibility of any event. Moreover, the Axiom of Constructibility does more than the Axiom of Foundation and constructivism does more than ontology because it does not have to know anything of the event or of the event's undecidability. A decision to accept or reject the existence of the event never arises because the event is never encountered. The question of nomination never arises. The event would have already been constructible if it was discerned or encountered at all. Not only is the event undecidable in the situation, but the question of decision and of naming the event is also completely foreclosed. To encounter the demand for a decision would mean that the event was already constructible and was already named in the first place, which directly negates the Axiom of Constructibility.

So a change can occur only in the form of either (1) an intervention declaring that the event exists, or (2) a simple exploration wholly immanent to the constructible situations themselves. In the latter case, '[t]he thought of the situation evolves, because the exploration of the effects of the state brings to light previously unnoticed but linguistically controllable new connections' (*BE*, 290), an exploration that is without end because of the infinity of the situation and the infinity of language.

The legalization of intervention via choice

How can there be a matter of deciding on that which subtracts itself from ontology? Does ontology allow for such a thing as intervention, such a thing as a concept without construction, a concept whose referent lies outside of Being?

Yes, because of the Axiom of Choice that legalizes intervention. '[W]ithin ontology, the Axiom of Choice formalizes the predicates of intervention' (*BE*, 227). Intervention is legitimized by ontology (via the Axiom of Choice) and by the constructivist orientation (via the Axiom of Constructibility that proves the Axiom of Choice). The rules of ontology allow there to be intervention, which it names as choice. It achieves this by suspending the Being of choice from the one while still allowing the existential concept of choice to remain. The Being of intervention, its form-multiple, is precisely the choice function that is legalized as an existence subtracted from any explicit presentation, from any explicit count. 'The Axiom of Choice – the Idea which postulates the existence, for every multiple, of a function of choice – has to do solely with existence in general' (*BE*, 226), an existence whose affirmation is submitted solely to an intrinsic condition without any connection to the internal structure of an explicit multiple. Badiou writes:

> The undecidability of the event's belonging is a vanishing point that leaves a trace in the ontological Idea in which the intervention-[B]eing is inscribed: a trace which is precisely the unassignable or quasi-non-one character of the function of choice. In other words, the Axiom of Choice thinks the form of [B]eing of intervention devoid of any event. What it finds therein is marked by this void in the shape of the unconstructibility of the function. (*BE*, 227)

Note that the Axiom of Choice guarantees not the existence of the event but the existence of the intervention grasped in its pure Being without necessarily referring to an event. It can only guarantee the form-multiple of a choice function that does not correspond to any presented multiple. It can only prove that there are interventions.

But the role of the Axiom of Choice in Badiou's metaontology of forcing is still not obvious. The naming of the event involves recognizing that which is without an explicit and constructed one-multiple: the event. Choice also involves guaranteeing the existence of that which is without an explicit and constructed one-multiple: the choice function. But it is not clear how the definition of the choice function – which selects undetermined elements from the members of some set – relates to the definition of the evental multiple – a self-belonging set that collects unpresented elements from a site. And we should be careful here as the Axiom of Choice has, in the past, often been abused by non-mathematicians as a figure to justify some unrelated philosophical conceit, often by ignoring the precise and rigorous meaning of the word 'Choice' within this Axiom. We delineate the exact role played by the choice function in Badiou's metaontology of intervention.

My hypothesis is that the choice function connects more directly to the aleatory construction of the procedure of fidelity. The domain of the choice function is precisely the set of all dominations (dense sets) in the fundamental situation. Intervention is involved in the aleatory weaving of the generic procedure that defines itself without the support of a construction given by the situation. The generic set ♀ is woven by choosing elements from within every domination and from the outside of every correct subset. The set in question contains all the dominations within the fundamental situation *M* relating to the situation *S*. The choice function selects an element from every domination – every element in the set – and inserts it into the generic set. The choice function exists without an explicit construction within the situation. As a result, ♀ is allowed to be generic. This process of diagonalization is guaranteed, because of the Axiom of Choice, to be new so long as we are prepared to leave the constructible universe, which reduces the existence of the choice function to a constructible orientation of knowledge.

Following through with intervention

Whatever name the subject tentatively gives to truth, to the imaginary generic set, is simply a name without any discernment. Since it belongs to itself, the event is not recognized by ontology. What stands for the event in ontology is the supernumerary and supplementary letter that anticipates its naming.

Fidelity then traverses all existent knowledge and diagonalizes across all encyclopaedic determinants, beginning from the supernumerary point, which is the name of the event. As previously established, knowledge, as determined by the state, cannot know of the event, which, being supernumerary, lies outside the language of the situation. The minimal determination of connection or un-connection of multiple to the supernumerary cannot be based on the encyclopaedia. Fidelity 'is not the work of an expert: it is the work of a militant' (*BE*, 329). Fidelity corresponds to a set of procedures that legalizes the randomness of the event by gathering all the elements of the situation whose existence is connected to the event. Despite the word 'procedure', fidelity is always specific to the event and its own results. It is in no way a general capacity or a subjective quality. The lack of necessary connection between fidelity and the event implies that there can be different fidelities for the same event within the same situation.

Subjectivation is the weaving of an operator that follows from interventional nomination. The subject lies in the two-without-one of the event and fidelity: $\{e_x, f\}$. The subject supplies the linking process between, on one side, the name of e_x and, on the other side, the operator f that determines the belonging relation of ♀. The subject is precisely what maps the name of the event to the operator of faithful connection. The subject is the juncture between the event and fidelity. Fidelity originates from the name of the event and the operator of subjectivation. The question then arises of the measurable relation between the two, of how the operator emerges from the name.

There should be no relation between intervention and the operator of connection, between the name of the event and the procedure of fidelity. This is why the operator f constitutes a second event supplementing the first event of e_x. The only thing we can say is that trajectory begins at the borders of the evental site. The subject does not belong solely to the fidelity – that would lead to a spontaneist orientation that says the event belongs only to those who intervene and links fidelity solely to a general faithful displosition. It does not belong solely to the event either – that would lead to dogmatic orientation, which says that all the enquired multiples depend on the specifics of the event. Badiou writes $s \square e_x$ to say that element s is connected to the event e_x. The statement $s \square e_x$ implies that the element s will belong to generic filter ♀, and so $s(+)$ and $s \in$ ♀. Likewise, $\sim(s \square e_x)$ means that s is not connected to the event, and so $s(-)$ and $s \notin$ ♀. To be in $\overline{G}$ is to have a positive connection to the event.

As a multiple, subjectivation counts what is faithful to the event. Subjectivation also subsumes its two under the one of a proper name. The name designates the

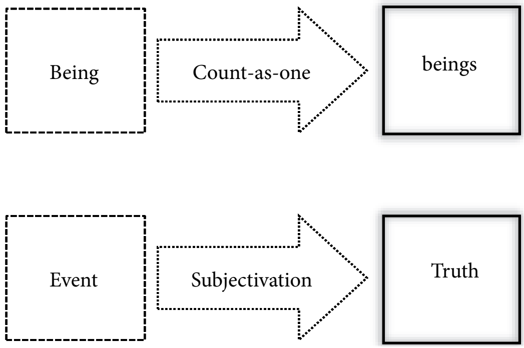

Figure 8.4 From Being to beings via the count-as-one; from the event to truth via subjectivation.

subject, which is the advent of heteroclite pairing that is the intervention and the operator, the incorporation of the event into the situation in the mode of a generic procedure. This pairing designated by the name is subtracted from any sense specified within the situation, thereby indicating the void which is the name of pure Being. Subjectivation is thus an occurrence of the void. In its opening, the generic procedure founds, at its limit, the weaving of a multiple of truth, the generic set. So subjectivation is that through which a truth is made possible by converting and orienting the event towards a truth. The proper name designates both the event and truth.

While the count-as-one lies between the inconsistency of Being and the consistency of beings, subjectivation lies between the undecidable non-Being of the event and the Being-to-come that is decided of truth (Figure 8.4). The count and the subject exist only as operations whose domains are subtracted from presentation. Each operation gives its own form of subtraction: the count draws the gap that subtracts Being from presentation of being, while the subject supports the infinite subtraction that diagonalizes through knowledge, language and the encyclopaedia. Moreover, each operation constitutes, in its subtractive moments, an essential abyss delimiting some real that lies outside the grasp of ontology while simultaneously forming its central concern. The relation between Being and being forms the unconscious of ontology, the unknown that, because of the Second Incompleteness Theorem, is undetermined with respect to the laws of Being. The relation between the event and truth is the gap that supports the basis for any undecidable decision with respect to Being-qua-Being, an undecidability whose encounter is guaranteed by the First Incompleteness Theorem so long as ontology is possible.

Epilogue

We summarize the metaontological vista given by *Being and Event*:

1. Badiou militantly decides to equate mathematics with ontology, with the science of Being-qua-Being. Metaontology, the part of philosophy directly conditioned by mathematics, constitutes the first philosophy. The four external conditions of philosophy are science, politics, art and love. Philosophy seizes the truths arising from its four conditions via a program of compossibilization.
2. Presentation takes the ontological form of the situation, the consistent multiple whose mathematical condition is the set forming the domain of a model. Consistent multiplicity is the result of the count-as-one. Prior to the count, and haunting the consistent closure of presentation as a void, is inconsistent multiplicity, the domain of Being.
3. For non-ontological situations, Being is one therein and the one is. For ontological situations, Badiou decides that Being is multiple and the one is not. *ZFC* constitutes a suitable axiomatic for a situation of ontology because it is able to formally speak of Being-qua-Being directly and without unifying it. The members of an ontological situation are the pure sets woven from the empty set, the void that is the nominal suture of Being to presentation. Every situation is embedded in a universe, the quasi-complete situation that is the name for a situation of ontology. Since V is a proper class, the complete universe of all presentations cannot exist. And it is undecidable with respect to ontology whether a complete universe exists, because of the Second Incompleteness Theorem.
4. The semantics of a situation, at least those involving predicates with one free variable, take the form-multiple of the situation's power set. The subsets are called representations and the power set is called a state, which dictates a regime of representation. The names for the elements in the situation are the singleton subsets. The semantics that involve higher-order predicates take the form-multiple of the power set to the relevant Cartesian products and powers. The quasi-complete universe of ontology does not have a

state, although it specifies the state of every presentation contained therein. Within a universe, every finite representation exists.

5. In relation to a specific situation, a term can be normal (a member and a subset), singular (a member but not a subset) or excrescent (a subset but not a member). Historical situations contain singular elements. Neutral and natural situations, which contain only normal elements, are transitive since they contain the elements of all their elements. Natural situations differ from neutral situations because they contain only normal elements that are themselves natural situations. The ordinal numbers, being pure sets that are natural situations, constitute the form-multiple of nature, though without providing a one-multiple of nature as such.
6. As implied by the Cohen-Easton Theorem, the complete identity of the representations collected by a state is generally undetermined with respect to ontology. In the constructivist-nominalist-programmatic-neoclassical orientation of thought, which corresponds ontologically to the constructible universe L, the representations are limited only to those that are constructible. Here, Being is pre-conditioned by language and every presentation must be constructible therein in terms of simpler constructible presentations. Moreover, the state of any denumerably infinite situation must, within the constructivist orientation of thought, be fixed to $\aleph_1$.

We also summarize Badiou's conceptualization of the construction of truth, the eruption of the event and the generic procedure of the subject:

1. Everything begins with an infinite historical situation S embedded within a countable, transitive and quasi-complete fundamental situation M. The situation S contains a totally singular, foundational, edge-of-the-void element X that is the evental site.
2. At the void that is the evental site X, at the inconsistent borders of the situation S, an event erupts, leaving a trace taking the inconsistent form-multiple of the matheme $e_x = X \cup \{e_x\}$.
3. The subject is born at the edges of the evental site through the decision to name this event and to recognize its existence and evental form. The nominal suture of the event e_x finds its basis in some anonymous unpresented element $x \in X$. The subject links the event to an imaginary truth whose form-multiple is the generic set ♀.
4. The subject enquires into each element of S and decides if it is connected to the name of the event. Elements connected to the name are committed

to belong to the truth ♀. The sequence through which the elements are encountered, and the faithful decision to insert them into ♀, constitutes a trajectory that is aleatory and subtracted from the dimensions of language, knowledge and the encyclopaedia of the situation *S*. The trace of the subject in *S* takes the form of this finite but continually expanding trajectory of parts to ♀.

5. The construction of ♀ is executed in tandem with the construction of the new situation, the generic extension *S*(♀). Every element in the extension is given a name that is constructed wholly within the resources of the situation *S*. Each name has, as its elements, other names that are tagged with a condition, a finite number of atomic commitments, about ♀. The referent of the name is the set collecting all the referents for its element that are tagged with conditions that are committed to belong to ♀. The veridicy of statements in *S*(♀) is determined by the truth given in ♀. The truth or falsehood of the statement can be determined by checking if they are conditioned by the corresponding condition in ♀. This determination is wholly local to the situation *S*.

It is worth reiterating the conditioning relationship between the set-theoretic mathematics and Badiou's metaontology:

1. The general concept of Being is conditioned by the anonymous set-theoretic concept of multiplicity. The form-multiple of beings, of situations, is conditioned by concrete and consistent sets. The Being of beings is forced by the retrospective excess that haunts the count. The theorem that the universe of sets *V* is a proper class forces the militant decision that the one is not and that the complete universe is not. The metaontological concept of situation is forced by the ontological concepts of set and model. The count-as-one operation is forced by the set-making operation, both of which, though entirely anonymous, are axiomatized by *ZFC*. Russell's Paradox conditions the metaontological proposition that multiplicity precedes language and the militant decision that the one is not.
2. The concept of void is forced by the empty set – or rather the emptiness within the empty set – and by the exterior inconsistency, the Being, of a set. The concepts of consistency and presentation are forced by the compositional consistency of a count and the veridical consistency, the lack of self-contradictions, within the domain of a set because of Gödel's Completeness Theorem.

3. Representation is forced by the concepts of inclusion and the subset, as well as the Axiom of Separation. The state, the regime of representation, and the semantics are forced by the power set. The forming-into-one operation, the naming operation, is forced by the operation of making a singleton from a single set. The Theorem of the Point of Excess conditions the proposition that the situation and state are separate. The Cohen-Easton Theorem and Skolem's Paradox condition the metaontological proposition that the state of every (infinite) situation is undetermined with respect to ontology.
4. Infinity is forced by the Axiom of Infinity and the infinite ordinals and cardinals. Nature is forced by the ordinal numbers, while quantity is forced by the cardinal numbers. The theorem that the universe of ordinals is a proper class conditions the proposition that nature, as a totality, does not exist.
5. The constructivist orientation of thought is forced by Gödel's Axiom of Constructibility and the hierarchical structure of the Constructible Universe *L*. This orientation limits the errant excess of the state by fixing the size to $\aleph_1$ for infinite situations, thereby implying the Continuum Hypothesis. The encyclopaedic determinant is forced by the concept of a non-generic filter (non-generic correct subset) within a constructible universe.
6. The subject is forced by the two Forcing Theorems (the second being the Law of the Subject) as well as the trajectory of constructing the generic filter, finite part by finite part, via the aleatory algorithm provided by the Rasiowa-Sikorski Lemma. The form-multiple of the event is forced by the concept of a self-belonging multiple defined in relation to a totally singular multiple (the evental site). The generic filter conditions the new domain of truth.
7. The concept of truth is conditioned by the two Forcing Theorems and the mathematical properties of being generic and being a filter. Indiscernibility is forced by the fact that the generic filter avoids every non-generic filter, each of which corresponds to a discernible predicate. In particular, the Indiscernibility Theorem conditions the indiscernibility and exteriority of truth, while the Rasiowa-Sikorski Lemma conditions the aleatory construction of truth within the immanence of a situation.
8. The concept of truth-conditions is forced by forcing conditions in some notion of forcing, some forcing poset, that satisfies the Countable Antichain Condition and the Requirement of Real Choice. In Badiou's deployment, the concept of truth-conditions is, more specifically, forced by those taking the mathematical form of finite partial functions.

The main objective of this book was to explicate the relevant mathematics and to rebuild the rudiments behind Badiou's metaontological architecture with respect to its mathematical conditioning. In the process, we hope that we have also contributed indirectly, in comparison to the existing secondary literature on the topic, a few notable things to the current state of understanding as concerns Badiou's metaphysical framework:

1. We suggest that Badiou's metaontological concept of the situation is informed by the mathematical concepts of both set and model. Through its relation to the concept of set, a situation is a consistent and presented multiple. Through its relation to the concept of model, a situation is a domain of presented multiples whose horizon of veracity is logically consistent. The semantics of the model will be the various power sets connected to the situation.
2. We emphasize the difference between situations and universes, the latter being a quasi-complete situation where all the basic rules of ontology, the *ZFC* axioms, are veridical. A situation is connected to its state, its semantics. A universe, however does not have a state but prescribes the regime of representation for every situation contained therein. Moreover, the process of forcing involves, at the outset, a situation embedded within a larger fundamental situation, which is a countable and transitive universe that is distinct from the first situation.
3. We recognize the background role of Gödel's Completeness Theorem in Badiou's thinking on the consistency of situations and states. This theorem implies that the compositional consistency of a situation, its consisting-together-as-one, is mathematically equivalent to its veridical consistency, to the absence of paradoxes in its horizon of veracity. I suggest that the Completeness Theorem can help us understand what Badiou says about the role of the state in avoiding any encounter with the inconsistent Being and the local void of the situation.
4. We recognize the background role of Gödel's Incompleteness Theorem in Badiou's thought concerning the decision that Being is multiple and that *ZFC* forms a suitable axiomatic for ontology. We can understand the proposition 'The one is not' not only because of the fact that a formal axiomatic system never directly prescribes its own semantics, but also because the specific axiomatic of ontology is undecidably incapable of prescribing its own situation. The *ZFC* axioms cannot prove it has a model. This is why it satisfies the requirement of being able to speak of Being-qua-Being without implicating particular objects, particular beings.

5. We examine how the power sets connected to a situation S relate to the interpretation of the corresponding model. The situation's power set $P(S)$ is an ontological register of all the multiples that are available as the product of separating elements from the situation with respect to some formula with one variable. I suggest that it is possible to expand on Badiou's framework by adding regimes of relation, corresponding to the power sets of Cartesian products and powers. Such power sets would also list all the multiples separated by formulas with more than one variable. For example, the power set $P(S^3)$ is the regime of ternary relations corresponding to all formulas with three variables.

The main intention of my book was not to provide a critique or an evaluation of *Being and Event*. However, in the process of our reconstruction of Badiou's metaontology and its relation to the mathematics, we were also able to pinpoint some specific gaps in the work:

1. Badiou declares that mathematics is ontology and that ontology is mathematics. But he does not expand on how he pursues the second portion of this equation, which is based on his decision that ontology is a situation and a scientific discourse. Perhaps mathematics is part of ontology, the part of ontology that can be implemented as a science. But it is not clear, at least from my reading, how every direct study of Being must be mathematical – even when we take into account that Badiou's equation must be taken as an unverifiable decision. And it is not clear how every philosophy of ontology, how every metaontology, must be conditioned solely by mathematics. The source of our confusion is, among other things, Badiou's acceptance of certain philosophical propositions about Being, particularly from Heidegger, whose origins do not appear to be directly mathematical. The question then is whether every work in the philosophy of ontology (at least those works whose ideas Badiou accepts) was unconsciously conditioned by some matheme – whether, for example, Heidegger's *Being and Time* contained some unconscious mathematical framework. If that is not the case, then we can only say that there is some part of ontology that does not belong to mathematical enquiry or, at the very least, that the excess relates to something otherwise than Being, some event.
2. In *Being and Event*, Badiou imposes a specific requirement on the situation S as a pre-condition to the process of forcing, a requirement that is not mathematically necessary. Unlike the universe M, the situation S cannot

be transitive. It must be historical and contain a non-empty foundational element that will act as the evental site. This is why the situation *S* and the universe *M* must be separate. The question is how to reconcile the specificity of this requirement with the generality in the mathematics of forcing, which need not involve historical situations at all. Moreover, as far as we can see, there is nothing in the mathematics that requires forcing to be triggered by the introduction of the name to some event, some self-belonging multiple implicating the non-empty foundational element in the situation. That is, unless mathematics has nothing to say about what activates a forcing procedure within some situation, even though the necessary exteriority of the subject can be understood in terms of some faith towards some exterior event and truth. Perhaps the philosophical justification for the introduction of an event is conditioned not by mathematics but by more empirical and, perhaps, political conditions.

3. Badiou's discussion of the relation between the concept of intervention and the Axiom of Choice is not fully detailed. In particular, he does not describe how exactly the choice function – whose existence is guaranteed by the axiom – relates to intervention. Both intervention and the choice function share the feature of involving a presentability-without-presentation. But the choice function involves a particular mathematical structure, one that chooses one element from every member of a particular set. To be sure, Badiou speaks of a concept of intervention that does not necessarily relate to the matheme of the event. But it is not clear how this general concept connects to the specific existence-without-presentation of choice.

Nevertheless, despite these gaps – which are unavoidable except possibly in an infinitely long work – there is enough in *Being and Event*, even when we remove Badiou's commanding and imposing voice, to confirm its reputation as a robust and philosophically provocative tour de force. In my opinion, the most inventive and provocative of Badiou's metaontological forcings – all of which have almost no precedent in previous philosophical literature – involve the conditioning relations between:

1. Being versus set-theoretic multiplicity
2. presentation of presentation versus the set-making operation
3. inconsistent Being versus the empty set
4. representation and the state versus subset and power set
5. nature versus ordinal numbers

6. the event versus self-belonging and total singularity
7. truth versus the generic filter
8. the subject versus the construction of the generic filter

All of these provocative forcings return to the originary forcing: Badiou's decision to equate mathematics with ontology and then to proceed from it by faithfully constructing a metaphysical schema based on the mathematical resources of modern set theory. The audacity and profundity of this equation is enough justification to call Badiou's philosophy radical and groundbreaking.

The task at hand, for anyone who wishes to follow through with the event of Badiou's equation, would be to develop on his program of compossibility by examining set theory in greater detail along with the mathematics of forcing, or even examine other mathematical fields outside of the science of sets. Perhaps one need not accept the specific proposals offered by *Being and Event*. Perhaps it is enough to be militant towards the possibility that mathematics can still contribute deeply and dramatically to the most basic philosophical and metaphysical problems. As I mentioned in Chapter 1, we are confronted with the prospect of what I hope could be the early stages of a new 'mathematical turn' initiated by Badiou's visionary recognition of a new role for mathematical thinking in philosophy. The enticing panorama of possibilities is almost endless, for many of the most revolutionary developments in mathematics have been overlooked by the philosophers for the last one hundred years. And the topic of forcing has evolved tremendously since Cohen's work. There are other alternatives for meta-mathematical foundations besides set theory, a field that has already been axiomatized in various manners other than *ZFC*. There are viable mathematical figures to think Being, multiplicity, the event, the subject and truth, other than what is forced by set theory under Badiou's metaphysical schema. Moreover, Badiou supplements his schema by looking into other mathematical fields, such as the meta-mathematical foundations to arithmetic in *Numbers and Numbers* and category theory in *Logics of Worlds*. In order to proceed with this 'mathematical turn', it would be convenient to have some methodological principles, or at least some guide, that can be gained by examining precedents in the work of other mathematical philosophers – in Badiou's *Being and Event*, for example. My hope is that, with this book, we have been provided with some of the possible means by which to carry forward this new procedure of fidelity. The question now is how many subjects are prepared to make this gamble.

Notes

Introduction

1 At the References, we list some of the texts that try to tackle the relation between the mathematics and the philosophy in *Being and Event*. These include Brassier's *Nihil Unbound* (2010) and 'Badiou's Materialist Epistemology of Mathematics' (2005); Feltham's *Alain Badiou: Live Theory* (2008); Fraser's 'The Law of the Subject' (2006); Gillespie's *The Mathematics of Novelty* (2004); Hallward's *A Subject to Truth* (2003); Livingston's *The Politics of Logic* (2012); Mount's 'The Cantorian Revolution' (2005); Norris's *Badiou's Being and Event: A Reader's Guide* (2009); Smith's 'Mathematics and the Theory of Multiplicities' (2004) and 'Badiou and Deleuze on the Ontology of Mathematics' (2003); and Tho's 'The Consistency of Inconsistency' (2008).

2 Badiou is clearly a 'tabular' thinker and it is possible to summarize each of his texts by drawing a table or diagram of the distinctions, systems, lists and inter-relationships that it mentions. He takes great delight in schematizations and admires, for its own sake, the provocatively elegant symmetry of a succinct philosophical expression. For example, he is pleased by the fact that the Dictionary in *Being of Event* begins with the word 'absolute' and finishes with 'void' (*BE*, 498). A single sentence in *Being and Event* can sometimes be read as organizing and compressing a whole catalogue of propositions, implications and paradoxes. We often feel like the imaginary 'readers' of a work by Jorge Luis Borges, readers who have been given only a structured précis to a lengthier manuscript. Instead of justifying his propositions with comprehensive elucidations, Badiou often only supplies distilled demonstrations whose explicatory principle is akin to reproducing the austere elegance of a short mathematical proof. As many mathematicians know, a succinct and elegant proof, which often fails to motivate the reader towards grasping the deeper truth therein, is sometimes uninformative when it comes to elucidating the more profound validity of the theorem in question. Badiou supplies 'proofs' to his 'theorems', but conceals the probably arduous experience of how he arrived at them. We are reminded of the mathematician Niels Abel's description of the writing style of the great mathematician Carl Gauss: '[He] is like the fox, who effaces his tracks in the sand with his tail' (Simmons 1992, 177).

3 Taleb (2007) in particular characterizes the event or the 'black swan' as a surprise with a major impact and whose occurrence is only rationalized retrospectively, as if it could have been expected by what erstwhile was unaccounted data.

4 Peter Drucker writes that 'the entrepreneur always searches for change, responds to it and exploits it as an opportunity' (1985, xiv). The entrepreneur's pursuit of innovation, as analysed in economic and management theory, can be compared with the philosopher's pursuit of the event, as analysed by Badiou. Besides the entrepreneur and the philosopher, other neophiles or 'seekers of the new' might include the reporter, the journalist, and what today's culture calls the 'hipster'.

Chapter 1

1 However, this definition of ontology needs to be qualified with respect to Badiou's specific methodological orientation in *Being and Event*, which differs from his later work. The obvious aporia is that ontology identifies itself partially through its negative connection, its 'relation without relation', to the ontic. At the heart of the austere and sovereign question of Being-*qua-Being* – what creates, makes and regulates it – is the question of Being-*qua-beings*. Any ontological account must simultaneously navigate within some operative relationship with what is otherwise than itself. Within the unavoidable dialectical arbitration between beings and Being, ontology can commence only by choosing to begin with one side and then by enquiring about its relation with the other side. Philosophy secures the site of ontological truths through investigations that start with either the problematic of existence or the problematic of essence. The two beginnings correspond to different styles of philosophical investigations into Being: the empiricist versus essentialist. A philosophy that starts with the phenomenal logic of appearances identifies ontology with onto-*logy*, while one that begins with Being as such, subtracted from the particularity of presentation, understands ontology as *onto*-logy. Badiou's later work, *Logics of Worlds*, privileges the former methodology, while *Being and Event* favours the latter style of ontological inquiry. Philosophy must then concern itself with the differential relation between the realm of concrete presentation and the realm of pure Being that is the 'there is' of presentation. It makes no sense to speak of Being without speaking of its relation to beings, and it makes no sense to speak of the ontological question of beings without looking into Being in general. This question of ontological difference is tackled in *Being and Event* by thinking what is called the operation of the 'count-as-one', which corresponds in mathematics to the anonymous operation of 'set-making'.

2 In an interview with Lauren Sedofsky in the *Artforum* magazine (October 1996) Badiou claims to have once tried to study quantum mechanics, though without much success because of his lack of advanced background knowledge on the topic. But he admits that many scientific fields, particularly in contemporary physics, are of great philosophical interest. Nevertheless, Badiou rejects biology as a 'wild empiricism disguised as science' (*TW*, 17). 'Biology, for time being, is . . . nothing but a collection of findings, an apparatus that enables, for example, the blind statistical testing of the effects of a given molecule or the role of a given protein in a particular physiological sequence' (*TA*, 235–6).

3 The oft-used candidates, for various reasons, seem to be Gödel's Incompleteness Theorems, Turing's Halting Problem, and the discovery of non-Euclidean geometry in the nineteenth century. These results have often been applied to justify unsophisticated claims within unrelated contexts. The mathematician Torkel Franzén bemoaned the fact that the Incompleteness Theorems had been abused to justify claims about the inconsistency or incompleteness of, among others, quantum mechanics, the philosophy of Ayn Rand, evolutionary biology and the legal system (2006).

4 Nevertheless, the relation between condition and analogy is more complex as it can be argued that the metaphor leaves its sedimented trace in every mode of language, knowledge and thought.

5 In his later book, *Logics of Worlds*, Badiou will nevertheless come to employ category theory as another meta-mathematical foundation, supplementing the truth procedure constructed by *Being and Event*.

6 It is not presently known whether Paul Cohen had read or was aware of Badiou's philosophical appropriation of his mathematics. But he would agree on the essential secularism of mathematics and the proximity between 'theological' concerns and the question of foundations. As Cohen writes in his essay 'The Discovery of Forcing':

> The early years of the twentieth century were marked by a good deal of polemics among prominent mathematicians about the foundations of mathematics. These were greatly concerned with methods of proof, and particular formalizations of mathematics. It seemed that various people thought that this was a matter of great interest, to show how various branches of conventional mathematics could be reduced to particular formal systems, or to investigate the limitations of certain methods of reasoning. All this was to illustrate, or convince one of, the correctness of a particular philosophical viewpoint. Thus, Russell and Whitehead, probably influenced by what appeared to be the very real threat of contradictions, developed painstakingly in their very long work, *Principa Mathematica*, a theory of 'types' and then did much of basic mathematics in their particular formal system. The result

is of course totally unreadable, and in my opinion, of very little interest. Similarly, I think most mathematicians, as distinct from philosophers, will not find much interest in the various polemical publications of even prominent mathematicians. My personal opinion is that this is a kind of 'religious debate'. (2002, 1080)

7 We speculate that they are also there to pre-emptively avoid 'Sokalist' claims that he is an 'intellectual imposture' who does not really understand the mathematics he uses.
8 It is worth noting that even professional mathematicians and mathematical scientists, some of whom like to project this bizarre superiority complex in front of their non-mathematical comrades, often hide a secret phobia for learning new and difficult mathematics.

Chapter 2

1 We might know that the words 'array', 'collection', 'class', 'category', 'group' and 'universe' each have precise definitions in mathematics, different from the mathematical notion of set. But let us ignore that for a moment. That the objects gathered already pre-exist is a requirement that appears trivial, but will prove to be crucial to the internal consistency of set theory.
2 For simplicity, we will ignore the difference between ordered sets and what are called 'tuples'.
3 Note that the standard mathematical terms in English are 'normal' and 'abnormal' for 'ordinary' and 'evental', respectively. We will stick to Badiou's usage, mainly because the term 'normal', which has a different meaning in Badiou's metaontology, shall later be applied with reference to a different predicate, for multiples that are both 'presented' and 'represented'. But we must bear in mind the likelihood of confusion among English readers who are already used to the standard mathematical terminology.
4 However, there are certain restrictions to this reduction if one is limited to a first-order language.

Chapter 3

1 We should mention that the words 'condition', 'state' and 'structure' have more precise meanings in Badiou's metaontology. But let us ignore that for the moment.
2 Badiou often uses the words 'multiple' [*multiple*] and 'multiplicity' [*multiplicité*] interchangeably. This is apparent when he juxtaposes the two words in the

definition he provides in the dictionary at the end of *Being and Event* (*BE*, 514). Depending on the context, the latter can also mean the substantivation of the former, i.e. the question concerning the quiddity and the multipleness of being-multiple. Since the question of Being is, for Badiou, essentially linked to the question of multiplicity, the form-multiple of a particular entity forms a means to accessing the Being of that entity. This separation between the multiple and its multiplicity attest to the subtractive orientation of Badiou's philosophy, which is unlike, for example, Deleuze's more materialist and univocal orientation.

3 Badiou has commented extensively on this choice in his essay "One, Multiple, Multiplicities" (*TW*, 68–82).

4 Once we understand that the *ZFC* axioms constitute the background rules to ontology, Badiou's use of the qualifier 'quasi-complete' should make sense in lieu of Gödel's Incompleteness Theorems.

5 In usual mathematical parlance, the words 'operation' and 'function' are sometimes synonymous.

Chapter 4

1 The term 'representation' leads to an obvious political metaphor, one that Badiou pursues in Meditation 9 when he explores historico-social situations.

Chapter 5

1 In regard to discussing in more detail the cardinals, ordinals and the foundation behind the idea of number, Badiou will later compose a separate philosophical book, *Number and Numbers* [*Le nombre et les nombres*] (1990).

2 Note that the use of the term normal is different from its usage in the standard set-theoretic terminology as non-self-belonging sets.

3 In the standard mathematics of ordinals and cardinals, the predicate 'singular' is also used to describe a certain class of ordinal and cardinal numbers. This class does not appear to play any conditioning role in the metaontology given in *Being and Event*, although it is mentioned in the book's appendices.

Chapter 7

1 For the Boolean-valued model understanding of forcing, please refer to Jech (2006). For Kripkean forcing, please refer to Goldblatt (2003). For an overview

of sheaf-theoretic forcing, please refer to Mac Lane and Moerdijk (1992). For a specific discussion on sheaves and the Continuum Hypothesis, please refer to Tierney (1972).

2 The connections between Badiou's and Kripke's philosophies of forcing are further explored by Z. L. Fraser in his paper 'The Law of the Subject: Alain Badiou, Luitzen Brouwer and the Kripkean Analyses of Forcing and the Heyting Calculus' (2006).

Chapter 8

1 It may be of some note that the Venus symbol, which is Badiou's symbolic mark for the generic filter within his metaontology, looks like a virus.

2 Kripke formulated his semantics in a series of papers between 1959 and 1965. For an overview of this development, see Goldblatt (2003).

3 In Meditation 27, Badiou calls his subject-supported understanding of forcing as the 'fourth orientation' of thought, which he distinguishes from the simpler understanding, the 'second orientation' (*BE*, 284–5). The 'first orientation' corresponds to the nominalism of the Axiom of Constructability, while the 'third orientation' corresponds to transcendentalism.

4 Such a route has already been taken on a more prominent level by Gilles Deleuze in *Difference and Repetition* (1994) with his study of differential calculus, and also by his more mathematically-minded readers such as Manuel DeLanda in *Intensive Science and Virtual Philosophy* (2002) with his study into complex dynamics and differential geometry.

5 Note that the word 'diagonalization' is sometimes used to name an entirely different technique by Cantor that proves the countability of all the rational numbers $\mathbb{Q}$.

6 This separation between the site and the event will be substantially revised by Badiou in *Logics of Worlds.*

7 The italicized emphasis is mine as it was not clear in the previous definition whether the event counts some or every unpresented element from the site.

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